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AN

ELEMENTARY TREATISE

ON

MECHANICS:

DESIGNED FOR THE USE OF STUDENTS IN THE UNIVERSITY.

By W. WHEWELL, M.A. F.R.S. M.G.S.

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THE FIFTH EDITION,

WITH CONSIDERABLE IMPROVEMENTS AND ADDITIONS.

Αναγκαῖον ἀγνοουμένης τῆς κινήσεως ἀγνοεῖσθαι καὶ τὴν φύσιν.

ARISTOT.

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PREFACE

TO THE FIFTH EDITION.

In preparing this edition for the press, I have made new endeavours to render the work fit for the circulation which it has obtained. I will briefly notice the principal instances of these.

I have added several Articles (85, &c.) upon the Theory of Arches. The theory of the equilibrated arch, which I had introduced in the earliest editions of the work, I rejected in subsequent editions, since, in the way in which it was then treated, it was quite inapplicable in practice. But though the theoretical arch of equilibration would not be of any practical value on the grounds formerly adopted, it is nevertheless, in fact, an important subject of consideration to the engineer. I have, in the Articles just referred to, shown the reason of this; and have, I hope, thus placed in a clear point of view the office which the resistance of the archstones, and the cohesion of the rest of the structure, respectively discharge.

Having in the Fourth Edition introduced several propositions concerning locomotive steam-engines, I have now borrowed from M. De Pambour's excellent work on that subject a new problem, highly important in its bearing on practice; namely, to determine the velocity of such an engine by means of that condition on which it mainly depends, the evaporating power of the boiler. Although such motions have not been so completely reduced to their elements as to enable us to calculate with precision the motion of steam-carriages under given conditions, it cannot be doubted that this investigation has put the problem in a shape in which we are more likely than we have hitherto been to arrive at that desirable result.

These additions are made with a view of giving the work that practical character, the absence of which in the common books on Mechanics has long been a serious evil. The want of such information, I have already said*, in our theoretical Treatises, produces an estrangement between the theoretical and practical cultivators of Mechanics which is prejudicial to both; it reduces the theory to a barren exercise of the intellect, and renders practical men careless of the philosophical correctness of their views.

I may add, that the introduction of these practical subjects into an Elementary Treatise on Mechanics is not without its value, even when the subject is considered mainly as an instrument for exercising and disciplining the mind. In the attempts to master these practical problems, we are called upon for a kind of intellectual exertion different from that which belongs to the usual course of mathematical reasoning. Instead of starting with given principles and tracing their consequences, we have to select out of several principles those which have the most important influence on the result; instead of finding our abstractions ready made, as *accelerating force*, *moving force*, and the like, we have to form new abstractions, as *mechanical power*, *work done*, such as may group together the effects which we wish to consider; instead of insisting upon rigorous precision of reasoning from the first, we must be content if we can approach to it

* Preface to Fourth Edition.

after several concessions. And thus the treatment of practical questions in Mechanics, though not so perfect an example of exact logic as the pure theory, is more of the nature of the exercise of the reason on common occasions; and may provide a discipline of a different kind from that which the study of pure Mathematics affords.

I have also introduced into the present volume several of the simpler propositions which respect the doctrine of the rotatory motion of a rigid body, transferring them from my Treatise on Dynamics. There can be no doubt that the principles which govern such motions are really the same as those which regulate motions of translation. To exclude these principles from an elementary treatise, looks like evidence of a disposition to ascribe to them an independent basis; and indeed, I believe many students have thus been led to suppose that "D'Alembert's principle" contains some peculiar law of motion or method of mechanical reasoning, in addition to the other foundations of Mechanics. By including this extension of the third law of motion, (for such it is) and its simpler applications, in this *Elementary* Treatise, the distinction between this and the portions of Dynamics which are to be afterwards studied, is made to consist solely in the simplicity of the pure mathematical processes which are here introduced; the Differential Calculus being excluded from this work, with the exception of one or two problems near the end. And thus the present volume contains all the *Principles* of Mechanics; and there is left for the succeeding volumes on Dynamics, only the application of these principles by means of the methods of the higher Mathematics.

The value of the science of Mechanics as a portion of academic study, depends principally on its being a good example both of physical and mathematical reasoning. If the science can have this character secured to it, it tends not only to cultivate logical habits of thought, as do all branches of Mathematics, but to show that the rigour of necessary reasoning is not confined to the domain of space and number. But such a beneficial influence cannot flow from this study, except the science be purified from all fallacies of reasoning and indistinctness of principle; and this has been one of the points at which I have laboured in the present as well as in former editions.

On this account I have always insisted earnestly upon the distinction of Statics and Dynamics, the doctrines of equilibrium and of motion. These two branches of the subject rest upon different fundamental principles; and the mixture of the two has been a fertile source of confused thought and vicious reasoning. It has given rise to many false or unphilosophical steps in mechanical treatises; as for instance, the attempt to prove the law of the composition of pressures by the consideration of motion; the attempt to prove the third law of motion by *defining* momentum gained and lost to be action and re-action; and the like. I trust that this error is now very generally recognised and avoided by the students of the subject.

Some persons appear to doubt* whether there are, in the physical sciences, other grounds of necessary truth than the intuitions of space and time. We might demand of such persons whether the properties of the pressures which balance each other on the lever, as proved by Archimedes, be not necessary truths? whether our conceptions of pressures, and the properties of pressures, are modifications of

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* *Edinburgh Review*, Note to No. 127, p. 271.

our conceptions of space and time? and if they are not, whether necessary truths concerning pressures must not have some other ground than the Axioms of Geometry and Number? We might ask them whether we do not, in fact, in works like this, show that there are such other grounds, by actually enunciating them? whether the Axiom, that the pressure on the fulcrum is equal to the sum of the weights, be not self-evident, and therefore necessary?

If it be said* that the establishment of such propositions as this "requires nothing but experience and the logical analysis of thought," we cannot help replying, that such a remark seems to betray confusion of thought and ignorance of the subject. For it would appear as if the author denied the character of necessary truth to such principles because they depend *only* on experience and analysis; and that if, besides these, they depended upon some additional grounds, he would allow them to be necessary. Again, it is clear that, in fact, such propositions do not depend at all upon experience; for, as has elsewhere been urged†, "Who supposes that Archimedes thought it necessary to verify this result by actual trial? Or if he had done so, by what more evident principle could he have tested the quality of the weights?" And if such propositions depend upon logical analysis only, how can they be otherwise than necessary? Does the objector hold that truths which resolve themselves into logical analysis, are empirical truths?

I conceive, therefore, that the cultivation of such a subject as this may be of great use both to the Students of this University and to other persons, not only in familiarizing them with the character of necessary truths, and the

* Ib. p. 275. † *Thoughts on the Study of Mathematics, &c.* p. 33.

processes of reasoning by which a system of such truths is built up; but also by shewing that such truths are not confined to the domain of space and number merely.

We must undoubtedly allow, or rather we must urge it as an additional recommendation of this study for the discipline of the mind, that the evidence of mechanical truths cannot be seen without a clear possession of the fundamental mechanical ideas; for example, the ideas of pressure, and of action and re-action. But in this respect the necessary truths of Mechanics do not differ from those of Geometry. The Propositions of Euclid cannot be seen to be necessarily true, except by persons who have distinct and steady ideas of the relations of space. Those properties of figures might be known, and in fact have been known, as matters of experience and tradition, by many persons who did not perceive them to be true *à priori*. But this is not enough for our purpose: it is only when the fundamental conceptions of space are clearly and distinctly developed, that a right apprehension of the nature of the evidence of geometrical truths can be attained. And just the same is the case in Mechanics. A person who rightly understands the axioms of Archimedes will see that they are not only true, but self-evident and necessarily true; and thus he will perceive that the whole structure of Statics, being built upon these axioms by the infallible operations of mathematical logic, is, no less than Geometry, a system of necessary truths.

If it be said that we cannot possess the ideas of pressure and mechanical action without the use of our senses, and that this is experience; it is sufficient to reply that the same may be said of the ideas of relations in space; and that thus Geometry depends upon experience in this sense, no less than Mechanics. But the distinction of necessary and empirical truth does not refer to experience *in this sense*, as I need not now stop to show.

I venture to repeat, therefore, that the study of the branch of Mathematics which is the subject of the present volume, is fitted to be a very useful instrument of education, not only as inuring the student to the salutary forms and habits of the strictest reasoning, but also as teaching him to extend to other subjects than position and number, that distinctness of ideas from which the evidence of his reasoning must spring. Till some other subjects, equally definite, are pointed out, by which the same result may be hoped for, we cannot cease to think the study of pure and mixed Mathematics deserving of special encouragement as a means of mental cultivation.

With reference to another assertion, made on the same occasion as those to which I have referred, I will merely remark, that the charge that the University of Cambridge bestows not only a special but a paramount and exclusive encouragement on these sciences* is not only unfounded, but is inexcusably so, because it is impossible to refer to any record of the prizes which the University bestows, without seeing that there is a much greater number offered and given in other subjects than in Mathematics.

TRINITY COLLEGE,
April 25, 1836.

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* *Edinburgh Review*, No. 127, p. 274.

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DYNAMICS.

AN

ELEMENTARY TREATISE

ON

MECHANICS.

INTRODUCTION.

1. *Mechanics is the science which treats of the laws of the motion and rest of bodies.*

In the science of Mechanics, as in every other branch of Natural Philosophy, we assume that the material world is governed by constant and determinate laws; and our object is to discover these laws, and to trace their consequences.

We shall, for example, have to investigate such subjects as the following;—the motion of a stone when it falls to the earth; the velocity with which it moves, and the path which it describes when thrown in any direction;—the mode in which a heavy body may be supported by any instrument, simple or complicated;—the motion of any machine by which weights are raised or removed;—the requisite form and adjustment, or the possible strength and stability, of any material structure;—and in general, any case in which bodies are pulled, or pushed, or struck, or stopped, or supported by other bodies in contact with them; or in which they are attracted or repelled by bodies at a distance.

Any cause which moves or tends to move a body, or which changes or tends to change its motion, is called Force.

The appearances and occurrences of the material world suggest to us the conception of *Motion*. Moreover we find that we can often, by our own volition and exertion, influence the motions of bodies; we perceive that they appear to influence each other's motions; and we thus obtain the conception of

Force. Motions, and change or prevention of motion, being considered as *effects*, and referred to their immediate *causes*, Force is the general term applied to such immediate cause.

Thus, when a man supports a stone in his hand, his hand is said to exert force upon the stone: and in the same manner if he moves a machine by turning a winch, he is said to exert force on the winch, and, by this, on the machine. If the machine be moved by the weight of a heavy body, this heavy body is said to exert force. When a stone falls, it is said to be moved by the force of gravity, or of the earth's attraction.

2. *Body or Matter is any thing extended and possessing the power of resisting the action of force.*

In the conception of force exerted, there is involved the notion of a certain power of resistance, residing in the object on which the force is exerted. This power of resistance shews itself by the object excluding other bodies from the space it occupies, by its transmitting force to other bodies, by its requiring a larger force to produce a quicker motion, and in other ways.

In this manner the solid bodies which are treated of in Mechanics differ from the solids treated of in Geometry. In the last mentioned science we conceive figures to possess extension only, without tangible solidity; they are mere modifications and limitations of our notion of space; they occupy space without excluding other figures from the same space; they have no material substance or mechanical attributes. In Mechanics we consider bodies as they really exist; not only as extended, but as impenetrable, stiff or flexible, inert, heavy.

In the science of Mechanics, as in all other sciences, our object is to establish general speculative truths; and we modify and refine our usual conceptions so that they may answer this purpose. The ideas which we obtain in the common practical employments of our faculties, are the original materials of our knowledge; but, in their familiar form, they are too vague and too partial to be the immediate objects of exact science. We must define and fix, purify and abstract, extend and generalise them. In this manner they may become capable of having their nature expressed by means of definitions, axioms and laws, and of being precisely measured; and they may then lead to scientific propositions.

This rectification of the original loose notion of force is requisite as the first step in our knowledge of Mechanics. Force must be conceived as a measurable quantity, and the truth of certain fundamental principles respecting it must be clearly seen before we can proceed in our reasonings.

The following considerations are to be attended to for this purpose.

3. *Forces may produce either rest or motion in bodies.*

It has already been stated that the science of which we have to treat includes the laws of the *rest* and *motion* of bodies. A single force will necessarily produce motion, but two or more forces may be so combined as to destroy each other's effects, and to produce rest. Thus if two persons pull with equal strength at two ends of a straight rope, as in fig. 3, or of a rope which goes round a peg, as in fig. 4; or if two weights, as P, Q in fig. 7, balance each other on a prop C; we have two forces producing rest. Again, if a person standing on the bank of a canal, as at P, fig. 1, pull a boat B which is in the water, by means of a rope BP, he will cause it to move in the direction BP; but if there be other persons, as at Q, R, also pulling the boat in the directions BQ, BR, it may happen, by properly adjusting the directions of the ropes and the strength exerted, that the boat shall remain at rest by the united action of the three forces.

Force is originally received as the cause of motion, and when it does not produce motion, this happens by its being counteracted by an *opposite* force or forces. In this case the forces are said to *balance* each other, or to be in *equilibrium*. They are also said to *destroy* each other.

4. Two opposite forces which thus balance or destroy each other are *equal* forces. Thus the forces Q, R, in fig. 3 or 4, (supposing the weight P removed,) are evidently equal. Each may be considered as measuring the other, since it exactly destroys its effects.

Forces are capable of *addition:* thus two men, pulling one way, exert a force which is the sum of their separate forces. It would require a man as strong as both together to resist them by pulling the opposite way.

Forces are capable of being *compounded:* thus the two forces which act in the directions BQ, BR, fig. 1, are compounded, and produce the same effect as a single force in the direction BS. For they are kept in equilibrium by a force in the direction BP, just as a single force in the direction BS would be.

When two or more forces are thus compounded into one, the one is said to be *equivalent* to the component forces. The force in BS is equivalent to the forces in BQ and BR.

The above principles lead to a measure of force; but it is proper for this purpose to consider particularly a kind of force of frequent occurrence in mechanical reasonings, namely that which arises from the *weight* of bodies.

5. All bodies within our observation fall or tend to fall to the earth; and the force which they exert in consequence of this tendency is called their *Weight.*

Weight is considered with reference to its mechanical effects, and two bodies which produce the same effect are said to have *equal weights.* Thus let MA, fig. 2, be a steel spring in its natural position, and let a mass of lead P bend it into the position MB; then a mass of iron Q, which, suspended in the same manner, bends it into the same position MB, has the same weight as P. If P be one pound, Q is one pound.

If we apply two bodies at once, in the same manner, to produce the same kind of mechanical effect by their weight, we *add* the forces arising from their weight. Thus if in fig. 2, P be one pound, and Q be one pound, the two together are *two pounds.* Let MC be the position into which the spring is bent by P and Q together, or by two such as P; then any weight which bends it into the position MC is called two pounds; and, in the same manner, if three weights such as P, bend it into the position MD, any weight which bends it to D is called three pounds; and so on for any number.

If instead of the bending of a spring we had taken any other mechanical effect, the mode of explaining the way in which weight is measured would have been the same.

6. *The weight of the whole of any body is the sum of the weights of its parts.*

It has appeared, in what has been said, that by joining bodies together we add their weights; and thus weight is a property which belongs to a body as consisting of parts. It appears, by closer examination, that the weight depends on nothing but the number of parts: their arrangement, or other circumstances, do not affect the weight of the whole. A basket of stones is of the same weight, however the stones are shaken into new positions. The same lump of material is of the same weight, into whatever new form it be moulded. If $2\frac{1}{2}$ cubic inches of lead be one pound, 25 cubic inches of lead will be 10 pounds, whatever be its shape.

Matter is originally apprehended, as we have said, by its *resistance* to force; but its properties are also manifested by any of its mechanical effects. The *Quantity of Matter* is considered to be the *same* when the mechanical effect is the same: and in order to measure a quantity of matter, we must take some property such that the amount of the property in the whole mass is the same as the sum of the amounts in the several parts. Since weight is such a property, *we may measure the quantity of matter by the weight.**

The *same* effect may be produced by masses of *different* magnitudes, when the materials are different. Thus $3\frac{1}{2}$ cubic inches of iron will produce the same effect by its weight as $2\frac{1}{2}$ cubic inches of lead. Now it is assumed that so long as the mechanical effect is the same, the *quantity of matter* is the same. Hence the quantity of matter in $3\frac{1}{2}$ cubic inches of iron is said to be the same as the quantity of matter in $2\frac{1}{2}$ cubic inches of lead; and the difference of magnitude is supposed to result from the different spaces in the two bodies which are porous, or not occupied by matter.

* The quantity of matter is sometimes said to be proportional to its *inertia*: that is, to its resistance to the communication of motion. But in practice the quantity of matter is always measured by the weight. It will be shewn in considering the third law of motion that the inertia is as the weight.

7. *A heavy body may act by means of a string, or by means of a rigid rod in the direction of its weight; and if we suppose the string or rod to have no weight, the effect of the heavy body will not be altered by the length of the string or rod.*

In fig. 3, let a weight P be supported by a hand at Q, by means of a string QP; the force exerted by the hand is equal to the weight itself: for instead of the weight P, let a hand, as at R, exert a force equal to the weight; then this force will be supported as before, because the force of the hand produces the same effect as was produced by the weight; but in this case it is clear that the forces exerted by Q and R must be equal, whatever be the length of the string, because they are similar forces and act upon it in exactly the same manner. Hence, the force exerted by Q is equal to the weight P, to which the force of R was assumed equal. And thus the force exerted at Q is the same, whatever be the length of the string.

The same reasoning would hold good, if QP were a rigid rod. In both cases the only property of the line QP which we consider is its capacity of transferring the action of the force from one point, P, to another, Q.

We may consider the string or rod to be without weight, for its weight has nothing to do with its office as transferring the force from one point to another. Its weight might be very small in fact, and in reasoning we need not consider it at all.

8. *A heavy body may act by means of a string which is capable of sliding over a fixed point,* (*as* O, *fig.* 4); *and if we suppose the string to have no weight, the heavy body will be supported by the same force, whatever be the length or direction of the string.*

Let a weight P be supported by a hand at Q, acting over a fixed point O; take $OR = OQ$, and let a hand act at R, with a force equal to the weight P; then by the last Article, this force will still be supported by Q; but the forces of Q and R must be equal, because they are similar forces and act on the string QOR in exactly the same manner; the only difference being that one of them is in a vertical direction, and the other not so; which difference cannot disturb their equilibrium, since neither of the forces depends at all upon gravity. Hence, the force exerted by Q is always equal to that exerted by R, and therefore to the weight P: it is therefore the same, whatever be the length or direction of the string OQ.

Instead of a fixed *point* at O, we may suppose a circular pulley moveable about its center, and the reasoning will be the same.

9. *The science of Mechanics is divided into two parts: Statics and Dynamics.*

Bodies have no natural preference for a state of rest or motion, as will be shewn hereafter. If they are at rest they remain at rest, so long as no force acts upon them; but the slightest force, acting and not counteracted, would produce motion. Hence when any number of forces act upon a body at rest, or upon any system of bodies, there are two possible cases:—the forces may be such as exactly to counteract or balance each other, or to produce equilibrium: (see Art. 3)—or they may produce motion. In the latter case we have to consider the direction, velocity and duration of the motion; in the former case we have only to consider the relations of the forces which thus balance each other. Hence these two cases may most conveniently be treated of separately.

We shall therefore divide our subject into the part which relates to the action of forces in equilibrium, and the part which relates to the action of force producing motion. The first part is called Statics, the second Dynamics.

Force being considered as balancing force in one case, and as producing motion in the other, is differently measured in the two divisions of the science.

10. *In Statics a force is measured by the weight which it would support.*

Whatever be the direction and magnitude of a force, we may suppose it, as Q, in fig. 4, to act by means of a string which, passing over a pulley O, supports a weight P; and the force will be measured by the weight so supported.

Statical forces are called *Pressures*. Thus, when a heavy body is supported, it exerts a pressure downwards on its supports, and is sustained by their pressure upwards.

The pressure exerted upwards *by* each support, is *equal* to the pressure downwards *upon* it; and the latter being called the *Action*, the former is called the *Re-action*.

The pressures exerted by strings pulled by any forces are called *Tensions*.

In the following articles we have to reduce all cases of equilibrium to the simplest principles by which they can be reasoned upon. And this may be done by various considerations. Thus in the case which has been already mentioned where, in fig. 1, three forces, *BP*, *BQ*, *BR* keep each other in equilibrium, each may be considered as balancing the other two, acting at the same point. In fig. 17, a force *AP* supports a wheel *AC* against an obstacle *C* and may be considered as acting at *A* to balance the weight which presses the wheel and which is supposed to act at the same point. But in this instance we might also, by Art. 7, consider the force *AP* as acting at *M*, and the weight as acting at *N*; and we should thus have to reason concerning forces acting upon different points of a rigid body, moveable about a point *C*. In the same manner, in fig. 41, we may consider the strings *AK* and *BC*, with the pulley *AB* and the weight *W*, which they support, as all exerting their force at the same point *n*, where the lines of their directions meet. But we may also consider these forces as exerted at *A*, *B*, and *o*, points in the rigid body *AB*, and as keeping it in equilibrium. By similar views the most complicated cases of equilibrium can be reduced to simple principles.

The simplest principles to which the doctrine of Statics can be reduced, are, the equilibrium of forces on the same point, and the equilibrium of forces on the lever. And these two principles are so connected that one can be deduced from the other. The most simple course appears to be, first to establish the latter principle, which is done in the following chapter, and then to deduce the composition of forces from it, which is the object of Chap. II.

STATICS.

CHAP. I.

THE LEVER.

11. A LEVER is a rigid rod, moveable, in one plane, about a point which is called the *fulcrum* or centre of motion, by means of forces which tend to turn it round the fulcrum.

The portions of the rod between the fulcrum and the points where the forces are applied, are called the *arms*.

When the arms are in the same straight line, it is called a *straight lever;* otherwise it is called a *bent lever*.

The forces which act upon it are supposed to act in the plane in which the lever is moveable.

The lever is generally considered to be without weight.

Its properties will be deduced from the following Axioms:

12. AXIOM I. *Equal forces acting perpendicularly at the extremities of equal arms of a lever to turn it opposite ways, will keep each other in equilibrium.*

For the forces act in a manner perfectly similar, and hence there can be no reason why one of them should prevail rather than the other; and as both cannot produce motion, neither of them will do so. Therefore there will be an equilibrium.

If one of the forces be greater, the arms remaining equal; or if one of them act at a longer arm, the forces being equal; the greater force or the longer arm preponderates, and the equilibrium is destroyed.

COR. 1. Hence, the converse propositions are true; namely,

If two equal and opposite forces, acting perpendicularly at the extremities of the arms of a lever, keep it at rest, the arms are equal. For if they were not, the longer would preponderate.

If two forces, acting oppositely and perpendicularly at the extremities of equal arms of a lever, keep it at rest, the forces are equal. For if they were not, the greater would preponderate.

Cor. 2. If a weight, as *W*, fig. 5, be supported upon a rod *AB*, on two fulcrums *A* and *B*, at equal distances from it, the pressures on the two fulcrums are equal. For there can be no reason why either of them should be the greater.

Cor. 3. If two equal weights, *P*, *Q*, fig. 5, be supported on two fulcrums, *A* and *B*, situated so that *PA*, *QB*, are equal, the pressures on *A* and *B* are equal. The reason is the same as in the last corollary.

13. Axiom II. *If two equal weights balance each other upon a straight lever, the pressure upon the fulcrum is equal to the sum of the weights, whatever be the length of the lever.*

Thus if, in fig. 5, *AB* be a lever resting on two fulcrums *A*, *B*, and supporting a weight *W* at its middle point, the pressures upon the two fulcrums *A*, *B* are together equal to the weight *W*.

Axiom III. *If two equal weights be supported upon a straight line on two fulcrums at equal distances from the weights, the pressures upon the two fulcrums are together equal to the sum of the two weights.*

Thus if, in fig. 5 or 6, the two equal weights *P*, *Q*, be supported on *A*, *B*, (*PA* being equal to *QB*) the pressures on *A* and *B* are together equal to the weights *P* and *Q*.

These two axioms are self-evident when we distinctly conceive pressure, weight, and equilibrium. The pressures which are entirely employed in supporting the whole weight must be equal to it (Art. 4); and the whole weight is equal to the weights of the parts (Art. 6).

It may be observed, that the axiom thus appears to be true for any weights, as well as for equal ones. For the sake of simplicity we confine our employment of it as an axiom to the latter case.

Cor. *If a weight be supported upon a lever which rests on two fulcrums at equal distances from the weight, the whole pressure upon the fulcrums is equal to the weight.*

The pressures of the fulcrums upwards in this case balance the pressure of the weight downwards, in exactly the same manner in which the pressures of the weights downwards in the case of the Axiom, balance the pressure of the fulcrum upwards. The pressures in each direction are *supported* by those in the opposite direction; and the notion of upwards and downwards does not at all affect the relation of the forces. Hence the relations of the pressures in this and in the former case will be the same; that is, the pressure which acts in one direction is, in this case as well as in the other, equal to the pressure which acts in the opposite direction.

14. Prop. *If two equal weights act perpendicularly on a straight lever, they may be kept in equilibrium round any fulcrum by the same force as if they were collected at the middle point between them.*

Let P, Q, fig. 5, or 6, be the two weights, A the fulcrum, and W the middle point. Take $WB = WA$, and suppose a fulcrum placed at B.

When equal weights P and Q are supported on the lever, the pressures on the two fulcrums are equal by Cor. 3. to Axiom 1; and the whole pressure is $P + Q$ by Axiom 3; therefore the pressure on each fulcrum is half $P + Q$.

When a weight W, equal to $P + Q$, is placed at the middle point, the pressure on each of the fulcrums is, by Cor. 2. to Axiom 1, equal to half the whole pressure; but the whole pressure is $P + Q$, by Axiom 3; therefore the pressure on each fulcrum is half $P + Q$.

Hence, the pressure on the fulcrum B is in each case equal to half $P + Q$: and therefore the lever will in both cases be kept in equilibrium by the same force applied at B.

Cor. Hence, a horizontal prism or cylinder of uniform thickness and material, will produce the same effect as if it were collected at its middle point.

Thus, a cylinder *BD*, fig. 7, will produce the same effect on a lever *CB* as if it were collected at its middle point *N*: for this cylinder may be considered as composed of pairs of equal small weights (as *d* and *b*) at equal distances from *N*, and each such pair will produce the same effect as if collected at *N*, and hence, the whole cylinder will produce the same effect as if it were collected there.

15. Prop. *If two weights, acting perpendicularly upon a lever, on opposite sides of the fulcrum, have their distances from the fulcrum inversely as the weights, they will balance each other.*

Let *P* and *Q*, fig. 7, be the weights; and let *AB* be a cylinder equal to the sum of the weights. Divide *AB* in *D* so that $AD : DB :: P : Q$. Then the weight of the portion *AD* of the cylinder will be equal to *P*, and the weight of the portion *DB* equal to *Q*. The cylinder *AB* will balance on its middle point *C*.

Let *M* be the middle of *AD*, and *N* of *BD*: then, by last Article, the cylinder *AD* will produce the same effect as if it were collected at *M*, and the cylinder *BD* as if it were collected at *N*. Hence, if we suppose *AB*, instead of being a cylinder, to be a rod without weight, and upon this rod a weight equal to *AD* to be placed at *M*, and a weight equal to *BD* to be placed at *N*, these weights will still balance each other on the point *C*.

That is, *P* placed at *M*, and *Q* placed at *N*, will balance each other on *C*. And hence, by Art. 7, they will balance when suspended by the strings *MP*, *NQ*.

Now, since *DM* is half *DA*, and *DN* half *DB*, *MN* is half *AB*.

$$\text{Also, } CM = CA - AM = \tfrac{1}{2}AB - \tfrac{1}{2}AD,$$
$$\text{and } DN = MN - DM = \tfrac{1}{2}AB - \tfrac{1}{2}AD;$$
$$\therefore CM = DN, \text{ and hence } DM = CN.$$
$$\text{Hence, } CM : CN :: DN : DM :: DB : DA$$
$$:: Q : P.$$

Hence, when the weights and distances have this proportion, they will balance each other.

Cor. 1. Conversely, if the weights P and Q balance each other on C, we have $P : Q :: CN : CM$; for if not, let $P : Q' :: CN : CM$, and therefore by the proposition, P and Q' will balance each other; and hence Q and Q' produce the same effect at N, and therefore must be equal.

Cor. 2. The pressure on the fulcrum C will be the sum of the weights P and Q.

For the pressure of the weights P, Q on the fulcrum C will be the same as the pressure of the cylinder AB on that fulcrum: and this pressure (Art. 14.) will be the same as if the cylinder AB, which is the same weight as $P+Q$, were collected at C.

Cor. 3. What has been proved of weights, is true of any forces whatever, for these may be represented by weights (Art. 10): for instance, it applies to forces acting in the directions CR and MP, fig. 8, about a fulcrum N.

16. Prop. *If two forces, acting perpendicularly on a straight lever on the same side of the fulcrum, are inversely as their distances from the fulcrum, they will balance each other.*

In this case the forces must act in opposite directions, as P and Q, in figures 8 and 9, acting at M and N.

If we suppose that there is at the fulcrum C a force R, acting parallel to that at M, and such that $R : P :: NM : NC$, the forces P and R will produce equilibrium about a fulcrum N, and the pressure on N will be $P + R$, by last Article and Cor. 2 and 3. Hence, if at N a force $= P + R$ act in the opposite direction, it will produce the effect of the fulcrum. If therefore $Q = P + R$, the three forces P, Q, R, will keep the lever in equilibrium. And this is true, if

$$R : P :: NM : NC,$$
$$\text{or if } R + P : P :: NM + NC : NC,$$
$$\text{or if } Q : P :: MC : NC.$$

And therefore, if Q and P have this proportion, they will balance each other.

Cor. 1. Conversely, if the weights balance on C,

$$P : Q :: CN : CM.$$

Cor. 2. The pressure on the fulcrum C will be the difference of the forces, for it will be R, and since

$$P + R = Q, \quad R = Q - P.$$

Also, it will be in the direction of the greater force Q.

Cor. 3. Multiplying extremes and means, we have, both in this Proposition and the last,

$$P \cdot CM = Q \cdot CN.$$

17. When a lever is used as a mechanical instrument, one of the forces, as Q, is a weight to be raised or supported, and the other, P, is employed to produce this effect. Hence P is called the *power*, and Q the *weight*.

Straight levers are divided into three kinds.

The lever of the first kind is that in which the power and the weight are on opposite sides of the fulcrum, as in fig. 7.

Scissars, *Pincers*, &c. are examples of double levers of this kind; the fulcrum being at the center of motion; the power being the force of the fingers; and the weight, the pressure exerted by the cutting or holding part.

The lever of the second kind is that in which the power and the weight are on the same side of the fulcrum, the power being more distant from it.

A stock-knife, *a pair of nutcrackers*, &c. are examples of this kind. *An oar* also belongs to this class, the fulcrum being that point of the blade of the oar which is for a moment stationary in the water while the boat is impelled forwards; the power being the pull of the rower; and the weight, the pressure of the oar upon the side of the boat.

In these cases, the lever always acts at a mechanical advantage, in consequence of the power acting at the longer arm; that is, the pressure produced is greater than the power exerted.

The lever of the third kind is that in which the power is on the same side as the weight, and nearer to the fulcrum, as in fig. 9.

In this case, the power always produces a pressure less than itself, and the instrument is employed not to obtain a mechanical advantage, but to enable the force to act at a greater distance.

Examples of this kind of lever are *a pair of tongs*, and *a pair of sheep-shears:* some *bones* of animals considered with respect to equilibrium only, are also levers of this kind, the *fulcrum* being the joint; the *power*, the muscle which is attached near the fulcrum; and the *weight*, the force exerted by the limb further from the joint.

18. Examples of the *Straight Lever*.

Ex. 1. On a lever of the first kind, 3 feet long, a weight of 100 pounds is suspended at the extremity, and $2\frac{2}{3}$ inches from this end is placed a fulcrum; what weight at the other end will preserve the equilibrium?

In fig. 7, $MN = 36$ inches; $CM = 2\frac{2}{3}$ inches; $\therefore CN = 33\frac{1}{3}$ inches,

$$P : Q :: 33\tfrac{1}{3} : 2\tfrac{2}{3} :: 100 : 8;$$

$$\text{and } P = 100\text{lbs.}; \therefore Q = 8\text{lbs.}$$

Ex. 2. On a straight lever MO, fig. 10, let MC, CN, NN', $N'N''$, &c. be all equal; then if a weight Q be slid along the arm CO, what are the weights at M, which it will balance when at N, N', N'', &c.?

Q at N balances Q at M; Q at N' balances $2Q$ at M; Q at N'' balances $3Q$ at M, &c.

Hence, excluding the weight of the lever, the weight at M might be known from knowing the place of Q. We shall see hereafter how the weight of the lever itself may be taken into account.

If $CM = CN$, the weights at M and N are equal, and one of them may be used to measure the other. This is the case in the common balance, but when the arms are unequal, it is called a *false balance.*

Ex. 3. In a false balance, to find the true weight of the substance weighed.

Let CM, CN, be unequal, and let x be the weight to be determined. Let x at N be balanced by a ounces at M, and let x at M be balanced by b ounces at N. Therefore,

$$x : a :: CM : CN,$$
$$x : b :: CN : CM;$$
$$\therefore x^2 : ab :: 1 : 1,$$
$$x^2 = ab, \text{ and } a : x :: x : b;$$

$\therefore x$ is a mean proportional between a and b, the apparent weights in opposite scales.

Ex. 4. When a weight is supported on a lever at two points, to compare the pressures supported at the two points.

Let a weight R be supported on a lever MN, fig. 11, by forces P, Q. The same force is exerted at M as if N were a fulcrum: hence,

$$P : R :: NC : NM.$$
$$\text{So } R : Q :: NM : MC;$$
$$\therefore P : Q :: NC : MC.$$

Or, the pressures supported are inversely as the distances from the weight.

19. Prop. *If two forces acting perpendicularly on the arms of a bent lever are inversely as the arms, they will balance each other.*

Let forces P, Q, fig. 12, act perpendicularly on the arms CM, CN, and be such that

$$P : Q :: CN : CM;$$

they will balance each other.

Produce NC to D, so that $CD = CM$, and at D let a force R, equal to P, act perpendicularly to CD: also take $CE = CM$, and at E let a force S, also equal to P, act perpendicularly to CN.

The forces P and S would balance each other, because they are equal, and act in a manner exactly similar upon the arms CM, CE. But the force R would balance S, because $CD = CE$. Therefore P and R produce the same effect on the lever.

$$\text{Now, since } P : Q :: CN : CM, \text{ we have}$$
$$R : Q :: CN : CD;$$

therefore, by Art. 15, R would balance Q on the straight lever. Hence, P will balance Q on the bent lever.

Cor. 1. Conversely, if the forces act perpendicularly and balance each other, they are inversely as the arms.

Cor. 2. If the arm CM or CN were bent so as to have any other form, CFM, its extremity being the same, the same proportions would be true.

For, CFM being perfectly rigid, if we join CM, the effect produced will be the same if, instead of the arm CFM, we suppose the rigid surface $CFMC$. And, in this surface, if we remove any portion of the surface by lines parallel to CM, so as to leave only a strip CM, the effect will manifestly be the same as before. Hence, whether we have the rigid rod CFM, or CM, the effect will be the same.

20. Prop. *If two forces, acting at any angles on the arms of any lever, are inversely as the perpendiculars from the fulcrum upon their directions, they will balance each other.*

Let ACB, fig. 13, be the lever moveable about C; P, Q, two forces acting in the lines AP, BQ, and CM, CN perpendiculars on those lines. And let $P : Q :: CN : CM$; the forces will balance each other.

Let AM and CM be considered as rigid rods; then by Cor. 2, to last Art., the same effect will be produced whether we suppose the force P to act by means of the crooked arm CAM, or the straight one CM. In the same manner the force Q, acting at B, will produce the same effect as if it acted at N. But the forces P and Q, acting at M and N, would produce equilibrium, by last Article, because

$$P : Q :: CN : CM;$$

hence, acting at A and B, they will produce equilibrium.

Cor. 1. $P \cdot CM = Q \cdot CN.$

Cor. 2. Conversely, if this proportion is true, or if these products are equal, the forces will balance.

Cor. 3. $P \cdot CA \cdot \sin A = Q \cdot CB \sin B.$

Cor. 4. The proposition is true, whatever be the angle made by the arms CA, CB, and hence it is true when this angle vanishes, and the arms coincide. That is, if two forces P, Q, fig. 14, act in directions AP, BQ, at points, A, B, of the *same straight lever*, when they are inversely as the perpendiculars on their directions, there will be an equilibrium.

Cor. 5. Hence, if two forces act at the *same point* of a lever to turn it in opposite directions; when they are inversely as the perpendiculars, there will be an equilibrium.

21. Examples of the *Bent Lever*.

Ex. 1. Fig. 13. P is 99 pounds, Q 100 pounds; $CA = 9$, $CB = 5$, and the angle $CAP = 30^0$; to find the angle CBN, that there may be an equilibrium;

$$P \cdot CA \cdot \sin. A = Q \cdot CB \cdot \sin. B;$$
$$\therefore \sin. B = \frac{P \cdot CA}{Q \cdot CB} \cdot \sin. A = \frac{99 \cdot 9}{100 \cdot 5} \cdot \frac{1}{2}$$
$$= .891 = \sin. 62^0,$$

as appears by the table of sines; whence $CBN = 62^0$.

Ex. 2. In a straight lever AB, fig. 15, acted on by weights P, Q; if there be an equilibrium when it is horizontal, there will be an equilibrium in every position.

Let AB be any position of the lever; MCN a horizontal line; PAM, QNB, vertical lines. If there be an equilibrium in the horizontal position,

$$P : Q :: CB : CA.$$

But, by similar triangles, $CB : CA :: CN : CM$; therefore

$$P : Q :: CN : CM;$$

and therefore the equilibrium subsists.

Ex. 3. In a bent lever ACB, (without weight) fig. 16, having given the lengths of the arms, the angle which they make, and the weights P, Q, appended to them; to find the position in which it will rest.

Draw MCN horizontal, meeting the vertical lines PA, QB, in M, N. Let $CA = a$, $CB = b$, $ACB = \omega$; and $ACM = \theta$, which is to be found. Therefore $BCN = \pi - \omega - \theta$, π being two right angles.

$$P \,.\, CA \,.\, \cos. ACM = Q \,.\, CB \,.\, \cos. BCN,$$

$$Pa \cos. \theta = Qb \cos. (\pi - \omega - \theta) = - Qb \cos. (\omega + \theta)$$

$$= - Qb \{\cos. \omega \cos. \theta - \sin. \omega \sin. \theta\},$$

$$(Pa + Qb \cos. \omega) \cos. \theta = Qb \sin. \omega \sin. \theta;$$

$$\therefore \tan. \theta = \frac{Pa + Qb \cos. \omega}{Qb \sin. \omega}.$$

Ex. 4. In the same case, having given P, to find Q, such that the arm CA may be horizontal.

In this case, $\theta = 0$;

$$\therefore Pa + Qb \cos. \omega = 0; \quad Q = - \frac{Pa}{b \cos. \omega}.$$

The problem will not be possible, except ω be greater than a right angle, in which case $\cos. \omega$ is negative, and Q is positive.

Ex. 5. To find the force requisite to draw a carriage wheel over an obstacle, supposing the weight of the carriage collected at the axis of the wheel.

Let A, fig. 17, be the axis of the wheel, CD the obstacle. Then if the wheel turn over the obstacle, it must turn round the point C; and the force which moves it being supposed to act in the line AP, and the weight in the vertical line AE, the wheel will be a lever such as that referred to in Cor. 5, Art. 20. Hence, in order that P may balance the weight Q,

$$P : Q :: CN : CM :: \text{sin. } CAE : \text{sin. } CAP,$$

$$P = Q \, . \, \frac{\text{sin. } CAE}{\text{sin. } CAP}.$$

Hence, P is least when sin. CAP is greatest, or when CAP is a right angle. In this case, $P = Q$ sin. CAE.

If the wheel be made larger, the obstacle being the same, the versed sine NE, or CD, remains the same; and the radius being increased, the angle CAE is diminished. Hence, *cæteris paribus*, P is diminished, and the larger the wheel, the smaller is the force requisite.

22. Prop. Fig. 18. *If any number of forces P, Q, &c. and P′, Q′, &c., acting upon the arms of a lever to turn it opposite ways, be such that*

$$P \, . \, CM + Q \, . \, CN + \text{\&c.} = P' \, . \, CM' + Q' \, . \, CN' + \text{\&c.}$$

there will be an equilibrium.

Here CM, CN, &c., and CM', CN', &c. are the perpendiculars on the directions of the forces; and the lever is supposed to have any number of arms inflexibly connected.

Draw any line OO' through C, and at O and O' let forces X, Y, &c., and X', Y', &c., act, perpendicularly to OO', to turn the system opposite ways. Let these be such that

$$X \, . \, CO = P \, . \, CM, \quad Y \, . \, CO = Q \, . \, CN, \text{ \&c.};$$

$$X' \, . \, CO' = P' \, . \, CM', \quad Y' \, . \, CO' = Q' \, . \, CN', \text{ \&c.};$$

$$\therefore \; X \, . \, CO + Y \, . \, CO + \text{\&c.} = P \, . \, CM + Q \, . \, CN + \text{\&c.},$$

$$X' \, . \, CO' + Y' \, . \, CO' + \text{\&c.} = P' \, . \, CM' + Q' \, . \, CN' + \text{\&c.}$$

But by supposition

$$P \, . \, CM + Q \, . \, CN + \text{\&c.} = P' \, . \, CM' + Q' \, . \, CN' + \text{\&c.}$$

therefore

$$X \cdot CO + Y \cdot CO + \&c. = X' \cdot CO' + Y' \cdot CO' + \&c.$$
$$\text{or, } (X + Y + \&c.)\, CO = (X' + Y' + \&c.)\, CO';$$

and hence, by Art. 16, Cor. 3, the force $X + Y + \&c.$ at O, and $X' + Y' + \&c.$ at O', produce equilibrium.

But, since $X \cdot CO = P \cdot CM$, P would balance X; similarly, Q would balance Y, &c.; and similarly, P' would balance X', Q' would balance Y', &c.

Hence, P, Q, &c. together, would balance X, Y, &c. together; and hence P, Q, &c. produce the same effect as X', Y', &c. which together balance X, Y, &c. But, in the same manner, X', Y', together, would balance P', Q' together; therefore P, Q, &c. will balance P', Q', &c.

Cor. 1. Conversely, if there be an equilibrium,

$$P \cdot CM + Q \cdot CN + \&c. = P' \cdot CM' + Q' \cdot CN' + \&c.$$

For, making the same construction, P, Q, &c. will balance X, Y, &c., and P', Q', will balance X', Y', &c. Hence, since P, Q, &c. balance P', Q', &c. X, Y, &c. will balance X', Y', &c.; and therefore,

$$(X + Y + \&c.)\, CO = (X' + Y' + \&c.)\, CO'.$$

But the first side

$$= X \cdot CO + Y \cdot CO + \&c. \text{ is } = P \cdot CM + Q \cdot CN + \&c.$$

by supposition.

And the second side, in the same manner,

$$= X' \cdot CO' + Y' \cdot CO' + \&c. \text{ is } = P' \cdot CM' + Q' \cdot CN' + \&c.;$$
$$\therefore \quad P \cdot CM + Q \cdot CN + \&c. = P' \cdot CM' + Q' \cdot CN' + \&c.$$

Cor. 2. If the forces be weights acting at points M, N, M', N', &c., on a horizontal lever, fig. 19, the equilibrium will subsist, when

$$P \cdot CM + Q \cdot CN + \&c. = P' \cdot CM' + Q' \cdot CN' + \&c.$$

If $P \cdot CM + Q \cdot CN + \&c.$ be greater than
$P' \cdot CM' + Q' \cdot CN' + \&c.$

the lever will turn in the direction in which P, Q, &c. would draw it, and *vice versâ*. Hence, the quantity $P \cdot CM + Q \cdot CN +$ &c. may be considered as the measure of the power or energy to turn the system in that direction. This quantity, viz., the sum of the products of each force into its perpendicular distance from the fulcrum, estimated in the same direction, is called the *moment* or *momentum* of the forces round C; and the product of one force P, by its perpendicular CM, is called the moment of that force.

23. Ex. 1. In fig. 19, let P, Q, P', Q', be weights of 3, 5, 7, 9 pounds respectively, and MN, NM', $M'N'$, equal distances of one foot: to find the point on which the weights will balance.

Let $MN = NM' = M'N' = a$, and $MC = x$;

$$\therefore\ NC = x - a,\ \ CM' = 2a - x,\ \ CN' = 3a - x;$$

and therefore, by last Corollary,

$$3x + 5(x - a) = 7(2a - x) + 9(3a - x);$$

$$\therefore\ x = \frac{5a + 7 \cdot 2a + 9 \cdot 3a}{3 + 5 + 7 + 9} = \frac{46a}{24} = \frac{23a}{12};$$

$\therefore\ x = 23$ inches, and C is one inch from M'.

Ex. 2. To shew how the steelyard must be graduated.

The steelyard is a lever AB, fig. 20, which is moveable about a center C, and on which substances to be weighed are suspended from the extremity B, as at Q. A known weight P, moveable along the arm CA, is placed at such a distance from C as to balance the body Q: then, from the place of A we may know the weight Q: and, if at different points of CA we place figures to represent the corresponding weights of Q, the arm CA is *graduated*.

The lever is now supposed to have weight, and the arm CA being longer and heavier than the other, will preponderate. Suppose, that when Q and P are removed, a weight equal to P, placed at D, would keep the beam horizontal. If we then take $CO = CD$, it appears that the whole beam AB produces the same effect as a weight P placed at O; for either of the two would balance P, placed at D. Now let P and Q balance at B and M: therefore, Q balances P at M, together with the beam; that is, Q balances P at M, together with a weight which produces the same effect as P at O does. Hence,

$$Q \cdot CB = P \cdot CM + P \cdot CO = P \cdot CM + P \cdot CD$$
$$= P \cdot DM.$$

Hence, if we make DE, DF, DG, &c. equal to CB, $2\,CB$, CB, &c. we shall have, when P is at E, at F, at G, &c.

$$Q = P, \quad Q = 2\,P, \quad Q = 3\,P, \quad \text{\&c.}$$

And therefore, the beam is graduated, by taking such equal distances from the point D, and numbering the points thus found 1, 2, 3, &c.

24. The reasonings of this chapter will apply if the arms of the lever, instead of being all in the same plane, are in any planes perpendicular to the axis of motion, the forces being supposed to act in these planes. The perpendiculars in these planes from the axis upon the force must be taken instead of CM, CN, &c. in the preceding Articles, and the same propositions are still true.

A *windlass* moved by handspikes, is an example of forces acting in this manner.

CHAP. II.

THE COMPOSITION AND RESOLUTION OF FORCES.

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25. Having considered the action of forces, and their equilibrium, upon a lever, we now proceed to consider the effect of the combined action of two or more forces on a point. If two forces act on a point, as the forces in BQ, and BR, fig. 1, they will produce the same effect as a single force acting in some intermediate direction as BS.

In this case, the force in BS is called the *Resultant* of the forces in BQ, BR; and the forces in BQ, BR, are called the *Components* of the force in BS.

When two forces act in the same direction, the combined effect is equivalent to the *sum* of the forces. Thus, if a force of 2 pounds, and another of 3 pounds, act upon the same point, in the same direction, the point will be drawn with a force of 5 pounds. And, in the same manner, if two forces act in opposite directions, the resultant will be the *difference* of the two, and in the direction of the greater. Thus, if a point be acted on by a force of 7 pounds in one direction, and 4 pounds in the opposite, it will be drawn with a force of 3 pounds in the first direction. For the force of 7 may be considered as composed of 4 and 3; of which the 4 will be destroyed by the opposite force, and the 3 will remain effective.

26. Prop. *If any two forces act at the same point, the force which is equivalent to the two is in the* direction *of the diagonal of the parallelogram, of which the sides represent the magnitude and direction of the component forces.*

Let Ap, Aq, (fig. 21.) represent in magnitude and direction the forces P, Q: complete the parallelogram $ApCq$; and draw AC. Draw also CM, CN perpendicular upon Ap, Aq produced. Now suppose CA to be a lever moveable about a point C, and acted on by the forces P, Q, in the directions Ap, Aq. The triangles CpM, CqN, have right angles at

M and N, and angles $CpM = qAp = CqN$. Hence they are similar, and

$$CM : CN :: Cp : Cq :: Aq : Ap$$
$$:: Q : P \text{ by supposition.}$$

Therefore the forces P, Q are inversely as the perpendiculars CM, CN, and would therefore together keep the lever CA at rest. (Art. 20, Cor. 5.)

And since the force equivalent to the two P, Q, would produce the same effect as they would together, this force also would keep the lever CA at rest. But manifestly no single force can keep the lever CA at rest, except it act in the direction AC*: for if it made an angle with CA on either side, it would turn CA round C in that direction. Hence the equivalent force acts in the direction AC; and AC is the diagonal of the parallelogram whose sides are Ap, Aq, which represent the forces.

Cor. 1. If a point acted on by two forces Ap, Aq, be kept at rest by a third force, this force must act in the direction CA.

Cor. 2. Hence if three forces act at a point and keep *each other* in equilibrium, each of them is in the direction of the diagonal of the parallelogram whose sides represent the other two.

27. Prop. *If any two forces act at the same point, the force which is equivalent to the two is expressed in* magnitude *by the diagonal of the parallelogram, of which the sides represent the magnitude and direction of the component forces.*

Let Ap, Aq, fig. 22, represent the two forces in magnitude and direction. Complete the parallelogram $Apqr$, and, by last Article, Cor. 2, the two forces Ap, Aq will be kept at rest by a force in the direction rA. Let Ay represent the force in magnitude; and complete the parallelogram $Apxy$. Since the

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* If the force were to act in the opposite direction CA it would keep the lever at rest, but in that case it manifestly could not be equivalent to forces AP, AQ which include an angle on the side towards C.

forces Ap, Aq, Ay, keep each other in equilibrium, we may consider Aq as counteracting Ap, Ay. Hence, by last Art. Aq will be in the direction of the diagonal of the parallelogram py, and xAq will be a straight line. Hence, in the triangles xAy, qAr, the angles $xAy = qAr$; as also $xyA = Arq$ by parallel lines; and therefore the triangles are equiangular. Also, by parallels, $qr = Ap = yx$; therefore the triangles xAy, qAr are equal: and $Ay = Ar$: that is, the force Ay is represented in magnitude by the diagonal Ar. And Ap, Aq would counteract Ay, and therefore their resultant in Ar would counteract Ay, and is therefore equal to it. Hence, the resultant is represented in magnitude by Ar.

Cor. If the components be represented by the sides of a parallelogram, the resultant is represented in magnitude and direction by the diagonal.

28. Prop. *Forces may be represented by lines parallel to their directions, and proportional to them in magnitude.*

For the direction of a line parallel to a force is the same as that of the force itself: and hence the force is properly represented in magnitude and direction.

Cor. 1. If AB, BC, fig. 23, represent two forces, AC will represent their resultant; for, completing the parallelogram $ABCD$, the force represented by BC is the same with the force represented by AD; and therefore AC, the resultant of AB, AD, is the resultant of AB, BC.

Cor. 2. If two forces be represented by two sides of a triangle proceeding from the same angle, as AB, AD; their resultant may be found by doubling the line which joins the angle and the bisection of the opposite side. For if the parallelogram be completed, its diagonals bisect each other, and therefore AC is twice AE.

Cor. 3. If three forces, represented in magnitude and direction by the three sides of a triangle taken in order, act on a point, they will keep it at rest. Let ABC, fig. 23, be the triangle; AB, BC are equivalent to AC by Cor. 1; therefore AB, BC, CA are equivalent to AC, CA, and therefore will keep the point at rest.

Cor. 4. Conversely, if three forces act on a point in the *directions* of the sides of a triangle, and keep it at rest, they are represented in *magnitude* by the sides of the triangle. For if one of these forces, as that in direction BC, be not represented by BC, let it be represented by BC'; then the two AB, BC' are equivalent to AC', and therefore cannot balance a force in direction CA: which is contrary to the supposition.

Cor. 5. Hence, knowing the directions of three forces which keep each other in equilibrium, we may find their relative magnitudes, by making a triangle whose three sides are parallel to these directions; these sides will be in the proportion of the forces.

Cor. 6. If three forces keep a body in equilibrium, and three lines be drawn making with the directions of the forces three equal angles towards the same parts; these three lines will form a triangle whose sides will represent the three forces respectively.

Let AB, BC, CA, fig. 24, and 25, be the directions of the forces; DM, EN, FO three lines such that the angles ADM, BEN, CFO are equal; these lines, produced if necessary, form a triangle abc. In the triangles aEM, ADM, the angle aME equals AMD, and by supposition aEM, that is BEN, equals ADM; hence the remaining angle MaE or bac equals MAD or BAC; similarly, the angle abc equals ABC, and bca equals BCA. Hence the triangles abc, ABC are equiangular, and therefore

$ab : bc :: AB : BC$

$:: \text{force in direction } AB : \text{force in direction } BC$,

by Cor. 4. And similarly of ca.

If therefore ab represent the force in direction AB; bc, ca will represent the forces in directions BC, CA.

Cor. 7. If three forces keep a point at rest, they are each as the sine of the angle contained by the other two.

Let P, Q, R, acting in Ap, Aq, Ay, fig. 22, keep a point A at rest. Produce yA, and draw pr parallel to Aq, and Ap, pr, rA will be as the forces, (Cor. 4.) Now

$$\begin{aligned} P : Q :: Ap : pr &:: \sin. Arp : \sin. pAr \\ &:: \sin. qAr : \sin. pAr \\ &:: \sin. qAy : \sin. pAy. \end{aligned}$$

And similarly, we should have

$$\begin{aligned} R : P &:: \sin. pAq : \sin. yAq, \\ \text{and } Q : R &:: \sin. yAp : \sin. qAp. \end{aligned}$$

COR. 8. If the angle between two given forces be diminished, the resultant is increased.

Let two forces Ap, Aq, fig. 26, act at the angle pAq; pr being equal and parallel to Aq, Ar is the resultant.

Let Ap, AQ, the same forces, act at the angle pAQ; pR being equal and parallel to AQ, AR is the resultant.

pR is equal to pr, and if the angle $pAQ < pAq$, we have $ApR > Apr$ and therefore $AR > Ar$. (EUC. XXIV. 1.)

29. We have instances of the composition of forces in such cases as the following.

Suppose a boat fastened to a fixed point by a rope, and acted on at the same time by the wind and the current. Then the direction of the rope will indicate the direction of the resultant of these actions.

In fig. 1, let BQ, BR, the directions of two forces which act at B, be at right angles, and let the forces exerted be 48 pounds, and 20 pounds: to find the magnitude and direction of the resultant.

If we make BRS a right angle, and $BR = 48$, $RS = 20$, BS will be the resultant. And $BS^2 = 48^2 + 20^2 = 2704$; $\therefore BS = 52$, and the resulting force is 52 pounds.

Also to find the angle SBR, we have $\sin. SBR = \frac{20}{52} = \frac{5}{13}$;

$\therefore SBR = 22^0\ 37'$ nearly.

We have many examples of the resolution of forces, in cases where the force exerted being resolved into two, one of them is somehow lost or counteracted, and the remaining part only is effective. Thus, if we drag an object along the ground by a rope attached to it, if we suppose this rope to be inclined to the horizon at an angle of 45°, the force which we exert is effective only in part. If we thus exert a force of 17 pounds, this force is equivalent to two equal forces, one in a horizontal and one in a vertical direction. And if each of these be called x, we shall have

$$x^2 + x^2 = 17^2,\ x = \frac{17}{\sqrt{2}} = 12 \text{ nearly.}$$

Hence the force with which we draw the body horizontally is 12 pounds.

30. Prop. *To find the resultant of any number of forces* AB, Ac, Ad, Ae, *acting in the same plane at a point* A, fig. 27.

Complete the parallelogram Bc, and draw the diagonal AC;
Complete the parallelogram Cd, and draw the diagonal AD;
Complete the parallelogram De, and draw the diagonal AE;

and so on, if there are more forces.

AB, Ac, are equivalent to AC;
$\therefore$ AB, Ac, Ad, are equivalent to AC, Ad, that is, to AD;
$\therefore$ AB, Ac, Ad, Ae, are equivalent to AD, Ae, that is, to AE.
That is, AE is the resultant of the forces AB, Ac, Ad, Ae.

Cor. 1. If any number of forces be represented by sides of a polygon taken in order, as AB, BC, CD, DE, their resultant will be represented by the line AE which completes the polygon. (See Art. 28.)

Cor. 2. A number of forces which are represented by *all* the sides of a polygon taken in order, as AB, BC, CD, DE, EA, acting upon a point, will keep it at rest.

For AB, BC, CD, DE are equivalent to AE: therefore, AB, BC, CD, DE, EA, are equivalent to AE, EA, and will keep a point at rest.

Cor. 3. It does not follow conversely, as in the case of three forces, that if they act in the direction of the sides of the polygon and are in equilibrium, they are proportional to the sides. For the directions of the sides may remain the same while their proportions are altered. Thus, if we draw $D'E'$

parallel to DE, forces parallel to the sides of the polygon will keep a point at rest, if they be proportional to AB, BC, CD', $D'E'$, $E'A$, as well as if they be proportional to AB, BC, CD, DE, EA.

31. Prop. *To find the resultant of forces which are not in the same plane.*

Let AB, AC, AD, fig. 28, be three forces not in the same plane. Let the planes BC, BD, CD, be drawn, and the planes DG, CG, BG, parallel to them, completing the parallelepiped, whose sides will be parallelograms. Join AF, DG; DF will be a parallelogram, as is evident; and by Art. 27,

AB, AC, are equivalent to AF;

$\therefore$ AB, AC, AD, are equivalent to AF, AD, that is, to AG. Hence, if the edges of a parallelepiped, drawn from the same point, represent the components, the diagonal will represent the resultant.

Cor. 1. If $ABEG$ be any four-sided figure, not all in the same plane, and if AB, BE, EG, represent three forces, AG will represent their resultant.

Cor. 2. If four forces acting upon a point, be represented by the sides of *any* four-sided figure, taken in order, they will keep the point at rest.

Cor. 3. If any number of forces be represented by sides, taken in order, of a polygon, which is not in the same plane, their resultant will be represented by the line which completes the polygon.

Cor. 4. If any number of forces be represented by all the sides, taken in order, of any polygon, they will keep a point at rest.

The three last Corollaries are proved from this Article, as those of Art. 30. are from Art. 30.

32. From the preceding principles, we may find the conditions under which a point will be kept in equilibrium, as will appear in the following Problems.

Prob. I. Fig. 29. A, B *are two points in the same horizontal line, and* AC, BC, *two strings from which, at the knot* C, *the weight* W *hangs: to find the forces exerted by the strings* CA, CB.

The point at which the equilibrium is produced is, in this case, the point C; and the forces which produce it are the forces of the strings CA, CB, and the weight W acting by the string CW. From any point d in the vertical line WC produced, draw db, da, parallel to CA, CB. In order to support the weight W the resultant of the forces of the strings CA, CB, must be in the direction Cd, and must be equal to the weight W. The forces must therefore be as Ca, Cb, and their resultant will then be as Cd by Art. 27. Hence if Cd represent the weight W, we have the forces of the strings represented by Ca, Cb. Or if P, Q represent the forces of the strings CA, CB; we have

$$\frac{P}{W} = \frac{Ca}{Cd} = \frac{\sin. Cda}{\sin. Cad} = \frac{\sin. dCb}{\sin. aCb} = \frac{\sin. DCB}{\sin. ACB};$$

$$\text{similarly, } \frac{Q}{W} = \frac{\sin. DCA}{\sin. ACB};$$

whence P and Q are known.

Cor. 1. The forces of the strings measure their *tensions*, (see Art. 10.) and these again are measured by the pressures exerted on the immoveable points A, B. But if instead of supposing the strings fixed at the points A, B, we suppose them to pass over those points, or over pullies placed there, and to have appended to them weights equal to the force P, Q; these weights will be just supported, that is, there will be an equilibrium. See Art. 8.

Prob. II. Fig. 30. *Two strings* CAP *and* CBQ *pass over pullies* A *and* B, *in the same horizontal line, and support a weight* W *by means of equal weights* P *and* Q *suspended at their other extremities: to find the position of the points* C.

Draw lines as in the preceding problem, and let Cd meet AB in E: then by the last corollary the weights W, P, Q will be as Cd, Ca, Cb; and since the weights P and Q are equal,

$Ca = Cb = ad$; $\therefore$ $\angle aCd = Cda = dCb$; $\therefore$ the triangles ACE, BCE are equal, and $AE = EB$. Hence E bisects AB, and C will be in the vertical line passing through E.

Join ab meeting cd in e; aec, AEC are right angles.

And $\dfrac{P}{W} = \dfrac{Ca}{Cd} = \dfrac{Ca}{2Ce} = \dfrac{CA}{2CE}$ by similar triangles.

Let $AE = EB = a$, $EC = x$; $\therefore$ $CA = (a^2 + x^2)^{\frac{1}{2}}$,

$$\frac{P}{W} = \frac{(a^2 + x^2)^{\frac{1}{2}}}{2x};$$

$$\therefore \frac{4P^2}{W^2} x^2 = a^2 + x^2; \quad \frac{4P^2 - W^2}{W^2} x^2 = a^2;$$

$$x = \frac{Wa}{(4P^2 - W^2)^{\frac{1}{2}}};$$

whence the position of C is known.

Cor. 1. In order that x may be possible, we must have the quantity under the radical sign positive, and therefore

$$W^2 < 4P^2,$$
$$\text{or} \quad W < 2P;$$

if W be equal to or greater than $2P$, it will descend, drawing up both the weights, and will never find a place where it will rest.

Cor. 2. In order that the string ACB may be drawn so as to be in the horizontal line, we must have $x = 0$,

$$\text{or} \quad \frac{Wa}{(4P^2 - W^2)^{\frac{1}{2}}} = 0;$$

which cannot be, except either W be indefinitely small or P indefinitely great. That is, no weights P, Q, however great, can draw up a weight W, so that the string ACB shall be a horizontal straight line. If ACB, instead of being a line without weight loaded with a weight at its middle, be a cord of which each part has weight, the same will be true.

Prob. III. Fig. 31. P, Q, *support* W *as in the last Problem, the values of* P, Q, *and the positions of the pullies* A, B, *being any whatever*; *to find the position of equilibrium of* C.

As before, let Cd be vertical, and da parallel to BC. Then P, Q, W, are as Ca, ad, dC. Hence, in the triangle Cad, we have the proportions of three sides given, to find the angles aCd, Cda.

$$\text{Also } BAC = BAP - CAP = BAP - aCd,$$

$$ABC = ABQ - CBQ = ABQ - bCd$$

$$= ABQ - Cda.$$

Hence, knowing the position of the points A, B, and therefore the angles BAP, ABQ, we know the angles BAC, ABC; and hence knowing the side AB, we may solve the triangle ABC, and calculate the position of C.

Prob. IV. *A string* ACDEB, *fig.* 32, *of which the extremities* A, B, *are fixed, is kept in a given position by weights* P, Q, R, *suspended at knots* C, D, E; *to compare the weights* P, Q, R.

Let the sides of the polygon AC, CD, DE, EB make with the horizontal line angles β, γ, δ, ϵ. Then it is easily seen that if AC be produced to c, $DCc = \beta - \gamma$. Similarly, $EDd = \gamma - \delta$, &c.

The point C is kept at rest by three forces; viz. the weight P, the *tension* of CA, and the *tension* of CD: let the latter be called C, and we shall have, by Cor. 7, Art. 28,

$$\frac{P}{C} = \frac{\sin. ACD}{\sin. ACP} = \frac{\sin. DCc}{\sin. ACx}$$

$$= \frac{\sin. (\beta - \gamma)}{\cos. \beta} = \frac{\sin. \beta \cos. \gamma - \cos. \beta \sin. \gamma}{\cos. \beta}$$

$$= \cos. \gamma\, (\tan. \beta - \tan. \gamma).$$

Also the point D is kept at rest by three forces; the weight Q, the tension of DE, and the tension of CD at D; and the last is the same as C, the tension of CD at C, because the string must be kept at rest by equal and opposite forces, and therefore must exert equal and opposite tensions at its two extremities.

$$\text{Hence, } \frac{Q}{C} = \frac{\sin. (\gamma - \delta)}{\cos. \delta}$$

$$= \frac{\sin. \gamma \cos. \delta - \cos. \gamma \sin. \delta}{\cos. \delta}$$

$$= \cos. \gamma (\tan. \gamma - \tan. \delta).$$

Hence, we have

$$\frac{Q}{P} = \frac{\tan. \gamma - \tan. \delta}{\tan. \beta - \tan. \gamma}.$$

Similarly, we shall have the proportions of the forces at the other angles.

Hence, P, Q, R, are proportional to the differences of the tangents of the angles which the supporting strings make with the horizon.

If one of the strings, as EB, have that end higher which is farther from the origin A, the corresponding angle ϵ is to be taken negative, and we shall still have, ($-\tan. \epsilon$ being a positive quantity,)

$$\frac{Q}{R} = \frac{\tan. \gamma - \tan. \delta}{\tan. \delta - \tan. \epsilon}.$$

Cor. If the forces P, Q, R, instead of being parallel, were to make any angles with each other, we should be able to compare them by the application of Cor. 7, Art. 28.

A cord kept in equilibrium in such a manner is called a *Funicular polygon.*

Prob. V. Fig. 33. *A cord* PAQ, *which passes round a fixed point* A, *is drawn in different directions by forces* P, Q ; *to find the pressure upon the point* A.

In the first place, the forces P, Q must necessarily be equal, for, as the string passes freely round A, the forces will balance each other in the same manner as if they acted at the two ends of a string which was in a straight line, and therefore they will be equal. Now if we suppose A, instead of being immoveable, to be retained in its place by a force, as AR, this force must manifestly, with the forces in AP and AQ, produce equilibrium at the point A. Hence, if we produce RA to any point r, and draw rp parallel to AQ; Ap, pr, rA will, by Art. 28, be proportional to the forces in AP, AQ, and AR. Also it has been shewn that the forces in AP, AQ are equal, and therefore Ap, pr are equal, and the angle $rAp = Arp = rAQ$. Hence Ar bisects the angle PAQ, and if po be perpendicular to Ar,

$$Ar = 2Ao = 2Ap \text{ cos. } pAr = 2Ap \text{ cos. } \tfrac{1}{2} PAQ.$$

Hence, if we put the forces P, Q, each $= P$, and the force in $AR = R$; also $< PAQ = A$,

$$\frac{R}{P} = \frac{Ar}{Ap} = \frac{2Ap \text{ cos. } \frac{1}{2}A}{Ap} = 2 \text{ cos. } \tfrac{1}{2}A; \quad \therefore R = 2P \text{ cos. } \tfrac{1}{2}A;$$

and R, the force which would keep A at rest, is evidently equal to the pressure upon that point produced by the chord PAQ: hence we have the pressure upon $A = 2P \text{ cos. } \frac{1}{2} A$.

Cor. 1. If A, instead of a point, be a pully round which the cord passes, the pressure on the pully will be the pressure at the center of the pulley. For in this case, fig. 34, the strings aP, bQ, touch the circle abd of the pulley, and would if produced meet in the line CA which passes through the center, and would make equal angles with it. Hence the resultant of the tensions in aP, bQ passes through the center, and is, as before, equal to $2P \text{ cos. } \frac{1}{2}A$.

Cor. 2. If a string pass over any number of fixed points $ABCD$, and be kept at rest by forces or weights P, Q drawing it in opposite directions, these forces or weights must be equal. And the pressure upon any one of the points, as B, will be $2P \text{ cos. } \frac{1}{2} ABC$.

Prob. VI. Fig. 35. *A given weight* W *is supported by two props* AC, BC, *upon a horizontal plane* AB. *To find the pressure upon each prop, their lengths and the distance at which they stand being given.*

If we take *Cd*, in the vertical line *CD*, to represent the weight of the body, and draw *da* parallel to *BC*, *Ca*, *ad* will represent the pressures (Art. 31.); but, to prepare the student for the solution of succeeding problems, we shall obtain them by a different method.

Let the *re-actions* of the props in the direction *AC*, *BC*, be *P*, *Q*, (see Art. 10.) Let *P* be resolved in the horizontal and vertical directions *AD*, *DC*. Then

$$\frac{\text{horizontal part of } P}{P} = \frac{AD}{AC} = \cos. A;$$

$$\frac{\text{vertical part of } P}{P} = \frac{DC}{AC} = \sin. A;$$

and similarly for *Q*;

$\therefore$ horizontal force of *AC* at $C = P . \cos. A$; of $BC = Q . \cos. B$;
vertical force of *AC* at $C = P . \sin. A$; of $BA = Q . \sin. B$.

And, since these forces support the weight, the horizontal parts must counteract each other, and the vertical parts must together $= W$;

$$\therefore \; P \cos. A = Q \cos. B;$$

$$P \sin. A + Q \sin. B = W.$$

By the first, $Q = \dfrac{P \cos. A}{\cos. B}$; hence, by the second,

$$P \sin. A + \frac{P \cos. A}{\cos. B} \sin. B = W;$$

$$\therefore \; P (\sin. A \cos. B + \cos. A . \sin. B) = W \cos. B;$$

$$\text{or } P . \sin. (A + B) = W \cos. B;$$

$$\text{or } P . \sin. C = W \cos. B;$$

$$\text{and } P = \frac{W \cos. B}{\sin. C}.$$

$$\text{Similarly, } Q = \frac{W \cos. A}{\sin C}.$$

Wherefore, as we can express cos. A, cos. B, sin. C, in terms of AC, BC, AB, we can thus obtain the forces or reactions P, Q. And the pressures upon the props are equal to these the re-actions which the props exert.

Cor. 1. If we make AC, BC, AB equal to a, b, c, respectively, we shall have

$$\cos. B = \frac{b^2 + c^2 - a^2}{2bc}, \text{ (Bridge's } Trig. \text{ p. 58.)}$$

$$\sin. C = \frac{c\sqrt{(2a^2b^2 + 2a^2c^2 + 2b^2c^2 - a^4 - b^4 - c^4)}}{2ab};$$

$$\therefore\ P = \frac{Wa.(b^2 + c^2 - a^2)}{c\sqrt{(2a^2b^2 + 2a^2c^2 + 2b^2c^2 - a^4 - b^4 - c^4)}};$$

$$\text{and } Q = \frac{Wb.(a^2 + c^2 - b^2)}{c\sqrt{(2a^2b^2 + 2a^2c^2 + 2b^2c^2 - a^4 - b^4 - c^4)}}.$$

Cor. 2. The props exert upon the plane at A and B pressures equal to those which are exerted on their upper extremities: these pressures at A and B may be resolved in directions perpendicular and parallel to the plane.

The parts perpendicular to the plane will be

$$\frac{W.\cos. B.\sin. A}{\sin C} \text{ at } A, \text{ and } \frac{W.\cos. A.\sin. B}{\sin. C} \text{ at } B,$$

and these are counteracted by the re-action of the plane.

The parts parallel to the plane will be

$$\frac{W.\cos. B.\cos. A}{\sin. C} \text{ at } A, \text{ and } \frac{W.\cos A.\cos. B}{\sin. C} \text{ at } B;$$

and these, if not counteracted, will make the props slide in opposite directions from A and B along the horizontal plane. They may be counteracted by immoveable obstacles placed behind the props at A and B. They will sometimes be counteracted by the friction of the plane.

PROB. VII. Fig. 36. *A Weight* W *is supported by three props,* AW, BW, CW, *upon a horizontal plane* ABC. *To find the pressure on each: the lengths of the props and the distances at which they stand being given.*

Draw WO perpendicular to the horizontal plane, join AO, and produce it to meet BC in K, and join WK.

The pressures of the three props in their own directions together support the weight, and therefore produce a pressure in the vertical direction OW; also the pressure of AW will not be altered if we substitute for the pressures of BW, CW a pressure equivalent to them both; and this equivalent pressure must, along with AW, produce a vertical pressure in OW; hence it must be in the plane AWO, and therefore in the line KW, for it must manifestly be in the plane BWC; hence the weight W may be supposed to be supported by two props AW, KW, and the pressure on AW found by the last problem. Let as before P, Q, R, represent the pressures of the props AW, BW, CW; then

$$P = W \cdot \frac{\cos. AKW}{\sin. AWK}; \text{ and similarly,}$$

$$Q = W \cdot \frac{\cos. BLW}{\sin. BWL},$$

$$R = W \cdot \frac{\cos. CMW}{\sin. CWM}.$$

COR. 1. Since WO is perpendicular to AK, we have

$$\cos. AKW = \frac{OK}{KW},$$

$$\sin. AWK = \frac{AK}{KW} \cdot \sin. KAW = \frac{KW}{AK} \cdot \frac{OW}{AW};$$

$$\text{hence, } P = W \cdot \frac{KW}{OK} \cdot \frac{KW}{AK} \cdot \frac{AW}{OW}$$

$$= W \cdot \frac{OK}{AK} \cdot \frac{AW}{OW}; \text{ and similarly.}$$

$$Q = W.\frac{OL}{BL}.\frac{BW}{OW};$$

$$R = W.\frac{OM}{CM}.\frac{CW}{OW}.$$

Cor. 2. If we draw AD, OE perpendicular to BC, we shall have

$$\frac{OK}{AK} = \frac{OE}{AD}; \text{ and hence } P = W.\frac{OE}{AD}.\frac{AW}{OW}$$

$$= W.\frac{OE}{OW}.\frac{AW}{AD}$$

$$= W.\frac{1}{\tan. OEW}.\frac{AW}{AD};$$

and similar expressions may be obtained for Q and R.

Cor. 3. It is easily shewn that WE is perpendicular to BC; hence OEW measures the inclination of the planes CBA, CBW. Hence if a sphere with radius = 1, and center C, cut the pyramid, and make a spherical triangle efg, the angle e will be equal to the angle OEW. And if the angles made by the lines CA, CB, CW are known, the sides ef, fg, ge are known, and e may be found.

Cor. 4. The horizontal pressures which are to be resisted by obstacles at the lower ends, A, B, C, are in the directions OA, OB, OC, and are equal to

$$P.\frac{OA}{AW} = W.\frac{OK}{AK}.\frac{OA}{OW};$$

$$Q.\frac{OB}{BW} = W.\frac{OL}{BL}.\frac{OB}{OW};$$

$$R.\frac{OC}{CW} = W.\frac{OM}{CM}.\frac{OC}{OW}.$$

Cor. 5. If the point O fall without the triangle ABC, the weight W cannot be supported.

CHAP. III.

ON THE MECHANICAL POWERS.

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33. *Machines*, or, as they are called in their simplest state, *the Mechanical Powers*, are contrivances to enable a smaller force to keep at rest, or to put in motion a larger weight, or to overcome a greater resistance. We shall at present only consider the case where *equilibrium* is produced; for, knowing the force which would, by means of any machine, just support a weight, it is manifest that a larger force would raise it.

In these cases, as in the case of a lever, the force applied is called *the power*, and the resistance overcome is called *the weight*, and is measured by a weight to which it is equivalent.

The mechanical powers may be reduced to THE LEVER, THE WHEEL AND AXLE, THE TOOTHED WHEEL, THE PULLY, THE INCLINED PLANE, THE WEDGE, AND THE SCREW.

The four first are, in the state of equilibrium, reducible to the lever. The screw may be reduced to the inclined plane, as may the wedge. The way in which the latter is considered is not immediately applicable to it in its common use: instead of being kept at rest by pressure, and put in motion by excess of pressure, as is supposed in our reasonings, it is practically kept at rest by friction, and put in motion by impact.

SECT. I.

MECHANICAL POWERS REDUCIBLE TO THE LEVER.

1. *The Lever.*

This instrument has already been considered in Chap. I.

2. *The Wheel and Axle.*

34. *The Wheel and Axle* consists of a cylinder or *axle* AB, fig. 37, and a concentric circle or *Wheel* EF, joined together, so that the whole is moveable about the axis of the cylinder: the weight W is attached to a chord NW, and will manifestly be raised or lowered as the wheel and axle are turned one way or the other. It is supported by a force applied at the circumference of the wheel EF, either by another weight P acting by means of a string wrapped the contrary way to that at N, or by some other force as P', acting at a point M' in the circumference of the wheel.

Prop. *In the Wheel and Axle the power is to the weight as the radius of the axle to the radius of the wheel**.

Let fig. 38 be a representation of the machine referred to the plane EF, which is perpendicular to the axis. It is evident that the equilibrium will continue to subsist, if we suppose P and W, retaining their distances from the axis, to act in this plane. Let them act in the vertical lines MP and NW, and let MCN be a horizontal line through the center. Hence, considering MCN as a lever,

$P : W :: CN : CM$ by Art. 15.

$:: \text{rad. of axle} : \text{rad. of wheel}.$

It is obvious, that in the state of equilibrium this is the same machine with the lever. When they are put in motion, the two machines differ. In the wheel and axle the weight W

* In this and the following Propositions of this Chapter, the machines are supposed to be in equilibrium.

ascends or descends in a vertical line; in the lever it describes a circular arc.

Cor. 1. The power may act by means of a bar CM', and the wheel may be removed; this is the case in the *capstan* and *windlass*.

Cor. 2. If the direction of the power be not perpendicular to CM, we must draw a perpendicular upon it from C, and the proportion will be

$$P : W :: \text{rad. of axle} : \text{per. on dir}^{n}. \text{ of power.}$$

3. *Toothed Wheels.*

35. If two circles, A, B, fig. 39, moveable about their centers, have their circumferences indented or cut into equal *teeth*, all the way round, and be so placed that their edges touch as at Q, the prominences of one of them at that part lying in the hollows of the other; then if one of them, as A, be turned round by any means, the other will be turned round also. Such circles are called *Toothed Wheels*.

If we suppose the two circles in fig. 39, to be in the same plane, and if, one of them A being turned by a power P acting on a winch CE, the other raise a weight W by means of an axle DF, we shall have the proportion of P and W by the following proposition.

Prop. *In Toothed Wheels, the moment of* P *about the center of the first wheel is to the moment of* W *about the center of the second wheel, as the perpendiculars from the centers of the wheels upon the line of direction of their mutual action.*

The edges of the teeth which act upon one another are conceived to be perfectly smooth; that is, they are supposed by their pressure to exert only a force perpendicular to their surface. All the effect produced to resist motion along a surface is supposed to arise from a defect of smoothness. If the pressure exerted at the point of contact were not perpendicular to the surface pressed, this pressure might be resolved into two forces, one perpendicular to the tangent, and the other in the direction of the tangent, and the latter force is understood to arise from friction, &c. and is at present left out of consideration.

Let the wheel A act upon the wheel B at Q; the action there exerted will be perpendicular to the surfaces which are in contact at that point: and the action of A on B, and the re-action of B on A will be equal and opposite: let this action be a pressure Q in the direction MQN. Then the force Q acting on the wheel B supports the weight W, and the re-action opposite to Q is supported by the power P. Hence, if CM, DN, be perpendiculars on MQN, we shall have, by Art. 20,

$$P : Q :: CM : CE,$$

$$Q : W :: DF : DN;$$

$$\therefore P : W :: CM . DF : DN . CE.$$

Hence multiplying the first and third terms by CE, and the second and fourth by DF, we shall have

$$P . CE : W . DF :: CM : DN,$$

$$\text{or mom. of } P : \text{mom. of } W :: CM : DN.$$

Cor. 1. If CD meet MN in O, we have, by similar triangles,

$$CM : DN :: CO : DO;$$

$$\therefore \text{mom. of } P : \text{mom. of } W :: CO : DO.$$

Cor. 2. If the form of the teeth be such, that the point O is fixed while the wheels revolve, the force continues the same during the motion.

This is the case when the form of the teeth is the involute of a circle.

Cor. 3. If the teeth be small in comparison with the radii of the wheels, Q will nearly coincide with O; and CO, DO will be very nearly the radii of the wheels measured to the point at which the contact takes place. Hence

$$\text{mom. of } P : \text{mom. of } W :: \text{rad. of } A : \text{rad. of } B.$$

Cor. 4. In order that the two wheels may work during a whole revolution, the intervals of their teeth must be equal;

hence the numbers of teeth in each wheel will be as the circumferences, and therefore as the radii: hence

mom. of P : mom. of W :: number of teeth of A : number of teeth of B.

Cor. 5. The case in the figure is a combination of a winch, two toothed wheels, and an axle. If we suppose the radius of the axle DF and the winch CE to be equal, the whole of the mechanical advantage will be owing to the toothed wheels. In this case, we have

$$P : W :: CO : DO.$$

When the number of teeth in A is very small, A is called a *Pinion*, and its teeth are called *Leaves*.

The teeth in which those of the wheel A work may be distributed along the edge of a straight bar instead of the circumference of a circle, the bar being restrained to move in the direction of its length.

Wheels are sometimes turned by simple contact with each other; sometimes by the intervention of cords, straps, or chains, passing over them; and in these cases the minute protuberances of the surfaces, or whatever else may be the cause of friction, prevents their sliding on each other. And at the points of contact an action and re-action are exerted corresponding to those which are supposed in the Proposition.

4. *Pullies.* (1.) *The Single Moveable Pully.*

36. A pully has already been mentioned, Art. 8, &c., as a means of changing the direction of part of the cord by which force is exerted; it is a small wheel which is moveable about its axis, and along part of the circumference of which the cord passes. So long as its axis is immoveable, it can produce only a change of direction; but when its axis is fixed in a *block* or *sheaf* which is moveable, it may produce a mechanical advantage.

Prop. *In the single moveable pully, the strings being parallel,*

$$P : W :: 1 : 2.$$

Let $CBAP$, fig. 40, be the cord passing round the pully AB; and let the force P act by this chord. By Art. 8, the tension of the string is the same throughout, and equal to the power P. Hence AB is supported by two equal and parallel forces in AP, BC; each equal to P; and hence, by Art. 25, the force W, which acts in the opposite direction upon AB, must be equal to their sum. Therefore $W = 2P$.

Cor. 1. If the strings be not parallel, as KA, CB, fig. 41, let them be produced and meet in n; and join on, o being the center of the pully. Then oA and oB, drawn to the points where the string touches the pully, are equal, because the pully is circular. And on is common, and oAn, oBn, right angles. Hence onA, onB, are equal.

The strings AK, BC will produce the same effect as if they acted at n. And the forces or tensions exerted by them are equal, each being equal to P. Hence the resultant bisects the angle AnB, and is in the direction no: and since the forces of the strings support the weight, no must be opposite to the direction in which the weight acts; and therefore vertical.

Let a horizontal line meet the strings in p, q, and the vertical line nm in m. np, nq, will be equal, and may be taken to represent the tensions of the strings AK, BC. And np is equivalent to nm, mp, and nq to nm, mq. And of these, the parts mp, mq destroy each other; and hence the force acting upwards in $2nm$. Therefore,

$$P : W :: np : 2nm$$

~~$:: \text{rad.} : 2\cos. pnm.$~~

If ~~rad. = 1, and~~ $pnm = a$, $W = 2P\cos. a$.

If $a = 0$, the strings are vertical, $\cos. a = 1$, and $W = 2P$, as before.

If $a = 60^0$, or $AnB = 120^0$, $\cos. a = \frac{1}{2}$, and $W = P$.

Cor. 2. When a weight is supported on a moveable pully, the two portions of the string make equal angles with the direction in which the weight acts.

Cor. 3. We may deduce the relation of P to W, in fig. 40, by considering BA as a lever. For if we suppose the point B to be a fulcrum, and the weight W to be supported by a force P acting vertically at A; we have

$$P : W :: Bo : BA :: 1 : 2,$$

as before.

Hence, the pully, in the state of equilibrium, may be reduced to the lever.

(2.) *First System of Pullies. Each Pully hanging by a separate String.*

37. The first system of pullies, fig. 42, is merely a repetition of the single moveable pully. The weight W is supported by the pully A_1; the string which passes round A_1 is supported by A_2; the string which passes round A_2 by A_3, and so on; and at the last string (which may pass over a fixed pully B) the power of P acts.

Prop. *In the First System of Pullies, where all the strings are parallel, and the weights of the pullies inconsiderable,*

$$P : W :: 1 : 2^n;$$

n *being the number of moveable pullies.*

By last Article,

$$\text{tension of } A_1 A_2 = \tfrac{1}{2} \text{ weight at } A_1 = \frac{W}{2},$$

$$\text{tension of } A_2 A_3 = \tfrac{1}{2} \text{ weight at } A_2 = \tfrac{1}{2} \text{ tension of } A_1 A_2 = \frac{W}{2^2},$$

$$\text{tension of } A_3 B = \tfrac{1}{2} \text{ weight at } A_3 = \tfrac{1}{2} \text{ tension of } A_2 A_3 = \frac{W}{2^3}.$$

And similarly, we should have, if A_n were the last of the moveable pullies,

$$\frac{W}{2^n} = \text{tension of } A_n B = \text{power } P,$$

for the tension of the string at which P acts is equal to P. Hence

$$W = 2^n P,$$

when n is the number of moveable pullies.

Cor. 1. If A_1, A_2, A_3, &c. be the weights of the pullies (including the blocks, &c.) respectively, we may consider each pully as a weight appended at that point: hence

$$\text{weight at } A_1 = W + A_1,$$

$$\text{tension of } A_1 A_2 = \tfrac{1}{2} \text{ weight at } A_1 = \frac{W}{2} + \frac{A_1}{2};$$

$$\therefore \text{ weight at } A_2 = \frac{W}{2} + \frac{A_1}{2} + A_2;$$

$$\therefore \text{ tension of } A_2 A_3 = \tfrac{1}{2} \text{ weight at } A_2 = \frac{W}{2^2} + \frac{A_1}{2^2} + \frac{A_2}{2};$$

$$\therefore \text{ weight at } A_3 = \frac{W}{2^2} + \frac{A_1}{2^2} + \frac{A_2}{2} + A_3;$$

$\therefore$ tension of $A_3 B = \frac{1}{2}$ weight at A_3

$$= \frac{W}{2^3} + \frac{A_1}{2^3} + \frac{A_2}{2^2} + \frac{A_3}{2} = P;$$

and so on; and if there be n moveable pullies,

$$\frac{W}{2^n} + \frac{A_1}{2^n} + \frac{A_2}{2^{n-1}} \ldots\ldots + \frac{A_n}{2} = P;$$

$$\therefore W + A_1 + 2A_2 + \ldots\ldots + 2^{n-1} A_n = 2^n P.$$

Cor. 2. If the pullies be all equal, and each equal to A,

$$W + A(1 + 2 \ldots\ldots + 2^{n-1}) = 2^n P,$$

$$W + A(2^n - 1) = 2^n P,$$

$$W = 2^n P - A(2^n - 1).$$

Cor. 3. Hence the weight W is less as A is greater. If we have $2^n P = A(2^n - 1)$, W will $= 0$, and the power will only just support the pullies.

Cor. 4. If the strings be not parallel, we must compare the tension of each with that of the preceding by Cor. 1, to last Article.

(3.) *Second System of Pullies. The same String passing round all the Pullies.*

36*. This system, fig. 43, consists of two blocks; an upper one $B_1 B_2$, and a lower one $A_1 A_2$: each contains a certain number of pullies, and the string passes round them alternately. The weight is hung to the lower block, and the power acts at the loose extremity of the string.

Prop. *In the Second System of Pullies, if the strings be parallel,*

$$P : W :: 1 : n;$$

n *being the number of strings at the lower block.*

Since the same string passes round all the pullies, its tension will be every where the same, and equal to the power P. And n being the number of strings at the lower block, since each of them supports a weight P, they will altogether, supposing them parallel, support a weight nP: hence

$$W = nP.$$

Cor. 1. If we consider the weight of the pullies, it is manifestly only requisite to add the weight of the lower block to W; hence if A be this block,

$$W + A = nP \; : \; W = nP - A.$$

Cor. 2. If the strings be inclined to the vertical we must take the resolved part of the force. But the angle made with the vertical is generally so small that this correction may be omitted.

(4.) *Third System of Pullies. Each String attached to the Weight.*

37*. In this system, fig. 45, each string, as PA_1C_1, supports the weight, partly by its action at C_1, where it is attached, and partly by its pressure on the next string, as A_1A_2.

Prop. *In the Third System of Pullies, the strings being parallel, and the weight of the pullies inconsiderable,*

$$P : W :: 1 : 2^n - 1;$$

n *being the number of pullies.*

For, tension of $PA_1 = P$;

$$\therefore \text{ weight supported at } C_1 = P;$$

$$\text{tension of } A_1A_2 = \text{pressure on } A_1 = 2P;$$

$$\therefore \text{ weight supported at } C_2 = 2P;$$

$$\text{tension of } A_2A_3 = \text{pressure on } A_2 = 2^2P;$$

$$\therefore \text{ weight supported at } C_3 = 2^2P;$$

and so on.

Hence the whole weight W, which is the sum of all those supported at C_1, C_2, C_3, &c. is $W = P + 2P + 2^2P + \dots\dots = (1 + 2 + 2^2 \dots\dots + 2^{n-1}) \cdot P$, if there be n pullies;

$$\therefore W = (2^n - 1) \cdot P.$$

Cor. 1. If we consider the weights of the pullies, and call them A_1, A_2, $A_3 \dots\dots A_n$, we shall have

tension of PA_1 = weight supported at $C_1 = P$;

$\therefore$ pressure on $A_1 = 2P$;

$\therefore$ tension of A_1A_2 = weight supported at $C_2 = 2P + A_1$;

$\therefore$ pressure on $A_2 = 2^2P + 2A_1$;

$\therefore$ tension of A_2A_3 = weight supported at $C_3 = 2^2P + 2A + A_2$;

$\therefore$ pressure on $A_3 = 2^3P + 2^2A_1 + 2A_2 + A_3$;

and so on.

Hence, since the weight W (including the hook, &c. at C_1) is equal to the sum of all the weights supported, if n be the number of pullies;

$$\begin{aligned} W = &(1 + 2 + 2^2 \ldots\ldots + 2^{n-1}) P \\ &+ (1 + 2 + 2^2 \ldots\ldots + 2^{n-2}) A_1 \\ &+ (1 + 2 + 2^2 \ldots\ldots + 2^{n-3}) A_2 \\ &\ldots\ldots\ldots \\ &+ (1 + 2) A_{n-2} \\ &+ A_{n-1} \end{aligned}$$

$$= (2^n - 1) P + (2^{n-1} - 1) A_1 + (2^{n-2} - 1) A_2 \ldots\ldots + A_{n-1}.$$

Cor. 2. Hence, contrary to the other cases, W becomes greater by giving weight to the pullies. If we make $P = 0$, we may find the weight which will be supported by the pullies alone.

Cor. 3. If all the pullies be equal and each $= A$,

$$\begin{aligned} W &= (2^n - 1) . P + (2^{n-1} - 1 + 2^{n-2} - 1 \ldots\ldots + 2 - 1) . A \\ &= (2^n - 1) . P + [2^n - 2 - (n - 1)] . A \\ &= (2^n - 1) . P + (2^n - n - 1) . A = (2^n - 1)(P + A) - nA. \end{aligned}$$

Cor. 4. If the strings be not parallel, we must use Cor. 1, of the single moveable pully.

SECT. II.

MECHANICAL POWERS REDUCIBLE TO THE RESOLUTION OF FORCES.

5. *The Inclined Plane.*

38. An Inclined Plane, that is, a plane inclined to the horizon, is sometimes used as a mechanical power. Let a weight W, fig. 46, be supported on an inclined plane AC, (inclined to the horizon at an angle CAB) by a power P acting in the direction WK.

PROP. *In the Inclined Plane, if* WK *in the direction of the power and* WN *perpendicular to the plane, be intercepted by a vertical line* KN;

$$P : W :: WK : KN.$$

The effect of the plane AC on the weight W will be in a direction WR perpendicular to the plane; for the plane cannot produce any pressure on W in a direction parallel to AC or CA. (See Art. 35.) Hence the weight will be supported in the same manner as if, instead of the plane AC, it were sustained by a string in the direction WR. It may therefore be considered as supported by a string WR, exerting a force equal to the re-action of the plane $(= R)$; a string WK exerting a force equal to the power P; and its weight $(= W)$ acting in the vertical direction WD.

Hence, since KN is parallel to WD, by Art. 28,

$$P : W :: WK : KN.$$

Similarly, $R : P :: WN : WK$.

COR. 1. If the force act parallel to the horizon, in the direction WE, fig. 47, K coincides with E, E is a right angle, and

$P : W :: WE : EN :: BC : AB$, by similar triangles.

In the same manner,

$$R : W :: WN : EN :: AC : AB.$$

COR. 2. If the force act parallel to the plane, K coincides with C,

$$P : W :: WC : NC :: BC : AC,$$
$$R : W :: WN : NC :: AB : AC.$$

COR. 3. If the force act perpendicular to the horizon, in the direction WZ, we must suppose the point K removed to an infinite distance, so that WK, NK, are infinite and equal. Hence

$$P = W, \quad R = 0.$$

COR. 4. If the force act so as to make an angle CWY below the plane equal to CWZ above it, take away from the right angles CWN, CWR, equal angles CWY, CWZ, and we have $YWN = ZWR$; and this $= YNW$ by parallels; therefore

$$YW = YN; \quad \therefore \; P = W.$$

$$\text{Also } \; YWC = YCW = CWZ; \quad \therefore \; YW = YC,$$

$$\text{and } \; NC = 2\,YC = 2\,YN;$$

$$\therefore \; R : W :: WN : NY :: 2\,WN : NC :: 2AB : AC.$$

Cor. 5. If the force act in a direction WS, situate between WZ the perpendicular to the horizon, and WR the perpendicular to the plane, the point K will fall below N, as at M, and the weight, the power, and the re-action of the plane, are represented in magnitude and direction by NM, MW, WN. Hence the re-action of the plane is in the direction WM, and the body W is supported on the underside of the plane.

Cor. 6. In fig. 46, if we draw CV parallel to KW, we have

$$P : W :: KW : KN :: CV : CN.$$

Hence, W being given, P is least when CV is least, that is, when CV coincides with CW, or the force is in the direction of the plane.

Cor. 7. Let two weights W, W', fig. 48, support each other on two inclined planes AC, $A'C$, by means of a string which is parallel to the planes. Let P be the tension of the string, which will be the same on each, then, by Cor. 2,

$$P : W :: BC : AC,$$

$$W' : P :: A'C : BC;$$

$$\therefore \; W' : W :: A'C : AC.$$

Cor. 8. If a body is supported on a curve surface to which AC is a tangent at W, fig. 46, the effect will be the same as if it were supported on the plane AC. For the equilibrium depends only upon the direction of the surface at the point W, which is the same in the plane and in the curve.

Cor. 9. If the weight W, instead of being in contact with the plane in one point only, touch it in a finite portion, or the whole, of its length AC, (as in fig. 48) the proportion of the power and weight will be the same as before, supposing the friction not to be considered. For the weight supported at each point will be in the same proportion to the part of the power which supports it; and hence the whole weight will have this proportion to the whole power.

Cor. 10. Let the angle of inclination of the plane to the horizon (CAB) be α; the angle which the string makes with the plane (KWC) be ϵ: then

$$WK : KN :: \sin. WNK : \sin. KWN,$$
$$:: \sin. CAB : \sin. KWR,$$
$$:: \sin. CAB : \cos. KWC;$$
$$\text{or } P : W :: \sin. \alpha : \cos. \epsilon.$$

6. *The Wedge.*

39. A Wedge is a triangular prism; and, when applied as a mechanical power, is generally used to separate obstacles, by introducing between them its edge and then thrusting it forwards. Thus, if ACc, fig. 49, be the end of a wedge, two objects EW, Ew, which have a tendency to rush together, may be separated by a force, (as a weight P,) applied at the back of the wedge, provided there be an immoveable obstacle at E. In the present Chapter we must consider the power as *in equilibrium* with the resistance, that is, P must be such a power as is just sufficient to prevent the wedge from being driven upwards, and not great enough to force it downwards.

In consequence of the immoveable obstacle at E, and of the nature of the object EW, the point W in the object will, if it move at all, be compelled to move in a certain direction WU*. Whatever force tends to produce motion in W will be effective only so far as it acts in this direction. Thus, if WV be a force acting on the object W, it will be equivalent to WU, and to UV perpendicular to WU: of which the latter is counteracted by the immoveable obstacle at E, and WU is effective in opposing the resistance.

* W will move in a curve to which WU is a tangent at W.

It is manifest, that the weight or resistance at W must be measured by the force which must be applied *immediately* at W to balance it. That is, if UW, uw, be the directions in which the points W, w will move if they move at all, and if we suppose W, w, to represent the forces which must be applied at those points in the directions WU, wu, to keep the parts EW, Ew in their present position when the wedge is removed; W, w, will also represent the resistances which are to be balanced by the wedge.

If we suppose the sides of the wedge to be perfectly smooth, their action at W, w, will necessarily be perpendicular to their surfaces, (Art. 38.)

This being premised, we can find the proportion of the power, and the weight or resistance. We shall take the case in which the wedge is isosceles, that is, when AW is equal to Aw, and the angles AWU, Awu, as also the resistances W, w, are equal. In this case the direction DA in which the power acts, must pass through the point A, and bisect the angle WAw.

Prop. *In the Wedge, to find the proportion of* P *and* W.

Draw OWV perpendicular to AW, and join Ow, which will be perpendicular to Aw, because the triangles OAW and OAw are equal. Join Ww meeting AO in M; therefore $WM = wM$.

Let WV, equal to WO, represent the action of the wedge perpendicular to its side; WV is equivalent to a force WU (which immediately opposes the resistance, and is therefore equal to it,) and a force UV, perpendicular to WU, which is counteracted by the obstacle at E. Hence, UW representing the resistance, $WO = WV$ may represent the re-action on the side of the wedge. Similarly, the re-action on the wedge at w, arising from an equal resistance similarly applied, may be represented by wO. Also WO is equivalent to WM, MO; and wO, to wM, MO; of which WM, wM balance each other; and if the remaining forces MO, MO be balanced by a power $2P$ represented by $2MO$, the whole will be in equilibrium,

$$\therefore 2P : W :: 2MO : WU.$$

Cor. 1. If WU coincide with WV, or the resisting body be to be moved perpendicularly to AC, WMO, ADC will be similar triangles,

$$\therefore P : W :: MO : WV :: MO : WO :: DC : AC;$$

$$\therefore 2P : W :: Cc : AC.$$

Cor. 2. If WU be perpendicular to AD;

$$\therefore P : W :: MO : WU :: MO : MW :: DC : AD;$$

$$\therefore 2P : W :: Cc : AD.$$

Cor. 3. The action of the resistance upon the side AC is necessarily perpendicularly to AC*. The reason why W does not move in that direction, is that it is also acted on by the resistance of an immoveable obstacle E.

Cor. 4. Let CAD, half the angle of the wedge, $= \alpha$, and UWV, the angle contained between WV perpendicular to AC, and WU the direction of the resistance, $= \iota$; W the resistance on each side; $2P$ the power,

$$2P : W :: \frac{2MO}{OW} : \frac{WU}{WV} \text{ because } OW = WV;$$

$$:: 2 \sin. OWM : \cos. UWY,$$

$$:: 2 \sin. \alpha : \cos. \iota,$$

and $2P : 2W :: \sin. \alpha : \cos. \iota$,

where $2W$ is the whole resistance.

7. *The Screw.*

40. The general form of a screw is well known. It consists of a cylinder, as CD, fig. 50, on the surface of which is a projecting rib or *thread* which runs round the cylinder, and at the same time proceeds uniformly along the cylinder

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* This is different from the way in which the wedge is sometimes considered, when the resistances are supposed to act in any direction, as for instance, parallel to AD. This is impossible; for if a body, as W, be pressed upon the side AC with a force parallel to AD, and with no lateral force, it will necessarily slide along AC, and the equilibrium cannot be established.

lengthways. This part of the instrument is inserted into a similar hollow cylinder *AB* which, with its thread, it exactly fits. In fig. 50, half of the internal and half of the external screw are supposed to be removed, for the purpose of shewing its construction.

It is manifest that if the external screw be fixed, the internal one can only move by turning on its axis, by which means it will also move lengthways. If we suppose the vertical cylinder *DC* to be urged in the direction of its length by a weight *W*, it will be clear, by considering the form of the machine, that *DC* will descend; each point of the thread which is in contact with the external screw descending upon the inclined surface of the external thread, as upon an inclined plane. And the weight may be prevented from descending by a force *P* acting at an arm *CM*, which prevents the screw from turning round.

The form of the screw is such that when its axis is vertical, the inclination of the thread to the horizon is at every point the same. The thread may be considered as an inclined plane wrapped round the cylinder. Let in fig. 51, fhf' be a right-angled triangle, of which the base fof' is equal to the circumference of a horizontal section FoF of the cylinder. If then this triangle be wrapped round the cylinder so that fof' coincides with the circle FoF, the hypotenuse fnh will coincide with FnH, the thread of the screw. And FH will be parallel to the axis, and is called the *distance of two contiguous threads*.

PROP. *In a vertical Screw, when a weight* W *is supported by a horizontal force* P *acting perpendicularly at the end of an arm* CM, P : W :: *distance of two contiguous threads* : *circumference of the circle whose radius is* CM.

In fig. 51, let the internal screw, which sustains the weight *W*, be supposed to be supported by its thread resting on the fixed thread *FGH* of the external screw. Then we may suppose a portion of the weight to be supported at each portion of the thread, and the whole weight will be the sum of these portions. Let a weight *w* be supported at n, by means of the arm CM; let cnm be an arm equal to CM, and let a force p, acting horizontally, and perpendicularly to cm, support w. Then w will be sustained in the same manner as if it were upon the inclined plane fnh, for this plane and the thread FnH are in the same direction at the point n. And the effect of the force p is to produce a horizontal pressure on n, which prevents it from descending; let this force be q. Then we have, by the property of the lever,

$p : q :: cn : cm$;

:: circumf. to rad. cn : circumf. to rad. cm;

and, by the property of the inclined plane,

$q : w :: fh : ff' :: FH$: circumf. to rad. cn;

$\therefore p : w :: FH$: circumf. to rad. $cm :: D : C$, suppose;

$$\therefore p = \frac{D}{C} w.$$

In the same manner, let the weight w' be supported at any other point by p' acting at the end of an arm $= CM$; w'' by p'', &c. And we shall have

$$p' = \frac{D}{C} w', \ p'' = \frac{D}{C} w'', \ \&c.$$

$$\therefore (p + p' + p'' + \&c.) = \frac{D}{C} (w + w' + w'' + \&c.).$$

And the sum of all the partial weights will be the whole weight supported; and the power $p + p' + p'' + \&c.$ acting at M will produce the effect of the separate powers p at cm, &c. (Art. 24.) Hence,

$$p + p' + p'' + \&c. = P, \ w + w' + w'' + \&c. = W;$$

$$\text{and } P = \frac{D}{C} W, \text{ or}$$

$P : W :: D : C$ :: dist. of two threads : circumf. to rad. CM.

Cor. 1. Instead of supposing the screw to support a weight W acting vertically, we may suppose it employed to produce a pressure W in any direction, and the proportion will be the same as before.

Cor. 2. In fig. 50, the form of the thread which is wrapped round the cylinder is such that its section through the axis of the screw gives a rectangular profile, with sides parallel and perpendicular to the axis. But the mechanical advantage will be the same, whatever be the form or depth of this profile, so long as the inclination of the thread is the same.

Cor. 3. The proportion of the power and weight would be the same, if the internal screw were fixed, and the external one, carrying the weight, were moveable.

Cor. 4. The diameter of the cylinder does not affect the proportion of P to W, so long as the distance of the threads remains the same.

8. *Combination of Mechanical Powers.*

41. The *advantage* of a simple machine is the number expressing the multiple which the weight or effect produced is of the power or force producing it. The advantages of the different simple mechanical powers are as follows; (see the preceding Articles).

Of the Lever, advantage = $\dfrac{\text{arm of the power}}{\text{arm of the weight}}$,

Wheel and axle............ $\dfrac{\text{radius of wheel}}{\text{radius of axle}}$,

Toothed wheels............ $\dfrac{\text{n}^{\text{r}}\text{. of teeth of wheel}}{\text{n}^{\text{r}}\text{. of teeth of pinion}}$ {nearly, when the teeth are small.}

Single moveable Pully...............	2	when the strings are parallel and the pullies without weight.
First System............................	2^n	
Second System..........................	$2n$	
Third System...........................	$2^n - 1$	

Inclined Plane..........	$\frac{\text{length of plane}}{\text{height of plane}}$	when the power acts parallel to the plane.
Wedge.................	$\frac{\text{side of wedge}}{\text{back of wedge}}$	when the resistance acts perpendicularly to the side.
Screw...........	$\frac{\text{circumf. desc}^{\text{d}}\text{. by power}}{\text{distance of threads}}$	when the power acts in a plane perpendicular to the axis.

From these the *advantage* of compound machines may be found.

42. Prop. *The* advantage *of a combination is found by multiplying together the* advantages *of the separate machines.*

This may be shewn without difficulty in any particular case.

Fig. 52, represents a combination of the screw, the wheel and axle, the pully, and the inclined plane. A winch *BC* turns a cylinder *CD*, on which is the thread of a screw. This thread works in the teeth of a wheel *ED*, which has an axle *EF*. The cord which passes round this axis acts on a system of pullies of the second kind, attached to the fixed point *G*. This system draws a mass *W* up the inclined plane *GH*.

Let P be the power at B, acting perpendicularly to CB;

$$\frac{\text{pressure at } D}{P} = \frac{\text{circ. desc}^{\text{d}}\text{. by } B}{\text{dist. of threads}} = n \text{ suppose,}$$

$$\frac{\text{pressure at } F}{\text{pressure at } D} = \frac{\text{rad. of wheel } ED}{\text{rad. of axle } EF} = n',$$

$$\frac{\text{force at } H}{\text{tension at } F} = \text{number of strings at } H = n'',$$

$$\frac{W}{\text{force in } GH} = \frac{\text{length of plane}}{\text{height of plane}} = n''';$$

$$\therefore \frac{W}{P} = nn'n''n'''.$$

For the sake of example, take the following numbers.

Let CB = 18 inches, distance of threads = 1 inch;

$\therefore$ circumf. by B = 113 inches nearly; $n = 113$.

ED = 2 feet, EF = 6 inches; $\therefore n' = 4$.

Number of strings at H = 4; $\therefore n'' = 4$.

Inclination of plane = 30^0; $\therefore n''' = 2$;

$$\therefore \frac{W}{P} = 113 . 4 . 4 . 2 = 3616.$$

Hence, on such a machine a force of 3 pounds would raise a weight of 10,000 pounds.

43. The following example of a Combination of Levers has some remarkable properties.

In fig. 59 *a*, let CA, AB, BD be three bars moveable in the plain of the paper about centers at C and D, and about joints at A and B. A force acts at E in the direction EF, and produces a pressure at B. Let this pressure be exerted in the direction CB, against a body placed between B and the immoveable obstacle G; and let it be required to determine the magnitude of the pressure. Draw CM perpendicular on EF; CN, DO on AB; DL on CB.

Let the force which acts in EF be P; and let W be the pressure produced at B in the direction CB. The lever CA communicates pressure to the lever DB by means of the bar AB; and the pressure thus communicated is in the direction of the length AB. Let Q be this pressure. The force Q acting on the lever CA in the direction BA balances the force P acting in EF: hence

$$\frac{P}{Q} = \frac{CN}{CM}.$$

Also the force Q acting in the direction AB on the lever DB produces the pressure W: hence

$$\frac{Q}{W} = \frac{DL}{DO};$$

$$\therefore \frac{P}{W} = \frac{CN \cdot DL}{CM \cdot DO}.$$

If we suppose CA, AB to be in the same straight line, CN will vanish, and W will be infinitely greater than P. And if, at the moment when the pressure W is exerted, CA, AB be nearly a straight line, the pressure will be very great in comparison of the force employed, and may be increased without limit.

A combination depending upon principles nearly similar is used in the Stanhope and the Columbian printing presses, for the purpose of pressing together the types and the paper. The considerations by which its convenience is shewn belong partly to the following articles. It will there be seen that when W is very great compared with P, the velocity of B's motion must in the same proportion be small compared with P's. But by the contrivance above described, B's velocity is not small compared with P's, till CA, AB are nearly a straight line. Hence B moves with a convenient rapidity while it is going toward the position in which the great pressure is to be exerted, and then only moves very slow, when it is come into this position and is actually exerting the pressure.

SECT. III.

GENERAL PROPERTY OF THE MECHANICAL POWERS.

44. By means of machines a given force may be made to overcome any resistance, or to raise any weight whatever: but it will be shewn in the following Propositions that what is gained in power is lost in time: that is, in proportion as the force which we exert to move a weight is increased by machinery, the velocity with which the weight moves is diminished.

When bodies move through spaces which have always the same proportion, their velocities have this proportion also. But when the proportion of the space is variable, we may suppose the bodies to describe very small spaces, and the ratio of these will be the ratio of the velocities *ultimately*, that is, by supposing the spaces to be diminished without limit.

PROP. *To find the velocity of a body estimated in a given direction.*

Let a point W, fig. 56, move in a direction Ww. Let WP, wp be parallel lines drawn in any other direction; and let wn be perpendicular on WP. If Ww represent the body's velocity in the direction of its motion, Wn will represent its velocity estimated in the direction WP.

Also we have $Wn = Ww \cdot \cos. PWw$.

45. PROP. *In any of the Mechanical Powers, we shall have power : weight :: weight's velocity in the direction of its action : power's velocity in the direction of its action.*

We shall prove this by an enumeration of the cases of the different mechanical powers.

1. *The Lever.*

46. Let ACB, fig. 53, be a Lever, acted on in directions AP, BW, by forces, PW: and let CM, CN be perpendiculars on the directions of the forces. Let the lever move through a small angle into the position aCb. A and B will describe circular arcs, Aa, Bb, which will be as the velocities of the points A and B, and being very small, may ultimately be taken for straight lines; and hence if am, bn be drawn perpendicular to AP, BW, Am, Bn will be as the velocities in the directions of the forces, by last Article.

Now, considering Aa as a straight line, CAa will ultimately be a right angle; hence,

$$CAM + aAM = \text{a right angle} = CAM + ACM,$$

and taking away CAM, $aAm = ACM$. Hence, the triangles CAM, Aam are similar. In the same way CBN and Bbn are similar. Also the angle aCb being equal to ACB, taking away

aCB, we have $ACa = BCb$; and $CA = Ca$, $CB = Cb$, therefore the triangles ACa, BCb, are similar. Hence, we have these proportions,

$$Am : Aa :: CM : CA,$$

$$Aa : Bb :: CA : CB,$$

$$Bb : Bn :: CB : CN.$$

Hence, compounding the proportions,

$$Am : Bn :: CM : CN$$

$$:: W : P, \text{ by Art. 20.}$$

$\therefore$ P's velocity : W's velocity :: W : P.

2. *The Wheel and Axle.*

47. If the Wheel and Axle, fig. 38, turn through any angle, it is manifest that the arcs described by the points M and N are as CM and CN. But the arcs described are equal to the length of string wrapped at one point and unwrapped at the other, and are therefore as the velocities of P and W. Hence

$$P\text{'s velocity} : W\text{'s velocity} :: CM : CN$$

$$:: W : P, \text{ by Art. 3'}$$

3. *Toothed Wheels.*

48. Let A, B, fig. 54, be Wheels which turn each other in any manner by means of their circumferences. If they are toothed wheels, we suppose the teeth small, so that the point of contact may be conceived to be at O, in the line joining their centers. We will suppose also that the power and weight hang from equal axles CE, DF. In this case $P : W :: CO : DO$, by Art. 35.

Now, let the wheels turn through a small angle, so that the points which were in contact at O, come to m and n. Om and On will be equal, because they have been applied to each other. And drawing meC meeting the circle CE in e, and nDf meeting the circle DF in f, Ee and Ff will be the spaces ascended and descended by P and W. And we have, by the similar sectors in the figure,

$$Ee : Om :: CE : CO,$$

$$On\,(= Om) : Ff :: DO : DF\,(= CE);$$

$$\therefore\ Ee : Ff :: DO : CO,$$

or P's velocity : W's velocity :: W : P.

Cor. Ee, Ff are as the angular velocities of the wheels A and B. Hence, in wheels which work in each other, the angular velocities are inversely as the radii. Hence also the number of revolutions in a given time will be inversely as the radii.

4. *Pullies.*

49. (1.) *In the Single Moveable Pully with parallel strings*, if the weight W, fig. 40, be raised through any space, as 1 inch, each of the strings, AP, BC, will be shortened one inch at the lower end, and hence the power P will move upwards through 2 inches. Hence,

P's velocity : W's velocity :: 2 : 1 :: W : P.

50. (2.) *In the single moveable pully with strings not parallel*; fig. 55, let the pully at A be considered as a point. Let CAK be the position of the string, and let it be moved into the position CaK, so that W ascends through the small space Aa, and P descends through Pp. Take Km, Cn equal to Ka, Ca respectively; and $Am + An$ is the quantity by which the string CAK is shortened, and therefore the quantity by which KP is lengthened, or $Pp = Am + An$. Now when the angle AKa is very small, am may be considered as ultimately perpendicular on AK, and an on AC: hence

$$Am = Aa \cos.\ aAm = Aa \cos.\ a, \text{ if } a = KAa.$$

$$\text{Similarly, } An = Aa \cos.\ a;$$

$$\therefore\ Pp = 2\,Aa \cos.\ a;$$

$$\therefore\ Pp : Aa :: 2 \cos.\ a : 1;$$

or P's velocity : W's velocity :: W : P, by Art. 36.

If the pully be of finite magnitude, as in fig. 41, since, when the change of position is small, the strings KA, CB, may be considered as remaining parallel to themselves, the part of the string AB which is wrapped round the pully is not altered; and hence the length of the space described by P is not altered on this account.

51. (3.) *In the First System of Pullies*, fig. 42, if the weight W be raised through any space, as 1 inch, the pully A_2 is, as in the single moveable pully, raised 2 inches; hence, for the same reason, the pully A_3 is raised 2×2 inches; and similarly, a succeeding pully would be raised $2 \times 2 \times 2$ inches; and so on to P, which will, by this reasoning be lowered 2^n inches: hence

$$P\text{'s velocity} : W\text{'s velocity} :: 2^n : 1 :: W : P.$$

52. (4.) *In the Second System of Pullies*, fig. 43, if the weight W be raised 1 inch, each of the strings by which the lower block hangs will be shortened 1 inch; and hence the whole length of the string between the blocks will be shortened n inches, and P will descend n inches; hence

$$P\text{'s velocity} : W\text{'s velocity} :: n : 1 :: W : P.$$

Cor. In this system, while 1 inch passes round the pully A_1, 2 inches pass round the pully B_1, 3 round A_2, 4 round B_2, &c.

Hence, if the radii of A_1, B_1, A_2, B_2, &c. be as 1, 2, 3, 4, the velocities of their circumferences will be as the radii, and therefore the angular velocities will be equal; and hence A_1, A_2 may be on the same axis, and may form one mass, and similarly B_1 and B_2 may be united on one axis, as in fig. 44.

53. (5.) *In the Third System of Pullies*, fig. 45, let the weight be raised 1 inch; then the pully A_2 will descend 1 inch: on this account the pully A_1 will descend 2 inches; and also on account of C_2 being raised 1 inch, A_1 will descend 1 inch; therefore it will descend $2 + 1$ inches. Again, on this account P will descend $2(2+1)$ or $2^2 + 2$ inches,

and 1 inch more in consequence of C_1 being raised 1 inch; hence, P will descend $2^2 + 2 + 1$ inches $= 2^3 - 1$ inches; hence,

$$P\text{'s velocity} : W\text{'s velocity} :: 2^3 - 1 : 1 :: W : P;$$

and similarly for any number of pullies.

5. *The Inclined Plane.*

54. Let W, fig. 56, be raised through a small space Ww, WP being supposed parallel to wp. Draw WE horizontal, and wm, wn perpendicular to WE, WP. Therefore Wn, wm are ultimately as the velocities in the directions of the power and weight. But if $CAB = wWm = a$, and $CWP = \epsilon$, we have

$$Wn : wn :: Ww . \cos. \epsilon : Ww . \sin. a$$
$$:: \cos. \epsilon : \sin. a;$$

or P's velocity : W's velocity :: $W : P$, Art. 38. Cor. 10.

6. *The Wedge.*

55. Let an isosceles Wedge ADC, fig. 57, in which AD is the line bisecting the back, move in the direction of the line DA through a small space AQ. Let the point W move through a space Wn, in the direction WU, making an angle ι with WV, which is perpendicular to the side AC. Then we shall have

$$Wn = \frac{Wm}{\cos. \iota} = \frac{Aa . \sin. a}{\cos. \iota}, \; a \text{ being} = DAC;$$

$$\therefore Aa \text{ or } Dd : Wn :: \cos. \iota : \sin. a,$$

or P's velocity : W's velocity :: $W : P$, by Art. 39. Cor.

7. *The Screw.*

56. If M, fig. 51, make a whole revolution with a uniform velocity, W will rise with a uniform velocity through the distance of two contiguous threads; and the space de-

scribed by P, estimated in a horizontal direction (in which direction the force is supposed to act) is the circle whose radius is CM; hence

P's velocity : W's velocity :: circle rad. $= DE$: distance of threads :: W : P.

8. *Any Combination of Machines.*

57. In any combination of these machines, the ratio of the power's velocity to the weight's velocity will be found by multiplying the ratios which obtain in the machines of which it is composed; and the ratio of the weight to the power is found by multiplying the ratios in each of the component machines, which ratios has been shewn to be the same as the former; hence the resulting ratios will be the same; and hence, in all combinations of machines by which a power P sustains a weight W, if the machine be put in motion through a very small space,

P's velocity in its direction : W's velocity in its direction :: W : P.

Cor. 1. Hence we have $P \cdot P$'s velocity $= W \cdot W$'s velocity.

A weight multiplied into its velocity is called its *Momentum:* hence P's momentum $=$ W's momentum.

Cor. 2. If $P \cdot P$'s velocity $= W \cdot W$'s velocity, P and W will balance: for if not, let P and W' balance on the same machine: then $P \cdot P$'s velocity $= W' \cdot W'$s velocity: and the velocity of W' is the same as that of W, so long as the machine remains the same. Hence $W' = W$, and therefore P and W balance.

CHAP. IV.

ON THE CENTER OF GRAVITY.

58. *The Center of Gravity of any Body or System of Bodies is a point upon which the Body or System, acted upon only by the force of Gravity, will balance itself in all positions.*

It will be made to appear that in every system there is such a point, by shewing how it may be found in every case. And it will also appear that there is only one point to which the definition is applicable.

Many of the properties of the point which we call the center of gravity, do not depend upon the action of gravity, and might be enunciated and proved without supposing that force to exist. This point has been by some authors called the *center of magnitude*, and by others the *center of parallel forces*.

The definition given above supposes the particles of the system to be connected inflexibly; but the point may be found by the same rule, when the particles are detached from one another.

It follows from our definition, that if a line or a plane which passes through the center of gravity be supported, the system will balance about the line or plane in all positions.

59. Prop. *If a System balance itself upon a line in all positions, the Center of Gravity is in that line.*

When a system of material points balances itself upon a straight line, the moments of the forces which tend to turn it one way and the other must be equal. The moment of the force of each particle to turn the system round the line, will be found by resolving the force into two, one parallel to the line or axis, and the other in a plane perpendicular to the axis. The latter force alone is effective, and its moment, in a given position of the axis, is proportional to the quantity of matter in the particle, and to the distance of the particle from a vertical plane passing through the axis, as appears by Art. 24.

If the center of gravity be not in the line on which the system balances in all positions, the system may be placed in such a position that the line on which it balances itself, and the center of gravity, are not in the same vertical plane. While the system is in this position, let the line on which the system balances be moved parallel to itself, till it passes through the center of gravity; then we have, on one side of the line, increased both the quantity of matter and the distance of each particle from the vertical plane; and on the other side we have diminished both of these. Hence, if the system balanced itself before we moved the line, the tendency of one side to descend will, in this position, be increased on both accounts, and the system will balance no longer, and, therefore, cannot now balance about the same line in all positions; but since the line passes through the center of gravity, the system should now balance upon it by the last article. Hence a contradiction follows from supposing the line about which the system balances itself in all positions to pass otherwise than through the center of gravity.

Cor. 1. By similar reasoning it appears that a system cannot have more than one center of gravity.

Cor. 2. If there be two lines, about each of which a system will balance in all positions, the center of gravity must be at their intersection.

Cor. 3. If a system balance itself upon a line in one position, the center of gravity will be in the vertical plane which in that position passes through the line.

For if not, we might draw a line through the center of gravity, and the system would balance on this line. And hence it would balance on two lines in two different vertical planes, which is impossible, by the reasoning of the Proposition.

60. Prop. *To find the Center of Gravity of two bodies* P, Q, *considered as points, fig.* 59.

Suppose PQ joined by an inflexible rod, and take $P + Q : P :: PQ : GQ$, and $\therefore Q : P :: PG : QG$; G will be the center of gravity. For let the horizontal line MGN meet the vertical lines PM, QN. And since $P : Q :: QG : PG :: GN : GM$ by similar triangles, we have $P \cdot GM = Q \cdot GN$; hence P and Q will balance on G, in every position. Therefore G is the center of gravity.

Cor. 1. The effect of the weights P, Q is the same as if they acted at the points M, N; but in this case, by Cor. 2, Art. 15, the pressure on the fulcrum G is $P + Q$; hence in every position of the two weights the pressure on the center of gravity is equal to their sum.

Cor. 2. To find the center of gravity of any number of bodies P_1, P_2, P_3, P_4, fig 60, considered as points.

Suppose $P_1 P_2$ joined by an inflexible rod, and take $P_1 P_2 : P_1 g_1 :: P_1 + P_2 : P_2$, and, as before, it will appear that P_1, P_2 will balance on g_1 in every position. Also by Cor. 1, the pressure on g_1 is $P_1 + P_2$.

Join $g_1 P_3$, and take $g_1 P_3 : g_1 g_2 :: P_1 + P_2 + P_3 : P_3$; whence $g_2 P_3 : g_1 g_2 :: P_1 + P_2 : P_3$; or $g_2 P_3 : g_2 g_1 ::$ pressure at g_1 : pressure at P_3; whence, as in the beginning of this Article, g_1 and P_3, that is, P_1, P_2, P_3, will balance in every position on g_2, which is therefore the center of gravity of P_1, P_2, P_3. Also, in the same way, the pressure on g_2 is $P_1 + P_2 + P_3$.

Join $g_2 P_4$, and take $g_2 P_4 : g_2 g_3 :: P_1 + P_2 + P_3 + P_4 : P_4$; whence, as before, P_1, P_2, P_3, P_4 will balance in every position on g_3, which is therefore their center of gravity. Also the pressure on g_3 will be $P_1 + P_2 + P_3 + P_4$.

And similarly, we might go on to any number of points.

This construction is applicable if the points be not in the same plane.

Cor. 3. If we take the points P_1, P_2, P_3, P_4, in any other order, we shall find the same point g_3. This appears from the last Article; for a system cannot have more than one center of gravity. It might also be shewn geometrically.

Cor. 4. It appears from the demonstration of Cor. 2, that the pressure on the center of gravity, when it is supported, is equal to the whole weight of the system.

Cor. 5. In order that the parts may *balance each other*, it is necessary that they should be connected, as we have supposed them to be; but the point found as in Cor. 2, is called the *center of gravity* when they are unconnected, and even in motion. Also if the force which acts upon them be not gravity, but any other uniform force acting in parallel lines, this point retains the same denomination.

61. Prop. *To find the Center of Gravity of any number of bodies*, P_1, P_2 P_3, P_4, *considered as points, in the same straight line, fig.* 61.

Let G be the point on which they will balance in a horizontal position; by Art. 46, G will be the centre of gravity. To find G, take any point A in the straight line; and since the weights P_1, P_2, P_3, P_4 balance, we have (Art. 22. Cor. 2.)

$$P_1 \cdot P_1G + P_2 \cdot P_2G = P_3 \cdot P_3G + P_4 \cdot P_4G;$$

$$\text{or } P_1 \cdot (AG - AP_1) + P_2(AG - AP_2)$$

$$= P_3(AP_3 - AG) + P_4(AP_4 - AG);$$

$$\text{or } P_1 \cdot AG - P_1 \cdot AP_1 + P_2 \cdot AG - P_2 \cdot AP_2$$

$$= P_3 \cdot AP_3 - P_3 \cdot AG + P_4 \cdot AP_4 - P_4 \cdot AG;$$

$$\therefore\ P_1 \cdot AG + P_2 \cdot AG + P_3 \cdot AG + P_4 \cdot AG$$

$$= P_1 \cdot AP_1 + P_2 \cdot AP_2 + P_3 \cdot AP_3 + P_4 \cdot AP_4;$$

and similarly, for any number of bodies;

$$\therefore\ AG = \frac{P_1 \cdot AP_1 + P_2 \cdot AP_2 + P_3 \cdot AP_3 + P_4 \cdot AP_4}{P_1 + P_2 + P_3 + P_4}.$$

Hence AG is known, and therefore G.

Cor. 1. If the center of gravity do not lie between P_2 and P_3, but, otherwise, as for instance, between P_1 and P_2; instead of having $P_2(AG - AP_2)$ on the first side of the above equation, we shall have $P_2(AP_2 - AG)$ on the second side, so that the result will be exactly the same.

COR. 2. If any of the points be on the other side of A, their distances from A are to be reckoned negative; thus, in this case, instead of a term $P_5(AG - AP_5)$, we shall have a term $P_5(AG + AP_5)$, or $P_5[AG - (-AP_5)]$.

62. PROP. *To find the Center of Gravity of any number of Bodies* P_1, P_2, P_3, P_4, ..., *considered as points, in the same plane, fig.* 60.

Let G be the center of gravity, found as in Art. 46, Cor. 2. Draw Ax any line in the plane, and draw on it perpendiculars P_1M_1, P_2M_2, P_3M_3 P_4M_4......, g_1h_1, g_2h_2, g_3h_3 ...... Also draw mg_1n parallel to Ax, meeting P_1M_1, P_2M_2, in m, n. Then the triangles P_1g_1m, P_2g_1n are similar; hence, by Art. 46,

$$\frac{P_1m}{P_2n} = \frac{P_1g_1}{P_2g_1} = \frac{P_2}{P_1}; \therefore P_1 \cdot P_1m = P_2 \cdot P_2n;$$

$$\text{or } P_1 \cdot (M_1m - M_1P_1) = P_2 \cdot (M_2P_2 - M_2n);$$

$$\text{or since } M_1m = M_2n = g_1h_1; \text{ transposing}$$

$$(P_1 + P_2) \cdot g_1h_1 = P_1 \cdot P_1M_1 + P_2 \cdot P_2M_2.$$

Similarly, since by Art. 60, $(P_1 + P_2)\ g_1g_2 = P_3 \cdot P_3g_2$, we should have

$$(P_1 + P_2 + P_3) \cdot g_2h_2 = (P_1 + P_2)\, g_1h_1 + P_3 \cdot P_3M_3$$
$$= P_1 \cdot P_1M + P_2 \cdot P_2M_2 + P_3 \cdot P_3M_3:$$

and

$$(P_1 + P_2 + P_3 + P_4)\, g_3h_3 = (P_1 + P_2 + P_3)\, g_2h_2 + P_4 \cdot P_4M_4$$
$$= P_1 \cdot P_1M_1 + P_2 \cdot P_2M_2 + P_3 \cdot P_3M_3 + P_4 \cdot P_4M_4.$$

And in like manner manner for any number of points,

$$(P_1 + P_2 + P_3 + \ldots\ldots)\, GH$$
$$= P_1 \cdot P_1M + P_2 \cdot P_2M_2 + P_3 \cdot P_3M_3 + \ldots\ldots;$$
$$\therefore GH = \frac{P_1 \cdot P_1M_1 + P_2\, P_2M_2 + P_3 \cdot P_3M_3 + \ldots\ldots}{P_1 + P_2 + P_3 + \ldots\ldots};$$

and is therefore known. Hence if we draw a line Kk parallel to Ax at this distance, G must be in this line Similarly, if we draw any other line Ay, we may find the distance of G from Ay, and drawing a line Hh parallel to Ay at this distance, the intersection of Hh with Kk will give the point G.

COR. If P_1M_1, P_2M_2, &c. and GH, instead of being drawn perpendicular to Ax, were drawn in any direction parallel to each other, and meeting Ax in M, M_2, &c. and H; we should still have $GH = \dfrac{P_1 . P_1M_1 + P_2 . P_2M_2 + \ldots\ldots}{P_1 + P_2 + \ldots\ldots}$.

63. PROP. *To find the Center of Gravity of any System of points whatever*, P_1, P_2, P_3 ...... *Fig.* 62.

If we draw any plane, yAx, and draw perpendiculars upon it, P_1M_1, P_2M_2, P_3M_3...... from the bodies, and g_1h_1, g_2h_2,..... from the centers of gravity of P_1, P_2, of P_1, P_2, P_3 ...... ; and GH from the center of gravity of the system: since P_1M_1, P_2M_2, g_1h_1 are in the same plane and perpendicular to M_1M_2, we shall have, by last Article,

$$(P_1 + P_2) . g_1h_1 = P_1 . P_1M_1 + P_2 . P_2M_2.$$

For the same reason, since g_1h_1, g_2h_2, P_3M_3 are in the same plane

$$(P_1 + P_2 + P_3)\, g_2h_2 = (P_1 + P_2)\, g_1h_1 + P_3 . P_3M_3$$
$$= P_1 . P_1M_1 + P_2 . P_2M_2 + P_3 . P_3M_3;$$

and, for any number of bodies,

$$(P_1 + P_2 + P_3 + \ldots)\, GH = P_1 . P_1M_1 + P_2 . P_2M_2 + P_3 . P_3M_3 + \ldots$$

$$\therefore\ GH = \frac{P_1 . P_1M_1 + P_2 . P_2M_2 + P_3 . P_3M_3 + \ldots.}{P_1 + P_2 + P_3 + \ldots.},$$

and is therefore known. Hence, if we draw a plane parallel to the plane xAy, at this known distance, G must be in this plane. Also if we take two other planes, as xAz and yAz, and find the distance of G from each of these, we shall be able to draw two other planes parallel to these, in each of which G must be: therefore it must be at the intersection of these three planes.

64. PROP. *The effect of any System* P_1, P_2, P_3, ... *to produce Equilibrium is the same as if it were collected at its Center of Gravity. Fig.* 60.

Let the system produce equilibrium about a point, or a line, and let a vertical plane pass through the point or line; and let P_1M_1, P_2M_2, P_3M_3 .... be perpendiculars on this plane: then the effect in producing equilibrium about this plane will be the same so long as the *moment* $P_1 . P_1M_1 \mp . P_2M_2 + P_3 . P_3M_3 + \dots$ remains the same. See Art. 22. But when all the system is collected at G, this *moment* becomes $(P_1 + P_2 + P_3 + \dots) \, GH$; and this, by Art. 62, is equal to the moment in the other case, however the plane be drawn. Therefore the effect remains the same as before.

It has been shewn (Cor. 4, Art. 60,) that when the system is supported at the center of gravity, the pressure there is the same as if the system were collected at that point.

Cor. Hence if P_1, P_2, P_3, P_4, fig. 60, instead of being points, be bodies of finite magnitude, we may find the center of gravity of the system, by supposing each body collected in its own center of gravity, and then proceeding as in Art. 60, 61, 62, or 63.

65. Examples *of finding the Center of Gravity.*

Ex. 1. To find the center of gravity of a *Straight Line;* supposed to be of uniform thickness and density.

A straight line will balance itself about its *middle point* in every position: this point is therefore the center of gravity.

Ex. 2. To find the center of gravity of *a Parallelogram,* as $ABCD$, fig. 63.

Bisect the opposite sides AB and DC in E and F, and the opposite sides AD and BC in H and K; and let the lines EF, HK meet in G; G is the center of gravity.

For the parallelogram may be conceived to be made up of lines parallel to AB, as for instance PMQ; and since $PM = AE = EB = MQ$, each of these lines, as PQ, will balance in every position on the point M, that is, on the line EF: hence the whole parallelogram will balance on the line EF. Similarly the whole parallelogram will balance on the line HK. Hence it will balance in every position on the point G; which is therefore the center of gravity.

Ex. 3. To find the center of gravity of *a Triangle;* as ABC, fig. 64.

Bisect AB in E, and AC in F; join CE, BF; the intersection G is the center of gravity.

For the triangle may be conceived to be made up of lines parallel to AB, as PQ: and we have by similar triangles,

$\frac{PM}{AE} = \frac{MC}{EC} = \frac{MQ}{EB}$, and since $AE = EB$, $PM = MQ$.

Hence each of the lines PQ will balance on the line CE in every position, and therefore the whole triangle will balance on that line. Similarly the whole triangle will balance on BF in every position; and hence it will balance in every position on the intersection G, which is therefore the center of gravity.

Join FE; and since $AE = \frac{1}{2} AB$, and $AF = \frac{1}{2} AC$, EF is parallel to BC; hence by similar triangles, AEF, ABC,

$$\frac{EF}{BC} = \frac{AE}{AB} = \frac{1}{2},$$

whence by similar triangles EFG, CBG,

$$\frac{EG}{GC} = \frac{EF}{BC} = \frac{1}{2}; \quad \therefore GC = 2EG; \quad \therefore EC = 3\,EG,$$

$$\text{hence } EG = \frac{1}{3} EC, \text{ and } GC = \frac{2}{3} EC.$$

Cor. 1. If we call the sides opposite to A, B, C, a, b, c respectively, and CE, e; since $AE = EB = \frac{c}{2}$,

$$e^2 = \frac{2a^2 + 2b^2 - c^2}{4} *.$$

* In any triangle ABC, fig. 64, if a side AB be bisected in E; retaining the letters in the text, we have

$$\text{in triangle } ACE,\ b^2 = \left(\frac{c}{2}\right)^2 + e^2 - 2 \cdot \frac{c}{2} \cdot e \cos. CEA,$$

$$\text{in triangle } BCE,\ a^2 = \left(\frac{c}{2}\right)^2 + e^2 + 2 \cdot \frac{c}{2} \cdot e \cos. CEA,$$

add, and we have

$$a^2 + b^2 = \frac{c^2}{2} + 2e^2;$$

whence the formula in the text.

Hence $CG = \frac{2}{3} e = \frac{\{2(a^2 + b^2) - c^2\}^{\frac{1}{2}}}{3}$.

Cor. 2. If we call GA, GB, GC, h, k, l, respectively, we shall have $l = \frac{2}{3} e$; whence

$$3l = \{2(a^2 + b^2) - c^2\}^{\frac{1}{2}}.$$

Hence $9l^2 = 2a^2 + 2b^2 - c^2$;

similarly, $9h^2 = 2b^2 + 2c^2 - a^2$;

$9k^2 = 2c^2 + 2a^2 - b^2$; and, by addition,

$$9(h^2 + k^2 + l^2) = 3(a^2 + b^2 + c^2);$$

or $3(h^2 + k^2 + l^2) = a^2 + b^2 + c^2$.

Cor. 3. If three equal bodies be placed in the angles of a triangle, the center of gravity of these bodies is the same as the center of gravity of the triangle.

Cor. 4. To find the center of gravity of any *Polygon*, divide it into triangles; and supposing each of these collected at its center of gravity, find the center of gravity of the whole; which, by Cor. to Art. 64, will be the center of gravity of the polygon.

Ex. 4. To find the center of gravity of *a Quadrilateral* $ACBC'$, fig. 65, which has two adjacent sides equal, and also the two other adjacent sides equal: $AC = BC$, and $AC' = BC'$.

Join CC', which will bisect AB in D, and will be perpendicular to AB. Let E be the center of gravity of ABC and F of ABC; if we take G so that

$$EG : FG :: ABC' : ABC :: DC' : DC,$$

G will be the center of gravity:

and hence $EG : EF :: DC' : CC'$, and $EG = \frac{DC' . EF}{CC'}$.

Let $DC = c$, $DC' = c'$; $\therefore DE = \frac{c}{3}$, $DF = \frac{c'}{3}$; $\therefore EF = \frac{c + c'}{3}$;

$\therefore EG = \frac{c'}{c + c'} . \frac{c + c'}{3} = \frac{c'}{3}$; $\therefore DG = DE - EG = \frac{c}{3} - \frac{c'}{3} = \frac{c - c'}{3}$.

Cor. Similarly if C and C' were both on the same side of AB, we should have

$$DG = \frac{c + c'}{3}.$$

Ex. 5. To find the center of gravity of *a quadrilateral* $ABDC$, fig. 66, of which two sides AB, CD are parallel.

Bisect AB, CD in H and K, and join HK; all lines parallel to AB will balance on HK, and therefore the center will be in that line. Join BC, CH, BK; and take $CE = \frac{2}{3} CH$, and $BF = \frac{2}{3} BK$; E and F will be the centers of gravity of the triangles ABC, DBC, which may, by Cor. to Art 64, be considered as collected at those points. Hence, by Cor. to Art. 62, if EM, FN be parallel to HK, and G be the center of gravity,

$$GH = \frac{\text{triangle } ABC \,.\, EM + \text{triangle } BCD \,.\, FN}{\text{triangle } ABC + \text{triangle } BCD}.$$

Let CL be parallel to KH, CI perpendicular to AB; therefore, by similar triangles,

$$\frac{EM}{CL} = \frac{HE}{HC} = \frac{1}{3}; \quad \frac{FN}{KH} = \frac{BF}{BK} = \frac{2}{3};$$

$$\therefore EM = \tfrac{1}{3} CL = \tfrac{1}{3} KH, \quad FN = \tfrac{2}{3} KH;$$

$$\therefore GH = \frac{\frac{1}{2} AB \,.\, CI \,.\, \frac{1}{3} KH + \frac{1}{2} CD \,.\, CI \,.\, \frac{2}{3} KH}{\frac{1}{2} AB \,.\, CI + \frac{1}{2} CD \,.\, CI}$$

$$= \frac{AB \,.\, KH + 2 CD \,.\, KH}{3 (AB + CD)}.$$

If $AB = a$, $CD = b$, $KH = c$,

$$GH = \frac{c}{3} \frac{a + 2b}{a + b}.$$

Cor. When $b = 0$, this gives $GH = \frac{c}{3}$, and the trapezium becomes a triangle.

Ex. 6. To find the center of gravity of *a Pyramid whose base is a triangle* ABC, fig. 67, and whose vertex is O.

Bisect BC in D, join AD, OD, and take $DE = \frac{1}{3} DA$, $DF = \frac{1}{3} DO$; join OE, AF; their intersection G will be the center of gravity of the pyramid.

The pyramid may be conceived to be made up of planes parallel to ABC, as PQR; E is the center of gravity of the triangle ABC, and N, where OE meets PQR, will be the center of gravity of PQR; as may easily be shewn. Hence each of the triangles PQR will balance on the line OE, and hence the whole pyramid will balance in any position about OE. Similarly, the whole pyramid will balance on the line AF: hence it will balance in every position on the intersection G, which is therefore the center of gravity.

By similar triangles,

$$\frac{EF}{AO} = \frac{ED}{AD} = \frac{1}{3}; \text{ and } \frac{EG}{GO} = \frac{EF}{AO} = \frac{1}{3}:$$

hence $GO = 3EG$; $EG = \frac{1}{4} EO$, and $GO = \frac{3}{4} EO$.

Cor. 1. Bisect AO in H, and draw HK parallel to OE; hence by similar triangles,

$$\text{since } AH = \frac{1}{2} AO, \therefore AK = \frac{1}{2} AE = DE; \therefore DK = 2DE.$$

$$\text{Also } HK = \frac{1}{2} OE, \text{ and } GE = \frac{1}{4} OE; \therefore HK = 2GE.$$

Hence $DE : DK :: GE : HK$, and DGH is a straight line bisected in G.

Hence we have this theorem: if in a triangular pyramid we bisect two edges which do not meet, and join the points of bisection, and bisect the joining line; the last bisection is the center of gravity of the pyramid.

Cor. 2. To find OG, let the edges of the pyramid adjacent to O, viz. OA, OB, OC be a, b, c; and the others BC, CA, AB, a', b', c', respectively: also let AD, OD, OE, be e, f, g.

Then we shall have *

$$g^2 = \frac{6f^2 + 3a^2 - 2e^2}{9}.$$

And by Cor. 1, to Ex. 3, we have

$$e^2 = \frac{2b'^2 + 2c'^2 - a'^2}{4},$$

$$f^2 = \frac{2b^2 + 2c^2 - a'^2}{4}.$$

Hence, by substitution,

$$g^2 = \frac{3(a^2 + b^2 + c^2) - (a'^2 + b'^2 + c'^2)}{9}.$$

And $OG = \frac{3}{4} \cdot g = \frac{1}{4} \left\{3(a^2 + b^2 + c^2) - (a'^2 + b'^2 + c'^2)\right\}^{\frac{1}{2}}.$

Cor. 3. If we join the center of gravity with each of the four angles O, A, B, C, and call the distances h, k, l, m, respectively, we shall have h^2, k^2, l^2, m^2, by formulæ easily derived from the preceding; and adding these together, we shall have

$$4(h^2 + k^2 + l^2 + m^2) = a^2 + b^2 + c^2 + a'^2 + b'^2 + c'^2.$$

Cor. 4. Since EG is $\frac{1}{4}$ of EO, it is manifest that if we draw parallel lines through G and O, meeting the base, the distance of G from this plane will be $\frac{1}{4}$ of the distance of O.

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* In any triangle AOD, fig. 67, if a side AD be divided so that DE is $\frac{1}{3}$ of DA; retaining the letters in the text, we have

in triangle DOE, $f^2 = \left(\frac{e}{3}\right)^2 + g^2 + 2 \cdot \frac{e}{3} \cdot g \cos. OEA$;

in triangle AOE, $a^2 = \left(\frac{2e}{3}\right)^2 + g^2 - 2 \cdot \frac{2e}{3} \cdot g \cos. OEA$.

Add twice the first to the second, and we have

$$2f^2 + a^2 = \frac{6e^2}{9} + 3g^2;$$

whence the formula in the text.

Ex. 7. To find the center of gravity of *any Pyramid*, whose base is a polygon $ABCDE$, fig. 68, and vertex O.

The polygon may be divided into triangles by lines drawn from one angle to another; and if planes pass through these lines and through the vertex, the pyramid will be divided into triangular pyramids. If a plane be drawn parallel to the base, at a distance equal to $\frac{1}{4}$ of the altitude of the pyramid, by Cor. 4 to last example, the center of gravity of each of the triangular pyramids, and therefore of the whole pyramid, will be in this plane. But if we join O with F the center of gravity of $ABCDE$, it will appear, as in the last Example, that the center of gravity will be in this line. Hence it will be in the point G where the line meets the plane. Also it is manifest that

$$FG = \tfrac{1}{4} FO, \text{ and } OG = \tfrac{3}{4} OF.$$

Cor. If the number of sides of the polygonal base of the pyramid be increased without limit, the method of finding the center of gravity remains the same. Hence it will be true in the case to which we thus approximate, that is, that of a *Conical body with a Curvilinear base.* In all such cases we must find the center of gravity by measuring from the vertex $\frac{3}{4}$ of the line which joins that point with the center of gravity of the base.

Ex. 8. To find the center of gravity of *a Frustum of a Pyramid;* cut off by a plane parallel to the base.

The two ends will be similar figures; let a, b, be homologous sides of the larger and smaller end. Also let the centers of gravity of the two ends be joined, and let the line which joins them be called the axis and be $= c$. Then the center of gravity will be in the axis, and it may be shewn, as in Ex. 5, that its distance from the larger end along this side will be

$$\frac{c}{4} \cdot \frac{a^2 + 2ab + 3b^2}{a^2 + ab + b^2}.$$

Cor. The same will be true of the *Frustum of a Cone;* a, b, representing the radii, or any homologous lines, in the two ends.

GENERAL PROPERTIES OF THE CENTER OF GRAVITY.

66. Prop. *If in a system consisting of any number of particles, a* point *be taken, and if each particle be multiplied into the square of its distance from the* point, *the sum of these products will be the least when the* point *is the center of gravity.*

Let O, fig. 69, be the point, and G the center of gravity of P_1, P_2, &c. Join GO, and draw $P_1 M_1$, $P_2 M_2$, &c. perpendicular on it, and join $P_1 G$, $P_2 G$, &c. and $P_1 O$, $P_2 O$, &c.

Then

$$\overline{P_1 O}^2 = \overline{P_1 G}^2 + \overline{GO}^2 - 2\, GO \,.\, GM_1,$$
$$\overline{P_2 O}^2 = \overline{P_2 G}^2 + \overline{GO}^2 - 2\, GO \,.\, GM_2,$$
$$\text{\&c.} = \text{\&c.}$$

Hence
$$P_1 \,.\, \overline{P_1 O}^2 + P_2 \,.\, \overline{P_2 O}^2 + \text{\&c.}$$
$$= P_1 \,.\, \overline{P_1 G}^2 + P_2 \,.\, \overline{P_2 G}^2 + \text{\&c.}$$
$$+ P_1 \,.\, \overline{GO}_2 + P_2 \,.\, \overline{GO}^2 + \text{\&c.}$$
$$- 2 P_1 \,.\, GO \,.\, GM_1 - 2 P_2 \,.\, GO \,.\, GM_2 - \text{\&c.}$$
$$= P_1 \,.\, \overline{P_1 G}^2 + P_2 \,.\, \overline{P_2 G}^2 + \text{\&c.}$$
$$+ (P_1 + P_2 + \text{\&c.})\, \overline{GO}^2$$
$$- 2\, GO\, (P_1 \,.\, GM_1 + P_2 \,.\, GM_2 - P_3 \,.\, GM_3 - P_4 \,.\, GM_4.)$$

But by the property of the center of gravity,

$$P_1 \,.\, GM_1 + P_2 \,.\, GM_2 - P_3 \,.\, GM_3 - P_4 \,.\, GM_4 = 0;$$
$$\therefore P_1 \,.\, \overline{P_1 O}^2 + P_2 \,.\, \overline{P_2 O}^2 + P_3 \,.\, \overline{P_3 O}^2 + P_4 \,.\, \overline{P_4 O}^2$$
$$= P_1 \,.\, \overline{P_1 G}^2 + P_2 \,.\, \overline{P_2 G}^2 + P_3 \,.\, \overline{P_3 G}^2 + P_4 \,.\, \overline{P_4 G}^2$$
$$+ (P_1 + P_2 + P_3 + P_4)\, GO^2.$$

And it is manifest that the second side will diminish as GO diminishes, and will be least when GO is 0, or when O coincides with G.

Cor. If with center G and radius GO we describe a circle, the sum of each particle into the square of its distance from O will be the same in whatever part of the circumference O is.

For GO will be the same in all the situations of O.

67. Prop. *In any machine kept in equilibrium by the action of two weights, if an indefinitely small motion be given to it, the center of gravity of the weights will neither ascend nor descend.*

It is easy to shew this independently, in each of the mechanical powers.

In the straight lever, the center of gravity is at the fulcrum, and remains fixed, however the lever be moved.

In the wheel and axle, fig. 38, the center of gravity of P and W is at G, in the vertical line passing through the center C, and if P descends, W ascends, and G remains fixed, as if PGW were a lever.

In the toothed wheels, fig. 54, if P ascends W descends; and the center of gravity G remains fixed in a point G, such that

$$PG : WG :: DO : CO.$$

In the systems of pullies, fig. 41, 42, 43, 44, 45, if we join P and W, and take $PG : WG :: W : P$, G will be the center of gravity; and if P descend W will ascend, so that P's descent : W's ascent :: $W : P :: PG : WG$; whence G remains fixed.

In the inclined plane, fig. 58, when the force is parallel to the plane, let P support W: and let P, W be their situations when they are in the same horizontal line. Let P descend to p, and W ascend to w; $\therefore Pp = Ww$: join wp meeting WP in g; draw wm perpendicular on WP; now by similar triangles,

$$wg : pg :: wm : Pp :: wm : Ww :: BC : AC :: P : W;$$

therefore g is the center of gravity of p, w. Hence the center has moved in the horizontal line Gg.

This is true, whatever be the space described.

The wedge and screw do not generally act by gravity; when they do, the same property is easily proved.

CHAP. V.

PROBLEMS CONCERNING THE EQUILIBRIUM OF RIGID BODIES.

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68. Bodies are hard or soft, rigid or flexible, extensible or inextensible, elastic or inelastic; and in all cases the conditions of their equilibrium may be deduced from the properties of the lever; which, as we have seen, leads to the properties of other mechanical combinations. For the present we shall consider only the case of rigid bodies; that is, of bodies which do not change the dimensions or figure of any of their parts by the action of any of the forces which we suppose to be applied to them.

Prop. *A lever is kept at rest by any two forces; it is required to find the pressure on the fulcrum.*

Let ACB, fig. 70, be a lever acted on by two forces P and Q; the lever and the two forces will be in the same plane. Let a portion of this plane, as EF, including the lever, be supposed to be material and rigid, moveable about C in its own plane, and acted on by the forces P and Q. Then this plane will be kept at rest in the same manner as the lever was; and if CM, CN be perpendicular upon the directions of the forces, we shall have

$$P : Q :: CN : CM.$$

Let the directions of the forces meet in D, and let Cp, Cq be parallel to BD, AD; then the angle CqN is equal to CpM, and the triangles CpM, CqN are similar; and

$$CN : CM :: Cq : Cp.$$

Therefore $P : Q :: Cq : Cp$, that is, $P : Q :: Dp : Dq$. Hence, Dp, Dq may represent the forces P and Q, and DC would, on the same scale, be their resultant, if they acted at D. But the force P produces the same effect as if it were applied at any other point of its direction AD, considering AD as a material line; and similarly of Q. Hence P and Q produce the same effect as if they acted at D; therefore they produce a pressure on C, which is equal to the resultant of the two forces.

Now the pressure on C will continue the same if any portion of the plane be removed. Suppose portions of the plane to be removed till nothing is left but the material line ACB composing the lever: then the pressure on C will be the same as before. Hence the pressure on the fulcrum of a lever agrees, in magnitude and direction, with the resultant of the two forces which act upon the lever, and keep it at rest.

This pressure acts in the direction of the line joining the intersection of the forces and the fulcrum.

Cor. 1. If the point C be acted upon by a force R, which is in the direction CD, and equal to the resultant of P and Q, the three forces P, Q, R, will keep the line ACB at rest, supposing no point of it to be fixed.

Cor. 2. If the forces be parallel, by Art. 18, the pressure on the fulcrum will be the sum of the forces, and also parallel to them. And by the same Article it appears that its distance from the two forces will be inversely as the forces.

Cor. 3. Hence it appears that two parallel forces produce the same effect as a force equal to their sum, acting in a direction parallel to them, and so situated that its distance from each force is inversely as the force.

This gives us the resultant of two parallel forces.

69. Prop. *In a Lever acted on by any number of forces in the same plane, the pressure on the Fulcrum is equal to the resultant of all the forces, supposing them applied at that point.*

Let any forces P, Q, and P', Q', &c. act on a lever CA, CB, &c. fig. 71.

Let P and P' meet in D, and let their resultant be R, in the direction DR: let DR meet Q in E, and let the resultant of R and Q be R', in the direction ER'; let ER' meet Q' in F, and let the resultant of R' and Q' be S; then S will be the pressure on the fulcrum. For since a force produces the same effect at whatever point of its direction it be supposed to act, P and P' produce the same effect as if they acted at D, and therefore the same effect as R; R and Q produce the same effect as if they acted at E, and therefore the same effect as R'; and R' and Q' produce the same effect as if they acted at F, and therefore the same effect as S. Hence P, P', Q, Q' produce the same effect as S; but P, P', Q, Q' keep the system in equilibrium round C: therefore S does so; and therefore it passes through C; and hence it produces on C a pressure S; therefore P, P', Q, Q', produce on C a pressure S.

Also, since R at D is equivalent to P, P'; R at E is also equivalent to P, P'; therefore R and Q at E, or R', is equivalent to P, P', Q; therefore also at F, R' is equivalent to P, P', Q; therefore R' and Q', or S, is equivalent to P, P', Q, Q' acting at the same point.

Hence the pressure on the fulcrum is the resultant of all the forces applied at one point.

It will be in the direction CF, for the force in FS produces the same effect as the forces P, Q, P', Q'. But these forces keep the lever at rest about C. Therefore the force in FS does not tend to turn the lever about C, and therefore passes through C.

Cor. Hence if $ABA'B'$ were a rigid body, and were acted on by the forces P, Q, P', Q', and also by a force S acting in FC, the body would be kept at rest, supposing no point to be fixed.

For the force S, acting thus, would produce the same effect as a fulcrum.

70. Prop. *When three forces act upon any body and keep it at rest, (1°), any one of them must be equal and opposite to the resultant of the other two; (2°), and must pass through the intersection of the other two.*

Let a body *EF*, fig. 70, be kept at rest by three forces *P*, *Q*, *R*. Take a point *C* in the direction of one of the forces *R*; and instead of a force *R*, suppose an immoveable fulcrum at *C*; then the re-action of this fulcrum will produce the same effect as the force *R*: but in this case, by Art. 68, the re-action will be equal to the resultant of the two forces *P* and *Q*, and will pass through their intersection. Hence the force *R* must fulfil these two conditions. And similarly, the Proposition is true of the forces *P* and *Q*.

Cor. In the same manner, by Art. 69, if a rigid body be kept at rest by any number of forces, as *P*, *Q*, &c. and *S*, fig. 68; any one of them, as *S*, must be equal to the resultant of all the others. Also it must pass through the point *F*, found as in Art. 69; and its direction must be opposite to the direction of the resultant of the other forces.

We shall proceed to give examples of the manner in which we may determine, in particular problems, the conditions of equilibrium of a rigid body.

1. *Equilibrium on a Point.*

71. When a rigid body is moveable about a fixed point, its conditions of equilibrium are reducible immediately to those of a lever, of which the fixed point is the fulcrum; including, amongst the forces which act upon the lever, the weight of the body, supposed to be collected in its center of gravity.

Prob. I. *In the Common Balance, the weights being unequal, to find the position in which it will rest.*

The common balance consists of a beam *AB*, fig. 72, which is moveable about an axis *C*, and from which the scales are suspended at points *A* and *B*. The axis *C* is so placed, that, in the horizontal position, it is a little above the straight line

AB joining the points of suspension. Let CD be perpendicular to AB; then the two arms DA, DB, must be equal in length and weight. Let $DA = DB = a$, $CD = b$. Also let G, a point in CD, be the center of gravity of the beam, and $CG = h$.

Draw MCN horizontal, meeting in H and E the vertical lines through G and D; and let θ be the angle which AB makes with the horizon, and therefore the angle which CD makes with the vertical. Then, since

$$AD = DB,\ EM = EN = a \cos. \theta\,;$$
$$CE = b \sin. \theta\,;\ CH = h \sin. \theta.$$

And if P and Q are the weights at A and B, and W the weight of the beam,

$$P\,.\,CM = Q\,.\,CN + W\,.\,CH\,;$$

or $P\,\{a \cos. \theta - b \sin. \theta\} = Q\,\{a \cos. \theta + b \sin. \theta\} + Wh \sin. \theta.$

$$\text{Hence, } \tan. \theta = \frac{(P - Q)\,a}{(P + Q)\,b + Wh}.$$

If we suppose D to be the difference of the weights, so that $P = Q + D$, we shall have

$$\frac{\tan. \theta.}{D} = \frac{a}{(2\,Q + D)\,b + Wh}.$$

The requisites of a good balance are the following: 1. It should rest in a horizontal position when loaded with equal weights. 2. It should have great *sensibility*; that is, the addition of a small weight in either scale should disturb the equilibrium, and make the beam incline sensibly from the horizontal position. 3. It should have great *stability;* that is, when disturbed from the position of equilibrium it should quickly return to it.

The first requisite will be obtained if the arms are equal, and the center of gravity lower than the point of suspension.

The sensibility is greater in proportion as for a given small difference of weights the inclination of the balance is greater, that is, in proportion as for a given small value of D,

θ is greater. It is greater also in proportion as for a given value of θ, D is less. It may therefore be conceived to be measured by $\frac{\theta}{D}$ or nearly by $\frac{\tan.\,\theta}{D}$. Hence the sensibility of a balance is as

$$\frac{a}{(2\,Q + D)\,b + Wh}.$$

But D is small compared with $2\,Q$, and may be neglected. Hence the sensibility is as

$$\frac{a}{2\,Qb + Wh}.$$

Hence the sensibility of a balance is increased—by increasing the length of the arms (a)—by diminishing the weight of the beam (W)—by diminishing the distance between the center of motion and the center of gravity of the beam (h)—by diminishing the distance between the center of motion and the line joining the points of suspension (b).

The stability is as the force which at a given angle of inclination urges the balance to the position of equilibrium. Let the weights be equal, and this force is

$$2\,Q\,.\,CE + W\,.\,CH = (2\,Qb + Wh)\ \sin.\ \theta.$$

Hence the measure of the stability is $2\,Qb + Wh$.

Cor. By increasing the lengths of the arms we increase the sensibility without diminishing the stability.

Prob. II. *Fig.* 73. *From a given rectangle* ABCD, *of uniform thickness, to cut off a triangle* CDO, *so that the remainder* ABCO, *when suspended at* O, *shall hang with the sides* AO, BC *horizontal**.

Let G be the center of gravity of BO, and H of CEO; OE, Gg, Hh, being vertical, and therefore perpendicular to AD.

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* This is Prop. 5, of Pappus's Mathematical Collections, Book 8.

Hence $Og = \frac{1}{2} OA$, $Oh = \frac{1}{3} OD$.

Let $AD = a$, $AB = b$, $DO = x$; $\therefore AO = a - x$.

Now

$$Og \,.\, \text{rectangle } AE = Oh \,.\, \text{triangle } CEO,$$

$$\text{or } \frac{a-x}{2} \,.\, b \,.\, (a-x) = \frac{x}{3} \,.\, b \,.\, \frac{x}{2};$$

$$\therefore 3(a-x)^2 = x^2;$$

$$\therefore 2x^2 - 6ax = -3a^2;$$

$$\therefore x = \frac{a}{2}(3 \pm \sqrt{3}).$$

$$= .634a.$$

The negative sign is to be taken: the positive sign would place O beyond A.

2. *Equilibrium on a Surface.*

72. When a body rests on a given surface it will touch it either in one point, or in several points, or with a finite portion of its surface. In all these cases the body must be supposed to be acted on by forces perpendicular to the surface at the points where it is in contact; that is, by the re-action of the surface at those points.

PROB. III. *Fig.* 74. *A Paraboloid* DAd *rests upon a horizontal plane; to find its position.*

If PK be a vertical line drawn through the point of contact, meeting the axis in K, this line must pass through the center of gravity; for the body may be supposed to be collected in its center of gravity, and it will then be supported by the re-action which acts in the line PK. And since the center of gravity is a point in the axis, K must be this center. Also, since PK is perpendicular to the tangent at P, PK is a normal; and hence if PN be perpendicular to the axis, by Conic Sections, $NK = \frac{1}{2}$ parameter $= \frac{1}{2} L$.

And

$$\tan. AKP = \frac{NP}{NK} = \frac{\sqrt{L \, . \, AN}}{\frac{1}{2}L} = 2\sqrt{\frac{AN}{L}} = 2\sqrt{\frac{AK - \frac{1}{2}L}{L}}$$

$$= 2\sqrt{\left\{\frac{AK}{L} - \frac{1}{2}\right\}}.$$

If AK be less than $\frac{1}{2}L$, this answer is impossible, that is, there will no longer be an oblique position of equilibrium, and the figure will not rest except when the axis is vertical.

It will be seen (in the Supplement,) that in a homogeneous paraboloid, if K be the center of gravity, $AK = \frac{2}{3}AC$. Hence the oblique position of equilibrium is possible so long as $\frac{2}{3}AC > \frac{1}{2}L$, or $AB > \frac{3}{4}L$.

Prob. IV. *Fig.* 75. *A solid composed of a Cone and a Hemisphere on the same base rests on a horizontal plane: to find its dimensions that it may rest on the hemispherical end in all positions.*

PC, the vertical line, will, in all positions, meet the axis in the center of the sphere; and hence this point must be the center of gravity of the whole figure. Let G be the center of gravity of the hemisphere, and H of the cone; and we must have,

$$\text{mass of cone} \times CH = \text{mass of hemisphere} \times CG.$$

The cone is $\frac{1}{3}$, and the hemisphere is $\frac{2}{3}$, of the circumscribing cylinder. Hence cone = base $DE \times \frac{1}{3}BC$, and the hemisphere = base $DE \times \frac{2}{3}AC$. Also by Art. 65, $CH = \frac{1}{4}BC$; and by the Supplement, $AG = \frac{5}{8}AC$, and $CG = \frac{3}{8}AC$. Hence

$$\text{base } DE \times \tfrac{1}{3}BC \times \tfrac{1}{4}BC = \text{base } DE \times \tfrac{2}{3}AC \times \tfrac{3}{8}AC;$$

$$\therefore \; BC^2 = 3AC^2.$$

Hence $BD^2 = BC^2 + CD^2 = 4AC^2$; and $BD = 2AC = DE$.

Hence the triangle DBE is equilateral.

Prob. V. *Fig.* 76. *When a body is supported on three vertical props* (A, B, C); *to find the pressure on each.*

Let G be the center of gravity of the body, Gg a vertical line meeting the plane ABC in g; join Ag, meeting BC in D; then, if we suppose the whole mass collected at the center of gravity, it may be considered as supported on a lever AD; and if W be the whole weight,

pressure at $A : W :: Dg : DA ::$ triangle BgC : triangle BAC.

In the same manner,

pressure at B (or C) : W :: triangle AgC (or AgB) : triangle BAC.

Hence the pressure on each prop is as the triangle opposite to it, made by joining the angles of the triangle ABC with the point g.

Cor. When a body is supported on four vertical props, as a table on its four legs, the pressures will be indeterminate, if we consider the body as perfectly rigid. For since it may be supported on three of these props, the fourth may support either nothing, or a finite portion of the weight. The only conditions are, that the pressures have their sum equal to the weight of the body, and that they be such, that if they be considered as weights, their center of gravity and the center of gravity of the body are in the same vertical line.

The same is true if the props be more than four.

Prob. VI. *Fig.* 77, 78. *A body* ABCD *rests with its base on a horizontal plane; to find when it will be supported.*

The effect of gravity will not be altered if the body be supposed collected at its center of gravity G; G being still supposed to be connected with the base AB, as for instance, by means of rigid lines GA and GB. In this case, if the vertical line Gg fall within the base, AB, fig. 77, G will have no tendency to turn round A in the direction BG, or round B in the direction AG: and consequently the body will manifestly be supported.

If Gg fall without AB, fig. 78, the body will fall over on the side on which Gg falls. In order that this may be the case, G must evidently turn in the direction Gh round B, which is supposed to be prevented from sliding. Now if a vertical line Gm represent the weight of the body at G, this force may be resolved into Gn, in the direction of the tangent to the circular arc Gh, which G would describe round B, and nm perpendicular to this, and therefore in the direction GB. Of these, the force Gn tends to cause motion round B, and is not at all counteracted: hence the point G will move in Gh, and the body will fall over.

The force nm is counteracted by the resistance at B, if B be prevented from sliding; but if the base AB and the plane on which it rests be supposed perfectly smooth, the body will slide as well as fall: and in fact, since there is no lateral force, G will descend in a vertical line whenever the body rests on a horizontal plane, as will be shewn when we consider the motion of such a body.

If we consider the base as an area, the same still holds; viz. that the body will be supported if the perpendicular from the center of gravity falls within, and will fall if this perpendicular falls without the base.

If the body be supported on several points, or on several portions of its surface, we may suppose a string to pass round all of them, and the area comprehended within this string is to be considered as the base.

3. *Equilibrium on a Point and a Surface.*

73. When a body rests with one part of it upon a point and another upon a surface, as in fig. 79, the forces by which it is supported are the re-actions of the point and of the surface. If A be the supporting point, the re-action will be in Ag, perpendicular to the surface of the body. And if PB be the supporting surface, and P the point of contact, the re-action

there will be in Pg perpendicular to both the surfaces. And by Art. 70, the point g of intersection of the two forces must be in the line in which the third force acts; that is, in the vertical line passing through G the centre of gravity. Hence Gg is vertical; and from this property the position of equilibrium may be determined.

PROB. VII. *Fig.* 80. *A beam* PQ, *considered as a line, rests upon a point* A, *with its end against a vertical plane* BC; *to find the position in which it will rest.*

Let G be the centre of gravity, Pg horizontal, Ag perpendicular to PA, Gg joined; and since the re-action of the plane is in Pg, and that of the point A in gA, the point of intersection g, of these forces, must be in the vertical line through the centre of gravity: hence Gg is vertical, and perpendicular to Pg. Draw AE vertical. By the Elements, the triangles PAE, PgA, PGg, are all similar; hence

$$\frac{PE}{PA} = \frac{PA}{Pg},$$

$$\frac{PE}{PA} = \frac{Pg}{PG}; \quad \therefore \frac{PE^2}{PA^2} = \frac{PA}{PG};$$

$\therefore PA^3 = PG \,.\, PE^2 = PG \,.\, AD^2$; $\therefore PA = \sqrt[3]{(PG \,.\, AD^2)}$, and is therefore known; for PG and AD are known.

COR. 1. To find the inclination of PQ, we have

$$\cos.\ PAD = \frac{AD}{PA} = \sqrt[3]{\frac{AD^3}{PG \,.\, AD^2}} = \sqrt[3]{\frac{AD}{PG}}.$$

COR. 2. Hence if AD be greater than PG the equilibrium is impossible.

PROB. VIII. *Fig.* 81. *The same suppositions remaining, except that the plane* BC *is now inclined to the horizon at any angle, to find the position of the equilibrium.*

Let, as before, G be the centre of gravity, Pg perpendicular to BC, and Ag to PA, and therefore Gg vertical, and also let AEF be vertical, and AD perpendicular to BC. We have by similar triangles,

$$\frac{FP}{DF} = \frac{PE}{AD},$$

$$\frac{PG}{Pg} = \frac{PA}{PE},$$

$$\frac{Pg}{PA} = \frac{PA}{AD}; \text{ and multiplying,}$$

$$\frac{FP \,.\, PG}{FD \,.\, PA} = \frac{PA^2}{AD^2}; \quad \therefore \frac{PA^3}{PF} = \frac{PG \,.\, AD^2}{FD}.$$

Let $PG = a$, $AD = b$, $FD = c$; $PA = x$;

$$\therefore PD = (x^2 - b^2)^{\frac{1}{2}}, \ PF = c - (x^2 - b^2)^{\frac{1}{2}};$$

hence we have

$$\frac{x^3}{c - (x^2 - b^2)^{\frac{1}{2}}} = \frac{ab^2}{c};$$

whence x must be found.

4. *Equilibrium on two Points.*

74. When a body is supported with its surface resting on two points, the re-action at each point will be in the direction of a perpendicular to the surface; and these perpendiculars must meet in the vertical line passing through the centre of gravity as before.

PROB. IX. *Fig.* 82. *A plane figure, two contiguous sides of which are straight lines forming a right angle, rests in a vertical plane with these two sides on two given fixed points: to find its position.*

Let A, B, be the fixed points, CP, CQ two sides of the figure, G its centre of gravity.

Let GH be a perpendicular to PC; draw AD, HL horizontal, BD, CFL, GKE vertical. And if Ag and Bg be perpendicular to the sides CA, CB, they will be in the directions of the pressures exerted at A and B on their sides; and these directions will meet in the vertical line passing through G. Draw also DM perpendicular to BC.

Let $AD = a$, $BD = b$, $CH = h$, $HG = k$; and let the angle CAD be θ: then DBM, CHL, HGK also $= \theta$.

And since $AgBC$ is a rectangle, $Ag = CB$; and hence it appears that

$$AE = DF; \;\therefore\; EF = AD - 2FD.$$

Also $BC = CM - BM = a \sin.\theta - b \cos.\theta$;

$$\therefore\; FD = BC \sin.\theta = a \sin.^2\theta - b \cos.\theta \sin.\theta;$$

$$\therefore\; EF = AD - 2FD = a - 2a \sin.^2\theta + 2b \sin.\theta \cos.\theta.$$

But $EF = KL = HL - HK = h \cos.\theta - k \sin.\theta$;

$$\therefore\; a - 2a \sin.^2\theta + 2b \sin.\theta \cos.\theta = h \cos.\theta - k \sin.\theta;$$

$$\text{or } a \cos.2\theta + b \sin.2\theta = h \cos.\theta - k \sin\theta;$$

from which equation θ is to be determined.

5. *Equilibrium on two Surfaces.*

75. When a body rests on two surfaces, the re-actions at the points of support will take place in lines perpendicular to these surfaces; these lines must meet, for otherwise the body cannot be supported. And as before, the point of concourse will be in the vertical passing through the center of gravity.

PROB. X. *Fig.* 83. *A given beam* PQ, *considered as a line, is supported on two given inclined planes* CP, CQ: *to find the position in which it will rest.*

Let Pg, Qg, perpendicular to the planes, meet in g, and G being the centre of gravity of PQ, Gg will be vertical. Let gG meet the horizontal line drawn through C in H, and the plane PC in K. The angle PgK is the complement of PKg, as is also KCH. Therefore PgG is equal to KCH or PCA; similarly, QgG is equal to QCB.

Let PCA, the inclination of the plane PA, $= \iota$, $QCB = \iota'$; $\therefore PgG = \iota$, $QgG = \iota'$; also let $PG = a$, $QG = a'$, and let QP produced meet the horizontal plane in D, and $PDC = \delta$:

$$\text{hence } CPQ = PCD + CDP = \iota + \delta,$$

$$CQP = QCB - QDC = \iota' - \delta.$$

Now

$$\frac{PG}{Gg} = \frac{\sin. PgG}{\sin. GPg},$$

$$\frac{Gg}{QG} = \frac{\sin. GQg}{\sin. QgG}.$$

$$\therefore \frac{PG}{QG} = \frac{\sin. PgG}{\sin. QgG} \cdot \frac{\sin. GQg}{\sin. GPg} = \frac{\sin. PgG}{\sin. QgG} \cdot \frac{\cos. PQC}{\cos. QPC},$$

$$\text{or } \frac{a}{a'} = \frac{\sin. \iota}{\sin. \iota'} \cdot \frac{\cos. (\iota' - \delta)}{\cos. (\iota + \delta)}$$

$$= \frac{\sin. \iota}{\sin. \iota'} \cdot \frac{\cos. \iota' . \cos. \delta + \sin. \iota' . \sin. \delta}{\cos. \iota . \cos. \delta - \sin. \iota . \sin. \delta}$$

$$= \frac{\tan. \iota}{\tan. \iota'} \cdot \frac{1 + \tan. \iota' . \tan. \delta}{1 - \tan. \iota . \tan. \delta}.$$

Whence

$$a \tan. \iota' - a \tan. \iota . \tan. \iota' . \tan. \delta = a' . \tan. \iota + a' . \tan. \iota . \tan. \iota' . \tan. \delta;$$

$$\therefore \tan. \delta = \frac{a \tan. \iota' - a' \tan. \iota}{(a + a') \tan. \iota \tan. \iota'} = \frac{a \operatorname{cotan}. \iota - a' \operatorname{cotan}. \iota'}{a + a'};$$

whence we know the inclination of PQ to the horizon.

Cor. 1. If $a = a'$, which it will be if the line PQ be of uniform thickness and density;

$$\tan. \delta = \frac{\tan. \iota' - \tan. \iota}{2 \tan. \iota \tan. \iota'} = \frac{1}{2} \cdot \left(\frac{1}{\tan. \iota} - \frac{1}{\tan. \iota'} \right);$$

$$= \frac{\operatorname{cotan}. \iota - \operatorname{cotan}. \iota'}{2}.$$

Cor. 2. If $\iota' = \iota$, or the planes be equally inclined,

$$\tan.\ \delta = \frac{a - a'}{a + a'} \cdot \text{co-tan.}\ \iota.$$

Cor. 3. In order that PQ may rest parallel to the horizon, we must have $\delta = 0$;

$$\therefore\ a \tan.\ \iota' - a' \tan.\ \iota = 0;$$

$$\therefore\ \frac{a}{a'} = \frac{\tan.\ \iota}{\tan.\ \iota'};$$

the segments GP, GQ must be as the tangents of the inclinations.

Prob. XI. *Fig.* 84. *Let* p, q *be two Spheres, touching each other and resting on two inclined Planes* CP, CQ; *to find their Position.*

Join p, q, their centers. In every position the distance of their centers is equal to the sum of their radii : and hence they have no tendency to change their point of contact with each other, and may be considered as one mass. Also the re-action is perpendicular to the planes which touch the spheres, and will therefore pass through the centers p, q. Hence pq will be supported in the same way as if it rested at p and q, on planes cp, cq, parallel to CP and CQ. Hence we may find its position by the last problem.

Let r and r' be the radii of the spheres, p and q their weights, and δ the inclination of pq to the horizon. Let G be the center of gravity of the mass pq, therefore we shall have, retaining the notation of the last problem,

$$pq = r + r',\ pG = \frac{(r + r')q}{p + q} = a,\ \ qG = \frac{(r + r')p}{p + q} = a';$$

$$\text{hence}\ \tan.\ \delta = \frac{a \tan.\ \iota' - a' \tan.\ \iota}{(a + a') \tan.\ \iota \tan.\ \iota'},$$

$$\text{will} = \frac{q \tan.\ \iota' - p \tan.\ \iota}{(p + q) \tan.\ \iota \tan.\ \iota'}.$$

Cor. Hence it appears that the inclination of pq is independent of the radii r, r', and depends only upon the weights of the spheres.

The effect will be exactly the same whether the body be supported by the re-action of a surface, or by the tension of a string perpendicular to the surface. If any point of it hang by a string of given length, it will be confined to the surface of a sphere, and the case will be the same as if it rested on a spherical surface.

Prob. XII. *Fig.* 85. *A given Beam* PQ *hangs by two Strings of given lengths* AP, BQ, *from two given fixed Points* A, B: *to find its Position when it rests.*

Let AP, BQ meet in g; therefore gG through the center of gravity G is vertical; let this meet AB in E, and let PN, QN be parallel to it; also let QP meet BA in D.

Let $AB = c$, and its inclination to the vertical, $AEG = \epsilon$; $AP = p$, $BQ = q$, $GP = a$, $GQ = b$; $PAB = \alpha$, $QBA = \beta$, $PDA = \delta$. Hence

$$gPQ = APD = PAB - PDA = \alpha - \delta,$$

$$gQP = QBD + QDB = \beta + \delta,$$

$$AgB = PgQ = \pi - (\alpha + \beta);$$

$$\therefore\ Ag = AB.\frac{\sin. ABg}{\sin. AgB} = c\ .\ \frac{\sin. B\beta}{\sin. (\alpha + \beta)},$$

$$Bg = AB\ .\ \frac{\sin. BAg}{\sin. BgA} = c\ .\ \frac{\sin. \alpha}{\sin. (\alpha + \beta)};$$

$$\therefore\ Pg = Ag - AP = c\ .\ \frac{\sin. \beta}{\sin. (\alpha + \beta)} - p;$$

$$Qg = Bg - BQ = c.\ \frac{\sin. \alpha}{\sin. (\alpha + \beta)} - q;$$

but

$$Pg = PQ\ .\ \frac{\sin. PQg}{\sin. PgQ} = (a + b)\,\frac{\sin. (\beta + \delta)}{\sin. (\alpha + \beta)};$$

$$Qg = QP \cdot \frac{\sin.\ QPg}{\sin.\ QgP} = (a+b)\,\frac{\sin.\ (\alpha-\delta)}{\sin.\ (\alpha+\beta)};$$

hence $$c \cdot \frac{\sin.\ \beta}{\sin.\ (\alpha+\beta)} - p = (a+b) \cdot \frac{\sin.\ (\beta+\delta)}{\sin.\ (\alpha+\beta)};$$

$$c \cdot \frac{\sin.\ \alpha}{\sin.\ (\alpha+\beta)} - q = (a+b) \cdot \frac{\sin.\ (\alpha-\delta)}{\sin.\ (\alpha+\beta)};$$

or $c \sin.\ \beta - p \cdot \sin.\ (\alpha+\beta) = (a+b) \cdot \sin.\ (\beta+\delta)$ ......(1),

$c \sin.\ \alpha - q \cdot \sin.\ (\alpha+\beta) = (a+b) \cdot \sin.\ (\alpha-\delta)$ ......(2).

To obtain α, β, δ, we must have a third equation; for this purpose we must find the tensions of the strings PA, QB; and as these tensions must be equivalent to the weight, which acts in a vertical direction, their components in a horizontal direction must destroy each other.

To find the tension of the string PA, we may suppose the point Q to be a fulcrum on which the beam PQ is sustained by the string PA; hence if we draw Qx and Qy perpendicular on Gg and Ag, we have

$$\frac{\text{tension of } PA}{\text{weight of } PQ} = \frac{Qx}{Qy}:$$

or, if we call the tensions of PA, QB, P, Q, and the weight of PQ, W; we shall have

$$\frac{P}{W} = \frac{Qx}{Qy} = \frac{QG \cdot \sin.\ QGx}{QP \cdot \sin.\ QPy}:$$

$$= \frac{QG \cdot \sin.\ (GDE + GED)}{QP \cdot \sin.\ (PAB - PDA)},$$

$$= \frac{b \cdot \sin.\ (\delta+\epsilon)}{(a+b) \cdot \sin.\ (\alpha-\delta)}.$$

Similarly, we should have

$$\frac{Q}{W} = \frac{a \cdot \sin.\ (\delta+\epsilon)}{(a+b) \cdot \sin.\ (\beta+\delta)}.$$

Hence $$\frac{P}{Q} = \frac{b \cdot \sin.\ (\beta+\delta)}{a \cdot \sin.\ (\alpha-\delta)}.$$

But the forces which draw the beam in the horizontal direction are the resolved parts of these tensions; that is, P sin. APM and Q sin. BQN; $\therefore P$ sin. $AMP = Q$ sin. BQN.

$$\text{But sin. } APM = \text{sin. } (AMP + PAM) = \text{sin. } (\epsilon + a)$$

$$\text{sin. } BQN = \text{sin. } (ANQ - QBN) = \text{sin. } (\epsilon - \beta);$$

$$\therefore P = Q \frac{\text{sin. } (\epsilon - \beta)}{\text{sin. } (\epsilon + a)};$$

$$\text{hence } \frac{b \text{ sin. } (\beta + \delta)}{a \text{ sin. } (a - \delta)} = \frac{\text{sin. } (\epsilon - \beta)}{\text{sin. } (\epsilon + a)} \dots\dots\dots (3).$$

And the three equations (1), (2), (3), will give the three unknown quantities a, β, δ.

Cor. 1. If the center of gravity of PQ be in its middle point, which it will be if the beam be of uniform thickness and density, $a = b$; hence

$$\frac{P}{Q} = \frac{\text{sin. } (\beta + \delta)}{\text{sin. } (a - \delta)} = \frac{\text{sin. } BQP}{\text{sin. } APQ},$$

or the tensions are inversely as the signs of the angles at P and Q.

Cor. 2. If A, B be in the same horizontal line, $\epsilon = \frac{\pi}{2}$, and equation (3) becomes

$$\frac{b \,.\, \text{sin. } (\beta + \delta)}{a \,.\, \text{sin. } (a - \delta)} = \frac{\text{cos. } \beta}{\text{cos. } a}.$$

Prob. XIII. *Fig.* 85. *A Beam* PQ *is supported by Strings which go over given Pullies* A, B *and have given Weights* P *and* Q *attached to them at* p *and* q: *to find its Position.*

Let $PAB = a$, $QBA = \beta$, and the rest of the notation as in the last problem: the tensions of the strings Ap, Bq must be equal to the weights P, Q: hence, by the expressions there found for the tensions;

$$\frac{P}{W} = \frac{b}{(a+b)} \cdot \frac{\text{sin. } (\delta + \epsilon)}{\text{sin. } (a - \delta)},$$

$$\frac{Q}{W} = \frac{a}{(a+b)} \cdot \frac{\text{sin. } (\delta + \epsilon)}{\text{sin. } (\beta + \delta)}.$$

Also, as before, the equation (3) of last problem must be satisfied;

$$\therefore \frac{b \sin. (\beta + \delta)}{a \sin. (a - \delta)} = \frac{\sin. (\epsilon - \beta)}{\sin. (\epsilon + a)}:$$

from which three equations a, β, δ, must be determined.

If a body be acted on by more than three forces in the same plane, we may suppose any two of them to be applied at their point of concourse. We may then suppose that at this point the resultant of the two forces is substituted for them: by this means the number of forces will be less by one than it was; and by successive operations of this kind we may reduce the forces to three, which is the case already considered.

6. *Stable and Unstable Equilibrium.*

76. In some cases if a body be made to deviate slightly from the position of equilibrium, it has a tendency to return to it, in consequence of the action of the forces. In other cases if the position of the body be altered ever so little, it has a tendency to recede further and further from the position of equilibrium, and to assume some new position. In this latter case therefore the equilibrium would subsist only till some disturbing force, however slight, acted on the body; in the former case, if a slight disturbing force were to act, the body would come back to its position of equilibrium, and would rest there, if by any means the oscillatory motion, which would be produced by its returning, were put an end to. In the former case the equilibrium is *stable*, in the latter it is *unstable*.

The following problems will serve to illustrate this distinction.

PROB. XIV. *Fig.* 86. *A Body, the lower surface of which is spherical, rests upon a Sphere: to find in what case the Equilibrium will be Stable.*

In the position of equilibrium, the body must rest with its spherical surface touching the sphere at the highest point, and its center of gravity in the vertical line passing through the point of contact. Let A be this point, G the center of gravity, C the center of the sphere, and D the center of the spherical surface.

Let the body come into any other position touching the sphere in P, so that A, G come to A', G': the plane $PA'G'$ being vertical. Draw PR vertical, meeting $A'G'$ in R: and since the whole mass of the body may be supposed to be collected at the center of gravity, it is manifest that if G' fall between R and A', the body will have a tendency to return to the position of equilibrium; and if G' fall beyond R, it will have a tendency to recede farther from it. Hence the equilibrium will be stable if $A'G'$ or AG be less than $A'R$.

PA' is obviously equal to the arc PA, because, in moving from one position to the other, each point of PA' has been applied to each point of PA.

Hence,

$$\text{angle } PD'A' = \frac{\text{arc } A'P}{PD'} = \frac{AP}{AD}; \text{ and angle } ACP = \frac{AP}{AC}.$$

$$\text{Now } D'R : RP :: \sin.\ D'PR : \sin.\ PD'R$$
$$:: \sin.\ PCA : \sin.\ PD'A'.$$

And when the angles become very small, the sines are as the angles; therefore when AP is very small

$$D'R : RP :: PCA : PD'A' :: \frac{AP}{AC} : \frac{AP}{AD} :: AD : AC;$$

$$\therefore D'R + RP : RP :: AC + AD : AC, \text{ ultimately.}$$

But ultimately, when the angle ACP is indefinitely diminished, $D'R + RP$ becomes $D'P$ or DA, and RP becomes RA';

$$\therefore DA : RA' :: AC + AD : AC; \therefore RA' = \frac{AD \,.\, AC}{AC + AD},$$

and the equilibrium will be stable, if AG be less than this.

If AC be infinite, we have the case of a body with a spherical surface resting on a horizontal plane, and the equilibrium will be stable if AG be less than AD.

If AD be infinite, we have the case of a body with its lower surface plane, resting upon a sphere; and the equilibrium will be stable if AG be less than AC.

If the body be a hemisphere, $AG = \frac{5}{8} AD$, (Supplement). Hence the equilibrium will be stable, if

$$\tfrac{5}{8} AD < \frac{AD \cdot AC}{AD + AC};$$

$$\text{if } 5AD + 5AC < 8AC;$$

$$\text{if } AD < \tfrac{3}{5} AC.$$

If the body rest on the concave surface of a sphere, we shall find in the same manner that the equilibrium will be stable, if

$$AG < \frac{AD \cdot AC}{AC - AD}.$$

If the lower part of the body, and the surface upon which it rests, instead of being circular, have any other curvilinear form, the stability of the equilibrium will be the same as if the surfaces where both spherical, with radii equal respectively to the radius of curvature of the body and of the surface of the point of contact.

Prob. XV. *A homogenous Elliptical Spheroid rests on its smaller end in a concave Hemisphere; to find what the Radius of the Hemisphere must be that the Equilibrium may be Stable.*

Let the radius of the hemisphere $= c$; and let a, b, be the semi-axes major and minor of the ellipse. Then, by conics, the radius of curvature at the extremity of the major-axis is $\frac{b^2}{a}$; which must be put for AD in the formula. Also the center of gravity is at the center of the ellipse: hence AG is $= a$. And the equilibrium will be stable, if

$$a < \frac{\frac{b^2}{a} \cdot c}{c - \frac{b^2}{a}}; \text{ or if } a < \frac{b^2 c}{ac - b^2};$$

$$\text{if } a^2 c - a b^2 < b^2 c, \text{ or if } c < \frac{a b^2}{a^2 - b^2}, \text{ or } c < \frac{\frac{b^2}{a}}{1 - \frac{b^2}{a^2}}.$$

Also, in order that the spheroid may be within the hemisphere, the radius must be greater than the radius of curvature of the ellipse at the point of contact. Therefore

$$c > \frac{b^2}{a}.$$

$$\text{Let } b = \tfrac{1}{2} a; \ \therefore c > \frac{a}{4}, \text{ and} < \frac{a}{3}.$$

7. *The Equilibrium of Roofs.*

77. We shall consider a few problems relating to the subject of the pressure or *thrust* which beams, combined so as to support themselves and other weights, exert in the direction of their length. The consideration of the strength of such structures, requires also an examination of the force which tends to produce fracture, and of the power which different materials and different forms have to resist this tendency; but this part of the subject does not belong to our present investigation.

Prop. *Fig.* 87. *A Roof* ACA′, *consisting of Beams forming an isosceles triangle with its base horizontal, supports a given weight at* C: *the weights of the beams being also given, it is required to find the Horizontal Force at* A *and* A′.

Let G be the center of gravity of AC, and Gg a vertical line: and let Cg be the direction of the force at C, arising both from the weight at C, and from the beam $A'C$. Then Ag must be the direction of the force exerted at A; for it is requisite that the three forces which support the beam AC should meet in the same point.

In the same manner if G' be the center of gravity of $A'C$, and $G'g'$ vertical, Cg' and $A'g'$ will be the directions of the forces at C and A'; and if the beams AC, $A'C$ be exactly similar, gg' will be horizontal: and if Ag and $A'g'$ be produced, they will meet in N, a point in the vertical line NC.

NC, Cg, gN, which are in the directions of the forces which support the beam AC, are therefore as these forces. In the same way NC, Cg', $g'N$ are as the forces which act on BC. Hence the weight at C is supported by the two re-actions gC, $g'C$. Let gg' meet NC in M, and the two forces gC, $g'C$, are equivalent to a vertical force $2MC$. Also the force at A being represented by gN, the horizontal part of it is represented by gM. Hence NC representing the weight of the beam AC, $2CM$ represents the weight at C, and Mg represents the horizontal force at A or A', which stretches the beam AA'.

Let G bisect AC; $\therefore Gg = \frac{1}{2}CN$. Hence, if we bisect CN in O, $CO = Gg$, and gO will be parallel to AC. And by what has been said, if B be the weight of the beam AC, C the weight at C, and H the horizontal pressure at A,

$$\frac{H}{B+C} = \frac{Mg}{CN + 2CM} = \frac{Mg}{2CO + 2CM} = \frac{Mg}{2MO} = \frac{AD}{2DC};$$

by similar triangles.

If therefore a be the tangent of the angle which AC makes with the horizon,

$$\frac{H}{B+C} = \frac{1}{2 \tan. a}; \quad H = \frac{B+C}{2 \tan. a}.$$

If the beam AA' were not there, so as to destroy the lateral pressure by *tying* the two points A, A', together, the horizontal pressure H must be counteracted by the supports on which the ends A, A', were placed.

If the roof ACA' support a covering of uniform thickness, the formula will still be true, including in the weight of B the weight of that portion of the covering which rests upon the beam.

The weight C, at the point C, may arise from a longitudinal beam perpendicular to the plane $AA'C$.

78. Prop. *Any number of given beams, arranged as sides of a polygon, in a vertical plane, support each other, and support also given weights at the angles; it is required to find the horizontal pressure at the points of support.*

Let AC, fig. 88, be any one of the beams; and, G being its center of gravity, let Gg be a vertical line. Then the pressures at A and C will converge to some point in Gg, as g; and their directions will be Ag, Cg. Produce Ag, meeting in N the vertical line through C. And since the beam AC is supported by three pressures in directions NC, Cg, gN, those forces are as these lines. Hence, NC representing the weight of the beam, Ng and gC represent its re-action at A and C. Also Ng is equivalent to NM, Mg, and gC to gM, MC. Hence Mg represents the horizontal pressure of the beam at A, and gM the equal horizontal pressure at C. NM its vertical pressure downwards at A, and MC its vertical pressure upwards at C.

Let O bisect CN; and suppose G to bisect AC; then $CO = Gg$, and therefore gO is parallel to GC; and OgM will be the angle which AC makes with the horizon: let this be called a, and let H be the horizontal pressure at A or C, and B the weight of the beam,

$$\frac{\text{pressure downwards at } A}{H} = \frac{NM}{Mg} = \frac{MO}{Mg} + \frac{ON}{Mg}$$

$$= \tan. a + \frac{\frac{1}{2}B}{H};$$

$$\therefore \text{pressure downwards at } A = H \tan. a + \tfrac{1}{2}B.$$

Similarly,

$$\frac{\text{pressure upwards at } C}{H} = \frac{MC}{Mg} = \frac{MO}{Mg} - \frac{OC}{Mg}$$

$$= \tan.\ \alpha - \frac{\frac{1}{2} B}{H};$$

$$\therefore \text{ pressure upwards at } C = H \tan.\ \alpha - \tfrac{1}{2} B.$$

In the same manner we should find, calling the weight of $CD = B_1$, and the angle which it makes with the horizon $= \alpha_1$;

$$\text{pressure downwards at } C = H \tan.\ \alpha_1 + \tfrac{1}{2} B_1;$$

$$\text{pressure upwards at } D = H \tan.\ \alpha_1 - \tfrac{1}{2} B_1;$$

and similarly for the other angles.

Now the pressure upwards at C must support the pressure downwards at C, together with the weight at C. Calling this weight C, we have

$$H \tan.\ \alpha - \tfrac{1}{2} B = H \tan \alpha_1 + \tfrac{1}{2} B_1 + C;$$

$$\therefore H (\tan \alpha - \tan.\ \alpha_1) = \tfrac{1}{2} (B + B_1) + C;$$

$$\therefore H = \frac{\frac{1}{2} (B + B_1) + C}{\tan.\ \alpha - \tan.\ \alpha_1},$$

whence the horizontal pressure is known.

It appears from the proof that the horizontal pressure is the same at each angle.

Cor. 1. If we suppose the weights of the beams = 0, we have

$$H = \frac{C}{\tan.\ \alpha - \tan.\ \alpha_1}.$$

Cor. 2. If we suppose that there are no weights except the beams, we have

$$H = \frac{\frac{1}{2} (B + B_1)}{\tan.\ \alpha - \tan.\ \alpha_1}$$

79. Prob. *To find the positions of the beams, having given their weights* B_1, B_2, B_3, *&c. the weights* C_1, C_2, *&c. and the positions of two of the beams.*

By the last Proposition we have the following equations; a_1, a_2, a_3, &c. being the angles which the beams make with the horizon,

$$H\,(\tan. a_1 - \tan. a_2) = \tfrac{1}{2}(B_1 + B_2) + C_1,$$

$$H\,(\tan. a_2 - \tan. a_3) = \tfrac{1}{2}(B_2 + B_3) + C_2,$$

$$\&c. = \&c.$$

If there be n beams, there will be $n-1$ weights C_1, C_2, &c. and $n-1$ equations. The number of unknown quantities is $n+1$; viz. the n tangents tan. a_1, tan. a_2, &c. and the pressure H. Hence if we know two of the angles a_1, a_2, &c. we can find the rest.

In this investigation, if any one of the beams have its farther end (beginning from A) lower than the other, it makes an angle below the horizon, and the corresponding value of a will be negative.

Cor. 1. If the weights of the beams be 0, we shall have

$$H = \frac{C_1}{\tan. a_1 - \tan. a_2} = \frac{C_2}{\tan. a_2 - \tan. a_3} = \&c.$$

Hence it appears that the weights C_1, C_2, &c. are as

$$\tan. a_1 - \tan. a_2,\ \tan. a_2 - \tan. a_3,\ \&c.;$$

which agrees with the proportion of the weights on a funicular polygon (Art. 32.); as it should do. For if each side of the funicular polygon were supposed to be rigid, and if the polygon were inverted, so that the vertical lines should remain vertical, the angles being upwards, it is clear that all the forces would act in the directions opposite to their former directions, and the equilibrium would continue to subsist.

Cor. 2. If we suppose the weights C_1, C_2, &c. to be each 0, we have

$$H = \frac{\frac{1}{2}(B_1 + B_2)}{\tan. a_1 - \tan. a_2} = \frac{\frac{1}{2}(B_2 + B_3)}{\tan. a_2 - \tan. a_3} = \&c.$$

Hence $\frac{1}{2}(B_1 + B_2)$, $\frac{1}{2}(B_2 + B_3)$, &c. are as the differences of the tangents of the angles which the beams make with the horizon.

Cor. 3. If the positions of the beams be all unknown, their lengths b_1, b_2, b_3, &c. and the positions of the extreme points being given, we shall have, in addition to the above $n - 1$ equations, these two, from which we must determine the $n + 1$ unknown quantities.

$$b_1 \cos. a_1 + b_2 \cos. a_2 + b_3 \cos. a_3 + \&c. = h,$$
$$b_1 \sin. a_1 + b_2 \sin. a_2 + b_3 \sin. a_3 + \&c. = k;$$

h and k being the horizontal and vertical co-ordinates AH, HB of B measured from A. For it is easily seen that the part of AH which corresponds to b_2, is $b_2 \cos. a_2$, and so of the rest.

80. Prop. *Four beams of equal length and weight are to be placed with their extreme points in the same horizontal line, so that they, with the horizontal line, may form an irregular pentagon, and may balance each other: to find their position, fig.* 89.

It is manifest that the beams must be placed so that the two halves of the pentagon are symmetrical. Let a, a_1 be the two angles made by the lower and upper beam on one side with the horizon. Then, by Prop. 78,

$$H (\tan. a - \tan a_1) = B.$$

And there is no weight at the highest point, therefore the pressure upwards at that point must $= 0$. Therefore, by Prop. 78, also,

$$H \tan. a_1 - \tfrac{1}{2}B = 0, \text{ or } H \tan. a_1 = \tfrac{1}{2}B.$$

Dividing the former equation by this, $\dfrac{\tan. a}{\tan. a_1} - 1 = 2$;

$$\tan. a = 3 \tan. a_1.$$

If the extremities A, A', and the vertex B be given, we may find the figure by the following construction.

BH, perpendicular to AA', bisects it. Bisect AH in E, and erect EF perpendicular to AH, meeting in F the circle passing through A, B, A'. Join AF, and this will be the position of the lower beam; and if we take BG equal to AF, meeting AF in C, AC, CB will be the positions of the two beams on one side of DH; and $A'C'B$, their position on the other side, will be a figure exactly similar to ACB.

For supposing BC' to meet the circle in G.

$$AG' = A'B - BG'$$
$$= AB - BG$$
$$= AB - AF = BF;$$
$$\therefore BG' \text{ is parallel to } AF;$$
$$\therefore \text{angle } BCK = BC'K = FA'E.$$

Hence

$$\tan. CAD = \frac{FE}{EA} = \frac{3FE}{EA'} = 3 \tan. FA'E = 3 \tan. BCK;$$

and the condition of equilibrium found above is satisfied.

Cor. If the beams be not in the position of equilibrium, there will be a horizontal pressure, which may be resisted by a horizontal beam CC', fastened at C and C.

If it be not resisted, the beams will fall, B descending or ascending as it is too low or too high. The position $ACBC'A'$ is one of unstable equilibrium.

8. *The Equilibrium of Arches.*

81. Suppose a number of bodies of the form of wedges with the points truncated, as C_3, C_2, C_1, C, c, c_1, c_2, c_3, fig. 90, to be arranged with their lateral planes PQ, P_1Q_1, &c. in contact, and all perpendicular to the same vertical plane, which may be supposed to be the plane of the paper. Let these wedges be pressed downwards by their gravity or any other forces: then, if their lateral planes be supposed perfectly smooth, they will have a tendency to slide past each other in consequence of the action of these forces; and if their efforts do not balance each other, (and if the friction be not consi-

dered) those which have the stronger tendency will descend, pushing up the others out of their places. But it is possible so to adjust the magnitudes and forms of these wedges that they shall exactly balance, and that the combination shall remain supported in its present situation by the mutual action of its parts. In this case it is called an *Arch*. The wedges C, C_1, C_2, &c. are called *Voussoirs:* and the voussoir which is at the top or *Crown* of the arch, is called the *Key-stone*. The surfaces PQ, P_1Q_1, P_2Q_2, &c. which separate the voussoirs, are called *Joints*.

82. PROP. *It is required to find what must be the proportion of the weights of the voussoirs that there may be an Equilibrium of the Voussoir-course alone.*

In most cases, a bridge consists of a course of voussoirs and of a superincumbent mass of masonry, a road, and other materials, which the voussoir-course supports. But we will in the first place consider the equilibrium of the voussoir-course by itself.

In order that there may be an equilibrium, each voussoir must be kept at rest by the forces which act upon it. Now these are, besides its own weight, the pressures of the two voussoirs with which it is in contact on each side. These pressures are necessarily perpendicular to the surfaces which act on each other. At each joint the pressure may be supposed to act over the whole surface in contact, but it will be equivalent to a single force, acting at a certain point of the surface. The point of application of this force is determined by the condition that the forces at two successive joints must meet in the vertical passing through the center of gravity of the voussoir which is between them.

Let the voussoirs PQ_1, P_1Q_2, &c. be called C_1, C_2, &c.

Now since the body C_2 is kept at rest by three forces, (its weight and the two pressures,) these forces must have the same proportion as if they acted on a point. Hence they will be as the sides of a triangle which are perpendicular to their directions, (Art. 28). Let OT be in the line PQ or parallel to it, and therefore perpendicular to the pressure on PQ; OT_1 parallel to P_1Q_1, and therefore perpendicular to the pressure on P_1Q_1; TT_1 horizontal, and therefore perpendicular to the direction of gravity. The triangle OTT_1 will therefore have its sides as the three forces; hence

$$\frac{\text{weight of } C_1}{\text{pressure on } P_1Q_1} = \frac{TT_1}{OT_1}.$$

Similarly if OT_2, parallel to P_2Q_2, meet the horizontal line TT_1 in T_2, the sides of the triangle OT_1T_2 will be as the forces which act on C_2; hence

$$\frac{\text{pressure on } P_1Q_1}{\text{weight of } C_2} = \frac{OT_1}{T_1T_2}.$$

But the pressures at the joint P_1Q_1 on C_1 and on C_2, arising from the action and re-action of the voussoirs, are equal. Hence, multiplying the above equations together,

$$\frac{\text{weight of } C_1}{\text{weight of } C_2} = \frac{TT_1}{T_1T_2}.$$

Similarly, if OT_3 be parallel to P_3Q_3, another joint, we have

$$\frac{\text{weight of } C_1}{\text{weight of } C_2} = \frac{TT_1}{T_1T_2}, \text{ and } \frac{\text{weight of } C_2}{\text{weight of } C_3} = \frac{T_1T_2}{T_2T_3}.$$

Hence the weights of the voussoirs are as the portions TT_1, T_1T_2, T_2T_3 *of a horizontal line, which are intercepted by lines drawn from any point* O *parallel to the joints.*

Cor. 1. If we draw a vertical line OX meeting the horizontal line in X, OX being made radius, XT, XT_1, XT_2, XT_3, are, to radius OX, the tangents of the angles which the joints PQ, P_1Q_1, P_2Q_2, P_3Q_3 make with the vertical. Hence we have this theorem.

In a Voussoir-course which is in Equilibrium the weights of the voussoirs are as the differences of the tangents of the angles which their joints make with the vertical.

Cor. 2. This agrees with what was proved of a system of beams, Art. 78, Cor. 2. For suppose each beam in fig. 88 to be bisected, and suppose the two halves contiguous to C to be collected at C, the two halves contiguous to D to be collected at D, and so on. And instead of a line CD, suppose a joint perpendicular to CD; then the pressure on this joint will be in the direction of CD, and therefore the equilibrium will subsist if we consider the system as an arch, with weights $\frac{1}{2}(B_1 + B_2)$, $\frac{1}{2}(B_2 + B_3)$, &c. at the points C, D, &c. But these weights are as tan. a_1, $-$ tan. a_2, tan. a_2 $-$ tan. a_3, &c.: and a_1, a_2, &c., the angles made by the beams with the horizon, are the angles made by the joints perpendicular to them with the vertical. Hence this agrees with the last Corollary.

Cor. 3. Let the pressure at any joint, as P_1Q_1, be resolved into forces parallel and perpendicular to the horizon; and since the pressure and its resolved parts are perpendicular respectively to OT_1, OX, XT_1, those forces will be as these lines. Hence the horizontal force is represented by OX, and is the same at each joint.

If H be the horizontal pressure at DE, H is the horizontal pressure at each joint.

Cor. 4. The pressures at the joints are as OT, OT_1, &c.; hence it appears that the pressures are as the secants of the angles which the joints make with the vertical.

If θ be the angle of any joint with the vertical, H sec. θ is the pressure at that joint.

Cor. 5. Since the weights of the voussoirs C, C_1, C_2, C_3, &c. are as XT, TT_1, T_1T_2, &c.; the line XT_2 will represent the whole weight of the mass between DE and P_2Q_2, and similarly for any other joint. Hence the weight of any portion, beginning from the vertex, is as the tangent of the angle made by the joint which bounds it with the vertical.

H tan. θ is the weight of any portion of which the angle is θ.

Cor. 6. If the joint be horizontal, the weight of the arch must become infinite, in order that it may be exactly in equilibrium.

The conclusions here obtained are greatly modified by introducing the consideration of friction, as will be seen hereafter.

83. The *Pier* or *Abutment*, is the solid mass which forms the lowest part of an arch on each side, and on which the lowest voussoirs rest.

PROP. *To find the horizontal pressure exerted on the pier of an arch.*

The pressure at the surface RS, fig. 90, acts at all its points; but it is equivalent to a single force acting at a certain point. Let this point be A, and let the force act in the direction KA. Also the pressure at the vertical joint DE is equivalent to a horizontal force acting at a certain point B: Let KH be the vertical line (passing through the center of gravity of the half arch) in which the weight of the half arch acts. Then the weight of the half arch acting in the line KH is supported by the two pressures in AK and BK. And if AH be horizontal, the forces are as KH, AK and HA. And KH representing the weight A of the half arch, KA will represent the pressure at A, and therefore HA the part of it which acts in a horizontal direction. Hence

$$\text{horizontal pressure at } A = A \cdot \frac{HA}{KH}.$$

This is true whether friction act or not, if the half arch be supported by a pressure in the direction AK.

If the joint RS be perfectly smooth, and make an angle β with the vertical, the pressure will be perpendicular to the surface RS;

$$\text{and horizontal pressure} = A\,\frac{HA}{KH} = A \tan.\ \beta.$$

84. PROP. *To find the pressure exerted to overturn the pier of an arch.*

If the pier AF be overturned by the pressure of the arch, it will turn about the point F of its base. The force to overturn it is the pressure (see last Article) acting at A. This pressure may be supposed to act at L, in the direction AL. Let it be resolved in AN, NL; the former part will be A, the weight; the latter force, acting in NL, will not tend to turn the pier round F. Hence the force to overturn the pier is A, acting perpendicularly at an arm FL.

The force which opposes this is the weight of the pier, which may be supposed to be collected at its center of gravity G, and to act in the vertical line GM. Let B be the weight of the pier; then, in order that it may stand, we must have $B \cdot FM > A \cdot FL$,

$$\text{also } FL = NL - NF.$$

If we draw AH horizontal, meeting the vertical line KH, $NL = AN \cdot \frac{AH}{KH}$.

By introducing this value for NL, the expression for the force becomes independent of the angle which RS makes with the horizon, provided we suppose that no sliding can take place.

85. If, instead of supposing the pressure to be distributed over the surface of each joint, we suppose an equivalent pressure to act at a single point of the surface in the same direction, the polygon formed by all the lines in which these forces act is called here *the line of pressure.*

Thus if BC, CC_1, C_1C_2, fig. 90, represent respectively the lines in which the pressures at the joints DE_1, PQ_1, P_1Q_1, P_2Q_2 act, the line BCC_1C_2 is the line of pressure.

Prop. *If we know any one point of the line of pressure we may determine the whole of the line.*

Thus if C be a point in the line, draw CC_1 perpendicular to the next joint PQ and meeting in C_1 the vertical line passing through the center of gravity of the voussoir PQ_1. From the point C_1 thus found, draw C_1C_2 perpendicular to the next joint P_1Q_1, and meeting in C_2 the vertical line passing through the center of gravity of the voussoir P_1Q_2. From C_2 draw a line perpendicular to the next joint; and so on.

For the three forces which keep the voussoir PQ_1 at rest must meet in a point (Art. 70.); the pressures at the joints which are perpendicular to the joints are two of these forces; the weight of the voussoir, which acts in the line Z_1C_1, is the third. Whence the construction is manifest.

85*a*. PROP. *The line of pressure must not fall without the voussoirs.*

Let, as in fig. 94, CC_1 be part of the line of pressure, meeting in C_1 the vertical line Z_1G_1 which passes through G_1 the center of gravity of PQ_1; and let the perpendicular C_1O_1 upon P_1Q_1, fall beyond Q_1. Hence the line of pressure CC_1C_2 falls without the voussoirs.

The pressure which acts at the joint P_1Q_1 must act in the line C_2C_1, in order that it may support the voussoir PQ_1. But no pressures, acting at the points of the line P_2Q_2, and all of them being pushing forces, can produce a force which lies beyond P_1Q_1, since the resultant of the forces at any two points of the line would fall between the two points. Hence there cannot be equilibrium in this case.

In such a case, the pressure at the joint P_1Q_1, which pushes upwards, would meet the vertical line Z_1G_1 below the point where the pressure at the joint PQ meets it; and the voussoir PQ_1 would turn over, the end P_1 coming inwards.

In like manner if the line of pressure fall beyond P_1, the equilibrium would be impossible; and the voussoir would turn, the end P_1 going outwards.

Each voussoir, and the extraneous materials which act upon it, produce a vertical force which acts in a vertical line passing through the voussoir. Let ZC, Z_1C_1, Z_2C_2, &c. fig. 90, be these lines. Now in order that each voussoir may be kept at rest, the three forces which act upon it must meet in one point (Art. 70.). Hence we shall obtain the lines in which the pressures at each joint must act, in the following manner.

Let B be the point in DE at which the pressure may be supposed to act. Draw BC horizontal, meeting the first vertical line in C. Draw CC_1 perpendicular to the joint PQ, meeting the next vertical line in C_1. Draw C_1C_2 perpendicular to P_1Q_1, meeting the next vertical line in C_2; and so on. Then BCC_1C_2, &c. is the line of pressure. And at any joint, as for instance, P_1Q_1, the pressures at different points of the surface P_1Q_1 must be such that their resultant may be in the line C_1C_2. And this is impossible if the line C_1C_2 do not fall between P_1 and Q_1. Hence the line of pressure must every where fall within the voussoirs.

This will be the case, when the voussoirs are small, if the lower surfaces of the voussoirs be perpendicular to the joints, and if the vertical forces pass through the centers of gravity of the voussoirs.

If the first condition of the equilibrium of an arch, (Art. 82.) be not satisfied, the voussoirs will tend to slide past one another. If the second condition (contained in this Article) be not satisfied, the voussoirs will turn round the inner or outer edges of the joints.

In the proofs of the preceding Propositions we have supposed a joint at the highest point of the arch D. In general there is not such a joint; but the reasoning is the same as if there were, because the line of pressure will there be necessarily perpendicular to the vertical line DE.

86. PROB. *The arch consisting of the Voussoir-course alone, and the Intrados being a circle with the joints in the direction of the radii, to find the Extrados so that there may be equilibrium.*

The *intrados* is the curve which bounds the arch internally, as DP, fig. 91; the *extrados* is the curve which bounds it externally, as EQ.

Let P be any point of the intrados, O its center; POP' a small angle $= \phi$; and let the whole arch be made up of voussoirs, such as $PQQ'P$, the angle of each being $= \phi$. Let there be n of these between DO and PO; therefore $DOP = n\phi = \theta$, $DOP' = (n+1)\phi$. Also let $OD = OP = l$, $OQ = r$, $OE = k$. And if, with center O and radius OQ, we describe an arc Qq, the area $PQqP' = \frac{1}{2}(r^2 - l^2)\phi$. Also if we take $DOD' = \phi$, and describe Ee with radius OE, the area $DEeD' = \frac{1}{2}(k^2 - l^2)\phi$. And we have, by Art. 82, considering $PQqP'$, $DEeD'$ as voussoirs;

$$\frac{\frac{1}{2}(r^2 - l^2)\phi}{\frac{1}{2}(k^2 - l^2)\phi} = \frac{\tan.(n+1)\phi - \tan.n\phi}{\tan.\phi};$$

$$\therefore \frac{r^2 - l^2}{k^2 - l^2} = \frac{\sin.\phi}{\tan.\phi.\cos.(n+1)\phi.\cos.n\phi} = \frac{\cos.\phi}{\cos.(n+1)\phi.\cos.n\phi}.$$

Now if we make ϕ indefinitely small, $DEeD'$, $PQqP'$ will approach indefinitely near to the portions $DEE'D'$, $PQQ'P'$, of the area contained between the curves. But in this case, $n\phi$ and $(n+1)\phi$ are indefinitely near to equality, and each equal to $DOP = \theta$; also cos. ϕ approaches to 1. Hence we shall have

$$\frac{r^2 - l^2}{k^2 - l} = \frac{1}{\cos.^2\theta};$$

$$\therefore r^2 = l^2 + (k^2 - l^2)\sec.^2\theta.$$

Cor. 1. We have the following construction:

Make OR horizontal, $RF = OE$, FG horizontal. Let OP meet FG in S; draw ST vertical, and take $OQ = ET$; the *locus* of Q will be the extrados.

For $OF^2 = RF^2 - RO^2 = k^2 - l^2$; therefore $FS^2 = (k - l^2)\tan.^2\theta$ and $ET^2 = EO^2 + OT^2 = EO^2 + FS^2 = k^2 + (k^2 - l^2)\tan.^2\theta = l^2 + (k^2 + l^2)\sec.^2\theta = r^2$.

Cor. 2. ET is always greater than FT or OS; hence P is always above FG; and the extrados has FG for an asymptote.

87. PROB. *To find the conditions of the equilibrium of a voussoir-course terminated by vertical planes and by a horizontal plane above and below, as in fig.* 92.

The weights of the voussoirs must be as the differences of the tangents of the angles: and the weight of each, as PQQ_1P_1, will be as the surface PQQ_1P_1, &c. Now if ODE be vertical, the surface

$$PQQ_1P_1 = \tfrac{1}{2}DE\,(PP_1 + QQ_2). \quad \text{And } QQ_1 = PP_1 \cdot \frac{OQ}{OP};$$

$$\therefore\ PQQ_1P_1 = \tfrac{1}{2}\,DE\,.\,PP_1\left(1 + \frac{OQ}{OP}\right).$$

Also the difference of the tangents is as $\dfrac{P_1D}{DO} - \dfrac{PD}{DO} = \dfrac{PP_1}{DO}$;

$$\therefore\ \frac{PP_1}{DO} \propto \tfrac{1}{2}DE\,.\,PP_1\left(1 + \frac{OQ}{OP}\right).$$

And since DE is the same for all the voussoirs,

$$\frac{1}{DO} \propto 1 + \frac{OQ}{OP} \propto 1 + \frac{OE}{OD} \propto 1 + \frac{OD + DE}{OD};$$

$$\therefore \propto 2 + \frac{DE}{OD};$$

$$\text{or } 1 \propto 2\,OD + DE;$$

and since DE is constant, OD is constant.

Hence it appears that the point O is constant, and all the joints must converge to the same point. If this be the case the weights of the voussoirs will be such as to produce equilibrium.

It is also requisite, as before, that the line of pressure DCC_1C_2A should cut the joints within the limits of the voussoirs.

87*a*. When the voussoir-course supports a superincumbent mass, the same reasoning still applies; provided we substitute for the weights of the voussoirs the forces which act upon them, arising both from their own weights and the superincumbent materials.

In this case, for any given system of pressures on the different parts of the voussoir-course, the voussoir-course must have a certain form, in order that there may be equilibrium; this form is sometimes called the *equilibrated arch.* It may be found, under given conditions, by the application of the differential calculus; as will be seen in the Analytical Statics.

87*b*. Prop. *When the voussoirs are short, the line of pressure nearly coincides with the intrados.*

The lines in which the forces on each voussoir act, pass through the respective voussoirs; therefore the line of pressure, which meets all these lines, and does not pass without the voussoirs, must lie upon the voussoir-course; and, since the voussoirs are short, must nearly coincide with the intrados.

This is true even if it be supposed impossible for the voussoirs to slide past each other. Its truth depends upon this, that the voussoir-course is supposed to be the only part of the structure which can resist pressure; the rest of the structure having no coherence.

On this supposition, the part of the arch incumbent on any part of the voussoir-course, as BP, fig. 95, is supported by the pressures which act at the joints at B and P. Hence if BC, PC be the directions in which these pressures act, these directions must meet in C, a point of ZG, the line of action of the pressure of the whole superincumbent mass $DQRP$.

If the pressure of each part of the superincumbent mass be supposed to be vertical, the line ZG in which the pressure of the whole mass acts will be the line passing through the center of gravity of the mass $BERP$. If the structure be formed of square stones or bricks, we may make this supposition without much error.

87*c*. Prop. *In the equilibrated arch, the pressure on the crown of the arch may be increased, so as to occasion a tendency to burst outwards at the haunches.*

The highest part of an arch is called the *crown*, and the parts of the arch on each side the crown, are called the *haunches*.

Let BP, fig. 95, be the curve which represents the voussoir-course of an equilibrated arch: then the tangents to the curve, BC, PC meet in C, a point of ZG, the line of action of the pressure of the whole mass superincumbent on BP.

This pressure is the resultant of the pressure of all the parts of the superincumbent mass $BERP$; and if the pressure at the crown of the arch B be increased, this resultant will be brought nearer to B, and will, for example, be in the line $Z'C'$ instead of ZC.

But in this case, if we draw $C'P'$ perpendicular to the joint at P, it is requisite that the force at the joint at P should act in the line $P'C'$ in order to support the superincumbent mass: that is, the line of pressure must now pass through P'.

Hence if P' fall beyond the surface of the voussoir-course, the joints at P will tend to open, the end P going outwards.

Cor. In the same manner if the pressure at the haunches be increased or that at the crown diminished, the line of pressure may be thrown beyond the voussoir-course and without it; and the joint P may tend to burst, the end P going inwards.

87*d*. Prop. *If the superincumbent mass have any cohesion, the arch may stand, though the line of pressure falls without the voussoirs.*

Suppose in fig. 95, the line of pressure to meet the joint QP at P'. The equilibrium requires a pushing force at this joint, in the line $P'C'$. Such a force cannot be produced by the resistance of the voussoir-course at P. But let the resistance which is required in the line $P'C'$ be R; and let also a cohesive force C act at the point Q in a direction opposite to R; such that $C \times QP = R \times P'P$. Then there will be supplied the proper pushing force in the line $P'C$, and the equilibrium will subsist.

Hence it appears, that when the voussoir stones are very short, it is necessary, in order to avoid the necessity of cohesive forces, that the voussoir-course should have the form of the equilibrated arch, although the sliding of the voussoirs on one another be considered to be impossible. The voussoir-course may be considered as a flexible arch, which, in order to retain its form, requires an equilibrium of the pressures acting upon it. This is the case if the superincumbent mass be altogether without coherence. But if the parts of the superincumbent wall adhere to each other, the arch may stand, although it do not possess the form of the equilibrated arch.

We have hitherto supposed the parts to be without friction. If friction be considered, it is no longer necessary that the joints be perpendicular to the line of pressure, as will appear in the next chapter. Friction may also operate as a cohesive force, and thus tend to sustain the structure, according to last Article.

87*e*. Prop. *Within certain limits, the equilibrium of an arch is stable.*

If the voussoirs be of evanescent length, and the mass perfectly incoherent, the equilibrium is unstable, and the slightest addition or subtraction of weight in any part would destroy the arch, by throwing the line of pressure to a place where there was no structure to supply pressure. But (1) the voussoirs are of some length, so that small alterations of the pressures may take place, and may somewhat alter the line of pressure, without altering it so much as to make it fall without the voussoirs.

(2). Even if the line of pressure do fall without the voussoirs, the cohesion of the superincumbent mass may be such as to produce, along with the thrust of the voussoirs, a pushing force in the line of pressure, and thus may sustain the mass.

It thus appears that the equilibrium of an arch is maintained by the thrust at the joints of the voussoir-course and by the cohesive forces in other parts of the mass. If these forces be slightly altered, the line of pressure passes into different positions, and the principal stress of the pressure at each joint is thrown near to the inner or the outer boundary of the voussoirs. And when the cohesive forces of the different points of the mass, or the thrusting force of the different points of the voussoirs, which are thus requisite for equilibrium, become larger than the nature of the material can supply, the material yields, by breaking in the one case and by crushing in the other, and the structure is destroyed.

87*f*. The following is the manner in which the late Professor Robison has illustrated this view of the subject.

An arch, when exposed to a great overload on the crown (or indeed on any part) divides of itself into a number of parts, each of which contains as many voussoirs as can be pierced by one straight line; and, it may then be considered as nearly in the same situation with a polygonal arch of long stones or beams abutting on each other. Thus the arch ABA', fig. 93, may be divided into four portions AC, CB, BC', $C'A'$ by straight lines AC, &c., drawn so as not to fall within the inner curve nor without the outer one. It may then be considered as resembling the roof $ACBC'A$, fig. 89. When pressed from above at B, it tends to break at the angles A, C, B, C', A'; and it is not sufficiently resisted there, because the materials with which the flanks are filled up have so little cohesion that the angle feels no load except what is immediately above it. Hence the arch will fail, the part B falling inwards, and the part C and C' bursting outwards.

In confirmation of this view of the subject, it was observed that an arch in which the arch-stones were too short, fell in this manner: splitting in several points; viz. in the middle *B*; at another point *C*, intermediate between the crown and the springing of the arch; and at the springing *A*. Also, about a fortnight before it fell, chips were observed to be dropping off from the joints of the arch-stones, at two points *E* and *F*, between *A* and *B*, and between *B* and *C*. This splintering may be conceived to have arisen from the lines of pressure *BC*, *CA* passing near the lower ends of the voussoirs at *C* and *F*, so as to throw almost all the pressure upon the inner edges. Upon making the experiment by means of models or arches in chalk, Professor Robison found that in all cases the overloaded arch always broke at some place *C*, considerably beyond another point *F* where the first splintering was observed. (Robison's System of Mech. Phil. Edinb. 1822. Vol. I. p. 639).

CHAP. VI.

ON FRICTION IN STATICS.

88. When a body tends to slide along a material surface, or tends to move so as to rub against the surface, there is a resistance exerted to this tendency. This resistance is called *Friction.* It may be measured and reasoned upon in the same way as other forces, but in the problems of equilibrium treated in the preceding pages its effect is neglected. We shall here consider its consequences in some of the cases of equilibrium.

Different surfaces exert more and less of this resistance or friction. When we suppose it to vanish, the surface is said to be *perfectly smooth.*

Prop. *When a body presses a perfectly smooth surface, the effect of the surface will be exerted in a direction perpendicular to the surface.*

A surface is perfectly smooth, as has been said, when it exerts no resistance in consequence of the tendency of a body to move *along* it. Hence the direction of the resistance which it does exert, must be similarly related to all the directions in which a body can move along the surface; that is, it must be equally inclined to all these directions; or it must be perpendicular to the surface.

89. When a surface is not perfectly smooth, it exerts, besides the resistance perpendicular to the surface, another resistance along the surface, or in the direction of a tangent, and directly opposed to the direction in which the body tends to move along the surface. This resistance is the friction.

And when this resistance is employed in producing equilibrium, it is to be considered as a pressure like any other statical force, and its effect with regard to the equilibrium is to be estimated in the same manner in which the effect of other forces has been estimated in the chapters on equilibrium. Hence

The friction of a body which is moveable along any surface, is *measured* by the force which would just put the body in motion, the force being supposed to act in the direction in which the body must move, if it does move.

Let a weight W, fig. 174, rest on a horizontal table AB, and let it be drawn horizontally by a force P. If the table be not perfectly smooth, P may be so small that its pressure will not overcome the friction, and W will remain at rest notwithstanding P's action. If P be gradually increased, W will at last begin to move, the friction being overcome. Hence there is a force P, such that any smaller force will not put W in motion, and any larger force will do so. This value of the force P measures the friction of W upon the table.

90. The friction in this case depends upon the *materials* of which the *table AB* and the body W consist; upon the degree of roughness of each; and upon the *weight* of W. It is not much affected by the *form* of W, or by the magnitude or form of the *surface MN*, which is *in contact* with the table. These laws of friction are proved by experiment.

91. Prop. *The friction is not altered by altering the surface of contact, so long as the pressure continues the same.*

Let LMN, fig. 174, be a rectangular parallelopiped, and let it be placed on its broader side MN on the horizontal table AB, and let the force P be ascertained, which acting horizontally will put it in motion. Again, let it be placed on its narrower side LM, and put in motion as before. It will be found that the force Q requisite for this purpose is the same as P which was requisite before. And in this case it is manifest that the pressure is the same, viz. the weight of the body, while the surface in contact is different.

92. In what follows we shall suppose the friction to be proportional to the pressure. This is not exactly true; for the friction corresponding to large pressures is less than it would be according to this law.

Hence if R be the pressure of the body against the surface, fR may represent the friction, f being a friction varying according to the nature of the substances, &c.

Friction operates in various ways, some of which are the following.

1. *Friction between finite Surfaces in contact.*

93. Let a weight W, fig. 174 or 175, of which the side MN is plane, be placed in contact with a plane AB. If R be its pressure on the plane, and fR the friction, the surfaces AB and LM being supposed to be made smooth by the usual processes of art, we shall have the value of f as follows.

When the surfaces are wood, the *grains* being in the same directions, $f = \frac{1}{2}$.

When the grains are placed across each other, $f = \frac{1}{4}$.

When one surface is wood and the other metal, $f = \frac{1}{5}$.

When both surfaces are metal, $f = \frac{1}{4}$.

Friction is diminished by greasing or oiling the surfaces in contact. Fresh tallow is said to diminish the resistance by one half.

94. Prop. *To determine by experiment the magnitude of the friction.*

Let AB, fig. 175, be an inclined plane, the angle of which can be altered at will. Let the weight W be placed upon it, and let the plane be gradually raised from a horizontal position CB, to a greater and greater angle, till W begins to slide down the plane. Let the inclination of the plane be measured at which this just does *not* take place, and let this be the position represented in the figure.

In this position the force of the friction, which necessarily acts along the inclined plane, just keeps in equilibrium the body upon the inclined plane. Hence if we draw CD perpendicular to AB, the body W is kept at rest by three forces, its weight, the resistance of the plane, and the friction, which are perpendicular respectively to CB, BD, and DC. Those forces are therefore as these lines: and, (as in Art. 38,)

$$\text{friction} : \text{pressure} :: CD : DB.$$

Cor. 1. If we draw CA vertical we have also

$$\text{friction} : \text{pressure} :: AC : CB.$$

Cor. 2. Using the same notation as before,

$$fR : R :: AC : CB \text{ and } f = \frac{AC}{BC}.$$

Cor. 3. If β be the angle ABC at which the body begins to slide, $f = \tan. \beta$.

Cor. 4. If the friction be proportional to the pressure, the angle β, at which sliding takes place, is the same for all weights.

Cor. 5. Hence we may ascertain experimentally whether the friction is proportional to the pressure, by observing whether this angle remains the same when we alter the weight.

Since the friction is proportionally somewhat less for larger pressures than for small ones, the angle at which sliding takes place will be less where the weight is large.

When a ship is launched, it slides upon planes of which the inclination is very small, and along which a small weight would not slide.

95. One property by which friction differs from other forces, is, that it is not exerted except there be a tendency to motion which it has to resist: and it increases as the tendency increases. Thus when the plane AB in fig. 175 is horizontal, the friction is nothing: it increases as the inclination increases, being always of such a magnitude as is requisite to prevent the

body from sliding, till it reaches its limit. And if the body had a tendency to slide in the opposite direction, the friction would also be exerted in the opposite direction.

This property of friction makes it modify in a remarkable manner all the conditions of equilibrium of bodies in actual practice. For it results from this effect of friction, that the equilibrium will not be destroyed although the conditions investigated in the preceding pages are violated. The force of friction, to a certain extent, will enter the system as far as it is wanted, and supply the deficiency which occurs: so that the balance will not be lost till the conditions of equilibrium have been transgressed to an amount deviating considerably from the calculated state.

We will take one remarkable example of this.

96. In deducing the conditions of equilibrium of an arch, we have (Art. 81) found the weights which its parts must have in order that they may have no tendency to *slide* past each other. But in point of fact the destruction of the equilibrium in this manner is what can never or scarcely ever happen.

PROP. *When two plane surfaces are in contact, with friction, to find what the direction of the surface of contact must be, in order to produce the same effect without friction.*

Let two bodies of which the centers are C and C_1, fig. 176, be in contact at the surface PQ. Let M be the point at which the mutual action of the surfaces may be supposed to take place. And let Mn be taken in MQ to represent the friction, and nc perpendicular to Mn to represent the pressure; so that $Mn : nc :: f : 1$. Then if the whole friction in the direction PQ be called into action, the force which acts at M will be compounded of Mn and nc, and will be represented in magnitude and direction by MC. And if Mc' be drawn similarly the other way from M, Mc' will be the direction of the force which acts at M, when the whole friction in the direction QP be called into action.

The degree of friction which will really be exerted will depend upon what is wanted to preserve the equilibrium; and according to this event the resulting force may have any direction between Mc and Mc'.

If we draw pq perpendicular to Mc, and suppose that pq is the plane of contact, the effect of the contact without friction would be a force in the direction Mc. And similarly if $p'q'$ be perpendicular to Mc', the effect of a surface $p'q'$ would be in Mc'.

Hence we may suppose the surface of contact to have any position between pq and $p'q'$. If any of these positions *without friction* will preserve the equilibrium, the surface PQ will preserve it with friction.

97. PROP. *To explain the effect of friction in supporting an arch.*

Without friction the conditions of equilibrium of an arch are, that the pressures on the voussoirs shall be proportional to the differences of the tangents of the angles which the joints make with the vertical, and that the line of pressure shall fall within the voussoirs. The action of friction does not affect the latter condition except so far as it gives cohesion to the mass (see Art. 87d). But with regard to the former condition, the effect is, that a joint, as PQ, fig. 176, may be supposed to occupy any position between the limiting positions pq and $p'q'$, without the equilibrium being destroyed; the tangent of the limiting angle PMp, or PMp' being f, the coefficient of friction.

If there were no friction, the line of pressure of the arch would be a polygon having an angle on the line of the force which acts on each voussoir, and having its sides perpendicular to the joints. But by the effect of friction, the line of pressure will be a polygon in which the sides may make with the joints any angles within the limiting angle on each side of a perpendicular.

If the positions of the joints be given, we may begin at any point and draw the line of pressure of the arch, making, at every joint, the limiting angle on one side of the perpendicular. We may also, from the same point, draw the line of pressure of the arch, making, at every joint, the limiting angle on the other side of the perpendicular. We have thus two limiting lines of pressure; and if either of these, or any line between them, fall in the voussoirs, we shall have equilibrium.

If the form of the voussoir-course be given (the course being narrow), the joints, which if there were no friction must be perpendicular to the curve of the course, may make any angle, not greater than the limiting angle, with this direction.

If the force on each voussoir be vertical and be given, then, one joint being known, all the others would be known if friction did not act. For in fig. 90, the lines XT, TT_1, T_1T_2, &c. would be given; and if one of the lines TO were known, the position of the point O would be known, and hence the positions of all the lines TO, T_1O, T_2O, &c. to which the joints are parallel, and the joints being given, the line of pressure might be found; and this line must fall within the voussoirs. But if friction act, each joint may deviate from the position thus found by any angle less than the limiting angle: and thus the limits of the conditions within which equilibrium is possible are extended according to the principles just explained.

2. *Friction of Cylindrical Surfaces in Contact.*

98. Let two circles cn, dn, fig. 177, be in contact internally in the point n. If these two circles be the ends of two cylinders, the cylindrical surfaces will touch each other in a straight line. And if one of the cylinders turn so as to slide upon the other, there will be a friction between them; which will be proportional to the pressure as in the preceding case, that is, friction $= fR$. But the fraction (f) of the pressure which expresses the friction will be different from what it was in that case.

Thus if both surfaces be of wood, $f = \frac{1}{12}$.

If iron turn in contact with brass, $f = \frac{1}{7}$.

This is the kind of friction which takes place at the axle of a lever, pully, &c. And we shall consider its effect in such a case.

99. Prop. *To investigate the limits within which friction will preserve the equilibrium of the lever.*

Let a lever AB, fig. 177, consist of a bar pierced with a cylindrical hole yn, by means of which it turns upon a solid cylinder xn, which is something smaller than the hole. The friction takes place at the point n, and is there in the direction of a common tangent to the two surfaces of the circles xn, yn.

Let the lever be acted upon by forces P, Q, acting at A, B, and let it tend to turn in the direction AP. Then the friction will act in the direction nF. All the forces P, Q which act upon the lever are equivalent to a single force acting in some direction as HG; and this force combined with the friction nF keeps the lever at rest at the point n on the surface of the cylinder xn. Hence the force arising from the composition of these forces must pass through the point n, and therefore HG must pass through n.

Also the resultant of the forces nF and nG must be perpendicular to the surface of the cylinder xn, and must therefore pass through the center c. Hence if GK be perpendicular to nc, the forces nF and nG must be as GK, nG.

But on the same scale the pressure on the cylinder xn must be as nK. Hence we have in the extreme case

$$GK = f \cdot nK, \text{ and } nG = \sqrt{1+f^2} \cdot nK.$$

If S be the resultant of all the forces P, Q, and R the pressure on the cylinder, we have

$$S = \sqrt{1+f^2} \cdot R, \text{ and } R = \frac{S}{\sqrt{1+f^2}}.$$

Let ch be perpendicular upon HQ, and let $ch = s$. Also let r be the radius cn. Then we have by similar triangles

$$ch : cn :: GK : Gn,$$

$$\text{or } s : r :: fR : \sqrt{1+f^2} \cdot R;$$

$$\therefore s = \frac{fr}{\sqrt{1+f^2}}.$$

This is the condition for the *limiting* case of equilibrium.

Hence if the resultant of the forces P, Q pass through c, there will be an equilibrium without the action of friction. But if this resultant pass at a distance s from c, the friction will preserve the equilibrium so long as s is not greater than $\frac{fr}{\sqrt{1+f^2}}$.

Cor. 1. Hence $\cos. Gnc = \frac{ch}{cn} = \frac{s}{r} = \frac{f}{\sqrt{1+f^2}}$.

Cor. 2. Also $\tan. Gnc = f$.

DYNAMICS.

CHAP. I.

DEFINITIONS AND PRINCIPLES.

100. DYNAMICS (see Art. 9) is the part of Mechanics which relates to the action of force producing motion. In any of the machines and mechanical combinations which we have described, if the forces be not in that relation which is requisite for the equilibrium, the excess of force will produce motion. Thus, in fig. 29, if the weights P, Q and W had not the proportion which their position makes requisite to the equilibrium, they would move in a manner depending upon their magnitudes: and the laws which we are about to lay down, are those by which their motions are to be calculated.

101. The *pressure* which produces motion is to be conceived to be of the same kind as pressure in Statics. Let two equal bodies A, B, fig. 136, hang over a pully E. They will balance each other, exerting equal and opposite pressures on the string AEB, and no motion will be produced. Let now a weight P be added to A, and the pressure of P will cause A to descend and B to ascend. In this case the string exerts its tension upon A and B equally and in opposite directions. The mass $A + P$ is pressed downwards by its weight $A + P$, and upwards by the tension of the string, and descends by the difference of these forces. And in the same manner B is drawn upwards by the tension, and downwards by its weight, and ascends by the difference of these forces.

In this case the bodies move in the directions of the forces; but if we suppose one of the weights to be compelled to move obliquely to the force, as would be the case with P in fig. 98, if P and Q were to move, the pressures will act obliquely, and must be resolved by the rules given for resolution of force in

Art. 27, in order to obtain the pressure which produces the motion.

In order to obtain the effect of pressures in producing motion under given circumstances, we shall establish the three laws of motion, defining also the measures of velocity, accelerating force, and moving force.

102. All motion is performed in *time:* and the time employed is measured by the number of units of time which it contains. The passage of time is marked by the events which take place in it, and those intervals in which there is no discoverable reason why they should be unequal, are supposed equal. The intervals thus taken as a standard are in all countries the natural day and its divisions. The unit of time may be any portion we choose: in Mechanics a second is generally taken for the unit.

103. VELOCITY is the measure of the degree in which a body moves quickly or slowly: that is, one body is said to have a greater velocity than another when it moves over a greater space in the same time, or an equal space in a less time.

When a body moves over equal spaces in equal successive times the motion is said to be *uniform.* And the velocity is *measured* by the space described in a unit of time, as for instance, in one second.

In variable motions it will be seen hereafter that the velocity is measured by the space which *would be* described in a unit of time, if the velocity were uniform.

104. PROP. *In uniform motions the space described in any time is equal to the product of the numbers which express velocity and the time.*

Let v be the velocity expressed in feet; then by the last article, v is the number of feet described in one second. And since the motion is uniform, $2v$ is the space described in two seconds; $3v$ in three seconds; and generally, tv in t seconds. If s be the space, $s = tv$.

If we suppose the space described in equal fractions of a second to be equal, this equation will also be true when t is a fractional or mixed number.

COR. Since $s = tv$, $v = \frac{s}{t}$.

Hence in uniform motions the quotient of the space by the time is constant, and measures the velocity.

Thus if a ship, sailing uniformly, move 10 miles in 1 hour, the velocity, measured by the space described in a second, is

$$\frac{10 \times 5280}{60 \times 60} = 14\tfrac{2}{3} \text{ feet.}$$

105. When the velocity is not uniform, it can no longer be measured by the quotient of the space divided by the time; for these quotients will be different for different times. Thus if we suppose *P*, *A* and *B*, fig. 136, to be such that *A* shall fall from rest 16 feet in the first 4 seconds, *A* will move, not with a uniform, but with an increasing velocity. And if we then measure the space described by this body in the 4 seconds succeeding, we shall find it 48 feet; in three seconds from the end of the first four, the space would be 33 feet; in two seconds, 20; in one second, 9; in the half-second immediately following the fourth, it will be $4\frac{1}{4}$ feet, and in the quarter-second after the fourth, it will be $2\frac{1}{16}$. Hence we shall have the following values of the quotient of the space by the time, measuring from the beginning of the fifth second.

Values of t,	$4''$	$3''$	$2''$	$1''$	$\frac{1}{2}''$	$\frac{1}{4}''$
of s,	48	33	20	9	$4\frac{1}{4}$	$2\frac{1}{16}$
of the quotient $\frac{s}{t}$,	12	11	10	9	$8\frac{1}{2}$	$8\frac{1}{4}$.

The quotients, commencing at the beginning of the fifth second, go on increasing, and are larger as we take the time larger. And this must always be the case with an increasing velocity: for the space described beginning from any time will depend both upon the velocity at that time, and upon the augmentation of velocity which takes place afterwards.

Also the portion of the space which is due to this augmentation is smaller as the time of the motion is smaller. And if we approach nearer and nearer to the initial point of time, we approach nearer and nearer also to the velocity at that point of time.

106. *Hence the* VELOCITY *at any point is measured by the* LIMIT *of the quotient of the space by the time beginning from that point ;*

the Limit being taken by supposing the space and the time indefinitely diminished.

Thus in the above instance, if we were to suppose more minute values of t to be taken, as $\frac{1}{8}$, $\frac{1}{16}$, it would appear that the value of $\frac{s}{t}$ would always be greater than 8. But the excess above 8 might be diminished, by diminishing s and t sufficiently, so as to be made smaller than any assigned quantity. Hence 8 is the *Limit* of the fraction $\frac{s}{t}$, and 8 feet measures the *velocity* of the body at the beginning of the 5th second.

Instead of taking the time immediately after the point considered, we may take the time immediately before it, and we shall have analogous results.

107. PROP. *In any motion, the velocity is measured by the space which would have been described in a unit of time, if the velocity had continued constant.*

Let the velocity be increasing, and let s' be the space from the given point, which would be described in the time t if the velocity were to continue constant from that time ; let $s' + s''$ be the space which is actually described in t. Then, by last Article, the limit of $\frac{s' + s''}{t}$ is the measure of the velocity. In this expression, s'' is the part which arises from the augmentation of the velocity *after* the body leaves the given point, and its effect diminishes as t diminishes. Hence in taking the limit, the effect of s'' cannot appear. Therefore the limit of $\frac{s' + s''}{t}$ is the same as the limit of $\frac{s'}{t}$; and $\frac{s'}{t}$ measures the velocity.

When $t = 1$, s' is the space described in a unit of time, supposing the velocity to become constant. Hence the space so described measures the velocity;

And similarly for a decreasing velocity.

108. PROP. *If* s *be the space*, v *the velocity*, t *the time*,

$$v = \frac{ds}{dt}.$$

For by the definition of the Differential Coefficient, the differential coefficient of s with respect to t is the limit of the quotient of the increments of s and t. Hence, by Art. 106, the differential coefficient, or $\frac{ds}{dt}$, is equal to the velocity.

Of the Laws of Motion.

109. It has already been stated that we conceive the world of matter and motion to be governed by constant and determinate laws. We may express this otherwise, by saying that, in the cases which we consider, *no change can take place without a cause*, and that *causes are measured by their effects*.

To these two principles, which are the basis of all physical philosophy, we must add another, belonging to the particular kind of cause with which we are here concerned: namely force, the cause of motion and of tendency to motion. This force is conceived as *pressure*, as we have already seen (Art. 10.) And it is a principle universally true, in all parts of the science of mechanics, that pressure at any point implies at the same point an equal and *opposite* pressure, which resists the first; or, as it is usually stated, *action is accompanied by an equal and opposite reaction*.

These principles are necessarily the basis of the doctrine of force producing motion; but they require to be defined and interpreted by observation and experiment.

110. The velocity and direction of a body's motion are regulated by the forces which act upon it; and the simplest

principles to which the relation of these quantities can be reduced, are called the Laws of Motion. We must first establish the law of the motion of a body when it is not acted upon by any force, but left to itself.

First Law of Motion. *A body in motion, not acted on by any force, will move on in a straight line, with a uniform velocity.*

This law consists of two parts: first, that the body will go on in a straight line; and second, that it will move with a uniform velocity.

First, it will move in a straight line: for there is no force acting upon it: that is, there is no cause in existence which tends to change its direction; therefore the direction will remain unchanged, and the path will be a straight line.

Secondly, it will move with a uniform velocity: for since no force acts upon the body, there is no cause which can change its velocity except there be some cause depending upon *time* which produces that effect. The question therefore occurs, is time alone a cause of change of velocity?

At first men believed that time alone did tend to diminish velocity. For all the bodies which we see in motion we observe to move more and more slowly, and finally to stop.

But on further consideration, we find that there is always some other cause of this diminution of velocity, besides the lapse of time alone. There is some impediment, resistance, friction, which operates to destroy the velocity of the body; and time causes the change, because time is requisite for those other causes to produce their effect.

If a ball be thrown along a level surface, as a bowling-green; or if a wheel, supported by its axis, have a rotatory motion given it; or if a pendulum, hanging freely, be made to oscillate*; these motions will, after a short time, cease.

* It is easy to see that in the first of these three cases the action of gravity does not, except by producing friction against the plane, tend to retard the velocity: and that in the other two cases, though the motion is not rectilinear, it would go on for

But this extinction of the motion arises from external causes which act in these cases. Thus the ball is resisted by the friction of the surface along which it runs; the wheel by the friction of the axis; and the bodies in all the cases, by the resistance of the air.

Hence the cause of the retardation in these cases is not the time, but certain external forces.

Also we find that in proportion as we remove the impediments to the continuance of the motion, we diminish the retardation; and this is true without limit.

Thus the ball will soon stop if thrown along rough ground; its motion will continue longer if it be projected along a smooth pavement; and if it be thrown along a sheet of smooth ice it will lose its velocity very slowly, and move a long way before it stops, though it is still retarded by the resistance of the air, and by the ice, which is never, mathematically speaking, devoid of friction.

In the same manner, if in a wheel we diminish friction by the employment of *friction-wheels*, we cause its motion to continue longer. And if we remove also the resistance of the air by making the experiment in a vacuum, the motion will continue apparently unabated for a great length of time.*

In the same manner a pendulum, which, in the air, ceases to oscillate after a short time, will, in a vacuum, continue its

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ever if rectilinear motion would do so. In the case of the wheel, the actions of gravity in the different parts counteract each other, so that there is no force to retard the velocity; and in the case of the pendulum, the quantities of velocity alternately generated and destroyed, would, if it were not for the impediments mentioned in the text, be perpetually the same.

* Experiments shewing how rotatory motion tends to become uniform by removing the resistance of the air, may be seen in accounts of the effects of the air-pump. An account of the effects of friction-wheels may be found in the *Phil. Trans.* Vol. III.

The undisputed authority which is now allowed to the laws of motion mentioned in this Chapter, is the result of innumerable experiments never recorded and discussions now forgotten, to which they were subjected during the seventeenth century. Great numbers of trials were made, both by individuals and before learned bodies, to prove almost every one of the propositions which are now considered as nearly self-evident. An experiment for proving this first law was made before the Royal Society by Hooke in 1669; of which there is an account in Birch's History of the Royal Society, Vol. II. p. 342.

oscillations for a long time; and the time will be longer as the vacuum is more perfect.

From these and similar instances we infer that if we could entirely remove the external forces which retard the motions of bodies, we should get rid of the whole of the retardation; therefore time alone is not a cause of retardation.

Hence when a body moves not acted upon by any external force its velocity will be uniform.

111. Hence, under such circumstances, a body would continue its motion for ever.

112. This first law of motion being proved, it follows, that if a body, considered as a point, move either in a curve line, or in a straight line with a velocity not constant, it is acted upon by some external force: and the deviation from rectilinear and uniform motion depends upon the *direction* and *magnitude* of the force which acts upon the body.

The *Direction* of a force is the straight line in which the force would cause a body to move if it acted on the body at rest. When a force acts on a body already in motion, the motion which the force would communicate to the body at rest will be combined with the other motion which the body has, according to laws which will be mentioned hereafter. If a force act upon a body in motion, so that the direction of the force coincides with the direction of the body's motion, the body manifestly will not be made to deviate on one side or the other of the direction, but will go on in this straight line with an altered velocity. If a force act so as to make an angle with the direction of the motion of the body, it will cause the body to describe a curvilinear path, the concavity of the path being on the side towards which the force tends.

Since causes are measured by their effects, the *Magnitude* of forces is measured by their effects, and the effect of forces which we consider in Dynamics is *Velocity*. Hence forces are greater or less as they produce a greater or less velocity in the same time.

113. Accelerating Force is force measured by the velocity which, in a given time, it would produce in a body.

If an accelerating force, acting upon a body in the direction of its motion, add equal velocities in each equal time, the force is called *uniform* or *constant*.

When a body is acted upon by a continuous force, as pressure or attraction, the velocity communicated to the body goes on increasing as the force acts for a longer time. Thus, if a stone fall from rest during one second, and another stone fall during two seconds, the velocity of the latter stone, upon which gravity has acted for a longer time, will be the greater of the two. Similarly, if we produce velocity by the continued action of the hand, as when we turn by hand a machine carrying a fly-wheel; or by means of a spring, as when a bow impels an arrow; the velocity goes on increasing so long as the operation of the force continues. Now we may, at any point of time, suppose the action of the force to cease; and, by the first law of motion, the body would then go on with the velocity already acquired: and if, after this, we suppose the force again to begin to act in the direction of the motion, an additional velocity will be communicated. Thus force *produces* a velocity in a body at rest, and *adds* velocity to the motion of a body already moving: and if the force be supposed to act for any time, it is adding velocity during the whole of that time; and the velocity produced at last is the aggregate of all the successive additions.

If, under these circumstances, the velocity *added* be equal in equal times, the force is said to be *uniform* or constant.

113*a*. *Gravity is a Uniform Force.*

This is proved by experiment and calculation. The laws of the space and time which follow from the above definition of uniform force being established, it is found that these laws prevail in the case of bodies falling by gravity. The laws will be deduced in the Chapter on "Uniformly Accelerated Motion." The agreement was shewn by Galileo by means of inclined planes, and by Atwood by means of his machine.

Historically speaking, however, the course of proof was not exactly what is here represented. Galileo *assumed* that gravity is a uniform force, and proved by experiment that for this uniform force the velocity increases in the proportion of the time.

113*b*. *Uniform Accelerating Force is* MEASURED *by the velocity added (or subtracted) in a given time, as for instance, one second.*

Thus gravity, which every second generates, in a body moving vertically downwards, a velocity of $32\frac{1}{5}$ feet, may be represented by this velocity (that is by $32\frac{1}{5}$ feet); and then any other uniform force, as for instance, one which would generate a velocity of 1 foot in a second, will be measured by this its velocity, and its proportion to gravity will be that of 1 to $32\frac{1}{5}$ or 10 to 322.

114. PROP. *With uniform accelerating forces, the velocity generated in any time is equal to the product of the force and the time.*

Let f be the accelerating force; then f is the velocity generated in one second. And since the force is uniform, f will also be the velocity added in the next second; and $2f$ will be the velocity at the end of 2 seconds. In the same manner $3f$ will be the velocity at the end of 3 seconds; and, generally, tf will be the velocity at the end of t seconds. If v be the velocity, $v = tf$.

If we suppose the velocity generated in equal fractions of a second to be equal, this equation will also be true when t is a fractional or mixed number.

COR. Since $v = tf$, $f = \frac{v}{t}$.

Hence, in uniform forces, the quotient of the velocity generated, by the time in which it is generated, is constant, and measures the force.

Thus if, as in Art. 106, a velocity of 8 feet be generated in 4 seconds, the accelerating force is $\frac{8}{4}$ or 2.

The velocity generated by gravity in one second is $32\frac{1}{5}$ feet. Hence the accelerating force of gravity is $32\frac{1}{5}$*.

115. *The* ACCELERATING FORCE *at any point of a body's motion is measured by the limit of the quotient of the velocity generated, (beginning from that point,) divided by the time in which it is generated.*

When the accelerating force is not uniform, it can no longer be measured by the quotient which results from dividing *any* velocity by the time in which it is generated. This quotient will vary with the time during which the force is supposed to act, in the same manner as the quotient of the space by the time in Art. 106 was shewn to be variable; and the quotient of the velocity by the time will have a limit, in the same manner as the quotient of the space by the time in that case was shewn to have a limit. And this limit will measure the *accelerating force* at the given point in the same manner as the limit of the quotient in Art. 106 measured the velocity. For, by taking the value of the quotient of the velocity generated by the time in which it is generated; and by taking the whole of this time smaller and smaller beginning at the given point; we approximate to the measure of the force *at* the given point. And hence by taking the ultimate limit of this quotient, we obtain the exact measure of the force at the point which is considered.

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* According to some writers a force is *proportional* to the velocity generated in 1″, but not equal to it. Forces are measured on such a scale that gravity is = 1. If we put $32\frac{1}{5}$ feet $= g$, and if F be any other force which generates a velocity f in 1″, measured on the scale now mentioned,

$$F : 1 \text{ (gravity)} :: f : g; \ \therefore F = \frac{f}{g}.$$

$$\text{Also } \frac{dv}{dt} = f = Fg.$$

If the velocity be diminished by the force, the force is a retarding force, and the same is true.

116. Prop. *In any motion, the force is measured by the velocity which would have been generated in a unit of time, if the force had continued constant.*

If the force be an increasing one, the augmentation of velocity in any time, beginning from a given point, will be due, partly to the force *at* that point, and partly to the increase of force *after* that point. And the latter portion of the augmented velocity must disappear when we consider only what belongs to the given point itself. Hence the force is to be measured as if it had continued constant from the given point; that is, it is measured by the velocity generated in a unit of time on that supposition. See Art. 107.

117. Prop. *If* f *be force,* v *the velocity,* t *the time,*

$$f = \frac{dv}{dt}.$$

For $\frac{dv}{dt}$ is the limit of the quotient of the increments of v and t, and therefore, by Art. 115, it is equal to the force.

Cor. 1. If s be the space, $\frac{ds}{dt} = v$; hence multiplying

$v\frac{dv}{dt} = f\frac{ds}{dt}$. And (Lac. D.C. Art. 9.) $\frac{dv}{dt} = \frac{dv}{ds}\cdot\frac{ds}{dt}$.

Hence substituting and omitting $\frac{ds}{dt}$ on both sides $v\frac{dv}{ds} = f$.

Cor. 2. We have also $f = \frac{dv}{dt} = \frac{d\left(\frac{ds}{dt}\right)}{dt} = \frac{d^2 s}{dt^2}$, t being the independent variable.

118. We have already seen that when a body in motion is acted on by a force which is not in the direction of its motion, it will no longer describe a straight line. We proceed to consider the law according to which the action of the force takes place in this case: for that purpose we shall first establish the following proposition.

119. *When two velocities are combined, if separately they be represented in magnitude and direction by the two sides of a parallelogram, when compounded they will be expressed by the diagonal.*

Let PQ, fig. 134, be a plane, as the deck of a ship, which moves parallel to itself, with a uniform motion, from the position PQ to the position pq. Let a body have, on this plane, a uniform motion, which would carry the body through BD, while the point B of the plane moves through Bb. If the parallelogram $BDbd$ be completed, and the diagonal Bd drawn, the body will describe the diagonal Bd uniformly by the composition of the two motions.

When B comes to b, BD comes to the position bd, and therefore the body will have moved from B to d. Also, at any intermediate time, let BD have come into the position $\beta\delta$, parallel to BD; and take $\beta\gamma : \beta\delta$ or bd :: time in $B\beta$: time in Bb; that is :: $B\beta : Bb$, because the motions are uniform.

But since $\beta\gamma : bd :: B\beta : Bb$, $B\gamma d$ is a straight line. Also $B\gamma : Bd :: B\beta : Bb$ :: time in $B\beta$: time in Bb; hence the motion in Bd is uniform.

We have here supposed that the moving point has and retains the two velocities; it retains the velocity represented by Bb, because it is carried along with the plane, and it has the velocity represented by BD, with which it moves on the plane, and relatively to it.

120. SECOND LAW OF MOTION. *When any force acts upon a body in motion, the change of motion which it produces is in the direction and proportional to the magnitude of the force which acts.*

This may be also thus expressed. When any force is exerted upon a body already in motion, the motion which the force would produce in a body at rest, is compounded with the previous motion, in such a way, that both produce their full effects parallel to their own directions.

Thus, suppose a body, considered as a point, to be moving in the direction AB, fig. 135, with such a velocity that it may describe AB uniformly in $1''$. Then by the first law of motion it would in the next $1''$ describe Bb in the same straight line equal to AB. But when it comes to B, let a force in the direction BM begin to act and act uniformly upon it for $1''$; the force being of such a magnitude that it would in $1''$ cause the body to describe BM from rest. Then at the end of $1''$ from the time when the body is at B, it will be found at C, so that MC and bC are equal and parallel to Bb and BM.

If when the body comes to C the force were to cease to act, the body would go on moving in the direction and with the velocity which it has at C. Let Cc be the space it would thus describe in $1''$. But now suppose a force to begin to act at C, which by its uniform action for $1''$ would carry the body through CN. Then its place at the end of $1''$ from C, will be D, DN and Dc being parallel and equal to cC and NC.

Similarly, if other forces act uniformly for successive seconds, we may find their effects. If the forces be not such that they can be considered as constant in magnitude and direction for $1''$, we must apply this law of motion to them for any small time during which they may be considered as constant. If they vary continuously, we must consider the *Limits* of Bb and BM, &c. as will be seen hereafter.

121. If a body were moving in the direction and with the velocity Bb, and if a velocity BD were impressed upon it at B, and it were then left to itself; it would, *by the second law of motion*, move in the direction and with the velocity Bd.

And in this case the body's motion, relatively to the moving space PQ, would be represented by BD, by Art. 119.

Thus it appears, as a consequence of the second law of motion, that if a body, which is moving along with a moving space, have any velocity impressed upon it, the motion of the body, relative to the moving space, will be the same as if the body and the space had been originally at rest.

Hence, if this second law of motion be true, all the mechanical actions, which take place in a space moving uniformly, will be the same, relatively to the parts of this space, as if the space were at rest.

Hence this law is confirmed by our finding that the relative motions and actions of bodies, in a space which moves uniformly, are exactly the same as in the space at rest.

Thus in a ship under way, a ball will go equal distances when thrown with equal force, whether towards the bow or the stern. The effects of the mutual pressures and impacts of bodies in such a case are the same in every direction. Also the motions of bodies on land are the same, under the same conditions, whether they take place east, west, north, south, or in any other direction, although in some of these cases the Earth's motion conspires with, in others is transverse or opposite to, the motion of the bodies. The oscillations of a pendulum are performed in the same time whether they take place east and west, or north and south. It may be shewn that a very small deviation from exactitude in the asserted law would produce a perceptible difference in the last experiment.

Let *PQ*, fig. 134, be a boat moving uniformly in the direction *Pp*: let *B* be a ball at rest upon the deck, carried along by the motion of the boat, (the deck being supposed to be horizontal.) Let *BD* be a line drawn upon the deck, and by the motion of the boat in any time, let *BD* come into the position *bd*. At the instant when the boat is in the situation *PQ* and the ball at *B*, let the ball be struck, so as to receive an impulse in the direction *BD*. Then it is found that the ball moves uniformly in the line *BD*, so that when *B* comes to *b*, the ball is at some point in *bd*, moving, relatively to the boat, in the line *BD*.

Now since the ball, if no impulse had been commmunicated to it, would have moved from B to b, and since it is found in d at the end of the same time, it appears that the effect of the impulse has been to compound a motion in bd with the motion in Bb; which is agreeable to the second law of motion.

Again, let BK, fig. 135, be the mast of a vessel, and let this in one second be transferred by the motion of the ship into the position bk. When it is at BK, suppose a body to be let fall from B, and let BM be the space through which a body would fall from rest in 1″. Then it is found by experiment that at the end of 1″ the body has fallen down the mast through a space bC, bC being equal to BM. Now at B the body had the velocity bB, and was then acted on by a force which would carry it over BM; and it appears that these motions are compounded so that $BMCb$ is a parallelogram, as by the law which we have enunciated it should be.

It appears that according to this second law of motion, all motions are compounded so as not to disturb each other; each remaining, relatively, the same as if there were no others.

If this law of motion were not true, it would follow that bodies placed upon a horizontal plane which is in motion, and struck in a given direction, would not move, relatively to the plane, in the direction of the impact, except when the impact was in the direction of the plane's motion. Hence if we suppose the Earth to revolve on its axis, a body struck in a direction north and south, would not move in that direction, but would deviate to the east or west; which is not found to be the case. In the same manner the oscillations of a pendulum would be performed in different times, according as it oscillated in a north and south plane, or in an east and west plane*: and similarly in other cases.

Besides the motion of the Earth on its axis, which is combined with all terrestrial motions according to this second law

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* According to Laplace, *Mec. Cel.* Tom. I. liv. i. No. 5, this experiment would indicate a deviation from the above law of motion if any existed, though it should be very small.

of motion, the motion of the Earth round the Sun, which is much more considerable, is also combined with them; and if the whole solar system be in motion, the motion of the solar system is also similarly combined.

122. We now proceed to the principles which regulate the motions of bodies, taking into account the *quantity of matter*.

Def. *The* Momentum *of a body is the product of the numbers which represent its quantity of matter and its velocity.*

Def. Moving Force *is measured by the momentum generated by the direct action of a force in a given time.*

The action of a force is *direct*, when the line of its direction passes through the body acted on.

123. Third Law of Motion. *When pressure communicates motion directly, the moving force is as the pressure.*

The most obvious and universal experience shews us that when we move a body by pressure; as for instance by pushing it, the motion is increased by increasing the pressure. Also it is clear that for a given pressure, the motion is less when the mass to be moved is greater. The question now is, according to what exact law does the velocity communicated depend upon the pressure and the quantity of matter?

When one body exerts pressure upon another, it suffers an equal and opposite pressure, for re-action is equal and opposite to action. Therefore if a body B press and accelerate a body A, A retards B by an equal opposite pressure. If we suppose a body A to be laid on a smooth horizontal table, and another body B fastened to A by a string, to hang over the edge of the table*, B will descend, drawing A along the table. Hence B accelerates A, and A retards B by a pressure equal to the tension of the string.

* It is supposed that the table is perfectly smooth, and the body A so small that the string may be considered to be parallel to the table.

If the momentum which B communicates to A be called *momentum gained*, and if the momentum which B has be less than it would have been if B had not been connected with it by a certain quantity which we call the *momentum lost*, the momentum gained and lost in any time are equal in all such cases.

Thus if A be 3 pounds and B 1 pound, there will be generated in one second a velocity of 8 feet per second. But if B had fallen freely, its velocity in one second would have been 32 feet. Hence the momentum gained by A is 8×3, and that lost by B is $32 \times 1 - 8 \times 1$; and these are equal.

If A be still 3 pounds, and B be also 3 pounds, the velocity will be 16 feet; the momentum gained and lost will be 16×3 and $32 \times 3 - 16 \times 3$, which are equal.

In this case, though the moving weight B is tripled, the velocity is only doubled; for the body B has to move itself, as well as A, and part of its weight is employed in doing this.

But since the momentum gained and the momentum lost in the same time are equal, the moving force which accelerates A and the moving force which retards B are equal by the definition above given. Now the pressure which accelerates A and the pressure which retards B are equal, being action and re-action. Therefore so long as the pressure which produces or destroys motion is the same, the moving force is the same.

124. Also, since two equal pressures acting for the same time will put two equal bodies in motion with equal velocities, we may suppose the two bodies to coalesce, and the pressures to be compounded into a single pressure; therefore a double pressure will produce a double momentum; and the same will be seen to be true in the same manner for any other multiple. Therefore when the pressure increases in any ratio, the momentum produced in a given time, and therefore the moving force, increases in the same ratio.

Therefore the moving force in such cases is as the pressure.

Since in virtue of the equality of the action and re-action between two bodies, the momentum gained and lost are always equal, the *momentum gained and lost are sometimes called action and re-action*; and the third law of motion is then expressed by saying, that *in the communication of motion re-action is equal and opposite to action.*

125. Experiments of the kind above mentioned, if they could be made with accuracy, would establish the proposition which we are considering. Instead of supposing the body A to rest on a horizontal plane, we might suppose A and B to hang over a pully, (as in fig. 136,) in which case it would be only the excess of B above A which would produce motion. A machine invented by Atwood enables us in this case to reduce the magnitudes of the velocities, while we retain their law; so that we can more easily measure their quantities. The experiments being made, are found uniformly to agree with this law*. See Atwood on Rectilinear and Rotatory Motion, Sec. 7; also Mr Smeaton's Experiments, Phil. Trans. Vol. LXVI.

If we should communicate motion to A by means of the pressure of the hand, or of a spring, as was before supposed, it would be more difficult to illustrate this law by experiments. In the case of pressure by the hand it is impossible to ascertain whether in two cases the effort be exactly the same, and still more impracticable to determine its ratio when different. In a spring this ratio might be ascertained by observing the weights which bend it to given curvatures. But in both cases the pressure would not be uniform, because it is perpetually

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* To compare the velocities observed in such machines as this, with the results of the third law, we must take into account the *rotatory inertia* of the machine, which is calculated upon principles depending also upon this third law of motion.

It is here assumed that gravity, which by its action upon B produces the pressure, and consequently the motion, acts with the same intensity whatever be the velocity with which B is moving. This is proved hereafter. (Chap. III.)

The result of Mr Smeaton's experiments was, that when pressure or weight, which he calls *Impulsive Force* or *Impelling Power*, produces velocity in a given mass, the velocity produced in a given time is as the pressure, consideration being had of the mechanical advantage at which the pressure acts.

diminished as the body acted upon recedes and moves away from the agent: and a part of the force, the amount of which it is not easy to ascertain, is employed in moving the hand itself in one case and the spring in the other.

126. Prop. *The Accelerating Force is as the Pressure directly, and the Quantity of Matter inversely.*

This follows from Article 124. For the accelerating force is as the velocity produced in a given time.

Ex. 1. To find the moving force of a body P, which falls freely by gravity.

Let g be the velocity generated in a time 1 by gravity. Then (Art. 116.) g represents the accelerating force on P. Also the moving force is the product of the accelerating force by the quantity of matter. Therefore the moving force $= Pg$.

Ex. 2. To find the accelerating force, when two equal bodies A, B are caused to move over a pully by a body P, fig. 136.

When a body falls freely by gravity, let the accelerating force $= g$. In this case, the pressure which produces motion is the weight of the body, and the quantity of matter moved is the mass of the body itself.

When P produces motion in P, A, and B, since A and B balance each other by their equal pressures in opposite directions, the weight of P only is in the pressure *which produces motion*. Also in this case the mass moved is P, A, and B, which all move with the same velocity, and therefore are moved in the same manner as if they were one mass*. Hence $P + A + B$, or $P + 2A$ is the mass moved. And accelerating force in fig. 136 : accelerating force of P falling

* If, instead of supposing the mass B to hang over the pully E, we suppose the string AE to be continued in a straight line, and B', equal to B, to be annexed to the string; and if we suppose A and B' to be destitute of gravity, so as to have no tendency to move, and to be put in motion entirely by the pressure of P; the case will manifestly be the same as that in the text; because it will make no difference whether A and B' be kept at rest by the absence of gravity, or by their weights balancing each other. But in the case thus supposed, it is clear that motion is produced in A, P and B' as if they were one mass at A. Hence the truth of the above reasoning is manifest.

freely (g) :: $\frac{\text{pressure}}{\text{mass moved}}$ in first case : $\frac{\text{pressure}}{\text{mass moved}}$ in second case

$$:: \frac{P}{P+2A} : \frac{P}{P} \text{ or } 1;$$

$$\therefore \text{ accelerating force in fig. 136, } = \frac{Pg}{P+2A}.$$

Ex. 3. The velocity generated by a gun in a bullet of 1 oz. is 1000 feet per second: supposing that the bullet described the length of the barrel in $\frac{1}{10}$ of a second, and that the force is uniform, to find the moving force.

Since the velocity generated in $\frac{1}{10}$ of a second is 1000 feet, if the force were uniform, the velocity generated in one second would be 10000. Hence

Moving force : force of gravity (1 oz.) :: 10000 : g. And putting $32\frac{1}{5}$ for g, the moving force is equal to that of a weight

$$= \frac{10000}{32\frac{1}{5}} \text{ ounces} = 310 \text{ ounces}.$$

It appears from the first example that the *moving force* of a *weight* P is Pg. In Statics we represented a weight P by the quantity of matter P, but in Dynamics it is requisite also to introduce the force of gravity g, and Pg represents the moving force of the weight.

127. Def. *The Inertia of a body is its Quantity of Matter, considered as resisting the communication of motion.*

If a force Pg produce motion in a mass A, the accelerating force is $\frac{Pg}{A}$, and therefore the velocity produced in a given time is proportional to $\frac{Pg}{A}$. Hence the greater A is, the less is the velocity produced. And hence A is sometimes considered as measuring the resistance or disinclination of the body to motion, and is called its *Inertia*.

It appears from what has been said, that this term implies a law of motion rather than a property of matter.

128. Besides pressure, *Impact or Collision* is a mode in which bodies act upon each other, and the laws of motion are applicable to this case also.

Prop. *Impact is really a pressure of short duration.*

All bodies are susceptible of a change of figure sensible or insensible; and this change occupies the time, apparently infinitely small, which bodies employ in changing their motions by impact. Thus, if an ivory ball in motion strike another at rest, they appear to separate as soon as they touch, and the second ball appears to have a certain finite velocity communicated to it instantaneously. Similarly, if there be two balls which do not separate when one strikes the other, either from their want of elasticity, as in balls of lead or clay, or from their adhering to each other when they come in contact; the alteration of velocity which is produced by the impact will appear to take place in an instant.

But in all cases, if it were not for the rapidity of the change, we should see that the communication of motion was gradual, and that the ball which was at rest was brought into motion by insensible degrees of velocity. As soon as one ball comes in contact with the other, their surfaces are compressed, and motion is, by the pressure, communicated to the ball which is at rest. The change of figure increases so long as the impinging ball has a tendency to move faster than the other, and during the whole of this time, the one is gaining and the other is losing velocity. When this action ceases, the bodies move on together, if inelastic; or if they are elastic, they separate by their elasticity; recovering their globular figure, and producing a further change of velocity, by the pressure they thus exert upon each other.

That this change of figure takes place in impact, is evident by trial in soft bodies which do not recover their shape; and may be made manifest in elastic balls by covering one of them with some substance, as ink, which will, in the impact, stain those parts of the other with which it comes in contact. The spot thus produced is found to be of a finite magnitude, which could not be if the balls retained their globular shape; and it is found to be larger as the force of the impact is greater.

That the communication of motion is thus gradual, is obvious also by considering that the action is *of the same kind*, whether the bodies, which undergo this change of shape in impact, are compressible easily or with difficulty. But in the case of a body which easily admits a certain change of figure, as for instance, a balloon filled with air, it would be manifest to the senses that any motion produced by impact would be generated by degrees, and the change and restitution of figure would employ a finite time. Hence in other cases where the magnitude and elasticity are different, the same is true.

129. Prop. *The Third Law of Motion is true in the case of Impact.*

Impact is a pressure continued for a short time; increasing from nothing to a finite magnitude, and then decreasing to nothing again. And hence, if the third law of motion be true for pressure, it will be true for impact.

We can easily see from this, the effects that impact would produce in generating, and consequently also in destroying, velocity: for the same force which would generate any velocity, would also, applied in the opposite direction, destroy it. Now when two bodies impinge on each other, the pressures on each, arising from the contact of their surfaces, must be *equal*, and in *opposite directions*. Hence, by this third law, the momenta which the impact would generate (and consequently destroy) in each, must be equal. Hence it appears, that if two bodies move in opposite directions with equal momenta, and meet, (being supposed not to separate after the impact,) the impact will destroy both their velocities, and the mass will remain at rest. Now this is found to agree with experiment*.

* The way in which the experiment may be made is described by Newton (Scholium to the laws of motion, *Principia*, Book I.). Two pieces of wood A, B, fig. 137, are hung up by equal strings from two points C, D, which are in the same horizontal line, so that the bodies can swing in arcs MA, NB. One of them, A, has a steel point, which when it strikes against the other, sticks into it, so that the two move on together. The bodies are drawn aside into positions M, N, and let fall so as to meet at the lowest points A, B. The weights may be made to bear any proportion

130. The following consideration may serve to shew that the third law of motion, as above stated, though not demonstrable *à priori*, is agreeable to the *most simple* suppositions.

Let two inelastic bodies, A, B, fig. 138, approach each other with velocities which are inversely as their quantities of matter. If C be taken so that $A : B :: BC : AC$, C will be the center of gravity of the bodies. At the end of a certain time suppose that by their motions they come into the positions a, b. Then, since the velocities are inversely as the bodies, we have $A : B :: Bb : Aa$. From this and the former proportion we have $A : B :: bC : aC$. Hence C is still the center of gravity. Hence it appears that during the whole time in which the bodies approach each other, the center of gravity remains at rest. But if they do not destroy each other's motions, they will move together after impact, and therefore their center of gravity will also move. Hence, if this third law be not true, it follows that the center of gravity, having remained at rest during the separate movements of the bodies, will start into motion as soon as the impact takes place. But it is more simple, and therefore more probable, to suppose that the center of gravity will continue at rest, that is, that the third law of motion is true.

The same reasoning may be applied to cases of continued pressure. Thus let A and B be a boat and ship afloat, and let a person in one of them pull the other by means of a rope AB. The force on each of the two is the same, namely, the tension of the rope; and hence the velocities produced should be inversely as the quantities of matter in A and B;

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to each other by loading one of them with lead; and the velocities may be made to bear the inverse proportion of the weights, by taking properly the arcs AM, BN, as will be seen hereafter. If this be the case, when they meet at A, B, they will stop each other.

Newton makes the experiment in a form a little different. He draws one of the weights, as A, into the position M, and letting it impinge on the other, which is at rest, he examines the velocities after impact, which he can do by observing the arcs through which the bodies rise. These velocities he finds to be those which, in the Chapter on Collision, will be shewn to result from the third law of motion.

In the place referred to, Newton shows how allowance may be made for the small errors which arise in this experiment from the resistance of the air.

in which case the center of gravity will remain at rest all the time the bodies are moving towards each other, and they will meet in this point. This is the most natural supposition; for after they have met, if we suppose the tension to continue, it is manifest that the center of gravity must remain at rest, because the tension will produce only a statical action and re-action which balance each other.

This may be applied also to attractions. If we suppose *A*, *B*, to be two bodies, as a magnet and a piece of iron, which are at liberty to approach each other*, their attraction will act in exactly the same manner as the tension of a cord by which one should be pulled to the other. Hence the pressures on each of the two, arising from the attraction, are equal; and therefore, by this third law, the velocities of the two will be inversely as the quantities of matter. The bodies will approach, their center of gravity remaining at rest all the while, and will meet in this point: and this agrees with experiment.

131. The motions of bodies under all circumstances are governed by the third law of motion. But in order to trace its consequences in general, we must consider it as extended from the case in which we have hitherto considered, in which the action is direct, to cases in which the action is *indirect*.

Thus, in fig. 99, a weight *W* moves certain bodies *m*, *n*, *p*, although the string *Bf* by which the action is com-

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* This experiment made be made by placing the magnet and the iron each on a piece of cork, and setting them to float in water.

There is another consideration which has sometimes been brought forwards as a confirmation of this third law of motion. It has been seen (Art. 43,) that if two *A*, *B*, balance each other on any machine, as for instance a straight lever, when they are supposed to move through small spaces, their velocities are inversely as their masses. Since, therefore, when the velocities of bodies acting *on a machine* would be inversely as the masses, they keep each other at rest; it is considered as agreeable to the uniformity of nature, that when bodies *meet* with velocities which are inversely as their masses, they will reduce each other to rest. This is nearly the same illustration as the one in the text, and like that, is only an analogy, not a demonstration.

municated does not act directly on any of these bodies. The force turns the system about a fixed axis AC, and the bodies turn as being part of the system.

In this case the pressure which produces motion is transmitted from f to m, by means of a lever, of which the arms are cf and cm.

THIRD LAW OF MOTION EXTENDED. *When Pressure, transmitted by any material system, produces motion, the Moving Force is proportional to the Pressure.*

The third law of motion, as already explained, asserts that the moving force is as the pressure, in cases of *direct* action; it is here asserted that the same rule applies to *indirect* action, produced by means of any machine or material combination whatever. The moving force on each body is still measured by the momentum generated in the body in a given time, as one second.

The proof of the law in this case must be of the same nature as the proof of it in the case of direct action. *In the first place*, this is the simplest law which we can conceive. If two equal pressures acting together in a similar manner did not produce a double momentum, it would follow that the action of the second pressure is different when it acts alone, and when it is combined with another pressure; and if this were so, there must be a law which should determine this difference, in addition to the other laws of motion; whereas, if the double pressure produce a double momentum, each pressure produces the same effect as if it acted independently, and no additional law is necessary.

In the second place, there is no such difference *obvious*, in common phenomena. Two men can turn a wheel twice as fast as one can, making allowance for the additional force expended in moving their own limbs when the speed increases. A very small overbalance of weight in a machine produces a very small velocity; with the amount of overbalance, the velocity increases; a very great overbalance produces a velocity approaching to that of a body falling freely. There are no common and general facts which contradict the assertion that the velocity produced in a given mass in a given time is as the pressure which produces it. Hence if this assertion be not exactly true, its falsity must be detected by experiments conducted with precise measures. And

In the third place, it is found that in such experiments, the facts agree with the results of calculation on the assumption of the truth of the third law of motion, thus extended. The experiments of Atwood and Smeaton, referred to as proofs of the third law of motion (see Art. 125), are, in fact, examples of the action of weights *transmitted* by means of certain machinery; and therefore prove the law in the extended sense here considered.

The application of the third law of motion, thus extended to cases of indirect action, will be *further* pursued in a succeeding chapter.

132. The preceding laws of motion are, it would seem, the fewest and most simple principles to which mechanical phenomena can be reduced. And it appears, from what has been said, that these laws depend upon self-evident truths concerning cause and effect, action and reaction, which, though they come *with* our experience, cannot properly be said to come *from* it, since experience is not possible without the conviction of these truths. But in order to deduce from these axiomatic truths the laws of motion, experience, in a more special sense, is necessary, to enable us to interpret and apply our fundamental conceptions. Thus a part of each law is necessarily true; while a part is proved, or at least learnt, only by attention to facts.

The nature of the truth which belongs to the laws of motion will perhaps appear still more clearly, if we state, in the following tabular form, the analysis of each law into the part which is necessary, and the part which is empirical.

Necessary.	*Empirical.*
FIRST LAW. Velocity does not change without a cause.	The time for which a body has already been in motion is not a cause of change of velocity.
SECOND LAW. The accelerating quantity of a force is measured by the acceleration produced.	The velocity and direction of the motion which a body already possesses are not, either of them, causes which change the acceleration produced.
THIRD LAW. Reaction is equal and opposite to action.	The connexion of the parts of a body, or of a system of bodies, and the action to which the body or system is already subject, are not, either of them, causes which change the effects of any additional action.

Of course, it will be understood that, when we assert that the connexion of the parts of a system does not change the effect of any action upon it, we mean that this connexion does not introduce any *new* cause of change, but leaves the effect to be determined by the previously established rules of equilibrium and motion. The connexion will modify the application of such rules; but it introduces no additional rule: and the same observation applies to all the above stated empirical propositions.

This being understood, it will be observed that the part of each law which is here stated as empirical, consists of a negation of the supposition that the condition of the moving body with respect to motion and action, is a cause of any change in the circumstances of its motion; and from this it follows that these circumstances are determined entirely by the forces extraneous to the body itself.

There are other propositions, some of which may occur hereafter, which have been called Mechanical *Principles;* and some of these have been brought forwards as elementary. Many of these are valuable, both as remarkable propositions in Dynamics, and as convenient steps in the solution of extensive and difficult classes of problems; but when distinctly stated and examined, they will be found, so far as they are true, to be consequences and combinations of the preceding three laws of motion.

CHAP. II.

UNIFORM MOTION AND COLLISION.

133. When two bodies which are in motion, acted upon by no extraneous forces, impinge upon each other, and then move on, together or separate, the motions of each before the impact, and also after it, will, by the first law of motion, be uniform and rectilinear. The change which takes place in consequence of the impact will depend upon the third law of motion, as we shall see hereafter. This is, in some respects, the most simple case of dynamical action. We have only to consider two states of each body with respect to velocity and direction; that *before*, and that *after* the collision. We have not, as in most other instances, a perpetual and continuous change of velocity, or a curvilinear path. The alteration which the concourse of the bodies produces, is supposed to be abrupt and instantaneous; so that, between their condition before and after that event, there is no intermediate one which requires to be contemplated.

It is true, as has already been observed, that, in fact, the change which collision produces in the motions of bodies does occupy a finite time; that the velocity is increased or diminished not at once, but by degrees; and that no body passes from one state of motion to another without going through all the intermediate states. But in the cases to which we shall apply our reasonings, the time which this change employs, that is, the time during which two impinging bodies continue in contact, is so small, that it may be neglected as inconsiderable, and all that we attend to are the preceding and succeeding states, which it divides. During the mutual action, however, of the two bodies, they exert a certain pressure upon each other, which will produce and destroy velocity according to the third law of motion.

134. If the bodies be not, as we have supposed them to be, of inconsiderable magnitude, the point in each which is taken to represent it, is its center of gravity; and its path is the line described by this point. The bodies are supposed in general to be homogeneous, and bounded by spherical surfaces, or, at least, by such convex surfaces that their contact may only take place in a point. The action which takes place at this point of contact will necessarily be exerted in the line which, passing through that point, is perpendicular to the surfaces which there touch; and this line is supposed to pass through the center of gravity of each body. That this may be the case in every position of the bodies, they must necessarily be spherical; but for a particular position it may happen with innumerable different forms.

If the line of the action of the bodies upon one another did not pass through the center of gravity of one of the bodies, the action would communicate to that body a rotatory motion; which is a case that we do not consider here.

If the line in which the action of the bodies takes place be the line in which they are moving, the impact is called *direct*. If either or both of them be moving in any other line, their impact is said to be *oblique*.

The *relative velocity* of two bodies is the velocity with which they approach to or recede from each other; and is therefore the difference of their velocities when they move in the same, and the sum of their velocities when they move in opposite directions.

Now in the direct impact of two bodies which move in opposite directions *with equal momenta*, as we have already said, the velocity of each is destroyed during the compression, (Art. 129) and a certain velocity is again generated during the restitution of the figures. The ratio of the velocities destroyed and generated may be taken as the measure of the proportion of the forces of compression and restitution, and we must then examine by experiment how these forces are related.

135. Prop. *When two bodies meet, moving in the same straight line, with equal momenta, in opposite directions, their velocities are destroyed by the force of compression, and new momenta, opposite and equal to each other, are generated by the force of restitution.*

It has been seen in the proof of the third law of motion, (Art. 129,) that if two inelastic bodies moving in opposite directions in the same straight line, meet with equal momenta, the collision will destroy the motions of both, and they will remain at rest. If they be not inelastic, they will, after the impact, separate with velocities differing according to the nature of the bodies.

In this case, the action between them will manifestly consist of two parts; the compression, or change of figure which their concourse and mutual pressure produce; and the restitution of figure which takes place in consequence of the elasticity, and makes them again rebound from each other.

It is obvious that the effect in the former part of the process is the same as if the bodies were inelastic; for when the compression is completed, and just before they begin to recover their figures, if we suppose the internal constitution of the bodies to undergo a sudden change by which they lose all their elasticity, they will remain in contact, and the laws of their motion will be the same as those of inelastic bodies. Hence in this first part of the collision they will lose their whole velocities, as inelastic bodies would have done.

When they separate by their elasticity, the momenta which are communicated to them afresh by their mutual pressure will be, by the third law of motion, equal. The actual magnitudes of the velocities, and consequently the relative velocity with which the bodies separate, depend upon the elasticity of the bodies, and the laws which regulate these circumstances are to be determined from experiments, as we will shew.

136. Prop. *In the direct impact of elastic bodies, the force of restitution is to the force of compression in a ratio which is constant for bodies of the same nature.*

That is, whatever be the velocities, magnitudes, and figures of the bodies in question, so long as the material continues the same, the ratio of the velocity of each after impact, to the velocity before impact, is the same. Hence also, since the velocity of each body after impact bears a certain proportion to its velocity before, the sum of those velocities, that is, the relative velocity with which the bodies separate, ought to have a given ratio to the velocity with which they approach; and this is found to be the case.

The experiments necessary for the proof of this proposition may be made in the manner already described in proving the third law of motion: see Note, Art. 129. Two balls *A*, *B*, fig. 137, are hung by vertical strings *CA*, *DB*; and being drawn out of their vertical position, are allowed to fall so as to come together at the lowest point, where they meet and recoil. The arcs down which they fall, and up which they rise, afford the means, as will be shewn hereafter, of knowing the velocities before and after the impact. And thus it was found by Newton and others, that the relative velocities before and after impact are always in a given ratio*.

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* Newton, *Principia*, Book I, Scholium to the Laws of Motion. "Propterea quod vis illa (elastica) certa ac determinata sit (quantum sentio) faciatque ut corpora redeant ab invicem cum velocitate relativâ quæ sit ad velocitatem relativam concursus in datâ ratione." He then goes on to mention in what manner and on what substances he made his observations. Experiments upon elastic bodies were made by Wren and Hooke before the Royal Society about 1670. Mariott, a French mathematician, also made experiments, of which he has given an account in his *Traité de Percussion.* Mr Smeaton (see *Phil. Trans.* Vol. LXXII.) repeated the fundamental experiments upon elastic bodies with an ingenious apparatus for separating the effects on soft and on elastic bodies. But all these mechanicians, except Newton, have considered their experiments as made on bodies *perfectly* elastic; and have taken, as approximations to such bodies, the most elastic bodies which occurred. The theory of imperfect elasticity has, it would appear, been taken for granted on the authority of Newton; and, if it were necessary to rest upon authority, there is none on which we might rely with less scruple. But, that elasticity, depending upon the internal constitution of bodies so completely different, metals, stones, ivory, cork, &c.,

If the bodies be made to meet with velocities which are not in the inverse proportion of their masses, we may find, as we shall shortly shew, what their motions ought to be after impact; and these are found to coincide with the results of observation. The case in which the experiment is most easily made, is when one of the bodies is at rest and is struck by the other.

Bodies are called *perfectly elastic* when the force of restitution is equal to the force of compression. When the force of restitution is less, the bodies are said to be *imperfectly elastic.*

The *elasticity* of imperfectly elastic bodies is the fraction which the force of restitution is of the force of compression.

Thus it appears from Newton's experiments that in the collision of balls of worsted, the relative velocity after impact is to that before as 5 to 9. Hence the fraction $\frac{5}{9}$ expresses the elasticity in this case. In balls of steel, the ratio was nearly the same; in cork, it was a little less; in ivory, it is 8 to 9; in glass, 15 to 16. According to this way of measuring, perfect elasticity will be represented by 1. In every case the value of the elasticity may be ascertained by a single experiment; and represented by a fraction e, which expresses the portion that the force of restitution is of the force of compression.

&c., should in all instances obey one general law, is, though not improbable, highly curious, and, if it be really and exactly true, worth establishing by repeated trials.

And even if further observation should prove the truth of Newton's results, there are still several obvious questions to which his experiments do not enable us to give any answer whatever. For instance, if two bodies of *different* degrees of elasticity impinge upon each other, how are their motions to be determined? Manifestly this and similar problems can only be resolved by obtaining from new experiments the principles on which they depend.

[*Addition to Fifth Edition.* Recent experiments by Mr. Eaton Hodgkinson, not yet published, appear to confirm the constancy of the ratio of elasticity.]

The authors of the common Theory of Collision, were Wren, Wallis, and Huyghens, who about the same time (1669) sent to the Royal Society papers on the subject. Wren appears to have confirmed his results by experiment; the attempts to establish the doctrine upon axioms independent of observation, have been, as they must be, very unsatisfactory.

We now proceed to determine from these principles the motions of bodies in every case of direct impact.

1. *Direct Impact.*

137. Prop. *Two inelastic* bodies, moving with given velocities, impinge directly upon each other: it is required to determine their common velocity after impact.*

Let A, B represent the magnitudes, and a, b the velocities of the bodies. And first, let them move in opposite directions, and let a and b be inversely as A and B; that is, let $Aa = Bb$, or let the momenta be equal. In this case, as has been seen in considering the third law of motion, the bodies will destroy each other's velocities by the impact; and they are supposed to be inelastic, and have therefore no tendency to separate; they will therefore remain at rest in contact.

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* The bodies here mentioned may be either soft bodies, which change their figure without recovering it, or bodies of any kind, which, when they meet, are prevented from separating by some such contrivance as that mentioned in the note to Art. 129. In the former case the effect will be the same, whatever be the quantity of compression which the bodies suffer, and the time employed in producing it, provided there be no elasticity. The compression, and the time during which it is produced, will be less as the inelastic body is harder; but we cannot conceive any action taking place between two bodies, which does not occupy some portion of time. Hence we cannot conceive the action of two bodies which are perfectly hard, that is, which are not susceptible of the smallest change of figure. If there were such substances, the actions which would take place between them would be of a nature entirely different from any thing with which we are acquainted, and therefore we should have no right to extend our laws of motion to them. Accordingly, those who have reasoned concerning such bodies have gone upon arbitrary or inconsistent assumptions. Wallis makes the motions of hard bodies to be such as we have shewn those of soft bodies to be. Huyghens and Wren suppose these motions to be the same as those of perfectly elastic bodies. Succeeding authors have generally followed the former theory with respect to the motions of two hard bodies, though the congress of a perfectly hard and a perfectly elastic body is supposed to follow the laws of two elastic bodies. It is clear, that since we do not find in nature any facts from which the principles of such cases can be deduced, the problems can only be solved by gratuitous hypotheses, and do not form any portion of mechanics considered as the science of the laws by which motions are *actually* regulated.

On the subject of the effect of the different degrees of hardness in modifying the circumstances of impact, see a succeeding chapter of this work.

Now if the motions and collision of these bodies take place in a limited space which is not at rest, but which is moving uniformly in any direction, these motions will, by the second law of motion, still continue to be the same as before, relatively to the parts of that space. Let the space move in the direction of A's motion with a velocity v; and in this space, let A and B meet with velocities α and β relative to the space, and such that $A\alpha = B\beta$. Hence they will, after the impact, be at rest relatively to this space.

But since the space which includes the two bodies is carried in the direction of A's motion with the velocity v, A has, before the impact, an absolute velocity $\alpha + v$; and B, which is carried in the same direction with a velocity v, and in the opposite direction by its own motion with a velocity β, has in the former direction an absolute velocity $v - \beta$, supposing v greater than β. Also, after the impact the bodies are at rest in the space, and are carried by its motion with a common velocity v.

Hence it appears that if a body A with a velocity $a = \alpha + v$ overtake a body B moving with a velocity $b = v - \beta$, they will after the impact move on with a common velocity v. Now since

$$a = \alpha + v, \text{ and } b = v - \beta,$$

$$\text{we have } \alpha = a - v, \text{ and } \beta = v - b;$$

and the equation $A\alpha = B\beta$ becomes $A(a - v) = B(v - b)$:

$$\text{hence } Aa + Bb = Av + Bv;$$

$$\text{and } v = \frac{Aa + Bb}{A + B}.$$

If v be not greater than β, the absolute velocity of B before the impact is $\beta - v$ in a direction opposite to that of A. If this be called b, we have

$$\beta = b + v, \text{ and } A(a - v) = B(b + v);$$

$$\text{whence } v = \frac{Aa - Bb}{A + B}:$$

which differs from the former case, only in having the sign of b negative.

Hence if a and b represent the velocities *in the same direction*, and if velocities in the opposite direction be considered to be *negative*; the first expression for v is general for all cases.

Cor. 1. If the resulting value of v be negative, the bodies will, after impact, move in a direction contrary to that which was supposed positive.

Cor. 2. If $b = 0$, or if A impinge on B at rest, we have for their velocity after impact;

$$v = \frac{Aa}{A + B}.$$

Cor. 3. The velocity lost by A is

$$a - v = a - \frac{Aa + Bb}{A + B} = \frac{B(a - b)}{A + B}.$$

The velocity gained by B in the direction of A's motion is

$$v - b = \frac{Aa + Bb}{A + B} - b = \frac{A(a - b)}{A + B};$$

$a - b$ is obviously the velocity with which A approaches B; that is, the relative velocity.

Cor. 4. If B before the impact be moving in a direction opposite to A, the velocity *gained* by B in the direction of A's motion, is not the *excess* of the velocity after impact above the velocity before, but the *sum* of the velocity destroyed in the opposite direction and of the velocity communicated in A's direction. This is also the result which we obtain from the expressions in the last Corollary, paying proper attention to the signs. The same is applicable to the velocity lost by A, when it moves in the opposite direction after the impact.

COR. 5. The *momentum* lost by A is

$$Aa - Av = \frac{AB(a-b)}{A+B}.$$

The momentum gained by B is

$$Bv - Bb = \frac{BA(a-b)}{A+B}.$$

Hence the momentum gained by B and the momentum lost by A are equal. This is what is meant by the equality of action and re-action in this case.

This equality of action and re-action is sometimes made the principle on which the theory is established. See Art. 124.

138. PROP. *Two bodies of which the common Elasticity is* e, *moving with given Velocities, impinge directly upon each other; it is required to determine their Velocities after Impact.*

Let A, B be the bodies, and a, b their velocities. And first, let their velocities be inversely as their masses, and opposite: that is, let $Aa = Bb$. As before, in the first part of the collision the velocities will be destroyed; and then, by the elasticity, there will be generated new velocities in the opposite directions, with which the bodies will separate. By Art. 136, these velocities will be to the velocities before impact in the ratio of e to 1. That is, A will return with a velocity ea, and B with a velocity eb, and thenceforth the bodies will move uniformly with these velocities.

Now let the same actions take place in a space which is moving with a velocity x in the direction of A's motion. Let A and B meet with velocities α, β, relative to this space, and such that $A\alpha = B\beta$. They will then separate with velocities $e\alpha$ and $e\beta$ relative to this moveable space. Hence if a, b, be the absolute velocities in the same direction before impact. and u, v the velocities after it; since A's velocity will be its velocity in the space together with the velocity of the space,

$$a = \alpha + x;\ \text{ similarly, } b = x - \beta;\ \therefore \alpha = a - x,\ \beta = x - b:$$

$$\text{also, } A\alpha = B\beta;\ \text{ hence } A(a - x) = B(x - b);$$

$$\therefore x = \frac{Aa + Bb}{A + B}, \quad \alpha = a - x = \frac{B(a - b)}{A + B}, \quad \beta = x - b = \frac{A(a - b)}{A + B}.$$

And since after impact A is carried forwards with the velocity x, and backwards with the velocity $e\alpha$:

$$u = x - e\alpha. \quad \text{Similarly, } v = x + e\beta.$$

Hence

$$u = \frac{Aa + Bb - eB(a - b)}{A + B}.$$

$$v = \frac{Aa + Bb + eA(a - b)}{A + B}.$$

If the bodies are not moving in the same direction before impact, attention to the signs of the velocities will preserve the truth of the formulæ.

Cor. 1. The velocity lost by A is

$$a - u = a + x - (x - e\alpha) = \alpha + e\alpha = \frac{(1 + e) \cdot B \cdot (a - b)}{A + B}.$$

The velocity gained by B is

$$v - b = x + e\beta - (x - \beta) = \beta + e\beta = \frac{(1 + e) \cdot A \cdot (a - b)}{A + B}.$$

Both these are greater than the velocities gained and lost in the case of inelastic bodies, in the ratio $1 + e : 1$.

Cor. 2. The momentum lost by A and that gained by B are each

$$\frac{(1 + e) AB (a - b)}{A + B}.$$

Hence the sum of the momenta is the same before and after impact: or $Aa + Bb = Au + Bv$.

Cor. 3. The relative velocity after impact is

$$v - u = x + e\beta - (x - e\alpha) = e(\alpha + \beta)$$

$$= e \cdot \frac{B(a - b) + A(a - b)}{A + B} = e(a - b).$$

Hence, for the same bodies, the velocity before impact is in a given ratio to the relative velocity after impact.

Cor. 4. If the bodies be equal, or $A = B$, we have for the velocities after impact,

$$u = \tfrac{1}{2}\{(1 - e)\, a + (1 + e)\, b\};$$
$$v = \tfrac{1}{2}\{(1 + e)\, a + (1 - e)\, b\}.$$

Cor. 5. If B be at rest before the impact, $b = 0$, and

$$u = \frac{(A - eB)a}{A + B};$$

$$v = \frac{(A + eA)\, a}{A + B}.$$

139. Prop. *When the Elasticity is perfect, to determine the Motions.*

We must here make $e = 1$ in the preceding expressions. Hence,

$$\text{the velocity lost by } A = \frac{2B(a - b)}{A + B}:$$

$$\text{the velocity gained by } B = \frac{2A(a - b)}{A + B}.$$

Also the relative velocity after impact $= a - b =$ the relative velocity before impact.

Cor. 1. If the bodies be equal, $B = A$. Hence

velocity lost by $A = a - b$; velocity gained by $B = a - b$.

Therefore A's velocity after impact $= a - (a - b) = b$,

B's........................ $= b + (a - b) = a$.

Hence the bodies in this case *interchange* velocities.

Cor. 2. If B be at rest when it is struck by A, $b = 0$.

Hence

$$\text{velocity lost by } A = \frac{2Ba}{A+B};$$

$$\text{therefore } A\text{'s velocity after impact} = a - \frac{2Ba}{A+B} = \frac{(A-B)a}{A+B},$$

$$B\text{'s} \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots = \frac{2Aa}{A+B}.$$

Hence if the bodies be equal, after the impact A will stop, and B will move on with A's velocity.

If A be the less body, it will move backwards, and B will move forwards with less than A's original velocity. If A be the greater, B will move forwards with a velocity greater than A's original velocity, and A will follow it more slowly.

Cor. 3. Hence if there be a row of perfectly elastic bodies at rest,

$$A,\ B,\ C,\ D,\ E, \ldots$$

and if the first, A, be made to impinge on the second, B, with a certain velocity; and B, with the motion thus acquired, on C; C on D; and so on; we see what will become of the bodies. If they be all equal, each will stop after impact, and the last will move off with the original velocity. If they go on increasing, each will, after it is struck, move forwards with a velocity less than the preceding; and after it has struck the next, will move backwards. If they are a decreasing series, each will move faster than the preceding, and after the impacts they will all move forwards.

140. Prop. *To compare the Velocity communicated immediately from* A *to* C *with that communicated by the intervention of* B *as in the last Corollary.*

Let a be A's velocity. Then, by Cor. 2, Art. 139, the velocity communicated by A to $C = \dfrac{2Aa}{A+C}$.

Also the velocity communicated by A to $B = \dfrac{2Aa}{A+B} = b$ suppose;

$\therefore$ the velocity communicated by B to $C = \dfrac{2Bb}{B+C} = \dfrac{4ABa}{(A+B)(B+C)}$.

The latter of these velocities communicated to C is greater than the former,

$$\text{if } \frac{2Aa}{A+C} < \frac{4ABa}{(A+B)(B+C)};$$
$$\text{if } (A+B)(B+C) < 2B(A+C);$$
$$\text{if } B^2 + AB + CB + AC < 2AB + 2CB;$$
$$\text{if } B^2 - AB - CB + AC < 0;$$
$$\text{if } (B-A)(B-C) < 0;$$

which will be the case if one of the factors $B-A$, $B-C$, be positive, and one negative; that is, if the body B be greater than one and less than the other of the bodies A, C. In this case the velocity communicated by the mediation of B is greater than that communicated immediately from A to C.

The velocity communicated through B is the greatest when B is a mean proportional between the other two bodies, as may easily be shewn*.

---

* To find what must be the magnitude of B that the velocity communicated by its interposition may be the greatest possible, we must make the expression for the velocity of C a maximum, or its reciprocal a minimum; that is,

$$\frac{(A+B)(B+C)}{4ABa} = \text{min.}; \text{ and omitting constant factors,}$$

$$B + A + C + \frac{AC}{B} = \text{min.}; \text{ and, differentiating with respect to the vari-}$$

able B, and putting the differential coefficient $= 0$:

$$1 - \frac{AC}{B^2} = 0:$$

$$\therefore B^2 = AC, B = \sqrt{(AC)}:$$

or B is a mean proportional between A and C.

COR. If we interpose in the same manner a body which is a mean proportional between A and B, or between B and C, the velocity communicated to C will be increased. By increasing perpetually the number of mean proportionals between the first and last body, we increase the velocity communicated to the last, and make it approach to a certain limit, which we shall find in the next Article.

141. PROP. *To find the Limit of the Velocity communicated in the last Corollary.*

Let there be $n + 1$ perfectly elastic bodies $A, B, C, D \ldots Z$, their magnitudes being in a geometrical progression of which the common ratio is $1 + r$. Therefore

$$B = (1 + r) A, \quad C = (1 + r)^2 A, \text{ \&c.} \quad Z = (1 + r)^n A.$$

Let A impinge with a velocity a upon B, and communicate a velocity b: and let B communicate to C a velocity c, and so on; z being the velocity of Z. Hence,

$$b = \frac{2Aa}{A + B}, \quad c = \frac{2Bb}{B + C}, \text{ \&c.}$$

$$\text{or } b = \frac{2a}{2 + r}, \quad c = \frac{4a}{(2 + r)^2}, \text{ \&c.}$$

$$\text{and } z = \frac{2^n a}{(2 + r)^n} = \frac{a}{\left(1 + \frac{r}{2}\right)^n}.$$

But since $Z = (1 + r)^n A$, $\sqrt{Z} = (1 + r)^{\frac{n}{2}} \sqrt{A}$; and multiplying this equation by the former one,

$$z\sqrt{Z} = a\sqrt{A} \cdot \frac{(1 + r)^{\frac{n}{2}}}{\left(1 + \frac{r}{2}\right)^n}.$$

$$\text{Now } \mathrm{l}\left(1 + \frac{r}{2}\right)^n = n\,\mathrm{l}\left(1 + \frac{r}{2}\right) = n\left\{\frac{r}{2} - \frac{1}{2} \cdot \frac{r^2}{4} + \text{\&c.}\right\}.$$

$$\mathrm{l}\,(1 + r)^{\frac{n}{2}} = \frac{n}{2}\,\mathrm{l}\,(1 + r) = \frac{n}{2}\left\{r - \frac{1}{2} r^2 + \text{\&c.}\right\};$$

$$\therefore \frac{\text{l}\left(1+\frac{r}{2}\right)^n}{\text{l}\,(1+r)^{\frac{n}{2}}} = \frac{1-\frac{1}{4}r + \&c.}{1-\frac{1}{2}r + \&c.}.$$

And as n becomes very large, r becomes very small, and ultimately may be neglected in comparison with 1. Hence the second side of this equation becomes 1, when n becomes indefinitely great. Therefore, ultimately,

$$\text{l}\left(1+\frac{r}{2}\right)^n = \text{l}\,(1+r)^{\frac{n}{2}},$$

$$\left(1+\frac{r}{2}\right)^n = (1+r)^{\frac{n}{2}};$$

$$\therefore z\sqrt{Z} = a\sqrt{A};$$

$\therefore \dfrac{a}{z} = \sqrt{\dfrac{Z}{A}}$ and $z = \dfrac{\sqrt{A}}{\sqrt{Z}}\cdot a$; which is the value to which the velocity approximates, by increasing indefinitely the number of mean proportionals between A and Z.

142. PROP. *In the direct impact of perfectly elastic bodies, the sum of each body into the square of its velocity is the same before and after impact.*

We have, by Cor. 2, Art. 138,

$$Aa + Bb = Au + Bv;$$

$$\therefore A(a-u) = B(v-b).$$

Also $a - b = v - u$, or $a + u = v + b$; (Art. 139.)

hence, multiplying the equations,

$$A(a^2 - u^2) = B(v^2 - b^2),$$

$$\text{or } Aa^2 + Bb^2 = Au^2 + Bv^2.$$

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143. PROP. *In the direct impact of imperfectly elastic bodies, to compare the sum of each body into the square of its velocity before and after impact.*

By Cor. 3, Art. 138,

$$e(a-b) = v - u;$$

$$\therefore ea + u = eb + v;$$

$$\text{and } 2ea + 2u = 2eb + 2v;$$

$$\text{or } (1+e)(a+u) - (1-e)(a-u) = (1+e)(v+b) + (1-e)(v-b).$$

Also by Cor. 2, Art. 138,

$$Aa + Bb = Au + Bv,$$

$$\text{or } A(a-u) = B(v-b).$$

Multiply the former equation by this, and we have

$$(1+e)A(a^2-u^2) - (1-e)A(a-u)^2 = (1+e)B(v^2-b)^2.$$
$$+ (1-e)B(v-b)^2.$$

Hence

$$Aa^2 - Au^2 + Bb^2 - Bv^2 = \frac{1-e}{1+e}\left\{A(a-u)^2 + B(v-b)^2\right\}.$$

Let the velocity lost by $A = a - u = p$, and the velocity gained by $B = v - b = q$, and we have

$$Aa^2 + Bb^2 = Au^2 + Bv^2 + \frac{1-e}{1+e}(Ap^2 + Bq^2).$$

2. *Oblique Impact.*

144. PROP. *In oblique impact the mutual action of the bodies is perpendicular to the surfaces at the point of contact; and it affects only the velocities resolved in this direction.*

Suppose a ball B, fig. 139, moving uniformly in a straight line bB, to be struck by a ball A, which is moving in a direction aA at right angles to bB. The impact is supposed to take place in such a way, that the line of A's motion passes through the centre of B, and is perpendicular to the surfaces at the point of contact. Therefore, neglecting the effects of friction*, the action of each of the bodies upon the other

* During the time that the bodies are in contact, the surface of B must by its lateral motion slide along the surface of A, and hence, if we suppose the surfaces not to be perfectly smooth, its motion in that direction will be, in some measure, retarded.

is entirely in this line. The quantity of the compression will not be altered by the lateral motion of *B*, and hence the action in this direction will be the same as if *A* impinged on *B* at rest. The motion of *B* in the direction *bB* will not be affected by this action; and the motion impressed by *A* will be combined with this original motion by the second law of motion. Let *Bn* represent the velocity which *A* would communicate to *B* at rest; and let *Bm* on the same scale represent the original velocity of *B*. Then if we complete the parallelogram *mn*, and draw the diagonal *Bq*; by Art. 121, the velocity of *B* after impact will be represented by *Bq*.

In the same way it will appear in every other case, that the mutual action of the bodies is wholly in a direction perpendicular to the surfaces in contact. The motion at right angles to this direction will not be affected; the motion in this direction will be regulated by the same laws as if the bodies had no lateral motion; and these motions, combined according to the second law of motion, give the motion after impact.

145. PROP. *Two given bodies of given elasticity meet with given velocities in given directions; it is required to find their motion after impact.*

It is supposed, as before, that at the instant of collision, the line joining the centers of gravity of the two bodies passes through the point of contact of the surfaces and is there perpendicular to them.

Let *PA*, *QB*, fig. 140, represent the velocities of the bodies which meet at *A* and *B*. Draw *AB* joining their centers, which will be perpendicular to their surfaces in the point of contact *c*. Produce *AB* and draw *PM*, *QN* perpendicular upon it. The velocities *PA*, *QB*, may be considered as compounded of *PM*, *MA*, and of *QN*, *NB*; and by what has just been said, it appears that the lateral velocities *PM*, *QN* are not affected by the collision, and continue the same after the impact. Also the action in the direction *AB* is the same as if the bodies had only the velocities *MA*, *NB*.

Suppose, therefore, the bodies to impinge with the velocities MA, NB; and let Am, Bn be their velocities after impact, found by Art. 138. Draw mp, nq perpendicular to AB, and equal to PM, QN respectively. Join Ap, Bq; these will be the velocities of A and B after impact: for they are compounded of the velocities PM, QN, which are not affected by the collision, and of Am, Bn, which will be the velocities in the direction AB.

Hence if we can find where the bodies meet, we can determine all the circumstances of the collision.

146. Prop. *When two spherical bodies move in the same straight line, to determine where they will meet.*

When A is at M, fig. 141, let B be at N, and let MO, NQ represent their velocities. Let A, B be the positions of the centers at the time of concourse; therefore AB is the sum of the radii of the spheres, and is known.

And since MA, NB, are described in the same time,

$$MA : NB :: MO : NQ;$$

$$\therefore MA : MA - NB :: MO : MO - NQ;$$

and $MA - NB = MN + NA - (NA + AB) = MN - AB$.

And hence the three last terms of the proportion are known, and therefore MA; which gives the position of A, and hence of B, at the time of concourse.

147. Prop. *When two spherical bodies move uniformly in any two straight lines in the same plane, to determine their concourse.*

In fig. 142, MO, NO being the directions of the motions, let MO be taken to represent A's velocity, and let NQ on the same scale represent the velocity of B. Join MN, and complete the parallelogram MP, and join PQ. With center O and radius equal to the distance of the centers of A and B when in contact, describe a circle meeting PQ in D. Join

DO; draw *DB* parallel to *OM*, and *BA* parallel to *DO*; *A*, *B* will be the positions of the centers at the instant of concourse.

By similar triangles,

$$NP : BD :: NQ : BQ;$$

$$\text{or } MO : AO :: NQ : BQ;$$

$$\therefore MO : MA :: NQ : NB;$$

$$\text{or } MA : NB :: MO : NQ :: \text{vel. of } A : \text{vel. of } B;$$

therefore when one body comes to *A*, the other comes to *B*. Also $AB = OD =$ the distance of the centers; therefore they will then be in contact. And if we divide *AB* in *c*, so that *Ac* and *Bc* may be the distances of the surfaces from the centers, a plane perpendicular to *AB* in *c* will touch both the surfaces at the instant of contact.

The circle described with center *O* will meet *PQ* in two points; of these we must take the one which is nearest to *P*.

When the two directions are not in the same plane, the problem may be solved in a manner nearly similar.

148. We may also find the position thus.

Let *MO*, *NO*, fig. 142, be the lines; *M*, *N* the positions at the beginning of the time t; a, b, the velocities; θ the angle *MON*; and c the distance, *AB*, of the centers, when the bodies meet; which will be the sum of the radii if the bodies are spherical. The bodies are supposed to move towards the point *O*; hence if $OM = m$, and $ON = n$, at the end of the time t the distances from *O* will be $OA = m - ta$, $OB = n = tb$. And we shall have

$$AB^2 = OA^2 + OB^2 - 2\,OA\,.\,OB\,.\cos.\theta;$$

or at the point of concourse.

$$c^2 = (m - ta)^2 + (n - tb^2) - 2\,(m - ta)\,(n - tb)\cos.\theta;$$

which would enable us to determine t by means of a quadratic. Of the two roots, we must manifestly take the less for the value of the time; for that will give the first time that the surfaces come in contact; and after that they will no longer go on in the same lines; so that the second root will not be applicable.

These processes will give the solution of the problem when it is possible. It is impossible, when the centers of the bodies in the course of their motions never approach within a distance sufficiently small to bring their surfaces into contact. This will happen when the roots of the quadratic become impossible; and (which is the same thing), when the circle in the construction does not meet the line PQ.

149. Prop. *When the bodies do not meet, to find at what instant they approach nearest to each other.*

It appears from the demonstration in Art. 147, that if any line OD be drawn, and DB parallel to OM, and BA parallel to DO; A, B, will be the positions of the bodies when their distance is equal to OD. Hence their distance will be least when OD is least. Therefore if we draw Od *perpendicular* to PQ, and db, ba, parallel to OM, OD; we shall have b and a the positions of the bodies when their distance is Od, the smallest possible.

150. When bodies are in motion, influenced by no forces but their mutual action, there are some remarkable properties of their center of gravity; which we now proceed to demonstrate.

Prop. *When two bodies move uniformly in straight lines, their center of gravity also moves uniformly, and in a straight line.*

Let MO, NO, fig. 143, be the paths; when the bodies are in any positions, A, B, let G be the center of gravity: when A is at O, let B be at Q, and the center of gravity at R. Take $Bb = QO$; $\therefore bO = BQ$, hence

$$OA : Ob :: OA : QB :: \text{velocity of } A : \text{velocity of } B;$$

a constant ratio: hence the triangle AOb is similar in all positions of A, B. Take g so that $Ag : bg :: B : A$, a constant ratio; therefore the triangle AOg is similar in all positions, and the locus of g is a straight line. But

$$Ag : bg :: B : A :: AG : BG; \therefore Gg \text{ is parallel to } Bb;$$

and

$$Gg : Bb :: AG : AB, \text{ or}$$

$$Gg : OQ :: B : A + B :: OR : OQ;$$

therefore Gg is equal and parallel to OR; and OG is a parallelogram, RG being parallel to Og. Hence the locus of G is a straight line. Also $RG = Og$, is in a given ratio to OA, and therefore, RG increases uniformly as OA does, and G moves with a uniform velocity.

It is easy to extend this demonstration to the case when the bodies are in different planes.

151. The proposition in last Article may also be proved in the following manner.

First, let the bodies A, B, move in the same straight line; and at any time t let x, x' be their distances from a given point O. Then the center of gravity will be in this line; and if g be its distance from O, we have, (Art. 47,)

$$g = \frac{Ax + Bx'}{A + B}.$$

Now in the uniform motion of A and B, if a and b be their velocities, and m and m' their distances from O at the beginning of the time t, we have, by Art. 114,

$$x = m + ta, \quad x' = m' + tb;$$

$$\text{hence } g = \frac{Am + Ata + Bm' + Btb}{A + B} = \frac{Am + Bm'}{A + B} + \frac{t(Aa + Bb)}{A + B};$$

whence it appears that the center of gravity is a point whose distance from O at the beginning of the time t is $\dfrac{Am + Bm'}{A + B}$, and whose velocity is $\dfrac{Aa + Bb}{A + B}$.

152. Now, let A and B move in any straight lines in the same plane, and, as before, let a and b be their velocities. Through a point O in this plane draw two lines at right angles, to represent the axes of x and y. Let the direction of A's motion make with the axis of x an angle α, and let the direction of B's motion make an angle β with the same line: consequently the angles which these directions make with the axis of y will be $\frac{1}{2}\pi - \alpha$ and $\frac{1}{2}\pi - \beta$. Now if x, y be the co-ordinates of A, and x', y' those of B; also g, h those of the center of gravity; we shall have, by Art. 48,

$$g = \frac{Ax + Bx'}{A + B}; \quad h = \frac{Ay + By'}{A + B}.$$

But if the velocity of A be resolved in the directions of x and y, it will be uniform in these directions; and its component parts will be $a \,.\, \cos.\, \alpha$, $a \,.\, \sin.\, \alpha$. Hence if m and n be the co-ordinates of A when $t = 0$, we have

$$x = m + ta \,.\, \cos.\, \alpha; \quad y = n + ta \,.\, \sin.\, \alpha;$$

similarly, $x' = m' + tb \,.\, \cos.\, \beta$; $y' = n' + tb \,.\, \sin.\, \beta$;

m', n', being the corresponding quantities for B. Hence

$$g = \frac{Am + Bm'}{A + B} + t\,\frac{(Aa \,.\, \cos.\, \alpha + Bb \,.\, \cos.\, \beta)}{A + B}:$$

$$h = \frac{An + Bn'}{A + B} + t\,\frac{(Aa \,.\, \sin.\, \alpha + Bb \,.\, \sin.\, \beta)}{A + B}.$$

From which it appears that g and h are spaces described with the uniform velocities,

$$\frac{Aa \,.\, \cos.\, \alpha + Bb \,.\, \cos \beta}{A + B}, \text{ and } \frac{Aa \,.\, \sin.\, \alpha + Bb \sin.\, \beta}{A + B};$$

which velocities are the components, in those directions, of the velocity of the center of gravity; which is therefore uniform, and its motion rectilinear, as would easily be shewn by compounding the two uniform velocities.

If the paths be not in the same plane, we must take *three* rectangular co-ordinates for each of the bodies, and it will easily be shewn, in the same way as before, that the motions of the center of gravity resolved in these three directions, and of course its whole motion, are uniform.

Cor. 1. The angle which the path of the center of gravity makes with the axis of x has its tangent

$$= \frac{Aa \,.\, \sin. \alpha + Bb \,.\, \sin. \beta}{Aa \,.\, \cos. \alpha + Bb \,.\, \cos. \beta}.$$

Cor. 2. It may be shewn, in the same way, that the motion of the center of gravity of any number of bodies is uniform.

153. Prop. *The direction and velocity of the motion of the center of gravity are not altered by the impact of the bodies.*

First, let the bodies move in the same straight line. The center of gravity will, both before and after impact, move in this line; and, by Art. 151, the velocity of this center will be

$$\text{before impact } \frac{Aa + Bb}{A + B}; \text{ after impact } \frac{Au + Bv}{A + B}.$$

And by Cor. 2, Art. 138, $Aa + Bb = Au + Bv$. Hence the velocity will not be affected by the impact.

Next, let the bodies move in different straight lines in the same plane. Let them be referred to rectangular co-ordinates as in the last Article; and let these be so taken that the axis of x is parallel to the surfaces which are in contact in the collision. Then the motion in the direction of x will not be altered by the collision, by Art. 145. Also the motions in the directions perpendicular to this will be affected as if there were no other motions; and therefore, by what has just been shewn, the motion of the center of gravity will not be altered. Since therefore the motion of the center of gravity in these two directions at right angles to each other remains the same, this point will manifestly go on describing the same straight line, and with the same velocity, as before the impact.

If the paths be not in the same plane, the demonstration is easily extended to that case, in the same way as before.

3. *Impact on Planes.*

154. When an elastic body impinges perpendicularly upon an immoveable surface, so as to be reflected, its elasticity is supposed to act according to the same laws as in the former case; that is, the velocity with which the body is reflected is supposed to bear a certain constant ratio to that with which it impinges, which ratio is independent both of the magnitude and of the velocity of the elastic body. Hence we shall be able to determine the result of oblique impact.

Prop. *A body of given elasticity impinges in a given direction upon a plane: to find the direction in which it will be reflected.*

It is supposed, as before, that the perpendicular to the plane at the point of contact passes through the center of gravity of the impinging body.

Let CD be the plane, and let PA, fig. 144, represent the velocity of the body before impact; and let CA be perpendicular to the surface at the point of impact. Draw PM perpendicular to CA. Then the velocity PA may be supposed to be resolved into the two PM, MA: and, in the same manner as in the oblique impact of bodies, the first of these will not be affected, if we leave out of consideration the momentary friction during the contact. The body will be impelled against the plane by the other part of the velocity MA, and will rebound with a velocity Am, which is to AM in the ratio of e to 1; e being the elasticity between the body and the plane. Hence if we take mp perpendicular to Am and equal to PM, the velocity after impact will be compounded of Am and mp, and will therefore be represented in quantity and direction by Ap.

Cor. 1. When $e = 1$, or the elasticity is perfect, produce PM to Q so that $MQ = PM$, and AQ will be the motion after impact.

Cor. 2. The angles which the directions of the body before and after impact make with the perpendicular to the plane are called the *Angles of Incidence* and *of Reflexion.*

In the case of perfect elasticity, these angles are PAM, QAM, which are manifestly equal.

Cor. 3. In any other case, PAM, pAm, are the angles of incidence and reflexion. Now we have

$$\tan. PAM : \tan. pAm :: \frac{PM}{AM} : \frac{pm}{Am}$$

$$:: Am : AM, \text{ because } pm = PM;$$

or tan. ang. incidence : tan. ang. reflexion :: e : 1.

Cor. 4. If we join Qp, it will be parallel to AM; hence

$$QA : Ap :: \sin. QpA : \sin. AQp$$

$$:: \sin. pAM : \sin. QAM,$$

or since $QA = PA$, and $QAM = PAM$,

vel. before impact : vel. after impact :: sin. ang. reflexion : sin. ang. incidence.

Cor. 5. When the body is perfectly inelastic and perfectly smooth, it will, after impact, move along the plane with its lateral velocity; and

vel. before impact : vel. after impact :: PA : PM

:: 1 : cos. APM,

where APM is the angle which the direction of incidence makes with the plane.

From the principles of this Chapter we can without difficulty solve such problems as the following.

155. Prob. I. A *and* B *are two bodies whose elasticity is* e: *to find their proportion, so that* A, *impinging directly upon* B, *may be at rest after the impact.*

By Cor. 5. to Art. 138, the velocity of A after the impact is $\frac{(A - eB)a}{A + B}$; and that this may be $= 0$, we must have $A = eB$; or A less than B in the ratio of e to 1.

This conclusion is independent of the velocity of A, and might therefore be made a means of trying the accuracy of the common hypothesis concerning elasticity, by observing whether, experimentally, in the collision of two bodies which have this proportion, A remains at rest.

156. Prob. II. *A perfectly elastic ball* A, *strikes an equal perfectly elastic ball* B *which is at rest; to find the conditions under which it is possible that after impact they may strike two given points* P *and* Q *respectively. Fig.* 145.

Join Q with B, the center of the second ball, and let QB produced meet the surface of the second ball in c. In order that B may move in the direction BQ after collision, the contact must manifestly take place in the point c: and hence the center A, of the other ball, must be in the line QB produced. Let aA be its velocity before impact; draw am perpendicular to QA; then, since the bodies are equal, the part mA of the velocity will, by Cor. 2. to Art. 139. be entirely destroyed. Hence the body A will retain only its velocity parallel to am; and, if AP be perpendicular to AQ, will move in the direction AP.

Since PAQ is a right angle, the locus of the possible positions of A at the moment of contact is a semicircle on PQ. If the balls be small, the locus of the positions of B for which the problem is possible, is nearly a semicircle.

157. Prob. III. *In the last Article, the elasticity of the balls being imperfect and* = e, *to find the conditions necessary to make the balls after impact strike two given points* P′ *and* Q. *Fig.* 145.

Let aA be the velocity of A before impact; as before, the center of A must, at the moment of contact, be in QB. And mA being the velocity in this direction before impact, we shall find the velocity An after impact in the same direction by Cor. 4, Art. 138. But by that Corollary we have, making $b = 0$, $An = \frac{1}{2}(1 - e) . Am$. and drawing np perpendicular to An and equal to am, Ap will be the direction of A's motion after impact, which, by the question, is to pass through P'.

Hence conversely, if we join AP', and from any point p in AP' draw pn perpendicular on AQ; and take Am so that

$$An = \tfrac{1}{2}(1-e)\,Am, \text{ or } Am = \frac{2\,An}{1-e};$$

and draw ma perpendicular to Am and equal to np; aA is the direction in which A must impinge.

In this case, the problem is always possible, when the point A is without the semicircle described on $P'Q$.

158. Prob. IV. *To find in what direction a perfectly elastic ball must be projected from a given point* P, *that after reflexion at a given plane* DE, *it may strike a given point* Q. *Fig.* 146.

Draw QN perpendicular on the given plane; produce it and make $Nq = NQ$; join Pq meeting DE in A; PA is the direction required.

Join AQ. The triangles QNA, qNA are manifestly equal; hence $PAD = NAq = NAQ$; and since in this case the angle of incidence is equal to the angle of reflexion; the ball projected in the direction PA, will be reflected in the direction AQ, and will strike the point Q.

We have here supposed the ball to be a point. If its magnitude be not inconsiderable, let de be the plane, and draw DE parallel to it, at a distance equal to the radius of the ball: DE will be the plane at which the reflexion of the *center* of the ball may be supposed to take place.

159. Prob. V. *The same things being given, and the ball being imperfectly elastic, to find the direction in which it must be projected in order to strike the point* Q. *Fig.* 147.

Draw QN perpendicular to the plane, and produce it to q, so that $Nq : NQ :: 1 : e$; e being the fraction which expresses the elasticity. Join Pq, and this will be the direction in which the body must be projected. For let Pq meet DE

in A, and join AQ; and draw Az perpendicular to the plane. Then

$$\tan. PAz : \tan. QAz :: \tan. AqN : \tan. AQN$$

$$:: \frac{AN}{Nq} : \frac{AN}{NQ} :: NQ : Nq :: e : 1.$$

Hence by Cor. 3, Art. 154, if the body impinge in the direction PA, it will be reflected in the direction AQ, and will strike the point Q.

If the ball be of finite magnitude, the construction must be modified as in the last problem.

Cor. If PM, perpendicular to the plane, meet QA in p, we have, as may easily be shewn,

$$PM : Mp :: qN : NQ :: 1 : e.$$

Hence the point A may be found by taking $PM : Mp :: 1 : e$ and joining pQ.

160. Prob. VI. *A ball of given elasticity, perfect or imperfect, is to be projected from a given point* P, *so that, being reflected at any number of given planes in a given order, it may afterwards strike a given point* Q. *Fig.* 148.

Let DE, EF, FG be the planes, and let them be produced when necessary.

Draw QN perpendicular on the last plane, and take $QN : Nq :: e : 1$.

Draw qn perpendicular on the next plane, and take $qn : nq' :: e : 1$.

Draw $q'n'$ perpendicular on the next plane, and take $q'n' : n'q'' :: e : 1$.

And so on if there be more planes.

Join Pq'', and this will be the direction in which the ball must be projected.

For let Pq'' meet the first plane in A; join Aq' meeting the second plane in A'; join $A'q$ meeting the third plane in A''; join $A''Q$.

It may be shewn, as in the last problem, that if the body strike the plane in the direction PA, it will be reflected in the direction Aq'; that if it strike the second plane in the direction AA' it will be reflected in the direction $A'q$; and so on. Hence its path will be $PAA'A''Q$, and it will strike the point Q as required.

If the elasticity be perfect, QN and Nq, qn and nq', &c. must be equal respectively.

Cor. If we had begun the construction from P, and made $PM : Mp :: 1 : e$, &c. we should have got the same result. Also if we had drawn perpendiculars on some of the planes beginning from Q, and on others beginning from P, so as to comprehend all the planes between the two extremes, we should still obtain the same solution, and the proof would be nearly the same.

CHAP. III.

UNIFORMLY ACCELERATED MOTION AND GRAVITY.

161. The simplest case of the action of a continuous force, is when the force is uniform, and acts in the straight line in which the body moves. We shall consider in the first place what will be the motion of a body under these circumstances; and in the next place to what cases these conclusions are applicable.

1. *Uniformly Accelerated Motion.*

When a body is accelerated in a straight line by a uniform force, the velocity is as the time from the beginning of the motion.

This is already proved in Art. 114.

If f be the accelerating force, t the time from the beginning of the motion, v the velocity, $v = tf$; therefore v is as t.

The force is represented by f, the velocity which it generates in 1 second.*

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* If gravity be called 1, of course a force which is F times gravity will be called F. Let g be the velocity generated by gravity in 1″, then Fg will be the velocity generated by the force F in 1″; and we shall have

$$v = Fgt.$$

162. PROP. *On the same supposition, the space from the beginning of the motion is as the square of the time.*

During the time t, the velocity of a body acted on by a force f begins from 0, and increases incessantly up to a certain finite magnitude v. It is manifest therefore that the space described in any portion of the time, with this increasing velocity, is less than the space which would have been described in the same portion of time, if the velocity had been, during the whole portion, as great as it is at the end of that portion. Let the time t be divided into n equal portions τ, τ, &c. so that $n\tau = t$. Then by the last Article, we know the velocities at the end of each of these portions of time; and the space which would have been described if these velocities had been respectively continued uniform through each portion of time will be found, by Art. 104, by multiplying the velocity by the time in each portion. Thus, at the end of

the 1st, 2d, 3d, 4th ...n^{th} of the portions τ,

the velocities are $f\tau$, $2f\tau$, $3f\tau$, $4f\tau$,...$nf\tau$.

Hence, *if these velocities had been uniform* through their respective times τ, the space described would have been

in the 1st, 2d, 3d, 4th ...n^{th},

$$f\tau^2,\ 2f\tau^2,\ 3f\tau^2,\ 4f\tau^2 \ldots nf\tau^2;$$

and the sum of all these is

$$f\tau^2 (1 + 2 + 3 + \ldots n) = f\tau^2 \cdot \frac{n \cdot (n+1)}{2}$$

$$= \frac{f\tau^2 n^2}{2} + \frac{f\tau^2 n^2}{2n} = \frac{ft^2}{2} + \frac{ft^2}{2n}, \text{ because } \tau n = t.$$

Now the space which is actually described by the uniformly accelerated body, is, as has been said, in each of these portions, *less* than the corresponding space just found.

Hence the whole space described, which we call s, is *less* than the sum of all these spaces; that is,

$$s < \frac{ft^2}{2} + \frac{ft^2}{2n}.$$

But the sum of all the spaces described with the uniform velocities will differ less and less from the actual space, as the portions of time are made smaller and smaller, and of course as their number is made larger and larger; and by increasing this number indefinitely, the aggregate of the spaces described with the successive velocities, approaches indefinitely near to the space described with the accelerated motion; that is

s approaches to $\frac{ft^2}{2} + \frac{ft^2}{2n}$ when n becomes indefinitely large.

Or $s = \frac{ft^2}{2}$ *, because the fraction $\frac{ft^2}{2n}$ becomes indefinitely small.

Hence s varies as t^2.

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* That this is the accurate value of s may perhaps be made more evident as follows.

The velocities at the *beginning* of the

1st, 2d, 3d ...... n^{th} of the portions τ,

are $0, \; f\tau, \; 2f\tau \ldots\ldots (n-1)f\tau$.

Hence if these initial velocities had been continued uniform through these portions respectively, the spaces described would have been

$$0, \; f\tau^2, \; 2f\tau^2 \ldots\ldots (n-1)f\tau^2.$$

And the sum of these (an arithmetical progression) is,

$$f\tau^2 . \frac{n.(n-1)}{2} = \frac{f\tau^2 n^2}{2} - \frac{f\tau^2 n}{2n} = \frac{ft^2}{2} - \frac{ft^2}{2n}.$$

Now the space described (s) when the velocity increases continually, is *greater* than this. Hence, (combining this with what is said in the text,) whatever be n,

$$s > \frac{ft^2}{2} - \frac{ft^2}{2n}, \text{ and } s < \frac{ft^2}{2} + \frac{ft^2}{2n}.$$

And as the fraction $\frac{ft^2}{2n}$ may become smaller than any assigned quantity by increasing n, this cannot be true except $s = \frac{ft^2}{2}$.

If we had taken gravity for our unit of force, the formula would have been $s = \frac{1}{2}Fgt^2$; or, if m be the space through which a body would fall in 1″, $s = mFt^2$.

 The

Cor. 1. In the two equations $v = ft$, $s = \frac{1}{2}ft^2$, we have four quantities, any two of which serve to determine the other two. By simple eliminations we obtain the following results;

$$\left.\begin{aligned} s &= \tfrac{1}{2}ft^2 = \tfrac{1}{2}tv = \frac{v^2}{2f};\\ v &= ft = \frac{2s}{t} = \sqrt{(2fs)};\\ t &= \frac{v}{f} = \frac{2s}{v} = \sqrt{\frac{2s}{f}};\\ f &= \frac{v}{t} = \frac{v^2}{2s} = \frac{2s}{t^2} \end{aligned}\right\}\ldots\ldots\ldots\ldots\ldots(A).$$

Cor. 2. Since $s = \frac{1}{2}tv$, and tv is the space described in the time t with the velocity v, it appears, that *the space described by a body uniformly accelerated from rest, is half the space described in the same time with the last acquired velocity.*

Hence also the space through which the body moves in the first second is the half of f, because f is the velocity acquired in 1″.

Cor. 3. The space described in t seconds $= \frac{1}{2}ft^2$;

...... in $t - 1$ seconds $= \frac{1}{2}f(t-1)^2 = \frac{1}{2}f(t^2 - 2t + 1)$;

therefore, subtracting, we have

the space in the t^{th} second $= \frac{1}{2}f(2t - 1)$.

The equations in the text may be immediately obtained from the equations $\frac{dv}{dt} = f$, and $\frac{ds}{dt} = v$, putting g for f, and integrating. We have thus

$$\frac{dv}{dt} = g, \quad \therefore\ v = gt;$$

$$\frac{ds}{dt} = v = gt, \quad \therefore\ s = \tfrac{1}{2}gt^2.$$

There are no corrections required in these integrations, for, when $t = 0$, $v = 0$, and $s = 0$.

Hence the space in the 1st, 2d, 3d, 4th, &c. seconds are $\frac{1}{2}f.1$, $\frac{1}{2}f.3$, $\frac{1}{2}f.5$, $\frac{1}{2}f.7$, &c. and are as the odd numbers 1, 3, 5, 7, &c.

163. PROP. *Let a body be projected with a given velocity* u, *and acted on in the same direction by a constant force* f; *it is required to determine the relation of the space, time, and velocity.*

It is manifest that if the body is, at a certain point, moving with a certain velocity, its motion after that point will be the same, however we suppose the velocity to have been acquired. Hence the motion will be the same, if we suppose that velocity to have been generated by the force accelerating the body from rest. Let the force f generate the velocity u by acting for a time t', through a space s'. Hence $u = ft'$. Let the body afterwards continue to be acted on by the same force, and describe a space s in a time t; so as to describe a space $s' + s$ from rest in a time $t' + t$. Hence we have, by the last Article,

$$s' + s = \tfrac{1}{2}f(t' + t)^2 = \tfrac{1}{2}f(t'^2 + 2t't + t^2),$$

$$\text{and, } s' = \tfrac{1}{2}ft'^2;$$

$$\therefore\ s = \tfrac{1}{2}f(2t't + t^2) = ft't + \tfrac{1}{2}ft^2;$$

$$\text{but } u = ft';\ \therefore\ s = tu + \tfrac{1}{2}ft^2.$$

COR. 1. Since tu is the space which the body would have described in the time t, with the uniform velocity u, and $\frac{1}{2}ft^2$ the space through which the force would have drawn it in the same time; it appears that *the space in any time is equal to the space described with the velocity of projection, plus the space described from rest by the action of the force.*

COR. 2. If v be the velocity at the end of the time t,

$$v = f(t' + t) = ft' + ft = u + ft.$$

Hence *the velocity after any time is equal to the velocity of projection plus the velocity generated by the force:* as is also manifest from the definition of uniform force.

Cor. 3. We have also, by equations (A),

$$v^2 = 2f(s' + s);\ \ u^2 = 2fs':$$

$$\text{hence } v^2 - u^2 = 2fs;\ \ v^2 = u^2 + 2fs.$$

164. Prop. *When a body is projected in a direction opposite to that in which the force acts, the same formulæ will be true as in the last two Articles,* s *being the space, and* t *the time, from the end of the motion.*

In this case the force will diminish the velocity; and, since it is uniform, will produce equal decrements in equal times. In a certain time, the body will be reduced to rest, and during this time, the velocity will go on decreasing by exactly the same degrees by which it increased when a body was accelerated from rest. Hence the spaces reckoned from the end of this motion will be the same as the spaces from the beginning of the former motion.

Cor. 1. Let the body be projected with the velocity u, and let t' be the time and s' the space in which the whole of the velocity would be destroyed by the action of the force in the opposite direction. In a time t let a space s be described; then in the remaining time $t' - t$ from the end of the motion in a direction opposite to the force, there would be described a space $s' - s$. Hence we have

$$s' = \tfrac{1}{2}ft'^2;$$

$$s' - s = \tfrac{1}{2}f(t' - t)^2 = \tfrac{1}{2}f(t'^2 - 2t't + t^2);$$

$$\therefore s = \tfrac{1}{2}f(2t't - t^2) = ft't - \tfrac{1}{2}ft^2,$$

and since $u = ft'$, $s = tu - \frac{1}{2}ft^2$.

Cor. 2. Hence, as in the last Article, *the space in any time is equal to the space described with the velocity of projection, minus the space described from rest by the action of the force in that time.*

Cor. 3. Similarly, *the velocity after any time is equal to the velocity of projection, minus the velocity generated by the force in that time.*

2. *Vertical Motion by Gravity.*

165. *Gravity, near the earth's surface is a uniform force.*

This has already been stated in Article 113*a*.

When a stone falls from rest by the action of gravity, its velocity goes on perpetually increasing so long as it falls freely. The law according to which this acceleration takes place, is, of course, to be determined from experiment; and it is found, that whatever be the material and mass of the falling body, and the other circumstances of the fall, if we make allowance for the effects produced by the resistance of the air and other impediments, the velocity generated by gravity is as the time, and consequently, from what has been said, that the force of gravity is constant.

This was first asserted by Galileo; some facts were adduced by him to prove the hypothesis; and all the experiments which were made afterwards, tended to confirm it. The motions of bodies which fall freely, are so rapid, that they cannot be observed with sufficient accuracy; and hence some contrivance is necessary which may diminish the velocity while it preserves the law of the acceleration. This effect may be obtained in different ways. Instead of allowing the body, the velocity of which we observe, to fall, unconnected with any other, we may cause it to descend, drawing up another body, or producing rotatory motion in a mass fixed upon an axis; by which means the motion will be so much retarded that it may be measured. Or we may make the body descend down a very smooth inclined plane, or other inclined surface; and by making the inclination small, the velocity will become sufficiently slow to be observed. Or we may cause bodies to descend down circular arcs by hanging them to strings of given lengths, and making them swing; which will be equivalent to letting them descend down perfectly smooth circular surfaces. This last method is susceptible of great accuracy. The times of oscillation of the pendulums depend upon the velocities with which the bodies move in the circular arcs; and these velocities have a known relation to the velocities with which the bodies would fall perpendicularly, as will be seen in a succeeding Chapter. Hence, since the times of the oscillations of pendulums agree with the theory as deduced from the supposition of constant gravity, that supposition is proved to be true. In the same way the other experiments confirm the proposition that *gravity is a constant force**.

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* The contrivance first mentioned is employed in the machine invented by Atwood, and an account of experiments made with it may be found in his Treatise of Rectilinear and Rotatory Motion, Sect. 7. The descent of bodies down inclined planes was the method used by Galileo. It does not admit of much accuracy, on account of the effects of friction, which causes the bodies to roll instead of sliding, and otherwise affects their motion.

The quantities on which we might expect the variation of gravity to depend, if it were not constant, might be the situation of places upon the earth's surface, and their elevation; the velocity of the body on which the force acts, and the size and substance of the body. Accurately speaking, it does vary with some of these. Gravity is a force arising from the attraction of the earth, tending at every point nearly to its center, and dependent on the distance from that center; at different distances from the equator, and at different altitudes, it is perceptibly different if the intervals be taken of sufficient magnitude: and it is only in consequence of the smallness of this variation in any spaces with which we are here concerned, that we may suppose gravity to be constant and to act in parallel lines. There are also other variations still more inconsiderable, arising from the irregularities of the form and materials in the structure of the earth, which in some measure influence its attraction, from which gravity arises.

In producing the same effect upon a body, whatever be its previous velocity, gravity differs remarkably from all the mechanical powers which we can exert. For instance, supposing that by turning a winch with a certain muscular exertion we could communicate to a wheel a certain angular velocity in 1″, we should not add an *equal* velocity, if we were to exert the same effort for the next second, because part of the muscular power must be employed in moving the hand so as to keep up with the winch. In the same way, if motion were communicated by a spring, the action of the spring would be less as the body receded faster from it; and the body might move with such a velocity that the effect of the spring should be only just sufficient to enable it to keep up with the body, and should not at all increase its motion. Gravity, on the contrary, acts with the same energy, whether the body acted on be at rest, or moving in the direction of its action, or in a contrary direction.

It was formerly supposed that heavier bodies descended faster than lighter ones in proportion to their weight. The falsity of this was shewn by Galileo from experiment.

166. Prop. *Gravity is the same in all bodies, whatever be the difference of material or magnitude.*

In bodies of different material this was proved by Newton from experiments upon pendulums: (see *Principia*, Book III. Prop. 6:) he inferred from his observations that all substances would descend to the earth with equal velocities.

That bodies of the *same* material and different magnitudes would descend with the same velocity, is easily seen. For if one body be 10 times the other, let the first be divided into 10 bodies, each equal to the second. If these were all to fall at the same time from the same point, but separate, they would descend each with the same velocity as the second body. Hence if they were supposed to be connected and united, they would not accelerate or retard each other's motions; and therefore the whole mass would still descend with the same velocity.

167. The intensity of gravity, or the space through which a body would fall in 1″, must be determined by experiment. The most accurate observations for this purpose are those that are made upon pendulums. It will be shewn hereafter, that, knowing the length of a pendulum which oscillates once in a second, we can find the space through which a body would fall in the same time. By the latest experiments of this kind it appears that in the latitude of London, and at the surface of the earth, a body would, *in vacuo*, fall through a space of 193.14 English inches, or $16\frac{1}{10}$ feet nearly. Consequently, (Art. 162. Cor. 2,) the velocity generated in that time would in the same time carry it through 386.28 inches; and thus this space measures the velocity generated in 1″ by gravity, and is therefore the value of that force, according to the way of measuring

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So far as gravity is concerned, the same velocity is communicated to all bodies, whatever be their mass; but in consequence of the resistance of the air, which is proportionally greater on smaller bodies, heavy ones do, in the atmosphere, descend with greater celerity.

accelerating forces (Art. 113). This quantity will generally be represented by g.

Hence we can easily solve all questions relating to the fall of bodies *in vacuo* by gravity. We have only to substitute the known quantity g for f in the formulæ (A) of Art. 162: as is seen in the following examples*.

Ex. 1. *To find how far a body will fall in vacuo in* $2\frac{1}{2}''$, *and the velocity acquired.*

By the first expression for s in (A), putting g for f;

$$s = \tfrac{1}{2}gt^2 = \tfrac{1}{2} \times 32.2 \times \left(\frac{5}{2}\right)^2 \text{ feet} = 100.6 \text{ feet}.$$

By the first expression for the velocity;

$$v = gt = 32.2 \times \frac{5}{2} \text{ feet} = 80.5 \text{ feet}.$$

Ex. 2. *A body is projected upwards with a velocity of* 100 *feet; to find how high it will rise, and in what time it will reach its greatest height.*

By Art. 164, the height to which it will rise will be the same as the height down which it must fall to acquire the velocity. Hence, by the third expression for s in (A),

$$s = \frac{v^2}{2g} = \frac{100^2}{2 \times 32.2} = \frac{10000}{64.4} = 155.28 \text{ feet}.$$

Similarly the time of the ascent is equal to the time of the descent; hence, by the first expression for t,

$$t = \frac{v}{g} = \frac{100}{32.2} = 3.1'.$$

* The descent of a body in the atmosphere will be nearly the same as *in vacuo*, so long as the velocity is small. But when bodies fall freely through great heights, the resistance becomes very large, and the effect arising from this may be made to bear any ratio to the whole.

Ex. 3. *On the same supposition, to find how high the body will ascend in 2″.*

Putting g for f in Art. 164, we have

$$s = tu - \tfrac{1}{2}gt^2 = 2 \times 100 - \tfrac{1}{2} \times 32.2 \times 4 = 200 - 64.4 = 135.6 \text{ feet.}$$

By means of our formulæ we can easily solve such problems as the following:

168. Prob. I. *A person drops a stone into a well, and after* t *seconds hears it strike the water; to find the depth to the surface of the water.*

We neglect the resistance of the air. The velocity of sound, as appears by experiment, is uniform, and equal to 1130 feet in a second. Now the time between dropping the stone and hearing the sound, is equal to the time of the stone falling the depth of the well, together with the time of the sound rising through the same distance. Let x be this depth, and n the velocity of sound. Then

$$\text{time of falling through } x = \sqrt{\frac{2x}{g}};$$

$$\text{time of sound's passage} = \frac{x}{n};$$

$$\therefore \frac{x}{n} + \sqrt{\frac{2x}{g}} = t;$$

$$\therefore \frac{2x}{g} = t^2 - \frac{2tx}{n} + \frac{x^2}{n^2};$$

$$\therefore x^2 - 2\left(tn + \frac{n^2}{g}\right)x = -t^2n^2;$$

$$x = tn + \frac{n^2}{g} \pm \sqrt{\left(\frac{2tn^3}{g} + \frac{n^4}{g^2}\right)}.$$

The negative sign must be taken*. Also it will be found that $\frac{n}{g} = 35$ nearly †;

$$\therefore \; x = n \,.\left\{t + \frac{n}{g} - \sqrt{\left(\frac{2tn}{g} + \frac{n^2}{g^2}\right)}\right\}$$

$$= n \,.\left\{t + 35 - \sqrt{(70t + 1225)}\right\}.$$

Thus, let the time t be $3''$; then

$$x = n \,.\left\{38 - \sqrt{1435}\right\} = n \,.\left\{38 - 37.88\right\}$$

$$= .12\, n = 138.6 \text{ feet.}$$

169. Prop. *When two bodies hang over a fixed pully, to determine their motion, neglecting the inertia of the pully and the string.*

Thus let two unequal bodies p and q, hang over a pully as in fig. 136; (p corresponding to $P + A$ and q to B). Let p be greater than q. If p were equal to q, it would just balance q, and there would be no motion; the weight which is employed in producing motion is the excess of p above q, or $p - q$. Also the two bodies move with equal velocities, and hence the mass in which motion is produced is $p + q$. The accelerating force is therefore equal to $\frac{p-q}{p+q}$ multiplied into some constant quantity. (Art. 125.)

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* For t, the whole time, must be greater than the time of the sound's motion, which is

$$\frac{x}{n}, \text{ or } t + \frac{n}{g} \pm \sqrt{\left(\frac{2tn}{g} + \frac{n^2}{g^2}\right)}.$$

But t is not greater than $t + \frac{n}{g} + \sqrt{\left(\frac{2tn}{g} + \frac{n^2}{g^2}\right)}$ with the positive sign.

† If n be equal to 1127 feet, $\frac{n}{g}$ is accurately equal to 35. The values generally taken for n have been 1142 and 1130 feet. Recent experiments would seem to shew that at the usual temperature of the air, the velocity is less; but the determinations are too various to entitle us to fix upon any particular value. See *Trans. of Camb. Phil. Soc.* Vol. II. Part I. p. 120.

Let the accelerating force $f = \frac{p-q}{p+q} c$: and when $q = 0$, or p descends freely, the force is g. Therefore $c = g$. Hence in other cases $f = \frac{p-q}{p+q} \cdot g$: which agrees with Ex. 2. Art. 126.

By substituting this value for f in equations (A) we can find the circumstances of the motion in any given case.

Ex. *Supposing* p = 81 *ounces and* q = 80; *to find the space descended by* p *in* 1″, *and the velocity acquired.*

$$s = \frac{g}{2} \cdot \frac{p-q}{p+q} t^2 = 16.1 \times \frac{1}{161} \times 1^2 = .1, \text{ or } \tfrac{1}{10} \text{ of a foot.}$$

$$v = g \cdot \frac{p-q}{p+q} - 32.2 \times \frac{1}{161} = .2, \text{ or } \tfrac{1}{5} \text{ of a foot.}$$

If instead of p drawing q vertically upwards, p draw q along a horizontal plane; as for instance, if q be laid upon a perfectly smooth table, and p, connected with it by a string, hang over the edge of the table; the whole weight of p is employed in producing motion; and, as before, the two bodies move with the same velocity, and therefore may be considered as one mass $p + q$. Hence the accelerating force

$$f = \frac{pg}{p+q}.$$

3. *Motion on Inclined Planes.*

170. Prop. *To find the force which accelerates a body down an inclined plane.*

When a body q is *supported* on an inclined plane whose height and length are h and l respectively, by a force p, acting parallel to the plane, we have, by Art. 38, Cor. 2,

$$p : q :: h : l; \quad \therefore p = \frac{qh}{l}.$$

Hence, when q is not supported, $\frac{qh}{l}$ is the pressure which it exerts *along* the plane, and which is employed in producing motion. Also if the body be suffered to descend by its weight, the mass moved is the body q itself*. Therefore, by Art. 126, the accelerating force will be proportional to $\frac{p}{q}$ or $\frac{h}{l}$; and, as in last Article, equal to $\frac{hg}{l}$. By substituting this value for f in equations (A), we obtain the circumstances of the descent of bodies down inclined planes.

Also if the body descend down the whole length of the plane, l may be put for s.

By this means, from the first value of s in (A), we have

$$l = \frac{1}{2} \cdot \frac{gh}{l} \cdot t^2; \quad \therefore t^2 = \frac{2l^2}{gh}; \quad t = \sqrt{\frac{2l^2}{gh}}.$$

Also by the third expression for the velocity,

$$v = \sqrt{(2fs)} = \sqrt{\left(\frac{2gh}{l} \cdot l\right)} = \sqrt{(2gh)}.$$

Since this expression for the velocity is independent of the length, it appears that *the velocity acquired down all planes whose perpendicular heights are equal, will be the same; and equal to the velocity acquired by falling down the perpendicular height.* This principle was *assumed* by Galileo as the basis of his reasonings on inclined planes.

Cor. If a be the inclination of the plane to the horizon, $\frac{h}{l} = \sin. a$, and $g \sin. a$ is the accelerating force upon the plane; which may be substituted for f in the formulæ (A).

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* The plane is supposed to be *perfectly* smooth, so as to exert no resistance to motion along it; in which case the body q will slide, and not roll, down the plane, even if it be spherical in its form. In actual cases the friction is almost always great enough to produce rotatory motion in round bodies. This circumstance changes the value of the accelerative force, as may be shewn hereafter.

Ex. *A smooth plane,* 10 *feet long, has one end* 1 *foot higher than the other: to find the time of a body sliding down it, and the velocity acquired.*

By the formula just obtained,

$$t = \sqrt{\frac{2l^2}{gh}} = \sqrt{\frac{200}{32.2}} = \frac{10}{4}, \text{ nearly}, = 2\tfrac{1}{2}''.$$

$$\text{Also } v = \sqrt{(2gh)} = \sqrt{(64.4)} = 80.2 \text{ feet.}$$

171. Prob. II. *A right-angled triangle being placed with its two sides horizontal and vertical respectively, it is required to determine their proportion, so that the time of the body falling down the perpendicular and describing the base with the velocity acquired, may be equal to the time of descent down the hypotenuse.*

Let x and y be the vertical and horizontal sides respectively; therefore the hypotenuse will be $= \sqrt{(x^2 + y^2)}$. And by formula (A),

$$\text{time down } x = \sqrt{\frac{2x}{g}}; \text{ velocity acquired} = \sqrt{(2gx)};$$

$$\therefore \text{ time through } y = \frac{y}{\sqrt{(2gx)}}.$$

Also by last Article, x and $\sqrt{(x^2 + y^2)}$ being the height and length on an inclined plane,

$$\text{time down the length} = \sqrt{\frac{2(x^2 + y^2)}{gx}}.$$

$$\text{Hence } \sqrt{\frac{2(x^2 + y^2)}{gx}} = \sqrt{\frac{2x}{g}} + \frac{y}{\sqrt{(2gx)}};$$

$$\text{squaring, } \frac{2(x^2 + y^2)}{gx} = \frac{2x}{g} + \frac{2y}{g} + \frac{y^2}{2gx};$$

$$\therefore 4x^2 + 4y^2 = 4x^2 + 4xy + y^2;$$

$$\therefore 3y = 4x, \text{ or } \frac{x}{y} = \frac{3}{4}.$$

Hence also $\frac{\sqrt{(x^2+y^2)}}{y} = \frac{5}{4}$; and the sides of the triangle are as 3, 4, and 5.

172. Prop. *To find the accelerating force, when a heavy body draws another along an inclined plane.*

Let, in fig. 46, the weight $W(=q)$ be fixed to a string WC, which is parallel to the inclined plane on which the weight rests, and, passing over a fixed pully at C, has a weight p appended to it and hanging freely. If $p = \frac{qh}{l}$; p and q will balance. And if p be greater than this value, it will descend and draw q up the inclined plane. If p be less than the value just mentioned, q will descend down the inclined plane and draw up p. In both cases the accelerating force will be constant.

When p draws up q, the part $\frac{qh}{l}$ of p is employed in supporting q, and the remainder, $p - \frac{qh}{l}$, is the pressure which produces motion in the two bodies p, q. And these two bodies move with the same velocity. Hence the accelerating force in this case is

$$\frac{p - \frac{qh}{l}}{p+q} \cdot g = \frac{pl - qh}{pl + ql} \cdot g;$$

and by substituting this value for f we can apply our formulæ.

Prob. III. *The notation remaining, to find the time in which* p *will draw* q *up the given plane whose length is* l *and height* h.

We have, as before, $l = \frac{1}{2}ft^2 = \frac{pl - qh}{pl + ql} \cdot \frac{t^2 g}{2}$,

$$\therefore\ t = \sqrt{\frac{2(p+q)l^2}{g(pl-ql)}}\ *.$$

* Prop. *On the same supposition,* h, p, *and* q *being given, to find* l *so that the time of drawing* q *up it may be the least possible.*

173. PROP. *If a circle be placed with its plane vertical, the times of descent down all chords drawn through the highest or lowest points are equal.*

Let ABP, fig. 149, be a circle, and AB a vertical diameter. Let PA be any chord drawn through A. By Article 170, we have

$$\text{time down } AP = \sqrt{\frac{2\,AP^2}{g \cdot AM}};$$

but by similar triangles $\dfrac{AP}{AM} = \dfrac{AB}{AP}$; $\therefore \dfrac{AP^2}{AM} = AB$;

$$\therefore \text{time down } AP = \sqrt{\frac{2\,AB}{g}}.$$

This is independent of the position of P: and hence the times down all chords AP, Ap, &c. are equal; and of course equal to the time of falling freely down AB.

In the same way it may be shewn that times down all chords PB, pB, &c. are equal.

COR. Also we have

$$\text{velocity acquired down } AP = \sqrt{(2g \cdot AM}; \text{ (Art. 170.)}$$

$$\text{and as before } AP^2 = AM \cdot AB; \therefore AM = \frac{AP^2}{AB};$$

$$\therefore \text{velocity} = \sqrt{\frac{2g \cdot AP^2}{AB}} = AP\sqrt{\frac{2g}{AB}}.$$

We must have the above expression for t a minimum, and therefore its square a minimum; and it will also be a minimum if we omit the constant factors $2(p+q)$ and g. Hence

$$\frac{l^2}{pl - qh} = \text{min.}; \quad \therefore \frac{pl - qh}{l^2} = \text{max.}$$

$$\therefore \frac{p}{l} - \frac{qh}{l^2} = \text{max.}; \quad \therefore \text{differentiating} -\frac{p}{l^2} + \frac{2qh}{l^3} = 0;$$

$$\therefore l = \frac{2qh}{p}.$$

Here $p = \dfrac{2 \cdot qh}{l}$; that is, p is twice as great as it is for equilibrium.

And as g and AB are constant, *the velocities acquired down planes* AP, Ap, *&c. are as the lengths* AP, Ap, &c.

174. Prop. *Let* APB, AQC, *fig.* 149, *be two circles with their diameters in the same vertical line* AB, *and with the highest point common;* Apq, APQ *any chords: the times down* PQ, pq *from rest at* P *and* p *are equal.*

On BC describe a semi-circle; join CQ, meeting this semicircle in R; join BR. The angles APB, AQC, BRC are right angles, and therefore AQ, BR, and PB, QC, are parallel. Hence $PBRQ$ is a parallelogram, and BR is equal and parallel to PQ; and hence the time down PQ is equal to the time down BR. Similarly if Br be drawn parallel to pq, the time down pq will be equal to the time down Br. But the times down BR, Br are equal; therefore the times down PQ pq are equal.

Cor. Similarly, if two circles touch each other at the lowest point, and chords be drawn through this point; it may be shewn that the times down those portions of the chords which are intercepted between the circles are all equal.

4. *Planes of Quickest and Slowest Descent.*

175. There are a number of Problems concerning the planes on which bodies would descend between given points, lines, and circles, so as to employ in their descent the longest or the shortest time possible. The constructions and demonstrations are very nearly similar for all of them, and the student will have no difficulty, after one or two specimens, in making out the rest. We shall give the Problems with their constructions, and the demonstrations in some of the most important cases, which will suggest them in the others.

It is required to find the plane of shortest descent.

Prob. IV. *From a given point* P *to a given straight line* AB, *fig.* 150.

From the given point P draw a horizontal line meeting the given line in A. Take AQ *downwards* along the given line, and equal to AP: PQ will be the plane required.

Prob. V. *From a given straight line* AB *to a given point* P, *fig.* 150.

Draw *PA* as before; and along the given line measure a distance *upwards* from *A*, equal to *AP*; the line joining the extremity of this distance with the point *P* is the plane required.

Prob. VI. *From a given point without a given circle to the circle.*

Join the given point with the *lowest* point of the given circle: the part of the joining line which lies without the circle is the plane required.

Prob. VII. *From a given circle to a given point without it.*

Join the given point with the *highest* point of the given circle: the part of the joining line which lies without the circle is the plane required.

Prob. VIII. *From a given point within a given circle to the circle.*

Join the given point and the *highest* point of the circle: the part of the joining line produced which is between the point and the circle is the plane required.

Prob. IX. *From a given circle to a given point within it.*

Join the given point and the *lowest* point of the circle: the part of the line produced which is between the circle and the point is the plane required.

Prob. X. *From a given straight line* (RM, *fig.* 151,) *without a given circle* (ASB) *to the circle.*

Through *B*, the *lowest* point of the circle, draw *BM* horizontal. Take *MR* *upwards* equal to *MB*, and join *RB*: *RS* is the plane required.

Prob. XI. *From a given circle to a given straight line without it.*

Draw a horizontal line through the *highest* point of the circle, terminated by the given line; and take *downwards* along the given line a distance equal to this horizontal line. Join the extremity of this distance with the highest point: a part of this joining line is the plane required.

Prob. XII. *From a given circle to another given circle without it.*

Join the *highest* point of the first circle with the *lowest* point of the second: the portion of the joining line which is between the circles is the plane required.

Prob. XIII. *From a given circle to another given circle within it.*

Join the *lowest* point of the first circle with the *lowest* point of the second: the part of the joining line produced which lies between the two circles is the plane required.

Prob. XIV. *From a given circle within another given circle to the other circle.*

Join the *highest* point of the first circle with the *highest* point of the second: the part of the joining line produced which lies between the two circles is the plane required.

It may also be required to find the plane of *longest* descent;—

Prob. XV. *From a given point without a given circle to the circle.*

Join the given point and the *highest* point of the circle: this joining line, produced till it again meets the circle, is the plane required.

Prob. XVI. *From a given circle to a given point without it.*

Join the given point and the *lowest* point of the circle: this joining line, produced till it again meets the circle, is the plane required.

Prob. XVII. *From a given circle to another given circle without it.*

Join the *lowest* point of the first circle with the *highest* point of the second: the joining line, produced both ways, till it again meets the circumferences, is the plane required.

The plane of longest descent cannot be determined in any case when there is a possibility of drawing a horizontal plane under the conditions: for as the plane approaches to this position, the time of descent increases without limit. Also the plane of shortest descent cannot be determined in any case when the circles, &c. between which it is to be drawn, intersect each other: for by bringing the extremities of the plane near this point, we may diminish the plane and the time down it indefinitely.

176. We shall now give the demonstrations of Prob. 4, Prob. 7, and Prob. 12.

Demonstration for Prob. 4. Fig. 150. Draw PO vertical and QO perpendicular to AQ. Since AQ was taken equal to AP, the angles AQP, APQ are equal. Also APO, AQO are equal, being right angles. Hence OPQ, OQP are equal, and therefore OP, OQ. With center O and radius OP describe a circle, which will pass through Q, and there touch AQ: also P will be the highest point. Draw any line Pr, meeting the circle in q. Now by the last Article the time down Pq is equal to that down PQ; hence the time down Pr, which is greater than that down Pq, is greater than that down PQ: and as this is true for every line Pr which does not coincide with PQ, the time down PQ is the shortest.

Demonstration for Prob. 7. Fig. 151. P being the given point and AB a vertical diameter of the given circle. AP is joined, and QP is the plane required. For C being the center of *AQB*, let *CQ* meet a vertical line *PO* in *O*. The angles *OPQ*, *QAC*, *CQA*, *OQP* are equal: hence *OP*, *OQ* are equal. With center *O* describe a circle *PQ*, which will touch *AQB*. As before, the time down *QP* may be shewn to be less than the time down any other plane *rP*.

Demonstration for Prob. 12. Fig. 152. *AB*, *ab* being vertical diameters of the given circles, *Ab* is joined, and *PQ* is the plane required. It appears from the demonstration for Prob. 7, that whatever be the point *to* which the plane is drawn, it must pass through the highest point *A* of the first circle, in order that the time may be less down the plane than down any other plane from the first circle to the point Q in the second. Hence, we have to determine down which of the planes *pr*, which produced pass through the point *A*, the time is least. The center of *aQb* being *c*, let *cQ* meet *AB* in *O*, and as before *OA*, *OQ* are equal. With center *O* describe a circle *AQ*. Then it follows from Cor. 1, to Art. 174, that the time down *PQ* is equal to the time down *pq*, and therefore less than the time down *pr*. Hence *PQ* is the plane of shortest descent of all that pass through *A*; and hence, by what has been said, of all that can be drawn from one circle to the other.

CHAP. IV.

THE MOTION OF PROJECTILES.

177. When a body is projected in any direction, not vertical, and acted upon by gravity, it will describe a curve line. The nature of this curve may be deduced from the principles laid down in Chap. I. It follows from the second law of motion, that if a body be projected in the direction AR, fig. 153, with any velocity, and if, in the time in which it would describe AR with this velocity continued uniform, it would, by the action of gravity, fall through the space Am from rest; its place at the end of this time will be P, so situated that RP is equal and parallel to Am.

It appears from this that the motion of the body will be in a vertical plane.

178. Prop. *A body is projected from a given point, in a given direction, with a given velocity; it is required to find where it will strike the horizontal plane passing through the point of projection.*

In fig. 153, let A be the point, AT the direction of projection, and APH the path; AH being horizontal. Let TAH, the angle of projection, $= \alpha$; the velocity of projection $= V$; and the time of describing $APH = T$. Then for the reasons mentioned in the last Article, in the time T, AT would have been described uniformly, and TH would have been fallen through by the force of gravity. Therefore

$$AT = T \, . \, V, \text{ and } TH = \tfrac{1}{2} g T^2, \text{ (Art. 162.)}$$

Also $TH = AT \, . \, \text{sin.}\, \alpha$, or $\frac{1}{2} g T^2 = T \, . \, V \, . \, \text{sin.}\, \alpha$;

$$\therefore \; T = \frac{2V \sin. a}{g}.$$

Hence $AT = T \,.\, V = \dfrac{2V^2 \sin. a}{g}$;

$$AH = AT \,.\, \cos. a = \frac{2V^2 \sin. a \cos. a}{g} = \frac{V^2}{g} \sin. 2a.$$

The distance AH is called the *range*, and T is called the *time of flight* of the projectile.

Cor. 1. If t be any other time, in which the arc AP is described, and if RPM be vertical; $AR = V.t$, $RP = \frac{1}{2}gt^2$, and

$$PM = MR - RP = Vt \sin. a - \tfrac{1}{2}gt^2.$$

Also $AM = Vt \cos. a$;

and therefore the point M moves uniformly in AM.

Cor. 2. Let $t = \frac{1}{2} T = \dfrac{V \sin. a}{g}$, and let V in the figure be the corresponding place of the body; then we have, by Cor. 1,

$$PM \text{ or } VG = \frac{V^2 \sin.^2 a}{g} - \frac{V^2 \sin.^2 a}{2g} = \frac{V^2 \sin.^2 a}{2g}.$$

Cor. 3. Let t be greater or less than $\frac{1}{2}T$. Suppose

$$t = \tfrac{1}{2} T (1 \pm m) = \frac{V \sin. a}{g} (1 \pm m).$$

Then by Cor. 1,

$$PM = \frac{V^2 \sin.^2 a}{g} (1 \pm m) - \frac{V^2 \sin.^2 a}{2g} (1 \pm m)^2$$

$$= \frac{V^2 \sin.^2 a}{2g} (1 - m^2).$$

Hence it appears that PM is greatest when $m = 0$, that is, when $t = \frac{1}{2} T$; or the greatest height VG occurs in the middle of the time of flight.

Cor. 4. It appears also that for equal values of m, whether they be positive or negative, we have the same value of PM; hence on the two sides of the highest point V, the points P, P', corresponding to equal times from V, are at the same height above the horizontal plane.

Also equal distances GM, GM', correspond to equal times from V (Cor. 1.). Hence the curve consists of two equal and similar arcs from V to A, and from V to H.

179. Prop. *On the same suppositions, it is required to find where the body will strike any given plane passing through the point of projection.*

Let the body be projected in the direction AT, fig. 154; and let AQ be the line in which the vertical plane passing through AT meets the given plane; AH horizontal. Let

$$TAH = \alpha,\ QAH = \iota; \quad \therefore\ TAQ = \alpha - \iota.$$

Also let T be the time of flight in AQ; R the *range* or distance AQ; V the velocity of projection. Hence, as in last Article,

$$AT = T \,.\, V,\ TQ = \tfrac{1}{2} g T^2.$$

But by Trig. $QT : AT :: \sin. QAT : \sin. AQT$,

and $\sin. AQT = \sin. AQH = \cos. QAH$.

$$\text{Hence } QT = AT \,.\, \frac{\sin. QAT}{\sin. AQT} = AT \,.\, \frac{\sin. QAT}{\cos. QAH};$$

$$\text{or } \tfrac{1}{2} g T^2 = T \,.\, V \,.\, \frac{\sin. (\alpha - \iota)}{\cos. \iota}; \quad \therefore T = \frac{2V}{g} \,.\, \frac{\sin. (\alpha - \iota)}{\cos. \iota}.$$

$$AT = T \,.\, V = \frac{2V^2}{g} \,.\, \frac{\sin. (\alpha - \iota)}{\cos. \iota}.$$

Again $AQ : AT :: \sin. ATQ : \sin. AQT$.

And $\sin. ATQ = \cos. TAH$; $\sin. AQT = \cos. QAH$, as before;

$$\therefore AQ = AT \cdot \frac{\sin.\ ATQ}{\sin.\ AQT} = AT \cdot \frac{\cos.\ TAH}{\cos.\ QAH};$$

$$\text{or } R = \frac{2V^2}{g} \cdot \frac{\sin.\ (a-\iota)}{\cos.\ \iota} \cdot \frac{\cos.\ a}{\cos.\ \iota} = \frac{2V^2}{g} \cdot \frac{\sin.\ (a-\iota)\cos.\ a}{\cos.^2 \iota}.$$

180. Prop. *To find in what direction a body must be projected with a given velocity, that its range upon a given plane may be the greatest possible.*

On a horizontal plane, the range $= \dfrac{V^2}{g} \cdot \sin.\ 2a$.

This will be greatest when $\sin.\ 2a$ is greatest, that is, when $2a =$ a right angle, and a, the angle of projection, = half a right angle.

On an inclined plane, the range $= \dfrac{V^2}{g} \cdot \dfrac{2\sin.\ (a-\iota)\cos.\ a}{\cos.^2 \iota}$.

Since ι is constant, the range will be greatest when

$$2\sin.\ (a-\iota)\cos.\ a$$

is greatest. But

$$2\sin.\ (a-\iota)\cos.\ a = \sin.\ \{a + (a-\iota)\} - \sin.\ \{a - (a-\iota)\}$$

$$= \sin.\ (2a-\iota) - \sin.\ \iota:$$

which, since ι is constant, is greatest when $\sin.\ 2a - \iota$ is greatest; that is, when $2a - \iota$ is a right angle; or,

$$2a - \iota = \frac{\pi}{2}; \quad \therefore a = \tfrac{1}{2}\left(\frac{\pi}{2} + \iota\right);$$

$$\therefore a - \iota = \tfrac{1}{2}\left(\frac{\pi}{2} - \iota\right) \text{ or } TAQ = \tfrac{1}{2} ZAQ; \ AZ \text{ being vertical.}$$

Hence in this case AT bisects the angle ZAQ.

Cor. Hence, on the inclined plane, the greatest range is

$$= \frac{V^2}{g} \cdot \frac{2 \sin.(\alpha - \iota) \cos.\alpha}{\cos.^2 \iota} = \frac{V^2}{g \cos.^2 \iota} \{\sin.(2\alpha - \iota) - \sin.\iota\}$$

$$= \frac{V^2}{g \cos.^2 \iota} \left\{ \sin.\frac{\pi}{2} - \sin.\iota \right\} = \frac{V^2 (1 - \sin.\iota)}{g(1 - \sin.^2 \iota)} = \frac{V^2}{g(1 + \sin.\iota)}.$$

181. PROP. *To express the formulæ for projectiles in terms of the height due to the velocity of projection.*

The *height due* to the velocity of projection is the height down which a body must fall so as to acquire that velocity. Let h be this height: then $V^2 = 2gh$; $h = \frac{V^2}{2g}$, $\frac{V^2}{g} = 2h$; and by the preceding Article we shall find,

On a horizontal plane,

$$\text{Range} \ldots\ldots\ldots = 2h \sin.2\alpha.$$

$$\text{Time of flight} = \sqrt{\frac{2h}{g}} \cdot 2\sin.\alpha.$$

$$\text{Greatest height} = h \sin.^2 \alpha.$$

$$\text{Greatest range} = 2h.$$

On an inclined plane,

$$\text{Range} \ldots\ldots\ldots = 4h \frac{\sin.(\alpha - \iota)\cos.\alpha}{\cos.^2 \iota}$$

$$= \frac{2h}{\cos.^2 \iota} \{\sin.(2\alpha - \iota) - \sin.\iota\}.$$

$$\text{Time of flight} = \sqrt{\frac{2h}{g}} \cdot \frac{2\sin.(\alpha - \iota)}{\cos.\iota}.$$

$$\text{Greatest range} = \frac{2h}{1 + \sin.\iota}.$$

182. PROP. *The curve described by a projectile is a parabola, and the velocity of the projectile at any point is that acquired by falling from the directrix of the parabola.*

In fig. 153, $AR = V.t$, and $RP = \frac{1}{2}gt^2$;

$$\therefore \frac{AR^2}{RP} = \frac{2V^2}{g} = 4h: \text{ and } AR^2 = 4h.RP;$$

or, Am being vertical, and mP parallel to AR,

$$(mP)^2 = 4h.Am.$$

Hence the curve AP is a parabola, of which Am is the abscissa, mP the ordinate, and $4h$ the parameter.

If AC be taken in mA produced, = one-fourth the parameter at A, and CK drawn at right angles to AC, CK will be the directrix of the parabola. And one-fourth the parameter at A is h, the height due to the velocity.

Hence the velocity at the point of projection A is equal to the velocity acquired in falling from the directrix. Also the velocity at any point P will be the same as if P were considered as the point of projection. Hence at any point the velocity is equal to that acquired in falling down DP, the distance from the directrix.

Cor. 1. It is manifest that AR will be a tangent to the curve at the point A. Now the tangent to the parabola makes equal angles with two lines, the one drawn to the focus, and the other perpendicular to the directrix. Hence if we make the angle $RAS = RAC$, the focus will be in the line AS.

Also the distance of a point from the focus is equal to its distance from the directrix. Hence if we take $AS = AC$, S will be the focus.

Cor. 2. To find the *principal parameter* or *latus rectum* of the parabola.

If we draw SK perpendicular to the directrix, and bisect SK in V, V will be the vertex of the parabola: and $4SV$ or $2SK$ will be the principal parameter. Now

$$SK = GK \pm GS = AC + AS \,.\, \cos. SAm = AC - AS \,.\, \cos. SAC$$

$$= AC - AS \,.\, \cos. 2\,TAC = AC \left\{1 - \cos. 2\,TAC\right\}$$

$$= AC \,.\, 2 \sin.^2 TAC, \text{ by Trigonometry;}$$

$\therefore$ the principal parameter $= 2\,SK = 4\,AC \,.\, \sin.^2 TAC = 4h \cos.^2 a$.

183. Prop. *To find an equation to the curve referred to horizontal and vertical co-ordinates.*

In fig. 153, let $AM = x$, $MP = y$; t any time; the rest of the notation as before.

$$AM = AR \,.\, \cos. a; \text{ or } x = V \,.\, t \,.\, \cos \,.\, a; \quad \therefore t = \frac{x}{V \,.\, \cos. a};$$

$$\therefore RP = \tfrac{1}{2} g t^2 = \frac{g x^2}{2 V^2 \cos.^2 a}.$$

Also $MR = AM \,.\, \tan. a$. And $MP = MR - RP$;

$$\therefore y = x \tan. a - \frac{g x^2}{2 V^2 \cos.^2 a},$$

the equation to the curve.

Cor. 1. If, as before, $h = \dfrac{V^2}{2g}$,

$$y = x \tan. a - \frac{x^2}{4h \cos.^2 a}.$$

Cor. 2. To find where the curve meets the horizontal plane.

For this point we must have $y = 0$;

$$\therefore x \tan. a - \frac{x^2}{4h \cos.^2 a} = 0.$$

This gives two values; viz. $x = 0$, which belongs to the point A; and

$$\tan. a - \frac{x}{4h \cos.^2 a} = 0, \text{ whence}$$

$$x = 4h \tan. a \,.\, \cos.^2 a = 4h \sin. a \,.\, \cos. a = 2h \sin. 2a;$$

which agrees with Article 181.

184. Prop. *To find the angle which the curve makes with the horizon at any point.*

The horizontal velocity with which P, or M moves is always the same, and is $= V \cos. \alpha$.

The vertical velocity of P is the velocity with which MP, or $MR - RP$, increases: that is, it is the velocity with which MR increases *minus* the velocity with which RP increases. The former of these velocities is $V \sin. \alpha$; the latter is gt, by last Chapter. Therefore the vertical velocity is $V \sin. \alpha - gt$. Hence if ϕ be the angle which P's path makes with a horizontal line

$$\tan. \phi = \frac{\text{vertical velocity}}{\text{horizontal velocity}} = \frac{V \sin \alpha - gt}{V \cos. \alpha} = \tan. \alpha - \frac{gt}{V \cos. \alpha}$$

$$= \tan \alpha - \frac{gx}{V^2 \cos.^2 \alpha} \text{ (Art. 183.)} = \tan \alpha - \frac{x}{2h \cos.^2 \alpha}.$$

Cor. To find the point V when the height of the projectile above a given plane AQ is the greatest. Fig. 154.

At this point it is evident that the direction of the motion must be parallel to AQ; hence $\tan. \phi = \tan. \iota$;

$$\therefore \tan. \iota = \tan. \alpha - \frac{x}{2h \cos.^2 \alpha};$$

$$\therefore \frac{x}{2h \cos.^2 \alpha} = \tan. \alpha - \tan. \iota = \frac{\sin. \alpha}{\cos. \alpha} - \frac{\sin. \iota}{\cos. \iota}$$

$$= \frac{\sin. (\alpha - \iota)}{\cos. \alpha \cdot \cos. \iota};$$

$$AL = x = 2h \frac{\sin. (\alpha - \iota) \cos. \alpha}{\cos. \iota}.$$

$$\text{Hence } AG = \frac{AL}{\cos. \iota} = 2h \frac{\sin. (\alpha - \iota) \cos. \alpha}{\cos.^2 \iota}.$$

By comparing this with the value of AQ, the range, Art. 181, it will be seen that $AG = \frac{1}{2} AQ$.

185. PROB. I. *A body is to be projected from a given point with a given velocity so as to strike another given point: to find the direction of projection.* *Fig.* 154.

Let Q be the point to be struck; then AQ and the angle QAH are known as before. Let $AQ = R$, $QAH = \iota$. Then by Articles 179 and 181;

$$R = \frac{2V^2}{g} \cdot \frac{\sin.(\alpha - \iota) \cdot \cos.\alpha}{\cos^2.\iota} = \frac{2h}{\cos.^2\iota} \cdot \{\sin.(2\alpha - \iota) - \sin.\iota\};$$

$$\therefore \sin.(2\alpha - \iota) = \frac{R\cos.^2\iota}{2h} + \sin.\iota.$$

When the problem is possible, this equation will necessarily give a value of $2\alpha - \iota$ less than $\frac{1}{2}\pi$; let this be θ: then since the sine of $\pi - \theta$ is the same as the sine of θ, the equation will also be satisfied if $\pi - \theta$ be the value of $2\alpha - \iota$. Let α', α'' be the two values of α; that is, let

$$2\alpha' - \iota = \theta; \quad 2\alpha'' - \iota = \pi - \theta;$$

$$\text{or } \alpha' = \frac{\theta + \iota}{2}, \quad \alpha'' = \frac{\pi - \theta + \iota}{2}.$$

Both these values are comprehended in the formula

$$\tfrac{1}{2}\left(\frac{\pi}{2} + \iota\right) \pm \tfrac{1}{2}\left(\frac{\pi}{2} - \theta\right).$$

If AI bisect the angle QAZ, $IAH = \iota + \frac{1}{2}\left(\frac{\pi}{2} - \iota\right)$ $= \frac{1}{2}\left(\frac{\pi}{2} + \iota\right)$. Hence, if, in fig. 154, TAH, tAH be the two values of α given by the formula, AT and At, which are the required directions of projection, make equal angles with AI.

We can easily find the limits within which this problem is possible. It is impossible if $\sin.(2\alpha - \iota)$ be greater than 1; that is,

$$\text{if } \frac{R\cos.^2\iota}{2h} + \sin.\iota > 1;$$

which may happen either from R becoming too large, or from V, and therefore h, becoming too small.

This problem might likewise have been solved by putting the known values of AN, NQ for x and y in the equation to the curve, Art. 183; by which means a, which determines the direction of projection, will be the only unknown quantity, and may be found.

It is easy also to obtain, from the properties of the parabola, geometrical constructions, which shall satisfy the question.

186. Prob. II. *A body is projected in a given direction with given velocity, from the summit of a hill whose form is an upright paraboloid: to find where the projectile will strike it.* *Fig.* 155.

Let AQ be the section of the hill which is in the vertical plane of projection; AQ is a parabola; and if we refer the curve AQ to horizontal and vertical co-ordinates x, y, its equation will be, if b be the parameter,

$$y = -\frac{x^2}{b};$$

y being negative, because ordinates measured upwards are positive. At the point Q where the projectile strikes the hill, the parabola AQ and the curve of the projectile must have the same co-ordinates. Hence, equating this value of y with that of the ordinate to the curve APQ, Art. 183, we have

$$-\frac{x^2}{b} = x \tan. a - \frac{x^2}{4h \cos.^2 a};$$

$$\frac{x}{4h \cos.^2 a} - \frac{x}{b} = \tan. a;$$

$$x = \frac{4bh \sin. a \,.\, \cos. a}{b - 4h \cos.^2 a}$$

$$= \frac{2bh \sin. 2a}{b - 4h \cos.^2 a}.$$

If $4h \cos.^2 \alpha = b$, or $2 V^2 \cos.^2 \alpha = bg$, x is infinite, and the projectile never meets the parabola. In this case, by Art. 182, the two parabolas have equal parameters, and are parallel. If b be less than this value, the parabolas diverge, and never meet.

In the same way we may find where a body, projected under given circumstances, meets any curve, given by an equation between its co-ordinates.

187. PROB. III. *A body is projected from a given point with a given velocity: to find the direction, that it may just touch a given plane. Fig.* 155.

Let Bb be the intersection of the given plane with the vertical plane of projection; Bb must necessarily be above A. Let $AB = b$, and the angle $ABb = \beta$. Hence the equation to the line Bb is

$y = (x + b) . \tan. \beta$. Also for the path of the projectile,

$$y = x . \tan. \alpha - \frac{x^2}{4h \cos.^2 \alpha}.$$

And since at the point P, where the projectile touches the plane, we must have the co-ordinates common, we have

$$(x + b) \tan. \beta = x . \tan. \alpha - \frac{x^2}{4h \cos.^2 \alpha}.$$

Also since the curve and the line *touch* at P, we must have the tangent of the angle which they make with the horizontal line the same for both. Now for the straight line Bb this tangent is $\tan. \beta$; and for the path of the projectile it is (Art. 184)

$$= \tan. \alpha - \frac{x}{2h \cos.^2 \alpha}.$$

$$\text{Hence } \tan. \beta = \tan. \alpha - \frac{x}{2h \cos.^2 \alpha}.$$

From this equation, $x = 2h \cos.^2 a (\tan. a - \tan. \beta)$,

$$= \frac{2h \,.\, \cos. a \,.\, \sin. (a - \beta)}{\cos. \beta},$$

which substituted in the former equation, or in

$$b \,.\, \tan. \beta = x (\tan. a - \tan. \beta) - \frac{a^2}{4h \cos.^2 a}$$

$$= \frac{x \,.\, \sin. (a - \beta)}{\cos. a \cos. \beta} - \frac{x^2}{4h \cos.^2 a},$$

gives $b \,.\, \tan. \beta = \frac{2h}{\cos.^2 \beta} \,.\, \sin.^2 (a - \beta) - \frac{h}{\cos.^2 \beta} \,.\, \sin.^2 (a - \beta)$

$$= \frac{h}{\cos.^2 \beta} \sin.^2 (a - \beta).$$

Hence $\sin. (a - \beta) = \pm \frac{(b \sin. \beta \cos. \beta)^{\frac{1}{2}}}{h^{\frac{1}{2}}} = \pm \left\{\frac{b \sin. 2\beta}{2h}\right\}^{\frac{1}{2}}$.

It appears from this, that there are *two* directions which answer the condition: and if Aa be drawn parallel to Bb, these directions make with Aa equal angles aAT, aAt.

In nearly the same way it may be shewn how to make the path of the projectile touch any given curve.

188. Prob. IV. *Several bodies being projected in different directions with the same velocity from the same point* A: *to find the locus of them all at the end of a given time. Fig.* 156.

Let AR, AR', AR'' be any directions in which the bodies are projected. If $AR = AR' = AR''$, be the space which the bodies would describe in any time with the uniform velocity of projection, and $RP = R'P' = R''P''$ the space through which a body would fall by gravity in the same time; it appears by what has been said, that P, P', P'' will be the places of the bodies at the end of this time.

Let AM be vertical and $= RP$. Hence $MP = AR$, $MP' = AR'$, $MP'' = AR''$. Therefore $MP = MP' = MP''$. Hence P, P', P'' are all in the circumference of a circle whose center is M and radius MP. The center of this circle descends according to the laws of a falling body, and the radius increases uniformly.

Cor. If the projections take place in different vertical planes, the bodies will, at any time, be situated in the surface of a sphere.

189. Prob. V. *On the same supposition, to find the locus of the vertices of the parabolas described, fig.* 157.

Let V be the vertex of any one of the parabolas; $AU = p$, $VU = q$, vertical and horizontal co-ordinates. It is manifest that $UV = AG = \frac{1}{2} AH$, AH being the horizontal range; and $AU = GV$, the greatest height above the horizontal plane.

Hence if h be the height due to the velocity of projection, a the angle which the direction of projection makes with the horizon, we have, by Art. 181;

$$q = h \,.\, \sin\, 2a = 2h \,.\, \sin.\, a \,.\, \cos.\, a; \;\; p = h \,.\, \sin.^2 a;$$

$$\therefore \sin.^2 a = \frac{p}{h}; \;\; q^2 = 4h^2 \,.\, \sin.^2 a \,.\, \cos.^2 a$$

$$= 4h^2 \,.\, \sin.^2 a \,.\, (1 - \sin.^2 a)$$

$$= 4h^2 \,.\, \left(1 - \frac{p}{h}\right)$$

$$= 4 \,.\, (hp - p^2).$$

And if $k = \frac{h}{2}$; we shall have $h = 2k$, 4, $4 = \frac{h^2}{k^2}$; whence

$$q^2 = \frac{h^2}{k^2}(2kp - p^2):$$

which is the equation to an ellipse whose minor axis is $2k = AC$, the height due to the velocity of projection; and whose major axis is $2h$, horizontal, and double the former.

190. Prob. VI. *Planes* AQ, AQ′, AQ″, *being drawn in every direction from the point* A, *and bodies projected from* A *with a given velocity, at such angles that the ranges on each of these planes shall be the greatest: to find the locus of all the extreme points,* Q, Q′, Q″. *Fig.* 157.

By Art. 181, if ι be the angle HAQ, we have

$$AQ = \frac{2h}{1 + \sin.\iota} = \frac{2h}{1 + \cos.\theta} \text{ putting } CAQ = \theta.$$

And this is the equation to a parabola whose parameter is $4h$. Hence Q, Q', Q'' is a parabola with focus A.

Cor. 1. This parabola circumscribes all those described by projectiles from the point A with the velocity $V = \sqrt{(2gh)}$. For these parabolas will meet any point in this curve; and they will reach no point without it, as would be clear by joining such point with the point A.

Cor. 2. It has been seen that AR bisects the angle CAQ: hence the focus S, of AVQ, is in AQ; and hence the parabolas AVQ, CQ have a common tangent at Q, for the tangent must bisect the angle AQq, Qq being parallel to the axis. Hence the parabola $CQQ'Q''$ *touches* all those described by projectiles from A with the velocity V.

Cor. 3. If the bodies be projected in different vertical planes, their paths will all be circumscribed by a parabolic conoid, formed by the revolution of QA round CA.

CHAP. V.

MOTION UPON A CURVE.

191. When a body is compelled to move along a curve, it is acted on at every point by the re-action, which is perpendicular to the curve, and therefore to the direction of the body's motion. Hence this force neither accelerates nor retards the body. To determine the velocity of the body, we must take the resolved part of the force in the direction of the curve, and consider the effect produced by this resolved force.

Prop. *If a body descend down any curve by the action of gravity, the velocity acquired at any point will be the same as if the body had descended down the same vertical space falling freely.*

In fig. 158, let a body which has any velocity at P descend down the curve PQ to Q. Let PM, QN be horizontal lines meeting a vertical line in M and N; and let a body, which has at M the same velocity as the body at P, fall to N. The velocity of the body falling freely at N will be equal to the velocity of the body descending on the curve at Q.

Let the arc PQ be divided into small portions PP', $P'P''$, &c. and let MM', $M'M''$, &c. be the corresponding portions of MN. Suppose the force which accelerates the body down the curve to be uniform throughout PP', and equal to its value at P; uniform through $P'P''$, and equal to its value at P; and so on. Then the velocity thus acquired at Q will approach to that acquired by the real action of the force down PQ, as the portions PP' &c. become smaller and more numerous.

Let PT be a tangent at P meeting $M'P'$ in T'; and TG, a perpendicular at T meeting the vertical line PG. If PG represent the force of gravity at P, this force may be resolved into PT, TG. Of these, TG is counteracted by the re-action of the curve, and PT is the force which accelerates the body's motion along the curve at P. If g be the force of gravity, $g \cdot \dfrac{PT}{PG}$ will be the force at P. Also if MT meet PG in U, we shall have

$$PU : PT :: PT : PG;$$

therefore $g \cdot \dfrac{PU}{PT}$, or $g \cdot \dfrac{MM'}{PT}$, is the force at P, in the direction of the curve.

When the force acts uniformly in PP', $P'P''$, &c. let u be the velocity at P, u' at P', u'' at P'', &c. and v at Q. By Art. 162, Cor. 3, we have, if the force f act uniformly from P to P',

$$u'^2 - u^2 = 2f \cdot PP' = 2g \cdot \frac{PP' \cdot MM'}{PT};$$

$$\text{similarly, } u''^2 - u'^2 = 2f' \cdot P'P'' = 2g \cdot \frac{P'P'' \cdot M'M''}{P'T'}, \text{ \&c.}$$

And adding all these equation together, observing that v^2 is the last of the values u, u', u'', &c.

$$v^2 - u^2 = 2g \left\{ MM' \frac{PP'}{PT} + M'M' \frac{P'P''}{P'T'} + \text{\&c.} \right\}.$$

Also as the portions PP', $P'P''$, &c. are taken smaller and smaller, the velocity thus generated approaches to that actually acquired in the curve. And on the same supposition, each of the fractions $\dfrac{PP'}{P'T}$, $\dfrac{P'P''}{P'T'}$, &c. approaches to unity. Hence taking the limits, and considering v and u as the actual velocities at Q and P,

$$v^2 - u^2 = 2g \{MM' + M'M'' + \text{\&c.}\} = 2g \cdot MN,$$

$$\text{and } v^2 = u^2 + 2g \cdot MN.$$

And this is (by Art. 162,) the value of the square of the velocity of N, on the supposition of the body falling freely from M, with the velocity at u. Hence the velocities acquired down MN and down PQ are the same.

Cor. 1. The same is true of any other force which acts in parallel lines upon a body, while the body moves upon a curve. If f be the force, u the velocity at one point, v the velocity at another point, and h the distance of these points in the direction of the force,

$$v^2 = u^2 + 2fh.$$

Cor. 2. If a body begin to move up the curve QP from Q with the same velocity with which another body is projected vertically upwards at N, they will have the same velocities when they arrive at P and M. For the velocity destroyed in ascending QP is equal to that generated in deconding PQ, that is, to that acquired down MN, and therefore to that destroyed up NM.

Cor. 3. If a body begin to ascend a curve surface with a certain velocity, it will rise to the same height above the initial horizontal line, whatever be the form of the curve.

For in each case it will ascend to the same height as if it had been projected vertically upwards.

192. Prop. *To find the time of falling down any arc of an inverted cycloid**.

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* If a circle EPF, fig. 160, 161, roll along a straight line CBc, a point P in the circumference of this circle will describe a curve which is called a *Cycloid*.

When the circle has made one complete revolution, the describing point which was in contact with the straight line at C, will return to it again at c, having described the curve $CPAc$.

If we bisect Cc in B, and draw BA at right angles to it, A will be the position of the describing point when the circle has made half a revolution; and the two branches AC, Ac will be equal and similar.

AB is called the axis of the cycloid; Cc its base; A its vertex; and the circle AQB is called the generating circle.

Prop. I. Fig. 160. *If an ordinate* MQP *be drawn perpendicular to the axis* QP = *arc* QA.

Let EPF be the position of the generating circle at the time when the describing point is at P. Then the arc PF has been applied to CF, so that all the points of each have

Let A, fig. 159, be the vertex of the cycloid, and L the point from which the body begins to fall: LH horizontal,

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have successively coincided: therefore the two are equal, that is, $CF = \text{arc } PF = \text{arc } QB$. For the same reason $CB = \text{semicircle } AQB$. Hence, taking away equals, $FB = \text{arc } AQ$. But evidently $PN = QM$, and therefore $PQ = NM = FB$. Hence $PQ = \text{arc } AQ$.

Prop. II. Fig. 160. *The tangent to the cycloid at the point* P *is parallel to the chord* AQ.

If the circle EPF be supposed for an instant to turn round a fixed point F, instead of rolling along FB, the motion of P will be ultimately in the same direction on either supposition. But on this supposition the motion of P will evidently be perpendicular to FP, or in the direction PE. Hence the direction of the curve CP at P is PE, and therefore PE is a tangent. And PE is parallel to QA: hence the tangent at P is parallel to QA.

Prop. III. Fig. 160. *The length of the arc of the cycloid* AP, *beginning from the vertex, is double of the chord of the circular arc* AQ *cut off by the same ordinate.*

Let $M'Q'P'$ be an ordinate very near to MQP. Let AQ meet $M'Q'$ in R, and draw at Q a tangent to the circle meeting $M'Q'$ in S, and meeting a tangent at A in T. Also draw SO perpendicular upon QR.

Since $TA = TQ$, angle $TAQ = TQA$. And $TAQ = QRS$, and $TQA = RQS$; therefore $QRS = RQS$ and $SQ = SR$. Hence the triangles SOQ and SOR are equal; $QO = RO$, and $QR = 2\,QO$.

Now when Q' approaches indefinitely near to Q, S approaches to Q', and OS coincides ultimately with a circular arc to radius AQ' and center A. Hence QO is ultimately the excess of BQ' above AQ or the quantity by which AQ is increased.

Also QR is parallel to the tangent at P, and hence QR is ultimately equal to PP', the quantity by which AP is increased.

Hence it appears, that, for corresponding points, AP is increased by a quantity twice as great as the increase of AQ; and, therefore, as AP and AQ begin together, AP will always be twice as great as the chord AQ.

Cor. $AP = 2\,PE$.

Prop. IV. *To make a pendulum oscillate in a given cycloid.*

Let APC, fig. 161, be a given semi-cycloid, AB being its axis. Produce AB to S, making $SB = AB$: complete the rectangle $SBCD$, and with an axis, CD, and base DS, describe a semi-cycloid CS.

Draw any line EFG parallel to ABS; and on opposite sides of this line describe the two semi-circles, EPF, FOG, of the generating circles of the cycloids AC, CS. Join OF, FP. Then arc $FP = FC$, and arc $FPE = BFC$; therefore arc $PE = BF$, and $BF = SG = \text{arc } GO$. Hence $PE = GO$; and therefore the angles EFP, GFO are equal. Hence OFP is a straight line. Hence also $OF = FP$; therefore $OP = 2\,OF = \text{arc } OC$, by Cor. to Prop. III. And by Prop. II, OP is a tangent to the cycloid at O.

Hence it appears, that if a string SOC, fixed at S, and wrapped along the semi-cycloid SOC, be unwrapped, beginning at C, its extremity will describe a semi-cycloid CPA. And if an equal and similar semi-cycloid Sc be placed with its base Sd in the same line with DS, the same string fixed at S and wrapping upon the semi-cycloid Sc, will, with its extremity, describe the semi-cycloid Ac, thus completing the cycloid CAc. Hence a body P, suspended by a string SOP between two such semi-cycloids in a vertical plane, will oscillate in an inverted cycloid.

meeting the axis in H; and PM horizontal meeting in q a semi-circle described on AH.

The velocity at P acquired down LP, is equal to the velocity acquired down HM, by Art. 191; hence

$$\text{velocity at } P = \sqrt{(2g \,.\, HM)}$$

$$= \sqrt{\left(2g \,.\, \frac{Hq^2}{AH}\right)} = Hq\sqrt{\frac{2g}{AH}}.$$

Also, if AQB be a semi-circle described on AB, $AP =$ 2 chord AQ, (see Note)

$$\therefore AP = 2\sqrt{(AB \,.\, AM)}$$

$$= 2\sqrt{\left(\frac{AB}{AH} \,.\, AH \,.\, AM\right)} = 2\sqrt{\left(\frac{AB}{AH} \,.\, Aq^2\right)}$$

$$= 2Aq\sqrt{\frac{AB}{AH}}.$$

And similarly, if $P'M'$ be near and parallel to PM,

$$AP' = 2Aq'\sqrt{\frac{AB}{AH}}.$$

$$\text{Therefore } PP' = 2(Aq - Aq')\sqrt{\frac{AB}{AH}}.$$

Hence, if PP' be described uniformly with the velocity at P,

$$\text{time in } PP' = \frac{PP'}{\text{vel. at } P} = \frac{Aq - Aq'}{Hq}\sqrt{\frac{2AB}{g}}.$$

And if we take the times, supposing PP' indefinitely diminished, the sum of all such intervals from L to P will approximate to the actual time of describing LP.

Join Hq' meeting Aq in o. And since AoH approximates to AqH, a right angle, oq approximates to $Aq - Aq'$. Hence taking the limit, we have

$$\text{time in } PP' = \frac{oq}{Hq}\sqrt{\frac{2AB}{g}}$$

$$= \text{angle } qHo\sqrt{\frac{2AB}{g}};$$

and angle $qHo = qHq' = \frac{1}{2}qCq'$;

$$\therefore \text{ time in } PP' = \tfrac{1}{2}qCq'\sqrt{\frac{2AB}{g}}.$$

And the whole time in LP will be found if we take the sum of all such intervals from L to P. Now the sum of all the angles qCq' is evidently HCq. Hence

$$\text{time in } LP = \tfrac{1}{2}HCq\sqrt{\frac{2AB}{g}}.$$

Cor. 1. To find the time of descending through the *whole* arc LA, we must put for the angle HCq its value for that case, which is two right angles. Hence we have

$$\text{time in } LA = \frac{\pi}{2}\sqrt{\frac{2AB}{g}}.$$

Cor. 2. If we suppose the body, after coming to the vertex A, to go on and to ascend the opposite semi-cycloid, (Ac, fig. 161.) it will ascend to a point l, having described an arc Al equal to AL. And the time of ascending through Al will be equal to that of descending through LA. Hence we shall have

$$\text{time in } LAl = 2 \text{ time in } LA = \pi\sqrt{\frac{2AB}{g}}.$$

Cor. 3. Since the time of descending down LA is independent of the position of the point L, it appears that the times down all arcs LA are the same, whatever be the magnitude. Hence the curve CA is said to be *isochronous.*

193. Prop. *When a body oscillates in a cycloid; to determine the time of oscillation.*

If two equal inverted semi-cycloids SC, Sc, fig. 161, be placed in contact at S, in the same vertical plane, and if a string SOP equal in length to either of them, be suspended from S and oscillate between them, its extremity P will describe a cycloid CAc. (Note to Art. 192.) And if a body be suspended by this string, it will move in the same manner as if it moved upon a curve PAp. After descending down

LA, it will, with the velocity acquired, ascend up *Al*, describing an arc equal and similar to *AL*. And after coming to *l* it will again descend through *lA* and ascend through *AL*, and so on continually.

Let l be the length of the pendulum *SA*. Then $l = 2AB$; and we shall have, by last Article, the time of oscillation

$$= \pi \sqrt{\frac{l}{g}}.$$

Cor. 1. Since $\sqrt{\frac{l}{g}} = \sqrt{\frac{2 \times \frac{1}{2} l}{g}}$ = time of falling down $\frac{1}{2} l$, (Art. 162,) we shall have time of oscillation $= \pi \times$ time of falling down $\frac{1}{2}$ pendulum.

Cor. 2. For very small distances from the point *A*, the cycloid will very nearly coincide with the circle whose center is *S*. Hence the motion of the cycloidal pendulum *SOP* will very nearly coincide with the motion of a body suspended by a string *SA* and oscillating *freely* through very small arcs. We shall suppose the times of oscillation of these two pendulums to be equal.

Hence we may determine all the circumstances of the small oscillations of pendulums in circular arcs from the expression above,

$$t = \pi \sqrt{\frac{l}{g}},$$

where t is the time of oscillation.

Cor. 3. It is manifest that t varies as the root of l, when g is constant:

Also that t varies inversely as the root of g, when l remains the same: And that g varies as l, when t remains the same.

Hence if L be the length of the pendulum which oscillates seconds, and t the time in seconds of the oscillation of a pendulum whose length is l,

$$t = \sqrt{\frac{l}{L}}; \text{ hence } l = Lt^2.$$

The value of L, the length of the seconds pendulum in the latitude of London, (*in vacuo*), is found by experiment to be 39.1386 inches.

From this value of L we can find the value of g; for making $t = 1$, we have

$$1 = \pi \sqrt{\frac{L}{g}}; \quad \therefore g = \pi^2 L = 386.28 \text{ inches.}$$

Ex. 1. To find the time of oscillation of a pendulum 20 feet long,

$$t = \sqrt{\frac{240}{39.1386}} = 2.5'', \text{ nearly.}$$

Ex. 2. To find the length of a pendulum which shall make its oscillations in half minutes,

$$l = L \,.\, (30)^2 = 39.1386 \times 900 \text{ inches} = 978.4 \text{ yards.}$$

194. Prop. *If a pendulum be slightly altered in length, to find the number of oscillations gained or lost in a day.*

If n be the daily number of oscillations of the pendulum in the latitude of London, (*in vacuo*,) and $N = 24 \times 60 \times 60 = 86400$, the number of seconds in 24 hours; we have

$$t = \frac{N}{n}; \quad \therefore \frac{N}{n} = \sqrt{\frac{l}{L}}; \quad \therefore l = \frac{LN^2}{n^2}.$$

If n and l be nearly equal to N and L, we may obtain approximations for the differences.

Suppose the length of the pendulum L to be increased by a small quantity p: to find q, the number of seconds it will lose in a day.

Here $L + p = \dfrac{LN^2}{(N-q)^2} = L\left(1 + \dfrac{2q}{N}\right)$; omitting powers of $\dfrac{q}{N}$;

$$\therefore p = \frac{2qL}{N}; \quad \text{and } q = \frac{pN}{2L}.$$

The same formula will apply when L is diminished, and consequently N is increased.

Ex. A seconds pendulum is lengthened $\frac{1}{100}$ of an inch: to find the number of seconds it will lose per day.

$$\text{Here } p = .01\,; \quad \therefore q = \frac{.01 \times 86400}{2 \times 39.13} = \frac{43200}{3913} = 11'' \text{ nearly.}$$

195. Prop. *If the force of gravity be slightly altered, to find the number of seconds gained or lost in a day by a seconds pendulum.*

Let G be the value of g at a given place; when l remains the same, t varies inversely as the root of g; hence

$$\frac{t}{1} = \frac{\sqrt{G}}{\sqrt{g}}\,; \quad g = \frac{G}{t^2} = \frac{Gn^2}{N^2}.$$

Hence if a seconds pendulum is taken to a place where the gravity is greater, n will be greater than N, and the pendulum will gain, and *vice versâ*. The increase of gravity is generally small, and hence we may approximate as before. Let $g = G\,(1 + h)$, and let the pendulum gain q seconds a day;

$$\therefore G\,(1 + h) = \frac{G\,(N + q)^2}{N^2} = \frac{G\,(N^2 + 2qN)}{N^2}, \text{ omitting } q^2;$$

$$\therefore h = \frac{2q}{N}.$$

Ex. 1. A pendulum which would oscillate seconds at the equator, would, if carried to the pole, gain 5 minutes a day: to find the proportion of the polar and equatoreal gravity,

$$h = \frac{2 \times 300''}{86400} = \frac{1}{144};$$

hence gravity at the equator : gravity at the pole :: 144 : 145.

Ex. 2. A pendulum which oscillates seconds, is carried to the top of a mountain whose height is m: to find the number of seconds which it will lose per day; gravity being supposed to vary inversely as the square of the distance from the center.

Let r be the distance from the center of the earth to the first station, and G the gravity at that station. Therefore $r+m$ is the distance of the second station from the center of the earth, and the gravity at that station is

$$\frac{G r^2}{(r+m)^2} = G\left(1 - \frac{2m}{r}\right), \text{ omitting } \frac{m^2}{r^2}, \text{ \&c.}$$

Hence, putting $\frac{2m}{r}$ for h in the formula, which will be the same for the diminution as for the increase of gravity,

$$\frac{2m}{r} = \frac{2q}{N}, \text{ and } q = \frac{Nm}{r}.$$

If the radius of the earth be 4000 miles, and the height of the mountain 1 mile,

$$q = \frac{86400}{4000} = 21''.6, \text{ the number of seconds lost per day.}$$

CHAP. VI.

ROTATION ABOUT A FIXED AXIS.

196. In the preceding Chapters we have considered the effect of forces acting directly upon points. This includes the case when a single material particle moves about a fixed axis with which it is connected by a rigid arm ; for the motion of a point in a circular arc, treated of in the last Chapter (Art. 193) would be the same whether the material point were connected with the center of the circle by means of a flexible string, or by means of a rigid rod ; supposing the string and the rod to be without weight and without inertia.

But when a system of several material particles connected by rigid rods, or any other rigid system, moves about a fixed axis, the moving forces of the separate particles modify one another; and the motion of the system is determined by the Third Law of Motion extended to indirect action, as explained in Article 131. Thus in figure 100, if *a*, *m*, *n*, *p*, be material particles, rigidly connected by the arms *ca*, *cm*, *cn*, *cp*, with the center of motion *p*, and with each other, no one of them, as *a*, can move, except all the others move with velocities which are to its velocity in a given ratio. And hence the motion of *a* will no longer be determined by the third law of motion applied to direct action, but requires us to consider the mutual action of the particles in producing and resisting motion.

This remark applies to the motion of rigid bodies in general; for any such body may be considered as a collection of material particles rigidly connected. When such a body has any rotatory motion, its parts act upon one another, and thus modify the effects of the other forces by which they are acted on. Any body, under such circumstances, may be considered as a machine; and by means of its rigidity or its other properties, the forces which are applied to one part of the body propagate their effect to another; so that each particle both presses with its own force, and serves to form levers and rods by which the pressures of other particles are communicated. The laws according to which this connexion of different particles modifies the effect of the forces which move them, are to be the subject of our consideration in the present and following Chapters.

197. The principle on which our reasonings must depend in such cases, is, as we have said, the third law of motion, extended to the case of indirect action. According to this law, the moving force is in all cases proportional to the Pressure (Art. 131). In order clearly to distinguish these two quantities, the moving force, as collected from the momentum actually impressed on each particle, is called the *Effective* (moving) *Force* of that particle; and the pressure, or other external force which operates on the system, is called the *Impressed* (moving) *Force;* and this latter force does not, in such cases, produce its whole effect on the point on which it acts, being modified by the connection of this with other points.

198. The third law of motion, extended to the case of indirect action, may be expressed in the following manner.

In the motion of any system of connected points, the Impressed Forces and the Effective Forces are statically *equivalent to each other.*

The *Impressed Forces*, that is, the pressures which act upon the system, produce the same effect as would be produced by any other pressures, which, according to the nature of the system, are statically equivalent to them. Thus a force P, acting perpendicularly at the distance CP (fig. 97) from the axis, to turn the body round C, exerts the same effort as a force $2P$ acting perpendicularly at half the distance, CP', and will produce the same effect.

This follows necessarily from the nature of material connexion as a mode of transmitting pressure. When we say that a moving body is rigid, we imply that any force applied at one part communicates its effect to other parts according to the same law as if the body were at rest.

Thus any force which acts to turn a body round an axis, acts effectively upon all the particles; the body itself transmitting the action after the manner, and according to the laws, of a lever.

The *Effective Forces* are the moving forces inferred from the momentum generated in each part of the system in a unit of time. Both the Impressed and Effective Forces are measured as Moving Forces; that is, as mass multiplied into Accelerating Force, or as mass multiplied into velocity generated.

It has been shown that in the case of direct action these Impressed and Effective Forces are equal to each other: and it has been stated already (Art. 131) that in passing to the case of indirect action, we do not find any ground, either in reason or in experiment, to suppose that the equality ceases to obtain; and that by experiments fitted to decide the point, it appears that this equality is still rigorously true.

199. The above law of motion is often enunciated and applied in the following form; in which form it is called D'Alembert's *Principle*.

Prop. *When any forces produce motion in any material system, the Moving Forces Lost by the different parts of the system must* balance *each other.*

Let m be one of the material points of the system, and let u be the velocity which the force that acts upon m would communicate to it in a unit of time if m were detached: then, if τ be a very small time, $u\tau$ would be the velocity which the force would communicate to m in the time τ. Let $q\tau$ be the velocity which in the actual state of the system is communicated to m in the time τ: and let the velocity $u\tau$ be resolved into two velocities $p\tau$ and $q\tau$. The accelerating force which would produce the velocity $u\tau$ in the time τ, or u in the unit of time, is u; therefore the moving force which would produce this velocity is mu. In like manner the moving forces which would produce the velocities $p\tau$ and $q\tau$ in the time τ, are mp and mq respectively: and since forces are resolved in the same manner as velocities, the force mu may be resolved into mp and mq. Of these forces, mq is effective, and mp is *the force lost* by the point m.

If in the same way m', u', p', q' represent the quantities analogous to m, u, p, q for another point; m'', u'', p'', q'', for another, and so on; $m'p'$, $m''p''$, and so on, will be the forces lost by these points m', m'', &c.

By the last Article, all the impressed forces

$$mu, \; m'u', \; m''u'', \; \&c.$$

are equivalent to the effective forces

$$mq, \; m'q', \; m''q'', \; \&c.$$

according to the statical conditions of equilibrium of the system.

That is,

$$mp, \; mq, \; m'p', \; m'q', \; m''p'', \; m''q'', \; \&c.$$

are equivalent to

$$mq, \; m'q', \; m''q'', \; \&c.$$

Therefore the forces

$$mp, \; m'p', \; m''p'', \; \&c.$$

are statically equivalent to nothing; that is, they balance each other on the machine.

Cor. In this case, some of the forces lost will be negative, and if the negative sign be omitted, these may be considered as *forces gained*.

Thus if a point m' be acted upon by no forces, $m'q$ being the effective force, $m'p'$ is $-m'q'$; and the force lost being $-m'q'$, the force gained is $m'q'$.

If we adopt this term, the proposition may be stated by saying that

In every system the moving forces lost and gained by the different points balance each other.

This is the application to the case now considered, of the general principle, that Action and Reaction are always equal and opposite to each other.

200. We proceed to determine the angular motion produced when forces act upon a body moveable about a fixed axis. We consider the effect in producing motion only: the other effects, of producing pressure upon the axis, and of affecting the motion of the axis when it is moveable, will be investigated afterwards.

PROP. *In a rigid system consisting of any number of points* m, n, p, q, *&c.* fig. 98, *in the same plane, moveable about an axis* C, *perpendicular to that plane, a force* F *acts to turn the system; to find the effective accelerating force on any point.*

Let F be a moving force which acts perpendicularly at an arm Cf. And let M, N, P, &c. be the effective accelerating forces on m, n, p, &c. Therefore Mm, Nn, Pp, &c. are the effective moving forces; and they are perpendicular to CM, CN, CP, &c. because the motion is so.

Hence, we have

Impressed force .... F acting perpendicularly at an arm CF,

Effective forces .... Mm, Nn, Pp, &c. acting perpendicularly at arms Cm, Cn, Cp, &c.

Hence, by Art. 198, and by the general proposition of the lever,

$$F \cdot Cf - M \cdot m \cdot Cm - N \cdot n \cdot Cn - P \cdot p \cdot Cp - \&c. = 0.$$

But since m, n, p, &c. must all move with the same angular velocity round C, their linear velocities must always be as Cm, Cn, Cp, &c. and therefore all alterations of the velocities must be in this ratio, and the accelerating forces which produce such alterations in the same time must be in the same ratio. Hence, we have $M : N :: Cm : Cn$; therefore $N = \frac{M \cdot Cm}{Cm}$; similarly $P = \frac{M \cdot Cp}{Cm}$, &c. And substituting these values in the above equation, it becomes

$$F \cdot Cf - M \cdot m \cdot Cm - M \cdot n \cdot \frac{Cn^2}{Cm} - M \cdot p \cdot \frac{Cp^2}{Cm} - \text{\&c.} = 0.$$

$$\text{Whence } M = \frac{F \cdot Cf \cdot Cm}{m \cdot Cm^2 + n \cdot Cn^2 + p \cdot Cp^2 + \text{\&c.}} {}^{*}.$$

$$\text{Similarly } N = \frac{F \cdot Cf \cdot Cn}{m \cdot Cm^2 + n \cdot Cn^2 + p \cdot Cp^2 + \text{\&c.}},$$

$$P = \frac{F \cdot Cf \cdot Cp}{m \cdot Cm^2 + n \cdot Cn^2 + p \cdot Cp^2 + \text{\&c.}},$$

and so on for any other point.

$$\text{Thus the effective force on } f = \frac{F \cdot Cf^2}{m \cdot Cm^2 + n \cdot Cn^2 + p \cdot Cp^2 + \text{\&c.}}.$$

The effective force at a distance 1 from the axis

$$= \frac{F \cdot Cf}{m \cdot Cm^2 + n \cdot Cn^2 + p \cdot Cp^2 + \text{\&c.}}.$$

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* As this Proposition is the foundation of the whole doctrine of rotatory motion, we shall shew how it may be deduced from elementary laws, independently of the general Principle.

Let F be statically equivalent to a number of accelerating forces, as M at m, N at n, &c. which act perpendicularly to Cm, Cn, &c., and are as Cm, Cn, &c. Therefore

$$F \cdot Cf - M \cdot m \cdot Cm - N \cdot n \cdot Cn, \text{\&c.} = 0; \text{ and } N = M \cdot \frac{Cn}{Cm}, \text{\&c.}$$

$$\therefore F \cdot Cf - M \cdot m \cdot Cm - M \cdot n \cdot \frac{Cn^2}{Cm}, \text{\&c.} = 0; \text{ and } M = \frac{F \cdot Cf \cdot Cm}{m \cdot Cm^2 + n \cdot Cn^2} + \text{\&c.}$$

Now, by Art. 198, M, N, &c. will produce the same effect as F. Also, if Cm, Cn be supposed to be unconnected and moveable independently about C, and *if the third law of motion be applicable here*, it will follow that when M, N, &c. act on m, n, the accelerations of m, n, will be proportional to Cm, Cn, and therefore the angular velocities of Cm, Cn will be equally increased, and the points m, n of the system will always retain the same position with respect to each other. Hence we may suppose Cm, Cn, &c. to be connected, and the system to become rigid, and the effect will still be the same as when m, n, &c. were unconnected. Hence, when F acts on the rigid system, the effective force in m is the same value of M which we have just found, and which agrees with the value in the text.

If instead of the force F, we have any forces acting in any manner upon the system, we must substitute, instead of $F.Cf$ the *moment* of these forces about the axis C, that is, the sum of each into the perpendicular upon it from C; those being taken negative which tend to turn the system in the opposite direction.

Cor. 1. Since the effective accelerating force on f is

$$= \frac{F}{m \cdot \frac{Cm^2}{Cf^2} + n \cdot \frac{Cn^2}{Cf^2} + p \cdot \frac{Cp^2}{Cf^2}}.$$

It appears, that the resistance which m opposes to the communication of motion, is the same as that of a mass $m \cdot \frac{Cm^2}{Cf^2}$ placed at f, and acted upon immediately; and similarly of the other particles.

Cor. 2. It appears by the demonstation, that the effective forces on different points are as their distances from the axis C.

Cor. 3. If the force F, and the radius Cf, be constant, the effective force on each point will be constant; the motions will be uniformly accelerated, and the formula for such motions may be applied. If F be variable, the formulæ for variable motions may be applied.

Cor. 4. If the force which acts be the *weight* of any body, this body must be included among the bodies m, n, p, &c. in the denominator.

Thus if a system of material points in horizontal planes, m, n, p, fig. 99, be moved about a vertical axis AC, by a weight W acting perpendicularly at the radius Cf, by means of a string passing over a pully B; W moves with the same velocity as a body at the extremity of the area Cf; and therefore the same effective force is employed in moving W, as if it were at f. Hence, we have

$$\text{effective force on } f = \frac{\text{weight of } W . Cf^2}{W . Cf^2 + m . Cm^2 + n . Cn^2 + \&c.},$$

and the effective force on W is the same as on f.

Cor. 5. The quantities W, m, n, &c. in the denominator in the last Corollary, are the *masses* of the bodies; the weight of W in the numerator is a *moving force*. If g represent the accelerating force of gravity, the weight of W is Wg.

Cor. 6. If the lines, m, n, p, &c. be not in the same plane perpendicular to the axis, let lines Cm, $C'n$, $C''p$, &c. be their perpendicular distances from the axis: the same formulæ as before will be true, putting these lines for Cm, Cn, Cp, &c.

Or, if we take a plane Cmn, perpendicular to the axis, and refer the points of the system to this plane, by lines parallel to the axis; m, n, p, &c. being the points as thus referred, the same formulæ will be true.

Cor. 7. If, in fig. 98, the body p be not fixed to the radius Cp, but be fastened to the extremity of a flexible string which is perpendicular to Cp at p, and which is kept stretched by the forces which act, the effect, for an instant, will be the same as if p were fixed at the extremity of Cp. For the moving force which the string exerts on the body will be the same as the reaction which it exerts upon the extremity of the radius Cp; and will be, for an instant, perpendicular to Cp. Hence the effective moving force on the body will be identical with the force which must act at the extremity of Cp in order to balance the reaction of the string; and the equilibrium of the impressed and the effective forces will still subsist, including the effective force on the body fastened to the string.

Cor. 8. Hence, if as in fig. 102, a body P be connected with the system by a string which is wrapped round a cylinder having the center of motion C for its center, the formulæ of this Article may be applied, in the same manner as if P were fixed to the cylinder at the point where the string leaves it: for the string will be *at every instant* perpendicular to the radius drawn to the point of its contact with the cylinder.

201. The denominator of the fractions which express the effective forces in the preceding formulæ, is a quantity which is described by a peculiar term, the moment of inertia.

DEF. *The sum of each particle multiplied into the square of its distance from the axis, is called the Moment of Inertia* with respect to this axis*. This quantity occurs perpetually in considering the subject of rotation.

If the system, instead of consisting of distinct material points, be a continuous body of finite magnitude, the Moment of Inertia will be the sum of *each* point into the square of its distance from the axis, and will consist of an indefinite number of terms. The sum of these terms may be found by the integral calculus, as will be shewn elsewhere.

If the points be m, m_1, m_2, m_3, &c. and their distances from the axis, Cm, Cm_1, Cm_2, Cm_3, &c. the moment of inertia may be represented by $\Sigma(m \cdot Cm^2)$. And if F be a moving force which acts perpendicularly at a distance Cf, we shall by Art. 200, have the accelerating force at the point where the force acts

$$= \frac{F \cdot Cf^2}{\Sigma(m \cdot Cm^2)}.$$

We may suppose k to be such a quantity that

$$k^2 \Sigma m = \Sigma(m \cdot Cm^2),$$

Σm indicating the sum of all the points of the system.

In this case k is called the *Radius of Gyration.*

If forces act upon every point of the system, the effect may be calculated by the same principle as before, as will be seen in the next Problem.

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* *The inertia* of a body is the measure of its effect in resisting the communication of motion (Art. 127): in a single point, it is as the mass simply; but in a body revolving about an axis, the effect of a particle in resisting motion depends on the distance from the axis, like the effect of the force acting on a lever. The effect on a lever is as the product of the force and distance, and this product is called *the moment;* the effect of the inertia of the mass in resisting rotatory motion, appears from the above investigation to be as the product of the mass and square of the distance, and hence, this product is called *the moment of inertia:* and the sum of these products is called the moment of inertia of the system.

202. Prop. *A rigid system of material points, moveable about a horizontal axis, has all its parts acted on by gravity; it is required to determine the accelerating force.*

Let C, fig. 100, be the axis, and m, n, p, the points. Draw a horizontal line through C, meeting vertical lines through m, n, p, in d, e, h. Then the moving forces impressed are the weights of m, n, p. Let M be the effective accelerating force upon m; therefore in the same way as before, the effective accelerating forces on n, p, are

$$\frac{M . Cn}{Cm}, \quad \frac{M . Cp}{Cm}.$$

And the effective moving forces are

$$M . m, \quad \frac{M . n . Cn}{Cm}, \quad \frac{M . p . Cp}{Cm}.$$

Now Cd, Ce, Ch are perpendicular to the directions of the former forces, and Cm, Cn, Cp are perpendicular to the directions of the latter forces; also if m be the mass of one of the bodies, its weight or moving force is mg, and so for the rest. Hence, by the principle of the equilibrium between the impressed and the effective forces, Art. 198, and by the property of the lever, Art. 22, we have,

$$m . g . Cd + n . g . Ce + p . g . Ch$$

$$= M . m . Cm + \frac{M . n . Cn^2}{Cm} + \frac{M . p . Cp^2}{Cm};$$

$$\therefore \; M = \frac{(m . Cd + n . Ce + p . Ch) \, Cm . g}{m . Cm^2 + n . Cn^2 + p . Cp^2}.$$

Cor. 1. If we had supposed that there were more bodies, we should have had a corresponding number of terms, both in the numerator and denominator.

Cor. 2. There will be negative terms in the numerator, when any of the bodies are on the other side of the vertical line drawn through C; the terms in the denominator will always be positive, because the bodies all move in the same direction round C; and therefore the effective accelerating forces are always in the same direction.

Cor. 3. The effective accelerating force on any other point of the system, as n, will be

$$N = \frac{(m \,.\, Cd + n \,.\, Ce + p \,.\, Ch)\, Cn \,.\, g}{m \,.\, Cm^2 + n \,.\, Cn^2 + p \,.\, Cp^2}.$$

Cor. 4. If G be the center of gravity of the system, and if a perpendicular from G meet Ch in H, we have, by Art. 62,

$$(m \,.\, Cd + n \,.\, Ce + p \,.\, Ch) = (m + n + p) \,.\, CH;$$

and if θ be the angle which CG makes with the vertical,

$$CH = CG \,.\, \sin.\, \theta.$$

Hence,

$$M = \frac{(m + n + p)\, CG \,.\, \sin.\, \theta \,.\, Cm \,.\, g}{m \,.\, Cm^2 + n \,.\, Cn^2 + p \,.\, Cp^2};$$

or, denoting $m + n + p$ by Σm, and the denominator by $\Sigma (m \,.\, Cm^2)$, whatever be the number of bodies,

$$M = \frac{Cm \,.\, CG \,.\, g \sin.\, \theta \,.\, \Sigma m}{\Sigma (m \,.\, Cm^2)}.$$

203. Prop. *To find a point of a rigid system, moveable about a fixed horizontal axis, which shall be accelerated by gravity, exactly as much as a single point, moveable about the same axis, would be accelerated in the same position.*

If O, fig. 100, be any point in CG, we shall have, by Art. 202,

$$\text{accelerating force on } O = \frac{CO \,.\, CG \,.\, g \,.\, \sin.\, \theta \,.\, \Sigma m}{\Sigma (m \,.\, Cm^2)}.$$

Now, if a single particle were placed in O, and all the rest removed, we should have

$$\text{accelerating force on particle in } O = g \sin.\, \theta;$$

because the particle may be considered as moving upon a circular arc, of which the radius is CO; and this arc will, at O, make the same angle θ with the horizontal line, which CO makes with the vertical. Hence, by Art. 170,

$$\text{Force} = \frac{gh}{l} = g \sin. \theta.$$

And we have to find O, so that these accelerating forces may be equal. For this purpose, we must have

$$CO . CG . \Sigma m = \Sigma (m . Cm^2);$$

$$\therefore\ CO = \frac{\Sigma . (m . Cm^2)}{CG . \Sigma m} \ldots\ldots\ldots\ldots (a).$$

The point O is called the *Center of Oscillation;* a single point placed in O, would, in any position of CG, be acted on by the same accelerating force as when O is a point in the system; and therefore, the oscillations of CO and of the system would be exactly the same as if we had but one particle O.

Cor. 1. The time of oscillation of the system, is the same as that of a simple pendulum, whose length is CO. Hence, if we make $CO = l$, we shall have the time of one of the small oscillations $= \pi \sqrt{\frac{l}{g}}$. (Art. 193.)

Cor. 2. When we know the moment of inertia, and the place of the center of gravity, the center of oscillation with respect to the axis C is found by the formula

$$CO = \frac{\Sigma (m . Cm^2)}{CG . \Sigma m};$$

and this is applicable, whether the system consist of distinct points, or of finite bodies.

204. Prop. *The moment of inertia of any system, with respect to any given axis, is equal to the moment of inertia about an axis parallel to this, passing through the center of gravity, plus the moment of inertia of the whole body, (collected in its center of gravity.) about the given axis.*

Let fig. 101 represent any system, moveable about an axis C; and let m, n, p, q, be the particles of it, referred to a plane perpendicular to the axis. Let G be the center of gravity of m, n, p, q. Draw md perpendicular on CG;

$$\text{Then, } Cm^2 = CG^2 + Gm^2 + 2CG \cdot Gd.$$

Similarly, if ne, ph, qk, be perpendicular on CG.

$$Cn^2 = CG^2 + Gn^2 - 2CG \cdot Ge,$$

$$\&c. = \&c.;$$

$$\therefore m \cdot Cm^2 + n \cdot Cn^2 + p \cdot Cp^2 + \&c.$$
$$= m \cdot CG^2 + n \cdot CG^2 + p \cdot CG^2 + \&c.$$
$$+ m \cdot Gm^2 + n \cdot Gn^2 + p \cdot Cp^2 + \&c.$$
$$+ 2CG(m \cdot Gd - n \cdot Ge + p \cdot Gh - \&c.)$$

And, since by the property of the center of gravity,

$$m \cdot Gd - n \cdot Ge + p \cdot Gh - \&c. = 0;$$

we have

$$m \cdot Cm^2 + n \cdot Cn^2 + p \cdot Cp^2 + \&c. = (m + n + p + \&c.)\, CG^2$$
$$+ m \cdot Gm^2 + n \cdot Gn^2 + p \cdot Gp^2 + \&c.$$

Or, moment of inertia about C

= moment of inertia of $(m + n + p + \&c.)$ at G about C

+ moment of inertia of m, n, p, &c. about G.

COR. 1. We may represent this theorem thus, whatever be the number of bodies;

$$\Sigma(m \cdot Cm^2) = CG^2 \Sigma m + \Sigma(m \cdot Gm^2);$$

or if $Cm = r$, $CG = a$, $\Sigma m = M$, $\Sigma(m \cdot Gm^2) = Mk^2$,

it will become $\Sigma \cdot mr^2 = M(a^2 + k^2)$.

COR. 2. Knowing the moment of inertia about G, we may, from this expression, find the moment of inertia about C.

COR. 3. The moment of inertia about G, the center of gravity, is less than that about any other axis C, parallel to G.

205. PROP. *The Centers of Suspension and Oscillation are reciprocal:*

That is, if the center of oscillation be made the point of suspension, the former point of suspension will become the center of oscillation.

$$\text{In fig. 100, } CO = \frac{\Sigma\,(m\,.\,Cm^2)}{CG\,.\,\Sigma m}, \text{ by Art. 203.}$$

But by Art. 204,

$$\Sigma\,(m\,.\,Cm^2) = \Sigma\,(m\,.\,Gm^2) + CG^2\,.\,\Sigma m\,;$$

or, putting M for Σm the mass, and $k^2 M$ for $\Sigma\,(m\,.\,Gm^2)$,

$$CO = \frac{k^2}{CG} + CG; \text{ whence } GO = \frac{k^2}{CG}, \text{ and } CG = \frac{k^2}{GO}.$$

If therefore, we suspend the body from O, C will be its center of oscillation, by the same formula.

COR. 1. CO depends on CG alone, and will be the same, so long as CG is the same. Hence, if with center G (fig. 100,) and radius GC, we describe a circle CC'; (in the plane of oscillation, that is, in a plane perpendicular to the axis of suspension;) CO is the same, from whatever point of this circumference we suspend the body. And therefore, the time of oscillation is the same, from whichever of such points the body is suspended.

COR. 2. Also, if we describe a circle OO' with radius GO, the time of oscillation will be the same, whether the body is suspended from any point in the circumference OO', or from any point in the circumference CC'.

CHAP. VII.

MOTION OF MACHINES.

206. We shall in the present Chapter apply the preceding principles to determine the accelerating forces of the mechanical powers, and of other simple combinations of bodies acting on each other.

The accelerating forces being known, the motions are known by the Integral Calculus; as will be seen in the Analytical Dynamics.

Sect. I. Motion about a fixed Axis.

Prop. *To determine the Accelerating Force of weights on a Wheel and Axle*, fig. 102.

Let P draw up Q by means of strings wrapping round two cylinders, A, B, which have a common horizontal axis. Let a, b, be the radii of the cylinders respectively; and as in Art. 206, let the moment of inertia of the machine AB, about its axis, be $k^2 \Sigma m$, or $k^2 M$, M here indicating the whole mass of the machine, and k its radius of gyration. We shall then have

impressed forces, Pg at distance a, $-Qg$ at distance b;

of which the moment is $Pga - Qgb$.

Hence, by Cor. 8, Art. 200, and Cor. 2, Art. 201, we have

$$\text{accelerating force on } P = \frac{(Pa - Qb)\,ga}{Pa^2 + Qb^2 + Mk^2} \text{ downwards;}$$

$$\text{accelerating force on } Q = \frac{(Pa - Qb)\,gb}{Pa^2 + Qb^2 + Mk^2} \text{ upwards;}$$

and since these are constant, the motion may be found by the formulæ for constant forces.

Cor. 1. If $Qb > Pa$, the force will act in the opposite direction, and if Q be at rest originally, it will descend.

Cor. 2. If Tg be the tension of the string by which P hangs, P is impelled downwards by its weight, and upwards by the tension. Hence, the moving force on P is $Pg - Tg$, and the accelerating force $\frac{(P-T)g}{P}$;

$$\therefore \frac{(P-T)}{P} g = \frac{(Pa - Qb) ga}{Pa^2 + Qb^2 + Mk^2},$$

$$T = \frac{P(Qb^2 + Qab + Mk^2)}{Pa^2 + Qb^2 + Mk^2}.$$

Similarly, if T' be the tension of the string by which Q hangs,

$$T' = \frac{Q(Pa^2 + Pab + Mk^2)}{Pa^2 + Qb^2 + Mk^2}.$$

Cor. 3. The pressure on the center of motion arising from P, Q, will be the sum of these tensions, and will be counteracted by a reaction equal to this sum, for the forces which act on the machine must be in equilibrium according to the principles of Statics:

$$\therefore \text{pressure on the center } T + T' = \frac{PQ(a+b)^2 + (P+Q)Mk^2}{Pa^2 + Qb^2 + Mk^2 g}.$$

207. Prop. *To determine the Accelerating Force of weights acting on a Combination of Wheels and Axles*, fig. 103.

The wheels and axles may act on each other, either by means of teeth, as at D, or by strings passing round both the wheel and the axle, and turning them by friction, as at D'; or in other ways: the mechanical conditions of the problem are the same in all these cases.

Let a, b be the radii of the first wheel, and of its axle, respectively; a', b' the radii of the second wheel and axle, a'',

b'' of the third, and so on. Then the impressed moving forces are Pg acting at A, and Qg acting in the opposite direction at B. By Statics, the latter would be counterbalanced by a force at P equal to $Qg \cdot \frac{bb'b''}{aa'a''}$. Hence, the moving force impressed, and employed in producing motion, is equivalent to

$$Pg - Qg\frac{bb'b''}{aa'a''} \text{ acting at } A, \text{ at distance } a \text{ from } C.$$

Let Mk^2, $M'k'^2$, $M''k''^2$ be the moments of inertia of the respective wheels M, M', M'' about their centers, (including the axles): and let x be the effective accelerating force on P or on A. Then, since the accelerating forces are as the velocities, the accelerating force at D or E will be $\frac{bx}{a}$; that at D' or E' will be $\frac{bb'x}{aa'}$; and that at B or Q, $\frac{bb'b''x}{aa'a''}$.

Now, since the effective accelerating force at P is x, that at any distance r from C, in the wheel M, is $\frac{bx}{a}$; and if m be a particle at that distance, the effective moving force is $\frac{mrx}{a}$. And this is equivalent, in its moment round C, to a force $\frac{mr^2x}{a^2}$, acting at A. And hence, the whole effective moving forces in M are equivalent to a force $\frac{x\Sigma . mr^2}{a^2}$ acting at A; that is, they are equivalent to $\frac{x . Mk^2}{a^2}$ at A.

Similarly, the effective moving forces in M' are equivalent to $\frac{bx}{a} \cdot \frac{M'k'^2}{a'^2}$ at E; which force is, by the property of the machine, equivalent in its effect to turn the system round C, to a force $\frac{b^2x}{a^2} \cdot \frac{M'k'^2}{a'^2}$ at A.

The effective accelerating force on Q is $\frac{bb'b''x}{aa'a''}$; which gives a moving force $\frac{bb'b''x}{aa'a''}Q$, equivalent to $\frac{b^2b'^2b''^2x}{a^2a'^2a''^2}$ at A.

Hence, we have the moment of the impressed forces, about C,

$$= ga\left(P - Q\,\frac{bb'b''}{aa'a''}\right).$$

And the moment of all the effective forces, about C,

$$= Pax + Qax\,\frac{b^2b'^2b''^2}{a^2a'^2a''^2}$$

$$+ Max\,.\,\frac{k^2}{a^2} + M'ax\,.\,\frac{b^2}{a^2}\,.\,\frac{k'^2}{a'^2} + M''ax\,.\,\frac{b^2b'^2}{a^2a'^2}\,.\,\frac{k''^2}{a''^2}$$

Equating these forces (by Art. 198,) and putting n, n', n'' for $\frac{b}{a}$, $\frac{b'}{a'}$, $\frac{b''}{a''}$, we have, for the accelerating force on P,

$$x = \frac{(P - Qnn'n'')g}{P + Qn^2n'^2n''^2 + M\frac{k^2}{a^2} + M'n\,\frac{k'^2}{a'^2} + M''nn\frac{k''^2}{a''^2}}\,.$$

The accelerating force on $Q = nn'n''x$. These accelerating forces are constant.

208. A machine was constructed by Atwood, to measure the spaces and velocities of bodies descending by gravity, in order to compare them with theory. It is represented in fig. 104. Two equal weights P, Q, hang by a fine string over a fixed pully M. One of them is made to descend, by placing upon it a small weight D, and the times and spaces of the motion are observed. The weights at P and Q are enclosed in equal and cylindrical boxes; so that the effect of the resistance of the air will be the same upon both, and will not affect the motion. And the effect of friction is nearly removed by making the axis of M very slender, and causing each end of it as C to rest upon *Friction Wheels*, as M, M'*. The

* To shew that these wheels will diminish the effect of friction, we may consider friction as a force acting in a tangent to the axle. If the axle C rested on immoveable

times are observed by means of a pendulum, and the spaces by a scale of inches BF, down which P descends. To determine the velocity, P is made to pass through an opening MN, in a stage fastened to the scale BF; and the weight D, which is too large to pass, is left resting on M, N. Therefore, since the weight of D produces the only accelerating force which exists in the machine, after passing the point E, P will move uniformly with the velocity acquired. When P has passed through a given space EF, it is stopped by striking the stage F, which is there fixed to the scale.

The body P being allowed to descend from rest at a given point B, descends till D is heard to strike the stage M, N, and the time is noted; it then descends till P is heard to strike the stage F, and the time is noted: the space EF, divided by the interval of the two times, gives the velocity; and the space BF, and the time of describing it, being known, we can compare our results with theory. The velocities of P, Q are small, both because D is small, and because the wheels F', M', M'' are to be moved, and their moment of inertia is a part of the denominator of the accelerating force.

Observing, that besides the friction wheels, M', M'', there are two others at the other end of the axis A; calling the moment of each of these $M'k'^2$, and that of M, Mk^2, and the radii of the wheels and axles a, b, a', b', we have

$$\text{accelerating force on } P = \frac{Dg}{2P + D + M \cdot \frac{k^2}{a^2} + 4M' \cdot \frac{b}{a}\frac{k'^2}{a'^2}}.$$

surfaces, and the amount of the friction were F, its effect at A would be $\frac{Fb}{a}$. But if the axle C rest upon friction wheels, their circumferences will turn with the circumferences of the axle, and between these surfaces there will be no friction. The friction will take place at the axles C, C'; and if we suppose it to be F at the surface of the axle C', this force will be equivalent to $F\frac{b'}{a'}$ at the circumferences of the axle C, and to $F\frac{bb'}{aa'}$ at A. And as there are four ends of the axles C' for one C, the effect of the friction, when such friction wheels are used, is $4F\frac{bb'}{aa'}$ acting at A. Hence, by means of such a contrivance, it is diminished in the ratio of $a' : 4b'$, supposing F to be the same in both.

The effect of the inertia of the wheels is the same as if a mass $M \cdot \frac{k^2}{a^2} + 4M' \cdot \frac{b\,k^2}{a\,a'^2}$ were collected at the circumference of M.

The reader will find, in Atwood's *Treatise on Rectilinear and Rotatory Motion*, Sect. 7, an account of experiments made with this machine. The results of all of them agreed accurately with the formulæ for constant forces.

209. PROP. *To determine the Accelerating Forces of two weights on a Lever*, fig. 105.

Let P, Q, be attached to the extremities of a lever whose arms are a, b; and let M be the mass of the lever, and h the distance of its center of gravity from the center of motion. Let PQ be any position in which the lever makes an angle θ with the vertical. Then $a \cos.\theta$, $b \cos.\theta$, $h \cos.\theta$ are the perpendiculars from the center of motion upon the vertical directions of the forces of P, Q, M. And the momont of the forces is $(Pa + Mh - Qb)\,g \cos.\theta$, which acts to make P descend. Hence, if Mk^2 be the moment of inertia of the lever itself, we have

$$\text{accelerating force on } P = \frac{(Pa + Mh - Qb)\,ga \cos.\theta}{Pa^2 + Qb^2 + Mk^2}$$

acting perpendicular to CP.

210. PROP. *A body moveable about an axis* C *is* struck *at a given point by a given mass with a given velocity; to determine its motion*, fig. 106.

Impact is, properly speaking, a violent pressure continued for a short time. Now if any force act at a distance a from the axis of a body whose moment of inertia is Mk^2, the effect produced at any instant will be the same as if a mass $\frac{Mk^2}{a^2}$ were collected at the distance a. Hence, the whole effect produced will be the same as if such a mass were substituted for the body, whatever be the time which the change employs. And hence, the effect of perpendicular impact at a distance a will be the same as if it took place upon a mass $\frac{Mk^2}{a^2}$, placed there.

In fig. 106, let a mass P impinge directly (that is, in a direction perpendicular to the surfaces where the contact takes place,) on a system CA, with a velocity V; and let CA be a perpendicular on P's direction. If $CA = a$, the effect on the system will be the same as if P impinged on a body $\frac{Mk^2}{a^2}$. Let the substances be supposed inelastic, and the bodies will both move with the same velocity after the impact; and since, by the third law of motion, the mass multiplied into the velocity will be the same before and after the blow, we shall have, if x be the velocity of A after the stroke,

$$x\left(P + \frac{Mk^2}{a^2}\right) = PV,$$

$$x = \frac{PVa^2}{Pa^2 + Mk^2}.$$

If the body be acted upon by no force after the impact, it will revolve uniformly. If it move about a horizontal axis, and be acted on by gravity, it will ascend till all the velocity be destroyed, and then descend, and so oscillate.

If the bodies be elastic, we must apply the rules for impact in that case. On this supposition, P and M will separate after the impact. And if the impact be not direct, we must, supposing the bodies perfectly smooth, take that part of it which is perpendicular to the surfaces at the point of contact.

211. An instrument depending upon these principles was constructed by Robins for the purpose of measuring the velocities of musquet and cannon bullets, and has been called the Ballistic Pendulum. It consisted in an iron plate CA, fig. 107, suspended from a horizontal axis at C, and faced

with a thick board DE. When this was at rest, a bullet was shot into the board as at P, which caused the pendulum to move through an arc MN. The chord of this arc was known by means of a ribbon fastened to the pendulum, as at N, and sliding through a slit at M, so that when drawn to the length MN it did not return. The ball stuck in the wood, and was prevented from going through by the iron.

Let O be the center of oscillation of the pendulum, including the bullet. Then the motion of the pendulum when left to itself, will be the same as if all the matter were collected in O. And hence the arc up which O will move will be that down which it would acquire the velocity which it has at the lowest point. If θ be this angle, the velocity acquired in describing it would be that acquired down the versed sine of θ; or down a perpendicular height $CO\,.\,\text{ver.}\sin.\theta$. Let $C = l$; $\therefore$ (velocity)^2 of O at lowest point

$$= \sqrt{(2gl\,\text{ver.}\sin.\theta)} = 2\sin.\frac{\theta}{2}\sqrt{(gl)}.$$

But since the velocity of P at the lowest point is, by last Article,

$$\frac{PVa^2}{Pa^2 + Mk^2},$$

the velocity of O, which is to this as CO to CP, will be

$$\frac{PVal}{Pa^2 + Mk^2}, \text{ which is } = 2\sin.\frac{\theta}{2}\sqrt{(gl)} \text{ by what has been said.}$$

If h be the distance from C of the center of gravity of the pendulum, including the ball, by Arts. 203, 204,

$$l = \frac{Pa^2 + Mk^2}{(P+M)h}; \quad Pa^2 + Mk^2 = (P+M)\,hl;$$

$$\therefore\ PVal = 2\sin.\frac{\theta}{2}(P+M)\,hl\sqrt{(gl)},$$

$$V = 2\sin.\frac{\theta}{2}\,.\,\frac{P+M}{P}\,\frac{h}{a}\sqrt{(gl)}.$$

If the pendulum, after being struck by the ball, makes n oscillations in a minute, we have

$$\text{time of oscillation} = \frac{60}{n} = \pi\sqrt{\frac{l}{g}}; \quad \therefore \sqrt{(gl)} = \frac{60g}{\pi n}.$$

$$\text{And, } V = 2\sin.\frac{\theta}{2}.\frac{P+M}{P}.\frac{60gh}{\pi na}.$$

This agrees with Dr Hutton's formula. We have $2\sin.\frac{\theta}{2}$ by dividing the chord MN by the radius CN.

Dr Hutton himself however, in his own experiments, found the velocity of balls, by suspending the cannon which he used, and observing the arc through which it was driven by the recoil. The same formula is still applicable, M now representing the weight of the cannon and its appendages without the ball. For the effect will be the same, whether a velocity be communicated to the pendulous body by the impact of the ball, or by its reaction. And the momentum communicated at the axis of the cannon will be PV, because the *momentum* communicated to the ball in one direction, and to the pendulum in the other, must be equal.

It is found by experiments of this kind, that the velocity of musquet and cannon bullets varies from 1600 to 2000 feet per second.

Sect. II. Motion of Bodies rolling and unrolling.

212. A body *rolls* when it moves in contact with a line or surface so that the parts of the surface of the body are applied continuously to the successive portions of the line or surface. In this case a motion of translation implies also a motion of rotation, and conversely; and a force which produces the one of these motions must produce the other.

When a string has its end fastened to a body, the body, by revolving in a proper direction, wraps the string on or unwraps it off the body, and produces a motion which resembles rolling; when the body thus unwraps itself we may term the motion *unrolling*.

In all such cases the body does not revolve about a permanent fixed axis, and therefore the proportions of the preceding chapter are not here immediately applicable. The following proposition will however enable us to reason concerning such cases.

In all these cases we shall suppose all the motions to take place in planes perpendicular to the axis of revolution: the consequence of this will be that the axis of instantaneous revolution will always be parallel to the axis of apparent revolution passing through the center of gravity, or through any other point.

213. Prop. *When a body rolls or unrolls, the center of gravity moving in a straight line, it is required to find the Resultant of the Effective Forces, and their Moment.*

Let the body move for two successive seconds, and in each of these seconds let it be supposed that the motion is resolved in the following manner;—the center of gravity moves out of its initial into its final position, the body moving parallel to itself; and then the body moves out of the position into which it is thus brought, into its final position, the center of gravity remaining at rest. In this manner the motion in each second will be resolved into a motion of translation and a motion of revolution.

The effective force (supposing it to be constant) is measured by the increment of velocity in one second. The velocity of each point being resolved in the manner just described, the increments of the velocity will be resolved in the same manner, and therefore the effective forces will likewise be so resolved. Hence the effective forces will be those which are due to the motion of translation, together with those which are due to the motion of rotation.

If f be the effective accelerating force which is due to the motion of the center of gravity, mf will be the moving force due to the motion of translation of any particle m; and the sum of all those forces will be $\Sigma . mf$, which, since they are all equal and parallel, will be $f\Sigma m$, or Mf, if M be the mass of the body.

Also the resultant of all the parallel forces mf will pass through the center of gravity of the body, and will be in the direction of the motion of the center of gravity: therefore its moment with respect to any point in that direction will be nothing.

The resultant of all the effective forces due to the rotatory motion will be nothing; for if ϕ be the effective accelerating force on a particle at the distance 1 from the axis of rotation passing through the center of gravity, $mr\phi$ will be the effective moving force on a particle m at distance r from the axis, and will act perpendicular to the radius r. If x, y be the co-ordinates of the point m as referred to the center of gravity by rectangular axes in the plane of rotation, the components of the effective moving force in the directions of x and y, will be proportional to the sides of the triangle y, x, of which the third side is r, by Art. 28, these three sides being respectively perpendicular to the force and its components; and hence, considering the directions of the forces, $my\phi$ and $-mx\phi$ will be the components of $mr\phi$ in the directions of these co-ordinates; and $\Sigma . my\phi$ and $\Sigma . -mx\phi$ will be the resulting forces in these directions. But $\Sigma . my\phi = \phi\Sigma my = 0$, by the property of the center of gravity; and in like manner $\Sigma . -mxy = \phi\Sigma mx = 0$. Therefore the resultant of the effective forces due to the rotatory motion is 0.

The moment of the force $mr\phi$ with reference to the axis passing through the center of gravity is $mr^2\phi$, and the sum of all such moments is $\Sigma . mr^2\phi = \phi\Sigma . mr^2 = Mk^2\phi$, Mk^2 being the moment of inertia for the axis now supposed.

Hence, it appears that the effective forces which act on the system M, are equivalent to a force Mf acting at the center of gravity, and in the direction of the motion of that center; and that their moment is equivalent to a moment $Mk^2\phi$, acting to move the body about the point where its center of gravity is at any instant; f being the effective force due to the motion of the center of gravity, and ϕ the effective force due to the rotatory motion of a particle at a distance 1 from the axis of rotation passing through the center of gravity.

214. Prop. *A cylindrical body* M (fig. 108,) *unrolls itself from a vertical string* ABP, *which passes over a fixed pully and has a weight* P *appended to its other extremity; to determine the motion.*

The tension of the string ABP will be the same throughout, if we neglect the inertia of the pully B; let this be supposed, and let the tension be Tg; and let, as in the last Article, Mk^2 be the moment of inertia, f the effective accelerating force on the center of gravity, ϕ the effective accelerating force on a particle at a distance 1 from the axis; and let a be the radius of the cylinder.

The impressed forces are Tg at the circumference of the cylinder, acting upwards, and the force of gravity on each of the particles of the body, which is equivalent to Mg at the center of gravity acting downwards. The impressed and effective forces must have their resultants equal and opposite, and also their moments equal and opposite, by the conditions of statical equilibrium. Therefore, by last Article,

$$Mg - Tg = Mf,$$

$$Tga = Mk^2\phi.$$

But if f' be the effective accelerating force which acts on P, we shall have, for a like reason,

$$Pg - Tg = Pf'.$$

Now it is clear that if, when any radius, as CA, is horizontal, the point C moves downwards with a velocity v, and the point A moves upwards with a velocity v', the relative velocity of A with respect to C, will be $v + v'$, and the angular velocity of A about C, will be $\frac{v + v'}{a}$, or $\frac{1}{a}(v + v')$.

Hence the effective accelerating force due to the rotatory motion will be

$\frac{1}{a}$ (force which increases v + force which increases v')

$= \frac{1}{a}(f+f)$.

Therefore $\phi = \frac{1}{a}(f+f')$, or $f+f' = a\phi$.

Taking the values of f, f' and ϕ from the above three equations, this gives

$$PMg - PTg + PMg - MTg = PTg\frac{a^2}{k^2}.$$

Hence, $2PMk^2 = \{Pa^2 + (M+P)k^2\}\, T$,

$$T = \frac{2PMk^2}{Pa^2 + (M+P)k^2}.$$

From this we find $f = \frac{Mg - Tg}{M}$

$$= \frac{Pa^2 + (M-P)k^2}{Pa^2 + (M+P)k^2}\cdot g$$

$$f' = \frac{Pg - Tg}{P}$$

$$= \frac{Pa^2 - (M-P)k^2}{Pa^2 + (P+M)k^2}\, g$$

$$\phi = \frac{Tga}{Mk^2}$$

$$= \frac{2Pga}{Pa^2 + (M-P)k^2}.$$

These forces are constant, and therefore the motion of the center of gravity downwards, the motion of P downwards, and the motion of rotation, will be uniformly accelerated.

Cor. 1. If f be negative, M will ascend, that is, if

$$Pa^2 + (M-P)k^2 < 0, \text{ or if } \frac{M}{P} < \frac{k^2 - a^2}{a^2}:$$

this cannot be, except a is less than k.

Cor. 2. If f' be negative, P will ascend, that is, if

$$Pa^2 - (M - P)k^2 < 0, \text{ or if } \frac{M}{P} > \frac{k^2 + a^2}{a^2}.$$

Cor. 3. It is not necessary that the whole body should be cylindrical, but only that the part of it from which the string unrolls should be a cylinder, of which the axis passes through the center of gravity. The vertical plane of the string must be perpendicular to the axis of the cylinder, and must pass through the center of gravity.

Cor. 4. If the figure be a cylindrical shell of small thickness, the whole matter is at the distance a from the center, and $k = a$;

$$\text{Hence, accelerating force on } C = \frac{Mg}{2P + M};$$

$$\text{accelerating force on } P = \frac{2P - M}{2P + M} \cdot g;$$

$$\text{tension} = \frac{2MP}{2P + M}.$$

Cor. 5. If the figure be a solid homogeneous cylinder, it will be seen in the Analytical Dynamics that $k^2 = \frac{a^2}{2}$.

$$\text{Hence, accelerating force on } C = \frac{P + M}{3P + M} \cdot g,$$

$$\text{accelerating force on } P = \frac{3P - M}{3P + M} \cdot g,$$

$$\text{tension} = \frac{2MP}{3P + M}.$$

215. Prop. *A cylindrical body unrolls itself from a vertical string, the other end of which is fixed; to determine the Accelerating Force*, fig. 109.

If we assume P, in last Article, to be such that it shall neither ascend nor descend, we may suppose the string AB to be fixed at the point B, and the motion will be the same as before. We must therefore in this case, have the accelerating force on $P = 0$;

$$\text{or, } Pa^2 - (M - P)k^2 = 0, \text{ whence, } P = \frac{Mk^2}{a^2 + k^2}.$$

$$\text{Hence also, } T = \frac{Mk^2}{a^2 + k^2}.$$

$$\text{And accelerating force on } C = \frac{(M - T)g}{M} = \frac{a^2 g}{a^2 + k^2}.$$

Cor. 1. If the figure be a cylindrical shell, $k = a$;

$$\text{accelerating force on } C = \frac{g}{2}, \quad T = \frac{M}{2}.$$

Cor. 2. If the figure be a solid cylinder, $k^2 = \frac{a^2}{2}$, (see Analytical Dynamics,)

$$\text{accelerating force on } C = \frac{2g}{3}, \quad T = \frac{M}{3}.$$

Cor. 3. If the figure be a globe, it will appear (Analytical Dynamics) that $k^2 = \frac{2a^2}{5}$;

$$\text{accelerating force on } C = \frac{5g}{7}, \quad T = \frac{2M}{7}.$$

Cor. 4. If the string, instead of being vertical, be laid along an inclined plane as BA, fig. 110, the same conclusions are manifestly true; putting for g the force of gravity down the plane, which is $g \times$ sin. inclination. The tension will also be $Tg \times$ sin. inclination.

Cor. 5. If M, instead of rolling by means of a string, roll down the plane in consequence of the friction entirely preventing its sliding, the results will be the same. The tension of the string is now replaced by the effort which the friction exercises to prevent sliding.

Hence, when a body rolls down an inclined plane, the accelerating force is $\frac{1}{2}$ if it be a hollow cylinder, $\frac{2}{3}$ if it be a solid cylinder, and $\frac{5}{7}$ if it be a globe, of the force with which a body would slide down the plane, if friction were removed.

Sect. III. MOTION OF PULLIES.

216. PROP. *One body draws another over a single fixed pully; to determine the Accelerating Force,* fig. 111.

Let Mk^2 be the moment of inertia of the pully, a its radius. And let x be the effective accelerating force on P downwards; which is therefore also the effective accelerating force on the circumference of the pully M, and the effective accelerating force on Q upwards. Let Tg be the tension of the string AP, and $T'g$ of BQ. Hence, the force impressed at the circumference of the pully is $Tg - T'g$, and therefore, by Art. 205,

$$x = \frac{(T - T')ga^2}{Mk^2} \dots\dots\dots\dots (1).$$

But the accelerating force on $P = x = \dfrac{(P - T)g}{P}$;

and the accelerating force on $Q = x = \dfrac{(T' - Q)g}{Q}$;

$$\therefore\ Px = (P - T)g, \quad Qx = (T' - Q)g;$$

$$\therefore\ (P + Q)x = (T' - T)g + (P - Q)g \dots\dots\dots (2).$$

Multiply (1) by Mk^2, and (2) by a^2, and add;

$$\therefore\ Mk^2x + (P + Q)a^2x = (P - Q)ga^2,$$

$$x = \frac{(P - Q)ga^2}{Mk^2 + (P + Q)a^2}.$$

COR. 1. The tensions of AP, and BQ, are respectively

$$\frac{(Mk^2 + 2Qa^2)Pg}{Mk^2 + (P + Q)a^2}, \quad \frac{(Mk^2 + 2Pa^2)Qg}{Mk^2 + (P + Q)a^2}.$$

COR. 2. Hence, when strings are in motion about pullies, the tension of each string is no longer the same throughout its length. A part of the tension of PA is employed in turning M; and it is only the remainder which is continued along the cord, so as to act in BQ.

The same results might have been obtained from Art. 210, by making the radii of the wheel and axle equal.

217. Prop. *In the single moveable pully with the strings parallel; to determine the Accelerating Force*, fig. 112.

Let P, Q, be the weights; Mk^2, $M'k'^2$, the moments of inertia of the fixed and moveable pullies AB, DE; a, a' their radii. And let the tension of $AP = Tg$, of $BD = T'g$, of $EF = t'g$. Then, if x be the accelerating force on P, $\frac{x}{2}$ will be the accelerating force on Q, because it moves with half the velocity. Also, the accelerating force at the circumference of M will be x: and since, while E remains fixed, the center of M' rises with half P's velocity, the relative motion of E *about the center*, is half P's velocity, and therefore, the effective accelerating force at the circumference of M' round C is $\frac{x}{2}$.

And if we consider the forces which act upon M', we have

Impressed forces, $T'g$, $t'g$ upwards, Qg downwards; Q including the weight of M'.

Effective forces, $\frac{x}{2}$ acting upwards on Q, and $\frac{x}{2}$ acting at the circumference, turning M' round C'; whence the force *about* C' is $\frac{x}{2a}$, at distance 1, and by Art. 213, the moment *about* C' is $\frac{x}{2}\frac{M'k'^2}{a'}$.

Hence, by Art. 202, we must have

$$(T' + t')g - Qg = Q\frac{x}{2};$$

and, considering the moments with respect to C',

$$(T' - t')a'g = \frac{M'k'^2}{a'} \cdot \frac{x}{2}; \text{ or } (T' - t')g = \frac{M'k'^2}{a'^2} \cdot \frac{x}{2};$$

$$\therefore \text{ adding, } 2T'g - Qg = Q\frac{x}{2} + \frac{M'k'^2}{a'^2}\cdot\frac{x}{2}.$$

Also, we have, as before,

$$x = \frac{P - T}{P}g, \text{ or } (P - T)g = Px,$$

$$x = \frac{(T - T')ga^2}{Mk^2}, \text{ or } (T - T')g = \frac{Mk^2}{a^2}x;$$

add these two, and the result of former one, namely,

$$T'g = \frac{Qg}{2} + \frac{Qx}{2^2} + \frac{M'k'^2}{a'^2}\cdot\frac{x}{2^2},$$

and we have

$$Pg = \frac{Qg}{2} + \left(P + \frac{Q}{2^2} + \frac{Mk^2}{a^2} + \frac{M'k'^2}{2^2a'^2}\right)x;$$

$$\therefore x = \frac{\left(P + \frac{Q}{2}\right)g}{P + \frac{Q}{2^3} + \frac{Mk^2}{a^2} + \frac{M'k'^2}{2^2a'^2}};$$

from this also the tensions might be found.

218. Prop. *In the system of moveable pullies, where each pully hangs by a separate string; to determine the Accelerating Force*, fig. 113.

The strings are supposed parallel.

Let M, M', M'', M''' be the pullies; Mk^2, $M'k'^2$, $M''k''^2$, &c. their moments; a, a', a'', &c. their radii. Let x be the effective accelerating force on P; then $\frac{x}{2}$ will be the accelerating force on M'; $\frac{x}{2^2}$ on M''; $\frac{x}{2^3}$ on M'''; and these will also, as in last Article, be the effective accelerating forces producing rotation at the circumferences of M, M', M'', &c. And, by reasoning with respect to each pully, as we have done for M' in last article, we have these equations,

$$(P - T)g = Px, \quad (T - T')g = \frac{Mk^2}{a^2}x.$$

$$(T' + t')g - M'g - T''g = M'\frac{x}{2}, \quad (T' - t')g = \frac{M'k'^2}{a'^2}\frac{x}{2},$$

$$(T'' + t'')g - M''g - T'''g = M''\frac{x}{2^2}, \quad (T'' - t'')g = \frac{M''k''^2}{a''^2}\frac{x}{2^2};$$

and so on.

Eliminating t', t'', &c. from each successive pair, we have

$$Pg = Px + Tg,$$

$$Tg = \frac{Mk^2}{a^2}x + T'g,$$

$$T'g = \frac{M'k'^2}{a'^2}\cdot\frac{x}{2^2} + \frac{M'x}{2^2} + \frac{M'g}{2} + \frac{T''g}{2},$$

$$T''g = \frac{M''k''^2}{a''^2}\cdot\frac{x}{2^3} + \frac{M''x}{2^3} + \frac{M''g}{2} + \frac{T'''g}{2},$$

and so on.

Therefore,

$$Pg = Px + \frac{Mk^2}{a^2}x + \frac{M'k'^2}{a'^2}\cdot\frac{x}{2^2} + \frac{M''k''^2}{a''^2}\cdot\frac{x}{2^4} + \&c.$$
$$+ \frac{M'x}{2^2} + \frac{M''x}{2^4} + \&c. + \frac{M'g}{2} + \frac{M''g}{2^2} + \&c. + \frac{T^{iv}g}{2^3}.$$

The law of continuation is manifest: and the last tension (as T^{iv} in the figure) is that which immediately raises Q. Hence, we have

$$\text{the effective accelerating force on } Q, = \frac{x}{2^3} = \frac{(T^{iv} - Q)g}{Q};$$

$$\therefore\ T^{iv}g = Qg + Q\frac{x}{2^3}.$$

Substituting this, and obtaining the value of x, we have

$$x = \frac{\left(P - \frac{M'}{2} - \frac{M''}{2^2} - \&c. - \frac{Q}{2^3}\right)}{P + \frac{Mk^2}{a^2} + \frac{M'}{2^2} + \frac{M'k'^2}{2^2a'^2} + \frac{M''}{2^4} + \frac{M''k''^2}{2^4a''^4} + \&c. + \frac{Q}{2^6}};$$

and similarly for any number of pullies.

By similar reasoning, we shall have the accelerating force in the system of pullies in which each is attached to the weight: but more immediately in all such cases by the next Proposition.

219. PROP. *To find the Accelerating Force resulting from the action of gravity on any machine whatever, in which the ratios of the velocities of different points are constant.*

Let P be one of the bodies of the machine, and let P' be the mass, which, placed at P, would preserve the equilibrium. Then the weight $(P - P')g$ is the *impressed force*, which produces the motion.

Let u be the velocity of P, and v, v', &c. the velocities of any other bodies m, m', &c. in the system. If x be the effective accelerating force on P, since the forces are in the ultimate ratio of the increments of the velocities, $x\frac{dv}{du}$ will be the accelerating force on m, and $mx\frac{dv}{du}$ the effective moving force. Therefore the forces which must balance each other according to Art. 199, are $(P - P')g$, &c. in one direction, and Px, $mx\frac{dv}{du}$, $m'x\frac{dv'}{du}$, &c. in the opposite direction.

Now u, v, v', &c. may be considered as the virtual velocities of the points where these forces are applied. Hence by the principle of virtual velocities, (Analytical Statics, Art. 22,)

$$(P - P')g \cdot u - Px \cdot u - mx\frac{dv}{du} \cdot v - m'x\frac{dv'}{du} \cdot v' - \&c. = 0;$$

$$\therefore x = \frac{(P - P')g}{P + \frac{mv}{u}\frac{dv}{du} + \frac{m'v'}{u}\frac{dv'}{du} + \&c.} = \frac{(P - P')g}{P + \Sigma \cdot \frac{mv}{u}\frac{dv}{du}}.$$

Let a mass M, which forms any part of the system, be considered as for an instant moving about some stationary axis. This is always a possible way of representing the motion.

Let the distance of the center of gravity of M from the imaginary stationary axis be a, and the velocity of this center be α. Then if m be any particle of M which is at the distance r from the axis, m's velocity $= \frac{r\alpha}{a}$;

$$\text{whence } v = \frac{r\alpha}{a},\ \frac{dv}{du} = \frac{r}{a}\frac{d\alpha}{du};\ \text{and for the whole of } M,$$

$$\Sigma . \frac{mv}{u}\frac{dv}{du} = \Sigma . \frac{mr^2\alpha}{a^2u}\frac{d\alpha}{du} = \frac{\alpha}{a^2u}\frac{d\alpha}{du}\Sigma . mr^2$$

$$\text{(by Article 204)} = \frac{\alpha}{a^2u}\frac{d\alpha}{du}M(a^2+k^2) = M\frac{\alpha}{u}\frac{d\alpha}{du} + M\frac{k^2\alpha}{a^2u}\frac{d\alpha}{du}.$$

Hence in the denominator of the accelerating force x, we shall have, for each mass M, two terms, such as we have just found: one of them depending on the quantity of matter M, and the other on the moment of inertia Mk^2.

If the machine be such that the velocities of the particles have to each other ratios constant for the same particle, let $\alpha = nu$, where n does not vary with the time; therefore

$$\frac{d\alpha}{du} = n = \frac{\alpha}{u},\ \text{and } \frac{\alpha}{u}\frac{d\alpha}{du} = \frac{\alpha^2}{u^2};$$

whence the two terms of the denominator of x, corresponding to the mass M, are

$$M\frac{\alpha^2}{u^2} + M\frac{k^2\alpha^2}{a^2u^2}.$$

Also $\frac{\alpha}{a}$ is the angular velocity of M about its centre: therefore if ω be the angular velocity, the two terms corresponding to M will be

$$M\frac{\alpha^2}{u^2} + M\frac{k^2\omega^2}{u^2}.$$

It will be seen by comparison, that this includes all the preceding propositions of this Chapter.

CHAP. VIII.

ON THE FRICTION OF BODIES IN MOTION.

220. When two bodies move so that their surfaces rub or roll against each other, there is exerted between them a force of friction which has a tendency to affect their motions. It diminishes the motion produced by a given force, and makes a greater force requisite to produce a given motion.

The friction of a body which is dragged along a road on a sledge, or on a wheel-carriage; the friction of a wheel revolving upon a fixed axis; the friction of the pins, joints, sliding rods and valves, of pumps, steam engines, and other machines; all operate in this manner.

The Friction of bodies in motion is of different kinds: for example;

1°. *The Friction of Sliding or Rubbing.*

221. This is experienced when a body moves along a surface, touching it always with the same part of its surface, as a sledge drawn along the surface of the snow, or a flat stone thrown so as to run flatways along a sheet of ice. We have a friction of the same kind when an axle turns in a hole, or on the sides of a notch; or when a body revolves upon a pivot, as a top upon its peg. In all such cases the friction would, in the course of time, destroy the motion, if it were not maintained by the action of some force; and in order to keep up a uniform motion, a greater force is requisite, in proportion as the friction is greater.

2°. *Friction of a Rolling Body.*

222. When a cylinder rolls upon a horizontal plane, the opposition to its motion is much less than if it were to slide, but the resistance does not absolutely vanish. It would seem that in this case the resistance must arise rather from the cohesion of the surfaces than from friction properly so called, since the surfaces do not all *rub* upon each other. This kind of friction may be neglected in statical problems. In the case of motion it is found that the friction of rolling cylinders is as the pressure *directly*, and as the diameter of the cylinder *inversely*. Hence we see one of the mechanical advantages of large wheels.

When a cylinder of mahogany, of diameter 3 inches, rolled on a plane of oak, the friction was $\frac{1}{16}$ the pressure; when the same cylinder rolled on a plane of elm, the friction was $\frac{1}{100}$ the pressure.

3°. *Friction of Wheels.*

223. The wheels of carriages *roll* on the road, while the axle slides round in the hole into which it is inserted. Hence the friction of wheels will depend on both circumstances; the rolling friction of the rim, and the sliding friction of the axle. And the whole friction will be diminished by any change which diminishes either of these portions of it; for instance, by making the road smoother, or by smearing the axle with a lubricating substance.

In all these cases the friction is equal to some certain pressure, acting in a direction opposite to the body's motion and retarding its velocity, or requiring to be counteracted by a force acting in the direction of the motion.

224. We do not in Statics consider the effect of forces in producing or changing motion, having for our immediate subject the forces which keep a body at *rest*. But the forces which will keep a body in *uniform* motion are exactly those which will destroy or balance the resistances which oppose its motion: for bodies are in themselves indifferent to rest

or motion, and will go on moving uniformly if the resistances be balanced or destroyed, as has been explained in speaking of the First Law of Motion. Hence the forces which produce uniform motion may be reasoned upon by means of statical principles; and their results may be traced by the help of those principles, with the addition of the First Law of Motion.

PROP. *If a body move along a horizontal plane, and be perpetually urged forwards by a force exactly equal to the force of friction upon this plane, it will go on moving uniformly.*

For the force which urges it forwards, and the resistance produced by friction, are two equal and opposite forces, and exactly destroy each other's effects; therefore the body will move as if it were not acted on by any force; and therefore, by the First Law of Motion, it will go on moving uniformly.

Thus if a body in motion upon a horizontal plane experience a friction of $\frac{1}{n}$ of its weight, a pressure of the amount of $\frac{n}{1}$ of the weight, acting horizontally on the body, will keep it in uniform motion.

225. The force which is requisite to keep a body in uniform motion is sometimes called *the power of traction.*

The power of traction differs, as has been said, according to the differences in the wheels of the carriage, and also according to differences in the road. Taking carriages of the usual kind under the usual circumstances, the following are the results of the most recent experiments upon the subject.

The power of traction over a level well-constructed pavement varies from 32 to 39 lbs. for every ton. A waggon weighing 21 cwt. 8 lbs., drawn over a well-laid pavement, in Piccadilly, required a power of traction varying from 33 to 40 lbs. In a place where the pavement was uneven and worked into holes, the power was increased to 48 lbs.; but it may be assumed that the power of traction on the best laid pavement, when newly laid, is at the rate of about 32 lbs. to the ton, or $\frac{1}{70}$ of the weight. On a broken-stone surface of old flint the traction is about 64 lbs. being double that of a pavement. On a gravel road, the power of traction is nearly 150 lbs. to the ton; or $\frac{1}{15}$ of the weight: on a broken-stone road having a rough pavement foundation, the traction is 45 lbs. to the ton, or $\frac{1}{50}$.

The power of traction required on a well-constructed level rail-way is found to be from $\frac{1}{240}$ to $\frac{1}{300}$ of the load drawn.

226. Prop. *The effect of friction upon bodies in motion is nearly independent of the velocity.*

This is proved by experiment. Coulomb, in a series of experiments made with great accuracy and on a large scale, found that bodies sliding on a plane, on which they were moved by a constant force of traction, (greater than the power of traction requisite for maintaining a uniform velocity) were uniformly accelerated. Therefore the accelerating force which actually affected their motion must have been constant. (Art 114.) Therefore the pressure which produced motion must have been constant. (Art. 126.) But the pressure which produced motion was the excess of the force of traction above the force of friction; and as the former force was constant, the latter must have been so likewise.

It appears also by experiment that the friction which opposes the motion of a body when once set a going is less than the friction which opposes its beginning to move. According to Euler, the friction is reduced to one-half when the body is actually put in motion. The reduction appears in some cases to be greater than this. Coulomb found that the friction of wood sliding upon wood became less when the body began to move, than it had been the instant before, in the ratio of 2 to 9. The intensity afterwards did not change

But the friction, so far as it depends on the softness of the surface as in roads, will increase with the velocity.

227. Prop. *When the friction of carriages on a road is diminished, the maximum slope must be proportionally diminished, in order that the power of traction requisite for travelling may be diminished in a corresponding degree.*

The sloping parts of a road may be considered as inclined planes; and the force requisite to sustain a weight on these inclined planes is a traction of the weight which has the vertical height of the plane for its numerator, and the length of the plane for its denominator. Thus if the rise of the road be 1 foot in 20 feet of length, the force which must act up the slope in order to prevent a carriage from running down the hill (omitting the consideration of friction) is $\frac{1}{20}$ of the weight of the load and carriage. (Art. 38.)

The force which would just sustain the load is, by the reasoning in Art. 225, equal to the power of traction which is requisite to keep the carriage in uniform motion. And the power of traction must overcome or balance both the power of friction and the resistance produced by the slope. The friction upon the slope may be supposed to be the same as that upon the horizontal road, for the pressure is very nearly the same*.

Hence on a road on which the horizontal friction is $\frac{1}{12}$ and the greatest slope $\frac{1}{20}$, the average power of traction required will be $\frac{1}{12}$, and the greatest power required will be $\frac{1}{12} + \frac{1}{20}$ or $\frac{2}{15}$. The greatest power requisite for travelling is $2\frac{2}{3}$ times of the average.

If the friction be reduced to $\frac{1}{60}$, the slope remaining the same, the greatest power now required is $\frac{1}{60} + \frac{1}{20}$, or $\frac{1}{15}$, which is 4 times the average. While the average is reduced to one-fifth of what it was, the greatest power is only one-half of what it was; and except we have an available power amounting to this greatest force, we cannot travel on the road.

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* By Art. 38, the pressure on the inclined plane is to the weight of the load as the cosine of the inclination of the plane is to the radius. For a slope of 1 in 20, the pressure in the inclined plane is only $\frac{1}{800}$ part less than on the horizontal plane.

But if the slopes be now reduced so that their maximum is $\frac{1}{45}$; the greatest power is now $\frac{1}{60} + \frac{1}{45}$ or $\frac{7}{180}$; which is $2\frac{1}{3}$ of the average, and $\frac{7}{24}$ of the former greatest power.

If the road become a rail-road, with a friction of only $\frac{1}{240}$ of the load, and if the slopes were still as much as 1 in 20, the carriage could not ascend these slopes without a force of $\frac{1}{240} + \frac{1}{20}$ or $\frac{13}{240}$; that is, we must be able to use in these parts of the road a force 13 times as great as the force which we want on the level parts; and thus the advantage which a reduction of the requisite force of traction in the latter cases would give us, is lost by the necessity of providing for the hills.

But if in these cases we can reduce the slope very much, for instance, to 1 in 300, the greatest force then requisite is $\frac{1}{240} + \frac{1}{300} = \frac{3}{400}$, which, compared with $\frac{1}{14}$, the force on a Macadamized road, with a slope of $\frac{1}{20}$, $(\frac{1}{50} + \frac{1}{20})$ is less in the ratio of 3 to 28, or 1 to $9\frac{1}{3}$.

This reasoning is applicable whatever be the kind of force employed. If the roads of a country in general be so much improved that a single horse can draw a double-bodied phaeton with ease along a level, such carriages may be used with one horse, provided the hills be also so far cut down or avoided, that one horse can draw the carriage up them with moderate effort: but if this is not done, the improvement of the surface of the road will not make such a kind of carriage fit for common use.

228. Prop. *With small velocities, canals are more advantageous than roads; but with large velocities, good roads are more advantageous than canals.*

In canals the force which opposes the motion is the resistance of the water; and the power of traction must be sufficient to counteract this resistance, in order to maintain the speed of the boat. But this resistance increases in a higher proportion than the square of the velocity; while on roads, as we have stated, the friction is the same whether the velocity be large or small. If a carriage travel on a road 10 miles in 5 hours, or 10 miles in 1 hour, the power of traction must be nearly the same*; but if a boat be propelled on a canal 10 miles in 1 hour, the power of traction must be more than 25 times that which would be necessary to carry it 10 miles in 5 hours.

A vessel moved on the Paddington Grand Junction Canal, at the rate of $2\frac{1}{2}$ miles an hour, loaded with 21 tons, required a force of traction amounting to 77lbs.; while the same vessel moving at a rate of something less than 4 miles an hour, required a force of traction amounting to 308lbs. Thus while the speed was increased in a somewhat less proportion than $2\frac{1}{2}$ to 4 or 10 to 16, the resistance was increased in proportion of 77 to 308 or 10 to 40. The proportion of the square of the velocities would have been 100 to 256 or 10 to 25.6: that of their cubes 10 to 40.196. Other experiments also give the resistance in a greater proportion than the square of the velocity.

Let the boat just mentioned be compared with a carriage or train of carriages of equal weight, moving upon a road, on which the friction is $\frac{1}{n}$ of the weight: and let the resistance be supposed to vary as the square of the velocity.

The resistance at 4 miles an hour was 308 lbs. for $21 \times 20 \times 112$ lbs, that is, $\frac{1}{152}$ of the weight. If the resistance vary as the square of the velocity, it will for a velocity of v miles an hour be $\frac{v^2}{16 \times 152}$ of the weight. The resistance on the road is $\frac{1}{n}$ the weight; and the resistance of the canal will be the less of the two so long as nv^2 is less than 16×152 or 2532.

Hence if n be given, there is a certain velocity upon the canal for which the resistance is equal to the friction on the road; for smaller velocities the resistance is less than that on the road, and may be made much less by diminishing

* The different speed depends principally, as we shall see in the next chapter, on the different rate at which the power of traction is supplied and applied. But the resistance of the air makes a greater power of traction requisite for the greater velocity.

the velocity. In larger velocities the resistance on the canal is greater than that on the road, and may become much greater by increasing the velocity.

Thus if the friction of the road be $\frac{1}{72}$; the canal will have the advantage in this respect, as long as v^2 is less than 33, that is, till the velocity is about $5\frac{3}{4}$ miles an hour.

But if the load be placed on a rail-road, when the friction is $\frac{1}{240}$, the canal requires a greater force when v^2 becomes $10\frac{1}{2}$, that is, when the velocity reaches $3\frac{1}{5}$ miles an hour.

229. A remarkable exception to the above rule of the connexion of the resistance and the velocity of boats drawn on a canal, is found to obtain when the velocity is very great. Beyond 8 miles an hour it appears that the resistance does not increase nearly so rapidly as the above rule would give it. It does not seem to be clearly ascertained whether this curious anomaly arises from the circumstance of the boat being made to rise in the water by the rapid motion, and thus exposing a less surface to the resistance; or from a difference in the kind of motion communicated to the parts of the water.

230. There is another point in which the advantages of canals and roads come in competition; namely, in their power of *supporting* loads. The pressure of the wheels of carriages on a rail-road is limited by the strength of the rail, and is seldom more than 3 tons upon each wheel. The pressure on the wheels of carriages on a turnpike road is limited by the strength of the crust of the road: on the broad wheels of the heaviest waggons the pressure never exceeds two tons. But the weight which a canal is capable of sustaining, is only limited by the magnitude of the boats which the breadth and depth of the canal allow to float upon it: the weight of the boat and its cargo being, by the principles of Hydrostatics, equal to the weight of the water which is displaced by the part of the boat immersed in the canal.

CHAP. IX.

ON THE MEASURE OF THE POWER OF MECHANICAL AGENTS AND OF WORK DONE BY MACHINES.

231. Various mechanical agents, as animal strength, weight, wind, water, steam, are employed by man: and their agency is made, by various contrivances, to produce certain desired effects. The nature of the effect may be infinitely varied according to the details of the machinery which is put in action: but in all cases there must be some force exerted in order to produce any effect; and there will be a certain relation between the force exerted and the effect produced, which relation we shall here consider.

As examples, we may take a horse drawing a cart, a water-mill or windmill grinding corn, a steam engine employed in raising ore from a mine, or in propelling a locomotive carriage along a rail-road; and we shall in all these cases suppose the rate of working to be uniform: we may then assert the following Proposition.

232. Prop. *When work is done by any machine, the pressure at the work may be considered to be statically equal to the resistance overcome.*

On the above supposition, there is in these, and in all other cases, a constant pressure exerted at the work. In the case of the carriage, this pressure is the amount of the friction on the road, and of the resolved part of the weight acting down the slope, if the road be inclined. In the mill, the corn is by various mechanical contrivances brought between the mill-stones, so that it is ground when one of them turns upon the other: and the resistance to turning which is produced by this abrasion, is the pressure which must be overcome in order to keep the machine working. In the case of ores, or any other weights raised, the force which supports or raises the weight must be equal to the weight. When a locomotive engine moves itself, it must produce an excess of pressure in the direction of the motion, equal to the friction and other resistances.

Such pressures, however, as are here described, would only just *balance* the resistance to motion, and therefore they would not produce motion. But the slightest excess of force above this amount would produce motion, and, if we suppose the excess to be permanent, would work the machine: and as we are not here considering the rate of working, we may suppose the excess, or accessory force, to be as small as we choose, and may neglect it in comparison of the definite and principal force which is requisite to balance the resistances.

Thus the force which will balance the resistances is sufficient to *keep* the machinery in a state of uniform motion But it is to be observed that in this assertion we suppose the resistances to be estimated for the state of motion.

In many cases however the action of a machine is not uniform, but intermitting or alternate. Thus in the common process of pumping, we draw up the piston and thus raise a column of water; but before we can make another effective stroke, the piston must be returned to the bottom of the chamber in which it moves.

In such cases we shall, at least at first, consider only the effective part of the working process. By improvements in the machinery, the force expended in the ineffective part of the operation may be diminished without any known limit.

This may be effected in various ways. For instance, in the case just quoted of the pump, something might be done by diminishing the inertia and friction of the piston and other moveable parts. But the effective working might be made continuous by connecting two pumps, so that the same movement should produce the downstroke of one, and the upstroke of the other.

A resource generally applicable to such cases is a Fly Wheel, which may be kept in motion by a force either continuous, interrupted or alternating; and may be made to exert a pressure at the work, either continuous or interrupted according to any law, provided there be a sufficiently great disproportion between the whole work which the wheel is capable of doing, and each separate effort.

233. Prop. *The amount of work done may be measured by the pressure exerted at the work, and the space through which it is exerted, jointly.*

If the pressure necessary to keep the machinery in motion be increased in any ratio, the work done will be increased in the same ratio. For instance, in the examples above mentioned, if the weight of the carriage and load be doubled, or if the mass raised be doubled, while the space through which the carriage travels or the weight rises remains the same, the work done is also doubled: and the same may be said of any other ratio.

Also if the space through which the weight be moved be increased in any ratio, the work done is increased in the same ratio. Thus the work done in drawing a carriage two miles, is double of the work done in drawing the same carriage one mile; the work done in raising a weight through 20 feet is double of the work done in raising the same weight 10 feet: the work done in producing two turns of a mill-stone is double of the work done in producing one turn, the resistance being the same; and the same is true of any other ratio.

Hence when both the pressure and the space through which it acts vary, the work done is as the quantities jointly: and may be *measured by the product of the pressure exerted* *at the work, multiplied into the space through which it is exerted.*

Cor. According to this mode of measurement the work done is neither increased nor diminished by dividing it into separate portions. If we have to raise 1000lbs. through 20 feet, the work done may be represented by 1000×20. And whether we raise the whole weight at once through the whole space; or raise it 100lbs. at a time; or raise it through 5 feet at a time; the whole work done will be the same according to the measure here stated. And thus our measure of work done agrees with the useful effect produced.

If 1000lbs. be carried 20 feet high, 10lbs. at a time, by a person or a machine which had to return for each successive load, the force employed in the returns is not a part of the useful effect, and is not taken into account in our measure.

It is universally true, that the work done, measured as above, is not increased or diminished in amount by the rate of working, nor by the substitution of any action through machinery for direct action, however much this substitution may modify the time of producing the effect and the kind of work done. To prove this is the object of the following proposition.

234. Prop. *The work done in the direct action of any mechanical agent, is equal to the work done by the same agent through any machine, whatever be the nature of the machine or the rate of working.*

Let f be the pressure exerted directly through the small space s, by the agent: and let F be the pressure exerted at the work, S the space there corresponding to s. By Art. 232, f and F balance each other on the machine; and s and S are small spaces through which the parts of the machine at which f and F act, would move in the same time. Therefore by Art. 43, $f \times s = F \times s$.

Now $f \times s$ is the work done by the agent acting directly in the time of describing the space s: and $F \times S$ is the work done when the agent acts through the intervention of the machine in the same time: and hence the proposition is manifest.

Thus a man can mount on a ladder 1000 feet in 1 hour. If his weight be 150lbs., the work done in this hour consists in raising 150lbs. through 1000 feet. And if the same man work on a tread-mill, and thus, by machinery, exert a pressure f through a space s, we shall have, for one hour, $f \times s = 150 \times 1000$, whatever be the machinery.

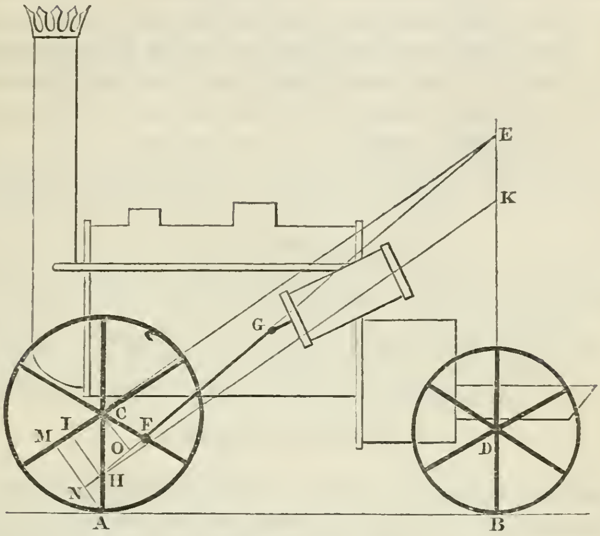
When rectilinear is converted into rotatory motion, as when the piston of a steam-engine is made to turn a fly-wheel by means of a crank, the same proposition is still true. Thus, if the engine make a double stroke during one revolution of the wheel, the pressure on the piston multiplied into the double length of the stroke, will be equal to the pressure exerted perpendicularly to any radius of the wheel, multiplied into the circumference of a circle corresponding to that radius. This might also be easily proved from elementary mechanical principles for any small portion of the motion, and hence for the whole motion.

The force here spoken of, as exerted perpendicularly to the radius of the wheel, is the sustaining force, which is requisite to counteract the resistances and to keep the machine in a state of regular motion. We do not here consider the force by which the momentum of the fly-wheel is generated from rest, but the force by which this momentum, whence once acquired, is preserved nearly uniform.

In the cases hitherto treated of, the work was done by the whole action of the machine upon the materials: but in some cases the useful effect is the excess of the action above the re-action: we shall now consider a case of this description—the Locomotive Engine.

235. Prop. *In a locomotive engine which is made to travel by turning a wheel on which it rests, the force required is the same as if the center of the wheel were fixed, and the resistance of the motion acted at the circumference of the wheel.*

The figure represents the "ROCKET" engine of Mr Stephenson.



Let C, D be the axles of the fore and hind wheels of a locomotive engine, which rests on a road at the points A, B. And let a force act from some point of the carriage, by the rod GF, upon the crank CF, which turns with the wheel CA round the axle C. The rod exerts a certain action upon the crank at F, and an equal re-action upon the carriage at G: if FG meet CA and DB in H and E, the action on the wheel CA may be supposed to take place at H, and the re-action on the carriage may be supposed to take place at E. Join CE, and draw HK parallel to CE: and let the equal action and re-action in the line GF or EH be represented by EH and HE.

The force EH is equivalent to KH, CH, of which the latter is supported and destroyed at A; the former is effective on the wheel CA.

The force HE is equivalent to KE, CE, of which the former diminishes the pressure at B, the latter draws C towards E.

Therefore the wheel CA is acted upon by two forces KH, CE; which tend to turn it about A. Also the resistance which the carriage experiences to its motion in the direction CD may be considered as a force acting at C perpendicular to CA. Let R be this force, and let ANM be perpendicular to CE, HK. Then, the forces and the resistance must balance each other, and by the property of the lever, if S be the force KH or CE,

$$R \times CA + S \times AN = S \times AM; \text{ or } R \times CA = S \times MN.$$

Let CO be drawn perpendicular from C upon EH, and HI upon CE; and let P be the pressure exerted by the rod GF. We then have

$$P : S :: EH : EC :: HI : CO :: MN : CO;$$

$$\text{therefore } P \times CO = S \times MN;$$

$$\text{and } R \times CA = P \times CO.$$

Hence the effect is the same as if C were fixed, and the force P acted by means of the rod GF and crank CF to produce motion in the wheel, in opposition to a force R acting at the circumference.

In order that the engine may be urged in the direction CD, the force which acts by means of the rod GF must be a pushing force while the line GF falls below C, and a pulling force when this line falls above C. If G be connected with the piston of a steam-engine, these two parts of the motion must correspond to the up and down stroke, or the backwards and forwards stroke.

The same reasoning which is here applied to locomotive steam-engines, is equally applicable to other locomotive engines; for instance a row-boat. In the case of such a boat the oar may be considered as a radius of a wheel, turning about the row-lock: the whole resistance of the water to the boat must then be considered as transferred to the blade of the oar, and balanced by the whole pull of the rower.

236. Prop. *In a locomotive engine moving uniformly, the pressure on the pistons multiplied into the length of the*

double stroke is equal to the resistance to the motion multiplied into the circumference of the driving wheels.

Let R be the resistance to the motion, Q the mean pressure on the piston, l the length of the stroke, D the diameter of the driving wheel. Then, while the piston moves up and down through $2l$, the wheel makes one revolution, and a point at the circumference moves through πD. Now the force Q and the resistance R balance each other: (Art. 234.) Hence, by the principle of virtual velocities,

$$R \times \pi D = Q \times 2l.$$

Cor. 1. We have here represented by Q the effective pressure on the piston: but if Q be the whole pressure on the piston and A the pressure of the atmosphere on the piston, the effective pressure is $Q - A$; whence

$$R \times \pi D = (Q - A) 2l.$$

Cor. 2. In the proof we have spoken of one piston only; but if Q be the *mean* pressure on both the pistons where there are two, the equation will still be true.

Hence the work done by an agent working a locomotive engine is determined in the same manner as the work done by any other machine.

237. Since in all circumstances the product of pressure and space moved through, when an agent acts by means of any machine, is equal to the product of the pressure exerted directly and of the space through which it is exerted, this product may be taken as representing the power of the agent to do work; and a particular term may be applied to it.

Mr Davies Gilbert (Phil. Trans. 1827, p. 25, "On the expediency of assigning specific names to all such functions of simple elements as represent definite properties; with the suggestion of a new term in mechanics,") has proposed to term this product the *efficiency* of the agent.

The term *duty* is used by engineers to express the work actually done by a steam-engine, estimated by the number of pounds raised one foot high by one bushel of coal. Perhaps it will be most convenient to call the product of the pressure and space moved through by the agent the *theoretical duty;* and the work really performed the *practical duty:* the former is what the machine *ought* to do; the latter is what it does.

238. We shall now consider the application of the preceding measure in some of the most important cases.

The agents, or sources of power which are commonly employed in doing work are, the weight of solid bodies; the weight of fluids; the elasticity of solids; the impulse of fluids either inelastic, as water, or elastic, as air; the elasticity of aerial fluids, as steam; the muscular power of men and animals. Fuel is mediately a source of power, because it may be used in converting water into steam.

In some of these agents the efficiency is capable of being accumulated and treasured up for any length of time, as weight and elasticity: in others the efficiency is necessarily expended immediately; it can only be produced at a certain rate, and must be employed when it is produced; as is the case with the strength of men and animals. In the former class of cases the preceding principles enable us to estimate the total efficiency of the agent; in the latter class we estimate the efficiency of the agent for a given time, as the efficiency of a man working for a minute.

1. *Power of Weights, or Gravity.*

239. This is a very extensive source of motion, as in the clock, in which it moves both the pendulum and the weight: in the heavy hammer, in which gravity is sometimes combined with other forces: in the pile-driver, in which gravity alone produces the impact.

When the weight of bodies is used as a moving power, the theoretical duty is the weight which we can command, multiplied into the space through which it can descend.

Thus a weight of 1000lbs. which can descend 10 feet, has a theoretical duty of 10000.

This weight may be made, by means of proper machinery, (neglecting friction, &c.) to raise 10000lbs. 1 foot high, or 1 pound 10000 feet high: in both cases the work done is the same.

And the total work done is the same whether the whole weight descend at once, or in parts; and whether it descend the whole height at once, or descend a small height at a time.

240. One of the instances in which the weight of bodies is used as a moving power on a larger scale is the case of "self-acting planes;" that is, sloping parts of a rail-road where the weight of loaded carriages descending is made to draw up unloaded ones.

In this case, however, the motion of the descending carriages is generally not uniform, but is, during a considerable portion of the descent, accelerated from rest by the force of gravity; and the equality between the maintaining force and the resistances, which is sufficient for the working of machines in uniform motion, is not sufficient for the proper effect of this contrivance.

In this case a part of the theoretical duty is employed in doing work in the sense above explained: and a part is expended in producing and increasing velocity.

Let there be a self-acting plane, the height of which is 100 feet, and the length 4000. On this plane let loaded carriages, of 3 tons each, draw up as many empty carriages of 1 ton each: and let in each case the friction be 1-200th part of the weight.

The *theoretical duty* is that corresponding to the difference of the descending and ascending weights: that is, for each carriage, to 2 tons descending through 100 feet: whence it is, in pounds,

$$2 \times 20 \times 112 \times 100 \text{ or } 448000.$$

The *work done* is the friction and resistance overcome on a length of 4000 feet (for the weight raised has already been taken account of).

The friction of the descending carriage is $\frac{3 \times 20 \times 112}{200}$ pounds; of the ascending one, $\frac{20 \times 112}{200}$; and the sum of these is 44.8lbs; which multiplied into 4000 feet gives 199200 for work done.

Hence there is an excess of more than one half the theoretical duty over the work done; and this excess is employed in generating and accelerating motion. Upon this excess of force depends the rapidity of the working.

The friction being 44.8lbs. and the excess of descending weight 4480lbs. these forces would balance if the ascent of the plane were 1 in 100: and if the slope were less than this, the plane would not work at all.

241. When weight is employed, altogether or in part, in generating velocity, the work done in generating velocity may be represented by the product of the weight moved, into the vertical space due to the velocity. This being assumed,

Prop. *The work done by any machine, working by weights, is equal to the theoretical duty; including in the estimate of the work done, the velocity generated in the weights.*

Let a body m descend to a given point from a height h; then, by what precedes, the work which it might have done in descending is mh.

But if it has done no work in descending, the whole of its weight has been employed in producing velocity: and in this case, if v be its velocity, $v^2 = 2gh$.

Hence the work which it might have done in descending is $\frac{mv^2}{2g}$, since $h = \frac{v^2}{2g}$.

If, during a part of the descent, some of the weight has been employed in doing work, as for instance in overcoming friction or resistance, let h' be the height which m has descended, when a part m' of the weight begins to be required to overcome resistance. Then the vertical accelerating force on m after this point is $\frac{m - m'}{m} g$ instead of g; and if v' be the velocity of m after it has fallen through h', and u the velocity after it has fallen through h,

$$u^2 = v'^2 + 2 \frac{m - m'}{m} g (h - h'), \quad \text{(Art. 163, Cor. 3.)}$$

or, since $v'^2 = 2gh'$,

$$u^2 = 2g \left\{ h - \frac{m'}{m} (h - h') \right\}.$$

$$\text{Hence } \frac{m u^2}{2g} + m' (h - h') = m h.$$

Let k be the height through which a body must fall to acquire the velocity u: then $u^2 = 2gk$; the above equation becomes

$$m k + m' (h - h') = m h.$$

That is, the work done is equal to the theoretical duty of the machine; including in the work done, the velocity generated, according to the above mode of estimation: for mk is the work done in generating this velocity, and $m' (h - h')$ is the work done in overcoming resistances.

242. Also the above proposition is true, however m' (the portion of the weight requisite to overcome resistances) may vary during the motion. For when $h - h'$ is made small, $h - h'$ may be considered as an element of the vertical height, and m' being constant during this element, the proposition is true for this element; and in like manner it is true for the next element; and so on to the end of the motion.

The proposition is also true for any number of bodies, and for any number of resistances: thus if m_1, m_2, m_3 be the weights; h_1, h_2, h_3 the spaces through which they can descend, k_1, k_2, k_3 the spaces due to the velocities acquired by m_1, m_2, m_3; m', m'' the weights requisite to overcome the resistances, and k', k'' the spaces through which these weights are required to act, we have

$$m_1 k_1 + m_2 k_2 + m_3 k_3 + m' k' + m'' k'' = m_1 h_1 + m_2 h_2 + m_3 h_3;$$

as will appear by the reasoning of the last Article.

If u_1, u_2, u_3 be the velocities actually acquired by m_1, m_2, m_3; v_1, v_2, v_3 the velocities which they would have acquired falling freely through h_1, h_2, h_3; u', u'' the velocities which bodies would acquire falling freely through k', k''; we have since $u_1^2 = 2gk_1$, &c.

$$m_1 u_1^2 + m_2 u_2^2 + m_3 u_3^2 + m' u'^2 + m'' u''^2 = m_1 v_1^2 + m_2 v_2^2 + m_3 v_3^2.$$

DEF. *The sum of all the bodies of the system, each multiplied into the square of its velocity, is called the* VIS VIVA *of the system.*

It appears by what precedes that the Vis Viva of a system acted on by gravity only is equal to that which it would have had if the parts had fallen freely; adding to the actual *vis viva*, that which corresponds to the resistances overcome.

It is, however, often very difficult to apply this proposition, for there are many cases in which we have no means of estimating the resistances overcome; for instance, when bodies affect each other's form by collision or mutual pressure, it is difficult to determine how much of the *vis viva* is thus absorbed.

2. *Power of Water.*

243. When water is used as a moving power and made to work by its weight, as in overshot or bucket-wheel, its theoretical duty and the work done by it are governed by the same rules as the corresponding quantities in the case of

solids. The *Duty* is the weight of the water which we can command, multiplied into the height through which we can cause it to descend.

When water works by means of the pressure or impulse arising from its motion (as in an undershot wheel), the duty is estimated by considering the height due to the velocity, as the space through which the weight of the water can move. And this would be a proper estimate, if all the velocity of the water could be employed in doing work. For if the velocity of each particle of water were to be turned in a vertical direction, each particle would lose the whole of its velocity in ascending to the height due to the velocity. And when each part of the water has thus been raised through the height due to the velocity, we may conceive the whole of the water to be at rest, and to have the power of descending through this space; and therefore the theoretical duty would be the weight of the water multiplied into the space.

But in the action of water in motion, the whole effect is much less than this, a large portion of the force being absorbed in the mutual action of the parts of the fluid.

The theoretical duty of a water-wheel of either kind would, by what has been said, be the gravity of water expended, multiplied into the *head* of water: the *head* meaning either the actual descent or the hypothetical descent due to the velocity.

In all cases the work done is less than this theoretical duty; some of the power being expended in unappreciated resistances, intestine motions of the fluid, and in other ways.

If a water-wheel or any other water-engine be employed to raise a weight, it follows from what has preceded that the weight of water expended, multiplied into the head of water, would be equal to the weight raised, multiplied into the height through which it is raised, if the whole power were effective.

Other things remaining the same, the quantity of water expended is as the velocity; and the *head* is as the square of the velocity: therefore the effect must vary as the cube of the velocity. This agrees with the result of experiments made by Mr. Smeaton, on models *. He collected from the same experiments, that in overshot wheels the proportion of the theoretical duty or power expended, to the work done or effect produced, is as 5 to 4 nearly; estimating, as work done, only the weight raised, and neglecting friction, &c.

He found also that in undershot wheels the proportion of the power expended to the effect produced, estimated in the same way, is as 10 to 3 nearly.

It is asserted that the machine in which the work done approaches most nearly to the power expended is the Hydraulic Ram.

3. *Power of Air.*

244. Air is employed as a power by taking advantage either of its elasticity or its motion. Air-guns act by the sudden expansion of air. The disturbance of the equilibrium of air, by the change of temperature and other causes, produces currents which are used as sources of motion, as in the smoke-jack; and on a larger scale, in the case of winds, which are employed to move windmills and ships.

The effect of winds, as moving powers, may be calculated theoretically in the same manner as the effect of streams of water. The theoretical duty of a given mass of air is equal to a weight, equivalent to the elasticity of the air, multiplied into the height due to its velocity.

Smeaton proved experimentally that in the case of windmills the effect of wind varied as the cube of its velocity: which agrees with the rule just stated*.

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* Phil. Trans. 1759.

4. *Power of Elastic Bodies.*

245. Elastic bodies, as springs, are used to produce continued motion; as in the watch, in various self-acting musical instruments, and other automata: sometimes also to produce a more sudden and intense effect, as in the bow and the trap.

The efficiency of a spring while it expands or contracts from a constrained position through any small space is measured by the pressure which it exerts at any point, multiplied into the space through which that point moves.

As the spring moves, it will generally happen that the pressure constantly varies. In this case the whole efficiency is the sum of all the elementary portions of the space described, each multiplied into the pressure exerted during the respective element of the motion.

5. *Power of Steam.*

246. Steam acts by means of it elasticity, and its capability of being generated and condensed by heat and cold. Its efficiency may be measured in the same manner as that of any other elastic body, but it has peculiar properties which make a more special consideration necessary.

Steam is used as a mechanical agent in various ways, of which the principal will be exemplified by considering, 1. *Atmospheric Engines*, in which the pressure of the atmosphere is made available by the condensation of steam: 2. *Condensing Engines*, in which the motion of the piston each way is produced by steam on one side of it, rendered available by the condensation on the other side: in this case the steam may have a higher elasticity than the atmospheric air; 3. *High-pressure Engines*, in which the steam *must* have a greater elasticity than the atmosphere, since it is made to urge the piston against the pressure of the atmosphere without condensation.

1. *Atmospheric Engines.*

In Atmosperic Engines, a piston, moveable in a vertical cylinder, rises to the top of the cylinder when steam is admitted below it, of the same elasticity as the atmosphere; the piston being slightly overbalanced. The steam is then condensed, and the piston is urged, by the whole pressure of the atmosphere, to the bottom of the cylinder. The steam being again admitted under the piston, the piston rises to the top of the cylinder; and so on.

In this case the efficiency of the machine is that corresponding to the descending stroke: the raising of the piston does not produce any power, but is necessary to the continuance of the alternating motion.

247. Prop. *To find the theoretical duty of an atmospheric steam-engine corresponding to 1 cubic foot of water.*

The space occupied by steam, of the temperature of boiling water, and consequently of the elasticity of the atmosphere, is 1711 times the bulk of the water which produces it. (*Tredgold on the Steam Engine, Art.* 302.) Hence 1 cubic foot of water produces 1711 cubic feet of steam of the pressure of the atmosphere: this pressure is, at its mean, 2120 pounds on a square foot. Hence the whole pressure is $1711 \times 2120 = 3627320$ lbs. supposing the steam to occupy a space a foot high and 1711 feet in horizontal surface. And when this is condensed, the whole of this pressure acts through 1 foot; and therefore the theoretical duty is 3627320 lbs.

If the steam occupy a prismatic space h feet high, the horizontal surface will be $\frac{1711}{h}$, and the pressure on the piston $\frac{1711}{h} \times 2120$; and when the steam is condensed, the piston will move through h feet: whence 1711×2120 will still be the duty of the machine.

If the piston be overbalanced by an excess of pressure p in its ascent, and if q be the pressure of the atmosphere on the piston, h the length of the stroke; there is an efficiency ph capable of doing work in the ascent of the piston; and the efficiency in the descent is $(q-p)h$: hence the whole efficiency of the double stroke is qh; which is the same as when there is no available excess of balancing force.

If p be $\frac{1}{2}q$, the efficiency in the ascent and descent will be equal, namely $\frac{1}{2}qh$ in each case. This is a mode in which atmospheric engines are often employed.

248. Prop. *To find the theoretical duty of an Atmospheric Steam-engine, corresponding to 1lb. of coal.*

It appears (*Tredgold*, Art. 191.) that it requires from 7 to 10lbs. of Newcastle coal to form a cubic foot of water into steam of the elasticity of the atmosphere. Mr Watts states that a bushel of coal (84lbs.) would evaporate 10 cubic feet of water (*Wood on Rail Roads*, p. 353.): Mr Davies Gilbert gives 14 cubic feet as the water evaporated by one bushel.

Taking Mr Watt's statement, we have 8.4lbs. for the coal requisite to evaporate 1 cubic foot.

Hence dividing the efficiency of 1 cubic foot of water by 8.4, we have 431824 for the efficiency of 1lb. of coal.

Or, since a bushel of coal reduces 10 cubic feet of water to steam, we have, by the last Article, 36273200 for the efficiency of a bushel of coal in an atmospheric engine; and this, divided by 84, gives 431824 for the efficiency of 1lb.

Mr Gilbert gives 39361000 for the efficiency of a bushel of coal employed in an atmospheric engine.

From this must be deducted the efficiency requisite to work the air pump, which is usually about one eighth of the whole; and the resistance arising from imperfection of the vacuum, as well as the friction.

2. *Condensing Engines.*

249. In these the motion of the piston to and fro is produced by the alternate action of steam upon each side of the piston, the steam on the other side being condensed. The steam here may be and often is of a greater elasticity than the atmosphere.

We shall not attempt here to find the theoretical duty of a given quantity of coal. It appears by the results that this is a very advantageous way of using steam.

The duty of such engines has been much increased by two expedients:—First, by raising the temperature of the steam above that of boiling water, by which its elasticity is made greater than that of the atmosphere; this proceeding being of course accompanied by an increased expenditure of fuel.

Secondly, by causing the steam to act *expansively;* that is, by stopping the influx of steam when the piston has moved along part of the length of the cylinder, so that the remainder of the stroke is produced by the expansive power of the steam already admitted.

By these and other improvements in the machinery and economy of condensing steam engines, the efficiency of a bushel of coals has been carried much beyond the limit calculated above for atmospheric engines. In 1829 an engine in Cornwall, with a cylinder of 80 inches diameter, gave an actual duty of 75628000, and several others approached this amount.

We may thus consider 70 millions as below the actual duty of a bushel of coal, and 840000 as the actual duty of a pound of coal, working to great advantage in a condensing engine.

Hence one pound of coal is capable of raising itself through 840000 feet, or about 160 miles.

3. *High-pressure Engines.*

250. If a piston be moveable in a cylinder as before, and if steam of a greater elasticity than the atmosphere be admitted

on one side of the piston, the cylinder on the other side having an opening into the atmosphere, the piston will be moved to the open end. If then this end be closed and steam of the same kind admitted into it, the other end being opened, the piston will move back again; and by a continuation of this process an alternating motion may be produced.

Prop. *To find the theoretical duty of a High-pressure Engine.*

In order to determine the theoretical duty of a high-pressure engine corresponding to a given quantity of coal, it is necessary to know the quantity of steam, of the elasticity employed in the engine, which the given fuel would produce. Moreover it appears that in the transmission of high-pressure steam from the boiler to the cylinder a portion of its elasticity is lost in its passage through the valves.

If we know the elasticity of the steam in the cylinder, we may determine the theoretical efficiency, as in the following example.

In a locomotive engine the surface of the pistons was 127.2 square inches; the elasticity of the steam in the boiler 50lbs. per square inch more than the pressure of the atmosphere; the length of the stroke 2 feet; the diameter of the travelling wheels 37 inches: to find the efficiency expended in travelling 388 yards. (*Wood on Rail Roads*, p. 346.)

The pressure on the pistons is 6360lbs, and the space described by the piston in a double stroke is 4 feet; and hence the efficiency for one such stroke is 6360×4.

The diameter of the wheels being 37 inches, the circumference is 116.24. And the length of path described is $3 \times 388 \times 12$ inches: hence the number of revolutions of the wheel is $3 \times 388 \times 12 \div 116.24 = 120$ revolutions. And to each revolution corresponds a double stroke of the piston. Therefore the whole *efficiency expended* is $6360 \times 4 \times 120 = 3052800$, supposing the steam in the cylinders to be of the same elasticity as that in the boiler.

It appeared in the experiment to which this calculation refers that the total amount of the resistances was 1829lbs. which was moved over 3×388 feet: therefore the work done was $1829 \times 3 \times 388 = 2128956$.

The excess of the efficiency expended over the work done is to be attributed partly to resistances which have been left out of the account, and partly to the different elasticity of the steam in the boiler and in the cylinders.

In engines of this kind it was found that it required from 18 to 21lbs. of coal to evaporate a cubic foot of water.

251. The following may be taken as another example of this calculation. In a small high-pressure engine the cylinder was 8 inches in diameter, with a stroke of $4\frac{1}{2}$ feet: it worked a pump $18\frac{1}{2}$ inches diameter and $4\frac{1}{2}$ feet stroke, which raised water 28 feet high. The engine consumed 80lbs. of coal per hour, working 18 strokes per minute.

The section of the column of water $= 18.5 \times 18.5 \times .7854 = 268.8$ square inches; and the pressure of a column of water 1 foot high on a square inch is .434lbs. Hence the weight of the column of water is $268.8 \times .434 \times 28 = 3266.5$lbs.

The motion of the piston per minute is $4\frac{1}{2} \times 18 = 81$ feet, or 4860 feet per hour: and this multiplied by 3266.5 gives 15875190 for the hourly efficiency.

And dividing by 80 we have, for the efficiency of one pound of coal, 198439.5.

In this case, since the stroke of the pump and of the piston are equal, the pressures must be equal, in order that they may balance each other. Hence the pressure on the piston is 3266.5lbs; and since the area is $8 \times 8 \times .7854 = 50\frac{1}{4}$ square inches; the effective pressure per inch is 65lbs. This is the *excess* of the elasticity of the steam above that of the atmosphere.

In locomotive engines moving uniformly, the pressure is employed in maintaining the velocity; and since the resistance is independent of the velocity, the pressure on the piston is also independent of the velocity. Hence when the same engine moves with different velocities, the elasticity of the steam in the cylinder is nearly the same: the principal difference is the rapidity of its generation.

252. Prop. *To compare the efficiency of a given quantity of coal in a Locomotive and in an Atmospheric engine.*

It appears that the best modern locomotive engines on rail roads require half a pound of coal per ton per mile.

If we suppose the friction to be $\frac{1}{240}$ of the weight, the friction for a ton is $9\frac{1}{4}$ lbs: and in one mile the work done is $9\frac{1}{4} \times 5280 = 48840$ for half a pound of coal, or 97680 for one pound of coal.

The duty for one pound of coal in an atmospheric engine is, as we have seen, 431824, which is $4\frac{1}{2}$ times as much as the other.

It appears that the fuel consumed is very nearly proportional to the work done in locomotive engines, at whatever rate they travel. The speed depends upon the degree of rapidity with which the water can be converted into steam of the given elasticity. Hence the speed is increased by increasing the surface of the boiler which is exposed to the fire; and by increasing the draft of the furnace.

One mode of producing the latter effect which has been found very efficacious in practice, is the throwing the waste steam up the chimney when it is driven out of the cylinder. There appears to be theoretically no limit to the velocity which may thus be attained.

253. Practically speaking, the velocity of a locomotive engine is, as we have said, limited by the rate at which the water can be converted into steam. Hence we shall add the following proposition.*

* De Pambour on Locomotive Engines, p. 187.

PROP. *Given the rate of evaporation, and the pressure of the steam in the boiler, to find the velocity of the engine, moving uniformly.*

Let the whole resistance to the motion of the carriage, arising from the friction of the road and of the engine, be R; let D be the diameter of the driving wheels, C the diameter of each of the two steam cylinders, l the length of the stroke, m the pressure of the atmosphere on a square inch.

Let q be the pressure of the steam in the cylinder on a square inch: then, since the area of the two pistons is $\frac{1}{2}\pi C^2$, $(q-m)\frac{1}{2}\pi C^2$ is the effective pressure on the piston. And by Art. 236, the principle of vertical velocities may be here applied; hence

$$R \times \pi D = (q-m)\frac{1}{2}\pi C^2 \times 2l$$

$$q = \frac{DR}{C^2 l} + m.$$

Let p be the pressure of the steam per inch in the boiler:

Let s represent the rate of evaporation, that is, the number of cubic feet the boiler is able to evaporate in a minute at the pressure p:

And let n be the ratio of the volume of steam at the pressure p, to the volume of water.

Hence ns will be the volume of the steam generated in the boiler in a minute.

Then since q is the pressure of the steam in the cylinder, the steam, in going from the boiler to the cylinder, passes from the pressure p to the pressure q, and (by known principles) changes its volume in the inverse ratio of the pressures. Hence, in the cylinder, the volume of the steam will be $\frac{nsp}{q}$.

This volume of steam passes through the cylinders in a minute. Hence if we divide it by the section of the cylinders, or $\frac{1}{2}\pi C^2$, we shall have its mean velocity. Hence the mean velocity of the steam, and therefore of the piston, is

$$\frac{2nsp}{\pi C^2 q}.$$

Now the velocity of the carriage is to that of the piston as πD to $2l$: therefore the velocity of the carriage is

$$\frac{nspD}{C^2ql};$$

or putting for q its value, the velocity is

$$\frac{nspD}{DR + mC^2l}.$$

Ex. In an engine of which the load was 100 tons, the cylinders were 11 inches diameter, the stroke 16 inches, the wheels 5 feet; the effective pressure of the steam in the boiler 50lbs. per square inch, the effective evaporating power 41.87 cubic feet of water per hour, or 0.7 cubic feet per minute. Find the rate of travelling.

This water is immediately transformed in the boiler into steam at the effective pressure of 50lbs. per square inch, or at the total pressure of 65 lbs. per square inch.

According to tables founded on experiment, steam generated under a total pressure of 65 lbs. per square inch occupies 435 times the space of the water which produced it. Thus the water expanded in each minute formed a volume of steam of $0.7 \times 435 = 304$ cubic feet.

It appeared by calculations respecting the engine in question, that with a load of 100 tons, the resistance was 46lbs. per square inch of the piston. Hence the steam, in passing from the boiler to the cylinders, was reduced from the pressure 65 to 46. The volume being increased in the same ratio becomes

$$304 \times \frac{65}{46} = 430 \text{ cubic feet},$$

which passes through the cylinders every minute.

Now the area of the two cylinders is 190 square inches, or 1.32 square feet. Hence the above steam passes through the cylinders with a velocity of

$$\frac{430}{1.32} = 326 \text{ feet per minute},$$

which is the mean velocity of the piston.

The velocity of the carriage, is to that of the pistons in the proportion of the circumference of the wheel to the double stroke; that is, 5.887 to 1. Hence the velocity of the carriage is

$$5.887 \times 326 \text{ feet per minute; or}$$

$$5.887 \times 326 \times \frac{60}{5280} = 21.83 \text{ miles per hour.}$$

6. *Power of Men and Horses.*

254. The strength of man may be employed in doing work in various ways; two obvious cases are those in which weights are raised by the use of the arms and of the legs.

If a man go up a staircase, or up a steep hill, he lifts his own body, and any additional weight which he carries, by the action of his legs. In like manner a man working on a tread-mill raises his body at every step, as much as the wheel sinks.

The following experiments afford an estimate of the efficiency of a man.

(1) It is found that a man working on a tread-mill raises himself 10000 feet in the course of one day. He has here his weight only to raise: if we assume this to be 150 pounds, he daily exerts an efficiency of 1500000.

(2) To ascend a hill of 10000 feet high would be a good day's labour: this gives the same result.

(3) In an experiment of Coulomb, porters, mounting a convenient staircase 12 metres high, made 66 journies in a day, carrying 68 chilograms each time. This gives 4488 chilograms raised 12 metres, for the daily efficiency, together with the man's weight raised 792 metres. The man's weight being estimated at 70 chilograms, this gives for the daily efficiency $4488 \times 12 + 70 \times 792 = 109296$ chilograms raised one metre.

A metre is 3.28052 feet, and a chilogram 2.2063 pounds. Hence the daily efficiency estimated as before is 791139.

This amount is much less than the former. The carrying of weights of 68 chilograms (150 pounds) is therefore a less advantageous mode of employing the strength than the raising the weight of the body alone. The alternate ascent and descent also will diminish the estimated efficiency, since the last is not reckoned.

The following are examples of the efficiency of men working with their arms.

(4) In pile-driving the strength of man is employed in raising a ram which then falls by its weight. Thirty-eight labourers, working 10 hours a day, made in each hour 12 efforts, each effort consisting of 30 pulls; and in each pull a ram of the weight of 587 chilograms was raised 1.45 metres.

The daily number of pulls was 3600. Hence the effect of each workman was $\frac{587}{38} \times 1.45 \times 3600 = 80635.3$ chilograms raised 1 metre.

This reduced to pounds and feet is 583678.

(5) Workmen employed in turning a wheel exerted a force of 7 chilograms and made 20 turns in a minute, the circumference described by the force being 2.3 metres. The effective time of work was 6 hours.

Hence the number of turns is 7200, and the daily effect $7 \times 2.3 \times 7200 = 115920$ in chilograms and metres, or 839087 in pounds and feet.

(6) Desaguliers considers that a man can raise a weight of 550 pounds 10 feet high in a minute, and continue to do so for 6 hours. This gives $360 \times 550 \times 10 = 1980000$ for the daily efficiency.

According to this estimate the efficiency of a man in a minute is 5500: but this is considered by other authors too high.

(7) Smeaton thinks that a good labourer can raise 370 pounds 10 feet high in a minute; this gives 3700 for the efficiency per minute.

In the case of the tread-mill, [see (1)] if we suppose the time of working to be 8 hours; the efficiency per minute is 3125.

Horses.

255. According to Desaguliers a horse drawing a weight out of a well over a pully, can raise 200 pounds for 8 hours together, at the rate of $2\frac{1}{2}$ miles an hour. This gives, for the efficiency, 200 pounds raised twenty miles, or 105600 feet: whence the efficiency of such a day's work is 21120000.

If a horse draw a ton on a common road, at the rate of $2\frac{1}{2}$ miles an hour, we may suppose the friction to be the 12^{th} part of a ton, or 186 pounds: whence the efficiency in one minute is

$$\frac{2\frac{1}{2} \times 5280 \times 2240}{60 \times 12} = \frac{220 \times 2240}{12} = \frac{492800}{12} = 41066.$$

The usual estimate of the power of a horse as employed by engineers in their calculation is 33000 per minute.

Mr Smeaton states the efficiency of a horse to be 550 pounds raised 40 feet in one minute: this gives 22000 for the measure of a horse's power.

256. Prop. *To compare the power of animals moving with different velocities.*

The strength of men and of animals is most powerful, as pressure, when directed against a resisting object which is at rest: when the animal is in motion, the pressure which it can exert is diminished; and with a certain velocity the animal can do no work, and can only keep up the motion of its own body.

The following formula is given for the power of a man as modified by this cause. Let v represent his velocity in miles per hour: then the force which he exerts in dragging forwards a load is $\frac{4}{5}(6-v)^2$ in pounds.

Thus when v is 0, or the man is at rest, he pulls with a force of 29 pounds: when he moves at the rate of 2 miles an hour, his power of traction is reduced to 13 pounds: and if he quicken his pace to 4 miles an hour, he can only draw with a force of 3 pounds. His extreme velocity is 6 miles an hour.

The efficiency will be the pressure exerted, multiplied into the space described: and therefore in 1 hour, it is $\frac{4v}{5}(6-v)^2$.

Hence, making v respectively 0, 1, 2, 3, 4, 5, 6, we have the efficiency in these cases

v	0	1	2	3	4	5	6
efficiency	0	20	$25\frac{3}{5}$	$21\frac{3}{5}$	$12\frac{4}{5}$	4	0

Hence the efficiency is greatest when he moves at the rate of two miles an hour.

257. The power of traction of a horse may be expressed nearly by $(12-v)^2$ pounds, when v is the number of miles per hour, which the horse is moving. If the rate be 4 miles an hour, the power of traction would be 64 pounds. Also the efficiency would be this pressure into the space described. At 4 miles an hour, the space described in one minute is $\frac{5280}{15}$ or 352 feet; and therefore the efficiency in a minute is 22528. If the horse were to move with a velocity of 1 mile an hour, the efficiency would be, in like manner, $121 \times 88 = 10648$; and with a velocity of 10 miles an hour, the efficiency would be only 4×880 or 3520.

A waggon on a turnpike road, loaded to the amount of 8 tons, may be drawn by 8 horses at the rate of $2\frac{1}{2}$ miles an hour, the horses working for 8 hours daily. Thus the performance of a horse in this way will amount to 1 ton transported 20 miles a day; or if the friction be 1-12[th] of the load, this amounts to a pressure of $1\frac{2}{3}$ ton through a space of 1 mile; or in pounds and feet to 19712000 per day, and 41066 for one minute.

A mail coach weighing 2 tons and travelling at the rate of 10 miles an hour, may be worked on a line of road in both directions by a number of horses equal to the number of miles. The performance of each horse would amount to 2 tons drawn 2 miles daily, or 1 ton drawn 4 miles. Thus with this great velocity, the work done is only one-fifth of what it was in the other case. The horse-power in this case is 8215.

258. Prop. *To express the efficiency of animals by the equivalent quantity of coal employed in a steam-engine.*

Ex. It has appeared (Art. 249) that the efficiency of 1 pound of coal is 840000; and the daily efficiency of a man is 1500000 (Art. 254): hence such a day's work is equivalent to something less than 2 pounds of coal.

259. In order to compare the efficiency of a man or horse, working for one minute, with the efficiency of a steam-engine *for the same time*, we must know the rate of working of the engine.

The pressure on the piston, multiplied into the space described in one minute, will by the preceding principles, be the efficiency in one minute.

The pressure on the piston will be as the square of the piston, in inches, and as the pressure on one circular inch. If the force employed be that of the atmosphere, the available part of this pressure is 5.9 pounds. Hence the pressure on any piston may be found.

The space described by the piston in one minute, by the action of the force of the atmosphere, will be the length of the stroke, multiplied into the number of double strokes made per minute.

Ex. The diameter of the cylinder of an atmospheric steam-engine was 72 inches, and the length of the stroke 9 feet, 9 strokes being made per minute.

In this case the pressure on the piston = $72^2 \times 5.9$ pounds, and the space described in a minute = 9×9. Hence the efficiency per minute is $72^2 \times 5.9 \times 81 = 2477433.6$.

Dividing by 33000, we have 75 for the number of horses' power to which the efficiency of the machine is equivalent.

On the Effect of Springs on the amount of Work done.

260. When a carriage runs with a uniform velocity on a smooth horizontal road, it must be constantly drawn in the direction of the motion, by a force equal to the friction. If the carriage runs up an inclined plane with a uniform velocity, the force of traction must be equal to the sum of the friction and of the resolved part of the weight of the carriage. In these cases all the parts of the carriage move uniformly in parallel lines; and any change which takes place in the velocity may be supposed to be produced by a force applied at the axle of the wheel, the whole inertia of the carriage being also supposed to be collected at the point where the force is applied.

If a carriage, running along a smooth horizontal road, pass suddenly over a small obstacle, as a stone, the parts of the carriage no longer move in straight lines. If we conceive the parts to be perfectly stiff, and suppose the obstacle to be of such a form that the wheel, in passing over it, touches it in one point only, it is evident, that the center of the wheel will, during this passage, describe a circular arc of which the point of contact is the center; the center of the wheel will therefore describe a path in which it passes suddenly from a straight line to this circular arc, and then, the obstacle being surmounted, resumes the straight line.

If the whole carriage be perfectly stiff, a portion of the velocity will be lost in thus passing on such an obstacle; and in order to keep up the velocity an additional force will be requisite.

261. Prop. *To find the efficiency expended in keeping up the velocity of a stiff carriage which surmounts a small rise on a horizontal road.*

Let a be the velocity, r the radius of the wheel, and h the height of the obstacle. When the wheel comes in contact with the obstacle, the radius drawn to the point of contact, makes with the vertical line an angle of which the versed sine is h. And this radius then begins to move about the point of contact, whence it appears that the new path of the center of the wheel makes with the horizontal line an angle of which the versed sine is h to radius r. Let this angle be α.

The center of the wheel having its direction suddenly changed from a horizontal direction, to a direction making an angle α with the horizon, loses a portion of its velocity: the velocity which it retains is $a \cos. \alpha$ (Art. 154. Cor. 5). Let e be the uniform resistance, arising from friction, &c. on the horizontal road, and let f be the force which, when the carriage passes over the obstacle, acts in the direction of the motion, and which is such, that the velocity when the obstacle has been surmounted is the same as it was before meeting the obstacle.

Suppose m to be the inertia of the carriage, collected at the center of the wheel, and suppose the resistance arising from friction, &c. to be the same in the curvilinear as in the horizontal portions of the path. At any point of the curvilinear path, there will be a resolved portion of the force of gravity acting in the direction of the motion. Let g' be this resolved portion: then the whole accelerating force in the ascent will be $\frac{f-e}{m} - g'$.

And the change produced in the velocity, during each elementary portion of the motion, will be the difference of the acceleration arising from the constant force, $\frac{f-e}{m}$, and of the retardation arising from the variable force g. Let u be the velocity at the highest point of the curvilinear path,

l the length from the limit of the curve to the highest point. By Art. 163, Cor. 3, the square of the velocity in passing through the space l is altered by the quantity $2\frac{f-e}{m}l$, by the action of the force $\frac{f-e}{m}$; and by Art. 191, Cor. 2, since h is the vertical height of the highest above the lowest point, the square of the velocity is altered by the quantity $2gh$, by the action of gravity. Therefore, since $a\cos. \alpha$ is the velocity at the lowest point,

$$u^2 = a^2 \cos.^2 \alpha + 2\frac{f-e}{m} l - 2gh.$$

Now if the velocity at the highest point is required to be equal to the velocity before contact with the obstacle, u is equal to a. Also

$$\cos. \alpha = \frac{r-h}{r}, \text{ whence } \cos.^2 \alpha = 1 - \frac{2h}{r} + \frac{h^2}{r^2};$$

$$\text{hence } a^2 = a^2\left(1 - \frac{2h}{r} + \frac{h^2}{r^2}\right) + 2\frac{f-e}{m} l - 2gh.$$

$$\text{And } (f-e)\, l = mgh + m\left\{a^2\frac{h}{r} - a^2\frac{h^2}{2r^2}\right\}.$$

But the horizontal path corresponding to the length of path l, is $r \sin. \alpha$; and l is the arc of which the horizontal path is the sine to radius r; and l is therefore $r\alpha$. Hence the requisite efficiency, if the road were perfectly horizontal with the same friction e, would be $er \sin. \alpha$.

Therefore the excess of the requisite efficiency in order to keep up the velocity is

$$fl - er\sin. \alpha = l(f-e) + er(\alpha - \sin. \alpha),$$

$$\text{or since } \alpha = \sin. \alpha + \frac{\sin.^3 \alpha}{6} + \&c.$$

$$\text{the excess is } l(f-e) + \frac{er}{6}\sin.^3 \alpha + \&c.,$$

and since α is small, $\sin.^3 \alpha$, &c. may be neglected.

Therefore the excess of requisite efficiency

$$= (f - e)\, l = mgh + m \left\{ a^2 \frac{h}{r} - a^2 \frac{h^2}{2r^2} \right\}.$$

The first term on the second side is the portion of this efficiency, which arises from the amount of the ascent; the second is the portion which arises from the suddenness of the change of direction, or from the *jerk*.

262. Prop. *To compare the two portions just mentioned, of the efficiency requisite to surmount an obstacle.*

The proportion of these two portions of the efficiency is

$$gh : a^2 \frac{h}{r} - a^2 \frac{h^2}{2r^2}; \text{ or } gr : a^2 - \frac{a^2 h}{2r^2}.$$

If h be small compared with r, this proportion becomes $gr : a^2$; and the hindrance arising from weight will be less than the hindrance arising from jerk, so long as gr is less than a^2; that is, so long as the velocity of the carriage is greater than the velocity acquired by falling down half the radius of the wheel.

Ex. A carriage travelling 10 miles an hour with wheels 4 feet diameter, surmounts a sudden rise in the road: compare the hindrance arising from jerk and from weight.

The velocity is $\frac{5280 \times 10}{60 \times 60}$ or 14 feet a second: and the proportion is therefore $14^2 : 32 \times 2$, or 3 to 1 nearly.

263. The hindrance arising from jerk may be removed in a great measure by the use of springs. In the part of the carriage which is suspended in springs there is no *sudden* change of the direction of the motion of the parts. When the wheel meets the obstacle, the suspended part is acted on by the pressure of the springs, and the path of its center of gravity thus deviates from a straight line; but it describes a curve to which the rectilinear portion of the path is a tangent, and there is no finite angle between two successive portions of the path. Hence the diminution of the velocity

from a to a cos. a, does not take place in the suspended part. If we suppose the action of the springs to be always perpendicular to the path described by the center of the suspended part, there will be no loss of velocity in consequence of the change of direction of the motion of that part. And the use of springs will on this supposition render unnecessary all that portion of the efficiency requisite to surmount the obstacle, which arises from the suddenness of the effect; which is, as we have seen, much the larger part.

264. If the carriage pass over the obstacle, and on leaving it go on in a continuation of its first path, some additional considerations are requisite.

PROP. *To find the efficiency expended on a stiff carriage in passing over a small obstacle in a smooth horizontal road.*

Let h be the height of the obstacle, and the rest of the notation as in Art. 259. We have for the ascent, as before,

$$u^2 = a^2 \cos.^2 a + 2\frac{f-e}{m} l - 2gh.$$

And if v be the velocity at the bottom of the descent, we have in like manner,

$$v^2 = u^2 + 2\frac{f-e}{m} l + 2gh = a^2 \cos.^2 a + 4\frac{f-e}{m} l.$$

The length of the arc of descent is the same as the length of the arc of ascent; and the angle a which these arcs make with the horizontal line is the same at the two extremities. The velocity v, when the motion of the carriage resumes the horizontal direction, is reduced by the change of direction to v cos. a. But it is required to be the same as at first. Therefore $v \cos. a = a$. And

$$a^2 = v^2 \cos.^2 a = a^2 \cos.^4 a + 4\frac{f-e}{m} l \cos^2. a.$$

$$\text{Hence } 2\,(f-e)\,l = m\,\frac{a^2}{2}\cdot\frac{1-\cos.^4 a}{\cos.^2 a}.$$

If, as before, we put $1 - \frac{h}{r}$ for $\cos. a$, h being small: we have

$$2(f - e)\, l = m \,.\, a^2 \frac{2h}{r}.$$

And, as before, this is the efficiency expended in passing over the obstacle, in addition to that which would be requisite if there were no obstacle.

In this case the hindrance arising from weight disappears, the carriage being as much accelerated by gravity in descending as it had been retarded in ascending.

265. If the road be paved, it may be considered as a row of obstacles close together, and the carriage never moves in a horizontal line. In this case we may suppose that the center of the wheel describes a series of circular arcs of which the centers are the summits of the paving-stones over which the wheel passes. Let it be supposed that these arcs are all equal, and that each of them makes an angle a with the horizontal line at its extremity. Then the center of the wheel in passing from one arc to another, changes its direction suddenly by an angle $2a$. In general a will be small.

Prop. *A stiff carriage travels along a horizontal regularly paved road; to find the efficiency, in addition to that due to the friction, requisite for uniform progression.*

Let u be the velocity at the highest point of each arc; the rest of the notation as before. Then

$$v^2 = u^2 + 2\frac{f - e}{m}\, l - 2gh.$$

And since the carriage, beginning the next ascent with the velocity $v \cos. 2a$, has, at the next summit, the same velocity u,

$$u^2 = v^2 \cos.^2 2a + 2\frac{f - e}{m}\, l - 2gh.$$

$$\text{Hence } v^2 = v^2 \cos.^2 2a + 4\frac{f-e}{m} l;$$

$$2(f-e)l = \tfrac{1}{2} m v^2 \sin.^2 2a.$$

And the excess of efficiency requisite is, as before,

$$2fl - 2er \sin. a = 2(f-e)l + 2er(a - \sin. a)$$
$$= \tfrac{1}{2} m v \sin.^2 2a + 2er(a - \sin. a),$$

$$\text{or, since } a = \sin. a + \frac{\sin.^3 a}{6} + \&c.$$

$$\text{excess} = 2 m v^2 \sin.^2 a \cos.^2 a + er\frac{\sin.^3 a}{6},$$

or, neglecting $\sin.^2 a$ and $\sin.^4 a$ as small,

$$\text{excess} = 2 m v^2 \sin.^2 a.$$

If we neglect $\frac{h^2}{r^2}$ as small, $\sin. a$ is $\sqrt{\frac{2h}{r}}$, and v^2 is equal to u^2; hence excess of efficiency $= \frac{2 m u^2 h}{r}$.

Cor. The efficiency which becomes requisite in consequence of the roughness of a road, is as the square of the velocity, and as the height of the paving-stones directly, and the radius of the wheels inversely.

266. The efficiency thus wanted is rendered less by the use of springs, by means of which the sudden change of direction of the motion of the suspended portion is avoided. If the springs be extremely flexible and perfectly elastic, the hindrance arising from the successive jerks will be entirely removed, so far as the suspended portion is concerned.

267. This is the effect of springs as connected with the subject of the present chapter. An effect however, no less important, is the preservation of the materials both of the carriage and of the roads, from the destruction arising from the collision of hard bodies. The following chapter shews the mode of calculating the effects of collision in some cases.

CHAP. X.

ON THE CONNEXION OF PRESSURE AND IMPACT.

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268. In Art. 128, it was observed that impact is really a pressure of short duration. The duration of the pressure depends, *cæteris paribus*, upon the materials of which the impinging bodies are composed, viz. upon their *hardness*. Various results are connected with the changes of this element, some of which we shall here consider.

Suppose a hammer moving with a considerable velocity impinges on a hard block. The block sustains a very great force, and the magnitude of this force depends *cæteris paribus* on its own hardness, and the hardness of the hammer. An iron hammer produces a greater effect than a soft ball of worsted; and an iron anvil sustains a greater force than a soft pillow. How is this to be accounted for? The answer is not difficult. The momentum of the hammer must be destroyed by a finite force continued for a finite time; and the shorter the time, the greater must be the force. But the time will evidently be shortest with those bodies which undergo the least compression from a given force, since the time in which the momentum of the hammer is destroyed, begins at the instant of the first contact, and terminates when the center of the impinging body is nearest to that of the body struck, that is, when the sum of their compressions has attained its greatest value. The less, then, that this compression is, the less will be the space that the impinging body describes before its momentum is destroyed, and therefore the greater will be the force which will have resisted it.

269. To subject this to mathematical calculation, it will be convenient to assume some law connecting the compression of a body with the force with which it endeavours to return to its former state. As one of the most probable hypotheses, suppose this force to be proportional to the compression: then if x be the space which the surface of the struck body has yielded, $\frac{x}{a}$ will be the force necessary to keep it in that state, or the force which it is then sustaining, a being a constant coefficient which is different for every different body.

In the same manner if b be the constant in the impinging body, and y the space through which its surface has yielded, $\frac{y}{b}$ will be the force which it is exerting.

270. Prop. *When one body impinges on another, the force exerted is greater in proportion as the bodies are harder.*

Let as above $\frac{x}{a}$ and $\frac{y}{b}$ be the forces exerted by the striking body, and the body struck. These must be equal, or $\frac{x}{a} = \frac{y}{b}$.

Let H be the weight of the striking body or *hammer*, V its velocity at the instant of first contact: s the space described by its center, since that time. This space is described in consequence of the compression of the two bodies, therefore $s = x + y$.

But since $\frac{x}{a} = \frac{y}{b}$, $y = \frac{bx}{a}$, $\frac{a+b}{a}x = x + y = s$;

$$\therefore x = \frac{as}{a+b}; \quad \text{similarly } y = \frac{bs}{a+b}.$$

Also if g be the force of gravity, $\frac{gy}{b}$ is the *pressure* on H; and the force which retards H is $\frac{gy}{Hb}$, or $\frac{1}{a+b} \cdot \frac{gs}{H}$.

Hence by the equation $v\dfrac{dv}{ds} = f$, Art. 117, we have (v being the velocity of H),

$$v\frac{dv}{ds} = \frac{1}{a+b}\cdot\frac{gs}{H}; \quad v^2 = C - \frac{1}{a+b}\cdot\frac{gs^2}{H};$$

and since $v = V$ when $s = 0$, $v^2 = V^2 - \dfrac{gs^2}{H(a+b)}$.

$$\text{When } v = 0, \quad s = V\sqrt{\frac{(a+b)H}{g}};$$

and the pressure at this moment $= \dfrac{y}{b} = \dfrac{s}{a+b} + \dfrac{V\sqrt{H}}{\sqrt{(a+b)g}}$.

At this moment the whole motion of the hammer is destroyed, and the compression is greatest and the force greatest; *i. e.* the force increases from the first contact, when it is 0, till it is

$$\frac{V\sqrt{H}}{\sqrt{(a+b)g}}.$$

Now the harder the bodies are, the greater is the force for a given compression, and the smaller are a and b. Hence this force is greater as the bodies are harder.

Cor. Upon the law here assumed, it appears that the force exerted is, *cæteris paribus*, as the square root of the weight of the hammer.

Since the body struck was in the preceding case supposed to be kept at rest by an immoveable obstacle, this is the greatest pressure which can take place from the impact of H upon the other body. If the other body can move, it is evident that the force will not be so great. We will consider the effect of impact in moving a body in opposition to a uniform force; and first we will consider the body struck as being so small that its weight and inertia may be neglected. This will be nearly the case of a hammer driving a nail, the friction being supposed uniform.

271. Prop. *To find how far a given hammer will drive a nail.*

Taking the same letters as before, and putting F for the friction, it is evident that the nail will not stir till the compression of the nail and hammer is sufficient to cause a force $f = F$. This will be the case when

$$\frac{y}{b} = \frac{1}{a+b} s = F, \text{ or } s = (a+b) F.$$

Hence by the last Article, if v be the velocity of the hammer, when the nail begins to move,

$$v^2 = V^2 - \frac{gs^2}{H(a+b)} = V^2 - \frac{g}{H}(a+b) F^2.$$

We must now consider that the hammer and nail move on, resisted by the uniform force (pressure) F, till the momentum of the hammer is destroyed. Let s' be the space through which they move: the retarding force being

$$\frac{Fg}{H},$$

$$\therefore \text{ (Art. 162) } s' = \frac{v^2}{2\frac{Fg}{H}} = \frac{H}{2g}\left(\frac{V^2}{F} - \frac{g}{H}\cdot(a+b)F\right).$$

This space will $= 0$, or the blow will not move the nail at all, if

$$V^2 = \frac{g}{H}\cdot(a+b) F^2, \text{ or less.}$$

If V be very great, the space through which the nail moves will be almost independent of the hardness: but if V be barely sufficient to move the nail, a small increase in the hardness of the hammer or nail will much increase the space through which it moves. If while friction prevents the nail from moving *in* the block into which it is driven, it move *with* the block on account of the yielding of the block, this must evidently produce the same effect as the softness of the nail, that is, b will be increased; hence it will be driven farther (*cæteris paribus*) into a substance that does not yield than into one that does.

Since at the limit of motion $V^2 H = gF^2 (a+b)$, when the bodies are very hard, or a and b very small, a small hammer and small velocity will produce the effect of a very great pressure F. Thus it is found that a sledge hammer driving hard oak pegs produces as great an effect as a pressure of 70 tons.

Suppose now we consider the weight of the body moved, as in this example.

272. Prop. *A pile, of weight* W, *is driven by a hammer, or a ram* H, *impinging with a velocity* V: *the friction being represented by* F, *to find the motion.*

We will here take account of the weight as well as the momentum of the ram. Let s' be the space through which the aggregate compression takes place before the pile moves. Then, as in the last Article, this will occur when the force downwards = the resistance, or $\frac{y}{b} + W = F$. Hence

$$\frac{s'}{a+b} = \frac{y}{b} = F - W,\ \ s' = (a+b)(F-W).$$

Also as before

$$v\frac{dv}{ds} = -\frac{1}{a+b}\frac{gs}{H} + g,$$

$$\therefore v^2 = V^2 - \frac{1}{a+b}\frac{gs^2}{H} 2gs,$$

because when $s = 0$, $v = V$.

And if V' be the velocity of the ram when the pile begins to move, putting s' for s,

$$V'^2 = V^2 - \frac{g}{H}(a+b)(F-W)^2 + 2g(a+b)F - W \ldots\ldots (1).$$

Now as the inertia of the pile resists the communication of motion, the compression will still go on increasing after this time. Let r be the space through which the ram has moved since the first contact; p that through which the pile has moved. Then $p = r - s$, and $s = r - p$.

For the motion of the ram, (Art. 117.)

$$\frac{d^2 r}{dt^2} = g - \frac{s}{a+b} \cdot \frac{g}{H};$$

and for that of the pile,

$$\frac{d^2 p}{dt^2} = g + \frac{s}{a+b} \frac{g}{W} - \frac{Fg}{W};$$

subtracting,

$$\frac{d^2 s}{dt^2} = -\frac{gs}{a+b}\left(\frac{1}{H} + \frac{1}{W}\right) + \frac{Fg}{W}.$$

$$= -n^2 s + \frac{Fg}{W}, \text{ where } n^2 = \frac{g}{a+b} \cdot \frac{H+W}{HW};$$

$$\therefore s = \frac{Fg}{n^2 W} + A \cos. nt + B \sin. nt.$$

Suppose $t = 0$ when the pile begins to move: then, as has already been shewn, at this point of time $s' = (a+b)(F-W)$; hence

$$(a+b)(F-W) = \frac{Fg}{n^2 W} + A = (a+b)\frac{FH}{H+W} + A;$$

$$\therefore A = (a+b)\left(\frac{FW}{H+W} - W\right).$$

Again, when $t = 0$, the velocity of the pile $= 0$ and that of the ram $= V'$. Therefore

$$\frac{ds}{dt} = \frac{dr}{dt} - \frac{dp}{dt} = V' - 0 = V'.$$

But generally

$$\frac{ds}{dt} = -An \sin. nt + Bn \cos. nt,$$

$$\therefore Bn = V', \quad B = \frac{V'}{n}.$$

Hence, substituting for A, B and n^2,

$$s = (a+b)\frac{FH}{H+W} + (a+b)\left(\frac{FW}{H+W} - W\right)\cos. nt + \frac{V'}{n}\sin. nt.$$

And substituting this,

$$\frac{d^2p}{dt^2} = g\left(1 - \frac{F}{H+W}\right)(1-\cos. nt) + \frac{V'g}{(a+b)nW}\sin. nt,$$

or, putting $g\left(\frac{F}{H+W} - 1\right) = m$,

$$\frac{d^2p}{dt^2} = -m + m\cos. nt + \frac{V'g}{(a+b)nW}\sin. nt.$$

Integrating twice, making $\frac{dp}{dt}$ and p both $= 0$ when $t = 0$, we have

$$\frac{dp}{dt} = -\frac{m}{n}(nt - \sin. nt) + \frac{V'g}{(a+b)n^2W}(1-\cos. nt)$$

$$p = \frac{m}{n^2}\left(1 - \tfrac{1}{2}n^2t^2 - \cos. nt\right) + \frac{V'g}{(a+b)n^3W}(nt - \sin. nt) \ldots\ldots (2).$$

Now when the pile ceases to move, we shall have $\frac{dp}{dt} = 0$.

Hence $\frac{V'g}{(a+b)mnW}(1-\cos. nt) + \sin. nt - nt = 0 \ldots (3)$,

whence t is determined. Also t being known, we have p, the whole space through which the pile is driven.

273. If we suppose V' to be small, that is, the velocity to be such as only just to stir the pile, we may approximate. In this case t, the time during which the pile moves, will be very small. Hence, expanding $\sin. nt$ and $\cos. nt$ in (3), and taking only the lowest two terms;

$$\frac{V'g}{(a+b)mnW}\cdot\frac{n^2t^2}{2} + \frac{n^3t^3}{2\,.\,3} = 0;$$

$$\therefore t = \frac{3V'g}{(a+b)n^2W},$$

or putting for $n^2(a+b)$ and m their values,

$$t = \frac{3V'H}{g(F-H-W)}.$$

Hence find p from (2), again taking the lowest term only;

$$p = \frac{V'g}{(a+b)n^3W} \cdot \frac{n^3t^3}{2 \cdot 3}$$

$$= \frac{gH^3}{2(a+b)W(F-H-W)} \cdot \frac{V'^4}{g^2}.$$

From what precedes we may draw the following conclusions:

1st. If the ram will just stir the pile, V' in (1) is small, and a slight increase of the hardness of the ram or pile (i. e. of a or b), or of the weight of the ram (i. e. of H), will very much increase V', and thus increase very considerably the distance p to which the pile is driven.

2d. The resistance of F being supposed great, the space p will be very nearly inversely as W the weight of the pile; consequently the lighter the pile is *cæteris paribus* the faster it will be driven.

3d. The space p varies directly as the cube of the weight of the ram, the velocity with which the pile begins to move being given. And since this velocity itself is much increased by increasing the ram, there is on both accounts a great advantage in making the ram heavy.

274. The quantities a and b in the preceding investigations depend upon the hardness and elasticity of the substances striking and struck. If we suppose that a body of which the diameter is D would, by the action of a force E, undergo a contraction or expansion D, in linear dimensions (the law by which the contraction x takes place being supposed to extend to this case;) we shall have $E = \frac{D}{a}$, and $a = \frac{D}{E}$.

Hence the force for a compression x is $\frac{Dx}{D}$. And if the force

E be expressed by means of the weight of a column of the substance itself of height E, E will be the same thing as the *modulus of elasticity*, in Analytical Statics, Art. 79. The harder the bodies the greater is this modulus, and the less is a.

The modulus of elasticity of iron or steel is about 9000000 feet. By means of this value we can obtain numerical values in the preceding propositions.

PROB. I. An iron hammer strikes an anvil with a velocity acquired down a height H; to find the compression.

The weight of the hammer H may be expressed by means of a column of iron of which the base is the surface in contact with the anvil. If the hammer be a parallelopiped, H will be its height. Also $V^2 = 2gh$. And if A be the diameter of the anvil in the direction of the stroke, we have by Art. 3 of this Chapter,

$$\left(\text{since } a = \frac{A}{E},\ \ b = \frac{H}{E}\right),$$

$$s = \sqrt{2h}\sqrt{\frac{(A+H)\,H}{E}}.$$

Thus if the hammer be $\frac{1}{4}$ foot high, and fall on an anvil 2 feet high from a height of 8 feet.

$$s = \sqrt{16}\sqrt{\frac{\frac{9}{4}\cdot\frac{1}{4}}{9000000}}$$

$$= \frac{1}{1000} \text{ of a foot},$$

which is the space through which both have been compressed; and the hammer and anvil share this compression in the proportion of 1 : 8.

Also to find the greatest pressure in this case, we have by the formulæ in p. 321.

$$\text{pressure} = \frac{\sqrt{2h}\,\sqrt{HE}}{\sqrt{A+H}}$$

$$= \frac{\sqrt{16} \times \sqrt{\frac{1}{4}} \times 9000000}{\sqrt{\frac{9}{4}}}$$

$$= 4000.$$

Hence in this case the pressure is equal to that of a column of iron 4000 feet high.

If we suppose the face of the hammer to be a square inch, the pressure will be 48000 inches of iron, which is above 12000 pounds.

PROB. II. The friction of a nail in wood being equal to the weight of a column of iron having its base the surface of the nail-head in contact with the hammer, and its height F; to find the space through which the nail is driven.

Using the same letters as in the last problem (A being now the length of the nail) we have by Art. 4 of this Chapter,

$$s' = \frac{Hh}{F} - \frac{1}{2}\frac{(A+H)F}{E};$$

or since A is small in comparison with H,

$$s' = \frac{Hh}{F} - \frac{1}{2}\frac{HF}{E}.$$

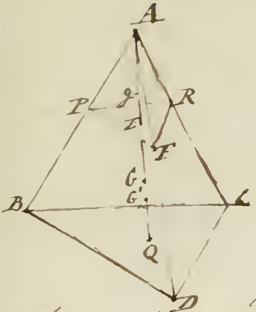
Hence the space through which the nail is driven is, by the defect from absolute hardness, diminished by the quantity $\frac{1}{2}\frac{HF}{E}$.

275. The same principles give easy and curious results in such problems as these. To find the effect of impact on the wedge, friction being taken into account. To find the weight which dropped from a given height into the empty scale of a balance, will raise any weight in the other, &c.

It is evident that impact cannot be conveniently used to overcome any continuous force, except that force cease as soon as the motion ceases. This is the case with the resistance of friction, and some others: and in these instances the effect of impact is greater than that of any pressure which it would be practicable to employ. A construction which is rather a modification than a direct application of this principle is used in punching iron plates for fenders, and a similar one is employed in coining. The long lever which turns a screw is loaded at its ends with heavy weights; a man gives it a considerable velocity; and when the screw is suddenly stopped by the punches, the force impressed is enormous. In the same way (though by a construction rather different) the holes in the nuts of screwbolts are punched in iron bars sometimes $\frac{3}{4}$ inch thick. All these, as well as the simplest cases of force produced by impact, depend on the same principles, viz. that to destroy momentum in a short time a great force is necessary.

The same principle may be used to explain some facts observed by workmen, which at first sight appear very strange. It is found that if a cylindrical hole be made in a block of granite, and an iron rod driven into it, of such a size that a few blows with a hammer will overcome the friction, the block may be raised by this rod: but if the same process be used with a block of soft stone, the block cannot be raised by it. The reason appears to be this; the granite yields so little that a blow of the hammer overcomes a very great friction; whereas in the soft stone, except the friction be very small, the iron yields with the stone to the blow of the hammer, and the friction which is really overcome in the granite, and which sustains the granite when it is raised, is much greater than that which the same blows overcome in the soft stone.

THE END.



$BC = a,\ PR = b,\ EQ = c$

Let G be the centre of gravity of the pyramid $ABCD$, and let g be the centre of gravity of the pyramid $APRE$.

Then the pyramids may be considered to be collected at these points. Then in Gg produced take $Gg : GG' :: (a^3 - b^3) : b^3$

Then G' is the centre of gravity of the frustum.

$$QG' = QG - GG'$$

but $GQ = \frac{1}{4}AQ$ & $AQ : AE :: a : b$

$$AQ : EQ :: a : a - b$$

but $EQ = c$ $\therefore AQ = \frac{ac}{a-b}$ $\therefore GQ = \frac{ac}{4(a-b)}$

$$GG' = \frac{Gg \cdot b^3}{a^3 - b^3}$$

but $Gg = AG - Ag = \frac{3}{4}AQ - \frac{3}{4}AE = \frac{3c}{4}$

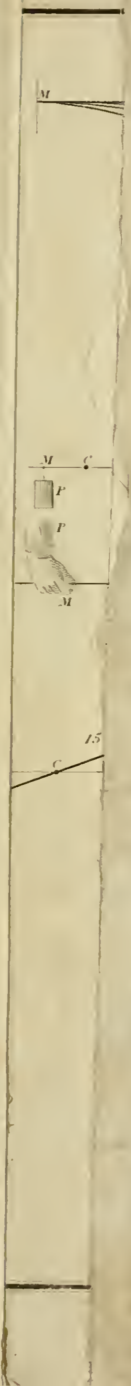
$$\therefore GG' = \frac{c}{4} \cdot \frac{3b^3}{(a^3 - b^3)}$$

$$\therefore QG' = \frac{c}{4}\left\{\frac{a}{a-b} - \frac{3b^3}{a^3-b^3}\right\}$$

$$= \frac{c}{4}\left\{\frac{a(a^2+ab+b^2) - 3b^3}{(a^3-b^3)}\right\}$$

$$= \frac{c}{4}\left\{\frac{a^3+a^2b+ab^2-3b^3}{(a^3-b^3)}\right\}$$

$$= \frac{c}{4}\left\{\frac{a^2+2ab+3b^2}{a^2+ab+b^2}\right\}$$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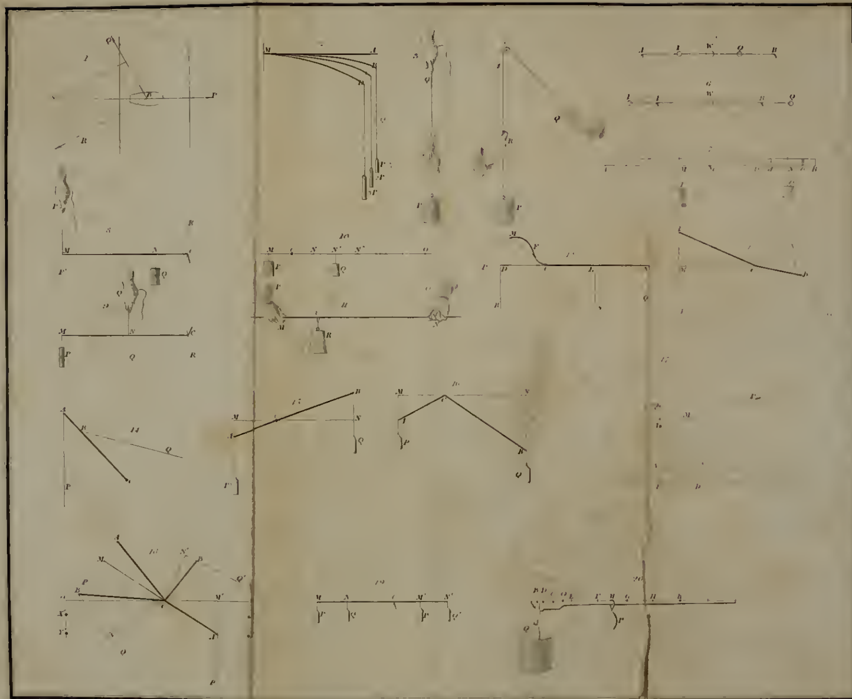

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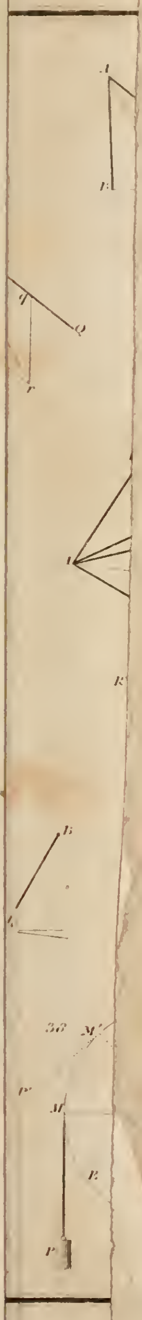

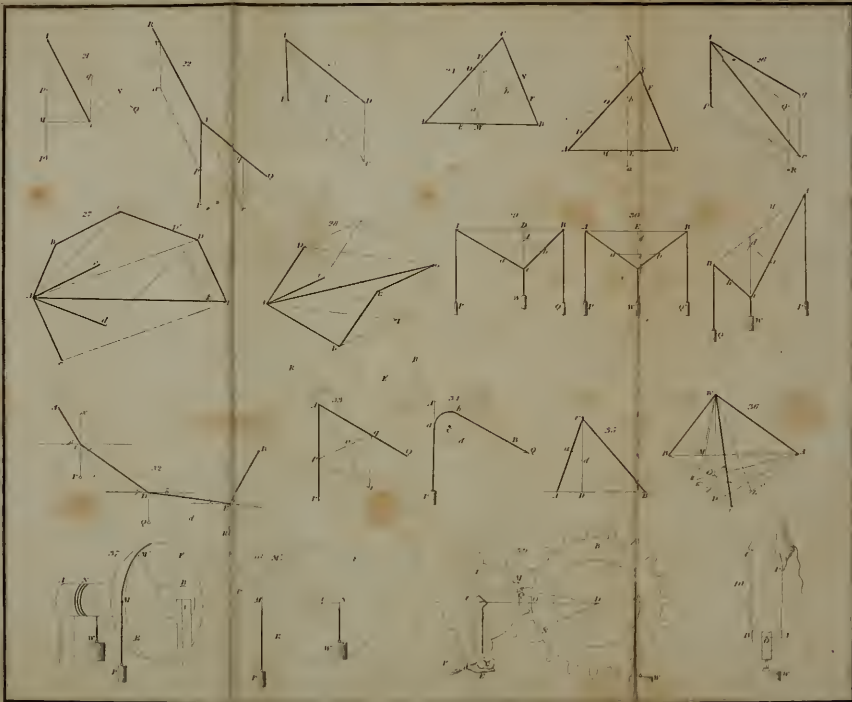

Whewell's Elementary Mechanics

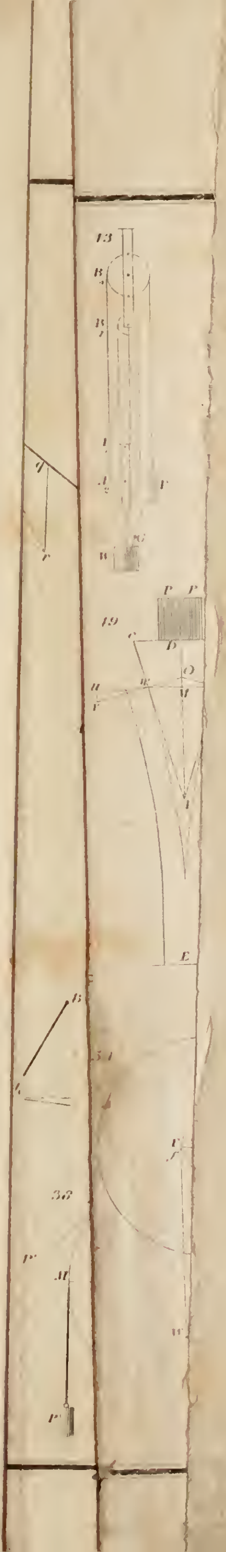
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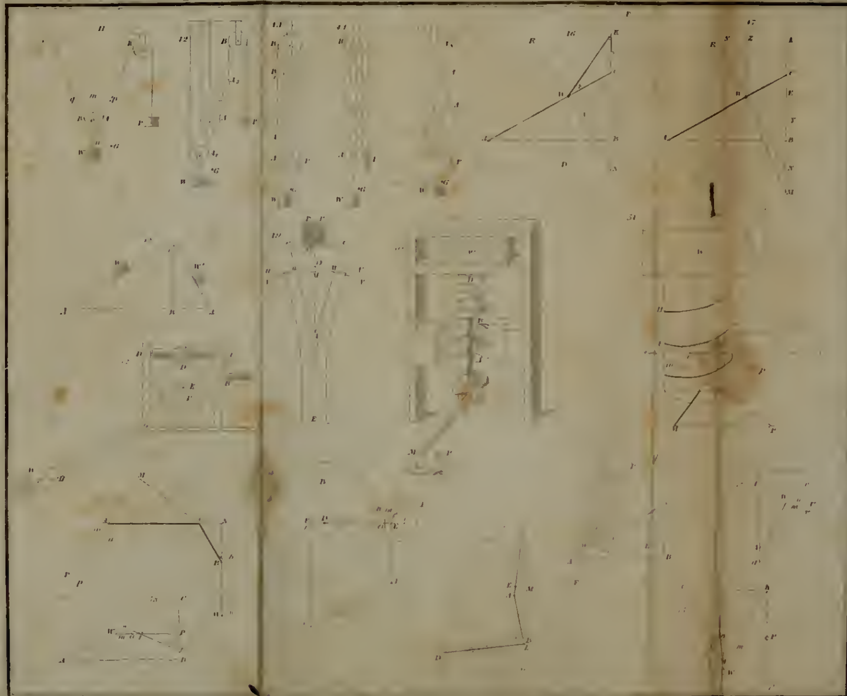
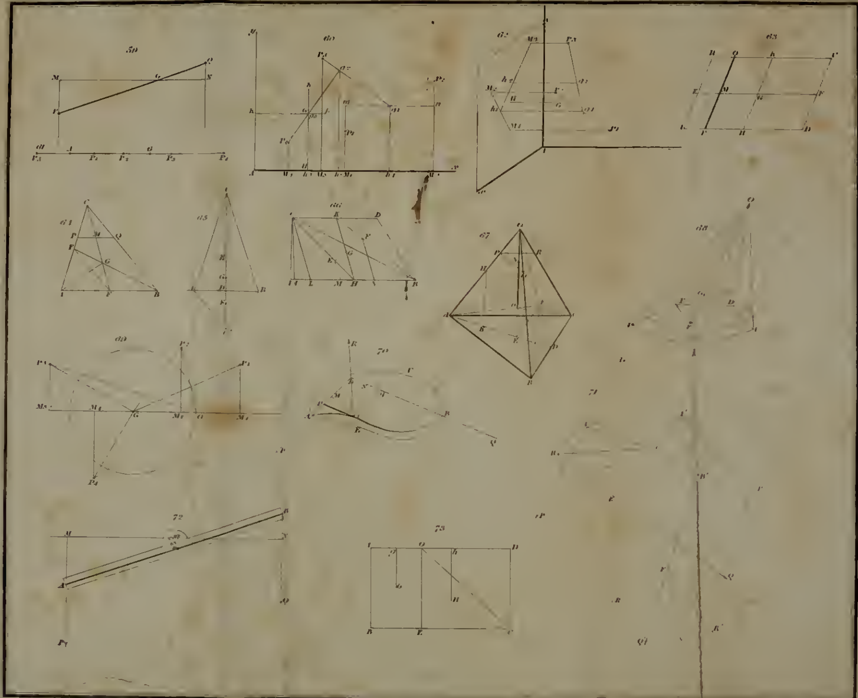
PLATE III





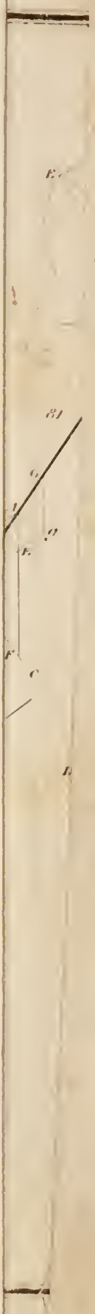
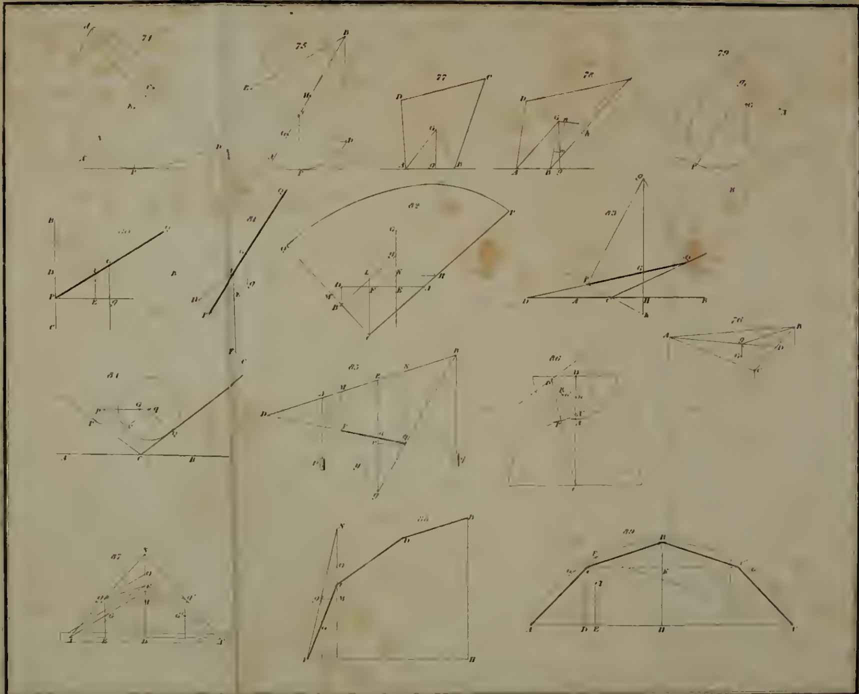
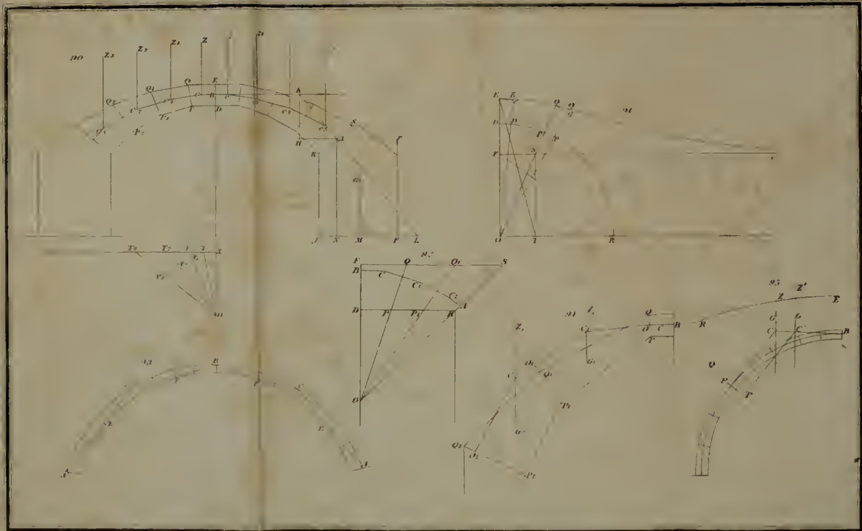


PLATE V.



Whewell Elementary Mechanics

PLATE VI



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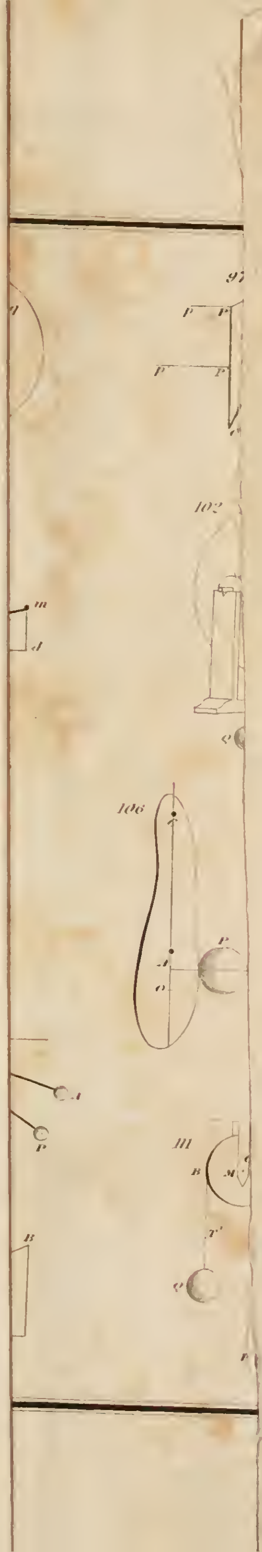
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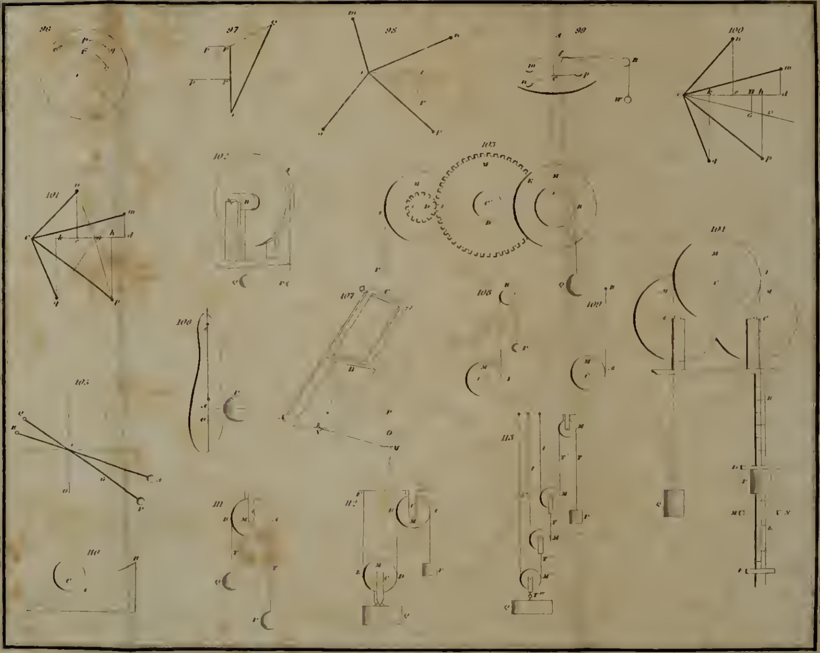

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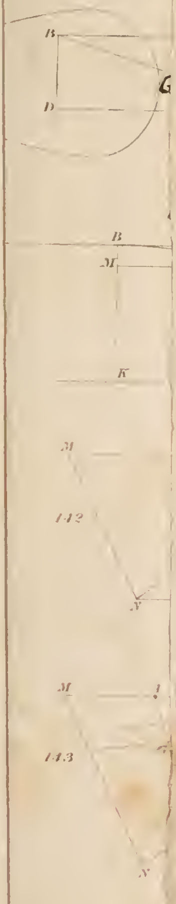
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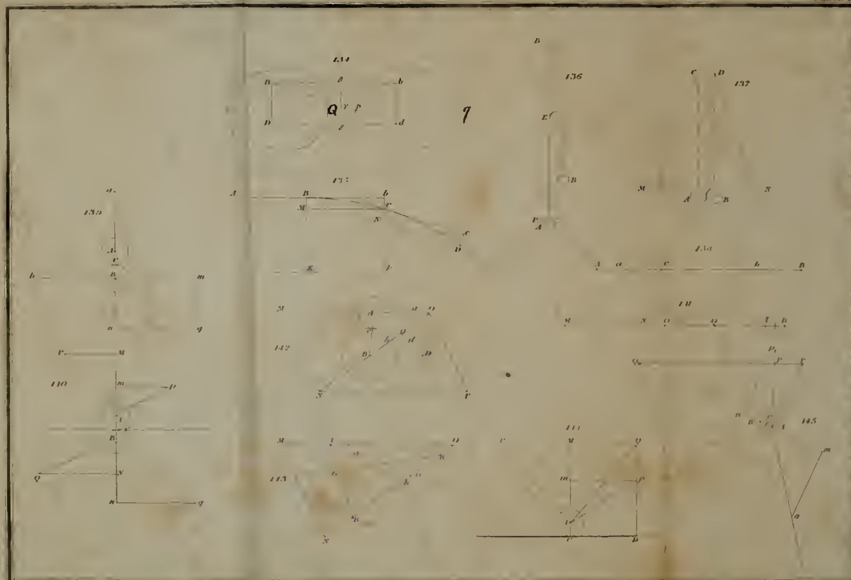
PLATE VIII



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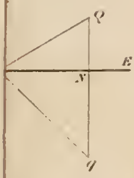

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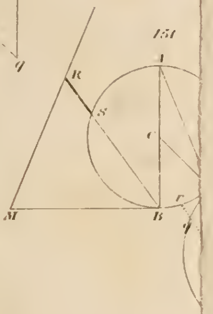

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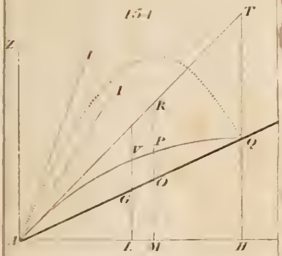

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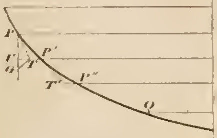

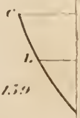

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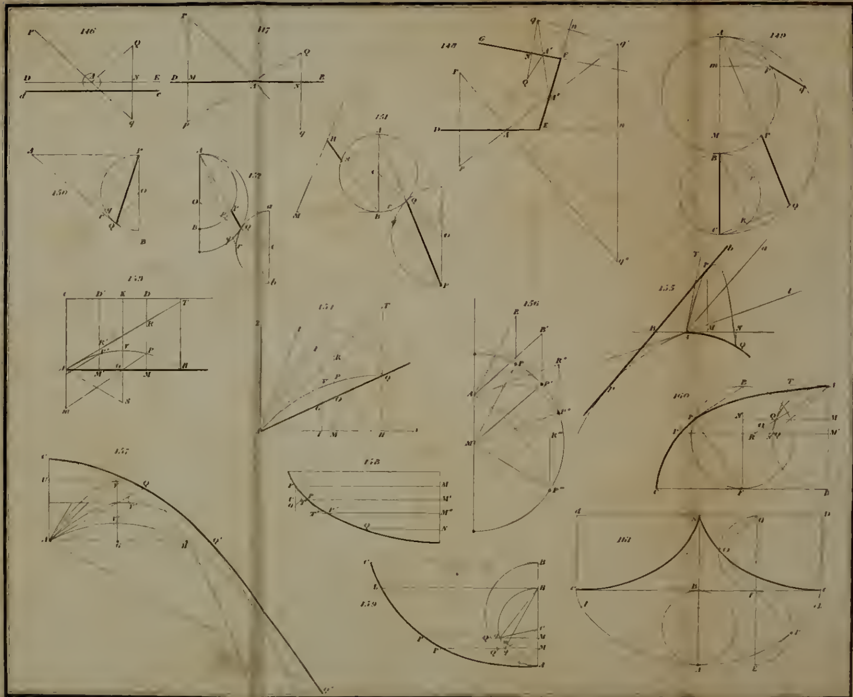

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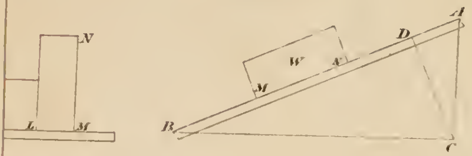

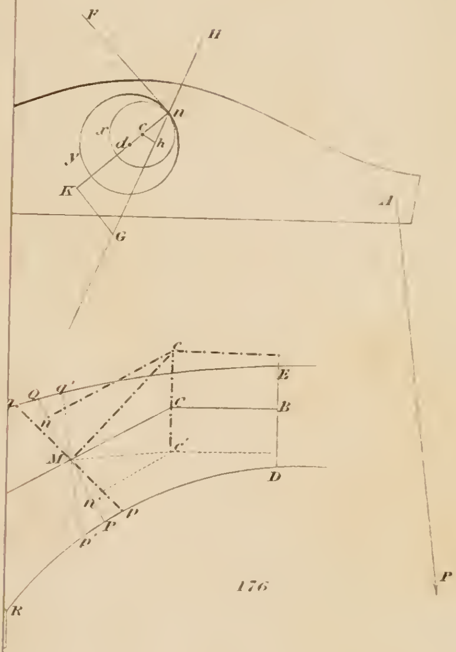

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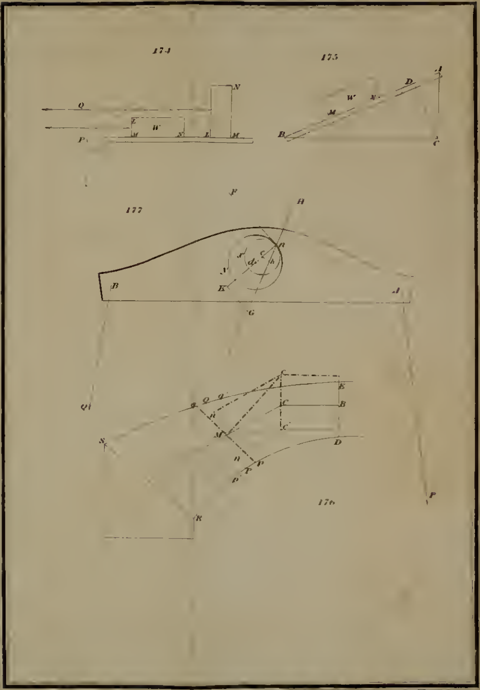
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