AN

ELEMENTARY TREATISE

ON

THEORETICAL MECHANICS

BY

ALEXANDER ZIWET
ASSISTANT PROFESSOR OF MATHEMATICS IN THE
UNIVERSITY OF MICHIGAN

Sew + Rease

IN THREE PARTS

New York
THE MACMILLAN COMPANY
LONDON: MACMILLAN & CO., LTD.
1901

First printed November, 1894.
Reprinted March, 1901.

Norwood Press
J. S. Cushing & Co. — Berwick & Smith
Norwood Mass. U.S.A.

PREFACE.

THE present work owes its existence mainly to the difficulty of finding a good modern text-book suited to the requirements of the American student.

In England it is customary to take a thorough course in elementary mechanics (comprising plane statics and kinetics of a particle) before entering upon the study of higher mathematics; and there is no lack of works of this character (Loney, Macgregor, Selby, Thomson and Tait's Elements, Hicks, Robinson, Browne, Blaikie, Parkinson, Wormell, Lodge, Laverty, etc.), some of which are very well adapted to the purpose. A good course in analytic geometry and the differential and integral calculus will then prepare the student for reading the more advanced English works on analytical statics (Todhunter, Minchin, Routh) and rigid dynamics (Williamson and Tarleton, Routh, Thomson and Tait, Price, Besant, etc.). A similar arrangement is presupposed by most of the French and German treatises.

In many American colleges and universities, however, the student takes up the study of mechanics at a later stage, after having acquired a knowledge of the elements of higher mathematics. A somewhat different treatment of the subject of mechanics is required in this case.

The present volume, which is devoted to kinematics, forms the first of three parts of nearly equal extent. The second part, after an introduction to dynamics in general, takes up statics; it will appear in the fall of this year. The third part, which will be ready in the fall of 1894, is devoted to kinetics.

While the work is intended, first of all, as an introduction to the science of theoretical mechanics as such, the author has constantly kept in mind the particular wants of engineering students, aiming to make it serve as a preparation for the practical applications of this science, and to bring out the utility and importance of the purely mathematical training. General theories are illustrated by special problems and applications in the text, and sets of exercises are inserted to be worked out by the student.

To keep the whole work within reasonable bounds, the more advanced parts of the subject had to be strictly excluded. Bibliographical references have therefore been given for the use of any who are desirous to pursue the subject farther. In accordance with the elementary character of the work, these references are not to original memoirs, but to such standard treatises as can be expected to be found in a well-assorted college library.

At a first reading, the Articles 57–87, 181–214, 221–244, 272–305, can be omitted, also some of the applications and the more difficult exercises.

ALEXANDER ZIWET.

ANN ARBOR, MICH.,
July, 1893.

CONTENTS.

THEORETICAL MECHANICS.

INTRODUCTION.

1. The science of theoretical mechanics has for its object the mathematical study of motion.

2. The idea of motion is intimately related to the fundamental ideas of **space, time,** and **mass.** It will be convenient to introduce these consecutively. Thus we shall begin with a purely geometrical study of motion, without regard to the time consumed in the motion and to the mass of the thing moved, the moving object being considered as a mere geometrical configuration. This introductory branch of mechanics may be called the **geometry of motion.**

3. The introduction of the idea of time will then lead us to study the velocity and acceleration of geometrical configurations. This constitutes the subject-matter of **Kinematics** proper. The name Kinematics is, however, used by many authors in a less restricted sense, so as to include the geometry of motion.

4. Finally, endowing our geometrical points, lines, and other configurations with mass, we are led to the ideas of momentum, force, energy, etc. This part of our subject, the most comprehensive of all, has been called **Dynamics**, owing to the importance of the idea of force in its investigation. For the sake of convenience it is usually divided into two branches, **Statics** and

Kinetics. In statics those cases are considered in which no change of motion is produced by the acting forces, or, as it is commonly expressed, in which the forces are in equilibrium. The investigations of statics are therefore independent of the element of time. Kinetics treats in the most general way of the changes of motion produced by forces.

CHAPTER I.

GEOMETRY OF MOTION.

I. *Linear Motion; Translation and Rotation.*

5. Motion consists in change of position.

6. We begin with the simple case of a point moving in a straight line. The position of a point P in a line is determined by its distance $OP = x$ from some fixed point or origin, O, assumed in the line, the length x being taken with the proper sign to express the *sense* (say forward or backward, to the right or to the left) in which it is to be measured on the line. This sense is also indicated by the order of the letters, so that $PO = -OP$, and $OP + PO = 0$.

The position of a point in a line is thus fully determined by a single algebraical quantity or co-ordinate; viz. by its abscissa $x = OP$.

7. Let the point P move in the line from any initial position P_0 (Fig. 1) to any other position P_1, and let $OP_0 = x_0$, $OP_1 = x_1$.

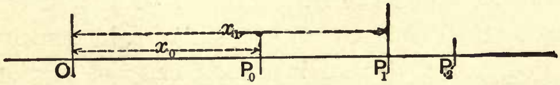

Fig. 1.

This change of position, or *displacement*, is fully determined by the distance $P_0P_1 = x_1 - x_0$ traversed by the point.

Now let this displacement P_0P_1 be followed by another displacement in the same line, from P_1 to P_2, in the same sense as the former, or in the opposite sense. In either case the total, or **resultant**, displacement is the algebraic sum of the two displacements P_0P_1, P_1P_2, which are called its **components**; *i.e.* we have $P_0P_2=P_0P_1+P_1P_2$, or $P_0P_1+P_1P_2+P_2P_0=0$, whatever may be the positions of the points P_0, P_1, P_2 in the line.

This reasoning is easily extended to any number of component displacements; that is, *the resultant of any number of consecutive displacements of a point in a line is a single displacement equal to the algebraic sum of the components.*

Similar considerations apply to the motion of a point in a curved line provided the displacements be always measured along the curve.

8. Let us next consider the motion of a rigid body. The term **rigid body**, or simply **body**, is used in kinematics to denote a figure of invariable size or shape, or an aggregate of points whose distances from each other remain unchanged. Examples are: a segment of a straight line, a triangle, a cube, an ellipsoid, etc.

Imagine such a body M brought in any manner from some initial position M_0 into any other position M_1. This displacement M_0M_1 is determined by the displacements of the various points of the body. We shall see that, even in the most general case, the displacements of three points of the body determine those of all other points, and consequently the displacement of the whole body.

There are, however, two special cases of motion, translation and rotation, in which the displacement of the body is fully determined by the displacement of a single point: such motions can be called **linear**. There is also a class of motions determined by the displacements of only two points of the body: this is called **plane** motion.

9. *The displacement of a rigid body is called a* **translation** *when the displacements of all of its points are parallel and equal.* It is evident that in this case the displacement of any one point of the body fully represents the displacement of the whole body. The translation M_0M_1 of a rigid body M from a position M_0 to a position M_1 is therefore measured by the rectilinear segment P_0P_1 that represents the displacement of any point P of the body M.

Two or more consecutive translations of a rigid body in the same direction produce a resultant displacement of translation equal to the algebraic sum of the components.

10. When a rigid body has two of its points fixed, the only motion it can have is a rotation about the line joining the fixed points as axis. *In a motion of* **rotation** *all points of the body excepting those on the axis describe arcs of circles whose centres lie on the axis while the points on the axis are at rest.*

The different positions of a rotating body may be referred to any fixed plane passing through the axis of rotation. Any plane of the body passing likewise through the axis will make with the fixed plane an angle θ which varies in the course of the motion. This angle, taken with the proper sign, fully determines the positions of the body.

Let the body rotate from a position θ_0 to a position θ_1; the angle $\theta_1 - \theta_0$ measures the corresponding displacement, or the rotation, just as (Art. 7) the distance $P_0P_1 = x_1 - x_0$ measures the displacement of a point, and hence (Art. 9) the translation of a rigid body.

Two or more consecutive rotations of a rigid body about the same axis give a resultant rotation whose angle is the algebraic sum of the angles of the component rotations.

11. The particular case when the rigid body is a plane figure whose motion is confined to its plane deserves special mention. If one point of such a figure be fixed, the figure can only have a motion of rotation, every other point of the figure describing an arc of a circle whose centre is the fixed point. This point is therefore called the **centre of rotation.** The positions of the figure are given by the angle that any line of the figure passing through the centre makes with any fixed line through the centre in the plane.

12. We have seen that a translation as well as a rotation is measured by a single algebraical quantity, the translation by a distance, the rotation by an angle. This is the reason why such motions may be called *linear* or of one dimension. The two fundamental forms of motion, translation and rotation, are thus seen to correspond to the two fundamental magnitudes of metrical geometry, viz. distance and angle.

It is to be noticed that both for translations in the same direction and for rotations about the same axis the resultant displacement is found by algebraic addition of the components, not only when the components are *consecutive* motions, but even when they are *simultaneous*. Thus we may imagine a point P displaced by the amount P_1P_2 along a straight line while this line itself is moved along in its own direction by an amount Q_1Q_2. The resultant displacement of P is the algebraic sum $P_1P_2+Q_1Q_2$.

13. Translations being measured by distances or lengths, and rotations by angles, we need in mechanics a unit of length and a unit of angle.

The two most important systems of measurement are the C. G. S. (*i.e.* centimetre-gramme-second) system, and the F. P. S. (*i.e.* foot-pound-second) system. The former is frequently called the scientific system; it is based on the international or metric system of weights and measures. The F. P. S., or British system, is still used in England and the United States almost universally in engineering practice.*

14. The unit of length in the C. G. S. system is the **centimetre** (cm.), *i.e.* $\frac{1}{100}$ of the metre. The original standard metre is a

* For fuller information on all questions relating to standards and units see J. D. EVERETT, *Illustrations of the C. G. S. system of units with tables of physical constants;* London, Macmillan, 1891.

platinum bar preserved in the *Palais des Archives* in Paris, a legalized copy of which has been deposited at Washington, D.C. The metre can be defined as the distance between two marks on the standard metre when at a temperature of 0° C.

In the F. P. S. system, the unit of length is the **foot**, *i.e.* $\frac{1}{3}$ of the standard yard. The original British standard yard is a bronze bar preserved in London. For the United States the yard is defined as the distance between the twenty-seventh and sixty-third divisions of the brass standard yard kept in the Bureau of Weights and Measures at Washington, when the bar is at a temperature of $16\frac{2}{3}$° C. or 62° F.

The relation between these two fundamental units of length is, according to the *United States Coast and Geodetic Survey Bulletin* No. 9, 1889,

$$1 \text{ cm.} = 0.032\,808\,2 \text{ ft.}$$

For practical use we have the following approximate relations:

$$1 \text{ m.} = 3.2809 \text{ ft.}, \qquad 1 \text{ ft.} = 30.48 \text{ cm.},$$

$$1 \text{ cm.} = 0.3937 \text{ in.}, \qquad 1 \text{ in.} = 2.54 \text{ cm.}$$

15. The unit of angle is either the **degree**, *i.e.* $\frac{1}{360}$ of one revolution, or the **radian**, *i.e.* the angle measured by an arc whose length is equal to the radius.

If α be any angle expressed in radians, and α°, α', α'' the same angle expressed respectively in degrees, minutes, seconds, we have the relations

$$\alpha = \frac{\pi}{180} \cdot \alpha^\circ = \frac{\pi}{10800} \cdot \alpha' = \frac{\pi}{648000} \cdot \alpha'',$$

or

$$\alpha = 0.017\,453\,\alpha^\circ = 0.000\,291\,\alpha' = 0.000\,004\,85\,\alpha''.$$

II. *Plane Motion.*

16. The position of a plane figure in its plane is fully determined by the positions of any two of its points since every other point of the figure forms with these two points an invariable triangle. But the position of the figure can of course be determined in other ways; for instance, by the position of one point and that of a line of the figure passing through the point; or by the position of two lines of the figure.

17. Let us now consider the motion of a plane figure F in its plane from any initial position F_0 to any other position F_1. The displacement F_0F_1 can be brought about in various ways. Thus, it would be sufficient to bring any two points A, B (Fig. 2) of the figure F from their initial positions A_0, B_0 in F_0 to their final positions A_1, B_1 in F_1. This can be done, for instance, by first giving the whole figure a translation through a distance A_0A_1 and then a rotation by an angle equal to the angle between A_0B_0 and A_1B_1; or by such a rotation followed by the translation.

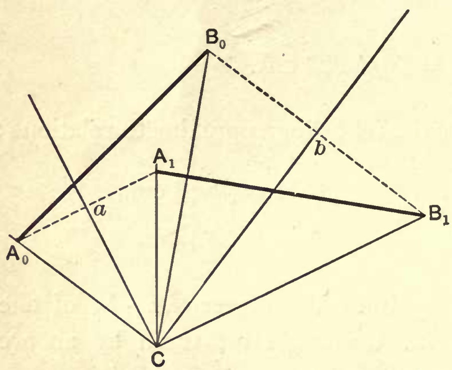

Fig. 2.

Instead of A we might have selected any other point of the figure. But it is important to notice that the angle of rotation required for a given displacement F_0F_1 is always the same, while the translation will differ according to the point selected as centre.

18. This leads us to inquire whether the centre of rotation cannot be so selected as to reduce the translation to zero. Now any rotation that is to bring A from A_0 to A_1 must have its centre on the perpendicular bisector of A_0A_1; similarly for B. Hence the intersection C of the perpendicular bisectors of A_0A_1 and B_0B_1 is the only point by rotation about which both A and B can be brought from their initial to their final positions. That they actually are so brought follows at once from the equality of the angles A_0CB_0 and A_1CB_1 (and hence of the angles A_0CA_1 and B_0CB_1) which are homologous angles in the equal triangles A_0CB_0 and A_1CB_1.

We thus have the proposition: *Any displacement of an invariable plane figure in its plane can be brought about by a single rotation about a certain point* which we may call the *centre* of the displacement.

19. The construction of the centre C given in the preceding article becomes impossible when the bisectors coincide (Fig. 3) and when they are parallel (Fig. 4). In the former case, C is readily found as the intersection of A_0B_0 and A_1B_1. In the latter, *i.e.* whenever $A_0A_1 = B_0B_1$, the centre lies at infinity, and the rotation becomes a translation.

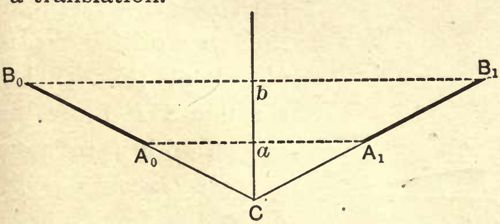

Fig. 3.

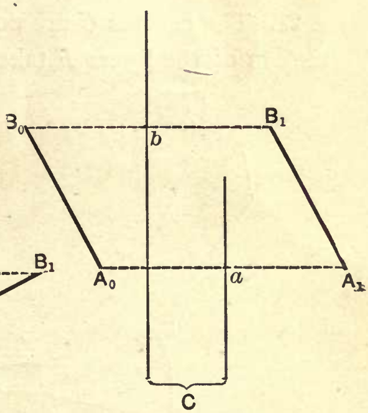

Fig. 4.

Any translation may therefore be regarded as a rotation about a centre at infinity.

20. Let the figure F pass through a series of displacements F_0F_1, F_1F_2, ... $F_{n-1}F_n$. Each displacement has its angle and its centre. If the successive positions F_0, F_1, ... F_n of the figure are taken each very near the preceding one, the angles of rotation will be very small, and the successive centres C_1, C_2, ... C_n will follow each other very closely. In the limit, *i.e.* when for the series of finite displacements we substitute a continuous motion of the figure, the centres C will form a continuous curve (c) and the angles become the infinitely small angles between the successive normals to the paths described by the points of the figure. The point C about which the figure rotates in any one of its positions during the motion is now called the **instantaneous centre**; the locus of the centres, that is the curve (c), is called the **centrode**, or path of the centre. It is apparent that in any position of the moving figure *the normals to the paths of all its points must pass through the instantaneous centre*, and *the direction of motion of any such point is therefore at right angles to the line joining it to the centre.*

21. The centres C are points of the fixed plane in which the motion of the figure F takes place. But in any position F_1 of this figure some point C'_1 of F will coincide with the point C_1 of the fixed plane. Thus, in the case of finite displacements (Fig. 5), let the figure F begin its motion with a rotation of angle θ_1 about a point C_1 of the fixed plane; let C'_1 be the point of the moving figure that coincides during this rotation with C_1.

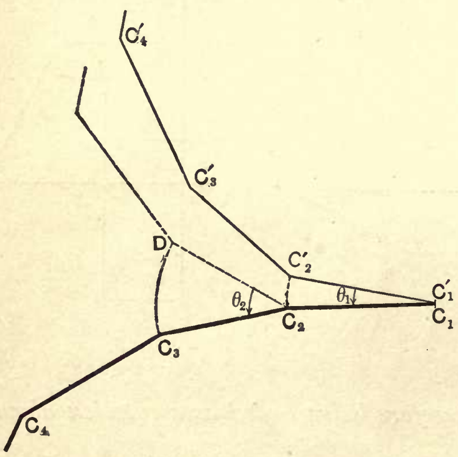

Fig. 5.

The next rotation, of angle θ_2, takes place about a point C_2 of the fixed plane. The point of the moving figure that now coincides with C_2 was brought into the position C_2 by the preceding rotation. Its original position is therefore obtained by turning C_1C_2 back by an angle $-\theta_1$ into the position $C_1C'_2$. The rotation of angle θ_2 about C_2 brings a new point C'_3 of the moving figure to coincidence with the fixed centre C_3; and the original position C'_3 of this point can be determined by first turning C_2C_3 back about C_2 by an angle $-\theta_2$ into the position C_2D, and then turning the broken line C_1C_2D by a rotation of angle $-\theta_1$ about C_1 back into the position $C'_1C'_2C'_3$. Continuing this process we obtain, besides the broken line $C_1C_2C_3\ldots$ formed by joining the successive centres of rotation in the fixed plane, a broken line $C'_1C'_2C'_3\ldots$ in the moving figure formed by joining those points of this figure which in the course of the motion come to coincide with the fixed centres. The whole motion may be regarded as a kind of rolling of the broken line $C'_1C'_2C'_3\ldots$ over the broken line $C_1C_2C_3\ldots$.

22. In the case of continuous motion each of the broken lines becomes a curve, and we have actual rolling of the curve (c'), or **body centrode**, over the curve (c), or **space centrode.** *The continuous motion of an invariable plane figure in its plane may therefore always be produced by the rolling* (without sliding) *of the body centrode over the space centrode.* The point of contact of the two curves is of course the instantaneous centre.

23. It appears from the preceding articles that the continuous motion of a plane figure in its plane is fully determined if we know the centre of rotation for every position of the figure. This centre can be found as the intersection of the normals of the paths of any two points of the figure, so that the motion of the figure will be known if the paths of any two of its points are given. This, however, is only one out of many ways of determining plane motion by two conditions.

Thus the motion may be determined by the condition that a curve of the moving figure should remain in contact with two fixed curves. In this case the instantaneous centre is found as the intersection of the common normals at the points of contact.

The condition that a curve of the moving figure should always pass through a fixed point may be regarded as a special case of the condition just mentioned, one of the fixed curves being reduced to a point.

24. Any curve of the moving figure forms during the motion an **envelope**, the points of the envelope being the intersections of the successive infinitely near positions of the moving curve. Let l, l' be two such successive positions of the curve, A their intersection, C the instantaneous centre; then CA is perpendicular to l as well as to l', and hence to the envelope. The envelope can therefore be constructed by letting fall normals from the instantaneous centres on the corresponding positions of the generating curve.

25. The following examples will illustrate the method of finding the centrodes and the path of any point of the moving figure in plane motion.

Elliptic motion: *Two points of a plane figure move along two fixed lines that are at right angles to each other.*

Let A, B (Fig. 6) be the points moving on the lines Ox, Oy; the perpendiculars to these lines erected at A and B intersect at the instantaneous centre C. Denoting by $2a$ the invariable distance of A and B, we have $OC = AB = 2a$ for all positions of the moving figure. The fixed centrode (c) is therefore a circle of radius $2a$ described about the intersection O of the fixed lines.

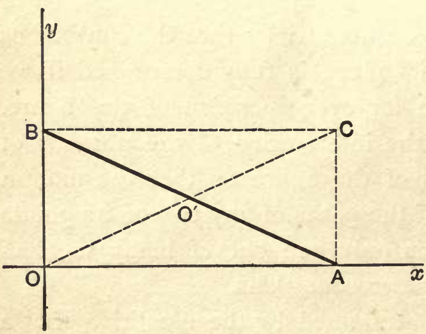

Fig. 6.

To find the body centrode (c') we must construct the triangle ABC for all possible positions of AB. As BCA is always a right angle, the body centrode will be a circle described on AB as diameter. Hence the whole motion can be produced by the rolling of a circle of radius a within a circle of radius $2a$.

The student is advised to carefully carry out the constructions indicated in this as well as the following problems. Thus, in the present case, draw the moving figure, *i.e.* the line AB, in a number of its successive positions in each of the four quadrants, and construct the instantaneous centre C in every case. This gives a number of points of the space centrode. Then take any one position of AB and transfer to it as base all the triangles ABC previously constructed. The vertices of these triangles all lie on the body centrode.

26. To find the equation of the path of any point P of the moving figure, let this point be referred to a co-ordinate system fixed in, and moving with, the figure (Fig. 7); let the middle point O' of AB be the origin, and $O'A$ the axis $O'x'$, of this system. Then the co-ordinates x', y' of P in this moving system are connected with its co-ordinates x, y in the fixed system Ox, Oy by the following equations,

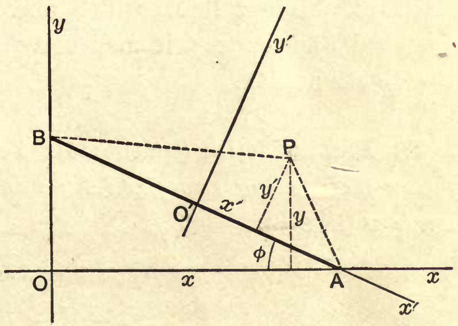

Fig. 7.

$$x=(a+x')\cos\phi+y'\sin\phi,$$
$$y=(a-x')\sin\phi+y'\cos\phi,$$

where ϕ is the angle OAB that determines the instantaneous position of AB. Solving these equations for $\sin\phi$ and $\cos\phi$, squaring and adding, we find for the equation of the path of P

$$\left(\frac{y'x-(a+x')y}{x'^2+y'^2-a^2}\right)^2+\left(\frac{y'y-(a-x')x}{x'^2+y'^2-a^2}\right)^2=1,$$

or $[(a-x')^2+y'^2]x^2-4ay'xy+[(a+x')^2+y'^2]y^2=(x'^2+y'^2-a^2)^2,$

which represents an ellipse, since the determinant

$$\begin{vmatrix} (a-x')^2+y'^2, & -2ay' \\ -2ay', & (a+x')^2+y'^2 \end{vmatrix}$$

$$=(x'^2+y'^2+a^2)^2-4a^2(x'^2+y'^2)=(x'^2+y'^2-a^2)^2$$

is necessarily positive.

In general, therefore, the points of the figure describe ellipses; O' describes a circle; A and B describe straight lines, and so does every point on the circle of diameter AB. It is this fact that by rolling a circle within a circle of double diameter the points of the smaller circle are made to describe segments of straight lines, which makes this form of motion of practical importance: it may serve to transform circular into rectilinear motion.

27. Elliptic Motion (continued): *Two points* A, B *of a plane figure move along two fixed lines inclined to each other at an angle* ω (Fig. 8).

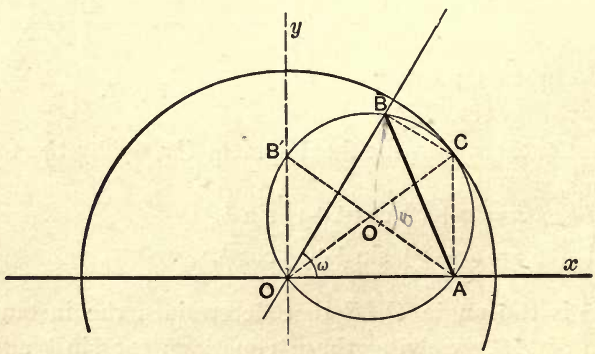

Fig. 8.

This case is readily reduced to the preceding one. The instantaneous centre is found as before; its distance OC from the intersection of the fixed lines OA, OB is again constant and $=AB/\sin\omega$; for O, A, C, B all lie on a circle whose centre O' bisects OC; hence

$$\measuredangle\, AO'B = 2\omega, \text{ and } AB = 2AO' \sin\omega = OC \sin\omega.$$

The motion is therefore produced by the rolling of this circle of diameter $AB/\sin\omega$ within a circle of twice this diameter described about O; it is not essentially different from the preceding case (Art. 26). This will also be seen if we take OA as axis of x, the perpendicular to it through O as axis of y. This perpendicular Oy intersects the circle OAB in a point B', which is the end of the diameter $AO'B'$ and moves along Oy during the motion. The points A, B' of the figure move, therefore, along the rectangular lines Ox, Oy, just as in the problem of Art. 26.

28. Connecting Rod Motion: *One point* A *of the figure describes a circle, while another point* B *moves on a straight line, passing through the centre* O *of the circle* (Fig. 9).

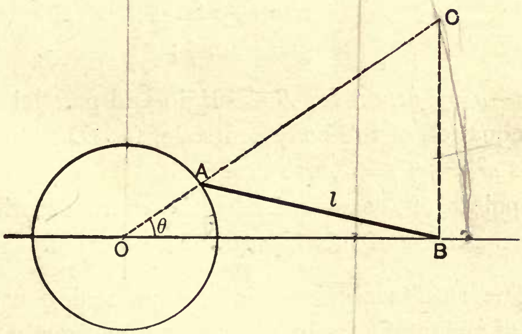

Fig. 9.

With OB as polar axis, the equation of the fixed centrode is

$$r^2 \cos^2\theta - 2ar\cos^2\theta + a^2 = l^2.$$

This, as well as the equation of the body centrode, is of the sixth degree in rectangular Cartesian co-ordinates. But the graphical construction presents no difficulties.

29. Conchoidal Motion: *A point* A *of the figure moves along a fixed straight line* l, *while a line of the figure*, l′, *containing the point* A *always passes through a fixed point* B (Fig. 10).

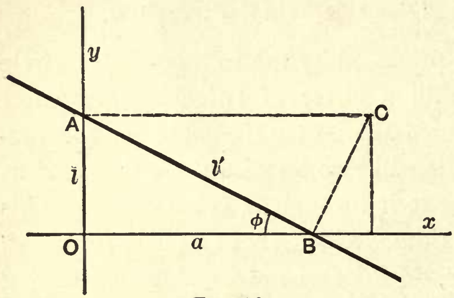

Fig. 10.

The fixed point B may be regarded as a circle of infinitely small radius, which the line l' is to touch. The instantaneous centre is therefore the intersection C of the perpendiculars erected at A on l and at B on l'.

The fixed centrode is a parabola whose vertex is B. To prove this we take the fixed line l as axis of y, the perpendicular OB to it drawn through the fixed point B as axis of x. Then, putting $\angle OBA = \phi$ and $OB = a$, we have for the co-ordinates of C

$$x = a + y \tan\phi,$$
$$y = a \tan\phi;$$

hence $x - a = y^2/a$, or, for B as origin and parallel axes, $y^2 = ax$.

The equation of the body centrode, for OB, OA as axes of x and y, is $a^2(x^2+y^2) = x^4$, or $r\cos^2\theta = a$.

The points of l' can easily be shown to describe conchoids, whence the name of this form of plane motion.

30. The results obtained in the preceding articles for the motion of a plane figure in its plane apply directly to the motion of a rigid body, if any one point of the body describes a plane curve while a line of the body remains parallel to itself. For in this case all points of the body move in parallel planes, and the motion in any one of these planes determines the motion of the whole figure.

The only modifications required would be that instead of an instantaneous centre we should have an **instantaneous axis**, viz.:

the perpendicular to the plane of motion of any point through the centre of motion of this point; and that the centrodes are now not curves, but cylindrical surfaces rolling one upon the other.

31. Exercises.

(1) Show how to find the direction of motion of any point P rigidly connected with the connecting rod of a steam engine.

(2) A wheel rolls on a straight track; find the direction of motion of any point on its rim. What are the centrodes in this case?

(3) Show how to construct the normal at any point of a conchoid.

(4) Find the equation of the fixed centrode when a line l' of a plane figure always touches a fixed circle O, while a point A of l' moves along a fixed line l.

(5) Show that, in (4), the fixed centrode is a parabola when the fixed circle touches the fixed line.

(6) Two straight lines l', l'' of a plane figure constantly pass each through a fixed point O', O''; investigate the motion.

(7) Four straight rods are jointed so as to form a plane quadrilateral $ABDE$ with invariable sides and variable angles. One side AB being fixed, investigate the motion of the opposite side; construct the centrodes graphically.

(8) Let a straight line l in a fixed plane be brought by a finite displacement from an initial position l_0 into a final position l_1; and let P be any point of l, P_0 its initial position (in l_0), P_1 its final position (in l_1). Then the following propositions can be proved:

(*a*) The middle points of the displacements P_0P_1 of all points P of l lie in a straight line;

(*b*) the lines P_0P_1 envelop a parabola;

(*c*) the projections of the displacements P_0P_1 on the line joining their middle points are all equal;

(*d*) if l have a continuous motion in the plane, the tangents to the paths of all its points envelop a parabola of which the instantaneous centre is the focus and l the tangent at the vertex.

III. *Spherical Motion.*

32. The motion of a spherical figure of invariable form on its sphere presents a close analogy to plane motion; in fact, plane motion is but a special case of spherical motion, since a plane may be regarded as a sphere of infinite radius.

33. By a generalization similar to that of Art. 30, the study of the motion of a spherical figure on its sphere leads directly to the laws of motion of a rigid body having one fixed point. For the motion of such a body is evidently determined by the spherical motion on any sphere described about the fixed point.

34. Let us consider any two positions F_0 and F_1 of a spherical figure F on its sphere, and let O be the centre of the sphere. Just as in the case of plane motion (Art. 18) the displacement F_0F_1 can always be brought about by a single rotation about a point C on the sphere, or what amounts to the same, by a single rotation about the *axis* OC. The proof is strictly analogous to that given in Art. 18. We first remark that the position of the figure on the sphere is fully determined by the position of two of its points, say A and B (Fig. 11), since any third point forms with these an invariable spherical triangle. Let A_0, B_0 be the positions of A, B in F_0; A_1, B_1 their positions in F_1; and draw the great circles A_0A_1 and B_0B_1. Their perpendicular bisectors intersect in two points C, D which are the ends of a diameter of the sphere. CD is the axis of the displacement F_0F_1, and the angle A_0CA_1, or B_0CB_1, gives the angle of the displacement.

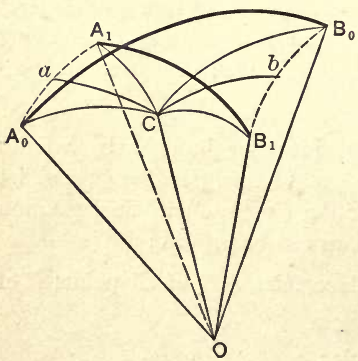

Fig. 11.

35. If we consider a series of positions of the moving figure, $F_0, F_1, F_2, \ldots$, we obtain a series of axes of rotation, say $c_1, c_2, \ldots$; and in the limit when these positions follow one another at infinitely near intervals, the axes $c_1, c_2, \ldots$ will form a cone fixed in space, with the vertex at the centre O of the sphere. The points $C_1, C_2, \ldots$ where these axes intersect the sphere form a curve (c) on the fixed sphere, while the points $C'_1, C'_2, \ldots$ of the moving figure with which these fixed points come to coincide form a spherical curve (c') invariably connected with the moving figure. The whole motion may be produced by the rolling of the curve (c') over the curve (c), or also by the rolling of the corresponding cones one over the other. We have thus the proposition that *any continuous motion of a rigid body having a fixed point can be produced by the rolling of a cone fixed in the body on a fixed cone, the vertices of both cones being at the fixed point.*

IV. *Screw Motion.*

36. The position of a rigid body in space is fully determined by the position of any three of its points not situated in the same straight line. For any fourth point of the body will form an invariable tetrahedron with these three points. As two points determine a straight line, the position of a rigid body may also be given by the position of a point and line or by the positions of two intersecting or parallel lines of the body.

37. The position of a point being determined by its three co-ordinates requires three conditions to be fixed. A point is therefore said to have three *degrees of freedom* when its position is not subject to any conditions. One conditional equation between its co-ordinates restricts the point to the surface represented by that equation; the point is then said to have but two degrees of freedom and one *constraint*. Two conditions would restrict the point to a line, the curve of intersection of the two surfaces represented by the equations of condition;

the point has then but one degree of freedom and two constraints.

A rigid body that is perfectly free to move has six degrees of freedom. For we have seen that its position is fully determined when three of its points not in the same line are fixed. The nine co-ordinates of these points are, however, not independent; they are connected by the three equations expressing that the three distances between the three points are invariable. Thus the number of independent conditions is $9-3=6$.

A rigid body with one fixed point has three degrees of freedom and therefore three constraints. For it takes two more points, *i.e.* six co-ordinates, to fix the position of the body; and the distances of these two points from each other and from the fixed point being invariable, there are again three conditional equations to which the six co-ordinates are subject. The three co-ordinates of the fixed point may be regarded as the three constraints.

A rigid body with two fixed points, i.e., *with a fixed axis, has one degree of freedom, and five constraints.* Indeed, the six co-ordinates of the two fixed points are equivalent to five constraining conditions, since the distance of these two points is invariable.*

38. Let us now consider any two positions M_0, M_1 of a rigid body M, given by the positions A_0, B_0, C_0 and A_1, B_1, C_1 of three points A, B, C of the body. The displacement M_0M_1 can be effected in various ways. Thus we might for instance begin by giving the whole body a translation equal to A_0A_1 which would bring the point A to its final position while all other points of the body would be displaced by distances parallel and equal to A_0A_1. As the body has now one of its points, A, in its final position, it will (by Art. 34) require only

* Interesting remarks on the mechanical means of producing constraints of various degrees will be found in THOMSON and TAIT, *Natural philosophy*, London, Macmillan, new edition, 1879, Art. 195 sq. (Part I., p. 149).

a single rotation about a certain axis passing through this point to bring the whole body into its final position. It thus appears that any displacement of a rigid body can be effected by subjecting the body first to a translation and then to a rotation (or *vice versa*, as is easily seen); and this can be done in an infinite number of ways, as the displacement of *any* point of the body may be selected for the translation.

39. It is to be noticed that for all these different ways of effecting the displacement M_0M_1 the *direction* of the axis of rotation and the *angle* of rotation are the same. To see this more clearly, let the displacement be effected first by the translation A_0A_1 and a rotation of angle α about the axis a_1 passing through A_1; and then let the same displacement be produced by the translation B_0B_1 of some other point B and a rotation of angle β about an axis b_1 passing through B_1. We wish to show that a_1 and b_1 are parallel and that the angles α and β are equal.

Consider a plane π of the rigid body which in its original position π_0 is perpendicular to the axis a_1. The translation A_0A_1 transfers it into a parallel position and the rotation α about a_1 turns it in itself into its final position π_1; hence π_0 and π_1 are parallel. The translation B_0B_1 likewise moves π into a position parallel to the original one; and as its final position, π_1, is parallel to π_0, the axis of rotation b_1 must necessarily be perpendicular to π_0 and π_1, that is b_1 must be parallel to a_1.

Again, any straight line l in π remains parallel to its original position l_0 after the translations A_0A_1 and B_0B_1. Its change of direction is due to the rotations alone; the angle of rotation must therefore be the same for both rotations, viz. equal to the angle (l_0l_1) formed by the initial and final positions of the line l.

40. Among the different combinations of a translation with a rotation effecting the displacement M_0M_1 there is one of particular importance; it is that for which the axis of rotation is parallel to the translation.

Let us again consider the plane π perpendicular to the common direction of the axes of rotation. To bring any three points of this plane into their final position it is only necessary to give the body a translation at right angles to π such as to bring π into its final position and then to add the necessary rotation for plane motion.

We have therefore the important proposition that *it is always possible to bring a rigid body* M *from any position* M_0 *into any other position* M_1 *by a translation combined with a rotation about an axis parallel to the direction of translation*, and this can be done in only one way. The axis so determined is called the **central axis** of the displacement.

The order of translation and rotation about the central axis is indifferent; indeed, translation and rotation might take place simultaneously.

41. A motion of a rigid body consisting of a rotation about an axis combined with a translation parallel to the axis is called a *screw motion*, or a *twist*. We have proved therefore, in Art. 40, that *the most general displacement of a rigid body can be brought about by a single twist.*

42. To construct the central axis and find the translation and angle of the twist when the displacement is given by the positions A_0, B_0, C_0 and A_1, B_1, C_1 of three points of the body, we first remark that the projection on the central axis of the displacement of any point, say A_0A_1, is equal to the translation of the twist, and hence the projections of the displacements of all points of the body (such as A_0A_1, B_0B_1, C_0C_1) are all equal. If therefore from any point O we draw lines OA, OB, OC equal and parallel to A_0A_1, B_0B_1, C_0C_1, their ends A, B, C will lie in a plane π perpendicular to the central axis, and the perpendicular p dropped from O on this plane π will represent in length and direction the translation of the twist.

The direction of the central axis being thus determined, we find its position in space by projecting the displacements of any two of the three given points, say A_0A_1 and B_0B_1, on the plane

π, and finding the intersection of the perpendicular bisectors of these projections. This intersection is evidently a point of the central axis, and a perpendicular through it to the plane π will give the central axis in position.

43. In the case of continuous motion there exists a central axis for every position of the body; but its position both in space and in the body in general varies in the course of the motion. The central axis at any moment is therefore called in this case the **instantaneous axis.**

44. The straight lines of space which during the progress of the motion become instantaneous axes for the infinitely small twists of the body form a ruled surface. Similarly, the lines of the moving body which in the course of the motion come to coincide with these axes generate another ruled surface. In any given position of the body these two surfaces are in contact along a line (the instantaneous axis) which is a generator in each of the two surfaces. The body has an infinitely small rotation about this line and at the same time slides along this line through an infinitely small distance.

Thus *the continuous motion of a rigid body in the most general case can be regarded as consisting of the combined rolling and sliding of one ruled surface over another.*

V. *Composition and Resolution of Displacements.*

I. TRANSLATIONS; VECTORS.

45. All the points of a rigid body subjected to a translation describe parallel and equal lines (Art. 9). The translation of the body is therefore fully determined by the displacement A_0A_1 of any one point A of the body (Fig. 12), and can be represented geometrically by A_0A_1 or any line equal and parallel to it, like 01.

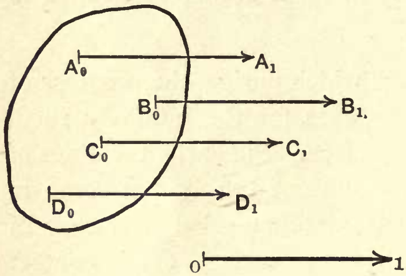

Fig. 12.

A segment of a straight line of definite length, direction, and sense is called a **vector.** The *sense* of the vector (see Art. 6) which expresses whether the translation is to take place from 0 to 1 or from 1 to 0, is indicated graphically by an arrow-head, and in naming the vector, by the order of the letters, 01 and 10 being vectors of opposite sense.

46. Imagine a rigid body subjected to two successive translations. From any point 0 (Fig. 13) draw a vector 01 representing the first translation, and from its end 1 a vector 12 representing the second translation. The vector 02 will then represent a translation that would bring the body directly from its initial to its final position. This vector 02 is called the **geometric sum,** or the **resultant,** of the vectors 01 and 12, which are called the **components.** The operation of combining the components into a resultant, or of finding the geometric sum of two vectors, is called **geometric addition,** or **composition,** of vectors.

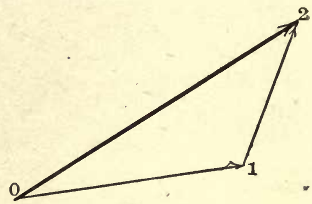

Fig. 13.

47. The process of geometric addition explained in Art. 46 for the case of *two* components is readily extended to the general case of n components. It thus appears that *the succession of any number of translations of a rigid body has for its resultant a single translation whose vector is found by geometrically adding the vectors of the component translations.* (Compare Art. 7.)

48. The order in which vectors are combined, or added, is indifferent for the result. This is directly apparent from a figure in the case of two vectors (Fig. 14). For the case of n vectors it follows from the consideration that any order of the vectors can be obtained by repeated interchanges of two successive vectors.

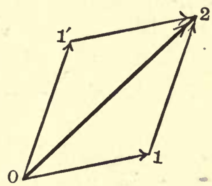

Fig. 14.

Geometric addition agrees, therefore, with algebraic addition in being *commutative.*

49. The vector, as the geometric symbol of a translation, has *length*, *direction*, and *sense;* but it is not restricted to any definite *position*, the same translation being represented by all equal and parallel vectors. We express this by saying that *two vectors are equal if they are of the same length, direction, and sense.*

Translations are not the only magnitudes in mechanics that can be represented by vectors. We shall see later that velocities, accelerations, moments of couples, etc., can all be represented by vectors and are therefore compounded into resultants and resolved into components by geometric addition and subtraction. In this lies the importance of this subject which in its special application to translations might appear too simple and self-evident to require extended presentation.

The case when the vectors represent concurrent forces is probably known to the student from elementary physics as the "parallelogram" or "polygon" of forces.

50. A translation may be *resolved* into two or more translations by resolving its vector into components.

When the resultant translation and one of its components are given by their vectors, the process of finding the other component is called **geometric subtraction.** It is effected, like algebraic subtraction, by reversing the *sense* of the component to be subtracted, and then geometrically adding it to the resultant (Fig. 15). In other words, the *geometric difference* of two vectors AB and CD is found by geometrically adding to AB a vector equal but opposite to CD. Thus, in Fig. 15, 02 is made equal and parallel to AB; 21 is equal and parallel to CD reversed, that is to DC; 01 is the required difference.

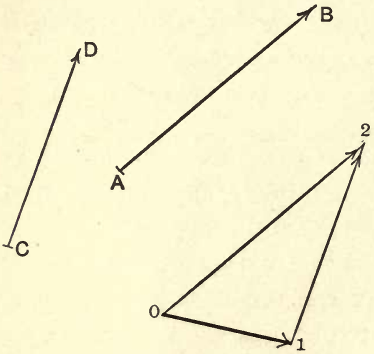

Fig. 15.

51. The composition of translations by geometric addition of their vectors (Art. 47) holds, not for *successive* translations only, but, owing to the commutative law (Art. 48), for *simultaneous* translations as well. This is easily seen by resolving the components into infinitesimal parts.

To obtain a clear idea of two simultaneous translations it is best to imagine the body as having one of these translations with respect to some other body, while the latter itself is subjected to the other translation. A man walking across the deck of a vessel in motion, an object let fall in a moving carriage, a spider running along a branch swayed by the wind, are familiar examples.

52. This leads us to the idea of **relative motion.**

Properly speaking, all motion is relative; that is, we can conceive of the motion of a body only with regard to some other body, called the **body of reference.** If the latter be regarded as fixed, the motion of the former is called its **absolute motion.**

Thus in speaking of the motion of a railway train, we usually regard the earth as fixed and can thus call the displacement of the train from one station to another an *absolute displacement.* If, however, the motion of the earth with regard to the sun be taken into account, the displacement of the train from station to station is the relative displacement of the train with respect to the earth; and its absolute displacement would be found by combining this relative displacement with the absolute displacement of the earth (with respect to the sun regarded as fixed).

53. It follows that when the two displacements are translations *the absolute displacement of the body will be found by geometrically adding its relative displacement to the absolute displacement of the body of reference.* And conversely, *the relative displacement of a body is found by geometrically subtracting from its absolute displacement the absolute displacement of the body of reference.*

54. Analytically, the composition and resolution of vectors is merely a problem of trigonometry. Thus, the resultant of two vectors is the diagonal of the parallelogram formed by the two vectors as adjacent sides; the resultant of three vectors is the diagonal of the parallelepiped having the three vectors as concurrent edges.

55. In the case of more than two or three vectors, however, the solution by ordinary trigonometry would become rather tedious, and it is best to proceed as follows:

Assume an origin O and three rectangular axes Ox, Oy, Oz, and project each vector on the three axes; let X, Y, Z be its projections. These projections X, Y, Z are three vectors whose geometrical sum is equal to the vector. If n vectors were originally given, we should now have them replaced by $3n$ components of which n lie in each axis. The components lying in the same axis can be added algebraically; let their respective sums be $\Sigma X, \Sigma Y, \Sigma Z$. The n vectors are therefore equivalent to the three vectors $\Sigma X, \Sigma Y, \Sigma Z$, which form the concurrent edges of a rectangular parallelepiped whose diagonal drawn through the origin O is the resultant vector $OR = R$, *i.e.*

$$R = \sqrt{(\Sigma X)^2 + (\Sigma Y)^2 + (\Sigma Z)^2}.$$

The direction of this vector is given by the equations

$$\cos\alpha = \frac{\Sigma X}{R}, \ \cos\beta = \frac{\Sigma Y}{R}, \ \cos\gamma = \frac{\Sigma Z}{R},$$

where α, β, γ are the angles made by OR with the axes Ox, Oy, Oz, respectively.

If all the vectors lie in the same plane, we have simply:

$$R = \sqrt{(\Sigma X)^2 + (\Sigma Y)^2}, \ \tan\alpha = \frac{\Sigma Y}{\Sigma X}.$$

56. Exercises.

(1) A ship sails first 5 miles N. 30° E., then 12 miles N. 60° E., and finally 25 miles E. 75° S. Find distance and bearing of the point reached: (*a*) graphically, (*b*) analytically.

(2) Is a scale of 8 miles to the inch sufficient to obtain the results of Ex. (1) correctly to whole miles and degrees?

(3) A rigid body undergoes three translations, of 1, 2, and 3 feet, whose directions are respectively parallel to the three sides of an equilateral triangle taken the same way round. Find the resulting displacement.

(4) A ship is carried by the current 2 miles due W., and at the same time by the wind 4 miles due N.E., and by her screw 11 miles E. 30° S. Find her resultant displacement.

(5) A ferry-boat crosses a river in a direction inclined at an angle of 60° to the direction of the current. If the width of the river be half a mile, what are the component displacements of the boat along the river and at right angles to it?

(6) Two vectors of equal length a are inclined to each other at an angle α. Find the resultant in magnitude and direction.

(7) For what angle α, in Ex. (6), is the resultant equal in magnitude: (*a*) to each component a? (*b*) to $\frac{1}{3}a$?

(8) Resolve a vector a into two components making with the vector angles of 30° and 45° on opposite sides.

(9) Steering his boat directly across a river whose current is due west, a man arrives on the opposite bank at a point from which the starting-point bears S.E.; the width of the river being 1200 feet, how far has he rowed? What is the absolute, and what the relative, displacement of the boat?

(10) Assuming a raindrop to fall 25 feet in a second in a vertical direction, find in what direction it appears to be falling to a man: (*a*) walking at the rate of 5 feet per second, (*b*) driving at the rate of 10 feet per second, (*c*) riding on a bicycle at 25 feet per second, (*d*) in a railroad car running 60 feet per second.

(11) Find in magnitude and direction the resultant of 8 translations of 1, 2, 3, 4, 5, 6, 7, 8 feet, respectively, each component making an angle of 45° with the preceding one: (*a*) graphically, (*b*) analytically.

(12) If a, b, c are three vectors whose geometric sum is 0, prove that $a/\sin(bc) = b/\sin(ca) = c/\sin(ab)$.

(13) Find the resultant of two translations represented in magnitude and direction by two rectangular chords of a circle drawn from a point on its circumference.

(14) From a point C in the plane of a circle whose centre is O, draw two lines at right angles to each other so as to intersect the circle in A, A' and B, B', respectively. Show that the resultant of the four vectors CA, CA', CB, CB' is equal to twice CO.

(15) Prove that the geometric sum of two vectors P_0P_1, P_0P_2 issuing from the same point P_0 passes through the middle point G of P_1P_2 and has a length $= 2\,P_0G$.

(16) Prove that the geometric sum of two vectors P_0P_1 and P_0P_2 is equal to $(n+1)P_0G$ if G be found as follows: on P_0P_1 take Q so that $P_0Q = \frac{1}{n}P_0P_1$, and on QP_2 take G so that $QG = \frac{1}{n+1}QP_2$.

(17) Show that Ex. (15) is a special case of Ex. (16).

(18) Prove the following rule for constructing the geometric sum of n vectors $P_0P_1, P_0P_2, P_0P_3, \cdots P_0P_n$ issuing from the same point P_0: on P_1P_2 take G_1 so that $P_1G_1 = \frac{1}{2}P_1P_2$; on G_1P_3 take G_2 so that $G_1G_2 = \frac{1}{3}G_1P_3$; on G_2P_4 take G_3 so that $G_2G_3 = \frac{1}{4}G_2P_4$; and so on. If G be the last point so determined, the geometric sum of the n vectors is $= nP_0G$.

2. ROTATIONS; ROTORS.

57. When a rigid body has a motion of rotation about a fixed axis, all its points with the exception of those on the axis describe circular arcs whose centres are situated on the axis (Art. 10).

The elements determining a *rotary displacement*, or a *rotation*, are the axis and the angle of rotation. These elements can be represented by a single geometrical symbol; we have only to lay off on the axis of rotation a length OI (Fig. 16) representing on some scale the magnitude of the angle θ. An arrow-head can be used to mark the *sense* of the angle. It is customary, at least in English works on mechanics, to adopt the counter-clockwise sense of rotation as positive. The arrow-head should then be placed at that end of the line representing the angle θ from which the rotation appears counter-clockwise in a plane through the other end at right angles to the axis. The arrow then points in the direction in which an ordinary screw moves when turned in the positive sense.

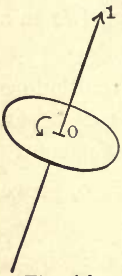

Fig. 16.

This geometrical symbol of a rotation, OI, has been called a **rotor.** It becomes of importance in the case of infinitesimal rotations, as we shall see later (Art. 68).

58. Two or more rotations about the same axis can evidently be combined into a single rotation about the same axis whose angle is the algebraic sum of the angles of the component rotations (Art. 12). As regards rotations about different axes, we have to distinguish three cases: intersecting axes, parallel axes, and crossing or skew axes.

It will be shown in the following articles that rotations about intersecting or parallel axes can always be combined into a single rotation which may happen to reduce to a translation.

Rotations about skew axes cannot in general be reduced to a single rotation or translation; it will be shown in the next section (Arts. 74–79) that they reduce to a twist, or screw motion.

59. Intersecting Axes. *The resultant of two successive rotations, θ_1 about l_1 and θ_2 about l_2, when the axes l_1 and l_2 intersect in a point O, is a single rotation of angle θ about an axis l passing through O.* The trihedral formed by l_1, l_2 and l has at l_1 a dihedral angle $=\frac{1}{2}\theta_1$, at l_2 a dihedral angle $=-\frac{1}{2}\theta_2$, while its exterior angle at l is $=\frac{1}{2}\theta$; that is, we have on a sphere of radius 1 described about O:

$$\cos\tfrac{1}{2}\theta=\cos\tfrac{1}{2}\theta_1\cos\tfrac{1}{2}\theta_2-\sin\tfrac{1}{2}\theta_1\sin\tfrac{1}{2}\theta_2\cos(l_1l_2), \tag{1}$$

$$\frac{\sin(l_1l)}{\sin\frac{1}{2}\theta_2}=\frac{\sin(ll_2)}{\sin\frac{1}{2}\theta_1}=\frac{\sin(l_1l_2)}{\sin\frac{1}{2}\theta}. \tag{2}$$

The truth of this proposition will appear by considering Fig. 17. The rotation θ_1 about the axis l_1 brings the axis l_2 into its final position l'_2. The rotation θ_2 about l'_2 brings l_1 into its final position l'_1. The planes bisecting the dihedral angles θ_1 at l_1 and θ_2 at l'_2 intersect in a line l which by the rotation θ_1 about l_1 is brought into the position l', and by the rotation θ_2 about l'_2 is brought back into its original position l. The effect of the two rotations taken in this order is therefore to leave the line l in its place; that is, the resultant of the two successive rotations is a single rotation about l as axis. Moreover, inspection of the figure shows that a rotation about l by an angle equal to twice the exterior angle of the trihedral ll_1l_2 at l brings l_1 and l_2 into their final positions l'_1 and l'_2.

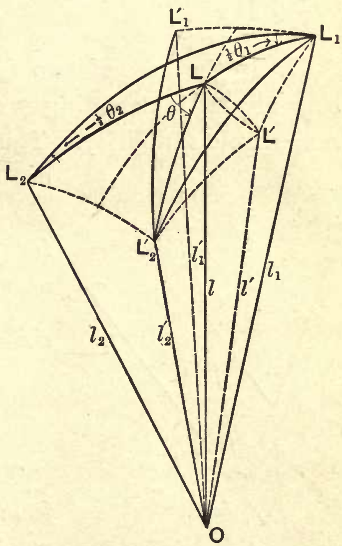

Fig. 17.

60. It is to be noticed that l_1 and l_2 are here regarded as lines of the rigid body; and while l_1 coincides with the position of the first axis of rotation in space, the second axis of rotation in space has the position l'_2, and not l_2. It follows that, *in general, the order of the two rotations is not indifferent.* But by repeating the construction, any number of rotations taken in a definite order can be combined into a single rotation provided every axis intersects the axis of the resultant of all preceding rotations.

61. Again, in finding l from l_1 and l_2, the positions of the axes in the rigid body, as we did in Art. 59, the angle $\frac{1}{2}\theta_1$ is to be applied to the plane $l_1 l_2$ at l_1 in its proper sense, *i.e.* on that side towards which the rotation about l_1 takes place; but $\frac{1}{2}\theta_2$ at l_2 is to be applied to this plane in the opposite sense. If, however, we wish to construct l from the *absolute* positions of the axes of rotation in space, l_1 and l'_2, we have to use $-\frac{1}{2}\theta_1$ and $+\frac{1}{2}\theta_2$.

62. In the case of two *infinitely small rotations*, say $d\theta_1$ and $d\theta_2$, about intersecting axes l_1, l_2, the construction gains remarkable simplicity. The resulting axis l falls into the plane of the given axes.

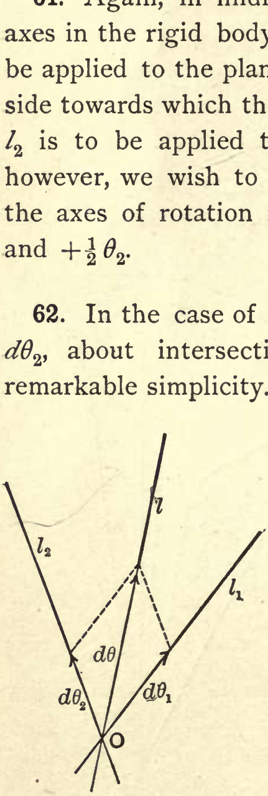

Fig. 18.

Substituting $d\theta$ for $\sin\theta$ and $1-\frac{1}{2}d\theta^2$ for $\cos\theta$, the equations of Art. 59 assume the form

$$d\theta^2 = d\theta_1^{\,2} + d\theta_2^{\,2} + 2\, d\theta_1 d\theta_2 \cos(l_1 l_2), \qquad (1')$$

$$\frac{\sin(l_1 l)}{d\theta_2} = \frac{\sin(l l_2)}{d\theta_1} = \frac{\sin(l_1 l_2)}{d\vartheta}. \qquad (2')$$

These equations show that $d\theta$ can be found by geometrically adding the rotors (Art. 57) representing the rotations $d\theta_1$ and $d\theta_2$. In other words, the components $d\theta_1$ and $d\theta_2$ (or lengths proportional to them) being laid off on their respective axes (Fig. 18), the resultant rotation

$d\theta$ will be found in magnitude and direction as the diagonal of the parallelogram whose adjacent sides are $d\theta_1$ and $d\theta_2$, just as in the case of translations (Art. 46). The importance of this proposition will appear later (Art. 276).

It is to be noticed that, in the case of infinitesimal rotations, the order of succession in which they take place is obviously indifferent; they can therefore be imagined to take place simultaneously.

63. Parallel Axes. The composition of two successive rotations about parallel axes is not essentially different from the composition of rotations about two intersecting axes. The

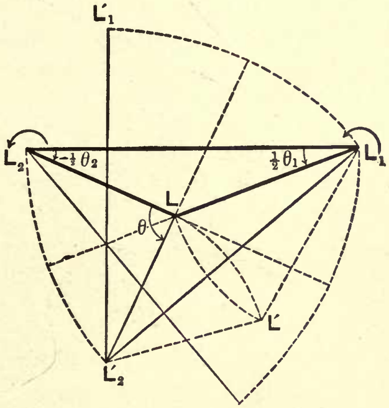

Fig. 19.

trihedral ll_1l_2 of Fig. 17, formed by the given axes l_1, l_2, and the resulting axis l, becomes now a triangular prism, and the spherical construction is replaced by a construction in a plane at right angles to the axes. Fig. 19 shows this construction for the case of two rotations having the same sense (θ_1 and θ_2 being of the same sign); Fig. 20 illustrates the case of two opposite rotations. The letters have the same meaning as in Fig. 17.

The signs of θ_1 and θ_2 being taken into account, the formulæ of Art. 59 are now replaced by the following:

$$\theta = \theta_1 + \theta_2, \tag{1''}$$

$$\frac{L_1 L}{\sin \frac{1}{2}\theta_2} = \frac{L L_2}{\sin \frac{1}{2}\theta_1} = \frac{L_1 L_2}{\sin \frac{1}{2}\theta}. \tag{2''}$$

The order of two finite rotations about parallel axes is not invertible.

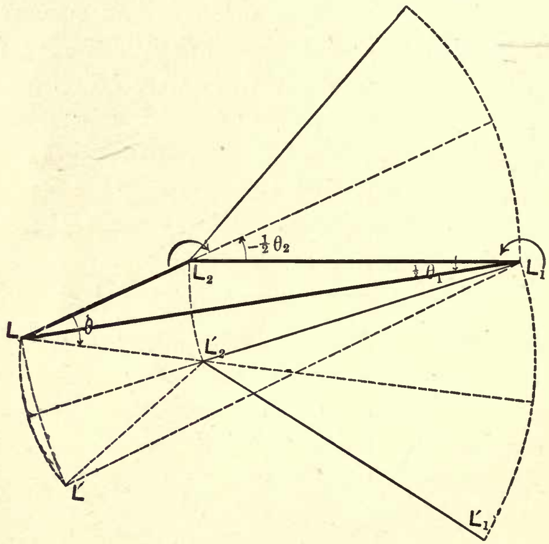

Fig. 20.

By repeating the above construction it is evidently possible to find the resultant of any number of successive rotations about parallel axes, the rotations being taken in a definite order.

64. The particular case of two *equal and opposite* rotations about parallel axes deserves special consideration. The point L lies at infinity; hence, the axis of rotation being at an infinite distance, the resulting motion is a translation (Art. 19). This will also appear from Fig. 21; the first rotation, about l_1, brings the plane $l_1 l_2$ into the position $l_1 l'_2$; the following rotation, about l'_2, brings it into the position $l'_2 l'_1$ which is parallel to the original position $l_2 l_1$. The whole body has thus been moved parallel to itself in the direction $L_1 L'_1$, and the magnitude of this translation is

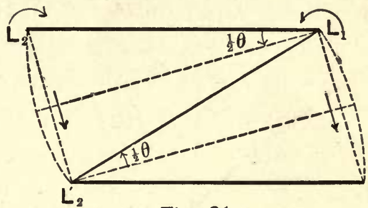

Fig. 21.

$$L_1 L'_1 = L_2 L'_2 = 2\, L_1 L_2 \sin\frac{\theta}{2}, \tag{3}$$

where θ is the angle of rotation about each axis, and $L_1 L_2$ is the distance of the axes.

The order of the rotations is evidently not invertible.

65. We have seen in the preceding article that *two equal and opposite rotations about parallel axes produce a translation at right angles to the axes of rotation.* A translation can therefore always be replaced by two such rotations. It follows that *a translation followed by a rotation about an axis at right angles to the direction of translation can be replaced by a single rotation about a parallel axis.* To find this resulting rotation it is only necessary to replace the translation by two parallel equal and opposite rotations having the same effect (Art. 64); the three rotations so obtained have parallel axes and can therefore (Art. 63) be combined into a single one.

66. The case of two *infinitely small rotations* (Fig. 22) is again of particular importance, as we shall see later on. The formulæ of Arts. 59 and 63 become in this case

$$d\theta = d\theta_1 + d\theta_2, \tag{1'''}$$

$$\frac{L_1 L}{d\theta_2} = \frac{L L_2}{d\theta_1} = \frac{L_1 L_2}{d\theta}. \tag{2'''}$$

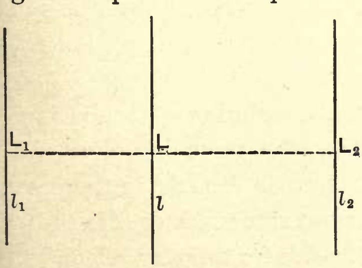

Fig. 22.

The axis l of the resulting rotation lies therefore in the plane of the given axes l_1, l_2 and divides their distance in the invratio of the angles of rotation. The sense of the segments L_1L, LL_2, L_1L_2 must be taken into account as well as the sense of the angles $d\theta_1$, $d\theta_2$, $d\theta$. The axis l lies between l_1 and l_2 if $d\theta_1$, $d\theta_2$ have the same sense; otherwise it lies outside the space between l_1, l_2 on the side of the axis having the greater angle.

67. Two equal and opposite infinitely small rotations about parallel axes produce an infinitely small translation equal to $L_1L_2 \cdot d\theta$ (see Art. 64, Formula (3)) directed at right angles to the plane of the axes l_1, l_2. Conversely, an infinitely small translation can always be replaced by two equal and opposite infinitesimal rotations.

68. An infinitesimal rotation of angle $d\theta$ about an axis l can be represented (Art. 57) by a rectilinear segment laid off on l equal to $d\theta$, or, to avoid infinitesimal lengths, proportional to $d\theta$. This geometrical symbol of an infinitesimal rotation has all the characteristics of a vector (compare Arts. 45, 49); but it has one more which distinguishes it from the vector representing a translation: it is *localized*, or attached to a definite line; for two equal and parallel rotations about different axes do not represent the same thing. Such a localized vector is called a **rotor.**

69. The theory of rotors is of just as great importance in mechanics as that of vectors (Art. 49). Angular velocities, momenta, forces, all have for their geometrical representatives rotors, *i.e.* rectilinear segments of definite direction, length, sense, and situated on a definite line.

The theory of the composition and resolution of rotors is a matter of pure geometry; it remains the same whatever the rotor may represent. Thus we have seen in Art. 62, in the case of infinitesimal rotations, that concurrent rotors are combined by geometrical addition. The same rule holds for angular velocities, momenta, and forces. In Art. 66 the rule for combining two parallel rotors is explained by the example of infinitesimal rotations. The student acquainted with elementary physics will recognize in this rule the so-called principle of the lever which is based on the composition of parallel forces.

70. Two rotors of equal length and opposite sense situated on parallel lines (Fig. 23) are said to form a **couple**. The two rotors P, P are called the **sides**, their perpendicular distance p the **arm**, and the product Pp the **moment** of the couple.

It has been proved in Art. 67 that a couple of infinitesimal rotations produces an infinitesimal translation. In general, a rotor couple is equivalent to a vector, as we shall see later.

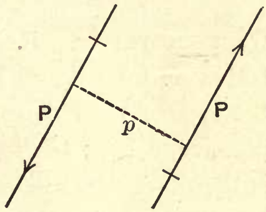

Fig. 23.

71. The converse proposition of Art. 67, viz. that an infinitesimal translation can always be replaced by a couple of infinitesimal rotations, requires a little further consideration.

Suppose we wish to replace the translation ds by a couple. According to Art. 67, the axes l_1, l_2 of the two rotations must be at right angles to ds; the distance L_1L_2 of the axes and the angle of rotation $d\theta$ are only subject to the condition that their product should equal ds, i.e.

$$L_1L_2 \cdot d\theta = ds.$$

There is, therefore, an infinite number of couples equivalent to ds, all having the same moment $L_1L_2 \cdot ds$ and all lying in a plane perpendicular to ds.

It thus appears that the characteristics of a couple are its moment and the aspect of its plane; in other words, a couple (P, p) is equivalent to any couple (P', p') provided (*a*) that they lie in parallel planes or in the same plane, and (*b*) that their moments are equal, i.e. $P \cdot p = P' \cdot p'$. This allows us to *represent a rotor couple* (P, p) *by a vector perpendicular to the plane of the couple and equal in magnitude to its moment* Pp.

The *sense* of the vector is determined as follows. In the case of infinitesimal rotations it appears from Arts. 67 and 64 that a couple of the type A, Fig. 24, produces a translation upwards from the plane of the figure, *i.e.* towards the reader; while a couple of the type B produces a downward translation, away from the reader. Regarding the couples as rigid figures, their rotors as forces, and the middle point of their arms as fixed, the type A tends to produce rotation in the counter-clockwise, positive, sense; the type B in the negative sense. The former is therefore regarded as positive, and its vector is drawn from its plane towards the reader.

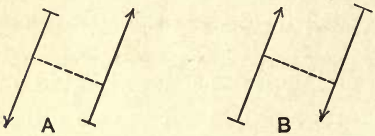

Fig. 24.

72. Let us now return to our infinitesimal displacements.

An infinitesimal translation ds *can be combined with an infinitesimal rotation* dθ *about an axis* l *at right angles to* ds (Fig. 25). To find the resultant single rotation we have only to replace the translation ds by an equivalent couple whose angle of rotation we select equal to that of the given rotation; that is, we put $ds = L_1L_2 \cdot d\theta$, whence

$$L_1L_2 = \frac{ds}{d\theta}.$$

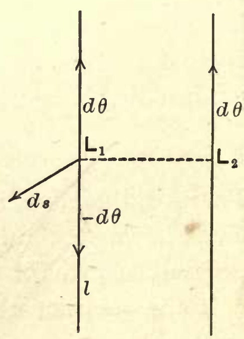

Fig. 25.

The plane of the couple, being perpendicular to ds, can be taken so as to contain the axis l of the given rotation $d\theta$; and in this plane the couple can be so placed that one of its sides (see Fig. 25) falls into this axis l. Selecting the proper side of the couple, we shall have on l two equal and opposite rotations $d\theta, -d\theta$, which destroy each other, leaving only the rotation $d\theta$, about an axis at the distance $L_1L_2 = ds/d\theta$ from l.

Thus it is seen that *the combination of an infinitely small rotation* $d\theta$, *with an infinitely small translation* ds *at right angles to the axis of rotation, produces a single rotation of the same angle about a parallel axis at a distance* $ds/d\theta$ *from the original axis in the plane through this axis perpendicular to the direction of translation.*

73. Exercises.

(1) The telescope of a theodolite, originally horizontal and pointing north, is tipped into an elevation of 60°, and then turned into the prime vertical so as to point west. What single rotation is equivalent to the two successive rotations?

(2) In the preceding example, what would be the result of inverting the order of the two rotations?

(3) The motion of a man in walking may be approximately described as consisting at every step of two rotations of the body about parallel axes perpendicular to the direction of motion, one axis passing through the hip-joint, the other through the foot that remains on the ground while the other foot is thrown forward. Find the angle of swing (assuming the two rotations to be equal and opposite) if the length of the step is 15 inches and the height of the hip-joint $3\frac{1}{4}$ feet.

3. SCREW MOTIONS; TWISTS.

74. We have seen in Arts. 40, 41 that a twist, *i.e.* a rotation combined with a translation parallel to the axis of rotation, constitutes the most general form of the displacement of a rigid body. We proceed to discuss the most important cases of the compositions of rotations and translations resulting in twists.

75. A rotation of angle θ about an axis l can be combined with a translation whose vector is s, by resolving s into two components; s_1 perpendicular to l, and s_2 parallel to l. The former component combines (by Art. 65) with the rotation into a single rotation of the same angle θ about an axis parallel to l. The result is therefore a rotation accompanied by a translation s_2 parallel to the axis of rotation, *i.e.* a twist.

76. If the rotation $d\theta$ and the translation ds are infinitesimal, the axis of the resulting twist has (by Art. 68) a distance $ds/d\theta$ from the axis l of the rotation $d\theta$ and lies in the plane laid through l at right angles to ds.

77. Skew Axes. *The resultant of two successive rotations* θ_1 *and* θ_2 *about two skew axes* l_1 *and* l_2 *is a twist.* This follows of course from the proposition of Art. 40. The axis of the resulting twist is the central axis of the displacement; its direction and position can be found as in Art. 42. Fig. 26 illustrates the process. L_1L_2 is the shortest distance of the axes l_1, l_2. The first rotation, θ_1, about l_1, brings l_2 into its final position l'_2, and L_2 into L'_2; the second rotation, θ_2 about l'_2, brings l_1 into its final position l'_1, and L into L'_1. The axis l of the resulting twist will evidently be the shortest distance of the bisectors of the angles $L_2L_1L'_2$ and $L_1L'_2L'_1$. For a rotation about this line l brings l_2 into l'_2 and l_1 into l'_1.

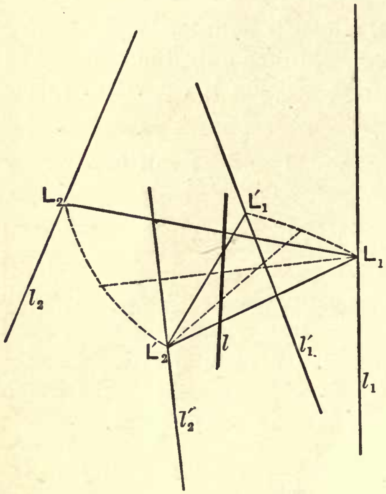

Fig. 26.

78. The angle of the resulting twist is the same as the angle of the rotation resulting from two rotations θ_1, θ_2 about two *intersecting* axes parallel to the given axes l_1, l_2. For (by Art. 65) either one of the rotations, say θ_2 about l_2, may be replaced by a rotation of the same angle θ_2 about an axis parallel to l_2 and intersecting l_1, combined with a translation at right angles to l_2. The two rotations about the intersecting axes can then be combined into a single rotation, and the angle and direction of the axis of this latter rotation are not changed by combination with the translation (Art. 74).

79. It follows from the two preceding articles that a twist can always be resolved into two rotations about skew axes, and this can be done in an infinite number of ways. It is also easy to see that two, or any number of, successive twists can be combined into a single twist by resolving each twist into its rotation and translation, and combining all rotations into a resulting twist and all translations into a resulting translation; the resulting twist combined with the resulting translation gives the twist equivalent to all the given twists.

80. For a more complete account of the geometry of motion the student is referred to A. SCHOENFLIES, *Geometrie der Bewegung*, Leipzig, Teubner, 1886; and to W. SCHELL, *Theorie der Bewegung und der Kräfte*, Leipzig, Teubner, Vol. I., 1879, pp. 144–187. See also R. S. BALL, *Theory of screws*, Dublin, Hodges, 1876; and H. GRAVELIUS, *Ball's theoretische Mechanik starrer Systeme*, Berlin, Reimer, 1889,— for the more advanced parts of the subject. Many authors treat the geometry of motion in connection with Kinematics; see the references in Chapter II., in particular the works of Burmester, Resal, Villié.

Applications to mechanism and machinery will be found in F. REULEAUX, *Kinematics of machinery, edited by A. B. W. Kennedy*, London, Macmillan, 1876; in J. H. COTTERILL, *Applied mechanics*, London, Macmillan, 1884, pp. 99–134; and in ALEX. B. W. KENNEDY, *The mechanics of machinery*, London, Macmillan, 1886.

CHAPTER II.

KINEMATICS.

I. *Time.*

81. Before introducing the idea of time into the study of motion, a word must be said on the measurement of time.

It is the province of astronomy to devise methods for measuring time; the usual method consists in transit observations. Thus the fundamental unit of time in astronomy, or the **sidereal day**, is the interval between two successive upper transits of the true vernal equinox over the same meridian.

82. For the purposes of every-day life, it is more convenient to make the measurement of time depend on the apparent revolution of the sun. But the interval between two successive upper transits of the sun over the same meridian, which is the *true*, or *apparent solar day*, is not constant throughout the year, owing to the inclination of the earth's axis to the plane of its orbit and to the ellipticity of this orbit. The true solar day is thus not well adapted to serve as a unit of time.

Astronomers imagine, therefore, a so-called *first mean sun* moving uniformly in the ecliptic so as to pass the perigee simultaneously with the real sun; and a *second mean sun* moving uniformly in the equator so as to pass the vernal equinox simultaneously with the first mean sun. The interval between two successive passages of the second mean sun over the same meridian is called the **mean solar day**. This may be regarded as the standard on which all time-determinations in mechanics are based.

The mean solar day is subdivided into 24 hours = 1440 minutes = 86 400 seconds. In theoretical mechanics the **second** is generally used as the unit of time.

83. To reduce mean time to apparent time, it is only necessary to subtract from mean time the so-called *equation of time*, whose value for any particular date is given in the Ephemeris.

84. The relation between mean solar time and sidereal time is readily found by considering that the tropical year, *i.e.* the interval between two successive passages of the sun through the mean vernal equinox, has 365.2422 mean solar days, and of course just one more sidereal day. Hence 1 solar day $= 366.2422/365.2422 = 1.002738$ sidereal day; in other words, the sidereal day contains 86 164.1 seconds of mean time, while the solar day contains 86 400 such seconds.*

85. It will have been noticed that all these methods of measuring time are ultimately based on the assumption that the rotation of the earth on its axis is perfectly uniform. Observation shows this assumption to be true, or at least to have a very high degree of approximation.

It might be asked how we can know, without using some unit of time for comparison, that the earth's rotation on its axis is uniform; in other words, that the mean solar day is constant. Our absolute unit of time would seem to be obtained by reasoning in a circle. This objection is not quite without foundation; and as similar difficulties arise in the case of other fundamental data of mechanics, it may be well to consider the matter a little more in detail.

86. The simplest answer is that we *assume* the constancy of the mean solar day and find this assumption fully justified by the fact that while the whole structure of the astronomical and physical sciences rests on this assumption, the theoretical predictions of these sciences are found to be in close agreement with the results of direct observation.

Historically, the assumption was originally adopted on account of its

---

* For further particulars see W. CHAUVENET, *Spherical and practical astronomy*, Vol. I., p. 52 sq. and pp. 651–654; also the *American Ephemeris and Nautical Almanac*.

simplicity, as a practical working hypothesis, and it was found to work well. From the logical point of view we may strengthen its probability by the following considerations.

87. The origin of our notion of time as a measurable quantity lies in the subjective sensation that teaches us instinctively to distinguish between shorter and longer intervals of time. This feeling of time is of course (just as in the analogous case of muscular force) far too vague and indefinite to admit of measurement. But it is sufficient to convince us that, approximately, the lengths of successive days are equal. With far greater approximation can we judge by our time-feeling that the oscillations of the pendulum of a clock are nearly isochronous. Let us combine these two entirely independent facts. Careful observation will show that the number of oscillations made by the pendulum in the interval between two culminations of the mean sun is almost precisely the same for every mean day. Moreover, the agreement becomes the more perfect the more we eliminate any causes that tend to disturb the isochronism of the pendulum. It will therefore be reasonable to conclude that the mean solar day must have a very nearly constant length.

But it is to be kept in mind that this is an empirical fact and hence not absolutely true, but only within the limits of the errors of observation. Indeed, certain considerations concerning the friction caused by the tides make it probable that the angular velocity of the earth is diminishing very slowly.*

---

* See O. RAUSENBERGER, *Analytische Mechanik*, I., Leipzig, Teubner, 1888, p. 14; H. STREINTZ, *Physikalische Grundlagen der Mechanik*, Leipzig, Teubner, 1883, p. 81 sq.; E. BUDDE, *Allgemeine Mechanik*, I., Berlin, Reimer, 1890, p. 33; THOMSON and TAIT, *Natural philosophy*, I., London, Macmillan, 1879, p. 460; J. C. MAXWELL, *Matter and motion*, New York, Van Nostrand, 1878, p. 27 and p. 60.; K. PEARSON, *Grammar of science*, London, Scott, 1892, pp. 217–230.

II. *Linear Kinematics.*

I. UNIFORM RECTILINEAR MOTION; VELOCITY.

88. Consider a point moving in a straight line. If throughout the whole motion equal spaces are always described in equal times, the motion is said to be **uniform.**

89. Next consider two points each moving uniformly in a straight line. The motions may still be different; for it is possible that while one of the points moves in a given time t over a space s_1, the other moves during the same time t over a different space s_2. The points are then said to have different velocities, and their velocities are said to be as s_1 is to s_2. The **velocity** v of uniform motion is therefore measured by the ratio of the space s described in any time t to this time; that is, $v = s/t$.

90. This equation written in the form

$$s = vt \tag{1}$$

is called the **equation of motion** of the point. It follows from Art. 89 that in uniform motion the velocity v is constant.

With t as abscissa and s as ordinate (or *vice versa*), the equation of uniform motion (1) represents a straight line; the tangent of the angle made by this line with the axis of t represents the velocity v.

91. Let the point P start at the time $t = 0$ from a point O (Fig. 27); let it reach the point P_0 at the time $t = t_0$ and the

Fig. 27.

point P_1 at the time $t = t$. Then, putting $OP_0 = s_0$, $OP_1 = s$, the space passed over in the time $t - t_0$ is $s - s_0$; hence the velocity $v = (s - s_0)/(t - t_0)$. The equation of uniform motion can therefore be written in the form

$$s - s_0 = v(t - t_0). \tag{1'}$$

If the times be counted from the instant when the moving point is at P_0, we have $t_0=0$, and the equation of motion is

$$s=s_0+vt. \qquad (1'')$$

Finally, if both times and spaces are counted from P_0 as origin, we have $s_0=0$, so that $(1'')$ reduces to (1).

92. To measure velocities we must adopt a **unit of velocity.**

In kinematics, the only **fundamental,** *i.e.* independent, units required are those of length and time. All other quantities can be expressed in terms of length and time, and their units are therefore called **derived** units.

Thus, the definition of the velocity of uniform motion as a length divided by a time (Art. 89) can be expressed by the symbolic equation

$$\mathbf{V}=\frac{\mathbf{L}}{\mathbf{T}}, \text{ or } \mathbf{V}=\mathbf{LT}^{-1},$$

and we say that the **dimensions** of velocity are 1 in length and −1 in time.

When $\mathbf{L}=1$ and $\mathbf{T}=1$, we have $\mathbf{V}=1$. We must therefore select for our unit of velocity that velocity with which unit length is described in unit time.

Hence in the C. G. S. system (see Arts. 13, 14) the unit velocity is a velocity of 1 cm. per second; in the F. P. S. system it is a velocity of 1 ft. per second.

93. In practice other units are often used, and the same concrete velocity can therefore be expressed by different numbers. Thus the same velocity of a railroad train can be described as 30 miles per hour, or 44 ft. per second, or (approximately) 13.41 metres per second.

The symbols s, v, t, etc., in the kinematical equations must be understood to represent the numerical **ratios** of the concrete quantities to their respective units. The symbol v, for instance, stands for the ratio V/V_1 of the concrete velocity V to its unit

V_1, and we have of course the proportion: 30 miles an hour is to 1 mile an hour as 44 ft. per second is to 1 ft. per second, etc.

94. The full meaning of the equation of dimensions $\mathbf{V}=\mathbf{LT}^{-1}$ is obtained if we substitute V/V_1 for **V**, L/L_1 for **L**, T/T_1 for **T**, where V, L, T are the concrete quantities and V_1, L_1, T_1 their units. We find

$$\frac{V}{V_1}=\frac{L}{L_1}\cdot\frac{T_1}{T};$$

and this equation shows two things which are of frequent application in reductions between different systems of units:

(*a*) The *numerical value* V/V_1 of a velocity varies directly as the unit of time and inversely as the unit of length;

(*b*) the *unit* of velocity V_1 varies directly as the unit of length and inversely as the unit of time.*

95. In speaking of velocities, the time unit (usually the second) is frequently understood without being mentioned. This has led to considering velocity as a length (viz. the length passed over in unit time); it can then be represented graphically by a segment of a straight line, and if in addition we combine with the idea of velocity that of the *direction* and *sense* of the motion, its geometrical representative will be a vector (see Art. 45). We shall see later that this view is of particular advantage in studying the velocity of curvilinear motion.

Some recent writers on mechanics use the term *velocity* exclusively in this meaning, *i.e.* as denoting a vector, and apply the term *speed* to denote the numerical magnitude of this vector. In linear kinematics the direction is given, and the "speed" alone is the subject of investigation. The + or − sign of the "speed" expresses the *sense* of the motion.†

96. Exercises.

(1) A train leaves the station A at 9 h. 5 m., passes (without stop-

---

* See J. D. Everett, *C. G. S. system of units*, 1891, p. 3.

† See *Syllabus of elementary dynamics*, Part I., prepared by the Association for the Improvement of Geometrical Teaching; London, Macmillan, 1890, p. 8.

ping) B at 9 h. 31 m., C at 9 h. 47 m., and arrives at D at 9 h. 59 m., the distance AD being 36.9 miles. Considering the motion as uniform:

(*a*) What is the velocity?

(*b*) What is the equation of motion?

(*c*) What are the distances BD and CD?

(*d*) If after stopping 5 minutes at D the train goes on with the same velocity as before, when will it reach E, $10\frac{1}{4}$ miles beyond D?

(*e*) Construct a graphical time-table, taking the times as abscissas and the distances as ordinates.

(2) Interpret equations (1′) and (1″) geometrically.

(3) A train leaves Detroit at 9 h. 5 m. A.M. and reaches Chicago at 4 h. 30 m. P.M.; another train leaves Chicago at 12 h. 20 m. and arrives in Detroit at 7 h. 25 m. P.M. The distance is 285 miles. Regarding the motion as uniform and neglecting the stops, find, both analytically and graphically, when and where the trains will meet.

(4) Reduce the following velocities to F. P. S. units: (*a*) Walking 4 miles an hour; (*b*) trotting a mile in 2 m. 10 s.; (*c*) railroad train, from 30 to 50 miles per hour; (*d*) bicyclist, 2 miles in 4 m. $59\frac{3}{5}$ s.; (*e*) sound in air, 333 metres per second.

(5) What is the numerical value of a velocity of 22 ft. per second when the hour is taken as unit of time and the mile as the unit of length?

(6) How is the unit of velocity changed if the minute be adopted as unit of time, the unit of length remaining unchanged?

(7) The mean distance of the sun being $92\frac{1}{3}$ million miles, what is the velocity of light if it takes light 16 m. 40 s. to cross the earth's orbit?

(8) Two trains are running on the same track at the rate of 25 and 15 miles per hour, respectively. If at a certain instant they are 10 miles apart, find when they will collide (*a*) if they are headed the same way; (*b*) if they run in opposite directions.

(9) In what latitude is a bullet shot west with a velocity of 1320 ft. per second at rest relatively to the earth's axis, the radius being taken as 4000 miles?

(10) Two trains, one 250, the other 440 ft. long, pass each other on parallel tracks in opposite directions with equal velocity. A passenger in the shorter train observes that it takes the longer train just 4 seconds to pass him. What is the velocity?

101. If v be given as function of t, say $v = \phi(t)$, we find from (2) $ds = vdt$, and hence by integration

$$s - s_0 = \int_{t_0}^{t} vdt, \qquad (3)$$

where s_0 is the space described during the time t_0. The equation $v = \phi(t)$ furnishes a graphical representation of the velocity, and formula (3) shows that the space $s - s_0$ described during the time $t - t_0$ is represented by the area included between the curve $v = \phi(t)$, the axis Ot, and the ordinates v_0 and v corresponding to t_0 and t, respectively (Fig. 29).

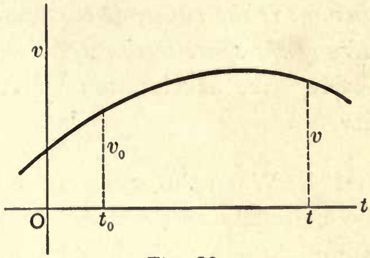

Fig. 29.

102. Similarly, if v be given as a function of s, say $v = \psi(s)$, we have from (2) $dt = ds/v$, and hence

$$t - t_0 = \int_{s_0}^{s} \frac{ds}{v}. \qquad (4)$$

The two velocity curves $v = \phi(t)$ and $v = \psi(s)$ are of course in general different, and must not be confounded with the *path* of the moving point, which is here supposed rectilinear.

103. We have seen (Art. 91, equation (1″)) that in the case of uniform motion the velocity $v = (s - s_0)/t$, *i.e.* the rate of change of space with time, is constant. The simplest case of variable motion is that in which the velocity varies uniformly. *The rate at which the velocity varies with the time* is called the **acceleration**; we shall denote it by j.

If the velocity vary uniformly, the acceleration is constant, and we have $j = (v - v_0)/t$, where t is the time during which the velocity changes from v_0 to v.

By reasoning analogous to that employed in Art. 99, we find for the acceleration of *any* rectilinear motion at the time t

$$j = \frac{dv}{dt} = \frac{d^2s}{dt^2}; \qquad (5)$$

that is, *in rectilinear motion the acceleration at any point or instant is the value, at that point or instant, of the second derivative of the space with respect to the time.*

Negative acceleration will thus indicate a decreasing velocity.

104. When the acceleration is constant, the motion is said to be **uniformly accelerated.** In the case of variable acceleration we might again consider its rate of change, which may be called the *acceleration of the second order;* and so on. Compare Art. 156.

105. Conformably to the definition of acceleration, its unit is the "cm. per second per second" in the C. G. S. system, and the "foot per second per second" in the F. P. S. system. As it can rarely be convenient to use two different time units in the unit of acceleration (say, for instance, mile per hour per second), it is customary to mention the time unit but once and to speak of an acceleration of so many feet per second, or cm. per second, it being understood that the other time unit is also the second.

For the *dimensions* of acceleration we have (see Art. 92)

$$\mathbf{J} = \mathbf{VT}^{-1} = \mathbf{LT}^{-2}.$$

Denoting, as in Arts. 93, 94, the concrete value of an acceleration by J, its unit by J_1, and similarly for length and time, we have the equation

$$\frac{J}{J_1} = \frac{L}{L_1} \cdot \frac{T_1^2}{T^2},$$

which shows that (*a*) the numerical value J/J_1 of an acceleration varies directly as the square of the unit of time, and inversely as the unit of length; and (*b*) the unit of acceleration, J_1, varies directly as the unit of length, and inversely as the square of the unit of time.

106. Exercises.

(1) A point moving with constant acceleration gains at the rate of 30 miles an hour in every minute. Express its acceleration in F. P. S. units.

(2) At a place where the acceleration of gravity is $g = 9.810$ metres per second, what is the value of g in feet per second?

(3) A railroad train, 10 minutes after starting, attains a velocity of 45 miles an hour; what was its average acceleration during these 10 minutes?

(4) If the acceleration of gravity, $g = 32$ feet per second, be taken as unit, what is the acceleration of the railroad train in Ex. (3)?

3. APPLICATIONS.

107. Uniformly Accelerated Motion. As in this case the acceleration j is constant (see Art. 103), the equation of motion (5)

$$\frac{d^2s}{dt^2} = j, \text{ or } \frac{dv}{dt} = j,$$

can readily be integrated:

$$v = jt + C.$$

To determine the constant of integration C, we must know the value of the velocity at some particular moment of time. Thus, if $v = v_0$ when $t = 0$, we find $v_0 = C$; hence, substituting this value for C,

$$v - v_0 = jt. \tag{6}$$

This equation, which agrees with the definition of j given in Art. 103, gives the velocity at any time t. Substituting ds/dt for v and integrating again, we find $s = v_0t + \frac{1}{2}jt^2 + C'$, where the constant of integration, C', must again be determined from given "initial conditions." Thus, if we know that $s = s_0$ when $t = 0$, we find $s_0 = C'$; hence

$$s - s_0 = v_0t + \tfrac{1}{2}jt^2. \tag{7}$$

This equation gives the space or distance passed over in terms of the time.

108. Eliminating j between (6) and (7), we obtain the relation

$$s - s_0 = \frac{1}{2}(v_0 + v)t,$$

which shows that in uniformly accelerated motion the space can be found as if it were described uniformly with the mean velocity $\frac{1}{2}(v_0 + v)$.

109. To obtain the velocity in terms of the space, we have only to eliminate t between (6) and (7); we find

$$\frac{1}{2}(v^2 - v_0^2) = j(s - s_0). \tag{8}$$

This relation can also be derived by eliminating dt between the differential equations $v = ds/dt$, $dv/dt = j$, which gives $vdv = jds$, and integrating. The same equation (8) is also obtained directly from the fundamental equation of motion $d^2s/dt^2 = j$ by a process very frequently used in mechanics, viz. by multiplying both members of the equation by ds/dt. This makes the left-hand member the exact derivative of $\frac{1}{2}(ds/dt)^2$ or $\frac{1}{2}v^2$, and the integration can therefore be performed.

110. The three equations (6), (7), (8) contain the complete solution of the problem of uniformly accelerated motion. For uniformly retarded motion, taking the direction of motion as positive, we have only to write $-j$ for $+j$.

If the spaces be counted from the position of the moving point at the time $t = 0$, we have $s_0 = 0$, and the equations become

$$v = v_0 + jt, \tag{6'}$$

$$s = v_0 t + \frac{1}{2}jt^2, \tag{7'}$$

$$\frac{1}{2}(v^2 - v_0^2) = js. \tag{8'}$$

111. If in addition the initial velocity v_0 be zero, the point starting from rest at the time $t=0$, the equations reduce to the following:

$$v=jt, \tag{6''}$$

$$s=\tfrac{1}{2}jt^2, \tag{7''}$$

$$\tfrac{1}{2}v^2=js. \tag{8''}$$

112. The most important example of uniformly accelerated motion is furnished by a body falling in vacuo near the earth's surface. Assuming that the body does not rotate during its fall, its motion relative to the earth is a mere translation, and it is sufficient to consider the motion of any one point of the body. It is known from observation and experiment that under these circumstances the acceleration of a falling body is constant at any given place and equal to about 980 cm., or 32 ft., per second per second.; the value of this so-called *acceleration of gravity* is usually denoted by g.

In the exercises on falling bodies (Art. 114) we make throughout the following simplifying assumptions: the falling body does not rotate; the resistance of the air is neglected, or the body falls in vacuo; the space fallen through is so small that g may be regarded as constant; the earth is regarded as fixed, *i.e.* we consider only the relative motion of the body with respect to the earth.

113. The velocity v acquired by a falling body after falling from rest through a height h is found from (8″) as

$$v=\sqrt{2gh}.$$

This is usually called the **velocity due to the height** (or **head**) **h,** while

$$h=\frac{v^2}{2g}$$

is called the **height** (or **head**) **due to the velocity v.**

114. Exercises.

(1) A body falls from rest at a place where $g = 32.2$. Find (a) the velocity at the end of the third second; (b) the space fallen through in 5 seconds; (c) the space fallen through in the fifth second.

(2) If a railroad train, at the end of 2 m. 40 s. after leaving the station, has acquired a velocity of 30 miles per hour, what was its acceleration (regarded as constant)?

(3) Galilei, who first discovered the laws of falling bodies, expressed them in the following form: (a) The velocities acquired at the end of the successive seconds increase as the natural numbers; (b) the spaces described during the successive seconds increase as the odd numbers; (c) the spaces described from the beginning of the motion to the end of the successive seconds increase as the squares of the natural numbers. Prove these statements.

(4) A stone dropped into the vertical shaft of a mine is heard to strike the bottom after t seconds; find the depth of the shaft, if the velocity of sound be given $= c$. Assume $t = 4$ s., $c = 332$ metres, $g = 980$.

(5) A railroad train approaches a station with uniformly retarded motion. During the first two minutes of its retarded motion it makes 3960 ft.; during the next minute 990 ft. (a) When will it come to rest? (b) What is the retardation? (c) What was the initial velocity? (d) When will its velocity be 4 miles an hour?

(6) Interpret equations (6) and (7) geometrically.

(7) A body being projected vertically upwards with an initial velocity v_0, (a) how long and (b) to what height will it rise? (c) When and (d) with what velocity does it reach the starting-point?

(8) A bullet is shot vertically upwards with an initial velocity of 1600 ft. per second. (a) How high will it ascend? (b) What is its velocity at the height of 32,000 ft.? (c) When will it reach the ground again? (d) With what velocity? (e) At what time is it 32,000 ft. above the ground? (f) Explain the meaning of the double signs wherever they occur in the answers.

(9) With what velocity must a ball be thrown vertically upwards to reach a height of 100 ft.?

(10) A body is dropped from a point A at a height $AB = h$ above the ground; at the same time another body is thrown vertically

upward from the point B, with an initial velocity v_0. (*a*) When and (*b*) where will they collide? (*c*) If they are to meet at the height $\frac{1}{2}h$, what must be the initial velocity?

115. The general problem of rectilinear motion requires the integration of the differential equation

$$\frac{d^2s}{dt^2}=j, \tag{5}$$

where j is a function of s, t, and v, in connection with the equation

$$\frac{ds}{dt}=v. \tag{2}$$

As these two equations involve four quantities t, s, v, j, a third relation between them, say

$$f(t, s, v, j)=0, \tag{9}$$

is always necessary in order to express three of these four quantities in terms of the fourth. Next to the case of uniformly accelerated motion where the relation (9) is simply $j=\text{const.}$, the most important cases are those when j is given as a function of s, or of v, or of both s and v.

116. Whenever in nature we observe a motion not to remain uniform, we try to account for the change in the character of the motion by imagining a special cause for such change. In rectilinear motion, the only change that can occur in the motion is a change in the velocity, *i.e.* an acceleration (or retardation). The cause producing acceleration or retardation we call **force** (attraction, repulsion, pressure, tension, friction, resistance of a medium, elasticity, cohesion, etc.), and assume it to be proportional to the acceleration. A fuller discussion of the nature of force and its relation to mass will be found in Chapter III., § II. The present remark is only intended to make more intelligible the physical meaning and applications of the problems to be discussed in the following articles.

117. Acceleration inversely proportional to the square of the distance, *i.e.* $j=\mu/s^2$ where μ is a constant (viz. the acceleration at the distance $s=1$) and s is the distance of the moving point from a fixed point in the line of motion.

The differential equation (5) becomes in this case

$$\frac{d^2s}{dt^2}=\frac{\mu}{s^2}; \tag{10}$$

the first integration is readily performed by multiplying both members by ds/dt so as to make the left-hand member the complete derivative of $\frac{1}{2}(ds/dt)^2$ or $\frac{1}{2}v^2$. Thus we find

$$\tfrac{1}{2}v^2=\mu\int\frac{ds}{s^2}+C=-\frac{\mu}{s}+C, \tag{11}$$

where the constant of integration, C, must be determined from the so-called initial conditions of the problem. For instance, if $v=v_0$ when $s=s_0$, we have $\frac{1}{2}v_0^2=-\mu/s_0+C$; hence, eliminating C between this relation and (11),

$$\tfrac{1}{2}(v^2-v_0^2)=-\mu\left(\frac{1}{s}-\frac{1}{s_0}\right). \tag{12}$$

To perform the second integration, we solve this equation for v and substitute ds/dt for v:

$$\frac{ds}{dt}=\sqrt{\frac{1}{s}[(v_0^2+2\mu/s_0)s-2\mu]},$$

or putting $v_0^2+2\mu/s_0=2\mu/\mu'$,

$$\frac{ds}{dt}=\sqrt{\frac{2\mu}{\mu'}}\cdot\sqrt{\frac{s-\mu'}{s}}. \tag{13}$$

Here the variables s and t can be separated, and we find

$$t=\sqrt{\frac{\mu'}{2\mu}}\int\sqrt{\frac{s}{s-\mu'}}\,ds+C'. \tag{14}$$

To integrate, put $s=x^2$. The result will be different according to the signs of μ, μ', and v, which must be determined from the nature of the particular problem.

118. It is an empirical fact that the acceleration of bodies falling in vacuo on the earth's surface is constant only for distances from the surface that are very small in comparison with the radius of the earth. For larger distances the acceleration is found inversely proportional to the square of the distance from the earth's centre.

By a bold generalization Newton assumed this law to hold generally between any two particles of matter; and this assumption has been verified by all subsequent observations. It can therefore be regarded as a general law of nature that any particle of matter produces in every other such particle, each particle being regarded as concentrated at a point, an acceleration inversely proportional to the square of the distance between these points. This is known as *Newton's law of universal gravitation*, the acceleration being regarded as caused by a force of attraction inherent in each particle of matter.

It is shown in the theory of attraction that the attraction of a spherical mass, such as the earth, on any particle *outside* the sphere is the same as if the mass of the sphere were concentrated at its centre. The acceleration produced by the earth on any particle outside it is therefore inversely proportional to the square of the distance of the particle from the centre of the earth.

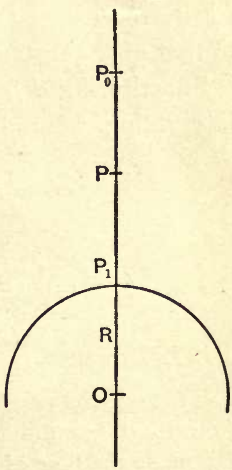

Fig. 30.

119. Let us now apply the general equations of Art. 117 to the particular case of a body falling from a great height towards the centre of the earth, the resistance of the air being neglected.

Let O be the centre of the earth (Fig. 30), P_1 a point on its surface, P_0 the initial position of the moving point at the time $t=0$, P its position at the time t; let $OP_1=R$, $OP_0=s_0$, $OP=s$; and let g be the acceleration at P_1, j the acceleration at P. Then, according to Newton's law, $j:g=R^2:s^2$. This relation determines the value of μ in (10), which becomes

$$\frac{d^2s}{dt^2}=-\frac{gR^2}{s^2}, \tag{15}$$

the minus sign indicating that the acceleration tends to diminish the distances counted from O as origin.

The integration can now be performed as in Art. 117. Multiplying by ds/dt and integrating, we find $\frac{1}{2}v^2=gR^2/s+C$. If the initial velocity be zero, we have $v=0$ for $s=s_0$; hence $C=-gR^2/s_0$, and

$$v=-R\sqrt{2g}\sqrt{\frac{1}{s}-\frac{1}{s_0}}=-R\sqrt{\frac{2g}{s_0}}\sqrt{\frac{s_0-s}{s}}. \tag{16}$$

Here again the minus sign is selected after extracting the square root, since the velocity v is directed in the sense opposite to that of the distance s.

Substituting ds/dt for v, separating the variables v and s, and integrating, we find

$$t=-\frac{1}{R}\sqrt{\frac{s_0}{2g}}\int_{s_0}^{s}\sqrt{\frac{s}{s_0-s}}\,ds$$

$$=\frac{1}{R}\sqrt{\frac{s_0}{2g}}\left\{\sqrt{s(s_0-s)}+s_0\cos^{-1}\sqrt{\frac{s}{s_0}}\right\}. \tag{17}$$

120. Exercises.

(1) Find the velocity with which the body arrives at the surface of the earth if it be dropped from a height equal to the earth's radius, and determine the time of falling through this height.

(2) Interpret equation (17) geometrically.

(3) Show that formula (16) reduces to $v=\sqrt{2gh}$ (Art. 113) when $s=R$ and $s_0-s=h$ is small in comparison with R.

(4) A particle is projected vertically upwards from the earth's surface with an initial velocity v_0. How far will it rise?

(5) If, in (4), the initial velocity be $v_0 = \sqrt{gR}$, how high and how long will the particle rise? How long will it take the particle to rise and fall back to the earth's surface?

(6) A body is projected vertically upwards. Find the least initial velocity that would prevent it from returning to the earth, taking $g = 32$, $R = 4000$ miles.

121. Acceleration directly proportional to the distance, *i.e.* $j = \kappa s$, where κ is a constant and s is the distance of the moving point from a fixed point in the line of motion.

The equation of motion

$$\frac{d^2 s}{dt^2} = \kappa s \tag{18}$$

can be integrated by the method used in Art. 117. The result of the second integration will again be different according to the sign of κ. We shall here study only a special case, reserving the general discussion of this law of acceleration for later (see Arts. 177 sq.).

122. It is shown in the theory of attraction that the attraction of a spherical mass such as the earth on any point *within* the mass produces an acceleration directed to the centre of the sphere and proportional to the distance from this centre. Thus, if we imagine a particle moving along a diameter of the earth, say in a straight narrow tube passing through the centre, we should have a case of the motion represented by equation (18).

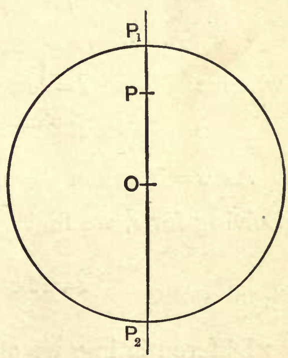

Fig. 31.

To determine the value of κ for our problem we notice that at the earth's surface, that is, at the distance $OP_1 = R$ from the centre O (Fig. 31), the acceleration must be $= g$. If, therefore, j denote the numerical value of the acceleration at any distance $OP=s\,(<R)$, we have $j:g=s:R$, or $j=gs/R$. But the acceleration tends to diminish the distance s, hence $\frac{d^2s}{dt^2}=-\frac{g}{R}s$. Denoting the positive constant g/R by μ^2, the equation of motion is

$$\frac{d^2s}{dt^2}=-\mu^2 s, \text{ where } \mu=\sqrt{\frac{g}{R}}. \tag{19}$$

Integrating as in Arts. 117 and 119, we find

$$\tfrac{1}{2}v^2=-\tfrac{1}{2}\mu s^2+C.$$

If the particle starts from rest at the surface, we have $v=0$ when $s=R$; hence $0=-\frac{1}{2}\mu^2R^2+C$; and subtracting this from the preceding equation, we find

$$v=-\mu\sqrt{R^2-s^2}, \tag{20}$$

where the minus sign of the square root is selected because s and v have opposite sense.

Writing ds/dt for v and separating the variables, we have

$$dt=-\frac{1}{\mu}\frac{ds}{\sqrt{R^2-s^2}},$$

whence $$t=\frac{1}{\mu}\cos^{-1}\frac{s}{R}+C'.$$

As $s=R$ when $t=0$, we have $0=\frac{1}{\mu}\cos^{-1}1+C'$, or $C'=0$. Solving for s, we find

$$s=R\cos\mu t. \tag{21}$$

Differentiating, we obtain v in terms of t:

$$v=-\mu R\sin\mu t. \tag{22}$$

123. The motion represented by equations (21) and (22) belongs to the important class of *simple harmonic motions* (see Arts. 177 sq.). The particle reaches the centre when $s=0$, *i.e.* when $\mu t=\pi/2$, or at the time $t=\pi/2\mu$. At this time the velocity has its maximum value. After passing through the centre the point moves on to the other end, P_2, of the diameter, reaches this point when $s=-R$, *i.e.* when $\mu t=\pi$, or at the time $t=\pi/\mu$. As the velocity then vanishes, the moving point begins the same motion in the opposite sense.

The time of performing one complete oscillation (back and forth) is called the **period** of the simple harmonic motion; it is evidently

$$T=4\cdot\frac{\pi}{2\mu}=\frac{2\pi}{\mu}.$$

124. Exercises.

(1) Equation (19) is a differential equation whose general integral is known to be of the form

$$s=C_1\sin\mu t+C_2\cos\mu t;$$

determine the constants C_1, C_2 and deduce equations (21) and (22).

(2) Find the velocity at the centre and the period, taking $g=32$ and $R=4000$ miles.

(3) If the acceleration, instead of being directed toward the centre, is directed away from it, the equation of motion would be $d^2s/dt^2=\mu^2 s$. Investigate this motion, which can be imagined as produced by a force of repulsion emanating from the centre.

125. Retardation Due to a Resisting Medium. We know from observation that the velocity of a body moving in a liquid or gas is continually diminished. The resistance of such a medium may be regarded as a force producing a retardation, or negative acceleration. The same may be said of the effect of friction. The law according to which such resistances retard the motion must of course be determined by experiment.

Careful experiments on the resistance offered by the air to the motion of projectiles have shown that this resistance increases with the quantity of air displaced; that is, with the density of the air, the cross-section of the projectile, and the velocity. The retardation due to the resistance of the air can therefore be expressed in the form

$$j = \kappa\rho f(v),$$

where ρ is the density of the air, while κ is a coefficient depending upon the shape, mass, and physical condition of the surface of the projectile. Its value may be regarded as inversely proportional to the mass and directly proportional to the cross-section of the body at right angles to the direction of motion.

The velocity function $f(v)$ may be taken $=cv^2$ for velocities not exceeding 250 metres per second; for greater velocities, up to about 420 metres per second, it is proportional to a higher power of v, or must be represented by a more complicated expression, such as av^3+bv+c; for velocities above 420 metres it seems to be again of the form $c'v^2$.*

126. Assuming the resistance of the air to be proportional to the square of the velocity, the motion of a body falling through air of uniform density is determined by the equation

$$\frac{d^2s}{dt^2}=g-\kappa v^2.$$

To simplify the resulting formulæ, it will be convenient to put $\kappa=\frac{\mu^2}{g}$, so that the equation of motion is

$$\frac{d^2s}{dt^2}=\frac{g^2-\mu^2v^2}{g}. \qquad (23)$$

Writing $\frac{dv}{dt}$ for $\frac{d^2s}{dt^2}$, the variables v and t can be separated:

$$\frac{gdv}{g^2-\mu^2v^2}=dt;$$

* For further particulars the reader is referred to special works on ballistics.

integrating, we find

$$t=\frac{1}{2\mu}\log\frac{g+\mu v}{g-\mu v}, \tag{24}$$

the constant of integration being 0 if the initial velocity be 0. Solving for v, we have

$$v=\frac{g}{\mu}\frac{e^{\mu t}-e^{-\mu t}}{e^{\mu t}+e^{-\mu t}}. \tag{25}$$

As the numerator, apart from a constant factor, is the derivative of the denominator, the second integration can at once be performed, giving

$$s=\frac{g}{\mu^2}\log\left(e^{\mu t}+e^{-\mu t}\right)+C.$$

For $t=0$, we have $s=0$; hence $0=\frac{g}{\mu^2}\log 2+C$. Hence

$$s=\frac{g}{\mu^2}\log\tfrac{1}{2}\left(e^{\mu t}+e^{-\mu t}\right). \tag{26}$$

To find s in terms of v, we may eliminate dt between the differential equations $ds=vdt$ and $dv=\frac{1}{g}(g^2-\mu^2v^2)\,dt$. The resulting equation

$$ds=g\frac{v}{g^2-\mu^2v^2}dv$$

is readily integrated; as $v=0$ when $s=0$, we find:

$$s=\frac{g}{2\mu^2}\log\frac{g^2}{g^2-\mu^2v^2}. \tag{27}$$

127. Exercises.

(1) Show that as t increases, the motion considered in Art. 126 approaches more and more a state of uniform motion without ever reaching it.

(2) Show that when μ, and hence κ, becomes 0, the equations of Art. 126 reduce to those for bodies falling in vacuo.

(3) Investigate the motion of a particle thrown vertically upwards in the air with a given initial velocity, the resistance of the air being proportional to the square of the velocity.

(4) Find the whole time of ascent in (3) and the height to which the particle rises.

(5) Show that owing to the resistance of the air a particle thrown vertically upwards returns to the starting point with a velocity less than the initial velocity of projection.

(6) A particle begins moving with an initial velocity v_0 in a medium of constant density whose resistance is proportional to the velocity. Express s and v in terms of t, and v in terms of s.

(7) A body falls from rest in a medium whose resistance is proportional to the velocity. Investigate its motion.

4. ROTATION; ANGULAR VELOCITY; ANGULAR ACCELERATION.

128. A motion of rotation about a fixed axis can be treated in precisely the same way in which we have treated rectilinear motion in the preceding sections. It is only to be remembered that rotations are measured by angles (see Arts. 11–15), while translations are measured by lengths.

129. The rotation of a rigid body (see Art. 8) about a fixed axis is said to be *uniform* if the circular arcs described by the same point in equal times are equal throughout the whole motion; in other words, if the angle of rotation is proportional to the time in which it is described. In this case of uniform rotation, the quotient obtained by dividing the angle of rotation, θ, by the corresponding time, t, is called the **angular velocity.** Denoting it by ω we have $\omega = \theta/t$; and the equation of motion is

$$\theta = \omega t. \tag{1}$$

Thus, expressing the time in seconds and the angle in radians (Art. 15), the angular velocity is equal to the number of radians described per second. (Compare Arts. 88–90.)

130. If the time of a whole revolution be denoted by T, we have, from (1), $2\pi = \omega T$; hence

$$\omega = \frac{2\pi}{T}. \tag{2}$$

In engineering practice it is customary to take a whole revolution as angular unit and to express the angular velocity of uniform motion by the number of revolutions made in the unit of time. Let n, N be the numbers of revolutions per second and per minute, respectively; then we have evidently

$$n = \frac{\omega}{2\pi}, \quad N = \frac{30\,\omega}{\pi}. \tag{3}$$

131. When the rotation is *not* uniform, the quotient obtained by dividing the angle of rotation by the time in which it is described, gives the *mean*, or *average*, *angular velocity* for that time.

The rate of change of the angle of rotation with the time at any particular moment is called the *angular velocity at that moment.* By reasoning in a similar way, as in Art. 99, it will be seen that its mathematical expression is

$$\omega = \frac{d\theta}{dt}. \tag{4}$$

132. The rate at which the angular velocity changes with the time is called the **angular acceleration**; denoting it by α, we have

$$\alpha = \frac{d\omega}{dt} = \frac{d^2\theta}{dt^2}. \tag{5}$$

133. The most important special case of variable angular velocity is that of uniformly accelerated (or retarded) rotation when the angular acceleration is constant. The formulæ for this case have precisely the same form as those given in Arts. 107–111 for uniformly accelerated rectilinear motion. Denoting the constant linear acceleration by j, we have, when the initial velocity is 0,

FOR TRANSLATION:	FOR ROTATION:	
$v = jt,$	$\omega = \alpha t,$	
$s = \frac{1}{2} jt^2,$	$\theta = \frac{1}{2}\alpha t^2,$	(6)
$\frac{1}{2} v^2 = js;$	$\frac{1}{2}\omega^2 = \alpha\theta;$	

and when the initial velocities are v_0 and ω_0, respectively:

FOR TRANSLATION:	FOR ROTATION:	
$v=v_0+jt,$	$\omega=\omega_0+\alpha t,$	
$s=v_0t+\frac{1}{2}jt^2,$	$\theta=\omega_0 t+\frac{1}{2}\alpha t^2,$	(7)
$\frac{1}{2}v^2-\frac{1}{2}v_0^2=js;$	$\frac{1}{2}\omega^2-\frac{1}{2}\omega_0^2=\alpha\theta.$	

134. Let a point P, whose perpendicular distance from the axis of rotation is $OP=r$, rotate about the axis with the angular velocity $\omega=d\theta/dt$. In the element of time, dt, it will describe an element of arc $ds=rd\theta=r\omega dt$. Its velocity $v=ds/dt$ (frequently called its **linear** velocity in contradistinction to the angular velocity) is therefore related to the angular velocity of rotation by the equation

$$v=\omega r. \tag{8}$$

135. The radius vector $OP=r$ sweeps over a circular sector which in uniform rotation has an area $S=\frac{1}{2}\theta r^2=\frac{1}{2}\omega t r^2$, while in variable rotation the infinitesimal sector described during the element of time dt is $dS=\frac{1}{2}r^2d\theta=\frac{1}{2}\omega r^2dt.$

The quotients

$$\frac{S}{t}=\frac{1}{2}r^2\frac{\theta}{t}=\frac{1}{2}\omega r^2, \tag{9}$$

for uniform rotation, and

$$\frac{dS}{dt}=\frac{1}{2}r^2\frac{d\theta}{dt}=\frac{1}{2}\omega r^2, \tag{10}$$

for variable rotation, represent, therefore, the *sectorial*, or *areal, velocity*, *i.e.* the rate of increase of area with the time.

The rate of change of this velocity with the time,

$$\frac{d^2S}{dt^2}=\frac{1}{2}\frac{d}{dt}\left(r^2\frac{d\theta}{dt}\right), \tag{11}$$

is called the *sectorial*, or *areal, acceleration.*

136. Exercises.

(1) If a fly-wheel of 12 ft. diameter makes 30 revolutions per minute, what is its angular velocity, and what is the linear velocity of a point on its rim?

(2) A pulley 5 ft. in diameter is driven by a belt travelling 500 ft. a minute. Neglecting the slipping of the belt, find (*a*) the angular velocity of the pulley in radians, and (*b*) its number of revolutions per minute.

(3) Find the constant acceleration (such as the retardation caused by a Prony brake) that would bring the fly-wheel in Ex. (1) to rest in $\frac{1}{3}$ minute.

(4) How many revolutions does the fly-wheel in Ex. (3) make during its retarded motion before it comes to rest?

(5) A wheel is running at a uniform speed of 32 turns a second when a resistance begins to retard its motion uniformly at the rate of 8 radians per second. (*a*) How many turns will it make before stopping? (*b*) In what time is it brought to rest?

(6) A belt runs over two pulleys turning about parallel axes. Show that the angular velocities of the pulleys are inversely proportional to their diameters. Do the pulleys rotate in the same or opposite sense?

III. *Plane Kinematics.*

I. VELOCITY; COMPOSITION OF VELOCITIES; RELATIVE VELOCITY.

137. The motion of a point in a curved path would not be completely characterized by its velocity and acceleration as defined in the preceding section; the varying direction of the motion, and the rate of change of direction, must be taken into account. It is convenient to incorporate these ideas in the definitions of velocity and acceleration. By this generalization of their original meaning, velocity and acceleration become *vectors*, *i.e.* magnitudes having both length and direction.

138. The generalized idea of velocity as a vector may be derived as follows:

Consider a point P moving in a curve (Fig. 32). Let P be its position at the time t, P' its position at the time $t+\Delta t$, and let $P_0P=s$, $PP'=\Delta s$. The space s described in any time t may be regarded as some function of the time t, say $s=f(t)$.

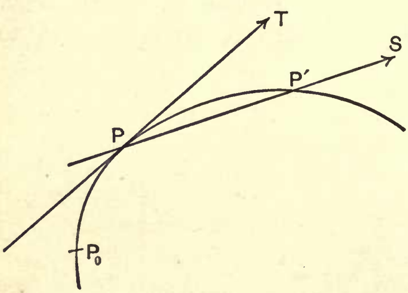

Fig. 32.

The mean velocity $\Delta s/\Delta t$ for the time Δt during which the point passes from P to P' may be represented by a length PS laid off on the chord PP' from P. As Δt diminishes, P' approaches P, and in the limit when $\Delta s/\Delta t$ becomes the derived function $ds/dt=f'(t)$, the chord merges into the tangent at P. This leads us to represent the velocity at the time t, or at the place P, by a length PT proportional to ds/dt laid off on the tangent at P from this point in the sense of the motion. The vector PT will then completely represent the velocity at the time t.

139. The vector PT may also be regarded as the limit of a vector PS laid off on the chord PP' as before, but proportional to the velocity with which the point would describe the *chord* PP' in the time Δt, *i.e.* to the velocity $PS = \frac{\text{chord } PP'}{\Delta t}$. For as Δt approaches the limit 0, PS approaches the direction of the tangent, and the ratio of the arc Δs to the chord PP' approaches the limit 1. Hence the equation $\frac{\Delta s}{\Delta t} = \frac{\Delta s}{\text{chord } PP'} \cdot PS$ gives in the limit $\lim \frac{\Delta s}{\Delta t} = \lim PS$, or $PT = \lim PS$.

It may be noticed here that, in view of the practical applications, the function $f(t) = s$ is in mechanics always supposed to be itself continuous and to possess continuous and finite derivatives of the first and second order.

140. Velocity having thus been defined as a vector, we may at once apply to it the rules for vector composition and vector resolution laid down in Arts. 45–55 for vectors representing displacements. Thus if a point be subjected to two or more simultaneous velocities, the velocity of the resulting motion will be represented by the vector found by geometrically adding the component velocities. A velocity may be resolved into any number of component velocities whose geometrical sum is equal to the given velocity.

141. We proceed to consider the most important cases of resolution of a velocity in a plane.

Let a point P move in a curve P_0P (Fig. 33) whose equation is referred to rectangular Cartesian co-ordinates x, y. It is usually convenient in this case to resolve the velocity v parallel to the axes into v_x and v_y.

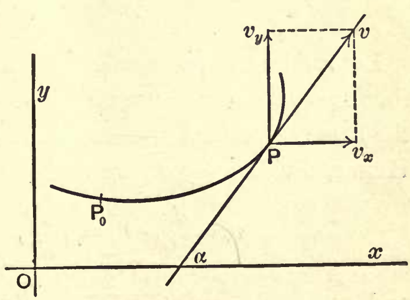

Fig. 33.

If α be the angle made by the vector v with the axis of x, we have $v_x = v\cos\alpha$, $v_y = v\sin\alpha$. And as the element ds of the curve at P makes the same angle α with the axis of x, we also have $dx = ds \cos \alpha$, $dy = ds \sin \alpha$. Dividing by dt and comparing with the preceding equations, we find

$$v_x = \frac{dx}{dt}, \quad v_y = \frac{dy}{dt}. \tag{1}$$

Conversely, knowing the velocities of the moving point parallel to the axes, we find its resulting velocity from the relation

$$v = \sqrt{v_x^2 + v_y^2} = \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}. \tag{2}$$

142. If the equation of the path be given in polar co-ordinates, it may be convenient to resolve the velocity v along the radius vector OP and at right angles to it (Fig. 34).

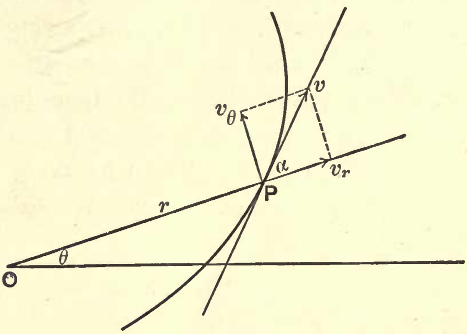

Fig. 34.

Let r, θ be the polar co-ordinates, α the angle between v and r; then $v_r = v \cos \alpha$, $v_\theta = v \sin \alpha$. The element ds of the curve has in the same directions the components $dr = ds \cos \alpha$, $rd\theta = ds \sin \alpha$. Hence, dividing by dt, we find

$$v_r = \frac{dr}{dt}, \quad v_\theta = r\frac{d\theta}{dt}, \tag{3}$$

and

$$v = \sqrt{v_r^2 + v_\theta^2} = \sqrt{\left(\frac{dr}{dt}\right)^2 + r^2\left(\frac{d\theta}{dt}\right)^2}. \tag{4}$$

143. In the case of **relative motion** we have to distinguish between the *absolute velocity* v of a point, its *relative velocity* v_1, and the *velocity of the body of reference* v_2.

To fix the ideas, imagine a man walking on deck of a steam-boat. His velocity of walking is his relative velocity v_1; the velocity of the boat (say with respect to the water or shore regarded as fixed), or more exactly speaking, the velocity of that point of the boat at which the man happens to be at the time, is the velocity v_2 of the body of reference; and the velocity with which the man is moving with respect to the water or shore, is his absolute velocity.

Representing these three velocities by means of their vectors, *we* evidently *find the absolute velocity* v *as the geometric sum of the relative velocity* v_1 *and the velocity* v_2 *of the body of reference*, just as in the case of displacements of translation (Art. 53). And conversely, *the relative velocity is found by geometrically subtracting from the absolute velocity the velocity of the body of reference.*

It is often convenient to state the last proposition in a somewhat different form. Imagine that we give the velocity $-v_2$ both to the man and to the boat; then the boat is brought to rest, and the resulting velocity of the man is what was before his relative velocity. Hence *the relative velocity is found as the resultant of the absolute velocity, and the velocity of the body of reference reversed.*

144. Exercises.

(1) A straight line in a plane turns with constant angular velocity ω about one of its points O, while a point P, starting from O, moves along the line with a constant velocity v_0. Determine the absolute path of P and its absolute velocity v.

(2) Show how to construct the tangent and normal to the spiral of Archimedes $r = a\theta$, where $\theta = \omega t$.

(3) A wheel of radius a rolls on a straight track with constant velocity (of its centre) v_0. Find the velocity v of a point P on the rim.

(4) Show that the tangent to the cycloid described by P, Ex. (3), passes through the highest point of the wheel.

(5) Show that the tangent to the ellipse bisects the angle between the radii vectores r, r' drawn from any point P on the ellipse to the foci S, S'.

(6) Construct the tangent to any conic section when a directrix and the corresponding focus are given.

(7) Two trains of equal length pass each other with equal velocity on parallel tracks. A man riding on a bicycle along the track at the rate of 8 miles an hour notices that the train meeting him takes 4 seconds to pass him, while the other takes 6 seconds. Find the velocity of the trains.

(8) A swimmer, starting from a point A on one bank of a river, wishes to reach a certain point B on the opposite bank. The velocity v_2 of the current and the angle θ made by AB with the direction of the current being given, determine the least relative velocity v_1 of the swimmer in magnitude and direction.

(9) Two men, A and B, walking at the rate of 3 and 4 miles an hour, respectively, cross each other at a rectangular street corner. Find the relative velocity of A with respect to B in magnitude and direction.

(10) A man jumps from a car at an angle of 60° with a velocity of 8 feet a second (relatively to the car). If the car be running 10 miles an hour, with what velocity and in what direction does the man strike the ground?

(11) The point P_1 moves with constant velocity v_1 along the line P_1Q. In what direction P_2Q must a point P_2 move with constant velocity v_2 in order to meet P_1? What is the locus of Q when the direction of P_1Q varies? When is the solution impossible?

(12) A point P moves uniformly in a circle, while another point Q moves with equal velocity along a tangent to the circle. Find the relative path of either point with respect to the other.

(13) The velocity of light being taken as 300,000 kilometres per second, and the velocity of the earth in its orbit as 30 kilometres, determine approximately the constant of the annual aberration of the fixed stars.

2. APPLICATIONS.

145. The motion of the **piston of a steam engine** furnishes interesting illustrations of the application of graphical methods in kinematics.

In Fig. 35, let $OQ = a$ be the crank arm, $PQ = l = ma$ the connecting rod, $P_1P_2 = s$ the "stroke," so that $l = ma = \frac{1}{2}ms$.

As $P_1P_2 = A_1A_2 = 2a$, we may regard A_1A_2 as representing the stroke. The position of the piston head P at the time when the crank pin is at Q will then be found as the intersection N of a circle of radius l described about P with the diameter A_1A_2

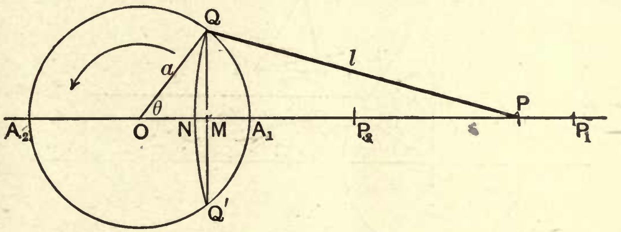

Fig. 35.

of the crank circle; in other words, N represents the position of the piston corresponding to the angle $A_1OQ = \theta$ in the forward stroke and to the angle $A_1OQ' = 2\pi - \theta$ in the return stroke.

146. The crank may generally be assumed to turn uniformly, making n revolutions per second. The linear velocity of the crank pin Q is therefore $u = 2\pi a \cdot n = \pi n s$.

For the piston head P, or for the point N, we must distinguish between its *mean*, or *average*, velocity V, and its variable *instantaneous* velocity v at any particular moment. For each revolution of the crank the piston head completes a double stroke so that its mean speed is $V = 2ns$. Hence we have

$$\frac{u}{V} = \frac{\pi n s}{2 n s} = \frac{\pi}{2}.$$

147. The instantaneous velocity v of the piston can be found graphically by considering the motion of the connecting rod PQ. The velocity u of the end Q is known, both in magnitude and direction; the velocity v of the other end is known in direction only. Now considering that the length of the rod PQ is invariable and hence the components of u and v along PQ must be equal, we can find the magnitude of v by drawing (Fig. 36) from any point M parallels to u and v, laying off u to scale and drawing through its extremity a perpendicular to the direction of PQ; this perpendicular will cut off the proper length on the direction of v.

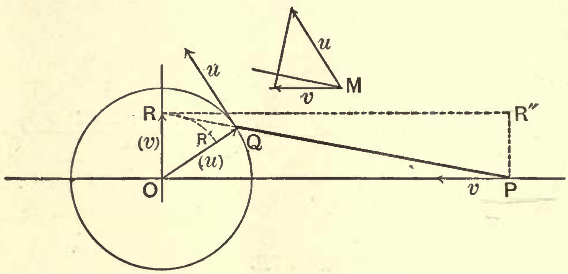

Fig. 36.

In applying this construction to our case it will be convenient to turn the auxiliary diagram of velocities by an angle of 90° and place it so as to make M coincide with O; u will then lie along OQ, and v at right angles to OP. Hence, if the scale of velocities be selected so as to have u represented in length by OQ, v will be represented on the same scale by OR, that is, by the segment cut off by PQ produced on the perpendicular to OP drawn through O.

148. The variation of the piston velocity in the course of the motion can best be exhibited graphically. Thus a *polar curve of piston velocity* is obtained by laying off on OQ a length $OR' = OR$, for a number of positions of OQ, and joining the points R' by a continuous curve.

Another convenient method consists in erecting perpendiculars to OP at the various positions of P and laying off, on these perpendiculars, $OR'' = OR = v$.

149. To derive an analytical expression for the piston velocity v, let ϕ be the angle OPQ which determines the position of the connecting rod.

It follows from the construction of the velocity v given in Art. 147 (see Fig. 36) that

$$\frac{v}{u}=\frac{OR}{OQ}=\frac{\sin(\theta+\phi)}{\cos\phi}=\sin\theta+\cos\theta\tan\phi.$$

If, as is usually the case, the connecting rod is much longer than the crank arm, ϕ will be a small angle, and we may substitute $\sin\phi$ for $\tan\phi$. But from the triangle OPQ we have

$$\frac{\sin\phi}{\sin\theta}=\frac{OQ}{PQ}=\frac{a}{l}=\frac{1}{m}.$$

Hence $v=u\left(\sin\theta+\cos\theta\cdot\frac{1}{m}\sin\theta\right)=u\left(\sin\theta+\frac{1}{2m}\sin 2\theta\right)$.

150. The motion of the piston head being rectilinear, we find its acceleration j by differentiating the expression for v found in the preceding article with respect to t:

$$j=\frac{dv}{dt}=\left(\sin\theta+\frac{1}{2m}\sin 2\theta\right)\frac{du}{dt}+u\left(\cos\theta+\frac{1}{m}\cos 2\theta\right)\frac{d\theta}{dt},$$

or, since $\frac{d\theta}{dt}=\omega=u/a$,

$$j=\left(\sin\theta+\frac{1}{2m}\sin 2\theta\right)\frac{du}{dt}+\left(\cos\theta+\frac{1}{m}\cos 2\theta\right)\frac{u^2}{a},$$

where $\frac{du}{dt}=0$ if the crank motion can be regarded as uniform.

151. If the connecting rod were of infinite length so as to make PQ (in Fig. 35) parallel to A_1A_2, the position of the piston corresponding to the position Q of the crank pin would be represented by the projection M of Q on A_1A_2; that is, NM would be $=0$. This length NM is therefore called the *deviation due to the obliquity* of the connecting rod.

With $NM=0$ the expression for the acceleration (Art. 150) would reduce to $dv/dt=(u^2/a)\cos\theta$, representing a simple harmonic motion (see Art. 179).

152. The slide valve of a steam engine is generally worked by an eccentric whose radius is set on the shaft at such an angle as to shut off the steam when the crank makes a certain angle θ with the direction of motion of the piston. It follows that the fraction of stroke completed before cut-off takes place is affected by the obliquity of the connecting rod. The rates of cut-off are therefore different in the forward and backward strokes. In the forward stroke, the effect of the obliquity is to put the piston in advance of the position it would have if the connecting rod were of infinite length; in the return stroke, *i.e.* when θ is greater than π, the piston lags behind.

153. An analytical expression for the deviation due to obliquity is readily obtained from Fig. 35. We have

$$MN = PN - PM = l(1 - \cos\phi)$$

$$= ms\sin^2\frac{\phi}{2} = \frac{ms}{4}\left(2\sin\frac{\phi}{2}\right)^2,$$

or approximately, since ϕ is small,

$$MN = \frac{ms}{4}\sin^2\phi.$$

Also, as in Art. 149, $\dfrac{\sin\phi}{\sin\theta} = \dfrac{1}{m}$;

hence $$MN = \frac{s}{4m}\sin^2\theta.$$

The greatest value of MN is thus seen to be $s/4m$; for instance, if the connecting rod be four times the length of the crank, the deviation due to obliquity cannot exceed 1/16 of the stroke.

154. Exercises.*

(1) Construct a polar diagram exhibiting the position of the piston for all angles θ by laying off on the crank arm OQ a length $ON' = ON$ and joining the points N' by a continuous curve.

(2) Construct the curves of piston velocity indicated in Art. 148.

---

* These problems are taken with slight modification from COTTERILL'S *Applied mechanics*, 1884, p. 112.

(3) Show that for a connecting rod of infinite length the two loops of the curve of Ex. 1 reduce to two equal circles.

(4) The driving wheels of a locomotive are 6 ft. in diameter; find the number of revolutions per minute and the angular velocity, when running at 50 miles per hour. If the stroke be 2 ft., find the speed of the piston.

(5) The pitch of a screw is 24 ft., and the number of revolutions 70 per minute. Find the speed in knots. If the stroke is 4 ft., find the speed of piston in feet per minute.

(6) The stroke of a piston is 4 ft., and the connecting rod is 9 ft. long. Find the position of the crank, when the piston has completed the first quarter of the forward and backward strokes respectively. Also find the position of the piston when the crank is upright.

(7) The valve gear is so arranged in the last question as to cut off the steam when the crank is 45° from the dead-points both in the forward and backward strokes. Find the point at which steam will be cut off in the two strokes. Also when the obliquity of the connecting rod is neglected.

3. ACCELERATION IN CURVILINEAR MOTION.

155. Let the velocity of a moving point be represented by the vector $v = PT$ at the time t, and by the vector $v' = P'T'$ at the time $t + \Delta t$ (Fig. 37). Then, drawing from any point O OV and OV' respectively equal and parallel to PT and $P'T'$, the vector VV' represents the geometrical difference between v' and v; in other words, VV' is the velocity which, geometrically added to v, produces v'. The vector VV' approaches the limit 0 at the same time with Δt and PP'. This limit of VV' for an infinitely small time dt may be called the *geometrical differential* or *vector differential*, of v.

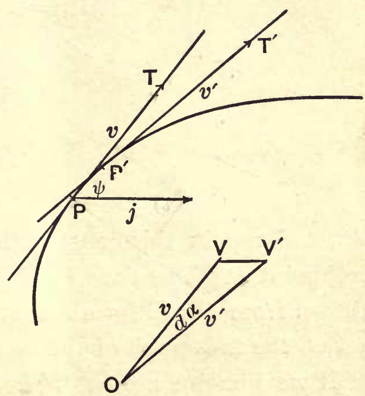

Fig. 37.

Dividing this infinitesimal vector by dt, we obtain in general a finite magnitude $\frac{\lim VV'}{dt}$, the *geometrical derivative* of the velocity with respect to the time, and that is what we call the **acceleration** at the time t or at the point P. We represent it geometrically by a vector j drawn from P parallel to the direction of $\lim VV'$.

It will be noticed that the sense of the acceleration will be towards that side of the tangent of the curve on which the centre of curvature is situated.

156. Suppose a point P to move along a curve $P_1P_2P_3 \ldots$ with variable velocity v (Fig. 38). From any fixed origin O draw a vector $OV_1 = v_1$, equal and parallel to the velocity v_1 of

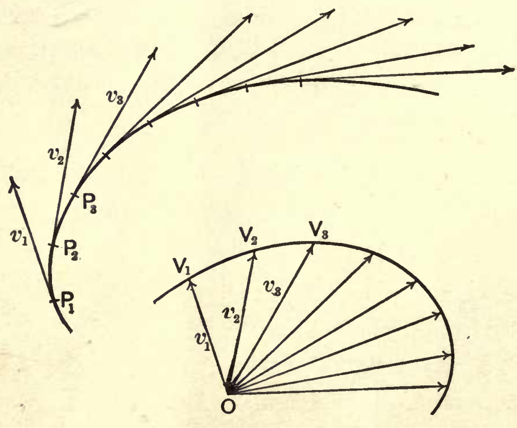

Fig. 38.

P_1, and repeat this construction for every position of the moving point P. The ends V_1, V_2, V_3, ... of all these radii vectores drawn from O will form a continuous curve $V_1V_2V_3 \ldots$ which is called the **hodograph** of the motion of P.

If we imagine a point V describing this curve $V_1V_2V_3 \ldots$ at the same time that P describes the curve $P_1P_2P_3 \ldots$, it is evident that the velocity of V, *i.e.* $\frac{\lim V_1V_2}{dt}$, laid off on the tangent of the curve $V_1V_2V_3 \ldots$, represents the acceleration of the point P

both in magnitude and direction; i.e. *the velocity of the hodograph is the acceleration of the original motion.*

It is easy to see how these considerations might be extended. We might construct the hodograph of the hodograph; its velocity might be called the acceleration of the second order for the motion of P; and so on.

It is sometimes convenient to draw the radii vectores of the hodograph not parallel to the velocities of the point P, but so as to make some constant angle (usually a right angle) with these velocities.

157. Exercises.

(1) Discuss rectilinear motion as a special case of plane motion.

(2) Show that the hodograph of rectilinear motion is a straight line.

(3) Show that the velocity of a moving point is increasing, constant, or diminishing, according to the value of the angle ψ between v and j (Fig. 37).

158. Acceleration having been defined as a vector, the rules for vector composition and resolution may be applied to acceleration just as they were before applied to displacements and to velocities. Thus, a point subjected to two or more simultaneous accelerations will have a resulting acceleration found by geometrically adding the component accelerations; and conversely, the acceleration of a point may be resolved in various ways.

159. Let the vector j which represents the acceleration of the point P at the time t, make an angle ψ with the vector representing the velocity v at the same time (see Fig. 37). Resolving the vector j along the tangent and normal at P, we obtain the **tangential acceleration** $j_t = j \cos\psi$ and the **normal acceleration** $j_n = j \sin\psi$.

To find expressions for these components, let us regard PP' in Fig. 37 as the element ds of the path described by P; then the length of $P'T'$, or of OV', is $v' = v + dv$, and the angle VOV', being equal to the angle between two consecutive tangents of the curve, is the angle of contingence $d\alpha$ at P. This angle being equal to the angle between the normals at P and P', we have $\rho d\alpha = ds$, where ρ is the radius of curvature at P.

Resolving the elementary acceleration, *i.e.* the infinitesimal vector VV', along OV and at right angles to OV, we find the components $VV' \cos\psi = dv$, $VV' \sin\psi = v d\alpha = v ds/\rho$. Dividing by dt and observing that $ds/dt = v$, we finally obtain

$$j_t = \frac{dv}{dt}, \tag{1}$$

$$j_n = v\frac{d\alpha}{dt} = \rho\left(\frac{d\alpha}{dt}\right)^2 = \frac{v^2}{\rho}. \tag{2}$$

By composition we have

$$j = \sqrt{j_t^2 + j_n^2} = \sqrt{\left(\frac{dv}{dt}\right)^2 + \frac{v^4}{\rho^2}}. \tag{3}$$

160. When rectangular Cartesian co-ordinates are used, we may resolve the acceleration j into two components $j_x = j\cos\phi$, $j_y = j\sin\phi$ parallel to the co-ordinate axes Ox, Oy; ϕ being the angle made by the vector j with the axis of x. We obtain an expression for j_x by projecting the infinitesimal triangle OVV' (Fig. 37) on the axis Ox and denoting, as before, the projections of the velocities OV, OV' by v_x, v'_x. This gives

$$VV' \cos\phi = v'_x - v_x = dv_x,$$

whence, dividing by dt, $j_x = dv_x/dt$. Similarly, we find $j_y = dv_y/dt$. Hence, by formulæ (1), Art 141,

$$j_x = \frac{dv_x}{dt} = \frac{d^2x}{dt^2}, \quad j_y = \frac{dv_y}{dt} = \frac{d^2y}{dt^2}. \tag{4}$$

These so-called *equations of motion* offer the advantage that the curvilinear motion is replaced by two rectilinear motions, thus avoiding the use of vectors.

By composition, we have of course

$$j=\sqrt{j_x^2+j_y^2}=\sqrt{\left(\frac{d^2x}{dt^2}\right)^2+\left(\frac{d^2y}{dt^2}\right)^2}. \qquad (5)$$

161. For polar co-ordinates r, θ, we may resolve the acceleration j into a component j_r along the radius vector r and a component j_θ at right angles to r. Expressions for these components are readily found by projecting the components

$$j_x=\frac{d^2x}{dt^2} \text{ and } j_y=\frac{d^2y}{dt^2}$$

on r and at right angles to r (Fig. 39):

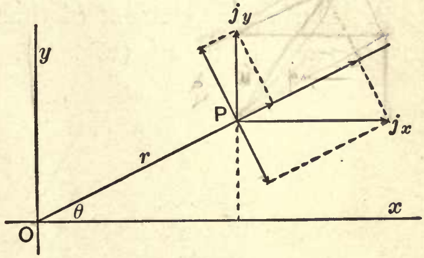

Fig. 39.

$$j_r=\frac{d^2x}{dt^2}\cos\theta+\frac{d^2y}{dt^2}\sin\theta,\quad j_\theta=-\frac{d^2x}{dt^2}\sin\theta+\frac{d^2y}{dt^2}\cos\theta.$$

Differentiating the relations $x=r\cos\theta$, $y=r\sin\theta$, we find

$$\frac{dx}{dt}=\frac{dr}{dt}\cos\theta-r\sin\theta\frac{d\theta}{dt},\quad \frac{dy}{dt}=\frac{dr}{dt}\sin\theta+r\cos\theta\frac{d\theta}{dt};$$

and differentiating again:

$$\frac{d^2x}{dt^2}=\left[\frac{d^2r}{dt^2}-r\left(\frac{d\theta}{dt}\right)^2\right]\cos\theta-\left[2\frac{dr}{dt}\frac{d\theta}{dt}+r\frac{d^2\theta}{dt^2}\right]\sin\theta,$$

$$\frac{d^2y}{dt^2}=\left[\frac{d^2r}{dt^2}-r\left(\frac{d\theta}{dt}\right)^2\right]\sin\theta+\left[2\frac{dr}{dt}\frac{d\theta}{dt}+r\frac{d^2\theta}{dt^2}\right]\cos\theta.$$

Substituting these expressions for $\frac{d^2x}{dt^2}$ and $\frac{d^2y}{dt^2}$ in the above equations for j_r, j_θ, we find:

$$j_r=\frac{d^2r}{dt^2}-r\left(\frac{d\theta}{dt}\right)^2,\quad j_\theta=2\frac{dr}{dt}\frac{d\theta}{dt}+r\frac{d^2\theta}{dt^2}=\frac{1}{r}\frac{d}{dt}\left(r^2\frac{d\theta}{dt}\right). \qquad (6)$$

162. The meaning of these expressions will perhaps be better understood by the following geometrical derivation. As shown in Art. 142, the velocity v has the components

$$v_r = \frac{dr}{dt}, \quad v_\theta = r\frac{d\theta}{dt},$$

the former along the radius vector, the latter at right angles to it. During the element of time dt, while the moving point passes from P to P' (Fig. 40), each of the vectors v_r, v_θ

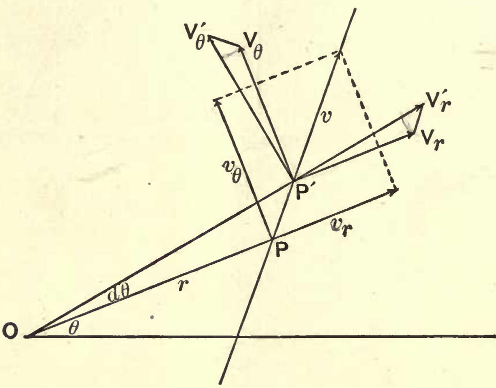

Fig. 40.

receives a geometrical increment $V_r V'_r$, $V_\theta V'_\theta$. Let us resolve each of these infinitesimal vectors along r and at right angles to r, and then combine the two components along r, and also the two components perpendicular to r; finally, dividing by dt, we obtain j_r and j_θ.

Thus v_r gives $\frac{d^2r}{dt^2}$ along r, and $\frac{dr}{dt}\frac{d\theta}{dt}$ at right angles to r, while v_θ, or $r\frac{d\theta}{dt}$, contributes $-r\left(\frac{d\theta}{dt}\right)^2$ along r and

$$\frac{d}{dt}\left(r\frac{d\theta}{dt}\right) = \frac{dr}{dt}\frac{d\theta}{dt} + r\frac{d^2\theta}{dt^2}$$

at right angles to r. We obtain in this way the same expressions for j_r, j_θ as in the formulæ (6) above.

163. Exercises.

(1) Show that the sectorial velocity (Art. 135) is constant whenever $j_\theta = 0$.

(2) Show that the normal component of the acceleration is the product of the radius of curvature into the square of the angular velocity about the centre of curvature.

(3) Show that the velocity is the mean proportional between the acceleration and half the chord intercepted by the direction of the acceleration on the osculating circle.

(4) If the acceleration of a point P be always directed to a fixed point A, show that the radius vector AP describes equal areas in equal times.

(5) Show that in uniform circular motion the acceleration is directed to the centre and proportional to the radius.

(6) A wheel rolls on a straight track; find the acceleration of its lowest point.

4. APPLICATIONS.

164. Inclined Plane. Imagine a body sliding down a smooth plane inclined at an angle θ to the horizon. In addition to the assumptions made in the case of falling bodies (see Art. 112) we assume that the motion takes place along a "line of greatest slope," *i.e.* in a vertical plane at right angles to the intersection of the inclined plane with a horizontal plane. A "smooth" plane means one that offers no frictional resistance. The body is therefore subject only to the acceleration of gravity, g; and it is sufficient to consider the motion of a single point of the body.

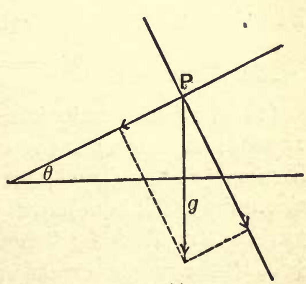

Fig. 41.

Resolving g into two components, $g\cos\theta$ perpendicular to the plane and $g\sin\theta$ along the plane (Fig. 41), it will be seen that the former component, being at right angles to the velocity, cannot change the magnitude of this velocity.

We have therefore simply a rectilinear motion with the constant acceleration $g \sin\theta$, so that all the formulæ of Art. 107–113 will here apply if for the acceleration j (or g) we substitute $g \sin\theta$.

Thus, if the initial velocity be 0, the motion is determined by the equations

$$v = g \sin\theta \cdot t, \tag{1}$$

$$s = \tfrac{1}{2} g \sin\theta \cdot t^2, \tag{2}$$

$$\tfrac{1}{2} v^2 = g \sin\theta \cdot s. \tag{3}$$

165. If there be an initial velocity v_0 parallel to the line of greatest slope of the inclined plane, the equations are

$$v = v_0 + g \sin\theta \cdot t, \tag{1'}$$

$$s = v_0 t + \tfrac{1}{2} g \sin\theta \cdot t^2, \tag{2'}$$

$$\tfrac{1}{2}(v^2 - v_0^2) = g \sin\theta \cdot s, \tag{3'}$$

where v_0 is to be regarded as positive if its direction is down the plane and negative when up the plane.

If the initial velocity v_0 be inclined to the plane at an angle β, it can be resolved into the components $v_0 \cos\beta$ and $v_0 \sin\beta$, the former alone being effective so that $v_0 \cos\beta$ must be substituted for v_0 in the above formulæ.

166. Exercises.

(1) A railroad train is running up a grade of 1 in 250 at the rate of 15 miles an hour when the coupling of the last car breaks. Neglecting friction, (*a*) how far will the car be after two minutes from the point where the break occurred? (*b*) When will it begin moving down the grade? (*c*) How far behind the train will it be at that moment? (*d*) If the grade extend 2000 ft. below the point where the break occurred, with what velocity will it arrive at the foot of the grade?

(2) Show that the final velocity is independent of the inclination of the plane; in other words, in sliding down a smooth inclined plane a

body acquires the same velocity as in falling vertically through the "height" of the plane.

(3) Show that it takes a body twice as long to slide down a plane of 30° inclination as it would take it to fall through the height of the plane.

(4) At what angle θ should the rafters of a roof of given span $2b$ be inclined to make the water run off in the shortest time?

(5) Prove that the times of descending from rest down the chords issuing from the highest (or lowest) point of a vertical circle are equal.

(6) If any number of points starting at the same time from the same point slide down different inclined planes, they will at any time t all be situated on a sphere passing through the starting point and having a diameter $= \frac{1}{2}gt^2$.

(7) Show how to construct geometrically the line of quickest descent from a given point: (a) to a given straight line, (b) to a given circle, situated in the same vertical plane.

(8) Analytically, the line of quickest or slowest descent from a given point to a curve in the same vertical plane is found by taking the equation of the curve in polar co-ordinates, $r = f(\theta)$, with the given point as origin and the axis horizontal. The time of descending the radius vector r is $t = \sqrt{2r/(g \sin\theta)}$. Show that this becomes a maximum or minimum when $\tan\theta = f(\theta)/f'(\theta)$, according as $f(\theta) + f''(\theta)$ is negative or positive.

(9) Show that the line of quickest descent to a parabola from its focus, the axis of the parabola being horizontal, is inclined at an angle of 60° to the horizon.

167. Projectiles. With the same assumptions as in Art. 112, the motion of a projectile reduces to that of a point subject to the constant acceleration of gravity and starting with an initial velocity v_0 inclined to the horizon at an angle ϵ different from 90°. The angle ϵ between the horizon and the initial velocity is called the **elevation** of the projectile.

Taking the horizontal line through the starting point O as axis of x, the vertical upwards as positive axis of y (Fig. 42), the

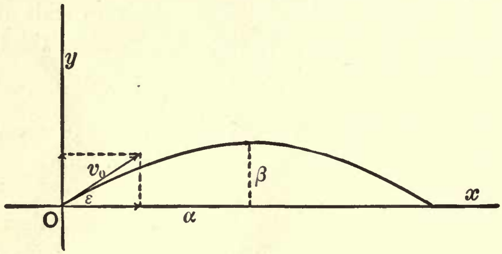

Fig. 42.

x-component of the acceleration is evidently 0, while the y-component is $-g$; hence, by (4), Art. 160, the equations of motion are

$$\frac{d^2x}{dt^2}=0, \quad \frac{d^2y}{dt^2}=-g. \tag{4}$$

The first integration gives

$$\frac{dx}{dt}=C_1, \quad \frac{dy}{dt}=-gt+C_2.$$

As $\frac{dx}{dt}=v_x$, $\frac{dy}{dt}=v_y$ are the components of the velocity v at the time t, the constants are determined from the values of v_x, v_y at the time $t=0$; viz. $v_0 \cos \epsilon = C_1$, $v_0 \sin \epsilon = 0 + C_2$. We have therefore

$$v_x \equiv \frac{dx}{dt}=v_0 \cos \epsilon, \quad v_y \equiv \frac{dy}{dt}=v_0 \sin \epsilon - gt. \tag{5}$$

Integrating again, we find

$$x=v_0 \cos \epsilon \cdot t, \qquad y=v_0 \sin \epsilon \cdot t - \tfrac{1}{2}gt^2, \tag{6}$$

the constants of integration being 0, since for $t=0$ we have $x=0$, $y=0$.

These values of x, y, v_x, v_y show that the motion in the horizontal direction is uniform while in the vertical direction it is uniformly accelerated. This is otherwise directly evident from the nature of the problem.

Eliminating t between the expressions for x and y, we find the *equation of the path*

$$y = \tan\epsilon \cdot x - \frac{g}{2\,v_0^2\cos^2\epsilon}\cdot x^2, \tag{7}$$

which represents a parabola passing through the origin. To find its vertex and latus rectum, divide by the coefficient of x^2 and rearrange:

$$x^2 - \frac{2\,v_0^2}{g}\sin\epsilon\cos\epsilon\cdot x = -\frac{2\,v_0^2}{g}\cos^2\epsilon\cdot y;$$

completing the square in x, the equation can be written in the form

$$\left(x - \frac{v_0^2}{2g}\sin 2\epsilon\right)^2 = -\frac{2\,v_0^2}{g}\cos^2\epsilon\left(y - \frac{v_0^2}{2g}\sin^2\epsilon\right). \tag{7'}$$

The co-ordinates of the vertex are therefore $\alpha = \frac{v_0^2}{2g}\sin 2\epsilon$, $\beta = \frac{v_0^2}{2g}\sin^2\epsilon$; the latus rectum $4a = \frac{2\,v_0^2}{g}\cos^2\epsilon$; the axis is vertical, and the directrix is a horizontal line at the distance $a = \frac{v_0^2}{2g}\cos^2\epsilon$ above the vertex.

168. Exercises.

(1) Show that the velocity at any time is $v = \sqrt{v_0^2 - 2gy}$.

(2) Prove that the velocity of the projectile is equal in magnitude to the velocity that it would acquire by falling from the directrix: (*a*) at the starting point, (*b*) at any point of the path (see Art. 113).

(3) Show that a body projected vertically upwards with the initial velocity v_0 would just reach the common directrix of all the parabolas described by bodies projected at different elevations ϵ with the same initial velocity v_0.

(4) The *range* of a projectile is the distance from the starting point to the point where it strikes the ground. Show that on a horizontal plane the range is $R = 2\alpha = \frac{v_0^2}{g} \sin 2\epsilon$.

(5) The *time of flight* is the whole time from the beginning of the motion to the instant when the projectile strikes the ground. It is best found by considering the horizontal motion of the projectile alone, which is uniform. Show that on a horizontal plane the time of flight is $T = \frac{2v_0}{g} \sin \epsilon$.

(6) Show that the time of flight and the range on a plane through the starting point inclined at an angle θ to the horizon are

$$T_\theta = \frac{2v_0}{g} \frac{\sin(\epsilon - \theta)}{\cos\theta} \text{ and } R_\theta = \frac{2v_0^2}{g} \frac{\sin(\epsilon - \theta)\cos\epsilon}{\cos^2\theta}.$$

(7) What elevation gives the greatest range on a horizontal plane?

(8) Show that on a plane rising at an angle θ to the horizon, to obtain the greatest range, the direction of the initial velocity should bisect the angle between the plane and the vertical.

(9) A stone is dropped from a balloon which, at a height of 1000 ft., is carried along by a horizontal air-current at the rate of 15 miles an hour. (*a*) Where, (*b*) when, and (*c*) with what velocity will it reach the ground?

(10) What must be the initial velocity v_0 of a projectile if with an elevation of 30° it is to strike an object 100 ft. above the horizontal plane of the starting point at a horizontal distance from the latter of 1200 ft.?

(11) What must be the elevation ϵ to strike an object 100 ft. above the horizontal plane of the starting point and 5000 ft. distant, if the initial velocity be 1200 ft. per second?

(12) Show that to strike an object situated in the horizontal plane of the starting point at a distance x from the latter, the elevation must be ϵ or $90° - \epsilon$, where $\epsilon = \frac{1}{2}\sin^{-1}(gx/v_0^2)$.

(13) The initial velocity v_0 being given in magnitude and direction, show how to construct the path graphically.

(14) If it be known that the path of a point is a parabola and that the acceleration is parallel to its axis, show that the acceleration is constant.

(15) Prove that a projectile whose elevation is 60° rises three times as high as when its elevation is 30°, the magnitude of the initial velocity being the same in both cases.

(16) Construct the hodograph for parabolic motion, taking the focus as pole and drawing the radii vectores at right angles to the velocities.

169. A projectile moving in the air or in any other **resisting medium** of uniform density is subject, in addition to the constant acceleration g of gravity, to the resistance of the medium which produces a retardation variable both in magnitude and direction (Art. 125). Experiment shows that this retardation can be expressed in the form cv^n, where c is constant for a given projectile and medium, and n must be determined by experiment for different initial velocities.

170. For $n=1$ the integrations can be readily effected. Resolving the retardation cv into its components $cv_x = cdx/dt$, $cv_y = cdy/dt$, the equations of motion are

$$\frac{d^2x}{dt^2} = -c\frac{dx}{dt}, \quad \frac{d^2y}{dt^2} = -c\frac{dy}{dt} - g. \tag{8}$$

Integrating, we find

$$v_x \equiv \frac{dx}{dt} = v_0 \cos\epsilon \cdot e^{-ct}, \quad v_y \equiv \frac{dy}{dt} = \frac{1}{c}[-g + (cv_0 \sin\epsilon + g)e^{-ct}], \tag{9}$$

since for $t=0$ we have $v_x = v_0 \cos\epsilon$, $v_y = v_0 \sin\epsilon$.

The second integration gives

$$x = \frac{v_0}{c}\cos\epsilon\,(1 - e^{-ct}), \quad y = -\frac{g}{c}t + \frac{cv_0 \sin\epsilon + g}{c^2}(1 - e^{-ct}), \tag{10}$$

since $x=0$, $y=0$ for $t=0$. Eliminating t, we find the equation of the path of the projectile:

$$y=\frac{cv_0\sin\epsilon+g}{cv_0\cos\epsilon}\cdot x+\frac{g}{c^2}\log\frac{v_0\cos\epsilon-cx}{v_0\cos\epsilon}. \qquad (11)$$

The curve has a vertical asymptote $x=\frac{v_0\cos\epsilon}{c}$; for this value of x, $t=\infty$.

171. Uniform Circular Motion. Let a point P (Fig. 43) describe a circle of radius a with constant angular velocity ω. Its linear

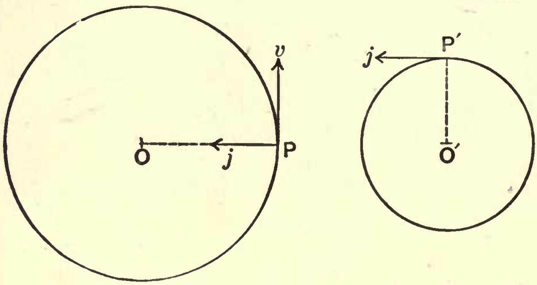

Fig. 43.

velocity $v=\omega a$ is of constant magnitude, but varies in direction. By the formulæ (1), (2) of Art. 159, the tangential acceleration $j_t=0$, while the normal acceleration $j_n=v^2/a=\omega^2 a$ represents the total acceleration. Hence, in uniform circular motion, the acceleration is

$$j=\frac{v^2}{a}=\omega^2 a, \qquad (12)$$

and is always directed toward the centre O of the circle.

This appears also by constructing the hodograph of the motion, which is evidently a circle of radius v (Fig. 43). As the acceleration of P is represented by the velocity of the point P' of the hodograph (see Art. 156), we have only to determine this velocity. Let T be the so-called **period**, or **periodic time**, *i.e.* the time in which the point P makes a whole revolution, so that $T=2\pi a/v$; then, since P' describes the circle of radius v in the same time T, we have for the velocity of P' the expression $2\pi v/T$, or substituting for T its value, v^2/a, as above.

172. **Simple harmonic motion** is a rectilinear motion in which the distance x of the moving point P_x (Fig. 44) from a fixed origin O in the line of motion is a simple harmonic function of the time, *i.e.* a function of the form

$$x=a\cos(\omega t+\epsilon), \text{ or } x=a\sin(\omega t+\epsilon), \tag{13}$$

where a, ω, ϵ are constants.

If the positions P of a point moving uniformly in a circle be projected at every instant on any diameter AA' of the circle, it is easy to see that the motion of the projection P_x along the diameter is simply harmonic. For denoting the constant angular velocity of P by ω, the angle AOP will be $=\omega t$ if the time be counted from the point A. Hence the distance $OP_x=x$ of the point P_x from the centre O, or the **displacement** of P_x at the time t, is

$$x=a\cos\omega t,$$

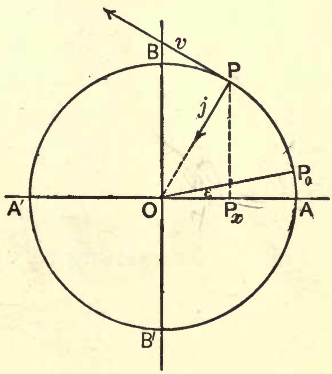

Fig. 44.

where a is the radius of the circle. This radius $a=OA$ is called the **amplitude** of the simple harmonic motion.

173. While P moves uniformly in the circle, its projection P_x evidently performs oscillations from A through O to A' and back through O to A.

The time T of completing one whole oscillation forward and backward is called the **period** of the simple harmonic motion; it is obviously equal to the period of the motion of P in the circle; *i.e.*

$$T=\frac{2\pi}{\omega}. \tag{14}$$

The period is therefore independent of the amplitude a. It follows that two simple harmonic motions resulting from two uniform circular motions of the same angular velocity on two concentric circles of different radii have the same period; such motions are called **isochronous.**

174. If the time t be counted, not from A, but from some other point P_0 on the circle for which $\measuredangle AOP_0 = \epsilon$, we have $\measuredangle AOP = \omega t + \epsilon$, and the equation of the simple harmonic motion is

$$x = a \cos(\omega t + \epsilon). \tag{15}$$

The angle $\omega t + \epsilon$ is called the **phase-angle,** while ϵ is the **epoch-angle,** of the motion. The names *phase* and *epoch* are sometimes applied to these angles, although, strictly speaking, the phase is the *time* (usually expressed as a fraction of the period T) of passing from the position A of maximum displacement to any position P_x, while the epoch is the phase corresponding to the time $t=0$.

175. Differentiating equation (15), we find the velocity

$$v_x \equiv \frac{dx}{dt} = -a\omega \sin(\omega t + \epsilon); \tag{16}$$

and differentiating again, we obtain the acceleration

$$j_x \equiv \frac{d^2x}{dt^2} = -a\omega^2 \cos(\omega t + \epsilon) = -\omega^2 x \tag{17}$$

of simple harmonic motion.

The same values can be derived by projecting the velocity and acceleration of the uniform circular motion of P on the diameter AA', as is readily seen from Fig. 44.

176. Equation (17) shows that *in simple harmonic motion the acceleration is directly proportional to the distance from the centre.*

Conversely, it can be shown that if the acceleration be proportional to the distance from a fixed point in the direction of the initial velocity, and if it be directed *towards* this point, the motion is simply harmonic. For we then have

$$\frac{d^2x}{dt^2} = -\mu^2 x,$$

where μ is constant. The general integral of this differential equation is (compare Art. 122)

$$x = C_1 \sin \mu t + C_2 \cos \mu t.$$

Differentiating, we find for the velocity

$$v = C_1 \mu \cos \mu t - C_2 \mu \sin \mu t.$$

To determine the constants of integration C_1, C_2, let $s = s_0$ and $v = v_0$ at the time $t = 0$. Substituting these values, we find $s_0 = C_2$ and $v_0 = \mu C_1$; hence

$$x = \frac{v_0}{\mu} \sin \mu t + s_0 \cos \mu t.$$

Putting $v_0/\mu = a \cos \epsilon$, $s_0 = a \sin \epsilon$, which is always allowable, we obtain

$$x = a (\sin \mu t \cos \epsilon + \cos \mu t \sin \epsilon)$$

$$= a \sin (\mu t + \epsilon).$$

This represents a simple harmonic motion whose amplitude is $a = \sqrt{v_0^2 + \mu^2 s_0^2}/\mu$, and whose epoch-angle is $\epsilon = \tan^{-1}(\mu s_0/v_0)$. As the angular velocity of the corresponding uniform circular motion is μ, the period is $T = 2\pi/\mu$.

177. If the uniform circular motion of P be projected on the diameter BB', which is at right angles to the diameter AA' (Fig. 44), we have $OP_y \equiv y = a \sin (\omega t + \epsilon)$. Writing this in the equivalent form

$$y = -a \cos \left(\omega t + \epsilon + \frac{\pi}{2}\right),$$

it appears that the motion of P_y is simply harmonic of the same period and amplitude with the motion of P_x, but differing by $\pi/2$ in phase.

178. Simple harmonic motions occur very frequently in applied mechanics and mathematical physics. A particular case has been treated in Arts. 121–124. As another example we may mention the apparent motion of a satellite about its primary as seen from any point in the plane of the motion, provided the satellite be regarded as moving uniformly in a circle relatively to its primary. Thus the moons of Jupiter, as seen from the earth, have approximately a simple harmonic motion.

179. A mechanism for producing simple harmonic motion can readily be constructed as follows. The end A (Fig. 45) of a crank rotating uniformly about the axis O, carries a pin running in the slot AB of a T bar ABD whose axis (produced) passes through the centre O of the crank circle. The T bar is constrained by guides to move back and forth along the line OD; its motion is evidently simply harmonic, the uniform circular motion of the crank being transformed into rectilinear motion. Compare Art. 151.

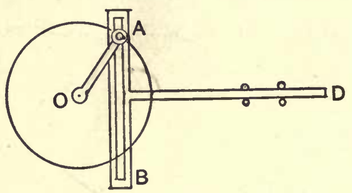

Fig. 45.

180. Exercises.

(1) Show that the maximum acceleration of the simple harmonic motion is numerically equal to the acceleration in the corresponding uniform circular motion.

(2) Find the time of one oscillation from equation (15) without reference to the circular motion.

(3) In the mechanism shown in Fig. 45, if the length of the crank be 2 feet and the number of revolutions 15 per minute, find the velocity and acceleration of the end D of the T bar: (a) when at elongation; (b) when at quarter stroke; (c) when at the middle of the stroke.

(4) Show that the period of a simple harmonic oscillation can be expressed in the form $T = 2\pi\sqrt{-x/j_x}$ where j_x is the acceleration of the oscillating point at the time when its distance from the centre, or its displacement, is x.

(5) P_x, P'_x being the positions of the oscillating point at the times t, t', respectively, and δ the angle POP', *i.e.* the difference of phase, show that $t' - t = \delta/\omega$.

(6) Show that $v_x = -\omega\sqrt{a^2 - x^2}$.

181. Compound Harmonic Motion. We have seen (Art. 176) that the motion of a point, whose acceleration is directly proportional to its distance from a fixed centre, and directed towards this centre, is simply harmonic, provided the centre lies in the line of the initial velocity. Removing this last restriction, we have the more general case of compound harmonic motion.

Let O (Fig. 46) be the centre, P the position of the moving point at the time t, $OP = s$ its distance from the centre, v its velocity, $j = -\mu^2 s$ its acceleration, at that time. Referring the motion to two rectangular axes Ox, Oy in the plane determined by v and O, we can resolve v and j into their components along these axes:

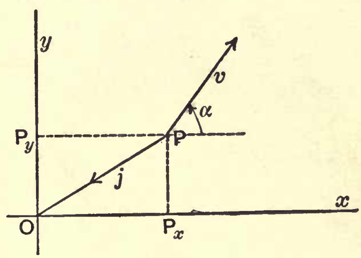

Fig. 46.

$$v_x = v \cos\alpha, \quad v_y = v \sin\alpha,$$

and $j_x = -\mu^2 x$, $j_y = -\mu^2 y$, where α is the angle made by v with the axis Ox, and x, y are the co-ordinates of P.

The projections P_x, P_y of P on the axes have therefore each a simple harmonic motion, and the motion of P may be regarded as the resultant of these component motions.

182. In general, the motion of P will be curvilinear. We proceed to examine somewhat more in detail the most important cases of the **composition** of two or more **simple harmonic motions,**

beginning with those cases in which the resultant motion is rectilinear.

As, according to Hooke's law, the particles of elastic bodies, after release from strain within the elastic limits, perform small oscillations for which the acceleration is proportional to the displacement from a middle position, the motions under discussion find a wide application in the theories of elasticity, sound, light, and electricity, and form the basis of the general theory of wave motion in an elastic medium.

183. *Two simple harmonic motions in the same line, of equal amplitude* a *and equal period* T, *but differing in phase by* δ, *compound into a simple harmonic motion in the same line, of the same period* T, *but having the amplitude* 2 a *cos* (δ/2).

For we have for the component displacements

$$x_1 = a \cos \omega t, \quad x_2 = a \cos(\omega t + \delta);$$

and as these are in the same line, they can be added algebraically giving the resultant displacement

$$x = x_1 + x_2 = a[\cos \omega t + \cos(\omega t + \delta)],$$

or, by the formula $\cos\alpha + \cos\beta = 2\cos\frac{\alpha+\beta}{2}\cos\frac{\alpha-\beta}{2}$,

$$x = 2a\cos\frac{\delta}{2}\cdot\cos\left(\omega t + \frac{\delta}{2}\right).$$

184. *Two simple harmonic motions in the same line, of equal period* T, *but differing both in amplitude and in phase, compound into a single simple harmonic motion in the same line and of the same period.*

For the component displacements

$$x_1 = a_1 \cos(\omega t + \epsilon_1), \quad x_2 = a_2 \cos(\omega t + \epsilon_2)$$

can again be added algebraically, and the resultant displacement is

$$x = x_1 + x_2 = a_1 \cos(\omega t + \epsilon_1) + a_2 \cos(\omega t + \epsilon_2)$$

$$= (a_1 \cos\epsilon_1 + a_2 \cos\epsilon_2)\cos\omega t - (a_1 \sin\epsilon_1 + a_2 \sin\epsilon_2)\sin\omega t.$$

Putting $a_1 \cos\epsilon_1 + a_2 \cos\epsilon_2 = a \cos\epsilon$, $a_1 \sin\epsilon_1 + a_2 \sin\epsilon_2 = a \sin\epsilon$, we have

$$x = a \cos\epsilon \cos\omega t - a \sin\epsilon \sin\omega t$$

$$= a \cos(\omega t + \epsilon),$$

where $a^2 = (a_1 \cos\epsilon_1 + a_2 \cos\epsilon_2)^2 + (a_1 \sin\epsilon_1 + a_2 \sin\epsilon_2)^2$

$$= a_1^2 + a_2^2 + 2 a_1 a_2 \cos(\epsilon_2 - \epsilon_1)$$

and $\tan\epsilon = (a_1 \sin\epsilon_1 + a_2 \sin\epsilon_2)/(a_1 \cos\epsilon_1 + a_2 \cos\epsilon_2)$.

185. A geometrical illustration of the preceding proposition is obtained by considering the uniform circular motions corresponding to the simple harmonic motions (Fig. 47).

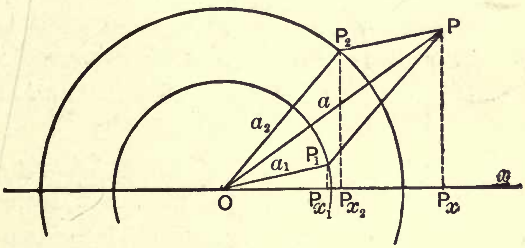

Fig. 47.

Drawing the radii $OP_1 = a_1$, $OP_2 = a_2$ so as to include an angle equal to the difference of phase $\epsilon_2 - \epsilon_1$ and completing the parallelogram OP_1PP_2, it appears from the figure that the diagonal OP of this parallelogram represents the resulting amplitude a. For since P_1P is equal and parallel to OP_2, we have for the projections on Ox the relation $OP_{x_1} + OP_{x_2} = OP_x$, or $x_1 + x_2 = x$.

Again, if the angle xOP_1 be taken equal to the epoch-angle ϵ_1, and hence $xOP_2 = \epsilon_2$, the angle xOP represents the epoch ϵ of the resulting motion.

We thus have a simple geometrical construction for the elements a, ϵ of the resulting motion from the elements a_1, ϵ_1 and a_2, ϵ_2 of the component motions. As the period is the same for the two component motions, the points P_1 and P_2 describe their respective circles with equal angular velocity so that the parallelogram OP_1PP_2 does not change its form in the course of the motion.

186. The construction given in the preceding article can be described briefly by saying that two simple harmonic motions of equal period in the same line are compounded by *geometrically adding* their amplitudes, it being understood that the phase-angles determine the directions in which the amplitudes are to be drawn.

It follows at once that not only two, but *any number of simple harmonic motions, of equal period in the same line, can be compounded by geometric addition of their amplitudes into a single simple harmonic motion in the same line and of the same period.*

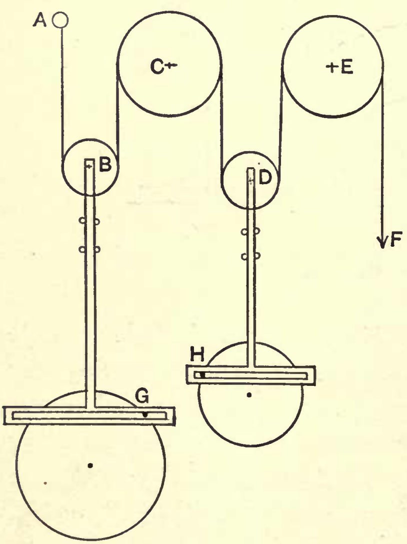

Fig. 48.

Conversely, any given simple harmonic motion can be resolved into two or more components in the same line and of the same period.

187. The kinematical meaning of this composition of simple harmonic motions of equal period in the same line will perhaps be best understood from the mechanism sketched in Fig. 48. A cord runs from the fixed point A over the movable pulleys B, D and the fixed pulleys C, E, and ends in F. Each of the movable pulleys receives a vertical simple harmonic motion from the T bars *BG* and *DH*, just as in Fig. 45 (Art. 179). If the free end *F* of the cord be just kept tight, its vertical displacement will be twice the sum of the vertical displacements of *B* and *D*, and as these points have simple harmonic motions, the motion of *F* will be twice the resultant simple harmonic motion.

The idea of this mechanism is due to Lord Kelvin.

188. Exercises.

(1) Find the resultant of three simple harmonic motions in the same line, and all of period $T = 10$ seconds, the amplitudes being 5, 3, and 4 cm., and the phase differences 30° and 60°, respectively, between the first and second, and the first and third motions.

(2) Apply the geometrical method of Art. 185 to the problem of Art. 183.

(3) Find the resultant of two simple harmonic motions in the same line and of equal period when the amplitudes are equal and the phases differ: (*a*) by an even multiple of π, (*b*) by an odd multiple of π.

(4) Resolve $x = 10 \cos(\pi t + 45°)$ into two components in the same line with a phase difference of 30°, one of the components having the epoch 0.

(5) Trace the curves representing the component motions as well as the resultant motion in Ex. (1), taking the time as abscissa and the displacement as ordinate.

(6) Show that the resultant of n simple harmonic motions of equal period T in the same line, viz.:

$$x_1 = a_1 \cos\left(\frac{2\pi}{T}t + \epsilon_1\right),\ x_2 = a_2 \cos\left(\frac{2\pi}{T}t + \epsilon_2\right),\ \cdots\ x_n = a_n \cos\left(\frac{2\pi}{T}t + \epsilon_n\right),$$

is the isochronous simple harmonic motion

$$x = a \cos\left(\frac{2\pi}{T}t + \epsilon\right),$$

where $$a^2 = \left(\sum_1^n a_i \cos\epsilon_i\right)^2 + \left(\sum_1^n a_i \sin\epsilon_i\right)^2$$

and $$\tan\epsilon = \sum_1^n a_i \sin\epsilon_i \Big/ \sum_1^n a_i \cos\epsilon_i.$$

189. The composition of two or more simple harmonic motions in the same line can readily be effected, even *when the components differ in period. But the resultant motion is not simply harmonic.*

Thus, for two components

$$x_1 = a_1 \cos(\omega_1 t + \epsilon_1), \quad x_2 = a_2 \cos(\omega_2 t + \epsilon_2),$$

putting $\omega_2 t + \epsilon_2 = \omega_1 t + (\omega_2 - \omega_1) t + \epsilon_2 = \omega_1 t + \epsilon_1 + \delta$, say, where $\delta = (\omega_2 - \omega_1) t + \epsilon_2 - \epsilon_1$ is the difference of phase at the time t, we have for the resulting motion

$$x = x_1 + x_2 = a_1 \cos(\omega_1 t + \epsilon_1) + a_2 \cos(\omega_1 t + \epsilon_1 + \delta);$$

and treating this similarly as in Art. 184 we find

$$x = (a_1 + a_2 \cos\delta) \cos(\omega_1 t + \epsilon_1) - a_2 \sin\delta \sin(\omega_1 t + \epsilon_1),$$

or putting $a_1 + a_2 \cos\delta = a \cos\epsilon, \quad a_2 \sin\delta = a \sin\epsilon,$

$$x = a \cos(\omega_1 t + \epsilon_1 + \epsilon),$$

where $a^2 = a_1^2 + a_2^2 + 2 a_1 a_2 \cos\delta$ and $\tan\epsilon = a_2 \sin\delta / (a_1 + a_2 \cos\delta)$.

190. These formulæ can be interpreted geometrically by Fig. 47, similarly as in Art. 185. But as in the present case the angle δ, and consequently the quantities a and ϵ in the expression for x, are variable, the parallelogram OP_1PP_2 while having constant sides has variable angles and changes its form in the course of the motion.

A mechanism similar to that of Fig. 48 (Art. 187), can be used to effect mechanically the composition of simple harmonic motions in the same line whether the periods be equal or not. This is the principle of the tide-predicting machine devised by Lord Kelvin.*

---

* See THOMSON and TAIT, *Natural philosophy*, Vol. I., Part I., new edition, 1879, p. 43 sq. and p. 479 sq. and J. D. EVERETT, *Vibratory motion and sound*, 1882.

191. To show the connection of the present subject with the theory of **wave motion,** imagine a flexible cord AB of which one end B is fixed while the other A is given a sudden jerk or transverse motion from A to C and back through A to D, etc. (Fig. 49). The displacement given to A will, so to speak, run along the cord, travelling from A to B and producing a wave. The figure exhibits the successive stages of the motion up to the time when a complete wave has been produced.

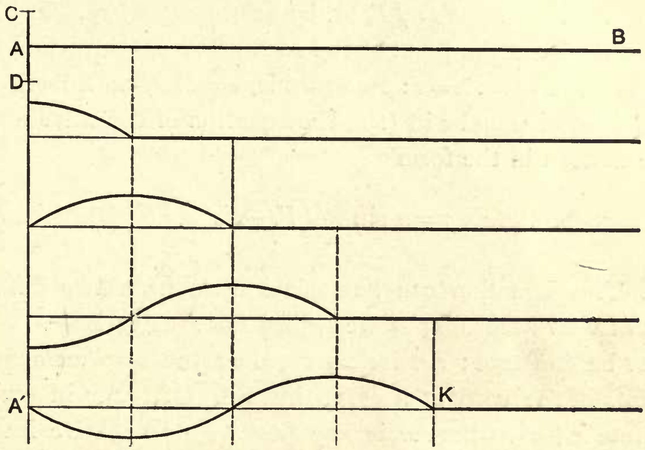

Fig. 49.

192. The distance $A'K$ (Fig. 49) is called the **length of the wave.** Denoting this length by λ, and the time in which the motion spreads from A' to K by T we have for the **velocity of propagation** of the wave

$$V=\frac{\lambda}{T}. \tag{18}$$

It is to be noticed that the motion of any particular point of the cord is supposed to be rectilinear and at right angles to AB; this is the case with the simple transverse vibrations in an elastic medium such as the luminiferous ether regarded as the vehicle of light waves.

193. If the motion of A be simply harmonic, say $y=a\sin\omega t$, the motions of the successive points of the cord will differ from the motion of A only in phase, and the displacements of all these points at any time t can be represented by

$$y=a\sin(\omega t-\epsilon), \tag{19}$$

where ϵ varies from 0 to 2π as we pass from A to K.

As the time T in which the motion spreads from A to K is equal to the period of a vibration of A (or of any other point of the cord), we have $\omega=2\pi/T$, or, by (18), $\omega=2\pi V/\lambda$. And if x be the distance of the point of the cord under consideration from A, we must have $x:\lambda=\epsilon:2\pi$; that is, $\epsilon=2\pi x/\lambda$. Substituting these values of ω and ϵ in (19), the equation of the wave motion can be written in the form

$$y=a\sin\frac{2\pi}{\lambda}(Vt-x). \tag{20}$$

194. This equation can be looked upon from two different points of view according as we regard t or x as variable.

Let t be constant; *i.e.* let us consider the displacements of all points of the cord at a given instant. If for x in (20) we substitute $x+n\lambda$, where n is any positive or negative integer, the angle $(Vt-x)\,2\pi/\lambda$ is changed by $2\pi n$, so that the value of y remains unchanged. The displacements of all particles whose distances from A differ by whole wave lengths are therefore the same; in other words, the state of motion at any instant is represented by a series of equal waves.

Now let x be constant, and t variable. Substituting for t in (20) the value $t+nT=t+n\lambda/V$, the angle $(Vt-x)\,2\pi/\lambda$ is again changed by $2\pi n$, and y remains the same. This shows the periodicity in the motion of any given particle.

195. If the point A (Fig. 49) be subjected simultaneously to more than one simple harmonic motion, the displacements resulting from each must be added algebraically, thus forming a compound wave which can readily be traced by first tracing the component waves and then adding their ordinates, or analytically by forming the equation of the resultant motion as in Art. 189.

196. Exercise.

(1) Trace the wave produced by the superposition of two simple harmonic motions in the same line of equal amplitudes, the periods being as 2 : 1, (*a*) when they do not differ in phase, (*b*) when their epochs differ by 7/16 of the period.

197. The idea of wave motion implies that the displacement y should be a periodic function of x and t such as to fulfil the following conditions: y must assume the same value (*a*) when x is changed into $n\lambda$, (*b*) when t is changed into $t+T$, (*c*) when both changes are made simultaneously; the constants λ and T being connected by the relation $\lambda=VT$.

The condition (*c*) requires y to be of the form $y=f(Vt-x)$; for $Vt-x$ remains unchanged when x is replaced by $x+VT$ and at the same time t by $t+T$.

A particular case of such a function is $y=a\sin c(Vt-x)$. As y should remain unchanged when t is replaced by $t+T$, we must have $c=2\pi/VT=2\pi/\lambda$. Thus the function

$$y=a\sin\frac{2\pi}{\lambda}(Vt-x)$$

fulfils the three conditions (*a*), (*b*), (*c*). Putting as before (Art. 193) $2\pi x/\lambda=-\epsilon$, we can write it

$$y=a\sin\left(\frac{2\pi}{T}t+\epsilon\right).$$

198. The importance of this particular solution of our problem lies in the fact that, according to **Fourier's theorem,** any single-valued periodic function of period T can be expanded, between definite limits of the variable, into a series of the form

$$f(t)=a_0+a_1\sin\left(\frac{2\pi}{T}\cdot t+\epsilon_1\right)+a_2\sin\left(\frac{2\pi}{T}\cdot 2t+\epsilon_2\right)$$
$$+a_3\sin\left(\frac{2\pi}{T}\cdot 3t+\epsilon_3\right)+\cdots. \qquad (21)$$

As applied to the theory of wave motion this means of course that any wave motion, however complex, can be regarded as made up of a series of superposed simple harmonic vibrations of periods $T, T/2, T/3, \cdots$, or since $T=\lambda/V$, of wave lengths $\lambda, \lambda/2, \lambda/3, \cdots$.

199. A full discussion of Fourier's theorem cannot be given in this place.* We wish, however, to show its practical application in an example.

The equation (21) can be written in the form

$$f(t)=a_1\cos\epsilon_1\sin\frac{2\pi}{T}\cdot t+a_2\cos\epsilon_2\sin\frac{2\pi}{T}\cdot 2t+a_3\cos\epsilon_3\sin\frac{2\pi}{T}\cdot 3t+\cdots$$

$$+a_0+a_1\sin\epsilon_1\cos\frac{2\pi}{T}\cdot t+a_2\sin\epsilon_2\cos\frac{2\pi}{T}\cdot 2t+a_3\sin\epsilon_3\cos\frac{2\pi}{T}\cdot 3t+\cdots,$$

or putting $2\pi t/T=x$, $a_1\sin\epsilon_1=A_1$, $a_2\sin\epsilon_2=A_2$, $\cdots$, $a_1\cos\epsilon_1=B_1$, $a_2\cos\epsilon_2=B_2$, $\cdots$,

$$f(x)=a_0+A_1\cos x+A_2\cos 2x+A_3\cos 3x+\cdots$$
$$+B_1\sin x+B_2\sin 2x+B_3\sin 3x+\cdots. \tag{22}$$

This is known as Fourier's series. According to the nature of the function to be expanded, it is often sufficient to use the sine series or the cosine series alone. As the method of determining the coefficients is always the same, it will be sufficient to consider the simple sine series:

$$f(x)=B_1\sin x+B_2\sin 2x+B_3\sin 3x+\cdots. \tag{23}$$

200. The problem before us can now be stated as follows: Given any single-valued function of x, either by its analytical expression or by the trace of the curve representing it, to determine the coefficients B in (23) so as to make the right-hand member of this equation represent the values of the given function between certain finite limits of x.

We shall assume these limits to be $x=0$ and $x=\pi$; and we shall

* The student is referred to THOMSON and TAIT, *Natural philosophy*, I. 1, 1879, pp. 55–60; also to B. RIEMANN, *Partielle Differentialgleichungen*, herausgegeben von K. Hattendorff, 3d ed., Braunschweig, Vieweg, 1882, pp. 44–95, and to G. M. MINCHIN, *Uniplanar kinematics*, Oxford, Clarendon Press, 1882, p. 13 sq.

first try to make $n-1$ points of the given curve taken between these limits coincide with as many points of the curve

$$f(x)=B_1\sin x+B_2\sin 2x+\cdots+B_{n-1}\sin(n-1)x. \qquad (24)$$

Then passing to the limit for $n=\infty$, the problem will be solved.

201. Dividing the interval from $x=0$ to $x=\pi$ into n equal parts and taking the $n-1$ points on the given curve so that their abscissæ are

$$\frac{\pi}{n},\ \frac{2\pi}{n},\ \frac{3\pi}{n},\ \cdots\ \frac{(n-1)\pi}{n},$$

the curve (24) will contain these points if the following $n-1$ equations are fulfilled:

$$\left.\begin{aligned}
f\left(\frac{\pi}{n}\right) &= B_1\sin 1\cdot\frac{\pi}{n}+B_2\sin 1\cdot\frac{2\pi}{n}+\cdots+B_{n-1}\sin 1\cdot\frac{(n-1)\pi}{n},\\
f\left(\frac{2\pi}{n}\right) &= B_1\sin 2\cdot\frac{\pi}{n}+B_2\sin 2\cdot\frac{2\pi}{n}+\cdots+B_{n-1}\sin 2\cdot\frac{(n-1)\pi}{n},\\
f\left(\frac{3\pi}{n}\right) &= B_1\sin 3\cdot\frac{\pi}{n}+B_2\sin 3\cdot\frac{2\pi}{n}+\cdots+B_{n-1}\sin 3\cdot\frac{(n-1)\pi}{n},\\
&\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\\
f\left(\frac{(n-1)\pi}{n}\right) &= B_1\sin(n-1)\frac{\pi}{n}+B_2\sin(n-1)\frac{2\pi}{n}+\cdots+B_{n-1}\sin(n-1)\frac{(n-1)\pi}{n}.
\end{aligned}\right\} \qquad (25)$$

These equations are sufficient to determine the $n-1$ unknown constants $B_1, B_2, \cdots, B_{n-1}$.

202. To solve the equations we multiply them by indeterminate coefficients and add. The coefficients can be so selected as to make all the unknown quantities disappear except one which is thus determined.

Thus, to find B_m multiply each equation by twice the coefficient of B_m in this equation, viz. the first equation by $2\sin(m\pi/n)$, the second by $2\sin(2m\pi/n)$, etc., the last by $2\sin[(n-1)m\pi/n]$.

After adding, the factor of B_k will be

$$\beta_k=2\left[\sin k\frac{\pi}{n}\sin m\frac{\pi}{n}+\sin k\frac{2\pi}{n}\sin m\frac{2\pi}{n}+\cdots\right.$$
$$\left.+\sin k(n-1)\frac{\pi}{n}\sin m(n-1)\frac{\pi}{n}\right].$$

Transforming every term by the formula $2\sin\alpha\sin\beta = \cos(\alpha-\beta) - \cos(\alpha+\beta)$, we find

$$\beta_k = \cos(k-m)\frac{\pi}{n} + \cos(k-m)\frac{2\pi}{n} + \cdots + \cos(k-m)\frac{(n-1)\pi}{n}$$

$$-\left[\cos(k+m)\frac{\pi}{n} + \cos(k+m)\frac{2\pi}{n} + \cdots + \cos(k+m)\frac{(n-1)\pi}{n}\right]. \quad . \quad . \quad . \quad (26)$$

Applying the trigonometrical formula

$$\cos\alpha + \cos 2\alpha + \cdots + \cos(n-1)\alpha = \frac{1}{2}\left(-1 + \frac{\sin(2n-1)\alpha/2}{\sin\alpha/2}\right),$$

we obtain

$$2\beta_k = \frac{\sin(2n-1)(k-m)\frac{\pi}{2n}}{\sin(k-m)\frac{\pi}{2n}} - \frac{\sin(2n-1)(k+m)\frac{\pi}{2n}}{\sin(k+m)\frac{\pi}{2n}}$$

$$= \frac{\sin(k-m)\pi\cos(k-m)\frac{\pi}{2n} - \cos(k-m)\pi\sin(k-m)\frac{\pi}{2n}}{\sin(k-m)\frac{\pi}{2n}}$$

$$- \frac{\sin(k+m)\pi\cos(k+m)\frac{\pi}{2n} - \cos(k+m)\pi\sin(k+m)\frac{\pi}{2n}}{\sin(k+m)\frac{\pi}{2n}}.$$

If k be different from m, this reduces to

$$\beta_k = \tfrac{1}{2}[-\cos(k-m)\pi + \cos(k+m)\pi],$$

and this is always $=0$, since $k+m$ and $k-m$ are either both odd or both even.

If $k=m$, we find from (26)

$$\beta_m = 1 + 1 + \cdots + 1$$

$$-\left[\cos 2m\frac{\pi}{n} + \cos 2m\frac{2\pi}{n} + \cdots + \cos 2m\frac{(n-1)\pi}{n}\right].$$

$$= n - 1 - \frac{1}{2}\left[-1 + \frac{\sin(2n-1)\dfrac{m\pi}{n}}{\sin\dfrac{m\pi}{n}}\right]$$

$$= n - \frac{1}{2} - \frac{1}{2}\,\frac{\sin 2m\pi\cos\dfrac{m\pi}{n} - \cos 2m\pi\sin\dfrac{m\pi}{n}}{\sin\dfrac{m\pi}{n}}$$

$$= n - \tfrac{1}{2} - \tfrac{1}{2}(-1) = n.$$

We have, therefore, finally

$$nB_m = 2f\left(\frac{\pi}{n}\right)\sin m\frac{\pi}{n} + 2f\left(\frac{2\pi}{n}\right)\sin m\frac{2\pi}{n} + \cdots$$
$$+ 2f\left(\frac{(n-1)\pi}{n}\right)\sin m\frac{(n-1)\pi}{n}, \qquad (27)$$

ith $m = 1, 2, 3, \cdots n-1$.

203. It remains to pass to the limit when $n = \infty$ and $\frac{\pi}{n}$ vanishes. Vriting (27) in the form

$$B_m = \frac{2}{\pi}\cdot\frac{\pi}{n}\left[f\left(\frac{\pi}{n}\right)\sin m\frac{\pi}{n} + f\left(\frac{2\pi}{n}\right)\sin m\frac{2\pi}{n} + \cdots \right.$$
$$\left. + f\left(\frac{(n-1)\pi}{n}\right)\sin m\frac{(n-1)\pi}{n}\right],$$

e obviously have in the limit

$$B_m = \frac{2}{\pi}\int_0^\pi f(x)\sin mx\,dx, \quad m = 1, 2, 3, \cdots. \qquad (28)$$

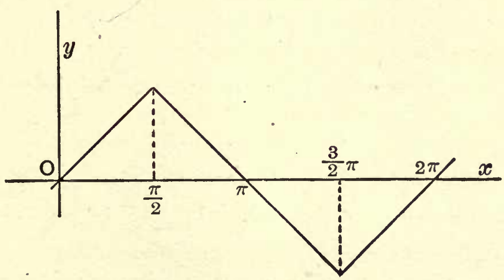

Fig. 50.

204. As an application let us determine the series representing the roken line formed by the two sides of an isosceles right-angled triangle hose hypotenuse lies in the axis of x (Fig. 50).

We assume the length of this hypotenuse $=\pi$; then the given function is $f(x)=x$ from $x=0$ to $x=\pi/2$, and $f(x)=\pi-x$ from $x=\pi/2$ to $x=\pi$.

On account of the discontinuity at the point $x=\pi/2$, the integral in (28) must be resolved into two, and we have

$$B_m=\frac{2}{\pi}\left[\int_0^{\frac{\pi}{2}} x\sin mx\,dx+\int_{\frac{\pi}{2}}^{\pi}(\pi-x)\sin mx\,dx\right]$$

$$=\frac{2}{\pi}\left[-\frac{1}{m}\frac{\pi}{2}\cos\frac{m\pi}{2}+\frac{1}{m^2}\sin\frac{m\pi}{2}+\frac{1}{m}\frac{\pi}{2}\cos\frac{m\pi}{2}+\frac{1}{m^2}\sin\frac{m\pi}{2}\right]$$

$$=\frac{4}{\pi m^2}\sin\frac{m\pi}{2}.$$

For even values of m, $\sin(m\pi/2)=0$; for odd values, $\sin(m\pi/2)$ is alternately positive or negative. Hence the series (23) becomes

$$f(x)=\frac{4}{\pi}\left[\frac{\sin x}{1^2}-\frac{\sin 3x}{3^2}+\frac{\sin 5x}{5^2}-\cdots\right]. \tag{29}$$

This expansion certainly holds when x lies between 0 and π. As every term of the series vanishes for $x=0$ as well as for $x=\pi$, the expansion holds even at these limits. Moreover, when x lies between π and 2π, all the terms of the series, with signs reversed, pass through the same succession of values as between 0 and π. The series represents therefore between these limits an equal triangle with its vertex below the axis of x (Fig. 50). Beyond the point $x=2\pi$, the same figure repeats itself owing to the periodicity of the sine.

It thus appears that the series represents an infinite zigzag line for all values of x.

205. We proceed to the *composition of simple harmonic motions not in the same line.* We shall, however, assume that all the component motions lie in the same plane.

It is evident that the projection of a simple harmonic motion on any line is again a simple harmonic motion of the same period and phase and with an amplitude equal to the projection of the original amplitude.

Hence, to compound any number of simple harmonic motions along lines lying in the same plane, we may project all these motions on any two rectangular axes Ox, Oy taken in this plane, and compound, by Art. 184 or 189, the components lying in the same axis. It then only remains to compound the two motions, one along Ox, the other along Oy, into a single motion.

206. Just as in Arts. 184, 189, we must distinguish two cases: (*a*) when the given motions have all the same period, and (*b*) when they have not.

In the former case, by Art. 184, the two components along Ox and Oy will have equal periods, *i.e.* they will be of the form

$$x = a \sin \omega t, \quad y = b \sin(\omega t + \delta). \tag{30}$$

The path of the resulting motion is obtained by eliminating t between these equations. We have

$$\frac{y}{b} = \sin \omega t \cos \delta + \cos \omega t \sin \delta$$

$$= \frac{x}{a} \cos \delta + \sqrt{1 - \frac{x^2}{a^2}} \sin \delta.$$

Writing this equation in the form

$$\left(\frac{y}{b} - \frac{x}{a} \cos \delta\right)^2 = \left(1 - \frac{x^2}{a^2}\right) \sin^2 \delta,$$

or

$$\frac{x^2}{a^2} - \frac{2xy}{ab} \cos \delta + \frac{y^2}{b^2} = \sin^2 \delta, \tag{31}$$

we see that it represents an ellipse (since $\frac{1}{a^2} \cdot \frac{1}{b^2} - \frac{\cos^2 \delta}{a^2 b^2} = \left(\frac{\sin \delta}{ab}\right)^2$ is positive) whose centre is at the origin. The resultant motion is therefore called **elliptic harmonic motion.**

207. Although in what precedes we have assumed the axes at right angles to each other, this is not essential. The same equation (31) is obtained for oblique axes Ox, Oy, and it is easy to show (say by transforming (31) to rectangular axes) that this equation still represents an ellipse. We have, therefore, the general result that *any number of simple harmonic motions of the same period and in the same plane, whatever may be their directions, amplitudes, and phases, compound into a single elliptic harmonic motion.*

208. A few particular cases may be noticed. The equation (31) will represent a (double) straight line, and hence the elliptic vibration will degenerate into a simple harmonic vibration, whenever $\sin^2\delta=0$, *i.e.* when $\delta=n\pi$, where n is a positive or negative integer. In this case $\cos\delta$ is $+1$ or -1, and (31) reduces to

$$\frac{x}{a}-\frac{y}{b}=0, \text{ if } \delta=2n\pi,$$

and to

$$\frac{x}{a}+\frac{y}{b}=0, \text{ if } \delta=(2n+1)\pi.$$

Thus two rectangular vibrations of the same period compound into a simple harmonic vibration when they differ in phase by an integral multiple of π, that is when one lags behind the other by half a wave length.

209. Again, the ellipse (31) reduces to a circle only when $\cos\delta=0$, *i.e.* $\delta=(2n+1)\pi/2$, and in addition $a=b$, the co-ordinates being assumed orthogonal.

Thus two rectangular vibrations of equal period and amplitude compound into a circular vibration if they differ in phase by $\pi/2$, *i.e.* if one is retarded behind the other by a quarter of a wave length.

This circular harmonic motion is evidently nothing but uniform motion in a circle; and we have seen in Art. 172 that, conversely, uniform circular motion can be resolved into two rectangular simple harmonic vibrations of equal period and amplitude, but differing in phase by $\pi/2$.

The results of Arts. 205–209 can also be established by purely geometrical methods of an elementary character.*

210. It remains to consider the case when the given simple harmonic motions do not all have the same period. It follows from Art. 189 that in this case, if we again project the given motions on two rectangular axes Ox, Oy, the resulting motions along Ox, Oy are in general not simply harmonic.

The elimination of t between the expressions for x and y may present difficulties. But, of course, the curve can always be traced by points, graphically.

We shall here consider only the case when the motions along Ox and Oy are simply harmonic.

211. *If two simple harmonic motions along the rectangular directions* Ox, Oy, viz.:

$$x=a_1 \cos\left(\frac{2\pi}{T_1}t+\epsilon_1\right), \quad y=a_2 \cos\left(\frac{2\pi}{T_2}t+\epsilon_2\right),$$

of different amplitudes, phases, and *periods* are to be compounded, the resulting motion will be confined within a rectangle whose sides are $2a_1$, $2a_2$, since these are the maximum values of $2x$ and $2y$.

The path of the moving point will be a *closed* curve only when the quotient T_1/T_2 is a commensurable number, say $= m/n$, where m is prime to n. The x co-ordinate of the curve will have m maxima, the y co-ordinate n, and the whole curve will be traversed after m vibrations along Ox and n along Oy.

The formation of the resulting curve will best be understood from the following example.

---

* See, for instance, J. G. MACGREGOR, *An elementary treatise on kinematics and dynamics*, London, Macmillan, 1887, pp. 115 sq.

212. Let $a_1 = a_2 = a$, $\epsilon_1 = 0$, $\epsilon_2 = \delta$, and let the ratio of the periods be $T_1/T_2 = 2/1$. The equations of the component simple harmonic vibrations are

$$x = a\cos\omega t,\ y = a\cos(2\omega t + \delta).$$

Here it is easy to eliminate t. We have

$$y = a\cos 2\omega t\cos\delta - a\sin 2\omega t\sin\delta$$

$$= a\left(2\frac{x^2}{a^2} - 1\right)\cos\delta - 2a\frac{x}{a}\sqrt{1 - \frac{x^2}{a^2}}\sin\delta.$$

Hence the equation of the path is:

$$ay = (2x^2 - a^2)\cos\delta - 2x\sqrt{a^2 - x^2}\sin\delta.$$

If there be no difference of phase between the components, *i.e.* if $\delta = 0$, this reduces to the equation of a parabola:

$$x^2 = \frac{a}{2}(y + a).$$

For $\delta = \pi/2$, the equation also assumes a simple form:

$$a^2y^2 = 4x^2(a^2 - x^2).$$

213. It is instructive to trace the resulting curves for a given ratio of periods and for a series of successive differences of phase (*Lissajous's Curves*).

Thus, in Fig. 51, the curve for $T_1/T_2 = 3/4$, and for a phase difference $\delta = 0$ is the fully drawn curve, while the dotted curve represents the path for the same ratio of the periods when the phase difference is one-twelfth of the smaller period. The equations of the components are for the full curve

$$x = 6\cos\frac{2\pi}{3}t,\ y = 5\cos\frac{2\pi}{4}t,$$

and for the dotted curve

$$x = 6\cos\left(\frac{2\pi}{3}t + \frac{2\pi}{12}\right),\ y = 5\cos\frac{2\pi}{4}t.$$

In tracing these curves, imagine the simple harmonic motions replaced by the corresponding uniform circular motions (Fig. 51). With the amplitudes 6, 5 as radii, describe the semi-circles *ADB*, *AEC*, so that *BC* is the rectangle within which the curves are confined; the intersection of the diagonals of this rectangle is the origin *O*, *AB* is parallel to the axis of x, *AC* to the axis of y. Next divide the circles over *AB*, *AC* into parts corresponding to equal intervals of time. In the present case, the periods for *AB*, *AC* being as 3 to 4, the circle over

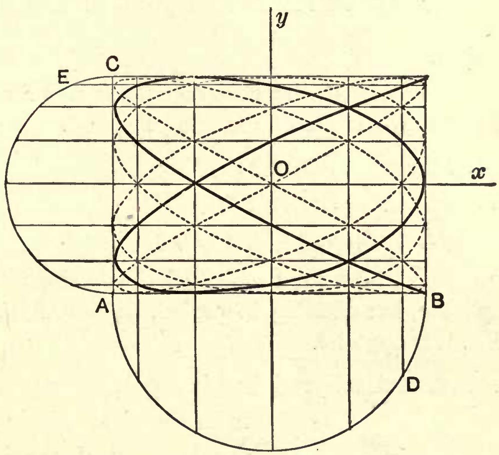

Fig. 51.

AB must be divided into $3n$ equal parts, that over *AC* into $4n$. In the figure, n is taken as 4, the circles being divided into 12 and 16 equal parts, respectively.

The first point of the full drawn curve corresponds to $t=0$, that is $x=6, y=5$; this gives the upper right hand corner of the rectangle. The next point is the intersection of the vertical line through *D* and the horizontal line through *E*, the arcs $BD=1/12$ of the circle over *AB*, and $CE=1/16$ of that over *AC* being described in the same time, so that the co-ordinates of the corresponding point are

$$x=6\cos\left(\frac{2\pi}{3}\cdot\frac{3}{12}\right)=6\cos\left(2\pi\cdot\frac{1}{12}\right)$$

$$y=5\cos\left(\frac{2\pi}{4}\cdot\frac{4}{16}\right)=5\cos\left(2\pi\cdot\frac{1}{16}\right).$$

Similarly the next point

$$x=6\cos\left(2\pi\cdot\frac{2}{12}\right),\ y=5\cos\left(2\pi\cdot\frac{2}{16}\right)$$

is found from the next two points of division on the circles, etc.

To construct the dotted curve, it is only necessary to begin on the circle over AB with D as first point of division.

214. Exercises.

(1) With the data of Art. 213 construct the curves for phase differences of 2/12, 3/12, ··· 11/12 of the smaller period.

(2) Construct the curves (Art. 212)

$$x=a\cos\omega t,\ y=a\cos(2\omega t+\delta)$$

for $\delta=0, \pi/4, \pi/2, 3\pi/4, \pi, 5\pi/4, 3\pi/2, 7\pi/4, 2\pi$.

(3) Trace the path of a point subjected to two circular vibrations of the same amplitude, but differing in period, (*a*) when the sense is the same; (*b*) when it is opposite.

215. The **mathematical pendulum** is a point compelled to move in a vertical circle under the acceleration of gravity.

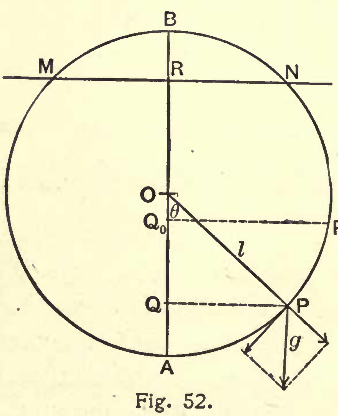

Fig. 52.

Let O be the centre (Fig. 52), A the lowest, and B the highest point of the circle. The radius $OA=l$ of the circle is called the length of the pendulum. Any position P of the moving point is determined by the angle $AOP=\theta$ counted from the vertical radius OA in the positive (counterclockwise) sense of rotation.

If P_0 be the initial position of the moving point at the time

$t=0$, and $\measuredangle AOP_0=\theta_0$, then the arc $P_0P=s$ described in the time t is $s=l(\theta_0-\theta)$; hence $v=ds/dt=-l d\theta/dt$, and $dv/dt = -l d^2\theta/dt^2$, the negative sign indicating that θ diminishes as s and t increase.

Resolving the acceleration of gravity, g, into its normal and tangential components $g\cos\theta$, $g\sin\theta$, and considering that the former is without effect owing to the condition that the point is constrained to move in a circle, we obtain the equation of motion in the form $dv/dt=g\sin\theta$, or

$$l\frac{d^2\theta}{dt^2}+g\sin\theta=0. \qquad (32)$$

216. The first integration is readily performed by multiplying the equation by $d\theta/dt$ which makes the left-hand member an exact derivative,

$$\frac{d}{dt}\left(\frac{l}{2}\left(\frac{d\theta}{dt}\right)^2-g\cos\theta\right);$$

hence integrating, we obtain

$$\tfrac{1}{2}l\left(\frac{d\theta}{dt}\right)^2-g\cos\theta=C,$$

or considering that $v=-l d\theta/dt$,

$$\tfrac{1}{2}v^2-gl\cos\theta=Cl.$$

To determine the constant C, the initial velocity v_0 at the time $t=0$ must be given. We then have $\frac{1}{2}v_0^2-gl\cos\theta_0=Cl$; hence

$$\tfrac{1}{2}v^2=\tfrac{1}{2}v_0^2-gl\cos\theta_0+gl\cos\theta=g\left(\frac{v_0^2}{2g}-l\cos\theta_0+l\cos\theta\right). \qquad (33)$$

The right-hand member can readily be interpreted geometrically; $v_0^2/2g$ is the height by falling through which the point would acquire the initial velocity v_0 (see Art. 113); $l\cos\theta$

$-l\cos\theta_0 = OQ - OQ_0 = Q_0Q$, if Q, Q_0 are the projections of P, P_0 on the vertical AB. If we draw a horizontal line MN at the height $v_0^2/2g$ above P_0 and if this line intersect the vertical AB in R, we have for the velocity v the expression:

$$\tfrac{1}{2}v^2 = g \cdot RQ. \tag{34}$$

If the initial velocity be $=0$, the equation would be

$$\tfrac{1}{2}v^2 = g \cdot Q_0Q. \tag{35}$$

At the points M, N where the horizontal line MN intersects the circle the velocity becomes 0. The point can therefore never rise above these points.

Now, according to the value of the initial velocity v_0, the line MN may intersect the circle in two real points M, N, or touch it at B, or not meet it at all. In the first case the point P performs oscillations, passing from its initial position P_0 through A up to M, then falling back to A and rising to N, etc. In the third case P makes complete revolutions.

217. The second integration of the equation of motion cannot be effected in finite terms, without introducing elliptic functions. But for the case of most practical importance, viz. for very small values of θ, it is easy to obtain an approximate solution. In this case θ can be substituted for $\sin\theta$, and the equation becomes:

$$\frac{d^2\theta}{dt^2} + \frac{g}{l}\theta = 0. \tag{36}$$

This is a well known differential equation (compare Art. 122, Eq. (19), and Art. 176), whose general integral is

$$\theta = C_1 \cos t\sqrt{\frac{g}{l}} + C_2 \sin t\sqrt{\frac{g}{l}}.$$

The constants C_1, C_2 can be determined from the initial conditions for which we shall now take $\theta=\theta_0$ and $v=0$ when $t=0$; this gives $C_1=\theta_0$, $C_2=0$; hence

$$\theta=\theta_0 \cos t\sqrt{\frac{g}{l}}, \quad t=\sqrt{\frac{l}{g}}\cos^{-1}\frac{\theta}{\theta_0}. \tag{37}$$

The last equation gives for $\theta=-\theta_0$ the time t_1 of one oscillation, or half the period T,

$$t_1=\tfrac{1}{2}T=\pi\sqrt{\frac{l}{g}}. \tag{38}$$

The time of a small oscillation is thus seen to be independent of the arc through which the pendulum swings; in other words, for all small arcs the times of oscillation of the same pendulum are the same; such oscillations are therefore called *isochronous.*

218. A pendulum whose length is so adjusted as to make it perform at a certain place just one oscillation in a second is called a **seconds pendulum.**

Putting $t_1=1$ in (38) we find for the length l_1 of the seconds pendulum at a place where the acceleration of gravity is g,

$$l_1=\frac{g}{\pi^2}. \tag{39}$$

As the length of the pendulum can be determined with great accuracy by measurement, the pendulum can be used to find the value of g.

The length of the seconds pendulum is very nearly a metre; it varies for points at sea level from $l_1=99.103$ cm. at the equator to $l_1=99.610$ at the poles.*

---

* Further numerical data for l_1 and g will be found in J. D. EVERETT, *C. G. S. system of units*, 1891, pp. 21–22.

219. Let n be the number of small oscillations made by a pendulum of length l in the time T. Then, by (38),

$$\frac{T}{n} = \pi \sqrt{\frac{l}{g}}. \tag{40}$$

If T and one of the three quantities n, l, g in this equation be regarded as constant, the small variations of the two others can be found approximately by differentiation. For instance, if the daily number of oscillations of a pendulum of constant length be observed at two different places, we have, since T and l are constant,

$$-\frac{T}{n^2} dn = -\frac{\pi\sqrt{l}}{2} \frac{dg}{g^{\frac{3}{2}}},$$

or, dividing by (40),

$$\frac{dn}{n} = \frac{1}{2}\frac{dg}{g}. \tag{41}$$

220. Exercises.

(1) Find the number of oscillations made in a second and in a day by a pendulum 1 metre long, at a place where $g = 981.0$.

(2) Find the length of the seconds pendulum at a place where $g = 32.12$.

(3) To determine the value of g at a given place, the length of a pendulum was adjusted until it would make 86 400 oscillations in 24 hours. Its length was then found to be 3.3031 feet. What was the value of g?

(4) A chandelier suspended from the ceiling of a theatre is seen to vibrate 24 times a minute. Find its distance from the ceiling.

(5) A pendulum adjusted so as to beat seconds at the equator ($g = 978.1$) is carried to another latitude and is there found to make 100 oscillations more per day; find the value of g at this place.

(6) Investigate whether the approximate process of Formula (41) is sufficiently accurate for the solution of Ex. (5).

(7) If the length of a pendulum be increased by a small amount dl, show that the daily number of oscillations, n, will be decreased so that

$$\frac{dn}{n} = -\frac{1}{2}\frac{dl}{l}.$$

(8) A clock is gaining 3 minutes a day. How much should the pendulum bob be screwed up or down?

(9) A clock regulated at a place where $g=32.19$ is carried to a place where $g=32.14$. How much will it gain or lose per day if the length of its pendulum be not changed?

(10) The acceleration of gravity being inversely proportional to the square of the distance from the earth's centre, show that the seconds pendulum will lose about 22 seconds per day if taken to a height one mile above sea level.

(11) A seconds pendulum loses 12 seconds per day, if taken to the top of a mountain. What is the height of the mountain?

(12) Show that for small oscillations the motion of a pendulum is nearly simply harmonic, and deduce from this fact the equation $t_1=\pi\sqrt{l/g}$.

221. When the oscillations of a pendulum are not so small that the arc can be substituted for its sine as was done in Art. 217, an expression for the time t_1 of one oscillation can be obtained as follows.

We have by (33), Art. 216,

$$\tfrac{1}{2}v^2-\tfrac{1}{2}v_0^2=gl(\cos\theta-\cos\theta_0). \tag{33}$$

Let the time be counted from the instant when the moving point has its highest position (N in Fig. 52), so that $v_0=0$. Substituting $v=-l d\theta/dt$ and applying the formula

$$\cos\theta=1-2\sin^2\tfrac{1}{2}\theta$$

we find:

$$\tfrac{1}{2}l\left(\frac{d\theta}{dt}\right)^2=2g\left(\sin^2\frac{\theta_0}{2}-\sin^2\frac{\theta}{2}\right),$$

whence

$$dt=\tfrac{1}{2}\sqrt{\frac{l}{g}}\frac{d\theta}{\sqrt{\sin^2\frac{\theta_0}{2}-\sin^2\frac{\theta}{2}}}. \tag{42}$$

Integrating from $\theta=0$ to $\theta=\theta_0$ and multiplying by 2 we find for the time t_1 of one oscillation:

$$t_1=\sqrt{\frac{l}{g}}\int_0^{\theta_0}\frac{d\theta}{\sqrt{\sin^2\frac{\theta_0}{2}-\sin^2\frac{\theta}{2}}}. \tag{43}$$

As θ cannot become greater than θ_0 we may put $\sin(\theta/2) = \sin(\theta_0/2)\sin\phi$, thus introducing a new variable ϕ for which the limits are 0 and $\pi/2$. Differentiating the equation of substitution, we have

$$\tfrac{1}{2}\cos\frac{\theta}{2}d\theta=\sin\frac{\theta_0}{2}\cos\phi\, d\phi,$$

or, as $\cos(\theta/2)=\sqrt{1-\sin^2(\theta_0/2)\sin^2\phi}$,

$$d\theta=\frac{2\sin\frac{\theta_0}{2}\cos\phi\, d\phi}{\sqrt{1-\sin^2\frac{\theta_0}{2}\sin^2\phi}}.$$

Substituting these values and putting for shortness

$$\sin\frac{\theta_0}{2}=\kappa, \tag{44}$$

we find for the time t_1 of one oscillation:

$$t_1=2\sqrt{\frac{l}{g}}\int_0^{\frac{\pi}{2}}\frac{d\phi}{\sqrt{1-\kappa^2\sin^2\phi}}. \tag{45}$$

The integral in this expression is called the complete elliptic integral of the first species, and is usually denoted by K. Its value can be found from tables of elliptic integrals or by expanding the argument into an infinite series by the binomial theorem (since $\kappa\sin\phi$ is less than 1), and then performing the integration. We have

$$(1-\kappa^2\sin^2\phi)^{-\frac{1}{2}}=1+\tfrac{1}{2}\kappa^2\sin^2\phi+\frac{1\cdot 3}{2\cdot 4}\kappa^4\sin^4\phi+\cdots;$$

hence

$$t_1=\pi\sqrt{\frac{l}{g}}\left[1+\left(\frac{1}{2}\right)^2\kappa^2+\left(\frac{1\cdot 3}{2\cdot 4}\right)^2\kappa^4+\cdots\right]. \tag{46}$$

If H be the height of the initial point $N(\theta=\theta_0)$ above the lowest point A of the circle, we have by (44)

$$\kappa^2 = \sin^2\frac{\theta_0}{2} = \frac{1-\cos\theta_0}{2} = \frac{H}{2l},$$

so that (46) can be written in the form

$$t_1=\pi\sqrt{\frac{l}{g}}\left[1+\left(\frac{1}{2}\right)^2\frac{H}{2l}+\left(\frac{1\cdot 3}{2\cdot 4}\right)^2\left(\frac{H}{2l}\right)^2+\cdots\right]. \qquad (47)$$

222. Exercises.

(1) Show that $t_1=\pi\sqrt{\frac{l}{g}}\left(1+\frac{1}{16}+\frac{9}{1024}+\frac{225}{147456}+\cdots\right)$ if the amplitude $2\theta_0$ of the oscillation is 120°.

(2) Show that as a second approximation to the time of a small oscillation we have $t_1=\pi\sqrt{l/g}(1+\theta_0^2/16)$.

(3) Find the time of oscillation of a pendulum whose length is 1 metre at a place where $g=980.8$, to four decimal places.

(4) A pendulum hanging at rest is given an initial velocity v_1. Find to what height h_1 it will rise.

(5) Discuss the pendulum problem in the particular case when MN (Fig. 52) touches the circle at B, that is when the initial velocity is due to falling from the highest point of the circle.

223. Central Motion. The motion of a point P is called **central** if the following two conditions are fulfilled: (1) the direction of the acceleration must pass constantly through a fixed point O; (2) the magnitude of the acceleration must be a function of the distance $OP=r$ only, say

EVA BRAMLET

$$j=f(r). \qquad (48)$$

The fixed point O is in this case usually regarded as the seat of an attractive or repulsive force producing the acceleration, and is therefore called the *centre of force*.

Harmonic motion as discussed in Arts. 172–214 is a special case of central motion, viz. the case in which the acceleration j is

women and study are contradictory

directly proportional to the distance from the fixed centre O, i.e. $f(r)=\mu r$.

Another very important particular case is that of *planetary motion* in which $f(r)=\mu/r^2$; this will be discussed below, Arts. 236, 239.

We proceed to establish the fundamental properties of central motion.

224. The motion is fully determined if in addition to the form of the function $f(r)$ we know the "initial conditions," say the initial distance $OP_0=r_0$ (Fig. 53) and the initial velocity v_0 of the point at the time $t=0$. As v_0 must be given both in magnitude and direction, the angle ψ_0 between r_0 and v_0 must be known.

225. It is evident, geometrically, that the motion is confined to the plane determined by O and v_0 since the acceleration

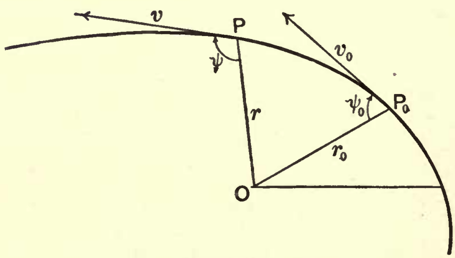

Fig. 53.

always lies in this plane. This fact *that the motion is plane* depends solely on the former of the two conditions of our problem (Art. 223); that is, any motion in which the acceleration passes constantly through a fixed point is plane.

226. With O as origin, let x, y be the rectangular Cartesian co-ordinates of the moving point P, and r, θ its polar co-ordinates, at any time t. Then $\cos\theta=x/r$, $\sin\theta=y/r$ are the direction cosines of $OP=r$, and, therefore, those of the acceleration j, provided the sense of j be away from the centre, *i.e.* provided the force causing the acceleration be *repulsive*. In the case of *attraction*, the direction cosines of j are of course $-x/r$, $-y/r$.

Thus the *equations of motion* are in the case of attraction,

$$j_x \equiv \frac{d^2x}{dt^2} = -f(r)\frac{x}{r}, \quad j_y \equiv \frac{d^2y}{dt^2} = -f(r)\frac{y}{r}. \tag{49}$$

For repulsion, it would only be necessary to change the sign of $f(r)$.

227. To perform a first integration, multiply the equations (49) by y, x and subtract when the left-hand member will be found an exact derivative, while the right-hand member vanishes. Hence, integrating and denoting the constant of integration by h, we find

$$x\frac{dy}{dt} - y\frac{dx}{dt} = h, \tag{50}$$

or, introducing polar co-ordinates,

$$r^2\frac{d\theta}{dt} = h. \tag{51}$$

These equations show that *the sectorial velocity is constant*, and $=\frac{1}{2}h$ for our problem (see Art. 135 and Art. 163, Ex. (4)).

228. Let S be the sector P_0OP described by the radius vector r in the time t, so that $dS = \frac{1}{2}r^2 d\theta$. Then (51) can be written in the form

$$\frac{dS}{dt} = \frac{1}{2}h, \tag{52}$$

whence integrating

$$S = \frac{1}{2}ht; \tag{53}$$

this expresses the fact that *the sector is proportional to the time in which it is described* which is of course only another way of stating the proposition of Art. 227.

The proof of the converse, viz. that if in a plane motion the areas swept out by the radius vector drawn from a fixed point be proportional to the time, the acceleration must constantly pass through that fixed point, is left to the student.

229. It is well known that Kepler had found by a careful examination of the observations available to him that *the orbits described by the planets are plane curves, and the sector described by the radius vector drawn from the sun to any planet is proportional to the time in which it is described.* This constitutes **Kepler's first law** of planetary motion.

He concluded from it that the acceleration must constantly pass through the sun.

230. To express the value of the constant of integration h in terms of the given initial conditions (Art. 224), *i.e.* by means of r_0, v_0, ψ_0, we notice that, at any time t,

$$h = r^2\frac{d\theta}{dt} = r\cdot\frac{rd\theta}{ds}\cdot\frac{ds}{dt} = r\sin\psi\cdot v; \qquad (54)$$

hence for the time $t=0$, we find

$$h = v_0 r_0 \sin\psi_0. \qquad (55)$$

Denoting the perpendiculars let fall from O on v_0 and v by p_0, p, we have $r_0\sin\psi_0 = p_0$, $r\sin\psi = p$; hence also

$$h = p_0 v_0 = pv, \qquad (56)$$

i.e. *the velocity at any time is inversely proportional to its distance from the centre.*

231. The equations of motions (49), if multiplied by dx/dt, dy/dt and added, give an equation in which both members are exact derivatives. On the left we find

$$\frac{d}{dt}\left[\frac{1}{2}\left(\frac{dx}{dt}\right)^2 + \frac{1}{2}\left(\frac{dy}{dt}\right)^2\right] = \frac{d}{dt}(\tfrac{1}{2}v^2);$$

on the right

$$-\frac{f(r)}{r}\left[x\frac{dx}{dt}+y\frac{dy}{dt}\right]=-\frac{f(r)}{2r}\frac{d}{dt}(x^2+y^2)=-\frac{f(r)}{2r}\frac{d(r^2)}{dt}=-f(r)\frac{dr}{dt}.$$

The equation

$$d(\tfrac{1}{2}v^2)=-f(r)\,dr \qquad (57)$$

can therefore be integrated and we obtain

$$\tfrac{1}{2}v^2-\tfrac{1}{2}v_0^2=-\int_{r_0}^{r}f(r)dr. \qquad (58)$$

232. The two methods of integrating the differential equations of motion used in Art. 227 and in Art. 231 are known, respectively, as the *principle of areas* and the *principle of energy* (or *vis viva*). The former name explains itself. The latter is due to the fact (to be more fully explained in kinetics) that if equation (58) be multiplied by the mass of the moving point, the left-hand member will represent the increase of the kinetic energy of the point during the motion.

Each of these methods of preparing the equations of motion for integration consists merely in combining the equations so as to obtain an exact derivative in the left-hand member of the resulting equation. If by this combination the right-hand member happens to vanish or to become likewise an exact derivative, an integration can at once be performed. This is the case in our problem.

233. The two equations (51) and (58) can be used to find the equation of the path. We have for *any* curvilinear motion (by (4), Art. 142)

$$v^2=\left(\frac{dr}{dt}\right)^2+r^2\left(\frac{d\theta}{dt}\right)^2=\left[\left(\frac{dr}{d\theta}\right)^2+r^2\right]\left(\frac{d\theta}{dt}\right)^2;$$

eliminating dt by means of (51) this becomes

$$v^2=\frac{h^2}{r^4}\left[\left(\frac{dr}{d\theta}\right)^2+r^2\right]=h^2\left[\left(\frac{du}{d\theta}\right)^2+u^2\right], \qquad (59)$$

where $u=1/r$. Substituting for v its expression in terms of r or u, from (58), we have the differential equation of the path which is directly integrable.

Shorter methods will often suggest themselves in particular cases.

234. To solve the converse problem, viz. to find the law of acceleration when the equation of the path is given, we have only to substitute in (57) the expression of v^2 from (59). We find, with $u=1/r$,

$$f(r)=-\frac{d(\frac{1}{2}v^2)}{dr}=-\frac{d(\frac{1}{2}v^2)}{du}\frac{du}{dr}=u^2\frac{d(\frac{1}{2}v^2)}{du}$$

$$=h^2u^2\left(\frac{d^2u}{d\theta^2}+u\right). \tag{60}$$

235. Kepler in his **second law** had established the empirical fact that *the orbits of the planets are ellipses, with the sun at one of the foci.*

From this, Newton concluded that the law of acceleration must be that of the inverse square of the distance from the sun.

Equation (60) allows us to draw this conclusion. The polar equation of an ellipse referred to focus and major axis is

$$r=\frac{l}{1+e\cos\theta},$$

where $l=b^2/a=a(1-e^2)$; a, b being the semi-axes, l the semi-latus rectum, and e the eccentricity of the ellipse. Hence

$$\frac{1}{r}\equiv u=\frac{1}{l}+\frac{e}{l}\cos\theta,\quad \frac{d^2u}{d\theta^2}=-\frac{e}{l}\cos\theta,$$

and (60) becomes

$$f(r)=\frac{h^2u^2}{l}=\frac{h^2}{a(1-e^2)}\cdot\frac{1}{r^2}. \tag{61}$$

236. The **third law of Kepler**, found by him likewise as an empirical fact, asserts that *the squares of the periodic times of different planets are as the cubes of the major axes of their orbits.*

From this fact Newton drew the conclusion that in the law of acceleration,

$$j \equiv f(r) = \frac{\mu}{r^2}, \tag{62}$$

the constant μ has the same value for all the planets.

Our formulæ show this as follows. Let T be the *periodic time* of any planet, *i.e.* the time of describing an ellipse whose semi-axes are a, b. Then, since the sector described in the time T is the area πab of the whole ellipse, we have by (53)

$$\pi ab = \tfrac{1}{2} hT.$$

Substituting in (61) the value of h found from this equation we have

$$f(r) = \frac{4\pi^2 a^2 b^2}{lT^2} \cdot \frac{1}{r^2} = \frac{4\pi^2 a^3}{T^2} \cdot \frac{1}{r^2}. \tag{63}$$

Hence
$$\mu = \frac{4\pi^2 a^3}{T^2} \tag{64}$$

is constant by Kepler's third law.

237. As mentioned in Art. 230, the velocity v can be expressed in terms of the perpendicular p let fall from the centre O on the tangent to the path:

$$v = \frac{h}{p} = \frac{v_0 p_0}{p}. \tag{65}$$

The acceleration j is also conveniently expressible in terms of p. We have by (57)

$$j \equiv f(r) = -\frac{d(\frac{1}{2}v^2)}{dr} = -\tfrac{1}{2} h^2 \frac{d}{dr}\left(\frac{1}{p^2}\right) = \frac{h^2}{p^3}\frac{dp}{dr}. \tag{66}$$

238. Finally, another expression for the acceleration is sometimes found convenient. In *any* motion, the component of the acceleration along the radius vector is (see Art. 161)

$$j = \frac{d^2 r}{dt^2} - r\left(\frac{d\theta}{dt}\right)^2.$$

As in our problem the total acceleration is along the radius vector, in the sense towards the origin, we have

$$j \equiv f(r) = -\frac{d^2r}{dt^2} + r\left(\frac{d\theta}{dt}\right)^2,$$

or since, by (51), $d\theta/dt = h/r^2$,

$$j \equiv f(r) = -\frac{d^2r}{dt^2} + \frac{h^2}{r^3}. \qquad (67)$$

The first term is what the acceleration would be if the motion were rectilinear along the radius vector; the second term represents what is due to the curvature of the path.

239. Planetary Motion, in its simplest form, is (see Art. 223) that particular case of central motion in which the acceleration is inversely proportional to the square of the distance from the centre O, so that

$$j \equiv f(r) = \frac{\mu}{r^2},$$

where μ is a constant, viz. the acceleration at the distance $r = 1$ from O.

The equations of motion (49) are in this case, with O as origin,

$$\frac{d^2x}{dt^2} = -\mu\frac{x}{r^3}, \quad \frac{d^2y}{dt^2} = -\mu\frac{y}{r^3}. \qquad (68)$$

Combining these by the principle of energy (Arts. 231, 232), we find

$$\frac{d(\frac{1}{2}v^2)}{dt} = -\frac{\mu}{r^3}\left(x\frac{dx}{dt} + y\frac{dy}{dt}\right) = -\frac{\mu}{r^3}\frac{d}{dt}\left(\frac{x^2+y^2}{2}\right)$$

$$= -\frac{\mu}{r^2}\frac{dr}{dt} = \mu\frac{d\left(\frac{1}{r}\right)}{dt};$$

hence integrating

$$\tfrac{1}{2}v^2 - \tfrac{1}{2}v_0^2 = \frac{\mu}{r} - \frac{\mu}{r_0}. \qquad (69)$$

240. To find the equation of the path, or *orbit*, we write the equations (68) in the form

$$\frac{d^2x}{dt^2} = -\frac{\mu}{r^2}\cos\theta, \quad \frac{d^2y}{dt^2} = -\frac{\mu}{r^2}\sin\theta,$$

and eliminate r^2 by means of (51):

$$\frac{d^2x}{dt^2} = -\frac{\mu}{h}\cos\theta\frac{d\theta}{dt}, \quad \frac{d^2y}{dt^2} = -\frac{\mu}{h}\sin\theta\frac{d\theta}{dt}.$$

These equations can be integrated separately:

$$\frac{dx}{dt} - v_1 = -\frac{\mu}{h}\sin\theta, \quad \frac{dy}{dt} - v_2 = \frac{\mu}{h}(\cos\theta - 1) \tag{70}$$

where v_1, v_2 are the components of the initial velocity.

Multiplying by y, x and subtracting, we find, owing to (50),

$$\left(\frac{\mu}{h} - v_2\right)x + v_1 y + h = \frac{\mu}{h}(x\cos\theta + y\sin\theta) = \frac{\mu}{h}\sqrt{x^2+y^2}. \tag{71}$$

241. The geometrical meaning of this equation is that the radius vector $r = \sqrt{x^2+y^2}$ drawn from the fixed point O to the moving point P is proportional to the distance of P from the fixed straight line

$$\left(\frac{\mu}{h} - v_2\right)x + v_1 y + h = 0. \tag{72}$$

It represents, therefore, a conic section having O for a focus and the line (72) for the corresponding directrix.

The character of the conic depends on the absolute value of the ratio of the radius vector to the distance from the directrix; according as this ratio

$$\frac{h}{\mu}\sqrt{\left(\frac{\mu}{h} - v_2\right)^2 + v_1{}^2} \lesseqgtr 1,$$

the conic will be an ellipse, a parabola, or a hyperbola. The criterion can be simplified. Multiplying by μ/h and squaring, we have

$$-\frac{2\mu v_2}{h} + v_2{}^2 + v_1{}^2 \lesseqgtr 0,$$

or since $v_1^2+v_2^2=v_0^2$ and $h=r_0v_0\sin\psi_0=r_0v_2$:

$$v_0^2 \lesseqgtr \frac{2\mu}{r_0}. \tag{73}$$

242. Introducing polar co-ordinates in (71), the equation of the orbit assumes the form

$$\frac{1}{r}=\frac{\mu}{h^2}+\left(\frac{v_2}{h}-\frac{\mu}{h^2}\right)\cos\theta-\frac{v_1}{h}\sin\theta,$$

or putting $(hv_2-\mu)/h^2=C\cos\alpha$, $v_1/h=C\sin\alpha$,

$$\frac{1}{r}=\frac{\mu}{h^2}+C\cos(\theta+\alpha). \tag{74}$$

This equation might have been obtained directly by integrating (60), which in our case, with $f(r)=\mu/r^2$, reduces to

$$\frac{d^2u}{d\theta^2}+u=\frac{\mu}{h^2};$$

the general integral of this differential equation is of the form (74), C and α being the constants of integration.

Equation (74) represents a conic section referred to the focus as origin and a line making an angle α with the focal axis as polar axis.

243. Exercises.

(1) If $2k$ be the chord intercepted by the osculating circle on the radius vector drawn from the fixed centre, show that $v^2=k\cdot f(r)$.

(2) A point moves in a circle; if the acceleration be constant in direction, what is its magnitude?

(3) A point moves in a circle; if the acceleration be constantly directed towards the centre, what is its magnitude?

(4) A point is subject to a central acceleration proportional to the distance from the centre and directed away from the centre; find the equation of the path.

(5) A point P is subject to two accelerations, $-\mu^2\cdot O_1P$ directed towards the fixed point O_1, and $-\mu^2\cdot O_2P$ directed away from the fixed point O_2. Show that its path is parabolic.

(6) A point P describes an ellipse owing to a central acceleration $f(r) = \mu/r^2$ directed toward the focus S. Its initial velocity v_0 makes an angle ψ_0 with the initial radius vector r_0. Determine the semi-axes a, b of the ellipse in magnitude and position.

244. The student will find numerous examples for further practice in the kinematics of a particle in the following works: P. G. TAIT and W. J. STEELE, *A treatise on dynamics of a particle*, 6th ed., London, Macmillan, 1889; W. H. BESANT, *A treatise on dynamics*, London, Bell, 1885; B. WILLIAMSON and F. A. TARLETON, *An elementary treatise on dynamics*, 2d ed., London, Longmans, 1889; W. WALTON, *Collection of problems in illustration of the principles of theoretical mechanics*, 3d ed., Cambridge, Deighton, 1876.

5. VELOCITIES IN THE RIGID BODY.

245. A rigid body is said to have plane motion when all its points move in parallel planes. Its motion is then fully determined by the motion of any plane section of the body in its plane.

It has been shown in Arts. 18–24 that the continuous motion of an invariable plane figure in its plane consists in a series of infinitesimal rotations about the successive instantaneous centres, *i.e.* about the points of the space centrode.

If at any instant the centre of rotation and the angular velocity ω about it be known, we can find *the velocity of any point of the plane figure.*

To show this let us first take the instantaneous centre as origin. Then the component velocities v_x, v_y of any point P whose co-ordinates are x, y, or r, θ, are found (Art. 141) by differentiating the expressions

$$x = r\cos\theta, \quad y = r\sin\theta$$

with respect to t. Considering that $d\theta/dt$ is the angular velocity ω about the instantaneous centre, we find

$$v_x = \frac{dx}{dt} = -r\sin\theta\frac{d\theta}{dt} = -\omega y,$$
$$v_y = \frac{dy}{dt} = \quad r\cos\theta\frac{d\theta}{dt} = \quad \omega x. \tag{1}$$

246. Next, taking an arbitrary origin O (Fig. 54), let x, y be the co-ordinates of P and x', y' those of any other point O' of the moving figure with respect to fixed rectangular axes through O; and let ξ, η be the co-ordinates of P with reference to rectangular axes through O' fixed in the figure but moving with it. Then, if θ be the angle between the axes Ox and $O'\xi$, we have

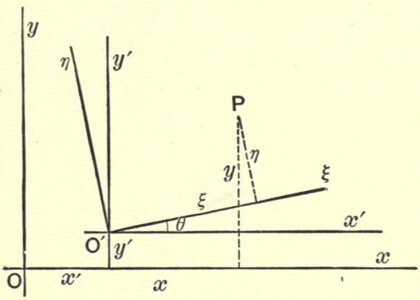

Fig. 54.

$$x = x' + \xi\cos\theta - \eta\sin\theta, \quad y = y' + \xi\sin\theta + \eta\cos\theta.$$

Differentiating we find for the component velocities of P parallel to the fixed axes Ox, Oy:

$$v_x = \frac{dx'}{dt} - (\xi\sin\theta + \eta\cos\theta)\frac{d\theta}{dt}, \quad v_y = \frac{dy'}{dt} + (\xi\cos\theta - \eta\sin\theta)\frac{d\theta}{dt}. \tag{2}$$

Now, $d\theta/dt$ is the angular velocity ω about the point O' while dx'/dt, dy'/dt are the velocities of O' parallel to the fixed axes, say $v_{x'}$, $v_{y'}$. Considering moreover that $\xi\sin\theta + \eta\cos\theta = y - y'$, $\xi\cos\theta - \eta\sin\theta = x - x'$, we have

$$v_x = v_{x'} - (y - y')\omega, \quad v_y = v_{y'} + (x - x')\omega. \tag{3}$$

The velocity of P *consists*, therefore, *of two parts, a velocity of translation equal to that of* O$'$ *and a velocity of rotation equal to that of* P *about* O$'$.

247. The instantaneous centre being the point whose velocity is zero at the given instant, we find its co-ordinates x_0, y_0 from the equations

$$0 = v_{x'} - (y_0 - y')\omega, \quad 0 = v_{y'} + (x_0 - x')\omega,$$

whence $$x_0 = x' - \frac{v_{y'}}{\omega}, \quad y_0 = y' + \frac{v_{x'}}{\omega}. \tag{4}$$

By eliminating t between these equations, the equation of the space centrode can be found.

The co-ordinates ξ_0, η_0 of the instantaneous centre referred to the moving axes are found in a similar way from the equations (2):

$$\xi_0 = \frac{1}{\omega}(v_{x'} \sin\theta - v_{y'} \cos\theta), \quad \eta_0 = \frac{1}{\omega}(v_{x'} \cos\theta + v_{y'} \sin\theta), \tag{5}$$

from which the body centrode can be found by eliminating t.

248. In Arts. 245 and 246 expressions were found for the component velocities v_x, v_y parallel to the fixed axes Ox, Oy. To find the component velocities v_ξ, v_η parallel to the moving axes $O\xi$, $O\eta$, let x, y be the co-ordinates of any point P with respect to the fixed axes (Fig. 55), ξ, η those with respect to the moving axes, and let θ be the angle $xO\xi$. The velocity of P parallel to the axes $O\xi$, $O\eta$ consists of two parts, that arising from the motion of P relative to $\xi O\eta$ whose components are of course $d\xi/dt$, $d\eta/dt$, and that due to the rotation of the moving axes. The components of the latter velocity are, by (1), Art. 245, $-\omega\eta$, $\omega\xi$. Hence

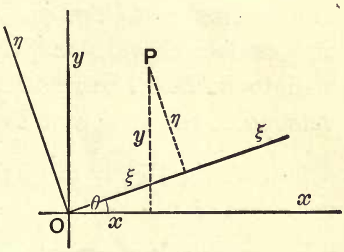

Fig. 55.

$$v_\xi = \frac{d\xi}{dt} - \omega\eta, \quad v_\eta = \frac{d\eta}{dt} + \omega\xi. \tag{6}$$

249. Exercises.

(1) Two points A, A' of a plane figure move on two fixed circles described with radii a, a' about O, O'; show that the angular velocities ω, ω' of OA, $O'A'$ about O, O' are inversely proportional to OM, $O'M$, M being the point of intersection of OO' with AA'.

(2) Given the magnitudes v, v' of the velocities of two points A, A' of an invariable plane figure and the angle (v, v') formed by their directions; find the instantaneous centre C and the angular velocity ω about C.

(3) Show that in the "elliptic motion" of a plane figure (Arts. 25–27) the velocity of any point (x', y') is

$$v = [a^2 + x'^2 + y'^2 - 2a(x' \cos 2\phi + y' \sin 2\phi)]^{\frac{1}{2}} \frac{d\phi}{dt}.$$

(4) In the same motion find the velocities of B and O' (Fig. 6, Art. 26) when A moves uniformly along the axis of x.

250. The continuous motion of a rigid body is called a **translation** when the velocities of all its points are equal and parallel at every moment (Art. 9). All points describe therefore equal and similar curves, and every line of the body remains parallel to itself. The velocity $v = ds/dt$ of any point is called the velocity of translation of the body.

251. A rigid body can be imagined to be subjected to several velocities of translation simultaneously; the resulting motion is a translation whose velocity is found by geometrically adding the component velocities.

Conversely, the velocity of translation of a rigid body can be resolved into components in given directions.

252. The continuous motion of a rigid body is called **rotation** when two points of the body are fixed; the line joining these points is the axis of rotation. All points excepting those on the axis describe arcs of circles whose centres lie on the axis.

The velocity of any point P of the body at the distance $OP = r$ from the axis is $v = \omega r = r\, d\theta/dt$, if $\omega = d\theta/dt$ is the angular velocity of the rigid body. The velocities of the different points of the body at any given moment are therefore directly proportional to their distances from the axis, and the velocities of all points at this moment are known if the instantaneous angular velocity ω is given. It is frequently convenient to imagine this angular velocity represented by its *rotor*, *i.e.* by a length ω laid off in the proper sense on the axis of rotation (see Arts. 68, 69).

253. The body may have several simultaneous rotations. Imagine, for instance, a top spinning about its axis placed on a table or disc which is made to rotate about an axis. The resulting motion can be found by compounding the rotors in the same way in which the rotors representing infinitesimal rotary displacements are compounded (Arts. 62, 66, 67); indeed, the rotor $\omega = d\theta/dt$ of an angular velocity is merely the rotor $d\theta$ divided by dt, and therefore identical with the rotor of the angular displacement $d\theta$.

254. As we are at present concerned with plane motion, we require only the rule for the composition of angular velocities about *parallel* axes.

Dividing the equations (1‴) and (2‴) of Art. 66 by dt, and putting $d\theta/dt = \omega$, $d\theta_1/dt = \omega_1$, $d\theta_2/dt = \omega_2$, we obtain:

$$\omega = \omega_1 + \omega_2, \quad \frac{L_1L}{\omega_2} = \frac{LL_2}{\omega_1} = \frac{L_1L_2}{\omega}. \qquad (7)$$

Thus, *the resultant of two angular velocities* ω_1, ω_2 *about parallel axes* l_1, l_2 *is an angular velocity* ω *equal to their algebraic sum*, $\omega = \omega_1 + \omega_2$, *about a parallel axis* l *that divides the distance between* l_1, l_2 *in the inverse ratio of* ω_1 *and* ω_2.

Conversely, an angular velocity ω about an axis l can always be replaced by two angular velocities ω_1, ω_2 whose sum is equal to ω and whose axes l_1, l_2 are parallel to l and so selected that l divides the distance between l_1, l_2 inversely as ω_1 is to ω_2.

255. It may be well to prove this important proposition independently. Any point P (Fig. 56) in a plane at right angles to the axes receives from ω_1 a linear velocity $\omega_1 r_1$ perpendicular to L_1P, and from ω_2 a linear velocity $\omega_2 r_2$ perpendicular to L_2P, if $L_1P=r_1$, $L_2P=r_2$. These linear velocities fall into the same straight line only for points situated on the line L_1L_2. A point L whose linear velocity is zero, must therefore lie on L_1L_2 so that $L_1L+LL_2=L_1L_2$; moreover, it must satisfy the condition $L_1L\cdot\omega_1=LL_2\cdot\omega_2$. This gives the above equations (7).

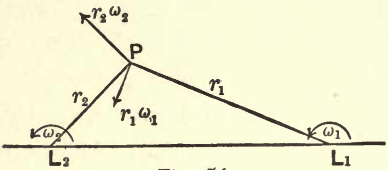

Fig. 56.

256. The resulting axis lies between L_1 and L_2 when the components ω_1, ω_2 have the same sense; when they are of opposite sense, it lies without, on the side of the greater one of these components.

If ω_1 and ω_2 are equal and opposite, say $\omega_1=\omega$, $\omega_2=-\omega$, the resulting axis lies at infinity (Art. 67). Two such equal and opposite angular velocities about parallel axes are said to form a **rotor-couple**; its effect on the rigid body is that of a velocity of translation $v=L_1L_2\cdot d\theta/dt=p\cdot\omega$ at right angles to the plane of the axes. The distance of the rotor, $L_1L_2=p$, is called the *arm* of the couple, and the product $p\omega=v$ its *moment.*

257. A velocity of translation v can therefore always be replaced by a rotor-couple $p\omega=v$, whose axes have the distance p and lie in a plane at right angles to v.

Again, an angular velocity ω about an axis l can be replaced by an equal angular velocity ω about a parallel axis l' at the distance p from l, in combination with a velocity of translation $v=\omega p$ at right angles to the plane determined by l and l'.

It easily follows from these propositions that *the resultant of any number of velocities of translation,* v, v′,..., *parallel to the same plane, and any number of angular velocities* ω, ω',..., *about* *axes perpendicular to this plane is always a single angular velocity about an axis perpendicular to the plane or a single velocity of translation parallel to the plane.*

6. APPLICATIONS.

258. Kinematics of Machinery. A large majority of the cases of motion that are of importance in mechanical engineering can be reduced to plane motion.

At first glance the application of theoretical kinematics to machines might seem to lead to rather complicated problems owing to the fact that a machine is never formed by a single rigid body, but always consists of an assemblage of several bodies some of which may even be not rigid (belting, springs, water, steam). The problem is, however, very much simplified by a characteristic of all machines, properly so called, that was first pointed out and insisted upon by recent writers on applied kinematics, in particular by Reuleaux.* This characteristic is the **constrainment** of the motions of the parts of a machine.

Thus Professor Kennedy defines a machine as "a combination of resistant bodies whose relative motions are completely constrained, and by means of which the natural energies at our disposal may be transformed into any special form of work."

With the latter clause of this definition we are not at present concerned; it will be considered in kinetics. To explain the former in detail would lead us too far into the domain of applied mechanics. A brief indication of the fundamental ideas must be sufficient.

259. By considering machines of various types it appears that the bodies, or *elements*, composing a machine always occur

* F. REULEAUX, *Theoretische Kinematik*, Berlin, 1875; translated into English and edited by ALEX. B. W. KENNEDY under the title *Kinematics of machinery*, London, Macmillan, 1876. Compare also R. WILLIS, *Principles of mechanism*, London, Longmans, 2d ed. 1870 (1st ed. 1841); F. GRASHOF, *Theoretische Maschinenlehre*, Vol. II., Leipzig, Voss, 1883; L. BURMESTER, *Lehrbuch der Kinematik*, Leipzig, Felix, 1888; ALEX. B. W. KENNEDY, *Mechanics of machinery*, London, Macmillan, 1886; J. H. COTTERILL, *Applied mechanics*, London, Macmillan, 1884.

in **pairs.** Thus a single rigid bar will form a lever only when taken in connection with a support, or fulcrum; a shaft to be used in a machine must rest in bearings; a screw must turn in a nut. To take a more complex illustration, consider the mechanism formed by the crank and connecting rod of a steam-engine (Fig. 57). It may be regarded as composed of three pairs, two so-called turning pairs at O and A, and a sliding pair at B; and these three pairs are connected by three rigid bars, called **links**, OA, AB, OB, the last of which is fixed.

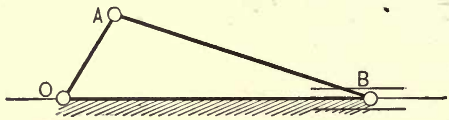

Fig. 57.

260. A **sliding pair** is formed by two bodies so connected that one is constrained to have a motion of translation relatively to the other. A pin moving in a groove or slot, a sleeve sliding along a shaft, are familiar examples.

A **turning pair** constrains one body to rotate about a fixed axis in another, as in the case of a shaft turning in its bearings.

A **twisting pair** makes one body have a screw motion about an axis fixed in the other.

These three pairs are the only so-called **lower pairs.** They are characterized as such by the fact that their elements have surface contact, and that, if either element be fixed, every point of the other is constrained to move in a definite line. In other words, the constraint effected by lower pairing is such as to leave but one degree of freedom (see Art. 37) to either element if the other be fixed.

261. All other pairs are called **higher pairs.** The contact in such pairs is usually line contact, and the two bodies have more than one degree of freedom relatively to each other, usually two degrees, so that if one element be fixed, any point of the other is constrained to a surface.

Higher pairs are of far less frequent occurrence in ordinary machines than lower pairs. The only very common example of higher pairing is found in toothed wheel gearing.

262. For the purposes of kinematics a machine may be regarded as consisting of a number of bodies (*links*) connected by pairs in such a way that when one of the links is fixed all other links are constrained in their motion. In most cases this constraint is such as to leave but one degree of freedom to every link.

A system of links of this kind forming, so to speak, a skeleton of the machine is called a **kinematic chain** (Reuleaux). When one link of such a chain is fixed, the chain becomes a **mechanism**. As a typical example we may take the "slider crank" in Fig. 57.

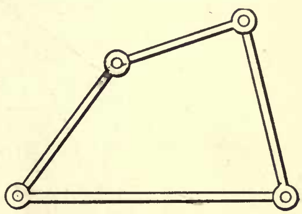

Fig. 58.

If the pairs are all turning pairs with parallel axes, the chain is called a **linkage** (Sylvester). A typical example is the four bar linkage in Fig. 58. A linkage with one link fixed has been called a **linkwork** (Sylvester). The four bar linkwork in Fig. 58 is also called a "lever crank" (Kennedy).

263. The Four Bar Linkage 1 2 3 4 (Fig. 59). Whatever may be its motion, each link considered separately moves as an invariable plane figure and has therefore at any moment an instantaneous centre C and an angular velocity ω about this centre.

The centre C_{12} of 1 2 and the centre C_{23} of 2 3 must always lie on a line passing through 2 since the velocity of 2 is perpendicular to both $C_{12}2$ and $C_{23}2$.

Similarly, 3 must lie on the line joining the centres C_{23} and C_{34}; and so on.

The quadrilateral 1 2 3 4 is therefore, and always remains, inscribed in the quadrangle $C_{12}C_{23}C_{34}C_{41}$. This can be shown to hold even for the *complete* quadrilateral and quadrangle. The complete quadrilateral, or four-side, 1 2 3 4 has six vertices, viz. the six intersections 1, 2, 3, 4, 5, 6 of its four sides; the complete quadrangle, or four-point, $C_{12}C_{23}C_{34}C_{41}$ has six sides, viz. the six lines $C_{41}C_{12}$, $C_{12}C_{23}$, $C_{23}C_{34}$, $C_{34}C_{41}$, $C_{12}C_{34}$, $C_{23}C_{41}$ joining its four vertices; and these six sides of the quadrangle pass through the six vertices of the quadrilateral, respectively.

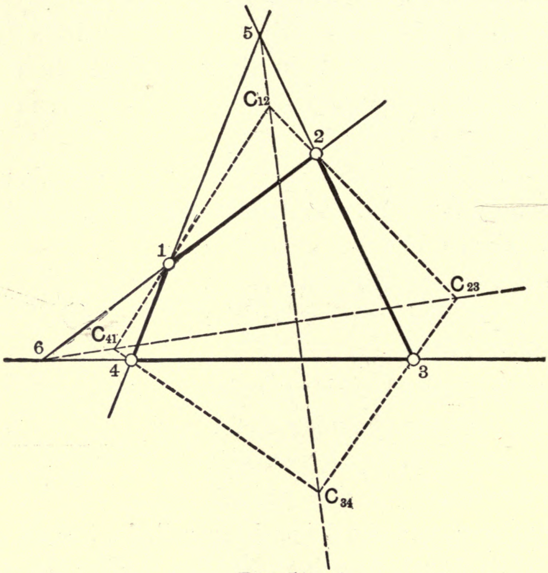

Fig. 59.

It remains to prove that $C_{12}C_{34}$ passes through 5 and that $C_{23}C_{41}$ passes through 6.

Now the velocity of 2 can be expressed by $\omega_1 \cdot C_{12}2$ and also by $\omega_2 \cdot C_{23}2$; hence $C_{12}2/C_{23}2 = \omega_2/\omega_1$; similarly $C_{23}3/C_{34}3 = \omega_3/\omega_2$. We have therefore, by the proposition of Menelaus,* for the intersection of 2 3 with $C_{12}C_{34}$:

$$\frac{5\,C_{12}}{5\,C_{34}} = \frac{\omega_3}{\omega_1}.$$

The same value is obtained by determining the intersection of 1 4 with $C_{12}C_{34}$; the two intersections must therefore coincide.

The proof for the point 6 is analogous.

---

* If the sides of a triangle ABC be cut by any transversal, in the points A', B', C', then $\frac{BA'}{A'C} \cdot \frac{CB'}{B'A} \cdot \frac{AC'}{C'B} = -1$. See J. Casey, *Sequel to Euclid*, London, Longmans, 1882, p. 69.

A corresponding proposition holds of course for four bar linkages with crossed bars 1 2 4 3, or 1 3 2 4.

264. Lever-crank. The linkage considered in the preceding article becomes a mechanism, or linkwork, as soon as one of its four links is fixed. It occurs in machines under a variety of forms some of which are referred to below.

Let the link 3 4 be fixed; then the centre C_{34} (Fig. 59) disappears; C_{41} falls into 4, C_{23} into 3, and C_{12} becomes the intersection 5 of 4 1 and 3 2. If 1 2 were fixed instead of 3 4, 3 4 would have its centre at 5.

Similarly, if either 4 1 or 2 3 be fixed, the centre of the other is 6.

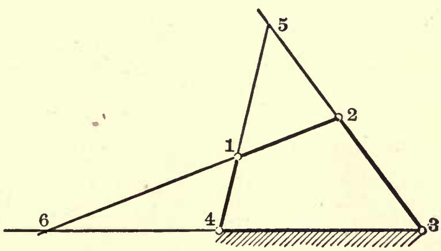

Fig. 60.

Hence whichever of the four links be fixed, the centres of all the links lie at some of the six vertices of the complete quadrilateral 1 2 3 4.

If 3 4 be the fixed link (Fig. 60), the ratio of the angular velocities ω_1 of 4 1 and ω_2 of 3 2 can be found. For if ω denote the angular velocity of 1 2 about 5, we have

$$41 \cdot \omega_1 = 51 \cdot \omega, \quad 32 \cdot \omega_2 = 52 \cdot \omega;$$

hence
$$\frac{\omega_2}{\omega_1} = \frac{41}{32} \cdot \frac{52}{51} = \frac{52}{32} \Big/ \frac{51}{41};$$

or, by the proposition of Menelaus:

$$\frac{\omega_2}{\omega_1} = \frac{46}{36}.$$

265. Parallelogram: 4 1 = 3 2 = a, 4 3 = 1 2 = b (Fig. 61). The link 1 2 has evidently a motion of translation, its instantaneous centre lying at the intersection of the parallel lines 4 1, 3 2.

The space centrode is the line at infinity; the body centrode may be regarded as a circle of infinite radius described about the midpoint of 3 4 as centre.

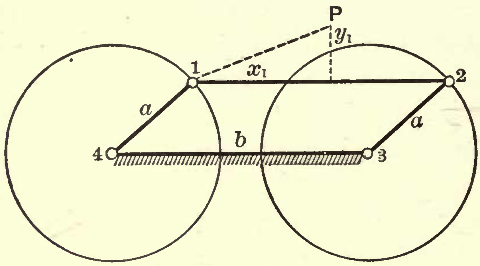

Fig. 61.

To find the equation of the path of any point P rigidly connected with 1 2, let x, y be the rectangular co-ordinates, with respect to 4 as origin and 4 3 as axis of x, and x_1, y_1 its co-ordinates for parallel axes through 1; then, putting $\measuredangle$ 3 4 1 = θ, we have

$$x = a\cos\theta + x_1, \quad y = a\sin\theta + y_1;$$

hence, eliminating θ,

$$(x - x_1)^2 + (y - y_1)^2 = a^2,$$

which represents a circle of radius a whose centre has the fixed co-ordinates x_1, y_1.

For the velocity of P we have $dx/dt = -a\omega\sin\theta$, $dy/dt = a\omega\cos\theta$; hence $v = a\omega$, as is otherwise apparent.

266. If in the parallelogram 1 2 3 4 the point 4 alone be fixed, we have a linkage called the **pantograph.**

It can serve to trace a curve similar to a given curve. Indeed, any line through 4 (Fig. 62) cuts the opposite links

1 2, 2 3 (produced if necessary) in points A, A' whose paths are homothetic (similar and similarly situated) curves. For the points 4, A, A' remain always in line and the ratio $4A/4A'$ remains constant. Hence if a pencil be attached to A' and A be made to trace a given curve, A' will trace a similar curve.

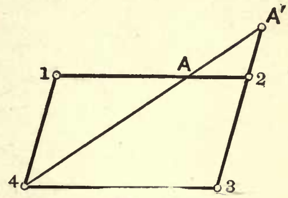

Fig. 62.

Instead of fixing 4, the point A' might be fixed; then 4 and A will describe similar curves. This property is utilised in Watt's parallel motion (see Art. 271).

The parallelogram linkage furnishes also a simple instrument for describing ellipses. Let the sides of the parallelogram be $23 = 41 = a$, $12 = 34 = b$; and let a point A' on 2 3 produced, at the distance b from 2, be fixed (Fig. 63). Then, if 1 be made to describe a straight line, passing through A', 4 will describe an ellipse. For, taking A' as origin and $A'1$ as axis of x, we

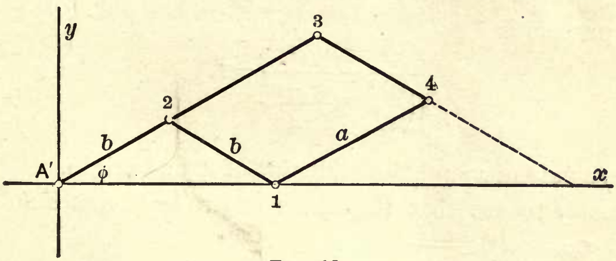

Fig. 63.

have for the co-ordinates of 4: $x = (a + 2b)\cos\phi$, $y = a\sin\phi$, whence

$$\frac{x^2}{(a+2b)^2} + \frac{y^2}{a^2} = 1.$$

267. In the parallelogram 1 2 3 4, let the link 1 2 be turned so as to coincide in direction with 4 3, and then give the links 4 1 and 3 2 rotations of opposite sense. We thus obtain a linkage with equal, but intersecting, opposite sides, which we may call **anti-parallelogram** (Fig. 64). If 3 4 be fixed, the instantaneous centre of 1 2 is the intersection 5 of 4 1 and 2 3.

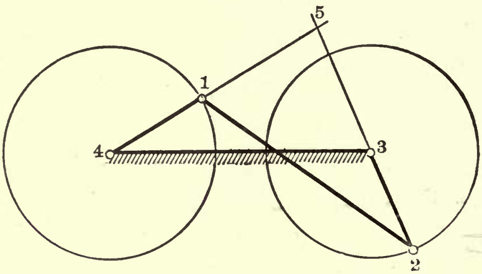

Fig. 64.

To obtain the centrodes in this case, notice that as the triangles 1 5 2 and 5 3 4 are equal, the triangle 5 4 2 is isosceles; hence 5 1 = 5 3, and 4 5 − 3 5 = 4 1 = a. The difference of the radii vectores of 5 drawn from 4 and 3 being thus constant, it follows that the space centrode is a hyperbola whose foci are 4, 3, and whose real axis = a. As 4 3 = 1 2 = b, the equation of this hyperbola is

$$\frac{x^2}{\left(\frac{a}{2}\right)^2} - \frac{y^2}{\frac{b^2-a^2}{4}} = 1,$$

for 4 3 as axis of x and the midpoint of 4 3 as origin.

It is easy to see that the space centrode becomes an ellipse when $b < a$.

As the triangles 1 5 2 and 3 5 4 are equal the body centrode is an equal hyperbola or ellipse. The two centrodes lie symmetrically with respect to their common tangent at 5.

For a given anti-parallelogram the centrodes are hyperbolas when one of the larger links is fixed; they are ellipses when one of the shorter links is fixed.

268. If in the anti-parallelogram only one point, say 4, be fixed, it can be used as an **inversor**, *i.e.* as an instrument for describing the inverse of a given curve.

Let $r = OP$ be the radius vector drawn from an arbitrary fixed origin, or pole, O to a given curve; on OP lay off a length $OP' \equiv r' = \kappa^2/r$, where κ is a constant; then P' is said to describe the *inverse* of the given curve.

The theory of inversors is based on the following geometrical proposition: if three lines $CA = a$, $CA' = a$, $CO = b$ (Fig. 65) turn about C so that O, A, A' are always in line, the product $OA \cdot OA'$ remains constant, viz. $OA \cdot OA' = b^2 - a^2$. For if the circle of radius a described about C intersect the line OC in B and B' we have $OA \cdot OA' = OB \cdot OB' = (b-a)(b+a)$.

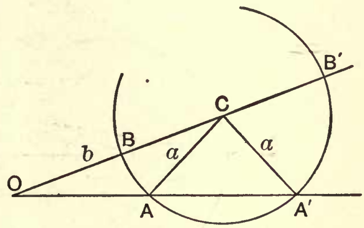

Fig. 65.

This proposition shows that in the anti-parallelogram 1 2 3 4 (Fig. 66), with the vertex 4 fixed, the line joining the vertices 4 and 2 intersects the circle described about 3 with radius 3 2 in a point 2′ such that 2 and 2′ describe inverse curves with respect to 4 as pole. For we have $4\,2' \cdot 4\,2 = 4\,3^2 - 2\,3^2 = b^2 - a^2$.

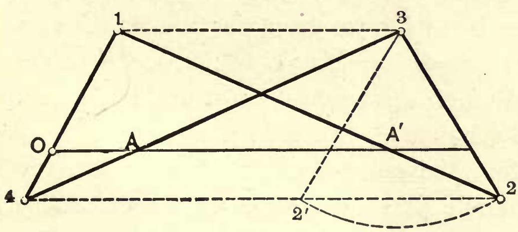

Fig. 66.

Moreover, any parallel to 4 2 will intersect the links 4 1, 4 3, 2 1 in points O, A, A' dividing the three lines in the same ratio; hence if O be fixed, A and A' will describe inverse curves for O as pole. This is the principle on which **Hart's inversor** is based.

269. Peaucellier's cell is another inversor (Fig. 67). It consists of the linked rhombus $ABA'B'$ whose side we denote by a, and the two equal links OB, OB' of length b. If O be fixed, A and A' evidently describe inverse curves for O as pole.

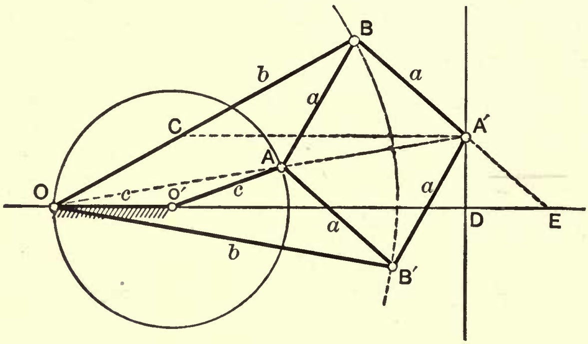

Fig. 67.

The practical application of inversors is based on the property that they enable us to transform circular motion into rectilinear motion (see Art. 271).

The inverse of a circle $r=2c\cos\theta$ passing through the pole is a straight line; for we have for the radius vector r' of the inverse curve $r'=\kappa^2/r=\kappa^2/2c\cos\theta$; hence $r'\cos\theta=\kappa^2/2c$ which is the equation of a straight line at right angles to the polar axis, at the distance $\kappa^2/2c$ from the pole.

If therefore the point A of an inversor be made to describe an arc of a circle passing through O, the point A' will describe a segment of a straight line. The vertex A (Fig. 67) can be compelled to describe a circle by inserting the additional link $O'A$ turning about the fixed point O'. If O' be selected so as to make $O'O=O'A$, say $=c$, the circle described by A will pass through O; and the motion of A' will be confined to the straight line $A'D$ perpendicular to OO', at the distance $OD=(b^2-a^2)/2c$ from O.

The linkage has thus become a linkwork, OO' being the fixed link.

270. To determine the linear velocity v of A' along DA' when the angular velocity ω of the link OB is given, we notice that the instantaneous centre C of the link BA' lies at the intersection of OB with the line drawn through A' parallel to OO'. Let ω' be the angular velocity of BA' about C. Then $v=\omega'\cdot CA'$; also since the point B describes a circle about O, $\omega b=\omega'\cdot CB$; hence

$$v=\omega\cdot\frac{CA'}{CB}b.$$

If BA' intersect OO' in E, we have from similar triangles $CA':CB=OE:OB$; hence

$$v=\omega\cdot OE.$$

The variable length OE depends on the angles $EOB=\theta$ and $BEO=\phi$ which are connected by the relation (Art. 269)

$$a\cos\phi+b\cos\theta=OD=\frac{b^2-a^2}{2c}.$$

The figure gives $OE=b\cos\theta+b\sin\theta\cot\phi$; hence, finally,

$$v=\omega b\sin\theta(\cot\theta+\cot\phi).$$

271. In the steam engine and other machines mechanisms are required for transforming the alternating rectilinear motion of the piston into the reciprocating circular motion of a crank, eccentric, or beam; a mechanism of this kind is called, rather inappropriately, a **parallel motion.** The problem of effecting this transformation has been solved in various ways. Peaucellier's inversor (1864) was the first *accurate* solution. Generally, an approximate solution is sufficient for practical purposes. The most common of such approximations is **Watt's parallel motion.** This mechanism is a combination of a linked parallelogram with a four bar linkwork with crossed links.

To fix the ideas, let 4 1 (Fig. 68) be the horizontal middle position of the beam of a beam engine; 4 is fixed and 1 describes an arc of a circle of radius $4\,1 = a$. We might place a counter-beam 3 2 of equal length turning about the fixed end 3 so as to be in its middle position parallel to 4 1 and so as to make the connecting link 1 2 nearly vertical. The middle point of 1 2 would then describe a looped curve whose central portion does

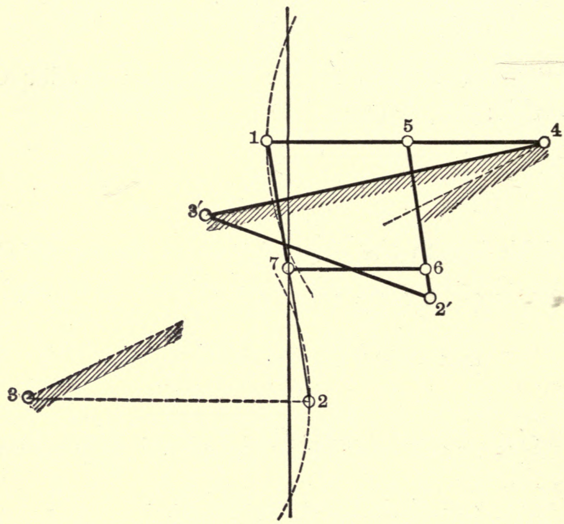

Fig. 68.

not differ very much from a straight line; connecting this middle point with the piston rod, the problem would be solved.

But the introduction of the large counter-beam 3 2 in the position indicated above would be very inconvenient. To reduce the size of the mechanism the counter-beam 3 2 is placed nearer to 4 1, into the position 3′ 2′, and the parallelogram 1 5 6 7 is introduced, the piston rod being attached at 7. Owing to the property of the linked parallelogram (Art. 266), the point 7 has a motion similar to that of the point of intersection of 4 7 with 5 6; it describes therefore approximately a straight line. The point of intersection of 47 with 56 can be used to connect with the pump rods of the engine.

7. ACCELERATIONS IN THE RIGID BODY.

272. To find the accelerations of the various points of a rigid body we must compare the velocities of these points during two consecutive elements of time; the change of the velocity divided by dt gives the acceleration.

In the case of **translation** (Art. 250) the accelerations of all points of the body are evidently equal so that the acceleration of any point may be called the acceleration of the body.

273. In the case of **rotation about a fixed axis** l, any point P of the body at the distance r from the axis describes during the element of time dt a space element $ds = rd\theta = \omega r dt$ proportional to this distance r, where $\omega = d\theta/dt$ is the angular velocity of the body about the axis l. The linear velocity of P is $v = \omega r$. The space element ds' described during the next element of time is an infinitesimal arc of the same circle of radius r, *i.e.*

$$ds' = rd\theta' = r(\omega + d\omega)\,dt.$$

Drawing from any point O (Fig. 69) the vectors $OV = ds/dt$, $OV' = ds'/dt$, and resolving the elementary acceleration VV' parallel to the tangent and normal of the path into $TV' = dv = rd\omega$ and $VT = vd\theta = r\omega d\theta = r\omega^2 dt$, we find the tangential and normal components of the acceleration of P by dividing these elements by dt. Hence denoting the angular acceleration $d\omega/dt$ by α, we have

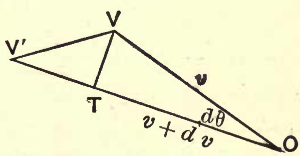

Fig. 69.

$$j_t = \alpha r, \quad j_n = \omega^2 r. \tag{1}$$

The total acceleration of P,

$$j = \sqrt{j_t^2 + j_n^2} = r\sqrt{\alpha^2 + \omega^4}, \tag{2}$$

is therefore proportional to the distance r of this point from the axis, so that the accelerations of all points can be found as soon as that of any one point is known.

274. We proceed next to the investigation of the accelerations of the various points of a rigid body having **plane motion.** The motion is determined by that of a plane section of the body parallel to the plane of motion, and this consists in the rolling of the body centrode over the space centrode (Art. 22).

During any element of time dt, every point P of the plane section rotates with angular velocity ω about the instantaneous centre of rotation C which is the point of contact of the two centrodes. During the next element of time dt, the angular velocity is $\omega + d\omega$, and the centre of rotation has changed to the infinitely near point C_1 on the space centrode, which has now become the point of contact of the two centrodes. The acceleration of a point P at the distance r from C evidently depends on this distance r, the angular velocity ω, the angular acceleration $\alpha = d\omega/dt$, and the element $CC_1 = d\sigma$ of the space centrode. This element divided by dt may be regarded as a velocity, $u = d\sigma/dt$, viz. the velocity with which the instantaneous centre changes its position. We may call it the *velocity of rolling* of the body centrode. The change in the state of motion during two consecutive elements of time depends on α and u.

275. The relation of the velocity of rolling u to the angular velocity ω depends on the relative curvature of the centrodes c, c'.

To fix the ideas imagine these curves to lie on the same side of their common tangent; let $d\alpha$, $d\alpha'$ be their angles of contingence, and let ρ, ρ' be their radii of curvature (Fig. 70).

The rotation about C brings the second element of c' to coincidence with the second element of c. The angle $d\theta$ of this rotation is therefore equal to the difference of the angles of contingence of the two curves, *i.e.*

$$d\theta = d\alpha' - d\alpha.$$

This angle is therefore called the *angle of relative contingence;* the quotient $d\theta/d\sigma = (d\alpha' - d\alpha)/d\sigma$, where $d\sigma = CC_1$, is called the *relative curvature,* and the reciprocal value, $d\sigma/d\theta$, is the *radius of relative curvature.*

Now $\omega = d\theta/dt$, $u = d\sigma/dt$; hence

$$\frac{\omega}{u} = \frac{d\theta}{d\sigma} = \frac{d\alpha' - d\alpha}{d\sigma},$$

or as $d\alpha/d\sigma = 1/\rho$, $d\alpha'/d\sigma = 1/\rho'$,

$$\frac{\omega}{u} = \frac{d\theta}{d\sigma} = \frac{1}{\rho'} - \frac{1}{\rho}; \tag{3}$$

i.e. *the ratio of the angular velocity to the velocity of rolling is equal to the relative curvature of the centrodes.*

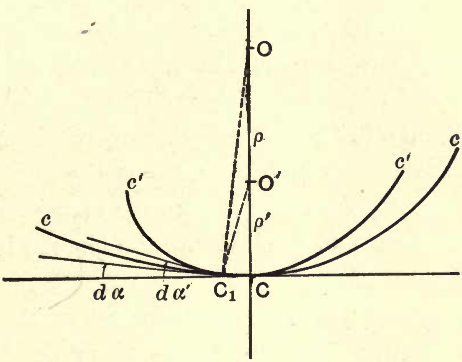

Fig. 70.

When $\rho < \rho'$, that is when $d\alpha > d\alpha'$, the relative curvature is negative. When the centrodes lie on opposite sides of the common tangent we should find in absolute value $d\theta = d\alpha' + d\alpha$. But taking into account the sense of the angles $d\alpha$, $d\alpha'$ we still have $d\theta = d\alpha' - d\alpha$. The formula (3) holds, therefore, generally if the radius of curvature ρ of c be taken as positive or negative according as it lies on the same side of the common tangent with the radius of curvature ρ' of c', or on the opposite side.

276. To determine the *components of the acceleration of any point* P *of the body*, it will be convenient to imagine the angular velocities represented by their rotors: the velocity ω about C by a line of length ω erected at C at right angles to the plane of motion, on that side of this plane from which the rotation appears counter-clockwise; similarly the angular velocity $\omega + d\omega$ by a parallel line of length $\omega + d\omega$ erected at C_1.

The rotor $\omega + d\omega$ through C_1 can be replaced by a parallel rotor of the same magnitude and sense through C, in combination with a rotor-couple whose moment is $(\omega + d\omega) \cdot CC_1 = \omega d\sigma$ (see Arts. 255, 256). This couple being equivalent to a vector $\omega d\sigma$ at right angles to the plane of the couple produces an infinitesimal velocity of translation.

Thus the body, during the first element of time dt, rotates about the axis through C with angular velocity ω; and during the second element of time dt, it can be regarded as having the angular velocity $\omega + d\omega$ about the same axis, and at the same time a velocity of translation $\omega d\sigma$ at right angles to the tangent at C. The change in the state of motion consists, therefore, in the angular acceleration $d\omega/dt = \alpha$ and in the linear acceleration $\omega d\sigma/dt = \omega u$, the former being due to the change in the magnitude of the acceleration, the latter to the change in the position of the axis of rotation.

While the acceleration of translation ωu is the same for all points of the figure, the angular acceleration α produces in every point P (Fig. 71) a linear acceleration proportional to its distance $r = CP$ from the centre C, just as in the case of rotation about a fixed axis (Art. 273). Resolving this acceleration into its tangential and normal components we have for the acceleration of P the following three components: αr at right angles to CP, $\omega^2 r$ along PC, and ωu at right angles to CC_1.

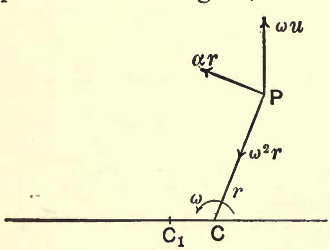

Fig. 71.

277. Another important method for finding the components of the acceleration of any point P of the body consists in resolving (according to Art. 254) the rotor $\omega+d\omega$ through C_1 into two parallel rotors, ω through C, and $d\omega$ through a point H (Fig. 72) on the tangent CC_1 whose distance $CH=h$ from C is given by the relations

$$\frac{CC_1}{d\omega}=\frac{C_1H}{\omega}=\frac{CH}{\omega+d\omega}.$$

Putting again $CC_1=d\sigma$, $d\sigma/dt=u$, $d\omega/dt=\alpha$, we find for the distance $CH=h$:

$$h=\frac{\omega u}{\alpha}. \qquad (4)$$

The body can therefore be regarded as having, during the second element of time dt, the same angular velocity ω about the same axis through C as during the first element of time, but in addition an angular velocity $d\omega$ about a parallel axis through H. As the magnitude of the angular velocity about C does not change, the rotation about C produces at any point P (Fig. 72) only a normal acceleration $\omega^2 r$ towards C, but no

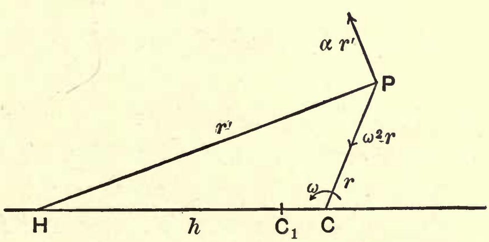

Fig. 72.

tangential acceleration. The infinitesimal angular velocity $d\omega$ about H, on the other hand, produces only a tangential acceleration $\alpha r'$, perpendicular to $HP=r'$.

The acceleration of any point P can therefore be resolved into two components, one $\omega^2 r$ directed towards the centre of rotation C and proportional to the distance r from this centre, the other $\alpha r'$ perpendicular and proportional to the distance r' of P from a point H on the tangent at C, such that $CH = \omega u/\alpha$.

The point H may be called the **centre of angular acceleration.**

278. The resolution of the acceleration given in the last article enables us to show the existence, at any time t, of a point having at this instant no acceleration. This point is called the instantaneous **centre of acceleration**; we shall denote it by the letter I, and its distances from C and H, respectively, by r_0 and r_0'.

For a point of acceleration zero the components $\alpha r'$ and $\omega^2 r$ must be equal and opposite. Now it is evident that these

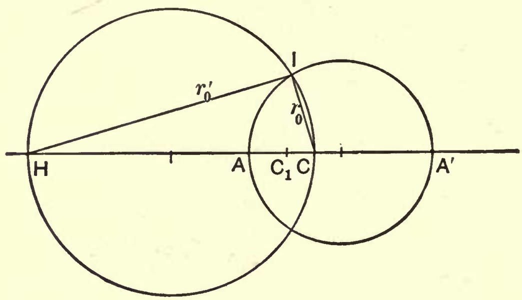

Fig. 73.

components fall into the same straight line only for points whose radii vectores r, r' are at right angles. The centre I must therefore lie on the circle described over CH as diameter (Fig. 73). In addition to this the radii vectores of I must fulfil the condition

$$\omega^2 r_0 = \alpha r_0'. \tag{5}$$

The locus of all points for which at any instant the ratio r/r' is constant and equal to α/ω^2 is a circle whose centre lies on CH and whose intersections A, A' with CH divide this distance internally and externally in the ratio α/ω^2.

The two circles intersect in two points; but only for one of these have the components $\omega^2 r$ and $\alpha r'$ opposite sense. There exists therefore only one centre of acceleration I, and its radii vectores satisfy the conditions

$$\omega^2 r_0 = \alpha r_0', \quad r_0^2 + r_0'^2 = h^2 = \frac{\omega^2 u^2}{\alpha^2}. \tag{6}$$

279. The appropriateness of the name centre of acceleration for the point I appears in particular when the acceleration of *any* point P is referred to this point I. For it can be shown that, if p be the distance of P from I, *the acceleration of* P *can be resolved into two components, one* ω^2p *along* PI, *the other* αp *at right angles to* IP (Fig. 74), similarly as in the case of rotation about a fixed axis (see Art. 273).

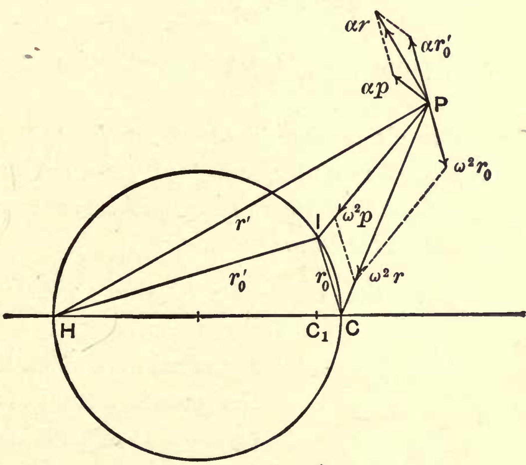

Fig. 74.

To prove this we resolve the component $\omega^2 r$ of the acceleration of P along PI and parallel to IC; it appears from the figure that these components are $\omega^2 p$ and $\omega^2 r_0$. The other component $\alpha r'$ of the acceleration of P is due to the infinitesimal angular velocity $d\omega$ about H. Replacing this $d\omega$ about H by an equal angular velocity $d\omega$ about I in combination with the infinitesimal velocity of translation $r_0' d\omega$ at right angles to

HI, we obtain, in the place of $\alpha r'$, the components αp at right angles to IP and $\alpha r_0'$ perpendicular to r_0'.

As of the four components $\omega^2 p$, αp, $\omega^2 r_0$, $\alpha r_0'$ the last two are, by (6), equal and opposite, it follows that the acceleration of P has only the two components, $\omega^2 p$ along PI, and αp perpendicular to IP.

280. The total acceleration of any point P is therefore proportional to the distance p of this point from the centre of acceleration I, viz.

$$j = p\sqrt{\alpha^2 + \omega^4}; \tag{7}$$

and the angle ψ it makes with this distance IP, being given by the relation

$$\tan\psi = \frac{\alpha}{\omega^2}, \tag{8}$$

is the same for all points. By (5), this angle ψ is equal to the angle CHI.

All points on a circle described about I as centre have accelerations of equal magnitude but of different directions. All points on a straight line drawn through I have accelerations that are parallel but differ in magnitude.

281. Returning to the resolution of the acceleration into three components $\omega^2 r$, αr, ωu, as given in Art. 276, let us take the common tangent of the centrodes as axis of x, their normal as axis of y (Fig. 75), and let x, y be the co-ordinates of any point P whose distance from C is $CP = r$.

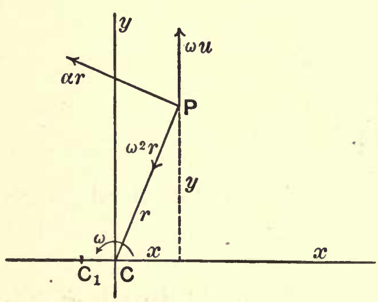

Fig. 75.

As the direction cosines of $\omega^2 r$, αr, ωu are respectively $-x/r$, $-y/r$; $-y/r$, x/r; 0, 1, we have for the components of the acceleration j parallel to the axes:

$$j_x = -\omega^2 x - \alpha y,$$
$$j_y = -\omega^2 y + \alpha x + \omega u. \tag{9}$$

The co-ordinates x_0, y_0 of the centre of acceleration I must fulfil the conditions

$$\omega^2 x_0 + \alpha y_0 = 0, \quad \alpha x_0 - \omega^2 y_0 + \omega u = 0, \tag{10}$$

whence
$$x_0 = \frac{\omega\alpha u}{\alpha^2+\omega^4}, \quad y_0 = -\frac{\omega^3 u}{\alpha^2+\omega^4}. \tag{11}$$

The equations (10) evidently represent the two lines CI and HI.

282. Let $\xi = x - x_0$, $\eta = y - y_0$ be the co-ordinates of P with respect to parallel axes through I; then, combining (10) and (9), we find

$$j_x = -\omega^2\xi - \alpha\eta, \; j_y = -\omega^2\eta + \alpha\xi. \tag{12}$$

These expressions show that the total acceleration j of P is

$$j = \sqrt{\alpha^2(\xi^2+\eta^2) + \omega^4(\xi^2+\eta^2)} = p\sqrt{\alpha^2+\omega^4},$$

since $\sqrt{\xi^2+\eta^2} = p = IP$, as in Art. 280.

283. The tangential and normal components of the acceleration j are readily obtained from Fig. 74, as follows:

$$j_t = \alpha r + \omega u \cdot \frac{x}{r}, \quad j_n = \omega^2 r - \omega u \cdot \frac{y}{r}. \tag{13}$$

The loci of the points having only normal, and only tangential, acceleration at any moment are therefore the circles:

$$\alpha(x^2+y^2) + \omega u x = 0, \quad \omega(x^2+y^2) - u y = 0. \tag{14}$$

284. Exercises.

(1) A wheel of radius a rolls on a straight track. Find the centre of angular acceleration H, (a) when the velocity v with which the axis of the wheel moves along the track is constant; (b) when v is uniformly accelerated as when the wheel rolls down an inclined plane; (c) when v is uniformly retarded, as in rolling up an inclined plane.

(2) Show that ωu is the total acceleration of the instantaneous centre C.

(3) Show that the points of the semi-circle described over CH as diameter and containing I have no tangential acceleration, and that for points without the circle about CH the velocity is increasing while for points within it is decreasing.

(4) Find the locus of the points of equal tangential acceleration.

(5) Show that the locus of the points having no normal acceleration at a given instant is a circle touching the common tangent of the centrodes at C and passing through I. This circle is called the *circle of inflexions;* give the reason for this name.

(6) Find the locus of the points having equal normal acceleration.

(7) Show that the diameter of the circle of inflexions is equal to the radius of relative curvature of the centrodes.

(8) Determine the locus of the points whose acceleration at any instant is parallel (a) to the common normal, (b) to the common tangent of the centrodes.

IV. *Solid Kinematics.*

I. MOTION OF A POINT IN A TWISTED CURVE.

285. We have so far considered only those cases of motion where the path of the point is a plane curve. In the most general case when the path is a so-called twisted or tortuous curve we may refer it to three rectangular axes and resolve the velocity v as well as the acceleration j each into three rectangular components parallel to these axes:

$$v_x = v \cos\alpha = \frac{dx}{dt}, \qquad j_x = j \cos\lambda = \frac{dv_x}{dt} = \frac{d^2x}{dt^2},$$

$$v_y = v \cos\beta = \frac{dy}{dt}, \qquad j_y = j \cos\mu = \frac{dv_y}{dt} = \frac{d^2y}{dt^2},$$

$$v_z = v \cos\gamma = \frac{dz}{dt}, \qquad j_z = j \cos\nu = \frac{dv_z}{dt} = \frac{d^2z}{dt^2},$$

$$v = \sqrt{v_x^2 + v_y^2 + v_z^2} = \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2};$$

$$j = \sqrt{j_x^2 + j_y^2 + j_z^2} = \sqrt{\left(\frac{d^2x}{dt^2}\right)^2 + \left(\frac{d^2y}{dt^2}\right)^2 + \left(\frac{d^2z}{dt^2}\right)^2}.$$

286. As polar co-ordinates of the point P we take the radius vector $OP = r$, the colatitude $xOP = \theta$, and the longitude $yOQ = \phi$, Q being the projection of P on the plane yOz (Fig. 76).

The velocity v can be resolved into three rectangular components: v_r along r, v_θ at right angles to r in the plane xOP of the angle θ, and v_ϕ at right angles to this plane. To find their values we take the element $PP' = ds$ of the curve described by the point P as diagonal of an infinitesimal parallelepiped having its edges in those three rectangular directions. The three edges concurring in P are evidently dr, $rd\theta$, $r\sin\theta\, d\phi$; hence the components of the velocity are

$$v_r = \frac{dr}{dt}, \quad v_\theta = r\frac{d\theta}{dt}, \quad v_\phi = r\sin\theta\frac{d\phi}{dt}.$$

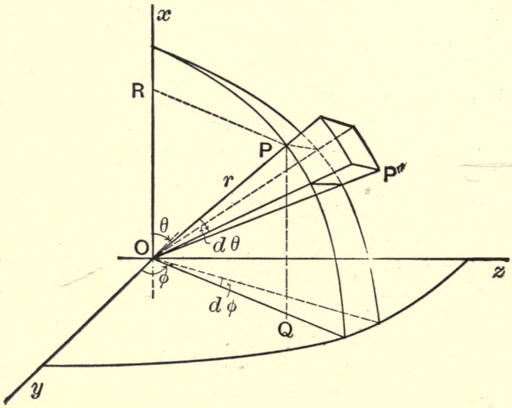

Fig. 76.

287. The components of the acceleration j in polar co-ordinates are readily obtained by considering that the accelerations of the point P in the direction at right angles to Ox in the plane xOP and in the direction at right angles to this plane are the same as the accelerations of the point Q (Fig. 76); they are therefore, by Art. 161, (6), since $RP = OQ = r\sin\theta$,

$$\frac{d^2(r\sin\theta)}{dt^2} - r\sin\theta\left(\frac{d\phi}{dt}\right)^2$$

in the direction RP, and

$$\frac{1}{r\sin\theta}\frac{d}{dt}\left(r^2\sin^2\theta\frac{d\phi}{dt}\right)$$

at right angles to the plane of the angle θ. The component of j parallel to the axis Oz is, of course, $d^2(r\cos\theta)/dt^2$. Resolving these three components parallel to the three rectangular directions along r, at right angles to r in the plane xOP, and at right angles to this plane, and collecting the terms, we obtain :

$$j_r = \frac{d^2r}{dt^2} - r\left(\frac{d\theta}{dt}\right)^2 - r\sin^2\theta\left(\frac{d\phi}{dt}\right)^2,$$

$$j_\theta = r\frac{d^2\theta}{dt^2} + 2\frac{dr}{dt}\frac{d\theta}{dt} - r\sin\theta\cos\theta\left(\frac{d\phi}{dt}\right)^2,$$

$$j_\phi = r\sin\theta\frac{d^2\phi}{dt^2} + 2\sin\theta\frac{dr}{dt}\frac{d\phi}{dt} + 2r\cos\theta\frac{d\theta}{dt}\frac{d\phi}{dt}.$$

288. It is to be noticed that the resolution of the acceleration j into a tangential component j_t and a normal component j_n,

$$j_t = \frac{dv}{dt}, \quad j_n = \frac{v^2}{\rho},$$

given in Art. 159, holds for twisted curves as well as for plane curves, provided the normal be understood to mean the principal normal of the curve, and ρ the radius of absolute curvature at P. For it follows from the definition given in Art. 155 that the acceleration lies in the plane of the tangent and principal normal at P, so that the component along the binormal is zero.

289. This can also be seen from the expressions for the components of j in Cartesian co-ordinates, $j_x = d^2x/dt^2$, $j_y = d^2y/dt^2$, $j_z = d^2z/dt^2$. For since $\frac{dx}{dt} = \frac{dx}{ds}\frac{ds}{dt}$, etc., we have

$$j_x = \frac{d^2x}{dt^2} = \frac{d^2s}{dt^2}\frac{dx}{ds} + \left(\frac{ds}{dt}\right)^2\frac{d^2x}{ds^2},$$

$$j_y = \frac{d^2y}{dt^2} = \frac{d^2s}{dt^2}\frac{dy}{ds} + \left(\frac{ds}{dt}\right)^2\frac{d^2y}{ds^2},$$

$$j_z = \frac{d^2z}{dt^2} = \frac{d^2s}{dt^2}\frac{dz}{ds} + \left(\frac{ds}{dt}\right)^2\frac{d^2z}{ds^2}.$$

Now, dx/ds, dy/ds, dz/ds are the direction cosines of the tangent of the curve, while $\rho\, d^2x/ds^2$, $\rho\, d^2y/ds^2$, $\rho\, d^2z/ds^2$ are the direction cosines of the principal normal. The formulæ show therefore that the acceleration j consists of two components, $\frac{d^2s}{dt^2}=\frac{dv}{dt}$ along the tangent, and $\frac{1}{\rho}\left(\frac{ds}{dt}\right)^2=\frac{v^2}{\rho}$ along the normal.

2. VELOCITIES IN THE RIGID BODY.

290. When the motion of a rigid body is a *translation*, all points of the body have at any instant equal and parallel velocities (Art. 250). The velocity $v=ds/dt$ of any one point can therefore be called the velocity of the body. The body can be subjected at a given instant to several velocities of translation, and the resultant velocity is found by the geometrical addition of the vectors representing the component velocities.

291. When a rigid body *rotates* at the time t about an instantaneous axis l, all its points (excepting those on the axis) describe infinitesimal arcs of circles of angle $d\theta$, and the angular velocity $\omega=d\theta/dt$ of any point of the body may be called the angular velocity of the body. This angular velocity can be represented geometrically by its rotor ω laid off on the axis l (Arts. 68, 69, 252).

As this rotor is proportional to the infinitesimal angle of rotation $d\theta$, the propositions proved in Arts. 62, 66, 67, 68, for the composition and resolution of infinitesimal rotations can be applied directly to angular velocities. The propositions referring to parallel axes have been discussed in Arts. 254–257.

292. If in Art. 62 we divide equation (1′) by dt^2 and divide the denominators of equation (2′) by dt, we obtain

$$\omega^2=\omega_1^2+\omega_2^2+2\,\omega_1\omega_2\cos(l_1l_2), \tag{1}$$

$$\frac{\sin(l_1l)}{\omega_2}=\frac{\sin(ll_2)}{\omega_1}=\frac{\sin(l_1l_2)}{\omega}. \tag{2}$$

The meaning of these equations can be stated as follows. Let a rigid body be subjected simultaneously to two angular velocities about intersecting axes, ω_1 about l_1 and ω_2 about l_2. Represent these angular velocities by their rotors ω_1, ω_2 laid off on the axes l_1, l_2 from their point of intersection O and construct their geometric sum ω; that is, form the diagonal of the parallelogram whose adjacent sides are ω_1, ω_2. Then ω is the rotor of the resulting angular velocity.

This proposition is known as the **parallelogram of angular velocities.**

It follows that the resultant of any number of simultaneous angular velocities whose axes all intersect in the same point is a single angular velocity whose rotor is found by geometrically adding the rotors of the components.

293. Conversely, an angular velocity ω about an axis l can always be replaced, in an infinite number of ways, by two (or more) angular velocities whose geometric sum is ω, about two (or more) axes passing through any point O of l and lying in the same plane with l.

Thus, for instance, the angular velocity ω about the instantaneous axis l can be resolved into three components ω_x, ω_y, ω_z about three rectangular axes Ox, Oy, Oz passing through any point O of l, and we have

$$\omega^2 = \omega_x^2 + \omega_y^2 + \omega_z^2. \tag{3}$$

The linear velocity v of any point P of a body rotating with angular velocity ω about the axis l can be expressed by means of the components ω_x, ω_y, ω_z of ω and the co-ordinates x, y, z of the point P. The component ω_x produces at P a velocity whose components along the axes Ox, Oy, Oz are 0, $-\omega_x z$, $\omega_x y$; similarly, ω_y gives the components $\omega_y z$, 0, $-\omega_y x$; and ω_z gives $-\omega_z y$, $\omega_z x$, 0. Hence, combining the terms that lie along the same axis, the components of the velocity v of the point P are

$$\frac{dx}{dt} = \omega_y z - \omega_z y, \quad \frac{dy}{dt} = \omega_z x - \omega_x z, \quad \frac{dz}{dt} = \omega_x y - \omega_y x. \tag{4}$$

294. If a rigid body be subjected at the time t to two simultaneous angular velocities ω_1, ω_2 about skew (or crossing, *i.e.* not intersecting and not parallel) axes l_1, l_2, or if it be subjected to an angular velocity ω about an axis l and a simultaneous linear velocity v not perpendicular to l, its state of motion during the time dt cannot be expressed by a single angular or linear velocity.

The body can be said to have in either case a *twist-*, or *screw-velocity*, *i.e.* an angular velocity ω about an axis l combined with a linear velocity v_0 parallel to this axis.

To prove this in the latter of the two cases it is only necessary to resolve v into a component v_0 parallel to l and a component v' perpendicular to l. The latter, being equivalent to a rotor couple $(\omega, -\omega)$ of moment $v' = p\omega$ (see Art. 256), combines with the given angular velocity ω about l into an angular velocity ω about a parallel axis l' at the distance $p = v'/\omega$ from l. The combination of the angular velocity ω about l with the simultaneous oblique linear velocity v is therefore equivalent to the angular velocity ω about l' with the simultaneous linear velocity v_0 parallel to l'.

295. When the rigid body has two simultaneous angular velocities ω_1, ω_2 about skew axes l_1, l_2, the reduction is best made by replacing ω_2 about l_2 by an equal angular velocity ω_2 about a parallel axis l' intersecting l_1, in combination with a linear velocity $v = p\omega_2$ perpendicular to the plane of l_2 and l' (Art. 257). The angular velocities ω_1 about l_1 and ω_2 about l' combine (by Art. 292) into a singular angular velocity whose rotor is the geometric sum of ω_1 and ω_2. The case is therefore reduced to the preceding one.

296. It follows from the preceding articles that any number of simultaneous linear and angular velocities can always be combined into a single twist-velocity about the central axis.

3. ACCELERATIONS IN THE RIGID BODY.

297. The accelerations of the points of a rigid body are found by comparing the velocities of these points during two successive elements of time.

If the motion of the rigid body be a pure **translation**, all points of the body describe equal and parallel curves. The accelerations of all points being equal and parallel (Art. 272), the acceleration j of any one point of the body can be spoken of as *the acceleration of the body.* It can be resolved into a *tangential component* j_t along the tangent to the path of any point and a *normal component* j_n along the normal to the path, and we have, just as in Art. 159,

$$j_t = \frac{dv}{dt}, \quad j_n = \frac{v^2}{\rho},$$
$$j = \sqrt{j_t^2 + j_n^2} = \sqrt{\left(\frac{dv}{dt}\right)^2 + \frac{v^4}{\rho^2}}. \tag{1}$$

298. If the motion of the rigid body be a pure **rotation** about the same axis l for at least two successive elements of time dt, all points describe arcs of circles whose centres lie on the fixed axis l. As shown in Art. 273, the acceleration j of any point P whose distance from l is r can be resolved into a *tangential component* j_t perpendicular to the plane (l, P) and a *normal component* j_n at right angles to the axis l; and we have (Art. 273)

$$j_t = \alpha r, \quad j_n = \omega^2 r,$$
$$j = \sqrt{j_t^2 + j_n^2} = r\sqrt{\alpha^2 + \omega^4}, \tag{2}$$

where ω is the angular velocity and $\alpha = d\omega/dt$ the angular acceleration of the body.

The normal component j_n being always directed towards the axis of rotation l is sometimes called the **centripetal acceleration.**

299. If the motion of the rigid body consists in a rotation about an axis l during the first element of time and a rotation about an infinitely near *parallel* axis l' during the second element of time, we have the case of **plane motion** of a rigid body which has been treated in Arts. 274–284.

It remains to discuss the case of *intersecting axes*, which is of fundamental importance in the kinetics of the rigid body.

When the axes about which the body rotates in the successive elements of time intersect at a point O, this point remains fixed during the motion and may be called the *centre of rotation*. The motion of a rigid body with a fixed point may be called **spherical motion.**

The accelerations of the points of a body in spherical motion can be studied in a manner strictly analogous to that used in the case of plane motion (Arts. 274–284).

300. Let the body rotate during the first element of time dt with angular velocity ω about an axis l, and during the second element of time dt with angular velocity $\omega + d\omega$ about an axis l' intersecting l in the point O and making with l the infinitesimal angle $(l, l') = d\sigma$. The angular velocities can be represented by their rotors, ω along l, $\omega + d\omega$ along l' (Fig. 77).

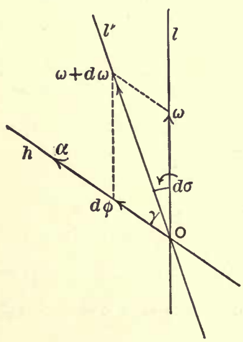

Fig. 77.

The rotor $\omega + d\omega$ along l' can be resolved into a rotor ω along l and an infinitesimal rotor $d\phi$ along an axis h that passes through O and lies in the plane (l, l'). The value of $d\phi$ and the angle $(l, h) = \gamma$ are given by the relations

$$\frac{\sin(l, l')}{d\phi} = \frac{\sin(l', h)}{\omega} = \frac{\sin(l, h)}{\omega + d\omega}, \tag{3}$$

whence
$$\sin(l, h) \equiv \sin\gamma = \omega \frac{d\sigma}{d\phi}. \tag{4}$$

Putting $d\sigma/dt = u$, $d\phi/dt = \alpha$, we may, similarly, as in Art. 274, call u the *velocity of rolling* of the cone of instantaneous axes and α the *angular acceleration.* With these notations

$$\sin\gamma = \frac{\omega u}{\alpha}. \qquad (5)$$

301. The appropriateness of these names will appear by considering that the body can now be regarded as having, for two successive elements of time, the same angular velocity ω about the same axis l, modified during the second element of time by the additional infinitesimal angular velocity $d\phi$ about the axis h, which is called the *axis of angular acceleration.*

Thus the rotation about l produces only *centripetal* (and no tangential) *acceleration* which at unit distance from l is $= \omega^2$ and is directed at right angles to l towards l (see Art. 298), while the rotation about h gives at unit distance from h the infinitesimal velocity $d\phi$ at right angles to the planes through h and thus produces the *angular acceleration* $\alpha = d\phi/dt$, which may be represented by a vector α along h.

The projection of $d\phi$ on l is evidently $d\omega$ (see Fig. 77), so that

$$\cos\gamma = \frac{d\omega}{d\phi} = \frac{1}{\alpha}\frac{d\omega}{dt}. \qquad (6)$$

Squaring and adding the equations (5) and (6), we find

$$\alpha = \sqrt{\left(\frac{d\omega}{dt}\right)^2 + \omega^2 u^2}. \qquad (7)$$

302. These results are further illustrated by another resolution analogous to that of Art. 276.

Imagine the body subjected, during the second element of time, to the equal and opposite angular velocities $\omega + d\omega$ and $-(\omega + d\omega)$ about l (Fig. 78); then combine $\omega + d\omega$ about l' with $-(\omega + d\omega)$ about l into the infinitesimal angular velocity $(\omega + d\omega)\sin d\sigma = \omega d\sigma$ about an axis n through O at right angles to l in the plane (l, l'). This is equivalent to resolving the rotor $\omega+d\omega$ along l' into the rotors $\omega+d\omega$ along l and $\omega d\sigma$ along n.

The body can now be regarded as rotating during both elements of time about the axis l, viz. during the first element with angular velocity ω, during the second with angular velocity $\omega+d\omega$, and in addition to that during the second element about the axis n with the infinitesimal angular velocity $\omega d\sigma$.

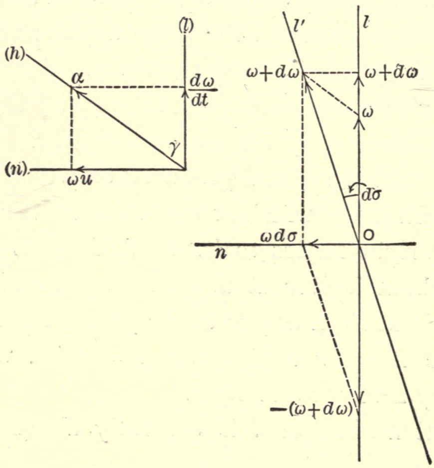

Fig. 78.

The rotation about l (Art. 298) produces, for points at unit distance from l, a *centripetal acceleration* ω^2 perpendicular to l and a *tangential acceleration* $d\omega/dt$ which may be represented by a rotor $d\omega/dt$ along l. The rotation about n gives to points at unit distance from n an infinitesimal velocity $\omega d\sigma$ at right angles to the planes through n and thus produces an acceleration $\omega d\sigma/dt=\omega u$ which may be represented by a rotor along n. The rotors $d\omega/dt$ along l and ωu along n being at right angles to each other (see Fig. 78), combine to form the *angular acceleration*

$$\alpha=\sqrt{\left(\frac{d\omega}{dt}\right)^2+\omega^2u^2}.$$

It is apparent that the component $d\omega/dt$ of α has the effect of changing the magnitude of ω by the amount $d\omega$, without affecting the direction of the axis, while the effect of the component ωu is to incline the axis l by the angle $d\sigma$.

303. To obtain analytical expressions for the components of the acceleration of any point P of a rigid body in spherical motion, let us take the centre of rotation O as origin of a system of fixed rectangular axes. Let x, y, z be the co-ordinates of P; α, β, γ the direction cosines of the instantaneous axis l; and λ, μ, ν those of the perpendicular $PQ=r$ let fall from P on this axis l.

The total acceleration of P is composed of the centripetal acceleration $\omega^2 r$, which is directed along PQ, and the component arising from the angular acceleration α (Art. 301).

The components of $\omega^2 r$ along the axes of x, y, z are $\lambda\omega^2 r$, $\mu\omega^2 r$, $\nu\omega^2 r$. Projecting the closed polygon $OQPO$ on each of the axes, we find

$$\alpha \cdot OQ=\lambda r+x, \quad \beta \cdot OQ=\mu r+y, \quad \gamma \cdot OQ=\nu r+z;$$

or, since OQ is the projection of OP on l, i.e. $OQ=\alpha x+\beta y+\gamma z$,

$$\lambda r=\alpha(\alpha x+\beta y+\gamma z)-x,$$
$$\mu r=\beta(\alpha x+\beta y+\gamma z)-y,$$
$$\nu r=\gamma(\alpha x+\beta y+\gamma z)-z.$$

Multiplying these equations by ω^2 and putting $\alpha\omega=\omega_x$, $\beta\omega=\omega_y$, $\gamma\omega=\omega_z$, we find for the components of the centripetal acceleration of the point (x, y, z):

$$\begin{aligned}\lambda\omega^2 r&=\omega_x(\omega_x x+\omega_y y+\omega_z z)-\omega^2 x,\\ \mu\omega^2 r&=\omega_y(\omega_x x+\omega_y y+\omega_z z)-\omega^2 y,\\ \nu\omega^2 r&=\omega_z(\omega_x x+\omega_y y+\omega_z z)-\omega^2 z.\end{aligned} \tag{8}$$

The angular acceleration $\alpha=d\phi/dt$ (Art. 301) has for its components along the axes of x, y, z

$$\alpha_x=\frac{d\omega_x}{dt}, \quad \alpha_y=\frac{d\omega_y}{dt}, \quad \alpha_z=\frac{d\omega_z}{dt}.$$

The component α_x produces an infinitesimal angular velocity $\alpha_x dt$ about the axis Ox; and hence gives to P the infinitesimal velocities 0, $-\alpha_x z dt$, $\alpha_x y dt$ along the axes Ox, Oy, Oz (see Art. 293); similarly, $\alpha_y dt$ produces the velocities $\alpha_y z dt$, 0, $-\alpha_y x dt$, and $\alpha_z dt$ produces $-\alpha_z y dt$, $\alpha_z x dt$, 0. Collecting the terms parallel to each axis and dividing by dt, we find the components of the acceleration of P due to the angular acceleration α:

$$\alpha_y z - \alpha_z y, \quad \alpha_z x - \alpha_x z, \quad \alpha_x y - \alpha_y x. \tag{9}$$

Finally, combining the corresponding terms in (8) and (9) and remembering that $\alpha_x = d\omega_x/dt$, $\alpha_y = d\omega_y/dt$, $\alpha_z = d\omega_z/dt$, we find the following expressions for the components of the total acceleration j of the point P (x, y, z):

$$\begin{aligned} j_x &= \omega_x(\omega_x x + \omega_y y + \omega_z z) - \omega^2 x + \frac{d\omega_y}{dt} z - \frac{d\omega_z}{dt} y, \\ j_y &= \omega_y(\omega_x x + \omega_y y + \omega_z z) - \omega^2 y + \frac{d\omega_z}{dt} x - \frac{d\omega_x}{dt} z, \\ j_z &= \omega_z(\omega_x x + \omega_y y + \omega_z z) - \omega^2 z + \frac{d\omega_x}{dt} y - \frac{d\omega_y}{dt} x. \end{aligned} \tag{10}$$

304. The formulas (10) for the components of the acceleration of any point (x, y, z) of a body rotating about a fixed point O can also be derived by differentiating the expressions (4) in Art. 293, which represent the components of the velocity of such a point. It is only necessary, after the differentiation, to substitute for dx/dt, dy/dt, dz/dt their values from (4), Art. 293, and to remember that $\omega^2 = \omega_x^2 + \omega_y^2 + \omega_z^2$.

305. The complete study of the motion of a rigid body in the most general case, in particular the investigation of its accelerations, is beyond the scope of the present work.

In addition to the works previously referred to, the following works on kinematics may here be mentioned.

An elementary introduction to kinematics, without the use of the infinitesimal calculus, will be found in J. G. MACGREGOR, *An elementary* *treatise on kinematics and dynamics*, London, Macmillan, 1887. This may be supplemented by W. K. CLIFFORD, *Elements of dynamic*, part 1, Kinematic, *ib.*, 1878. For more advanced study see G. M. MINCHIN, *Uniplanar kinematics of solids and fluids*, Oxford, Clarendon Press, 1882; THOMSON and TAIT, *Natural philosophy*, new edition, part 1, *ib.*, 1879; W. SCHELL, *Theorie der Bewegung und der Kräfte*, vol. 1, 1879, Leipzig, Teubner; J. SOMOFF, *Theoretische Mechanik*, übersetzt von A. Ziwet, part 1, Kinematik, Leipzig, Teubner, 1878; E. BUDDE, *Allgemeine Mechanik der Punkte und starren Systeme*, Berlin, Reimer, 1890; H. RESAL, *Traité de cinématique pure*, Paris, Mallet-Bachelier, 1862; E. BOUR, *Cours de mécanique et machines*, part 1, *Cinématique*, 2d ed., Paris, Gauthier-Villars, 1887; E. COLLIGNON, *Traité de mécanique*, part 1, *Cinématique*, 3d ed., Paris, Hachette, 1885; E. VILLIÉ, *Traité de cinématique*, Paris, Gauthier-Villars, 1888.

ANSWERS.

Page 17.

(1) Join the point P to the instantaneous centre C; the direction of motion is perpendicular to CP.

(3) See Art. 29.

(4) See Art. 29. With O as origin and a parallel to l as axis of y, the fixed centrode is $(y^2-cx)^2=a^2(x^2+y^2)$, where a is the radius of the circle about O, and c the distance of O from l.

(5) $y^2=2a(x+\frac{1}{2}a)$.

(6) The fixed centrode is a circle passing through O', O''; the body centrode is a circle of twice the radius. The path of any point in the fixed plane is in general a limaçon of Pascal.

(8) Consider the initial and final positions of the point of intersection of l_0 and l_1.

Page 28.

(1) 24 miles; E. 35° S.

(3) $\sqrt{3}$.

(4) 10.7 miles; E. $14\frac{1}{2}°$ S.

(6) $2a\cos(\alpha/2)$.

(7) (*a*) 120°; (*b*) 160°48′.6.

(8) $(\sqrt{3}-1)a$; $\frac{1}{2}\sqrt{2}(\sqrt{3}-1)a$.

(10) Inclination to vertical: (*a*) 11°.3; (*b*) 21°.8; (*c*) 45°; (*d*) 67°.4.

(11) $10\frac{1}{4}$ ft.; $\alpha=247\frac{1}{2}°$.

(16) On P_0P_1, P_0P_2 construct the parallelogram $P_0P_1RP_2$ and draw P_1S parallel to QP_2, S being the intersection with the diagonal P_0R.

(18) Apply (16).

Page 39.

(1) For the angle θ of the resulting rotation we have $\sin(\theta/2) = \frac{1}{2}\sqrt{5/2}$; for the position of its axis l, $\sin(l_1 l) = 2/\sqrt{5}$, $\sin(ll_2) = \sqrt{2/5}$.

(3) 22°.

Page 47.

(1) (*a*) 41 miles per hour; (*c*) 19.1, 8.2; (*d*) 10 h. 19 m.

(3) At 2 h. 22 m.; 203 miles from Detroit.

(4) (*a*) 5.9; (*b*) 40.6; (*c*) 44; (*d*) 35.25; (*e*) 1093.

(5) 15.

(7) 185,000 miles per second.

(8) (*a*) 1 h.; (*b*) 15 m.

(9) $30\frac{3}{4}°$.

(10) $37\frac{1}{2}$ miles per hour.

Page 53.

(1) $\frac{11}{15}$.

(2) 32.186.

(3) Nearly $\frac{1}{10}$ ft. per second per second.

(4) 0.0034.

Page 56.

(1) (*a*) 96.6; (*b*) 402.5; (*c*) 144.9.

(2) 0.275.

(4) $h = c\left[t + \frac{c}{g} - \sqrt{\frac{c}{g}\left(2t + \frac{c}{g}\right)}\right]$; an approximate value is

$h = \dfrac{t^2gc}{2(c+gt)}$. For a direct numerical computation, the method of successive approximations may be used. Thus, neglecting the time

t_3 required by the sound, find the depth s approximately from $s=\frac{1}{2}gt^2$, with $t=4$; with this value of s find t_2; hence the time of fall t_1, with which correct s; etc. Result: $s=70.4$ metres.

(5) (*a*) 4 min.; (*b*) 11/60; (*c*) 30 miles per h.; (*d*) after 3 m. 28 s.

(8) (*a*) 40,000 ft.; (*b*) ± 715.5 ft. per second; (*c*) 1 m. 40 s.; (*d*) 1600 ft. per second; (*e*) 1 m. 12.4 s. and 27.6 s.

(9) 80 ft. per second.

(10) (*a*) $t=h/v_0$; (*b*) $h-s=\frac{1}{2}gh^2/v_0^2$; (*c*) $v_0=\sqrt{gh}$.

Page 60.

(1) (*a*) 26,000 ft. per second; (*b*) 34 m. 48 s.

(2) It represents a cycloid.

(4) $v_0^2R/(2gR-v_0^2)$. If $v_0^2 \gtreqless 2gR$, the particle will not fall back.

(5) Height $=R$; time of ascending $=\sqrt{\frac{R}{g}}\left(1+\frac{\pi}{2}\right)=$ time of falling back $=34$ m. 48 s.; hence whole time $=1$ h. 9 m. 36 s.

(6) 7 miles per second.

Page 63.

(2) $v=26,000$ ft. per second; $t=1$ h. 25 m. 4.5 s.

(3) $2s=R(e^{\mu t}+e^{-\mu t})$, or $s=R\cosh \mu t$.

Page 65.

(1) $\lim v=g/\mu$ for $\lim t=\infty$.

(3) $$v=\frac{\sqrt{gk}\,v_0\cos\sqrt{gk}\,t-g\sin\sqrt{gk}\,t}{\sqrt{gk}\cos\sqrt{gk}\,t+kv_0\sin\sqrt{gk}\,t},$$

$$s=\frac{1}{k}\log\left(\cos\sqrt{gk}\,t+\sqrt{\frac{k}{g}}v_0\sin\sqrt{gk}\,t\right)=\frac{1}{2k}\log\frac{g+kv_0^2}{g+kv^2}.$$

(4) Time of ascent $$T=\frac{1}{\sqrt{gk}}\tan^{-1}\sqrt{\frac{k}{g}}v_0;$$

height of ascent $$H=\frac{1}{2k}\log\left(1+\frac{k}{g}v_0^2\right).$$

(5) Compare the height of ascent in Ex. (4) to the distance fallen through as obtained in (27), Art. 126. If v_1 be the velocity with which the particle returns to the starting point, we find

$$v_1 : v_0 = \sqrt{g} : \sqrt{g + kv_0^2}.$$

(6) $v = v_0 e^{-kt}$, $s = \frac{v_0}{k}(1 - e^{-kt})$, $v = v_0 - ks$.

(7) $v = \frac{g}{k}(1 - e^{-kt})$, $s = \frac{g}{k^2}(kt + e^{-kt} - 1) = \frac{1}{k^2}\left(g \log \frac{g}{g - kv} - kv\right)$.

Page 69.

(1) $\omega = \pi$ radians; $v = 18.8$ ft. per second.

(2) (*a*) $3\frac{1}{3}$; (*b*) 32.

(3) -0.157.

(4) 5.

(5) (*a*) 402.1; (*b*) 25.1 seconds.

Page 73.

(1) $r = v_0 t$, $\theta = \omega t$; hence $r = v_0 \theta / \omega$, a spiral of Archimedes.

(2) About the pole O describe a circle of radius a and find its intersection Q with the perpendicular to the radius vector OP drawn through O; then QP is the normal. Proof by Ex. (1).

(3) For the direction of v see Art. 31, Ex. (2). Resolving v into v_0 parallel to the track and v_1 along the tangent to the wheel, it appears that v bisects the angle between these components; hence $v = 2 v_0 \cos CAP$, where C is the centre of the wheel, and A its lowest point.

(5) For the ellipse, $r + r' =$ const.; hence $\frac{dr}{dt} = -\frac{dr'}{dt}$, *i.e.* the projections of the velocity on the radii vectores are equal.

(6) The projections of the velocity on the radius vector and on the focal axis are in the constant ratio e of the focal radius vector to the distance to the directrix. It follows that the tangent intersects the directrix in the same point as does the perpendicular to the radius vector through O.

(7) 40.

(10) $v_1 = 20$ ft. per second, nearly; angle $= 20\frac{1}{4}°$.

(11) The relative velocity of P_2 with respect to P_1 must always pass through P_1. The locus of Q is a circle.

(12) A cycloid.

(13) About 20″.

Page 79.

(4) $233\frac{1}{3}$; $24\frac{4}{9}$; $933\frac{1}{3}$ ft. per minute.

(5) 16.6 knots; 560 ft. per minute.

(6) 55°; 66°; $2\frac{2}{3}$ in.

(7) 0.174, 0.119, 0.146 of the stroke.

Page 85.

(2) By (2), Art. 159, $j_n = \rho\left(\frac{d\alpha}{dt}\right)^2$.

(3) By Art. 159, $j_n = j \sin\psi = v^2/\rho$; hence $v^2 = j \cdot \rho \sin\psi$.

(4) Since j is directed towards A, taking A as origin, we have $j_\theta = 0$, *i.e.* $r^2\frac{d\theta}{dt} =$ const.; comp. Art. 135.

(5) $\frac{d\alpha}{dt} = \omega =$ const., $r =$ const.; hence, by (6), Art. 161, $j = j_r = -r\omega^2$.

(6) $j = r\omega^2$.

Page 86.

(1) (*a*) 1718 ft. above the point; (*b*) after 2 m. 52 s.; (*c*) 1891 ft.; (*d*) 21.2 miles an hour.

(4) 45°.

(7) Construct a circle having the given point as its highest point and touching (*a*) the straight line, (*b*) the circle.

Page 90.

(9) (*a*) 174 ft.; (*b*) in about 8 seconds; (*c*) 254 ft. per second, inclined at an angle of about 5° to the vertical.

(10) 227 ft. per second.

PART II:

INTRODUCTION TO DYNAMICS; STATICS

PREFACE.

THE subject of statics is here developed only so far as is absolutely necessary in order to lay the foundation on the one hand for the study of elementary kinetics, on the other for applied mechanics. From the former point of view it was desirable to bring out clearly the connection of the subject with the general science of mechanics and to determine its place as a subdivision of the larger science. The second section of Chapter III should be considered only as preliminary; the fundamental laws of dynamics can of course be fully understood only by studying kinetics. Prominence is given throughout to geometrical methods and graphical constructions because these seem to conform best to the nature of the subject. The applications given here and there are to be regarded merely as illustrations of the general principles.

The following articles might be omitted at first reading: 18, 19, 20, 34, 43, 44, 48, 52, 113, 117–127, 152–164, 180, 181, 209, 210, 214, 220–225, 257–285.

ALEXANDER ZIWET.

ANN ARBOR, MICH.,
October, 1893.

CONTENTS.

CHAPTER III.

INTRODUCTION TO DYNAMICS.

CHAPTER IV.

STATICS.

THEORETICAL MECHANICS.

CHAPTER III.

INTRODUCTION TO DYNAMICS.

I. *Mass; Moments of Mass; Centroids.*

I. MASS; DENSITY.

1. In the first part of this work only the geometrical and kinematical properties of motion have been considered, the moving object being regarded as a mere point or as a geometrical configuration. It is, however, known, from observation and experiment, that the motions of actual physical bodies are not fully described and determined by those properties alone.

Physical bodies are distinguished from geometrical configurations by being possessed of **mass**; and this property as affecting their motion must be taken into account in dynamics.

2. In physics the mass of a body is usually defined as *the quantity of matter contained in the body.* Postponing for the present the full discussion of the idea of mass in its relation to acceleration and force, and of the methods for comparing and measuring masses, it will suffice for our present purpose to think of the mass of a body as a certain constant quantity, independent of the body's position or motion with respect to the earth or other bodies, as an indestructible something underlying every physical body.

The student must be warned not to confound mass with weight. The weight of a body, as we shall see later, is the force with which the body is attracted by the earth; it varies, therefore, with the distance of the body from the earth's centre, and would vanish completely if the earth were suddenly annihilated; while the *indestructibility of mass* is the first fundamental principle of chemistry and physics.

3. To compare the masses of different bodies, we may adopt any given body as a standard.

Thus in the F. P. S. system, the *standard mass* is a certain bar of platinum marked "P. S., 1844, 1 lb.," and preserved at the Office of the Exchequer, London, England. This is called the "imperial standard pound avoirdupois"; any mass equal to it is a *unit of mass* in this system.

In the C. G. S. system, the standard of mass is the "Kilogramme des archives," a bar of platinum kept in the Palais des archives, in Paris, France. A mass equal to one-thousandth of this standard is the unit of mass in this system; this unit is called the *gram.*

The numerical relation between the British and metric units of mass is as follows:

$$1 \text{ lb.} = 453.592\,65 \text{ gm.}$$

$$1 \text{ gm.} = 0.002\,204\,621\,2 \text{ lb.} = 15.432 \text{ grains.}$$

4. The three units of *space, time,* and *mass* are called *the fundamental units of mechanics,* because with the aid of these three, the units of all other quantities occurring in mechanics can be expressed. Thus we have seen how the units of velocity and acceleration are based on those of space and time, and we shall have many more illustrations in what follows. Any unit that can be expressed mathematically by means of one or more of the fundamental units is called a *derived unit.*

5. From the mathematical point of view, mass appears in our dynamical equations as a coefficient, generally to be regarded as an absolute, positive constant. It serves to give different values (different valency, or "weight" in the meaning of the theory of least squares) to the moving points, lines, areas, volumes, apart from their geometrical extension.

6. Thus, a geometrical point endowed with mass is called a *material particle.* We may regard such a mass-point, or particle, as the limit to which a physical body approaches if its volume be imagined to decrease indefinitely, approaching the limit zero, while its mass may remain a finite quantity. From the physical point of view a particle must be regarded as much an abstraction as a geometrical point, since every finite physical mass occupies a finite space and cannot be identified with a point. We shall see, however, that in dynamics this idea of the mass-point, or particle, is of the greatest importance not only because physical matter is usually considered as made up of an aggregation of such points or centres possessing mass (molecules, atoms), but principally because in many cases the motion of a solid body can be fully represented by the motion of a certain point in it, called its *centre of mass* or **centroid**, the whole mass being regarded as concentrated at this point.

7. It is also customary in dynamics to speak of *material lines* and *material surfaces*, which may be regarded as the limits of physical bodies in which two dimensions or one dimension have been reduced to zero. Thus a material line represents the limit of a wire, chain, or bar, in which two dimensions are neglected; a material surface can be imagined as the limit of a thin shell, or lamina, with one dimension reduced to zero.

8. A continuous mass of one, two, or three dimensions, is said to be *homogeneous* if the masses contained in *any* two equal lengths, areas, or volumes (as the case may be), are equal. The mass is then said to be distributed *uniformly.* In all other cases the mass is said to be *heterogeneous.*

9. The whole mass M of a homogeneous body divided by the space V it fills is called the **density** of the mass or body; denoting density by ρ we have therefore

$$\rho = \frac{M}{V},$$

for homogeneous bodies.

In a heterogeneous body, this quotient is called the *average*, or *mean*, density. In this case *the density at any point*, or the density of any space-element dV, is defined as the derivative

$$\rho = \frac{dM}{dV}.$$

10. The *unit of density* is the density of a substance such that the unit of volume contains the unit of mass. If the units of volume and mass are selected arbitrarily, there need not of course necessarily exist any physical substance having unit density exactly. Thus in the F.P.S. system, unit density is the density of an ideal substance 1 pound of which would just fill a cubic-foot. As a cubic foot of water has a mass of $62\frac{1}{2}$ pounds, or 1000 ounces, the density of water is $62\frac{1}{2}$ times the unit density.

The *specific density*, or *specific gravity*, of a substance, is the ratio of its density to that of water at 4° C. Let ρ be the density, ρ' the specific density, M the mass, V the volume of a homogeneous mass, then in British units

$$M = \rho V = 62.5\, \rho' V.$$

In the C.G.S. system, the unit of mass has been so selected as to make the density of water equal to 1 very nearly; in other words, the unit mass (1 gramme) of water, at the temperature of 4° C., fills one cubic centimetre.

In the metric system, then, there is no difference between density and specific density or specific gravity.

2. MOMENTS AND CENTRES OF MASS.

11. The product of a mass m, concentrated at a point P, into the distance of the point P from any given point, line, or plane, is called the **moment** of this mass with respect to the point, line, or plane.

Thus, denoting by r, q, p, the distance of the point P from the point O, the line l, and the plane π, respectively, we have for the moments of m with respect to O, l, π, the expressions mr, mq, mp.

12. Let a system of n points, or particles, $P_1, P_2, \ldots P_n$ be given; let $m_1, m_2, \ldots m_n$ be their masses, and $p_1, p_2, \ldots p_n$ their distances from a given plane π. Then we call **moment of the system** with respect to the plane π the algebraic sum

$$m_1 p_1 + m_2 p_2 + \cdots + m_n p_n = \Sigma mp,$$

the distances $p_1, p_2, \ldots p_n$ being taken with the same sign or opposite signs according as they lie on the same side or on opposite sides of the plane π.

It is always possible to determine one and only one distance $\bar{p}$ such that $\Sigma mp = M\bar{p}$, where $M = \Sigma m = m_1 + m_2 + \cdots + m_n$ is the total mass of the system. If this distance $\bar{p}$ should happen to be equal to zero, the moment of the system would evidently vanish with respect to the plane π.

13. Let us now refer the points P to a rectangular system of co-ordinates, and let x, y, z be their co-ordinates. Then we have for the moments of the system with respect to the co-ordinate planes yz, zx, xy, respectively

$$m_1 x_1 + m_2 x_2 + \cdots + m_n x_n = \Sigma mx = M\bar{x},$$

$$m_1 y_1 + m_2 y_2 + \cdots + m_n y_n = \Sigma my = M\bar{y},$$

$$m_1 z_1 + m_2 z_2 + \cdots + m_n z_n = \Sigma mz = M\bar{z}.$$

The point G whose co-ordinates are

$$\bar{x} = \frac{\Sigma mx}{M}, \quad \bar{y} = \frac{\Sigma my}{M}, \quad \bar{z} = \frac{\Sigma mz}{M} \tag{1}$$

is called the *centre of mass*, or the **centroid**, of the system.

The centroid is, therefore, defined as *a point such that if the whole mass* M *of the system be concentrated at this point, its* *moment with respect to any one of the co-ordinate planes is equal to the moment of the system.*

14. It is easy to see that this holds not only for the co-ordinate planes but for any plane whatever. Let

$$\alpha x+\beta y+\gamma z-p_0=0$$

be the equation of any plane in the normal form;

$$\bar{p},\ p_1,\ p_2,\ \ldots,\ p_n,$$

the distances of the points $G, P_1, P_2, \ldots, P_n$ from this plane. Then we wish to prove that $\Sigma mp = M\bar{p}$.

Now $$\bar{p}=\alpha\bar{x}+\beta\bar{y}+\gamma\bar{z}-p_0,$$

$$p_1=\alpha x_1+\beta y_1+\gamma z_1-p_0,\ \cdots$$

hence $$\Sigma mp=\alpha\Sigma mx+\beta\Sigma my+\gamma\Sigma mz-p_0\Sigma m$$

$$=M(\alpha\bar{x}+\beta\bar{y}+\gamma\bar{z}-p_0)$$

$$=M\bar{p}.$$

The centroid can therefore be defined as *a point such that its moment with respect to any plane is equal to that of the whole system, with respect to the same plane.*

It follows that *the moment of the system vanishes for any plane passing through the centroid.*

15. In the case of a continuous mass, whether it be of one, two, or three dimensions, the same reasoning will apply if we imagine the mass divided up into elements dM of one, two, or three infinitesimal dimensions, respectively. The summations indicated above by Σ will then become integrations, so that the centroid of a continuous mass has the co-ordinates

$$\bar{x}=\frac{\int xdM}{\int dM},\quad \bar{y}=\frac{\int ydM}{\int dM},\quad \bar{z}=\frac{\int zdM}{\int dM}. \qquad (2)$$

According as the mass is of one, two, or three dimensions, a single, double, or triple integration over the whole mass will in general be required for the determination of the moments $\int xdM$, $\int ydM$, $\int zdM$ of the mass with respect to the co-ordinate planes, as well as of the total mass $\int dM = M$.

Thus, for a mass distributed along a line or a curve we have, if ds be the line-element,

$$dM = \rho ds;$$

for a mass distributed over a surface-area we have, with dS as a surface-element,

$$dM = \rho dS;$$

finally, for a mass distributed throughout a volume whose element is dV,

$$dM = \rho dV.$$

If the mass be distributed along a straight line, the centroid lies of course on this line, and one co-ordinate is sufficient to determine the position of the centroid. In the case of a plane area, the centroid lies in the plane and two co-ordinates determine its position; we then speak of moments with respect to lines, instead of planes.

16. If the mass be homogeneous (Art. 8), *i.e.* if the density ρ be constant, it will be noticed that ρ cancels from the numerator and denominator in the equations (2), and does not enter into the problem. Instead of speaking of a centre of mass, we may then speak of a centre of arc, of area, of volume. The term *centroid* is, however, to be preferred to *centre*, the latter term having a recognised geometrical meaning different from that of the former.

The geometrical centre of a curve or surface is a point such that any chord through it is bisected by the point; there are but few curves and surfaces possessing a centre.

The centroid (Art. 14) is a point such that, for any plane passing through it, the moment of the system is equal to zero. Such a point exists for every mass, volume, area, or arc. The centroid coincides, of course, with the centre, when such a centre exists and the distribution of mass is uniform.

17. As soon as ρ is given either as a constant or as a function of the co-ordinates, the problem of determining the centroid of a continuous mass is merely a problem in integration. To simplify the integrations, it is of importance to select the element in a convenient way conformably to the nature of the particular problem.

Considerations of symmetry and other geometrical properties will frequently make it possible to determine the centroid without resorting to integration. Thus, in a homogeneous mass, any plane of symmetry, or any axis of symmetry, must contain the centroid, since for such a plane or line the sum of the moments is evidently zero (see Art. 47).

It is to be observed that the whole discussion is entirely independent of the physical nature of the masses m which appear here simply as numerical coefficients, or "weights," attached to the points (comp. Art. 5). Some of the masses might even be negative.

It will be shown later that the *centre of gravity*, as well as the *centre of inertia*, of a body coincides with its centroid.

18. The centroid can be defined without any reference to a co-ordinate system as follows.

As in Art. 12, let there be given a system of n points $P_1, P_2, \dots P_n$ (Fig. 1) whose masses are $m_1, m_2, \dots m_n$. Taking an arbitrary origin O and putting $OP_1 = r_1$, $OP_2 = r_2, \dots OP_n = r_n$, we may represent the moments m_1r_1, m_2r_2, $\dots m_nr_n$ of the given masses with respect to O (Art. 11) by lengths (vectors) laid off on OP_1, $OP_2, \dots OP_n$. *The moment of the system* can then be defined as the geometric sum of these vectors. It is therefore found by geometrically adding these vectors; *i.e.* we have to lay off from O, on OP_1, $Op_1 = m_1r_1$; from p_1, parallel to OP_2, $p_1p_2 = m_2r_2$, etc.; and finally join O to the end p_n of the polygon so formed; then Op_n is the geometric sum, or resultant, of the

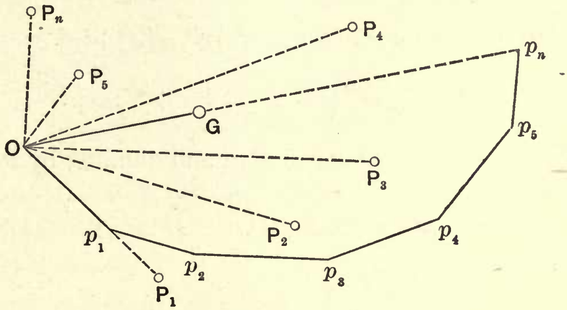

Fig. 1.

vectors m_1r_1, m_2r_2, ... m_nr_n. Using square brackets to indicate geometric addition, we have $Op_n = \Sigma[mr]$. A point G taken on the line Op_n so that

$$M \cdot OG = Op_n = \Sigma[mr], \tag{3}$$

where $M = \Sigma m$, is the *centroid of the system.*

19. It is easy to see that this definition of the centroid agrees with the one previously given (Art. 13). For, to form the geometric sum, or resultant, of the vectors m_1r_1, m_2r_2, ... m_nr_n, we may resolve each of these vectors along three rectangular axes drawn through O. The components of m_1r_1 are evidently m_1x_1, m_1y_1, m_1z_1, if x_1, y_1, z_1 are the co-ordinates of P_1, since x_1/r_1, y_1/r_1, z_1/r_1 are the direction cosines of the line OP_1. We find therefore for the components of Op_n the values Σmx, Σmy, Σmz; and hence for the co-ordinates of G,

$$\bar{x} = \Sigma mx/M, \ \bar{y} = \Sigma my/M, \ \bar{z} = \Sigma mz/M.$$

20. The position of the centroid G of a given system of masses is independent of the point O selected as origin. For let another point O' at the distance d from O be selected as origin, and let G' be the point obtained as centroid from this origin, so that

$$M \cdot OG = \Sigma[mr], \quad M \cdot O'G' = \Sigma[mr'].$$

As we have the geometric equation $[r'] = [d] + [r]$, we find

$$M \cdot O'G' = \Sigma[md] + \Sigma[mr] = Md + \Sigma[mr].$$

Hence subtracting the first equation and dividing by M,

$$[O'G'] - [OG] = [d], \text{ or } [O'G'] = [d] + [OG] = [O'G]$$

so that G and G' coincide.

It follows from this consideration that a given system has only one centroid.

21. Regarding again the mass of the centroid as equal to that of the whole system, we may now define the **centroid** of a system as *a point such that its moment with respect to any point or plane is equal to the sum of the moments of all the points constituting the system;* the sum being understood to be a geometric sum for moments with respect to a point, and an algebraic sum for moments with respect to a plane.

Taking the centroid itself as origin, we have the proposition that *the geometric sum of the moments of a system with respect to the centroid is equal to zero.* It has been proved before (Art. 14) that *the algebraic sum of the moments of a system vanishes for any plane passing through the centroid.*

22. In determining the centroid of a given system it will often be found convenient to break the system up into a number of partial systems whose centroids are either known or can be found more readily. *The moment of the whole system is* obviously *equal to the sum of the moments of the partial systems.*

Thus let the given mass M be divided into k partial masses $M_1, M_2, \dots M_k$, so that $M = M_1 + M_2 + \cdots + M$; let $G, G_1, G_2, \dots$

G_k be the centroids of M, M_1, M_2, ... M_k, and $\bar{p}$, $\bar{p}_1$, $\bar{p}_2$, ... $\bar{p}_k$ their distances from some fixed plane. Then we have

$$M\bar{p} = M_1 p_1 + M_2 p_2 + \cdots + M_k p_k.$$

23. The particular case of *two* partial systems occurs most frequently. We then have with reference to any plane

$$M\bar{p} = M_1 p_1 + M_2 p_2, \quad M = M_1 + M_2.$$

Letting the plane coincide successively with the three co-ordinate planes, it will be seen that G must lie on the line joining G_1, G_2. Now taking the plane at right angles to $G_1 G_2$ through G_1, we have

$$M \cdot G_1 G = M_2 \cdot G_1 G_2;$$

similarly for a plane through G_2,

$$M \cdot GG_2 = M_1 \cdot G_1 G_2;$$

whence $$\frac{G_1 G}{M_2} = \frac{GG_2}{M_1} = \frac{G_1 G_2}{M};$$

i.e. *the centroid of the whole system divides the distance of the centroids of the two partial systems in the inverse ratio of their masses.*

3. EXAMPLES OF THE DETERMINATION OF CENTROIDS.

24. Two Particles. The centroid G of two particles of masses m_1, m_2 concentrated at two points P_1, P_2 lies on the line $P_1 P_2$ and divides the distance $P_1 P_2$ in the inverse ratio of their masses, *i.e.* so that

$$\frac{P_1 G}{m_2} = \frac{GP_2}{m_1} = \frac{P_1 P_2}{m_1 + m_2}.$$

(See Art. 23.) These formulæ hold even when one of the masses is positive and the other negative, in which case the sense of the segments must be attended to.

25. Three Particles. We find first the centroid P' of m_2 at P_2 and m_3 at P_3 (Fig. 2) by Art. 24; then, by the same rule, the centroid G of m_2+m_3 at P' and m_1 at P_1. We might have begun with P_3 and P_1, finding P''; or with P_1 and P_2, finding P'''. G lies at the intersection of the three lines P_1P', P_2P'', P_3P''', and can therefore be constructed graphically.

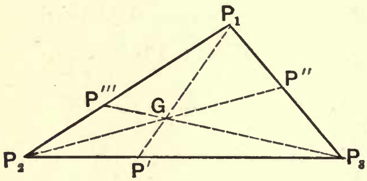

Fig. 2.

26. Four Particles. Find the centroid P' of m_1 at P_1 and m_2 at P_2; also the centroid P'' of m_3 at P_3 and m_4 at P_4; then the centroid G of m_1+m_2 at P' and m_3+m_4 at P''.

The four particles can be arranged in groups of two in three different ways. There are therefore three lines, like $P'P''$, on each of which G lies. Any two of these are sufficient to construct G geometrically.

27. The centroid of a **homogeneous rectilinear segment** (thin rod or wire of constant cross-section) is evidently at its middle point.

28. *If the density of a rectilinear segment be proportional to the* n*th power of the distance from one end*, say $\rho=kx^n$, we have

$$\bar{x}=\frac{\int_0^l \rho x dx}{\int_0^l \rho dx}=\frac{k\int_0^l x^{n+1}dx}{k\int_0^l x^n dx}=\frac{n+1}{n+2}l,$$

where l is the length of the segment.

(*a*) For $n=0$, this gives $\bar{x}=\frac{1}{2}l$ which determines the centroid of a *homogeneous straight segment* (see Art. 27).

(*b*) For $n=1$, we have $x=\frac{2}{3}l$. This determines the distance, from the vertex, of the centroid of a *homogeneous triangular area.* For such an area can be resolved (Fig. 3) by parallels to the base into elements each of which may be regarded as a homogeneous segment PQ. If we imagine the mass of every such element concentrated at its middle point, the homogeneous triangle is replaced by its median CC' in which the density is proportional to the distance from the vertex C.

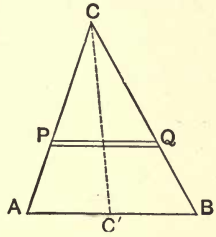

Fig. 3.

The centroid of a homogeneous triangular area lies therefore on the median at two-thirds of its length from the vertex; as this holds for each median, the intersection of the three medians is the centroid (see Art. 32).

(*c*) For $n=2$, we have $\bar{x}=\frac{3}{4}l$. This gives the position of the centroid of a *homogeneous pyramid* or *cone*, by reasoning precisely similar to that used in (*b*).

Thus, to find the centroid of any homogeneous pyramid or cone, join the vertex to the centroid of the area of the base; the required centroid lies on this line at a distance equal to $\frac{3}{4}$ of its length from the vertex.

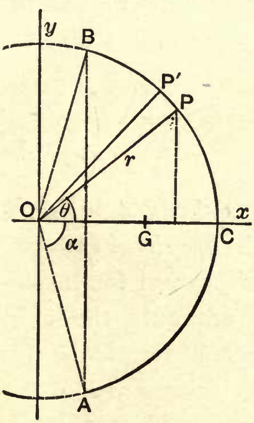

Fig. 4.

29. Homogeneous Circular Arc (Fig. 4). Let O be the centre, r the radius of the circle; $ACB=s$ the arc, C its middle point. The centroid G must lie on the bisecting radius OC, since this being a line of symmetry, the sum of the moments of the elements of the arc is $=0$ with respect to this line (Art. 17). To find the distance $\bar{x}=OG$, we take moments with respect to the diameter perpendicular to OC. With OC as axis of x, we have

$$s\cdot\bar{x}=\int x ds=r\int\frac{x}{r}ds=r\int ds\cos COP=r\int dy.$$

Hence, $s\cdot\bar{x}=r\cdot c$, if c be the length of the chord AB.

If the angle $AOB=2\alpha$ of the arc AB were given, we might obtain the result by taking the angle $COP=\theta$ as independent variable. We have then

$$s \cdot \bar{x} = \int_{-\alpha}^{+\alpha} r \cos\theta \cdot r d\theta = 2 r^2 \sin\alpha,$$

whence $$\bar{x} = r \cdot \frac{\sin\alpha}{\alpha}.$$

This can be written $\bar{x} = r \cdot \frac{2 r \sin\alpha}{2 r\alpha} = r \cdot \frac{c}{s}$, which agrees with the expression found above.

30. The First Proposition of Pappus and Guldinus. If an arc of a plane curve be made to rotate about an axis situated in its plane, it generates a surface of revolution whose surface-area is $S = 2\pi \int y ds$, where ds is the element of the curve and the axis of rotation is taken as axis of x. On the other hand we have, if s be the length of the generating arc and y the ordinate of its centroid, $s \cdot \bar{y} = \int y ds$; hence

$$S = 2\pi \cdot s\bar{y} = 2\pi\bar{y} \cdot s,$$

i.e. *the surface-area of a solid of revolution is obtained by multiplying the generating arc into the path described by its centroid.*

It is easy to see that this proposition holds even for incomplete revolutions. When the generating arc cuts the axis, proper regard must be had for signs and sense of rotation.

31. It follows from symmetry that the centroid of a **homogeneous circular** or **elliptic area** (plate, lamina) is at the geometrical centre of figure. Similarly, the centroid of a **homogeneous parallelogram** is at the intersection of its diagonals.

In general, if a homogeneous plane figure have two axes of symmetry, the centroid must be at the intersection of these lines since the sum of the moments is zero for each of these lines.

32. It has been shown in Art. 28 (*b*) how the centroid of a **homogeneous triangular area** ABC can be found.

Dividing the area into linear elements by drawing lines parallel to one of the sides, say AB (Fig. 3, p. 13), it appears that the centroid of each element, such as PQ, lies at its middle point. The locus of these middle points is the median CC' of the triangle; on this line, then, the centroid G of the triangle must be situated. Resolving the triangle into linear elements parallel to the side BC, or to CA, it follows in the same way that G must lie on each of the other two medians of the triangle. The intersection of these medians is therefore the centroid G.

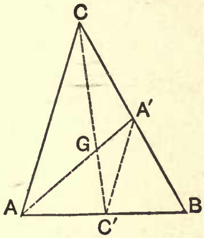

Fig. 5.

The point G trisects each median so that $CG/GC'=2$. For if AA' (Fig. 5) is another median, the triangles AGC and $A'GC'$ are similar, and $A'C'=\frac{1}{2}AC$; hence $C'G=\frac{1}{2}CG$.

It follows from Art. 25, that the centroid of the homogeneous triangular area coincides with that of three particles of equal mass placed at the vertices.

33. Homogeneous Quadrilateral. The centroid is found graphically by resolving the quadrilateral into triangles, finding their centroids, and deducing from them the centroid of the quadrilateral. This process applies generally to *any polygon* and can be carried out in various ways.

Thus for the quadrilateral $ABCD$ (Fig. 6) drawing the diagonal AC and determining the centroids of the triangles ABC and ADC, we obtain by joining these centroids one line on which the required centroid of the quadrilateral must lie. Repeating the same construction for the triangles obtained by drawing the other diagonal BD, we find a second line on which the centroid

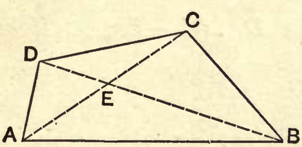

Fig. 6.

must lie. The intersection of these lines gives the centroid of the quadrilateral.

34. For some purposes it is convenient to find a system of particles whose centroid shall be the same as that of a quadrilateral. The problem is of course indeterminate and may be solved in various ways.

Let m be the mass of the quadrilateral $ABCD$; m_1, m_2 the masses of the triangles ABC, ADC. By Art. 32, each of these triangles can be replaced by three equal particles $\frac{1}{3}m_1$, $\frac{1}{3}m_2$, placed at the vertices. We thus have at A, as well as at C, a mass $\frac{1}{3}(m_1+m_2)=\frac{1}{3}m$.

The masses $\frac{1}{3}m_1$ at B and $\frac{1}{3}m_2$ at D, whose sum is also $=\frac{1}{3}m$, are proportional to the areas of the triangles ABC, ADC, or to the lengths EB, ED, if E be the intersection of the diagonals. Now these two different masses at B and D can be replaced by a system of three masses, $\frac{1}{3}m$ at B, $\frac{1}{3}m$ at D, and $-\frac{1}{3}m$ at E. For (1) the total mass evidently remains the same, and (2) the centroids of the two systems coincide as is easily seen by taking moments with respect to E.

Indeed, the centroid G' of $\frac{1}{3}m_1$ at B and $\frac{1}{3}m_2$ at D is determined by the equation

$$(m_1+m_2)\cdot EG'=m_1\cdot EB-m_2\cdot ED;$$

substituting for m_1, m_2 their values as found from the relations $m_1+m_2=m$, $m_1/m_2=EB/ED$, this reduces to

$$m\cdot EG'=m\cdot(EB-ED).$$

The centroid G'' of $\frac{1}{3}m$ at B, $\frac{1}{3}m$ at D, and $-\frac{1}{3}m$ at E is given by

$$m\cdot EG''=m\cdot EB-m\cdot ED-m\cdot 0.$$

Hence G' and G'' coincide.

The centroid of the area of a homogeneous quadrilateral is therefore the same as that of four equal particles placed at its vertices together with a fifth particle of equal but negative mass, placed at the intersection of the diagonals.

35. In the particular case of a **homogeneous trapezoid** (Fig. 7), it may be noticed that the figure can be divided into rectilinear elements by lines drawn parallel to the parallel sides of the trapezoid. Every such element has its centroid at its middle point; the locus of all these points is the so-called median; and the centroid G of the trapezoid must lie on this median, *i.e.* on the line joining the middle points E, F of the parallel sides.

To find the ratio in which G divides the length EF, we use again the method of taking moments. We divide the trapezoid

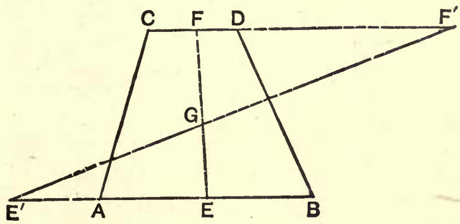

Fig. 7.

into two triangles by the diagonal BC and remember that the distance of the centroid of a triangle from its base is equal to one-third of its height; then taking moments with respect to the two parallel sides $AB=a$, $CD=b$, denoting the height of the trapezoid by h, and the distances of G from a and b by $\bar{y}$ and $\bar{y}'$, we obtain

$$\tfrac{1}{2}(a+b)h\cdot\bar{y}=\tfrac{1}{2}ah\cdot\tfrac{1}{3}h+\tfrac{1}{2}bh\cdot\tfrac{2}{3}h,$$
$$\tfrac{1}{2}(a+b)h\cdot\bar{y}'=\tfrac{1}{2}ah\cdot\tfrac{2}{3}h+\tfrac{1}{2}bh\cdot\tfrac{1}{3}h.$$

Dividing, we find

$$\frac{\bar{y}}{\bar{y}'}=\frac{EG}{GF}=\frac{a+2b}{2a+b}=\frac{\frac{1}{2}a+b}{a+\frac{1}{2}b}.$$

This gives the following construction: Make $AE'=b$ on the prolongation of a, and $DF'=a$ on the prolongation of b, in the opposite sense; then $E'F'$ will intersect EF in G.

36. To find the centroid of the **cross-section of a T-iron** (Fig. 8) it is only necessary to find its distance $\bar{x}$ from the lower side AB; for it must lie on the axis of symmetry CD. Taking moments with respect to AB we obtain with the notation indicated in the figure:

$$[2a\beta+2\alpha(b-\beta)]\cdot\bar{x}=2a\beta\cdot\frac{a}{2}+2(b-\beta)\alpha\cdot\frac{\alpha}{2},$$

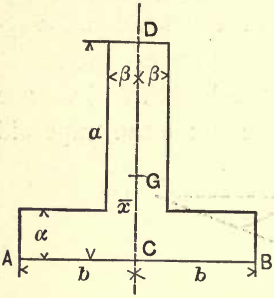

Fig. 8.

hence $$\bar{x}=\frac{1}{2}\frac{a^2\beta+b\alpha^2-\alpha^2\beta}{a\beta+b\alpha-\alpha\beta}.$$

If α, β are nearly equal and very small in comparison with a, b, we have approximately

$$\bar{x}=\frac{1}{2}\frac{a^2+b\alpha}{a+b}.$$

37. The area of a **homogeneous circular sector** (Fig. 4, p. 13) of radius r and angle $AOB=2\alpha$ can be resolved into triangular elements $POP'=\frac{1}{2}r^2d\theta$, the bisecting radius OC being taken as polar axis. The centroid of such an element lies, by Art. 32, at the distance $\frac{2}{3}r$ from the centre O. Regarding the mass, $\rho\cdot\frac{1}{2}r^2d\theta$, of each element as concentrated at its centroid, the sector is replaced by a homogeneous circular arc of radius $\frac{2}{3}r$ and density $\frac{1}{2}\rho r^2d\theta$. By Art. 29, the centroid of such an arc, which is the required centroid of the sector, lies on the bisecting radius OC at the distance $\frac{2}{3}r\cdot\frac{\sin\alpha}{\alpha}$ from the centre O. Hence

$$\bar{x}=\frac{2}{3}r\frac{\sin\alpha}{\alpha}.$$

38. In general, for **areas bounded by curves** we must resort to integration, using the general formulæ of Art. 15.

If the area S be plane, we have in rectangular co-ordinates

$$M=\int_{x_1}^{x_2}\int_{y_1}^{y_2}\rho dxdy,$$

$$M\cdot\bar{x}=\int_{x_1}^{x_2}\int_{y_1}^{y_2}\rho x dx dy,\quad M\cdot\bar{y}=\int_{x_1}^{x_2}\int_{y_1}^{y_2}\rho y dx dy;$$

and if the mass be homogeneous, *i.e.* $\rho=$const., since then the first integration can at once be effected:

$$S=\int_{x_1}^{x_2}(y_2-y_1)dx,$$

$$S\cdot\bar{x}=\int_{x_1}^{x_2}x(y_2-y_1)dx,\quad S\cdot\bar{y}=\tfrac{1}{2}\int_{x_1}^{x_2}(y_2^2-y_1^2)dx,$$

or similar expressions for y as independent variable.

In polar co-ordinates, the element of area is $rdrd\theta$, and we have $x=r\cos\theta$, $y=r\sin\theta$; hence

$$S=\int\int rdrd\theta,$$

$$S\cdot\bar{x}=\int\int r^2\cos\theta drd\theta,\quad S\cdot\bar{y}=\int\int r^2\sin\theta drd\theta;$$

or, performing the first integration,

$$S=\tfrac{1}{2}\int_{\theta_1}^{\theta_2}r^2d\theta,$$

$$S\cdot\bar{x}=\tfrac{1}{3}\int_{\theta_1}^{\theta_2}r^3\cos\theta d\theta,\quad S\cdot\bar{y}=\tfrac{1}{3}\int_{\theta_1}^{\theta_2}r^3\sin\theta d\theta.$$

It will be noticed that these last formulæ express also that the infinitesimal sector $\frac{1}{2}r^2d\theta$ is taken as element, the centroid of this element having the co-ordinates $\frac{2}{3}r\cos\theta$, $\frac{2}{3}r\sin\theta$.

39. As a somewhat more complicated example let us consider a circular disc of radius a, in which the density varies directly as the distance from the centre (Fig. 9). Let a circle described upon a radius as diameter be cut out of this disc; it is required to find the centroid of the remainder.

Let O be the centre of the disc of radius a, C that of the disc of radius $\frac{1}{2}a$; G_1 the centroid of the latter, G the required centroid; and put $OG_1=\bar{x}_1$, $OG=\bar{x}$. Then if M_1 be the mass of the smaller disc, M_2 that of the larger, we must have $(M_2 - M_1)\cdot\bar{x} = M_1\bar{x}_1$.

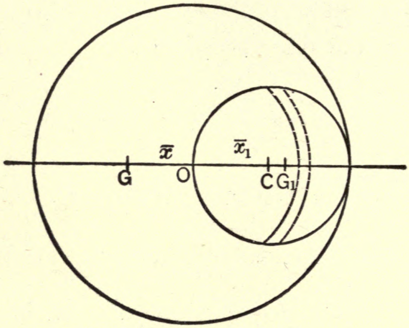

Fig. 9.

The equation of the smaller circle is $r = a\cos\theta$. Taking as element of the mass of the smaller disc the mass contained between two arcs of radii r and $r+dr$, we have for this element:

$$dM_1 = \rho\cdot 2\theta r dr,$$

or since $\rho = kr$, $r = a\cos\theta$,

$$dM_1 = 2ka^3\theta\cos^2\theta d(\cos\theta).$$

Hence $$M_1 = \tfrac{2}{3}ka^3\int_{\frac{\pi}{2}}^{0}\theta d(\cos^3\theta)$$

$$= \tfrac{2}{3}ka^3\left(\theta\cos^3\theta - \int\cos^3\theta d\theta\right)_{\frac{\pi}{2}}^{0}$$

$$= \tfrac{2}{3}ka^3\int_0^{\frac{\pi}{2}}\cos^3\theta d\theta = \tfrac{2}{3}ka^3\cdot\tfrac{2}{3} = \tfrac{4}{9}ka^3.$$

The centroid of the element dM_1 lies, according to Art. 29, at the distance $r\dfrac{\sin\theta}{\theta}$ from O. We have therefore

$$M_1\bar{x}_1 = -2ka^3\int_{\frac{\pi}{2}}^{0}\theta\sin\theta\cos^2\theta d\theta\cdot r\frac{\sin\theta}{\theta}$$

$$= 2ka^4\int_0^{\frac{\pi}{2}}\sin^2\theta\cos^3\theta d\theta = \tfrac{4}{15}ka^4.$$

The mass of the larger disc is

$$M_2 = \int_0^a kr \cdot 2\pi r \cdot dr = 2\pi k \int_0^a r^2 dr = \tfrac{2}{3}\pi k a^3.$$

Substituting these values into the equation of moments we find

$$\bar{x} = \frac{M_1 x_1}{M_2 - M_1} = \frac{6}{5(3\pi - 2)} a = 0.1616..a.$$

40. Proceeding to the determination of the centroids of *curved surface-areas*, we begin with the special case of the homogeneous area of a **surface of revolution.** If the axis of x coincide with the axis of revolution and $R = r\sin\theta$ be the distance of any point P of the surface from this axis (Fig. 10), the equation of the surface, or of its meridian section, is $x = f(R)$; and the element of area is

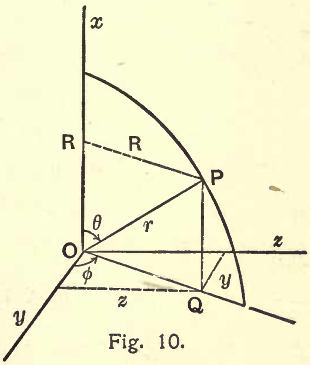

Fig. 10.

$$dS = Rd\phi\sqrt{dx^2 + dR^2} = R\sqrt{1 + [f'(R)]^2}\,dR\,d\phi.$$

We have therefore for the centroid of the portion of the surface contained between two sections at right angles to the axis and two meridian planes (*i.e.* planes through the axis) including an angle ϕ:

$$S = \int_{R_1}^{R_2}\int_0^{\phi} R\sqrt{1 + f'^2}\,dR\,d\phi = \phi\int_{R_1}^{R_2} R\sqrt{1 + f'^2}\,dR,$$

$$S\cdot\bar{x} = \int_{R_1}^{R_2}\int_0^{\phi} Rf(R)\sqrt{1 + f'^2}\,dR\,d\phi = \phi\int_{R_1}^{R_2} Rf(R)\sqrt{1 + f'^2}\,dR,$$

$$S\cdot\bar{y} = \int_{R_1}^{R_2}\int_0^{\phi} R^2\cos\phi\sqrt{1 + f'^2}\,dR\,d\phi = \sin\phi\int_{R_1}^{R_2} R^2\sqrt{1 + f'^2}\,dR,$$

$$S\cdot\bar{z} = \int_{R_1}^{R_2}\int_0^{\phi} R^2\sin\phi\sqrt{1 + f'^2}\,dR\,d\phi = (1 - \cos\phi)\int_{R_1}^{R_2} R^2\sqrt{1 + f'^2}\,dR.$$

Similar formulæ result when x is taken as independent variable instead of R. For a complete surface of revolution $\phi = 2\pi$ so that $\bar{y} = 0$, $\bar{z} = 0$, as is otherwise evident.

41. In the case of *spherical surfaces*, although the preceding formulæ can of course be used, it is often more convenient to make use of the geometrical property of the sphere that any spherical area is equal to the area of its projection on a cylinder circumscribed about the sphere.

Thus the *area on the sphere contained between two parallel planes* is equal to the area cut out by the same two planes from the circumscribed cylinder whose axis is perpendicular to the planes. The centroid of such a spherical area is therefore on the radius at right angles to the bounding planes midway between these planes.

42. The Second Proposition of Pappus and Guldinus (compare Art. 30).

A plane area S (Fig. 11) rotating about any axis situated in in its plane generates a solid of revolution whose volume is $V = \pi \int (y_2^2 - y_1^2)\, dx$, if the axis of revolution is taken as axis of x and y_1, y_2 are the two ordinates of the curve bounding the area. On the other hand, if $\bar{y}$ be the distance of the centroid G of the plane area from the axis, we have

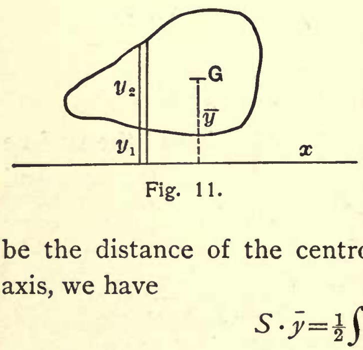

Fig. 11.

$$S \cdot \bar{y} = \tfrac{1}{2} \int (y_2^2 - y_1^2)\, dx,$$

by Art. 38. Combining these two results, we find

$$V = 2\pi \bar{y} \cdot S,$$

i.e. *the volume of a solid of revolution is obtained by multiplying the generating area into the path described by its centroid.*

The proposition evidently holds even for a partial revolution.

43. To find the centroid of a **portion of any curved surface** $F(x, y, z)=0$, we have only to substitute $dM=\rho dS$ in the general formulæ of Art. 15, and then express dS by the ordinary methods of analytic geometry.

Denoting by l, m, n the direction cosines of the normal to the surface at the point (x, y, z), and putting for shortness $\partial F/\partial x=F_x$, $\partial F/\partial y=F_y$, $\partial F/\partial z=F_z$, we have

$$dS=\frac{dydz}{l}=\frac{dzdx}{m}=\frac{dxdy}{n},$$

$$\frac{l}{F_x}=\frac{m}{F_y}=\frac{n}{F_z}=\frac{1}{\sqrt{F_x^2+F_y^2+F_z^2}}.$$

Hence, substituting

$$dS=dxdy\cdot\frac{\sqrt{F_x^2+F_y^2+F_z^2}}{F_z}$$

in the formulæ of Art. 15, we find

$$M=\int_{y_1}^{y_2}\int_{x_1}^{x_2}\rho dxdy\cdot\frac{\sqrt{F_x^2+F_y^2+F_z^2}}{F_z},$$

where the integration is to be extended over the projection of the portion of surface under consideration on the plane xy. The equation of the curve bounding this projection must be given: it determines the limits of integration. It is obvious how the formula has to be modified when the projection of the area on either of the other co-ordinate planes be given.

The expressions for $M\cdot\bar{x}$, $M\cdot\bar{y}$, $M\cdot\bar{z}$ differ from the above expression for M only in containing the additional factor x, y, z, respectively, under the integral sign.

44. If the equation of the surface be given in the form $z=f(x, y)$, as is frequently the case, we have

$$F(x, y, z)\equiv z-f(x, y);$$

hence with the usual Gaussian notation

$$\frac{\partial z}{\partial x} \equiv \frac{\partial f}{\partial x} \equiv p, \quad \frac{\partial z}{\partial y} \equiv \frac{\partial f}{\partial y} \equiv q,$$

$$F_x = -p, \quad F_y = -q, \quad F_z = 1,$$

which gives
$$M = \int_{y_1}^{y_2}\int_{x_1}^{x_2} \rho\sqrt{1+p^2+q^2}\, dxdy,$$

$$M \cdot \bar{x} = \int_{y_1}^{y_2}\int_{x_1}^{x_2} \rho x\sqrt{1+p^2+q^2}\, dxdy,$$

$$M \cdot \bar{y} = \int_{y_1}^{y_2}\int_{x_1}^{x_2} \rho y\sqrt{1+p^2+q^2}\, dxdy,$$

$$M \cdot \bar{z} = \int_{y_1}^{y_2}\int_{x_1}^{x_2} \rho z\sqrt{1+p^2+q^2}\, dxdy.$$

In the case of a homogeneous *spherical surface* $x^2+y^2+z^2=a^2$, we have $p=\partial z/\partial x=-x/z$, $q=\partial z/\partial y=-y/z$; hence $z\sqrt{1+p^2+q^2}=a$, so that the last of the above formulæ gives

$$S \cdot \bar{z} = a\int\int dxdy = a \cdot S_z,$$

where S is the area of the surface and S_z the area of its projection on the plane xy. The formula shows that the distance $\bar{z}$ of the centroid of any spherical area S from a plane passing through the centre is equal to the radius a multiplied by the ratio of the projection S_z of the area on the plane to the area itself.

45. We proceed to the methods of finding the centroids of volumes or solids.

Considerations of symmetry make it clear that the centroid of a **homogeneous parallelepiped** lies at the intersection of its diagonals; similarly, that of a **homogeneous prism** or **cylinder** coincides with the centroid of the area of its middle section (*i.e.* a plane section parallel to, and equally distant from, the bases).

46. For a **homogeneous pyramid** or **cone**, we have found in Art. 28 (*c*) that the centroid lies on the line joining the vertex to the centroid of the area of the base, at a distance equal to $\frac{1}{4}$ of this line from the base. This is, of course, easily shown directly by resolving the pyramid or cone into plane elements parallel to the base, in a manner analogous to that used for the triangular area in Art. 32.

47. It may, perhaps, be well to formally state the principal **laws of symmetry** for homogeneous solids, although they present themselves so naturally that they are used almost instinctively. For however simple and obvious these propositions may appear, the beginner may be led into error if he does not use them cautiously. The proof rests on the fundamental definition of the centroid as a point such that for any plane through it the sum of the moments is zero.

(a) *If the surface of the solid have a plane of symmetry*, i.e. a plane such that every line perpendicular to it intersects the surface in two points equidistant from the plane, *the centroid lies in this plane.* Hence, the centroid of a homogeneous solid is at once known if its surface possesses three planes of symmetry. If the surface has two planes of symmetry, the centroid lies on their line of intersection.

(b) *If the surface have an axis of symmetry*, i.e. a line such that every line perpendicular to it intersects the surface in two points equidistant from the line, *the centroid must lie on this axis.* Two axes of symmetry in the same homogeneous solid determine its centroid by their intersection.

(c) *If the surface have a centre*, i.e. a point such that every line through it intersects the surface in two points equidistant from it, *the centroid coincides with this centre.*

(d) *If the surface have a diametral plane*, i.e. a plane bisecting all chords that are parallel to a certain direction, *the centroid lies in this plane.*

48. Homogeneous spherical solids can be treated by a method analogous to that used for circular areas (see Art. 37). Thus a **homogeneous spherical sector** can be resolved into infinitesimal elements, each of which is a pyramid whose vertex lies at the centre of the sphere and whose base is an infinitesimal element of the spherical surface area of the sector. Such an element, regarded as a pyramid (Art. 46), has its centroid at the distance $\frac{3}{4}a$ from the centre, if a be the radius of the sphere. We may regard its mass as concentrated at its centroid and have thus the solid sector replaced by a homogeneous segment of a spherical area, of radius $\frac{3}{4}a$. It has been shown in Art. 41 that the centroid of such a segment bisects its height.

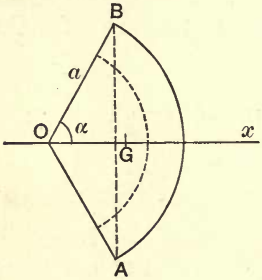

Fig. 12.

Let 2α be the angle at the vertex of the given sector (Fig. 12); then the height of the segment of radius $\frac{3}{4}a$ is $\frac{3}{4}a(1-\cos\alpha)$; hence the distance $\bar{x}$ of the centroid of the solid spherical sector from the centre is

$$\bar{x}=\tfrac{3}{4}a\cos\alpha+\tfrac{3}{8}a(1-\cos\alpha)=\tfrac{3}{8}a(1+\cos\alpha)=\tfrac{3}{4}a\cos^2\frac{\alpha}{2}.$$

49. In a **homogeneous solid of revolution** the centroid lies on the axis of revolution, since this line is an axis of symmetry (Art. 47 (*b*)). Taking this line as the axis of x, the equation of the surface of the solid is determined by that of the curve bounding the generating area, say $y=f(x)$.

We select as element the circular or ring-shaped plate of thickness dx contained between two sections of the solid at right angles to the axis of revolution (Fig. 11, p. 22). The centroid of each such element lies on the axis, and the volume of the element is $\pi(y_2^2-y_1^2)\,dx$, if y_1, y_2, are the ordinates of the curve corresponding to the same value of x.

We have, therefore,

$$\bar{x}=\frac{\int_{x_1}^{x_2} x\,(y_2^2-y_1^2)\,dx}{\int_{x_1}^{x_2}(y_2^2-y_1^2)\,dx},\quad \bar{y}=0,\quad \bar{z}=0.$$

It is easy to see how the formula has to be modified when only one value or more than two values of y correspond to a given value of x.

50. In the most general case of **any solid** whatever the formulæ of Art. 15 assume different forms according to the system of co-ordinates used. Thus for rectangular Cartesian co-ordinates the element of volume is $dv=dxdydz$, and we have:

$$M=\int\int\int\rho\,dxdydz,\quad M\cdot\bar{x}=\int\int\int\rho x\,dxdydz,$$

$$M\cdot\bar{y}=\int\int\int\rho y\,dxdydz,\quad M\cdot\bar{z}=\int\int\int\rho z\,dxdydz.$$

51. In polar co-ordinates, *i.e.* for the radius vector r, the co-latitude θ and the longitude ϕ (Fig. 10, p. 21), the element of volume is an infinitesimal rectangular parallelepiped having the concurrent edges dr, $rd\theta$, $r\sin\theta d\phi$; hence

$$dv=r^2\sin\theta drd\theta d\phi.$$

As $x=r\cos\theta$, $y=r\sin\theta\cos\phi$, $z=r\sin\theta\sin\phi$, the centroid is determined by the equations:

$$M=\int\int\int\rho r^2\sin\theta drd\theta d\phi,$$

$$M\cdot\bar{x}=\int\int\int\rho r^3\sin\theta\cos\theta\,drd\theta d\phi,$$

$$M\cdot\bar{y}=\int\int\int\rho r^3\sin^2\theta\cos\phi drd\theta d\phi,$$

$$M\cdot\bar{z}=\int\int\int\rho r^3\sin^2\theta\sin\phi drd\theta d\phi.$$

52. As an illustration let us determine the centroid of the volume $OABCD$ (Fig. 13), bounded by the three co-ordinate planes and the warped quadrilateral (hyperbolic paraboloid) $ABCD$. The latter is generated by the line LM gliding along AB and CD so as to remain parallel to the plane yz. The data are $OA = CD = a$, $OB = b$, $OC = AD = c$.

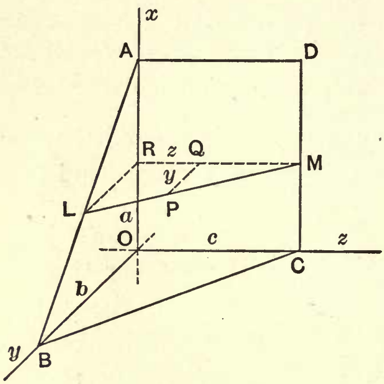

Fig. 13.

We take as element an infinitesimal prism PQ of base $dxdz$ and height y. From similar triangles we have $y/RL = (c-z)/c$, and $RL/b = (a-x)/a$; hence

$$y = b\frac{a-x}{a}\cdot\frac{c-z}{c}.$$

Thus we find, rejecting the constants which cancel in numerator and denominator,

$$\bar{x} = \frac{\int_0^a\int_0^c x(a-x)(c-z)dxdz}{\int_0^a\int_0^c (a-x)(c-z)dxdz} = \frac{\int_0^a x(a-x)\,dx\cdot\left(c^2-\frac{c^2}{2}\right)}{\int_0^a (a-x)\,dx\cdot\left(c^2-\frac{c^2}{2}\right)}$$

$$= \frac{\int_0^a x(a-x)\,dx}{\int_0^a (a-x)\,dx} = \frac{\frac{a^3}{2}-\frac{a^3}{3}}{a^2-\frac{a^2}{2}} = \frac{\frac{1}{6}}{\frac{1}{2}}a = \tfrac{1}{3}a;$$

$$\bar{y} = \frac{\frac{1}{2}\int_0^a\int_0^c b^2\frac{(a-x)^2}{a^2}\cdot\frac{(c-z)^2}{c^2}dxdz}{\int_0^a\int_0^c b\frac{a-x}{a}\cdot\frac{c-z}{c}dxdz} = \frac{b}{2ac}\,\frac{\int_0^a (a-x)^2dx\cdot\frac{1}{3}c^3}{\int_0^a (a-x)\,dx\cdot\frac{1}{2}c^2}$$

$$= \frac{1}{3}\cdot\frac{b}{a}\cdot\frac{\frac{a^3}{3}}{\frac{a^2}{2}} = \tfrac{2}{9}b.$$

Finally, $\bar{z} = \frac{1}{3}c$, by analogy with $\bar{x}$.

53. Exercises.

(1) Three beads of masses 3, 5, 12, are strung on a straight wire whose mass is neglected, the bead of mass 5 being midway between the other two. Find the centroid. (Take moments about the middle point.)

(2) Show that the centroid of three equal particles placed at the vertices of a triangle is at the intersection of the medians of the triangle.

(3) Show that the centroid of three masses m_1, m_2, m_3 situated at the vertices of a triangle and proportional to the opposite sides, is at the centre of the inscribed circle.

(4) Equal particles are placed at five of the six vertices of a regular hexagon. Find the distance of the centroid from the centre of figure.

(5) Find the centroid of a homogeneous triangular frame.

(6) Show that the centroid of a homogeneous semicircular wire lies at the distance $\frac{2}{\pi}r$ from the centre, r being the radius.

(7) Find the co-ordinates of the centroid of the arc of a quadrant of a circle by using the first proposition of Pappus (Art. 30).

(8) Find the centroid of a circular arc AB of angle $AOB = \alpha$, whose density varies as the length of the arc measured from A.

Find the centroids of the following homogeneous arcs of curves:

(9) Parabola $y^2 = 4ax$ from the vertex to the end of the latus rectum.

(10) Cycloid $x = a(\theta - \sin\theta)$, $y = a(1 - \cos\theta)$, from cusp to cusp.

(11) Half the cardioid $r = a(1 + \cos\theta)$.

(12) Catenary $y = \frac{c}{2}(e^{\frac{x}{c}} + e^{-\frac{x}{c}})$ between two points equally distant from the axis of x.

(13) Common helix: $x = r\cos\theta$, $y = r\sin\theta$, $z = kr\theta$, from $\theta = 0$ to $\theta = \theta$.

(14) The sides of a right-angled triangle are a and b. Find the distances of the centroid of the triangular area from the vertices.

(15) From a square $ABCD$ one corner EAF is cut off so that $AE = \frac{3}{4}a$, $AF = \frac{1}{4}a$, a being the side of the square. Find the centroid of the remaining area.

(16) In a trapezoid the parallel sides are a, b, the height is h, and one of the non-parallel sides is perpendicular to the parallel sides; show that the co-ordinates of the centroid with a as axis of x and the perpendicular side as axis of y are $\bar{x} = \frac{a^2 + ab + b^2}{3(a+b)}$, $\bar{y} = \frac{(a+2b) \cdot h}{3(a+b)}$.

(17) Find the centroid of the cross-section of a bar formed by placing four angle-irons with their edges together, two of the irons having the dimensions a, b, α, β, as in Fig. 8, Art. 36, while the other two have the dimension a different, say a'.

(18) Find the centroid of the cross-section of a U- iron, the length of the flanges being $a = 12$ in., that of the web $2b = 8$ in., and the thickness $\delta = 1$ in. Deduce the general formula for $\bar{x}$, and an approximate formula for a small δ, and compare the numerical results.

(19) In the cross-section of an unsymmetrical double T, the flanges are $2b = 12$ in., $2b' = 8$ in.; the web is $a = 10$ in.; and the thickness of each of the two channel-irons forming the bar is $\delta = 1$ in. throughout; find the centroid.

(20) In a T-iron the width of the flange is b, its thickness α; the depth of the web is a, its thickness β. Find the distance of the centroid from the outer side of the flange; give an approximate expression and investigate it for $a = b$, $\alpha = \beta = \frac{1}{5}a$.

(21) If one-fourth be cut away from a triangle by a parallel to the base, show that in the remaining area the centroid divides the median in the ratio 4 : 5.

(22) Prove that the centroid of any plane quadrilateral $ABCD$ coincides with that of the triangle ACF, if the point F be constructed by laying off $BF = DE$ on the diagonal BD, E being the intersection of the diagonals.

(23) The centroid of a homogeneous semicircular area of radius r lies at the distance $\bar{x} = \frac{4}{3\pi} r$ from the centre.

(24) The centroid of the area of a homogeneous circular segment of radius r subtending at the centre an angle 2α is at the distance $\bar{x} = \frac{2}{3} r \cdot \frac{\sin^3 \alpha}{\alpha - \sin\alpha \cos\alpha}$, or, $\bar{x} = \frac{1}{6} \frac{c^3}{rs - ch}$, if c is the chord, h its distance from the centre, and s the arc.

(25) A painter's palette is formed by cutting a small circle of radius b out of a circular disc of radius a, the distance between the centres being c. It is required to find the distance of the centroid of the remainder from the centre of the larger circle. (Routh.)

(26) The arch constructed of brick over a door is in the form of a quadrant of a circular ring. The door is 5 ft. wide; $1\frac{1}{2}$ lengths of brick are used (say 12 in.). Find the centroid of the arch.

Find the co-ordinates of the centroid for the following plane areas:

(27) Area bounded by the parabola $y^2 = 4ax$, the axis of x, and the ordinate y.

(28) Area bounded by the curve $y = \sin x$ from $x = 0$ to $x = \pi$ and the axis of x.

(29) Quadrant of an ellipse.

(30) Elliptic segment bounded by the chord joining the ends of the major and minor axes.

(31) Show, by Art. 28, that the centroid of the surface of a right circular cone lies at a distance from the base equal to one-third of the height.

(32) Find the centroid of the portion of the surface of a right circular cone cut out by two planes through the axis inclined at an angle ϕ.

(33) Find the centroid of the area of the earth's surface contained between the tropic of Cancer (latitude $= 23° 28'$) and the arctic circle (polar distance $= 23° 28'$).

(34) Regarding the earth as a homogeneous sphere of density $\rho = 5.5$, how much would its centroid be displaced by superimposing over the area bounded by the arctic circle an ice-cap of a uniform thickness of 10 miles?

(35) A bowl in the form of a hemisphere is closed by a circular lid of a material whose density is three times that of the bowl. Find the centroid.

(36) Determine the centroid of a homogeneous solid hemisphere.

(37) Find the centroid of a frustum of a cone, the radii of the bases being r_1, r_2; the height of the frustum, h.

(38) Show that the formula for the frustum of the cone applies likewise to the frustum of any pyramid of the same height h if r_1, r_2 are any two homologous linear dimensions of the two bases.

(39) Find the centroid of a solid segment of a sphere of radius a, the height of the segment being h.

(40) Show that, both for a triangular area and for a tetrahedral volume, the distance of the centroid from any plane is the arithmetic mean of the distances of the vertices from the same plane.

(41) Find the centroid of the paraboloid of revolution of height h, generated by the complete revolution of the parabola $y^2 = 4ax$ about its axis.

(42) The area bounded by the parabola $y^2 = 4ax$, the axis of x, and the ordinate $y = y_1$, revolves about the tangent at the vertex. Find the centroid of the solid of revolution so generated.

(43) The same area as in problem (42) revolves about the ordinate y_1. Find the centroid.

(44) Find the centroid of an octant of an ellipsoid $x^2/a^2 + y^2/b^2 + z^2/c^2 = 1$.

(45) The equations of the common cycloid referred to a cusp as origin and the base as axis of x are $x = a(\theta - \sin\theta)$, $y = a(1 - \cos\theta)$. Find the centroid: (a) of the arc of the semi-cycloid (*i.e.* from cusp to vertex); (b) of the plane area included between the semi-cycloid and the base; (c) of the surface generated by the revolution of the semi-cycloid about the base; (d) of the volume generated in the same case; (e) of the surface generated by the revolution of the whole cycloid (from cusp to cusp) about its axis, *i.e.* the line through the vertex at right angles to the base; (f) of the volume so generated.

(46) Find the centroid of a solid hemisphere whose density varies as the nth power of the distance from the centre.

(47) From out of the right cone ABC a cone ABD is cut of the same base and axis, but of smaller height. Find the centroid of the remaining solid.

(48) A triangle ABC, whose sides are a, b, c, revolves about an axis situated in its plane. Find the surface area and volume of the solid so generated, if p, q, r are the distances of A, B, C from the axis.

(49) "Water is poured gently into a cylindrical cup of uniform thickness and density. Prove that the locus of the centre of gravity of the water, the cup, and its handle is a hyperbola." (Routh.)

(50) Prove that the volume of a truncated right cylinder (*i.e.* a right cylinder cut by a plane inclined at any angle to its base) is equal to the product of the area of its base into the height of the truncated cylinder at the centroid of its base.

(51) Prove that the volume of a doubly truncated cylinder is equal to the product of the area of the section at right angles to the axis into the distance of the centroids of the bases.

54. For the theory of moments and centres of mass the student is referred to W. SCHELL, *Theorie der Bewegung und der Kräfte*, Leipzig, Teubner, Vol. I., 1879, pp. 81–100; E. J. ROUTH, *Analytical statics*, Cambridge, University Press, Vol. I., 1891, pp. 270–314; J. SOMOFF, *Theoretische Mechanik*, übersetzt von A. Ziwet, Leipzig, Teubner, Vol. II., 1879, pp. 1–72. For problems see in particular W. WALTON, *Problems in illustration of the principles of theoretical mechanics*, Cambridge, Deighton, 1876, pp. 1–45; M. JULLIEN, *Problèmes de mécanique rationnelle*, Paris, Gauthier-Villars, Vol. I., 1866, pp. 1–46; F. KRAFT, *Probleme der analytischen Mechanik*, Stuttgart, Metzler, Vol. I., 1884, pp. 527–617. Compare, also, B. PRICE, *Infinitesimal calculus*, Oxford, Clarendon Press, Vol. III., 1868, pp. 163–206; MOIGNO, *Leçons de mécanique analytique, Statique*, Paris, Gauthier-Villars, 1868, pp. 106–206; G. MINCHIN, *Treatise on statics*, Oxford, Clarendon Press, Vol. I., 1884, pp. 261–305; I. TODHUNTER, *Analytical statics*, edited by J. D. Everett, London, Macmillan, 1887, pp. 115–189; W. WALTON, *Problems in elementary mechanics*, London, Bell, 1880, pp. 56–78; and for geometrical methods, the works on graphical statics.

II. *Momentum; Force; Energy.*

55. Let us consider a point moving with constant acceleration from rest in a straight line. We know from *Kinematics* (Art. 111) that its motion is determined by the equations

$$v = jt, \quad s = \tfrac{1}{2}jt^2, \quad \tfrac{1}{2}v^2 = js, \tag{1}$$

where s is the distance passed over in the time t, v the velocity, and j the acceleration at the time t.

If, now, for the single point we substitute an m-tuple point, *i.e.* if we endow our point with the mass m, and thus make it a *particle* (see Art. 6), the equations (1) must be multiplied by m, and we obtain

$$mv = mjt, \quad ms = \tfrac{1}{2}mjt^2, \quad \tfrac{1}{2}mv^2 = mjs. \tag{2}$$

The quantities mv, mj, $\frac{1}{2}mv^2$ occurring in these equations have received special names because they correspond to certain physical conceptions of great importance.

56. *The product* mv *of the mass* m *of a particle into its velocity* v is called the **momentum**, or the **quantity of motion**, of the particle.

57. In observing the behaviour of a physical body in motion, we notice that the effect it produces — for instance, when impinging on another body, or more generally, whenever its velocity is changed — depends not only on its velocity, but also on its mass. Familiar examples are the following: a loaded railroad car is not so easily stopped as an empty one; the destructive effect of a cannon-ball depends both on its velocity and on its mass; the larger a fly-wheel, the more difficult is it to give it a certain velocity; etc.

It is from experiences of this kind that the physical idea of mass is derived.

The fact that any change of motion in a physical body is affected by its mass is sometimes ascribed to the so-called "*inertia*," or "force of inertia," of matter, which means, however, nothing else but the property of possessing mass.

58. Momentum, being by definition (Art. 56) the product of mass and velocity, has for its *dimensions* (see *Kinematics*, Art. 92)

$$\mathbf{MV} = \mathbf{MLT^{-1}}.$$

The *unit of momentum* is the momentum of the unit of mass having the unit of velocity.

Thus in the C.G.S. system the unit of momentum is the momentum of a particle of 1 gramme moving with a velocity of 1 cm. per second. There is no generally accepted name for this unit, although the name **bole** was proposed by the Committee of the British Association.

In the F.P.S. system, the unit is the momentum of a particle of one pound mass moving with a velocity of 1 ft. per second.

To find the relations between these two units, let there be x C.G.S. units in the F.P.S. unit; then

$$x \cdot \frac{\text{gm. cm.}}{\text{sec.}} = 1 \cdot \frac{\text{lb. ft.}}{\text{sec.}};$$

hence

$$x = \frac{\text{lb.}}{\text{gm.}} \cdot \frac{\text{ft.}}{\text{cm.}},$$

or, by Art. 3 and *Kinematics*, Art. 14,

$$x = 453.59 \times 30.48 = 13\,825.3;$$

i.e. 1 F.P.S. unit of momentum $= 13\,825.3$ C.G.S. units, and 1 C.G.S. unit $= 0.000\,072\,331$ F.P.S. units.

59. Exercises.

(1) What is the momentum of a cannon-ball weighing 200 lbs. when moving with a velocity of 1500 ft. per second?

(2) With what velocity must a railroad-truck weighing 3 tons move to have the same momentum as the cannon-ball in Ex. (1)?

(3) Determine the momentum of a one-ton ram after falling through 20 feet.

60. *The product* mj *of the mass* m *of a particle into its acceleration* j is called **force.** Denoting it by F, we may write our equations (2) in the form

$$mv = Ft, \quad s = \frac{1}{2}\frac{F}{m}t^2, \quad \frac{1}{2}mv^2 = Fs. \tag{3}$$

As long as the velocity of a particle of constant mass remains constant, its momentum remains unchanged. If the velocity changes uniformly from the value v at the time t to v' at the time t', the corresponding change of momentum is

$$mv' - mv = mjt' - mjt = F(t' - t); \tag{4}$$

hence

$$F = \frac{mv' - mv}{t' - t}. \tag{5}$$

Here the acceleration, and hence the force, was assumed constant. If F be variable, we have in the limit when $t' - t$ becomes dt,

$$F = \frac{d(mv)}{dt} = m\frac{dv}{dt}. \tag{6}$$

Instead of defining force as the product of mass and acceleration, we may therefore define it as the *rate of change of momentum with the time.*

61. Integrating equation (6), we find

$$\int_t^{t'} F dt = mv' - mv. \tag{7}$$

The product $F(t' - t)$ *of a constant force into the time* $t' - t$ *during which it acts,* and in the case of a variable force, *the time-integral* $\int_t^{t'} F dt$, is called the **impulse** of the force during this time.

It appears from the equations (4) and (7) that *the impulse of a force during a given time is equal to the change of momentum during that time.*

62. The idea of force is no doubt primarily derived from the sensation produced in a person by the exertion of his "muscular force."

Like the sensations of light, sound, heat, etc., the sensation of exerting force is capable, in a rough way, of measurement. But the physiological and psychological phenomena attending the exertion of muscular force when analysed more carefully are very complicated.

In ordinary language the term "force" is applied in a great variety of meanings. For scientific purposes it is of course necessary to attach a single definite meaning to it.

63. In physics it is customary to speak of force as *producing* or *generating velocity*, and to define force as the *cause of acceleration*. Thus observation shows that the velocity of a falling body increases during the fall; the cause of the observed change in the velocity, *i.e.* of the acceleration, is called the force of attraction, and is supposed to be exerted by the earth. Again, a body falling in the air, or in some other medium, is observed to increase its velocity less rapidly than a body falling *in vacuo*; a force of resistance is therefore ascribed to the medium as the cause of this change. In a similar way we speak of the expansive force of steam, of electric and magnetic forces, etc., because all these agencies produce changes of velocity.

Now, any change in the velocity v of a body of given mass m implies a change in its momentum mv; and it is this change of momentum, or rather the rate at which the momentum changes with the time, which is of prime importance in all the applications of mechanics. It is therefore convenient to have a special name for this rate of change, and that is what is called *force*.

It is, however, well to remember that in using this term "force," it is not intended to assert anything as to the objective reality or actual nature of force and matter in the ordinary acceptation of these terms. Our knowledge comes to us through our sense-impressions, and these would all seem to reduce finally to changes of motion and changes of momentum: these alone we can perceive directly.

64. The definition of force (Art. 60) as the product of mass and acceleration gives the *dimensions* of force as

$$\mathbf{F} = \mathbf{MJ} = \mathbf{MLT}^{-2}.$$

The *unit of force* is therefore the force of a particle of unit mass moving with unit acceleration.

Hence, in the C.G.S. system, it is the force of a particle of

1 gramme moving with an acceleration of 1 cm. per second per second. This unit force is called a **dyne.**

The definition is sometimes expressed in a slightly different form.* We may say the dyne is the force which, acting on a gramme uniformly for one second, would generate in it a velocity of 1 cm. per second; or would give it the C.G.S. unit of acceleration; or it is the force which, acting on *any* mass uniformly for one second, would produce in it the C.G.S. unit of momentum.

That these various statements mean the same thing follows from the fundamental formulæ $F=mj$, $j=vt$, if F, m, t, v, j be expressed in C.G.S. units.

65. In the F.P.S. system, the unit of force is the force of a mass of 1 lb. moving with an acceleration of 1 ft. per second per second. It is called the **poundal.**

66. The dyne and the poundal are called the **absolute**, or scientific, units of force.

To find the relation between these two units, let x be the number of dynes in the poundal; then we have

$$x \cdot \frac{\text{gm. cm.}}{\text{sec.}^2} = 1 \cdot \frac{\text{lb. ft.}}{\text{sec.}^2};$$

hence, just as in Art. 58,

$$x = 13\,825.3;$$

i.e. 1 poundal = 13 825.3 dynes, and 1 dyne = 0.000 072 331 poundals.

67. Another system of measuring force, the so-called **gravitation** (or engineering) system, is in very common use, and must here be explained.

Among the forces of nature the most common is the *force of gravity*, or the *weight*, *i.e.* the force with which any physical body is attracted by the earth. As we have convenient and

---

* J. D. Everett, *C.G.S. system of units*, 1891, p. 23, 24.

accurate appliances for comparing the weights of different bodies at the same place, the idea suggests itself of selecting as unit force the weight of a certain standard mass.

In the metric gravitation system the *weight of a kilogramme* has been selected as unit force; in the British gravitation system, the *weight of a pound* is the unit force.

68. There are two serious objections to the gravitation system of measuring force, one of a practical nature, the other theoretical. The former is that the words "kilogramme" and "pound" are thus used in two different meanings, sometimes, and more correctly, as denoting a mass, sometimes as denoting a force. Wherever an ambiguity might arise from this double use, the word "mass" or "weight" must be added.

The other objection is more serious. The weight of a body, and hence the gravitation unit of force, is not a constant quantity; it changes from place to place as it depends on the value of g, the acceleration of gravity.

For, the weight W of any mass m being the force with which this mass is attracted by the earth, we have

$$W = mg,$$

where g is the acceleration produced by the earth's attraction. Now it is known from experiment that this acceleration varies from place to place; according to the law of gravitation, it is inversely proportional to the square of the distance from the centre of the earth.

The weight of a body is therefore a meaningless term unless the place be specified where the body is situated, and the value of g at that place be given.

It is true, however, that the value of g for different points on the earth's surface varies but little, so that for most practical purposes the gravitation system is accurate enough.

In the equations of theoretical dynamics, in particular in kinetics, the use of absolute units is always understood. In statics, however, where we are mainly concerned with the *ratios* of forces and not with their absolute values, gravitation units will generally be used in the present work in view of the practical applications.

69. The numerical relation between the absolute and gravitation measures of force is expressed by the equations

1 kilogramme (force) = 1000 g dynes,
1 pound (force) = g poundals,

where g is about 981 in metric units, and about 32.2 in British units. In most cases the more convenient values 980 and 32 may be used.

70. Exercises.

(1) What is the exact meaning of "a force of 10 tons"? Express this force in poundals and in dynes.

(2) Reduce 2 000 000 dynes to British gravitation measure.

(3) Express a pressure of 2 lbs. per square inch in kilogrammes per square centimetre.

(4) Prove that a poundal is very nearly half an ounce, and a dyne a little over a milligramme, in gravitation measure.

(5) The numerical value of a force being 100 in (absolute) F.P.S. units, find its value for the yard as unit of length, the ton as unit of mass, and the minute as unit of time (see Art. 66).

71. *The quantity* $\frac{1}{2}mv^2$, i.e. *half the product of the mass of a particle into the square of its velocity*, is called the **kinetic energy** of the particle.

Let us consider again a particle of constant mass m moving with a constant acceleration, and hence with a constant force; let v be the velocity, s the space described at the time t; v', s' the corresponding values at the time t'. Then the last of the three fundamental equations (see Arts. 55 and 60) gives

$$\tfrac{1}{2}mv'^2 - \tfrac{1}{2}mv^2 = F(s' - s); \qquad (8)$$

hence

$$F = \frac{\tfrac{1}{2}mv'^2 - \tfrac{1}{2}mv^2}{s' - s}. \qquad (9)$$

If F be variable, we have in the limit

$$F = \frac{d\left(\tfrac{1}{2}mv^2\right)}{ds} = mv\frac{dv}{ds}. \qquad (10)$$

Force can therefore be defined as *the rate at which the kinetic energy changes with the space.* (Compare the end of Art. 60.)

72. Integrating the last equation (10), we find

$$\int_s^{s'} Fds = \tfrac{1}{2}mv'^2 - \tfrac{1}{2}mv^2. \tag{11}$$

The product F (s′ − s) *of a constant force* F *into the space* s′ − s *described in the direction of the force*, and in the case of a variable force, *the space-integral* $\int_s^{s'} Fds$, is called the **work** of the force for this space.

The equations (8) and (11) show that *the work of a force is equal to the corresponding change of the kinetic energy.*

We have here assumed that the force acts in the direction of motion of the particle. A more general definition of work including the above as a special case will be given later (Art. 232 *sq.*).

The ideas of energy and work have attained the highest importance in mechanics and mathematical physics within comparatively recent times. Their full discussion belongs to Kinetics.

73. According to their definitions, both *momentum* (Art. 56) and *force* (Art. 60) may be regarded mathematically as mere numerical multiples of velocity and acceleration, respectively. They are therefore so-called vector-quantities; *i.e.* a momentum as well as a force can be represented geometrically by a segment of a straight line of definite length, direction, and sense. Moreover, as they are referred to a particular point, viz. to the point whose mass is *m*, the line representing a momentum or a force must be drawn through this point; the line has therefore not only direction, but also position; *i.e. a momentum as well as a force is represented geometrically by a* **rotor** (compare *Kinematics,* Arts. 57, 68, 291 *sq.*).

It follows that concurrent forces, for instance, can be compounded by geometrical addition, as will be explained more fully in Chapter IV.

On the other hand, kinetic energy and work are not vector-quantities.

74. The ideas of momentum, force, energy, work, with the fundamental equations connecting them, as given in the preceding articles, form the groundwork of the whole science of theoretical dynamics. The application of this science to the interpretation of natural phenomena gives results in exact agreement with observation and experiment. It is therefore important to inquire what are the physical assumptions and experimental data on which this application of dynamics is based.

These assumptions were formulated with remarkable clearness by Sir Isaac Newton in his *Philosophiæ naturalis principia mathematica*, first published in 1687, and have since been known as **Newton's laws of motion.** As these three *axiomata sive leges motus*, as Newton terms them, are very often referred to and, at least by English writers on dynamics, are usually laid down as the foundation of the science,* they are given here in a literal translation:

I. Every body persists in its state of rest or of uniform motion along a straight line, except in so far as it is compelled by impressed (*i.e.* external) forces to change that state.

II. Change of motion is proportional to the impressed moving force and takes place along the straight line in which that force acts.

III. To every action there is an equal and contrary reaction; or, the mutual actions of two bodies on one another are always equal and directed in contrary senses.

75. Some explanation is necessary to correctly understand the meaning of these laws; indeed, Newton's laws should not be studied by themselves. They become intelligible only if taken in connection with the definitions preceding them in the *Principia*, and with the explanations and corollaries that Newton himself has appended to them.

The word "body" must be taken to mean particle; the word "motion" in the second law means what is now called momentum.

All three laws imply the idea of *force as the cause of any change of momentum* in a particle.

---

* See the *Syllabus of elementary dynamics*, Part I., London, Macmillan, 1890, p. 13 *sq.*, prepared by the Association for the Improvement of Geometrical Teaching.

76. With this definition of force the first law, at least in the ordinary form of statement, for a single particle, merely states that where there is no cause there is no effect. While this law may appear superfluous to us, it was not so in the time of Newton. Kepler and Galilei, less than a century before Newton, were the first to insist more or less clearly on this so-called **law of inertia**, viz. that there is no intrinsic power or tendency in moving matter to come to rest or to change its motion in any way.

77. The second law gives as the measure of a constant force the amount of momentum generated in a given time (see Art. 60); it can be called the **law of force.** If force be defined as the cause of any change of momentum, the second law follows naturally by assuming, as is always done, that the effect is proportional to the cause.

The first two laws may thus be regarded from the mathematical point of view as nothing but a definition of force; but they are certainly meant to emphasize the physical fact that the assumed definition of force is not arbitrary, but based on the characteristics of motion as observed in nature.

In the corollaries to his laws Newton shows how the composition and resolution of forces by the parallelogram rule follows from his definition. In deriving this result he tacitly assumes that the action of any force on a particle takes place independently of the action of any other forces that may be acting on the particle at the same time, a principle that would seem to deserve explicit statement. Some writers on mechanics, in particular French authors, prefer to replace Newton's second law by this *principle of the independence of the action of forces.*

78. The third law expresses the physical fact that in nature all forces occur in pairs of equal and opposite forces. In modern phraseology, two such equal and opposite forces in the same line are said to constitute a *stress.* Newton's third law is therefore called the **law of stress.**

This law, which was first clearly conceived in Newton's time, involves what may be regarded as the second fundamental property of matter or mass (the first being its indestructibility); viz. that *any two particles of matter determine in each other oppositely directed accelerations along the line joining them.*

79. For a more complete discussion of the physical laws underlying the applications of theoretical mechanics, the student is referred to THOMSON and TAIT, *Natural philosophy*, London, Macmillan, 1879, Part I., Chapter II., p. 219 *sq.;* E. MACH, *Die Mechanik in ihrer Entwickelung*, Leipzig, Brockhaus, 1889, p. 203; K. PEARSON, *The grammar of science*, London, Scott, 1892, p. 357 *sq.;* J. D. EVERETT, *C.G.S. system of units*, London, Macmillan, 1891, p. 73; P. G. TAIT, article, "Mechanics," in the *Encyclopædia Britannica*, 9th ed.; J. CLERK MAXWELL, *Matter and Motion*, New York, Van Nostrand, 1878; P. G. TAIT, *Properties of matter*, Edinburgh, Black, 1885.

CHAPTER IV.

STATICS.

I. *Introduction.*

80. When a particle has two equal and opposite accelerations j, $-j$, its motion will not be changed. The same result must follow when a particle is acted on by two equal and opposite forces $F = mj$, $F' = -mj$. Their combined effect on the particle is *nil*, so that the particle, if originally at rest, will remain at rest; if originally moving with constant velocity in a straight line, it will continue to do so; and if originally moving under the action of any other forces in any way whatever, the introduction of the two equal and opposite forces will have no effect on its motion.

We say that two equal and opposite forces acting on a particle *balance*, or *are equivalent to* 0, or *are in equilibrium*. If no other forces act on the particle, the particle itself is said to be in equilibrium. It must be kept in mind that equilibrium is not synonymous with rest.

81. Let us next consider *any* two forces F_1, F_2 acting simultaneously on the same particle P of mass m, and let j_1, j_2 be the accelerations produced by these forces so that

$$F_1 = mj_1, \quad F_2 = mj_2.$$

The resultant acceleration of the particle is found by geometrically adding the vectors j_1, j_2; let j be their geometric sum. Then the force

$$F = mj$$

producing the resultant acceleration is called the **resultant** of the forces F_1, F_2; these, or any other two or more forces having the same resultant F, are called the **components** of F.

82. In many investigations we are not so much concerned with the actual accelerations produced as with the effects that *might be* produced by any particular force or system of forces if the particle or body were perfectly free to move, *i.e.* not subject to other forces or restraints.

We proceed to study the composition and resolution of forces from this point of view, *i.e.* without reference to the accelerations produced, but with particular attention to the conditions under which the given system of forces is in equilibrium. This study forms the subject of **Statics.**

83. The geometrical characteristics of a force are (*a*) its *line* of action, (*b*) its *magnitude* or intensity, (*c*) its *sense.* Properly speaking, two forces should be called equal only when they agree in these three characteristics. But it is customary to call two forces *equal* even when they have only equal magnitude; we shall call them *geometrically equal*, when they agree in all three characteristics.

84. A force acting on a particle P is said to have its *point of application* at P, and the line representing it is usually drawn from P as origin. But the point of application is not an

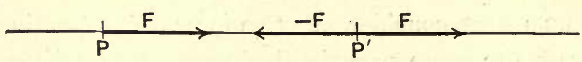

Fig. 14.

essential characteristic of the force; it may be taken at any point of its line if this line be regarded as rigid. Thus the force F acting on the particle P (Fig. 14) can be transferred, without changing its effect, to any point P' of its line; and two equal and opposite forces in the same line, such as F at P and

$-F$ at P', are in equilibrium; provided always that P and P' may be regarded as belonging to the same rigid body.

85. It follows from Arts. 81 and 84 that any two forces F_1 F_2 whose lines intersect, say at O' (Fig. 15), are equivalent to,

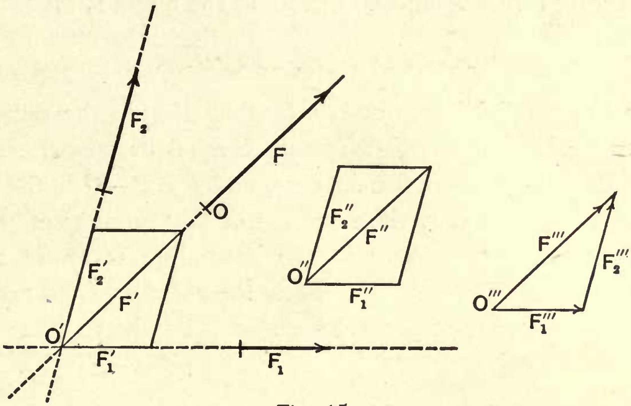

Fig. 15.

i.e. can be replaced by, a single force F called their *resultant.* This resultant can be found by replacing the forces F_1, F_2 by the equal forces F_1', F_2' at O', and forming the parallelogram having F_1', F_2' as adjacent sides. The diagonal F' through O' is the required resultant; it can be replaced by any force F of equal length and sense in the same line with this diagonal.

The parallelogram construction need not be made at O'; we may select any origin O'' (Fig. 15), draw through it two vectors F_1'', F_2'' equal (in direction, length, and sense) to F_1, F_2, find the diagonal F'' through O'', and transfer it to a parallel line drawn through O'.

Finally, it is not necessary to draw the whole parallelogram; we have only to add the vectors F_1, F_2 geometrically from any origin O''' (Fig. 15) and transfer their sum F''' to the parallel through O'.

86. Conversely, any force may be resolved into two components along any two lines intersecting the line of the force at the same point and lying in the same plane with it. These components are together equivalent to the force, *i.e.* they may be substituted for the force.

87. It follows from Art. 85 that the resultant R of two intersecting forces P and Q, including the angle θ, is

$$R=\sqrt{P^2+Q^2+2PQ\cos\theta}.$$

For two parallel forces or two forces acting in the same line, $\theta=0$ or $180°$, according as they are of equal or opposite sense; hence $R=P+Q$ in the former case, and $R=P-Q$ in the latter. It is also apparent that the resultant of any number of parallel forces or of forces acting in the same line is found as the algebraic sum of these forces. How the *position* of the resultant is found in the case of parallel forces will be shown later (Arts. 104, 106).

88. By Art. 86, to resolve a force R (Fig. 16) into two components P, Q along two lines making the angles α, β with the line of R, we have only to draw through the ends of a vector

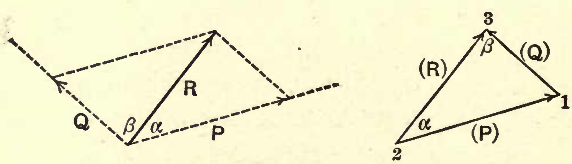

Fig. 16.

$2\,3=R$ lines $2\,1$, $3\,1$ making angles α, β with $2\,3$; then $2\,1=P$, $1\,3=Q$. The triangle $1\,2\,3$ gives the relations

$$\frac{P}{\sin\beta}=\frac{Q}{\sin\alpha}=\frac{R}{\sin(\alpha+\beta)}.$$

When the components are at right angles, we have $P=R\cos\alpha$, $Q=R\sin\alpha$.

89. The projection of a closed polygon on any line being evidently zero, and the resultant being by definition the geometric sum of its components, it follows that *the projection of the resultant on any line equals the algebraic sum of the projections of its components.* This proposition is sometimes expressed in the following form: the resolved part of the resultant in any direction is equal to the algebraic sum of the resolved parts of the components.

Let l be the line on which we project (Fig. 17), and let (l, R) (l, P), (l, Q) denote the angles it makes with the resultant R and the components P, Q, respectively; then

$$R \cos (l, R) = P \cos (l, P) + Q \cos (l, Q).$$

90. Varignon's Theorem. Multiplying the last equation by any length $OS = s$ taken through the initial point O of R and at right angles to l, we obtain

$$R \cdot s \cos (l, R) = P \cdot s \cos (l, P) + Q \cdot s \cos (l, Q),$$

or since $s \cos (l, R) = r$, $s \cos (l, P) = p$, $s \cos (l, Q) = q$, where r, p, q are the perpendiculars let fall from S, on R, P, Q, respectively,

$$Rr = Pp + Qq.$$

In this form the proposition is independent of the direction of the line l and holds for any point S in the plane of the parallelogram.

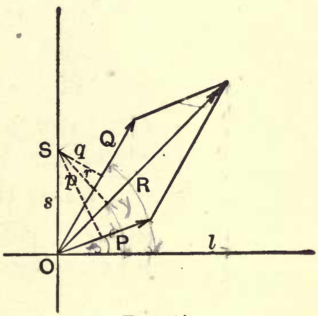

Fig. 17.

91. Moment of a Force. The product of a force into its perpendicular distance from a point is called the **moment** of the force about the point. It is taken with the positive or negative sign according as the force as seen from the point is directed counter-clockwise or clockwise.

The proposition of Art. 90, $Pp + Qq = Rr$, can now be stated in the following form: *the algebraic sum of the moments of any two intersecting forces about any point in their plane is equal to the moment of their resultant about the same point.*

92. The product Rr represents twice the area of the triangle having R for its base and S for its vertex; Pp, Qq can be interpreted similarly. This remark leads to another simple proof of Varignon's theorem, which may serve to make its meaning better understood. With the notation of Fig. 18 we have

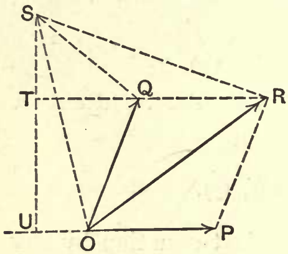

Fig. 18.

$$SOR = SOQ + SQR + QOR,$$

or $$R \cdot r = Q \cdot q + P \cdot ST + P \cdot TU;$$

or since $$ST + TU = SU = p,$$

$$Rr = Qq + Pp.$$

93. If the point S be taken on the resultant R, we have $r = 0$, hence $Pp = -Qq$; i.e. *the sum of the moments of two forces about any point on their resultant is zero.*

94. The forces of nature receive various special names according to the circumstances under which they occur. Thus the **weight** of a mass has already been defined (Art. 67) as the force with which the mass is attracted by the mass of the earth.

When a string carrying a mass at one end is suspended with its other end from a fixed point, it will be *stretched, i.e.* subjected to a certain tension. This means that if the string were cut it would require the application of a force along the line of the string to keep the weight in equilibrium. This force, which may thus serve to replace the action of the string, is called its **tension.**

When the surfaces of two physical bodies A, B are in contact, a **pressure** may exist between them; that is, if one of the bodies, say B, be removed, it may require the introduction of a force to keep A in the same state of rest or motion that it had before the removal of B. This force, which will obviously act along the common normal of the surfaces at the point of contact, is called the **resistance** of B, and a force equal and opposite to it is called the **pressure** exerted by A on B.

95. Exercises.

(1) Find the resultant of two equal forces acting at right angles to each other.

(2) Show that the resultant R of two equal forces P including an angle θ is $R = 2P\cos(\theta/2)$.

(3) If the resultant of two equal forces P be equal to P, what is the angle between the components?

(4) Find the magnitude and direction of the resultant of two forces of 100 and 200 lbs., including an angle of 60°.

(5) Let R be the effective piston pressure of a steam engine and ϕ the angle between the direction of motion of the piston and the connecting rod at any moment; show that the thrust in the connecting rod is $R \sec\phi$ and the pressure on the guide-bars $R \tan\phi$. For what position of the crank is the pressure on the guides greatest?

(6) A weight W is suspended from two fixed points A, B by means of a string ACB, C being the point of the string where the weight W is attached. If AC, BC be inclined to the vertical at angles α, β, find the tensions in AC, BC: (*a*) analytically; (*b*) graphically.

(7) Resolve a force of 20 lbs. into two components making angles of 45° and 30° with the given force: (*a*) analytically; (*b*) graphically.

(8) Find the rectangular components of a force P if one of the components is to make an angle of 30° with P.

(9) The resultant R, one of the components P, and the angle between the two components, $\theta = 60°$, being given, find the other component Q.

(10) A particle is acted on by two forces P, Q lying in the same vertical plane and inclined to the horizon at angles p, q. Find their resultant in magnitude and direction, if $P = 527$ lbs., $Q = 272$ lbs., $p = 127° \, 52'$, $q = 32° \, 13'$.

(11) Prove that the moments of the two components of a force about any point on the line of the force are equal and opposite.

(12) Two forces acting on a point are represented in magnitude and direction by the tangent and normal of a parabola passing through the point. Find their resultant, and show that it passes through the focus of the parabola.

(13) The magnitudes of two forces acting on a point are as 2 to 3. If their resultant be equal to their arithmetic mean, what is the angle between the forces?

(14) What is the angle between a force of 1 ton and a force of $\sqrt{3}$ tons if their resultant is 2 tons?

(15) A string with equal weights W attached to its ends is hung over two smooth pegs A, B fixed in a vertical wall. Find the pressure on the pegs: (*a*) when the line AB is horizontal; (*b*) when it is inclined to the horizon at an angle θ. The weight of the string, its extensibility and stiffness, and the friction on the pegs are neglected in this problem as well as in those immediately following.

(16) The string being hung over three pegs A, B, C, determine graphically the pressures on the pegs. Let the vertical line through B lie between the vertical lines drawn through A and C; there will be a pressure on B only if B lies above the line AC. If B lies below AC, the pressure may be distributed over the three pegs by passing the string around the peg B from below.

(17) In Ex. (15), for what position of the line AB are the pressures equal?

(18) In Ex. (16), let AC be horizontal, and let α, β, γ denote the angles of the triangle ABC. What are the pressures on the pegs?

(19) In Ex. (18), what must be the position of B to make the pressures on the three pegs equal: (*a*) when B lies above AC; (*b*) when B lies below AC?

(20) If the string with the equal weights W attached to its ends be strung over any number of pegs, the pressures on the pegs are readily determined, either graphically or analytically, in magnitude and direction; these pressures depend only on the value of W and on the angles between the successive sides of the polygon formed by the string, but not on the distances between the pegs.

(21) Suppose the string be closed, its ends being fastened together. Let this string be hung over three pegs A, B, C forming an isosceles triangle in a vertical plane with its base AC horizontal, and let a weight

W be suspended from the lowest point D of the string. If $AC = 4$ ft., $AB = BC = 2.5$ ft., and the length of the string $2l = 14$ ft., find the tension of the string and the pressures on the pegs.

(22) If, in Ex. (21), the triangles ABC and ADC be equilateral, what would be the tension and the pressures on the pegs?

(23) In Ex. (21), the triangles ABC and ADC being isosceles and their common base AC horizontal, what must be the relation between the angles 2β at B and 2δ at D to make the pressures on the three pegs A, B, C equal? The pressures being made equal, what angle gives the least pressure?

(24) Show, both analytically and geometrically, that a force whose components P_1, P_2 make an angle θ can be resolved into two rectangular components $(P_1 + P_2)\cos(\theta/2)$, $(P_1 - P_2)\sin(\theta/2)$.

(25) In the toggle-joint press two equal rods CA, CB are hinged at C; a force F bisecting the angle 2α between the rods forces the ends A, B apart. If A be fixed, find the pressure exerted at B at right angles to F if $F = 100$ lbs. and $\alpha = 15°, 35°, 65°, 85°, 90°$.

II. *Concurrent Forces.*

96. Let there be given any number n of forces F_1, F_2, F_3, ..., F_n, whose directions all pass through the same point. By Art. 85, we can find the resultant R_1 of F_1 and F_2, next the resultant R_2 of R_1 and F_3, then the resultant R_3 of R_2 and F_4, and so on. The resultant R of R_{n-2} and F_n is evidently equivalent to the whole system F_1, F_2, F_3, ..., F_n, and is called its **resultant.** We thus have the proposition that *a system consisting of any number of concurrent forces is equivalent to a single resultant.*

97. It may of course happen that this resultant is zero. In this case, the system is said to be *in equilibrium. The condition of equilibrium of a system of concurrent forces* is therefore $R=0$.

98. In practice, the process of finding the resultant indicated in Art. 96 is inconvenient when the number of forces is large.

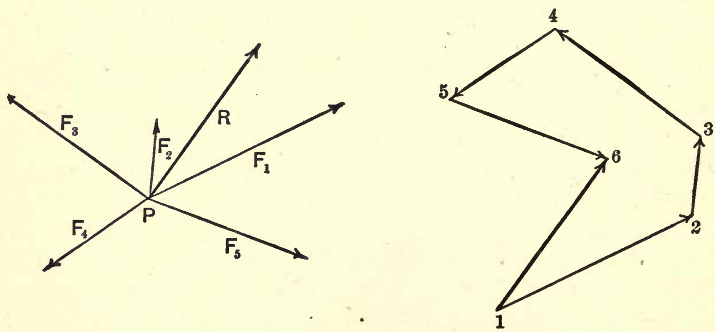

Fig. 19.

If the forces are given graphically, by their vectors, we have only to add these vectors geometrically (see *Kinematics*, Art. 46), and this can best be done in a separate diagram, called the **force polygon,** or **stress diagram.** Thus, in Fig. 19, 1 2 is drawn equal and parallel to F_1, 2 3 equal and parallel to F_2, 3 4 to F_3, 4 5 to F_4, 5 6 to F_5. The closing line of the force polygon, viz. 1 6 in the figure, is equal and parallel to the resultant R, which is therefore obtained by drawing through the point of intersection of the forces a line equal and parallel to 1 6.

The graphical condition of equilibrium consists in the closing of the force polygon, that is, in the coincidence of its terminal point (6) with its initial point (1).

99. Analytically, a systematic solution is obtained by resolving each force F into three components X, Y, Z, along three rectangular axes passing through the point of intersection of the given forces. All components lying in the direction of the same axis can then be added algebraically, and the whole system of forces is found to be equivalent to three rectangular forces $\Sigma X, \Sigma Y, \Sigma Z$, which, by the parallelogram law, can be combined into a single resultant

$$R = \sqrt{(\Sigma X)^2 + (\Sigma Y)^2 + (\Sigma Z)^2}.$$

The angles α, β, γ made by this resultant with the axes are given by the relations

$$\frac{\cos\alpha}{\Sigma X} = \frac{\cos\beta}{\Sigma Y} = \frac{\cos\gamma}{\Sigma Z} = \frac{1}{R}.$$

100. If the forces all lie in the same plane, only two axes are required, and we have

$$R = \sqrt{(\Sigma X)^2 + (\Sigma Y)^2}, \quad \tan\theta = \frac{\Sigma Y}{\Sigma X},$$

where θ is the angle between the axis of X and R.

101. The condition of equilibrium (Art. 97) $R=0$ becomes, by Art. 99, $(\Sigma X)^2 + (\Sigma Y)^2 + (\Sigma Z)^2 = 0$. As all terms in the left-hand member are positive, their sum can vanish only when each term is $=0$. *The analytical conditions of the equilibrium of any number of concurrent forces* are therefore:

$$\Sigma X = 0, \quad \Sigma Y = 0, \quad \Sigma Z = 0.$$

102. As the projection on any line of any closed polygon, even when its sides do not all lie in the same plane, is equal to 0, it follows that the proposition of Art. 89 holds for any number of concurrent forces.

103. Exercises.

(1) Show that three forces that are in equilibrium must lie in the same plane and pass through the same point.

(2) Six forces of 1, 2, 3, 4, 5, 6 lbs., respectively, act in the same plane on the same point, making angles of 60° with each other. Find their resultant in magnitude and direction: (*a*) graphically; (*b*) analytically.

(3) Let $AB = c$ (Fig. 20) be the vertical post, $AC = b$ the jib, of a crane, the ends BC being connected by a chain of length a. If a weight W be suspended from C, find the tension T produced by it in the chain and the thrust P in AC.

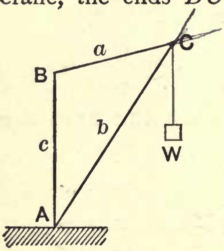

Fig. 20.

(4) Let AC be hinged at A (Fig. 20) so as to turn freely in a vertical plane, and let the chain pass over a pulley at C and carry the weight W. In what position of AC will there be equilibrium?

(5) Find the resultant R of three concurrent forces A, B, C lying in the same plane and making angles α, β, γ with each other.

(6) Prove that the moment of the resultant of any number of concurrent forces lying in the same plane about any point in this plane is equal to the sum of the moments of the forces about the same point.

(7) By means of Ex. (6), express the conditions of equilibrium of any number of concurrent forces in the same plane.

(8) When three forces are in equilibrium, show that they are proportional and parallel to the sides of a triangle.

(9) When any number of concurrent forces are in equilibrium, show that any one of them reversed is the resultant of all the others.

(10) A weightless rod AC (Fig. 21), hinged at one end A so as to be free to turn in a vertical plane, is held in a horizontal position by means of the chain BC. If a weight W be suspended at C, find the thrust P in AC and the tension T of the chain. Assume $AC = 8$ ft., $AB = 6$ ft.

(11) In Ex. (10), suppose the rod AC, instead of being hinged at A, to be set firmly into the wall in a horizontal position; and let the chain fastened at B run at C over a smooth pulley and carry the weight W. Find the tension of the chain and the magnitude and direction of the pressure on the pulley at C.

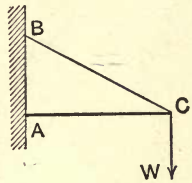

Fig. 21.

(12) In "tacking against the wind," let W be the force of the wind; α, β the angles made by the axis of the boat with the direction in which the wind blows, and with the sail, respectively. Determine the force that drives the boat forward and find for what position of the sail it is greatest.

(13) A cylinder of weight W rests on two inclined planes whose intersection is horizontal and parallel to the axis of the cylinder. Find the pressures on these planes.

(14) Find the tensions in the string $ABCD$, fixed at A and D, and carrying equal weights W at B and C, if $AD = c$ is horizontal, $AB = BC = CD$, and the length of the string is $3l$.

(15) One of the vertices A of a regular hexagon is acted upon by 5 forces represented in magnitude and direction by the lines drawn from A to the other vertices of the hexagon. Find their resultant.

(16) Find the resultant of three equal forces P acting on a point, the angle between the first and second as well as that between the second and third being 45°.

(17) A mass m rests on a plane inclined to the horizon at an angle θ; it is kept in equilibrium (*a*) by a force P_1 parallel to the plane; (*b*) by a horizontal force P_2; (*c*) by a force P_3 inclined to the horizon at an angle $\theta + \alpha$. Determine in each case the force P and the pressure R on the plane.

(18) Show that the three forces represented by the vectors OA, OB, OC are in equilibrium if O is the centroid of the triangular area ABC.

(19) Show that the three vectors OA, OB, OC have the same resultant as the three vectors OA', OB', OC', if A', B', C' are the middle points of the sides of the triangle ABC.

(20) Show that the resultant of the vectors OA, OB, OC is OO', if O is the centre of the circle circumscribed to the triangle ABC and O' the intersection of the altitudes of the same triangle.

III. *Parallel Forces.*

104. Resultant of Two Parallel Forces. The graphical construction of the resultant (Art. 85) fails in the case of parallel forces.

As an expedient, we may resolve one of the two given forces into two components and then combine these successively with the other force. Thus, resolving P (Fig. 22) into P' and P'' along the lines I and II respectively, we may compound P'' with Q, and their resultant (acting along III) with P'. The resolution of P into two arbitrary components P', P'' is best done in a separate diagram, the **force polygon**, by taking 1 2 equal and parallel to P, and drawing from any arbitrary point O,

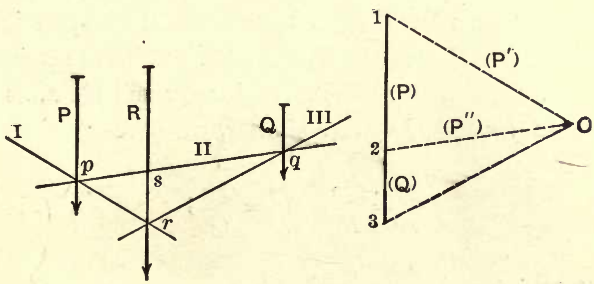

Fig. 22.

called the **pole**, $O1$, $O2$, which will represent the components P', P'' in magnitude and direction. Then drawing 2 3 equal and parallel to Q, we find $O3$ as the resultant of P'' and Q.

The whole operation of finding the resultant R of two parallel forces P, Q is therefore as follows. First construct the *force polygon* by making 1 2 equal and parallel to P, 2 3 equal and parallel to Q; 1 3 gives the magnitude and direction of the resultant R. Then assume a pole O and draw $O1$, $O2$, $O3$. Now construct the so-called **funicular polygon** (or **equilibrium polygon**) by drawing in the original figure a line I parallel to $O1$ intersecting P say in p; through p a line II parallel to $O2$ intersecting Q say in q; through q a line III parallel to $O3$. The intersection r of I and III is a point of the resultant R which is therefore obtained in position by drawing through r a line equal and parallel to 1 3.

105. In Fig. 22 the two given parallel forces P, Q were assumed of the same sense. The construction applies, however, equally well to the case when they are of opposite sense. The resultant R will then be found to lie not between P and Q, but outside, on the side of the larger force. The construction fails only when the two given forces are equal and of opposite sense, a case that will be considered later (see Art. 112 and Arts. 128–138).

106. To determine the position of R analytically, we may find the ratio in which it divides the distance (perpendicular or oblique) between P and Q. Let s (Fig. 22) be the point where R meets pq. Then, since the triangles prs and $O12$, as well as the triangles qsr and $O23$, are similar, we have

$$\frac{ps}{sr}=\frac{O2}{P}, \quad \frac{sq}{sr}=\frac{O2}{Q};$$

hence, dividing,

$$\frac{ps}{sq}=\frac{Q}{P}.$$

This means that *the resultant of two parallel forces divides their distance in the inverse ratio of the forces.* As this proposition finds application in the theory of the lever, it is commonly referred to as the **principle of the lever.**

Dropping perpendiculars p, q from any point of the resultant R on the components P, Q, the relation can be expressed in the form

$$Pp=-Qq,$$

which shows that Varignon's proposition of moments (Arts. 89–93) applies to parallel forces.

107. The resultant of two parallel forces can also be found by the following simple process. Intersect the two parallel forces P, Q by any transversal in p and q (Fig. 23) and apply at these points along pq two equal and opposite forces F, $-F$; find the resultant P' of F and P and the resultant P'' of $-F$ and Q; these resultants P' and P'' will intersect (unless P and Q be equal and opposite) and their resultant R can be found.

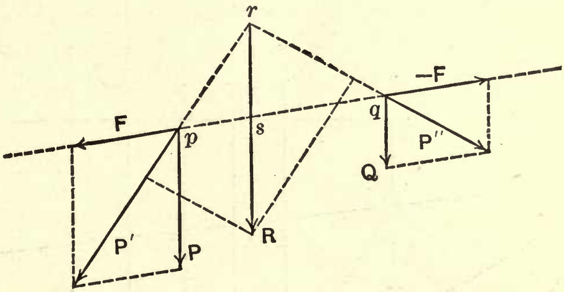

Fig. 23.

It will be noticed that this construction reduces to that given in Art. 104 if for F we select the force $2\,O$ in the force polygon, Fig. 22, p. 59.

108. Resultant of Any Number of Parallel Forces. The graphical method of Art. 104 is readily extended to the general case of any number of parallel forces lying in the same plane. Whatever the number of the forces, the force polygon gives magnitude, direction, and sense of the resultant, which is simply the algebraic sum of the given forces; while the funicular polygon (formed by the lines I, II, III, etc.) gives the position of the resultant by furnishing one of its points, viz. the intersection of the first and last sides of the funicular polygon.

The process will best be understood from the following example.

The horizontal beam AB (Fig. 24) resting freely on the fixed supports A, B carries four weights W_1, W_2, W_3, W_4.

To determine the position of the resultant and the reactions A, B of the supports, construct the *force polygon* by laying off in succession on a vertical line 1 2 = W_1, 2 3 = W_2, 3 4 = W_3, 4 5 = W_4; select any point O as pole and join it to the points 1, 2, 3, 4, 5.

Now we may regard 1 O and O 2 as components into which W_1 has been resolved; similarly 2 O and O 3 as components of W_2, 3 O and O 4 as components of W_3, and 4 O and O 5 as components of W_4. This resolution of the weights into components is transferred into the

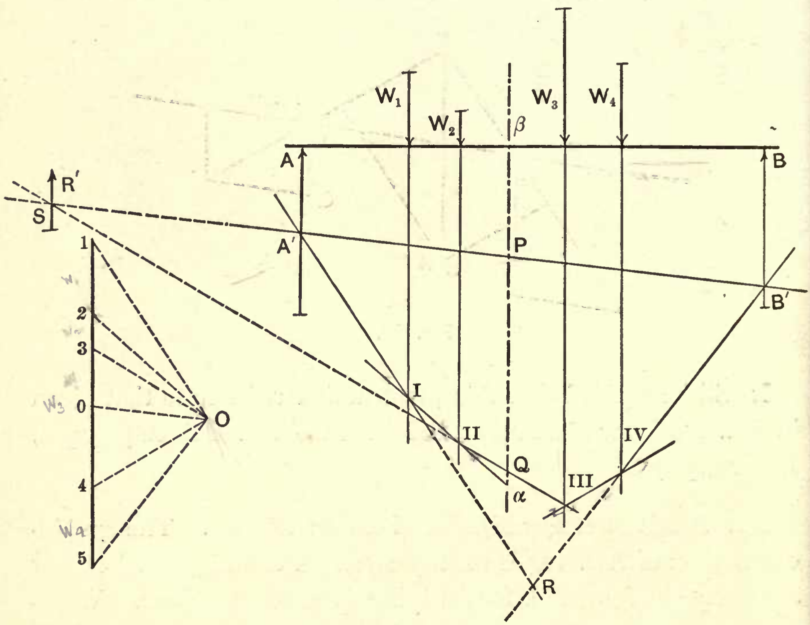

Fig. 24.

main figure by constructing the *funicular polygon* as follows: through any point A' on the direction of the reaction A draw a parallel to O 1 and let it meet W_1 in I; through I draw I II parallel to O 2; through II draw II III parallel to O 3; through III draw III IV parallel to O 4; and through IV draw IV B' parallel to O 5; the point B' being on the direction of the reaction B.

If now each weight be regarded as resolved along the sides adjacent to it in the funicular polygon, since the two components falling into I II are equal and opposite, and also those falling into II III and III IV, the system of weights is reduced to the two components along

A'I and IV B'. The intersection of these lines, *i.e.* of the first and last sides of the funicular polygon, gives a point, R, of the resultant of W_1, W_2, W_3, W_4.

Moreover, if the component in A'I be resolved along $A'B'$ and the vertical through A', and similarly the component in B' IV along $B'A'$ and the vertical through B', the two components along $A'B'$ will be equal and opposite, each being equal to the parallel Oo drawn to $A'B'$ in the force polygon. This parallel furnishes, therefore, the magnitudes of the reactions $A = 0\,1$, $B = 5\,0$.

109. Analytically, the resultant of n parallel forces F_1, F_2, ... F_n, whether in the same plane or not, can be found as follows:

The resultant of F_1 and F_2 is a force F_1+F_2 situated in the plane (F_1, F_2), so that $F_1p_1 = F_2p_2$ (Art. 106), where p_1, p_2 are the (perpendicular or oblique) distances of the resultant from F_1 and F_2, respectively. This resultant F_1+F_2 can now be combined with F_3 to form a resultant $F_1+F_2+F_3$, whose distances from F_1+F_2 and F_3 in the plane determined by these two forces are as F_3 is to F_1+F_2. This process can be continued until all forces have been combined; the final resultant is

$$F_1 + F_2 + \ldots + F_n.$$

Any number of parallel forces are, therefore, *in general equivalent to a single resultant equal to their algebraic sum.*

110. To find the *position* of this resultant analytically, let the points of application of the forces F_1, F_2, ... F_n be (x_1, y_1, z_1), (x_2, y_2, z_2), ... (x_n, y_n, z_n). The point of application of the resultant F_1+F_2 of F_1 and F_2 may be taken so as to divide the distance of the points of application of F_1 and F_2 in the ratio $F_2 : F_1$; hence, denoting its co-ordinates by x', y', z', we have $F_1(x'-x_1) = F_2(x_2-x')$, or

$$(F_1+F_2)x' = F_1x_1 + F_2x_2,$$

and similarly for y' and z'.

The force $F_1 + F_2$ combines with F_3 to form a resultant $F_1+F_2+F_3$, whose point of application x'', y'', z'' is given by

$$(F_1+F_2+F_3)\,x''=F_1x_1+F_2x_2+F_3x_3,$$

and similar expressions for y'', z''.

Proceeding in this way, we find for the point of application $(\bar{x}, \bar{y}, \bar{z})$ of the resultant of all the given forces

$$(F_1+F_2+\cdots+F_n)\,\bar{x}=F_1x_1+F_2x_2+\cdots+F_nx_n,$$

with corresponding equations for $\bar{y}$ and $\bar{z}$. We may write these equations in the form:

$$\bar{x}=\frac{\Sigma Fx}{\Sigma F},\quad \bar{y}=\frac{\Sigma Fy}{\Sigma F},\quad \bar{z}=\frac{\Sigma Fz}{\Sigma F}. \qquad (1)$$

As these expressions for $\bar{x}, \bar{y}, \bar{z}$ are independent of the direction of the parallel forces, it follows that the same point $(\bar{x}, \bar{y}, \bar{z})$ would be found if the forces were all turned in any way about their points of application, provided they remain parallel. The point $(\bar{x}, \bar{y}, \bar{z})$ is for this reason called the **centre** of the system of parallel forces. It is nothing but what in geometry is called the *mean point*, or *mean centre*, of the points of application if the forces are regarded as coefficients or "weights" (in the meaning of the theory of least squares) of these points.

111. As the origin of co-ordinates in the last article is arbitrary, the equations (1) evidently express the proposition that *in any system of parallel forces the sum of their moments about any point is equal to the moment of their resultant about the same point.* In particular, *the sum of the moments about any point on the resultant is zero.*

This proposition may be regarded as a generalisation of the *principle of the lever* referred to in Art. 106. It furnishes the convenient method of "taking moments" for the purpose of determining the position of the resultant.

112. Couple of Forces. The construction given in Art. 104 for the resultant of two parallel forces fails only when the two given forces are equal and of opposite sense. In this case, the lines I and III of the funicular polygon become parallel, so that their intersection r lies at infinity. The magnitude of the resultant is of course $=0$.

The combination of two equal and opposite parallel forces $(F, -F)$ is called a **couple**. *A couple is,* therefore, properly speaking, *not equivalent to a single force,* although it may be said to be equivalent to a force of magnitude 0 at an infinite distance. The theory of couples will be considered in detail in Arts. 128–138.

113. Conditions of Equilibrium. We have seen (Art. 109) that a system of n parallel forces is, in general, equivalent to a single force; but, as appears from the preceding article, it may happen to reduce to a couple. It follows that *for the equilibrium of a system of parallel forces the condition* $R=0$, *though always necessary, is not sufficient.*

Now, if the resultant R of the n parallel forces $F_1, F_2, \ldots F_n$ be $=0$, the resultant R' of the $n-1$ forces $F_1, F_2, \ldots F_{n-1}$ cannot be 0, and its point of application is found (by Art. 110) from $\bar{x}=(F_1x_1+F_2x_2+\cdots+F_{n-1}x_{n-1})/(F_1+F_2+\cdots+F_{n-1})$ and similar expressions for $\bar{y}$ and $\bar{z}$. The whole system of parallel forces is therefore equivalent to the two parallel forces R' and F_n. Two such forces can be in equilibrium only when they lie in the same straight line; *i.e.* F_n must coincide with R' and must therefore pass through the point $(\bar{x}, \bar{y}, \bar{z})$, which is a point of R'.

The additional condition of equilibrium is, therefore,

$$\frac{\bar{x}-x_n}{\cos\alpha}=\frac{\bar{y}-y_n}{\text{osc}\,\beta}=\frac{\bar{z}-z_n}{\cos\gamma},$$

where α, β, γ are the angles made by the direction of the forces with the axes.

114. For practical application it is usually best to replace the last condition by taking moments about a convenient point.

Thus, the analytical conditions of equilibrium can be written in the form

$$\Sigma F = 0, \quad \Sigma Fp = 0.$$

Graphically, to the former corresponds the closing of the force-polygon, to the latter the closing of the funicular polygon.

115. Weight; Centre of Gravity. The most important special case of parallel forces is that of the force of gravity which acts at any given place near the earth's surface in approximately parallel lines on every particle of matter.

If g be the acceleration of gravity, the force of gravity on a particle of mass m is

$$w = mg,$$

and is called the **weight** of the particle or of the mass m.

For a system of particles of masses $m_1, m_2, \ldots m_n$ we have

$$w_1 = m_1 g, \quad w_2 = m_2 g, \ldots w_n = m_n g.$$

The resultant W of these parallel forces,

$$W = w_1 + w_2 + \cdots + w_n = (m_1 + m_2 + \cdots + m_n) g = Mg,$$

where M is the mass of the system, is called the weight of the system.

The centre of the parallel forces of gravity of a system of particles has, by Art. 110, the co-ordinates

$$\bar{x} = \frac{\Sigma mgx}{\Sigma mg}, \quad \bar{y} = \frac{\Sigma mgy}{\Sigma mg}, \quad \bar{z} = \frac{\Sigma mgz}{\Sigma mg},$$

or since the constant g cancels,

$$\bar{x} = \frac{\Sigma mx}{\Sigma m}, \quad \bar{y} = \frac{\Sigma my}{\Sigma m}, \quad \bar{z} = \frac{\Sigma mz}{\Sigma m}.$$

This point is called the **centre of gravity** of the system, and is evidently identical with the *centre of mass*, or **centroid** (see Art. 13).

For continuous masses the same formulæ hold, except that the summations become integrations.

The *weight* W of a physical body of mass M is therefore a vertical force passing through the centroid of its mass.

116. Exercises.

(1) A straight rod (*lever*) of length $2l = 5$ ft. has suspended from its ends masses of 12 and 27 pounds, respectively. Find the point (*fulcrum*) on which it balances in a horizontal position: (*a*) if its own weight be neglected; (*b*) if it is homogeneous and weighs 2.2 pounds per running foot.

(2) A straight beam rests in a horizontal position on two supports A, B. The distance between the supports (the *span*) is $2l = 24$ ft. The beam carries a weight of 14 tons at a distance of 8 ft. from A, and a weight of 10 tons at 16 ft. from A. Find the pressures on the supports (or the *reactions* of the supports): (*a*) when the proper weight of the beam is neglected; (*b*) when the beam weighs $\frac{1}{4}$ ton per running foot; (*c*) when the first third of the beam (from A) weighs $\frac{1}{8}$ ton, the second 1 ton, the third $\frac{1}{2}$ ton per running foot.

(3) A homogeneous circular plate weighing W pounds rests in a horizontal position on three equidistant supports near its edge. (*a*) What is the least weight P that will upset it when placed on the plate? (*b*) If there be four equidistant supports near the edge, what is the least weight that will upset the plate?

(4) Construct the resultant of two parallel forces of opposite sense by the graphical method of Arts. 104, 105.

(5) Solve exercises (1) and (2) by the graphical method.

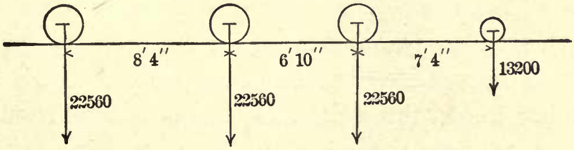

Fig. 25.

(6) Find the reactions of the supports of a bridge truss of 50 ft. span, produced by a freight locomotive whose weight is distributed over the three pairs of driving wheels and the front truck, as indicated in Fig. 25: (*a*) when it stands in the middle of the span; (*b*) when its front truck stands over one support.

(7) Explain how the centroid of a plane area can be found graphically by dividing the area into narrow parallel strips.

(8) A homogeneous rectangular plate is pivoted on a horizontal axis through its centre so as to turn freely in a vertical plane. If weights W_1, W_2, W_3, W_4 be suspended from its vertices, what is its position of equilibrium?

(9) The ends of a straight lever of length l are acted upon by two forces F_1, F_2 in the same plane with it, but inclined to the lever at angles α_1, α_2. Determine the position of the fulcrum.

117. Funicular Polygons and Catenaries. The *funicular polygon* in its original meaning represents the form of equilibrium assumed by a string or cord suspended from two fixed points and acted upon by any forces in the same plane. The "cord" is supposed to be perfectly flexible, inextensible, inelastic, and without weight. When the number of forces is made infinite, the polygon becomes a continuous curve called a *catenary.*

The present discussion is confined to the case when the forces are all vertical so that they can be regarded as weights.

118. Let A, B (Fig. 26) be the fixed points, and let there be five weights, W_1, W_2, W_3, W_4, W_5, suspended from the points I, II, III, IV, V, of the cord.

If the cord be cut on both sides of the point I and the corresponding tensions T_1, T_2 be introduced, the point I must be in equilibrium under the action of the three forces W_1, T_1, T_2. Hence drawing a line 1 2 to represent the weight W_1 and drawing through its ends 1, 2 parallels to AI and I II, respectively, we have the force polygon of the point I. Its sides O 1 and 2 O represent in magnitude, direction, and sense the tensions T_1, T_2; in other words, the weight W_1 has thus been resolved into its components along the adjacent sides.

The same can be done at every vertex of the polygon I II III IV V, and all tensions can thus be found. But as the the tension T_2 in I II occurs again (with sense reversed) in the force polygon for the point II, and so on, the successive force polygons can be fitted together, every triangle having one side in common with the next one. Thus the complete force polygon of the whole cord is formed, as shown on the right in Fig. 26. Its vertical line represents the successive weights $W_1 = 1\,2$, $W_2 = 2\,3$, $W_3 = 3\,4$, $W_4 = 4\,5$, $W_5 = 5\,6$, while the lines

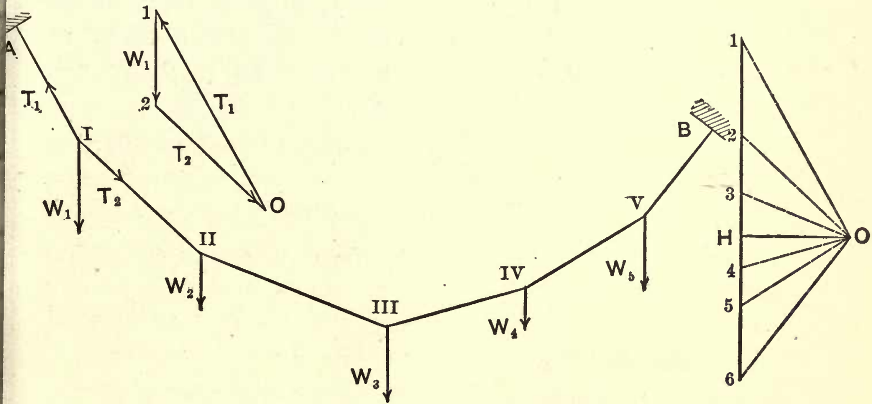

Fig. 26.

radiating from the point O, or pole, represent on the same scale the tensions in AI, I II, II III, III IV, IV V, V B.

119. The polygon AI II III IV VB is called the *funicular polygon.* It will be noticed that if we have given the fixed points A, B, the magnitudes of the weights, their horizontal distances, say from A, and the directions of the first and last sides AI, VB (whatever may be the number of the forces), the remaining sides of the funicular polygon can be found by laying off on a vertical line the weights $W_1 = 1\,2$, $W_2 = 2\,3$, etc., in succession, drawing through 1 a parallel to the first side, through the end of the last weight (6 in Fig. 26) a parallel to the last side, and joining the intersection O of these parallels to the points 2, 3, etc. The sides of the funicular polygon must be parallel to the lines radiating from O; at the same time these lines represent the tensions in these sides.

120. For the analytical investigation, let P_i be that vertex of a funicular polygon of any number of sides at which the ith and

$(i+1)$th sides intersect; let α_i, α_{i+1} be the angles at which these sides are inclined to the horizon, and W_i the weight suspended from the vertex P_i (Fig. 27).

Cutting the cord on both sides of P_i, and introducing the tensions T_i and T_{i+1}, the conditions of equilibrium of the point P_i are found by resolving the three forces W_i, T_i, T_{i+1} horizontally and vertically (Art. 100):

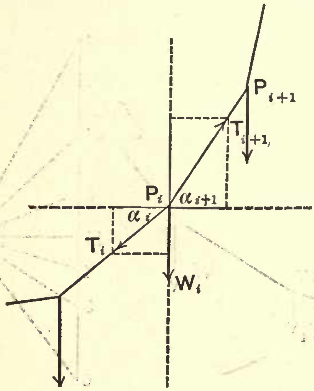

Fig. 27.

$$T_{i+1} \cos \alpha_{i+1} = T_i \cos \alpha_i, \qquad (1)$$

$$T_{i+1} \sin \alpha_{i+1} = T_i \sin \alpha_i + W_i. \qquad (2)$$

The former of these equations shows that, whatever the weights W and the lengths and inclinations of the sides, *the horizontal components of the tensions* T *are all equal.* Denoting this constant value by H, we have

$$T_1 \cos \alpha_1 = T_2 \cos \alpha_2 = \cdots = T_i \cos \alpha_i = \cdots = H. \qquad (3)$$

Substituting the values of T_i and T_{i+1} as obtained from these relations, into (2), this equation becomes

$$\tan \alpha_{i+1} = \tan \alpha_i + \frac{W_i}{H}, \qquad (4)$$

which shows that as soon as all the weights and the inclination and tension of any one side are given, the inclinations and tensions of all the other sides can be found.

121. Let us now assume that the weights W are all equal. Then the values of $\tan \alpha_{i+1}$ given by (4) form an arithmetical progression. If, in addition, we assume that *the sides of the polygon are such as to have equal horizontal projections, i.e.* if we assume the weights to be equally spaced horizontally, *the vertices of the polygon will lie on a parabola* whose axis is vertical.

To find its equation, let us suppose, for the sake of simplicity, that one side of the polygon, say the kth, is horizontal so that $\alpha_k = 0$. Taking this side as axis of x, its middle point O as origin, the co-ordinates of the vertex P_k are $\frac{1}{2}a$, 0, if a be the length of the horizontal side and hence also that of the horizontal projection of every side.

Putting $W/H = \tau$, we have $\tan \alpha_k = 0$, $\tan \alpha_{k+1} = \tau$, $\tan \alpha_{k+2} = 2\tau$, $\cdots$; hence the co-ordinates of P_{k+1} are $x = \frac{3}{2}a$, $y = a\tau$; those of P_{k+2} are $x = \frac{5}{2}a$, $y = a\tau + 2a\tau = 3a\tau$; those of P_{k+3} are $x = \frac{7}{2}a$, $y = a\tau + 2a\tau + 3a\tau = 6a\tau$, etc.; those of the nth vertex after P_k are

$$x = \frac{2n+1}{2}a, \quad y = \frac{n(n+1)}{2}a\tau.$$

Eliminating n, we find the equation

$$x^2 = \frac{2a}{\tau}\left(y + \frac{a\tau}{8}\right),$$

which represents a parabola whose axis is the axis of y, and whose vertex lies at the distance $\frac{1}{8}a\tau = \frac{1}{8}aW/H$ below the origin O.

122. Let the number of sides be increased indefinitely, the length a and the weight W approaching the limit 0, but so that the quotient a/W remains finite, say $\lim (a/W) = 1/w$. Then $\lim (a/\tau) = H/w$, $\lim (a\tau) = 0$; so that the equation of the parabola becomes

$$x^2 = \frac{2H}{w}y,$$

where w is evidently the weight of the cord, or chain, per unit length.

The parabola is, therefore, *the form of equilibrium of a cord suspended from two points when the weight of the cord is uniformly distributed over its horizontal projection.* This is, for instance, the case approximately in a suspension bridge with uniformly loaded roadbed, the proper weight of the chains being neglected.

123. This result can easily be derived independently of Art. 121, by considering the equilibrium of any portion OP of the chain beginning at the lowest point O (Fig. 28). The forces acting on this portion are the horizontal tension H at O, the tension T along the tangent at P, and the proper weight W of the chain. As this weight is assumed to be uniformly distributed over the horizontal projection $OP'=x$ of OP, the weight is $W=wx$, and bisects OP'.

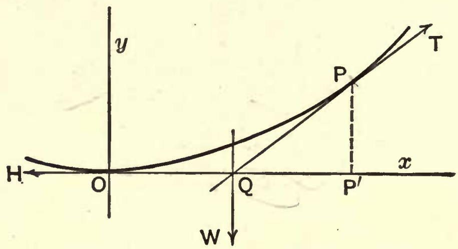

Fig. 28.

Resolving the forces in the horizontal and vertical directions, we find, as conditions of equilibrium,

$$-H+T\frac{dx}{ds}=0, \quad -wx+T\frac{dy}{ds}=0;$$

whence, eliminating ds,

$$\frac{dy}{dx}=\frac{w}{H}x.$$

Integrating and considering that $x=0$ when $y=0$, we find the equation of the parabola as above,

$$y=\frac{w}{2H}x^2.$$

124. The three forces H, T, W are in equilibrium; they must intersect in a point Q which bisects OP', and the force polygon must be similar to the triangle QPP'.

Hence, if the height of a suspension bridge be h, its span $2l$, its total weight $2W$, we have for the horizontal tension H, and the tension T at the point of support

$$\frac{H}{l/2}=\frac{T}{\sqrt{h^2+l^2/4}}=\frac{W}{h}.$$

125. *The form of equilibrium assumed by a homogeneous cord is an ordinary catenary.*

To find its equation, we again consider the equilibrium of a

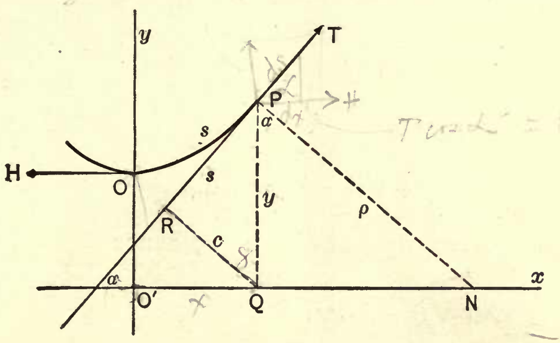

Fig. 29.

portion $OP=s$ (Fig. 29) of the cord, beginning at the lowest point O.

The weight of this portion is now $W=ws$, and if α be the angle made by the tangent at P with a horizontal line, we have the conditions of equilibrium

$$T\cos\alpha=T\frac{dx}{ds}=H,\quad T\sin\alpha=T\frac{dy}{ds}=ws.$$

Dividing and putting $H/w=c$, we have the differential equation of the curve in the form

$$\frac{dx}{dy}=\frac{c}{s}.$$

Substituting this value of dx/dy in the relation $ds^2=dx^2+dy^2$, we obtain

$$\left(\frac{ds}{dy}\right)^2=1+\frac{c^2}{s^2},\quad\text{or}\quad dy=\pm\frac{sds}{\sqrt{s^2+c^2}},$$

which gives by integration $y+C=\sqrt{s^2+c^2}$, the minus sign being rejected since y increases with s.

The constant C can be made to disappear by taking the origin O' on the vertical through O at the distance $O'O=c$ below the lowest point O. We have, therefore,

$$y^2=s^2+c^2.$$

By means of this relation, s can be eliminated from the original differential equation, and the result,

$$\frac{c\,dy}{\sqrt{y^2-c^2}}=dx,$$

can be integrated:

$$c\log(y+\sqrt{y^2-c^2})=x+C.$$

As $y=c$ when $x=0$, we find $C=c\log c$; hence

$$y+\sqrt{y^2-c^2}=ce^{\frac{x}{c}}.$$

Taking reciprocals and rationalising the denominator, we find

$$y-\sqrt{y^2-c^2}=ce^{-\frac{x}{c}};$$

hence, adding and subtracting,

$$y=\frac{c}{2}(e^{\frac{x}{c}}+e^{-\frac{x}{c}}),\quad s=\frac{c}{2}(e^{\frac{x}{c}}-e^{-\frac{x}{c}}).$$

126. The first equations of Art. 125, $T\cos\alpha=H=wc$, $T\sin\alpha=ws$, give for the total tension T at any point P

$$T^2=w^2(c^2+s^2)=(wy)^2.$$

Thus, *while the horizontal component is constant, the vertical component at any point* P *is equal to the weight of the portion of the cord from the lowest point* O *to the point* P, *and the total tension is equal to the weight of a portion of the cord equal to the ordinate of the point* P.

Let Q be the foot of the ordinate of P (Fig. 29), N the intersection of the normal with the axis $O'x$, and draw QR perpendicular to the tangent. Then $PR = y\sin\alpha = s$, since $T\sin\alpha = ws$ and $T = wy$; also $QR = y\cos\alpha = c$. Dividing, we have $\tan\alpha = s/c$; hence, differentiating,

$$\frac{1}{\cos^2\alpha}\frac{d\alpha}{ds} = \frac{1}{c}; \quad \therefore \rho = \frac{ds}{d\alpha} = \frac{c}{\cos^2\alpha}.$$

The figure shows that the radius of curvature ρ is equal to the length of the normal PN.

The relation $\rho\cos^2\alpha = c$ shows further that at the vertex $(\alpha = 0)$ the radius of curvature is $\rho_0 = c$. It follows that for a cord or chain suspended from two points B, C in the same horizontal line, c (and consequently H) is large when ρ_0 is large, *i.e.* when the curve is flat at the vertex; in other words, when B and C are far apart.

127. Exercises.

(1) A weightless cord $ABCDEF$ is suspended from the fixed points A, F, and carries weights at the intermediate points B, C, D, E. Taking A as origin, the axis of x horizontal, the axis of y vertically upwards, the co-ordinates of the points B, C, D, E, F are $(2, -1)$, $(4, -1.5)$, $(7, -1.5)$, $(8.5, -1)$, $(10, 2)$. If the weight at B be one pound, what are the weights at C, D, E? What are the tensions of the sections of the cord? What are the reactions of the fixed points A, F?

(2) The total weight of a suspension bridge is $2W = 50$ tons; the span is $2l = 200$ ft.; the height is $h = 18$ ft. Find the tension of the chain at the ends and in the middle, both graphically and analytically.

(3) A uniform wire of length $2s$ is stretched between two points in the same horizontal line whose distance $2x$ is very nearly equal to $2s$. Find an approximate expression for the parameter c of the catenary and thence for the tension of the wire.

IV. *Theory of Couples.*

128. *The combination of two equal forces of opposite sense* F, −F, *acting along parallel lines, is called a couple of forces,* or simply a **couple** (Art. 112).

The perpendicular distance $AB=p$ (Fig. 30) of the forces of the couple is called the **arm**, and the product Fp of the force F into the arm p is called the **moment** of the couple.

If we imagine the couple (F, p) to act upon an invariable plane figure in its plane, and if the middle point of its arm be a fixed point of this figure, the couple will evidently tend to turn the figure about this middle point. (It is to be observed that it is *not* true, in general, that a couple acting on a rigid body produces rotation about an axis at right angles to its plane.) A couple of the type (F, p) or (F', p')

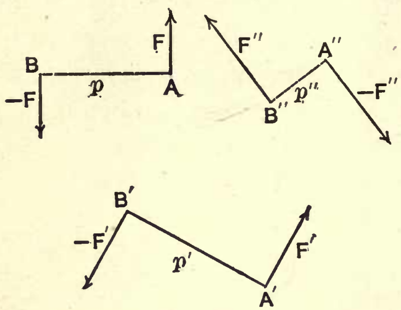

Fig. 30.

(see Fig. 30) will tend to rotate counter-clockwise, while a couple of the type (F'', p'') tends to turn clockwise. Couples in the same plane, or in parallel planes, are therefore distinguished as to their **sense**; and this sense is expressed by the algebraic sign attributed to the moment. Thus, the moment of the couple (F, p) in Fig. 30, is $+Fp$, that of the couple (F'', p'') is $-F''p''$.

129. *The effect of a couple is not changed by translation.*

Let $AB=p$ (Fig. 31) be the arm of the couple (F, p) in its original position, and $A'B'$ the same arm in a new position parallel to the original one in the same plane, or in any parallel plane. By introducing at each end of the new arm $A'B'$ two opposite forces F, $-F$, each equal and parallel to the original forces F, the given system is not changed (Art. 80). But the two equal and parallel forces F at A and B' form a resultant $2F$ at the middle point O of the diagonal AB' of the parallelogram $ABB'A'$. Similarly, the two forces $-F$ at B and A' are together equivalent to a resultant $-2F$ at the same point O. These two resultants, being equal and opposite and acting in the same line, are together equivalent to o. Hence the whole system reduces to the force F at A' and the force $-F$ at B', which form, therefore, a couple equivalent to the original couple at AB.

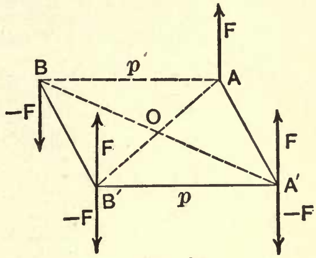

Fig. 31.

130. *The effect of a couple is not changed by rotation in its plane.*

Let AB (Fig. 32) be the arm of the couple in the original position, C its middle point, and let the couple be turned about C into the position $A'B'$. Applying again at A', B' equal and opposite forces each equal to F, the forces $-F$ at A' and F at A will form a resultant acting along CD, while F at B' and $-F$ at B give an equal and opposite resultant along CE. These two resultants destroy each other and leave nothing but the couple formed by F at A' and $-F$ at B', which is therefore equivalent to the original couple.

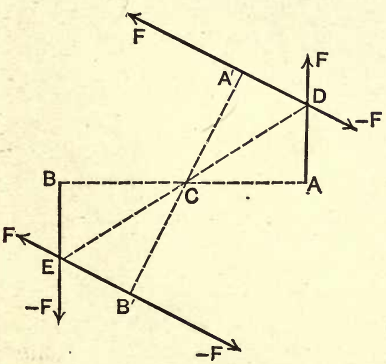

Fig. 32.

Any other displacement of the couple in its plane, or to a parallel plane, can be effected by a translation combined with a rotation about the middle point of its arm in its plane. The effect of a couple is therefore not changed by any displacement in its plane or to a parallel plane.

131. *The effect of a couple is not changed if its force* F *and its arm* p *be changed simultaneously in any way, provided their product* Fp *remain the same.*

Let $AB=p$ be the original arm (Fig. 33), F the original force of the couple; and let $A'B'=p'$ be the new arm. The introduction of two equal and opposite forces F' at A', and also at B', will not change the given system F, $-F$. Now, selecting for F' a magnitude such that $F'p'=Fp$, the force F at A and the force $-F'$ and A' combine (Arts. 104–106) to form a parallel resultant through C, the middle point of the arm, since for this point $F\cdot\frac{1}{2}p+(-F')\cdot\frac{1}{2}p'=0$. Similarly, $-F$ at B and F' at B' give a resultant of the same magnitude, in the same line through C, but of opposite sense. These two resultants thus destroying each other, there remains only the couple formed by F' at A' and $-F'$ at B', for which $Fp=F'p'$.

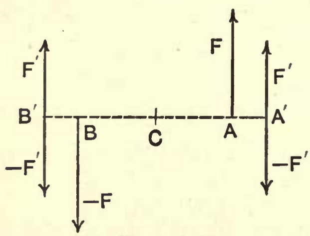

Fig. 33.

132. It results from the last three articles that the only essential characteristics of a couple are (*a*) the numerical value of the moment; (*b*) the sense, or direction of rotation; and (*c*) what has been called the "aspect" of its plane, *i.e.* the direction of any normal to this plane.

It is to be noticed that the plane of the two forces forming the couple is not an essential characteristic of the couple; just as the point of application of a force is not an essential characteristic of the force (see Art. 84).

Now the three characteristics enumerated above can all be indicated by a *vector* which can therefore serve as the geometrical representative of the couple. Thus, the couple formed by the forces F, $-F$ (Fig. 34), whose perpendicular distance is p, is represented by the vector $AB=Fp$ laid off on any normal to the plane of the couple. The sense is indicated by drawing the vector toward that side of the plane from which the couple is seen to rotate counter-clockwise.

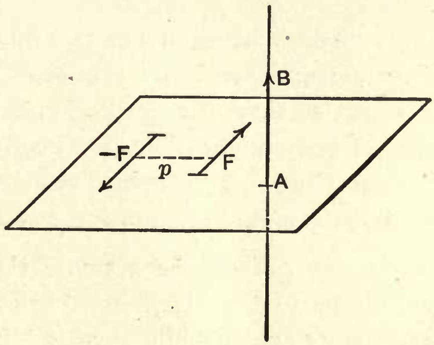

Fig. 34.

We shall call this geometrical representative *AB* of the couple simply the **vector** of the couple. It is sometimes called its *moment*, or its *axis*, or its *axial moment*.

133. As was pointed out in Art. 112, a couple is equivalent to a single force acting along a line at infinity. Couples are, therefore, used in statics to avoid the introduction of such forces whose line of action is at an infinite distance, just as in kinematics a rotation about an axis at infinity receives the special name of *translation*, and an angular velocity about an axis at infinity is called a *velocity of translation*.

It has been shown in *Kinematics*, Arts. 64, 65, that two equal and opposite rotations about parallel axes produce a translation, and in *Kinematics*, Art. 256, that two equal and opposite angular velocities about parallel axes produce a velocity of translation; similarly, two equal and opposite forces along parallel lines form a new kind of quantity called a *couple of forces*, or simply a *couple*.

While rotations, angular velocities, and forces are represented by *rotors*, *i.e.* by vectors confined to definite lines, translations, velocities of translation, and couples have for their geometrical representatives vectors not confined to particular lines.

Just as in the case of couples of infinitesimal rotations and of couples of angular velocities, the vector representing a couple of forces has for its magnitude and sense those of the moment of the couple, and for its direction that perpendicular to the plane of the couple.

It is due to this analogy between the two fundamental conceptions that a certain dualism exists between the theories of statics and kinematics, so that a large portion of the theory of kinematics of a rigid body might be made directly available for statics by simply substituting for angular velocity and velocity of translation the corresponding ideas of force and couple.

134. When any number of couples act on a rigid body their resultant can readily be found. Representing each couple by its vector, we have only to combine these vectors by geometrical addition. In the particular case when the couples all lie in parallel planes, or in the same plane, their vectors may be taken in the same line, and add, therefore, algebraically.

Hence, *the resultant of any number of couples is a single couple whose vector is the geometric sum of the vectors of the given couples.*

Conversely, a couple can be resolved into components by resolving its vector into components.

135. To combine a single force P with a couple (F, p) lying in the same plane it is only necessary to place the couple in its plane into such a position (Fig. 35) that one of its forces, say $-F$, shall lie in the same line and in opposite sense with the single force P, and to transform the couple (F, p) into a couple (P, p'), by Art. 131, so that $Fp = Pp'$. The original single force P and the force $-P$ of the transformed couple destroying each other at A, there remains only the other force P, at A', of the transformed couple which is par-

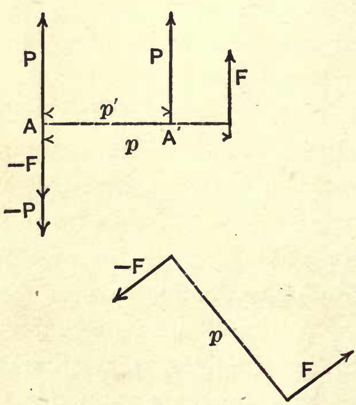

Fig. 35.

allel and equal to the original single force P, and has the distance

$$p' = \frac{F}{P} p$$

from it.

Hence, *a couple and a single force in the same plane are together equivalent to a single force equal and parallel to, and of the same sense with, the given force, but at a distance from it which is found by dividing the moment of the couple by the single force.*

136. Conversely, *a single force* P *applied at a point* A *of a rigid body can always be replaced by an equal and parallel force* P *of the same sense, applied at any other point* A′ *of the same body, in connection with the couple formed by* P *at* A *and* $-$P *at* A′.

137. The proposition of Art. 135 applies even when the force lies in a plane parallel to that of the couple, since the couple can be transferred to any parallel plane without changing its effect.

If the single force intersects the plane of the couple, it can be resolved into two components, one lying in the plane of the couple, while the other is at right angles to this plane. On the former component the couple has, according to Art. 135, the effect of transferring it to a parallel line. We thus obtain *two non-intersecting,* or *skew, forces at right angles to each other.*

Let P be the given force, and let it make the angle α with the plane of the given couple, whose force is F and whose arm is p. Then $P \sin \alpha$ is the component at right angles to the plane of the couple, while $P \cos \alpha$ combines with the couple whose moment is Fp to a force $P \cos \alpha$ in the plane of the couple; this force $P \cos \alpha$ is parallel to the projection of P on the plane, and has the distance $\frac{Fp}{P \cos \alpha}$ from this projection.

Hence, in the most general case, *the combination of a single force and a couple can be replaced by the combination of two single forces crossing each other at right angles;* it can be reduced to a single force only when the force is parallel to the plane of the couple.

138. Exercises.

(1) Show that the moment of a couple can be represented by the area of the parallelogram formed by the two forces of the couple, or by twice the area of the triangle formed by joining any point on the line of one of the forces to the ends of the other force.

(2) Show that the sum of the moments of two forces forming a couple is the same for any point in the plane of the couple.

(3) Show, by means of Arts. 129–131, how to combine any number of couples situated in the same plane, or in parallel planes.

(4) Find the resultant of two couples situated in non-parallel planes, without using the vectors of the couples.

V. *Plane Statics.*

I. THE CONDITIONS OF EQUILIBRIUM.

139. Suppose a rigid body to be acted upon by any number of forces, all of which are situated in the same plane. To reduce such a *plane system of forces* to its simplest form the proposition of Art. 136 may be used. This proposition allows us to transfer all the forces to a common origin, by introducing, in addition to each force, a certain couple in the same plane. The concurrent forces can then be combined into their resultant by geometric addition, or by forming their force polygon (Art. 98); and the couples lying all in the same plane combine by algebraic addition of their moments into a resultant couple (Art. 134).

Thus, let F (Fig. 36) be one of the forces of the given plane system, P its point of application. Selecting any point O in the plane as origin, apply at O two equal and opposite forces F, $-F$, each equal and parallel to the given force F; and let p be the perpendicular distance of the origin O from the line of action of the given force F. The force F at P is equivalent to the force F at O in connection with the couple formed by F at P and $-F$ at O; the moment of this couple is Fp, its vector is perpendicular to the plane of the system.

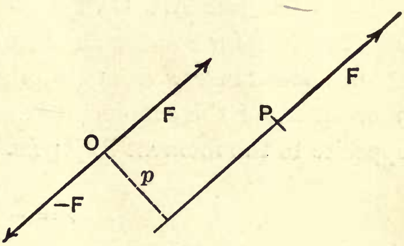

Fig. 36.

Proceeding in the same way with every force of the given system, all forces are transferred to the common origin O. The whole system is therefore equivalent to their resultant R passing through O, in connection with the resulting couple $H = \Sigma Fp$.

140. The given system of forces will be in equilibrium if the following two **conditions of equilibrium** are fulfilled:

$$R=0, \quad H=0.$$

It will be noticed that the moment Fp of the couple introduced by transferring the force F to the point O is the moment of the force F with respect to this point O.

Hence, *a plane system of forces is in equilibrium if* (a) *its resultant is zero, and* (b) *the algebraic sum of the moments of all its forces is zero with respect to any point in its plane.*

141. It is evident that the magnitude and direction of the resultant R do not depend on the selection of the origin O. But the position of this resultant and the magnitude of the resulting couple H will in general differ for different points selected as origin. Indeed, the origin can be so taken as to make the couple H vanish (unless the resultant R be zero); that is, the whole system can be reduced to a single resultant.

To do this (see Art. 135), it is only necessary, after determining R and H for some point O, to transfer R to a parallel line at such a distance r from its original position as to make the moment Rr of the couple introduced by the transfer equal and opposite to the moment ΣFp; *i.e.* we must take (Art. 135)

$$r=-\frac{H}{R}.$$

The line along which this single resultant acts is called the **central axis** of the given system of forces.

142. For a purely analytical reduction of a plane system of forces the system is referred to rectangular axes Ox, Oy, arbitrarily assumed in the plane (Fig. 37). Every force F is resolved at its point of application $P(x, y)$ into two components X, Y, parallel to the axes, so that

$$X=F\cos\alpha, \quad Y=F\sin\alpha,$$

α being the angle made by F with the axis Ox. At the origin O two equal and opposite forces X, $-X$ are applied along Ox, and two equal and opposite forces Y, $-Y$ along Oy. Thus, X at P is equivalent to X at O in combination with the couple formed by X at P and $-X$ at O; the moment of this couple is evidently $-yX$. Similarly, Y at P is replaced by Y at O in combination with a couple whose moment is xY.

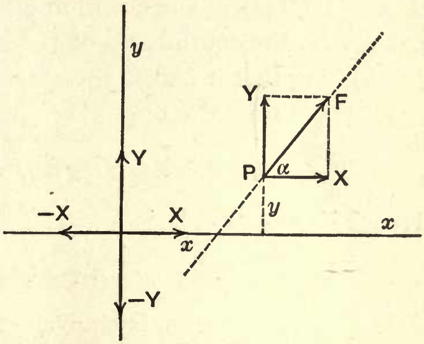

Fig. 37.

The force F at P is therefore equivalent to the two forces X, Y at O in combination with a couple whose moment is $xY-yX$.

Proceeding in the same way with every given force, we obtain a number of forces X along Ox which can be added algebraically into ΣX, and a number of forces Y along Oy which give ΣY. These two rectangular forces form the resultant

$$R=\sqrt{(\Sigma X)^2+(\Sigma Y)^2}$$

whose direction is given by

$$\tan\alpha=\frac{\Sigma Y}{\Sigma X},$$

where α is the angle between Ox and R.

In addition to this, we obtain a number of couples $xY-yX$ whose algebraic sum forms the resulting couple

$$H=\Sigma(xY-yX).$$

143. The whole system is thus found equivalent to a resultant force R in combination with a resultant couple H in the same plane with R. The *conditions of equilibrium* $R=0$, $H=0$ (Art. 140) can therefore be expressed analytically by the three equations

$$\Sigma X=0,\quad \Sigma Y=0,\quad \Sigma(xY-yX)=0.$$

144. If R be not zero, R and H can be combined into a single resultant R' equal and parallel to R at the distance $-H/R$ from it (see Art 141). The equation of the line of this single resultant R', *i.e.* the central axis of the system of forces, is found by considering that it makes the angle α with the axis of x and that its distance from the origin is

$$H/R = \Sigma(xY - yX)/\sqrt{(\Sigma X)^2 + (\Sigma Y)^2}.$$

Hence its equation is

$$\xi \cdot \Sigma Y - \eta \cdot \Sigma X - \Sigma(xY - yX) = 0.$$

If $R = 0$, the system is equivalent to the couple

$$H = \Sigma(xY - yX),$$

unless H itself be also zero, in which case the system is in equilibrium.

145. The same results can be obtained by a transformation of co-ordinates. Let $R = \sqrt{(\Sigma X)^2 + (\Sigma Y)^2}$ and $H = \Sigma(xY - yX)$ be the resultant force and couple for a point O as origin. If some other point O', whose co-ordinates with respect to O are ξ, η, be taken as new origin and x', y' be the co-ordinates of the point of application P of the force F for parallel axes through O', the resultant R remains the same while the resulting couple becomes

$$H' = \Sigma(x'Y - y'X) = \Sigma[(x - \xi)Y - (y - \eta)X]$$
$$= H - \xi\Sigma Y + \eta\Sigma X.$$

Hence this new couple will vanish whenever the origin $O'(\xi, \eta)$ is taken on the straight line whose equation referred to the original axes is

$$\Sigma Y \cdot \xi - \Sigma X \cdot \eta - H = 0.$$

This equation of the central axis agrees with the equation found in Art. 144; it represents the line of action of the single resultant to which the system can be reduced.

146. The following examples will illustrate the application of the conditions of equilibrium. To establish these conditions in any particular problem it will generally be found best to resolve the forces along two rectangular directions and equate the sums of the components to zero; and then to "take moments," *i.e.* equate to zero the sum of the moments of all the forces with respect to some point conveniently selected as origin.

147. *A homogeneous straight rod* AB = 2l (Fig. 38) *of weight* W *rests with one end* A *on a smooth horizontal plane* AH, *and with the point* E(AE = e) *on a cylindrical support, the axis of the cylinder being at right angles to the vertical plane containing the rod. Determine what horizontal force* F *must be applied at a given point* F *of the rod* (AF = f > e) *to keep the rod in equilibrium when inclined to the horizon at an angle* θ.

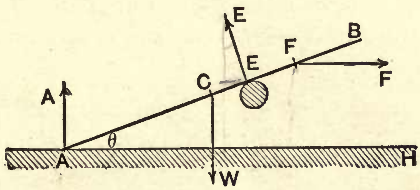

Fig. 38.

The rod exerts a certain unknown pressure on each of the supports at A and E, in the direction of the normals to the surfaces of contact, provided there be no friction, as is here assumed. The supports may therefore be imagined removed if forces A, E, equal and opposite to these pressures, be introduced; these forces A, E are called the *reactions* of the supports. The rod itself is here regarded as a straight line; its weight W is applied at its middle point C.

Taking A as origin and AH as axis of x, the resolution of the forces gives

$$\Sigma X \equiv F - E\sin\theta = 0, \tag{1}$$

$$\Sigma Y \equiv A - W + E\cos\theta = 0. \tag{2}$$

Taking moments about A, we find

$$E \cdot e - W \cdot l\cos\theta - F \cdot f\sin\theta = 0. \tag{3}$$

Eliminating F from (1) and (3), we have

$$E = \frac{l\cos\theta}{e - f\sin^2\theta} W;$$

hence from (2),

$$A = \left(1 - \frac{l\cos^2\theta}{e - f\sin^2\theta}\right) W$$

and finally from (1),

$$F = \frac{l\sin\theta\cos\theta}{e - f\sin^2\theta} W.$$

148. *A cylinder of length* 2l *and radius* r *rests with the point* A *of the circumference of its lower base on a horizontal plane and with the point* B *of the circumference of its upper base against a vertical wall. The vertical plane through the axis of the cylinder contains the points* A, B, *and is perpendicular to the intersection of the vertical wall and the horizontal plane. If there be no friction at* A *and* B, *what horizontal force* F *applied at* A *will keep the cylinder in equilibrium? When is this force* F = 0?

Let G be the centre of gravity of the cylinder; W its weight; A, B the reactions at A, B; and θ the given angle between AB and the horizontal plane. Then $B - F = 0$, $A - W = 0$, and taking moments about A,

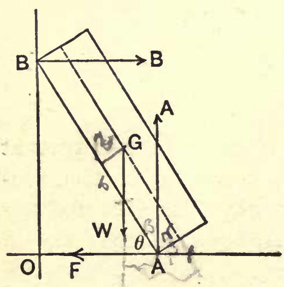

Fig. 39.

$$W(l\cos\theta - r\sin\theta) = B \cdot 2l\sin\theta.$$

Hence $A = W$,

$$B = F = W \cdot \frac{l\cos\theta - r\sin\theta}{2l\sin\theta}$$

$$F = \tfrac{1}{2}\left(\cot\theta - \frac{r}{l}\right) \cdot W.$$

If either the dimensions of the cylinder, or the angle θ, be such as to make $\tan\theta = l/r$, no force F will be required to maintain equilibrium; G and A will then lie in the same vertical line.

149. *The homogeneous rod* AB = 2l *of weight* W *is jointed at* A, *so as to turn about* A *in a vertical plane. A string* BC *attached to the end* B *of the rod runs at* C *over a smooth pulley, and carries a weight* P. *The axis of the pulley* C *is parallel to, and in the same vertical plane* *with, the axis of the joint* A ; AC = h. *Find the position of equilibrium and the pressure on the axis of the joint* A. (Fig. 40.)

To reduce to a purely statical problem, cut the string between B and C and introduce the tension, which is $= P$; also, replace the pressure A by its horizontal and vertical components A_x, A_y. Then, if $\measuredangle ACB = \phi$, $\measuredangle BAC = \theta$, the conditions of equilibrium give

$$A_x = P \sin\phi, \quad A_y = W - P\cos\phi,$$

$$P \cdot h \sin\phi = W \cdot l \sin\theta.$$

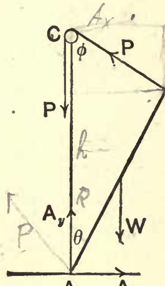

Fig. 40.

From the last equation,

$$\frac{\sin\phi}{\sin\theta} = \frac{l}{h} \cdot \frac{W}{P},$$

while from the triangle ABC,

$$\frac{\sin\phi}{\sin\theta} = \frac{2l}{BC};$$

hence $BC = 2hP/W$, *i.e.* if we take h to represent W, P will be represented by $\frac{1}{2}BC$.

For the total pressure A we have

$$A^2 = A_x^2 + A_y^2 = W^2 + P^2 - 2WP\cos\phi,$$

i.e. A is the third side of a triangle having W and P for the two other sides, and ϕ for the included angle. The magnitude of A is therefore represented by the median from A in the triangle ABC on the same scale on which W is represented by h. But this median gives also the direction of A; for we have

$$\frac{A_y}{A_x} = \frac{W - P\cos\phi}{P\sin\phi} = \frac{h - \frac{BC}{2}\cos\phi}{\frac{BC}{2}\sin\phi}.$$

150. *A weightless rod* AB *rests without friction on two planes inclined to the horizon at angles* α, β, *and carries a weight* W *at the point* D. *The intersection* (C) *of these planes is horizontal and at right angles to the vertical plane through* AB. *Find the inclination* θ *of* AB *to the horizon, and the pressures at* A *and* B. (Fig. 41.)

As there are only three forces, viz. the weight W and the reactions A and B, their lines must intersect in a point E. Resolving horizontally and vertically, we have

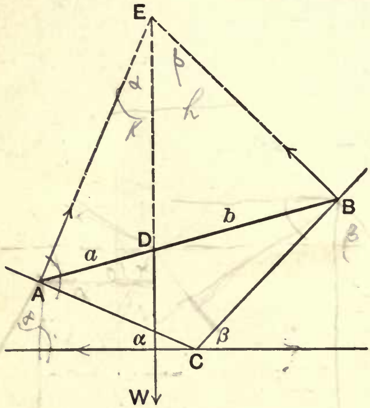

Fig. 41.

$$A \sin\alpha = B \sin\beta,$$

$$A \cos\alpha + B \cos\beta = W,$$

whence $$A = \frac{\sin\beta}{\sin(\alpha+\beta)} W,$$

$$B = \frac{\sin\alpha}{\sin(\alpha+\beta)} W.$$

Taking moments about D, we find with $AD = a$, $DB = b$,

$$A \cdot a \sin DAE = B \cdot b \sin DBE,$$

or $$Aa \cos(\alpha+\theta) = Bb \cos(\beta-\theta);$$

to eliminate A and B, divide by the first equation above:

$$a\frac{\cos(\alpha+\theta)}{\sin\alpha} = b\frac{\cos(\beta-\theta)}{\sin\beta};$$

solving for θ, we finally obtain

$$\tan\theta = \frac{a\cot\alpha - b\cot\beta}{a+b}.$$

151. Exercises.

(1) A homogeneous rod $AB = 2l = 8$ ft., weighing $W = 20$ lbs., rests with one end A on a horizontal plane AH, and with the point E on a support whose height above AH is $DE = h = 3$ ft. A horizontal cord $AD = d = 4$ ft. holds the rod in equilibrium. Find the tension T of this cord, and the reactions at A and E.

(2) A weightless rod AB of length l can turn freely about one end A in a vertical plane. A weight W is suspended from a point C of the rod; $AC = c$. A string BD attached to the end B of the rod holds it in equilibrium in a horizontal position, the angle ABD being $\alpha = 150°$. Find the tension T of the string and the resulting pressure A on the hinge at A.

(3) A uniform rod $AB = 2l$ of weight W rests with its upper end A against a smooth vertical wall, while its lower end B is fastened by a string of given length, $BC = 2b$, to a point C in the wall. The rod and the string are in the vertical plane at right angles to the wall. Find the position of equilibrium, *i.e.* the angle $\phi = ACB$, the tension T of the string, and the pressure A against the wall.

(4) A uniform rod $AB = 2l$ of weight W rests with one end A on a smooth horizontal plane AC, with the other end B against a smooth vertical wall BC, the vertical plane through AB being at right angles to the intersection C of the wall with the horizontal plane. The rod is kept in equilibrium by a string EC. Find the tension T of this string if the angles $CAB = \theta$ and $ECA = \phi$ are given.

(5) A weightless rod $AB = l$ can revolve in a vertical plane about a hinge at A; its other end B leans against a smooth vertical wall whose distance from A is $AD = a$. At the distance $AC = c$ from A, a weight W is suspended. Find the horizontal thrust A_x at A and the normal pressures A_y and B at A and B.

(6) The same as (5) except that at B the rod rests on a smooth horizontal cylinder whose axis is at right angles to the vertical plane through AB. In which of the two problems is the horizontal thrust A_x at A least?

(7) The lower end A of a smooth uniform rod AB of weight W rests on a smooth horizontal plane making an angle θ with it. At the point C it rests on a smooth cylinder whose axis is horizontal and at right angles to the vertical plane through the rod; at D the rod is pressed upon by another smooth cylinder whose axis is parallel to that of the cylinder at C. Determine the reactions at A, C, D, if W, θ, $AB = 2l$, $CD = a$ are given.

(8) A smooth weightless rod $AB = l$ rests at C on a smooth horizontal cylinder whose axis is at right angles to the vertical plane through the rod; its lower end A leans against a smooth vertical wall whose distance from C is $CD = a$; from its upper end B a weight W is suspended. Determine the distance $AC = x$ for equilibrium, and the reactions at A and C.

(9) A uniform rod of weight W is hinged at its lower end A, while its upper end B leans against a smooth vertical wall. The rod is inclined at an angle θ to the vertical, and carries three weights, each equal to w, at three points dividing the rod into four equal parts. Determine the pressure on the wall and the reaction of the hinge.

(10) A homogeneous rod $AB = 2l$ of weight W rests with one end A on the inside of a fixed hemispherical bowl of diameter $2a$ and leans at C on the horizontal rim of the bowl, so that the other end B is outside. Determine the inclination to the horizon θ in the position of equilibrium.

2. STABILITY.

152. The equilibrium of the forces acting on a rigid body may subsist while the body is in motion. Thus, if the motion consist in a mere translation with constant velocity, the equilibrium will not be disturbed during the motion if the forces remain equal and parallel to themselves.

If, however, the body be subjected to a rotation, this will in general not be the case. The present considerations are restricted to the case of plane motion; the forces are supposed to lie in the plane of the motion and to remain equal and parallel to themselves and applied at the same points of the body.

153. Let A_1A_2 (Figs. 42 and 43) be a rigid rod having two equal and opposite forces F_1, F_2 applied at its extremities in the

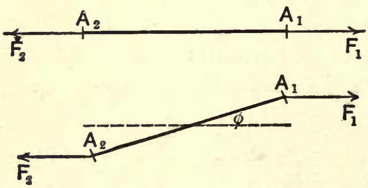

Fig. 42.

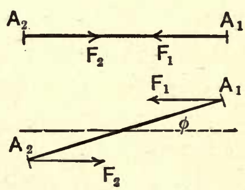

Fig. 43.

direction of the line A_1A_2. Let this rod be turned by an angle ϕ about an axis at right angles to A_1A_2. In the new position the forces F_1, F_2, instead of being in equilibrium, form a couple whose moment is $\pm F_1 \cdot A_1A_2 \sin\phi$.

If in the original position of the rod the forces tend to increase the distance A_1A_2 (Fig. 42), the couple in the new position will tend to bring the rod back to its position of equilibrium. In this case the original position of the rod is said to be a position of **stable equilibrium.** The effect of the earth's magnetism on the needle of a compass offers a familiar example.

If, however, in the original position the forces tend to diminish the distance A_1A_2 (Fig. 43), the couple arising after displacement tends to increase the displacement and thus to remove the rod still farther from equilibrium. The original position in this case is said to be one of **unstable equilibrium.** The weight resting on a vertical post and the reaction of the support on which the rod stands may be taken as an illustration.

Finally, a third case would arise if the forces F_1, F_2, being still equal and opposite, were applied at one and the same point of the rod. The forces would then remain in equilibrium after any displacement of the rod; such equilibrium is called **neutral** or **astatic.**

154. These different cases of equilibrium can be distinguished by the algebraic sign of the product $A_1A_2 \cdot F_1 = A_2A_1 \cdot F_2$, which is negative for stable equilibrium, since A_1A_2 and F_1 have opposite sense (Fig. 42), positive for unstable equilibrium (Fig. 43), and indeterminate (since $A_1A_2 = 0$) for neutral equilibrium.

It is to be noticed that these considerations will hold whether the rotation of angle ϕ take place in the positive or negative sense. But they hold only within certain limits for the angle of rotation. Thus, in the example illustrated by Figs. 42 and 43, when ϕ reaches the value π, the nature of the equilibrium is changed.

155. Strictly speaking, investigations of stability are not purely statical, but require a kinetic examination of the subsequent motion. However, the principles of statics are sufficient to determine the nature of the equilibrium for infinitesimal displacements, *i.e.* when only the initial motion of the body is considered.

The theory of astatic equilibrium forms a special branch of mechanics called **astatics**; its object is to determine the conditions under which a system of forces acting on a rigid body remains in equilibrium when the body is subjected to any displacement while the forces remain applied at the same points of the body and retain their magnitude, direction, and sense.

156. The equilibrium of forces acting on one and the same point is evidently always astatic.

In the case of a plane system of forces acting on a plane figure in its plane, the only displacement that need be considered is a rotation about an axis at right angles to the plane. For every displacement of a plane figure in its plane can be reduced to a rotation about a certain centre in the plane.

Instead of turning the body or plane figure by an angle ϕ, we may turn all the forces about their points of application by the same angle in the opposite sense.

157. If the plane system consists of two forces in equilibrium, they must be equal and opposite, and act in the same line; this case has been considered in Art. 154.

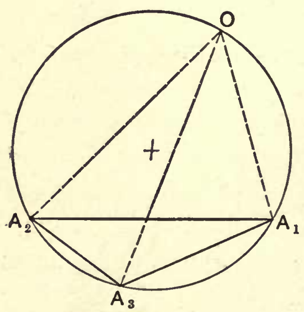

Fig. 44.

If there be three forces F_1, F_2, F_3 in the same plane in equilibrium, applied at the points A_1, A_2, A_3, they must meet in a point O, and fulfil the parallelogram law.

After turning each force about its point of application by the same angle, the forces will, in general, cease to intersect in a point, and hence to be in equilibrium. If, however, the original meeting point O of the forces be situated on the circle described through A_1, A_2, A_3 (Fig. 44), the forces will continue to intersect at some point of this circle when turned through some angle, because the angles between the forces remain constant.

Hence, *three forces* F_1, F_2, F_3 *in the same plane, applied at points* A_1, A_2, A_3, *are in astatic equilibrium if they meet in a point* O *situated on the circle passing through* A_1, A_2, A_3.

The condition of equilibrium of the three forces also requires that

$$\frac{F_1}{\sin(F_2F_3)} = \frac{F_2}{\sin(F_3F_1)} = \frac{F_3}{\sin(F_1F_2)};$$

by the property of the circle (Fig. 44), we have $\measuredangle(F_2F_3) = A_1$, $\measuredangle(F_3F_1) = A_2$, $\measuredangle(F_1F_2) = \pi - A_3$; and as the sines of these angles of the triangle $A_1A_2A_3$ are proportional to the opposite sides, we have

$$\frac{F_1}{A_2A_3} = \frac{F_2}{A_3A_1} = \frac{F_3}{A_1A_2};$$

i.e. *three forces in astatic equilibrium are to each other as the sides of the triangle formed by their points of application.*

158. The results of the preceding article can be interpreted from a somewhat different point of view. Let two of the forces, F_1 and F_2, be given, and let it be required to determine their resultant for astatic equilibrium. This resultant F_3 must evidently pass through a definite point A_3 of the circle described through the points of application A_1, A_2 of the given forces and their intersection O. This point A_3, through which the resultant must pass, howsoever the two given forces be turned about A_1, A_2, is called the **centre of the forces.**

If the two given forces be parallel, the point O lies at infinity, and the circle through A_1, A_2, O becomes the straight line A_1A_2. The point A_3 is therefore situated on this line and divides the distance A_1A_2 in the inverse ratio of the forces F_1, F_2, by Art. 157. Compare Art. 110.

159. These results are readily generalised. Any plane system of forces has a centre unless the resultant be zero. To find the centre we have only to combine the forces in succession, *i.e.* to find the centre of two of the forces, then the centre of their resultant and a third force, etc.

It has been shown in Arts. 141 and 144 that a plane system of forces whose resultant does not vanish can always be reduced to a single resultant R whose line is called the *central axis* of the system. It appears now that if the forces be all turned by the same angle θ about their points of application, the line of the resultant, or the central axis, will turn about a certain fixed point called the **centre** of the system. For a system of parallel forces the existence of such a centre has already been proved in Art. 110.

160. Analytically, the centre of a plane system of forces is found as the intersection of the two positions of the central axis before and after any displacement of the plane figure, or body, on which the forces act.

By Arts. 144 and 145 the equation of the central axis is

$$\Sigma Y\cdot\xi-\Sigma x\cdot\eta-\Sigma(xY-yX)=0. \tag{1}$$

Let the figure with the axes of co-ordinates be turned through an angle ϕ about an axis through the origin perpendicular to its plane, while the forces keep their original directions. The central axis of the forces in the new position will have an equation of the same form as before, in which, however, x, y, ξ, η are referred to the new system of co-ordinates. To find the equation of the central axis in the old co-ordinates, we have to substitute $x\cos\phi-y\sin\phi$ for x, $x\sin\phi+y\cos\phi$ for y, and similarly for ξ, η. This gives

$$\Sigma Y\cdot(\xi\cos\phi-\eta\sin\phi)-\Sigma X\cdot(\xi\sin\phi+\eta\cos\phi)$$
$$-\Sigma\left[(x\cos\phi-y\sin\phi)\,Y-(x\sin\phi+y\cos\phi)X\right]=0,$$

or collecting the terms containing $\cos\phi$ and $\sin\phi$, respectively,

$$[\Sigma Y\cdot\xi-\Sigma X\cdot\eta-\Sigma(xY-yX)]\cos\phi$$
$$-[\Sigma x\cdot\xi+\Sigma Y\cdot\eta-\Sigma(xX+yY)]\sin\phi=0. \tag{2}$$

The centre being the intersection of the lines (1) and (2), its co-ordinates are found by solving these equation for ξ and η, or ...e the coefficient of $\cos\phi$ in (2) vanishes by (1), by solving the equations

$$\Sigma Y\cdot\xi-\Sigma X\cdot\eta=\Sigma(xY-yX), \tag{3}$$

$$\Sigma x\cdot\xi+\Sigma Y\cdot\eta=\Sigma(xX+yY). \tag{4}$$

Putting, for shortness, $\sqrt{(\Sigma X)^2+(\Sigma Y)^2}=R$, $\Sigma(xY-yX)=H$, $\Sigma(xX+yY)=K$, we find the co-ordinates of the centre,

$$\xi=\frac{\Sigma Y\cdot H+\Sigma X\cdot K}{R^2},\quad \eta=\frac{\Sigma Y\cdot K-\Sigma X\cdot H}{R^2}. \tag{5}$$

161. By the rotation of the figure, the magnitude of the resultant R of the system is of course not changed. But the resulting couple H for the origin, or what amounts to the same, the moment of the system about the origin, is changed and becomes, by Art. 160,

$$\begin{aligned} H'&=\Sigma[(x\cos\phi-y\sin\phi)Y-(x\sin\phi+y\cos\phi)X] \\ &=\Sigma(xY-yX)\cdot\cos\phi-\Sigma(xX+yY)\cdot\sin\phi \\ &=H\cos\phi-K\sin\phi. \end{aligned} \tag{6}$$

This couple H' vanishes if the figure be turned through an angle ϕ determined by the equation

$$\tan\phi=\frac{H}{K}. \tag{7}$$

162. If the system of forces be originally in equilibrium, we have $\Sigma X=0$, $\Sigma Y=0$, $\Sigma(xY-yX)=0$ (Art. 143). Hence after turning the figure through an angle ϕ, the forces will be equivalent to the couple

$$H'=-K\sin\phi. \tag{8}$$

This couple has its greatest value when $\phi=\pi/2$; it vanishes only when $\phi=\pi$, in which case the system will again be in equilibrium.

163. The stability of a plane system in equilibrium depends on the algebraic sign of the quantity,

$$K=\Sigma(xX+yY), \quad (9)$$

which can therefore be called the **stability function.** If this function be positive, the equilibrium is stable; if it be negative, the equilibrium is unstable; finally, if $K=0$, the system is astatic, and the equilibrium is neutral.

The proof follows at once from equation (8). This equation shows that, for a positive K, the moment of the couple to which the system becomes equivalent when the figure is turned through an angle ϕ has a sign opposite to that of the angle ϕ; hence this couple will tend to turn the body back into the position of equilibrium. Similarly, if K be negative, H' agrees in sign with ϕ and tends therefore to increase this angle.

164. Exercises.

(1) Explain the nature of the equilibrium of a body of weight W supported at a single point according to the position of that point above the centroid G, below G, and at G (*common balance*).

(2) A homogeneous rod $AB=2l$ of weight W leans with the lower end A against a vertical wall and rests with the point C ($AC=c>l$) on a cylindrical support. Show that the equilibrium is unstable.

(3) A body of weight W is placed on a horizontal plane. Show that the equilibrium is stable if W meets the horizontal plane at a point A within the area of contact and that it is unstable if A lies on the contour of this area. If the actual area of contact have re-entrant angles, or consist of several detached portions, the area bounded by a thread drawn tightly around the actual area, or areas, of contact must be substituted.

(4) An oblique cylinder rests with its circular base on a horizontal plane in unstable equilibrium. If the length of its axis be twice the diameter of its base, what is the inclination of the axis to the horizon?

(5) Show how to determine graphically the stability of a retaining wall against toppling over the front edge of the base, the pressure of the earth behind the wall being given in magnitude, direction, and position.

3. JOINTED FRAMES.

165. The equations of equilibrium are derived on the suppositions that all the forces of the given system act on one and the same rigid body and that this body is perfectly free to move. Hence, in applying these equations to determine the equilibrium of an engineering structure, a machine, etc., each rigid body must be considered separately, and the reactions required to make the body free must be introduced. It will be shown in a subsequent section how the principle of work makes it possible to dispense with some of these precautions.

When two rigid rods are connected by a pin-joint whose axis is perpendicular to the plane of the rods, the action of either rod on the other at the joint is represented by a single force whose direction is in general unknown. Sometimes considerations of symmetry will allow to determine this direction.

If a rigid rod, in equilibrium, be hinged at both ends and not acted upon by any other forces, the reactions of the hinges must of course be along the rod, and must be equal and opposite.

166. *Two rods* AC, BC (Fig. 45) *in a vertical plane, hinged together at* C, *rest with the ends* A, B *on a horizontal plane, and carry a weight* W *suspended from the joint* C. *If the proper weight of the rods be neglected, determine the normal pressures* A_y, B_y *and the horizontal thrusts* A_x, B_x *at* A, B.

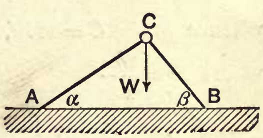

Fig. 45.

Resolving the weight W along CA, CB into W_A, W_B and considering the rod AC alone it appears that the total reaction at A is along AC and $= W_A$; hence resolving W_A in the horizontal and vertical directions, A_x and A_y are found; similarly for BC. If α, β be the angles at A and B in the triangle ABC, we find

$$W_A = \frac{\cos\beta}{\sin(\alpha+\beta)} W, \quad W_B = \frac{\cos\alpha}{\sin(\alpha+\beta)} W;$$

$$A_x = B_x = \frac{\cos\alpha\cos\beta}{\sin(\alpha+\beta)}W, \quad A_y = \frac{\sin\alpha\cos\beta}{\sin(\alpha+\beta)}W, \quad B_y = \frac{\cos\alpha\sin\beta}{\sin(\alpha+\beta)}W.$$

As the horizontal thrusts at A and B are equal, it makes no difference whether the rods be hinged to the support at A and B, or whether the thrust is taken up by lateral supports, or by a string connecting the ends A, B of the rods.

167. *Two equal homogeneous rods* AC, BC (Fig. 46) *are hinged at* A, B, C *so as to form a triangle whose height* h *is vertical and whose base* AB = 2b *is horizontal. The weight of each rod being* W, *find the reactions at the joints.*

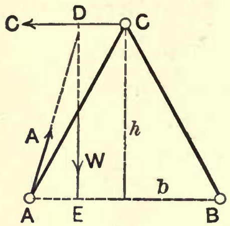

Fig. 46.

Owing to the symmetry of the figure, the reactions at C must be equal and opposite and horizontal. The rod AC is subject to three forces only, viz. the horizontal reaction C, the weight W, and the reaction A; the latter must therefore pass through the intersection D of C and W.

If the direction of W intersect AB at E and the scale of forces be taken so as to have W represented by $DE = h$, DEA will be the force polygon; hence EA represents C and AD represents A on the same scale on which W is represented by h.

Analytically, the reactions are found by resolving the forces horizontally and vertically and taking moments about A:

$$A_x = C, \quad A_y = W, \quad C \cdot h = W \cdot \frac{b}{2};$$

whence $$C = mW, \quad A = \sqrt{A_x^2 + A_y^2} = \sqrt{m^2+1} \cdot W,$$

where $m = \dfrac{b}{2h}$.

168. *Two equal homogeneous rods* AC, BC, *each of weight* W, *are hinged at* C; *their ends* A, B *rest on a smooth horizontal plane; a third rod* DE *is hinged to them, connecting their middle points* (Fig. 47).

The plane AB being smooth, the reaction at A is vertical; the reaction at C is horizontal owing to the symmetry; that at D is likewise horizontal if the weight of the rod DE be neglected, for then this rod is subject only to the reactions at its ends.

Resolving horizontally and vertically and taking moments about D, we find in this case

$A = W, \quad C = D = W \cot \alpha,$

where $\alpha = \measuredangle BAC$.

If, however, the weight w of the rod DE cannot be neglected, we have at D a horizontal reaction D_x and a vertical reaction D_y. The equilibrium of DE requires that $2D_y = w$. Hence resolving and taking moments as before, we find

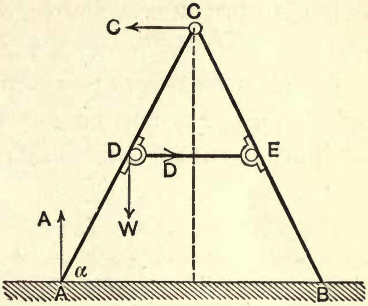

Fig. 47.

$$A = W + \tfrac{1}{2}w, \quad C = D_x = (W + \tfrac{1}{2}w) \cot \alpha, \quad D_y = \tfrac{1}{2}w.$$

169. Exercise.

(1) *Two homogeneous rods* AC, BC *of equal weight, but unequal length, are hinged together at* C *while their other ends are attached to fixed hinges* A, B *in the same vertical line. Show that the line of action of the reaction at* C *bisects* AB.

170. A triangular frame formed of rigid rods is rigid as a whole, even when the connections are pin-joints. A quadrangular frame with pin-joints becomes rigid only by the insertion of a diagonal.

The iron and steel trusses used for roofs and bridges generally consist of a system of triangles, or quadrangles with diagonals, so that the whole truss can be regarded as one rigid body, at least in first approximation.

Any one rod, or *member*, of the frame-work is thus acted upon by two equal and opposite forces, *i.e.* by a *stress*, in the direction of its length, the external forces, including the proper weight, being regarded as applied at the joints only. If the stress be a *tension*, *i.e.* if the forces tend to stretch or elongate the member, the latter is called a **tie**; a member subject to *compression* or crushing is called a **strut**.

171. For the purpose of dimensioning the members, it is necessary to know the stress in every member. The following example illustrates a simple method for finding these stresses when the external forces are given.

Let the frame-work represented in Fig. 48 be cut in two along any line $\alpha\beta$; the portion on either side of this line must be in equilibrium under the action of its external forces and the

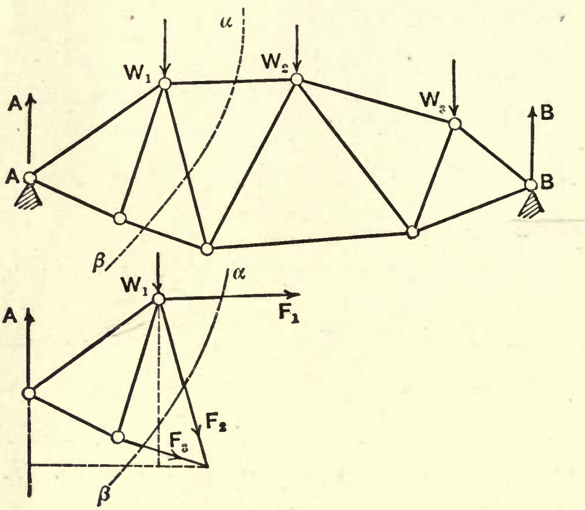

Fig. 48.

stresses in the members intersected by $\alpha\beta$. Thus, in the figure, the forces A, W_1, F_1, F_2, F_3 form a system in equilibrium; hence, the sum of the moments of these forces with respect to any point must vanish.

To determine F_1, take moments about the intersection of F_2 and F_3; thus F_2 and F_3 are eliminated from the equation of moments, and F_1 is found. Similarly F_2 is obtained by taking moments about the intersection of F_3 and F_1. The arms of the moments are best taken from a correctly drawn diagram of the frame-work.

If only two members be intersected by $\alpha\beta$, the origin for the moments is taken first on one, then on the other, of the two members intersected.

By beginning at one of the supports and taking sections through the successive panels, it will in the more simple cases be possible to draw the line $\alpha\beta$ so as to intersect not more than three members whose stresses are unknown. Thus the stresses in all the members can be determined.

172. Exercises.

(1) Find the stresses in the braced beam *AB* (Fig. 49), carrying a weight of 5 tons at each joint of the upper chord. The horizontal width of the panels is 10 ft., the middle vertical is 8 ft.

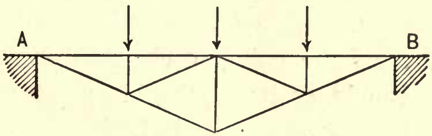

Fig. 49.

(2) In Fig. 50, the dimensions are in feet, the loads in tons. After the first panel the sections cannot be so taken as to intersect not more than three unknown stresses. But the girder can be regarded as obtained by the superposition of two girders (each carrying half the load), in one of which the diagonals *CF*, *EH* are wanting, while in the other *DE*, *FG* are wanting. Each of these can readily be computed.

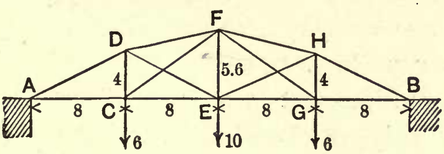

Fig. 50.

4. GRAPHICAL METHODS.

173. The graphical method explained in Art. 108 for determining the resultant of a system of parallel forces can be extended without difficulty to the general case of a plane system of forces. The only difference will appear in the form of the force polygon, which for parallel forces collapses into a straight line, while in the general case it is an ordinary (unclosed) polygon whose closing line represents the resultant in magnitude and direction. In other words, when the forces are not parallel, they must be added geometrically, and not algebraically.

The construction of the funicular polygon and its properties are the same as for parallel forces.

If the force polygon does not close, the given system is equivalent to a single resultant represented in magnitude, direction, and sense by the closing line; its position is obtained from the funicular polygon whose initial and final lines must intersect on the resultant.

If, however, the force polygon closes, the system may be equivalent to a couple, or it may be in equilibrium. The distinction between these two cases is indicated by the funicular polygon. If the initial and final lines of this polygon coincide, the system is in equilibrium; if they are merely parallel, these lines are the directions of the forces of the couple to which the whole system reduces. The magnitude and sense of the forces of the resulting couple are obtained from the force polygon.

174. Thus it follows from the graphical as well as from the analytical method that *a plane system may be equivalent to a single force, or to a couple, or to zero.* In the first case, the force polygon does not close, and the initial and final sides of the funicular polygon intersect at a finite distance. In the second case, the force polygon closes, and the initial and final lines of the funicular polygon are parallel. In the third case, the force polygon closes, and the initial and final sides of the funicular polygon coincide.

The *graphical conditions of equilibrium* of a plane system are, therefore, two: (1) the force polygon must close; (2) the funicular polygon must have its initial and final sides coincident.

175. To every *vertex* of the force polygon corresponds a *side* of the funicular polygon, and *vice versa.* The force polygon is said to close if the last vertex coincides with the first; similarly, the funicular polygon might be said to close when its last side coincides with the first. With this convention, we may say that *the conditions of equilibrium of a plane system require the closing of both the force polygon and the funicular polygon.*

176. One of the most important applications of the graphical methods is found in the *determination of the stresses in the frame-works* used for bridges, roofs, cranes, etc. The following example will illustrate the method.

Fig. 51 represents the skeleton frame of a roof truss subjected to the "loads" W_1, W_2, W_3 and the reactions of the supports A, B. The members of the frame in connection with the lines of action of these forces (imagined as drawn from infinity up to the points of application) divide the whole plane into a number of compartments marked in the figure by the letters $a, b, c, d, \cdots$. The external forces as well as the members of the frame (or the stresses acting along them) can thus be designated by the two letters of the two portions of the plane separated by the force or stress. For instance, the reaction A is denoted ab, and the stresses in the two members concurring at A are bc and ca. The figure just described may be called the *frame diagram;* and we proceed now to construct its *stress diagram.**

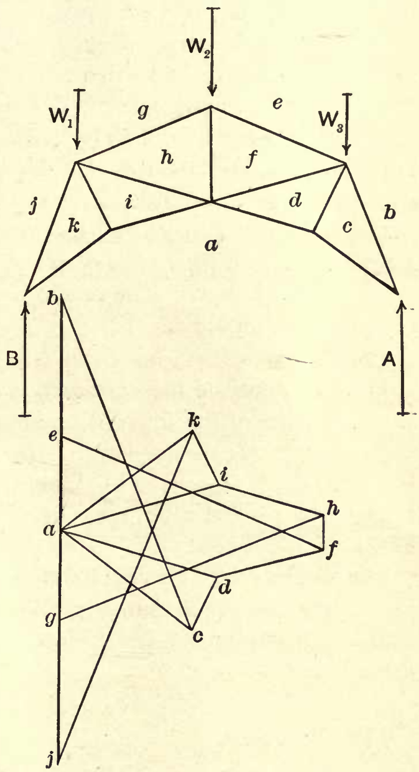

Fig. 51.

Laying off on a vertical line $gj = W_1$, $eg = W_2$, $be = W_3$, and bisecting bj at a, we have the polygon of the external forces which gives the reactions $A = ab$, $B = ja$.

* The student is advised to draw the stress diagram himself step by step as indicated in the text.

Next, beginning at the vertex *A* the stresses in the two members intersecting at *A* are found by resolving the reaction *A* along the directions of these members; and this is done in the stress diagram by drawing parallels to these directions through the points *a* and *b*. The intersection is denoted by *c*.

177. It will be noticed that the three lines meeting at *A* have corresponding to them, in the stress diagram, the three sides *ab*, *bc*, *ca* of a triangle. The force $A = ab$ is represented by *ab*; the stress in the member *bc* (*i.e.* in the member separating the compartments *b*, *c* in the frame diagram) is represented in magnitude, direction, and sense by the side *bc* in the stress diagram; and the stress in the member *ca* is given by the side *ca* of the triangle *abc*. To obtain the sense of each stress correctly, the triangle *abc* in the stress diagram must be traversed in the sense of the known force $A = ab$; this shows that the member *bc* is compressed, the stress at *A* acting towards *A*, while *ca* is subject to tension.

It will be found in general that *the lines of the stress diagram corresponding to all the lines meeting at any one vertex of the frame diagram form a closed polygon.* The reason is obvious: the forces at the vertex must be in equilibrium.

178. To continue the construction of the stress diagram, we pass to another vertex of the frame diagram, selecting one at which not more than two stresses are unknown. Thus at the vertex *acd* the stress in *ac* is known, being represented by *ac* in the stress diagram. Hence drawing through *a* a parallel to *da*, through *c* a parallel to *cd*, we find the point *d* of the stress diagram.

The vertex *dcbef* can now be attacked; *dc*, *cb*, *be* are already drawn, and it only remains to draw *ef* parallel to *ef* and *df* parallel to *df*.

The rest explains itself. Considerations of symmetry are frequently helpful in affording checks.

179. Exercises.

(1) Check the computed stresses of Exercises (1) and (2), Art. 172, by constructing the stress diagrams.

(2) Find the stresses in the frame (Fig. 52), if the load consists of seven equal weights, of 2 tons each, applied at the joints of the upper chord. Owing to the symmetry of the figure, it is sufficient to construct

the stress diagram for half the frame. At the vertex *F*, a difficulty arises, there being apparently three members whose stresses are not

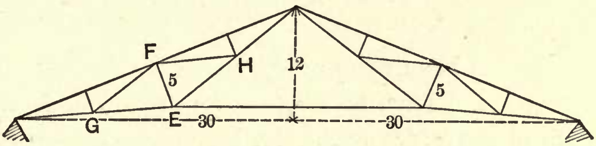

Fig. 52.

known from the previous construction; but on account of the symmetry with respect to *EF*, the members *FG* and *FH* must have equal stresses.

180. Shearing Force and Bending Moment. Consider a horizontal beam fixed at one end *A* (Fig. 53), and acted upon at the other end *B* by a vertical force *F*. If the beam be cut at any point *C* of its length, and the equilibrium of the portion *AC* be considered, the action on *AC* of the portion removed must be replaced by its equivalent. Now the force *F* at *B* is equivalent, by Art. 136, to an equal and parallel force *F* at *C* in connection with a couple whose moment is $F \cdot BC$.

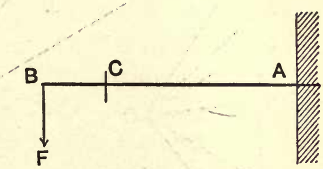

Fig. 53.

The force *F* at *C* is called the **shearing force** of the cross-section *C*, and the moment $F \cdot BC$ the **bending moment** at *C*. Both are of great importance in engineering, as their combined effect represents what must be overcome by the resistance of the material of the beam, *i.e.* by the internal forces holding together its fibres.

These definitions are readily generalised. Let any beam or girder, supported in any manner, and acted upon by any number of vertical forces, be divided by a vertical cross-section into two portions *A* and *B*. For the portion *A* the shearing force at the cross-section is the sum of all the external forces acting on *B*; and the bending moment is the sum of the moments of all these forces with respect to some point in the cross-section.

181. According to its definition the bending moment of a beam at any cross-section is found by adding the moments, with respect to the cross-section, of all the external forces on one side of the section.

Graphically, the bending moment is readily derived from the funicular polygon. Thus in Fig. 54, for the cross-section $\alpha\beta$, the resultant of the forces on the left is $R' = A - W_1 - W_2 = 0\,3$ in

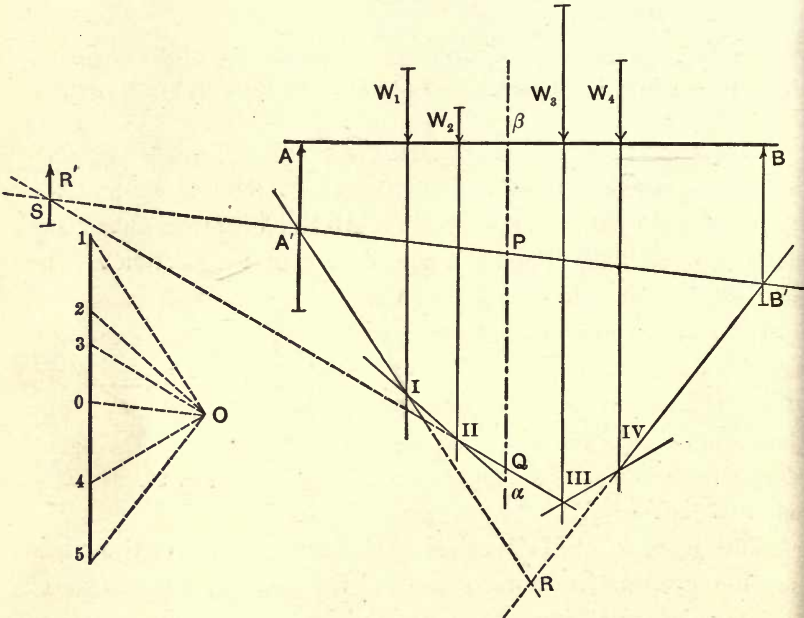

Fig. 54.

the force polygon. Its position is found by bringing to intersection the two sides $A'B'$ and II III of the funicular polygon met by the section $\alpha\beta$. For the funicular polygon resolves A along $A'B'$ and A'I, W_1 along IA' and I II, W_2 along II I and II III. The components falling into the same line being equal and opposite (as appears from the force polygon), the forces A, W_1, W_2 are together equivalent to the components along $A'B'$ and II III; their resultant R' must therefore pass through the intersection S of these lines.

Now if p be the horizontal distance of the point S from $\alpha\beta$, the bending moment at $\alpha\beta$ is $R' \cdot p = 03 \cdot p$. If $\alpha\beta$ intersect $A'B'$ in P, II III in Q, the triangles SPQ and $O03$ are similar, so that their altitudes p and H are as the homologous sides PQ and 03; hence

$$p = \frac{PQ}{03} \cdot H,$$

and the value of the bending moment is $H \cdot PQ$. As H is constant, we find that *the bending moment is proportional to the vertical height, or ordinate, of the funicular polygon.*

5. FRICTION.

182. The reaction between two surfaces in contact has so far been regarded as directed along the common normal of the surfaces. This is true when the surfaces are *perfectly smooth.*

The surfaces of physical bodies are *rough, i.e.* they present small elevations and depressions; when two such surfaces are "in contact" the projections of one will more or less enter into depressions of the other; the greater the normal pressure between the surfaces, the more will this be the case. Hence when a tangential force acting on one of the bodies tends to *slide* its surface over that of the other body, a resistance will be developed whose magnitude must depend on the roughness of the surfaces and on the normal pressure between them. This resistance is called the **force of friction.**

The study of friction belongs properly to applied mechanics, and will here only be touched upon very briefly.

183. Imagine a body resting with a plane surface on a horizontal plane. Let a small horizontal force P be applied at its centroid (which is supposed to be situated so low that the body is not overturned), and let the force P be gradually increased until motion ensues. The value of P when motion just begins is equal and opposite to the *frictional resistance* F between the surfaces at this moment, and this resistance is called the **limiting static friction.**

Careful experiments have shown this force to be subject to the following laws:

(1) *The magnitude of the limiting friction* F *bears a constant ratio to the normal pressure* N *between the surfaces in contact;* that is

$$F = \mu N,$$

where μ is a constant depending on the condition and nature of the surfaces in contact. This constant which must be determined experimentally for different substances and surface conditions is called the **coefficient of static friction.** It is in general a proper fraction; for perfectly smooth surfaces $\mu = 0$.

(2) *For a given normal pressure the limiting static friction, and hence the coefficient of static friction, is independent of the area of contact.*

184. The frictional resistance between two surfaces in relative motion is called **kinetic friction.** It is subject, in addition to the two laws just mentioned, to the third law:

(3) *Kinetic friction is independent of the velocities of the bodies in contact.*

The coefficient of static friction is generally slightly greater than that of kinetic friction.

It must not be forgotten that these so-called **laws of friction** are experimental laws, and therefore true only approximately, and within the limits of the experiments from which they were deduced. When the relative velocity of the surfaces in contact is very high, and when, as is usually the case in machinery, a lubricating material is introduced between the two surfaces, the frictional resistance is found to depend on a number of other circumstances, such as the temperature, the form of the surfaces, the velocity, the nature of the lubricator, etc.

185. Consider again a body resting on a horizontal plane (Fig. 55), and acted upon by a horizontal force P just large enough to equal the limiting friction F. The normal reaction N of the plane is equal and opposite to the weight W. The body is thus in equilibrium under the action of the two pairs of equal and opposite forces; but motion will ensue as soon as P is increased. If P be decreased, F will decrease at the same rate, so that the equilibrium remains undisturbed.

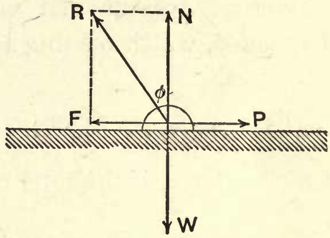

Fig. 55.

The force of friction F can be combined with the normal reaction N to form a resultant,

$$R=\sqrt{F^2+N^2}=\sqrt{P^2+W^2},$$

which represents the *total reaction* of the horizontal plane.

If ϕ be the angle between N and R when F has its limiting value $F=\mu N$ (Art. 183), we have, since $\tan\phi=F/N$,

$$\tan\phi=\mu.$$

The angle ϕ thus gives a kind of graphical representation for the coefficient of friction μ; it is called the **angle of friction.**

186. If the plane be not horizontal, but inclined to the horizon at an angle θ, the weight W of the body (regarded as a particle) resting on the plane can be resolved into a component $W\sin\theta$ along the plane, and a component $W\cos\theta$ perpendicular to it (Fig. 56). Hence, if no other forces act on the body it will be in equilibrium, provided the component $W\sin\theta$ be not greater than the limiting friction $F=\mu W\cos\theta$. The limiting condition of equilibrium is, therefore,

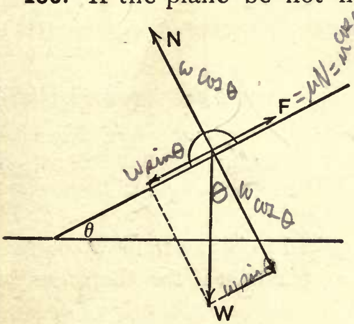

Fig. 56.

$$\mu W\cos\theta=W\sin\theta, \quad \text{or} \quad \mu=\tan\theta;$$

in other words, if the angle θ be gradually increased, the body will not slide down the plane until $\theta > \phi$. This furnishes an experimental method of determining the angle of friction ϕ, which on this account is sometimes called the **angle of repose.**

187. A particle P (Fig. 57) will be in equilibrium on any rough surface, if the total reaction of the surface, *i.e.* the resultant R of the normal reaction N and the friction F, is equal and opposite to the resultant R' of all the other forces acting on the particle.

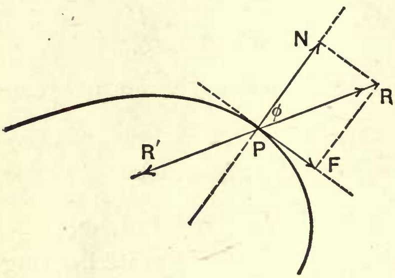

Fig. 57.

The limiting value of the angle between N and R is ϕ, so that the particle can be in equilibrium only if the resultant R' makes with the normal an angle $\overline{\gtrless}\,\phi$. Hence, if about the normal PN as axis, and with P as vertex, a cone be described whose vertical angle is 2ϕ, the condition of equilibrium is that R' must lie within this cone.

The cone is called the **cone of friction.**

188. Exercises.

(1) A particle of weight W is in equilibrium on a rough plane inclined to the horizon at an angle θ, under the action of a force P parallel to the plane along its greatest slope. Determine P: (a) when $\theta > \phi$, (b) when $\theta = \phi$, (c) when $\theta < \phi$, $\phi = \tan^{-1}\mu$ being the angle of friction.

(2) Determine the tractive force required to haul a train of 100 tons with constant velocity up a grade of 2.5 per cent if the coefficient of friction is 1/200.

(3) A weight W is to be hauled along a horizontal plane, the coefficient of friction being $\mu = \tan\phi$. Determine the required tractive force P if it is to act at an inclination α to the horizon, and show that this force is least when $\alpha = \phi$.

(4) A particle of weight W is kept in equilibrium on a plane inclined at an angle θ to the horizon by a force P making an angle α with the line of greatest slope (in the vertical plane at right angles to the intersection of the inclined plane with the horizon). Find the conditions of equilibrium when the particle is on the point of moving (a) down the plane, (b) up the plane.

(5) A homogeneous straight rod $AB = 2l$ of weight W rests with one end A on the horizontal floor, with the other end B against a vertical wall whose plane is at right angles to the vertical plane of the rod. If there be friction of angle ϕ at both ends, determine the limiting position of equilibrium.

(6) Two particles whose weights are W, W' are in equilibrium on an inclined plane, being connected by a string directed along the line of greatest slope. If the coefficients of friction are μ, μ', determine the inclination of the plane.

189. The idea of the angle of friction suggests a *graphical method* for problems on equilibrium with friction.

The case of a rod resting on two inclined planes, Art. 150, Fig. 41, may serve as an example. If the intersection E of the normal reactions A and B lies on the vertical through D, the rod will be in equilibrium whether there be friction at A and B or not. When this condition is not fulfilled, the rod may still be in equilibrium if there be sufficient friction between the ends of the rod and the supporting planes.

Let $\mu = \tan\phi$ be the coefficient of friction on the plane CA, $\mu' = \tan\phi'$ that on CB; then the total reactions at A and B will, by Art. 185, make angles not greater than ϕ and ϕ', respectively, with the normals to the planes. Hence the two limiting positions of equilibrium for the weight W, in a given position of the rod, can be found by bringing the lines of these total reactions to intersection; the limiting position of W is the vertical through this intersection. Thus, to prevent the rod from sliding up the plane CA and down the plane CB, the friction angles ϕ, ϕ' must be applied in the negative sense (clockwise) to the normals at A and B; this gives one limiting position D' for the point D. The other position D'' is found by applying the friction angles in the positive sense. Equilibrium will therefore subsist if the weight be placed anywhere between D' and D''.

The construction is somewhat simplified when $\phi = \phi'$ since then the intersections of the total reactions lie on the circle described about ABC (Fig. 58).

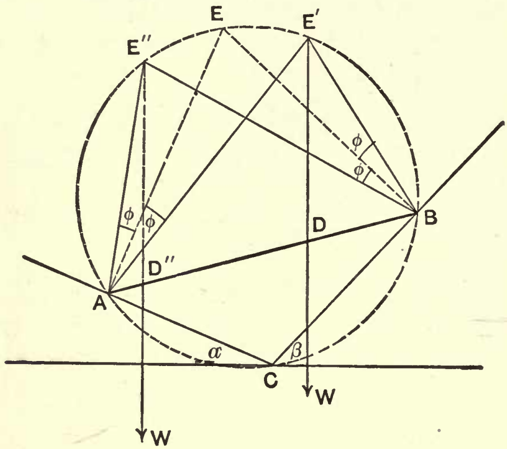

Fig. 58.

190. As another example consider the ordinary jack intended to raise an eccentric load W acting vertically downwards through A (Fig. 59) by a force P passing vertically upwards through the pitch-line B of the rack. Near C and D the rack is pressed against the casing. The directions of the total reactions C, D at these points are found by applying the friction angle to the normals.

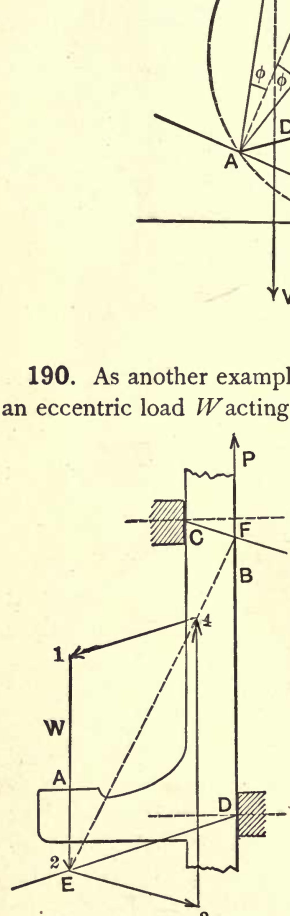

Fig. 59.

The four forces W, P, C, D can be in equilibrium only if the resultant of W and D is equal and opposite to the resultant of P and C; hence, if E be the intersection of W and D, F that of P and C, each of these resultants must act along EF.

If the load W be known, the other forces can now be found by constructing the force polygon. Draw $1\,2 = W$ in position (*i.e.* through A); draw $2\,3$ parallel to C; $4\,1$ parallel to D; and through the intersection 4 of $4\,1$ with EF draw the vertical $3\,4$ to the intersection 3 with $2\,3$.

191. Journal Friction. A journal, or trunnion, is the cylindrical end of a horizontal shaft, by means of which the shaft is supported in its bearing. The larger circle in Fig. 60 represents a cross-section of the journal at right angles to the axis of the shaft.

The shaft and journal may be regarded as rotating uniformly about their common horizontal axis under the action of a driving force whose moment with respect to a point O on the axis would have to be exactly equal and opposite to that of the resistance, or load, if there were no journal friction. For, in this case, the reaction of the bearing to the weight W of the shaft would act vertically upwards through the axis of the shaft, so that its moment would be zero.

The existence of friction at the place of contact A between journal and bearing requires an increase of the driving force, which may be regarded as a small tangential force P applied at any point B, such that its moment $P \cdot OB$ equals the moment about O of the frictional resistance at A.

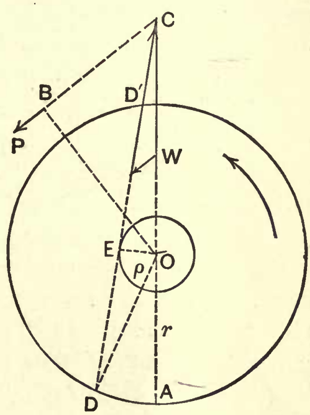

Fig. 60.

192. Let C be the intersection of the direction of this force P with the vertical through O and A, which is the line of action of the weight W of the shaft. The resultant of P and W passes through C, and intersects the circumference of the journal in a point D near A; the total reaction of the bearing is equal and opposite to this resultant. As the total reaction must make an angle equal to the angle of friction ϕ with the normal at D which passes through the centre O, we have for the perpendicular OE dropped from O on CD,

$$OE = \rho = r \sin \phi,$$

where r is the radius of the journal. A circle described about O, with ρ as radius, has the total reaction of the bearing as a tangent. This circle is called the **friction circle**. As ϕ is generally very small in the case of journal friction, $\mu = \tan \phi$ can be substituted for $\sin \phi$, and we have for the radius ρ of the friction circle

$$\rho = \mu r.$$

As soon as any one point is known through which the total reaction must pass (as the point C in Fig. 60), its direction is found by drawing through this point a tangent to the friction circle.

193. If the shaft revolved in the opposite sense, *i.e.* clockwise (instead of counter-clockwise, as assumed in Fig. 60), the tangent to the friction circle would have to be drawn through C on the *other* side of the friction circle.

In the case of **axle-friction,** *i.e.* when the journal, or axle, is fixed, while the bearing, or hub, revolves about it, the same considerations would apply, except that the point of application of the total reaction would now be at the top, at D', instead of D.

194. Pin-friction, as it occurs in link-work and jointed frames that are not absolutely stiff, is not different from journal friction or axle-friction, and can be treated in the same way. Thus, a link connected to other parts of a machine by means of a pin at each end would transmit the force along the line joining the centres of the pins if there were no friction. To take account of pin-friction, we have only to draw the friction circles about the centre of each pin; the direction in which the force is transmitted by the link is tangent to both these circles.

Which one of the four common tangents represents this direction must be decided in each particular case by considering that the reaction exerted by one link on another connected with it by a pin is in the direction of the motion of the former relative to the other. Thus if the link AB (Fig. 61) be subject to tension, and its motion relative to

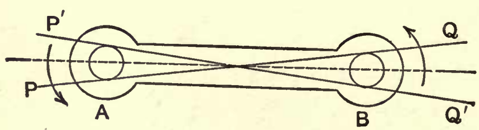

Fig. 61.

the adjoining links at A and B be as indicated by the arrows in the figure, the contact between the link and pin will be on the outside both at A and at B; the friction is, therefore, directed downwards at A and upwards at B, and the line PQ along which the force is transmitted touches the friction circle at A below, at B above.

If the link were under compression, with the same relative motions, the line of force would have the direction $P'Q'$.

195. The simplest case of **pivot friction** is that of a vertical shaft of weight W resting with its circular end on a plane horizontal support. If a be the radius of the end of the shaft, the pressure per unit of area is $W/\pi a^2$, and the pressure on a polar element of area is $\frac{W}{\pi a^2} \cdot r dr d\theta$.

The friction at this element, $\mu \frac{W}{\pi a^2} \cdot r dr d\theta$, is directed along the tangent to the circle of radius r; its moment with respect to the centre O of the circle is therefore $\frac{\mu W}{\pi a^2} r^2 dr d\theta$. Hence the whole moment of friction about O is

$$\frac{\mu W}{\pi a^2}\int_0^{2\pi} d\theta \int_0^a r^2 dr = \tfrac{2}{3}\mu Wa = \mu W \cdot \tfrac{2}{3}a.$$

This may be regarded as the moment of a force μW applied at a distance $\frac{2}{3}a$ from the centre.

196. Belt-friction. A belt running over two pulleys and stretched so tight as to prevent slipping is a common means of transferring the rotary motion about the axis of one pulley, say A, to the axis of the other pulley B; A is called the driver, B is the driven pulley. We assume the axes parallel and the rotation counter-clockwise.

When the pulleys are at rest the tension in CE (Fig. 62) is of course

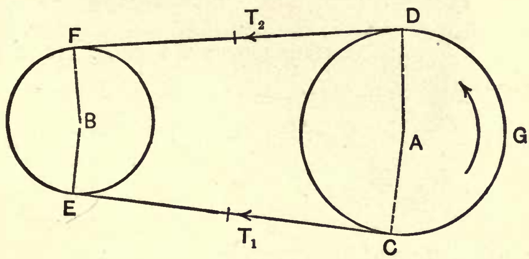

Fig. 62.

equal to the tension in DF. But if the pulley A be set in motion, say by a tangential driving force P acting at a lever-arm p, while the pulley B experiences a resistance Q whose arm is q, the tension in CE will increase to a certain value T_1, and the tension in DF will decrease to a value T_2 until the difference $T_1 - T_2$ is sufficient to overcome the resistance Q. This difference is due to the friction along the surface CGD. If the resistance Q be too great this friction might not be sufficient, and slipping of the belt on the driver would occur.

197. Let us try to determine the condition which T_1 and T_2 must satisfy to prevent slipping. To do this we determine the equilibrium of the belt at the moment when slipping is just on the point of taking place.

The tension of the belt decreases gradually along the arc CGD from the value T_1 at C to the value T_2 at D. Let it be T at the point P and $T+dT$ at the near point P' (Fig. 63). The portion PP' of the belt is in equilibrium under the action of the forces T, $T+dT$ and the reaction dR of the pulley; hence dR must pass through the intersection of T and $T+dT$ and must make with the radius AP an angle equal to the friction angle ϕ. Resolving these forces along T and at right angles to it, we have, if $\measuredangle PAP' = d\theta$,

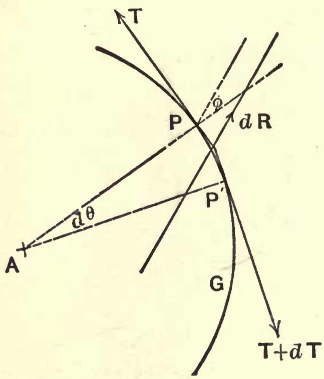

Fig. 63.

$$T + dR \sin\phi = T + dT,$$

$$dR \cos\phi = (T + dT)\, d\theta,$$

or

$$dR \sin\phi = dT,$$

$$dR \cos\phi = T d\theta;$$

hence, dividing,

$$\tan\phi = \frac{1}{T}\frac{dT}{d\theta}.$$

Putting μ for $\tan\phi$ and integrating over the whole arc of contact, we find, if θ be the angle of this arc,

$$\log T_1 - \log T_2 = \mu\theta,$$

or

$$\frac{T_1}{T_2} = e^{\mu\theta}.$$

For the common system of logarithms this becomes

$$\log \frac{T_1}{T_2} = 0.4343\,\mu\theta,$$

where θ must be expressed in circular measure.

198. Rolling Friction. The resistance offered by a surface to the *rolling* of another surface over it is of a somewhat different nature from that of ordinary or *sliding* friction. In sliding friction, the same point or surface area of one body comes in contact with different points or areas of the other. In the case of rolling friction, the points that come successively in contact are different for *both* bodies.

Let us examine the simplest case, viz. that of a cylinder rolling over a horizontal plane. If both cylinder and plane were perfectly rigid, there could be no resistance to rolling. This

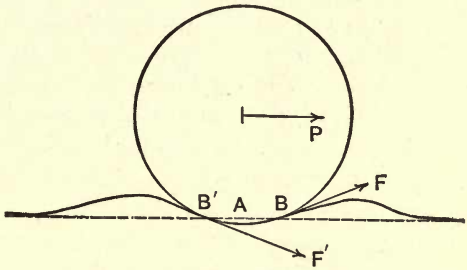

Fig. 64.

resistance is due to the compression both of the lower part of the cylinder and of the plane. Experiments made with a heavy roller on india-rubber have shown that the supporting surface when elastic is not only compressed under the roller but bulges out in front and behind, as indicated in Fig. 64. Thus, the area of contact is considerably increased, and as the roller advances, the portion AB of its surface rubs over the surface of the support, while the elastic material of the support in trying to regain its horizontal surface causes friction over the area $B'A$ also.

The experiments indicate for the value F of rolling friction an expression of the form

$$F = \mu' \frac{W}{r},$$

where W is the weight, r the radius of the cylinder, and μ' a constant depending on the nature of the materials in contact. For hard surfaces, this constant of rolling friction μ' is very much smaller than the constant of sliding friction μ.

199. On the subject of plane statics the student may consult in particular the recent work: E. J. Routh, *A treatise on analytical statics,* with numerous examples, Vol. I., Cambridge, University Press, 1891; also G. M. Minchin, *A treatise on statics with applications to physics*, Vol. I., 3d ed., Oxford, Clarendon Press, 1884; I. Todhunter, *Analytical statics*, 5th ed. by J. D. Everett, London, Macmillan, 1887; B. Price, *Infinitesimal calculus*, Vol. III.: Statics and dynamics of material particles, 2d ed., Oxford, Clarendon Press, 1868. For problems, see also W. Walton, *Collection of problems in illustration of the principles of elementary mechanics*, 2d ed., Cambridge, Deighton, 1880.

Numerous applications to civil and mechanical engineering will be found in J. H. Cotterill, *Applied mechanics*, London, Macmillan, 1884; W. J. M. Rankine, *A manual of applied mechanics*, 9th ed. by E. F. Bamber, London, Griffin, 1877; A. Ritter, *Lehrbuch der technischen Mechanik*, 5th ed., Leipzig, Baumgärtner, 1884; J. Weisbach, *Mechanics of engineering*, Vol. I., translated by E. B. Coxe, New York, Van Nostrand, 1875; and in works on graphical statics.

On friction, see in particular: G. Herrmann, *The graphical statics of mechanism*, translated by A. P. Smith, 2d ed., New York, Van Nostrand, 1892; R. H. Thurston, *Treatise on friction and lost work in machinery and mill work*, New York, Wiley, 1885; J. H. Jellett, *The theory of friction*, Dublin, Hodges, 1872.

VI. *Solid Statics.*

I. THE CONDITIONS OF EQUILIBRIUM.

200. The equilibrium of a rigid body in the most general case, that is, when acted upon by any number of forces F in a space of three dimensions, can be investigated in a manner similar to that adopted for the plane system in Art. 139.

Selecting as origin any point O rigidly connected with the body, let two equal and opposite forces F, $-F$ be applied at O, for every one of the given forces F (Fig. 65). The effect of the given system of forces on the body is not changed by the introduction of these forces at O. But we may now regard the given force F acting at its point of application P as replaced by the equal and parallel force F at O, in combination with the couple formed by the original force F at P and the force $-F$ at O. All the forces of the given system are thus transferred to a common point of application O, and can therefore be compounded into a single resultant R, passing through O and represented in magnitude and direction by the geometric sum of the forces. In addition to this resultant R, we obtain as many couples (F, $-F$) as there were forces given; and their resultant is found by geometrically adding the vectors of the couples (Art. 134).

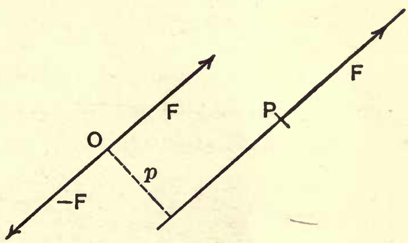

Fig. 65.

Thus the given system of forces is seen to be equivalent to a resultant R in combination with a couple whose vector we shall call H; in other words, it has been proved that *any system of forces acting on a rigid body can be reduced to a single resultant force in combination with a single resultant couple.*

201. A further reduction is in general not possible. The general *conditions of equilibrium* are, therefore,

$$R = 0, \quad H = 0.$$

202. Under special conditions it may of course happen that R is perpendicular to the vector H. In this case R and H combine to a single force R (Art. 135), and if the origin be taken on the line of this force, the whole system reduces to a single resultant.

203. It is to be noticed that in the general reduction of forces (Art. 200), the magnitude, direction, and sense of the resultant force R are entirely independent of the position of the origin O, the resultant being simply the geometric sum of all the given forces. The resultant couple H, on the other hand, will in general differ according to the origin selected.

To investigate this dependence, let R, H (Fig. 66) be the *elements of reduction* for the origin O; *i.e.* let R be the resultant, H the vector of the resulting couple of a given system of forces when O is selected as origin. To find the elements of reduction of the same system of forces when some other point O' is taken as origin, it is only necessary to apply at O' two equal and opposite forces R, $-R$, each equal and parallel to the original resultant R. The given system of forces being equivalent to R and H at O will also be equivalent to the resultant R at O', the couple whose vector is H (which may be drawn through O' without changing its effect), and the couple formed by R at O and $-R$ at O'. If l be the line of R through O, l' the line of R through O', and r the distance of these parallels, the moment of the latter couple is Rr and its vector is at right angles to the plane (l, O'). Combining the vectors H

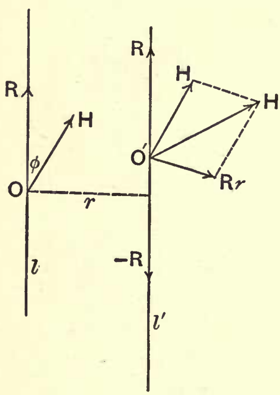

Fig. 66.

and Rr into a resultant vector H' by geometric addition, we have found the elements of reduction R, H' for the origin O'.

204. If the new origin O' had been selected on the line l of the original resultant, no new couple (R, r) would have been introduced, and H would not have been changed. But whenever the line of action l of the resultant is changed, the vector of the resultant couple H is changed.

By increasing the distance r between l and l' the moment Rr of the additional couple is increased. The effect of combining this additional couple Rr with H is, in general, to vary both the magnitude of the resulting couple H' and the angle ϕ it makes with the direction of the resultant R. It can be shown that the line l' of the new resultant can always be selected so as to reduce the angle ϕ to zero. The line l_0 for which $\phi = 0$, *i.e.* for which the vector H of the resultant couple is parallel to the resultant force R, is called the **central axis** of the given system of forces. We proceed to show how it can be found.

205. Let the vector H be resolved at O into a component $H_0 = H\cos\phi$ along l, and a component $H_1 = H\sin\phi$, at right angles to l (Fig. 67). In the plane passing through l at right angles to H_1, it is always possible to find a line l_0 parallel to l at a distance r_0 from l, such as to make $Rr_0 = -H_1$.

The line l_0 so determined is the central axis. For, if this line be taken as the line of the resultant R, the additional couple Rr_0 destroys the component H_1, so that the resulting couple H_0 has its vector parallel to R.

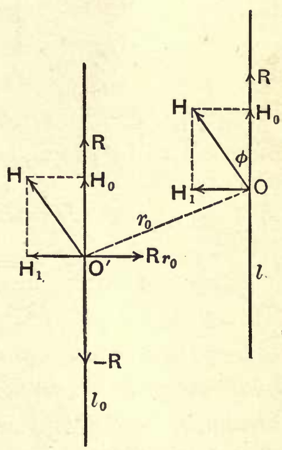

Fig. 67.

206. As the direction of the vector H is always changed in passing from line to line, there can be but one central axis for a given system of forces.

It appears from the construction of the central axis given in Art. 205, that the vector of the resulting couple for this axis l_0 is $H_0 = H\cos\phi$; it is, therefore, less than for any other line.

It is instructive to observe how the vector H increases and changes its direction as we pass from the central axis l_0 to any parallel line l.

The transformation from l_0 to l requires the introduction of a couple (R, r_0) whose vector Rr_0 (Fig. 68) is at right angles to the plane (l_0, l) and combines with H_0 to form the resulting couple H for l. As the distance r_0 of l from l_0 is increased, both the magnitude of H and the angle ϕ it makes with l increase until, for an infinite r_0, the angle ϕ becomes a right angle.

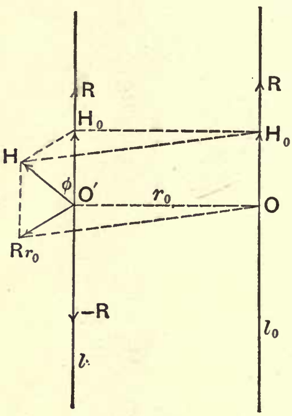

Fig. 68.

207. It is evident that since $H_0 = H\cos\phi$, the product $RH\cos\phi$ is a constant quantity for a given system of forces. It has been called the **invariant** of the system.

If the elements of reduction for the central axis (R, H_0) be given, those for any parallel line l at the distance r_0 from the central axis are determined by the equations

$$H^2 = H_0^2 + R^2 r_0^2, \quad \tan\phi = \frac{Rr_0}{H_0}.$$

208. To sum up the results of the preceding articles, it has been shown that *any system of forces acting on a rigid body can be reduced, in an infinite number of ways, to a resultant R in combination with a couple H.* For all these reductions the magnitude, direction, and sense of the resultant R are the same, but the vector H of the couple changes according to the *position* assumed for the line of R. There is one, and only one, position of R, called the central axis of the system, for which the vector

H is parallel to R, and has at the same time its least value, H_0; this value H_0 is equal to the projection of any other vector H on the direction of the resultant R.

209. While, in general, a system of forces cannot be reduced to a single resultant, it can always be reduced to *two non-intersecting forces*. This easily follows by considering the system reduced to its resultant R and resulting couple H for any origin O (Fig. 69). Let F, $-F$ be the forces, p the arm of the couple H, and place this couple so that one of the forces, say $-F$, intersects R at O. Then, combining R and $-F$ to their resultant F', the given system of forces is evidently equivalent to the two non-intersecting forces F, F' (compare Art. 137).

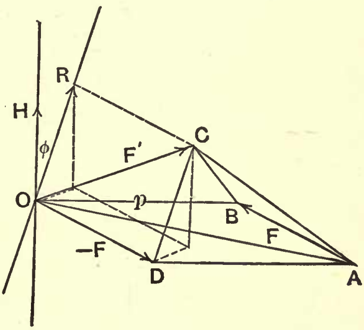

Fig. 69.

210. The two forces F, F' determine a tetrahedron $OABC$; and it can be shown that *the volume of this tetrahedron is constant and equal to one sixth of the invariant of the system* (Art. 207). The proof readily appears from Fig. 69. The volume of the tetrahedron $OABC$ is evidently one half of the volume of the quadrangular pyramid whose vertex is C and whose base is the parallelogram $OBAD$. The area of this parallelogram is $Fp = H$; and the altitude of the pyramid is $= R \cos \phi$, being equal to the perpendicular let fall from the extremity of R on the plane of the couple; hence the volume of the tetrahedron

$$= \tfrac{1}{6} RH \cos \phi = \tfrac{1}{6} RH_0.$$

211. To effect the reduction of a given system of forces analytically, it is usually best to refer the forces F and their points of application P to a rectangular system of co-ordinates

Ox, Oy, Oz (Fig. 70). Let x, y, z be the co-ordinates of P and X, Y, Z the components of F parallel to the axes.

To transfer these components to O as common origin, we proceed similarly as in Art. 142. Thus to transfer, say X, we introduce at P', the foot of the perpendicular let fall from P on the plane zx, two equal and opposite forces X, $-X$; and we do the same thing at O. Then the single force X at P is replaced by the force X at O in combination with the two couples formed by X at P, $-X$ at P', and X at P', $-X$ at O. The vector of the former couple is parallel to Oz, its moment is $-yX$; the negative sign being used because for a person looking on the plane of the couple from the positive side of the axis Oz the

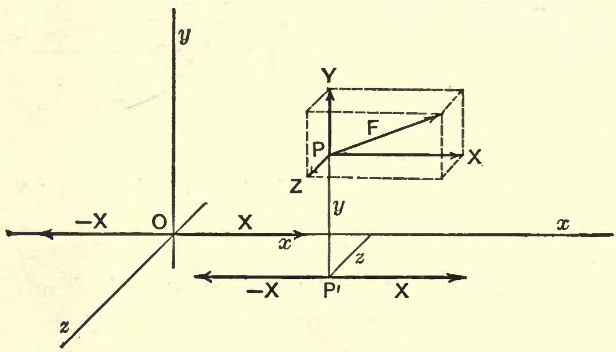

Fig. 70.

couple rotates clockwise. The vector of the latter couple is parallel to Oy, and its moment is zX.

The transfer of Y to the origin O requires the introduction of two couples, $-zY$ having its vector parallel to Ox, and xY having its vector parallel to Oz.

Finally, transferring Z to O, we have to introduce the couples $-xZ$ with a vector parallel to Oy, and yZ with a vector parallel to Ox.

Thus each force F is replaced by three forces X, Y, Z along the axes of co-ordinates and applied at O, in combination with three couples whose vectors are $yZ-zY$ parallel to Ox, $zX-xZ$ parallel to Oy, $xY-yX$ parallel to Oz.

212. Doing the same thing for every force of the given system and adding the components having the same direction, the system will be found equivalent to the three rectangular forces

$$\Sigma X, \quad \Sigma Y, \quad \Sigma Z$$

applied at O, together with the three couples

$$\Sigma(yZ-zY), \quad \Sigma(zX-xZ), \quad \Sigma(xY-yX),$$

whose vectors are at right angles.

The three forces can now be compounded into a single resultant

$$R=\sqrt{(\Sigma X)^2+(\Sigma Y)^2+(\Sigma Z)^2},$$

whose direction is determined by the angles α, β, γ which it makes with the axes Ox, Oy, Oz,

$$\cos\alpha=\frac{\Sigma X}{R}, \quad \cos\beta=\frac{\Sigma Y}{R}, \quad \cos\gamma=\frac{\Sigma Z}{R}.$$

In the same way the three couples can be compounded into a single resulting couple whose moment is

$$H=\sqrt{[\Sigma(yZ-zY)]^2+[\Sigma(zX-xZ)]^2+[\Sigma(xY-yX)]^2}.$$

213. Since R^2, as well as H^2, is thus found as the sum of three squares, each of these quantities can vanish only if the three squares composing it vanish separately. The **conditions of equilibrium of a rigid body** (Art. 201) are therefore expressed analytically by the following six equations:

$$\Sigma X=0, \quad \Sigma Y=0, \quad \Sigma Z=0,$$

$$\Sigma(yZ-zY)=0, \quad \Sigma(zX-xZ)=0, \quad \Sigma(xY-yX)=0.$$

As the system of co-ordinates can be selected arbitrarily, the meaning of the first three equations is that the sum of the components of all the forces along any three lines not in the same plane must vanish. The last three equations express that the sum of the moments of all the forces about any three axes not in the same plane must also vanish. The *moment of a force about an axis* must be understood as meaning the moment of its projection on a plane at right angles to the axis with respect to the point of intersection of the axis with the plane. This definition is in accordance with the somewhat vague notion of the moment of a force as representing its "turning effect." For, regarding the force as acting on a rigid body with a fixed axis, the force can be resolved into two components, one parallel, the other perpendicular, to the axis; the former component does evidently not contribute to the turning effect, which is therefore measured by the moment of the latter alone.

214. The *equations of the central axis* (Art. 204) can be found by a transformation of co-ordinates.

Let the system be reduced for any origin O to its resultant R, whose rectangular components we denote by

$$A=\Sigma X, \quad B=\Sigma Y, \quad C=\Sigma Z,$$

and to the vector H of its resulting couple with the components

$$L=\Sigma(yZ-zY), \quad M=\Sigma(zX-xZ), \quad N=\Sigma(xY-yX).$$

If a point O' whose co-ordinates are ξ, η, ζ be taken as new origin and the co-ordinates of any point with respect to parallel axes through O' be denoted by x', y', z', we have $x=\xi+x'$, $y=\eta+y'$, $z=\zeta+z'$. Substituting these values, we find

$$L=\Sigma[(\eta+y')Z-(\zeta+z')Y]=\eta\Sigma Z-\zeta\Sigma Y+\Sigma(y'Z-z'Y)$$
$$=\eta C-\zeta B+L',$$

where L' is the x-component of the couple H' resulting for O' as origin. Similar expressions hold for M and N. The components of H' are therefore

$$L'=L-\eta C+\zeta B, \quad M'=M-\zeta A+\xi C, \quad N'=N-\xi B+\eta A;$$

and its direction cosines are

$$\lambda=\frac{L'}{H'},\quad \mu=\frac{M'}{H'},\quad \nu=\frac{N'}{H'}.$$

The central axis being defined (Art. 204), as that line for which the vector of the resulting couple is parallel to the direction of the resultant, the point $O'(\xi, \eta, \zeta)$ will lie on the central axis if the direction cosines of H' are proportional to those of R, viz., to

$$\alpha=\frac{A}{R},\quad \beta=\frac{B}{R},\quad \gamma=\frac{C}{R}.$$

Hence the equations of the central axis are

$$\frac{L'}{A}=\frac{M'}{B}=\frac{N'}{C},$$

or
$$\frac{L-\eta C+\zeta B}{A}=\frac{M-\zeta A+\xi C}{B}=\frac{N-\xi B+\eta A}{C}.$$

215. To show the application of the conditions of equilibrium, let us consider the simple machine called the *wheel and axle.* It consists of a horizontal shaft (Fig. 71) resting with its ends on the supports or bearings A, B, and is intended to raise a weight W, suspended vertically by means of a rope wound around the shaft. The driving force F is applied

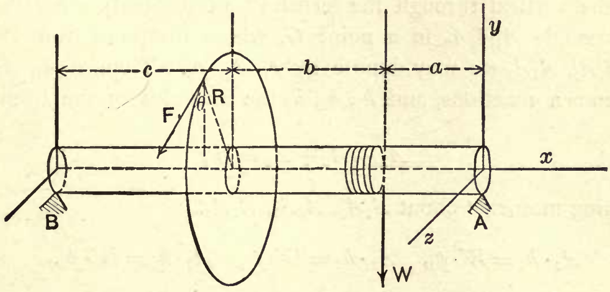

Fig. 71.

on the circumference of the "wheel," *i.e.* in a vertical plane at right angles to the axis of the shaft. It is required to find the relation between F and W for equilibrium, and the pressures on the bearings A, B.

Let r be the radius of the shaft, R that of the wheel, *i.e.* the lever-arm of the force F, and let F be inclined to the vertical at an angle θ; then, with the co-ordinates and notations of the figure, the conditions $\Sigma X = 0$, $\Sigma Y = 0$, $\Sigma Z = 0$, give

$$A_x + B_x = 0, \quad A_y + B_y - W - F\cos\theta = 0, \quad A_z + B_z + F\sin\theta = 0,$$

where A_x, A_y, A_z are the components of the unknown reaction at A; B_x, B_y, B_z, those at B.

Taking moments about each of the co-ordinate axes, we find

$$FR = Wr, \quad (a+b)F\sin\theta + lB_z = 0, \quad aW + (a+b)F\cos\theta - lB_y = 0,$$

where $l = a + b + c$ is the length of the shaft.

A_x and B_x must evidently be separately zero. Solving the equations, we find

$$F = \frac{r}{R} \cdot W,$$

$$A_y = \frac{1}{l}\left(b + c + \frac{cr}{R}\cos\theta\right) \cdot W, \qquad B_y = \frac{1}{l}\left(a + \frac{(a+b)r}{R}\cos\theta\right) \cdot W,$$

$$A_z = -\frac{1}{l}\frac{cr}{R}\sin\theta \cdot W, \qquad B_z = -\frac{1}{l}\frac{(a+b)r}{R}\sin\theta \cdot W.$$

216. As another example, consider *a rigid body of weight* W, *supported at three points* A_1, A_2, A_3; and let it be required *to determine the distribution of the pressure between the three supports.*

Let the vertical through the centroid of the body meet the plane of the triangle $A_1A_2A_3$ in a point G, whose distances from the sides A_2A_3, A_3A_1, A_1A_2 we may denote by p_1, p_2, p_3. Then, if A_1, A_2, A_3 be the unknown reactions, and h_1, h_2, h_3 the altitudes of the triangle, we have

$$A_1 + A_2 + A_3 = W,$$

and, taking moments about A_2A_3, A_3A_1, A_1A_2,

$$A_1 \cdot h_1 = W \cdot p_1, \quad A_2 \cdot h_2 = W \cdot p_2, \quad A_3 \cdot h_3 = W \cdot p_3.$$

Hence, $$A_1 = \frac{p_1}{h_1} W, \qquad A_2 = \frac{p_2}{h_2} W, \qquad A_3 = \frac{p_3}{h_3} W.$$

Substituting these values into the first equation, we find the condition,

$$\frac{p_1}{h_1} + \frac{p_2}{h_2} + \frac{p_3}{h_3} = 1.$$

If G falls outside the triangle, one or two of the points A_1, A_2, A_3 will be subject to pressures vertically upwards. If G be the centroid of the triangular area $A_1A_2A_3$, we have $p_1/h_1 = p_2/h_2 = p_3/h_3 = 1/3$; hence in this case the three reactions are equal.

217. *The axis of the hinges of a door is inclined at an angle θ to the horizon. The door is turned out of its position of equilibrium by an angle ϕ, and held in this position by a force* F *perpendicular to the plane of the door. Determine* F *and the reaction of the hinges* A, B (Fig. 72).

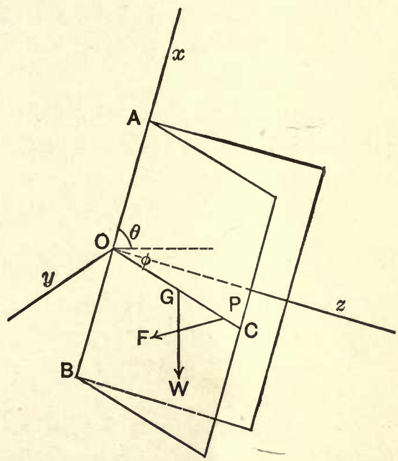

Fig. 72.

Let the axis of the hinges be taken as the axis of x, the vertical plane through it as the plane zx, and the point midway between the hinges A, B as the origin O. Regarding the door as a homogeneous rectangular plate whose dimensions are $AB = 2a$, $OC = 2b$, the co-ordinates of its centroid G are 0, $b \sin\phi$, $b\cos\phi$. If the force F be applied at a point P on the middle line OC at the distance $OP = p$ from O, the co-ordinates of its point of application P are 0, $p\sin\phi$, $p\cos\phi$.

To proceed systematically, we may tabulate the components of the forces, and the co-ordinates of their points of application, and then form the component couples, as shown below. The components of the unknown reactions A, B of the hinges are called A_x, A_y, A_z, B_x, B_y, B_z.

Forces.	Components.			Co-ordinates.			Couples.		
	X	Y	Z	x	y	z	$yZ - zY$	$zX - xZ$	$xY - yX$
W	$-W\sin\theta$	0	$W\cos\theta$	0	$b\sin\phi$	$b\cos\phi$	$Wb\cos\theta\sin\phi$	$-Wb\sin\theta\cos\phi$	$Wb\sin\theta\sin\phi$
F	0	$F\cos\phi$	$-F\sin\phi$	0	$p\sin\phi$	$p\cos\phi$	$-Fp(\sin^2\phi + \cos^2\phi)$	0	0
A	A_x	A_y	A_z	a	0	0	0	$-A_z a$	$A_y a$
B	B_x	B_y	B_z	$-a$	0	0	0	$B_z a$	$-B_y a$

From this table the six conditions of equilibrium are at once obtained:

$$-W\sin\theta \qquad\qquad + A_x + B_x = 0, \tag{1}$$

$$F\cos\phi + A_y + B_y = 0, \tag{2}$$

$$W\cos\theta - F\sin\phi + A_z + B_z = 0, \tag{3}$$

$$Wb\cos\theta\sin\phi - Fp \qquad = 0, \tag{4}$$

$$-Wb\sin\theta\cos\phi + (-A_z + B_z)a = 0, \tag{5}$$

$$Wb\sin\theta\sin\phi + (A_y - B_y)a \quad = 0. \tag{6}$$

If the reactions were not required, equation (4) alone would be sufficient, as it furnishes the value of F, viz.,

$$F = \frac{b}{p}\cos\theta\sin\phi\cdot W.$$

This relation can of course be found directly by taking moments about the axis of the hinges. It shows that, for a given inclination of the hinges, F is greatest when $\phi = \pi/2$.

The remaining five equations are sufficient to determine $A_x + B_x$, A_y, A_z, B_y, B_z.

To find the reactions for a door with vertical axis, we have to put $\theta = \pi/2$, which gives, of course, $F = 0$, and

$$A_x + B_x = W, \quad A_y + B_y = 0, \quad A_z + B_z = 0,$$

$$A_y - B_y = -\frac{b}{a}W\sin\phi, \quad A_z - B_z = -\frac{b}{a}W\cos\phi;$$

as ϕ may be assumed $= 0$ in this case, we find

$$A_y = -B_y = 0, \quad A_z = -B_z = -\frac{b}{2a}W.$$

The signs indicate that the upper hinge A is pulled out while the lower one B is pressed in.

2. CONSTRAINTS.

218. It has been shown in Art. 213 that the number of the conditions of equilibrium is six, for a rigid body that is perfectly free. This number will be diminished whenever the body is subject to conditions restricting its possible motions. Such conditions, or **constraints**, may be of various kinds; the body may have a fixed point, or a fixed axis, or one of its points may be constrained to move along a given curve or to remain on a given surface, etc.

As explained in *Kinematics*, Art. 37, a free rigid body is said to have six degrees of freedom. The most general form of motion that it can have is a screw-motion, or twist, consisting of a rotation about a certain axis, and a translation along this axis; each of these resolves itself analytically into three rectangular components, and these six components may be regarded as constituting the six possible motions of the body, on account of which it is said to have six degrees of freedom.

Equilibrium will exist only when these six possible motions are prevented; hence there must be six conditions of equilibrium.

219. We proceed to consider some forms of constraint and the corresponding changes in the equations of equilibrium.

It is generally convenient in dynamics to replace such restraining conditions by forces, usually called **reactions.** Whenever it is possible to introduce such forces having the same effect as the given conditions, the body may be regarded as free, and the general equations of equilibrium can be applied.

Before considering the constraints of a rigid body, those of a single particle, or point, must be briefly discussed.

220. Particle constrained to a Surface. A free particle has three degrees of freedom; and accordingly its equilibrium is determined by three conditions (Art. 101):

$$\Sigma X=0, \quad \Sigma Y=0, \quad \Sigma Z=0. \tag{1}$$

If the co-ordinates determining the position of the particle be subject to *one condition*, expressed by an equation between these co-ordinates, the particle is said to have two degrees of freedom and one constraint. Its motion is restricted to the surface represented by the equation between its co-ordinates, say

$$\phi(x, y, z) = 0. \tag{2}$$

The condition that the particle should remain on this surface can be replaced by introducing the *reaction of the surface, i.e.* a force that is always so directed as not to allow the particle to leave the surface. Combining this force with the given forces acting on the particle, this particle can be regarded as free, and the general conditions of equilibrium must hold.

221. If the surface be *smooth, i.e.* if the particle move along it without friction, the reaction of the surface must be directed along the normal to the surface (2). Let N denote this normal reaction; N_x, N_y, N_z its components; then the conditions of equilibrium are

$$\Sigma X + N_x = 0, \quad \Sigma Y + N_y = 0, \quad \Sigma Z + N_z = 0. \tag{3}$$

The condition that N has the direction of the normal is expressed by the relations

$$\frac{N_x}{\phi_x} = \frac{N_y}{\phi_y} = \frac{N_z}{\phi_z} = \frac{N}{\sqrt{\phi_x^2 + \phi_y^2 + \phi_z^2}}, \tag{4}$$

where $\phi_x = \frac{\partial \phi}{\partial x}$, $\phi_y = \frac{\partial \phi}{\partial y}$, $\phi_z = \frac{\partial \phi}{\partial z}$ are obtained from (2).

Eliminating the reactions by means of (3), we find the *two conditions of equilibrium,*

$$\frac{\Sigma X}{\phi_x} = \frac{\Sigma Y}{\phi_y} = \frac{\Sigma Z}{\phi_z}. \tag{5}$$

The meaning of these equations is obvious; they express that the resultant of the given forces must have the direction of the normal to the surface.

The problem generally consists in finding the positions of equilibrium of the particle on the surface. The two equations (5) represent a curve whose intersections with the surface (2) give the required positions.

The magnitude of the reaction N is found from (3):

$$N = \sqrt{(\Sigma X)^2 + (\Sigma Y)^2 + (\Sigma Z)^2}.$$

222. If the surface be *rough*, the total reaction of the surface lies within the cone of friction (Art. 187), and the resultant R of all other forces acting on the particle must therefore also fall within this cone.

The boundaries of the regions on the surface within which equilibrium is possible are found by considering the total reaction in its limiting position, *i.e.* when it makes the friction angle $\tan^{-1}\mu$ with the normal to the surface.

Let N represent the normal component of the total reaction; N_x, N_y, N_z its components; the force of friction μN lies in the tangent plane, and has, therefore, the components $\mu N dx/ds$, $\mu N dy/ds$, $\mu N dz/ds$, for motion along any curve s on the surface (2). These components of the friction must be given the double sign $\mp$, because the force of friction may act in either sense along the curve s. Thus, the conditions of equilibrium are

$$\Sigma X + N_x \mp \mu N \frac{dx}{ds} = 0,$$

$$\Sigma Y + N_y \mp \mu N \frac{dy}{ds} = 0, \qquad (6)$$

$$\Sigma Z + N_z \mp \mu N \frac{dz}{ds} = 0.$$

To eliminate the reactions, multiply these equations by ϕ_x, ϕ_y, ϕ_z and add; this gives

$$\Sigma X \cdot \phi_x + \Sigma Y \cdot \phi_y + \Sigma Z \cdot \phi_z + N_x\phi_x + N_y\phi_y + N_z\phi_z = 0,$$

since the differentiation of the equation of the surface (2) gives $\phi_x dx + \phi_y dy + \phi_z dz = 0$. Substituting for N_x, N_y, N_z their values from (4), the equation becomes

$$\Sigma X \cdot \phi_x + \Sigma Y \cdot \phi_y + \Sigma Z \cdot \phi_z + N\sqrt{\phi_x^2 + \phi_y^2 + \phi_z^2} = 0. \qquad (7)$$

This equation determines the normal reaction N of the surface.

To obtain an expression for μN, multiply the second of the equations (6) by ϕ_z, the third by ϕ_y and subtract; owing to the relations (4) this gives

$$\Sigma Y\cdot\phi_z-\Sigma Z\cdot\phi_y=\pm\,\mu N\left(\phi_z\frac{dy}{ds}-\phi_y\frac{dz}{ds}\right)\cdot$$

Similarly, we find

$$\Sigma Z\cdot\phi_x-\Sigma X\cdot\phi_z=\pm\,\mu N\left(\phi_x\frac{dz}{ds}-\phi_z\frac{dx}{ds}\right),$$

$$\Sigma X\cdot\phi_y-\Sigma Y\cdot\phi_x=\pm\,\mu N\left(\phi_y\frac{dx}{ds}-\phi_x\frac{dy}{ds}\right)\cdot$$

The left-hand members as well as the parentheses on the right are determinants of the second order; hence, squaring and adding, we find

$$\begin{aligned}[(\Sigma X)^2+(\Sigma Y)^2+(\Sigma Z)^2](\phi_x^{\,2}+\phi_y^{\,2}+\phi_z^{\,2})&\\ -(\Sigma X\cdot\phi_x+\Sigma Y\cdot\phi_y+\Sigma Z\cdot\phi_z)^2&\qquad(8)\\ =\mu^2N^2(\phi_x^{\,2}+\phi_y^{\,2}+\phi_z^{\,2}).&\end{aligned}$$

If N be now eliminated between (7) and (8), we find the *final condition of equilibrium* that must be fulfilled by the given forces independently of the reaction of the surface:

$$\begin{aligned}[(\Sigma X)^2+(\Sigma Y)^2+(\Sigma Z)^2](\phi_x^{\,2}+\phi_y^{\,2}+\phi_z^{\,2})&\\ =(1+\mu^2)(\Sigma X\cdot\phi_x+\Sigma Y\cdot\phi_y+\Sigma Z\cdot\phi_z)^2.&\qquad(9)\end{aligned}$$

Putting this equation into the form

$$\frac{1}{\sqrt{1+\mu^2}}=\frac{\Sigma X}{R}\cdot\frac{\phi_x}{R'}+\frac{\Sigma Y}{R}\cdot\frac{\phi_y}{R'}+\frac{\Sigma Z}{R}\cdot\frac{\phi_z}{R'},$$

where $R^2=(\Sigma X)^2+(\Sigma Y)^2+(\Sigma Z)^2$ and $R'^2=\phi_x^{\,2}+\phi_y^{\,2}+\phi_z^{\,2}$, it is seen to express the fact that the resultant R of the given forces makes the friction angle ϕ with the normal, each member of the equation being an expression for the cosine of this angle.

The regions of the surface (2) on which the particle is in equilibrium are cut out of this surface by the surface (9).

223. Particle constrained to a Curve. If the particle be subject to two conditions,

$$\phi(x, y, z)=0, \quad \psi(x, y, z)=0, \tag{10}$$

so that it has two constraints and but one degree of freedom, its motion is restricted to the curve of intersection of the two surfaces (10). The particle may be imagined as a small sphere moving within a tube, or as a small ring or bead sliding along a thin wire.

224. Let the curve be *smooth*, so that its total reaction is along the normal to the curve. Denoting this normal reaction again by N, its components by N_x, N_y, N_z, the conditions of equilibrium are

$$\Sigma X+N_x=0, \quad \Sigma Y+N_y=0, \quad \Sigma Z+N_z=0, \tag{11}$$

or

$$\frac{\Sigma X}{N_x}=\frac{\Sigma Y}{N_y}=\frac{\Sigma Z}{N_z}. \tag{12}$$

The condition that N has the direction of the normal can be expressed in the form

$$N_x dx+N_y dy+N_z dz=0,$$

which, by (12), reduces to

$$\Sigma X\cdot dx+\Sigma Y\cdot dy+\Sigma Z\cdot dz=0.$$

Differentiating the equations of the curve (10), we find

$$\phi_x dx + \phi_y dy + \phi_z dz=0,$$

$$\psi_x dx+\psi_y dy+\psi_z dz=0;$$

and eliminating the differentials between the last three equations, the single condition of equilibrium, independent of the reactions, is found in the form

$$\begin{vmatrix} \Sigma X & \Sigma Y & \Sigma Z \\ \phi_x & \phi_y & \phi_z \\ \psi_x & \psi_y & \psi_z \end{vmatrix} = 0. \qquad (13)$$

The intersections of the surface (13) with the curve (10) give the positions of equilibrium of the particle on the curve.

The reaction of the curve, or the pressure on the curve which is equal and opposite to this reaction, can then be found from the equations (11).

225. For a *rough* curve, the total reaction resolves itself into a normal component N and a tangential component μN, which represents the frictional resistance. The equations of equilibrium are

$$\begin{aligned} \Sigma X + N_x \mp \mu N \frac{dx}{ds} &= 0, \\ \Sigma Y + N_y \mp \mu N \frac{dy}{ds} &= 0, \\ \Sigma Z + N_z \mp \mu N \frac{dz}{ds} &= 0. \end{aligned} \qquad (14)$$

Transposing the third terms, multiplying by dx/ds, dy/ds, dz/ds, and adding, we find, since $N_x dx + N_y dy + N_z dz = 0$,

$$\Sigma X \cdot \frac{dx}{ds} + \Sigma Y \cdot \frac{dy}{ds} + \Sigma Z \cdot \frac{dz}{ds} = \pm \mu N. \qquad (15)$$

Multiplying the second of the equations (14) by dz/ds, the third by dy/ds, and subtracting, we have

$$\Sigma Y \cdot \frac{dz}{ds} - \Sigma Z \cdot \frac{dy}{ds} = -\left(N_y \frac{dz}{ds} - N_z \frac{dy}{ds}\right);$$

similarly,
$$\Sigma Z \cdot \frac{dx}{ds} - \Sigma X \cdot \frac{dz}{ds} = -\left(N_z \frac{dx}{ds} - N_x \frac{dz}{ds}\right),$$

$$\Sigma X \cdot \frac{dy}{ds} - \Sigma Y \cdot \frac{dx}{ds} = -\left(N_x \frac{dy}{ds} - N_y \frac{dx}{ds}\right).$$

Each member being a determinant of the second order, we find by squaring and adding the three equations,

$$(\Sigma X)^2+(\Sigma Y)^2+(\Sigma Z)^2-\left(\Sigma X\cdot\frac{dx}{ds}+\Sigma Y\cdot\frac{dy}{ds}+\Sigma Z\cdot\frac{dz}{ds}\right)^2=N^2. \quad (16)$$

The reaction N can now be eliminated between (15) and (16), and we obtain the single condition of equilibrium independent of the reaction:

$$\Sigma X\cdot\frac{dx}{ds}+\Sigma Y\cdot\frac{dy}{ds}+\Sigma Z\cdot\frac{dz}{ds}=\pm\frac{\mu}{\sqrt{1+\mu^2}}\sqrt{(\Sigma X)^2+(\Sigma Y)^2+(\Sigma Z)^2}. \quad (17)$$

The differential coefficients dx/ds, dy/ds, dz/ds must satisfy the differential equations of the curve (10), viz.:

$$\phi_x\frac{dx}{ds}+\phi_y\frac{dy}{ds}+\phi_z\frac{dz}{ds}=0,$$

$$\psi_x\frac{dx}{ds}+\psi_y\frac{dy}{ds}+\psi_z\frac{dz}{ds}=0.$$

If the values of dx/ds, dy/ds, dz/ds be determined from the last three equations and substituted into the relation

$$\left(\frac{dx}{ds}\right)^2+\left(\frac{dy}{ds}\right)^2+\left(\frac{dz}{ds}\right)^2=1,$$

the equation of a surface will result, which cuts out, on the curve (10), the limits between which equilibrium is possible.

226. Rigid Body with a Fixed Point. A body that is free to turn about a fixed point A can be regarded as free if the reaction A of this point be introduced and combined with the other forces acting on the body.

Let A_x, A_y, A_z be the components of A; then, taking the fixed point A as origin, the six equations of equilibrium (Art. 213) are

$$\Sigma X+A_x=0, \qquad \Sigma Y+A_y=0, \qquad \Sigma Z+A_z=0,$$

$$\Sigma(yZ-zY)=0, \quad \Sigma(zX-xZ)=0, \quad \Sigma(xY-yX)=0.$$

The first three of these equations serve to determine the reaction of the fixed point; the last three are the actual conditions of equilibrium corresponding to the three degrees of freedom of a body with a fixed point.

Hence, *a rigid body having a fixed point is in equilibrium if the sum of the moments of all the forces vanishes for any three axes passing through the fixed point and not situated in the same plane.*

227. Rigid Body with a Fixed Axis. A body with a fixed axis has but one degree of freedom; indeed, the only possible motion consists in rotation about this axis.

An axis is fixed as soon as two of its points, say A, B, are fixed. Hence, introducing the reactions A_x, A_y, A_z, B_x, B_y, B_z of these points, the body can be regarded as free. If the point B be taken as origin, the line BA as axis of z (Fig. 73), the equations of equilibrium become

$$\Sigma X + A_x + B_x = 0, \qquad \Sigma Y + A_y + B_y = 0, \qquad \Sigma Z + A_z + B_z = 0,$$

$$\Sigma(yZ - zY) - A_y a = 0, \quad \Sigma(zX - xZ) - A_x a = 0, \quad \Sigma(xY - yX) = 0,$$

where $a = BA$.

The last of the six equations is the only independent condition of equilibrium of the constrained body; the first five determine A_x, B_x, A_y, B_y, $A_z + B_z$. The two z-components cannot be found separately, since they act in the same straight line.

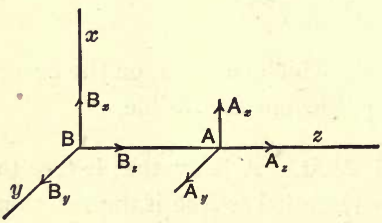

Fig. 73.

Hence, *a rigid body having a fixed axis is in equilibrium if the sum of the moments of all the forces vanishes for the fixed axis.*

228. If, in the preceding article, the axis be not absolutely fixed, but only fixed in direction so that *the body can rotate about the axis and also slide along it,* we have evidently

$$A_z = 0, \quad B_z = 0;$$

hence, by the third equation of equilibrium,

$$\Sigma Z = 0,$$

as an additional condition of equilibrium.

The body has in this case two degrees of freedom.

229. Rigid Body with a Fixed Plane. A body constrained to slide on a fixed plane has three degrees of freedom. At every point of contact between the body and the plane, the latter exerts a reaction. As all these reactions are parallel, they can be combined into a single resultant N. Taking the fixed plane as the plane xy, N will be parallel to the axis of z; hence, if a, b, 0 be the co-ordinates of its point of application, the six equations of equilibrium are

$$\Sigma X = 0, \quad \Sigma Y = 0, \quad \Sigma Z + N = 0,$$

$$\Sigma(yZ - zY) + bN = 0, \quad \Sigma(zX - xZ) - aN = 0, \quad \Sigma(xY - yX) = 0.$$

The third, fourth, and fifth equations determine the reaction N and the co-ordinates a, b of its point of application. The three other equations are the actual conditions of equilibrium; they agree, of course, with the three conditions of equilibrium of a plane system as found in Art. 143.

If there be not more than three points of contact (or supports) between the body and the fixed plane, the reactions of these points can be found separately. Let A_1, A_2, A_3 be the three points of contact; N_1, N_2, N_3 the required reactions; a_1, b_1, a_2, b_2, a_3, b_3 the co-ordinates of A_1, A_2, A_3; then N must be resolved into three parallel forces passing through these points, and the conditions are

$$N_1 + N_2 + N_3 = N,$$

$$a_1 N_1 + a_2 N_2 + a_3 N_3 = aN,$$

$$b_1 N_1 + b_2 N_2 + b_3 N_3 = bN.$$

These three equations determine N_1, N_2, N_3, unless the three points A_1, A_2, A_3 be situated in a straight line; for in this case the determinant of the coefficients of N_1, N_2, N_3 vanishes,

$$\begin{vmatrix} 1 & 1 & 1 \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix} = 0.$$

The reactions become almost indeterminate whenever there are more than three points of contact.

230. In addition to the works of ROUTH, MINCHIN, PRICE, TODHUNTER, mentioned in Art. 199, the student is referred, in particular for the more advanced parts of the subject, to W. SCHELL, *Theorie der Bewegung und der Kräfte*, Vol. II., Leipzig, Teubner, 1880; A. F. MÖBIUS, *Lehrbuch der Statik*, Leipzig, Göschen, 1837, reprinted in MÖBIUS's Gesammelte Werke, Vol. III., Leipzig, Hirzel, 1886; MOIGNO, *Statique*, Paris, Gauthier-Villars, 1868; J. SOMOFF, *Theoretische Mechanik*, übersetzt von A. Ziwet, Vol. II., Leipzig, Teubner, 1879; E. COLLIGNON, *Statique*, Paris, Hachette, 1889; THOMSON AND TAIT, *Natural Philosophy*, Part. II., Cambridge, University Press, 1890.

VII. *The Principle of Virtual Work.*

231. Work has been defined in Art. 72 as the product of a force into the displacement of its point of application in the direction of the force.

Thus the expansive force F of the steam in the cylinder of a steam-engine, in pushing the piston through a distance s, is said to *do work*, and this work is measured by the product Fs. Similarly the force of gravity, *i.e.* the attractive force of the earth's mass, does work on a falling body.

The resistance to be overcome by the engine, in the former case, and the resistance of the air in the latter, are also forces acting on the body during its displacement. But as the sense of the displacement is opposite to that of these forces, their work is negative; *work is done against these forces.* Thus the muscular force of a man who raises a weight does work against gravity; if the weight he holds is so heavy as to pull him down, gravity does work against his force; if he merely tugs at a weight without being able to lift it, the work is zero, because the displacement is zero.

232. In general, the point of application of a force F will be acted upon by a number of different forces, so that the displacement s of this point will not necessarily take place in the direction of F. In this general case *the work of a force is defined as the product of the force into the projection of the displacement of its point of application on the direction of the force.*

In Fig. 74, for instance, the particle P while acted upon by the force F (and any number of other forces) is displaced from P to P'; hence if $PP' = s$, and $\measuredangle P'PQ = \phi$, the work of the force F is

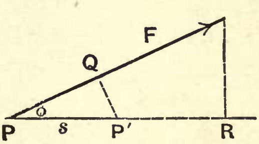

Fig. 74.

$$W = Fs\cos\phi. \qquad (1)$$

It is obvious that this work might also be defined as the product of the displacement into the projection of the force on the displacement; for we have

$$Fs \cos \phi = F \cdot PQ = s \cdot PR.$$

The work of a force is evidently positive or negative according as the angle ϕ is less or greater than $\pi/2$, provided we select for ϕ always that angle between F and s which is not greater than π.

233. The above definition of work assumes that the force F remains constant, both in magnitude and direction, while the displacement s takes place, and that this displacement is rectilinear. If either, or both, of these conditions be not fulfilled, the definition can be applied only to infinitesimal displacements ds. As the work done by a finite force F during such a displacement ds is infinitesimal, we have

$$dW = Fds \cos \phi; \tag{2}$$

and the total work done by any variable force F while its point of application is displaced along any straight or curvilinear path PQ, is obtained by integrating from P to Q:

$$W = \int_P^Q F \cos \phi \, ds. \tag{3}$$

234. Since work can always be regarded as the product of a force into a length, its dimensions are found by multiplying those of force, $\mathbf{MLT^{-2}}$ (Art. 64), by $\mathbf{L}$; hence, the *dimensions of work* are

$$\mathbf{W = ML^2T^{-2}}.$$

The unit of work is the work of a unit force (poundal, dyne) through a unit distance (foot, centimetre). The unit of work in the F.P.S. system is called the **foot-poundal**; in the C.G.S. system, the **erg**. Thus, the erg is the amount of work done by a force of one dyne acting through a distance of one centimetre. These are the *absolute* units.

In the gravitation system where the pound, or the kilogramme, is taken as unit of force, the British unit of work is the **foot-pound**, while in the metric system it is customary to use the **kilogramme-metre** as unit.

235. The numerical relations between these units are obtained as follows. Let x be the number of ergs in the foot-poundal, then (comp. Art. 66),

$$x \cdot \frac{\text{gm. cm.}^2}{\text{sec.}^2} = 1 \cdot \frac{\text{lb. ft.}^2}{\text{sec.}^2},$$

hence $$x = \frac{\text{lb.}}{\text{gm.}} \cdot \left(\frac{\text{ft.}}{\text{cm.}}\right)^2 = 453.59 \times 30.4797^2 = 4.2139 \times 10^5;$$

i.e. 1 foot-poundal $= 4.2139 \times 10^5$ ergs, and 1 erg $= 2.3721 \times 10^{-6}$ $= 0.000\,002\,372\,1$ foot-poundal.

Again, let x be the number of kilogramme-metres in 1 foot-pound, then

$$x \text{ kg. m.} = 1 \text{ ft. lb.},$$

hence $$x = \frac{\text{lb.}}{\text{kg.}} \cdot \frac{\text{ft.}}{\text{m.}} = 0.45359 \times 0.3048 = 0.13825,$$

i.e. 1 foot-pound $= 0.13825$ kilogramme-metres.

Finally, 1 foot-pound $= g$ foot-poundals (Art. 69); hence 1 foot-pound $= 1.356 \times 10^7$ ergs, and 1 erg $= 7.3737 \times 10^{-8}$ foot-pounds, if $g = 32.2$.

236. Exercises.

(1) A *joule* being defined as 10^7 ergs, show that 1 foot-pound $= 1.356$ joules, and that 1 joule is about 3/4 foot-pound.

(2) Show that a kilogramme-metre is nearly 10^8 ergs.

(3) What is the work done against gravity in raising 300 lbs. through a height of 25 ft.: (*a*) in foot-pounds, (*b*) in ergs?

(4) Find the work done against friction in moving a car weighing 3 tons through a distance of fifty yards on a level road, the coefficient of friction being 0.02.

(5) A mass of 12 lbs. slides down a smooth plane inclined at an angle of 30° to the horizon, through a distance of 25 ft.; what is the work done by gravity?

237. It follows from the definition of work that, if any number of forces $F_1, F_2, \ldots, F_n$ act on a particle P, *the sum of their works for any displacement* PP′ = ds *is equal to the work of their resultant* R *for the same displacement.* For, the resultant R being the closing line of the polygon constructed by adding the forces $F_1, F_2, \ldots, F_n$ geometrically, the projection of R on any direction, such as PP', is equal to the sum of the projections of the forces F on the same line (Art. 89); that is, if $\alpha_1, \alpha_2, \ldots, \alpha_n$ be the angles made by $F_1, F_2, \ldots, F_n$ with PP', and α the angle between R and PP', we have

$$F_1 \cos \alpha_1 + F_2 \cos \alpha_2 + \cdots + F_n \cos \alpha_n = R \cos \alpha;$$

multiplying this equation by ds, we obtain the above proposition

$$F_1 \cos \alpha_1 ds + F_2 \cos \alpha_2 ds + \cdots + F_n \cos \alpha_n ds = R \cos \alpha ds,$$

which expresses the so-called **principle of work** for a single particle.

238. When the particle is in equilibrium, so that the forces do not actually change the motion, we may derive from this proposition a convenient expression for the conditions of equilibrium by considering displacements that *might be* given to the particle. Such displacements are called *virtual*, and the corresponding work of any of the forces is called **virtual work.**

It is customary to denote a virtual displacement by δs, the letter δ being used to distinguish from an actual displacement ds; this distinction becomes of importance in kinetics.

239. The resultant being zero in the case of equilibrium, *the sum of the virtual works of all forces acting on the particle must be zero for any virtual displacement,* i.e.

$$F_1 \cos \alpha_1 \delta s + F_2 \cos \alpha_2 \delta s + \cdots + F_n \cos \alpha_n \delta s = 0. \qquad (4)$$

As the resultant must vanish if its three projections vanish for any three axes not lying in the same plane, the necessary and sufficient conditions of equilibrium of a single particle are that *the sum of the virtual works of all forces must be zero for any three virtual displacements not all in the same plane.*

This is the **principle of virtual work** for a single particle.

If the particle be referred to a rectangular system of co-ordinates, its displacement δs can be resolved into three component displacements δx, δy, δz, parallel to the axes. The forces acting on the particle being replaced by their components X, Y, Z, the sum of their virtual works, for the displacement δs is $\Sigma X \cdot \delta x + \Sigma Y \cdot \delta y + \Sigma Z \cdot \delta z$. Hence the analytical expression for the principle of virtual work:

$$\Sigma X \cdot \delta x + \Sigma Y \cdot \delta y + \Sigma Z \cdot \delta z = 0. \qquad (5)$$

As the displacements δx, δy, δz are independent of each other, and perfectly arbitrary, this single equation is equivalent to the three equations

$$\Sigma X = 0, \quad \Sigma Y = 0, \quad \Sigma Z = 0,$$

which are the ordinary conditions of equilibrium of a single particle.

240. The principle of virtual work is particularly useful in eliminating the unknown reactions arising from constraints.

Suppose the particle be constrained to a smooth surface or curve. After introducing the normal reaction of the surface or curve the particle can be regarded as free; and the equation of virtual work can be used to express the conditions of equilibrium. This equation will, in general, contain the unknown reaction. But as this reaction has the direction of the normal, it will be eliminated if the virtual displacement be selected along a tangent. Hence, *in the case of constrainment to a surface, the two conditions of equilibrium independent of the reaction are found by forming the equation of virtual work for virtual displacements along any two tangents to the surface; and in the case of constrainment to a curve the one such condition is found from a virtual displacement along the tangent.*

If it be required to find the normal pressure on the surface or curve, which is of course equal and opposite to the reaction, it can be found from a virtual displacement along the normal.

241. If the equation (4) which expresses the principle of virtual work be divided by the element of time δt, during which the displacement δs would take place, the factor $\delta s/\delta t = v$ represents a *virtual velocity*, and the equation becomes

$$F_1 \cos \alpha_1 \cdot v + F_2 \cos \alpha_2 \cdot v + \cdots + F_n \cos \alpha_n \cdot v = 0.$$

On account of this form, the proposition is often called the *principle of virtual velocities.*

The product of a force into the virtual velocity of its point of application in the direction of the force, $F \cos \alpha \cdot v$, is sometimes called the *virtual moment* of the force.

242. The principle of virtual work can readily be extended to the case of a rigid body acted upon by any number of forces.

The forces acting on a rigid body can always be reduced to a resultant R and a resulting couple H (Art. 200). This reduction is based on the supposition (Art. 84) that the point of application of a force can be displaced arbitrarily along the line of the force. It can be shown that such a displacement of the point of application P of a force F (Fig. 75), from P to Q along the line of the force, does not affect the work done by the force in any infinitesimal displacement of the body. Let $PP' = \delta s$ be the displacement of P, $QQ' = \delta s'$ that of Q; let p and q be the projections of P and Q on the line of the force F; then, since the body is rigid, $P'Q' = PQ$; and consequently Qq will differ from Pp only by an infinitesimal of an order higher than the order of the displacement $PP' = \delta s$. Hence,

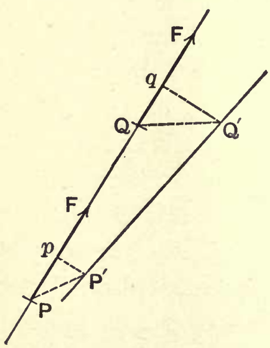

Fig. 75.

$$F \cdot Pp = F \cdot Qq.$$

It may here be noted that, in general, the principle of virtual work must be understood to mean that the sum of the works of the forces differs from the work of their resultant by an infinitesimal of an order higher than that of the virtual displacement. It does not mean that the difference is absolutely zero.

243. Owing to the proposition proved in the preceding article, the sum of the works of all the forces acting on a rigid body is equal to the sum of the works of the resultant R and the resulting couple H for any infinitesimal displacement of the body, and the work of the forces is not changed by such a displacement.

It follows that the necessary and sufficient conditions of equilibrium of a rigid body (Art. 201), viz.

$$R=0,\ H=0,$$

can be expressed by saying that *the sum of the virtual works of all the forces must be zero for any infinitesimal displacement of the body.*

For when the forces are in equilibrium, this condition is evidently fulfilled. To prove that there must be equilibrium whenever this condition is fulfilled, it is only necessary to show that both R and H must vanish if the sum of their works is zero for any infinitesimal displacement.

To see this, consider first a displacement of translation, δs, parallel to R. The work of R will be $R\delta s$ while the works of the two forces constituting H are equal and opposite, so that the work of H is zero. As the sum of the works of R and H must vanish by hypothesis, it follows that $R=0$.

Next consider a displacement of rotation $\delta\theta$ about an axis parallel to the vector H. Taking this axis so as to intersect R and bisect the arm p of the couple H, the work of R will be zero while that of each of the forces F of the couple H will be $\frac{1}{2}p\,\delta\theta\cdot F$; hence the whole work of H is $Fp\delta\theta=H\delta\theta$. As the sum of the works of R and H must vanish by hypothesis, it follows that $H=0$.

The two conditions $R=0$, $H=0$, are, therefore, both fulfilled.

244. The following examples may serve to illustrate the application of the principle of virtual work.

To find the force just necessary to move a cylinder of radius r *and weight* W *up a plane inclined at an angle* α *to the horizon by means of a crow-bar of length* l *set at an angle* β *to the horizon* (Fig. 76).

Let s be the distance from the fulcrum A of the crow-bar to the point of contact B of the cylinder with the plane.

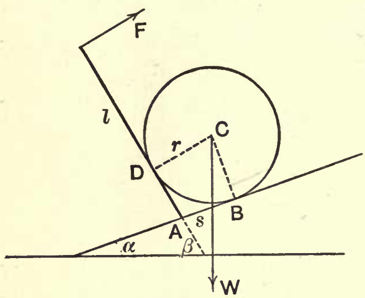

Fig. 76.

Turning the crow-bar about A by an angle $\delta\beta$, the work of the force F acting at the end of the bar is $F\cdot l\,\delta\beta$. The corresponding displacement of the centre C of the circle, which is the point of application of the force W, is parallel to the inclined plane, and may be regarded as the differential δs of the distance $AB=s$. The work of W is, therefore, $W\delta s\sin\alpha$. This gives the equation of work

$$F\cdot l\delta\beta = W\cdot\delta s\sin\alpha;$$

hence

$$F=\frac{W}{l}\sin\alpha\cdot\frac{\delta s}{\delta\beta}.$$

The relation between s and β can be found by projecting $ABCDA$ on the vertical line; this gives

$$r\cos\alpha + s\sin\alpha = r\cos\beta + s\sin\beta,$$

whence

$$s=r\frac{\cos\beta-\cos\alpha}{\sin\alpha-\sin\beta}.$$

Differentiating the former equation, we find

$$\sin\alpha\,\delta s = -r\sin\beta\,\delta\beta + s\cos\beta\,\delta\beta + \sin\beta\,\delta s,$$

$$\therefore\ \frac{\delta s}{\delta\beta}=\frac{s\cos\beta - r\sin\beta}{\sin\alpha-\sin\beta}=r\frac{\cos^2\beta-\cos\alpha\cos\beta-\sin\alpha\sin\beta+\sin^2\beta}{(\sin\alpha-\sin\beta)^2},$$

or
$$\frac{1}{r}\frac{\delta s}{\delta\beta}=\frac{1-\cos(\alpha-\beta)}{(\sin\alpha-\sin\beta)^2}=\frac{1}{1+\cos(\alpha+\beta)}$$

Hence, finally,
$$F=\frac{r}{l}\frac{\sin\alpha}{1+\cos(\alpha+\beta)}\cdot W.$$

245. *A weightless rod of length* AB = l *rests at* C *on a horizontal cylinder whose axis is at right angles to the vertical plane through the rod; its lower end* A *leans against a vertical wall, and from its upper end* B *a weight* W *is suspended. Determine the reactions at* A *and* C, *and the distance* AC = x *for equilibrium, if the distance* CD = a *of the point of support from the vertical wall is given* (Fig. 75).

(*a*) Let A glide vertically upwards, C remaining in contact. At A as well as at C the forces are perpendicular to the displacements; hence, putting $EB=y$, we have $W\delta y=0$.

$$\therefore\ \delta y=0.$$

Also,
$$\frac{y}{l-x}=\frac{\sqrt{x^2-a^2}}{x}.$$

$$\therefore\ \delta y=\delta\left\{\left(\frac{l}{x}-1\right)\sqrt{x^2-a^2}\right\}=0,$$

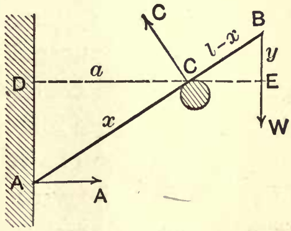

Fig. 77.

whence, $(l-x)x^2-l(x^2-a^2)=0$,

or
$$x^3=a^2l.$$

(*b*) Give the rod a vertical displacement to a parallel position:

$$-W\delta y+C\frac{a}{x}\delta y=0;\ \therefore\ C=\frac{x}{a}W=\sqrt[3]{\frac{l}{a}}\cdot W.$$

(*c*) Give the rod a displacement in its own direction:

$$A\frac{a}{x}\delta s+C\cos\frac{\pi}{2}\cdot\delta s-W\frac{\sqrt{x^2-a^2}}{x}\cdot\delta s=0.$$

$$\therefore\ A=\frac{\sqrt{x^2-a^2}}{a}\cdot W=\sqrt{\left(\frac{l}{a}\right)^{\frac{2}{3}}-1}\cdot W.$$

246. *In a parallelogram formed by four rods with hinges at the vertices, elastic strings are stretched along the diagonals. Determine the ratio of the tensions in these strings.*

Let m, m' be the lengths of the diagonals, T, T' the tensions, and δm, $\delta m'$ the changes of length of the diagonals when the parallelogram is slightly deformed; then by the principle of virtual work

$$T\,\delta m + T'\,\delta m' = 0. \tag{1}$$

From geometry we have, if a, b are the sides of the parallelogram,

$$m^2 + m'^2 = 2\,a^2 + 2\,b^2,$$

hence, differentiating, $$m\,\delta m + m'\,\delta m' = 0. \tag{2}$$

From (1) and (2) we find

$$T/T' = m/m'. \tag{3}$$

247. For the purposes of statics a body is regarded as rigid if the points of application of all the forces acting on the body have invariable distances from each other; these points may be imagined connected by a framework of rigid rods. The tensions in these connecting rods, since they occur in pairs of equal and opposite forces acting along rigid lines, do not enter into the equation of virtual work. One of the chief advantages of the principle of virtual work consists in this elimination of these *internal forces*.

Let us now generalize the idea of the rigid body by assuming the points of application of the forces to be connected by rods or threads which may even be elastic; the points may also be constrained to move on smooth surfaces or curves; friction, however, is to be excluded.

Let x, y, z be the co-ordinates of one of the points, P; x', y', z', those of another point, Q; let l be the length of the connecting thread or rod, PQ; α, β, γ, its direction cosines; and let T be the tension or stress in PQ. If the whole system be subjected to any infinitesimal displacement, for which δx, δy, δz are the component displacements of P, $\delta x'$, $\delta y'$, $\delta z'$ those of Q, the sum of the works of the two equal and opposite forces T for this displacement will be

$$T\alpha\,\delta x + T\beta\,\delta y + T\gamma\,\delta z - T\alpha\,\delta x' - T\beta\,\delta y' - T\gamma\,\delta z',$$

or, since $\alpha=(x-x')/l$, $\beta=(y-y')/l$, $\gamma=(z-z')/l$,

$$\frac{T}{l}[(x-x')(\delta x-\delta x')+(y-y')(\delta y-\delta y')+(z-z')(\delta z-\delta z')].$$

Differentiating the relation

$$(x-x')^2+(y-y')^2+(z-z')^2=l^2,$$

we have

$$(x-x')(\delta x-\delta x')+(y-y')(\delta y-\delta y')+(z-z')(\delta z-\delta z')=l\,\delta l;$$

hence the sum of the virtual works of the two tensions T reduces to

$$T\delta l,$$

which is of course zero when the connecting rod is rigid.

It thus appears that the internal reactions of a system of points connected as above described, are eliminated from the equation of virtual work by selecting the virtual displacements so as to leave the lengths of the connecting rods or threads unchanged. This can always be done when the rods and threads are not elastic. When they are elastic, the equation of virtual work will contain terms of the form $T\delta l$. These terms must then be determined from the known relation between the tension and the length of an elastic rod or thread.

248. It is somewhat difficult to prove the principle of virtual work for the most general case of any system of bodies although this is the case in which it finds its most important application. It is evident, however, that the principle will be true in this general case provided that all the connections and reactions between the different bodies constituting the system be expressed by means of forces and introduced into the equation of virtual work. The difficulty lies in expressing the connections existing between the parts of the system by means of forces.

But most of these internal reactions can be shown to disappear from the equation of virtual work, so that they need not be taken into account.

Thus the tension of an inelastic thread or rod being composed of two equal and opposite forces does no work in any infinitesimal displacement; the work of the reaction of a fixed point or fixed axis is zero, because the point of application cannot move in the direction of the force; the normal reaction of a surface along which a body is constrained to slide does no work, because any possible displacement is at right angles to the force. If, however, the surface be rough, the friction will in general do work.

249. A full discussion of the principle of virtual work for the most general case cannot be given in this place. It must suffice to state it and to illustrate its application in a few special cases.

The necessary and sufficient condition of equilibrium of a system of bodies the connections between which can be expressed by forces is this, that the sum of the virtual works of all the forces must vanish for all displacements consistent with the geometrical conditions to which the system may be subject.

The internal reactions between the different parts of the system, with the exception of friction and elasticity, will in general not enter into the equation of virtual work. Such reactions can therefore be neglected while friction and elastic tensions must be included among the forces acting on the system.

250. Let P be one of the forces, δp the projection on its direction of the virtual displacement of its point of application; then the principle of virtual work requires that

$$\Sigma P\delta p = 0.$$

If X, Y, Z be the rectangular components of P, and x, y, z the co-ordinates of its point of application, the same condition can be expressed in the form:

$$\Sigma (X\delta x + Y\delta y + Z\delta z) = 0.$$

251. Whether the system be in equilibrium or not, the quantity $\Sigma P\delta p = \Sigma(X\delta x + Y\delta y + Z\delta z)$ represents the element of work δW done by the forces in a virtual displacement. For an *actual* displacement of the system δ should be replaced by d, and we have

$$dW = \Sigma P dp = \Sigma(Xdx + Ydy + Zdz)$$

for the work done in the infinitesimal displacement.

Most of the forces occurring in nature are such as to make this quantity an exact differential. In this case the forces are said to form a *conservative system*, and the equation can be integrated from any initial position or configuration of the system to any final position or configuration without reference to the intermediate positions.

Taking the initial position as the standard of reference, the result can be written in the form

$$W = \int P dp = U - U_0,$$

where U is a function of the co-ordinates (or other quantities) determining the position and configuration of the system, while U_0 is the initial value of U.

Leaving the standard of reference indeterminate, the equation can be written in the form

$$W = U + C,$$

C being merely a symbol for the constant of integration. The function W is called the *work function* or *force function.*

If the final position of the system be taken as standard of reference, and U_1 be the value of U in the final position, the equation takes the form

$$V = U_1 - U,$$

where V is called the *potential energy* of the forces with reference to the final position.

252. If the work function

$$W = U + C$$

be given as a known function of the co-ordinates determining the configuration of the system, the positions of equilibrium of the system can be found from the condition

$$dW = 0,$$

which expresses that *the work function* W *is a maximum*, or *a minimum* (or stationary). It can be shown without difficulty that the equilibrium is stable when the work function is a maximum, and unstable when the work function is a minimum.

As the potential energy by the formulæ of Art. 248 is equal to the work function, but of opposite sign, it follows that the equilibrium is stable or unstable according as the potential energy is a minimum or a maximum.

253. The special case when the only forces acting are the weights of the particles constituting the system is worth mentioning.

Let m be the mass of one of the particles, mg its weight, and z its height above a horizontal plane of reference. Then the virtual work of the weights is

$$\delta W = -\Sigma mg\,\delta z = -g\Sigma m\,\delta z.$$

If $\bar{z}$ be the height of the centroid of the system above the plane of reference, we have $\Sigma mz = \Sigma m \cdot \bar{z}$; hence $\Sigma m\,\delta z = \Sigma m \cdot \delta\bar{z}$. The work function is, therefore,

$$W = -g\Sigma m \cdot \bar{z} + C,$$

and this becomes a maximum or minimum according as $\bar{z}$ is a minimum or maximum; *i.e. the equilibrium is stable or unstable according as the centroid of the system is at its least or greatest height.*

254. It is the object of every *machine* to do work in a certain prescribed way, *i.e.* to exert force, or overcome a resistance, through a certain distance. The various forces of nature, such as the muscular force of man and other animals, the force of gravity, the pressure of the wind, electricity, the expansive force of steam or gas, etc., are called upon for this purpose. In most cases it would not do to apply these forces directly; they must be controlled, guided, and transformed in various ways to become useful, and this is done by interposing the machine between the given *driving force*, commonly called the *power*, and the force which is to do the final work, usually called the *resistance*, *load*, or *weight*. We shall in general denote the "power" by P, the "weight" by Q.

The term *power* is somewhat objectionable in this connection, being here used to denote a force, while in Kinetics it is used for the *rate of doing work*.

255. *The ratio* Q/P *of the weight to the power is called the* **mechanical advantage** *of the machine.*

Under the action of the power P its point of application as well as that of the weight Q is displaced. The corresponding work of the force P may be called the *available* or *total work;* that of the force Q is called the *useful work.*

The ratio of the useful work to the total work is called the **efficiency** of the machine.

In all machines this efficiency is a proper fraction, owing to the fact that the work done by P must balance not only the useful work, but also the so-called wasteful work due to friction, stiffness of ropes, slipping of belts, lack of rigidity, etc.

256. For a more complete discussion of the principle of virtual work the student is referred to MINCHIN'S *Statics*, Vol. I., pp. 78–96, 160–180, and Vol. II., pp. 98–188; ROUTH'S *Statics*, Vol. I., pp. 146–197; SCHELL'S *Theorie der Bewegung und der Kräfte*, Vol. II., pp. 166–211; and J. PETERSEN, *Statik fester Körper*, übersetzt von R. von Fischer-Benzon, Kopenhagen, Höst, 1882, pp. 114–124.

VIII. *Theory of Attractive Forces.*

I. ATTRACTION.

257. Among the various kinds of forces introduced in physics for describing and interpreting natural phenomena, forces of attraction and repulsion occupy a most prominent place.

According to *Newton's law* (the law of universal or cosmical gravitation, the law of nature), every particle of matter attracts every other such particle with a force proportional to the masses and inversely proportional to the square of the distance of the particles.

If m, m' be the masses of the two particles, r their distance, and κ a constant, the mathematical expression for the force of mutual attraction exerted by each particle on the other, or for the stress between them, is, therefore,

$$F = \kappa \frac{mm'}{r^2}. \qquad (1)$$

258. Each particle is here regarded as a mathematical point at which its mass is concentrated. The attractive force would, therefore, approach the limit ∞ as the distance between the points approaches the value 0. To prevent the introduction of infinite forces, we may in such limiting cases regard the particles as very small homogeneous spheres formed of an impenetrable substance. If r, r' be the radii, ρ, ρ' the densities of the spheres, the attraction reaches a finite maximum value F_{max}, when the spheres are in contact, viz.

$$F_{max} = \kappa \frac{\frac{4}{3}\pi\rho r^3 \cdot \frac{4}{3}\pi\rho' r'^3}{(r+r')^2} = \frac{16}{9}\pi^2\kappa\rho\rho' \cdot \frac{r^3 r'^3}{(r+r')^2},$$

which is very small of the fourth order if r, r' be very small of the first order. Thus, for $r=r'$,

$$F_{max} = \frac{4}{9}\pi^2\kappa\rho\rho' \cdot r^4.$$

259. In many applications of the theory of attraction, in particular in electricity and magnetism, it is convenient to consider forces of *repulsion*. This only requires a change of sign in the expression for the force, and this sign may be regarded as attaching to the mass of one of the particles; in other words, if the mass of a centre of attraction be taken as positive, that of a centre of repulsion is taken to be negative, or *vice versa*.

260. While according to Newton's law (1) the force is inversely proportional to the square of the distance, it is often convenient to use forces depending upon the distance r in a different way. Thus the theory of Newtonian attraction can be generalized by assuming for the force between two particles m, m' the law

$$F = \kappa m m' f(r), \tag{2}$$

where $f(r)$ represents any function of the distance r.

When nothing is said to the contrary, we shall here always assume that $f(r) = 1/r^2$, as in Newton's law (1).

261. The constant κ evidently represents the force with which two particles, each of mass 1, attract each other when at the distance 1. It is a physical constant to be determined by experiment, and its numerical value depends on the units of measurement adopted. What can be directly observed is of course not the force itself, but the acceleration it produces. Dividing the force F, as given by formula (1), by the mass m' of the particle on which it acts, we find for the *acceleration* j produced by the attraction of the mass m in the mass m' at the distance r from m:

$$j = \kappa \frac{m}{r^2}. \tag{3}$$

This quantity may also be regarded as the *force* of attraction exerted by the mass m on a mass 1 at the distance r from m, and is therefore called briefly *the attraction at the point* where the mass 1 is situated.

262. It will be shown later (Art. 273) that the attraction of a homogeneous sphere on an external point is the same as if the mass of the sphere were concentrated at its centre. Thus, if m be the mass of the earth (here assumed as a homogeneous sphere), the attraction it exerts on a mass 1 situated at a point P above its surface, at the distance $OP=r$ from the centre O, is $=\kappa m/r^2$; and this is also the acceleration j that it would cause in any mass m' at P.

Now for points P near the earth's surface this acceleration j is known from experiments; it is the acceleration of gravity, usually denoted by g. As the radius of the earth, $r=6.37 \times 10^8$ centimetres, and its mean density $\rho=5\frac{2}{3}$, are also known, the value of the constant κ can be found from the formula

$$g=\kappa\frac{m}{r^2},$$

or

$$\kappa=\frac{3}{4}\,\frac{g}{\pi\rho r}.$$

With $g=980$ we find in C.G.S. units

$$\kappa=\frac{1}{1.543\times 10^7}=0.000\,000\,0648.$$

This, then, is the force in dynes with which two masses of 1 gramme each would attract each other if concentrated at two points 1 centimetre apart.

263. Exercises.

(1) Show that the value of κ in the F.P.S. system is $\frac{1}{9.8\times 10^8}$.

(2) When the units are so selected as to make the constant κ equal to 1, they are called *astronomical units*. Show that the astronomical unit of mass, *i.e.* a mass which when concentrated at a point produces unit acceleration at unit distance, is $=1/\kappa$.

264. Let a mass or a system of masses be given, and let it be required to determine the *attraction at any point P* (Art. 261) produced by it. The given masses may consist of discrete particles, or they may be continuous of one, two, or three dimensions. Continuous masses must be resolved into elements; the attraction at P produced by each element must be determined, and then all these forces must be compounded into a single resultant. This is always possible because the forces all pass through the point P.

Let dm be the element of mass situated at the point Q, P the attracted point of mass 1, $PQ=r$ the distance between them, and α, β, γ the direction cosines of r; then the attraction at P due to dm is dm/r^2; its components are $\alpha dm/r^2$, $\beta dm/r^2$, $\gamma dm/r^2$; and the components of the resultant attraction R at P are

$$X=\int\frac{\alpha dm}{r^2}, \quad Y=\int\frac{\beta dm}{r^2}, \quad Z=\int\frac{\gamma dm}{r^2}, \tag{4}$$

where the integrations must be extended over the whole given mass. The resultant R itself, and its direction cosines a, b, c, are finally found from the formulæ

$$R=\sqrt{X^2+Y^2+Z^2}, \quad a=\frac{X}{R}, \quad b=\frac{Y}{R}, \quad c=\frac{Z}{R}. \tag{5}$$

The following examples will illustrate the process.

265. Homogeneous Circular Arc. To determine the attraction exerted by a mass distributed uniformly along a circular arc ACB (Fig. 78) of angle 2α and radius a on a mass 1 situated at the centre P of the circle, let $QQ'=ds$ be an element of the arc, $dm=\rho ds$ its mass; then

$$\kappa\frac{\rho ds}{a^2}$$

is its attraction at P, and this force has the direction PQ.

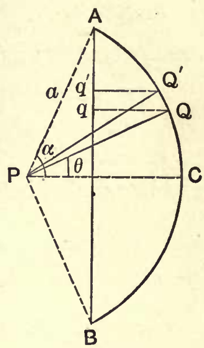

Fig. 78.

Resolving this force parallel to the bisecting radius PC of the arc and at right angles to it, it will be seen that the latter component need not be considered, since it is balanced by an equal and opposite component arising from the element situated symmetrically to QQ' with respect to PC. If $\measuredangle\, CPQ=\theta$, the component along PC is $\kappa\rho ds\cos\theta/a^2$, or since $ds=a d\theta$, $\kappa\rho\cos\theta d\theta/a$. Hence the resultant attraction at P is

$$R=\frac{\kappa\rho}{a}\int_{-\alpha}^{+\alpha}\cos\theta d\theta=2\kappa\rho\frac{\sin\alpha}{a}. \tag{6}$$

Denoting the chord AB of the given arc by c, the result can be put into the form

$$R = \kappa\rho \cdot \frac{c}{a^2}, \tag{6'}$$

which might have been found directly from Fig. 78, without integration, since $ds \cos\theta = qq'$.

Formula (6') shows that, if the chord AB were covered with mass of the same density ρ as the arc, and if this mass were concentrated at the middle point C of the arc, it would produce at P the same attraction as the arc ACB.

266. Homogeneous Straight Rod. To determine the attraction at any point P produced by a mass distributed uniformly along a segment of a straight line, AB, let $QQ' = ds$ (Fig. 79) be an element, $OQ = s$ its distance from the foot of the perpendicular $PO = p$ dropped from P on AB, and let r, θ be the polar co-ordinates of Q with respect to P as pole and PO as polar axis. Resolving the attraction $\kappa\rho ds/r^2$ of the element QQ' along PO and at right angles to it, we find the components

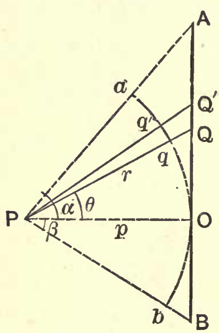

Fig. 79.

$$dX = \kappa\rho\frac{\cos\theta ds}{r^2}, \quad dY = \kappa\rho\frac{\sin\theta ds}{r^2}.$$

The figure gives $r = p/\cos\theta$, $s = p\tan\theta$; hence $ds = pd\theta/\cos^2\theta$; substituting these values, we have

$$dX = \frac{\kappa\rho}{p}\cos\theta d\theta, \quad dY = \frac{\kappa\rho}{p}\sin\theta d\theta;$$

and integrating between the limits $-\beta = OPB$ and $\alpha = OPA$:

$$X = \frac{\kappa\rho}{p}(\sin\beta + \sin\alpha), \quad Y = \frac{\kappa\rho}{p}(\cos\beta - \cos\alpha). \tag{7}$$

The resultant attraction

$$R = \sqrt{X^2 + Y^2} = \frac{2\kappa\rho}{p}\sin\frac{\alpha+\beta}{2} \tag{8}$$

makes with PO an angle ϕ, for which we have

$$\tan\phi = \frac{Y}{X} = \frac{\cos\beta - \cos\alpha}{\sin\beta + \sin\alpha} = \tan\frac{\alpha - \beta}{2}; \tag{8}$$

hence

$$\phi = \frac{\alpha - \beta}{2}, \tag{9}$$

i.e. the attraction at P bisects the angle APB subtended at P by the rod.

267. These results might have been derived from the problem of Art. 265. For it is easy to see that the attraction exerted at P by the straight rod AB is the same as that exerted by the circular arc ab (Fig. 79) described about P with radius p and bounded by PA, PB. This will follow if it can be shown that the attraction at P of the element QQ' of the rod is equal to that of the element qq' of the arc contained between the same radii vectores. Now the attraction of QQ' is $\kappa\rho \cdot QQ'/r^2$, while that of qq' is $\kappa\rho \cdot qq'/p^2$. Projecting QQ' on the circle of radius r, we have

$$\frac{qq'}{QQ' \cos\theta} = \frac{p}{r},$$

or since the triangle POQ gives $\cos\theta = p/r$,

$$\frac{qq'}{QQ'} = \frac{p^2}{r^2};$$

which proves the proposition.

268. It has been shown in Art. 266 that the attraction at any point P exerted by a straight rod AB bisects the angle APB; it is therefore tangent to the hyperbola passing through P and having A, B as its foci. Hence if in any plane through AB the system of confocal hyperbolas be constructed with A, B as foci, the direction of the attraction at any point P in the plane is along the tangent to the hyperbola that passes through P. These hyperbolas having everywhere the direction of the resulting attractive force, are called the **lines of force.**

An ellipse passing through P and having the same foci A, B would have the bisector of the angle APB as its normal. The confocal ellipses about A, B as foci form the so-called *orthogonal system* of the lines of force. If such an ellipse be regarded as offering a normal resistance, the point P would be kept in equilibrium under the action of the attraction of the rod and the reaction of the curve. The confocal ellipses are therefore called *equilibrium*, or *level*, *lines*, or also for a reason that will appear later **equipotential lines.**

Rotating the whole figure about AB as axis, the ellipses describe confocal ellipsoids of revolution which are *level*, or *equipotential, surfaces*.

269. Exercises.

(1) A segment AB is cut out of an infinite straight line along which mass is distributed uniformly. If the mass on the ray issuing from A be repulsive, that on the ray issuing from B (in the opposite sense), attractive, determine the resultant attraction at any point P by the method of Art. 267, and show that the lines of force are confocal ellipses, while the equipotential surfaces are confocal hyperboloids.

(2) Three rods of constant density form a triangle. Find the point at which the resultant attraction is zero.

(3) Find the attraction of a straight rod AB of constant density on a point P situated on the line AB so that $AP = a$, $BP = b$.

(4) Two rods of lengths $2a$, $2b$, and of equal constant density, are placed parallel to each other, at a distance c, so that the line joining their middle points is at right angles to them. Find their *mutual attraction*, *i.e.* the force required to keep them apart.

(5) Show that the attraction of a homogeneous rod of infinite length on a point at the distance p from it, is $2\kappa\rho/p$.

270. The formula of the last exercise (5) can be used to determine the attraction of an **infinitely long homogeneous cylinder** of finite cross-section on an external point P, by resolving the cylinder into filaments

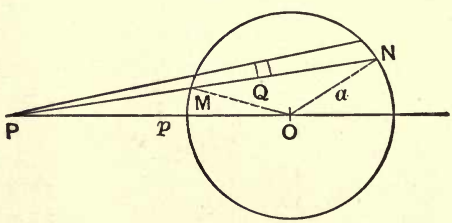

Fig. 80.

parallel to the axis. Fig. 80 represents the cross-section of the cylinder passing through P. The polar element of area at Q, $rd\theta \cdot dr$, can be regarded as the cross-section of a filament, whose attraction at P is, by Ex. (5),

$$2\kappa\rho\frac{rdrd\theta}{r} = 2\kappa\rho drd\theta.$$

Resolving this force along and at right angles to PO, and considering that owing to symmetry the resultant R must pass through O, we have

$$R = 2\kappa\rho\int\int\cos\theta d\theta dr.$$

If the radius vector PQ meet the surface of the cylinder at M and N, the integration with respect to r gives

$$R = 2\kappa\rho\int MN\cos\theta d\theta.$$

With $PO = p$, we have $MN = 2\sqrt{a^2 - p^2\sin^2\theta}$, and the limits of the integration are $\pm\sin^{-1}(a/p)$. Hence

$$R = 2\pi\kappa\rho\frac{a^2}{p}. \qquad (10)$$

271. Homogeneous Circular Plate. To determine the attraction of a homogeneous circular area on a point P situated on the axis through the centre O at right angles to its plane at the distance $PO = p$, we may resolve the plate into ring-shaped elements of radii r and $r + dr$. The mass of such a ring is $2\pi\rho r dr$; all points of the ring are at the same distance $\sqrt{r^2 + p^2}$ from P, and their attractions make the same angle $\phi = \tan^{-1}(r/p)$ with the axis PO. Hence the attraction of the ring is

$$2\pi\kappa\rho\cdot\frac{rdr}{r^2 + p^2}\cos\phi = 2\pi\kappa\rho\sin\phi d\phi,$$

since $dr = pd\phi/\cos^2\phi = (p^2 + r^2)d\phi/p$.

Let 2α be the vertical angle of the cone that subtends the plate at P; then

$$R = 2\pi\kappa\rho\int_0^\alpha\sin\phi d\phi = 4\pi\kappa\rho\sin^2\frac{\alpha}{2}, \qquad (11)$$

or, in terms of p,

$$R = 2\pi\kappa\rho\left(1 - \frac{p}{\sqrt{r^2 + p^2}}\right). \qquad (11')$$

272. Homogeneous Spherical Shell: *Geometrical Method.* (a) *Attraction at an internal point* P. Let C be the centre, a the radius of the sphere (Fig. 81). A thin double cone having its vertex at P cuts the sphere in two elements, $AB = dS$, $A'B' = dS'$, which can be shown to exert equal and opposite attractions at P.

Let $PA = r$, $PA' = r'$, and let $d\omega$ denote the *solid angle of the cone* (*i.e.* the area it cuts out of a sphere of radius 1 described about P as centre); then $r^2 d\omega$ is the area cut out of a sphere of radius r with the same centre P. Hence the element of mass at A is $\rho \cdot r^2 d\omega / \cos PAC$, and its attraction at P is $= \kappa\rho d\omega/\cos PAC$. Similarly, the attraction of the mass element at A' is

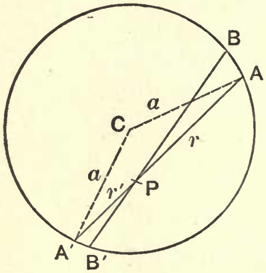

Fig. 81.

$$= \kappa\rho r'^2 d\omega/(r'^2 \cos PA'C) = \kappa\rho d\omega/\cos PA'C.$$

These attractions are equal, since for the sphere $\measuredangle PAC = \measuredangle PA'C$.

The whole sphere can thus be divided up into elements exerting equal and opposite attractions at P; *the resultant attraction of the whole shell at any internal point is, therefore, zero.*

273. (b) *Attraction at an external point* P. The investigation can be made similar to that for an internal point by introducing the point P' (Fig. 82), which is inverse to P with respect to the sphere, *i.e.* the point P' on CP for which $CP \cdot CP' = CA^2$, or putting $CP = p$, $CP' = p'$, $CA = a$, the point for which

$$pp' = a^2. \qquad (12)$$

Any chord HH' through P' determines two pairs of similar triangles: CHP' and CPH, $CH'P'$ and CPH'; for each pair has the angle at C

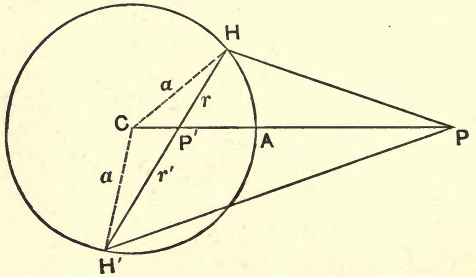

Fig. 82.

in common, and the sides including the equal angles proportional by (12), since $CH = CH' = a$. It follows that $\measuredangle CHP' = \measuredangle CPH$, and $\measuredangle CH'P' = \measuredangle CPH'$; hence, as the triangle HCH' is isosceles, the line CP bisects the angle HPH'.

With the aid of these geometrical properties it can be shown that equal attractions are produced at P by the elements dS at H and dS' at H' cut out by any thin cone whose vertex is the inverse point P'. We have, as in Art. 272, for the mass elements at H and H' cut out by the cone

$$dm = \rho dS = \rho \frac{r^2 d\omega}{\cos P'HC}, \quad dm' = \rho dS' = \rho \frac{r'^2 d\omega}{\cos P'H'C},$$

and for the corresponding attractions at P

$$\kappa\rho \frac{r^2 d\omega}{PH^2 \cos P'HC}, \quad \kappa\rho \frac{r'^2 d\omega}{PH'^2 \cos P'H'C};$$

but these expressions are equal since $\measuredangle P'HC = \measuredangle P'H'C$ and the similar triangles give $r/PH = a/p$, $r'/PH' = a/p$.

As, moreover, these attractions make equal angles with CP, their projections on this line are equal, and their resultant is

$$2\kappa\rho \frac{a^2 d\omega}{p^2}.$$

To form the final resultant, this expression must be integrated over the whole sphere, and as the summation of the double cone gives $\int d\omega = 2\pi$, we find

$$R = 4\pi\kappa\rho \frac{a^2}{p^2} = \kappa \frac{M}{p^2}, \tag{13}$$

where M denotes the whole mass of the shell. Hence, *the attraction of a homogeneous shell on an external point is the same as if the whole mass of the shell were concentrated at the centre of the shell.*

274. (c) *Attraction at the surface.* If the point P approaches the surface from within, the attraction remains constantly zero; if P approaches the surface from without, the attraction $\kappa M/p^2$ approaches the limit $\kappa M/a^2$. For a point on the surface the attraction is the arithmetic mean of these two values, viz.

$$R = 2\pi\kappa\rho. \tag{14}$$

This can be shown as follows (Fig. 83). The element of mass at H is

$$\rho dS = \rho \cdot r^2 d\omega / \cos PHC;$$

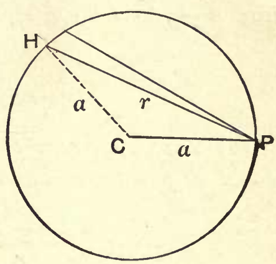

Fig. 83.

its attraction at P is $\kappa\rho\, d\omega/\cos PHC$, and as the angles at P and H are equal, the projection of the attraction on $PC = \kappa\rho d\omega$. For a point on the surface $\int d\omega = 2\pi$. Hence the total attraction at P is $= 2\pi\kappa\rho$.

275. Homogeneous Spherical Shell: *Analytical method.* Let Q (Fig. 84) be any point on the sphere; $PQ = r$, $CQ = a$, $CP = p$, $\measuredangle PCQ = \theta$. Through Q lay a plane at right angles to CP, and take as element the mass contained between this and an infinitely near parallel plane. This mass element is

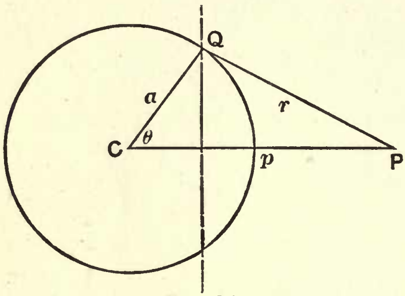

Fig. 84.

$$= \rho \cdot 2\pi a \sin\theta \cdot a d\theta,$$

and its attraction at P along CP is

$$= 2\pi\kappa\rho a^2 \sin\theta d\theta \cdot \frac{1}{r^2} \cdot \cos CPQ = 2\pi\kappa\rho a^2 \sin\theta d\theta \cdot \frac{p - a\cos\theta}{r^3}.$$

The relation between r and θ follows from the triangle CPQ, which gives

$$r^2 = a^2 + p^2 - 2ap\cos\theta,$$

hence $$rdr = ap\sin\theta d\theta.$$

Substituting for $a\sin\theta d\theta$ and for $a\cos\theta$ their values from the last two relations, the expression for the attraction of the elementary ring becomes

$$2\pi\kappa\rho a \cdot \frac{rdr}{p} \cdot \frac{p - \dfrac{a^2 + p^2 - r^2}{2p}}{r^3} = \pi\kappa\rho\frac{a}{p^2} \cdot \frac{p^2 - a^2 + r^2}{r^2} \cdot dr.$$

(*a*) For an *internal point P*, we have $p < a$, and the limits for r are from $a - p$ to $a + p$. Hence the resultant attraction is

$$R = \pi\kappa\rho\frac{a}{p^2}\int_{a-p}^{a+p}\left(\frac{p^2 - a^2}{r^2} + 1\right)dr = \pi\kappa\rho\frac{a}{p^2}\left[\frac{a^2 - p^2}{r} + r\right]_{a-p}^{a+p} = 0.$$

(*b*) For an *external point P*, we have $p > a$, and the limits are from $p - a$ to $p + a$. Hence the attraction becomes

$$R = \pi\kappa\rho\frac{a}{p^2}\left[\frac{a^2 - p^2}{r} + r\right]_{p-a}^{p+a} = 4\pi\kappa\rho\frac{a^2}{p^2} = \kappa\frac{M}{p^2}.$$

276. Exercises.

(1) Show that the attraction exerted by a right circular cone of vertical angle 2α and height h, at the vertex, is $= 2\pi\kappa\rho(1 - \cos\alpha)h$.

(2) Show that the attraction of a circular cylinder of radius a and of length l, at a point on its axis at the distance x from the nearest base, is $= 2\pi\kappa\rho[l + \sqrt{a^2 + x^2} - \sqrt{(l + x)^2 + a^2}]$.

(3) From the result of Ex. (2) show that the attraction of a cylinder extending in one sense to infinity is $= 2\pi\kappa\rho a$ at its base.

(4) Show that for a spherical shell of finite thickness, if the density be either constant or a function of the distance from the centre only, the attraction is zero at any point within the hollow of the shell, and that it is the same as if the whole mass were concentrated at the centre at any external point.

(5) Show how to find the attraction of a homogeneous spherical shell of finite thickness, at any point within the mass of the shell.

(6) Show that the attraction of a solid sphere of mass M, the density being any function of the distance from the centre, is $= \kappa M/p^2$ at any external point P, having the distance p from the centre, and that it is directly proportional to the distance of the point P from the centre when P lies within the mass.

(7) Show that the attraction of a solid homogeneous hemisphere at a point in its edge is $= \frac{2}{3}\kappa\rho a\sqrt{\pi^2 + 4}$, and that it makes with the plane of the base an angle of about $32\frac{1}{2}°$.

2. THE POTENTIAL.

277. The configuration and density of any attracting masses being given, the force of attraction R exerted by these masses on a mass 1 situated at any point P can be determined both in magnitude and direction. The method illustrated on some simple examples in the preceding articles, while theoretically quite general, becomes very laborious in more complicated cases. Moreover, the required resultant R, *i.e.* the "attraction at the point P," depends as to its magnitude and direction on the position of the point P; and it is of interest to investigate its variation from point to point throughout space, in a similar way as was done for the example of the straight rod in Art. 268.

Investigations of this kind are greatly facilitated by the aid of a certain function called the *potential*, whose meaning and use we proceed to discuss very briefly in the following articles.

278. The attraction at the point P exerted by a single particle Q of mass m (Fig. 85) is $=m/r^2$ if the units be so selected as to make the constant $\kappa=1$ (Art. 262 and Art. 263, Ex. 2). (This assumption is generally made in theoretical investigations, as there is nothing to be gained by carrying the constant factor κ through all the formulæ. The factor κ can always be re-introduced when numerical results are required.)

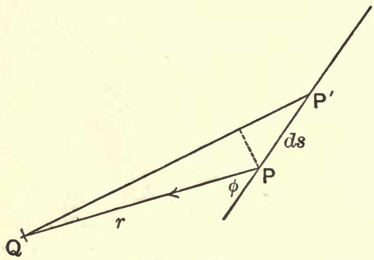

Fig. 85.

Let the particle P be displaced through an infinitesimal distance $PP'=ds$ in any direction, and let ϕ be the angle between $QP=r$ and PP'. The element of work done by the force m/r^2 in this displacement is

$$dW=-\frac{m}{r^2}\cos\phi ds=-\frac{m}{r^2}\frac{dr}{ds}ds=\frac{d}{ds}\left(\frac{m}{r}\right)ds.$$

The quantity m/r occurring in the last expression is called the *potential* of the mass m at the point P; it is usually denoted by V.

279. If the particle continue to move along some curve from its initial position P to some final position P_1, the total work done by the attraction of Q is evidently

$$W=\int_P^{P_1}\frac{dV}{ds}ds=V_1-V,$$

where $V=m/r$ is the potential at P, and $V_1=m/r_1$ is the potential at P_1. Hence, *the difference of the potentials at any two points is equal to the work done by the attraction*, whatever may have been the path along which the displacement has taken place.

As the potential $V=m/r$ becomes zero when $r=\infty$, it appears that *the potential* V_1 *at any point* P_1 *is the work that would be* *done by the attraction on a particle of mass* 1 *if it were brought up to the point* P_1 *along any path from infinity.*

The relations of Art. 278 can be written in the form

$$\frac{dV}{ds} = -\frac{m}{r^2}\cos\phi;$$

i.e. *the derivative of the potential with respect to any displacement is equal to the component of the attraction in the direction of the displacement.*

280. When there are given several masses $m, m', m'', \ldots$, concentrated at points $Q, Q', Q'', \ldots$, their potential at any point P is defined as the sum

$$V = \frac{m}{r} + \frac{m'}{r'} + \frac{m''}{r''} + \cdots = \Sigma\frac{m}{r};$$

when the given masses are continuous, the sign of summation must be replaced by an integral, and we have

$$V = \int\frac{dm}{r}.$$

The fundamental properties proved in Art. 279 remain the same.

281. Let there be given a continuous mass m, referred to a rectangular system of co-ordinates. The attraction at any point P (x, y, z) due to this mass has three components X, Y, Z, which can be found as follows. The attraction produced at P by an element dm at a point Q (x', y', z') of the mass is dm/r^2, where $r = PQ$, and its direction cosines are $(x'-x)/r$, $(y'-y)/r$, $(z'-z)/r$; hence its components are

$$dX = \frac{(x'-x)\,dm}{r^3}, \quad dY = \frac{(y'-y)\,dm}{r^3}, \quad dZ = \frac{(z'-z)\,dm}{r^3}.$$

Integrating, we find the components of the total attraction at P:

$$X = \int\frac{(x'-x)\,dm}{r^3}, \quad Y = \int\frac{(y'-y)\,dm}{r^3}, \quad Z = \int\frac{(z'-z)\,dm}{r^3}.$$

Differentiating the relation

$$r^2=(x'-x)^2+(y'-y)^2+(z'-z)^2$$

partially with respect to x, y, and z, we have

$$r\frac{\partial r}{\partial x}=-(x'-x),\quad r\frac{\partial r}{\partial y}=-(y'-y),\quad r\frac{\partial r}{\partial z}=-(z'-z).$$

Substituting these values in the above integrals, we find

$$X=-\int\frac{1}{r^2}\frac{\partial r}{\partial x}dm=\int\frac{\partial}{\partial x}\left(\frac{1}{r}\right)dm=\frac{\partial}{\partial x}\int\frac{dm}{r},$$

and similarly

$$Y=\frac{\partial}{\partial y}\int\frac{dm}{r},\quad Z=\frac{\partial}{\partial z}\int\frac{dm}{r}.$$

As $\int\frac{dm}{r}$ is the potential V of the given mass, we have

$$X=\frac{\partial V}{\partial x},\quad Y=\frac{\partial V}{\partial y},\quad Z=\frac{\partial V}{\partial z};$$

i.e. *the components of the attraction at any point are the derivatives of the potential at that point in the direction of these components.* This may be regarded as a special case of the last proposition of Art. 279.

282. It is to be noticed that the proof given in the preceding article can easily be extended to the case of forces of the form (2), Art. 260. In other words, even in the case of forces not following the Newtonian law of the inverse square, but expressed by any function $f(r)$ of the distance, there exists a function corresponding to the potential of Newtonian attractions; it is called the *force function.*

We have, just as in Art. 281,

$$r^2=(x'-x)^2+(y'-y)^2+(z'-z)^2;$$

hence $$-\frac{\partial r}{\partial x}=\frac{x'-x}{r},\quad -\frac{\partial r}{\partial y}=\frac{y'-y}{r},\quad -\frac{\partial r}{\partial z}=\frac{z'-z}{r}.$$

These are the direction cosines of the force $f(r)$ with which the mass m at $Q(x', y', z')$ attracts the mass 1 at $P(x, y, z)$. The components of this force are, therefore,

$$X=-f(r)\frac{\partial r}{\partial x},\quad Y=-f(r)\frac{\partial r}{\partial y},\quad Z=-f(r)\frac{\partial r}{\partial z}.$$

These expressions show that there exists a function

$$F(r)=-\int f(r)dr,$$

of which the components of the force at P are the partial derivatives:

$$X=\frac{\partial F(r)}{\partial x},\quad Y=\frac{\partial F(r)}{\partial y},\quad Z=\frac{\partial F(r)}{\partial z}.$$

283. The potential

$$V=\int\frac{dm}{r}$$

at a point P for a given system of masses is a function of the co-ordinates of the point P. If this function be known, the attraction at any point P produced by the given masses can at once be found; for the components of this attraction are

$$X=\frac{\partial V}{\partial x},\quad Y=\frac{\partial V}{\partial y},\quad Z=\frac{\partial V}{\partial z}.$$

Hence,

$$dV=Xdx+Ydy+Zdz.$$

If the function V be equated to any constant V_1, the resulting equation

$$V=V_1$$

represents a surface that is the focus of all points at which the potential of the given masses has one and the same value V_1. Such a surface is called an **equipotential surface,** or a *level,* or *equilibrium surface.*

284. Differentiating the equation of the equipotential surface, and dividing by ds and by the attraction R whose components are X, Y, Z, we find

$$\frac{\frac{\partial V}{\partial x}}{R} \cdot \frac{dx}{ds} + \frac{\frac{\partial V}{\partial y}}{R} \cdot \frac{dy}{ds} + \frac{\frac{\partial V}{\partial z}}{R} \cdot \frac{dz}{ds} = 0.$$

The first factors in each term are the direction cosines of R, the second factors are those of a tangent to the surface; the equation expresses, therefore, the fact that *the attraction* R *at any point of an equipotential surface is normal to the surface.*

The attraction at any point P of an equipotential surface is, therefore, equal to dV/dn, where dn is the element of the normal at P between this and the next equipotential surface. Consequently, the attraction is inversely proportional to the distance between the successive equipotential surfaces.

Let the normal of an equipotential surface at any point P intersect the next equipotential surface at a point P'; let the normal at P' intersect the next surface at P''; and so on. The elements PP', $P'P''$, etc., will form a curve which is at every point normal to the equipotential surface passing through that point. Such a curve is called a **line of force**, since its tangent at any point indicates the direction of the resultant attraction at that point. The lines of force cut the equipotential surfaces orthogonally.

285. Potential of a Homogeneous Spherical Shell. (a) For an *internal point*, we may proceed similarly as in Art. 272. The element of mass cut out at A (Fig. 81) by the small cone whose solid angle is $d\omega$ is again $\rho \cdot r^2 d\omega/\cos\alpha$ if $\measuredangle PAC = \measuredangle PA'C = \alpha$; the corresponding potential at P is $\kappa\rho r d\omega/\cos\alpha$; similarly, the potential due to the mass at A' is $\kappa\rho r' d\omega/\cos\alpha$. Their sum is

$$\kappa\rho \frac{r + r'}{\cos\alpha} d\omega = 2\,\kappa\rho a d\omega,$$

since $r + r' = 2\,a\cos\alpha$.

As $\int d\omega = 2\,\pi$, we find $\qquad V_i = 4\,\pi\kappa\rho a$;

i.e. the potential has the same constant value for all points within the hollow of the shell. It follows that the attraction is zero, as found in Art. 272.

(*b*) For an *external point* P, we might also proceed geometrically, making use of the inverse point P', as in Art. 273. But we shall use the analytical method.

Just as in Art. 275, Fig. 84, we have for the mass of the ring-shaped element $dm = \rho \cdot 2\pi a^2 \sin\theta d\theta$, or as $ap \sin\theta d\theta = rdr$, $dm = 2\pi\rho a r dr/p$. Hence the element of potential is $dV = 2\pi\kappa\rho a dr/p$, which integrated between the limits from $p - a$ to $p + a$, gives

$$V_e = \frac{4\pi\kappa\rho a^2}{p} = \kappa\frac{M}{p}.$$

Hence the potential is the same as if the whole mass were concentrated at the centre.

ANSWERS.

Pages 29–33.

(4) $\frac{1}{5}a$, where a is the side of the hexagon.

(5) At the centre of the incircle of the triangle formed by the mid-points of the sides.

(7) $\bar{x} = \bar{y} = \frac{2}{\pi} r.$

(8) Taking OA as axis of x,

$$\bar{x} = \frac{2r}{\alpha^2}(\alpha \sin\alpha + \cos\alpha - 1), \quad \bar{y} = \frac{2r}{\alpha^2}(\sin\alpha - \alpha\cos\alpha).$$

(9) $\bar{x} = \dfrac{3\sqrt{2} - \log(1+\sqrt{2})}{\sqrt{2} + \log(1+\sqrt{2})} \cdot \dfrac{a}{4}, \quad \bar{y} = \dfrac{2\sqrt{2} - 1}{\sqrt{2} + \log(1+\sqrt{2})} \cdot \frac{4}{3}a.$

(10) $\bar{x} = \pi a, \quad \bar{y} = \frac{4}{3}a.$

(11) $\bar{x} = \bar{y} = \frac{4}{5}a.$

(12) $\bar{x} = 0, \quad \bar{y} = \dfrac{cx}{s} + \frac{1}{2}y$, where $s = c(e^{\frac{x}{c}} - e^{-\frac{x}{c}}).$

(13) $\bar{x} = r \cdot \dfrac{\sin\theta}{\theta}, \quad \bar{y} = r \cdot \dfrac{1 - \cos\theta}{\theta}, \quad \bar{z} = \frac{1}{2}kr\theta.$

(14) $GC = \frac{1}{3}\sqrt{a^2 + b^2}, \quad GA = \frac{1}{3}\sqrt{a^2 + 4b^2}, \quad GB = \frac{1}{3}\sqrt{4a^2 + b^2}.$

(15) With A as origin, $\bar{x} = \frac{61}{116}a, \quad \bar{y} = \frac{63}{116}a.$

(17) $\bar{x} = \frac{1}{2}\dfrac{(a + a')^2\beta + 4a'(b - \beta)\alpha}{(a + a')\beta + 2(b - \beta)\alpha}.$

(18) $\bar{x} = \frac{1}{2}\dfrac{a^2 + b\delta - \delta^2}{a + b - \delta} = 4.90$ in.;

first approximation $\bar{x} = \frac{1}{2}\dfrac{a^2 + b\delta}{a + b} = 4.62$ in.;

second approximation $\bar{x} = \frac{1}{2}\dfrac{a^2}{a + b} = 4.50$ in.

(19) $\bar{x} = \frac{1}{2}\frac{a(a+2b') - (2a - b + b')\delta}{a + b + b' - 2\delta} = 4.5$ in.

(20) $\bar{x} = \frac{1}{2}\frac{a_2\beta + b\alpha^2 - \alpha^2\beta}{a\beta + b\alpha - \alpha\beta}, \quad \frac{1}{2}\frac{a^2\beta + b\alpha^2}{a\beta + b\alpha - \alpha\beta}, \quad \frac{1}{2}\frac{a^2\beta}{a\beta + b\alpha}$; 0.32 a, 0.33 a, 0.25 a.

(25) $\bar{x} = -\frac{b^2c}{a^2 - b^2}$.

(26) For a segment of a ring of angle 2α and radii r_1, r_2, the distance of the centroid from the centre is $\bar{x} = \frac{2}{3}\cdot\frac{\sin\alpha}{\alpha}\cdot\frac{r_1^2 + r_1r_2 + r_2^2}{r_1 + r_2}$. Hence $\bar{x} = \frac{2}{147\pi}(740 + 73\sqrt{2}) = 3.22$ ft.; *i.e.* the centroid lies about $8\frac{1}{2}$ in. above the door.

(27) $\bar{x} = \frac{3}{5}x$, $\bar{y} = \frac{3}{8}y$.

(28) $\bar{x} = \frac{\pi}{2}$, $\bar{y} = \frac{\pi}{8}$.

(29) $\bar{x} = \frac{4}{3\pi}a$, $\bar{y} = \frac{4}{3\pi}b$.

(30) $\bar{x} = \frac{2}{3(\pi - 2)}a$, $\bar{y} = \frac{2}{3(\pi - 2)}b$.

(32) Take the vertex as origin, the axis of the cone as axis of x, and one of the bounding planes as plane of xy. Then, if a be the radius of the base, h the height, and 2α the angle at the vertex of the cone, the formulæ of Art. 40 give

$$x = f(R) = \frac{h}{a}R, \quad \sqrt{1 + [f'(R)]^2} = \sqrt{1 + \frac{h^2}{a^2}} = \csc\alpha,$$

$$S = \tfrac{1}{2}\phi a^2 \csc\alpha;$$

hence $$\bar{x} = \tfrac{2}{3}h, \quad \bar{y} = \tfrac{2}{3}a\cdot\frac{\sin\phi}{\phi}, \quad \bar{z} = \tfrac{2}{3}a\cdot\frac{1 - \cos\phi}{\phi}.$$

(33) About 2631 miles from the centre.

(34) Regard the ice-cap as a surface mass of density δ; let $\bar{x}_1$ be the distance of its centroid from the earth's centre, m_1 the mass of the ice-cap, m that of the earth alone, and ϕ' the polar distance of the arctic circle; then the equation of moments $(m + m_1)\bar{x} = m_1\bar{x}_1$ gives, if m_1 in the parenthesis be neglected, $\bar{x} = (m_1/m)\bar{x}_1 = \frac{3}{22}\delta\sin^2\phi' = 0.216$ mile.

(35) At the distance $\frac{1}{5}r$ from the centre.

(36) $\bar{x} = \frac{3}{8}a$.

(37) $\bar{x} = \frac{h}{4} \frac{r_1^2 + 2 r_1 r_2 + 3 r_2^2}{r_1^2 + r_1 r_2 + r_2^2}.$

(38) Let V_1 be the volume of the whole pyramid, V_2 that of the top cut off, V that of the frustum ; $\bar{x}_1$, $\bar{x}_2$, $\bar{x}$ the distances of their centroids from the lower base ; h_1, h_2, h_3, their heights. Then the equation of moments is $(V_1 - V_2)\bar{x} = V_1\bar{x}_1 - V_2\bar{x}_2$. By geometry, we have $V_1/V_2 = r_1^3/r_2^3$; hence, $(r_1^3 - r_2^3)\bar{x} = r_1^3\bar{x}_1 - r_2^3\bar{x}_2$; also, $\bar{x}_1 = \frac{1}{4}h_1$, $\bar{x}_2 = h + \frac{1}{4}h_2$, $h_1 - h_2 = h$, $h_1/h_2 = r_1/r_2$. By means of these relations, we find

$$(r_1^3 - r_2^3)\bar{x} = \frac{h}{4} \frac{r_1^4 - 4 r_1 r_2^3 + 3 r_2^4}{r_1 - r_2}, \text{ or } \bar{x} = \frac{h}{4} \frac{r_1^2 + 2 r_1 r_2 + 3 r_2^2}{r_1^2 + r_1 r_2 + r_2^2}.$$

(39) $\bar{x} = \frac{3}{4} \frac{(2a - h)^2}{3a - h}.$

(40) Let A, B, C, D be the vertices of the tetrahedron, G its centroid, G_1 that of the face ABC; let a, b, c, d, $\bar{x}$, $\bar{x}_1$ be the distances of these points from the plane; and let the projections of these points on the plane be denoted by A', B', C', D', G', G_1'. Then, since $GG_1/DG_1 = 1/4$, and $GG_1/DG_1 = (\bar{x} - \bar{x}_1)/(d - \bar{x}_1)$, we have $(\bar{x} - \bar{x}_1)/(d - \bar{x}_1) = 1/4$; hence $\bar{x} = \frac{1}{4}(3\bar{x}_1 + d)$. Let E be the middle point of AB, e its distance from the plane; then, applying a similar method to the triangle ABC, we find $\bar{x}_1 = \frac{1}{3}(2e + c) = \frac{1}{3}(a + b + c)$. Hence, finally, $\bar{x} = \frac{1}{4}(a + b + c + d)$.

(41) $\bar{x} = \frac{2}{3}h.$

(42) $\bar{y} = \frac{5}{12}y_1.$

(43) $\bar{y} = \frac{5}{16}y_1.$

(44) $\bar{x} = \frac{3}{8}a$, $\bar{y} = \frac{3}{8}b$, $\bar{z} = \frac{3}{8}c.$

(45) (*a*) $\bar{x} = \bar{y} = \frac{4}{3}a.$

(*b*) $\bar{x} = \frac{9\pi^2 + 16}{18\pi}a$, $\bar{y} = \frac{5}{6}a.$

(*c*) $\bar{x} = \frac{26}{15}a.$

(*d*) $\bar{x} = \frac{45\pi^2 + 128}{90\pi}a.$

(*e*) $\bar{x} = \frac{2(15\pi - 8)}{15(3\pi - 4)}a.$

(*f*) $\bar{x} = \frac{63\pi^2 - 64}{6(9\pi^2 - 16)}a.$

(46) Take as element a hemispherical shell of radius r and thickness dr; $\bar{x} = \frac{n + 3}{2(n + 4)}a.$

(47) $\bar{x} = \frac{1}{4}(H + h).$

(48) Compare problems (40) and (5), and apply the propositions of Pappus, Arts. 30 and 42; $V = \frac{2}{3}\pi(p + q + r)A$, where A is the area of the triangle; $S = \pi[a(q + r) + b(r + p) + c(p + q)].$

(49) Taking the axis of the cup as axis of x, let (a, b) be the centroid of cup and handle, m their mass; $(x_1, 0)$ the centroid of the water whose mass can be expressed by $(m/c)x_1$, where c is a constant. Then the co-ordinates x, y of the centroid of cup, handle, and water together fulfil the equation $(a+c)y^2 - bxy - 2bcy + b^2c = 0$, which represents a hyperbola.

(50) Taking the axis of z parallel to the axis of the cylinder, and the origin in the line of intersection of the bases, we have $V = \int\int z\,dx\,dy$, or if ϕ be the angle of inclination of the bases:

$$V = \tan\phi \int\int y\,dx\,dy = \tan\phi \cdot \bar{y} \int\int dx\,dy.$$

(51) Apply (50) twice.

Page 35.

(1) 300 000 F.P.S. units.

(2) $34\frac{1}{11}$ miles per hour.

(3) 71 600.

Page 40.

(1) 6.4×10^5 poundals $= 8.9 \times 10^9$ dynes.

(2) 4.5 pounds.

(3) 0.14.

(5) 60.

Pages 51–53.

(4) $100\sqrt{7}$; $\tan^{-1}\frac{1}{5}\sqrt{3}$.

(7) $10\sqrt{2}(\sqrt{3}-1)$; $20(\sqrt{3}-1)$.

(9) $Q = \frac{1}{2}(-P + \sqrt{4R^2 - 3P^2})$.

(10) $R = 569$; angle with horizon $= 99° 27'$.

(12) Twice the focal distance.

(13) $124° 12'.5$.

(14) $90°$.

(15) (a) $\sqrt{2} \cdot W$; (b) $2W \cos\frac{1}{2}(\pi/2 - \theta)$.

(18) $2W \cos\frac{1}{2}(\pi/2 + \alpha)$, etc.

(19) (a) $\alpha = 30°$, $\beta = 120°$, $\gamma = 30°$; (b) impossible.

(21) $T = \frac{9}{16}W$, nearly; $A = C = 0.86\,W$, $B = 0.675\,W$.

(22) $T=\frac{1}{3}\sqrt{3}\,W$; $A=C=\frac{1}{3}\sqrt{3}\,W$; $B=W$.

(23) $\delta+3\beta=\pi$, $30^\circ<\beta<60^\circ$; $\beta=60^\circ$.

(24) Resolve the components P_1, P_2 along the bisectors of θ.

(25) $\frac{1}{2}F\tan\alpha$; 13.4, 35.0, 107.2, 572, ∞ pounds.

Pages 56–58.

(2) $R=6$, and acts along 5.

(3) $T=W\cdot a/c$, $P=W\cdot b/c$.

(4) AC must bisect the angle BCW.

(5) $R^2=A^2+B^2+C^2+2BC\cos\alpha+2CA\cos\beta+2AB\cos\gamma$.

(6) Compare Arts. 90, 92.

(7) The sum of their moments must vanish for two points in the plane not in line with their point of intersection.

(10) $P=\frac{4}{3}W$; $T=\frac{5}{3}W$.

(11) $T=W$; $P=0.89\,W$ along the bisector of $\measuredangle BCW$.

(12) $P=W\sin(\alpha+\beta)\sin\beta$ becomes a maximum for $\beta=(\pi-\alpha)/2$, *i.e.* when the sail bisects the angle between boat and wind.

(13) $W\dfrac{\sin\beta}{\sin(\alpha+\beta)}$; $W\dfrac{\sin\alpha}{\sin(\alpha+\beta)}$.

(14) Tension in AB and $CD=W\cdot l/d$, tension in $BC=W\cdot(c-l)/2d$, where $d=\sqrt{l^2-\frac{1}{4}(c-l)^2}$.

(15) The resultant acts along the diameter through A, and is in magnitude equal to the perimeter.

(16) $P(1+\sqrt{2})$.

(17) (*a*) $W\sin\theta$, $W\cos\theta$; (*b*) $W\tan\theta$, $W/\cos\theta$; (*c*) $W\sin\theta/\cos\alpha$, $W\cos(\theta+\alpha)/\cos\alpha$.

(20) Produce BO to the intersection D with the circumscribed circle; then DA is equal and parallel to the resultant of OA, OB. $DAO'C$ is a parallelogram; hence $DA=CO'$.

Pages 67–68.

(1) Take moments about the fulcrum. The distance of this point from the end carrying the mass 12 is (*a*) $3\frac{6}{13}$ ft.; (*b*) $3\frac{1}{4}$ ft.

(2) (*a*) $A = 12\frac{2}{3}$ tons, $B = 11\frac{1}{3}$ tons; (*c*) $18\frac{1}{6}$, $18\frac{5}{6}$.

(3) (*a*) $P = W$; (*b*) $P = (1 + \sqrt{2})\,W$.

(6) (*a*) 19.4 tons and 21.1 tons; (*b*) 30.5, 9.9.

(8) Let α be the angle subtended at the centre by the side 12, and θ the angle at which the diagonal 13 is inclined to the horizon; then

$$\tan\theta = \frac{W_3 - W_1}{W_4 - W_2}\csc\alpha + \cot\alpha.$$

(9) $x = F_2 l \sin\alpha_2 / (F_1 \sin\alpha_1 + F_2 \sin\alpha_2)$.

Page 75.

(1) $C = 1$, $D = 1\frac{1}{3}$, $E = 6\frac{2}{3}$, $AB = 4.5$, $BC = 4.1$, $CD = 4.0$, $DE = 4.2$, $EF = 8.9$; reaction at $A = 4.5$, at $F = 8.9$.

(2) $H = 69.4$, $T = 73.8$.

(3) $c = \sqrt{\dfrac{x^3}{6(s - x)}}$; $H = wc$.

Pages 90–92.

(1) $T = 7.68$, $A = 9.76$, $E = 12.80$ pounds.

(2) $T = 2mW$, $A = \sqrt{4m^2 - 2m + 1}\,W$, where $m = c/l$.

(3) The three forces W, T, A must pass through a point; $\cos\phi = 2\sqrt{\frac{1}{3}(1 - m^2)}$, where $m = l/b$; $T = W\sec\phi$, $A = W\tan\phi$.

(4) $T = \frac{1}{2} W \cos\theta / \sin(\theta - \phi)$.

(5) $A_x = B = (c/l)\cot\alpha W$, $A_y = W$.

(6) $A_x = (c/2l)\sin 2\alpha W$, $A_y = [1 - (c/l)\cos^2\alpha]\,W$, $B = (c/l)\cos\alpha W$; the thrust A_x in this case is to that in Ex. (5) as $\sin^2\alpha$ is to 1.

(7) $A = W$, $C = D = (l/a)\cos\theta W$.

(8) $x = am$, $A = \sqrt{m^2 - 1}\,W$, $C = mW$, where $m = (l/a)^{\frac{1}{3}}$.

(9) $B = \frac{1}{2}(3w + W)\tan\alpha$, $A = (3w + W)\sqrt{\frac{1}{4}\tan^2\alpha + 1}$.

(10) $\cos\theta = \frac{1}{8}(m + \sqrt{m^2 + 32})$, where $m = l/a$.

Page 101.

(1) The horizontal and vertical components of the reaction at C are $C_x = -\frac{h}{a+b}W$, $C_y = \frac{1}{2}\frac{b-a}{a+b}W$, where h is the perpendicular distance of C from AB, and a, b are the segments into which this perpendicular divides AB.

Pages 112–113.

(1) (*a*) $\frac{\sin(\theta - \phi)}{\cos\phi}W < P < \frac{\sin(\theta + \phi)}{\cos\phi}W$; (*b*) $P = 0$, or $P \leqq 2W\sin\theta$; (*c*) if P act up the plane, $P \leqq \frac{\sin(\phi + \theta)}{\cos\phi}W$; if P act down the plane, $P \leqq \frac{\sin(\phi - \theta)}{\cos\phi}W$.

(2) 3 tons.

(3) $P = \frac{\sin\phi}{\cos(\phi - \alpha)}W$.

(4) (*a*) $P = \frac{\sin(\theta - \phi)}{\cos(\alpha + \phi)}W$; (*b*) $P = \frac{\sin(\theta + \phi)}{\cos(\alpha - \phi)}W$.

(5) $\theta = \frac{\pi}{2} - 2\phi$.

(6) $\theta = \tan^{-1}\frac{\mu W + \mu' W'}{W + W'}$.

Pages 145–146.

(3) (*a*) 7500; (*b*) 3 160 425 000.

(4) 18 000 ft.-pounds.

(5) 150 ft.-pounds.

Page 164.

(2) The centre of the inscribed circle.

(3) $\kappa\rho\frac{a-b}{ab}$.

(4) $\frac{2\kappa\rho^2}{c}\left(\sqrt{c^2 + (a+b)^2} - \sqrt{c^2 + (a-b)^2}\right)$.

THEORETICAL MECHANICS

PART III:

KINETICS

PREFACE.

ABOUT one-half of this volume is devoted to the kinetics of a particle, the remainder being given to the study of the kinetics of a rigid body and a brief discussion of the fundamental principles of the kinetics of a system.

The first part of the chapter on the motion of a particle (impact, rectilinear motion) gradually introduces and illustrates in an elementary way such fundamental ideas as momentum, impulse, kinetic energy, force, work, potential energy, power. Then the general equations of motion of a particle are discussed; and the principle of kinetic energy (or *vis viva*), that of angular momentum (or of areas), and the principle of d'Alembert are explained and applied, first to the motion of a free particle (central forces), then to constrained motion. The example of such recent writers as Budde and Appell has been followed in treating the constraints of a particle with more than usual fulness, introducing generalized co-ordinates, and establishing the equations of motion of a particle in the Lagrangian form. It is believed that this will materially aid the student in understanding the use of these methods in the general case of the motion of a system.

The chapter on the motion of a rigid body, after a discussion of the fundamental principles and of the theory of moments and ellipsoids of inertia, takes up separately the action of impulses and the motion under continuous forces. The last chapter, on the motion of a system, is necessarily brief, owing to the elementary character of the treatise. A sketch of the theory of Lagrange's generalized co-ordinates and of Hamilton's principle is, however, included.

For a shorter course, the Articles 104–159, 180–188, 190–217, 225, 262, 268–272, 274–290, 304–310, 320–323, 327, 329–332, 336–356, 391–397 may be omitted.

ALEXANDER ZIWET.

ANN ARBOR, MICH.,
October, 1894.

CONTENTS.

CHAPTER V.

Kinetics of a Particle.

CHAPTER VI.

Kinetics of a Rigid Body.

CHAPTER VII.

Motion of a Variable System.

THEORETICAL MECHANICS.

CHAPTER V.

KINETICS OF A PARTICLE.

I. *Impulses; Impact of Homogeneous Spheres.*

1. Momentum and Impulse. A particle of mass m, moving with the velocity v, is said to have the *momentum* mv (see Part II., Art. 56). As long as this momentum remains constant, the particle will move in a straight line with constant velocity v (Newton's first law of motion, Part II., Art. 74). Any change occurring in the momentum is ascribed to the action of a force F on the particle.

2. If the rate of change of momentum is constant during the time $t'-t$, the force F is constant, and is measured by the change of momentum in the unit of time; that is,

$$F(t'-t) = mv' - mv, \tag{1}$$

where v is the velocity at the time t, and v' the velocity at the time t' (Newton's second law of motion). As the product $F(t'-t)$ of a constant force into the time during which it acts is called the *impulse* of the force during this time (Part II., Art. 61), equation (1) can be expressed in words by saying that *the impulse of the force is equal to the change of momentum.*

This proposition is easily seen to hold even for a variable force. For such a force, we have

$$Fdt = d(mv);$$

hence, by integration,

$$\int_t^{t'} F dt = mv' - mv, \qquad (2)$$

where the time-integral in the left-hand member is the impulse of the variable force F during the time $t' - t$.

3. It appears, from equations (1) and (2), that a very large force may produce a finite change of momentum in a very short interval of time, but that it would require an infinite force to produce an *instantaneous* change of momentum of finite amount. The impact of one billiard ball on another, the blow of a hammer, the stroke of the ram of a pile-driver, the shock imparted by a falling body, by a projectile, by a railway train in motion, by the explosion of the powder in a gun, are familiar instances of large forces acting for only a very short time and yet producing a very appreciable change of velocity. The time of action, $t' - t$, of such a force is the very brief period during which the colliding bodies are in contact. The force, F, is a pressure or an elastic stress exerted by either body on the other during this time.

Forces of this kind are called *impulsive*, or *instantaneous, forces*.

4. In the case of such impulsive forces, it is generally difficult or impossible by direct observation or experiment to determine separately the very brief time of action, $t' - t$, as well as the magnitude F of the impulsive force. Moreover, what is of most practical importance and interest in such cases of impact is, generally, not the force itself, but the change of momentum produced, *i.e.* the impulse of the impulsive force.

In the present section, which is devoted to the study of the simplest cases of impact, we shall therefore deal with impulses and momenta, and not with forces.

5. It should be observed that many authors use the name *impulsive, or instantaneous, force* for what has here been called the impulse of the impulsive force. They define an impulsive force as the limiting value of the integral $\int_t^{t'} Fdt$ when F increases indefinitely, while at the same time the difference of the limits, $t' - t$, is indefinitely diminished; in other words, as the impulse of an infinite force producing a finite change of momentum in an infinitesimal time.

According to this definition, an impulsive or instantaneous force is a magnitude of a character different from that of an ordinary force, and is measured by a different unit. Its dimensions are $\mathbf{MLT}^{-1}$, and not $\mathbf{MLT}^{-2}$. Its unit is the same as the unit of momentum. Indeed, it is not a force, but an impulse.

We can arrive at this idea of an instantaneous force from a somewhat different point of view. Just as in kinematics (Part I., Arts. 104 and 156) we may distinguish accelerations of different orders, regarding velocity as acceleration of order zero, so in dynamics instantaneous forces may be regarded as forces of order 0, ordinary (continuous) forces as forces of order 1, the product of mass into the acceleration of the second order as a force of the second order, and so on.

In the present elementary treatise, no use is made of these considerations. The word force is always used as meaning the product of mass into acceleration of the first order, and the time-integral $\int_t^{t'} Fdt$ is always called impulse, and not impulsive force.

6. The momentum mv of a particle P of mass m, moving with the velocity v, can be represented geometrically by a vector (more exactly by a localized vector, or rotor), *i.e.* by a segment of a straight line drawn through P and representing by its length the magnitude of the momentum, by its direction and sense the direction and sense of the velocity. Hence, *the composition and resolution of momenta follows the same rules that hold for forces.*

7. Let us consider two particles P_1, P_2 of masses m_1, m_2, having equal and parallel velocities v, and let their momenta m_1v, m_2v be represented by their vectors (Fig. 1). The two particles may be regarded as forming a single moving system; as the velocities are equal in magnitude, direction, and sense, the system has a motion of translation. According to the rule for compounding parallel rotors (explained for rotors representing angular velocities in Part I., Arts. 253–255, and for rotors representing forces in Part II., Arts. 104–107), the resultant momentum is parallel to the given momenta and equal in magnitude to their algebraic sum $(m_1 + m_2)v$; and its line divides the distance P_1P_2 in the inverse ratio of the momenta m_1v, m_2v, or of the masses m_1, m_2. *The resultant passes, therefore, through the centroid* P *of the masses* m_1, m_2.

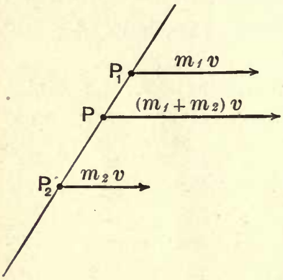

Fig. 1.

8. It is easy to see how this proposition can be generalized. If any number of particles, all having equal and parallel velocities, be given, the resultant momentum, or *the momentum of the system,* is equal to the mass of the system multiplied by the common velocity, and passes through the centroid of the system.

Thus, in the case of a rigid body having a velocity of translation v, but no rotation, the whole mass M of the body may be regarded as concentrated at the centroid, and the momentum of the centroid, Mv, is then equal to that of the body.

9. But we can speak of the momentum of a system of particles even when their velocities are not of equal magnitude but only parallel.

Let x be the distance, at the time t, of any particle P of mass m from some fixed plane, which, for the sake of simplicity, we may take at right angles to the direction of the velocity. Then the distance $\bar{x}$ of the centroid G of the system at the time t from the same plane is (Part II., Art. 13)

$$\bar{x} = \frac{\Sigma mx}{\Sigma m} = \frac{\Sigma mx}{M}. \tag{3}$$

Differentiating this equation with respect to the time, and remembering that $dx/dt=v$ is the velocity of the particle P, we find for the velocity $d\bar{x}/dt=\bar{v}$ of the centroid

$$\bar{v}=\frac{\Sigma mv}{M}. \tag{4}$$

10. In the special case of two particles P_1, P_2 of masses m_1, m_2, moving with the velocities v_1, v_2 in the same straight line, we have

$$\bar{v}=\frac{m_1 v_1+m_2 v_2}{m_1+m_2}. \tag{5}$$

If the velocities v_1, v_2 be constant, this equation shows that the centroid moves with constant velocity and constant momentum in the same line.

Similarly, the more general equation (4) of the preceding article,

$$M\bar{v}=\Sigma mv, \tag{6}$$

shows that *the momentum of a system of particles moving with constant velocities in the same direction remains constant*, *i.e.* the centroid of such a system moves with constant velocity in a straight line. It is to be noticed that the velocities need not be all of the same sense; that is, v may be positive for some particles and negative for others.

This proposition may be regarded as a generalization of Newton's first law of motion.

11. Direct Impact. We proceed to consider the particular case of two homogeneous spheres of masses m, m', whose centres C, C' move with velocities u, u' in the same straight line. The spheres are supposed not to rotate but to have a motion of pure translation; then their momenta are mu, $m'u'$, and can be represented by two vectors drawn from the centres C, C' along the line CC' (Fig. 2). To fix the ideas we assume the velocities

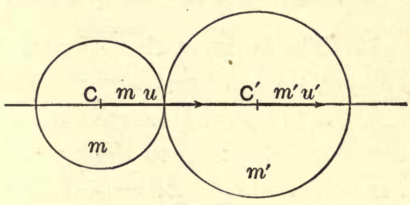

Fig. 2.

u, u' to have the same sense and $u>u'$, so that m will finally impinge upon m'. The case when the velocities are of opposite sense will not require special investigation, as only the sign of u' would have to be changed.

It is our object to determine the velocities v, v′ *of* m, m′, *immediately after impact,* when the velocities u, u' immediately before impact are given.

The results here derived for homogeneous spheres hold, generally, whatever the shape of the impinging bodies, provided that they do not rotate, and that the common normal at the point of contact passes through both centroids.

12. If the spheres were perfectly rigid, the problem would be indeterminate, for there is no way of deciding how the velocities would be affected by the collision.

Natural bodies are not perfectly rigid. The effect of the impact will, in general, consist in a compression of the portions of the bodies brought into contact. Moreover, all natural bodies possess a certain degree of elasticity; the compression will therefore be followed by an extension, each sphere tending to regain its shape at least partially.

The compression acts as a retarding force on the impinging sphere m, and as an accelerating force on m'. It will last until the velocities u, u' have become equal, say $=w$. During the subsequent period of extension, or *restitution,* the elastic stress still further diminishes the velocity of m, and increases that of m', until they become, say, v, v'.

13. The stress varies, of course, during the whole time τ of compression and restitution. But, according to Newton's third law, the pressure F exerted at any instant by m on m' must be equal and opposite to the pressure F' exerted by m' on m at the same instant. Since $F=mdu/dt$, $F'=m'du'/dt$, and $F=-F'$ at any instant during the time τ, we have

$$\int_0^\tau Fdt=-\int_0^\tau F'dt, \text{ or } m\int_0^\tau du=-m'\int_0^\tau du',$$

whence $$mv - mu = -(m'v' - m'u'),$$

or $$mv + m'v' = mu + m'u'; \tag{7}$$

that is, *the total momentum after impact is equal to that before impact.*

14. This proposition will evidently hold for any number of spheres whose centres move in the same line, and can then be expressed in the form

$$\Sigma mv = \Sigma mu. \tag{8}$$

It can be regarded as a special case of the so-called principle of the conservation of the motion of the centroid to be proved hereafter for any system not acted upon by external forces. On the other hand, the proposition can be looked upon as a further generalization of Newton's first law of motion. While the latter asserts that the momentum of a *particle* remains unchanged as long as no external forces act upon it, our law of impact asserts the same thing for the momentum of a *system*.

15. If the spheres were *perfectly non-elastic*, there would be only compression and no subsequent extension. As at the end of the period of compression, the velocities u, u' have both become equal, viz. $=w$ (Art. 12), the spheres after impact would have the common velocity

$$w = \frac{mu + m'u'}{m + m'}. \tag{9}$$

16. If the spheres were *perfectly elastic*, *i.e.* if the elastic stress following the compression, or the so-called *force of restitution*, were just equal to the preceding stress of compression, the spheres would completely regain their original shape. In this case, the elastic stress causes the impinging sphere m to lose during the period of restitution an amount of momentum $m(w-u)$ equal to that lost during the period of compression. Hence, the final velocity of m after impact would be

$$v = w - (u - w) = 2w - u.$$

Similarly, we have for the other sphere m'

$$v' = w + (w - u') = 2w - u'.$$

As w is known from (9), the velocities after impact can be determined by means of these formulæ for perfectly elastic spheres.

17. In general, physical bodies are *imperfectly elastic*, the force of restitution being less than that of the original compression; that is, we have

$$(w - v) = e(u - w),$$
$$(v' - w) = e(w - u'),$$

where e is a proper fraction whose limiting values are 0 for perfectly inelastic bodies and 1 for perfectly elastic bodies. This fraction e, whose value for different materials must be determined experimentally, is called the *coefficient of restitution* (or less properly, the coefficient of elasticity).

18. To eliminate w we have only to add the last two equations; this gives

$$v' - v = e(u - u'); \tag{10}$$

that is, *the ratio of the relative velocity after impact to the relative velocity before impact is constant and equal to the coefficient of restitution.*

This proposition, in connection with the proposition of Art. 13, expressed by formula (7), is sufficient to solve all problems of so-called *direct* impact, *i.e.* when the centres of the spheres move in the same line.

19. As the coefficient e is frequently difficult to determine, the limiting cases $e=0$, $e=1$ are important as giving approximate solutions for certain classes of substances.

Thus, for nearly inelastic bodies (such as clay, lead, etc.) we may put $e=0$, whence, by (10), $v'=v$, *i.e.* the velocities of the spheres become equal after impact; and the value of the common velocity is found from (7) as

$$v = \frac{mu + m'u'}{m + m'},$$

which agrees with the result (9) found in Art. 15. For perfectly elastic bodies $e=1$, and formula (10) shows that in this case the relative velocity after impact is numerically equal to that before impact, but of opposite sense.

20. Exercises.

(1) Two balls of clay ($e=0$) weighing 2 and 3 oz. move in the same direction. The heavier ball impinges from behind upon the lighter ball at the moment when the latter moves at the rate of 15 ft. per second. If the velocity of the lighter ball is doubled by the impact, what was the original velocity of the heavier ball?

(2) Two glass balls ($e=1$) weighing 1 lb. and 12 oz., respectively, move in the same line with velocities of 5 and 4 ft. per second. What are their velocities after impact (*a*) if their original velocities were of the same sense, (*b*) if they were of opposite sense?

(3) A ball weighing 5 lbs., while moving with a speed of 51 ft. per second, overtakes a ball of 7 lbs. moving in the same line at the rate of 40 ft. per second. If the coefficient of restitution be $\frac{1}{3}$, what are the velocities of the two balls after impact?

(4) With the data of Ex. (3), show that the velocities after impact would be equal if the balls were perfectly inelastic, and that these velocities would differ more than in Ex. (3) if the balls were perfectly elastic.

(5) Find the velocity with which an elastic ball rebounds from a fixed surface after impinging upon it perpendicularly with a velocity u.

(6) To determine the coefficient of restitution, a ball is dropped from a height H on a fixed horizontal plate of the same material, and the height of rebound h is measured. Show that $e=\sqrt{h/H}$.

(7) A ball is dropped from a height $H=12$ ft. on a fixed horizontal plate. Find the height h to which it will rebound if $e=\frac{5}{6}$.

(8) If not disturbed, the ball in Ex. (7) will continue to fall and rebound alternately. (*a*) What height does it reach at the tenth rebound? (*b*) In what time does it come to rest? (*c*) What is the whole space described?

(9) A number of equal, perfectly elastic balls are placed in contact so that their centres are in a straight line. An equal ball impinges with a velocity u along this line on the first ball of the row. Show that the

last ball of the row will move off with the velocity u, while all the other balls will remain at rest.

(10) Find the velocity of the last (nth) ball in Ex. (9), when the coefficient of restitution is e.

(11) An inelastic ball of 8 lbs. is moving with a velocity of 12 ft. per second. (*a*) With what velocity must a ball of 24 lbs. meet it to arrest its motion? (*b*) With what velocity would the ball of 24 lbs. have to impinge from behind on the ball of 8 lbs. to double its velocity?

(12) A ball m impinges upon a ball m' from behind with a velocity u. Determine the velocities after impact, both for inelastic and for perfectly elastic balls: (*a*) when m' is originally at rest; (*b*) when m' is at rest and very large in comparison with m; (*c*) when m' has the initial velocity u', and is very large in comparison with m.

21. Kinetic Energy. A particle of mass m, moving with the velocity v, has the kinetic energy $\frac{1}{2}mv^2$ (Part II., Art. 71). As this is not a vector-quantity, the kinetic energy of a system consisting of any number of free particles is simply the algebraic sum, $\Sigma\frac{1}{2}mv^2$, of the kinetic energies of these particles. It is an essentially positive quantity, provided the masses are all positive.

The kinetic energy of a rigid body having a motion of pure translation is evidently $=\frac{1}{2}mv^2$, if m be the mass of the body and v the common velocity of all its points.

22. Change of kinetic energy is brought about by the action of force, and we have (Part II., Art. 72) for a constant force F,

$$\frac{1}{2}mv'^2-\frac{1}{2}mv^2=F(s'-s); \qquad (11)$$

and for a variable force F,

$$\frac{1}{2}mv'^2-\frac{1}{2}mv^2=\int_s^{s'}Fds, \qquad (12)$$

where the quantity in the right-hand member is called the *work* of the force. Thus a particle of mass m, falling from rest through a distance s, acquires its kinetic energy owing to the work done upon it by the constant attractive force, $F=mg$, of the earth, and we have

$$\tfrac{1}{2}mv^2=Fs=mgs.$$

The kinetic energy $\frac{1}{2}mv^2$, possessed by a particle of mass m, moving with the velocity v, can therefore always be regarded as equivalent to a certain amount of work. If the motion of this particle be *opposed* by a constant force or resistance F, the distance s through which it will go on moving until it comes to rest is of course determined from the same equation,

$$\tfrac{1}{2}mv^2=Fs. \tag{13}$$

It is then said that the kinetic energy of the particle is spent in overcoming the resistance F, or in doing work against the force F (see Part II., Art. 231).

23. In the case of direct impact of spheres, as considered in Art. 11, the velocity, and hence also the kinetic energy, of each sphere is in general changed by the impact; a transfer of kinetic energy can be said to take place. Thus, when a sphere at rest is struck by a moving sphere, kinetic energy is imparted to the former by the impulsive force, and this energy can then be spent in doing work against a resistance. Impact is therefore frequently used for the purpose of performing useful work.

24. For instance, to drive a nail into a wooden plank, the resistance F of the wood must be overcome through a certain distance s. This might be done by applying a pressure equal to F; as, however, this pressure would have to be very large, it is more convenient to impart to the nail, by striking it with a hammer, an amount of kinetic energy, $\frac{1}{2}mv^2$, equivalent to the work Fs that is to be done. Neglecting elasticity, and denoting the mass of the hammer by m, that of the nail by m', the velocity of the hammer at the moment when it strikes the head of the nail by u, we have, by (7),

$$mv+m'v'=mu,$$

or since, by (10), for inelastic impact $v'=v$,

$$v=\frac{m}{m+m'}u.$$

This is the common velocity of hammer and nail after the stroke. We find, therefore, by (13),

$$\tfrac{1}{2}(m+m')\cdot\left(\frac{mu}{m+m'}\right)^2=Fs,$$

or

$$\frac{m}{m+m'}\cdot\tfrac{1}{2}mu^2=Fs. \tag{14}$$

25. It will be noticed that while the total kinetic energy of hammer and nail just before striking was $\frac{1}{2}mu^2+0$, the kinetic energy utilized for driving the nail is only the fraction $m/(m+m')$ of this total kinetic energy. The remaining portion of the original kinetic energy, viz.

$$\frac{m'}{m+m'}\cdot\tfrac{1}{2}mu^2, \tag{15}$$

must be regarded as spent in producing the slight deformations of hammer and nail and such accompanying phenomena as vibrations of the plank, sound, heat, etc. For it is an experimental result of modern physical research that, wherever kinetic energy disappears as such, there is done an exactly equivalent amount of work. The apparently disappearing kinetic energy may either be transferred to some other body, as in the case of the vibrations of the plank, or it may reappear in the form of molecular vibrations, causing sound or heat; or it may be transformed into an equivalent amount of so-called potential energy. This physical fact is known as *the principle of the conservation of energy*.

26. In our example the total original kinetic energy, $\frac{1}{2}mu^2$, resolves itself into two portions, the portion (14) used for driving the nail, and the "wasted" or, as it is often called, "lost" portion (15). It may, however, happen that the portion (15) does the useful work, while (14) is wasteful. This would be the case, for instance, in molding a rivet with a hammer, or in forging a piece of iron under the blows of a steam-hammer. The useful work here consists in the deformation of the bolt or piece of iron.

It appears from the expressions (14) and (15) that, for the purpose of driving the nail, m should be large in comparison with m', while for molding a rivet it is of advantage to have m' large in comparison with m.

27. In applying the formulæ (11) to (15), and in general all formulæ of theoretical kinetics, it should be noticed that the forces are supposed to be expressed in absolute measure, the unit being the poundal or the dyne. Hence, to find the force F in pounds the numerical result obtained from one of these formulæ must be divided by the value of g.

28. Let us now consider the *change of the total kinetic energy* produced by direct impact in two partially elastic spheres. With the notations of Art. 11, we have for the excess of the kinetic energy after impact over that before impact :—

$$\tfrac{1}{2}(mv^2+m'v'^2)-\tfrac{1}{2}(mu^2+m'u'^2).$$

To eliminate v and v' from this expression, square the equations (7) and (10),

$$(mv+m'v')^2=(mu+m'u')^2,$$

$$(v-v')^2=e^2(u-u')^2;$$

multiply the latter by mm', and write it in the form

$$mm'(v-v')^2+(1-e^2)mm'(u-u')^2=mm'(u-u')^2;$$

finally add the former equation,

$$(m+m')(mv^2+m'v'^2)+(1-e^2)mm'(u-u')^2 = (m+m')(mu^2+m'u'^2),$$

whence

$$\tfrac{1}{2}(mv^2+m'v'^2)-\tfrac{1}{2}(mu^2+m'u'^2)=-\tfrac{1}{2}(1-e^2)\frac{mm'}{m+m'}(u-u')^2. \quad (16)$$

As the right-hand member of this equation is essentially negative, it appears that *while in impact the total momentum remains unchanged, the total kinetic energy is in general diminished;* only in the limiting case of perfectly elastic bodies ($e = 1$) does the kinetic energy remain the same after as before impact. The "lost" kinetic energy (16) mainly represents the amount of energy spent in producing the permanent deformation of the impinging bodies.

29. Exercises.

(1) A hammer weighing 1.5 lbs. strikes a nail weighing $\frac{1}{2}$ oz. with a velocity of 20 ft. per second, and drives it $\frac{1}{4}$ in. Find the mean resistance of the wood, and determine the useful and wasteful work.

(2) In Art. 20, Ex. (3), find the loss of kinetic energy due to the impact.

(3) A train of 120 tons runs, with a speed of 15 miles an hour, into a train of 80 tons at rest. Neglecting elasticity, determine the destructive work of the collision, and the velocity along the track after impact.

(4) A pile weighing m' lbs. is driven into the ground by a ram of m lbs., falling from a height of h ft. If the pile sinks s in. into the ground after n falls of the ram, show that the resistance of the ground (assumed as uniform) is $= \frac{12n}{s}\left(\frac{1+e}{1+m'/m}\right)^2 m'h$ pounds.

(5) If, in Ex. (4), the elasticity of ram and pile be neglected, ram and pile will have equal velocities after impact, and move together. Hence, the factor m' should be replaced by $m + m'$, and the resistance is $= \frac{12n}{s} \cdot \frac{1}{1+m'/m} \cdot mh$ pounds.

(6) 10 blows of a ram of 500 lbs., falling from a height of 5 ft., sink a pile of 400 lbs. 4 in. If the permanent load of a pile be taken as one-fifth of the resistance, what permanent load can the pile bear?

(7) A steam-hammer of 3 tons is used in forging. It has a fall of 5 ft. If the weight of the anvil be 20 tons, what is the useful and what the wasteful work?

30. Recoil. The explosion of the powder in a gun produces an impulsive pressure both on the shot and on the body of the gun. Assuming the line of motion of the centroid of the shot to pass through the centroid of the gun, we may apply equation (7), with $u=0$, $u'=0$. Hence, denoting by m the mass of the gun, by m' that of the shot, we find for the velocity of recoil

$$v=-\frac{m'}{m}v'. \tag{17}$$

The kinetic energies $\frac{1}{2}mv^2$ and $\frac{1}{2}m'v'^2$ of gun and shot are in the ratio m'/m; hence, the energy of recoil is the fraction $m'/(m+m')$ of the total energy $\frac{1}{2}mv^2+\frac{1}{2}m'v'^2$ of the explosion of the powder, while the energy of the shot is $=m/(m+m')$ of the total energy. In large guns the recoil is diminished by a special elastic cushion or "compressor." Moreover, the mass of the powder gases cannot be entirely neglected in all cases.

31. Oblique Impact. In the case of oblique impact, *i.e.* when the centres of the colliding spheres do not move in the same straight line, the velocities after impact can be found without difficulty, provided that the velocities of the centres before impact lie in the same plane and that the spheres are perfectly smooth.

With these assumptions, let m, m' be the masses of the two spheres; C, C' their centres (Fig. 3); u, u' the velocities before impact; α, α' the angles made by u, u' with the line CC'; v, v' the velocities after impact; and β, β' the angles they make with CC'.

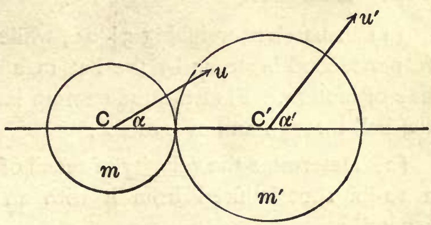

Fig. 3.

As there is no friction, the forces of impact act along the line CC' that joins the centres. Hence, resolving each velocity along and perpendicular to CC', the components at right angles to CC' must remain unchanged by the collision; that is, we must have

$$v\sin\beta=u\sin\alpha,\quad v'\sin\beta'=u'\sin\alpha', \tag{18}$$

The components of the velocities along CC' must satisfy the equations (7) and (10). Hence, substituting $u \cos \alpha$, $u' \cos \alpha'$, $v \cos \beta$, $v' \cos \beta'$ for u, u', v, v', respectively, we must have

$$mv \cos \beta + m'v' \cos \beta' = mu \cos \alpha + m'u' \cos \alpha', \tag{19}$$

$$v' \cos \beta' - v \cos \beta = e(u \cos \alpha - u' \cos \alpha'). \tag{20}$$

32. The particular case of the oblique impact of a homogeneous sphere against a smooth fixed plane deserves special mention. In this case, u' and v' are zero; and the angles α, β made by the velocities u, v with the normal to the plane, are called the *angle of incidence* and *of reflection*, respectively.

The equations (18) and (20) reduce to the following:

$$v \sin \beta = u \sin \alpha, \quad v \cos \beta = -eu \cos \alpha, \tag{21}$$

where the minus sign indicates that the projections of u and v on the normal have opposite sense. Dividing the former of these equations by the latter, we find

$$\tan \alpha = -e \tan \beta, \tag{22}$$

where the minus sign merely indicates that the angles α, β lie on opposite sides of the normal.

For perfectly elastic bodies, the last equation shows that the angles of incidence and reflection are equal.

33. Exercises.

(1) A baseball weighing $5\frac{1}{4}$ oz., while moving with a velocity of 100 ft. per second is struck by the bat in a direction at right angles to its line of motion. Find the momentum imparted by the blow if it deflects the ball through an angle of 60°.

(2) Determine the velocity of recoil of a gun weighing 1500 lbs. when a 12-lb. shot is fired from it with an initial velocity of 2000 ft. per second.

(3) The heavier one of two ivory balls ($e = 0.88$), whose centroids are C, C' and whose masses are 1 lb. and $\frac{3}{4}$ lb., impinges upon the lighter. The velocity of the heavier ball is 15 ft. per second and makes an angle of 30° with the line CC', while the velocity of the lighter ball is 5 ft. per second and makes an angle of 60° with the line CC' (produced). Find the velocities after impact in magnitude and direction.

34. As a more careful study of the theory of impact requires some knowledge of the theory of elasticity, it is generally treated more at length in works on applied mechanics. See, for instance, J. WEISBACH, *Mechanics of engineering*, translated by E. B. Coxe, New York, Van Nostrand, 1875, Vol. I., pp. 667–711; A. RITTER, *Technische Mechanik*, Leipzig, Baumgärtner, 1884, pp. 585–618; J. H. COTTERILL, *Applied Mechanics*, London, Macmillan, 1884, pp. 274–280 and 374–386; THOMSON and TAIT, *Natural philosophy*, I., Part 1, pp. 274–284. The general theory of impulsive forces will be given in Chapter VI.

II. *Rectilinear Motion.*

35. The motion of a single particle presents a comparatively simple problem, because the forces, being in this case all applied at one and the same point, have a single resultant which is readily found by geometrically adding the forces (Part II., Art. 96). Let this resultant be denoted by F, the mass of the particle by m, and its acceleration by j; then, according to the definition of force (Part II., Art. 60), we must have

$$mj = F.$$

This equation merely expresses the fact that the force F produces in the mass m an acceleration j, which agrees with F in direction and sense, and is inversely proportional to m.

36. The forces, whose resultant is F, are usually called the **impressed forces.** Both F and j are, in general, variable. If at any time t the particle m were acted upon by a force $= -mj$, in addition to the impressed forces, it would evidently be in equilibrium. The product mj of the mass of the particle into its acceleration at any instant is called the **effective force** of the particle at this instant. It can, therefore, be said that *the impressed forces are at any instant in equilibrium with the effective force reversed.*

This obvious proposition forms the fundamental idea of a most important method of treating the dynamical equations of motion, known as **d'Alembert's principle**, which will be discussed more fully later on (see Arts. 97–103, 383–386). It makes it possible to apply to kinetic problems the statical conditions of equilibrium. Thus, in the case of a single particle, if the reversed effective force, $-mj$, be combined with the impressed forces, we have a system of forces acting on the particle which, at the instant considered, is in equilibrium, and must satisfy the conditions of equilibrium for concurrent forces (Part II., Arts. 97, 101), viz. $-mj + F = 0$; or, resolving j into its components d^2x/dt^2, d^2y/dt^2, d^2z/dt^2, and F into the components X, Y, Z,

$$-m\frac{d^2x}{dt^2}+X=0,\quad -m\frac{d^2y}{dt^2}+Y=0,\quad -m\frac{d^2z}{dt^2}+Z=0.$$

37. To familiarize the student with the idea of force and its use in kinetics, we shall now study in some detail the simple case of rectilinear motion. The next section will be devoted to the general problem of the curvilinear motion of a *free* particle. This will be illustrated by the important case of motion due to central forces. Finally, the motion of a particle subject to conditions, or *constraints*, will be treated.

38. When a particle of mass m moves in a straight line, both its velocity v and the resultant force F must be directed along this line. The acceleration in rectilinear motion (see Part I., Art. 103) is $j=dv/dt=d^2s/dt^2$; hence *the dynamical equation of rectilinear motion,*

$$m\frac{dv}{dt}=F, \text{ or } m\frac{d^2s}{dt^2}=F. \tag{1}$$

It differs from the kinematical equation (Part I., Art. 115) only by the factor m, and can be treated in the same way.

Thus, if the law of force be given, *i.e.* if F be known as a function of t, s, v, or of only one or two of these quantities, the equation can be integrated; and if, moreover, the initial position and velocity of the particle be given, the constants of integration can be determined, and all the circumstances of the motion can be found.

If the mass m of the moving particle were not a constant quantity, the equation (1) should be written in the form

$$\frac{d(mv)}{dt}=F,$$

since the resultant force is the rate at which the momentum of the particle changes with the time (see Part II., Art. 60).

39. As long as a single free particle only is considered, there is generally no advantage in introducing the idea of force; the equation of motion can be divided by m, and this reduces it to a purely kinematical form.

Thus, for a particle of mass m falling *in vacuo*, the dynamical equation of motion is

$$m\frac{d^2s}{dt^2}=W,$$

where $W=mg$ is the weight of the particle; *i.e.* the force of attraction exerted by the earth on the particle (in poundals or dynes, if m be expressed in pounds or grammes, see Part II., Art. 115). Dividing by m, we find the kinematical equation

$$\frac{d^2s}{dt^2}=g,$$

which has been treated in Part I., Arts. 107–114.

The following articles give examples in which it is more convenient to retain the idea of force.

40. Let us consider a mass m that is being raised or lowered by means of a rope or chain (Fig. 4), such as a building stone suspended from a derrick. The rope acts as a *constraint*, conditioning the motion of the stone. To make the stone free we may imagine the rope cut just above the stone and the **tension** of the rope, T, introduced as a substitute. The stone then moves under the action of two forces, viz. its weight $W=mg$ and the tension T of the rope. Taking the downward sense as positive, we have the equation of motion,

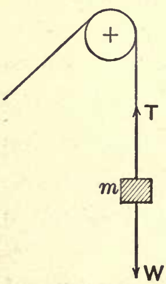

Fig. 4.

$$m\frac{d^2s}{dt^2}=mg-T. \qquad (2)$$

41. Writing j for the acceleration d^2s/dt^2 with which the stone is being lowered or raised, we find for the tension T of the rope

$$T=m(g-j). \qquad (3)$$

This equation shows that the tension is equal to the weight of the stone, not only when it is hanging at rest, but also whenever it is raised or lowered with constant velocity; and that the tension is zero if the stone is lowered with an acceleration equal to that of gravity, as is otherwise evident.

42. The above formula will give the tension T in poundals (or dynes), if the mass m be expressed in pounds (or grammes), and the accelerations in feet (or centimetres) per second per second.

In engineering practice, gravitation measure is commonly used for weights as well as for the forces that replace constraints (tensions, pressures, friction, etc.). The engineer would, therefore, divide by g the numerical value of T just found, so as to reduce it to pounds.

It should be noticed that the general equations of theoretical mechanics are of course independent of the system of units adopted, and that in applying them to numerical examples it is only necessary to use one and the same system of units consistently throughout. As modern physics has settled upon mass as a fundamental unit (see Part II., Art. 68), regarding the unit of force as derived from and based upon the unit of mass, this absolute system will always be adopted in this book, unless the contrary be specified. In other words, *it will always be assumed that mass is expressed in pounds* (or grammes), *and consequently force in poundals* (or dynes).

43. Let us next consider two particles, m_1, m_2, connected by a cord hung over a vertical fixed pulley, as in the apparatus known as *Atwood's machine* (Fig. 5). If $m_1 > m_2$, m_1 will descend while m_2 ascends. The effective force of the system formed by the two particles is evidently the difference of the weights of the particles, viz. $W_1 - W_2 = (m_1 - m_2)g$, while the whole mass moved (neglecting the mass of the cord and of the pulley) is m_1+m_2. Hence, we have for the acceleration j of the system,

$$j=\frac{m_1-m_2}{m_1+m_2}g. \tag{4}$$

This acceleration is constant, and the relations between space, time, and velocity are found just as for a single particle falling freely, except that the acceleration of gravity g is replaced by the fraction $(m_1-m_2)/(m_1+m_2)$ of g. It follows that if the masses m_1, m_2 be selected nearly equal, the acceleration will be small, and the motion can be observed more conveniently than that of a freely falling body.

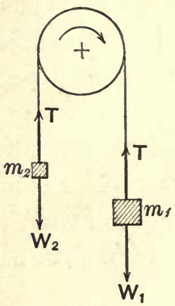

Fig. 5.

44. The tension T of the cord is, of course, the same at every point of the cord if, as is here assumed, the weight of the cord and the axle-friction of the pulley be neglected. To determine this tension, we have only to consider either particle separately.

If the cord be cut just above m_1, and the tension T be introduced, the particle m_1 will move like a free particle under the action of the resultant force $W_1-T=m_1g-T$. Hence, as the sense of the acceleration j of m_1 agrees with that of g,

$$j=\frac{m_1g-T}{m_1}. \tag{5}$$

Similarly, we have for the acceleration of m_2, whose sense is opposite to that of g,

$$j=-\frac{m_2g-T}{m_2}. \tag{6}$$

Eliminating j between any two of the equations (4), (5), (6), we find the tension

$$T=\frac{2\,m_1m_2}{m_1+m_2}g. \tag{7}$$

45. If the two particles of Art. 43 move on inclined planes intersecting in the horizontal axis of the pulley (Fig. 6), it is only necessary to resolve the weights m_1g and m_2g into two components, one parallel, the other perpendicular, to the inclined

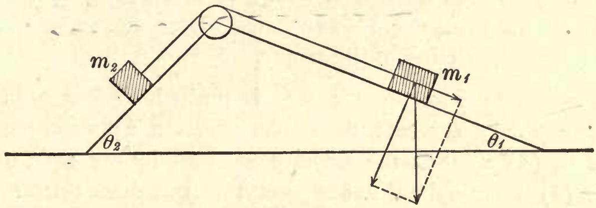

Fig. 6.

plane. If the planes be smooth, the system formed by the two particles is made free by introducing the normal reactions of the planes which counterbalance the perpendicular components of the weights. The effective force is therefore the difference of the parallel components, and the acceleration is

$$j = \frac{m_1 \sin\theta_1 - m_2 \sin\theta_2}{m_1 + m_2} g, \tag{8}$$

where θ_1, θ_2 are the angles of inclination of the planes to the horizon.

The tension T of the connecting cord is again determined by equating this value of j to the one obtained by considering either of the two particles separately. Thus, m_1 taken by itself, becomes free if we introduce not only the normal reaction of the plane, but also the tension of the string. This gives

$$j = \frac{m_1 g \sin\theta_1 - T}{m_1}. \tag{9}$$

Similarly, we have for m_2

$$j = \frac{T - m_2 g \sin\theta_2}{m_2}. \tag{10}$$

Hence,
$$T = \frac{m_1 m_2}{m_1 + m_2} g (\sin\theta_1 + \sin\theta_2). \tag{11}$$

With $\theta_1 = \theta_2 = \pi/2$ the formulæ (8) to (11) reduce, of course, to the formulæ (4) to (7).

46. Exercises.

(1) A stone weighing 200 lbs. is raised vertically by means of a chain running over a fixed pulley. Determine the tension of the chain: (*a*) when the motion is uniform; (*b*) when the motion is uniformly accelerated upwards at the rate of 8 ft. per second; (*c*) when the acceleration is 32 ft. per second downwards. Neglect the weight of the chain and the axle-friction of the pulley.

(2) A railroad car weighing 4 tons is pushed by four men over a smooth horizontal track. If each man exerts a constant pressure of 100 pounds, (*a*) what is the velocity acquired by the car at the end of 5 sec.? (*b*) what is the distance passed over in these 5 sec.?

(3) (*a*) Determine the constant force required to give a train of 90 tons a velocity of 30 miles an hour in 5 min. after starting from rest. (*b*) How far does the train go in this time? (*c*) If the same velocity is to be acquired at the end of the first mile, what must be the tractive force of the engine?

(4) If in Atwood's machine (Fig. 5) the two masses are each 2 lbs., and an additional mass of 1 oz. be placed on one of these masses, how long will it take this mass to descend 6 ft.? ($g = 32.2$.)

(5) If in Ex. (4) an additional mass of half an ounce be placed on *each* of the 2 lb. masses, how would the tension in the cord differ from the tension in Ex. (4)?

(6) Solve the problem of Art. 45 when the inclined planes are rough, the coefficients of friction being μ_1, μ_2.

(7) A mass of 5 lbs. rests on a smooth horizontal table, and has a cord attached which runs over a smooth pulley on the edge of the table. If a mass of 1 lb. be suspended from the cord, find the acceleration and the tension of the cord.

(8) A sleigh weighing 500 lbs. is drawn over a horizontal road, the coefficient of friction being $\frac{1}{70}$. Find the pull exerted by the horses when the motion is uniform.

(9) When the U.S.S. *Raritan* was launched she was observed to pass in 11 sec. over an incline of $3° 40'$, 54 ft. long. Find the coefficient of friction.

(10) A coaster, after coming down a hill, runs up another hill a distance of 200 ft. (from its foot) in 10 sec., when it stops. If the slope of the second hill be 6°, find the coefficient of friction.

(11) A train of 120 tons is running 25 miles an hour. Find what constant force is required to bring it to rest: (*a*) in 3 min.; (*b*) in half a mile.

(12) If it takes 1 min. to coast down a hill on a uniformly sloping road of 1 mile length, and the coefficient of friction be 0.02, what is the height of the hill?

47. Kinetic Energy and Work. The dynamical equation of motion, (1), Art. 38, can often be integrated after multiplying both members by $vdt = ds$; this makes the left-hand member an exact differential, viz. the differential of the *kinetic energy* $\frac{1}{2}mv^2$, while the right-hand member, Fds, represents the *elementary work* done by the force F:

$$d(\tfrac{1}{2}mv^2) = Fds.$$

If F be given as a function of s alone, this equation can be integrated, say from the time t_0 to the time t. Denoting by s_0 and v_0 the values of s and v at the time t_0 (Fig. 7), we find

$$\tfrac{1}{2}mv^2 - \tfrac{1}{2}mv_0^2 = \int_{s_0}^{s} Fds. \qquad (12)$$

This equation gives the velocity v as a function of the distance s, counted from the arbitrary origin O. As $v = ds/dt$, a second

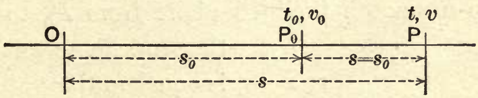

Fig. 7.

integration will give s as a function of t. Examples of this method have been given in Part I., Arts. 109, 117, 119, 121, 122; it will here only be necessary to call attention to the dynamical meaning of the quantities involved.

48. The left-hand member of equation (12) represents evidently the increase in the kinetic energy of the moving particle, while the right-hand member expresses the work done by the force F during the passage of the particle from the point P_0 to the point P (see Part II., Arts. 71, 72). Hence, the meaning of the equation is that *the increase in the kinetic energy is equal to the work done by the resultant force.* This is the **principal of work** or **of kinetic energy** (or of *vis viva*) for the case of the rectilinear motion of a particle.

Thus, for a falling body, F is constant and equal to the weight mg of the body; hence, equation (12) gives, if s be counted positive downwards,

$$\tfrac{1}{2}mv^2 - \tfrac{1}{2}mv_0^2 = mg(s - s_0),$$

where the right-hand member represents the work done by the weight of the body, *i.e.* by the attractive force of the earth during the fall of the body through the distance $s - s_0$.

For a body thrown vertically upwards with an initial velocity v_0, we have

$$\tfrac{1}{2}mv^2 - \tfrac{1}{2}mv_0^2 = -mgs,$$

if s be counted from the starting point and positive upwards. The kinetic energy here decreases, the initial kinetic energy, $\tfrac{1}{2}mv_0^2$, being, so to speak, consumed by the work done against the force of gravity.

49. Inclined Plane. When a particle of mass m is moved uniformly up a smooth inclined plane from P_0 to P_1 (Fig. 8), the work done against gravity is equal to the work that would have to be done in raising the particle m through the vertical height PP_1 of P_1 above the initial point P_0. For, putting $P_0P_1 = s$, $PP_1 = h$, and denoting the inclination of the plane to the horizon by θ, we have for the work,

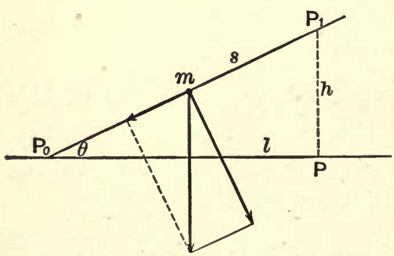

Fig. 8.

$$mg \sin\theta \cdot s = mg \cdot s \sin\theta = mg \cdot h.$$

If the plane be rough, the coefficient of friction being μ, the effective force for motion upwards is $=mg\sin\theta+\mu mg\cos\theta$; hence, the work done in moving the mass m from P_0 to P_1 is

$$mgs\sin\theta+\mu mgs\cos\theta=mg\cdot h+\mu mg\cdot l=mg(h+\mu l),$$

where $l=P_0P$ is the horizontal distance of the final position P_1 from the starting point P_0. The total work is, therefore, the sum of the work of overcoming gravity through the *vertical* distance h and the work of overcoming friction through the *horizontal* distance l.

50. Work done on a System of Particles. Let there be given any number of particles of masses $m_1, m_2, \ldots m_n$ at the distances $s_1, s_2, \ldots s_n$ above a fixed horizontal plane; and let these masses be raised vertically against gravity so that their distances from the same plane become $s_1', s_2', \ldots s_n'$. The centroid of the masses in their original position has a distance $\bar{s}=\Sigma ms/\Sigma m$ from the fixed plane, while in the final position it has the distance $\bar{s}'=\Sigma ms'/\Sigma m$ from the same plane. It has, therefore, been raised through a distance $\bar{s}'-\bar{s}$. It follows that *the total work done in raising the separate masses*, viz.

$$m_1g(s_1'-s_1)+m_2g(s_2'-s_2)+\cdots+m_ng(s_n'-s_n)=g(\Sigma ms'-\Sigma ms),$$

is equal to the work that would be done in raising the total mass Σm *through the distance* $\bar{\text{s}}'-\bar{\text{s}}$ *traversed by the centroid, i.e.* to

$$g\Sigma m\cdot(\bar{s}'-\bar{s}).$$

51. The Work of a Variable Force is well illustrated by the expansion of gas or steam in a cylinder with a movable piston (Fig. 9). Let r be the radius of the cylinder, p the pressure (in pounds) at any instant of the gas per square inch of surface; then the total pressure of the gas on the inside of the piston is $P=\pi r^2p$ pounds, and if P_0 be the pressure on the outside (say the atmospheric pressure), the effective force acting on the piston is $F=P-P_0$, friction being neglected.

The force F is variable, since the pressure p varies with the volume v occupied by the gas. This volume being in the present case proportional to the distance s of the piston from the fixed base of the cylinder, the force F is a function of s. The variation of F can therefore be represented graphically by a curve having s for abscissa and F for ordinate (Fig. 9); and *the area of this curve, i.e.* the area contained between the curve, the axis of s, and two ordinates whose abscissas are s_0 and s, being given by the integral $\int_{s_0}^{s} Fds$, *represents the work done on the piston when pushed through the distance* $s-s_0$.

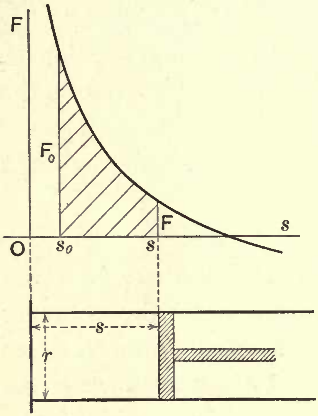

Fig. 9.

52. In the case of a perfect gas, Boyle's law gives the relation $pv=k$, where k is constant if the temperature remains constant. Hence,

$$F=\frac{K}{s}-P_0,$$

where K and P_0 are constants. This equation represents an equilateral hyperbola, whose asymptotes are the axis of F and a line parallel to the axis of s. For steam, the law connecting pressure and volume is more complicated, but the curve may be taken as very nearly hyperbolic.

53. The Steam-engine Indicator is an apparatus for measuring the pressure of the steam in the cylinder and at the same time recording it automatically on a drum revolving as the piston moves. Thus, if the indicator be put in connection with the interior of the cylinder, the curve traced by the indicator has for its abscissas the distances s of the piston from this end, and for its ordinates the corresponding pressures F of the steam on the inside of the piston.

At the beginning of the stroke, steam is admitted and acts with nearly constant pressure on the piston; the line AB (Fig. 10) traced by the indicator will therefore be nearly parallel to the axis of s. As soon as the steam is shut off by the slide-valve, the steam, being now confined within the cylinder, begins to expand nearly according to the law $pv =$ const., or $Fs =$ const.; the curve traced by the indicator is therefore approximately an equilateral hyperbola BC, having the axes as asymptotes. When the slide-valve connects the cylinder with the condenser, a partial vacuum is established behind the piston, and the pressure curve is approximately a line CD, parallel to the axis of F.

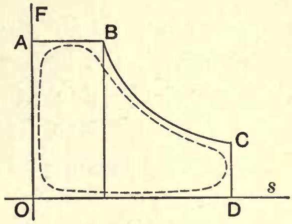

Fig. 10.

54. The area $ABCDO$ evidently represents approximately the work of the pressure on the inside of the piston in one complete (forward and backward) stroke. In reality, a large number of circumstances produce deviations from the regular shape $ABCDO$, and the actual trace, obtained by means of an indicator for one (forward and backward) stroke, usually called the *indicator diagram*, forms a loop somewhat like that indicated by the dotted curve in Fig. 10. The area of this loop, which represents the work in question, can readily be found by dividing it up into narrow rectangular strips, or with the aid of a planimeter.

55. The *effective* piston pressure is of course the difference between the pressures on the two sides of the piston. A diagram should therefore be obtained for each side of the piston; from these two diagrams the curve of effective piston pressure is then derived by constructing the curve whose ordinates are the differences of the corresponding pressures on the two sides. By dividing the area contained between this curve and the axes by the length of the stroke, the average, or *mean, piston pressure* is finally found.

For details the student is referred to special works on the steam engine, such as G. C. V. Holmes, *The steam engine*, New York, Appleton, 1887, pp. 317–345.

56. Attractive and Repulsive Forces. Let us consider the motion of a particle acted upon by a so-called *central force, i.e.*

a force whose direction constantly passes through a fixed centre O, while its magnitude is a function of the distance s from the centre alone. If the initial velocity be zero, or if its direction pass through the centre O, the motion of the particle will be rectilinear, the line of motion passing through the centre of force, O. The most important special cases of this kind have been treated in kinematics (Part I., Arts. 117–124, 176).

57. Let the force be due to a mass m' concentrated at the centre O, and attracting according to *Newton's law of the inverse square of the distance* (Part II., Art. 257). Counting the distances s from the centre O as origin (Fig. 11), we have, for the force acting on the particle m,

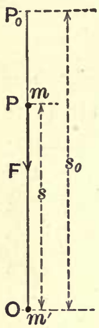

Fig. 11.

$$F = -\kappa \frac{mm'}{s^2},$$

where κ is a constant whose value may be determined, as indicated in Part I., Art. 119, and Part II., Arts. 262, 263.

The principle of kinetic energy, equation (12), Art. 47, gives at once

$$\tfrac{1}{2} mv^2 - \tfrac{1}{2} mv_0^2 = -\kappa mm' \int_{s_0}^{s} \frac{ds}{s^2} = \kappa m \left(\frac{m'}{s} - \frac{m'}{s_0} \right).$$

The quantity m'/s, or in absolute measure $\kappa m'/s$, is the potential at P due to m' (Part II., Art. 278), and $\kappa m'/s_0$ is the potential at P_0 due to the same mass m'. The increase in kinetic energy is, therefore, proportional to the decrease in potential.

The quantity $\kappa mm'/s$ is sometimes called the *mutual potential* of the masses m and m'; hence, the increase of the kinetic energy can be said to be equal to the difference of the mutual potentials in the final and initial positions.

The negative of the mutual potential is designated as the

potential energy of the moving particle m. Denoting this by V, and the kinetic energy by T, the last equation becomes

$$T - T_0 = -V + V_0,$$

or
$$T + V = T_0 + V_0 = \text{const.};$$

i.e. the sum of the kinetic and potential energies remains constant during the motion. This is **the principle of the conservation of energy** for this particular problem.

58. It is easy to see that the principle of the conservation of energy holds generally whenever the resulting force F is a function of the distance s alone.

Indeed, if $F = F(s)$, the principle of kinetic energy gives

$$\tfrac{1}{2} mv^2 - \tfrac{1}{2} mv_0^2 = \int_{s_0}^{s} F(s)\,ds; \tag{13}$$

hence, putting $\int F(s)ds = f(s)$, where $f(s)$ is called the *force-function*, or potential function, while $-f(s)$ is the *potential energy*, we have

$$\tfrac{1}{2} mv^2 - \tfrac{1}{2} mv_0^2 = f(s) - f(s_0), \tag{14}$$

or, with the notation of Art. 57,

$$T + V = T_0 + V_0 = \text{const.} \tag{15}$$

59. When the resultant force F is an *attraction directly proportional to the distance* s from a fixed centre O, say

$$F = -m\kappa^2 s,$$

the potential energy is, by Art. 58,

$$V = -f(s) = \tfrac{1}{2} m\kappa^2 s^2.$$

Hence, the principle of the conservation of energy gives

$$v^2 + \kappa^2 s^2 = \text{const.};$$

or, if the initial velocity is zero when $s = s_0$,

$$v = -\kappa\sqrt{s_0^2 - s^2}.$$

60. Tension of an Elastic String. According to Hooke's law, the tension of an elastic string is, within the limits of elasticity (*i.e.* as long as no *permanent* deformation is produced), directly proportional to the extension or change of length produced.

Thus, let an elastic string, whose natural length is l, assume the length s when its tension is T; then Hooke's law can be expressed in the form

$$T = k(s - l),$$

where k is a constant. To determine this constant for a given string, we may observe the length l_1 assumed by the string under a known tension, say the tension $T_1 = mg$, produced by suspending a given mass m from the string (the weight of the string itself being neglected). We then have

$$T_1 = k(l_1 - l).$$

Hence, dividing,

$$\frac{T}{mg} = \frac{s - l}{l_1 - l},$$

or, denoting by e the extension $l_1 - l$ due to the weight mg,

$$T = \frac{mg}{e}(s - l). \qquad (16)$$

61. By means of this relation we can determine the motion of a particle of mass m attached to a fixed point O by means of an elastic string, if the string be stretched and then let go. We shall assume the particle and string to lie on a smooth horizontal table, so as to eliminate the effect of the weight of the particle.

The equation of motion is

$$m\frac{d^2s}{dt^2} = -\frac{mg}{e}(s - l), \qquad (17)$$

whence, putting for shortness $\sqrt{g/e} = \kappa$,

$$s = l + C_1 \cos \kappa t + C_2 \sin \kappa t,$$

$$v = \frac{ds}{dt} = -\kappa C_1 \sin \kappa t + \kappa C_2 \cos \kappa t.$$

If the initial length of the string at the time $t = 0$ be s_0, the constants are readily determined, and we find

$$s = l + (s_0 - l) \cos \kappa t,$$

$$v = -\kappa (s_0 - l) \sin \kappa t. \qquad (18)$$

It should be noticed that these equations hold only as long as the string is actually stretched, *i.e.* as long as $s > l$; the motion that ensues when s becomes less than l is, however, easily determined from the velocity for $s = l$.

62. Exercises.

(1) In a steam engine, let $p = 15$ lbs. per square inch be the mean piston pressure during one stroke, $s = 4$ ft. the length of the stroke, and $d = 1.5$ ft. the diameter of the cylinder. (*a*) What is the work per stroke? (*b*) To what height could a mass of 500 lbs. be raised by this work?

(2) A train of 80 tons starting from rest acquires a velocity of 30 miles an hour on a level road at the end of the first mile. Determine the average tractive force of the engine: (*a*) if the frictional resistances be neglected; (*b*) if these resistances be estimated at 8 lbs. per ton. (*c*) What tractive force is required to haul the same train over a level road at a constant speed?

(3) A train of 60 tons runs one mile with constant speed; if the resistances be 8 lbs. per ton, find the work done by the engine: (*a*) on a level track; (*b*) on an average grade of 1%. (*c*) On a 1% grade, what is the ratio of the work done against gravity to that done against the resistances?

(4) Determine the work expended in raising from the ground the materials for a brick wall 30 ft. high, 40 ft. long, and 2 ft. thick, the weight of a cubic foot of brickwork being 112 lbs.

(5) Knowing that on the surface of the earth the attraction per unit of mass is $g = 32$, find what it would be on the sun if the density of the sun be $\frac{1}{4}$ of that of the earth, and its diameter 108 times that of the earth.

(6) Show that the velocity acquired by a body in falling to the surface of the earth from an infinite distance, under the action of the earth's attraction alone, would be $v = \sqrt{2gR}$, or about 7 miles per second (with $R = 4000$ miles).

(7) A homogeneous straight rod, $AB = l$, of constant density ρ, attracts a particle P of mass 1 according to the law of the inverse square of the distance. The initial position P_0 of P is on AB produced beyond B, at the distance $BP_0 = s_0$, and the initial velocity is zero.

(*a*) Determine the velocity v of P at any distance $BP = s$, and its velocity v_1 at B. (*b*) How is the solution to be modified if the linear mass BA extends from B to infinity?

(8) A circular wire of radius a and constant density ρ attracts, according to Newton's law, a particle P of mass 1, situated on the *axis* of the circle; *i.e.* on the perpendicular to its plane passing through the centre O. If the velocity is zero when the particle is at the distance $OP_0 = s_0$, determine the velocity of the particle at any distance s, and show that the motion is oscillatory.

(9) Determine the motion of two free particles of masses m_1, m_2, attracting each other according to Newton's law, and starting at the distance s_0 with zero velocity.

(10) Show that the motion of the particle in Art. 61 is oscillatory, and that the period, *i.e.* the time of one complete oscillation, is

$$= 2\sqrt{e/g}[\pi + 2\,l/(s_0 - l)].$$

(11) A particle of mass m is suspended from a fixed point by means of an elastic string whose weight is neglected. The natural length of the string is l. Its length, when the mass m is suspended at its end, is l_1. If the particle be pulled down so as to make the length of the string $= s_0$, and then released, the particle will perform oscillations. Determine their period: (*a*) if $s_0 - l_1 < l_1 - l$; (*b*) if $s_0 - l_1 > l_1 - l$.

(12) The particle in Ex. (11) is raised through a height h, so as to loosen the string, and then dropped. Determine the greatest extension of the string.

(13) An elastic string, whose natural length is $= l$, is suspended from a fixed point. A mass m_1 attached to its lower end stretches it to a length l_1; another mass m_2 stretches it to a length l_2. If both these masses be attached and then the mass m_2 be cut off, what will be the motion of m_1?

(14) A particle performs rectilinear oscillations owing to a centre of force in the line of motion attracting the particle with a force directly proportional to the distance. The motion of the particle is impeded by a resistance directly proportional to the velocity. Investigate the motion.

63. Power. It has been shown that the time-effect of a force is measured by its *impulse* (Art. 2), while the space-effect is measured by its *work* (Arts. 47, 48). In applied mechanics it is of great importance to take time and space into account simultaneously. *The time-rate at which work is performed by a force* has therefore received a special name, **power.** The source from which the force for doing useful work is derived is commonly called the *agent;* and it is customary to speak of the power of an agent, this meaning the rate at which the agent is capable of supplying useful work.

64. The *dimensions* of power are evidently ML^2T^{-3}. The *unit of power* is the power of an agent that does unit work in unit time. Hence, in absolute measure, it is the power of an agent doing one erg per second in the C.G.S. system, and one foot-poundal per second in the F.P.S. system. As, however, the idea of power is of importance mainly in engineering practice, power is usually measured in gravitation units. In this case, the unit of power is the power of an agent doing one foot-pound per second in the F.P.S. system, and one kilogramme-metre in the metric system.

A larger unit is frequently found more convenient. For this reason, the name **horse-power** (H.P.) is given to the power of doing 550 foot-pounds of work per second, or $550 \times 60 = 33{,}000$ foot-pounds per minute.

65. Efficiency of Machines. While the principle of the conservation of energy was proved in Arts. 57 and 58 only for a special case, it is known to be of almost universal application to the forces occurring in nature. Thus, in particular in the case of machines it is found to be verified with a degree of approximation corresponding to the precision of the investigation.

The principle can here be expressed in the form

$$W = W_0 + W_1,$$

if W denote the *total work* done by the agent driving the machine (such as animal force, the expansive force of steam, the pressure of the wind, etc.); W_0 the so-called *lost* or *wasteful work* spent in overcoming friction and other passive resistances of the machine; and W_1 the *useful* work done by the machine.

While W and W_1 allow of precise determination, it is in general difficult to determine W_0 accurately; but it is found that the more exactly in any given machine W_0 is determined, the more nearly will the equation $W = W_0 + W_1$ be fulfilled.

As explained in Part II., Art. 255, the ratio W_1/W of the useful work to the total work is called the *efficiency* of the machine. The term *modulus* is used sometimes for efficiency.

66. Exercises.

(1) In electrical engineering a *watt* is defined as the power of doing one joule, *i.e.* 10^7 ergs, per second. Find the relation between the watt and the horse-power.

(2) In countries using the metric system of weights and measures the horse-power is defined as 75 kilogramme-metres per second. Find its relation to the watt and to the British horse-power.

(3) Find the horse-power of the engine in Art. 62, Ex. (1), if it make 1 stroke per second.

(4) The cylinder of a steam engine has a diameter of 15 in.; the stroke is 3 ft.; the number of strokes per minute is 77; the mean pressure of the steam is 40 lbs. per square inch. What is the horse-power of the engine?

(5) Find the horse-power required of the locomotive to haul a train of 100 tons at the rate of 30 miles an hour, the resistances amounting to 8 lbs. per ton: (*a*) on a level road; (*b*) up a 1% grade; (*c*) up a 2% grade.

(6) How much water can an engine furnishing 50 H.P. raise per minute from the bottom of a mine 1000 ft. deep?

(7) The diameter of the cylinder of a steam engine is 30 in.; the stroke 4 ft.; the mean pressure 15 lbs. per square inch; the number of revolutions 24 per minute. If the efficiency of the engine be $\frac{3}{5}$, what is the amount of water raised per hour from a depth of 250 ft.?

(8) In what time would an engine yielding 2 H.P. perform the work of raising the brickwork in Art. 62, Ex. (4)?

(9) A shaft of 8 ft. diameter is to be sunk to a depth of 420 ft. through a material whose specific gravity is 2·2. Determine: (*a*) the total work of raising the material to the surface; (*b*) the time in which it can be done by an engine yielding 3·5 H.P.; (*c*) the time in which it can be done by 4 men working in a capstan, if each laborer does 2500 ft.-lbs. per minute, working 8 hours per day.

III. *Free Curvilinear Motion.*

I. GENERAL PRINCIPLES.

67. Let j be the acceleration of a particle of mass m at the time t; F the resultant of all the forces acting on the particle; then its equation of motion is (Art. 35)

$$mj = F.$$

In curvilinear motion (Fig. 12) the direction of j and F differs from the direction of the velocity v; and the angle ψ between j and v varies in general in the course of time. As shown in kinematics (Part I., Art. 159), the acceleration can be resolved into a tangential component $j_t = dv/dt = d^2s/dt^2$ and a normal component $j_n = v^2/\rho$, where ρ is the radius of curvature of the path. Hence, if the resultant force F which has the direction of j be resolved into a *tangential force* $F_t = F\cos\psi$, and a *normal force* $F_n = F\sin\psi$, the above equation of motion will be replaced by the following two equations:

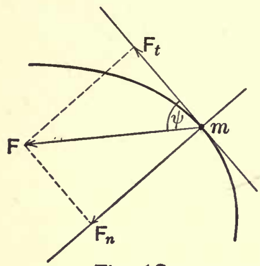

Fig. 12.

$$m\frac{dv}{dt} = F_t, \quad m\frac{v^2}{\rho} = F_n. \tag{1}$$

In the particular case when the normal component F_n is constantly directed towards a fixed point it is called *centripetal force.*

68. The formulæ (1) show how the force F affects the velocity of the particle and the curvature of the path. The change of the *magnitude* of the velocity is due to the tangential force F_t alone. If this component be zero, *i.e.* if the resultant force F be constantly normal to the path, the velocity v will remain of constant magnitude. The curvature of the path, $1/\rho$, and hence the *direction* of v, depends on the normal component F_n. If this component be zero, the curvature is zero; *i.e.* the path is rectilinear.

69. Instead of resolving the resultant force F along the tangent and normal, it is often more convenient to resolve it into three components, $F\cos\alpha = X$, $F\cos\beta = Y$, $F\cos\gamma = Z$, parallel to three fixed rectangular axes of co-ordinates Ox, Oy, Oz, to which the whole motion is then referred. If x, y, z be the co-ordinates of the particle m at the time t, the equations of motion assume the form

$$m\frac{d^2x}{dt^2} = X, \quad m\frac{d^2y}{dt^2} = Y, \quad m\frac{d^2z}{dt^2} = Z. \qquad (2)$$

Thus, the curvilinear motion is replaced by three rectilinear motions.

70. If the components X, Y, Z were given as functions of the time t alone, each of the three equations (2) could be integrated separately. In general, however, these components will be functions of the co-ordinates, and perhaps also of the velocity and time. No general rules can be given for integrating the equations in this case. By combining the equations (2) in such a way as to produce exact derivatives in the resulting equation, it is sometimes possible to effect an integration. Two methods of this kind have been indicated for the case of two dimensions in a particular example in Part I., Art. 232. We now proceed to study these *principles* of integration from a more general point of view, and to point out the physical meaning of the expressions involved.

71. The Principle of Kinetic Energy. Let us combine the equations of motion (2) by multiplying them by dx/dt, dy/dt, dz/dt respectively, and then adding. The left-hand member of the resulting equation will be the derivative with respect to t of

$$\tfrac{1}{2}m\left[\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2\right] = \tfrac{1}{2}mv^2.$$

We find, therefore,

$$\frac{d(\frac{1}{2}mv^2)}{dt}=X\frac{dx}{dt}+Y\frac{dy}{dt}+Z\frac{dz}{dt},$$

or, multiplying by dt and integrating,

$$\tfrac{1}{2}mv^2-\tfrac{1}{2}mv_0{}^2=\int(Xdx+Ydy+Zdz), \tag{3}$$

where v_0 is the initial velocity.

The left-hand member represents the increase in the kinetic energy of the particle; the right-hand member represents the work done by the resultant force; and equation (3) expresses the equality between the work done and the change in the kinetic energy, that is, the *principle of work* or *of kinetic energy* for the curvilinear motion of a particle (comp. Art. 47). Sometimes the name *principle of vis viva* is given to this proposition, the term vis viva, or living force, meaning the same as kinetic energy, or, in older works, twice the kinetic energy.

72. The principle of work can be deduced still more directly from the equations (1). Multiplying the former of these equations by $vdt=ds$, we find

$$d(\tfrac{1}{2}mv^2)=Fds\cos\psi;$$

hence, integrating,

$$\tfrac{1}{2}mv^2-\tfrac{1}{2}mv_0{}^2=\int_{s_0}^{s}Fds\cos\psi, \tag{4}$$

where v_0 is the velocity of the particle at the place specified by s_0 (comp. Part II., Art. 72).

73. The principle of kinetic energy gives a first integral of the equations of motion whenever the integration indicated in the right-hand member of (3) or (4) can be performed. We proceed to investigate under what conditions this integration becomes possible.

In the most general case the components X, Y, Z, in (3), as well as the tangential force $F\cos\psi$ in (4), are functions of the co-ordinates x, y, z, of the velocity, *i.e.* of the time-derivatives of x, y, z, and of the time t. If the motion of the particle were completely known, that is, if we knew its position at every instant, the co-ordinates would be known functions of the time, say

$$x=f_1(t), \quad y=f_2(t), \quad z=f_3(t).$$

By differentiation the velocities $v_x=dx/dt$, $v_y=dy/dt$, $v_z=dz/dt$ could be found; and, substituting in (3), the integral would assume the form $\int_{t_0}^{t}\phi(t)dt$, so that the work could be determined by evaluating this integral. As, however, the motion of the particle is generally not known beforehand, this motion being just the thing to be determined, the integral cannot be evaluated in the most general case.

74. *If the forces acting on the particle depend only on the position of the particle, i.e.* if X, Y, Z are functions of x, y, z alone, *the integral* $\int_{s_0}^{s}(Xdx+Ydy+Zdz)$ *can be determined whenever the path of the particle is given.* For the equations of the path, say

$$f_1(x, y, z)=0, \quad f_2(x, y, z)=0,$$

make it possible to eliminate two of the three variables x, y, z from under the integral sign, or to express all three in terms of a fourth variable. In either case the function under the integral sign becomes a function of a single variable, and the work of the forces can be found.

75. *If the forces are such as to make the expression* $Xdx+Ydy+Zdz$ *an exact differential,* say dU, *the integration can evidently be performed without any knowledge of the path of the particle between its initial and final positions.* In this case equation (3) becomes

$$\tfrac{1}{2}mv^2-\tfrac{1}{2}mv_0^2=U-U_0, \qquad (5)$$

U_0 being the value of U at the initial position, where $v = v_0$. As most of the forces occurring in nature are of this character, this particular case is of great importance, and deserves careful study.

76. The expression $Xdx + Ydy + Zdz$ will be an exact differential whenever there exists a function U of the co-ordinates x, y, z alone (*i.e.* not involving the time or the velocities explicitly), such that

$$\frac{\partial U}{\partial x} = X, \quad \frac{\partial U}{\partial y} = Y, \quad \frac{\partial U}{\partial z} = Z. \tag{6}$$

If these conditions are fulfilled, we have evidently

$$Xdx + Ydy + Zdz = dU.$$

The function U is called the **force-function**, and forces for which a force-function exists are called *conservative forces.*

Hence, *if the forces acting on a particle are conservative*, in other words, *if they have a force-function, the principle of work gives a first integral of the equations of motion.*

77. The conditions (6) for the existence of a force-function U can be put into a different analytical form which is frequently useful. Differentiating the second of the equations (6) with respect to z, the third with respect to y, we find

$$\frac{\partial^2 U}{\partial y \partial z} = \frac{\partial Y}{\partial z}, \quad \frac{\partial^2 U}{\partial z \partial y} = \frac{\partial Z}{\partial y},$$

whence $\partial Y/\partial z = \partial Z/\partial y$. If we proceed in a similar way with the other equations (6), it appears that they can be replaced by the following conditions:

$$\frac{\partial Y}{\partial z} = \frac{\partial Z}{\partial y}, \quad \frac{\partial Z}{\partial x} = \frac{\partial X}{\partial z}, \quad \frac{\partial X}{\partial y} = \frac{\partial Y}{\partial x}. \tag{7}$$

It is shown in works on the differential calculus and differential equations that these equations (7), or the equations (6), which are equivalent to them, are not only the sufficient, but also the necessary, conditions that must be fulfilled to make $Xdx + Ydy + Zdz$ an exact differential.

78. The dynamical meaning of the existence of a force-function U lies mainly in the fact that, if a force-function exists, the work done by the forces as the particle passes from its initial to its final position depends only on these positions, and not on the intervening path. This is at once apparent from equation (5), in which $U - U_0$ represents this work.

It follows that the work of conservative forces is zero if the particle returns finally to its original position, that is, if it describes a closed path, provided that the force-function U is single-valued, an assumption which will here always be made.

79. In the case of central forces inversely proportional to the square of the distance, for which a force-function can always be shown to exist (see Part II., Arts. 278–281), the force-function is usually called the *potential*. The negative of the force-function, say

$$V = -U,$$

is called the **potential energy**. If this quantity be introduced, and the kinetic energy be denoted by T (as in Art. 57), the equation (5) assumes the form

$$T + V = T_0 + V_0, \tag{8}$$

which expresses the **principle of the conservation of energy** for a particle: *the total energy*, i.e. *the sum of the kinetic and potential energies, remains constant throughout the motion whenever there exists a force-function*. In other words, whatever is gained in kinetic is lost in potential energy, and *vice versa*.

80. The name *force-function* is due to Sir William Rowan Hamilton. Some authors use it for $V = - U$, and not for U. With regard to the term *potential*, the usage is still less settled. Some writers use it for U, others for $-U$, nor is its use always restricted to Newtonian forces. Green was the first to give the name *potential function* to the function U; Gauss brought the expression *potential* into common use. Clausius uses "potential function" for what is called above "potential," reserving the latter name for the potential of a system on another system, or on itself. He also uses the term *ergal* for what is called above "potential energy." Several writers have followed him in this terminology.

81. As the force-function U is a function of the co-ordinates x, y, z alone, an equation of the form

$$U = c, \tag{9}$$

where c is a constant, represents a surface which is the locus of all points of space at which the force-function has the same value c. By giving to c different values, a system of surfaces is obtained, and these surfaces are called **level**, or **equipotential, surfaces.**

82. The values of the derivatives of U at any point $P(x, y, z)$ are proportional to the direction-cosines of the normal to the equipotential surface (9) at P. But, by (6), they are also proportional to the direction-cosines of the resultant force F at this point. It follows that *the resultant force* F *at any point* P *is always normal to the equipotential surface passing through* P.

If the equation of the equipotential surfaces be given, the resultant force F at any point (x, y, z) is readily found, both in magnitude and direction, from its components (6):

$$F^2 = X^2 + Y^2 + Z^2 = \left(\frac{\partial U}{\partial x}\right)^2 + \left(\frac{\partial U}{\partial y}\right)^2 + \left(\frac{\partial U}{\partial z}\right)^2. \tag{10}$$

83. As the particle moves in its path from any point P to an infinitely near point P' (Fig. 13), it passes from one equipotential surface $U = c$ to another $U = c'$. Its velocity meets these surfaces at a varying angle, while its acceleration, which has the direction of the resultant force F, is always normal to these surfaces. The work done by F as the particle moves from P to P' is

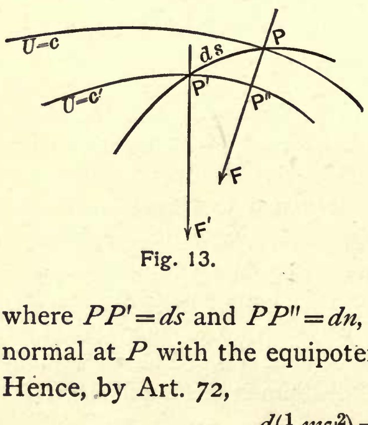

Fig. 13.

$$Fds \cos(F, ds) = Fdn,$$

where $PP' = ds$ and $PP'' = dn$, P'' being the intersection of the normal at P with the equipotential surface passing through P'. Hence, by Art. 72,

$$d(\tfrac{1}{2} mv^2) = F \cdot dn = dU. \tag{11}$$

The normal distance $PP''=dn$ between two equipotential surfaces is therefore inversely proportional to the force F.

It also appears that whenever the particle in its path returns to the same equipotential surface, the work done by F is zero, and hence, by (5), the velocity assumes the initial value v_0.

84. Let α, β, γ be the direction cosines of F at any point P; λ, μ, ν those of any straight line s drawn through P; and let ϕ be the angle between F and s, so that $\cos\phi=\alpha\lambda+\beta\mu+\gamma\nu$. Then the projection of F on s is

$$F_s=F\cos\phi=F(\alpha\lambda+\beta\mu+\gamma\nu),$$

or, since by (6) $\alpha F=\partial U/\partial x$, $\beta F=\partial U/\partial y$, $\gamma F=\partial U/\partial z$,

$$F_s=\frac{\partial U}{\partial x}\frac{dx}{ds}+\frac{\partial U}{\partial y}\frac{dy}{ds}+\frac{\partial U}{\partial z}\frac{dz}{ds}=\frac{dU}{ds}; \qquad (12)$$

i.e. *the projection of the resultant force on any direction is the derivative of the force-function with respect to that direction.*

This follows also from the equations (6), since the directions of the axes are arbitrary.

If s be taken tangent to the equipotential surface passing through P, we have $F_s=dU/ds=0$; if it be taken normal to this surface, we find $F_s=F=dU/dn$, which agrees with (11).

85. The force-function U determines, as has been shown, a system of equipotential surfaces $U=$const. Starting from a point P on one of these surfaces, say $U=c$ (Fig. 14), let us draw through P the direction of the resultant force, which is normal to the surface $U=c$ (Art. 82). Let this direction intersect in P' the next surface, $U=c'$. At P' draw the normal to $U=c'$, and let it intersect the next surface, $U=c''$, in P''. Proceeding in this way, we obtain a series of points P, P', P'',

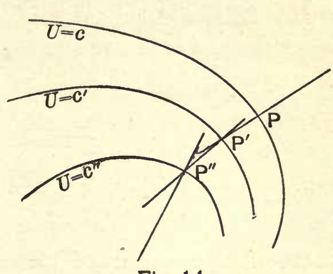

Fig. 14.

P''', ..., which in the limit will form a continuous curve whose direction at any point coincides with the direction of the resultant force at that point. Such a line is called a **line of force.**

The lines of force evidently form the orthogonal system to the system of equipotential surfaces. The differential equations of the lines of force are therefore:

$$\frac{dx}{\frac{\partial U}{\partial x}} = \frac{dy}{\frac{\partial U}{\partial y}} = \frac{dz}{\frac{\partial U}{\partial z}}. \tag{13}$$

86. Exercises.

(1) Show that a force-function exists when the resultant force is constant in magnitude and direction.

(2) Find the force-function in the case of a free particle moving under the action of the constant force of gravity alone (projectile *in vacuo*); determine the equipotential surfaces and the potential energy.

(3) Show the existence of a force-function when the direction of the resultant force is constantly perpendicular to a fixed plane, say the xy-plane, and its magnitude is a given function $f(z)$ of the distance z from the plane.

(4) Find the force-function, the equipotential surfaces, and the kinetic energy when the force is a function $f(r)$ of the perpendicular distance r from a fixed line, and is directed towards this line at right angles to it.

(5) Show that a force-function always exists for a *central force*, *i.e.* a force passing through a fixed point and depending only on the distance from this point.

(6) Show the existence of a force-function when a particle moves under the action of any number of central forces.

(7) A homogeneous sphere of mass m' attracts a free particle P of mass m with a force $F = \kappa mm'/r^2$, where κ is a constant, and $r = OP$ is the distance of P from the centre of the sphere. Show that the potential is $V = -\kappa mm'/r$, and that the equipotential surfaces are spheres whose common centre is at O.

(8) In Ex. (7), assume $\kappa mm' = 1$, and draw the intersections of the equipotential surfaces with a plane passing through O, from $r = 1$ centimetre to $r = 2$ centimetres, with a difference of potential $= \frac{1}{10}$.

(9) Two spheres, whose masses are as 1 to 2, attract a particle of mass 1 according to Newton's law; the distance of the centres of the spheres is $=4$. Construct the equipotential lines in a plane passing through the centres, by first constructing the equipotential lines for each sphere separately, and then joining the points of intersection whose potential is the same.

(10) A particle of mass m is subject to the force of gravity and to the actions of two fixed centres C_1, C_2, one attracting with a force inversely proportional to the square of the distance, the other repelling with a force directly proportional to the distance. Find the equipotential surfaces.

87. The Principle of Angular Momentum or of Areas. Let us begin with the case of plane motion, the equations of motion being

$$m\frac{d^2x}{dt^2}=X, \quad m\frac{d^2y}{dt^2}=Y.$$

If we combine these equations by multiplying the former by y, the latter by x, and subtracting the former from the latter, we find

$$mx\frac{d^2y}{dt^2}-my\frac{d^2x}{dt^2}=xY-yX. \tag{14}$$

The right-hand member is the moment (with respect to the origin) of the resultant force F whose components are X, Y (see Part II., Art. 91), while the left-hand member is an exact derivative, viz. the derivative with respect to the time of

$$mxdy/dt-mydx/dt,$$

as is easily verified by differentiating this quantity. The result can therefore be written in the form

$$\frac{d}{dt}\left(mx\frac{dy}{dt}-my\frac{dx}{dt}\right)=xY-yX, \tag{15}$$

and gives, if multiplied by dt and integrated,

$$mx\frac{dy}{dt}-my\frac{dx}{dt}=\int(xY-yX)dt. \tag{16}$$

These equations express the principle of angular momentum, or of areas, for plane motion.

88. The name *principle of areas* is due to the kinematical meaning of the left-hand member (comp. Part I., Arts. 227–232). As $xdy-ydx$ represents twice the infinitesimal sector described by the radius vector of the point (x, y) during the element of time, the quantity $xdy/dt-ydx/dt$ is twice the sectorial velocity about the origin. Introducing polar co-ordinates by putting $x=r\cos\theta$, $y=r\sin\theta$, we have $xdy-ydx=r^2d\theta$, and denoting by S the sector described in the time t,

$$2\frac{dS}{dt}=x\frac{dy}{dt}-y\frac{dx}{dt}=r^2\frac{d\theta}{dt}.$$

The kinematical meaning of equation (15), after dividing it by m, can therefore be stated as follows: *the time-rate of change of twice the sectorial velocity about any point is equal to the moment of the acceleration about the same point.*

89. The dynamical meaning of equation (15) appears by considering that mdx/dt, mdy/dt are the components of the momentum mv of the moving particle (Fig. 15). The product mvp of the momentum and its perpendicular distance from the origin is called the **moment of momentum**, or the **angular momentum**, of the particle about the origin.

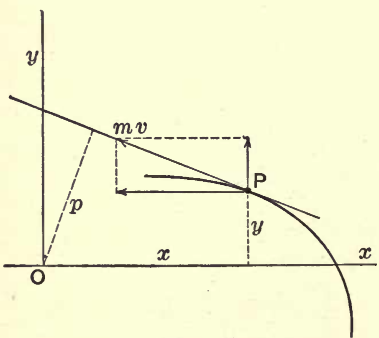

Fig. 15.

It appears from Fig. 15 that we have

$$mvp=mx\frac{dy}{dt}-my\frac{dx}{dt}.$$

The angular momentum is evidently nothing but twice the sectorial velocity multiplied by the mass, just as momentum is linear velocity times mass.

The dynamical meaning of equation (15) can therefore be expressed as follows: *the time-rate of change of angular momen-*

tum about any fixed point is equal to the moment of the resultant force about the same point.

90. The most important case in which the integration in (16) can be performed is the case when

$$xY - yX = 0, \qquad (17)$$

which evidently means that the direction of the resultant force F passes through the origin. If this condition be fulfilled, equation (16) reduces to the form

$$mx\frac{dy}{dt} - my\frac{dx}{dt} = c, \qquad (18)$$

where c is a constant of integration to be determined from the initial position and velocity.

Kinematically, this equation means that the sectorial velocity remains constant. It can be put into the form

$$\frac{dS}{dt} = \frac{c}{2m} = c',$$

whence, by integration, we find

$$S - S_0 = c'(t - t_0). \qquad (19)$$

Hence, *if the acceleration passes constantly through a fixed point, the sector* $S - S_0$ *described about this point in any time* $t - t_0$ *is proportional to this time.*

This is the *principle of the conservation of area* for plane motion.

Dynamically, equation (18) means that *if the resultant force passes constantly through a fixed point, the angular momentum about this point remains constant.* The proposition can also be called the *principle of the conservation of angular momentum.*

If v_0 be the initial velocity, p_0 the perpendicular to v_0 from the fixed point, equation (18) can be written in the form

$$vp = v_0 p_0. \qquad (20)$$

91. In the general case of three dimensions any two of the equations of motion,

$$m\frac{d^2x}{dt^2}=X, \quad m\frac{d^2y}{dt^2}=Y, \quad m\frac{d^2z}{dt^2}=Z,$$

can be combined by the method of Art. 87, and we find thus

$$\begin{aligned} my\frac{d^2z}{dt^2}-mz\frac{d^2y}{dt^2}&=\frac{d}{dt}\left(my\frac{dz}{dt}-mz\frac{dy}{dt}\right)=yZ-zY,\\ mz\frac{d^2x}{dt^2}-mx\frac{d^2z}{dt^2}&=\frac{d}{dt}\left(mz\frac{dx}{dt}-mx\frac{dz}{dt}\right)=zX-xZ, \qquad (21)\\ mx\frac{d^2y}{dt^2}-my\frac{d^2x}{dt^2}&=\frac{d}{dt}\left(mx\frac{dy}{dt}-my\frac{dx}{dt}\right)=xY-yX. \end{aligned}$$

The expression $xdy-ydx$ now represents the projection on the xy-plane of the infinitesimal sector described by the radius vector of the particle during the time dt; similarly, $ydz-zdy$ and $zdx-xdz$ are the projections of the same sector on the planes yz and zx, respectively.

92. The right-hand members of the equations (21) are easily seen to represent the moments of the resultant force about the axes of x, y, z, if it be remembered that the moment of a force with respect to an axis is the moment of its projection on a plane perpendicular to the axis about the point of intersection of the axis with the plane (Part II., Art. 213). If the moment of the momentum mv of the particle be defined in the same way, the quantities $mydz/dt-mzdy/dt$, $mzdx/dt-mxdz/dt$, $mxdy/dt-mydx/dt$ are the *moments of momentum*, or, as they are also called, the *angular momenta*, about the three axes of co-ordinates Ox, Oy, Oz.

As the axes are arbitrary, the equations (21) express the statement that *the moment of the resultant about any fixed axis is equal to the time-rate of change of the angular momentum about the same axis.*

93. The most important case of the application of the equations (21) arises when one or more of the conditions

$$yZ-zY=0, \quad zX-xZ=0, \quad xY-yX=0 \qquad .(22)$$

are fulfilled. The first of these conditions means that the projection of the resultant force on the yz-plane passes always through a fixed point, viz. the origin of co-ordinates; or what amounts to the same thing, that the resultant force always intersects the axis of x. Similarly, the second condition means that the resultant intersects the axis of y. Hence, if both these conditions are fulfilled, the resultant force passes constantly through a fixed point, the origin of co-ordinates.

It follows that if any two of the conditions (22) are fulfilled, the third must also be fulfilled. This is also evident analytically, as any one of the three equations can be derived from the two others.

94. If the conditions (22) are fulfilled, the integration of the equations (21) gives

$$\begin{aligned} my\frac{dz}{dt}-mz\frac{dy}{dt}&=c_1, \\ mz\frac{dx}{dt}-mx\frac{dz}{dt}&=c_2, \\ mx\frac{dy}{dt}-my\frac{dx}{dt}&=c_3, \end{aligned} \qquad (23)$$

where c_1, c_2, c_3 are constants depending on the initial conditions. These equations express the proposition that *if the resultant force passes constantly through a fixed point, the angular momentum about any axis passing through this point remains constant.*

Multiplying the equations (23) respectively by x, y, z and adding, we find

$$c_1x+c_2y+c_3z=0, \qquad (24)$$

which is the equation of a plane passing through the origin. As the co-ordinates x, y, z of the moving particle fulfil this equation independently of the time, it follows that *the motion is necessarily plane whenever the conditions* (22) *are satisfied.* The constants of integration c_1, c_2, c_3 are evidently proportional to the direction-cosines of the normal to the plane of motion.

95. If the equations (23) be written in the form

$$\frac{1}{2}\frac{ydz-zdy}{dt}=c_1', \quad \frac{1}{2}\frac{zdx-xdz}{dt}=c_2', \quad \frac{1}{2}\frac{xdy-ydx}{dt}=c_3',$$

they show that the projections of the motion on the three co-ordinate planes have constant sectorial velocities c_1', c_2', c_3'; hence, the sectorial velocity of the motion itself is constant, viz.

$$\frac{dS}{dt}=\sqrt{c_1'^2+c_2'^2+c_3'^2}.$$

It follows in this case that the sector $S-S_0$, described during the time $t-t_0$, is proportional to this time:

$$S-S_0=\sqrt{c_1'^2+c_2'^2+c_3'^2}\,(t-t_0). \tag{25}$$

96. Exercises.

(1) A particle of mass m is attracted, according to Newton's law, by a mass m' concentrated at a fixed point O. If x_0, y_0, z_0 be the initial co-ordinates, and $\dot{x}_0$, $\dot{y}_0$, $\dot{z}_0$ the initial velocities of the particle, find the equation of the plane in which it moves, and show that this plane passes through O and the initial velocity.

(2) A particle is attracted by n fixed centres, whose forces are directly proportional to the masses of the centres and to the distances from them. Show that there is one position of equilibrium for the particle, and that the motion takes place as if the total mass of all the centres were concentrated at this point. Find also the equation of the plane of the motion.

(3) A particle is acted upon by a central force, *i.e.* by a force whose direction passes through a fixed point, and whose magnitude is a function of the distance from this point, say $F=mf(r)$. Show that the path is a plane curve, and find the equation of the plane of the motion.

(4) The equation (15) can, by Art. 89, be written $d(mvp)/dt=xY-yX$. Show that the two terms of $d(mvp)/dt=mpdv/dt+mvdp/dt$ are equal

respectively to the moments of the tangential and normal components of the resultant force F.

97. The Principle of d'Alembert. Let us consider a particle of mass m moving under the action of any forces $F_1, F_2, \cdots F_n$, whose resultant is F. The total acceleration j of the particle has the components d^2x/dt^2, d^2y/dt^2, d^2z/dt^2 parallel to the rectangular axes Ox, Oy, Oz. If the forces $F_1, F_2, \cdots F_n$ be imagined removed, a force equal to mj would be required to give the particle the same acceleration j that it had under the action of the forces $F_1, F_2, \cdots F_n$. This fictitious force, mj, whose components are md^2x/dt^2, md^2y/dt^2, md^2z/dt^2, is called the *effective force*. For the sake of distinction, the forces $F_1, F_2, \cdots F_n$, which actually produce the motion, are called the *impressed forces* (comp. Art. 36).

98. The ordinary equations of motion of a particle,

$$m\frac{d^2x}{dt^2} = X, \quad m\frac{d^2y}{dt^2} = Y, \quad m\frac{d^2z}{dt^2} = Z, \qquad (26)$$

where X, Y, Z are the components of the resultant F of the impressed forces, express merely the equality between the effective force mj and the resultant impressed force F. It follows that, *if the reversed effective force* $-mj$, or its components, $-md^2x/dt^2$, $-md^2y/dt^2$, $-md^2z/dt^2$, *be combined with the impressed forces* $F_1, F_2, \cdots F_n$, *we have a system in equilibrium.* This is the fundamental idea of d'Alembert's principle.

99. The reversed effective force, $-mj$, is sometimes called the *force of inertia* of the particle. To understand the idea underlying this expression, imagine the impressed forces to be removed, and then push the particle with the hand so as to give it the same motion that it had under the action of the impressed forces. The pressure of the hand on the particle must at every instant be equal to the resultant F, or to the effective force mj, while the equal and opposite pressure of the particle on the hand represents the force of inertia. It must, however, be clearly understood that this force of inertia, or inertia-resistance, is a force exerted on the hand and not on the particle.

100. Owing to the fact that, by combining with the impressed forces the reversed effective force, we obtain at any given instant a system in equilibrium, it becomes possible to apply to kinetical problems the statical conditions of equilibrium.

Since in the case of a single particle the forces are all concurrent, the conditions of equilibrium are obtained by equating to zero the sums of the components of the forces along each axis. This gives

$$X - m\frac{d^2x}{dt^2} = 0, \quad Y - m\frac{d^2y}{dt^2} = 0, \quad Z - m\frac{d^2z}{dt^2} = 0,$$

and these are the ordinary dynamical equations of motion (see (26), Art. 98).

101. The conditions of equilibrium of a system of forces can also be expressed by means of the principle of virtual work (Part II., Art. 239). Thus, let δx, δy, δz be the components of any virtual displacement δs of the particle; then the principle of virtual work applied to our system of forces gives the single condition

$$\left(-m\frac{d^2x}{dt^2} + X\right)\delta x + \left(-m\frac{d^2y}{dt^2} + Y\right)\delta y + \left(-m\frac{d^2z}{dt^2} + Z\right)\delta z = 0, \quad (27)$$

which is of course equivalent to the three equations (26) on account of the abitrariness of the displacement δs.

The equation (27), which may also be written in the form

$$m\left(\frac{d^2x}{dt^2}\delta x + \frac{d^2y}{dt^2}\delta y + \frac{d^2z}{dt^2}\delta z\right) = X\delta x + Y\delta y + Z\delta z, \quad (28)$$

expresses d'Alembert's principle for a single particle: *for any virtual displacement the sum of the virtual works of the impressed forces is equal to that of the effective force.*

102. The advantage of using the equations of motion in the form given to them by d'Alembert arises mainly from the application of the principle of virtual work which thus becomes possible; this will be seen more clearly later on, in the treatment of constrained motion. For the present it may suffice to notice that, if the actual displacement ds of the particle in its path be selected as the virtual displacement δs, equation (28) becomes

$$m\left(\frac{d^2x}{dt^2}dx+\frac{d^2y}{dt^2}dy+\frac{d^2z}{dt^2}dz\right)=Xdx+Ydy+Zdz. \qquad (29)$$

This is the equation of kinetic energy (Art. 71); for the left-hand member is the exact differential $d(\frac{1}{2}mv^2)$ of the kinetic energy, while the right-hand member represents the element of work of the impressed forces.

In the particular, but very common, case of *conservative* impressed forces, the right-hand member is likewise an exact differential dU; hence, in this case a first integration can at once be performed, and we find, as in Art. 75,

$$\tfrac{1}{2}mv^2-\tfrac{1}{2}mv_0^2=\int(Xdx+Ydy+Zdz)=U-U_0. \qquad (30)$$

103. There is an essential distinction between the principle of d'Alembert on the one hand, and the principles of kinetic energy and of areas on the other. D'Alembert's principle merely gives a convenient form and interpretation to the dynamical equations of motion, through the application of the principle of virtual work; but it does not show how to integrate these equations.

The principle of kinetic energy and the principle of areas are really methods for integrating the equations of motion under certain conditions. If we enquire why these particular methods of combining the differential equations so frequently furnish the solution of physical problems, we are led to the conclusion that the quantities whose exact differentials are introduced by the combination correspond to something really existing in nature. It is thus made probable on purely theoretical grounds that kinetic and potential energy are not mere abstractions, but have an objective reality, and that the conservation of energy is a law of nature.

2. CENTRAL FORCES.

104. We proceed to apply the general principles developed in the preceding articles to the motion of a particle under the action of **central forces.**

The term *central force* is generally understood to imply two conditions, viz. (*a*) that *the direction of the force passes constantly through a fixed point, usually called the centre of force;* and (*b*) that *the magnitude of the force is a function of the distance from the centre alone* (comp. Art. 56).

Let O be the centre of force, P the position of the moving particle at any time t, m the mass of the particle, and $OP=r$ its distance from the centre; then the general expression for a central force F is

$$F=F(r)=mf(r),$$

where the function $F(r)$ represents the law of force, and the function $f(r)$ the law of the acceleration produced by this force in the particle m.

105. The most important special case is that of a force proportional to some power of the distance r, say

$$F(r)=\mu r^n,$$

where μ and n are constants. The constant μ, which represents the value of the force at unit distance from the centre, is often called the *intensity* of the force, or of the centre.

In the case of Newton's law of universal gravitation (Part II., Art. 257) we have $n=-2$, $\mu=\kappa mm'$, where κ is a constant, viz. the acceleration produced by a unit of mass acting on a unit of mass at unit distance, while m is the mass of the attracted particle, and m' that of the attracting centre; that is, Newton's law is expressed by the formula

$$F=\kappa\frac{mm'}{r^2}.$$

106. From the physical point of view, attractions following Newton's law, and indeed, central forces generally, are usually regarded as due to the presence of mass, not only in the moving particle, but also at the centre of force; and the action between these two masses is then a mutual action, being of the nature of a *stress*, *i.e.* consisting of two equal and opposite forces. It follows that what we have called the centre of force is not a fixed point.

It will, however, be shown later (Arts. 150–157) that a simple modification allows us to apply to this case the results deduced on the assumption that the centre is fixed.

Again, the attracting or repelling masses will here be regarded as concentrated at points. It should be remembered that a homogeneous sphere, according to Newton's law, attracts a particle outside of its mass as if the whole mass of the sphere were concentrated at the centre of the sphere (Part II., Arts. 272–276). The attraction of any other mass on a particle can, of course, always be reduced to a single force; but as the particle moves, the direction of this force will not in general pass through a fixed point; such a force is, therefore, not central.

107. If a particle P of mass m be acted upon by a single central force

$$F = mf(r),$$

its acceleration $j = F/m = f(r)$ will pass through the centre of force and be a function of r alone. The problem reduces, therefore, at once to the kinematical problem of *central motion* (Part I., Art. 223). Although the leading ideas of the solution of this problem have been indicated in kinematics (Part I., Arts. 225–234, 237–238), the importance of the subject of central forces demands a restatement in this place of the principal methods in the language of kinetics, and a more complete exposition of some special cases.

108. A particle of mass m acted upon by a single central force $F = mf(r)$ will describe a curvilinear path whenever the initial velocity is different from zero and does not pass through the centre of force (see Art. 56). As shown in kinematics (Part I., Art. 225), the path of the particle, here usually called the **orbit**, is always a plane curve.

Taking the plane of motion as xy-plane and the centre O as origin (Fig. 16), the direction cosines of the force F are $\mp x/r$, $\mp y/r$, the upper sign corresponding to an attractive force, the lower to a repulsion. Hence, the dynamical equations of motion are

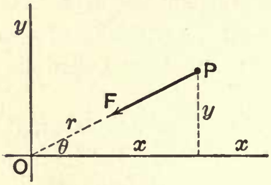

Fig. 16.

$$m\frac{d^2x}{dt^2} = \mp F\frac{x}{r}, \quad m\frac{d^2y}{dt^2} = \mp F\frac{y}{r}. \tag{1}$$

If $mf(r)$ be substituted for F, the factor m disappears, and the equations become purely kinematical.

109. To avoid the use of the double sign, we shall give the equations in the form corresponding to the more important case of attraction; for a repulsive force it will only be necessary to change throughout the sign of F or $f(r)$. Thus the fundamental equations of motion are (comp. Part I., Art. 226):

$$\frac{d^2x}{dt^2} = -f(r)\frac{x}{r}, \quad \frac{d^2y}{dt^2} = -f(r)\frac{y}{r}. \tag{2}$$

If polar co-ordinates r, θ (Fig. 16), with the centre as pole, be used, the equations of motion are, since the total acceleration is along the radius vector :

$$j_r \equiv \frac{d^2r}{dt^2} - r\left(\frac{d\theta}{dt}\right)^2 = -f(r), \quad j_\theta \equiv \frac{1}{r}\frac{d}{dt}\left(r^2\frac{d\theta}{dt}\right) = 0. \tag{3}$$

110. Two principal problems present themselves: (*a*) the problem of finding the orbit for a given law of force, and (*b*) the converse problem of determining the law of force, *i.e.* the function $f(r)$, when the orbit is given. The solution of the former problem is effected by obtaining first integrals of the equations of motion from the principle of areas and from the principle of kinetic energy, and by combining these integrals so as to effect a second integration. Formulæ for the solution of the latter problem will be found incidentally.

111. The second of the polar equations (3) gives immediately, if c denote the constant of integration:

$$r^2\frac{d\theta}{dt}=c; \tag{4}$$

and the meaning of this equation is that *the sectorial velocity is constant* and equal to $\frac{1}{2}c$. The same result can be obtained from the equations (2) by applying the principle of areas (see Arts. 87, 88).

To express the constant c in terms of the initial conditions, let v denote the velocity, p the perpendicular to it from the centre of force, and ψ the angle between the radius vector r and the velocity v, all at the time t (Fig. 17); and let the initial values of these quantities, at the time $t=0$, be distinguished by

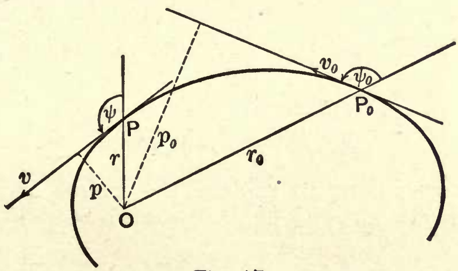

Fig. 17.

zero-subscripts. Then it follows from the equation (4) that we have (see Art. 89 and Part I., Art. 230)

$$c=pv=p_0v_0=vr\sin\psi=v_0r_0\sin\psi_0; \tag{5}$$

i.e. *the velocity is inversely proportional to its perpendicular distance from the centre*, or, as it is sometimes expressed, the moment of the velocity about the centre of force is constant.

112. Another first integral of the equations of motion is obtained by combining the equations (1) according to the principle of kinetic energy (Art. 71). This gives

$$d(\tfrac{1}{2}mv^2)=-Fdr, \text{ or } d(\tfrac{1}{2}v^2)=-f(r)dr, \tag{6}$$

whence
$$v^2 = v_0^2 - 2\int_{r_0}^{r} f(r)\,dr; \tag{7}$$

i.e. *the velocity at any distance* r *depends only on this distance* (besides the initial radius vector and velocity), and is independent of the path described, being the same as if the particle had been projected with the initial velocity along the straight line joining the initial position to the centre.

113. To perform the second integration we have only to substitute in (7) for v its value in terms of r and t or r and θ. Now the general expression for the velocity in any curvilinear motion is (Part I., Art. 142)

$$v^2 = \left(\frac{dr}{dt}\right)^2 + r^2\left(\frac{d\theta}{dt}\right)^2 = \left(\frac{d\theta}{dt}\right)^2\left[\left(\frac{dr}{d\theta}\right)^2 + r^2\right].$$

From these expressions one of the variables θ and t can be eliminated by substituting for $d\theta/dt$ its value c/r^2 from (4); this gives

$$v^2 = \left(\frac{dr}{dt}\right)^2 + \frac{c^2}{r^2} = \frac{c^2}{r^4}\left[\left(\frac{dr}{d\theta}\right)^2 + r^2\right]. \tag{8}$$

It is often convenient to replace the radius vector r by its reciprocal $u = 1/r$; we then have

$$v^2 = \frac{1}{u^4}\left(\frac{du}{dt}\right)^2 + c^2u^2 = c^2\left[\left(\frac{du}{d\theta}\right)^2 + u^2\right]. \tag{9}$$

114. The formulæ (4) and (7), together with the expressions (8) or (9), contain the complete solution of the two principal problems mentioned in Art. 110. Thus, if the law of force be given, the form of the function $f(r)$ is known, and v can be found from (7) in function of r or u; substituting this value of v in either (8) or (9), we have a differential equation of the first order between r and t, or between r and θ. The integration of the latter equation gives the integral equation of the orbit.

On the other hand, if the equation of the path be given, the expressions (8) or (9) furnish the value of v^2, which, substituted in (6), determines the law of force $f(r)$.

When the equation of the orbit is known, *i.e.* when r is known as a function of θ or *vice versa*, the time t of the motion can be found by integrating equation (4), viz.

$$dt = \frac{1}{c} r^2 d\theta.$$

115. If the second expression for v^2 in (9) be introduced into the differential equation of kinetic energy (6), we find

$$c^2\left(\frac{d^2u}{d\theta^2} + u\right)du = -f(r)dr,$$

or

$$f(r) = c^2u^2\left(\frac{d^2u}{d\theta^2} + u\right). \tag{10}$$

This will generally be found the most convenient form for finding the law of force when the polar equation of the orbit is given. Again, when $f(r)$ is given, the integration of this differential equation of the second order is often more convenient for finding the equation of the orbit than the method indicated in Art. 114.

It may be noted that the important relation (10) can be derived directly from the equations of motion (3), by eliminating t by means of (4) and introducing u for $1/r$. We have

$$\frac{dr}{dt} = \frac{dr}{d\theta}\frac{d\theta}{dt} = \frac{c}{r^2}\frac{dr}{d\theta} = -c\frac{du}{d\theta},$$

$$\frac{d^2r}{dt^2} = -c\frac{d^2u}{d\theta^2}\frac{d\theta}{dt} = -c^2u^2\frac{d^2u}{d\theta^2};$$

if these values be substituted into the first of the equations (3), the relation (10) will result.

116. When the equation of the orbit can be expressed conveniently in terms of r and p, as is, for instance, the case for the conic sections, it is of advantage to combine the equation of kinetic energy, $d(\frac{1}{2}v^2) = -f(r)dr$, directly with the equation resulting from the principle of areas, $pv = c$. This gives

$$-f(r) = \frac{1}{2}\frac{d(v^2)}{dr} = \frac{c^2}{2}\frac{d\left(\frac{1}{p^2}\right)}{dr} = -\frac{c^2}{p^3}\frac{dp}{dr}. \tag{11}$$

117. It is easy to see that the methods here explained would apply even to the more general problem *when the force* F, *while passing always through a fixed centre, is not a function of the distance* r *alone, but a function of both co-ordinates* r *and* θ. The principal difference will appear in the impossibility of performing directly the integration indicated in (7).

With $F=mf(r, \theta)$, the equation of kinetic energy is, for attraction,

$$d(\tfrac{1}{2}v^2) = -f(r, \theta)dr;$$

and substituting for v^2 the first of the values given in (8), we find

$$\frac{d^2r}{dt^2} = -f(r, \theta) + \frac{c^2}{r^3}. \tag{12}$$

This equation shows that the motion relative to the radius vector takes place as if the actual resulting force $F=mf(r, \theta)$ were increased by an additional force mc^2/r^3.

For the law of force we have, as in Art. 115:

$$f(r, \theta) = c^2u^2\left(\frac{d^2u}{d\theta^2} + u\right).$$

118. We proceed to the consideration of some special cases. The most important of these are the case of a force directly proportional to the distance, and that of a force inversely proportional to the square of the distance.

119. Force Proportional to the Distance: $f(r) = \kappa^2 r$. The equations of motions (2) are in this case

$$\frac{d^2x}{dt^2} = \mp\kappa^2 x, \quad \frac{d^2y}{dt^2} = \mp\kappa^2 y,$$

the upper sign holding for attraction, the lower for repulsion. Their solution is very simple, because each equation can be integrated separately. We find, in the case of *attraction*,

$$x = a_1 \cos\kappa t + a_2 \sin\kappa t, \quad y = b_1 \cos\kappa t + b_2 \sin\kappa t,$$

and in the case of *repulsion*,

$$x = a_1 e^{\kappa t} + a_2 e^{-\kappa t}, \quad y = b_1 e^{\kappa t} + b_2 e^{-\kappa t};$$

a_1, a_2, b_1, b_2 being the constants of integration.

120. To find the equation of the orbit, it is only necessary to eliminate t in each case.

In the case of attraction, this elimination can be performed by solving for $\cos \kappa t$, $\sin \kappa t$, squaring and adding. The result is

$$(a_1 y - b_1 x)^2 + (a_2 y - b_2 x)^2 = (a_1 b_2 - a_2 b_1)^2,$$

and this represents an ellipse, since

$$(a_1^2 + a_2^2)(b_1^2 + b_2^2) - (a_1 b_1 + a_2 b_2)^2 = (a_1 b_2 - a_2 b_1)^2$$

is always positive. The centre of the ellipse is at the origin, and the lines $a_1 y = b_1 x$, $a_2 y = b_2 x$ are a pair of conjugate diameters.

121. In the case of repulsion, solve for $e^{\kappa t}$ and $e^{-\kappa t}$, and multiply. The resulting equation,

$$(a_1 y - b_1 x)(b_2 x - a_2 y) = (a_1 b_2 - a_2 b_1)^2,$$

represents a hyperbola whose asymptotes are the lines $a_1 y = b_1 x$, $a_2 y = b_2 x$.

122. It is worthy of notice that the more general problem of *the motion of a particle attracted by any number of fixed centres, with forces directly proportional to the distances from these centres*, can be reduced to the problem of Art. 119.

Let x, y, z be the co-ordinates of the particle, r_i its distance from the centre O_i; x_i, y_i, z_i the co-ordinates of O_i; and $-\kappa_i^2 r_i$ the acceleration produced by O_i. Then the x-component of the resultant acceleration is

$$= - \Sigma \kappa_i^2 r_i \cdot \frac{x - x_i}{r_i} = - \Sigma \kappa_i^2 (x - x_i) = - x \Sigma \kappa_i^2 + \Sigma \kappa_i^2 x_i;$$

and similar expressions obtain for the y and z components. Hence, the equations of motion are

$$\frac{d^2x}{dt^2} = - x \Sigma \kappa_i^2 + \Sigma \kappa_i^2 x_i, \quad \frac{d^2y}{dt^2} = - y \Sigma \kappa_i^2 + \Sigma \kappa_i^2 y_i, \quad \frac{d^2z}{dt^2} = - z \Sigma \kappa_i^2 + \Sigma \kappa_i^2 z_i.$$

As these expressions are linear in x, y, z, there is one, and only one, point at which the resultant acceleration is zero. Denoting its co-ordinates by $\bar{x}, \bar{y}, \bar{z}$, we have

$$\bar{x} = \frac{\Sigma \kappa_i^2 x_i}{\Sigma \kappa_i^2}, \quad \bar{y} = \frac{\Sigma \kappa_i^2 y}{\Sigma \kappa_i^2}, \quad \bar{z} = \frac{\Sigma \kappa_i^2 z_i}{\Sigma \kappa_i^2}.$$

The form of these equations shows that this point of zero acceleration, which is sometimes called the *mean centre*, is the centroid of the centres of force, if these centres be regarded as containing masses equal to κ_i^2. It is evidently a fixed point.

123. By introducing the co-ordinates of the mean centre, we can now reduce the equations of motion to the simple form

$$\frac{d^2x}{dt^2} = -\kappa^2(x-\bar{x}),\quad \frac{d^2y}{dt^2} = -\kappa^2(y-\bar{y}),\quad \frac{d^2z}{dt^2} = -\kappa^2(z-\bar{z}),$$

where $\kappa^2 = \Sigma\kappa_i^2$. Finally, taking the mean centre as origin, we have

$$\frac{d^2x}{dt^2} = -\kappa^2 x,\quad \frac{d^2y}{dt^2} = -\kappa^2 y,\quad \frac{d^2z}{dt^2} = -\kappa^2 z.$$

It thus appears that *the motion of the particle is the same as if there were only a single centre of force*, viz. the mean centre $(\bar{x}, \bar{y}, \bar{z})$, *attracting with a force proportional to the distance from this centre.*

The plane of the orbit is, of course, determined by the mean centre and the initial velocity. The equation of this plane can be found by applying the principle of areas (Art. 94).

124. It is easy to see that most of the considerations of Art. 122 apply even when some or all of the centres *repel* the particle with forces proportional to the distance. It may, however, happen in this case that the mean centre lies at infinity, in which case it can, of course, not be taken as origin.

Simple geometrical considerations can also be used to solve the problem. Thus, in the case of two attractive centres O_1, O_2 (Fig. 18) of equal intensity κ^2, the forces can evidently be represented by the distances $PO_1 = r_1$, $PO_2 = r_2$ of the particle P from the centres. Their resultant is therefore $= 2\,PO$, if O denotes the point midway between O_1 and O_2; and this resultant always passes through this fixed point O, and is proportional to the distance PO from this point.

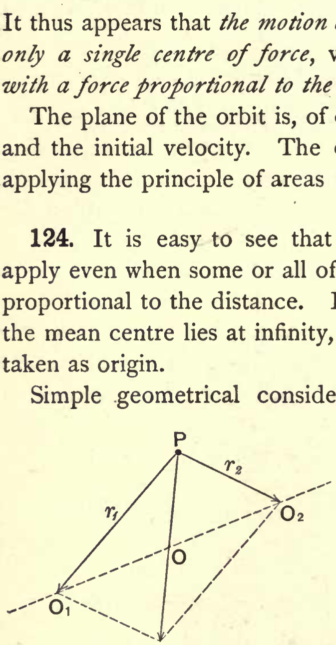

Fig. 18.

125. Exercises.

(1) Determine the constants of integration in Art. 119, if x_0, y_0 are the co-ordinates of the particle at the time $t = 0$ and $\dot{x}_0$, $\dot{y}_0$ the com-

ponents of its velocity v_0 at the same time. The equation of the orbit will assume the form

$$\kappa^2(x_0 y - y_0 x)^2 + (\dot{x}_0 y - \dot{y}_0 x)^2 = (x_0 \dot{y}_0 - y_0 \dot{x}_0)^2$$

for attraction, and

$$\kappa^2(x_0 y - y_0 x)^2 - (\dot{x}_0 y - \dot{y}_0 x)^2 = -(x_0 \dot{y}_0 - y_0 \dot{x}_0)^2$$

for repulsion.

(2) Show that the semi-diameter conjugate to the initial radius vector has the length v_0/κ, where $v_0^2 = \dot{x}_0^2 + \dot{y}_0^2$. As any point of the orbit can be regarded as initial point, it follows that *the velocity at any point is proportional to the parallel diameter of the orbit.*

(3) Find what the initial velocity must be to make the orbit a circle in the case of attraction, and an equilateral hyperbola in the case of repulsion.

(4) The initial radius vector r_0 and the initial velocity v_0 being given geometrically, show how to construct the axes of the orbit described under the action of a central force (of given intensity κ^2) proportional to the distance from the origin.

(5) A particle describes an ellipse under the action of a central force proportional to the distance; show that the eccentric angle is proportional to the time, and find the corresponding relation for a hyperbolic orbit.

(6) A particle of mass m describes a conic under the action of a central force $F = \mp m\kappa^2 r$. Show that the sectorial velocity is $\frac{1}{2}c = \frac{1}{2}\kappa ab$, a and b being the semi-axes of the conic.

(7) In Ex. (6) show that the time of revolution is $T = 2\pi/\kappa$, if the conic is an ellipse.

(8) A particle describes a conic under the action of a force whose direction passes through the centre of the conic. Show that the force is proportional to the distance from the centre.

(9) A particle is acted upon by two central forces of the same intensity (κ^2), each proportional to the distance from a fixed centre. Determine the orbit: (a) when both forces are attractive; (b) when both are repulsive; (c) when one is an attraction, the other a repulsion.

(10) A particle of mass m is attracted by two centres O_1, O_2 of equal mass m' and repelled by a third centre O_3, whose mass is $m'' = 2m'$. If the forces are all directly proportional to the respective distances, determine and construct the orbit.

(11) When a particle moves in an ellipse under a force directed towards the centre, find the time of moving from the end of the major axis to a point whose polar angle is θ.

126. Force Inversely Proportional to the Square of the Distance: $f(r) = \mu/r^2$ (Newton's law).

It has been shown in kinematics (Part I., Arts. 229–236) how this law of acceleration can be deduced from Kepler's laws of planetary motion. From Kepler's first law Newton concluded that the acceleration of a planet (regarded as a point of mass m) is constantly directed towards the sun; from the second he found that this acceleration is inversely proportional to the square of the distance. The motion of a planet can therefore be explained on the hypothesis of an attractive force,

$$F = m\frac{\mu}{r^2},$$

issuing from the sun.

The value of μ, which represents the acceleration at unit distance or the so-called intensity of the force, was found to be (Part I., Art. 236; or below, Art. 139)

$$\mu = 4\pi^2\frac{a^3}{T^2};$$

and as, according to Kepler's third law, the quantity a^3/T^2 has the same value for all the planets, Newton inferred that the intensity of the attracting force is the same for all planets; in other words, that it is one and the same central force that keeps the different planets in their orbits.

127. It was further shown by Newton and Halley that the motions of the comets are due to the same attractive force. The orbits of the comets are generally ellipses of great eccentricity, with the sun at one of the foci. As a comet is within range of observation only while in that portion of its path which lies nearest to the sun, a portion of a parabola, with the same focus and vertex, can be substituted for this portion of the elliptic orbit, at least as a first approximation.

It is also found from observation that the motions of the moons or satellites around the planets follow very nearly Kepler's law. A planet can therefore be regarded as attracting each of its satellites with a force proportional to the mass of the satellite and inversely proportional to the distance.

128. All these facts led Newton to suspect that the force of terrestrial gravitation, as observed in the case of falling bodies on the earth's surface, might be the same as the force that keeps the moon in its orbit around the earth. This inference could easily be tested, since the acceleration g of falling bodies as well as the moon's distance and time of revolution were known.

Let m be the mass of the moon, a the major semi-axis of its orbit, T the time of revolution, r the distance between the centres of earth and moon; then the earth's attraction on the moon is (Art. 126)

$$F = 4\pi^2 m \frac{a^3}{T^2 r^2},$$

or, since the eccentricity of the moon's orbit is so small that the orbit can be regarded as nearly circular,

$$F = 4\pi^2 m \frac{a}{T^2}.$$

On the other hand, the attraction exerted by the earth on a mass m on its surface, *i.e.* at the distance $R = 3963$ miles from the centre, must be

$$F' = mg.$$

Now, if these forces are actually in the inverse ratio of the squares of the distances, we must have

$$\frac{F'}{F} = \frac{a^2}{R^2},$$

or, since the distance of the moon is nearly $= 60 R$,

$$F' = 60^2 F.$$

Substituting the above values of F and F', we find

$$g = 4\pi^2 \times 60^3 \times \frac{R}{T^2}.$$

With $R = 3963$ miles, $T = 27^d\ 7^h\ 43^m$, this gives

$$g = 32 \cdot 0,$$

a value which agrees sufficiently with the observed value of g, considering the rough degree of approximation used.

129. In this way Newton was finally led to his **law of universal gravitation**, which asserts that *every particle of mass* m *attracts every other particle of mass* m′ *with a force*

$$F = \kappa \frac{mm'}{r^2},$$

where r is the distance of the particles and κ a constant, viz. the acceleration produced by a unit of mass in a unit of mass at unit distance (see Part II., Art. 257, 261–262).

The best proof of this hypothesis as an actual law of physical nature is found in the close agreement of the results of theoretical astronomy based on this law with the observed celestial phenomena.

It may be noticed that, according to this law, the path of a projectile *in vacuo* is only approximately parabolic, the actual path being a very elongated ellipse or hyperbola, one of whose foci is at the earth's centre.

130. Taking Newton's law as a basis, let us now turn to the converse problem of *determining the motion of a particle acted upon by a single central force for which* $f(r) = \mu / r^2$ (problem of planetary motion).

It has been shown in kinematics (Part I., Arts. 239–242) that *if the force be attractive*, the particle will describe a conic section with one of the foci at the centre of force, the conic being an ellipse, parabola, or hyperbola, according as

$$v_0^2 \gtreqless \frac{2\mu}{r_0}. \tag{13}$$

If the force be repulsive, the same reasoning will apply, except that μ is then a negative quantity. The orbit is, therefore, in this case always hyperbolic; the branch of the hyperbola that forms the orbit must evidently turn its convex side towards the focus at which the centre of force is situated, since the force always lies on the concave side of the path.

131. To exhibit fully the determination of the constants and the dependence of the nature of the orbit on the initial conditions, a solution somewhat different from that given in kinematics will here be given for the problem of planetary motion in its simplest form.

With $f(r)=\mu/r^2$, the equation of kinetic energy, (7), Art. 112, gives

$$v^2=v_0^2-2\mu\int_{r_0}^{r}\frac{dr}{r^2}=v_0^2+2\frac{\mu}{r}-\frac{2\mu}{r_0},$$

or, if the constant of integration be denoted briefly by h and $u=1/r$ be introduced:

$$v^2=2\mu u+h, \text{ where } h=v_0^2-\frac{2\mu}{r_0}. \tag{14}$$

Substituting this expression of v^2 into the equation (9), Art. 113, we find the differential equation of the orbit in the form

$$\left(\frac{du}{d\theta}\right)^2+u^2=\frac{1}{c^2}(2\mu u+h), \tag{15}$$

or

$$\left(\frac{du}{d\theta}\right)^2=-\left(u-\frac{\mu}{c^2}\right)^2+\frac{\mu^2}{c^4}+\frac{h}{c^2}.$$

To integrate, we introduce a new variable u' by putting

$$u-\frac{\mu}{c^2}=u'\sqrt{\frac{\mu^2}{c^4}+\frac{h}{c^2}};$$

the resulting equation,

$$\left(\frac{du'}{d\theta}\right)^2=1-u'^2, \text{ or } d\theta=\pm\frac{du'}{\sqrt{1-u'^2}},$$

has the general integral

$$\theta - \alpha = \mp \cos^{-1} u', \text{ or } u' = \cos(\theta - \alpha),$$

where α is the constant of integration. The orbit has, therefore, the equation

$$\frac{1}{r} = \frac{\mu}{c^2} + \sqrt{\frac{\mu^2}{c^4} + \frac{h}{c^2}} \cos(\theta - \alpha), \quad (16)$$

which agrees in form with the equation (74) given in kinematics (Part I., Art. 242), excepting the different notation used for the constants.

132. The equation (16) represents a conic section referred to its focus as origin. The general focal equation of a conic is

$$\frac{1}{r} = \frac{1}{l} + \frac{e}{l} \cos(\theta - \alpha), \quad (17)$$

where l is the semi-latus rectum, or parameter, e the eccentricity, and α the angle made with the polar axis by the line joining the focus to the nearest vertex.

In a planetary orbit (Fig. 19), the sun S being at one of the foci, the nearest vertex A is called the *perihelion*, the other vertex A' the *aphelion*, and the angle $\theta - \alpha$ made by any radius vector $SP = r$ with the perihelion distance SA is called the *true anomaly*.

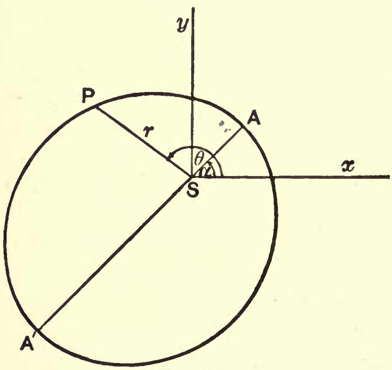

Fig. 19.

Comparing equations (17) and (16), we find, for the determination of the constants:

$$\frac{1}{l} = \frac{\mu}{c^2}, \quad \frac{e}{l} = \sqrt{\frac{\mu^2}{c^4} + \frac{h}{c^2}};$$

hence,

$$l = \frac{c^2}{\mu}, \quad e = \sqrt{1 + \frac{hc^2}{\mu^2}}, \quad (18)$$

or, solving for c and h,

$$c = \sqrt{\mu l}, \quad h = \mu \frac{e^2 - 1}{l}. \quad (19)$$

133. The expression for the eccentricity e in (18) determines the nature of the conic; the orbit is an ellipse, parabola, or hyperbola, according as $e \lesseqgtr 1$; hence, by (18), according as the constant h of the equation of kinetic energy is negative, zero, or positive. Owing to the value of h given in (14), this criterion agrees with the form (13), Art. 130.

It should be observed that it follows from (13) that *the nature* of the conic is independent of the *direction* of the initial velocity.

134. The criterion (13) can be given the following interpretation. Consider a particle attracted by a fixed centre according to Newton's law. If it move in a straight line passing through the centre, the principle of kinetic energy gives for its velocity, at the distance r,

$$v^2 = v_0^2 - 2\mu \int_{r_0}^{r} \frac{dr}{r^2} = \frac{2\mu}{r} + v_0^2 - \frac{2\mu}{r_0};$$

hence, if it start from rest at an infinite distance from the centre, it would acquire the velocity $\sqrt{2\mu/r}$ at the distance r. The criterion (13) is therefore equivalent to saying that *the orbit is an ellipse, a parabola, or a hyperbola, according as the velocity at any point is less than, equal to, or greater than the velocity which the particle would have acquired at that point by falling towards the centre from infinity* (comp. Art. 57).

135. For a *central conic*, whose axes are $2a$, $2b$, we have $l = b^2/a$, $e = \sqrt{a^2 \mp b^2}/a$ (the upper sign relating to the ellipse, the lower to the hyperbola), so that the equations (19) reduce to the following:

$$c = b\sqrt{\frac{\mu}{a}}, \quad h = \mp\frac{\mu}{a}. \tag{20}$$

The latter relation, with the value of h from (14), gives for the major or real semi-axis a:

$$\pm\frac{1}{a} = \frac{2}{r_0} - \frac{v_0^2}{\mu}; \tag{21}$$

while the former, with the value of c as given in (5), Art. 111, determines the minor or transverse axis b:

$$b = c\sqrt{\frac{a}{\mu}} = r_0 v_0 \sin\psi_0 \sqrt{\frac{a}{\mu}}. \tag{22}$$

136. The magnitudes of the axes having thus been found, their directions can be determined by a simple construction which furnishes the second focus.

In the *ellipse*, the focal radii have a constant sum $= 2a$, and lie on the same side of the tangent, making equal angles with it. In the *hyperbola*, they have a constant difference $= 2a$, and lie on opposite sides of the tangent.

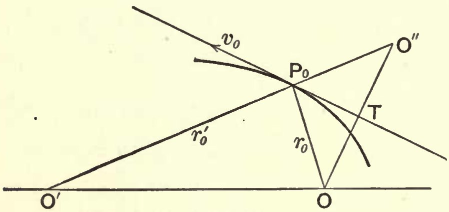

Fig. 20.

Hence, determining the point O'' (Fig. 20), which is symmetrical to the centre of force O with respect to the initial velocity, and drawing the line P_0O'', we have only to lay off on this line from P_0 a length $P_0O' = \pm (2a - r_0)$; then O' is the second focus, which for an elliptic orbit must be taken with O on the same side of the tangent PT, and for a hyperbolic orbit on the opposite side.

137. For a *parabola*, since $e = 1$, we find, from (19),

$$h = 0, \quad l = \frac{c^2}{\mu} = \frac{v_0^2 r_0^2 \sin^2\psi_0}{\mu}. \tag{23}$$

The axis of the parabola is readily found by remembering that the perpendicular let fall from the focus on the tangent bisects the tangent (*i.e.* the segment of the tangent between the point of contact and the axis). Hence, if OT (Fig. 21) be the perpendicular let fall from the centre O on the velocity v_0, it is only necessary to make $TT' = P_0T$, and T' will be a point of the axis. Moreover, the perpendicular let fall from T on OT' will meet the axis at the vertex A of the parabola, so that $OA = \frac{1}{2} l$.

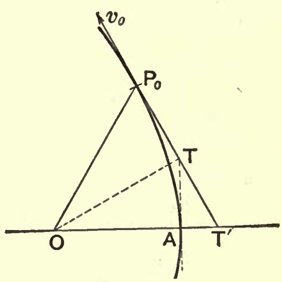

Fig. 21.

138. The relation (21), which must evidently hold at *any* point of the orbit, can be written in the form

$$v^2 = 2\mu\left(\frac{1}{r} \mp \frac{1}{2a}\right), \tag{24}$$

the upper sign relating to the ellipse, the lower to the hyperbola, while for the parabola, the second term in the parenthesis vanishes (since $a = \infty$).

This convenient expression for the velocity in terms of the radius vector might have been derived directly from the fundamental relation (5), $v = c/p$, the first of the equations (19), $c^2 = \mu l$, and the geometrical properties of the conic sections ($r \pm r' = 2a$, $pp' = b^2$, $p'r = pr'$, where r, r' are the focal radii, and p, p' the perpendiculars let fall from the foci on the tangent). The proof is left to the student.

139. Time. In the case of an elliptic orbit, the time T of a complete revolution, usually called the **periodic time**, is found by remembering that the sectorial velocity is constant and $= \frac{1}{2}c$ (Art. 111), whence

$$T = \frac{2\pi ab}{c},$$

or, by (20),

$$T = 2\pi\sqrt{\frac{a^3}{\mu}} = \frac{2\pi}{n}. \tag{25}$$

The constant

$$n = \sqrt{\frac{\mu}{a^3}},$$

which evidently represents the mean angular velocity in one revolution, is called the **mean motion** of the planet. It should be noticed that it depends not only on the intensity of the force, but also on the major axis of the orbit, while in the case of a force directly proportional to the distance it is independent of the size of the orbit (see Art. 125, Ex. 7).

The periodic time T and the major axis a of a planetary orbit determine the intensity μ of the force

$$\mu = 4\pi^2\frac{a^3}{T^2}, \tag{26}$$

whence

$$F = mf(r) = m\frac{\mu}{r^2} = 4\pi^2 m\frac{a^3}{T^2 r^2}, \tag{27}$$

where m is the mass of the planet.

140. To find generally the time t in terms of θ or r, we can, of course, proceed as indicated in Art. 114; but the resulting expressions are somewhat complicated, and it is best to introduce the eccentric angle ϕ of the ellipse as a new variable, and to express t, r, and θ in terms of ϕ. In astronomy, the polar angle θ is known as the *true anomaly*, and the eccentric angle ϕ as the *eccentric anomaly*.

141. The relation of the eccentric angle ϕ to the polar co-ordinates r, θ will appear from Fig. 22, in which P is the position of the planet

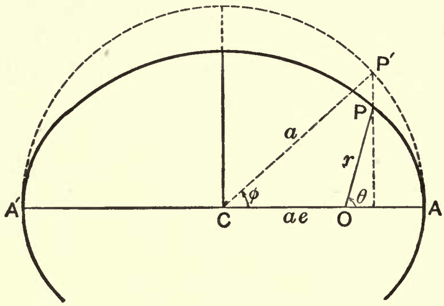

Fig. 22.

at the time t, P' the corresponding point on the circumscribed circle, $\angle AOP = \theta$ the true anomaly, and $\angle ACP' = \phi$ the eccentric anomaly. The focal equation of the ellipse

$$r = \frac{l}{1 + e\cos\theta} = \frac{a(1-e^2)}{1+e\cos\theta}$$

gives $r + er\cos\theta = a - ae^2$; and the figure shows that $r\cos\theta = a\cos\phi - ae$; hence,

$$r = a(1 - e\cos\phi), \text{ or } a - r = ae\cos\phi. \qquad (28)$$

Equating this value of r to that given by the polar equation of the ellipse, we have

$$1 - e\cos\phi = \frac{1-e^2}{1+e\cos\theta}, \quad \text{or} \quad \cos\theta = \frac{\cos\phi - e}{1 - e\cos\phi}.$$

A more symmetrical form can be given to this relation by computing

$$1 - \cos\theta \equiv 2\sin^2\frac{\theta}{2} = (1+e)\frac{1-\cos\phi}{1-e\cos\phi},$$

$$1 + \cos\theta \equiv 2\cos^2\frac{\theta}{2} = (1-e)\frac{1+\cos\phi}{1-e\cos\phi};$$

whence, by division, $$\tan\frac{\theta}{2} = \sqrt{\frac{1+e}{1-e}}\tan\frac{\phi}{2}. \qquad (29)$$

142. To find t in terms of r, we have only to substitute in (24) for v^2 its value from (8), Art. 113, and to integrate the resulting differential equation

$$\left(\frac{dr}{dt}\right)^2 + \frac{c^2}{r^2} = \frac{2\mu}{r} - \frac{\mu}{a}.$$

As, by (20), Art. 135, $c^2 = \mu b^2/a = \mu a(1-e^2)$, this equation becomes

$$r^2\left(\frac{dr}{dt}\right)^2 = \frac{\mu}{a}\left[a^2e^2-(a-r)^2\right],$$

or
$$dt = \sqrt{\frac{a}{\mu}}\frac{rdr}{\sqrt{a^2e^2-(a-r)^2}}.$$

The integration is easily performed by introducing the eccentric angle ϕ as variable by means of (28); this gives

$$dt = \sqrt{\frac{a}{\mu}}\cdot a(1-e\cos\phi)d\phi.$$

If the time be counted from the perihelion passage of the planet, we have $t=0$ when $r=a-ae$, *i.e.* when $\phi=0$; hence, putting $\sqrt{\mu/a^3}=n$, as in Art. 139, we find

$$nt = \phi - e\sin\phi. \qquad (30)$$

This relation is known as *Kepler's equation;* the quantity nt is called the *mean anomaly*.

143. Kepler's equation (30) can be derived directly by considering that the ellipse APA' (Fig. 22) can be regarded as the projection of the circle $AP'A'$, after turning this circle about AA' through an angle $=\cos^{-1}(b/a)$. For it follows that the elliptic sector AOP is to the circular sector AOP' as b is to a. Now, for the circular sector we have

$$AOP' = ACP' - OCP' = \tfrac{1}{2}a^2\phi - \tfrac{1}{2}ae\cdot a\sin\phi = \frac{a^2}{2}(\phi - e\sin\phi);$$

hence, the elliptic sector described in the time t is

$$AOP = \frac{b}{a}\cdot AOP' = \frac{ab}{2}(\phi - e\sin\phi).$$

The sectorial velocity being constant by Kepler's first law, we have

$$\frac{AOP}{t} = \frac{\pi ab}{T};$$

hence,
$$t = \frac{T}{2\pi}(\phi - e\sin\phi),$$

and this agrees with (30) since, by (25), $2\pi/T = n$.

144. Kepler's equation (30) gives the time as a function of ϕ; by means of (28), it establishes the relation between t and r; by means of (29), it connects t with θ. It is, however, a transcendental equation and cannot be solved for ϕ in a finite form.

For orbits with a small eccentricity e, an approximate solution can be obtained by writing the equation in the form

$$\phi = nt + e \sin \phi,$$

and substituting under the sine for ϕ its approximate value nt:

$$\phi = nt + e \sin nt. \tag{31}$$

This amounts to neglecting terms containing powers of e above the first power.

Substituting this value of ϕ in (28), we have with the same approximation

$$r = a(1 - e \cos nt). \tag{32}$$

To find θ in terms of t, we have, from the equation of the ellipse, $r = a(1 - e^2)(1 + e \cos \theta)^{-1} = a(1 - e \cos \theta)$, neglecting again terms in e^2; hence, $r^2 = a^2(1 - 2e \cos \theta)$. Substituting this value in the equation of areas, $r^2 d\theta = c dt = \sqrt{\mu a(1 - e^2)}\, dt$, we find

$$(1 - 2e \cos \theta) d\theta = \sqrt{\frac{\mu}{a^3}} dt = n dt;$$

whence, by integration, since $\theta = 0$ for $t = 0$,

$$\theta - 2e \sin \theta = nt,$$

or finally,

$$\theta = nt + 2e \sin nt. \tag{33}$$

Thus we have in (31), (32), (33) approximate expressions for ϕ, r, and θ directly in terms of the time. The quantity $2e \sin nt$, by which the true anomaly θ exceeds the mean anomaly nt, is called the *equation of the centre*.

145. Exercises.

(1) A particle describes an ellipse under the action of a central force. Determine the law of force by means of formula (11), Art. 116: (*a*) when the centre of force is at the centre of the ellipse; (*b*) when it is at a focus.

(2) A particle is attracted by a fixed centre according to Newton's law. What must be the initial velocity if the orbit is to be circular?

(3) A number of particles are projected, from the same point in the field of a force following Newton's law, with the same velocity, but in different directions. Show that the periodic times are the same for all the particles.

(4) The mean distance of Mars from the sun being 1.5237 times that of the earth, what is the time of revolution of Mars about the sun?

(5) A particle describes a conic under the action of a central force following Newton's law; if the intensity μ of the force be suddenly changed to μ', what is the effect on the orbit?

(6) In Ex. (5), if the original orbit was a parabola and the intensity be doubled, what is the new orbit?

(7) Regarding the moon's orbit about the earth as circular, what would it become: (*a*) if the earth's mass were suddenly doubled? (*b*) if it were reduced to one-half?

(8) In Ex. (5), determine the effect on the major semi-axis (or "mean distance") a and on the periodic time T, of a *small* change in the intensity μ of the force.

(9) If the mass of the sun be suddenly increased by a small amount while the earth is at the end of the minor axis of its orbit, what would be the effect on the earth's mean distance and on the period of revolution T?

(10) Find the equation of the hodograph of planetary motion, derive from it the expression for the velocity in terms of the radius vector, and show that the velocity is a maximum in perihelion and a minimum in aphelion.

(11) Show that the greatest velocity of a planet in its orbit about the sun is to its least velocity as $\sqrt{1+e}$ is to $\sqrt{1-e}$; and find this ratio for the earth, whose orbit has the eccentricity $e = 0.01677120$.

(12) Find the time exactly as a function of θ, for a parabolic orbit.

146. Force any Function of the Distance. The general methods have been given in Arts. 108–116. The equation of energy, (6), Art. 112, gives, with $u = 1/r$,

$$v^2 = 2\int \frac{f(u)\,du}{u^2} + h; \tag{34}$$

hence, substituting for $\dot{v}$ its value from (9), Art. 113, we find, for the differential equation of the orbit,

$$\left(\frac{du}{d\theta}\right)^2+u^2=\frac{2}{c^2}\int\frac{f(u)du}{u^2}+h'. \qquad (35)$$

As it is often difficult or impossible to perform the integration in finite form, it is of importance to determine the apses and apsidal distances of the orbit.

147. An **apse** is a point of the orbit at which the velocity is at right angles to the radius vector drawn from the centre of force; the length of the radius vector of an apse is called the **apsidal distance**, and its direction an **apsidal line**.

The importance of the apsidal lines lies in the fact that they are lines of symmetry of the orbit, while the apsidal distances are maximum or minimum values of the radius vector. This will be seen from the following considerations.

By the above formula (34) the velocity is a function of the radius vector alone; and by (5), Art. 111, since $\sin\psi=c/vr$, the angle ψ between radius vector and velocity is also a function of the radius vector alone. It follows that, if the velocity be reversed in direction at any point of the orbit, the same orbit will be described in the opposite sense; and as at an apse the velocity is perpendicular to the apsidal line, the two portions of the orbit on opposite sides of an apsidal line must be symmetrical with respect to this line.

148. The condition for an apse is therefore

$$\frac{du}{d\theta}=0.$$

Substituting this value in the above equation (35), the apsidal distances $1/u$ can be found by solving the equation for u. The value of $du/d\theta$ should also change sign as the particle passes through the apse.

If the law of force is given as a single-valued function $f(u)$, there can exist only two different apsidal distances (although there may be any number of apses). The angle between these two different apsidal distances is called the **apsidal angle.**

149. Exercises.

(1) Find the law of force when the equation of the orbit is $r^n = a\cos n\theta + b$, and investigate the particular cases $n=-1$, $n=-2$, $n=1$, $n=2$.

(2) A particle moves in a circle under the action of a single force of constant direction; determine the law of force and discuss the motion.

(3) Find the law of the central force directed to the origin under whose action a particle will describe the following curves: (a) the spiral of Archimedes $r = a\theta$; (b) the hyperbolic spiral $\theta r = a$; (c) the logarithmic or equiangular spiral $r = ae^{\alpha\theta}$; (d) the curve $r = a\cos n\theta$.

(4) A particle moves in a circle under the action of a central force directed towards a point on the circumference; find the law of force.

(5) A particle is acted upon by a force perpendicular to a given plane and inversely proportional to the cube of the distance from the plane. Determine its motion.

(6) A particle moves in a semi-ellipse under the action of a force perpendicular to the axis joining the ends of the semi-ellipse. Determine the law of force and the velocity at the ends.

3. THE PROBLEM OF TWO BODIES.

150. In the preceding discussion of the motion of a particle under the action of a central force, it has been assumed that the centre of force is fixed. In the applications of the theory of central forces this assumption is in general not satisfied. Thus, in considering the motion of a planet around the sun, the force of attraction is, according to Newton's law of universal gravitation (Art. 129), regarded as due to the presence of a mass M at the centre (sun), and of a mass m at the attracted point (planet); and the action between these two masses is a mutual action, being of the nature of a *stress*, *i.e.* consisting of two equal and opposite forces, each equal to

$$F = \kappa \frac{mM}{r^2}.$$

Hence, the mass m of the planet attracts the mass M of the sun with precisely the same force with which the mass M of the sun attracts the mass m of the planet. The attraction affects, therefore, the motions of both bodies.

151. The *accelerations* produced by the two forces are, of course, not equal. Indeed, the acceleration $F/m = \kappa M/r^2$, produced in the planet by the sun, is very much greater than the acceleration $F/M = \kappa m/r^2$, produced by the planet in the sun; for the mass of even the largest planet (Jupiter) is less than one thousandth of that of the sun. The assumption of a fixed centre can therefore be regarded as a first approximation in the problem of the motion of a planet about the sun.

In the case of the earth and moon, the difference of the masses is not so great, the mass of the moon being nearly one eightieth of that of the earth.

It can, however, be shown that the results deduced on the assumption of a fixed centre can, by a simple modification, be made available for the solution of *the general problem of the motions of two particles of masses*, m, M, *subject to no forces besides their mutual attraction.* In astronomy, this is called the **problem of two bodies.** In the solution below we assume the attraction to follow Newton's law of the inverse square of the distance. It will be convenient to speak of the two particles, or bodies, as planet (m) and sun (M).

152. With regard to any fixed system of rectangular axes, let x, y, z be the co-ordinates of the planet (m), at the time t; x', y', z' those of the sun (M), at the same time; so that for their distance r we have

$$r^2 = (x - x')^2 + (y - y')^2 + (z - z')^2.$$

Then the equations of motion of the planet are

$$\begin{aligned} m\frac{d^2x}{dt^2} &= \kappa\frac{Mm}{r^2}\cdot\frac{x'-x}{r}, \\ m\frac{d^2y}{dt^2} &= \kappa\frac{Mm}{r^2}\cdot\frac{y'-y}{r}, \\ m\frac{d^2z}{dt^2} &= \kappa\frac{Mm}{r^2}\cdot\frac{z-z}{r}; \end{aligned} \qquad (1)$$

while the equations of motion of the sun are

$$\begin{aligned} M\frac{d^2x'}{dt^2} &= \kappa\frac{mM}{r^2}\cdot\frac{x-x'}{r}, \\ M\frac{d^2y'}{dt^2} &= \kappa\frac{mM}{r^2}\cdot\frac{y-y'}{r}, \\ M\frac{d^2z'}{dt^2} &= \kappa\frac{mM}{r^2}\cdot\frac{z-z'}{r}. \end{aligned} \qquad (2)$$

153. By adding the corresponding equations of the two sets, we find

$$\frac{d^2}{dt^2}(mx+Mx')=0, \quad \frac{d^2}{dt^2}(my+My')=0, \quad \frac{d^2}{dt^2}(mz+Mz')=0.$$

If it be remembered that the centroid of the two masses m, M has the co-ordinates

$$\bar{x}=\frac{mx+Mx'}{m+M}, \quad \bar{y}=\frac{my+My'}{m+M}, \quad \bar{z}=\frac{mz+Mz'}{m+M},$$

it appears that these equations can be written in the form

$$\frac{d^2\bar{x}}{dt^2}=0, \quad \frac{d^2\bar{y}}{dt^2}=0, \quad \frac{d^2\bar{z}}{dt^2}=0;$$

in words: *the acceleration of the common centroid of planet and sun is zero;* i.e. *this centroid moves with constant velocity in a straight line.*

It may be noticed that this result is merely a special case of a more general proposition to be proved hereafter, viz. that the centroid of any system acted upon by no forces external to the system moves uniformly in a straight line (Art. 381).

154. The integration of the equations (1) would give the absolute path of the planet. But the constants could not be determined, because the absolute initial position and velocity of the planet are, of course, not known. The same holds for the absolute path of the sun. All we can do is to determine the *relative* motion, and we proceed to find the motion of the planet relative to the sun.

Taking the sun's centre as new origin for parallel axes, we have for the co-ordinates ξ, η, ζ of the planet in this new system,

$$\xi = x - x', \quad \eta = y - y', \quad \zeta = z - z'.$$

Now, dividing the equations (1) by m, the equations (2) by M, and subtracting the equations of set (2) from the corresponding equations of set (1), we find for the relative accelerations of the planet

$$\frac{d^2\xi}{dt^2} = -\kappa \frac{M+m}{r^2} \cdot \frac{\xi}{r},$$
$$\frac{d^2\eta}{dt^2} = -\kappa \frac{M+m}{r^2} \cdot \frac{\eta}{r}, \qquad (3)$$
$$\frac{d^2\zeta}{dt^2} = -\kappa \frac{M+m}{r^2} \cdot \frac{\zeta}{r}.$$

The form of these equations shows that *the relative motion of the planet with respect to the sun is the same as if the sun were fixed and contained the mass* M + m. Thus the problem is reduced to that of a fixed centre, the only modification being that the mass of the centre M should be increased by that of the attracted particle m.

155. This result can also be obtained by the following simple consideration. The *relative* motion of the planet with respect to the sun would obviously not be altered if geometrically equal accelerations were applied to both. Let us, therefore, subject each body to an additional acceleration equal and opposite to the actual acceleration of the sun (whose components are obtained by dividing the equations (2) by M).

Then the sun will be reduced to equilibrium, while the resulting acceleration of the planet, which is its relative acceleration with respect to the sun, will evidently be the sum of the acceleration exerted on it by the sun, and the acceleration exerted on the sun by the planet. This is just the result expressed by the equations (3).

156. It can here only be mentioned in passing that, while the problem of two bodies thus leads to equations that can easily be integrated, *the problem of three bodies* is one of exceeding difficulty, and has been solved only in a few very special cases. Much less has it been possible to integrate the $3n$ equations of the problem of n bodies.

157. According to the equations (3), the first and second laws of Kepler can be said to hold for the *relative* motion of a planet about the sun (or of a satellite about its primary). The third law of Kepler requires some modification, since the intensity of the centre μ should not be $=\kappa M$, but $=\kappa(M+m)$. Thus we have, by (26), Art. 139,

$$\mu=\kappa(M+m)=4\pi^2\frac{a^3}{T^2};$$

in other words, the quotient a^3/T^2 is not independent of the mass m of the planet.

Thus, if m_1, m_2 be the masses of two planets, a_1, a_2 the major semi-axes of their orbits, and T_1, T_2 their periodic times, we have

$$\frac{a_1^3/T_1^2}{a_2^3/T_2^2}=\frac{M+m_1}{M+m_2}=\frac{1+m_1/M}{1+m_2/M}.$$

This quotient is approximately equal to one if M is very large in comparison with both m_1 and m_2; hence, for the orbits of the planets about the sun, Kepler's law is very nearly true.

158. Exercises.

(1) Two particles of masses m_1, m_2 attract each other with a force which is any function of the distance r between them, say $F=m_1m_2f(r)$. Show that their common centroid moves uniformly in a straight line, and find the equations of this line.

(2) In Ex. (1), write out the equations for the relative motion of either particle with respect to the common centroid.

159. The theory of central forces is treated with considerable elaboration in most works on theoretical mechanics; a few references only will here be given: P. APPELL, *Traité de mécanique rationnelle*, Paris, Gauthier-Villars, 1893, Vol. I., pp. 354–405; B. WILLIAMSON and F. A. TARLETON, *An elementary treatise on dynamics*, London, Longmans (New York, Appleton), 2d edition, 1889, pp. 147–205; P. G. TAIT and W. J. STEELE, *A treatise on dynamics of a particle*, London and New York, Macmillan, 6th edition, 1889, pp. 113–166; W. H. BESANT, *A treatise on dynamics*, Cambridge, Bell (New York, Macmillan), 2d edition, 1893, pp. 120–166, and 267–275; W. WALTON, *A collection of problems in illustration of the principles of theoretical mechanics*, Cambridge, Bell, 3d edition, 1876, pp. 248–297. All these works contain numerous examples for practice. The theory of planetary motion is, of course, treated in works on theoretical astronomy. The student will also consult with advantage: W. SCHELL, *Theorie der Bewegung und der Kräfte*, Leipzig, Teubner, 2d edition, Vol. I., 1879, pp. 373–387; E. BUDDE, *Allgemeine Mechanik der Punkte und starren Systeme*, Berlin, Reimer, Vol. I., 1890, pp. 170–181; B. PRICE, *A treatise on analytical mechanics*, Oxford, Clarendon Press (New York, Macmillan), Vol. I. (= Vol. III. of *A treatise on infinitesimal calculus*), 2d edition, 1868, pp. 508–574; O. RAUSENBERGER, *Lehrbuch der analytischen Mechanik*, Leipzig, Teubner, Vol. I., 1888, pp. 32–102, where the problem of planetary motion is very fully discussed; T. DESPEYROUS, *Cours de mécanique, avec des notes par G. Darboux*, Paris, Hermann, 1884, Vol. I., pp. 336–369, 427–440, and Vol. II., pp. 38–57, 461–466; and others.

IV. *Constrained Motion.*

I. INTRODUCTION.

160. It has been shown, in the preceding sections, that the motion of a free particle is fully determined if all the forces acting upon the particle, as well as the so-called initial conditions, are given. The motion of a particle may, however, depend not only on given forces, but on other conditions not directly expressed in terms of forces. The motion is then said to be *constrained.*

Some of the more important forms of constraint have been considered in Part II., Arts. 218–225. To mention some more concrete examples: a heavy particle sliding down a smooth inclined plane is subject not only to the force of gravity, but also to the condition that it cannot pass through the plane; a railway train running on the rails, a piece of machinery sliding in a groove or between guides, can, for many purposes, be regarded as a particle constrained to a curve; the bob of a pendulum, a stone attached to a cord and swung around by the hand, may be regarded as constrained to a surface.

161. Sometimes these constraining conditions can be easily replaced by forces. Thus, in the first illustration above, the condition that the particle cannot pass through the inclined plane can be expressed by introducing the reaction of the plane, *i.e.* a force acting on the particle at right angles to the plane, so as to prevent it from passing through the plane. Similarly, in the case of the stone attached to the cord, we may imagine the cord cut and its tension introduced so as to replace the condition by a force.

Whenever the constraints to which a particle is subjected can thus be expressed by means of forces, these forces can be combined with the other impressed forces, and then, of course, the equations of motion for a free particle can be applied.

Thus, let X', Y', Z' be the components of the resultant of all the constraints; X, Y, Z those of the resultant of all the other impressed forces. Then the equations of motion are:

$$m\frac{d^2x}{dt^2}=X+X', \quad m\frac{d^2y}{dt^2}=Y+Y', \quad m\frac{d^2z}{dt^2}=Z+Z'. \qquad (1)$$

162. It must, however, be noticed that the reactions representing the constraints, such as the tension of the string in the example referred to, are generally not given beforehand. Moreover, the constraints are often expressed more conveniently by conditional equations. Thus, if the motion of a particle be restricted to a surface, the equation of this surface, say

$$\phi(x, y, z)=0, \qquad (2)$$

may be given as a constraining condition to be fulfilled by the co-ordinates of the moving particle.

163. As a particle has but three degrees of freedom, it can be subjected to only one or two conditions of the form (2). One such condition confines it to a surface; two to the curve of intersection of the two surfaces represented by the two conditional equations; three conditions would evidently prevent it entirely from moving.

164. The curve or surface to which a particle is constrained may vary its position and even its shape in the course of time. In this case the conditional equations, referred to fixed axes, will contain not only the co-ordinates, but also the time. That is, they will be of the more general form

$$\phi(x, y, z, t)=0. \qquad (3)$$

165. To constrain a particle *completely* to a surface, we may imagine it confined between two infinitely near impenetrable surfaces. The complete constraint to a curve might be realized by confining the particle to an infinitely narrow tube having the shape of the curve, or by regarding it as a ring sliding along a wire.

In many cases, however, the constraint is not complete, but only partial, or one-sided. Thus, the rails compelling the train to move in a definite curve do not prevent its being lifted vertically out of this curve, nor does the cord that confines the motion of the stone to a sphere prevent it from moving towards the inside of the spherical surface.

While complete constraints are generally expressed by equations, one-sided constraints should properly be expressed by inequalities. Thus, in the case of the stone, the condition is really that its distance r from the hand is not greater than the length l of the cord, *i.e.*

$$r \leqq l;$$

but as soon as r becomes less than l, the constraining action ceases, and the stone becomes free. It is, therefore, in general sufficient to consider conditional *equations;* but the nature of the constraint, whether complete or partial, must be taken into account to determine when and where the constraint ceases to exist.

166. We now proceed to consider separately the motion of a particle constrained to a fixed curve and that of a particle constrained to a fixed surface. After these special cases, the general problem of motion on a movable curve or surface will be discussed.

2. MOTION ON A FIXED CURVE.

167. The condition that a particle should move on a given fixed curve can always be replaced by introducing a single additional force F' called the *constraining force*, or the *constraint.* An example will best show how this force can be determined.

Let us consider a particle of mass m, subject to the force of gravity $F = mg$ alone; in general it will describe a parabola whose equation can be found if the initial conditions are known. To compel the particle to describe some other curve, say a vertical circle, a constraining force F' (Fig. 23) must be introduced such that the resultant R of F and F' shall produce the acceleration required for motion in the circle. Thus, for instance, for *uniform* motion in a circle the resulting acceleration must be directed towards the centre and must be $=\omega^2 a$, if a is the radius and ω the constant angular velocity. We have, therefore, in this case $R=m\omega^2 a$ along the radius, $F=mg$ vertically downwards; and hence, denoting by θ the angle made by the radius CP with the vertical (Fig. 23),

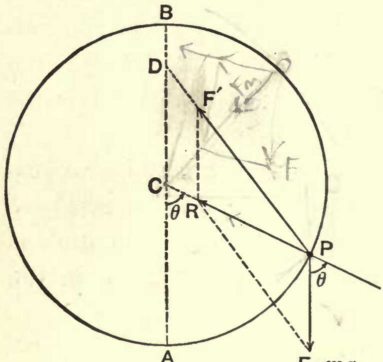

Fig. 23.

$$F'^2 = F^2 + R^2 + 2\,FR\cos\theta$$
$$= m^2(g^2 + \omega^4 a^2 + 2\,g\omega^2 a\cos\theta).$$

The constraint F', which is thus seen to vary with the angle θ, can be resolved into a tangential component F_t' and a normal component F_n'. As in our problem the velocity is to remain of constant magnitude, the tangential constraint must just counterbalance the tangential component $F_t = mg\sin\theta$ of gravity. The normal constraint F_n' not only counterbalances the normal component $F_n = mg\cos\theta$ of gravity, but also furnishes the centripetal force $R=m\omega^2 a$ required for motion in the circle; *i.e.*

$$F_n' = R + F\cos\theta = m(\omega^2 a + g\cos\theta).$$

168. In the general case of a particle of mass m acted upon by any given forces and constrained to any fixed curve, it is convenient to resolve both the resultant F of the given forces and the constraint F' along the tangent and the normal plane.

The equations of motion (see Art. 67) can then be written in the form

$$m\frac{dv}{dt}=F_t-F_t',$$

$$m\frac{v^2}{\rho}=\text{resultant of } F_n \text{ and } F_n',$$

where v is the velocity and ρ the radius of curvature of the path at the time t. It should be noticed that the components F_n and F_n', though both situated in the normal plane, do not in general have the same direction. But in the important special case of plane motion, *i.e.* when the path is a plane curve and the resultant F of the given forces lies in this plane, F_n and F_n' are both directed along the radius of curvature so that the right-hand member of the second equation becomes the sum or difference of F_n and F_n'.

169. The normal component F_n' of the constraining force is generally denoted by the letter N and is called the *resistance* or *reaction* of the curve; a force $-N$, equal and opposite to this reaction, represents the *pressure* exerted by the particle on the curve.

The tangential component F_t' of the constraint will exist only when the constraining curve is rough, *i.e.* offers frictional resistance; we have then, denoting the coefficient of friction by μ,

$$F_t'=\mu N.$$

We shall therefore write the equations of motion as follows:

$$m\frac{dv}{dt}=F_t-\mu N, \qquad (1)$$

$$m\frac{v^2}{\rho}=\text{resultant of } F_n \text{ and } N. \qquad (2)$$

170. The normal component, mv^2/ρ, of the effective force is sometimes called the *centripetal force* (see Art. 67); it is directed along the principal normal of the path towards the centre of curvature. A force equal and opposite to this centripetal force, *i.e.* $= -mv^2/r$, is called **centrifugal force.** It should be noticed that this is a force exerted not *on* the moving particle, but *by* it.

It appears from equation (2) that the normal reaction N is the resultant of the centripetal force mv^2/ρ and the reversed normal component of the given forces, $-F_n$. Changing all the signs, we can express the same thing by saying that *the pressure on the curve,* $-\mathrm{N}$, *is the resultant of the centrifugal force,* $-\mathrm{mv}^2/\rho$, *and of the normal component* F_n *of the given forces.*

If, in particular, this normal component F_n is zero, the pressure on the curve is equal to the centrifugal force. This case is of frequent occurrence. Thus, if a small stone attached to a cord be whirled around rapidly, the action of gravity on the stone can be neglected in comparison with the centripetal force due to rotation; hence the centrifugal force measures approximately the tension of the string, and may cause it to break. Again, when a railway train runs in a curve, the centrifugal force produces the horizontal pressure on the rails, which tends to displace and deform the rails.

171. It may happen that at a certain time t the pressure $-N$ vanishes. If the constraint be complete (Art. 165), this would merely indicate that the pressure in passing through zero inverts its sense. If, however, the constraint be one-sided, the consequence will be that the particle at this time leaves the constraining curve; for at the next moment the pressure will be exerted in a direction in which the particle is free to move.

Now N vanishes when its components $-F_n$ and mv^2/ρ become equal and opposite. The conditions under which the particle would leave the curve are, therefore, that the resultant F of the given forces should lie in the osculating plane of the path, and that $F_n = mv^2/\rho$.

172. To obtain the equations of motion expressed in rectangular Cartesian co-ordinates, let X, Y, Z be the components of the resultant F of the given forces, and N_x, N_y, N_z those of the normal reaction N of the curve. If there be friction, the frictional resistance μN, being directed along the tangent to the path opposite to the sense of the motion, has the direction cosines $-dx/ds$, $-dy/ds$, $-dz/ds$, so that the components of the force of friction are $-\mu N dx/ds$, $-\mu N dy/ds$, $-\mu N dz/ds$. The general equations of motion are, therefore,

$$m\frac{d^2x}{dt^2}=X+N_x-\mu N\frac{dx}{ds},$$
$$m\frac{d^2y}{dt^2}=Y+N_y-\mu N\frac{dy}{ds}, \qquad (3)$$
$$m\frac{d^2z}{dt^2}=Z+N_z-\mu N\frac{dz}{ds}.$$

If the acceleration of the particle be zero, the left-hand members are all $=0$, and the equations reduce to the conditions of equilibrium of a particle on a curve, as given in Statics (Part II., p. 138, (14)).

In addition to the equations (3) we have of course the equations of the curve, say

$$\phi(x, y, z)=0, \quad \psi(x, y, z)=0, \qquad (4)$$

and the relations

$$N^2=N_x^2+N_y^2+N_z^2, \qquad (5)$$

$$N_x\frac{dx}{ds}+N_y\frac{dy}{ds}+N_z\frac{dz}{ds}=0, \qquad (6)$$

the latter expressing that N is perpendicular to the element ds of the path.

173. Multiplying the equations (3) by dx, dy, dz, and adding, we find the equation of kinetic energy

$$d(\tfrac{1}{2}mv^2)=Xdx+Ydy+Zdz-\mu N ds. \qquad (7)$$

This relation might have been obtained directly from the consideration that for a displacement ds along the fixed curve the normal reaction N does no work, while the work of friction is $-\mu N ds$.

174. Exercises.

(1) Show that when the given forces are zero and there is no friction, the particle moves uniformly on the curve, and the pressure on the curve is proportional to the curvature of the path.

(2) *A particle of mass* m *moves down a straight line inclined to the horizon at an angle* θ, *under the action of gravity alone.*

(*a*) If there be no friction, we have by Art. 169, since $\rho = \infty$ (see Fig. 24).

$$m\frac{dv}{dt} = mg\sin\theta,$$

$$0 = mg\cos\theta - N.$$

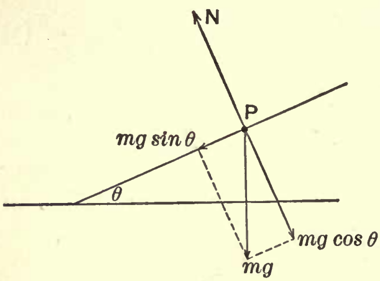

Fig. 24.

The first of these equations is the same as that derived in kinematics for motion down an inclined plane (see Part I., Arts. 164–166). The second equation gives the normal reaction of the line $N = mg\cos\theta$; hence, the pressure on the line, $-N$, is constant.

(*b*) If the line be rough, the second equation remains the same, while the first must be replaced by the following,

$$m\frac{dv}{dt} = mg\sin\theta - \mu N = mg(\sin\theta - \mu\cos\theta).$$

As the acceleration is constant whether there be friction or not, the motion is uniformly accelerated, unless $\sin\theta - \mu\cos\theta = 0$, *i.e.* $\mu = \tan\theta$.

Find v and s; show that, in the exceptional case $\mu = \tan\theta$, the motion is uniform unless the initial velocity be zero; show that, for motion *up* the plane, the first equation becomes $dv/dt = -g(\sin\theta + \mu\cos\theta)$, the motion being uniformly retarded until $t = v_0/g(\sin\theta + \mu\cos\theta)$ when the particle either begins to move down the line or remains at rest.

(3) A string of length l (ft.) carries at one end a mass of m lbs. while the other end is fixed at a point O on a smooth horizontal table. The mass m is made to describe a circle of radius l about O on the table, with constant velocity $= v$ ft. per second. Show that the tension of the string is $= mv^2/l$ poundals.

(4) In Ex. (3), let $m = 2$ lbs.; $l = 3$ ft.; find the tension in pounds: (*a*) when the mass makes one revolution per second; (*b*) when it makes 10 revolutions per second. (*c*) If the string cannot stand a tension of more than 300 pounds, what is the greatest allowable velocity?

(5) A locomotive weighing 32 tons moves in a curve of 800 ft. radius with a velocity of 30 miles an hour; find the horizontal pressure on the rails.

(6) To prevent the lateral pressure on the rails in a curve, the track is inclined inwards. Determine the required elevation e of the outer above the inner rail for a given velocity v and radius R if the gauge (*i.e.* the distance between the rails) is 4 ft. 8 in.

(7) A plummet is suspended from the roof of a railroad car; how much will it be deflected from the vertical when the train is running 45 miles an hour in a curve of 300 yards' radius?

(8) A body on the surface of the earth partakes of the earth's daily rotation on its axis. The constraint holding it in its circular path is due to the attractive force of the earth. Taking the earth's equatorial radius as 3963 miles, show that the centripetal acceleration of a particle at the equator is about $\frac{1}{9}$ ft. per second, or about $\frac{1}{290}$ of the actually observed acceleration $g = 32{\cdot}09$ of a body falling *in vacuo*.

(9) If the earth were at rest, what would be the acceleration of a body falling *in vacuo* at the equator?

(10) Show that if the velocity of the earth's rotation were over 17 times as large as it actually is, the force of gravity would not be sufficient to detain a body near the surface at the equator.

(11) Show that in latitude ϕ the acceleration of a falling body, if the earth were at rest, would be $g_1 = g + j\cos^2\phi$, where g is the observed acceleration of a falling body on the rotating earth and j the centripetal acceleration at the equator. Thus, in latitude $\phi = 45°$, $g = 980{\cdot}6$ cm.; hence $g_1 = 982{\cdot}3$.

(12) A chandelier weighing 75 lbs. is suspended from the ceiling of a hall by means of a chain $12\frac{1}{2}$ ft. long whose weight is neglected. By how much is the tension of the chain increased if it be set swinging so that the velocity at the lowest point is 5 ft. per second?

(13) A cord of 2 ft. length passes at its middle point through a hole in a smooth horizontal table. It carries at its lower end a mass of 2 lbs., at its other end a mass of 1 lb. The latter is set to revolve in a

circle about the hole so as to stretch the cord and just prevent the mass of 2 lbs. from descending. (*a*) How many revolutions must it make? (*b*) If only one-fourth of the cord lie on the table while three-fourths hang down, how many revolutions must be made?

(14) Show that, when a particle moves with constant velocity in a vertical circle, the constraining force F' (Art. 167) is always directed towards a fixed point on the vertical diameter.

175. *A particle of mass* m *subject to gravity alone is constrained to move in a vertical circle of radius* l. If there be no friction on the curve and the constraint be produced by a weightless rod or string joining the particle to the centre of the circle, we have the problem of the **simple mathematical pendulum.**

Equation (1), Art. 169, is readily seen to reduce in this case (see Fig. 25) to the form

$$l\frac{d^2\theta}{dt^2}+g\sin\theta=0. \qquad (8)$$

A first integration gives, as shown in kinematics (Part I., Arts. 215, 216),

$$\tfrac{1}{2}v^2=g\left(l\cos\theta+\frac{v_0^2}{2g}-l\cos\theta_0\right), \qquad (9)$$

where v_0 is the velocity which the particle has at the time $t=0$ when its radius makes the angle $AOP_0=\theta_0$ with the vertical. Multiplying by m, we have, for the kinetic energy of the particle,

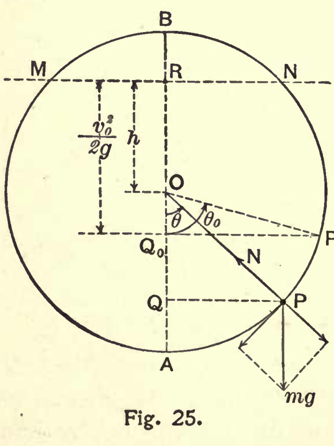

Fig. 25.

$$\tfrac{1}{2}mv^2=mg(l\cos\theta+h), \qquad (10)$$

where $h=v_0^2/2g-l\cos\theta_0$ is a constant. If the horizontal line MN, drawn at the height $v_0^2/2g$ above the initial point P_0, intersect the vertical diameter AB at R, it appears from the figure that $h=RO$.

176. Taking R as origin and the axis of z vertically downwards, we have $RQ=z=l\cos\theta+h$; hence the force-function U has the simple expression

$$U=mgz;$$

and the velocity $v=\sqrt{2gz}$ is seen to become zero when the particle reaches the horizontal line MN.

For the further treatment of the problem, three cases must be distinguished according as this line of zero-velocity MN intersects the circle, touches it, or does not meet it at all; *i.e.* according as

$$h \lesseqgtr l, \text{ or } \frac{v^2}{2g} \lesseqgtr 2l\sin^2\frac{\theta}{2}. \tag{11}$$

177. Equation (2), Art. 169, serves to determine the reaction N of the circle, or the pressure $-N$ on the circle. We have

$$m\frac{v^2}{l}=-mg\cos\theta+N,$$

whence
$$N=m\left(\frac{v^2}{l}+g\cos\theta\right).$$

Substituting for v^2 its value from (10), we find

$$N=mg\left(2\frac{h}{l}+3\cos\theta\right). \tag{12}$$

The pressure on the curve has therefore its greatest value when $\theta=0$, *i.e.* at the lowest point A. It becomes zero for $l\cos\theta_1=-\frac{2}{3}h$, which is easily constructed.

178. If the constraint be complete as for a bead sliding along a circular wire, or a small ball moving within a tube, the pressure merely changes sign at the point $\theta=\theta_1$. But if the constraint be one-sided, the particle may at this point leave the circle. The one-sided constraint may be such that $OP\leq l$, as when the particle runs in a groove cut on the inside of a ring, or when it is joined to the centre by a string; in this case the particle may leave the circle at some point of its upper half. Again, the one-sided constraint may be such that $OP \geqq l$, as when the particle runs in a groove cut on the rim of a disc; in this case the particle can of course only move on the upper half of the circle.

179. Exercises.

(1) For $h = l$, equation (10) can be integrated in finite terms. Show that in this limiting case the particle approaches the highest point B of the circle asymptotically, reaching it only in an infinite time.

(2) Derive the equations of motion for the problem of the simple pendulum (Art. 175) from the general equations of Arts. 172, 173.

(3) For $\theta_0 = 60°$, $l = 1$ ft., $v_0 = 9$ ft. per second, show that the particle will leave the circle very nearly at the point $\theta_1 = 120°$, if the constraint be such that $OP \leqq l$ (Art. 178).

(4) For $v_0 = 10$ ft. per second, everything else being as in Ex. (3), show that the particle will leave the circle at the point $\theta_1 = 134\frac{1}{2}°$, nearly.

(5) A particle, subject to gravity and constrained to the inside of a vertical circle ($OP \leqq l$), makes complete revolutions. Show that it cannot leave the circle at any point, if $\frac{2}{3}h > l$; and that it will leave the circle at the point for which $\cos\theta = -\frac{2}{3}h/l$, if $\frac{2}{3}h < l$.

(6) In the experiment of swinging in a vertical circle a glass containing water, and suspended by means of a string, if the string be 2 ft. long, what must be the velocity at the lowest point if the experiment is to succeed?

(7) A particle subject to gravity moves on the outside of a vertical circle; determine where it will leave the circle: (*a*) if MN (Fig. 25) intersects the circle; (*b*) if MN touches the circle; (*c*) if MN does not meet the circle.

(8) A particle subject to gravity is compelled to move on any vertical curve $z = f(x)$ without friction. Show that the velocity at any point is $v = \sqrt{2gz}$ (comp. Art. 176) if the horizontal axis of x be taken at a height above the initial point equal to the "height due to the initial velocity," *i.e.* $v_0^2/2g$.

(9) Investigate the motion of a particle subject to gravity, and compelled to move on a circle whose plane is inclined to the horizon at an angle α.

(10) A particle constrained to a straight line is attracted to a fixed centre outside this line, the attraction being proportional to the distance from the centre. Determine its motion.

180. Motion on Any Fixed Curve without Friction. The position of a point on a curve can always be determined by a single variable. Thus, for instance, the length s of the curve counted from some origin on the curve might be taken as this variable; if the curve be a circle, the polar angle θ might be selected; on an ellipse, the eccentric angle ϕ; on a cycloid, the angle through which the generating circle has rolled, etc. We shall designate this variable by q, and write the equations of the curve in the form

$$x=f_1(q),\quad y=f_2(q),\quad z=f_3(q). \tag{13}$$

The expression for the velocity v is in this case

$$v^2=\left(\frac{dx}{dt}\right)^2+\left(\frac{dy}{dt}\right)^2+\left(\frac{dz}{dt}\right)^2=\left[\left(\frac{df_1}{dq}\right)^2+\left(\frac{df_2}{dq}\right)^2+\left(\frac{df_3}{dq}\right)^2\right]\left(\frac{dq}{dt}\right)^2. \tag{14}$$

If there is no friction, the real equation of motion is the equation (1) of Art. 169, which is equivalent to the equation of kinetic energy (7), Art. 173; when the variable q is introduced, this equation becomes

$$d(\tfrac{1}{2}mv^2)=\left(X\frac{df_1}{dq}+Y\frac{df_2}{dq}+Z\frac{df_3}{dq}\right)dq, \tag{15}$$

where v^2 is given by (14).

Putting, for shortness,

$$X\frac{df_1}{dq}+Y\frac{df_2}{dq}+Z\frac{df_3}{dq}=Q, \tag{16}$$

we can write the equation of motion in the simple form

$$d(\tfrac{1}{2}mv^2)=Qdq. \tag{17}$$

181. In the most general case, the given forces X, Y, Z will depend not only on the position of the particle, but also on its velocity and on the time. In this case, Q would be a function of q, dq/dt, and t; and equation (17) represents a differential equation of the second order between q and t.

If, however, the resultant F of the given forces depends only on the position of the particle so that Q is a function of q alone, the right-hand member of (17) is an exact differential, and a first integration can at once be performed. Then, substituting for v^2 its value from (14) in terms of q and dq/dt, we find a differential equation of the first order whose integration gives t in function of q.

182. Exercise.

A particle of mass m is constrained to a common helix $x = a\cos\theta$, $y = a\sin\theta$, $z = \kappa\theta$, whose axis is vertical. The particle is subject to gravity and is attracted by a centre situated on the axis, with a force directly proportional to the distance. Determine the motion.

3. MOTION ON A FIXED SURFACE.

183. Just as for motion on a curve (Art. 172), we find the general equations of motion

$$m\frac{d^2x}{dt^2} = X + N_x - \mu N\frac{dx}{ds},$$

$$m\frac{d^2y}{dt^2} = Y + N_y - \mu N\frac{dy}{ds}, \qquad (1)$$

$$m\frac{d^2z}{dt^2} = Z + N_z - \mu N\frac{dz}{ds}.$$

The normal reaction

$$N = \sqrt{N_x^2 + N_y^2 + N_z^2} \qquad (2)$$

being at right angles to the constraining surface

$$\phi(x, y, z) = 0, \qquad (3)$$

the following condition must be satisfied:

$$\frac{N_x}{\phi_x} = \frac{N_y}{\phi_y} = \frac{N_z}{\phi_z}, \tag{4}$$

where ϕ_x, ϕ_y, ϕ_z denote, as usual, the partial derivatives of $\phi(x, y, z)$ with regard to x, y, z, respectively.[1]

If the acceleration of the particle be zero, the equations (1) reduce to the conditions of equilibrium of a particle on a surface, as given in Statics (Part II., Art. 222).

184. *A particle of mass* m, *subject to gravity, is constrained to remain on the surface of a sphere of radius* r. If the constraint is produced by a weightless rod or string joining the particle to the centre of the sphere, the rod or string describes a cone, and the apparatus is called a **conical** or **spherical pendulum**.

Taking the centre O of the sphere as origin (Fig. 26), and the axis of z vertically downwards, we have for the equation of the sphere

$$x^2 + y^2 + z^2 - r^2 = 0, \tag{5}$$

whence $\phi_x/x = \phi_y/y = \phi_z/z$. The direction cosines of N are $-x/r$, $-y/r$, $-z/r$. Hence, the equations of motion, as there is no friction:

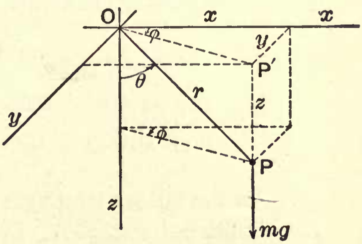

Fig. 26.

$$m\ddot{x} = -N\frac{x}{r}, \quad m\ddot{y} = -N\frac{y}{r}, \quad m\ddot{z} = mg - N\frac{z}{r}. \tag{6}$$

---

[1] This abridged notation is readily extended to the second and higher derivatives: $\phi_{xx} = \dfrac{\partial^2\phi}{\partial x^2}$, $\phi_{xy} = \dfrac{\partial^2\phi}{\partial x\partial y}$, etc. It will also sometimes be convenient to use the fluxional notation for derivatives with respect to the time

$$\frac{dx}{dt} = \dot{x}, \quad \frac{dy}{dt} = \dot{y}, \quad \frac{dz}{dt} = \dot{z}; \quad \frac{d^2x}{dt^2} = \ddot{x}, \quad \frac{d^2y}{dt^2} = \ddot{y}, \quad \frac{d^2z}{dt^2} = \ddot{z};$$

$$\frac{dr}{dt} = \dot{r}, \quad \frac{d\theta}{dt} = \dot{\theta}; \quad \frac{d^2r}{dt^2} = \ddot{r}, \text{ etc.}$$

In mechanics, this notation is of particular advantage, not only because the time so often appears as the independent variable, but also because the initial values of these derivatives (*i.e.* the components of the initial velocity and acceleration) can then be indicated by zero subscripts. Thus, the components of the initial velocity would be $\dot{x}_0$, $\dot{y}_0$, $\dot{z}_0$.

As the resistance N does no work, the principle of kinetic energy gives

$$\tfrac{1}{2} mv^2 = mgz + C,$$

or, dividing by $\frac{1}{2}m$,

$$v^2 = 2(gz + h). \tag{7}$$

To determine the constant of integration h, we have $v = v_0$ when $z = z_0$; hence

$$v^2 = v_0^2 + 2g(z - z_0). \tag{8}$$

185. Another first integral is found by applying the principle of areas which holds for the projection of the motion on the horizontal xy-plane. This appears by considering that N is always directed along the radius of the sphere so that the resultant of N and the weight mg of the particle always intersects the axis of z (see Art. 93). We have therefore

$$x\dot{y} - y\dot{x} = c, \tag{9}$$

where $\frac{1}{2}c$ is the sectorial velocity of the projection OP' of the radius $OP = r$ on the xy-plane.

186. For the further treatment of the problem it is best to introduce polar co-ordinates (Fig. 26). Let θ be the angle between r and the axis of z, ϕ that between the projection OP' of r on the xy-plane and the axis of x; then

$$x = r \sin\theta \cos\phi, \quad y = r \sin\theta \sin\phi, \quad z = r\cos\theta;$$

and

$$\dot{x} = r\cos\theta\cos\phi\cdot\dot{\theta} - r\sin\theta\sin\phi\cdot\dot{\phi},$$

$$\dot{y} = r\cos\theta\sin\phi\cdot\dot{\theta} + r\sin\theta\cos\phi\cdot\dot{\phi},$$

$$\dot{z} = -r\sin\theta\cdot\dot{\theta}.$$

Hence

$$v^2 = \dot{x}^2 + \dot{y}^2 + \dot{z}^2 = r^2(\dot{\theta}^2 + \sin^2\theta\cdot\dot{\phi}^2),$$

$$x\dot{y} - yx = r^2 \sin^2\theta\cdot\dot{\phi}.$$

The first integrals (7) and (9) thus become in polar co-ordinates

$$r^2(\dot{\theta}^2 + \sin^2\theta\cdot\dot{\phi}^2) = 2(gr\cos\theta + h), \tag{10}$$

$$r^2 \sin^2\theta\cdot\dot{\phi} = c. \tag{11}$$

187. The constants of integration h, c can now be determined if the initial values of θ, $\dot{\theta}$, $\dot{\phi}$ are given.

Eliminating dt between the equations (10) and (11), we find the differential equation of the path

$$d\phi = \frac{c\,d\theta}{\sin\theta\sqrt{2r^2\sin^2\theta(gr\cos\theta + h) - c^2}}, \tag{12}$$

whose integration gives the equation of the path in the spherical coordinates θ, ϕ (colatitude and longitude).

On the other hand, if $\dot{\phi}$ be eliminated between (10) and (11), we find the relation

$$dt = \frac{r^2 \sin\theta\, d\theta}{\sqrt{2r^2\sin^2\theta(gr\cos\theta + h) - c^2}}, \tag{13}$$

which, upon integration, determines the time as a function of θ, or the position of the point in its path at any given time.

The equations (12) and (13) contain therefore the complete solution of our problem, with the exception of the determination of the resistance N. Their discussion cannot here be given, as they lead to elliptic integrals.

188. To find the resistance N, multiply the equations (6) by x, y, z and add; this gives

$$m(x\ddot{x} + y\ddot{y} + z\ddot{z}) = mgz - Nr. \tag{14}$$

Differentiating twice the equation of the sphere (5), we find

$$x\ddot{x} + y\ddot{y} + z\ddot{z} + \dot{x}^2 + \dot{y}^2 + \dot{z}^2 = 0,$$

or since $\dot{x}^2 + \dot{y}^2 + \dot{z}^2 = v^2$,

$$x\ddot{x} + y\ddot{y} + z\ddot{z} = -v^2.$$

Substituting this value in (14), we find

$$N = \frac{m(gz + v^2)}{r}$$

or, by (8),

$$N = \frac{m}{r}(3gz + v_0^2 - 2gz_0). \tag{15}$$

If the constraint be one-sided as in the case of a string pendulum, the particle will leave the sphere whenever in its upper half z becomes $\overline{\overline{>}}\ \frac{2}{3}z_0 - v_0^2/3g$.

189. That particular case of the problem of the conical pendulum in which the particle moves in a horizontal circle can be treated directly in an elementary manner. It finds its application in the theory of the *governor of a steam engine.*

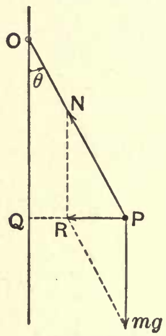

Fig. 27.

Let O (Fig. 27) be the point of suspension, $OP = l$ the length of the pendulum rod, $\measuredangle\, QOP = \theta$ the constant angle of inclination of the rod to the vertical OQ. The only forces acting on the particle are its weight mg and the tension of the rod. As both these forces lie in the normal plane of the path, the tangential acceleration is zero, and the particle moves uniformly in the circle.

The radius of curvature of the path is the radius $QP = l \sin\theta$ of the horizontal circle. The resultant R of mg and N must act along the radius; its magnitude is seen from the figure to be $R = mg\tan\theta$. Hence the equation (2) of Art. 169 gives

$$m\frac{v^2}{l\sin\theta} = mg\tan\theta,$$

or,

$$v^2 = gl\frac{\sin^2\theta}{\cos\theta}. \tag{16}$$

The figure also shows that the tension of the rod is

$$N = \frac{mg}{\cos\theta}. \tag{17}$$

190. Exercises.

(1) Show that the time of revolution T of the conical pendulum (Art. 186) is the same as the time of one complete oscillation of a simple pendulum of length $l\cos\theta$.

(2) Show that the angular velocity with which the vertical plane of the rod turns about the vertical axis OQ (Fig. 27) is inversely proportional to the cosine of the angle θ.

(3) A conical pendulum makes $n = 60$ revolutions per minute: (*a*) What is the height of the cone? (*b*) If the mass of the bob be $m = 1$ oz., and the length of the rod $l = 1$ ft., what is the tension of the rod? ($g = 32{\cdot}2$.)

(4) From the equations (5) and (6), Art. 184, derive the approximate path of the bob of a simple pendulum when the oscillations are very small.

191. Motion on Any Fixed Surface without Friction. The position of a point on a surface can always be determined by two variables, say q_1, q_2. Thus, on a sphere, the latitude and longitude of a point determine its position; and on any surface the two systems of curves known as the curves of curvature of the surface might serve as a system of co-ordinates. In other words, the motion of a point on a surface is really a problem of motion in two dimensions, just as the motion on a curve takes place in one dimension (Art. 180).

Let $$x=f_1(q_1, q_2),\ y=f_2(q_1, q_2),\ z=f_3(q_1, q_2) \tag{18}$$

be the equations of the given surface, so that the elimination of q_1, q_2 from these equations would give the ordinary equation $\phi(x, y, z)=0$ of the surface. Then we have for the velocity v the expression

$$v^2=\dot{x}^2+\dot{y}^2+\dot{z}^2$$

$$=\left(\frac{\partial f_1}{\partial q_1}\dot{q}_1+\frac{\partial f_1}{\partial q_2}\dot{q}_2\right)^2+\left(\frac{\partial f_2}{\partial q_1}\dot{q}_1+\frac{\partial f_2}{\partial q_2}\dot{q}_2\right)^2+\left(\frac{\partial f_3}{\partial q_1}\dot{q}_1+\frac{\partial f_3}{\partial q_2}\dot{q}_2\right)^2. \tag{19}$$

If there be no friction, the equation of kinetic energy gives

$$d(\tfrac{1}{2}mv^2)=$$

$$\left(X\frac{\partial f_1}{\partial q_1}+Y\frac{\partial f_2}{dq_1}+Z\frac{\partial f_3}{\partial q_1}\right)dq_1+\left(X\frac{\partial f_1}{\partial q_2}+Y\frac{\partial f_2}{\partial q_2}+Z\frac{\partial f_3}{\partial q_2}\right)dq_2,$$

or say

$$d(\tfrac{1}{2}mv^2)=Q_1dq_1+Q_2dq_2. \tag{20}$$

If the forces depend only on the position of the particle, Q_1 and Q_2 are functions of q_1, q_2 alone; if, moreover, the expression $Q_1dq_1+Q_2dq_2$ is an exact differential of a function of q_1 and q_2, say $dU(q_1, q_2)$, the equation (20) gives at once a first integral

$$\tfrac{1}{2}mv^2=U(q_1, q_2)+h. \tag{21}$$

4. MOTION ON A MOVING OR VARIABLE CURVE OR SURFACE.

192. If the constraining curve or surface be not fixed and invariable, the conditional equations will contain the time t explicitly, besides the co-ordinates x, y, z of the moving particle (Art. 164). For the sake of simplicity we here assume the curve or surface to be smooth, so as to offer only a *normal* resistance N; if there be friction, the components of the frictional resistance may be regarded as included in the components X, Y, Z of the resultant force acting on the particle.

The treatment of this general problem of constrained motion of a particle is here presented not so much on account of its application to the solution of particular problems, as for the reason that it offers an opportunity of explaining the meaning of d'Alembert's principle and illustrating its application in a comparatively simple case.

193. Two Constraints. Let the equations of the curve to which the particle is constrained be

$$\phi(x, y, z, t) = 0, \quad \psi(x, y, z, t) = 0. \tag{1}$$

To apply d'Alembert's principle (Arts. 97–102), let the particle be subjected, at any given time t, to an infinitesimal displacement δs. If this displacement be selected along the curve (1), the reaction N of the curve, being at right angles to δs, will do no work during the displacement; hence the equation of motion will be the same as that for a free particle (see Art. 101), viz.

$$(-m\ddot{x} + X)\delta x + (-m\ddot{y} + Y)\delta y + (-m\ddot{z} + Z)\delta z = 0. \tag{2}$$

In this equation, then, the forces X, Y, Z do not involve the normal reaction of the curve; but the components $\delta x, \delta y, \delta z$ of the displacement δs must be selected so that δs should lie on the curve (1) at the time t; this is usually expressed by saying that *the displacement should be compatible with the conditions* (1).

Some authors confine the term *virtual displacement* to displacements compatible with the given conditions.

The displacement $\delta s=\sqrt{\delta x^2+\delta y^2+\delta z^2}$ will be compatible with the conditions (1) at the given time t if the following conditions, obtained by differentiating the equations (1), are satisfied :

$$\begin{aligned}\phi_x\delta x+\phi_y\delta y+\phi_z\delta z=0,\\ \psi_x\delta x+\psi_y\delta y+\psi_z\delta z=0.\end{aligned}\tag{3}$$

It should be noticed that in this differentiation the time t is regarded as constant, the displacement being taken to occur *at a given time.*

The equations (2) and (3), which must be fulfilled simultaneously, constitute the equations of motion of our problem.

By means of the equations (3), two of the component displacements δx, δy, δz can be eliminated from the equation (2); the coefficient of the third equated to zero gives the actual equation of motion.

194. To perform this elimination systematically *the method of indeterminate multipliers* can be used as follows. Multiplying the equations (3) respectively by the indeterminate factors λ and μ and adding them to the equation (2), we obtain the single equation

$$(-m\ddot{x}+X+\lambda\phi_x+\mu\psi_x)\delta x+(-m\ddot{y}+Y+\lambda\phi_y+\mu\psi_y)\delta y \\ +(-m\ddot{z}+Z+\lambda\phi_z+\mu\psi_z)\delta z=0,$$

in which the arbitrary quantities λ, μ can be so selected as to make the coefficients of two of the three displacements δx, δy, δz vanish; the coefficient of the third must then also vanish. The equation is therefore equivalent to the following three equations :

$$\begin{aligned}m\ddot{x}&=X+\lambda\phi_x+\mu\psi_x,\\ m\ddot{y}&=Y+\lambda\phi_y+\mu\psi_y,\\ m\ddot{z}&=Z+\lambda\phi_z+\mu\psi_z.\end{aligned}\tag{4}$$

These equations (4), in connection with the two conditions (1), are sufficient to determine the five quantities x, y, z, λ, μ as functions of the time; the values of x, y, z so found give the position of the particle at any time, while λ, μ can be shown to determine the pressure on the curve.

195. To find the reaction N of the curve, let us compare the equations (4) with the equations of Art. 161. It appears at once that the forces X', Y', Z' that would replace the conditions (1), *i.e.* the components of the reaction N of the constraining curve, are

$$X' = \lambda\phi_x + \mu\psi_x, \quad Y' = \lambda\phi_y + \mu\psi_y, \quad Z' = \lambda\phi_z + \mu\psi_z,$$

whence

$$N^2 = (\lambda\phi_x + \mu\psi_x)^2 + (\lambda\phi_y + \mu\psi_y)^2 + (\lambda\phi_z + \mu\psi_z)^2. \tag{5}$$

These equations determine the magnitude and direction of the reaction N, as soon as λ and μ are found.

196. Let us now combine the equations (4) according to the principle of kinetic energy; that is, multiply them by dx, dy, dz, and add. The left-hand member becomes, of course, the exact differential $d(\frac{1}{2}mv^2)$. The right-hand member,

$$Xdx + Ydy + Zdz + \lambda(\phi_x dx + \phi_y dy + \phi_z dz) + \mu(\psi_x dx + \psi_y dy + \psi_z dz),$$

will in general contain terms depending on the reaction of the surface; in other words, in the *actual* displacement $ds = (dx^2 + dy^2 + dz^2)^{\frac{1}{2}}$ of the particle the reaction of the moving curve will in general do work.

197. Only in the particular case when the curve is fixed will the work of the reaction be zero; for in this case the conditional equations (1) do not contain the time explicitly, and their complete differentiation gives the relations

$$\phi_x dx + \phi_y dy + \phi_z dz = 0, \quad \psi_x dx + \psi_y dy + \psi_z dz = 0,$$

which show that the coefficients of λ and μ in the equation of kinetic energy vanish.

Hence, *for motion on a fixed curve* we have

$$d(\tfrac{1}{2}mv^2) = Xdx + Ydy + Zdz, \tag{6}$$

which agrees with the equation (7) of Art. 173, considering that the frictional resistance is supposed to be included among the forces X, Y, Z.

198. In the general case, the complete differentiation of the equations (1) gives

$$\phi_x dx + \phi_y dy + \phi_z dz + \phi_t dt = 0,$$
$$\psi_x dx + \psi_y dy + \psi_z dz + \psi_t dt = 0;$$

and the equation of kinetic energy *for motion on a moving or variable curve* becomes

$$d(\tfrac{1}{2}mv^2) = Xdx + Ydy + Zdz - (\lambda\phi_t + \mu\psi_t)dt. \tag{7}$$

The distinction between the *virtual* displacement δs along the curve in its position at the time t and the *actual* displacement ds of the particle along the moving curve should be clearly understood. The virtual displacement $\delta s = PP'$ joins the position P (x, y, z) of the particle at the time t to a point P' $(x+\delta x, y+\delta y, z+\delta z)$, which is on the curve, and infinitely near to P at the time t, while the actual displacement $ds = PP''$ joins P (x, y, z) to the position P'' $(x+dx, y+dy, z+dz)$ of the particle at the time $t+dt$; P'' lies, therefore, on the position that the curve has, not at the time t, but at the time $t+dt$. The reaction N of the curve at the time t is normal to δs, but not to ds.

199. One Constraint. Let

$$\phi(x, y, z, t) = 0 \tag{8}$$

be the equation of the surface on which the particle is assumed to remain throughout its motion. The reaction N of this surface will do no work if the displacement δs be taken along the position of the surface at a given time t. In other words, to obtain a displacement δs compatible with the condition (8), its components δx, δy, δz should satisfy the condition

$$\phi_x \delta x + \phi_y \delta y + \phi_z \delta z = 0, \tag{9}$$

obtained by differentiating the equation (8) with respect to the co-ordinates.

200. By means of the relation (9), one of the displacements δx, δy, δz can be eliminated from the general equation of motion (2); the two remaining displacements will then be independent, and their coefficients can therefore be equated to zero separately.

The elimination is again conveniently effected by the method of indeterminate multipliers. Multiplying equation (9) by an indeterminate factor λ, and adding it to equation (2), we find the single equation

$$\begin{aligned} &(-m\ddot{x} + X + \lambda\phi_x)\,\delta x \\ +&(-m\ddot{y} + Y + \lambda\phi_y)\,\delta y \\ +&(-m\ddot{z} + Z + \lambda\phi_z)\,\delta z = 0, \end{aligned}$$

in which the arbitrary quantity λ can be so selected as to make the coefficient of one of the three displacements vanish. The other two displacements being arbitrary, their coefficients must also vanish. The last equation can therefore be replaced by the following three:

$$m\ddot{x} = X + \lambda\phi_x, \quad m\ddot{y} = Y + \lambda\phi_y, \quad m\ddot{z} = Z + \lambda\phi_z, \tag{10}$$

which, in connection with the given condition (8), fully determine the problem; for they are sufficient for finding x, y, z, and λ as functions of t.

201. Just as in Art. 195, it follows that the components of the reaction N of the surface are

$$X' = \lambda\phi_x, \quad Y' = \lambda\phi_y, \quad Z' = \lambda\phi_z,$$

whence
$$N = \lambda\sqrt{\phi_x^2 + \phi_y^2 + \phi_z^2}. \tag{11}$$

202. If the equations (10) be combined according to the principle of kinetic energy, we find

$$d(\tfrac{1}{2}mv^2) = Xdx + Ydy + Zdz + \lambda(\phi_x dx + \phi_y dy + \phi_z dz),$$

where again the coefficient of λ vanishes only when the surface is fixed, in which case

$$d(\tfrac{1}{2}mv^2) = Xdx + Ydy + Zdz; \qquad (12)$$

while in the general case of a moving or variable surface we have

$$d(\tfrac{1}{2}mv^2) = Xdx + Ydy + Zdz - \lambda\phi_t dt. \qquad (13)$$

203. Plane Motion. If a particle be constrained to move in a plane curve under the action of forces lying in the plane of the curve, d'Alembert's principle gives the equation of motion

$$(-m\ddot{x} + X)\delta x + (-m\ddot{y} + Y)\delta y = 0; \qquad (14)$$

and the equation of the curve

$$\phi(x, y, t) = 0 \qquad (15)$$

gives by differentiation for a virtual displacement δs on the curve at a given time t,

$$\phi_x \delta x + \phi_y \delta y = 0. \qquad (16)$$

Hence, proceeding as in Art. 200, the equations of motion can be written in the form

$$m\ddot{x} = X + \lambda\phi_x, \quad m\ddot{y} = Y + \lambda\phi_y, \qquad (17)$$

while the normal reaction of the curve is

$$N = \lambda\sqrt{\phi_x^2 + \phi_y^2}. \qquad (18)$$

204. The process of solution is now as follows for the case of plane motion. Differentiate the equation of condition (15), which holds at any time, with respect to the time, remembering that x and y are functions of the time; this gives:

$$\frac{d\phi}{dt} \equiv \dot{x}\phi_x + \dot{y}\phi_y + \phi_t = 0. \qquad (19)$$

Differentiating again, we find

$$\frac{d^2\phi}{dt^2} \equiv \ddot{x}\phi_x + \ddot{y}\phi_y + \dot{x}^2\phi_{xx} + 2\dot{x}\dot{y}\phi_{xy} + \dot{y}^2\phi_{yy}$$
$$+ 2\dot{x}\phi_{tx} + 2\dot{y}\phi_{ty} + \phi_{tt} = 0. \quad (20)$$

If in this last equation the values of $\ddot{x}$, $\ddot{y}$ be substituted from (17), we have a linear equation for λ. The value of λ thus obtained can then be introduced into the equations of motion (17); and it only remains to integrate these equations.

This integration will often be facilitated by introducing new variables for x, y.

205. *A particle moves without friction in a straight tube which revolves uniformly in a horizontal plane about one of its points. Determine its motion.*

To illustrate the application of the general methods, we shall solve this problem completely, first without the use of indeterminate multipliers, and then with their aid, although the problem is so simple that it might be solved without applying these general methods, as will be pointed out below.

As the weight of the particle is balanced by the vertical reaction of the tube, we have a case of plane motion with $X = 0$, $Y = 0$. Hence d'Alembert's equation (14) becomes

$$\ddot{x}\delta x + \ddot{y}\delta y = 0. \quad (21)$$

If we take as origin the point O about which the tube rotates, the constraining curve is a straight line through the origin $y = x \tan\theta$, where $\theta = \omega t$, ω being the constant angular velocity of the tube and the axis of x coinciding with the initial position of the tube at the time $t = 0$. Hence

$$x = r\cos\omega t, \quad y = r\sin\omega t;$$
$$\delta x = \delta r\cos\omega t, \quad \delta y = \delta r\sin\omega t;$$
$$\dot{x} = \dot{r}\cos\omega t - \omega r\sin\omega t, \quad \dot{y} = \dot{r}\sin\omega t + \omega r\cos\omega t; \quad (22)$$
$$\ddot{x} = \ddot{r}\cos\omega t - 2\omega\dot{r}\sin\omega t - \omega^2 r\cos\omega t, \quad \ddot{y} = \ddot{r}\sin\omega t + 2\omega\dot{r}\cos\omega t - \omega^2 r\sin\omega t.$$

Substituting these values of $\ddot{x}$, $\ddot{y}$ and δx, δy into the equation of motion (21), we find after reduction

$$\ddot{r} - \omega^2 r = 0. \quad (23)$$

As mentioned above, this equation might have been derived directly by considering that the acceleration along the tube is due to the centrifugal force alone (see Art. 170).

206. The general integral of equation (23) is

$$r = c_1 e^{\omega t} + c_2 e^{-\omega t}.$$

If $r = r_0$ and $\dot{r} = v_0$ when $t = 0$, we have $r_0 = c_1 + c_2$, $v_0 = \omega(c_1 - c_2)$; hence

$$2\,\omega r = (\omega r_0 + v_0) e^{\omega t} + (\omega r_0 - v_0) e^{-\omega t}. \qquad (24)$$

With $v_0 = 0$, $r_0 = 1$, this equation represents a common catenary. If $v_0 = r_0 \omega$, the equation reduces to the form $r = r_0 e^{\omega t}$, whence $t = (1/\omega) \log (r/r_0)$.

The minimum of r in (24) occurs for

$$t = \frac{1}{2\omega} \log \frac{\omega r_0 - v_0}{\omega r_0 + v_0};$$

its value is $r_1 = \sqrt{r_0^2 - (v_0/\omega)^2}$. It is easy to see that such a minimum can occur only when v_0 is negative and $> \omega r_0$ numerically.

207. To apply the method of indeterminate multipliers to our problem, let the equation of the tube be written in the form

$$\phi(x, y, t) \equiv x \cos \omega t - y \sin \omega t = 0. \qquad (25)$$

Then we have $\phi_x = \cos \omega t$, $\phi_y = -\sin \omega t$, $\phi_t = -\omega(x \sin \omega t + y \cos \omega t)$; hence equation (16) assumes the form

$$\delta x \cos \omega t - \delta y \sin \omega t = 0;$$

and the equations of motion (17) are

$$m\ddot{x} = \lambda \cos \omega t, \quad m\ddot{y} = -\lambda \sin \omega t. \qquad (26)$$

We have also $\phi_{xx} = 0$, $\phi_{xy} = \phi_{yx} = 0$, $\phi_{yy} = 0$, $\phi_{tx} = \phi_{xt} = -\omega \sin \omega t$, $\phi_{ty} = \phi_{yt} = -\omega \cos \omega t$, $\phi_{tt} = -\omega^2(x \cos \omega t - y \sin \omega t) = 0$. Hence, by (20),

$$\frac{d^2\phi}{dt^2} \equiv \ddot{x} \cos \omega t - \ddot{y} \sin \omega t - 2\,\omega \dot{x} \sin \omega t - 2\,\omega \dot{y} \cos \omega t = 0.$$

Substituting in this equation the values of $\ddot{x}, \ddot{y}$ from (26), we find the linear equation for λ which gives

$$\frac{\lambda}{m} = 2\,\omega(\dot{x} \sin \omega t + \dot{y} \cos \omega t) \sec 2\,\omega t.$$

Introducing this value into the equations (26), we have the differential equations of our problem in the form

$$\ddot{x} = 2\omega(\dot{x} \sin \omega t + \dot{y} \cos \omega t) \frac{\cos \omega t}{\cos 2\omega t}, \quad y = -2\omega(\dot{x} \sin \omega t + \dot{y} \cos \omega t) \frac{\sin \omega t}{\cos 2\omega t}.$$

Their integration can best be performed by introducing the radius vector r by means of the relations (22). Multiplying the equations respectively by $\cos\omega t$ and $\sin\omega t$, and adding, we find that the right-hand member vanishes, and we have

$$\ddot{x}\cos\omega t + \ddot{y}\sin\omega t = 0,$$

or, substituting for $\ddot{x}$ and $\ddot{y}$ their values from (22),

$$\ddot{r} - \omega^2 r = 0,$$

which agrees with (23), Art. 205.

208. For the pressure on the curve, we have, by (18), since $\phi_x^2 + \phi_y^2 = 1$,

$$N = \lambda = \frac{2\,m\omega}{\cos 2\,\omega t}(\dot{x}\sin\omega t + \dot{y}\cos\omega t).$$

Substituting from (22) and (24), and reducing, we find

$$N = m\omega[(\omega r_0 + v_0)e^{\omega t}(1 + \tan 2\,\omega t) + (\omega r_0 - v_0)e^{-\omega t}(1 - \tan 2\,\omega t)].$$

209. Exercises.*

(1) A particle subject to gravity moves without friction in a straight tube which revolves uniformly in a vertical circle. Find the distance r of the particle from the centre of rotation at any time t.

(2) A particle moves without friction in a circular tube which rotates uniformly in a horizontal plane about a point O in its circumference. If the particle is at the time $t = 0$ at rest at the end of the diameter passing through O, what is its position at any time t?

(3) A particle moves in a horizontal circular tube whose radius increases proportionally to the time. At the time $t = 0$ the radius is a, and the particle has a velocity v_0 perpendicular to the radius. Find the position and velocity at any time t.

---

* These examples are taken from Walton's Collection (referred to in Art. 159), pp. 401–406.

V. *Lagrange's Form of the Equations of Motion.*

210. It has been shown in Arts. 180, 181, how the equations of motion on a fixed curve can be made to depend on a single variable q, and in Art. 191 how the motion on a fixed surface can be expressed by means of two variables q_1, q_2. By applying this idea, and by introducing the kinetic energy T and its derivatives, the equations of motion of a particle with or without conditions can be put into a remarkably compact form, which was first devised by Lagrange for the general equations of motion of a system of n particles (comp. Arts. 385–394). We proceed to establish these equations, first for the case of motion on a variable curve, then for motion on a variable surface, and finally for a free particle.

211. Particle Subject to Two Conditions. As shown in Art. 194 (comp. Art. 192), the equations of motion in Cartesian co-ordinates can be written in the form

$$\begin{aligned} m\ddot{x} &= X + \lambda\phi_x + \mu\psi_x, \\ m\ddot{y} &= Y + \lambda\phi_y + \mu\psi_y, \\ m\ddot{z} &= Z + \lambda\phi_z + \mu\psi_z, \end{aligned} \qquad (1)$$

if the equations of condition are

$$\phi(x, y, z, t) = 0, \qquad \psi(x, y, z, t) = 0. \qquad (2)$$

The single variable q, that determines the position of the particle on the curve, is called the *Lagrangian*, or *generalized, co-ordinate* of the particle. The Cartesian co-ordinates, x, y, z, are functions of the Lagrangian co-ordinate q, and of the time t, say

$$x = f_1(t, q), \quad y = f_2(t, q), \quad z = f_3(t, q). \qquad (3)$$

To introduce q in the place of x, y, z, we shall need the derivatives $\dot{x}, \dot{y}, \dot{z}$. The first of the equations (3) gives

$$\dot{x} = \frac{\partial f_1}{\partial t} + \frac{\partial f_1}{\partial q}\dot{q}.$$

so that $\dot{x}$ can be regarded as a function of t, q, and $\dot{q}=dq/dt$. Hence we have

$$\frac{\partial \dot{x}}{\partial q}=\frac{\partial^2 f_1}{\partial q \partial t}+\frac{\partial^2 f_1}{\partial q^2}\dot{q}, \quad \frac{\partial \dot{x}}{\partial \dot{q}}=\frac{\partial f_1}{\partial q}.$$

In the former of these expressions, the right-hand member can be put into the more compact form $\frac{d}{dt}\frac{\partial f_1}{\partial q}$, as is easily verified by carrying out the indicated differentiation with respect to t. As similar results hold for y and z, we have

$$\frac{\partial \dot{x}}{\partial q}=\frac{d}{dt}\frac{\partial f_1}{\partial q}, \quad \frac{\partial \dot{y}}{\partial q}=\frac{d}{dt}\frac{\partial f_2}{\partial q}, \quad \frac{\partial \dot{z}}{\partial q}=\frac{d}{dt}\frac{\partial f_3}{\partial q}, \tag{4}$$

$$\frac{\partial \dot{x}}{\partial \dot{q}}=\frac{\partial f_1}{\partial q}, \quad \frac{\partial \dot{y}}{\partial \dot{q}}=\frac{\partial f_2}{\partial q}, \quad \frac{\partial \dot{z}}{\partial \dot{q}}=\frac{\partial f_3}{\partial q}. \tag{5}$$

212. Let us now add the equations of motion (1) after multiplying them by $\partial f_1/\partial q$, $\partial f_2/\partial q$, $\partial f_3/\partial q$. The coefficient of λ in the resulting equation, viz.

$$\frac{\partial \phi}{\partial x}\frac{\partial f_1}{\partial q}+\frac{\partial \phi}{\partial y}\frac{\partial f_2}{\partial q}+\frac{\partial \phi}{\partial z}\frac{\partial f_3}{\partial q},$$

is equal to zero, since it is evidently proportional to the cosine of the angle made at a given time by the tangent to the curve (3) with the normal to the surface $\phi=0$. For a similar reason, the coefficient of μ vanishes; and the resulting equation is

$$m\left(\ddot{x}\frac{\partial f_1}{\partial q}+\ddot{y}\frac{\partial f_2}{\partial q}+\ddot{z}\frac{\partial f_3}{\partial q}\right)=Q, \tag{6}$$

if, as in Arts. 180, 181, we put for shortness

$$X\frac{\partial f_1}{\partial q}+Y\frac{\partial f_2}{\partial q}+Z\frac{\partial f_3}{\partial q}=Q.$$

This quantity Q can evidently be expressed as a function of t, q, and $\dot{q}$.

The equation (6) can also be written in the form

$$m\frac{d}{dt}\left(\dot{x}\frac{\partial f_1}{\partial q}+\dot{y}\frac{\partial f_2}{\partial q}+\dot{z}\frac{\partial f_3}{\partial q}\right)-m\left(\dot{x}\frac{d}{dt}\frac{\partial f_1}{\partial q}+\dot{y}\frac{d}{dt}\frac{\partial f_2}{\partial q}+\dot{z}\frac{d}{dt}\frac{\partial f_3}{\partial q}\right)=Q,$$

as appears by carrying out the indicated differentiations with respect to t, and if we now make use of the relations (4) and (5), our equation assumes the form

$$m\frac{d}{dt}\left(\dot{x}\frac{\partial\dot{x}}{\partial\dot{q}}+\dot{y}\frac{\partial\dot{y}}{\partial\dot{q}}+\dot{z}\frac{\partial\dot{z}}{\partial\dot{q}}\right)-m\left(\dot{x}\frac{\partial\dot{x}}{\partial q}+\dot{y}\frac{\partial\dot{y}}{\partial q}+\dot{z}\frac{\partial\dot{z}}{\partial q}\right)=Q.$$

The quantities in the two parentheses, each multiplied by m, are easily recognized as the partial derivatives with respect to $\dot{q}$ and q of the kinetic energy

$$T=\tfrac{1}{2}m(\dot{x}^2+\dot{y}^2+\dot{z}^2);$$

hence the equation reduces to the form

$$\frac{d}{dt}\frac{\partial T}{\partial\dot{q}}-\frac{\partial T}{\partial q}=Q, \tag{7}$$

known as the (second) *Lagrangian form* of the equation of motion of a particle constrained to a curve.

213. Particle Subject to One Condition. By Art. 200, the equations of motion are

$$m\ddot{x}=X+\lambda\phi_x,\quad m\ddot{y}=Y+\lambda\phi_y,\quad m\ddot{z}=Z+\lambda\phi_z, \tag{8}$$

with the condition

$$\phi(x, y, z, t)=0. \tag{9}$$

Let the two generalized co-ordinates q_1, q_2 be connected with the Cartesian co-ordinates by the equations

$$x=f_1(t, q_1, q_2),\quad y=f_2(t, q_1, q_2),\quad z=f_3(t, q_1, q_2). \tag{10}$$

The first of these equations gives

$$\dot{x}=\frac{\partial f_1}{\partial t}+\frac{\partial f_1}{\partial q_1}\dot{q}_1+\frac{\partial f_1}{\partial q_2}\dot{q}_2;$$

hence, regarding $\dot{x}$ as a function of t, q_1, q_2, $\dot{q}_1=dq_1/dt$, $\dot{q}_2=dq_2/dt$, we find:

$$\frac{\partial\dot{x}}{\partial q_1}=\frac{\partial^2 f_1}{\partial q_1\partial t}+\frac{\partial^2 f_1}{\partial q_1^2}\dot{q}_1+\frac{\partial^2 f_1}{\partial q_1\partial q_2}\dot{q}_2,\qquad \frac{\partial\dot{x}}{\partial\dot{q}_1}=\frac{\partial f_1}{\partial q_1}.$$

The right-hand member of the former of these equations is equivalent to $\frac{d}{dt}\frac{\partial f_1}{\partial q_1}$. As similar relations hold for y and z, we find again the relations (4) and (5), with q_1 substituted for q. It can be shown in the same way that these relations also hold if q_2 be written for q.

214. Let us now multiply the equations (8) by $\partial f_1/\partial q_1$, $\partial f_2/\partial q_1$, $\partial f_3/\partial q_1$, and add them. This gives

$$m\left(\ddot{x}\frac{\partial f_1}{\partial q_1}+\ddot{y}\frac{\partial f_2}{\partial q_1}+\ddot{z}\frac{\partial f_3}{\partial q_1}\right)=Q_1, \tag{11}$$

where $$Q_1=X\frac{\partial f_1}{\partial q_1}+Y\frac{\partial f_2}{\partial q_1}+Z\frac{\partial f_3}{\partial q_1}.$$

Similarly, multiplying (8) by $\partial f_1/\partial q_2$, $\partial f_2/\partial q_2$, $\partial f_3/\partial q_2$, and adding, we find

$$m\left(\ddot{x}\frac{\partial f_1}{\partial q_2}+\ddot{y}\frac{\partial f_2}{\partial q_2}+\ddot{z}\frac{\partial f_3}{\partial q_2}\right)=Q_2, \tag{12}$$

where $$Q_2=X\frac{\partial f_1}{\partial q_2}+Y\frac{\partial f_2}{\partial q_2}+Z\frac{\partial f_3}{\partial q_2}.$$

Each of these equations (11) and (12) can be treated by the method used in Art. 212, and we find as the final equations of motion on the surface (9) in the Lagrangian form:

$$\frac{d}{dt}\frac{\partial T}{\partial \dot{q}_1}-\frac{\partial T}{\partial q_1}=Q_1, \quad \frac{d}{dt}\frac{\partial T}{\partial \dot{q}_2}-\frac{\partial T}{\partial q_2}=Q_2. \tag{13}$$

215. Free Particle. In this case three variables q_1, q_2, q_3 are required to determine the position of the particle. If the expressions of x, y, z in terms of these new variables do not contain the time explicitly, the introduction of the new variables consists merely in a change of co-ordinates. If they do contain the time, *i.e.* if we have

$$x=f_1(t, q_1, q_2, q_3), \quad y=f_2(t, q_1, q_2, q_3), \quad z=f_3(t, q_1, q_2, q_3), \tag{14}$$

the new system of co-ordinates is a moving system.

The relations (4), (5) can again be shown to hold for each of the three Lagrangian co-ordinates q_1, q_2, q_3.

If the equations of motion

$$m\ddot{x}=X, \quad m\ddot{y}=Y, \quad m\ddot{z}=Z \tag{15}$$

be multiplied by $\partial f_1/\partial q_1$, $\partial f_2/\partial q_1$, $\partial f_3/\partial q_1$ and added, we find

$$m\left(\ddot{x}\frac{\partial f_1}{\partial q_1}+\ddot{y}\frac{\partial f_2}{\partial q_1}+\ddot{z}\frac{\partial f_3}{\partial q_1}\right)=Q_1, \tag{16}$$

where

$$Q_1=X\frac{\partial f_1}{\partial q_1}+Y\frac{\partial f_2}{\partial q_1}+Z\frac{\partial f_3}{\partial q_1}.$$

By the method of Art. 212, equation (16) reduces to the form

$$\frac{d}{dt}\frac{\partial T}{\partial \dot{q}_1}-\frac{\partial T}{\partial q_1}=Q_1. \tag{17}$$

Similarly we find

$$\left.\begin{aligned}\frac{d}{dt}\frac{\partial T}{\partial \dot{q}_2}-\frac{\partial T}{\partial q_2}=Q_2,\\ \frac{d}{dt}\frac{\partial T}{\partial \dot{q}_3}-\frac{\partial T}{\partial q_3}=Q_3.\end{aligned}\right. \tag{18}$$

The three equations (17) and (18) are the Lagrangian equations of motion of a free particle.

216. If there exists a force-function U for the forces X, Y, Z, *i.e.* if

$$X=\frac{\partial U}{\partial x}, \quad Y=\frac{\partial U}{\partial y}, \quad Z=\frac{\partial U}{\partial z},$$

we have

$$Q_1=\frac{\partial U}{\partial x}\frac{\partial f_1}{\partial q_1}+\frac{\partial U}{\partial y}\frac{\partial f_2}{\partial q_1}+\frac{\partial U}{\partial z}\frac{\partial f_3}{\partial q_1}=\frac{\partial U}{\partial q_1},$$

and similarly

$$Q_2=\frac{\partial U}{\partial q_2}, \quad Q_3=\frac{\partial U}{\partial q_3}.$$

In this case, one of the three equations (17), (18) can be replaced by the equation of kinetic energy

$$T=U+h,$$

where h is a constant.

217. The general theory of the constrained motion of a particle is treated with special care in the works of Schell, Budde, and Appell, referred to in Art. 159. In Appell's first volume, pp. 445–517, the student will find instructive examples of the application of Lagrange's equations. For more elementary problems, as also for the interesting theories of brachistochrones and tautochrones, the reader is referred, besides the works just named, to the text-books of Tait and Steele, Besant, Price, and Walton's Problems (see Art. 159).

CHAPTER VI.

KINETICS OF A RIGID BODY.

I. *General Principles.*

218. In kinetics the term *rigid body* means any system or aggregate of mass-particles whose mutual distances remain invariable. A rigid body may therefore consist of a finite number of rigidly connected particles or of a continuous mass of one, two, or three dimensions. Its motion depends not only on the forces acting on the body, but also on the way in which the mass is distributed throughout the body.

In the present section the rigid body is assumed to be free unless the contrary be stated explicitly.

219. Let us consider any one particle m of the body; at any time t, let j be its acceleration and F the resultant of all the forces acting on the particle. Then the motion of this particle (see Arts. 35, 67) is determined by the equation

$$mj = F. \tag{1}$$

It should be noticed that among the forces acting on the particle are included not only those external forces acting on the rigid body that happen to be applied at m, but also the so-called *internal forces* which would replace the rigid connection of the particle m with the rest of the body.

If, at the time t, x, y, z are the co-ordinates of the particle m with respect to a fixed set of rectangular axes, then the components of its velocity v may be denoted by $\dot{x}, \dot{y}, \dot{z}$; those of its acceleration j by $\ddot{x}, \ddot{y}, \ddot{z}$.* And if the components of F

---

* Here again we shall use this so-called *fluxional notation*, according to which derivatives with respect to the time are denoted by dots; see the foot-note to Art. 183.

along the same axes are X, Y, Z, the equation (1) can be replaced by the following three:

$$-m\ddot{x}+X=0, \quad -m\ddot{y}+Y=0, \quad -m\ddot{z}+Z=0. \tag{2}$$

Such a set of three equations can be written down for each particle; hence, if the body consist of n particles, there would be in all $3n$ equations.

220. For the solution of particular problems these $3n$ equations are of little use, not only because their number would in general be very great and may even be infinite, but mainly because the forces X, Y, Z include the unknown reactions between the particles. It is, however, possible to deduce certain general propositions from these equations.

The $3n$ equations express the equilibrium of the system formed by *all* the forces, both internal and external, acting on the particles, and the reversed effective forces. To apply the principle of virtual work to this system, let us multiply the three equations (2) by the components δx, δy, δz of some virtual displacement of the particle m; let the same thing be done for every other particle of the body; and let all the resulting equations be added:

$$\Sigma(-m\ddot{x}+X)\delta x+\Sigma(-m\ddot{y}+Y)\delta y+\Sigma(-m\ddot{z}+Z)\delta z=0. \tag{3}$$

221. It is important to notice that the internal reactions between the particles which make the body rigid occur in pairs of equal and opposite forces, and form, therefore, a system which is in equilibrium by itself. Hence, while these internal forces enter into the equations (2), they do not appear in equation (3), since the equal and opposite forces cancel in the summation. Thus, equation (3) expresses that *the external*

Derivatives were called *fluxes* by Newton; thus the component of the acceleration of a point in any direction is the time-flux of its velocity in that direction; the component of its effective force in any direction is the time-flux of its momentum; and so on.

forces acting on the rigid body and the reversed effective forces form a system in equilibrium; and this is **d'Alembert's Principle** for the rigid body.

It must, however, not be forgotten that the displacements δx, δy, δz should be so selected as to be compatible with the nature of the rigid body; *i.e.* with the conditions that the distances between the particles should not be disturbed.

222. The number of conditions expressing the invariability of the distances between n particles is $3n-6$. For if there were but 3 particles, the number of independent conditions would evidently be 3; for every additional particle, 3 additional conditions are required. Hence, the total number of conditions is $3+3(n-3)=3n-6$.

It follows that if a rigid body be subject to no other constraining conditions, the number of its equations of motion must be $3n-(3n-6)=6$. Hence, *a free rigid body has six independent equations of motion.* (Comp. Part I., Art. 37.)

223. The six equations of motion of the rigid body can be obtained as follows.

Imagine the equations (2), viz.

$$m\ddot{x}=X, \quad m\ddot{y}=Y, \quad m\ddot{z}=Z,$$

written down for every particle, and add the corresponding equations. This gives the first 3 of the 6 equations of motion:

$$\Sigma m\ddot{x}=\Sigma X, \quad \Sigma m\ddot{y}=\Sigma Y, \quad \Sigma m\ddot{z}=\Sigma Z. \tag{4}$$

As the internal forces cancel in the summation, the right-hand members of these equations represent the components R_x, R_y, R_z of the resultant R of all the external forces acting on the body. The left-hand members can be put into the form $d(\Sigma m\dot{x})/dt$, $d(\Sigma m\dot{y})/dt$, $d(\Sigma m\dot{z})/dt$; these are the time derivatives or fluxes of the sums of the linear momenta of all the particles parallel to the axes. The equations (4) can therefore be written in the form

$$\frac{d}{dt}\Sigma m\dot{x}=R_x, \quad \frac{d}{dt}\Sigma m\dot{y}=R_y, \quad \frac{d}{dt}\Sigma m\dot{z}=R_z. \tag{5}$$

The axes of co-ordinates are arbitrary. Hence, if we agree to call *linear momentum of the body in any direction* the algebraic sum of the linear momenta of all the particles in that direction, the equations (5) express the proposition that *the rate at which the linear momentum of a rigid body in any direction changes with the time is equal to the sum of the components of all the external forces in that direction.*

224. Let us now combine the second and third of the equations (2) by multiplying the former by z, the latter by y, and subtracting the former from the latter. If this be done for each particle, and the resulting equations be added, we find $\Sigma m(y\ddot{z}-z\ddot{y})=\Sigma(yZ-zY)$. Similarly, we can proceed with the third and first, and with the first and second of the equations (2). The result is:

$$\Sigma m(y\ddot{z}-z\ddot{y})=\Sigma(yZ-zY),\quad \Sigma m(z\ddot{x}-x\ddot{z})=\Sigma(zX-xZ),$$
$$\Sigma m(x\ddot{y}-y\ddot{x})=\Sigma(xY-yX). \tag{6}$$

Here again the internal forces disappear in the summation, so that the right-hand members are the components H_x, H_y, H_z of the vector H of the resultant couple, found by reducing all the external forces for the origin of co-ordinates. The left-hand members are the components of the resultant couple of the effective forces for the same origin.

We can also say that the right-hand members are the sums of the moments of the external forces about the co-ordinate axes (Part II., Art. 213), while the left-hand members represent the moments of the effective forces about the same axes. The latter quantities are exact derivatives, as shown in Arts. 87 and 91. The equations (6) can therefore be written in the form

$$\frac{d}{dt}\Sigma m(y\dot{z}-z\dot{y})=H_x,\quad \frac{d}{dt}\Sigma m(z\dot{x}-x\dot{z})=H_y,\quad \frac{d}{dt}\Sigma m(x\dot{y}-y\dot{x})=H_z. \tag{7}$$

As explained in Arts. 89 and 92, the quantity $m(y\dot{z}-z\dot{y})$ is called the *angular momentum* (or the moment of momentum) of the particle m about the axis of x. We may now agree to call the quantity $\Sigma m(y\dot{z}-z\dot{y})$ the *angular momentum of the body* about the axis of x, just as $\Sigma m\dot{x}$ is the linear momentum of the body along this axis; and similarly for the other axes. The meaning of the equations (7) can then be stated as follows: *The rate at which the angular momentum of a rigid body about any axis changes with the time is equal to the sum of the moments of all the external forces about this line.*

The equations (4) and (6), or (5) and (7), are the **six equations of motion of the rigid body**. The three equations (4) or (5) may be called the *equations of linear momentum*, while (6) or (7) are the *equations of angular momentum.*

225. The equations (4) and (6) can also be derived from the equation (3), which expresses d'Alembert's principle, by selecting for δx, δy, δz convenient displacements.

Thus, the rigidity of the body will evidently not be disturbed if we give to all its points equal and parallel infinitesimal displacements, since this merely amounts to subjecting the whole body to an infinitesimal translation. Equation (3) can in this case be written

$$\delta x\Sigma(-m\ddot{x}+X)+\delta y\Sigma(-m\ddot{y}+Y)+\delta z\Sigma(-m\ddot{z}+Z)=0,$$

and is therefore equivalent to the three equations (4), since δx, δy, δz are independent and arbitrary.

Again, let the body be subjected to an infinitesimal rotation of angle $\delta\theta$ about any line l.

As shown in Art. 293 of Part I., the linear velocities of any point (x, y, z) of a rigid body, due to a rotation of angular velocity $\omega=\delta\theta/\delta t$ about any line l are, if ω_x, ω_y, ω_z denote the components of ω:

$$\dot{x}=\omega_y z-\omega_z y,\quad \dot{y}=\omega_z x-\omega_x z,\quad \dot{z}=\omega_x y-\omega_y x.$$

Hence, putting $\omega_x\delta t=\delta\theta_x$, $\omega_y\delta t=\delta\theta_y$, $\omega_z\delta t=\delta\theta_z$, we have for the displacements of the point (x, y, z), due to a rotation of angle $\delta\theta$,

$$\delta x=z\,\delta\theta_y-y\,\delta\theta_z,\quad \delta y=x\,\delta\theta_z-z\,\delta\theta_x,\quad \delta z=y\,\delta\theta_x-x\,\delta\theta_y.$$

If these values be introduced into d'Alembert's equation (3) and the terms in $\delta\theta_x$, $\delta\theta_y$, $\delta\theta_z$ be collected, it assumes the form

$$\delta\theta_x\Sigma[-m(y\ddot{z}-z\ddot{y})+yZ-zY]+\delta\theta_y\Sigma[-m(z\ddot{x}-x\ddot{z})+zX-xZ]$$
$$+\delta\theta_z\Sigma[-m(x\ddot{y}-y\ddot{x})+xY-yX]=0;$$

as $\delta\theta_x$, $\delta\theta_y$, $\delta\theta_z$ are independent and arbitrary, their coefficients must vanish separately, and this gives the equations (6).

226. The equations of linear momentum, (4) or (5), admit of a further simplification, owing to the fundamental property of the centroid. By Part II., Art. 13, the co-ordinates $\bar{x}$, $\bar{y}$, $\bar{z}$ of the centroid satisfy the relations

$$M\bar{x}=\Sigma mx,\quad M\bar{y}=\Sigma my,\quad M\bar{z}=\Sigma mz,$$

where $M=\Sigma m$ is the whole mass of the body. Differentiating these equations, we find

$$M\dot{\bar{x}}=\Sigma m\dot{x},\quad M\dot{\bar{y}}=\Sigma m\dot{y},\quad M\dot{\bar{z}}=\Sigma m\dot{z},$$

and $$M\ddot{\bar{x}}=\Sigma m\ddot{x},\quad M\ddot{\bar{y}}=\Sigma m\ddot{y},\quad M\ddot{\bar{z}}=\Sigma m\ddot{z},$$

where $\dot{\bar{x}}$, $\dot{\bar{y}}$, $\dot{\bar{z}}$ are the components of the velocity $\bar{v}$, and $\ddot{\bar{x}}$, $\ddot{\bar{y}}$, $\ddot{\bar{z}}$ those of the acceleration $\bar{j}$, of the centroid.

The equations (4) or (5) can therefore be reduced to the form

$$M\ddot{\bar{x}}=\frac{d}{dt}M\dot{\bar{x}}=R_x,\quad M\ddot{\bar{y}}=\frac{d}{dt}M\dot{\bar{y}}=R_y,\quad M\ddot{\bar{z}}=\frac{d}{dt}M\dot{\bar{z}}=R_z,\qquad (8)$$

whence $$M\bar{j}=\frac{d}{dt}M\bar{v}=R\,;\qquad (9)$$

i.e. if the whole mass of the body be regarded as concentrated at the centroid, the effective force of the centroid, or the time-rate of change of its momentum, is equal to the resultant of all the external forces. It follows that *the centroid of a rigid body moves as if it contained the whole mass, and all the external forces were applied at this point parallel to their original directions.*

227. If, in particular, the resultant R vanish (while there may be a couple H acting on the body), we have by (8) and (9) $\bar{j}=0$; hence $\bar{v}=$ const.; i.e. *if the resultant force be zero, the centroid moves uniformly in a straight line.*

This proposition, which can also be expressed by saying that, if $R=0$, the momentum $M\bar{v}$ of the centroid remains constant, or, using the form (5) of the equations of motion, that the linear momentum of the body in any direction is constant, is known he **principle of the conservation of linear momentum**, or the principle of the conservation of the motion of the centroid.

228. Let us next consider the equations of angular momentum, (6) or (7). To introduce the properties of the centroid, let us put $x-\bar{x}=\xi$, $y-\bar{y}=\eta$, $z-\bar{z}=\zeta$, so that ξ, η, ζ are the co-ordinates of the point (x, y, z) with respect to parallel axes through the centroid. The substitution of $x=\bar{x}+\xi$, $y=\bar{y}+\eta$, $z=\bar{z}+\zeta$ and their derivatives into the expression $y\dot{z}-z\dot{y}$ gives

$$\bar{y}\dot{\bar{z}}-\bar{z}\dot{\bar{y}}+\bar{y}\dot{\zeta}-\bar{z}\dot{\eta}+\eta\dot{\bar{z}}-\zeta\dot{\bar{y}}+\eta\dot{\zeta}-\zeta\dot{\eta}.$$

To form $\Sigma m(y\dot{z}-z\dot{y})$ we must multiply by m and sum throughout the body; in this summation, $\bar{y}$, $\bar{z}$, $\dot{\bar{y}}$, $\dot{\bar{z}}$ are constant and, by the property of the centroid, $\Sigma m\eta=0$, $\Sigma m\zeta=0$, $\Sigma m\dot{\eta}=0$, $\Sigma m\dot{\zeta}=0$. Hence we find

$$\Sigma m(y\dot{z}-z\dot{y})=\Sigma m(\eta\dot{\zeta}-\zeta\dot{\eta})+M(\bar{y}\dot{\bar{z}}-\bar{z}\dot{\bar{y}}).$$

The second term in the right-hand member is the angular momentum of the centroid about the axis of x (the whole mass M of the body being regarded as concentrated at this point), while the first term is the angular momentum of the body about a parallel to the axis of x, drawn through the centroid.

Similar relations hold for the angular momenta about the axes of y and z; and as these axes are arbitrary, we conclude that *the angular momentum of a rigid body about any line is equal to its angular momentum about a parallel through the centroid plus the angular momentum of the centroid about the former line.*

229. Differentiating the above expression, we find

$$\frac{d}{dt}\Sigma m(y\dot{z}-z\dot{y})=\frac{d}{dt}\Sigma m(\eta\dot{\zeta}-\zeta\dot{\eta})+M(\bar{y}\ddot{\bar{z}}-\bar{z}\ddot{\bar{y}}).$$

The first of the equations (7) can therefore be written

$$\frac{d}{dt}\Sigma m(\eta\dot{\zeta}-\zeta\dot{\eta})+M(\bar{y}\ddot{\bar{z}}-\bar{z}\ddot{\bar{y}})=H_x.$$

Now, if at any time t the centroid were taken as origin, so that $\bar{y}=0$, $\bar{z}=0$, this equation would reduce to the form

$$\frac{d}{dt}\Sigma m(\eta\dot{\zeta}-\zeta\dot{\eta})=H_x,$$

which is entirely independent of the co-ordinates of the centroid. On the other hand, wherever the origin is taken, if the centroid were a fixed point, the same equation would be obtained.

Similar considerations apply of course to the other two equations (7). It follows that *the motion of a rigid body relative to the centroid is the same as if the centroid were fixed.*

230. If, in particular, the resultant couple H be zero for any particular origin O (which will be the case not only when all external forces are zero, but also when the directions of all forces pass through the point O), the equations (7) can be integrated and give

$$\Sigma m(y\dot{z}-z\dot{y})=C_1,\quad \Sigma m(z\dot{x}-x\dot{z})=C_2,\quad \Sigma m(x\dot{y}-y\dot{x})=C_3, \quad (10)$$

where C_1, C_2, C_3 are constants of integration (comp. Art. 94). Hence, *if the external forces pass through a fixed point, the angular momentum of the body about any line through this point is constant; if there are no external forces, the angular momentum is constant for any line whatever.* This is the **principle of the conservation of angular momentum.**

231. Another interpretation can be given to these equations. As shown in Arts. 88 and 91, the quantities $y\dot{z}-z\dot{y}$, $z\dot{x}-x\dot{z}$, $x\dot{y}-y\dot{x}$ can be regarded as sectorial velocities. Thus, if the radius vector, drawn from the origin to the particle m, be projected on the yz-plane, $y\dot{z}-z\dot{y}$ is twice the sectorial velocity of this radius vector in the yz-plane, $\frac{1}{2}(ydz-zdy)$ being the elementary sector described in the element of time dt. Let us denote by dS_x the sum of all these elementary sectors for the various particles, each multiplied by the mass of the particle; and similarly by dS_y, dS_z the corresponding sums of the projections on the other co-ordinate planes. Then the equations (10) can be written in the form

$$2\dot{S}_x = C_1, \quad 2\dot{S}_y = C_2, \quad 2\dot{S}_z = C_3. \tag{11}$$

Hence the proposition of Art. 230 might be called the principle of the conservation of sectorial velocities; it is more commonly called the **principle of the conservation of areas.**

The equations (11) can be integrated again and give, if the sectors be measured from the positions of the radii vectores at the time $t=0$,

$$S_x = \tfrac{1}{2} C_1 t, \quad S_y = \tfrac{1}{2} C_2 t, \quad S_z = \tfrac{1}{2} C_3 t.$$

232. If the radii vectores be projected on any plane through the origin whose normal has the direction cosines α, β, γ, the sum of the elementary sectors described in this plane, each multiplied by the mass, will be

$$dS = \tfrac{1}{2}(C_1\alpha + C_2\beta + C_3\gamma)dt;$$

hence

$$S = \tfrac{1}{2}(C_1\alpha + C_2\beta + C_3\gamma)t.$$

On the other hand, by (10), the angular momentum of the body about the normal of this plane has the expression $C_1\alpha + C_2\beta + C_3\gamma$, as it must be equal to the sum of the projections on this normal of the angular momenta about the axes of co-ordinates, which can be regarded as vectors laid off on these axes.

Now it is easy to see that this angular momentum $C_1\alpha + C_2\beta + C_3\gamma$, and hence the quantity S at a given time t, is greatest for the diagonal of the parallelepiped, whose edges are equal to C_1, C_2, C_3 along the axes, *i.e.* for the normal to the plane

$$C_1 x + C_2 y + C_3 z = 0. \tag{12}$$

For, the direction cosines of this normal are $\alpha' = C_1/D$, $\beta' = C_2/D$, $\gamma' = C_3/D$, where $D = \sqrt{C_1^2 + C_2^2 + C_3^2}$; and the quantity $C_1\alpha + C_2\beta + C_3\gamma$ can be put into the form

$$D\left(\frac{C_1}{D}\alpha + \frac{C_2}{D}\beta + \frac{C_3}{D}\gamma\right) = D(\alpha'\alpha + \beta'\beta + \gamma'\gamma),$$

where the quantity in parenthesis is the cosine of the angle between the directions $(\alpha', \beta', \gamma')$ and (α, β, γ), and is therefore greatest when these directions coincide.

The plane (12) about whose normal the angular momentum is greatest, and by projection on which the area S is made greatest, is called Laplace's **invariable plane**. As its equation is independent of t, it remains fixed. The normal of this plane is sometimes called the *invariable line or direction.*

233. Let us now return to the general case of the motion of a rigid body acted upon by any forces whatever.

The propositions of Arts. 226 and 229 together establish the so-called **principle of the independence of the motions of translation and rotation.** In studying the motion of a rigid body it is possible, according to this principle, to consider separately the motion of translation of the centroid, and the rotation of the body about the centroid.

By Art. 226, the motion of the centroid is the same as that of a particle of mass M acted upon by all the external forces transferred parallel to themselves to the centroid. As the motion of a particle has been discussed in Chapter V., nothing further need be said about this part of the problem.

By Art. 229, the motion of the body about the centroid is the same as if the centroid were fixed. The problem of the motion of a rigid body with a fixed point is therefore of great importance; it will be discussed in Section IV. The more simple special case of a rigid body with a fixed axis is treated in Section III. The solution of both these problems depends on the equations (6) or (7).

234. In d'Alembert's equation (3) it is of course allowable to substitute for the virtual displacements δx, δy, δz the actual displacements dx, dy, dz of the particles in any motion of a free rigid body, since these actual displacements are certainly compatible with the condition of rigidity. The equation can then be written

$$\Sigma m(\ddot{x}dx + \ddot{y}dy + \ddot{z}dz) = \Sigma(Xdx + Ydy + Zdz). \qquad (13)$$

The left-hand member of this equation evidently represents the differential of the kinetic energy

$$T = \Sigma \tfrac{1}{2} m v^2 = \Sigma \tfrac{1}{2} m (\dot{x}^2 + \dot{y}^2 + \dot{z}^2) \qquad (14)$$

of the body, while the right-hand member is the elementary work of the external forces. Hence equation (13) expresses the **principle of kinetic energy** for a free rigid body, viz. the proposition that, *for any infinitesimal displacement of the body, the increase of the kinetic energy is equal to the sum of the works done by all the external forces.*

235. By introducing the co-ordinates of the centroid, *i.e.* by putting $x = \bar{x} + \xi$, $y = \bar{y} + \eta$, $z = \bar{z} + \zeta$, as in Art. 228, the expression for the kinetic energy assumes the form (since $\Sigma m \dot{\xi} = 0$, $\Sigma m \dot{\eta} = 0$, $\Sigma m \dot{\zeta} = 0$):

$$T = \Sigma \tfrac{1}{2} m (\dot{\bar{x}}^2 + \dot{\bar{y}}^2 + \dot{\bar{z}}^2) + \Sigma \tfrac{1}{2} m (\dot{\xi}^2 + \dot{\eta}^2 + \dot{\zeta}^2)$$
$$= \tfrac{1}{2} M \bar{v}^2 + \Sigma \tfrac{1}{2} m u^2, \qquad (15)$$

where $\bar{v}$ is the velocity of the centroid and u the *relative* velocity of any particle m with respect to the centroid.

Thus, it appears that *the kinetic energy of a free rigid body consists of two parts, one of which is the kinetic energy of the centroid* (the whole mass being regarded as concentrated at this point), *while the other may be called the relative kinetic energy with respect to the centroid.*

236. By the same substitution the right-hand member of equation (13), *i.e.* the elementary work $\Sigma (X dx + Y dy + Z dz)$, resolves itself into the two parts

$$(d\bar{x} \Sigma X + d\bar{y} \Sigma Y + d\bar{z} \Sigma Z) + \Sigma (X d\xi + Y d\eta + Z d\zeta).$$

The first parenthesis contains the work that would be done by all the external forces if they were applied at the centroid; it is therefore equal to the kinetic energy of the centroid, that is to $d(\frac{1}{2} M \bar{v}^2)$. The equation of kinetic energy (13) reduces, therefore, to the following:

$$d(\Sigma \tfrac{1}{2} m u^2) = \Sigma (X d\xi + Y d\eta + Z d\zeta); \qquad (16)$$

in other words, *the principle of kinetic energy holds for the relative motion with respect to the centroid.*

237. Impulses. The equations determining the effect of a system of impulses (see Arts. 2–5) on a rigid body are readily obtained from the general equations of motion (4) and (6). We shall denote the impulse of a force F by $\underset{.}{F}$. It will be remembered that the impulse $\underset{.}{F}$ of a force F is its time integral; *i.e.*

$$\underset{.}{F} = \int_t^{t'} F dt.$$

We confine ourselves to the case when $t' - t$ is very small and F very large, in which case the action of the impulsive force F is measured by its impulse $\underset{.}{F}$.

If all the forces acting on a rigid body are of this nature, and the impulses of X, Y, Z during the short interval $t' - t$ be denoted by $\underset{.}{X}$, $\underset{.}{Y}$, $\underset{.}{Z}$, the integration of the equations (4) from $t = t$ to $t = t'$ gives

$$\Sigma m(\dot{x}' - \dot{x}) = \Sigma \underset{.}{X}, \quad \Sigma m(\dot{y}' - \dot{y}) = \Sigma \underset{.}{Y}, \quad \Sigma m(\dot{z}' - \dot{z}) = \Sigma \underset{.}{Z}, \qquad (17)$$

where $\dot{x}$, $\dot{y}$, $\dot{z}$ denote the velocities of the particle m at the time t just before the impulse, and $\dot{x}'$, $\dot{y}'$, $\dot{z}'$ those at the time t' just after the action of the impulse.

Similarly the equations (6) give

$$\begin{aligned} \Sigma m[y(\dot{z}' - \dot{z}) - z(\dot{y}' - \dot{y})] &= \Sigma(y\underset{.}{Z} - z\underset{.}{Y}), \\ \Sigma m[z(\dot{x}' - \dot{x}) - x(\dot{z}' - \dot{z})] &= \Sigma(z\underset{.}{X} - x\underset{.}{Z}), \qquad (18) \\ \Sigma m[x(\dot{y}' - \dot{y}) - y(\dot{x}' - \dot{x})] &= \Sigma(x\underset{.}{Y} - y\underset{.}{X}). \end{aligned}$$

238. In determining the effect on a rigid body of a system of such impulses, any ordinary forces acting on the body at the same time are neglected because the changes of velocity produced by them during the very short time τ are small in comparison with the changes of velocity $\dot{x}' - \dot{x}$, $\dot{y}' - \dot{y}$, $\dot{z}' - \dot{z}$ produced by the impulses. For the mathematical treatment it is generally most convenient to define the impulse $\underset{.}{F}$ of an impulsive force F as the limit of the integral $\int_t^{t'} F dt$ when $t' - t$ approaches 0 and

F approaches ∞ (Art. 5); in this case it is strictly true that the effect of ordinary forces can be neglected when impulsive forces act on the body.

If the rigid body be originally at rest, it will be convenient to denote by $\dot{x}$, $\dot{y}$, $\dot{z}$ the components of the velocity of the particle m just after the action of the impulses. We may also denote by $\underset{.}{R}$ the resultant of all the impulses, by $\underset{.}{H}$ the resultant impulsive couple for the reduction to the origin of coordinates, and mark the components of $\underset{.}{R}$ and $\underset{.}{H}$ by subscripts, as in the case of forces. With these notations the effect of a system of impulses on a body at rest is given by the equations

$$\Sigma m\dot{x} = \underset{.}{R}_x, \quad \Sigma m\dot{y} = \underset{.}{R}_y, \quad \Sigma m\dot{z} = \underset{.}{R}_z, \tag{19}$$

$$\Sigma m(y\dot{z} - z\dot{y}) = \underset{.}{H}_x, \quad \Sigma m(z\dot{x} - x\dot{z}) = \underset{.}{H}_y, \quad \Sigma m(x\dot{y} - y\dot{x}) = \underset{.}{H}_z. \tag{20}$$

In the equations (19) we have, of course, $\Sigma m\dot{x} = M\dot{\bar{x}}$, $\Sigma m\dot{y} = M\dot{\bar{y}}$, $\Sigma m\dot{z} = M\dot{\bar{z}}$, where $\dot{\bar{x}}, \dot{\bar{y}}, \dot{\bar{z}}$ are the components of the velocity of the centroid, and M is the mass of the body; i.e. *the momentum of the centroid is equal to the resultant impulse.* The meaning of the equations (20) can be stated by saying that *the angular momentum of the body about any axis is equal to the moment of all the impulses about the same axis.*

II. *Moments of Inertia and Principal Axes.*

1. INTRODUCTION.

239. As will be shown in Sections III. and IV., the rotation of a rigid body about any axis depends not only on the forces acting on the body, but also on the way in which the mass is distributed throughout the body. This distribution of mass is characterized by the position of the centroid and by that of certain lines in the body called *principal axes.*

It has been shown in Part II., Art. 13, that the centroid of a mass is found by determining the *moments*, or more precisely, the *moments of the first order*, Σmx, Σmy, Σmz, of the mass with respect to the co-ordinate planes, *i.e.* the sums of all mass-particles m each multiplied by its distance from the co-ordinate plane.

The principal axes of a mass or body can be found by determining the *moments of the second order*, Σmx^2, Σmy^2, Σmz^2, Σmyz, Σmzx, Σmxy of the mass with respect to the same planes. We proceed, therefore, to study the theory of such moments.

240. If in a rigid body the mass m of each particle be multiplied by the square of its distance r from a given point, plane, or line, the sum

$$\Sigma mr^2 = m_1 r_1^2 + m_2 r_2^2 + \cdots,$$

extended over the whole body, is called the *quadratic moment*, or, more commonly, the **moment of inertia** of the body for that point, plane, or line.

If the body is not composed of discrete particles, but forms a continuous mass of one, two, or three dimensions, this mass can be resolved into elements of mass dm, and the sum Σmr^2 becomes a single, double, or triple integral $\int r^2 dm$.

Expressions of the form $\Sigma m r_1 r_2$, or $\int r_1 r_2 dm$, where r_1, r_2 are the distances of m or of dm from two planes (usually at right angles), are called *moments of deviation* or **products of inertia.**

241. The determination of the moment of inertia of a continuous mass is a mere problem of integration; the methods are similar to those for finding the moments of mass of the first order required for determining centroids (see Part II., Chapter III.), the only difference being that each element of mass must be multiplied by the square, instead of the first power, of the distance.

A moment of inertia is not a directed quantity; it is not a vector, but a scalar; indeed, it is a positive quantity, provided the masses are all positive, as we shall here assume.

The moment of inertia of any number of bodies or masses for any given point, plane, or line is obviously the sum of the moments of inertia of the separate bodies or masses for the same point, plane, or line.

242. The moment of inertia $\Sigma m r^2$ of any body whose mass is $M = \Sigma m$ can always be expressed in the form

$$\Sigma m r^2 = M \cdot r_0^2,$$

where r_0 is a length called the **radius of inertia**, arm of inertia, or *radius of gyration.* This length r_0 is evidently a kind of average value of the distances r, its value being intermediate between the greatest r' and least r'' of these distances r. For we have $\Sigma m r'^2 > \Sigma m r^2 > \Sigma m r''^2$, or, since $\Sigma m r'^2 = M r'^2$, $\Sigma m r^2 = M r_0^2$, $\Sigma m r''^2 = M r''^2$,

$$r' > r_0 > r''.$$

243. As an example, let us determine the moment of inertia of a *homogeneous rectilinear segment* (straight rod or wire of constant cross-section and density) for its middle point (or, what amounts to the same thing, for a line or plane through this point at right angles to the segment).

Let $2l$ be the length of the rod (Fig. 28), O its middle point, ρ its density (*i.e.* the mass of unit length), x the distance OP

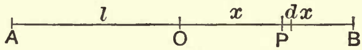

Fig. 28.

of any element $dm = \rho dx$ from the middle point. Then we have, for the moment of inertia I,

$$I = \int_{-l}^{l} x^2 \cdot \rho dx = \tfrac{2}{3}\rho l^3,$$

and for the radius of inertia r_0, since the whole mass is $M = 2\rho l$,

$$r_0^2 = \frac{I}{M} = \tfrac{1}{3} l^2.$$

244. Exercises.

Determine the radius of inertia in the following cases. When nothing is said to the contrary, the masses are supposed to be homogeneous.

(1) Segment of straight line of length l, for a perpendicular through one end.

(2) Rectangular area of length l and width h: (*a*) for the side h; (*b*) for the side l; (*c*) for a line through the centroid parallel to the side h; (*d*) for a line through the centroid parallel to the side l.

(3) Triangular area of base b and height h, for a line through the vertex parallel to the base.

(4) Square of side a, for a diagonal.

(5) Regular hexagon, for a diagonal.

(6) Right cylinder or prism of height $2h$, for the plane bisecting the height at right angles.

(7) Segment of straight line of length l, for one end, when the density is proportional to the nth power of the distance from this end. Deduce from this: (*a*) the result of Ex. (1); (*b*) that of Ex. (3); (*c*) the radius of inertia of a homogeneous pyramid or cone (right or oblique) of height h, for a plane through the vertex parallel to the base.

(8) Circular area (plate, disc, lamina) of radius a, for any diameter.

(9) Circular line (wire) of radius a, for a diameter.

(10) Solid sphere, for a diametral plane.

(11) Solid ellipsoid, for the three principal planes.

(12) Area of ring bounded by concentric circles of radii a_1, a_2, for a diameter.

(13) Area of the cross-section of a ⊥-iron: (*a*) for its line of symmetry; (*b*) for its base. (Dimensions as in Fig. 8, Part II., p. 18.)

(14) A rectangular door of width b and height h has a thickness δ to a distance a from the edges, while the rectangular panel (whose dimensions are $b-2a$, $h-2a$) has half this thickness. Find the moment of inertia for a line through the centroid parallel to the side b.

245. The moment of inertia of any mass *M for a point* can easily be found if the moments of inertia of the same mass are known for any line passing through the point, and for the plane through the point perpendicular to the line. Let O (Fig. 29) be the point, l the line, π the plane; r, q, p the perpendicular distances of any particle of mass m from O, l, π, respectively. Then we have, evidently, $r^2=q^2+p^2$. Hence, multiplying by m, and summing over the whole mass M,

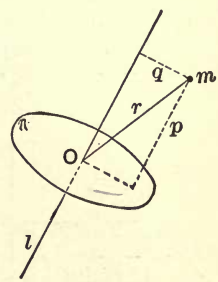

Fig. 29.

$$\Sigma mr^2=\Sigma mq^2+\Sigma mp^2; \tag{1}$$

or, putting $\Sigma mr^2=Mr_0^2$, $\Sigma mq^2=Mq_0^2$, $\Sigma mp^2=Mp_0^2$, where r_0, q_0, p_0 are the radii of inertia for O, l, π,

$$r_0^2=q_0^2+p_0^2. \tag{1'}$$

246. The moment of inertia of any mass *M for a line* is equal to the sum of the moments of inertia of the same mass for any two rectangular planes passing through the line. Thus, in particular, the moment of inertia for the axis of x in a rectangular system of co-ordinates is equal to the sum of the moments of inertia for the zx-plane and xy-plane. This follows at once by considering that the square of the distance of any point from the line is equal to the sum of the squares of the distances of the same point from the two planes. Thus, if q be the distance of any point (x, y, z) from the axis of x, we have $q^2 = y^2 + z^2$; whence

$$\Sigma mq^2 = \Sigma my^2 + \Sigma mz^2.$$

247. It follows, from the last article, that *the moment of inertia* I_x *of a plane area, for any line perpendicular to its plane*, is

$$I_x = I_y + I_z,$$

if I_y, I_z are the moments of inertia of the area for any two rectangular lines in the plane through the foot of the perpendicular line.

248. The problem of *finding the moment of inertia of a given mass for a line* l', *when it is known for a parallel line* l, is of great importance.

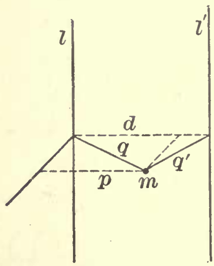

Fig. 30.

Let Σmq^2 be the moment of inertia of the given mass for the line l (Fig. 30), $\Sigma mq'^2$ that for a parallel line l' at the distance d from l. The distances q, q' of any particle m from l, l' form with d a triangle which gives the relation

$$q'^2 = q^2 + d^2 - 2qd \cos(q, d).$$

Multiplying by m, and summing over the whole mass M, we find

$$\Sigma mq'^2 = \Sigma mq^2 + Md^2 - 2d\Sigma mq \cos(q, d).$$

Now the figure shows that the product $q \cos(q, d)$ in the last term is the distance p of the particle m from a plane through l at right angles to the plane determined by l and l'. We have, therefore,

$$\Sigma mq'^2 = \Sigma mq^2 + Md^2 - 2d\Sigma mp, \tag{2}$$

where the last term contains the moment of the first order $\Sigma mp = M\bar{p}$ of the given mass M for the plane just mentioned.

If, in particular, this plane contains the centroid G of the mass M, we have $\Sigma mp=0$, so that the formula reduces to

$$\Sigma mq'^2=\Sigma mq^2+Md^2. \qquad (3)$$

Introducing the radii of inertia q_0', q_0, this can be written

$$q_0'^2=q_0^2+d^2. \qquad (3')$$

249. Similar considerations hold for the moments of inertia Σmp^2, $\Sigma mp'^2$ with respect to two parallel planes π, π' at the distance d from each other. We have, in this case, $p'=p-d$; hence,

$$\Sigma mp'^2=\Sigma mp^2+Md^2-2d\Sigma mp, \qquad (4)$$

and if the plane π contain the centroid G,

$$\Sigma mp'^2=\Sigma mp^2+Md^2. \qquad (5)$$

250. Of special importance is the case in which one of the lines (or planes), say l (π), contains the centroid. The formulæ (3), (3′), and (5) hold in this case; and if we agree to designate any line (plane) passing through the centroid as a *centroidal* line (plane), our proposition can be expressed as follows: *The moment of inertia for any line (plane) is found from the moment of inertia for the parallel centroidal line (plane) by adding to the latter the product* Md² *of the whole mass into the square of the distance of the lines (planes).*

It will be noticed that of all parallel lines (planes) the centroidal line (plane) has the least moment of inertia.

251. Exercises.

Determine the radius of inertia of the following homogeneous masses:

(1) Rectangular plate of length l and width h, for a centroidal line perpendicular to its plane.

(2) Area of equilateral triangle of side a: (a) for a centroidal line parallel to the base; (b) for an altitude; (c) for a centroidal line perpendicular to its plane.

(3) Circular disc of radius a: (a) for a tangent; (b) for a line through the centre perpendicular to the plane of the disc; (c) for a perpendicular to its plane through a point in the circumference.

(4) Solid sphere, for a diameter.

(5) Area of ring bounded by concentric circles of radii a_1, a_2, for a line through the centre perpendicular to the plane of the ring. For a ring whose thickness $a_2 - a_1$ is infinitesimal, the result can also be obtained by differentiation from Ex. (3) (*b*).

(6) Spherical shell of infinitesimal thickness, for a diameter.

(7) Right circular cylinder, of radius a and height $2h$: (*a*) for its axis; (*b*) for a generating line; (*c*) for a centroidal line in the middle cross-section.

(8) Prove that, in a right prism or cylinder of any cross-section, we have $q^2 = q_a^2 + q_c^2$, where q is the radius of inertia of the prism or cylinder for a line bisecting the axis at right angles, q_a the radius of inertia of the axis, q_c that of the middle cross-section, for the same line.

(9) Area of ellipse: (*a*) for the major axis; (*b*) for the minor axis; (*c*) for the perpendicular to its plane through the centre.

(10) Solid ellipsoid, for each of the three axes.

(11) Area of the cross-section of a ⊥-iron, for a centroidal line parallel to the flange. (Compare Art. 244, Ex. (13).)

(12) Area of the cross-section of a symmetrical double T-iron, width of flanges b, thickness of flanges δ, height of web h, thickness of web 2δ; for the two axes of symmetry, and for a centroidal line perpendicular to its plane.

(13) Wire bent into an equilateral triangle of side a, for a centroidal line at right angles to the plane of the triangle.

252. Routh's Rule. In the case of homogeneous masses with axes of symmetry, the radius of inertia for an axis of symmetry can readily be derived by the following mnemonical rule: *The square of the radius of inertia is* $\frac{1}{3}$, $\frac{1}{4}$, *or* $\frac{1}{5}$ *of the sum of the squares of the perpendicular semi-axes, according as the mass is rectangular, elliptic, or ellipsoidal.*

The proof rests on the following typical cases which are easily proved directly (comp. Art. 251, Ex. (1), (9), (10)):

(1) Rectangular area whose sides are $2a$, $2b$, for a centroidal line perpendicular to its plane: $q^2 = \frac{1}{3}(a^2 + b^2)$.

(2) Elliptic area whose axes are $2a$, $2b$, for a centroidal line perpendicular to its plane: $q^2 = \frac{1}{4}(a^2 + b^2)$.

(3) Solid ellipsoid whose axes are $2a$, $2b$, $2c$, for its axes:

$$q_a^2 = \tfrac{1}{5}(b^2 + c^2), \quad q_b^2 = \tfrac{1}{5}(c^2 + a^2), \quad q_c^2 = \tfrac{1}{5}(a^2 + b^2).$$

A large number of special cases can be brought under this rule, as will be seen from the following exercises. It should be remembered that the radius of inertia of a homogeneous right prism or cylinder for its axis is the same as that of its cross-section.

253. Exercises. Apply Routh's rule to find the radius of inertia in the following cases:

(1) Solid sphere of radius a, for a diameter.

(2) Right circular cylinder, for its axis.

(3) Thin straight rod of length $2a$, for a perpendicular through its middle point.

(4) Rectangular disc whose sides are $2a$, $2b$, for a line in its plane bisecting the sides $2a$.

(5) Circular disc, for a diameter.

2. ELLIPSOIDS OF INERTIA.

254. The moments of inertia of a given mass for the different lines of space are not independent of each other. Several examples of this have already been given. It has been shown, in particular (Art. 248), that if the moment of inertia be known for any line, it can be found for any parallel line. It follows that if the moments be known for all lines through any given point, the moments for all lines of space can be found. We now proceed to study the relations between the moments of inertia for all the lines passing through any given point O.

255. It will here be convenient to refer the given mass M to a rectangular system of co-ordinates with the origin at the point O. Let x, y, z be the co-ordinates of any particle m of the mass; and let us denote by A, B, C the moments of inertia of M for the axes of x, y, z; by A', B', C' those for the planes yz, zx, xy; by D, E, F the products of inertia (Art. 240) for the co-ordinate planes; *i.e.* let us put:

$$\begin{aligned} A&=\Sigma m(y^2+z^2), & A'&=\Sigma mx^2, & D&=\Sigma myz,\\ B&=\Sigma m(z^2+x^2), & B'&=\Sigma my^2, & E&=\Sigma mzx,\\ C&=\Sigma m(x^2+y^2), & C'&=\Sigma mz^2, & F&=\Sigma mxy. \end{aligned} \tag{6}$$

256. These nine quantities are not independent of each other. We have evidently

$$A=B'+C', \qquad B=C'+A', \qquad C=A'+B';$$

hence, solving for A', B', C',

$$A'=\tfrac{1}{2}(B+C-A), \qquad B'=\tfrac{1}{2}(C+A-B), \qquad C'=\tfrac{1}{2}(A+B-C).$$

The moment of inertia for the origin O is

$$\Sigma mr^2=\Sigma m(x^2+y^2+z^2)=A'+B'+C'=\tfrac{1}{2}(A+B+C). \tag{7}$$

257. The moment of inertia I for any line through O can be expressed by means of the six quantities A, B, C, D, E, F; and the moment of inertia I' for any plane through O can be expressed by means of A', B', C', D, E, F.

Let π (Fig. 31) be any plane passing through O; l its normal; α, β, γ the direction cosines of l; and, as before (Art. 245), p, q, r the distances of any point (x, y, z) of the given mass from π, l, and O, respectively. Then, projecting the closed polygon formed by r, x, y, z on the line l, we have

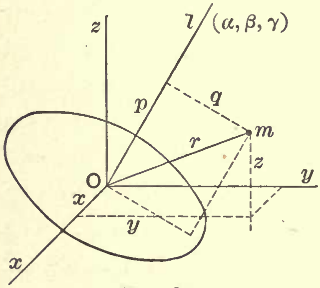

Fig. 31.

$$p=\alpha x+\beta y+\gamma z;$$

hence, squaring, multiplying by m, and summing over the whole mass, we find

$$\Sigma mp^2 = \alpha^2\Sigma mx^2+\beta^2\Sigma my^2+\gamma^2\Sigma mz^2+2\,\beta\gamma\Sigma myz+2\,\gamma\alpha\Sigma mzx+2\,\alpha\beta\Sigma mxy,$$

or, with the notations (6),

$$I'=A'\alpha^2+B'\beta^2+C'\gamma^2+2\,D\beta\gamma+2\,E\gamma\alpha+2\,F\alpha\beta. \tag{8}$$

Thus *the moment of inertia for any plane through the origin is expressed as a homogeneous quadratic function of the direction cosines of the normal of the plane.*

258. The moment of inertia $I=\Sigma mq^2$ for the line l can now be found from equation (1), Art. 245, by substituting for Σmr^2 and Σmp^2 their values from (7) and (8):

$$I=\Sigma mr^2-I'=A'+B'+C'-I'$$
$$=A'(1-\alpha^2)+B'(1-\beta^2)+C'(1-\gamma^2)-2D\beta\gamma-2E\gamma\alpha-2F\alpha\beta,$$

or, since $\alpha^2+\beta^2+\gamma^2=1$,

$$I=A'(\beta^2+\gamma^2)+B'(\gamma^2+\alpha^2)+C'(\alpha^2+\beta^2)-2D\beta\gamma-2E\gamma\alpha-2F\alpha\beta$$
$$=\alpha^2(B'+C')+\beta^2(C'+A')+\gamma^2(A'+B')-2D\beta\gamma-2E\gamma\alpha-2F\alpha\beta;$$

hence, finally, applying the relations of Art. 256,

$$I=A\alpha^2+B\beta^2+C\gamma^2-2D\beta\gamma-2E\gamma\alpha-2F\alpha\beta. \qquad (9)$$

The moment of inertia for any line through the origin is, therefore, also *a homogeneous quadratic function of the direction cosines of the line.*

259. These results suggest a geometrical interpretation. Imagine an arbitrary length $OP=\rho$ laid off from the origin O on the line l whose direction cosines are α, β, γ; the co-ordinates of the extremity P of this length will be $x=\rho\alpha$, $y=\rho\beta$, $z=\rho\gamma$. Now, if equation (9) be multiplied by ρ^2, it assumes the form

$$Ax^2+By^2+Cz^2-2Dyz-2Ezx-2Fxy=\rho^2 I,$$

which represents a quadratic surface provided that ρ be so selected for the different lines through O as to make $\rho^2 I$ constant, say $\rho^2 I=\kappa^2$. Hence, *if on every line* l *through the origin a length* $OP=\rho=\kappa/\sqrt{I}$ *be laid off,* i.e. *a length inversely proportional to the square root of the moment of inertia* I *for this line* l, *the points* P *will lie on the quadric surface*

$$Ax^2+By^2+Cz^2-2Dyz-2Ezx-2Fxy=\kappa^2.$$

The constant κ^2 may be selected arbitrarily; to preserve the homogeneity of the equation it will be convenient to put it into the form $\kappa^2 = M\epsilon^4$, where ϵ is still arbitrary.

260. As moments of inertia are essentially positive quantities, the radii vectores of the surface

$$Ax^2 + By^2 + Cz^2 - 2\,Dyz - 2\,Ezx - 2\,Fxy = M\epsilon^4 \qquad (10)$$

are all real, and the surface is an ellipsoid. It is called the *ellipsoid of inertia,* or the **momental ellipsoid**, of the point O. This point O is the centre; the axes of the ellipsoid are called the **principal axes** at the point O; and the moments of inertia for these axes are called the *principal moments of inertia* at the point O. Among these there will evidently be the greatest and least of all the moments of the point O, the greatest moment corresponding to the shortest, the least to the longest axis of the ellipsoid.

It may be observed that, owing to the relations of Art. 256, which show that the sum of any two of the quantities A, B, C is always greater than the third, not every ellipsoid can be regarded as the momental ellipsoid of some mass. An ellipsoid can be a momental ellipsoid only when a triangle can be constructed of its semi-axes.

261. If the axes of the ellipsoid (10) be taken as axes of co-ordinates, the equation assumes the form

$$I_1x^2 + I_2y^2 + I_3z^2 = M\epsilon^4, \qquad (11)$$

where I_1, I_2, I_3 are the principal moments at the point O.

By Art. 259 we have $\rho^2 = \kappa^2/I = M\epsilon^4/I$; hence $I = M\epsilon^4/\rho^2$. If, therefore, equation (11) be divided by ρ^2, the following simple expression is obtained for finding the moment of inertia, I, for a line whose direction cosines referred to the principal axes are α, β, γ,

$$I = I_1\alpha^2 + I_2\beta^2 + I_3\gamma^2. \qquad (12)$$

262. To make use of this form for I, the principal axes at the point O, *i.e.* the axes of the momental ellipsoid (10), must be known. The determination of the axes of an ellipsoid whose equation referred to the centre is given is a well-known problem of analytic geometry. It can be solved by considering that the semi-axes are those radii vectores of the surface that are normal to it. The direction cosines of the normal of any surface $F(x, y, z) = 0$ are proportional to the partial derivatives $\partial F/\partial x$, $\partial F/\partial y$, $\partial F/\partial z$. If, therefore, the radius vector ρ is a semi-axis, its direction-cosines α, β, γ must be proportional to the partial derivatives of (10); *i.e.* we must have

$$\frac{Ax - Fy - Ez}{\alpha} = \frac{-Fx + By - Dz}{\beta} = \frac{-Ex - Dy + Cz}{\gamma},$$

or dividing the numerators by ρ,

$$\frac{A\alpha - F\beta - E\gamma}{\alpha} = \frac{-F\alpha + B\beta - D\gamma}{\beta} = \frac{-E\alpha - D\beta + C\gamma}{\gamma}.$$

Denoting the common value of these fractions by I, we have

$$\alpha I = A\alpha - F\beta - E\gamma,\ \beta I = -F\alpha + B\beta - D\gamma,\ \gamma I = -E\alpha - D\beta + C\gamma;$$

multiplying these equations by α, β, γ, and adding, we find

$$I = A\alpha^2 + B\beta^2 + C\gamma^2 - 2D\beta\gamma - 2E\gamma\alpha - 2F\alpha\beta,$$

which, compared with (9), shows that I is the moment of inertia for the axis (α, β, γ). To obtain it in function of A, B, C, D, E, F, we write the preceding three equations in the form

$$\begin{aligned} (I-A)\alpha + F\beta + E\gamma &= 0, \\ F\alpha + (I-B)\beta + D\gamma &= 0, \\ E\alpha + D\beta + (I-C)\gamma &= 0, \end{aligned} \qquad (13)$$

whence, eliminating α, β, γ, we find I determined by the cubic equation

$$\begin{vmatrix} I-A, & F, & E \\ F, & I-B, & D \\ E, & D, & I-C \end{vmatrix} = 0. \qquad (14)$$

The roots of this cubic are the three principal moments I_1, I_2, I_3 of the point O. The direction-cosines of the principal axes are then found by substituting successively I_1, I_2, I_3 in (13) and solving for α, β, γ.

263. The geometrical representation of the moments of inertia for all lines passing through a point by means of the radii vectores of the momental ellipsoid at the point, gives at once a number of propositions about these moments. It is only necessary to interpret properly the geometrical properties of the ellipsoid. Thus, it is known that the sum of the squares of the reciprocals of any three rectangular semi-diameters of an ellipsoid is constant. It follows that the sum of the three moments of inertia for any three rectangular lines passing through the same point has a constant value.

In general, the three principal moments of inertia I_1, I_2, I_3 at a point O are different. If, however, two of them are equal, say $I_2 = I_3$, the momental ellipsoid becomes an ellipsoid of revolution about the third, I_1, as axis; and it follows that the moments of inertia for all lines through O lying in the plane of the two equal axes are equal.

If $I_1 = I_2 = I_3$, the ellipsoid becomes a sphere, and the moments of inertia are the same for all lines passing through O.

264. If the equation of the momental ellipsoid at a point O be of the form $Ax^2 + By^2 + Cz^2 - 2Dyz = M\epsilon^4$; *i.e.* if the two conditions

$$E \equiv \Sigma mzx = 0, \quad F \equiv \Sigma mxy = 0$$

be fulfilled, the axis of x coincides with one of the three axes of the ellipsoid, the surface being symmetrical with respect to the yz-plane. Hence, *if the conditions* E = 0, F = 0 *are satisfied, the axis of* x *is a principal axis at the origin.* The converse is evidently also true; *i.e.* if a line is a principal axis at one of its points, then, taking this point as origin and the line as axis of x, the conditions $\Sigma mzx = 0$, $\Sigma mxy = 0$ must be satisfied.

It is easy to see that if a line be a principal axis at one of its points, say O, it will in general not be a principal axis at any other one of its points. For, taking the line as axis of x and O as origin, we have $\Sigma mzx = 0$, $\Sigma mxy = 0$. If now for a point O' on this line at the distance a from O the line is likewise a principal axis, the conditions

$$\Sigma mz(x - a) = 0, \quad \Sigma m(x - a)y = 0$$

must be fulfilled. These reduce to

$$\Sigma mz = 0, \quad \Sigma my = 0,$$

and show that the line must pass through the centroid. And as for a centroidal line these conditions are satisfied independently of the value of a, it appears that a centroidal principal axis is principal axis at every one of its points. Hence *a line cannot be principal axis at more than one of its points unless it pass through the centroid; in the latter case it is principal axis at every one of its points.*

265. All those lines passing through a given point O for which the moments of inertia have the same value I can be shown to form a cone of the second order whose principal diameters coincide with the axes of the momental ellipsoid at O. This cone is called an **equimomental cone.** Its equation is obtained by regarding I as constant in equation (12) and introducing rectangular co-ordinates. Multiplying (12) by $\alpha^2+\beta^2+\gamma^2=1$, we find

$$(I_1-I)\alpha^2+(I_2-I)\beta^2+(I_3-I)\gamma^2=0;$$

and multiplying by ρ^2, we obtain the equation of the equimomental cone in the form

$$(I_1-I)x^2+(I_2-I)y^2+(I_3-I)z^2=0. \tag{15}$$

266. A slightly different form of the equations (11), (12), (15) is often more convenient; it is obtained by introducing the three **principal radii of inertia** q_1, q_2, q_3 defined by the relations

$$I_1=Mq_1^2, \quad I_2=Mq_2^2, \quad I_3=Mq_3^2.$$

The equation (11) of the momental ellipsoid at the point O then assumes the form

$$q_1^2x^2+q_2^2y^2+q_3^2z^2=\epsilon^4. \tag{11'}$$

The expression of the radius of inertia q for any line (α, β, γ) through O becomes

$$q^2=q_1^2\alpha^2+q_2^2\beta^2+q_3^2\gamma^2. \tag{12'}$$

Dividing (11′) by the square of the radius vector, ρ^2, and comparing with (12′), we find

$$q=\frac{\epsilon^2}{\rho},\ \rho=\frac{\epsilon^2}{q}, \tag{16}$$

as is otherwise apparent from the fundamental property of the momental ellipsoid (Art. 259).

The equation of the sphere of radius q described about O as centre, $x^2+y^2+z^2=q^2$, together with (11′), represents the curve of intersection of the ellipsoid with the sphere. Through this sphero-conic passes the equimomental cone, all of whose lines have the moment of inertia $I=Mq^2$. Hence, the equation of this cone can be written in the form

$$\left(\frac{q_1^{\,2}}{\epsilon^4}-\frac{1}{q^2}\right)x^2+\left(\frac{q_2^{\,2}}{\epsilon^4}-\frac{1}{q^2}\right)y^2+\left(\frac{q_3^{\,2}}{\epsilon^4}-\frac{1}{q^2}\right)z^2=0. \tag{15′}$$

267. If we assume $I_1>I_2>I_3$, and hence $q_1>q_2>q_3$, q must be $<\epsilon^2/q_3$ and $>\epsilon^2/q_1$. As long as q is less than the middle semi-axis ϵ^2/q_2 of the ellipsoid, the axis of the cone coincides with the axis of x, but when $q>\epsilon^2/q_2$, the axis of z is the axis of the cone. For $q=\epsilon^2/q_2$ the cone degenerates into the pair of planes $(q_1^{\,2}-q_2^{\,2})x^2-(q_2^{\,2}-q_3^{\,2})z^2=0$. These are the planes of the central circular (or *cyclic*) sections of the ellipsoid; they divide the ellipsoid into four wedges, of which one pair contains all the equimomental cones whose axes coincide with the greatest axis of the ellipsoid, while the other pair contains all those whose axes lie along the least axis of the ellipsoid.

268. There is another ellipsoid closely connected with the theory of principal axes; it is obtained from the momental ellipsoid by the process of reciprocation.

About any point O (Fig. 32) taken as centre let us describe a sphere of radius ϵ, and construct for every point P its polar plane π with regard to the sphere. If P describe any surface, the plane π will envelop another surface which is called the *polar reciprocal* of the former surface with regard to the sphere.

Let Q be the intersection of OP with π, and put $OP=\rho$, $OQ=q$; then it appears from the figure that

$$\rho q=\epsilon^2. \tag{16}$$

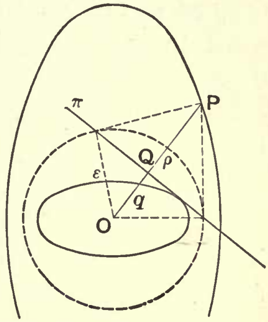

Fig. 32.

269. It is easy to see that the polar reciprocal of the momental ellipsoid (11′) with respect to the sphere of radius ϵ is the ellipsoid

$$\frac{x^2}{q_1^{\,2}}+\frac{y^2}{q_2^{\,2}}+\frac{z^2}{q_3^{\,2}}=1. \tag{17}$$

To prove this it is only necessary to show that the relation (16) is fulfilled for ρ as radius vector of (11′), and q as perpendicular to the tangent plane of (17). Now this tangent plane has the equation

$$\frac{x}{q_1^{\,2}}X+\frac{y}{q_2^{\,2}}Y+\frac{z}{q_3^{\,2}}Z=1;$$

hence we have for the direction cosines α, β, γ, and for the length q, of the perpendicular to the tangent plane

$$\frac{\alpha}{x/q_1^{\,2}}=\frac{\beta}{y/q_2^{\,2}}=\frac{\gamma}{z/q_3^{\,2}}=\frac{1}{[x^2/q_1^{\,4}+y^2/q_2^{\,4}+z^2/q_3^{\,4}]^{\frac{1}{2}}}=q.$$

These relations give $q_1\alpha=(x/q_1)q$, $q_2\beta=(y/q_2)q$, $q_3\gamma=(z/q_3)q$, whence

$$q_1^{\,2}\alpha^2+q_2^{\,2}\beta^2+q_3^{\,2}\gamma^2=\left(\frac{x^2}{q_1^{\,2}}+\frac{y^2}{q_2^{\,2}}+\frac{z^2}{q_3^{\,2}}\right)q^2=q^2. \tag{18}$$

For the radius vector ρ of (11′) whose direction cosines α, β, γ are the same as those of q we have by (11′):

$$\rho^2=\frac{\epsilon^4}{q_1^{\,2}\alpha^2+q_2^{\,2}\beta^2+q_3^{\,2}\gamma^2}.$$

Hence $\rho^2q^2=\epsilon^4$; and this is what we wished to prove.

270. The surface (17) has variously been called the *ellipsoid of gyration*, the *ellipsoid of inertia*, the **reciprocal ellipsoid.** We shall adopt the last name. The semi-axes of this ellipsoid are equal to the principal radii of inertia at the point O. The directions of its axes coincide with those of the momental ellipsoid; but the greatest axis of the former coincides with the least of the latter, and *vice versa.*

By comparing the equations (12′) and (18) it will be seen that q is the radius of inertia of the line (α, β, γ) on which it lies. Thus, *while the radius vector* $OP=\rho$ *of the momental ellipsoid is inversely proportional to the radius of inertia,* i.e. $\rho=\epsilon^2/q$, *the reciprocal ellipsoid gives the radius of inertia* q *for a line* l *as the segment cut off on this line by the perpendicular tangent plane.*

271. We are now prepared to determine the moment of inertia for any line in space. Let us construct at the centroid G of the given mass or body both the momental ellipsoid and its polar reciprocal. The former is usually called the **central ellipsoid** of the body; the latter we may call the **fundamental ellipsoid** of the body. As soon as this fundamental ellipsoid

$$\frac{x^2}{q_1^2}+\frac{y^2}{q_2^2}+\frac{z^2}{q_3^2}=1$$

is known, the moment of inertia of the body for any line whatever can readily be found. For, by Art. 270, the radius of inertia q for any line l_0 passing through the centroid is equal to the segment OQ cut off on the line l_0 by the perpendicular tangent plane of the fundamental ellipsoid; and for any line l not passing through the centroid the square of the radius of inertia can be determined by first finding the square of the radius of inertia for the parallel centroidal line l_0 and then, by Art. 250, adding to it the square of the distance d of the centroid from the line l.

272. In the problem of determining the ellipsoids of inertia for a given body at any point, considerations of symmetry are of great assistance, similarly as in the problem of finding the centroid (compare Part II., Art. 47).

Suppose a given mass to have a plane of symmetry; then taking this plane as the yz-plane, and a perpendicular to it as the axis of x, there must be, for every particle of mass m, whose co-ordinates are x, y, z, another particle of equal mass m, whose co-ordinates are $-x, y, z$. It follows that the two products of inertia Σmzx and Σmxy both vanish, whatever the position of the other two co-ordinate planes. Hence any perpendicular to the plane of symmetry is a principal axis at its point of intersection with this plane.

If the mass have two planes of symmetry at right angles to each other, then taking one as yz-plane, the other as zx-plane, and hence their intersection as axis of x, it is evident that all three products of inertia vanish,

$$\Sigma myz = 0, \qquad \Sigma mzx = 0, \qquad \Sigma mxy = 0,$$

wherever the origin be taken on the intersection of the two planes. Hence, for any point on this intersection, the principal axes are the line of intersection of the two planes of symmetry, and the two perpendiculars to it, drawn in each plane.

If there be three planes of symmetry, their point of intersection is the centroid, and their lines of intersection are the principal axes at the centroid.

273. Exercises.

Determine the principal axes and radii at the centroid, the central and fundamental ellipsoids, and show how to find the moment of inertia for any line, in the following Exercises (1), (2), (3).

(1) Rectangular parallelepiped, the edges being $2a$, $2b$, $2c$. Find also the moments of inertia for the edges and diagonals, and specialize for the cube.

(2) Ellipsoid of semi-axes a, b, c. Determine also the radius of inertia for a parallel l to the shortest axis passing through the extremity of the longest axis.

(3) Right circular cone of height h and radius of base a. Find first the principal moments at the vertex; then transfer to the centroid.

(4) Determine the momental ellipsoid and the principal axes at a vertex of a cube whose edge is a.

(5) Determine the radius of inertia of a thin wire bent into a circle, for a line through the centre inclined at an angle α to the plane of the circle.

(6) A peg-top is composed of a cone of height H and radius a, and a hemispherical cap of the same radius. The point, to a distance h from the vertex of the cone, is made of a material three times as heavy as the rest. Find the moment of inertia for the axis of rotation; specialize for $h = a = \frac{1}{3} H$.

(7) Show that the principal axes at any point A, situated on one of the principal axes of a body, are parallel to the centroidal principal axes, and find their moments of inertia.

(8) For a given body of mass M find the points at which the momental ellipsoid reduces to a sphere.

(9) Determine a homogeneous ellipsoid having the same mass as a given body, and such that its moment of inertia for every line shall be the same as that of the given body.

3. DISTRIBUTION OF PRINCIPAL AXES IN SPACE.

274. It has been shown in the preceding articles how the principal axes can be determined at any particular point. The distribution of the principal axes throughout space and their position at the different points is brought out very graphically by means of the theory of confocal quadrics. It can be shown that the directions of the principal axes at any point are those of the principal diameters of the tangent cone drawn from this point as vertex to the fundamental ellipsoid; or, what amounts to the same thing, they are the directions of the normals of the three quadric surfaces passing through the point and confocal to the fundamental ellipsoid.

In order to explain and prove these propositions it will be necessary to give a short sketch of the theory of confocal conics and quadrics.

275. *Two conic sections are said to be* **confocal** *when they have the same foci.* The directions of the axes of all conics having the same two points S, S' as foci must evidently coincide, and the equation of such conics can be written in the form

$$\frac{x^2}{a^2 + \lambda} + \frac{y^2}{b^2 + \lambda} = 1, \qquad (19)$$

where λ is an arbitrary parameter. For, whatever value may be assigned in this equation to λ, the distance of the centre O from either focus will always be $\sqrt{a^2+\lambda-(b^2+\lambda)}=\sqrt{a^2-b^2}$; it is therefore constant.

276. The individual curves of the whole system of confocal conics represented by (19) are obtained by giving to λ any particular value between $-\infty$ and $+\infty$; thus we may speak of the conic λ of the system.

For $\lambda = 0$ we have the so-called fundamental conic $x^2/a^2+y^2/b^2=1$; this is an ellipse. To fix the ideas let us assume $a > b$. For all values of $\lambda > -b^2$, *i.e.* as long as $-b^2 < \lambda < \infty$, the conics (19) are ellipses, beginning with the rectilinear segment SS' (which may be regarded as a degenerated ellipse $\lambda = -b^2$ whose minor axis is 0), expanding gradually, passing through the fundamental ellipse $\lambda = 0$, and finally verging into a circle of infinite radius for $\lambda = \infty$.

It is thus geometrically evident that through every point in the plane will pass one, and only one, of these ellipses.

277. Let us next consider what the equation (19) represents when λ is algebraically less than $-b^2$. The values of λ that are $< -a^2$ give imaginary curves, and are of no importance for our purpose. But as long as $-a^2 < \lambda < -b^2$, the curves are hyperbolas. The curve $\lambda = -b^2$ may now be regarded as a degenerated hyperbola collapsed into the two rays issuing in opposite directions from S and S' along the line SS'. The degenerated ellipse together with this degenerated hyperbola thus represents the whole axis of x.

As λ decreases, the hyperbola expands, and finally, for $\lambda = -a^2$, verges into the axis of y, which may be regarded as another degenerated hyperbola.

The system of confocal hyperbolas is thus seen to cover likewise the whole plane so that one, and only one, hyperbola of the system passes through every point of the plane.

278. The fact that every point of the plane has one ellipse and one hyperbola of the confocal system (19) passing through it allows us to regard the two values of the parameter λ that determine these two curves as co-ordinates of the point; they are called *elliptic co-ordinates*. If x, y be the rectangular Cartesian co-ordinates of the point, its elliptic co-ordinates λ_1, λ_2 are found as the roots of the equation (19) which is quadratic in λ. Conversely, to transform from elliptic to

Cartesian co-ordinates, that is, to express x and y in terms of λ_1 and λ_2, we have only to solve for x and y the two equations

$$\frac{x^2}{a^2+\lambda_1}+\frac{y^2}{b^2+\lambda_1}=1,\quad \frac{x^2}{a^2+\lambda_2}+\frac{y^2}{b^2+\lambda_2}=1.$$

279. The two confocal conics that pass through the same point P intersect at right angles. For the tangent to the ellipse at P bisects the exterior angle at P in the triangle SPS', while the tangent to the hyperbola bisects the interior angle at the same point; in other words, the tangent to one curve is normal to the other, and *vice versa*. The elliptic system of co-ordinates is, therefore, an *orthogonal* system; the infinitesimal elements $d\lambda_1 \cdot d\lambda_2$ into which the two series of confocal conics (19) divide the plane are rectangular, though curvilinear.

280. These considerations are easily extended to space of three dimensions.

An ellipsoid

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1, \text{ where } a>b>c,$$

has six real foci in its principal planes; two, S_1, S_1', in the xy-plane, on the axis of x, at a distance $OS_1=\sqrt{a^2-b^2}$ from the centre O; two, S_2, S_2', in the yz-plane, on the axis of y, at the distance $OS_2=\sqrt{b^2-c^2}$ from the centre; and two, S_3, S_3', in the zx-plane, on the axis of x, at the distance $OS_3=\sqrt{a^2-c^2}$ from the centre. It should be noticed that, since $b>c$, we have $OS_3>OS_1$; *i.e.* S_1, S_1' lie between S_3, S_3' on the axis of x.

The same holds for hyperboloids.

281. *Two quadric surfaces are said to be confocal when their principal sections are confocal conics.* Now this will be the case for two quadric surfaces whose semi-axes are a_1, b_1, c_1, and a_2, b_2, c_2, if the directions of their axes coincide and if

$$a_1^2-b_1^2=a_2^2-b_2^2,\quad b_1^2-c_1^2=b_2^2-c_2^2,\quad a_1^2-c_1^2=a_2^2-c_2^2.$$

Writing these conditions in the form

$$a_2^2-a_1^2=b_2^2-b_1^2=c_2^2-c_1^2, \text{ say } =\lambda,$$

we find $a_2^2=a_1^2+\lambda$, $b_2^2=b_1^2+\lambda$, $c_2^2=c_1^2+\lambda$. Hence the equation

$$\frac{x^2}{a^2+\lambda}+\frac{y^2}{b^2+\lambda}+\frac{z^2}{c^2+\lambda}=1, \qquad (20)$$

where λ is a variable parameter, represents a system of confocal quadric surfaces.

282. As long as λ is algebraically greater than $-c^2$, the equation (20) represents ellipsoids. For $\lambda = -c^2$ the surface collapses into the interior area of the ellipse in the xy-plane whose vertices are the foci S_2, S_2' and S_3, S_3'. For as λ approaches the limit $-c^2$, the three semi-axes of (20) approach the limits $\sqrt{a^2-c^2}$, $\sqrt{b^2-c^2}$, 0, respectively. This limiting ellipse is called the *focal ellipse*. Its foci are the points S_1, S_1', since $a^2 - c^2 - (b^2 - c^2) = a^2 - b^2$.

When λ is algebraically $< -c^2$, but $> -a^2$, the equation (20) represents hyperboloids; for values of $\lambda < -a^2$ it is not satisfied by any real points. As long as $-b^2 < \lambda < -c^2$, the surfaces are hyperboloids of one sheet. The limiting surface $\lambda = -c^2$ now represents the exterior area of the focal ellipse in the xy-plane. The limiting hyperboloid of one sheet for $\lambda = -b^2$ is the area in the zx-plane bounded by the hyperbola whose vertices are S_1, S_1', and whose foci are S_3, S_3'. This is called the *focal hyperbola*.

Finally, when $-a^2 < \lambda < -b^2$, the surfaces are hyperboloids of two sheets, the limiting hyperboloid $\lambda = -a^2$ collapsing into the yz-plane.

283. It appears from these geometrical considerations, that there are passing through every point of space three surfaces confocal to the fundamental ellipsoid $x^2/a^2 + y^2/b^2 + z^2/c^2 = 1$ and to each other, viz.: an ellipsoid, a hyperboloid of one sheet, and a hyperboloid of two sheets. This can also be shown analytically, as there is no difficulty in proving that the equation (20) has three real roots, say λ_1, λ_2, λ_3, for every set of real values of x, y, z, and that these roots are confined between such limits as to give the three surfaces just mentioned.

The quantities λ_1, λ_2, λ_3 can therefore be taken as co-ordinates of the point (x, y, z); and these *elliptic co-ordinates* of the point are, geometrically, the parameters of the three quadric surfaces passing through the point and confocal to the fundamental ellipsoid; while, analytically, they are the three roots of the cubic (20). To express x, y, z in terms of the elliptic co-ordinates, it is only necessary to solve for x, y, z the three equations obtained by substituting in (20) successively λ_1, λ_2, λ_3 for λ.

284. The geometrical meaning of the parameter λ will appear by considering two parallel tangent planes π_0 and π_λ (on the same side of the origin), the former (π_0) tangent to the fundamental ellipsoid $x^2/a^2+y^2/b^2+z^2/c^2=1$, the latter ($\pi_\lambda$) tangent to any confocal surface λ or $x^2/(a^2+\lambda)+y^2/(b^2+\lambda)+z^2/(c^2+\lambda)=1$. The perpendiculars q_0, q_λ, let fall from the origin O on these tangent planes π_0, π_λ, are given by the relations (the proof being the same as in Art. 269).

$$q_0^2 = a^2\alpha^2 + b^2\beta^2 + c^2\gamma^2, \tag{21}$$

$$q_\lambda^2 = (a^2+\lambda)\alpha^2 + (b^2+\lambda)\beta^2 + (c^2+\lambda)\gamma^2, \tag{22}$$

where α, β, γ are the direction-cosines of the common normal of the planes π_0, π_λ. Subtracting (21) from (22), we find, since $\alpha^2+\beta^2+\gamma^2=1$,

$$q_\lambda^2 - q_0^2 = \lambda; \tag{23}$$

i.e. *the parameter* λ *of any one of the confocal surfaces* (20) *is equal to the difference of the squares of the perpendiculars let fall from the common centre on any tangent plane to the surface* λ, *and on the parallel tangent plane to the fundamental ellipsoid* $\lambda = 0$.

285. Let us now apply these results to the question of the distribution of the principal axes throughout space.

We take the centroid G of the given body as origin, and select as fundamental ellipsoid of our confocal system the polar reciprocal of the central ellipsoid, *i.e.* the ellipsoid (17) formed for the centroid, for which the name "fundamental ellipsoid of the body" was introduced in Art. 271. Its equation is

$$\frac{x^2}{q_1^2} + \frac{y^2}{q_2^2} + \frac{z^2}{q_3^2} = 1,$$

if q_1, q_2, q_3 are the principal radii of inertia of the body.

The radius of inertia q_0 for any centroidal line l_0 can be constructed (Art. 270) by laying a tangent plane to this ellipsoid perpendicular to the line l_0; if this line meets the tangent plane in Q_0 (Fig. 33), then $q_0 = GQ_0$. Analytically, if α, β, γ be the direction-cosines of l_0, q_0 is given by formula (21) or (12′).

286. To find the radius of inertia q for a line l, parallel to l_0, and passing through any point P, we lay through P a plane π_λ, perpendicular to l, and a parallel plane π_0, tangent to the fundamental ellipsoid; let Q_λ, Q_0 be the intersections of these planes with the centroidal line l_0. Then, putting $GQ_0=q_0$, $GQ_\lambda=q_\lambda$, $GP=r$, $PQ_\lambda=d$, we have, by Art. 250,

$$q^2 = q_0^2 + d^2.$$

The figure gives the relation $d^2 = r^2 - q_\lambda^2$, which, in combination with (23), reduces the expression for the radius of inertia for the line l to the simple form:

$$q^2 = r^2 - \lambda. \qquad (24)$$

287. The value of $r^2 - \lambda$, and hence the value of q, remains the same for the perpendiculars to all planes through P, tangent to the same quadric surface λ: these perpendiculars form, therefore, an equimomental cone at P. By varying λ we thus obtain all the equimomental cones at P. The principal diameters of all these cones coincide in direction, since they coincide with the directions of the principal axes of the momental ellipsoid at P (see Art. 265); but they also coincide with the principal diameters of the cones enveloped by the tangent planes π_λ. It thus appears that *the principal axes at the point* P *coincide in direction with the principal diameters of the tangent cone from* P *as vertex to the fundamental ellipsoid* $x^2/q_1^2 + y^2/q_2^2 + z^2/q_3^2 = 1$.

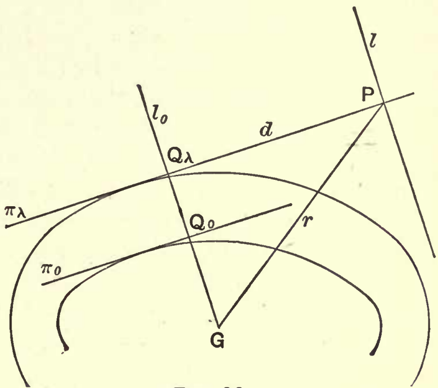

Fig. 33.

288. Instead of the fundamental ellipsoid, we might have used any quadric surface λ confocal to it. In particular, we may select the confocal surfaces $\lambda_1, \lambda_2, \lambda_3$ that pass through P. For each of these the cone of the tangent planes collapses into a plane, viz. the tangent plane to the surface at P, while the cone of the perpendiculars reduces to a single line, viz. the normal to the surface at P. Thus we find that *the principal axes at any point* P *coincide in direction with the normals to the three quadric surfaces, confocal to the fundamental ellipsoid and passing through* P.

For the magnitudes of the principal radii q_x, q_y, q_z at P, we evidently have

$$q_x^2 = r^2 - \lambda_1, \qquad q_y^2 = r^2 - \lambda_2, \qquad q_z^2 = r^2 - \lambda_3.$$

289. Exercise.

(1) The principal radii q_1, q_2, q_3 of a body being given, find the equation of the momental ellipsoid at any point P, referred to axes

through this point P parallel to the principal axes of the body; determine the directions of the principal axes at P, and show that these directions coincide with the normals of the three surfaces passing through P and confocal to the fundamental ellipsoid of the body.

290. A brief account of the theory of moments of inertia will be found in B. WILLIAMSON, *Integral Calculus*, 6th ed., London, Longmans, 1891, pp. 291–312. The subject is discussed very fully in E. J. ROUTH, *Dynamics of a system of rigid bodies*, Part I., 5th ed., London, Macmillan, 1891, pp. 1–49; and in B. PRICE, *Analytical mechanics*, Vol. II., 2d ed., Oxford, Clarendon Press, 1889, Chapter IV. The student will also consult with advantage W. SCHELL, *Theorie der Bewegung und der Kräfte*, Vol. I., 2d ed., Leipzig, Teubner, 1879, pp. 100–143, and A. CAYLEY, *Report on the progress of the solution of certain special problems of dynamics*, in the Report of the 32d meeting of the British Association, for 1862, London, Murray, 1863, pp. 223–229.

III. *Rigid Body with a Fixed Axis.*

291. A rigid body with a fixed axis has but one degree of freedom. Its motion is fully determined by the motion of any one of its points (not situated on the axis), and any such point must move in a circle about the axis. Any particular position of the body is, therefore, determined by a single variable, or co-ordinate, such as the angle of rotation. Just as the equilibrium of such a body depends on a single condition (see Part II., Art. 227), so its motion is given by a single equation.

292. The equation of motion can be derived directly from the proposition of angular momentum (Art. 224). Let r be the distance of any particle m of the body from the fixed axis, ω the angular velocity at the time t; then $m\omega r$ is the momentum of the particle, and $m\omega r^2$ its moment, or the angular momentum of the particle, about the axis. At any given instant t, ω has the same value for all particles. Hence, the angular momentum of the body is $\omega\Sigma mr^2 = \omega I$, where $I = \Sigma mr^2$ is the moment of inertia of the body for the fixed axis.

Now, by Art. 224, the rate at which the angular momentum of the body about the axis changes with the time is equal to the sum of the moments of all the external forces about the same axis. Denoting this resulting moment by H, and considering that the moment of inertia for the fixed axis is independent of the time, we have the **equation of motion**

$$\frac{d\omega}{dt} = \frac{H}{I}; \qquad (1)$$

i.e. *the angular acceleration about the fixed axis is equal to the moment of all the external forces about this axis, divided by the moment of inertia of the body for the same axis.*

293. The same result can of course be obtained from any one of the equations (6) or (7), Art. 224. Thus, taking the fixed line as axis of z, the third of the equation (7), viz.

$$\frac{d}{dt}\Sigma m(x\dot{y} - y\dot{x}) = H_z,$$

must be used. Now, for rotation of angular velocity ω about the axis of z, we have $\dot{x} = -\omega y$, $\dot{y} = \omega x$. Hence

$$\Sigma m(x\dot{y} - y\dot{x}) = \omega \Sigma m(x^2 + y^2) = \omega \Sigma m r^2 = \omega I.$$

The equation assumes, therefore, the form (1).

294. The reactions of the fixed axis do not enter into the composition of the resulting moment H. As they intersect the axis, their moments about this axis are zero.

The student should notice the close analogy between equation (1) and the equation for the rectilinear motion of a particle,

$$\frac{dv}{dt} = \frac{F}{m},$$

where v is the velocity and F the resultant of all the forces acting on the particle.

The expression for the *kinetic energy* of a body rotating about a fixed axis is

$$T = \Sigma \tfrac{1}{2} m v^2 = \Sigma \tfrac{1}{2} m \omega^2 r^2 = \tfrac{1}{2} I \omega^2, \qquad (2)$$

and has also a form similar to that for the kinetic energy of a particle m moving with velocity v in a straight line, viz.

$$T = \tfrac{1}{2} m v^2.$$

295. Let us denote the angle of rotation by θ, so that $\omega = d\theta/dt$, $d\omega/dt = d^2\theta/dt^2$. If the resulting moment be constant or a given function of θ, say $H = f(\theta)$, the equation of motion

$$I \frac{d^2\theta}{dt^2} = f(\theta)$$

can be integrated once, and gives

$$\tfrac{1}{2} I (\omega^2 - \omega_0^2) = \int_{\theta_0}^{\theta} f(\theta) d\theta, \qquad (3)$$

where ω_0 is the angular velocity corresponding to the angle θ_0.

This is the *equation of kinetic energy*. It might have been derived directly, according to Art. 234, by expressing that the increase of the kinetic energy equals the work of the forces.

The kinetic energy is given by (2). The work of a force F in a plane perpendicular to the axis, at the distance p from the axis, is $F\cdot pd\theta$ for an infinitesimal rotation of angle $d\theta$; hence, the sum of the elementary works of all the forces $= \Sigma Fpd\theta = Hd\theta = f(\theta)d\theta$.

296. While thus the motion of a rigid body about a fixed axis is given by a single equation, the other equations of motion of a rigid body are required to determine the *reactions of the fixed axis* (comp. Part II., Art. 227).

The axis will be fixed if any two of its points A, B are fixed. The reaction of the fixed point A can be resolved into three components A_x, A_y, A_z, that of B into B_x, B_y, B_z. By introducing these reactions the body becomes free; and the system composed of these reactions, of the external forces, and of the reversed effective forces must be in equilibrium. We take again the axis of rotation as axis of z (Fig. 34) so that the z-co-ordinates of the particles are constant, and hence $\dot{z}=0$, $\ddot{z}=0$; and we put $OA=a$, $OB=b$. Then the six equations of motion are (see Art. 223 (4) and Art. 224 (6)):

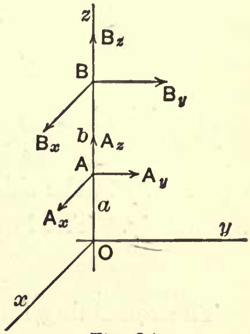

Fig. 34.

$$\begin{aligned} \Sigma m\ddot{x} &= \Sigma X + A_x + B_x, \\ \Sigma m\ddot{y} &= \Sigma Y + A_y + B_y, \\ 0 &= \Sigma Z + A_z + B_z, \\ -\Sigma mz\ddot{y} &= \Sigma(yZ - zY) - aA_y - bB_y, \\ \Sigma mz\ddot{x} &= \Sigma(zX - xZ) + aA_x + bB_x, \\ \Sigma m(x\ddot{y} - y\ddot{x}) &= \Sigma(xY - yX). \end{aligned}$$

297. It remains to introduce into these equations the values for $\ddot{x}, \ddot{y}$. As the motion is a pure rotation, we have (see Part I., Art. 245) $\dot{x} = -\omega y$, $\dot{y} = \omega x$; hence, $\ddot{x} = -\dot{\omega}y - \omega^2 x$, $\ddot{y} = \dot{\omega}x - \omega^2 y$. Summing over the whole body, we find

$$\begin{aligned} \Sigma m\ddot{x} &= -\dot{\omega}\Sigma my - \omega^2\Sigma mx = -M\dot{\omega}\bar{y} - M\omega^2\bar{x}, \\ \Sigma m\ddot{y} &= \dot{\omega}\Sigma mx - \omega^2\Sigma my = M\dot{\omega}\bar{x} - M\omega^2\bar{y}, \end{aligned}$$

where $\bar{x}, \bar{y}$ are the co-ordinates of the centroid; and

$$-\Sigma mz\ddot{y} = -\dot{\omega}\Sigma mzx + \omega^2\Sigma myz = -E\dot{\omega} + D\omega^2,$$
$$\Sigma mz\ddot{x} = -\dot{\omega}\Sigma myz - \omega^2\Sigma mzx = -D\dot{\omega} - E\omega^2,$$
$$\Sigma m(x\ddot{y} - y\ddot{x}) = \dot{\omega}\Sigma mx^2 - \omega^2\Sigma mxy + \dot{\omega}\Sigma my^2 + \omega^2\Sigma mxy = C\dot{\omega},$$

where $C = \Sigma m(x^2+y^2)$, $D = \Sigma myz$, $E = \Sigma mzx$ are the notations introduced in Art. 255.

With these values the equations of motion assume the form:

$$\left\{\begin{aligned} -M\bar{x}\omega^2 - M\bar{y}\dot{\omega} &= \Sigma X + A_x + B_x, \\ -M\bar{y}\omega^2 + M\bar{x}\dot{\omega} &= \Sigma Y + A_y + B_y, \\ 0 &= \Sigma Z + A_z + B_z, \\ D\omega^2 - E\dot{\omega} &= \Sigma(yZ - zY) - aA_y - bB_y, \\ -E\omega^2 - D\dot{\omega} &= \Sigma(zX - xZ) + aA_x + bB_x, \\ C\dot{\omega} &= \Sigma(xY - yX). \end{aligned}\right. \qquad (4)$$

298. The last equation is identical with equation (1).

The components of the reactions along the axis of rotation occur only in the third equation, and can therefore not be found separately. The longitudinal pressure on the axis is

$$= -A_z - B_z = \Sigma Z.$$

The remaining four equations are sufficient to determine A_x, A_y, B_x, B_y.

The total stress to which the axis is subject, instead of being resolved into two forces, at A and B, can be reduced for the origin O to a force and a couple (see Fig. 34). The equations (4) give for the components of the force

$$\begin{aligned} -A_x - B_x &= \Sigma X + M\bar{x}\omega^2 + M\bar{y}\dot{\omega}, \\ -A_y - B_y &= \Sigma Y + M\bar{y}\omega^2 - M\bar{x}\dot{\omega}, \\ -A_z - B_z &= \Sigma Z. \end{aligned} \qquad (5)$$

This force consists of the resultant of the external forces,

$$R = \sqrt{(\Sigma X)^2 + (\Sigma Y)^2 + (\Sigma Z)^2},$$

and two forces in the xy-plane which form the reversed effective force of the centroid; for $M\bar{x}\omega^2$ and $M\bar{y}\omega^2$ give as resultant the centrifugal force $M\omega^2\sqrt{\bar{x}^2+\bar{y}^2}=M\omega^2\bar{r}$, directed from the axis towards the projection of the centroid on the xy-plane, while $M\bar{y}\dot{\omega}, -M\bar{x}\dot{\omega}$ form the tangential resultant $M\dot{\omega}\bar{r}$, perpendicular to the plane through axis and centroid.

The couple has a component in the yz-plane, and one in the zx-plane, viz.:

$$\begin{aligned} aA_y+bB_y &= \Sigma(yZ-zY)-D\omega^2+E\dot{\omega}, \\ -aA_x-bB_x &= \Sigma(zX-xZ)+E\omega^2+D\dot{\omega}, \end{aligned} \qquad (6)$$

while the component in the xy-plane is zero. The resultant couple lies, therefore, in a plane passing through the axis of rotation.

299. In the particular case *when no forces* X, Y, Z *are acting on the body*, the last of the equations (4), or equation (1), shows that *the angular velocity* ω *remains constant.* The stress on the axis of rotation will, however, exist; and the axis will in general tend to change both its direction, owing to the couple (6), and its position, owing to the force (5).

If the axis be not fixed as a whole, but only one of its points, the origin, be fixed, the force (5) is taken up by the fixed point, while the couple (6) will change the direction of the axis. Now this couple vanishes if, in addition to the absence of external forces, the conditions

$$D\equiv\Sigma myz=0, \quad E\equiv\Sigma mzx=0 \qquad (7)$$

are fulfilled. In this case the body would continue to rotate about the axis of z even if this axis were not fixed, provided that the origin is a fixed point. A line having this property is called a **permanent axis of rotation.**

As the meaning of the conditions (7) is that the axis of z is a principal axis of inertia at the origin (see Art. 264), we have the proposition that *if a rigid body with a fixed point, not acted upon by any forces, begin to rotate about one of the principal axes at this point, it will continue to rotate uniformly about the same axis.* In other words, the principal axes at any point are always, and are the only, permanent axes of rotation. This can be regarded as the dynamical definition of principal axes.

300. It appears from the equations (5) that the *position* of the axis of rotation will remain the same if, in addition to the absence of external forces, the conditions

$$\bar{x}=0, \quad \bar{y}=0 \tag{8}$$

be fulfilled; for in this case the components of the force (5) all vanish. If, moreover, the axis of rotation be a principal axis, the rotation will continue to take place about the same line even when the body has no fixed point.

The conditions (8) mean that the centroid lies on the axis of z; and it is known (Art. 264) that a centroidal principal axis is a principal axis at every one of its points. The axis of z must therefore be a principal axis of the body, *i.e.* a principal axis at the centroid. We have, therefore, the proposition: *If a free rigid body, not acted upon by any forces, begin to rotate about one of its centroidal principal axes, it will continue to rotate uniformly about the same line.*

301. A rigid body with a fixed *horizontal* axis is called a **compound pendulum** if the only external force acting is the weight of the body.

The plane through axis and centroid will make, with the vertical plane (downwards) through the axis, an angle θ, which we may take as angle of rotation, so that $\omega=d\theta/dt$ (Fig. 35). The weights of the particles, being all parallel and proportional to their masses, have a single resultant Mg passing through the centroid G. Hence, if h be the perpendicular distance OG of the centroid from the axis, the moment of the external forces is $H=-Mgh \sin\theta$; and if the radius of inertia of the body for the centroidal axis parallel to the axis of rotation be q, the moment of inertia for the latter axis is $I=M(q^2+h^2)$.

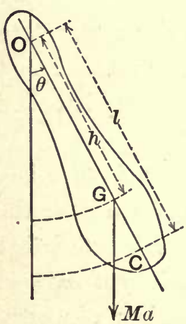

Fig. 35.

With these values the equation of motion (1) assumes the simple form

$$\frac{d^2\theta}{dt^2} = -\frac{gh}{q^2+h^2}\sin\theta. \tag{9}$$

As shown in Art. 175, the equation of the *simple* pendulum of length l is

$$\frac{d^2\theta}{dt^2} = -\frac{g}{l}\sin\theta.$$

The two equations differ only in the constant factor of $\sin\theta$, and it appears that *the motion of a compound pendulum is the same as that of a simple pendulum whose length is*

$$l = h + \frac{q^2}{h}. \tag{10}$$

302. The problem of the compound pendulum has thus been reduced to that of the simple pendulum. The length l is called *the length of the equivalent simple pendulum.* The foot O (Fig. 35) of the perpendicular let fall from the centroid on the axis is called *the centre of suspension.* If on the line OG a length $OC = l$ be laid off, the point C is called *the centre of oscillation.* It appears, from (10), that G lies between O and C.

The relation (10) can be written in the form

$$h(l-h) = q^2, \text{ or } OG \cdot GC = \text{const.}$$

As this relation is not altered by interchanging O and C, it follows that *the centres of oscillation and suspension are interchangeable; i.e.* the period of a compound pendulum remains the same if it be made to swing about a parallel axis through the centre of oscillation.

303. Exercises.

(1) A pendulum, formed of a cylindrical rod of radius a and length L, swings about a diameter of one of the bases. Find the time of a small oscillation.

(2) A cube, whose edge is a, swings as a pendulum about an edge. Find the length of the equivalent simple pendulum.

(3) A circular disc of radius r revolves uniformly about its axis, making 100 revolutions per minute. What is its kinetic energy?

(4) A fly-wheel of radius r, in which a mass, equal to that of the disc in Ex. (3), is distributed uniformly along the rim, has the same angular velocity as the disc. Neglecting the mass of the nave and spokes, determine its kinetic energy, and compare it with that of the disc.

(5) A fly-wheel of 12 ft. diameter, whose rim weighs 12 tons, makes 50 revolutions per minute. Find its kinetic energy in foot-pounds.

(6) A fly-wheel of radius r and mass m is making N revolutions per minute when the steam is shut off. If the radius of the shaft be r', and the coefficient of friction μ, find after how many revolutions the wheel will come to rest owing to the axle friction.

(7) A fly-wheel of 10 ft. diameter, weighing 5 tons, is making 40 revolutions when thrown out of gear. In what time does it come to rest if the diameter of the axle is 6 in. and the coefficient of friction $\mu = 0.05$?

(8) A uniform straight rod of length l is hinged at one end so as to turn freely in a vertical plane. If it be dropped from a horizontal position, with what angular velocity does it pass through the vertical position? (Equate the kinetic energy to the work of gravity.)

304. Impulses. Suppose a rigid body with a fixed axis is acted upon, when at rest, by a single impulse F, in a plane perpendicular to the axis and at the distance p from the axis. It is required to determine the initial motion of the body just after impact.

As the impulsive reactions of the fixed axis have no moment about this axis, the initial angular momentum of the body about the fixed axis must be equal to the moment of the impulse F about the same axis; *i.e.* to Fp. If ω is the initial angular velocity, the momentum of a particle m at the distance r from the axis is $m\omega r$; hence the angular momentum of the body $= \Sigma m\omega r^2 = \omega\Sigma mr^2 = \omega I$, where I is the moment of inertia of the body for the fixed axis. Hence we have

$$\omega = \frac{Fp}{I}. \qquad (11)$$

305. Let the impulse F be produced by the inelastic impact of a particle of mass m moving with a velocity u. It would not be correct to put $F=mu$ in (11); as the particle after impact continues to move with the body with a certain velocity v, it does not actually give up to the body its whole momentum, but only the amount $F=m(u-v)$, provided that u and v have the same direction. With this assumption, which evidently means that the particle meets the body at some point of the plane passing through the axis and perpendicular to u, the velocity after impact is $v=\omega p$. With the value (11) of ω this gives

$$F=m(u-v)=mu-\frac{mp^2}{I}F,$$

whence $F=muI/(I+mp^2)$, and finally, by (11),

$$\omega=\frac{mup}{I+mp^2}. \tag{12}$$

As mp^2 is the moment of inertia of the particle for the fixed axis, this formula shows that we may substitute in (11) the whole momentum mu for F if we increase the moment of inertia of the body by that of the particle; in other words, that the particle may be regarded as giving up its whole momentum if it be taken into account that after impact it forms part of the body.

306. It is easy to see how the considerations of the last two articles can be generalized. When any number of impulses act in various directions on a rigid body with a fixed axis, the initial angular velocity will be determined by

$$\omega=\frac{H}{I}, \tag{13}$$

where H is the sum of the moments of all the impulses about the fixed axis.

307. To determine the impulsive stress produced on the axis by a single impulse F, let us write out the general equations of the impulsive motion.

Take the fixed axis as the axis of z and the zx-plane through the centroid G (Fig. 36), and let $\bar{x}$, 0, 0 be the co-ordinates of G, and x_1, y_1, z_1 those of the point of application P of the impulse F. The components of F may be denoted by X, Y, Z; those of the reactions of the axis by A_x, A_y, A_z, B_x, B_y, B_z, similarly as in Art. 296.

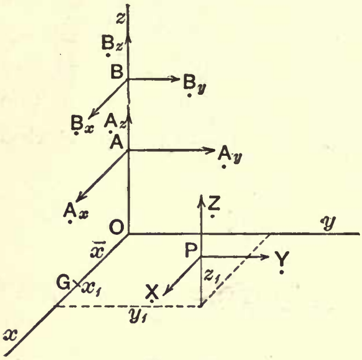

Fig. 36.

As the initial motion after impact is a rotation about the axis of z, we have $\dot{x} = -\omega y$, $\dot{y} = \omega x$, $\dot{z} = 0$, so that the momentum of a particle of mass m has the components $-m\omega y$, $m\omega x$, 0. Reducing these momenta to the origin O, we find a resultant momentum whose components are $-\omega\Sigma my = 0$, $\omega\Sigma mx = M\omega\bar{x}$, 0; and a resulting couple whose vector has the components $-\omega\Sigma mzx = -E\omega$, $-\omega\Sigma myz = -D\omega$, $\omega\Sigma m(x^2+y^2) = C\omega$, where C, D, E have the same meaning as in Art. 297.

The six equations of motion just after the impulse are therefore, if the body was originally at rest:

$$\begin{aligned} 0 &= X + A_x + B_x, \\ M\bar{x}\omega &= Y + A_y + B_y, \\ 0 &= Z + A_z + B_z, \\ -E\omega &= y_1 Z - z_1 Y - aA_y - bB_y, \\ -D\omega &= z_1 X - x_1 Z + aA_x + bB_x, \\ C\omega &= x_1 Y - y_1 X. \end{aligned} \qquad (14)$$

308. The last of these equations is nothing but the equation (11). The components A_z, B_z along the axis cannot be determined separately; the other components of the reactions can be found from the first, second, fourth, and fifth equations.

The impulsive stress to which the axis is subjected by the impulse, or the so-called *percussion of the axis*, instead of being represented by two impulses A, B as above, can also be regarded as composed of an impulse whose components are

$$-A_x - B_x = X, \quad -A_y - B_y = Y - M\bar{x}\omega, \quad -A_z - B_z = Z,$$

and an impulsive couple whose vector has the components

$$aA_y + bB_y = y_1 Z - z_1 Y + E\omega, \qquad -aA_x - bB_x = z_1 X - x_1 Z + D\omega, \ 0.$$

The last component being zero, the resulting couple lies in a plane passing through the axis of z.

If there were any number of impulses acting on the body simultaneously, the effect on the axis could be determined in the same way, except that the quantities X, Y, Z, $y_1 Z - z_1 Y$, $z_1 X - x_1 Z$, must be replaced by the corresponding sums.

309. It follows from the preceding article that the conditions under which a single impulse acting on a rigid body with a fixed axis will produce no stress on the axis are

$$X = 0, \quad Y = M\bar{x}\omega, \quad Z = 0, \quad -z_1 M\bar{x} + E = 0, \quad D = 0. \qquad (15)$$

If these conditions are fulfilled, the resulting motion will be the same even when the axis is free.

The first and third equations show that *the impulse must be perpendicular to the plane passing through axis and centroid.* The meaning of the fourth and fifth conditions becomes apparent if the xy-plane be taken so as to pass through the point of application P of the impulse. The new origin O' is the foot of the perpendicular let fall from P on the fixed axis. To transform the conditions (15) to the new system it is only necessary to substitute $z + z_1$ for z; the first three conditions are not affected, and the last two become

$$-z_1 M\bar{x} + \Sigma mzx + z_1 \Sigma mx = 0, \quad \Sigma myz + z_1 \Sigma my = 0,$$

or, since $\Sigma mx = M\bar{x}$, $\Sigma my = 0$,

$$E' = 0, \quad D' = 0,$$

where E', D' are the products of inertia at O'.

It thus appears that *the axis of* z *must be a principal axis at the foot of the perpendicular let fall on this axis from the point of application of the impulse.*

310. It should be noticed that a line taken at random in a body is not necessarily a principal axis at any one of its points. But if a line is a principal axis at a point O', then it is always possible to determine an impulse that will produce no stress on this line so that the body will begin to rotate about it as axis even though it be not fixed. As shown in the last article, the impulse must be $= M\bar{x}\omega$, and must be directed at right angles to the plane through axis and centroid. The point where it meets this plane is called the **centre of percussion.** Its distance x_1 from the axis is found from the equation of motion, viz. the last of the equations (14) which, owing to the conditions (15), reduces to

$$C = M\bar{x}x_1.$$

If q' be the radius of inertia of the body for a parallel centroidal axis, we have $C = M(q'^2 + \bar{x}^2)$; hence

$$x_1 = \bar{x} + \frac{q'^2}{\bar{x}}. \tag{16}$$

Hence, if a given line l be principal axis for one of its points O', there exists a centre of percussion; it lies on the intersection of the plane (l, G) with the plane through O' perpendicular to l, at the distance x_1, given by (16), from the line l. An impulse $M\bar{x}\omega$ through the centre of percussion at right angles to the plane through axis and centroid, while producing no percussion on the axis, sets the body rotating with angular velocity ω if it was originally at rest; on the other hand, if the body was originally in rotation about the axis, such an impulse can bring the body to rest without affecting the axis.

IV. *Rigid Body with a Fixed Point.*

311. A rigid body with a fixed point has three degrees of freedom. Any one of its points, with the exception of the fixed point O, is constrained to the surface of a sphere and has therefore two degrees of freedom; and the body itself can turn about the line joining this point to O. The motion consists, at any instant, of an infinitesimal rotation about an axis passing through O (see Part I., Arts. 32–35). Both the angular velocity and the direction of the instantaneous axis vary in the course of time.

312. We begin with the study of the *instantaneous motion* of the body, which may be regarded as due to the action of a system of impulses on the body at rest. This will lead to the solution of the converse problem, viz. *to determine the initial motion produced by a given system of impulses.*

I. INITIAL MOTION DUE TO IMPULSES.

313. The body rotates at the time t with angular velocity ω about the instantaneous axis l which passes through the fixed point O. It is required to determine a system of impulses that would produce this motion if acting on the body at rest.

For O as origin, let $\underset{\cdot}{R}$ be the resultant and $\underset{\cdot}{H}$ the resulting couple of these impulses. If the impulsive reaction $\underset{\cdot}{A}$ of the fixed point O be combined with them, the body can be regarded as free, and its instantaneous motion is determined by the equations (19) and (20), Art. 238. It is only necessary, in the equations (19), to add to the components $\underset{\cdot}{R}_x$, $\underset{\cdot}{R}_y$, $\underset{\cdot}{R}_z$ of $\underset{\cdot}{R}$ those of $\underset{\cdot}{A}$, while the right-hand members of (20) are not affected by $\underset{\cdot}{A}$, since its moment is zero for every axis through O.

314. It remains to form the sums in the left-hand members of (19) and (20) for our case; *i.e.* to reduce the system of momenta $m\dot{x}$, $m\dot{y}$, $m\dot{z}$ of the particles to its resultant and resultant couple for a fixed rectangular system of axes through O.

The resultant momentum has evidently the components

$$\Sigma m\dot{x} = M\dot{\bar{x}},\quad \Sigma m\dot{y} = M\dot{\bar{y}},\quad \Sigma m\dot{z} = M\dot{\bar{z}},$$

where $\dot{\bar{x}}$, $\dot{\bar{y}}$, $\dot{\bar{z}}$ are the components of the velocity of the centroid at the time t, and M is the mass of the body. Hence the equations (19) become

$$M\dot{\bar{x}} = R_x + A_x,\quad M\dot{\bar{y}} = R_y + A_y,\quad M\dot{\bar{z}} = R_z + A_z. \tag{1}$$

These equations serve to determine the impulsive pressure $-A = -\sqrt{A_x^2 + A_y^2 + A_z^2}$ on the fixed point O in magnitude and direction.

315. To form the moment $\Sigma m(y\dot{z} - z\dot{y})$ of the momenta of the particles, *i.e.* the angular momentum of the body, about the axis of x, we resolve the angular velocity ω into its components ω_x, ω_y, ω_z along the axes and observe that the components of the linear velocity of any point (x, y, z) arising from the rotation are (Part I., Art. 293):

$$\dot{x} = \omega_y z - \omega_z y,\quad \dot{y} = \omega_z x - \omega_x z,\quad \dot{z} = \omega_x y - \omega_y x.$$

Substituting these values, we find

$$\Sigma m(y\dot{z} - z\dot{y}) = \omega_x \Sigma m y^2 - \omega_y \Sigma m x y - \omega_z \Sigma m z x + \omega_x \Sigma m z^2,$$

or with the notation of Art. 255,

$$\Sigma m(y\dot{z} - z\dot{y}) = A\omega_x - F\omega_y - E\omega_z.$$

Forming in the same way the angular momenta about the axes of y and z, we find the equations (20) in the form

$$\begin{aligned} A\omega_x - F\omega_y - E\omega_z &= H_x, \\ -F\omega_x + B\omega_y - D\omega_z &= H_y, \\ -E\omega_x - D\omega_y + C\omega_z &= H_z. \end{aligned} \tag{2}$$

316. It appears, then, that *the rotation of angular velocity ω about the axis* l *can be regarded as due to an impulsive couple* H *whose components are given by* (2). Conversely, the effect of a couple $\underset{.}{H}$ on a rigid body at rest, with a fixed point, is to impart to the body a rotation ω whose magnitude and axis can be found by determining ω_x, ω_y, ω_z from (2).

Any system of impulses acting on the body can be reduced, for the fixed point O as origin, to a resultant $\underset{.}{R}$ and a couple $\underset{.}{H}$; the effect of the couple has just been stated; that of $\underset{.}{R}$ consists merely in producing a pressure on the fixed point. To find this pressure, determine $\omega = \sqrt{\omega_x^2 + \omega_y^2 + \omega_z^2}$ from (2); the velocity of the centroid can then be found and its components substituted in (1), Art. 314.

317. The axis l of the rotation produced by a given couple $\underset{.}{H}$ is not, in general, perpendicular to the plane of the couple. Imagine the angular velocity ω to be represented by its rotor, *i.e.* by a length ω laid off from O on the axis l, and the couple $\underset{.}{H}$ by its vector, *i.e.* by a length $\underset{.}{H}$ laid off from O on the perpendicular to the plane of the couple. The relation between the rotor ω and the vector $\underset{.}{H}$ producing it will best appear if we take the axis of rotation l as axis of z. We then have $\dot{x} = -\omega y$, $\dot{y} = \omega x$, $\dot{z} = 0$, and the momenta $-m\omega y$, $m\omega x$, 0 of the particles reduce to a resultant and couple at O as follows. The resultant momentum has the components:

$$-\omega\Sigma my = -M\bar{y}\omega, \quad \omega\Sigma mx = M\bar{x}\omega, \quad 0;$$

it is equal to $M\omega\sqrt{\bar{x}^2+\bar{y}^2} = M\omega\bar{r}$, where $\bar{r}$ is the distance of the centroid from the axis l, and is perpendicular to the plane through axis and centroid. The couple has the components

$$-\omega\Sigma mzx = -E\omega, \quad -\omega\Sigma myz = -D\omega, \quad \omega\Sigma m(x^2+y^2) = C\omega.$$

The equations (19) and (20) of Art. 238 reduce therefore to

$$-M\bar{y}\omega = \underset{.}{R}_x + \underset{.}{A}_x, \quad M\bar{x}\omega = \underset{.}{R}_y + \underset{.}{A}_y, \quad 0 = \underset{.}{R}_z + \underset{.}{A}_z, \tag{3}$$

$$-E\omega = \underset{.}{H}_x, \quad -D\omega = \underset{.}{H}_y, \quad C\omega = \underset{.}{H}_z. \tag{4}$$

These equations can also be derived directly from the equations (1) and (2) above, since in the present case we have $\dot{\bar{x}} = -\omega\bar{y}$, $\dot{\bar{y}} = \omega\bar{x}$, $\dot{\bar{z}} = 0$, $\omega_x = 0$, $\omega_y = 0$, $\omega_z = \omega$.

318. The equations (4) show that, in general, the couple H has three components (see Fig. 37); H_x and H_y can be combined into a partial resultant $H_{xy}=\omega\sqrt{D^2+E^2}$ in the xy-plane; and the total resultant $H=\omega\sqrt{C^2+D^2+E^2}$ makes with the axis l an angle ϕ such that $\cos\phi=C/\sqrt{C^2+D^2+E^2}$. As C is always positive, this angle is always acute; it vanishes only if $D=0$ and $E=0$, *i.e.* if the instantaneous axis l is a principal axis at O.

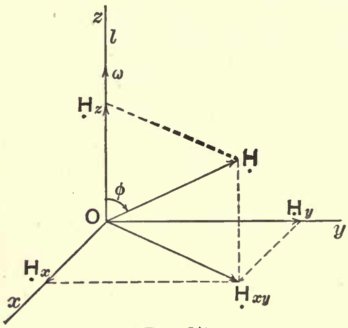

Fig. 37.

This result that *H and ω coincide only along a principal axis* is very important. It shows that *the vector* H *of the couple that produces a rotation* ω *has the direction of the axis of rotation* l *only, and always, if this axis* l *is a principal axis at the fixed point* O ; *in this case we have* $H=I\omega$, *where* I *is the moment of inertia for* l.

Conversely, a couple H acting on a rigid body with a fixed point O produces rotation about an instantaneous axis l, which is, in general, inclined to the vector of the couple at an acute angle ϕ. This angle reduces to zero, *i.e.* the instantaneous axis l coincides in direction with the vector of the couple, only, and always, when the plane of the couple is perpendicular to a principal axis at O.

319. Let us now take the principal axes at O as axes of co-ordinates. Let ω_1, ω_2, ω_3 be the components of ω along these axes; H_1, H_2, H_3 those of H; and let I_1, I_2, I_3 be the principal moments, q_1, q_2, q_3 the principal radii of inertia at O. Then we must have

$$H_1=I_1\omega_1=Mq_1^2\omega_1,\quad H_2=I_2\omega_2=Mq_2^2\omega_2,\quad H_3=I_3\omega_3=Mq_3^2\omega_3. \quad (5)$$

These relations follow also from (2), since $A=I_1$, $B=I_2$, $C=I_3$, $D=0$, $E=0$, $F=0$; they determine the relation between H and ω in the general case.

320. The relation between the vectors $\underset{\cdot}{H}$ and ω is very clearly brought out by making use of the ellipsoids of inertia at the point O.

The reciprocal ellipsoid at O has the equation (see Arts. 269, 270)

$$\frac{x^2}{q_1^{\,2}}+\frac{y^2}{q_2^{\,2}}+\frac{z^2}{q_3^{\,2}}=1.$$

Let P (Fig. 38) be the point where it is met by the vector $\underset{\cdot}{H}$; x, y, z the co-ordinates, ρ the radius vector of P; hence

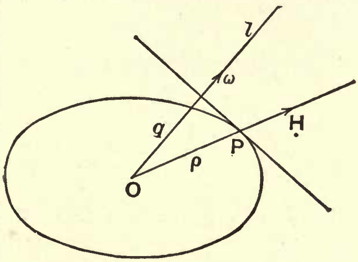

Fig. 38.

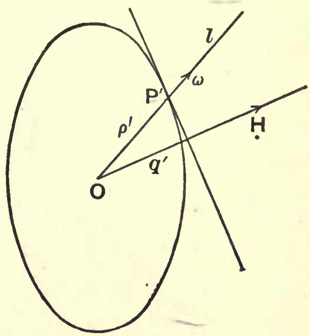

Fig. 39.

x/ρ, y/ρ, z/ρ the direction-cosines of $\underset{\cdot}{H}$, so that $\underset{\cdot}{H}_1=\underset{\cdot}{H}x/\rho$, $\underset{\cdot}{H}_2=\underset{\cdot}{H}y/\rho$, $\underset{\cdot}{H}_3=\underset{\cdot}{H}z/\rho$. The equations (5) give, therefore,

$$\omega_1=\frac{\underset{\cdot}{H}}{M\rho}\cdot\frac{x}{q_1^{\,2}},\quad \omega_2=\frac{\underset{\cdot}{H}}{M\rho}\cdot\frac{y}{q_2^{\,2}},\quad \omega_3=\frac{\underset{\cdot}{H}}{M\rho}\cdot\frac{z}{q_3^{\,2}},$$

whence
$$\omega=\frac{\underset{\cdot}{H}}{M\rho}\sqrt{\frac{x^2}{q_1^{\,4}}+\frac{y^2}{q_2^{\,4}}+\frac{z^2}{q_3^{\,4}}}=\frac{\underset{\cdot}{H}}{M\rho}\cdot\frac{1}{q}, \tag{6}$$

where q is the perpendicular let fall from O on the tangent plane at P (see Art. 269). The direction-cosines of ω,

$$\frac{\omega_1}{\omega}=q\frac{x}{q_1^{\,2}},\quad \frac{\omega_2}{\omega}=q\frac{y}{q_2^{\,2}},\quad \frac{\omega_3}{\omega}=q\frac{z}{q_3^{\,2}},$$

are the same as those of this perpendicular (*ib.*).

It thus appears that *the plane through* O *at right angles to the instantaneous axis* l *is conjugate to the direction of the vector* $\underset{\cdot}{\mathrm{H}}$ *with respect to the reciprocal ellipsoid at* O.

321. Again, the equation of the momental ellipsoid at O is (see Art. 266)

$$q_1^2x^2+q_2^2y^2+q_3^2z^2=\epsilon^4,$$

the semi-axes being $a=\epsilon^2/q_1$, $b=\epsilon^2/q_2$, $c=\epsilon^2/q_3$.

Let the instantaneous axis l meet this ellipsoid at a point P' (Fig. 39) whose co-ordinates and radius vector are x', y', z', ρ', so that x'/ρ', y'/ρ', z'/ρ' are the direction-cosines of l and $\omega_1=\omega x'/\rho'$, $\omega_2=\omega y'/\rho'$, $\omega_3=\omega z'/\rho'$. Substituting these values and introducing the semi-axes a, b, c, we find form (5)

$$\underset{\cdot}{H}_1=\frac{M\epsilon^4\omega}{\rho'}\cdot\frac{x'}{a^2},\quad \underset{\cdot}{H}_2=\frac{M\epsilon^4\omega}{\rho'}\cdot\frac{y'}{b^2},\quad \underset{\cdot}{H}_3=\frac{M\epsilon^4\omega}{\rho'}\cdot\frac{z'}{c^2},$$

whence
$$\underset{\cdot}{H}=\frac{M\epsilon^4\omega}{\rho'}\sqrt{\frac{x'^2}{a^4}+\frac{y'^2}{b^4}+\frac{z'^2}{c^4}}=\frac{M\epsilon^4\omega}{\rho'}\cdot\frac{1}{q'},\tag{7}$$

where q' is the perpendicular let fall from O on the tangent plane at P'. The direction-cosines of $\underset{\cdot}{H}$,

$$\frac{\underset{\cdot}{H}_1}{\underset{\cdot}{H}}=q'\frac{x'}{a^2},\quad \frac{\underset{\cdot}{H}_2}{\underset{\cdot}{H}}=q'\frac{y'}{b^2},\quad \frac{\underset{\cdot}{H}_3}{\underset{\cdot}{H}}=q'\frac{z'}{c^2},$$

agree with those of q'.

It follows that *the plane of the couple* $\underset{\cdot}{\mathrm{H}}$ *is conjugate to the direction of the instantaneous axis* l *with respect to the momental ellipsoid at* O.

322. The **kinetic energy** of a rigid body with a fixed point O has the expression

$$T=\Sigma\tfrac{1}{2}mv^2=\tfrac{1}{2}\omega^2\Sigma mr^2,$$

where Σmr^2 is the moment of inertia for the instantaneous axis l.

Now, by Art. 270, the radius of inertia for the line l is equal to the distance of O from the perpendicular tangent plane to the reciprocal ellipsoid, *i.e.* to q (Fig. 38). Hence

$$T=\tfrac{1}{2}Mq^2\omega^2.\tag{8}$$

As, according to the fundamental property of the momental ellipsoid (see (16), Art. 266), we have $q = \epsilon^2/\rho'$, this becomes

$$T = \tfrac{1}{2} M\epsilon^4 \cdot \left(\frac{\omega}{\rho'}\right)^2. \tag{9}$$

On the other hand, by Art. 258, if α, β, γ be the direction-cosines of l, *i.e.* of the rotor ω, we have

$$Mq^2 = A\alpha^2 + B\beta^2 + C\gamma^2 - 2D\beta\gamma - 2E\gamma\alpha - 2F\alpha\beta;$$

hence

$$T = \tfrac{1}{2}(A\omega_x^2 + B\omega_y^2 + C\omega_z^2 - 2D\omega_y\omega_z - 2E\omega_z\omega_x - 2F\omega_x\omega_y). \tag{10}$$

Differentiating with respect to ω_x, ω_y, ω_z, and comparing with the equations (2), we find

$$\begin{aligned} \frac{\partial T}{\partial \omega_x} &= \quad A\omega_x - F\omega_y - E\omega_z = H_x, \\ \frac{\partial T}{\partial \omega_y} &= -F\omega_x + B\omega_y - D\omega_z = H_y, \\ \frac{\partial T}{\partial \omega_z} &= -E\omega_x - D\omega_y + C\omega_z = H_z. \end{aligned} \tag{11}$$

If these relations be multiplied by ω_x, ω_y, ω_z and added, they give

$$2T = \frac{\partial T}{\partial \omega_x}\omega_x + \frac{\partial T}{\partial \omega_y}\omega_y + \frac{\partial T}{\partial \omega_z}\omega_z = H_x\omega_x + H_y\omega_y + H_z\omega_z. \tag{12}$$

Substituting in the last expression $\alpha\omega = \omega_x$, $\beta\omega = \omega_y$, $\gamma\omega = \omega_z$, or $\lambda H = H_x$, $\mu H = H_y$, $\nu H = H_z$, where λ, μ, ν are the direction-cosines of the vector H, we find

$$T = \tfrac{1}{2}\omega(H_x\alpha + H_y\beta + H_z\gamma) = \tfrac{1}{2}H(\omega_x\lambda + \omega_y\mu + \omega_z\nu). \tag{13}$$

We have, therefore, for the projection of H on the instantaneous axis l,

$$H\cos\phi = H_x\alpha + H_y\beta + H_z\gamma = \frac{2T}{\omega} = Mq^2\omega;$$

for the projection of ω on the direction of the vector H,

$$\omega\cos\phi = \omega_x\lambda + \omega_y\mu + \omega_z\nu = \frac{2T}{H} = Mq^2\frac{\omega^2}{H};$$

and finally,

$$T = \tfrac{1}{2}H\omega\cos\phi. \tag{14}$$

323. If the principal axes at O be taken as axes of co-ordinates, we have to write $\omega_1, \omega_2, \omega_3$ for $\omega_x, \omega_y, \omega_z$; H_1, H_2, H_3 for H_x, H_y, H_z; I_1, I_2, I_3 for A, B, C, while $D=0$, $E=0$, $F=0$. Thus the relations (5) give $H_1=I_1\omega_1$, $H_2=I_2\omega_2$, $H_3=I_3\omega_3$, whence

$$H^2=I_1^2\omega_1^2+I_2^2\omega_2^2+I_3^2\omega_3^2, \tag{15}$$

$$\omega^2=\frac{H_1^2}{I_1^2}+\frac{H_2^2}{I_2^2}+\frac{H_3^2}{I_3^2}; \tag{16}$$

and

$$T=\tfrac{1}{2}(I_1\omega_1^2+I_2\omega_2^2+I_3\omega_3^2) \tag{17}$$

$$=\tfrac{1}{2}\left(\frac{H_1^2}{I_1}+\frac{H_2^2}{I_2}+\frac{H_3^2}{I_3}\right). \tag{18}$$

For the angle ϕ between H and ω, we have

$$\cos\phi=\frac{H_1\omega_1+H_2\omega_2+H_3\omega_3}{H\omega}=\frac{I_1\omega_1^2+I_2\omega_2^2+I_3\omega_3^2}{H\omega}. \tag{19}$$

2. CONTINUOUS MOTION UNDER ANY FORCES.

324. We now proceed to consider the motion of a rigid body with a fixed point when acted upon by any forces.

For the fixed point O as origin, the external forces reduce to a resultant R and a couple H. While the force R is taken up by the fixed point, the effect of the couple consists in changing the angular velocity ω about the instantaneous axis l, which exists at the time t, to the angular velocity $\omega+d\omega$ about another instantaneous axis l', which determines the motion of the body at the time $t+dt$. The point O being fixed, both axes, l and l', pass through it; and by Part I., Art. 303, the acceleration of any point (x, y, z) of the body has the following components parallel to rectangular axes fixed in the body and moving with it:

$$\begin{aligned}\ddot{x}&=\omega_x(\omega_x x+\omega_y y+\omega_z z)-\omega^2 x+\dot{\omega}_y z-\dot{\omega}_z y,\\ \ddot{y}&=\omega_y(\omega_x x+\omega_y y+\omega_z z)-\omega^2 y+\dot{\omega}_z x-\dot{\omega}_x z,\\ \ddot{z}&=\omega_z(\omega_x x+\omega_y y+\omega_z z)-\omega^2 z+\dot{\omega}_x y-\dot{\omega}^y x.\end{aligned} \tag{1}$$

Multiplying these expressions by the mass m of the particle situated at the point (x, y, z), we have the components of the effective force of this particle.

325. To form the equations of motion (4), Art. 223, and (6), Art. 224, for our case, we must reduce the system of the effective forces to its resultant and resulting couple; or, what amounts to the same thing, we must form the sums occurring in the left-hand members of these equations.

The summation of the components of the effective forces throughout the body gives, as usual,

$$\Sigma m\ddot{x} = M\ddot{\bar{x}}, \quad \Sigma m\ddot{y} = M\ddot{\bar{y}}, \quad \Sigma m\ddot{z} = M\ddot{\bar{z}},$$

where $\ddot{\bar{x}}$, $\ddot{\bar{y}}$, $\ddot{\bar{z}}$ are the components of the acceleration of the centroid. The resultant is therefore equal to the effective force of the centroid, the whole mass M of the body being regarded as concentrated at this point.

To make the body free, the reaction A of the fixed point should be introduced. Denoting its components by A_x, A_y, A_z, those of the resultant R of the external forces by R_x, R_y, R_z, the equations (4), Art. 223, assume the form

$$M\ddot{\bar{x}} = R_x + A_x, \quad M\ddot{\bar{y}} = R_y + A_y, \quad M\ddot{\bar{z}} = R_z + A_z. \tag{2}$$

The left-hand members evidently vanish if the origin be the centroid. The equations (2) can serve to determine the pressure $-A$ on the fixed point in magnitude and direction.

326. To form the moment $\Sigma m(y\ddot{z} - z\ddot{y})$ of the effective forces about the axis of x, we have to multiply the second of the expressions (1) by z, and subtract the product from the third multiplied by y; then multiply the difference by m, and sum throughout the body.

Performing this operation first on the last two terms which were shown in Part I., Art. 302, to be due to the angular acceleration, we find

$$\dot{\omega}_x \Sigma m(y^2 + z^2) - \dot{\omega}_y \Sigma mxy - \dot{\omega}_z \Sigma mzx = A\dot{\omega}_x - F\dot{\omega}_y - E\dot{\omega}_z,$$

with the notation of Art. 255. As the axes are fixed in the body (Art. 324), the moments and products of inertia are constant; and it appears from the equations (2), Art. 315, that this expression is the derivative with respect to the time of the component $\underset{\cdot}{H}_x$ of the impulsive couple $\underset{\cdot}{H}$ that produces the rotation ω at the time t.

Next operating in the same way on the remaining terms of the component accelerations (1), viz. those arising from the centripetal acceleration, we find

$$
\begin{aligned}
&\omega_z(\omega_x\Sigma mxy+\omega_y\Sigma my^2+\omega_z\Sigma myz)-\omega^2\Sigma myz\\
&-\omega_y(\omega_x\Sigma mzx+\omega_y\Sigma myz+\omega_z\Sigma mz^2)+\omega^2\Sigma myz\\
&=\omega_z(F\omega_x+C\omega_y+D\omega_z)-\omega_y(E\omega_x+D\omega_y+B\omega_z)\\
&=\omega_y(-E\omega_x-D\omega_y+C\omega_z)-\omega_z(-F\omega_x+B\omega_y-D\omega_z)\\
&=\omega_y\underset{\cdot}{H}_z-\omega_z\underset{\cdot}{H}_y,
\end{aligned}
$$

by (2), Art. 315.

The moments of the effective forces about the other two axes can now be obtained by cyclical permutation of the subscripts x, y, z. Thus we find that the equations (6), Art. 224, assume the form

$$
\begin{aligned}
&\frac{d\underset{\cdot}{H}_x}{dt}+\omega_y\underset{\cdot}{H}_z-\omega_z\underset{\cdot}{H}_y=H_x,\\
&\frac{d\underset{\cdot}{H}_y}{dt}+\omega_z\underset{\cdot}{H}_x-\omega_x\underset{\cdot}{H}_z=H_y, \qquad (3)\\
&\frac{d\underset{\cdot}{H}_z}{dt}+\omega_x\underset{\cdot}{H}_y-\omega_y\underset{\cdot}{H}_x=H_z.
\end{aligned}
$$

The reaction A of the fixed point does not enter into these equations; as it intersects every one of the axes, its moments about these axes are zero.

327. Geometrically the equations (3) mean that the vector H of the resultant couple of the external forces has two components one of which resolves itself along the axes into $d\underset{\cdot}{H}_x/dt$, $d\underset{\cdot}{H}_y/dt$, $d\underset{\cdot}{H}_z/dt$, while the other has the components $\omega_y \underset{.}{H}_z - \omega_z \underset{.}{H}_y$, $\omega_z \underset{.}{H}_x - \omega_x \underset{.}{H}_z$, $\omega_x \underset{.}{H}_y - \omega_y \underset{.}{H}_x$. Each of these components can be interpreted geometrically, if we imagine the vector $\underset{.}{H}$ of the impulsive couple drawn from O as origin, so that the co-ordinates of its extremity are $\underset{.}{H}_x$, $\underset{.}{H}_y$, $\underset{.}{H}_z$ (Fig. 40). The time-derivatives of these co-ordinates are the velocities of the extremity of the vector $\underset{.}{H}$ with respect to the axes of co-ordinates which, it will be remembered, are fixed in the body. Hence that component of H which is due to the angular acceleration is the relative velocity of the extremity of $\underset{.}{H}$ with respect to the body.

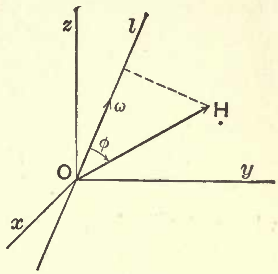

Fig. 40.

The other component, which is due to the centripetal acceleration, evidently represents the linear velocity, arising from the angular velocity ω, of the point of the body that coincides at the time t with the same extremity of the vector $\underset{.}{H}$.

It follows that the vector H represents in magnitude and direction the absolute velocity of the extremity of the vector $\underset{.}{H}$; in other words, H is geometrically equal to the geometrical increment of $\underset{.}{H}$ divided by the element of time. This was to be expected, and might indeed be taken as starting-point for deriving the equations (3).

328. Let us now select as axes of co-ordinates the principal axes at O. According to our usual notation, we have then to exchange the subscripts x, y, z for 1, 2, 3. Moreover, as shown in Art. 319, $\underset{.}{H}_1 = I_1\omega_1$, $\underset{.}{H}_2 = I_2\omega_2$, $\underset{.}{H}_3 = I_3\omega_3$, where I_1, I_2, I_3 are the principal moments of inertia at O. Thus the equations (3) reduce to the following:

$$\begin{aligned} I_1\dot{\omega}_1 + (I_3 - I_2)\omega_2\omega_3 &= H_1, \\ I_2\dot{\omega}_2 + (I_1 - I_3)\omega_3\omega_1 &= H_2, \\ I_3\dot{\omega}_3 + (I_2 - I_1)\omega_1\omega_2 &= H_3. \end{aligned} \tag{4}$$

These are **Euler's equations of motion.** Their solution gives ω_1, ω_2, ω_3 as functions of the time t.

It may be noted that the equations (4) are often written with the following notation:

$$A\frac{dp}{dt}+(C-B)qr=L,$$

$$B\frac{dq}{dt}+(A-C)rp=M, \qquad (4')$$

$$C\frac{dr}{dt}+(B-A)pq=N,$$

where A, B, C are the principal moments of inertia; p, q, r the components of the angular velocity ω along the principal axes; L, M, N the components of the resulting couple H of the external forces along the same axes.

329. Owing to the importance of the equations (4) it may be well to indicate another way of deriving them.

The rotation of angular velocity ω about the instantaneous axis l during the first element of time can be regarded as due to an impulsive couple H (Art. 316). Even if there were no external forces acting, the body would not in general continue to turn with the same velocity about the same axis. For if this were the case, any particle m of the body, at the distance r from the axis l would be moving uniformly in a circle of radius r, with a velocity ωr, and such uniform circular motion requires for its maintenance the action of a centripetal force.

Let us therefore introduce at every particle m two equal and opposite forces (Fig. 41), the centripetal force $m\omega^2 r$ directed towards the axis l, and the centrifugal force $-m\omega^2 r$; the introduction of these forces does not change the state of motion of the body.

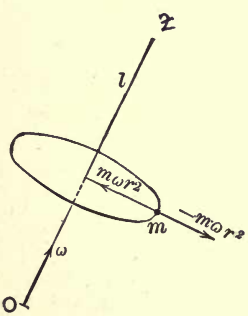

Fig. 41.

330. If the system of centripetal forces $m\omega^2 r$ alone were introduced and no other forces were acting, the body would continue to turn with the same angular velocity ω about the same axis l. The effect of the system of centrifugal forces $-m\omega^2 r$ represents therefore the change that would take place in the motion if no external forces were acting.

Let us reduce these centrifugal forces to their resultant and resulting couple, the fixed point O being taken as origin and the axis l as axis of z. In doing this we can make use of the reduction of momenta in Art. 317. For, evidently, the vector representing the centrifugal force $-m\omega^2 r$ can be obtained by multiplying the momentum $m\omega r$ of the particle m by ω and turning it through an angle of 90° in a sense opposite to that of the rotation ω. The reduction to O gives therefore a resultant force $M\omega^2\bar{r}$, in the xy-plane, directed toward the projection of the centroid on this plane. The resulting couple has its z-component equal to zero since all the centrifugal forces intersect the axis of z; the vector of the resulting couple lies therefore in the xy-plane, has the magnitude ωH_{xy}, and is perpendicular to the H_{xy} in Fig. 37.

331. The resultant vanishes only if $\bar{r}=0$, *i.e.* if the centroid lies on the axis l; the couple vanishes if $\omega H_{xy} \equiv \omega^2\sqrt{D^2+E^2}=0$, *i.e.* if the axis l is a principal axis at O. It follows that the centrifugal forces reduce to zero only if the axis of rotation is a principal centroidal axis; in this case the direction of the axis remains unchanged.

By Art. 318 (see Fig. 37) we have $H_{xy}=H\sin\phi$; hence the resulting couple of the centrifugal forces $=\omega H\sin\phi$, that is, its magnitude is represented by the area of the parallelogram formed by the vectors H and ω; the vector of this couple as shown above is perpendicular to this area. Projecting this parallelogram on any three rectangular co-ordinate planes, with O as origin, we find, since ω_x, ω_y, ω_z are the co-ordinates of the extremity of the rotor ω, H_x, H_y, H_y those of the extremity of the vector H drawn from O:

$$\omega_z H_y - \omega_y H_z, \quad \omega_x H_z - \omega_z H_x, \quad \omega_y H_x - \omega_x H_y.$$

This agrees with the results found in Arts. 326, 327.

332. If the principal axes at O be taken as axes of co-ordinates, the components of the resultant couple of the centrifugal forces become

$$\omega_3 H_2 - \omega_2 H_3, \quad \omega_1 H_3 - \omega_3 H_1, \quad \omega_2 H_1 - \omega_1 H_2,$$

or, since $H_1 = I_1\omega_1$, $H_2 = I_2\omega_2$, $H_3 = I_3\omega_3$,

$$(I_2 - I_3)\,\omega_2\omega_3, \quad (I_3 - I_1)\,\omega_3\omega_1, \quad (I_1 - I_2)\,\omega_1\omega_2.$$

As the planes of these couples are perpendicular to the principal axes at O, they produce during the element of time infinitesimal rotations about these axes, whose angles are, by Art. 318:

$$\frac{I_2 - I_3}{I_1}\,\omega_2\omega_3 dt, \quad \frac{I_3 - I_1}{I_2}\,\omega_3\omega_1 dt, \quad \frac{I_1 - I_2}{I_3}\,\omega_1\omega_2 dt.$$

These are the only increments of ω_1, ω_2, ω_3 if there are no forces acting on the body; hence, in this case we must have

$$I_1\frac{d\omega_1}{dt}=(I_2-I_3)\,\omega_2\omega_3,\quad I_2\frac{d\omega_2}{dt}=(I_3-I_1)\,\omega_3\omega_1,\quad I_3\frac{d\omega_3}{dt}=(I_1-I_2)\,\omega_1\omega_2.$$

If, however, there are external forces acting on the body, whose resulting couple for O is H, with the components H_1, H_2, H_3 along the principal axes at O, these couples produce infinitesimal rotations

$$\frac{H_1}{I_1}dt,\quad \frac{H_2}{I_2}dt,\quad \frac{H_3}{I_3}dt,$$

and the equations of motion are therefore

$$I_1\dot{\omega}_1=(I_2-I_3)\,\omega_2\omega_3+H_1,$$
$$I_2\dot{\omega}_2=(I_3-I_1)\,\omega_3\omega_1+H_2,$$
$$I_3\dot{\omega}_3=(I_1-I_2)\,\omega_1\omega_2+H_3.$$

These are Euler's equations (4).

333. Euler's equations determine the angular velocities of the body about the principal axes which move with the body. The position of these moving axes with respect to a system of fixed rectangular axes through the fixed point O can be expressed by means of three angles.

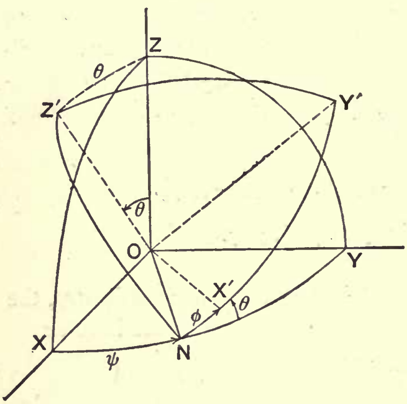

Fig. 42.

Let X, Y, Z (Fig. 42) be the intersections of the fixed axes, with a sphere of radius one, described about O as centre; X', Y', Z' those of the moving principal axes; N the intersection with the same sphere of the so-called *line of nodes*, *i.e.* the line in which the planes XOY and $X'OY'$ intersect. Then the angles

$$ZZ'=\theta,\quad NX'=\phi,\quad XN=\psi,$$

usually called **Euler's angles**, may serve to determine the relation between the two systems of axes.

334. The *sense* in which these angles are counted is best remembered by imagining the two trihedral angles XYZ and $X'Y'Z'$ originally coincident. Now turn the system $X'Y'Z'$ about the axis OZ in the positive sense (counter-clockwise) until the axis OX' coincides with the assumed positive sense of the nodal line ON, *i.e.* the final intersection of the planes XOY and $X'OY'$; the amount of this rotation gives the angle ψ. Next turn the trihedral $X'Y'Z'$ about this line of nodes in the positive sense until the plane $X'OY'$ falls into its final position; this gives the angle θ, as the angle between the planes XOY and $X'OY'$ at N, or as the angle ZOZ' between their normals. Finally a rotation of $X'Y'Z'$ about the axis OZ', which has reached its final position, in the positive sense until OX' comes into its final position, determines the angle ϕ.

335. The angular velocity, represented by its rotor ω, whose components along OX', OY', OZ' are ω_1, ω_2, ω_3, can be resolved along ON, OZ', OZ into three components which are evidently $\dot{\theta}$, $\dot{\phi}$, $\dot{\psi}$, respectively. The sum of the projections of these three components on the line OX' should give ω_1; hence

$$\omega_1=\dot{\theta}\cos\phi+\dot{\phi}\cos\tfrac{1}{2}\pi+\dot{\psi}\cos ZX'.$$

Similarly $$\omega_2=\dot{\theta}\cos(\phi+\tfrac{1}{2}\pi)+\dot{\phi}\cos\tfrac{1}{2}\pi+\dot{\psi}\cos ZY',$$

$$\omega_3=\dot{\theta}\cos\tfrac{1}{2}\pi+\dot{\phi}\cos 0+\dot{\psi}\cos\theta.$$

The spherical triangle ZNX' gives (by the fundamental formula of spherical trigonometry, $\cos c=\cos a\cos b+\sin a\sin b\cos\gamma$) $\cos ZX'=\sin\phi\cos(\frac{1}{2}\pi-\theta)=\sin\phi\sin\theta$; and the triangle ZNY' gives $\cos ZY'=\sin(\phi+\frac{1}{2}\pi)\cos(\frac{1}{2}\pi-\theta)=\cos\phi\sin\theta$. Hence, finally

$$\begin{aligned}\omega_1&=\dot{\theta}\cos\phi+\dot{\psi}\sin\phi\sin\theta,\\ \omega_2&=-\dot{\theta}\sin\phi+\dot{\psi}\cos\phi\sin\theta,\\ \omega_3&=\dot{\phi}+\dot{\psi}\cos\theta.\end{aligned}\tag{5}$$

Solving these equations for $\dot{\theta}$, $\dot{\phi}$, $\dot{\psi}$, we find

$$\begin{aligned}\dot{\theta}&=\omega_1\cos\phi-\omega_2\sin\phi,\\ \dot{\phi}&=-\omega_1\sin\phi\cot\theta-\omega_2\cos\phi\cot\theta+\omega_3,\\ \dot{\psi}&=\omega_1\sin\phi\csc\theta+\omega_2\cos\phi\csc\theta.\end{aligned}\tag{6}$$

336. The relation between two rectangular systems of axes with the same origin can also be expressed by means of the 9 cosines of the angles between the axes.

Let O be the common origin, x, y, z the co-ordinates of any point with respect to the fixed system, x', y', z' its co-ordinates in the moving system; then we have, evidently,

$$\begin{aligned} x &= a_1 x' + a_2 y' + a_3 z', \\ y &= b_1 x' + b_2 y' + b_3 z', \\ z &= c_1 x' + c_2 y' + c_3 z', \end{aligned} \tag{7}$$

where the coefficients of x', y', z' are the cosines of the angles between the axes, which can best be remembered in the form

	x'	y'	z'
x	a_1	a_2	a_3
y	b_1	b_2	b_3
z	c_1	c_2	c_3

(8)

Thus, 9 angles are used in order to fix the position of the moving axes with respect to the fixed axes, instead of Euler's 3 angles. But their 9 cosines (8) are connected by 6 independent relations, which can be written in either one of the equivalent forms:

$$\begin{aligned} a_1^2 + b_1^2 + c_1^2 &= 1, & a_2 a_3 + b_2 b_3 + c_2 c_3 &= 0, \\ a_2^2 + b_2^2 + c_2^2 &= 1, & a_3 a_1 + b_3 b_1 + c_3 c_1 &= 0, \\ a_3^2 + b_3^2 + c_3^2 &= 1, & a_1 a_2 + b_1 b_2 + c_1 c_2 &= 0, \end{aligned} \tag{9}$$

or

$$\begin{aligned} a_1^2 + a_2^2 + a_3^2 &= 1, & b_1 c_1 + b_2 c_2 + b_3 c_3 &= 0, \\ b_1^2 + b_2^2 + b_3^2 &= 1, & c_1 a_1 + c_2 a_2 + c_3 a_3 &= 0, \\ c_1^2 + c_2^2 + c_3^2 &= 1, & a_1 b_1 + a_2 b_2 + a_3 b_3 &= 0. \end{aligned} \tag{10}$$

The meaning of these equations is easily perceived from the meaning of the angles involved. Thus, the first of the equations, (9) expresses the fact that a_1, b_1, c_1 are the direction-cosines of a line, viz. the axis Ox'; the last of the equations (10) expresses the perpendicularity of the axes Ox and Oy; and similarly for the others.

337. The relations between the 9 angles, whose cosines are given in (8), and Euler's 3 angles θ, ϕ, ψ are readily found from Fig. 42, by applying the formula $\cos c = \cos a \cos b + \sin a \sin b \cos \gamma$ successively to the triangles

$$\begin{array}{ccc} XNX', & XNY', & XNZ', \\ YNX', & YNY', & YNZ', \\ ZNX', & ZNY', & ZNZ'. \end{array}$$

In this way the following relations are found:

$$a_1 = \cos\psi\cos\phi - \sin\psi\sin\phi\cos\theta,$$
$$b_1 = \sin\psi\cos\phi + \cos\psi\sin\phi\cos\theta,$$
$$c_1 = \sin\phi\sin\theta,$$

$$a_2 = -\cos\psi\sin\phi - \sin\psi\cos\phi\cos\theta, \qquad a_3 = \sin\psi\sin\theta,$$
$$b_2 = -\sin\psi\sin\phi + \cos\psi\cos\phi\cos\theta, \qquad b_3 = -\cos\psi\sin\theta,$$
$$c_2 = \cos\phi\sin\theta, \qquad c_3 = \cos\theta.$$

338. It is evident, geometrically, from (8), that we must have

$$\begin{aligned} x' &= a_1x + b_1y + c_1z, \\ y' &= a_2x + b_2y + c_2z, \\ z' &= a_3x + b_3y + c_3z. \end{aligned} \tag{11}$$

For just as the first of the equations (7) expresses that the sum of the projections on Ox of the co-ordinates x', y', z' is equal to x, so the first of the equations (11) expresses the equality of x' to the sum of the projections of x, y, z on the axis Ox'; and similarly for the other equations.

Now the solution of the equations (7) for x', y', z' should give the values (11). Putting

$$\begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} = \Delta,$$

solving the equations (7) for x', and comparing the coefficients of x, y, z to those in (11), we find the following relations:

$$a_1=\frac{b_2c_3-b_3c_2}{\Delta},\quad b_1=\frac{c_2a_3-c_3a_2}{\Delta},\quad c_1=\frac{a_2b_3-a_3b_2}{\Delta}. \tag{12}$$

339. Squaring and adding these equations and applying the relations (9), we find after reduction

$$\Delta^2=1.$$

The two values of Δ, $+1$ and -1, correspond to the two different relations between the two rectangular systems, which might perhaps be called like and unlike. Two systems are alike if their positive axes can be brought to coincidence; they are unlike if this cannot be done. It is, of course, always possible to bring the axes Ox' and Oy' to coincidence with Ox and Oy, respectively. But after having accomplished this, the axis Oz' may fall along Oz, in which case the systems are alike, or it may fall into the opposite direction, when the systems are unlike.

Now if Ox' coincides with Ox, Oy' with Oy, we have $a_1=1$, $b_2=1$; and $c_3=+1$ for like systems, $c_3=-1$ for unlike systems; as the other 6 cosines are zero, we find that $\Delta=+1$ corresponds to like systems, and $\Delta=-1$ to unlike systems. For it is evident that the motion of one system with respect to the other cannot affect Δ so as to change from one of these values to the other.

In mechanics the systems should generally be such that they can be brought to coincidence. We assume therefore $\Delta=1$.

With this value of Δ, the equations (12) and the similar relations obtained by cyclical permutation of the subscripts give the identities:

$$\begin{aligned} a_1&=b_2c_3-b_3c_2, & b_1&=c_2a_3-c_3a_2, & c_1&=a_2b_3-a_3b_2,\\ a_2&=b_3c_1-b_1c_3, & b_2&=c_3a_1-c_1a_3, & c_2&=a_3b_1-a_1b_3,\\ a_3&=b_1c_2-b_2c_1, & b_3&=c_1a_2-c_2a_1, & c_3&=a_1b_2-a_2b_1. \end{aligned} \tag{13}$$

340. If the axes Ox', Oy', Oz' be the principal axes at O, the equations (7) exhibit the relations between the system of the principal axes and a fixed system with the same origin O, by means of the cosines of the 9 angles between the axes of the two systems. They can be used to derive Euler's equations by a purely analytical process from the equations (7), Art. 224.

To accomplish this we must form the quantities $\Sigma m(y\dot{z}-z\dot{y})$, $\Sigma m(z\dot{x}-x\dot{z})$, $\Sigma m(x\dot{y}-y\dot{x})$. We need therefore $\dot{x}$, $\dot{y}$, $\dot{z}$. Now, differentiating the expressions (7) with respect to the time and remembering that x', y', z' are independent of the time, we find:

$$\dot{x}=\dot{a}_1x'+\dot{a}_2y'+\dot{a}_3z', \quad \dot{y}=\dot{b}_1x'+\dot{b}_2y'+\dot{b}_3z',$$
$$\dot{z}=\dot{c}_1x'+\dot{c}_2y'+\dot{c}_3z'. \tag{14}$$

To introduce the angular velocities ω_1, ω_2, ω_3 about the principal axes at O, we observe that the direction-cosines a_1, a_2, a_3 of the axis Ox can be regarded as the co-ordinates of the point situated on Ox at unit distance from O. The components of the linear velocity of this point, arising from ω_1, ω_2, ω_3 are

$$\dot{a}_1=a_2\omega_3-a_3\omega_2, \quad \dot{a}_2=a_3\omega_1-a_1\omega_3, \quad \dot{a}_3=a_1\omega_2-a_2\omega_1;$$

and similarly we have for points at unit distance from O on Oy and Oz:

$$\dot{b}_1=b_2\omega_3-b_3\omega_2, \quad \dot{b}_2=b_3\omega_1-b_1\omega_3, \quad \dot{b}_3=b_1\omega_2-b_2\omega_1,$$
$$\dot{c}_1=c_2\omega_3-c_3\omega_2, \quad \dot{c}_2=c_3\omega_1-c_1\omega_3, \quad \dot{c}_3=c_1\omega_2-c_2\omega_1.$$

It should be noticed that the motion of a body with a fixed point is fully determined by the motion of two of its points, not in the same line with the fixed point; the third point is here only introduced to preserve the symmetry.

Substituting these values in (14), we find

$$\begin{aligned}\dot{x}&=(a_2\omega_3-a_3\omega_2)x'+(a_3\omega_1-a_1\omega_3)y'+(a_1\omega_2-a_2\omega_1)z',\\ \dot{y}&=(b_2\omega_3-b_3\omega_2)x'+(b_3\omega_1-b_1\omega_3)y'+(b_1\omega_2-b_2\omega_1)z',\\ \dot{z}&=(c_2\omega_3-c_3\omega_2)x'+(c_3\omega_1-c_1\omega_3)y'+(c_1\omega_2-c_2\omega_1)z'.\end{aligned} \tag{15}$$

341. From (7) and (15) we now find, if we remember that $\Sigma my'z'=0$, $\Sigma mz'x'=0$, $\Sigma mx'y'=0$, since Ox', Oy', Oz' are principal axes at O:

$$\Sigma m(y\dot{z}-z\dot{y})=(b_2c_3-b_3c_2)\omega_1\Sigma m(y'^2+z'^2)$$
$$+(b_3c_1-b_1c_3)\omega_2\Sigma m(z'^2+x'^2)+(b_1c_2-b_2c_1)\omega_3\Sigma m(x'^2+y'^2),$$

or applying the relations (13) and denoting the principal moments of inertia at O by I_1, I_2, I_3:

$$\Sigma m(y\dot{z}-z\dot{y})=a_1I_1\omega_1+a_2I_2\omega_2+a_3I_3\omega_3.$$

The quantities $\Sigma m(z\dot{x}-x\dot{z})$ and $\Sigma m(x\dot{y}-y\dot{x})$ are obtained from this result by cyclical permutation of the letters a, b, c.

Thus the equations (7), Art. 224, assume the form:

$$\frac{d}{dt}(a_1I_1\omega_1+a_2I_2\omega_2+a_3I_3\omega_3)=H_x,$$

$$\frac{d}{dt}(b_1I_1\omega_1+b_2I_2\omega_2+b_3I_3\omega_3)=H_y, \qquad (16)$$

$$\frac{d}{dt}(c_1I_1\omega_1+c_2I_2\omega_2+c_3I_3\omega_3)=H_z.$$

The geometrical meaning of these equations is apparent. $I_1\omega_1$, $I_2\omega_2$, $I_3\omega_3$ are the components along the principal axes of the vector of the resultant impulsive couple $\underset{.}{H}$ (Art. 319); hence $a_1I_1\omega_1+a_2I_2\omega_2+a_3I_3\omega_3$ is the component of $\underset{.}{H}$ along the fixed axis Ox; the equations (16) express therefore the fact that H is geometrically equal to the geometrical derivative of $\underset{.}{H}$ with respect to the time (see Art. 327); they can be written in the form

$$\frac{d\underset{.}{H}_x}{dt}=H_x, \quad \frac{d\underset{.}{H}_y}{dt}=H_y, \quad \frac{d\underset{.}{H}_z}{dt}=H_z.$$

342. If the equations (16) be multiplied first by a_1, b_1, c_1, then by a_2, b_2, c_2, finally by a_3, b_3, c_3, and each time added, the right-hand members of the resulting equations will evidently represent H_1, H_2, H_3, respectively, *i.e.* the components of H

along the principal axes at O. The left-hand members reduce also to a simple form if the differentiations indicated in (16) be performed, the values for $\dot{a}_1$, $\dot{a}_2$, $\dot{a}_3$, $\dot{b}_1$, ... be substituted from Art. 340, and the relations (9) be applied. As final result we find Euler's equations:

$$I_1\dot{\omega}_1 + (I_3 - I_2)\omega_2\omega_3 = H_1,$$
$$I_2\dot{\omega}_2 + (I_1 - I_3)\omega_3\omega_1 = H_2,$$
$$I_3\dot{\omega}_3 + (I_2 - I_1)\omega_1\omega_2 = H_3.$$

3. CONTINUOUS MOTION WITHOUT FORCES.

343. In the particular case when no external forces are acting, the motion of a rigid body about a fixed point admits of an elegant geometrical interpretation which is due to Poinsot.

As there are no external forces, we have $\dot{H} = 0$, and hence H is constant in magnitude and direction. The plane of this couple is the *invariable plane* (see Arts. 230–232) which always exists in the case of no forces; its vector indicates the *invariable direction.*

The body can be replaced by its momental ellipsoid at O, and the invariable plane can be imagined placed so as to be tangent to this ellipsoid at a point P' (Fig. 43). The radius vector $OP' = \rho'$ of the point of contact P' is the diameter of the ellipsoid conjugate to the invariable plane; hence the line OP' is the instantaneous axis l of the rotation (Art. 321).

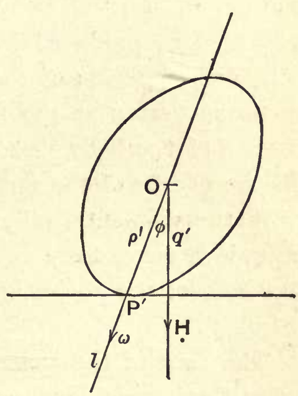

Fig. 43.

Now it can be shown that the perpendicular distance q' of O from the invariable plane (this plane being always placed so as to be tangent to the varying positions of the momental ellipsoid) is constant; it then follows at once that *the motion of the body consists in the rolling of its momental ellipsoid over the invariable plane.*

344. To prove that q' is constant, it should be remembered that, by (7), Art. 321, we have

$$q' = M\epsilon^4 \cdot \frac{\omega}{\rho'} \cdot \frac{1}{\underset{\cdot}{H}}.$$

As $\underset{\cdot}{H}$ is constant in our case, it only remains to show that ω/ρ' is constant. This follows from the expression (9) for the kinetic energy T, given in Art. 322, viz.

$$T = \tfrac{1}{2} M\epsilon^4 \left(\frac{\omega}{\rho'}\right)^2;$$

for, as there are no external forces, no work is done, and the kinetic energy must remain constant; hence ω/ρ' is constant, and ω is directly proportional to ρ'.

Moreover, the expression (14) of Art. 322 shows that

$$\omega \cos \phi = \frac{2T}{\underset{\cdot}{H}} = \text{const.}, \qquad (1)$$

that is, the projection $\omega \cos \phi$ of the angular velocity ω on the invariable direction remains the same throughout the motion.

345. It has been pointed out in Part I., Art. 35, that the motion of a rigid body with a fixed point can always be regarded as produced by the rolling of the cone of the body axes over the cone of the space axes, these cones having their common vertex at the fixed point O. The body axes, *i.e.* the lines l' of the body that become instantaneous axes of rotation in the course of the motion, form a cone, invariably connected with the momental ellipsoid at O, and intersecting this ellipsoid in a curve fixed in the body. This curve has been called by Poinsot the *polhode* (or path of the instantaneous pole P', Fig. 43).

The cone of the space axes l, which is fixed in space, intersects the invariable plane in a curve called *herpolhode* (or creeping path of the pole). During the motion of the body, the polhode rolls over the herpolhode.

346. The equations of the polhode are easily obtained by considering that this curve is the locus of those points of the ellipsoid whose tangent plane has the constant distance q' from the centre O. Hence, denoting the semi-axes of the momental ellipsoid by a, b, c, the *equations of the polhode* are

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1, \quad \frac{x^2}{a^4}+\frac{y^2}{b^4}+\frac{z^2}{c^4}=\frac{1}{q'^2}. \tag{2}$$

It can, therefore, be regarded as the intersection of the momental ellipsoid with a coaxial ellipsoid whose semi-axes are a^2/q', b^2/q', c^2/q'.

Multiplying the second equation by q'^2, and subtracting the result from the first equation, we find the equation of the *cone of the body axes*

$$\left(1-\frac{q'^2}{a^2}\right)\frac{x^2}{a^2}+\left(1-\frac{q'^2}{b^2}\right)\frac{y^2}{b^2}+\left(1-\frac{q'^2}{c^2}\right)\frac{z^2}{c^2}=0. \tag{3}$$

This is a cone of the second order, concentric and coaxial with the momental ellipsoid.

The polhode evidently consists of two equal separate branches, of which it is sufficient to consider one. Each branch has four vertices situated in the principal planes of the ellipsoid.

The herpolhode is confined between two concentric circles whose centre is the projection of O on the invariable plane. It is a transcendental curve and is in general not closed.

347. If the momental ellipsoid is an ellipsoid of revolution, the polhode consists of two circles, and the herpolhode is also a circle; as ρ' is in this case constant, it follows that ω remains constant.

If we assume in the general case $a > b > c$, the polhode reduces to two points whenever $q' = a$ or $q' = c$. The rotation then takes place about a principal axis and is *permanent*. If $q' = b$ (which does not necessarily mean that the axis of rotation coincides with the middle axis b), the cone of body axes reduces to two planes

$$\left(1-\frac{b^2}{a^2}\right)\frac{x^2}{a^2}+\left(1-\frac{b^2}{c^2}\right)\frac{z^2}{c^2}=0,$$

each of which intersects the ellipsoid in an ellipse. These ellipses divide the surface of the ellipsoid into two pairs of opposite regions, one about the greatest axis a, the other about the least c.

As long as $a > q' > b$, the polhode lies in the former region, and the cone of body axes has a as its axis. If $b > q' > c$, the polhode lies in the other region, and c is the axis of the cone.

Two polhodes cannot intersect; for if they did, the tangent plane at the point of intersection would have two different distances from the centre, which is impossible.

348. The motion of a body is called *stable* if after a slight disturbance the body tends to resume the original motion. In our case a slight disturbance displaces the instantaneous axis from one polhode to another near by. Hence if the polhode be situated very near to one of the bounding ellipses, the motion is not stable, because a slight disturbance might change the polhode to one in the other region. The motion is therefore the more stable the more closely the polhode surrounds either the greatest or the least axis of the ellipsoid.

If, however, one of the regions between the ellipses be very narrow, which will be the case if two of the axes of the ellipsoid are nearly equal, a polhode in this region, though close to the vertex, may still approach very near to the ellipses so as to make the motion unstable.

349. Integration of Euler's Equations. As $H=0$, Euler's equations (4), Art. 328, are

$$I_1\frac{d\omega_1}{dt}=(I_2-I_3)\,\omega_2\omega_3,\quad I_2\frac{d\omega_2}{dt}=(I_3-I_1)\,\omega_3\omega_1,$$

$$I_3\frac{d\omega_3}{dt}=(I_1-I_2)\omega_1\omega_2. \tag{4}$$

Multiplying by ω_1, ω_2, ω_3, and adding, we find

$$\frac{d}{dt}\cdot\tfrac{1}{2}(I_1\omega_1^2+I_2\omega_2^2+I_3\omega_3^2)=0,$$

whence, by (17), Art. 323,

$$\tfrac{1}{2}(I_1\omega_1^2+I_2\omega_2^2+I_3\omega_3^2)\equiv T=\text{const.} \tag{5}$$

This is nothing but the *equation of kinetic energy.*

Again, multiplying the equations (4) by $I_1\omega_1$, $I_2\omega_2$, $I_3\omega_3$, and adding, we find similarly, by (15), Art. 323,

$$I_1^2\omega_1^2+I_2^2\omega_2^2+I_3^2\omega_3^2 \equiv H^2 = \text{const.} \tag{6}$$

This is the *principle of areas* or *of the invariable plane.*

As, moreover,

$$\omega_1^2+\omega_2^2+\omega_3^2=\omega^2, \tag{7}$$

we have three equations (5), (6), (7) for determining ω_1^2, ω_2^2, ω_3^2. Their solution gives, after some reductions,

$$\omega_1^2=\frac{(I_2+I_3)\cdot 2T-H^2-I_2I_3\omega^2}{(I_1-I_2)(I_3-I_1)},\quad \omega_2^2=\frac{(I_3+I_1)\cdot 2T-H^2-I_3I_1\omega^2}{(I_2-I_3)(I_1-I_2)},$$

$$\omega_3^2=\frac{(I_1+I_2)\cdot 2T-H^2-I_1I_2\omega^2}{(I_3-I_1)(I_2-I_3)}. \tag{8}$$

350. To find the *time*, multiply the equations (4) by ω_1/I_1, ω_2/I_2, ω_3/I_3, and add. This gives

$$\frac{d}{dt}\tfrac{1}{2}(\omega_1^2+\omega_2^2+\omega_3^2)=\omega_1\omega_2\omega_3\left(\frac{I_2-I_3}{I_1}+\frac{I_3-I_1}{I_2}+\frac{I_1-I_2}{I_3}\right),$$

or

$$\frac{d}{dt}\tfrac{1}{2}\omega^2=-\frac{(I_2-I_3)(I_3-I_1)(I_1-I_2)}{I_1I_2I_3}\omega_1\omega_2\omega_3.$$

In this equation the values (8) should be substituted for ω_1, ω_2, ω_3. For the sake of brevity, let us put

$$(I_2+I_3)\cdot 2T-H^2=I_2I_3\alpha^2,\quad (I_3+I_1)\cdot 2T-H^2=I_3I_1\beta^2,$$

$$(I_1+I_2)\cdot 2T-H^2=I_1I_2\gamma^2;$$

we then find

$$dt=\pm\tfrac{1}{2}\int\frac{d(\omega^2)}{\sqrt{(\omega^2-\alpha^2)(\beta^2-\omega^2)(\omega^2-\gamma^2)}} \tag{9}$$

This is an elliptical integral whose discussion is beyond the scope of the present treatise.

351. It remains to determine the position of the moving system formed by the principal axes, with respect to a fixed system of axes through O, by means of Euler's angles θ, ϕ, ψ (Art. 333). After finding ω as a function of t from (9), we have, by (8), ω_1, ω_2, ω_3 as functions of t. Substituting these values into the equations (5) or (6), Art. 335, we have a system of differential equations of the first order whose integration gives θ, ϕ, ψ as functions of t.

352. To illustrate the method by a simple example, let us consider the case of *a body whose momental ellipsoid is an ellipsoid of revolution.*

Let $I_1 = I_2$; then, putting $(I_2 - I_3)/I_1 = -(I_3 - I_1)/I_2 = \lambda$, Euler's equations (4) become

$$\frac{d\omega_1}{dt} - \lambda\omega_2\omega_3 = 0, \quad \frac{d\omega_2}{dt} + \lambda\omega_3\omega_1 = 0, \quad \frac{d\omega_3}{dt} = 0. \tag{10}$$

The last of these equations shows that the component of the angular velocity about the axis of revolution of the body is constant. The other two equations give

$$\omega_1\frac{d\omega_1}{dt} + \omega_2\frac{d\omega_2}{dt} = 0,$$

whence
$$\omega_1^2 + \omega_2^2 = \text{const.} = \omega_0^2, \tag{11}$$

where ω_0 denotes the constant angular velocity about the projection of the instantaneous axis on the equatorial plane of the body. The resulting angular velocity is, therefore, constant, viz.

$$\omega = \sqrt{\omega_0^2 + \omega_3^2}. \tag{12}$$

353. The inclination of the instantaneous axis to the principal axes a, b varies, but its inclination to the axis c is constant, viz. $= \cos^{-1}(\omega_3/\omega_0)$. The cone of the body axes is, therefore, a cone of revolution about the axis c, and the polhode is a circle. The herpolhode is, therefore, likewise a circle, and the space axes form a cone of revolution (comp. Art. 347). As the two cones are always in contact along the instantaneous axis, this axis lies in the same plane with the vector $\underline{H}$ and the axis of revolution of the body.

354. To find the angular velocities ω_1, ω_2 as functions of the time, differentiate the first of the equations (10), and eliminate $d\omega_2/dt$ with the aid of the second. This gives

$$\frac{d^2\omega_1}{dt^2} + \lambda^2\omega_3^2\omega_1 = 0,$$

whence
$$\omega_1 = C_1 \cos \lambda\omega_3 t + C_2 \sin \lambda\omega_3 t. \tag{13}$$

The other component, ω_2, can now be found from the first of the equations (10):

$$\omega_2 = \frac{1}{\lambda\omega_3}\frac{d\omega_1}{dt} = -C_1 \sin \lambda\omega_3 t + C_2 \cos \lambda\omega_3 t. \tag{14}$$

To determine the constants C_1, C_2, the initial values of ω_1, ω_2, say at the time $t=0$, should be known. Let ϵ be the angle made at this time by ω_0 with the principal axis b. Then the initial values of ω_1, ω_2 are $\omega_0 \sin\epsilon$, $\omega_0 \cos\epsilon$, and the substitution of $t=0$ in (13) and (14) shows that $C_1 = \omega_0 \sin\epsilon$, $C_2 = \omega_0 \cos\epsilon$. Hence we have, finally,

$$\omega_1 = \omega_0 \sin(\lambda\omega_3 t + \epsilon), \quad \omega_2 = \omega_0 \cos(\lambda\omega_3 t + \epsilon), \quad \omega_3 = \text{const.} \qquad (15)$$

355. To determine the position of the body at any time t with respect to fixed axes through O, let us take as axis of z the fixed direction of $\underset{.}{H}$, which is perpendicular to the invariable plane. The cosines of the angles made by this axis with the principal axes are found similarly as in Art. 335 (see Fig. 42):

$$\cos ZX' = \sin\phi \sin\theta, \quad \cos ZY' = \cos\phi \sin\theta, \quad \cos ZZ' = \cos\theta.$$

Hence the components $\underset{.}{H}_1 = I_1\omega_1$, $\underset{.}{H}_2 = I_2\omega_2$, $\underset{.}{H}_3 = I_3\omega_3$ of $\underset{.}{H}$ along the principal axes are

$$I_1\omega_1 = \underset{.}{H}\sin\phi\sin\theta, \; I_2\omega_2 = \underset{.}{H}\cos\phi\sin\theta, \; I_3\omega_3 = \underset{.}{H}\cos\theta.$$

These equations give

$$\cos\theta = \frac{I_3\omega_3}{\underset{.}{H}}, \qquad (16)$$

and, as $I_1 = I_2$,
$$\tan\phi = \frac{\omega_1}{\omega_2} = \tan(\lambda\omega_3 t + \epsilon),$$

by (15); hence
$$\phi = \lambda\omega_3 t + \phi_0, \qquad (17)$$

where ϕ_0 is the value of ϕ for $t=0$. Thus it appears that the angle θ is constant, while ϕ increases proportionally to the time.

356. To find ψ, we may use the third of the equations (5), Art. 335, viz. $\omega_3 = \dot\phi + \dot\psi\cos\theta$. As $\cos\theta = I_3\omega_3/\underset{.}{H}$, $\dot\phi = \lambda\omega_3$, $\lambda = (I_1 - I_3)/I_1$, we find

$$\frac{d\psi}{dt} = \frac{(1-\lambda)\underset{.}{H}}{I_3} = \frac{\underset{.}{H}}{I_1},$$

whence
$$\psi = \frac{\underset{.}{H}}{I_1}t + \psi_0. \qquad (18)$$

It appears then that the equatorial plane $X'Y'$ of the body remains at a constant inclination θ to the invariable plane, while the nodal line ON (Fig. 42) turns uniformly in this invariable plane and a radius of the body in the equatorial plane turns also uniformly in the equatorial plane.

V. *Free Rigid Body.*

I. INITIAL MOTION DUE TO IMPULSES.

357. Kinematically, the most general motion of a rigid body consists, at every instant, of a twist, or screw-motion about a certain line, called the *instantaneous axis* l; that is, the body has, for an element of time, an angular velocity ω about l and at the same time a velocity of translation v along this axis (see Part I., Arts. 43, 44; 294). During the next element of time the body will, in general, rotate about a different axis with a different angular velocity and will have a different linear velocity along the new axis.

358. It has also been shown in kinematics (Part I., Art. 257) that the angular velocity ω about l can be replaced by an equal angular velocity about any parallel axis l', in connection with a certain velocity of translation. For without changing the state of motion we can give the body two equal and opposite rotations about l'; *i.e.* we can introduce along l' (Fig. 44) two equal and opposite rotors $\omega, -\omega$; and $-\omega$ about l' combines with ω about l to a rotor couple, which is equivalent to a velocity of translation $p\omega$, perpendicular to the plane (l, l'), $p = OO'$ being the distance of the parallel axes. The velocity of translation $p\omega$ can now be combined with v to a resultant velocity of translation $v' = \sqrt{v^2 + p^2\omega^2}$ inclined to l' at an angle $\phi = \tan^{-1}(p\omega/v)$.

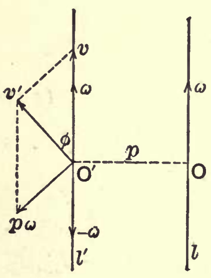

Fig. 44.

It thus appears that the instantaneous motion of a free rigid body can be regarded as a rotation about *any* line parallel to the instantaneous axis, in combination with a certain velocity of translation inclined to this line.

On account of the dynamical properties of the centroid of a rigid body, it will generally be found convenient to select the axis of rotation so as to pass through the centroid; we shall then call it the *centroidal instantaneous axis* $\bar{\text{l}}$.

359. Dynamically, the instantaneous motion of a free rigid body is determined by the momenta of its particles. These momenta can be reduced, for any point O as origin, to a resultant momentum and a resultant couple, or angular momentum, and these can be regarded as due to a certain system of impulses. This reduction will at the same time lead to the solution of the converse problem, viz. to determine the initial motion produced by a system of impulses acting on a rigid body at rest, and the change in the instantaneous motion due to such a system when the body is not at rest.

360. Translation. The velocities u of all points being equal and parallel in the case of translation, the momenta mu of all particles are parallel and have (see Arts. 6–8) a single resultant

$$\Sigma mu = Mu,$$

passing through the centroid G of the body. If the whole mass M be regarded as concentrated at the centroid, Mu is the momentum of the centroid. This momentum can be produced by applying at the centroid a single impulse $\underset{\cdot}{R} = Mu$. Hence *to impart to a free rigid body of mass* M *a velocity of translation* u, *it is sufficient to apply at the centroid an impulse* $\underset{\cdot}{\mathrm{R}} = \mathrm{Mu}$.

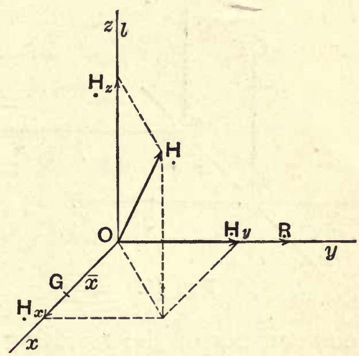

Fig. 45.

361. Rotation. Let us take the instantaneous axis l as axis of z, and the axis of x so as to pass through the centroid G (Fig. 45). The momentum $m\omega r$ of any particle of mass m, at the distance r from l, has the components $-m\omega y$, $m\omega x$, 0; and as $\Sigma my = 0$, $\Sigma mx = M\omega\bar{x}$, the resultant momentum has the components 0, $M\omega\bar{x}$, 0; it is therefore perpendicular to the plane through axis and centroid. *Hence the resultant impulse* R *at* O *must be equal in magnitude and direction to the momentum* $\mathrm{M\bar{v} = M\omega\bar{x}}$ *of the centroid, due to the rotation* ω *about the instantaneous axis* l.

The resultant angular momentum is found just as in Art. 317; it is $=\omega\sqrt{C^2+D^2+E^2}$ and has the components $-E\omega$ along Ox, $-D\omega$ along Oy, and $C\omega$ along the instantaneous axis Oz.

It follows that *a pure rotation of angular velocity* ω *about an axis* l *can be imparted to a free rigid body by the combined action of an impulse* R *and an impulsive couple* H. The impulse $R = M\omega\bar{x}$ is perpendicular to the plane (l, G), and passes through the foot O of the perpendicular let fall from the centroid G on the axis; it vanishes only when $\bar{x} \equiv OG = 0$; *i.e.* when the instantaneous axis passes through the centroid. The remarks of Art. 318 apply to the couple without change.

362. As mentioned above, it is often more convenient to take the centroid G as origin for the reduction of the impulses. To reduce the system of impulses R, H, determined in the preceding article, to G as origin and to parallel axes (Fig. 46), it is only necessary to apply R and $-R$ at G; we then have the resultant impulse $R = M\omega\bar{x}$ at G, and the couple formed by R at O, and $-R$ at G. The moment of this couple is $-R\bar{x} = -M\omega\bar{x}^2$; its vector is parallel to the instantaneous axis l, and can therefore be added algebraically to H_z, while H_x and H_y remain unchanged. Thus the components of the resulting couple for the reduction to the centroid are $H_x = -E\omega$, $H_y = -D\omega$, $H_z = (C - M\bar{x}^2)\omega = C'\omega$, where $C' = C - M\bar{x}^2$ is the moment of inertia about the cen-

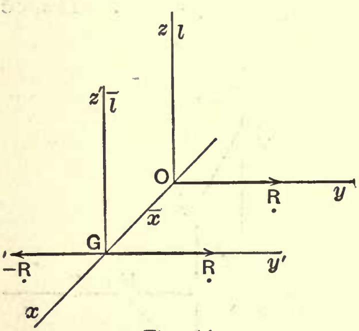

Fig. 46.

troidal axis $\bar{l}$ whose distance from l is $\bar{x}$ (see Art. 250), while D and E are the products of inertia for the new co-ordinate planes through G.

These results can, of course, also be derived directly by reducing the momenta for G as origin, the centroidal instantaneous axis $\bar{l}$ as axis of z, and the plane through G and the instantaneous axis l as the zx-plane.

363. It thus appears that the *form* of the results for this new system of co-ordinates is exactly the same as in Art. 354; but C, D, E refer now to the new co-ordinate axes and planes. Hence *a pure rotation about any instantaneous axis* l, *at the distance* $\bar{x}$ *from the centroid* G, *can be produced by an impulse* $\underset{.}{R}$ *and a couple* $\underset{.}{H}$, *the impulse* $\underset{.}{R}$ *being equal to the momentum* $M\omega\bar{x}$ *of the centroid, due to the rotation, and passing through* G *at right angles to the plane* (l, G), *while the vector of the couple* $\underset{.}{H}$ *has in general three components* $\underset{.}{H}_x = -E\omega$, $\underset{.}{H}_y = -D\omega$, $\underset{.}{H}_z = C\omega$.

The geometrical relation between the vector $\underset{.}{H}$ and the rotor ω can again be illustrated by means of the ellipsoids of inertia, as in Arts. 320, 321. The developments of these articles apply without change if the foot O of the perpendicular let fall from the centroid on the instantaneous axis l_0 be substituted for the fixed point; they apply likewise if the centroid G be substituted for the fixed point, in which case the momental ellipsoid becomes the central, and the reciprocal, the fundamental ellipsoid.

364. The resulting impulse $\underset{.}{R} = M\omega\bar{x}$ vanishes only for $\bar{x} = 0$; *i.e.* when the instantaneous axis l passes through the centroid. In other words, pure rotation about an axis not passing through the centroid cannot be produced by an impulsive couple alone.

On the other hand, pure rotation about a *centroidal* axis can always be regarded as due to an impulsive couple alone; and conversely, *the effect of a single impulsive couple on a free rigid body is to produce pure rotation about a centroidal axis.* But it should always be remembered (see Art. 318) that the axis of rotation l is parallel to the vector $\underset{.}{H}$ of the couple only, and always, if $D=0$ and $E=0$; *i.e.* if the vector $\underset{.}{H}$ is parallel to a principal axis at G. Hence *pure rotation about a centroidal principal axis can be produced by a single couple whose plane is perpendicular to the axis;* and conversely, *a couple whose plane is perpendicular to a centroidal principal axis produces pure rotation about this axis.* The relation between the moment $\underset{.}{H}$ of the couple and the angular velocity ω produced is in this case $\underset{.}{H}=I\omega=M\omega q^2$, where I is the moment, q the radius of inertia for the principal axis.

365. To find the condition under which the system of impulses producing pure rotation may reduce to a single impulse $\underset{.}{R}$, we have only to reduce the system of impulses to its central axis (comp. Part II., Arts. 204–206). For this line which is parallel to $\underset{.}{R}$ has the property that if any point on it be taken as origin of reduction, the couple has its vector parallel to $\underset{.}{R}$ and has its least value $\underset{.}{H}_0$, which is equal to the projection on this line (*i.e.* on the direction of $\underset{.}{R}$) of the vector $\underset{.}{H}$ for *any* reduction. Now, as in our case the components $\underset{.}{H}_x$ and $\underset{.}{H}_z$ are both perpendicular to $\underset{.}{R}$ (see Fig. 45), it follows that

$$\underset{.}{H}_0 = H = -D\omega.$$

This vanishes only with the product of inertia $D=\Sigma myz$.

Hence *pure rotation about an instantaneous axis* l *can be produced by a single impulse* $\underset{.}{\mathrm{R}}=\mathrm{M}\omega\bar{\mathrm{x}}$ *only, and always, if* l *is so situated that the product of inertia* $\mathrm{D}=\Sigma\mathrm{myz}$ *vanishes for the planes through* G *and* l *and through* G *perpendicular to* l. In particular, this is evidently the case *whenever* l *is a principal axis at the foot* O *of the perpendicular let fall on it from the centroid.* (Comp. Arts. 309, 310.)

366. It remains to find the position of the central axis, *i.e.* of the line of action of the single impulse $\underset{.}{R}$ capable of producing pure rotation about the instantaneous axis l provided it satisfies the condition just mentioned. This can be done, if $\underset{.}{R}$ and $\underset{.}{H}$ are known for the centroid, by transferring $\underset{.}{R}$ to parallel positions so as to reduce the components $\underset{.}{H}_z$ and $\underset{.}{H}_x$ to zero. Thus, to destroy $\underset{.}{H}_z = C\omega$ we have only to transfer $\underset{.}{R}$ from G along the axis of x to a point O' (Fig. 47) at a distance $GO' = x_1$ from G such that $\bar{x}x_1 = q^2$, where $\bar{x} = OG$, and q is the radius of inertia for the centroidal instantaneous axis $\bar{l}$. For then the couple resulting from the transfer has a vector along the axis of z equal to $-\underset{.}{R}x_1 = -M\omega\bar{x}x_1 = -M\omega q^2 = -\underset{.}{H}_z$.

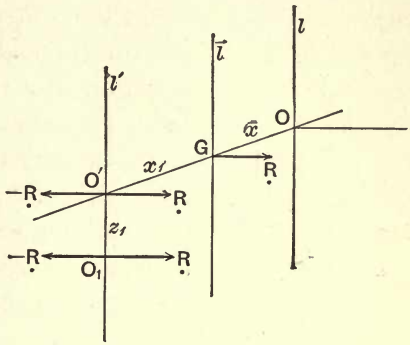

Fig. 47.

Next to destroy $\underset{.}{H}_x = -E\omega = -\omega\Sigma mzx$, we transfer the point of application of $\underset{.}{R}$ parallel to the instantaneous axis l to a point O_1, at a distance $O'O_1 = z_1$ from O', such that $\underset{.}{R}z_1 = -\underset{.}{H}_x$, whence $z_1 = \Sigma mzx / M\bar{x}$.

If the point O_1 be taken as origin of reduction, the system of impulses reduces to $\underset{.}{R}$ at O_1, and the couple $\underset{.}{H}_0 = \underset{.}{H}_y = -D\omega$, whose vector is parallel to $\underset{.}{R}$. Thus *the central axis*, which has of course the direction of $\underset{.}{R}$, and is therefore perpendicular to the plane (l, G), *meets this plane at a point* O_1 *whose co-ordinates* are $x_1 = q^2/\bar{x}$, $z_1 = E/m\bar{x}$. It is easy to see that these results agree with the developments of Arts. 309, 310, the centre of percussion, if it exists, being situated on the central axis.

367. Twist or Screw Motion. In the most general case the motion of a rigid body consists of an angular velocity ω about the instantaneous axis l and a simultaneous velocity of translation u along this axis (Art. 357).

Now, by Art. 360, the velocity of translation, u, can be regarded as due to a single impulse $R'=Mu$, passing through the centroid G and parallel to u, *i.e.* to l (Fig. 48). Again, by Art. 361, the angular velocity ω about l can be regarded as due to an impulse $R''=M\omega\bar{x}$ through G at right angles to the plane $(l,\ G)$, in combination with a couple whose vector H has the components $H_x=-E\omega$, $H_y=-D\omega$, $H_z=C\omega$. The two impulses R', R'' combine to form a single resultant impulse,

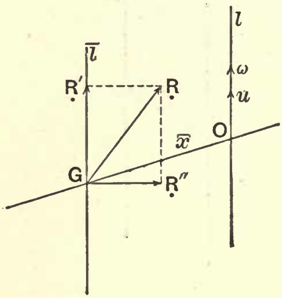

Fig. 48.

$$R=\sqrt{R'^2+R''^2}=M\sqrt{u^2+\omega^2\bar{x}^2},$$

inclined to l at an angle $\phi=\tan^{-1}(\omega\bar{x}/u)$. It should be noticed that the factor $\sqrt{u^2+\omega^2\bar{x}^2}$ is the velocity $\bar{v}$ of the centroid due to the twist, so that *the resultant impulse* $\mathrm{R}=\mathrm{M}\bar{\mathrm{v}}$ *is equal to the momentum of the centroid.* The resultant couple $H=\omega\sqrt{C^2+D^2+E^2}$ is the same as in the case of pure rotation.

368. The problem of *determining the initial motion produced in a free rigid body at rest by a given system of impulses* finds its geometrical solution in the preceding articles. It should also be remembered that the motion about the centroid takes place as if the centroid were fixed so that all the developments of Arts. 313–323 can be applied by substituting the centroid G for the fixed point O.

It will generally be best to reduce the given impulses to a resultant R, passing through the centroid, and to a couple H. By Art. 360, the impulse R at G produces a velocity of translation,

$$u=\frac{R}{M}. \qquad (1)$$

By Arts. 361, 363, 320, 321, the couple H produces an angular velocity,

$$\omega=\frac{H}{\sqrt{C^2+D^2+E^2}}=\frac{H}{M\rho q}=\frac{H\rho' q'}{M\epsilon^4}, \qquad (2)$$

about a centroidal axis $\bar{l}$ whose direction is conjugate to the plane of the couple H in the central ellipsoid, while a plane perpendicular to $\bar{l}$ is conjugate to the direction of the vector H in the fundamental ellipsoid. The components of ω along the principal centroidal axes are given by (5), Art. 319, viz.:

$$\omega_1 = \frac{H_1}{I_1}, \quad \omega_2 = \frac{H_2}{I_2}, \quad \omega_3 = \frac{H_3}{I_3},$$

where I_1, I_2, I_3 are the principal centroidal moments of inertia and H_1, H_2, H_3 the components of H along the centroidal principal axes.

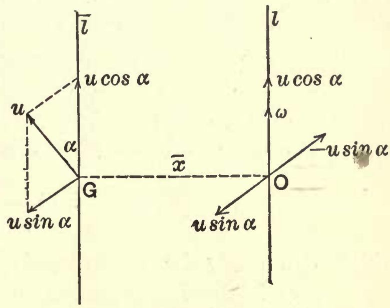

Fig. 49.

The direction of the instantaneous axis having thus been determined, its position can be found by resolving u into a component $u\cos\alpha$ along $\bar{l}$ and a component $u\sin\alpha$ perpendicular to $\bar{l}$ (Fig. 49). The latter component combines with ω about $\bar{l}$ to an equal angular velocity ω about an axis l parallel to $\bar{l}$ at the distance $\bar{x} = -u\sin\alpha/\omega$ from $\bar{l}$.

The initial motion produced by the impulses consists, therefore, of the angular velocity ω about l, and the linear velocity $u\cos\alpha$ along l. These together constitute the resulting twist.

369. Exercises.*

(1) *A homogeneous straight rod* AB = 2a (Fig. 50) *is acted upon by an impulse* F, *at the distance* c *from the centroid* G, *at an angle* α *with the rod. Determine the initial motion.*

The reduction to the centroid G gives the impulse F at G, which produces a velocity of translation $u = F/M$ in the direction of F, and the couple H formed by F at C and $-F$ at G. The moment of this couple is $Fc\sin\alpha$, and its vector H is at right angles to the plane deter-

* Most of these problems, as well as the discussions of Arts. 357–368, are adapted from W. SCHELL, *Theorie der Bewegung und der Kräfte*, Vol. II., pp. 352–386.

mined by AB and F. As any perpendicular to the rod is a principal axis, the radius of inertia at G being $q = \sqrt{\frac{1}{3}}\, a$, the couple $Fc \sin \alpha$ produces pure rotation about an axis $\bar{l}$ through G at right angles to the plane (AB, F) (see Art. 364), and we have

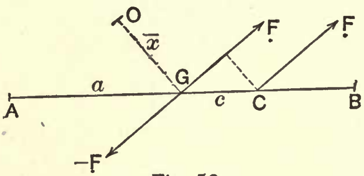

Fig. 50.

$$\omega = \frac{H}{Mq^2} = \frac{3Fc \sin \alpha}{Ma^2}.$$

As the axis $\bar{l}$ of this rotation is perpendicular to the direction of the velocity u of translation, their combined effect is a pure rotation of the same angular velocity ω about a parallel axis l whose position is found as follows (Art. 368): Draw through G, in the plane (AB, F), a perpendicular to F, and on this perpendicular lay off $GO \equiv \bar{x} = u/\omega$. With the values of u and ω given above, we have

$$\bar{x} = \frac{a^2}{3c \sin \alpha},$$

which can easily be constructed geometrically. The parallel to $\bar{l}$ through O is the instantaneous axis about which the rod begins to rotate with the angular velocity ω.

In what direction and at what distance from G must the rod be struck if it is to rotate about a perpendicular through the end A?

(2) *A homogeneous plane lamina of mass* M *receives an impulse* F *in its plane, the distance of the centroid from the direction of* F *being* x_1. *Determine the initial motion.*

The reduction to the centroid G (Fig. 51) gives F at G, and a couple $H = Fx_1$ whose vector is parallel to a principal centroidal axis. The couple produces, therefore, rotation about the perpendicular $\bar{l}$ through G to the plane of the lamina, and this rotation combines with the translation due to F to a single rotation about a parallel axis l. To find the position of l, lay off on the direction of F, drawn through G, a length GK equal to the radius of inertia for $\bar{l}$; join K to the foot O' of the perpendicular let fall from G on F, and draw KO at right angles to KO'. The instantaneous axis l passes through O, since $GO \cdot GO' = q^2$. The angular velocity is $\omega = Fx_1/Mq^2$.

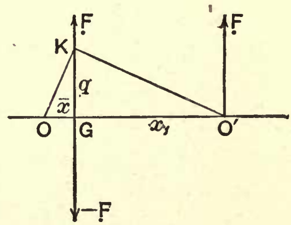

Fig. 51.

(3) *In Ex.* (2), *if the lamina be an ellipse of semi-axes* a, $\frac{1}{2}$a, *at what point of the major axis must it be struck at right angles to this axis in order to rotate initially about a focus?*

(4) *A rigid body of mass* M *receives an impulse* $\underset{\cdot}{F}$ *parallel to the principal axis* Gy, *and meeting the principal axis* Gx *at the distance* $GO' = x_1$ *from the centroid. Determine the initial motion* (Fig. 52).

The impulse $\underset{\cdot}{F}$ at O' is equivalent to an equal and parallel impulse $\underset{\cdot}{F}$ through the centroid G, in combination with the couple $\underset{\cdot}{H} = \underset{\cdot}{F}x_1$ whose vector is parallel to the principal axis Gz. This couple produces, therefore, rotation about the centroidal principal axis Gz, or $\bar{l}$, of angular velocity $\omega = \underset{\cdot}{F}x_1/Mq_3^2$, where q_3 is the radius of inertia for $\bar{l}$. The instantaneous axis l is parallel to $\bar{l}$, and meets the axis Gx at a point O such that $GO \cdot GO' = q_3^2$, or putting $GO = \bar{x}$, $GO' = x_1$, such that $\bar{x}x_1 = q_3^2$; it can be constructed as in Ex. (2).

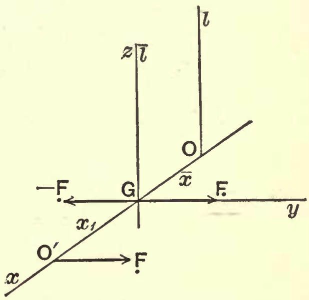

Fig. 52.

(5) *Determine the impulse* $\underset{\cdot}{F}$ *imparted to a body of mass* M *when struck at the point* O′ (Fig. 53) *of a principal axis* Gx *by a particle of mass* m *moving with a velocity* u *parallel to another principal axis* Gy. *The impact is assumed to be inelastic* (compare Art. 305).

It has been shown in Art. 19 that if a particle of mass m moving with a velocity u impinges upon a particle of mass M at rest, the two particles will, after inelastic impact, move on together with the common velocity $v = mu/(m+M)$. Similarly in our case, as soon as the impact has taken place, the two masses m and M may be regarded as forming a whole, and as moving together. The impulse mu acting on this mass $M+m$ at O' imparts to this point a certain velocity v which can be determined. As the particle m assumes this velocity v after impact, it loses the momentum $mu - mv$, owing to the impact; and this is the impulse

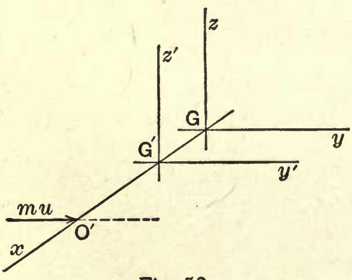

Fig. 53.

$$= m(u - v)$$

of the blow transmitted to the body.

To find v, let G' be the centroid of $M+m$, so that $GO'=x_1$, $GG'=mx_1/(M+m)$, $G'O'=Mx_1/(M+m)$. The principal axes of $M+m$ are $G'x$ and the parallels $G'y'$, $G'z'$ to Gy, Gz. The impulse mu at O' imparts to the mass $M+m$ (see Ex. (4)) a velocity of translation $mu/(M+m)$ parallel to $G'y'$, and an angular velocity $\omega = mu\cdot G'O'/(M+m)q^2$ about $G'z'$, q being the radius of inertia of $M+m$ for $G'z'$. The velocity v of O' is, therefore,

$$v=\frac{mu}{M+m}+\frac{M^2mux_1^2}{(M+m)^3q^2}.$$

For q we have the relation

$$(M+m)q^2=Mq_3^2+M\cdot GG'^2+m\cdot G'O'^2=M\cdot\frac{(M+m)q_3^2+mx_1^2}{M+m},$$

where q_3 is the radius of inertia of M for Gz. Substituting this value of q, we find

$$v=mu\cdot\frac{q_3^2+x_1^2}{(M+m)q_3^2+mx_1^2},$$

and hence

$$F=m(u-v)=mu\cdot\frac{Mq_3^2}{(M+m)q_3^2+mx_1^2}.$$

It thus appears that F equals mu only in the limiting case when $m=0$, $u=\infty$, while $\lim mu=\text{const.}$ For given values of m and u, F is a maximum for $x_1=0$; *i.e.* when m strikes the body M at the centroid. In this case $F=mMu/(M+m)$, as it should be, since for direct impact, we have

$$F=m(u-v)=mu-m\cdot mu/(M+m)=mu\cdot M/(M+m).$$

(6) *A free rigid body turns with angular velocity* ω *about an instantaneous axis* l, *which is parallel to a centroidal principal axis and meets another centroidal principal axis at a distance* GO $=\bar{x}$ *from the centroid* G (Fig. 54). *A point* P *of the body, situated on the principal axis* GO *at the distance* GP $=$ x *from the centroid, strikes a fixed obstacle; what is the reaction* P *of the obstacle?*

The system of impulses to which the angular velocity ω is due reduces to an impulse $F=Mu=M\omega\bar{x}$ through G, at right angles to the plane (l, G), and a couple $Fx_1=Fq^2/\bar{x}$, where q is the radius of inertia for the centroidal axis $\bar{l}$ parallel to l. The vector of this couple is parallel to l (see Ex. (4)). Just after impact, we have, in addition, the impulsive reaction P of the fixed point; hence the resultant impulse $= F + P$, the resultant couple $= Fx_1 + Px$.

As the point P of the body is reduced to rest by the impact, we have only to express the velocity of P and equate it to zero. This gives the condition

$$\frac{F+P}{M} + \frac{Fx_1 + Pp}{Mq^2} \cdot x = 0,$$

whence

$$P = -\frac{q^2 + x_1 x}{q^2 + x^2} F = -\frac{\bar{x} + x}{q^2 + x^2} \cdot Fx_1,$$

since $\bar{x}x_1 = q^2$. This becomes $= -F$ for $x = 0$ and for $x = x_1$.

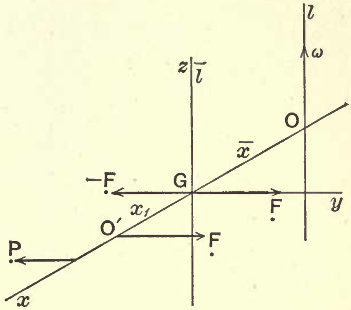

Fig. 54.

Show that there are two points of maximum impact on GO at equal distances from l on opposite sides, and that the maximum impulse is $= -\frac{1}{2} F(1 \pm \sqrt{1 + x_1/\bar{x}})$.

(7) *A free rigid body turns with angular velocity* ω *about a centroidal principal axis* $\bar{l}$ *when one of its points* P, *situated at the distance* x *from* $\bar{l}$ *in the centroidal plane perpendicular to* $\bar{l}$, *strikes a fixed obstacle. Determine the impulse on this obstacle, and show that it is greatest when* $x = q$, *where* q *is the radius of inertia for* $\bar{l}$.

(8) *In Ex.* (6), *determine the initial motion of the body after striking the fixed obstacle.*

2. CONTINUOUS MOTION.

370. In the preceding articles (357–367) it has been shown how to determine a system of impulses capable of producing any given instantaneous state of motion of a free rigid body. Any change in the state of motion can be regarded as due to a system of forces; and by reducing the effective forces of the particles, in a similar way as has been done for the momenta, this system of forces can be determined. This geometrical study of the continuous motion produced by forces is here omitted, as it would require a more complete exposition of the theory of acceleration than has been given in the first part of the present work.

371. Analytically, the continuous motion of a free rigid body is given by the six equations of motion, (4) or (5), Art. 223, and (6) or (7), Art. 224. As pointed out in Art. 233, the motion of the centroid and the motion of the body about the centroid can be considered separately. The former is given by the equations (8), Art. 226, viz.:

$$M\ddot{x}=R_x,\quad M\ddot{y}=R_y,\quad M\ddot{z}=R_z, \tag{1}$$

where M is the mass of the body; $\ddot{x}$, $\ddot{y}$, $\ddot{z}$ are the components of the accelerations of the centroid along any three fixed rectangular axes; and R_x, R_y, R_z are the components along the same axes of the resultant R of all the external forces acting on the body.

The motion of the body about the centroid is the same as if the centroid were fixed (Art. 229). It is therefore best studied by taking the centroid G as origin; all the developments of Arts. 324–356 will then apply without change, except that the centroid G must be substituted for the fixed point O. The general equations (3), Art. 326, or Euler's equations (4), Art. 328, can be used to determine the motion about the centroid.

The integration of Euler's equations gives the angular velocities ω_1, ω_2, ω_3 about the three centroidal principal axes of the body. The position of the body, *i.e.* the relation of this system of principal axes to a system of axes through the centroid, parallel to a fixed system, can be determined by means of Euler's angles θ, ϕ, ψ (see Arts. 333–335), or by means of the 9 cosines a_1, a_2, a_3, b_1, b_2, b_3, c_1, c_2, c_3 (Arts. 336, 337).

372. Kinetic Energy. As the instantaneous motion consists of an angular velocity ω about the instantaneous axis l and a velocity of translation u along this axis, the velocity v of any point of the body, at the distance r from l, is $v=\sqrt{u^2+\omega^2r^2}$. Hence the kinetic energy (comp. Art. 235) has the expression

$$T=\Sigma\tfrac{1}{2}mv^2=\Sigma\tfrac{1}{2}m(u^2+\omega^2r^2)=\tfrac{1}{2}Mu^2+\tfrac{1}{2}\omega^2\Sigma mr^2.$$

If q, $\bar{q}$ denote the radii of inertia of the body for the instantaneous axis l and the parallel centroidal axis $\bar{l}$, we have

$$\Sigma mr^2 = Mq^2 = M\bar{q}^2 + M\bar{x}^2,$$

where $\bar{x}$ is the distance between l and $\bar{l}$. Hence denoting by v the velocity of the centroid, $\bar{v} = \sqrt{u^2 + \omega^2\bar{x}^2}$, we find

$$T = \tfrac{1}{2}M(u^2 + \omega^2\bar{x}^2) + \tfrac{1}{2}M\omega^2\bar{q}^2 = \tfrac{1}{2}M\bar{v}^2 + \tfrac{1}{2}M\bar{q}^2\omega^2. \qquad (2)$$

It thus appears that the kinetic energy consists of two parts, $T = T_1 + T_2$, one of which,

$$T_1 = \tfrac{1}{2}M\bar{v}^2,$$

may be called the *kinetic energy of the centroid* (the whole mass M being regarded as concentrated at the centroid), while the other,

$$T_2 = \tfrac{1}{2}M\bar{q}^2\omega^2,$$

the so-called *kinetic energy of rotation*, is the kinetic energy which the body would possess if it were rotating with the angular velocity ω, not about the instantaneous axis l, but about the parallel centroidal axis $\bar{l}$. The developments of Arts. 322 and 323 apply without change to T_2.

373. Numerous exercises and applications will be found in the works of Price, Besant, Williamson and Tarleton, Walton, quoted in Art. 159; but above all in E. J. Routh, *Dynamics of a system of rigid bodies*, Elementary part, fifth edition, 1891; Advanced part, fourth edition, 1884; London and New York, Macmillan. Illustrations and examples, as well as further developments of the theory, will also be found in the works of Schell and Budde (Art. 159); in the French collections of problems by M. Jullien and by A. de Saint-Germain; in J. Petersen, *Lehrbuch der Dynamik fester Körper*, deutsch von R. von Fischer-Benzon, Kopenhagen, Höst, 1887; E. Bour, *Cours de mécanique et machines*, III[e] fascicule, Paris, Gauthier-Villars, 1874; and the original memoirs of L. Poinsot, in particular his *Théorie nouvelle de la rotation des corps*, Paris, Bachelier, 1852 (also in Liouville's *Journal de mathématiques*, Vol. XVI.), and his *Précession des équinoxes*, Paris, Mallet-Bachelier, 1857. Among the numerous French treatises on theoretical mechanics those of Poisson, Sturm, Resal, Collignon deserve especially to be mentioned here.

CHAPTER VII.

MOTION OF A VARIABLE SYSTEM.

374. In the present chapter we shall consider very briefly the motion of a general system of n particles, connected in any way whatever, subject to any conditions or constraints, and acted upon by any forces.

The forces can be distinguished as *external* and *internal.* The latter are exerted by certain particles, or groups of particles, of the system on other particles of the same system, while the former proceed from without the system. Thus, in considering our solar system, the attractions between its various members are internal forces, while the attractions of the fixed stars on the sun or planets would represent external forces.

Besides these two kinds of forces there may be forces replacing constraints, such as reactions of fixed points, lines, or surfaces, friction, etc.

I. *Free System.*

375. If the system be free, *i.e.* if it be only acted upon by external and internal forces, while there are no constraints or conditions prescribed for it, the establishment of the general equations of motion is simple, although their integration generally presents insuperable difficulties. The problem of two bodies (Arts. 150–158) is the simplest special case.

The general principles laid down in Arts. 218–238 for the motion of a rigid body apply almost without change to a free system of particles; indeed, they apply even to a constrained system, provided that all conditions and constraints are replaced by forces and that these constraining forces are included among the forces X, Y, Z, acting on the particles. Thus in particular, the general equations of motion of a rigid body, viz. (4) or (5), Art. 223, and (6) or (7), Art. 224, hold for a variable system. For they express the necessary, though not in general sufficient, conditions of equilibrium of the forces acting on the particles with the reversed effective forces of these particles; and this equilibrium is not changed by making the distances between the particles invariable; *i.e.* by what is sometimes called *solidifying* the system. But it should be observed that the reductions of the systems of momenta and effective forces, given in the chapter on the rigid body, do not in general hold for a variable system.

376. Let F be the resultant of all the external and internal forces acting on one of the n particles; X, Y, Z its components along a system of fixed rectangular axes; x, y, z the co-ordinates of the particle, and m its mass. Just as in Arts. 219, 220, we have the equations of motion of the particle

$$m\ddot{x} = X, \quad m\ddot{y} = Y, \quad m\ddot{z} = Z. \tag{1}$$

There are 3 such equations for each particle, and hence $3n$ for the whole system. These $3n$ equations express the equilibrium of the system of forces composed of the external, internal, and reversed effective forces.

377. Applying the principle of virtual work to this system of forces, we find *d'Alembert's equation*

$$\Sigma(-m\ddot{x}+X)\delta x + \Sigma(-m\ddot{y}+Y)\delta y + \Sigma(-m\ddot{z}+Z)\delta z = 0, \tag{2}$$

in which δx, δy, δz are the components of an arbitrary displacement δs of the particle m. As there are $3n$ such arbitrary component displacements, the equation (2) is equivalent to the $3n$ equations (1).

If written in the form

$$\Sigma m(\ddot{x}\delta x + \ddot{y}\delta y + \ddot{z}\delta z) = \Sigma(X\delta x + Y\delta y + Z\delta z), \tag{3}$$

it expresses the equality of the sum of the virtual works of the effective forces to the sum of the virtual works of the external and internal forces, for any infinitesimal displacement of the system. The internal forces do not enter into this equation if they occur in pairs of equal and opposite forces, as will usually be the case.

378. As there are no conditions, we may select for δs the actual displacement ds of every particle, so that the equation (3) becomes

$$\Sigma m(\ddot{x}dx+\ddot{y}dy+\ddot{z}dz)=\Sigma(Xdx+Ydy+Zdz).$$

The left-hand member is the exact differential $\Sigma\frac{1}{2}md(\dot{x}^2+\dot{y}^2+\dot{z}^2)=d\Sigma\frac{1}{2}mv^2$. Hence, integrating between the limits 0 and t, and denoting by v_0 the velocity of the particle m at the time 0, we find

$$\Sigma\tfrac{1}{2}mv^2-\Sigma\tfrac{1}{2}mv_0^{\,2}=\Sigma\int_0^t(Xdx+Ydy+Zdz). \tag{4}$$

This is the *equation of kinetic energy*. The right-hand member represents the work done by the forces during the time t.

379. If there exists a force function or potential U for the forces X, Y, Z, *i.e.* if these forces are the partial derivatives with respect to x, y, z of one and the same function U, the system is said to be *conservative*. We have then

$$\Sigma(Xdx+Ydy+Zdz)=dU,$$

and the integration of (4) gives

$$\Sigma\tfrac{1}{2}mv^2-\Sigma\tfrac{1}{2}mv_0^{\,2}=U-U_0, \tag{5}$$

where U_0 is the value of U for $t=0$.

Denoting as usual the kinetic energy by T, the potential energy $-U$ by V, this equation can be written

$$T+V=T_0+V_0=\text{const.}; \tag{6}$$

it expresses the **principle of the conservation of energy.** (Comp. Art. 79).

380. Exercises. Show the existence of a force-function and find its expression in the following cases (comp. Art. 86):

(1) When the resulting force F at each particle m is constant in magnitude and direction (gravity).

(2) When the forces F are all attractions, each being directed to some fixed centre O and a function of the distance r from this centre.

(3) When the forces F are the mutual attractions of the particles constituting the system.

381. A variable system of n particles possesses a centroid whose co-ordinates $\bar{x}, \bar{y}, \bar{z}$ satisfy the equations

$$M \cdot \bar{x} = \Sigma mx, \quad M \cdot \bar{y} = \Sigma my, \quad M \cdot \bar{z} = \Sigma mz.$$

The developments of Arts. 226, 227, in particular *the principle of the conservation of linear momentum*, or the principle of the conservation of the motion of the centroid, hold for a variable system just as well as for a rigid body. The position of the centroid in the system is of course variable with the time.

The principle just referred to asserts that, *if* $\Sigma X = 0$, $\Sigma Y = 0$, $\Sigma Z = 0$, *the centroid of the system is at rest, or moves with constant velocity in a straight line.* It should be noticed that the conditions $\Sigma X = 0$, $\Sigma Y = 0$, $\Sigma Z = 0$ do not mean that there are no forces at all acting on the system; they only mean that the resultant of these forces reduces to zero while there may be a resulting couple different from zero. The principle would, for instance, apply to the solar system if the action of the fixed stars be regarded as vanishing or as reducing to a couple; the mutual attractions of the various members of the system occur in pairs of equal and opposite forces, and have, therefore, a resultant zero.

382. Similarly, the developments of Arts. 228–232, in particular *the principle of the conservation of angular momentum*, or of areas, and the properties of the invariable plane, apply without change to the free variable system.

II. *System Subject to Conditions.*

383. The constraints and conditions to which a variable system is subject may be of very different kinds. In general, however, they can be imagined as replaced by certain forces, called *constraining forces* or *reactions*, by the introduction of which the system becomes free. On the other hand, it may be noticed that internal forces, such as tensions of connecting rods or strings, can sometimes be regarded as constraining conditions.

If all conditions and constraints be expressed by means of forces, and these forces be included among the forces X, Y, Z, the equations of motion of the particle m have again the form (1), Art. 376, and the principle of virtual work gives the equation of d'Alembert (2), Art. 377. But it should be noticed that, in general, the constraints will do no work if the displacements δx, δy, δz are properly selected; in other words, if the displacements be taken so as to be compatible with the conditions to which the system is subject, the constraining forces will not enter into the equation (2). This is **d'Alembert's principle.**

384. Before further developing this idea it may be well to indicate here the considerations by which d'Alembert himself (and, in more exact language, Poisson) explained his celebrated principle.

Any particle m of the system is acted upon at any time t by two kinds of forces, the given external and internal forces, whose resultant we denote by F, and the internal reactions and constraining forces whose resultant we call F' (Fig. 55). The resultant of F and F' must be geometrically equal to the effective force mj, where j is the acceleration of the particle at the time t.

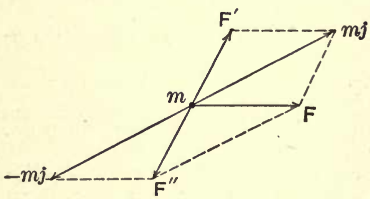

Fig. 55.

Now, if we introduce at m the equal and opposite forces mj, $-mj$, the action of F and F', and hence the motion of the particle, will not be changed. But we can now combine F and $-mj$ to a resultant F''. Since F, F', $-mj$ are in equilibrium, the forces F' and F'' are in equilibrium; *i.e.* F'' is equal and opposite to F', as appears from the figure.

The figure also shows that F can be resolved into the components mj and F''; the former component, mj, produces the actual motion of the particle, while the latter, F'', is consumed in overcoming the internal reactions and constraints represented by F'. This component F'' of F is therefore called by d'Alembert the *lost force*. As $F'+F''=0$ at every particle of the system, d'Alembert's principle can be expressed by saying that, at every moment during the motion, *the lost forces are in equilibrium with the constraints of the system.*

If the constraints, instead of being expressed by means of forces, are given by equations of condition, we may express the same idea by saying that, *owing to the given conditions, the lost forces form a system in equilibrium.*

385. We shall now assume that the constraints or conditions to which the system is subject are expressed by means of equations (the case of conditions expressed by inequalities is excluded) between the co-ordinates x, y, z of the particles and the time t. In the most general case these equations might also contain the velocities of the particles; this case, however, will not be considered here.

Let there be k conditions

$$\phi(t, x_1, y_1, z_1, x_2, \cdots)=0, \quad \psi(t, x_1, y_1, z_1, x_2, \cdots), \cdots \qquad (1)$$

for a system of n points. Then the number of the independent equations of motion will be $3n-k$. For these equations must express the equilibrium of the given forces, together with the reversed effective forces, under the given conditions; and for this equilibrium it is sufficient that *the virtual work should vanish for any displacement compatible with the conditions*, the work of the reactions and constraining forces being zero for such virtual displacement. In other words, in d'Alembert's equation

$$\Sigma(-m\ddot{x}+X)\,\delta x+\Sigma(-m\ddot{y}+Y)\,\delta y+\Sigma(-m\ddot{z}+Z)\,\delta z=0, \qquad (2)$$

the constraining forces due to the conditions will not appear if the displacements δx, δy, δz be so selected as to be compatible with the k conditions (1).

Now this will be the case if these displacements be made to satisfy the equations that result from differentiating the conditions (1), viz.

$$\Sigma(\phi_x\delta x+\phi_y\delta y+\phi_z\delta z)=0,$$
$$\Sigma(\psi_x\delta x+\psi_y\delta y+\psi_z\delta z)=0, \qquad (3)$$
$$\cdots\cdots\cdots\cdots$$

It should be noticed that in this differentiation, or rather variation, the time t is regarded as constant. If, for instance, one of the conditions (1) constrain a particle to a curve or surface varying with the time, say the surface of the moving earth, or that of a projectile in motion, the displacement is called *virtual*, or compatible with the condition, if it takes place on the curve or surface regarded momentarily as fixed (comp. Art. 193). Indeed, when the conditions contain the time, the statement that a virtual displacement is one compatible with the conditions has no definite meaning; virtual displacements are then *defined* as displacements satisfying the equations (3).

386. The k equations (3) make it possible to eliminate k of the $3n$ displacements from d'Alembert's equation (2). There will remain $3n-k$ independent arbitrary displacements, whose coefficients equated to zero give the $3n-k$ equations of motion.

Applying the method of indeterminate multipliers (comp. Art. 194) to perform this elimination in a systematic way, we have to multiply the k equations (1) by indeterminate factors λ, μ, $\cdots$, and to add them to equation (2). The k multipliers λ, μ, $\cdots$ can then be so selected as to make the coefficients of k of the $3n$ displacements δx, δy, δz vanish. As the remaining $3n-k$

displacements are arbitrary, their coefficients must also vanish separately. Thus it follows that the coefficients of *all* the displacements in the resulting equation must vanish, and we have n sets of 3 equations of the type

$$\begin{aligned} m\ddot{x} &= X + \lambda\phi_x + \mu\psi_x + \cdots, \\ m\ddot{y} &= Y + \lambda\phi_y + \mu\psi_y + \cdots, \\ m\ddot{z} &= Z + \lambda\phi_z + \mu\psi_z + \cdots. \end{aligned} \tag{4}$$

It is apparent from these equations that the constraining force acting on the particle m has the components

$$\begin{aligned} X' &= \lambda\phi_x + \mu\psi_x + \cdots, \\ Y' &= \lambda\phi_y + \mu\psi_y + \cdots, \\ Z' &= \lambda\phi_z + \mu\psi_z + \cdots. \end{aligned} \tag{5}$$

387. It has thus been shown that a system of n particles subject to k conditions has $3n-k$ independent equations of motion. The equations can be obtained either by eliminating from d'Alembert's equation (2) k of the $3n$ displacements δx, δy, δz by means of the k equations (3), and then equating to zero the coefficients of the remaining $3n-k$ arbitrary displacements, or they may be regarded as represented by the $3n$ equations (4), since these equations contain k arbitrary quantities λ, μ, $\cdots$. In this latter form they are sometimes denoted as *Lagrange's first form of the equations of motion.*

388. It follows from the remark at the end of Art. 385, that the actual displacements dx, dy, dz of the particles can be selected as virtual displacements only, and always, when the conditional equations (1) do not contain the time. If this condition be fulfilled, d'Alembert's equation (2) can be written

$$\Sigma m(\ddot{x}dx + \ddot{y}dy + \ddot{z}dz) = \Sigma(Xdx + Ydy + Zdz),$$

or

$$d\Sigma\tfrac{1}{2}mv^2 = \Sigma(Xdx + Ydy + Zdz). \tag{6}$$

This relation can also be deduced from the equations (4) by multiplying them by $\dot{x}dt$, $\dot{y}dt$, $\dot{z}dt$, and summing the equations for all the particles. The left-hand member of the resulting equation is again $d\Sigma\frac{1}{2}mv^2$. In the right-hand member we find, besides the term $\Sigma(Xdx+Ydy+Zdz)$, such terms as

$$\lambda\Sigma(\phi_x\dot{x}+\phi_y\dot{y}+\phi_z\dot{z})dt,\ \mu\Sigma(\psi_x\dot{x}+\psi_y\dot{y}+\psi_z\dot{z})dt, \cdots.$$

The coefficients of λ, μ, $\cdots$ vanish only when the conditions (1) are independent of the time, for then the differentiation of these equations (1) gives

$$\Sigma(\phi_x\dot{x}+\phi_y\dot{y}+\phi_z\dot{z})=0,\quad \Sigma(\psi_x\dot{x}+\psi_y\dot{y}+\psi_z\dot{z})=0, \cdots.$$

In other words, in this case the constraining forces do no work during the actual displacement of the system, as they are all perpendicular to the paths of the particles, and we find equation (6).

If, however, the conditional equations (1) contain the time, their differentiation gives

$$\Sigma(\phi_x\dot{x}+\phi_y\dot{y}+\phi_z\dot{z})+\phi_t=0,\quad \Sigma(\psi_x\dot{x}+\psi_y\dot{y}+\psi_z\dot{z})+\psi_t=0, \cdots,$$

and we find in the place of equation (6):

$$d\Sigma\tfrac{1}{2}mv^2=\Sigma(Xdx+Ydy+Zdz)-\lambda\phi_t-\mu\psi_t \cdots. \tag{7}$$

389. If the conditional equations do not contain the time, and if, moreover, there exists a force-function U for all the forces, equation (6) can be put into the form

$$d\Sigma\tfrac{1}{2}mv^2=dU,$$

which gives, by integration,

$$\Sigma\tfrac{1}{2}mv^2-\Sigma\tfrac{1}{2}mv_0{}^2=U-U_0, \tag{8}$$

or, by putting $U=-V$,

$$T+V=T_0+V_0. \tag{9}$$

This equation expresses the *principle of the conservation of energy.*

It should be noticed that, even when there exists no force-function, the elementary work $\Sigma(Xdx+Ydy+Zdz)$ is a quantity independent of the co-ordinate system, and the sum of these elementary works for a finite time, say from $t=0$ to $t=t$, represents a certain finite amount of work $W=\int_0^t \Sigma(Xdx+Ydy+Zdz)$, so that equation (6) gives always

$$\Sigma\tfrac{1}{2}mv^2-\Sigma\tfrac{1}{2}mv_0^2=W.$$

This means that *if the conditions are independent of the time, the increase of kinetic energy during any interval of time is equal to the work done during this time by all the external and internal forces.*

But when a force-function exists, this work is $W=U-U_0$, where U is a function of the co-ordinates only. The work done by the forces depends therefore only on the initial and final values of these co-ordinates; *i.e.* on the initial and final configuration of the system, but not on the character of the motion by which the system is brought from the initial to the final position.

390. It has been shown in Art. 222 that, for an *invariable* system of n points, *i.e.* for a free rigid body, the number of conditions is $k=3n-6$; hence the number of independent equations of motions of a free rigid body is $3n-(3n-6)=6$.

A rigid body with a fixed axis (Art. 291) has but one degree of freedom and 5 constraints; *i.e.* its position is given by a single variable, say the angle of rotation, θ, about the fixed axis. The motion of such a body is therefore given by a single equation.

A rigid body that can turn about and also slide along a fixed axis has 4 constraints and 2 degrees of freedom; it has therefore 2 equations of motion, and 2 variables are sufficient to determine any particular position of the body, say the angle θ and the distance x measured along the axis of rotation.

A rigid body with one fixed point (Art. 311) is an example of an invariable system with 3 constraints and 3 degrees of freedom. Three variables are necessary and sufficient to determine a particular position, and the number of independent equations of motion is 3.

Similarly, it will be seen in every other case that a rigid body has as many independent equations of motion as it has degrees of freedom, or as it requires variables to fix its position. These variables may be called the *co-ordinates of the rigid body.* Thus a free rigid body has 6 co-ordinates corresponding to its 6 degrees of freedom and 6 equations of motion; we might take as such co-ordinates the co-ordinates $\bar{x}, \bar{y}, \bar{z}$ of the centroid and Euler's angles θ, ϕ, ψ.

391. These considerations can be generalized so as to apply to a general variable system of n points with k conditions. Such a system is said to have $3n-k=m$ co-ordinates because it has $3n-k=m$ independent equations of motion (Art. 385). In other words, in the place of the $3n$ Cartesian co-ordinates x, y, z of the n points, subject to k conditional equations, we may introduce $3n-k=m$ independent variables, say q_1, q_2, $\cdots q_m$, which are so selected as to satisfy the k conditions (1) identically. These variables are called the **Lagrangian**, or **generalized, co-ordinates** of the system.

By the introduction of these new variables the equations of motion (4) assume a form which is known as the *second Lagrangian form.*

Suppose, for instance, that the system is subject to only one condition, viz. that one point P_1 of the system should remain on the surface of the ellipsoid

$$\phi \equiv \frac{x_1^2}{a^2}+\frac{y_1^2}{b^2}+\frac{z_1^2}{c^2}-1=0.$$

If we select two new variables q_1, q_2 connected with x_1, y_1, z_1 by the equations $x_1=a\cos q_1$, $y_1=b\sin q_1\cos q_2$, $z_1=c\sin q_1\sin q_2$, the condition $\phi=0$ is satisfied identically in the new co-ordinates q_1, q_2. Hence, by introducing q_1, q_2 in the place of x_1, y_1, z_1, the condition $\phi=0$ is eliminated from the problem.

We now proceed to establish the equations of motion in the second Lagrangian form, for a variable system of n points with the k conditions (1), *i.e.* to introduce $3n-k=m$ new variables or generalized co-ordinates, $q_1, q_2, \cdots q_m$ in the place of the $3n$ Cartesian co-ordinates $x_1, y_1, z_1, x_2, \cdots z_n$, selecting the new co-ordinates so as to satisfy the conditions (1) identically (comp. Arts. 210–216).

392. The Cartesian co-ordinates x, y, z of any one of the n points, as well as their time derivatives $\dot{x}, \dot{y}, \dot{z}$, are functions of $q_1, q_2, \cdots q_m$ and of the time t. We have therefore

$$\dot{x}=\frac{\partial x}{\partial t}+\frac{\partial x}{\partial q_1}\dot{q}_1+\frac{\partial x}{\partial q_2}\dot{q}_2+\cdots+\frac{\partial x}{\partial q_m}\dot{q}_m, \tag{10}$$

with similar expressions for $\dot{y}$ and $\dot{z}$. Thus $\dot{x}, \dot{y}, \dot{z}$ are represented as functions of the independent variables $t, q_1, q_2, \cdots q_m, \dot{q}_1, \dot{q}_2, \cdots \dot{q}_m$.

Differentiating $\dot{x}$ partially with respect to any one, q, of the quantities $q_1, q_2, \cdots q_m$, we find

$$\frac{\partial \dot{x}}{\partial q}=\frac{\partial^2 x}{\partial q \partial t}+\frac{\partial^2 x}{\partial q \partial q_1}\dot{q}_1+\frac{\partial^2 x}{\partial q \partial q_2}\dot{q}_2+\cdots+\frac{\partial^2 x}{\partial q \partial q_m}\dot{q}_m$$

$$=\frac{\partial}{\partial t}\frac{\partial x}{\partial q}+\frac{\partial}{\partial q_1}\frac{\partial x}{\partial q}\cdot\dot{q}_1+\frac{\partial}{\partial q_2}\frac{\partial x}{\partial q}\cdot\dot{q}_2+\cdots+\frac{\partial}{\partial q_m}\frac{\partial x}{\partial q}\cdot\dot{q}_m.$$

We have therefore,

$$\frac{\partial \dot{x}}{\partial q}=\frac{d}{dt}\frac{\partial x}{\partial q},\quad \frac{\partial \dot{y}}{\partial q}=\frac{d}{dt}\frac{\partial y}{\partial q},\quad \frac{\partial \dot{z}}{\partial q}=\frac{d}{dt}\frac{\partial z}{\partial q}. \tag{11}$$

Again, differentiating (10) partially with respect to $\dot{q}$, we have

$$\frac{\partial \dot{x}}{\partial \dot{q}}=\frac{\partial x}{\partial q},\quad \frac{\partial \dot{y}}{\partial \dot{q}}=\frac{\partial y}{\partial q},\quad \frac{\partial \dot{z}}{\partial \dot{q}}=\frac{\partial z}{\partial q}. \tag{12}$$

Let us also form the derivatives of the kinetic energy

$$T=\Sigma\tfrac{1}{2}m(\dot{x}^2+\dot{y}^2+\dot{z}^2), \tag{13}$$

viz.

$$\frac{\partial T}{\partial q}=\Sigma m\left(\dot{x}\frac{\partial \dot{x}}{\partial q}+\dot{y}\frac{\partial \dot{y}}{\partial q}+\dot{z}\frac{\partial \dot{z}}{\partial q}\right),$$

which by (11) and (12) becomes

$$\frac{\partial T}{\partial q}=\Sigma m\left(\dot{x}\frac{d}{dt}\frac{\partial \dot{x}}{\partial \dot{q}}+\dot{y}\frac{d}{dt}\frac{\partial \dot{y}}{\partial \dot{q}}+\dot{z}\frac{d}{dt}\frac{\partial \dot{z}}{\partial \dot{q}}\right); \tag{14}$$

and $$\frac{\partial T}{\partial \dot q} = \Sigma m\left(\dot x \frac{\partial \dot x}{\partial \dot q} + \dot y \frac{\partial \dot y}{\partial \dot q} + \dot z \frac{\partial \dot z}{\partial \dot q}\right). \tag{15}$$

From (15) and (14) we find

$$\frac{d}{dt}\frac{\partial T}{\partial \dot q} = \Sigma m\left(\ddot x \frac{\partial \dot x}{\partial \dot q} + \ddot y \frac{\partial \dot y}{\partial \dot q} + \ddot z \frac{\partial \dot z}{\partial \dot q}\right) + \frac{\partial T}{\partial q}. \tag{16}$$

393. Thus prepared we can introduce the new co-ordinates q into the equations of motion (4) by multiplying these equations by $\partial x/\partial q$, $\partial y/\partial q$, $\partial z/\partial q$, and adding them throughout the whole system; this gives:

$$\Sigma m\left(\ddot x \frac{\partial x}{\partial q} + \ddot y \frac{\partial y}{\partial q} + \ddot z \frac{\partial z}{\partial q}\right) = \Sigma\left(X\frac{\partial x}{\partial q} + Y\frac{\partial y}{\partial q} + Z\frac{\partial z}{\partial q}\right); \tag{17}$$

the coefficients of λ, μ, $\cdots$ all disappear in the summation, since, by hypothesis, the new co-ordinates satisfy the conditional equations (1) identically.

The right-hand member of (17) we shall denote by Q (comp. Arts. 180, 211); the left-hand member can be put into a more convenient form by means of (16) and (12). Thus we find finally *the equations of motion in the second Lagrangian form:*

$$\frac{d}{dt}\frac{\partial T}{\partial \dot q} - \frac{\partial T}{\partial q} = Q. \tag{18}$$

As there is one such equation for every one of the Lagrangian co-ordinates $q_1, q_2, \cdots q_m$, the number of such equations is $m = 3n - k$. They are obtained from the type (18) by attaching successively the subscripts 1, 2, $\cdots m$ to each of the symbols q, $\dot q$, Q.

394. In the particular case of a *conservative system, i.e.* when there exists a force function U such that

$$\Sigma X = \frac{\partial U}{\partial x}, \quad \Sigma Y = \frac{\partial U}{\partial y}, \quad \Sigma Z = \frac{\partial U}{\partial z},$$

the quantity Q in (18) is evidently $= \partial U/\partial q$, so that the equations of motion assume the form

$$\frac{d}{dt}\frac{\partial T}{\partial \dot q} = \frac{\partial}{\partial q}(T + U). \tag{19}$$

This equation can be derived more directly from equation (16) by considering an infinitesimal displacement of the system. If δq be the change of the co-ordinate q in such a displacement, the partial changes, or variations, of x, y, z will be

$$\frac{\partial x}{\partial q}\delta q, \quad \frac{\partial y}{\partial q}\delta q, \quad \frac{\partial z}{\partial q}\delta q;$$

hence the work of the effective forces $m\ddot{x}$, $m\ddot{y}$, $m\ddot{z}$, for the whole system, is

$$\Sigma m\delta q\left(\ddot{x}\frac{\partial x}{\partial q}+\ddot{y}\frac{\partial y}{\partial q}+\ddot{z}\frac{\partial z}{\partial q}\right)=\delta q\Sigma m\left(\ddot{x}\frac{\partial \dot{x}}{\partial \dot{q}}+\ddot{y}\frac{\partial \dot{y}}{\partial \dot{q}}+\ddot{z}\frac{\partial \dot{z}}{\partial \dot{q}}\right).$$

This is the amount by which the *potential energy* $V=-U$ is diminished; it is, therefore, equal to $(\partial U/\partial q)\delta q$. Hence the first term in the right-hand member of (16) can be replaced by $\partial U/\partial q$; this at once gives equation (19).

395. From Lagrange's equations it is easy to derive **Hamilton's principle.**

Let each of the equations (18) be multiplied by the infinitesimal displacement, or variation, δq; let the equations be added, multiplied by dt, and integrated from t_1 to t_2:

$$\int_{t_1}^{t_2}\Sigma\left(\frac{d}{dt}\frac{\partial T}{\partial \dot{q}}-\frac{\partial T}{\partial q}-Q\right)\delta q\,dt=0. \qquad (20)$$

The first term can be transformed by partial integration; remembering that $d(\delta q)/dt=\delta(dq)/dt$, we have

$$\int_{t_1}^{t_2}\frac{d}{dt}\frac{\partial T}{\partial \dot{q}}\delta q\,dt=\left(\frac{\partial T}{\partial \dot{q}}\delta q\right)_{t_1}^{t_2}-\int_{t_1}^{t_2}\frac{\partial T}{\partial \dot{q}}\delta \dot{q}\,dt.$$

If now the variations δq be so selected as to vanish both at the time t_1 and at the time t_2, the first term vanishes at both limits. Hence equation (20) assumes the form

$$\int_{t_1}^{t_2}\Sigma\left(\frac{\partial T}{\partial \dot{q}}\delta \dot{q}+\frac{\partial T}{\partial q}\delta q+Q\delta q\right)dt=0.$$

As $\Sigma\left(\frac{\partial T}{\partial \dot{q}}\delta \dot{q}+\frac{\partial T}{\partial q}\delta q\right)=\delta T$ and $\Sigma Q\delta q=\delta U$ for a conservative system, and $=\delta W$ for a general system (Art. 389), the equation reduces to the simple form

$$\int_{t_1}^{t_2}(\delta T+\delta W)dt=0, \tag{21}$$

in the general case, and to

$$\delta\int_{t_1}^{t_2}(T+U)dt=0, \text{ or } \delta\int_{t_1}^{t_2}(T-V)dt=0, \tag{22}$$

in the case of a conservative system.

396. Hamilton's principle consists in the proposition that the equation (21) or (22) holds for any displacements of the system compatible with the conditions (1), provided these displacements be zero at the times t_1 and t_2. Assuming the existence of a force-function, *i.e.* taking (22) as the expression of Hamilton's principle, its meaning can be expressed as follows. If we consider any two positions of the moving system, say the positions which it occupies at the times t_1 and t_2, the motion by which the system actually passes from the former to the latter position is characterized, and distinguished from any other imaginable ways of passing from the former to the latter, by the property that the variation of the time-integral of the difference between kinetic and potential energy vanishes. In other words, *for the actual motion the average value of the difference between kinetic and potential energy during any time is a minimum.*

The chief advantage of Hamilton's principle lies in the fact that it is independent of any co-ordinate system, and can therefore be used as a convenient starting-point for introducing the variables best adapted to the needs of the particular problem.

397. A more complete discussion of Lagrange's equations of motion and of Hamilton's as well as other similar principles of dynamics, such as the principle of least action, of least constraint, etc., will be found in the work of E. J. Routh (see Art. 373) and in W. Schell's *Theorie der Bewegung und der Kräfte*, Vol. II., pp. 544–571. The kinetics of the variable system forms the basis for the investigations of mathematical physics, *i.e.* for the theory of elastic bodies, of fluid and liquid motion, of heat, light, electricity, and magnetism. The following works are particularly recommended as introductions to this subject: THOMSON AND TAIT, *Natural Philosophy*, Vol. I., parts 1 and 2, Cambridge, University Press (New York, Macmillan), new edition, 1890; W. VOIGT, *Elementare Mechanik als Einleitung in das Studium der theoretischen Physik*, Leipzig, Veit, 1889; G. KIRCHHOFF, *Vorlesungen über mathematische Physik;* Mechanik, 3te Auflage, Leipzig, Teubner, 1883.

ANSWERS.

Pages 9, 10.

(1) 40.

(2) (*a*) $4\frac{1}{7}$ and $5\frac{1}{7}$; (*b*) $-2\frac{5}{7}$, $6\frac{2}{7}$.

(3) If the original velocities are of the same sense, $v=42\frac{4}{9}$, $v'=46\frac{1}{9}$; if not, $v=-19\frac{7}{9}$, $v'=10\frac{5}{9}$.

(4) $e=0$ gives (*a*) $44\frac{7}{12}$, (*b*) $-2\frac{1}{12}$; $e=1$ gives (*a*) $v=38\frac{1}{6}$, $v'=49\frac{1}{6}$, (*b*) $v=-55\frac{1}{6}$, $v'=35\frac{5}{6}$.

(5) $v=-eu$.

(7) $8\frac{1}{3}$ ft.

(8) (*a*) 0·31 ft.; (*b*) $9\frac{1}{2}$ sec.; (*c*) $66\frac{6}{11}$ ft.

(10) $\left(\frac{1+e}{2}\right)^n u$.

(11) (*a*) 4 ft. per second; (*b*) 28 ft. per second.

(12) For $e=0$: (*a*) $v=\frac{m}{m+m'}u$; (*b*) $\lim v=0$; (*c*) $\lim v=u'$. For $e=1$: (*a*) $v=\frac{m-m'}{m+m'}u$, $v'=\frac{2m}{m+m'}u$; (*b*) $\lim v=-u$, $\lim v'=0$; (*c*) $\lim v=2u'-u$, $\lim v'=u'$. Interpret these results.

Page 14.

(1) About 450 pounds; 9·375 and 0·191 foot-pounds.

(2) 156·85 foot-poundals.

(3) 363 foot-tons; 9 miles per hour.

(6) $4\frac{1}{6}$ tons.

(7) 13 and 2 foot-tons.

Page 16.

(1) 56·83 F.P.S. units.

(2) 16 ft. per second.

(3) $v=10$, $\beta=48\frac{1}{3}°$, $v'=16\frac{1}{2}$, $\beta'=15\frac{1}{5}°$.

Pages 24, 25.

(1) (*b*) 250 pounds.

(2) (*a*) 8 ft. per second; (*b*) 20 ft.

(3) (*a*) 825 pounds; (*b*) $1\frac{1}{4}$ miles; (*c*) about 1000 pounds.

(4) 4·9 sec.

(5) It would be greater by $\frac{1}{130}$ oz.

(6) $j = (m_1 \sin\theta_1 - m_2 \sin\theta_2 - \mu_1 m_1 \cos\theta_1 - \mu_2 m_2 \cos\theta_2) g/(m_1 + m_2)$; $T = (\sin\theta_1 + \sin\theta_2 - \mu_1 \cos\theta_1 + \mu_2 \cos\theta_2) m_1 m_2 g/(m_1 + m_2)$.

(7) $j = 5{\cdot}4$ ft. per second; $T = \frac{5}{6}$ pound.

(9) 0·0363.

(10) 0·025.

(11) (*a*) 1528 pounds; (*b*) 1910 pounds.

(12) 589 ft.

Pages 33, 34.

(1) (*a*) 15 270 foot-pounds; (*b*) $30\frac{1}{2}$ ft.

(2) (*a*) 917 pounds; (*b*) 1557 pounds; (*c*) 640 pounds.

(3) (*a*) 1267 foot-tons; (*b*) 4435 foot-tons; (*c*) 5 : 2.

(4) 2016 foot-tons.

(5) 864 ft. per second.

(7) (*a*) $v^2 = 2\kappa\rho \log \dfrac{s_0(s+l)}{s(s_0+l)}$, $v_1 = \infty$; (*b*) $v^2 = 2\kappa\rho \log \dfrac{s_0}{s}$.

(8) $v^2 = 2\kappa M\left(\dfrac{1}{\sqrt{a^2+s^2}} - \dfrac{1}{\sqrt{a^2+s_0^2}}\right)$.

(9) At the time t, let s be the distance of m_1 and m_2; s_1, s_2 their distances from their initial positions, so that $s_1 + s + s_2 = s_0$. Then we find $s_1 = \dfrac{m_2}{m_1+m_2}(s_0 - s)$, $s_2 = \dfrac{m_1}{m_1+m_2}(s_0 - s)$, and $\left(\dfrac{ds}{dt}\right)^2 = 2\kappa(m_1+m_2)\left(\dfrac{1}{s} - \dfrac{1}{s_0}\right)$. To integrate, put $s = s_0 \cos^2 \frac{1}{2}\phi$; then we find $t = \dfrac{s_0}{2}\sqrt{\dfrac{s_0}{2\kappa(m_1+m_2)}}(\phi + \sin\phi)$. The particles meet at the time $t_1 = \dfrac{\pi s_0}{2}\sqrt{\dfrac{s_0}{2\kappa(m_1+m_2)}}$.

(11) (*a*) $2\pi\sqrt{\dfrac{e}{g}}$; (*b*) $2\sqrt{\dfrac{e}{g}}\left[\cos^{-1}\dfrac{e}{s_0 - l_1} + \sqrt{\left(\dfrac{s_0 - l_1}{e}\right)^2 - 1}\right]$.

(12) $s_0 - l_1 = \sqrt{e(2h+e)}$.

(14) The equation $\frac{d^2x}{dt^2} = -\mu^2 x - 2\kappa\frac{dx}{dt}$ gives: (*a*) when $\kappa^2 > \mu^2$, $x = e^{-\kappa t}(C_1 e^{\sqrt{\kappa^2-\mu^2}t} + C_2 e^{-\sqrt{\kappa^2-\mu^2}t})$, from which it can be shown that the particle approaches the centre asymptotically, reaching it only in an infinite time; if C_1 and C_2 have opposite signs, the particle will do so after first reaching a maximum elongation and then returning. (*b*) When $\kappa^2 = \mu^2$, $x = e^{-\kappa t}(C_1 + C_2 t)$. (*c*) When $\kappa^2 < \mu^2$, $x = C_1 e^{-\kappa t}\sin(\sqrt{\mu^2-\kappa^2}t + C_2)$, and the particle performs oscillations of period $2\pi/\sqrt{\mu^2-\kappa^2}$ and of decreasing amplitude $C_1 e^{-\kappa t}$.

Pages 36, 37.

(1) 1 watt = 0·73737 foot-pounds per second = 0·001 341 H.P., 1 H.P. = 745·9 watts.

(2) 1 metric H.P. = 735·75 watts = 0·9863 British H.P.

(3) $27\frac{3}{4}$ H.P.

(4) $49\frac{1}{2}$ H.P.

(5) (*a*) 64; (*b*) 224 384.

(6) nearly 200 gallons.

(7) 35154 gallons.

(8) 1 hour.

(9) (*b*) 88 hours; (*c*) about 21 weeks.

Pages 46, 47.

(2) Taking the axis of z vertically upwards, $U = U_0 - mg(z - z_0)$; the equipotential surfaces are the horizontal planes $z = \text{const.}$; the potential energy is $V = mg(z - z_0)$.

(4) Taking the fixed line as axis of z, $U = -\int f(r)\,dr$; the equipotential surfaces are circular cylinders about the axis of z.

(5) Let $r = \sqrt{(x-x_0)^2 + (y-y_0)^2 + (z-z_0)^2}$ be the distance of the moving point (x, y, z) from the fixed centre (x_0, y_0, z_0); then the direction cosines of the central force $F = f(r)$ are

$$\alpha = \pm\frac{x-x_0}{r} = \pm\frac{\partial r}{\partial x},\quad \beta = \pm\frac{y-y_0}{r} = \pm\frac{\partial r}{\partial y},\quad \gamma = \pm\frac{z-z_0}{r} = \pm\frac{\partial r}{\partial z},$$

where the + sign holds for a repulsion, the − sign for an attraction; hence $X = \pm f(r)\frac{\partial r}{\partial x}$, $Y = \pm f(r)\frac{\partial r}{\partial y}$, $Z = \pm f(r)\frac{\partial r}{\partial z}$, or, putting $\int f(r)\,dr = F(r)$:

$$X = \pm\partial F(r)/\partial x,\quad Y = \pm\partial F(r)/\partial y,\quad Z = \pm\partial F(r)/\partial z;$$

and finally $U = \pm F(r)$.

(6) $U = \Sigma \int f(r)\,dr$.

(9) Compare J. C. MAXWELL, *Electricity and magnetism*, Vol. I., Arts. 118–123, and the plates I.–III.

(10) Taking the axis of z vertically upwards and denoting by r_1, r_2 the distances from C_1, C_2, we have by the principle of kinetic energy $d(\frac{1}{2}mv^2) = -mgdz - \kappa m dr_1/r_1^2 + \kappa' m r_2 dr_2$; hence the equipotential surfaces are $(c + gz - \frac{1}{2}\kappa' r_2^2)^2 r_1^2 = \kappa^2$. If $\kappa' = 0$, the equation becomes for C_1 as origin $(x^2 + y^2 + z^2)(c + gz)^2 = \kappa^2$; if $\kappa = 0$, we find for C_2 as origin the concentric spheres $x^2 + y^2 + \left(z - \frac{g}{\kappa'^2}\right)^2 = \frac{2c\kappa'^2 + g^2}{\kappa'^4}$.

Pages 52, 53.

(1) $(y_0\dot{z}_0 - z_0\dot{y}_0)x + (z_0\dot{x}_0 - x_0\dot{z}_0)y + (x_0\dot{y}_0 - y_0\dot{x}_0)z = 0$.

(2) $F_i = \kappa m m_i r_i$ has the x-component $X_i = -\kappa m m_i r_i \cdot (x - x_i)/r_i = -\kappa m m_i(x - x_i)$; hence $\Sigma X = -\kappa m \Sigma m_i(x - x_i)$, $\Sigma Y = -\kappa m \Sigma m_i(y - y_i)$, $\Sigma Z = -\kappa m \Sigma m_i(z - z_i)$. Equating these to zero, the position of equilibrium is found as the centroid

$$\bar{x} = \frac{\Sigma m_i x_i}{\Sigma m_i}, \quad \bar{y} = \frac{\Sigma m_i y_i}{\Sigma m_i}, \quad \bar{z} = \frac{\Sigma m_i z_i}{\Sigma m_i}$$

of the masses m_i. Taking this point as origin, the equation of the plane of motion is the same as in Ex. (1); and the resultant force has the components $-\kappa m \Sigma m_i \cdot x$, $-\kappa m \Sigma m_i \cdot y$, $-\kappa m \Sigma m_i \cdot z$.

Pages 65, 66.

(2) The equation of the orbit given in Ex. (1) is satisfied not only by (x_0, y_0), but also by $(\dot{x}_0/\kappa, \dot{y}_0/\kappa)$; *i.e.* the orbit passes not only through the initial point P_0, but also through the point Q, which is the extremity of the radius vector $OQ = v_0/\kappa$ parallel to v_0; OP_0 and OQ are the conjugate semi-diameters whose equations are $x_0 y = y_0 x$, $\dot{x}_0 y = \dot{y}_0 x$.

(4) The problem reduces to that of constructing the axes of a conic from a pair of conjugate diameters.

(8) The equation of a central conic can be written in the form

$$\frac{1}{p^2} = \frac{1}{a^2} \pm \frac{1}{b^2} \mp \frac{1}{a^2 b^2} r^2,$$

where p is the perpendicular from the centre to the tangent; the upper sign gives the ellipse, the lower the hyperbola. Apply (11), Art. 116.

(9) (*a*) Ellipse; (*b*) hyperbola; (*c*) parabola.

(10) Parabola: $x - x_0 = \frac{\dot{x}_0}{y_0}(y - y_0) - \frac{\kappa m' a}{2 y_0^2}(y - y_0)^2$, where $2a$ is the distance of O_3 from the point O that bisects O_1O_2; the point midway between O and O_3 is taken as origin, and OO_3 as axis of x.

(11) $t = \frac{1}{\kappa}\tan^{-1}\left(\frac{a}{b}\tan\theta\right)$.

Pages 76, 77.

(1) (*a*) $f(r) = \frac{c^2}{a^2b^2} \cdot r$; (*b*) $f(r) = \frac{c^2 a}{b^2} \cdot \frac{1}{r^2}$.

(2) $v_0 = \sqrt{\mu / r_0}$.

(4) 687 days.

(5) By (24), Art. 138, $v^2 = \frac{2\mu}{r} \mp \frac{\mu}{a}$; as the velocity is not changed instantaneously, we must have $\frac{2\mu}{r} \mp \frac{\mu}{a} = \frac{2\mu'}{r} \mp \frac{\mu'}{a'}$, whence the new major semi-axis a' can be found.

(6) An ellipse with the end of its minor axis at the point where the change takes place.

(7) (*a*) Ellipse with $a = \frac{2}{3}r$; (*b*) parabola.

(8) Differentiate (24), Art. 138, with respect to μ and a.

(9) The periodic time T would be diminished by $\frac{2}{m}T$.

(10) $r = \frac{l}{1 + e\cos\theta}$; hence $x = \frac{l\sin\theta}{1 + e\cos\theta}$, $y = \frac{l\sin\theta}{1 + e\cos\theta}$; differentiating and remembering that $r^2 d\theta/dt = c$, we find

$$\frac{dx}{dt} = -\frac{c}{l}\sin\theta, \quad \frac{dy}{dt} = \frac{c}{l}(\cos\theta + e);$$

eliminating θ, we find the equation of the hodograph

$$x^2 + \left(y - \frac{ec}{l}\right)^2 = \left(\frac{c}{l}\right)^2, \text{ or since } c = \sqrt{\mu l},\ x^2 + \left(y - \frac{\mu e}{c}\right)^2 = \left(\frac{\mu}{c}\right)^2.$$

(11) 1.016 914.

(12) $t = \sqrt{\frac{2a^3}{\mu}}\left(\tan\frac{\theta}{2} + \frac{1}{3}\tan^3\frac{\theta}{2}\right)$.

Page 79.

(1) $f(r) = c^2\left[\frac{(n+2)b}{r^{n+3}} + \frac{(n+1)(a^2-b^2)}{r^{2n+3}}\right]$. For $n = -1$, $f(r) = \frac{c^2}{1/b} \cdot \frac{1}{r^2}$; for $n = -2$, $f(r) = c^2(b^2 - a^2) \cdot r$; for $n = 1$ and $a = b$, $f(r) = \frac{3bc^2}{r^4}$; for $n = 2$ and $b = 0$, $f(r) = \frac{3a^2c^2}{r^7}$.

(2) Taking the diameter perpendicular to the force as axis of x, we find $F = -\frac{a^2c^2}{y^3}$, where a is the radius, c the x-component of the initial velocity.

(3) (a) $\frac{c^2}{r^3}\left(\frac{2a^2}{r^2} + 1\right)$; (b) $\frac{e^2}{r^3}$; (c) $\frac{c^2(1+\alpha^2)}{r^3}$; (d) $\frac{c^2}{r^3}\left(\frac{2n^2a^2}{r^2} + 1 - n^2\right)$.

(4) $\frac{8a^2c^2}{r^5}$.

(5) Ellipse, parabola, or hyperbola, according as $\mu \lesseqgtr y_0^2\dot{y}_0^2$, where y_0 is the initial distance from the plane, $\dot{y}_0$ the initial velocity perpendicular to the plane.

(6) $f(r) = -\frac{b^4}{a^2}\frac{\dot{x}_0^2}{y^3}$.

Page 83.

(2) Let ρ_1, ρ_2 be the distances of m_1, m_2 from the common centroid at any time t; $\xi_1 = x_1 - \bar{x}$, etc.; then the equations of the relative motion are

$$\frac{d^2\xi_1}{dt^2} = m_2 f\left(\frac{m_1+m_2}{m_2}\rho_1\right) \cdot \frac{\xi_1}{\rho_1}, \text{ etc.}; \quad \frac{d^2\xi_2}{dt^2} = m_1 f\left(\frac{m_1+m_2}{m_1}\rho_2\right) \cdot \frac{\xi_2}{\rho_2}, \text{ etc.}$$

Pages 93, 94.

(4) (a) $7\frac{1}{2}$ pounds; (b) 735·6 pounds; (c) 120 ft. per sec.

(5) 4840 pounds.

(6) $e = \frac{7}{4}\frac{v^2}{R}$ inches.

(7) 8°·6.

(9) 32·20 ft. per sec.

(12) 4·7 pounds.

(13) (a) 76 per min.; (b) 108.

(14) In Fig. 23, $CD : RF = PC : PR$, hence $CD = \frac{mg \cdot a}{\omega^2 a} =$ const.

Pages 96, 97.

(1) The integration gives $\tan\frac{1}{4}(\pi+\theta)=\tan\frac{1}{4}(\pi+\theta_0)\cdot e^{\sqrt{\frac{g}{l}}t}$, which gives $t=\infty$ for $\theta=\pi$.

(5) The particle will not leave the circle if N remains positive; $N=0$ if $\cos\theta_1=-\frac{2}{3}\frac{h}{l}$.

(6) Greater than 17·94 ft. per sec.

(8) The tangential force is $mg\sin\theta$, if θ is the inclination of the tangent to the horizon; as $\sin\theta=dz/ds$, we have

$$\frac{dv}{dt}=g\frac{dz}{ds},\quad\text{or}\quad ds\frac{d^2s}{dt^2}=gdz,$$

whence $\frac{1}{2}v^2-\frac{1}{2}v_0^2=g(z-z_0)$. The result also follows from the equation of kinetic energy, since the force-function is $U=mgz+C$.

(9) The equation of motion is the same as (8), Art. 175, except that g is replaced by $g\sin\alpha$.

(10) The particle performs oscillations of period $2\pi/\mu$ if $F=\mu^2r$.

Page 98.

Taking θ for q, we find $Q=-(mg+\mu\kappa\theta)\kappa$; hence the integration of (17) gives $v^2=h-2\kappa g\theta-\frac{\mu\kappa^2}{m}\theta^2$. On the other hand, (14) gives $v^2=(a^2+\kappa^2)\left(\frac{d\theta}{dt}\right)^2$. Hence

$$\frac{d\theta}{\sqrt{h-2\kappa g\theta-\frac{\mu\kappa^2}{m}\theta^2}}=\frac{dt}{\sqrt{a^2+\kappa^2}}.$$

Pages 102, 103.

(3) (*a*) 9·8 in.; (*b*) 1·23 oz.

(4) As x and y are small of the first order, z differs from r, by (5), only by a small quantity of the second order; hence $\ddot{z}=0$, $z/r=1$, so that the third of the equations (6) gives $N=mg$; hence $\ddot{x}=-gx/r$, $\ddot{y}=-gy/r$. Integrating and putting $\sqrt{g/r}=\mu$, we find $x=C_1\sin\mu t+C_2\cos\mu t$, $y=D_1\sin\mu t+D_2\cos\mu t$. Solving for sin and cos, squaring and adding, we find an ellipse as the required path, just as in Art. 121.

Page 112.

(1) The differential equation is $\ddot{r} - \omega^2 r + g \sin \omega t = 0$. If $r = r_0$, $dr/dt = v_0$ at the time $t = 0$, we find

$$2\,\omega r = \left(\omega r_0 + v_0 - \frac{g}{\omega}\right)e^{\omega t} + \left(\omega r_0 - v_0 + \frac{g}{\omega}\right)e^{-\omega t} + \frac{g}{\omega}\sin \omega t.$$

(2) Let θ be the angle, at the time t, between the radius CP, drawn from the centre C of the circle to the particle P, and the diameter OCA through the fixed point O. Then, taking O as origin, and the initial position OC_0A_0 of the diameter OCA (when P is at A_0) as axis of x, we find

$$\frac{d^2\theta}{dt^2} + \omega^2 \sin\theta = 0, \text{ whence } \left(\frac{d\theta}{dt}\right)^2 = 2\,\omega^2 \cos\theta + C.$$

As the absolute initial velocity is zero, we find $\dot{\theta}_0 = 2\,\omega$. Hence, $\dot{\theta}^2 = 4\,\omega^2 \cos^2 \frac{1}{2}\theta$, and finally, $\sin\frac{1}{2}\theta = \dfrac{e^{\omega t} - e^{-\omega t}}{e^{\omega t} + e^{-\omega t}}$, or $\tan\frac{1}{4}(\pi + \theta) = e^{\omega t}$.

(3) Let $x^2 + y^2 = a^2(1 + \alpha t)^2$ be the equation of the circle, whence $x\delta x + y\delta y = 0$; the equation of motion is $\ddot{x}\delta x + \ddot{y}\delta y = 0$; eliminating δx, δy, and integrating, we find $x\dot{y} - y\dot{x} = av_0$. The equation of the circle gives $x\dot{x} + y\dot{y} = a^2\alpha(1 + \alpha t)$; hence,

$$a(1 + \alpha t)^2\dot{x} = a\alpha(1 + \alpha t)x - v_0\sqrt{a^2(1 + \alpha t)^2 - x^2}.$$

To integrate, put $1 + \alpha t = \alpha\tau$, and then put $x = \xi\tau$. The result is

$$x = a(1 + \alpha t)\cos\frac{v_0 t}{a(1 + \alpha t)},\quad y = a(1 + \alpha t)\sin\frac{v_0 t}{a(1 + \alpha t)},$$

$$v^2 = a^2\alpha^2 + \frac{v_0^2}{(1 + \alpha t)^2}.$$

Pages 134, 135.

The square of the radius of inertia is:

(1) $\frac{1}{3}l^2$.

(2) (*a*) $\frac{1}{3}l^2$; (*b*) $\frac{1}{3}h^2$; (*c*) $\frac{1}{12}l^2$; (*d*) $\frac{1}{12}h^2$.

(3) $\frac{1}{2}h^2$.

(4) $\frac{1}{12}a^2$.

(5) $\frac{5}{24}a^2$.

(6) $\frac{1}{3}h^2$.

(7) $\dfrac{n+1}{n+3}l^2$.

(8) $\frac{1}{4}a^2$.

(9) $\frac{1}{2}a^2$.

(10) $\frac{1}{5}a^2$.

(11) $\frac{1}{5}a^2$, $\frac{1}{5}b^2$, $\frac{1}{5}c^2$.

(12) $\frac{1}{4}(a_1^2 + a_2^2)$.

(13) (a) $I = \frac{2}{3}[b^3\alpha + (a-\alpha)\beta^3]$; (b) $I = \frac{2}{3}[a^3\beta + (b-\beta)\alpha^3]$.

(14) $I = \frac{1}{24}\rho\delta[2bh^3 - (b-2a)(h-2a)^3]$.

Pages 137, 138.

The square of the radius of inertia is:

(1) $\frac{1}{12}(h^2 + l^2)$.

(2) (a) $\frac{1}{24}a^2$; (b) $\frac{1}{24}a^2$; (c) $\frac{1}{12}a^2$.

(3) (a) $\frac{5}{4}a^2$; (b) $\frac{1}{2}a^2$; $\frac{3}{2}a^2$.

(4) $\frac{2}{5}a^2$.

(5) $\frac{1}{2}(a_1^2 + a_2^2)$; in the limit, $I = Ma^2$.

(6) Differentiating the moment of inertia in Ex. (4), we find $I = M \cdot \frac{2}{3}a^2$.

(7) (a) $\frac{1}{2}a^2$; (b) $\frac{3}{2}a^2$; (c) $\frac{1}{4}a^2 + \frac{1}{3}h^2$.

(9) (a) $\frac{1}{4}b^2$; (b) $\frac{1}{4}a^2$; (c) $\frac{1}{4}(a^2 + b^2)$.

(10) $\frac{1}{5}(b^2 + c^2)$; $\frac{1}{5}(c^2 + a^2)$; $\frac{1}{5}(a^2 + b^2)$.

(11) $I = \frac{2}{3}[a^3\beta + (b-\beta)\alpha^3] - \frac{1}{2}\dfrac{(a^2\beta + b\alpha^2 - \alpha^2\beta)^2}{a\beta + b\alpha - \alpha\beta}$.

(12) For axis parallel to b, $I = \frac{1}{2}\delta[\frac{1}{3}(h^3 + b\delta^2) + b(h+\delta)^2]$; for axis parallel to h, $I = \frac{2}{3}\delta(\frac{1}{4}b^3 + h\delta^2)$; for perpendicular axis,

$$I = \frac{1}{6}\delta[h^3 + b^3 + 3bh^2 + 6bh\delta + 4(b+h)\delta^2].$$

(13) $\frac{1}{2}a^2$.

Pages 149, 150.

(1) The centroidal principal axes are perpendicular to the faces. The moments for these axes are $\frac{1}{3}M(b^2+c^2)$, $\frac{1}{3}M(c^2+a^2)$, $\frac{1}{3}M(a^2+b^2)$. The central ellipsoid is $(b^2+c^2)x^2 + (c^2+a^2)y^2 + (a^2+b^2)z^2 = 3e^4$. For an edge $2a$, $I = \frac{4}{3}M(b^2+c^2)$; for a diagonal $I = \frac{2}{3}M(b^2c^2 + c^2a^2 + a^2b^2)/(a^2+b^2+c^2)$.

For the cube the central ellipsoid becomes a sphere of radius $\frac{1}{3}\sqrt{6}\,a$; for an edge of the cube, $I = \frac{64}{3}a^5$.

(2) Central ellipsoid: $(b^2+c^2)x^2 + (c^2+a^2)y^2 + (a^2+b^2)z^2 = 5\epsilon^4$; for l, $q^2 = \frac{1}{5}(6a^2 + b^2)$.

(3) Take the vertex as origin, the axis of the cone as axis of x; then $I_1 = \frac{3}{10}Ma^2$; I_1', *i.e.* the moment of inertia for the yz-plane, $= \frac{3}{5}Mh^2$. As for a solid of revolution about the axis of x $B' = C'$ and $B = C$, we

have $I_2' = I_3' = \frac{1}{2} I_1$, and $I_2 = I_3 = I_1' + \frac{1}{2} I_1$. Hence, $I_2 = I_3 = \frac{3}{5} M(h^2 + \frac{1}{4} a^2)$. At the centroid the squares of the principal radii are $\frac{3}{10} a^2$, $\frac{3}{80}(4a^2 + h^2)$.

(4) $A = B = C = \frac{2}{3} Ma^2$, $D = E = F = \frac{1}{4} Ma^2$; hence momental ellipsoid: $4(x^2 + y^2 + z^2) - 3(yz + zx + xy) = 6\frac{\epsilon^4}{a^2}$; squares of principal radii: $\frac{1}{6} a^2$, $\frac{11}{12} a^2$, $\frac{11}{12} a^2$.

(5) $q^2 = \frac{1}{2} a^2(1 + \sin^2\alpha)$.

(6) $I = \frac{1}{10} \rho\pi a^4\left(\frac{8}{3}a + H - h + 9\frac{h^5}{H^4}\right)$; for $h = a = \frac{1}{3} H$, $q^2 = \frac{27}{106} a^2$.

(7) $A = I_1$, $B = I_2 + Mx_1^2$, $C = I_3 + Mx_1^2$.

(8) The centroid may be such a point; or if the central ellipsoid be an oblate spheroid, the two points on the axis of revolution at the distance $\pm\sqrt{(I_1 - I_2)/M}$ from the centroid.

(9) The ellipsoid must have the same central ellipsoid as the given body; its equation is $\frac{x^2}{A'} + \frac{y^2}{B'} + \frac{z^2}{C'} = \frac{5}{M}$, where M is the mass, and A', B', C' are the moments of inertia for the principal *planes* of the body at the centroid.

Pages 163, 164.

(1) $\pi\sqrt{\frac{3a^2 + 4L^2}{6gL}}$.

(2) $\frac{2}{3}\sqrt{2}\,a$.

(3) $\frac{25}{9} M\pi^2 r^2$.

(4) $\frac{50}{9} M\pi^2 r^2$.

(5) 375 000 foot-pounds.

(6) $\frac{\pi N^2}{3600\,\mu g} \cdot \frac{r^2}{r'}$, where r and r' are expressed in feet.

(7) 4 m. 22 s.

(8) $\sqrt{\frac{3g}{l}}$.

Pages 203–207.

(1) $c = \frac{1}{3} a$, $\alpha = \frac{1}{2}\pi$, $\omega = F/Ma$.

(3) $\frac{5\sqrt{3}}{24} a$.

(8) The body begins to turn about an axis through the fixed point, parallel to the instantaneous axis before impact, with angular velocity $\omega' = \frac{F}{M} \cdot \frac{x - x_1}{q^2 + x^2}$.