ELEMENTARY TRIGONOMETRY

BY

W. E. PATERSON, M.A., B.Sc.

MATHEMATICAL MASTER, MERCERS' SCHOOL
AUTHOR OF 'SCHOOL ALGEBRA

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PREFACE

The common examination syllabus 'Trigonometry up to and including solution of triangles' has caused most textbooks to treat the subject as if the sole use of Trigonometry were to solve triangles, and the practical examples deal almost exclusively with various forms of triangle-solving under the heading 'Heights and Distances'. Further it is customary to define the trigonometrical ratios by means of a right-angled triangle; this encourages the mistaken idea that the ratios are fundamentally attached to a triangle, and does not impress upon the pupil the fact that they are the property of an angle and of an angle only.

In this book the trigonometrical ratios are introduced as functions of the angle. The trigonometrical properties of the single angle are treated fully in the early chapters, and from the beginning the examples apply Trigonometry wherever it may be useful, to Geometry, Mensuration, Analytical Geometry, Physical formulae, &c. The right-angled triangle definitions are given in Chapter V. This chapter contains, in addition to the usual matter, a short treatment of Plane Sailing in Navigation. It is hoped that the examples in Navigation will provide practice in the use of tables, and at the same time be of interest to the pupil. Other examples in this chapter lead up to the formulae dealing with the ordinary triangle.

The formulae for $\frac{1}{2}A$ are proved by Geometry, independently of the addition formulae; thus the triangle is treated fully without breaking the sequence with a discussion of the $A+B$ and allied formulae.

A chapter is devoted to Projection. This includes a discussion of Vector Quantities, their composition and resolution, and finishes with a geometrical treatment of Demoivre's Theorem. In the last chapter, the addition formulae and the allied formulae are treated fully; the projection proofs are used and recommended but the old-fashioned proofs are also given

Throughout the book the student is given every opportunity of developing the subject for himself. A large portion of the bookwork first occurs among the examples of earlier chapters. Also, when a formula has been proved, the proofs of others of the same kind are left for the student to supply. Thus, when $\sin^2 A + \cos^2 A = 1$ has been proved, the student should have no difficulty in proving the connexion between $\sec^2 A$ and $\tan^2 A$; when $\sin(A+B)$ has been found, the student should himself find the expanded form of $\cos(A+B)$, &c.

The sets of examples in the body of a chapter are numbered IV. a, IV. b, &c.; these deal only with the matter immediately preceding them. The last set of examples in a chapter has no distinguishing letter and serves for revision of the whole chapter. There are also three sets of Revision Examples. Bookwork is frequently set as an example, both in the Revision Sets and elsewhere; only by constant repetition, oral or written, can the bookwork be learnt. There are a few sets of oral examples; these are intended to fill up spare minutes at the end of a lesson and often bring out the weak points in a pupil's knowledge. The book contains nearly 1,000 examples; it is not intended that any one should attempt all these, but it is hoped that they include a sufficient variety of types and a sufficient number of each type to meet all requirements.

Many examples are taken from Examination Papers by kind permission of the following authorities:—

The Controller of His Majesty's Stationery Office.

The University of Cambridge.

The Joint Matriculation Board of the Scottish Universities.

The Intermediate Education Board for Ireland.

The Oxford and Cambridge Schools Examination Board.

The Delegacy for Oxford Local Examinations.

The Syndicate for Cambridge Local Examinations.

The College of Preceptors.

I am indebted to Mr. Norman Chignell, B.A., of Charterhouse, for many suggestions and for assistance in correcting the proof-sheets. It is too much to hope that the answers are wholly free from mistakes, and I shall be grateful to receive early intimation of any corrections that may be found necessary.

W. E. P.

April, 1911.

CONTENTS

The following course of reading is suggested for beginners:—

Chapter I, §§ 1–4, 9–11.
Chapter II.
Chapter III, §§ 21–31.
Chapter IV, §§ 34–7, 40–2.
Chapter V, §§ 43–5, 51–3.
Chapter VI, §§ 54–8.
Chapter VII.

PRELIMINARY CHAPTER

GEOMETRY

A KNOWLEDGE of the following geometrical facts is required.* In this book these propositions are referred to by the numbers given below.

Angles.

Prop. 1. If a straight line meets another straight line, the adjacent angles are together equal to two right angles.

Prop. 2. If two straight lines cut, the vertically opposite angles are equal.

Prop. 3. The angle at the centre of a circle is double an angle at the circumference standing on the same arc.

Prop. 4. Angles in the same segment of a circle are equal.

Prop. 5. Angles at the centre of a circle standing on different arcs are in the same ratio as the lengths of the arcs.

Triangles.

Prop. 6. (*a*) The three angles of a triangle are together equal to two right angles.

(*b*) If one side be produced the exterior angle equals the sum of the two interior opposite angles.

Prop. 7. Any two sides of a triangle are together greater than the third.

Prop. 8. Two triangles are congruent (i. e. are equal in every respect) if they have—

(*a*) two sides of the one equal to two sides of the other, each to each, and the angle contained by the two sides of the one equal to the angle contained by the two corresponding sides of the other;

or (*b*) three sides of the one equal to three sides of the other, each to each;

or (*c*) two angles of the one equal to two angles of the other,

* For proofs see Warren's *Experimental and Theoretical Geometry* (Clarendon Press), or any standard textbook.

each to each, and a side of the one equal to the corresponding side of the other.

Prop. 9. If two triangles have an angle of the one equal to an angle of the other, and the sides about another pair of angles equal, each to each, then the third angles are either equal or supplementary.

Prop. 10. (*a*) If two sides of a triangle are equal, the opposite angles are equal.

(*b*) If two sides are unequal, the greater side is opposite a greater angle.

(*c*) If all the sides of a triangle are equal, all the angles are equal.

Prop. 11. (*a*) If two angles of a triangle are equal, the opposite sides are equal.

(*b*) If two angles are unequal, the greater angle is opposite a greater side.

(*c*) If all the angles of a triangle are equal, all the sides are equal.

Prop. 12. Two triangles are similar (i. e. their angles are equal, each to each, and the ratio of pairs of sides opposite equal angles is the same for all three angles) if they have—

(*a*) their angles equal each to each;

(*b*) their sides in the same ratio;

(*c*) an angle of the one equal to an angle of the other, and the sides about the equal angles in the same ratio.

Prop. 13. (Pythagoras' Theorem.) In a right-angled triangle the square on the hypotenuse is equal to the sum of the squares on the other two sides.

Parallel Lines.

Prop. 14. (*a*) If a line is drawn to cut two parallel lines, it makes (i) the alternate angles equal, (ii) the interior angles on the same side of it together equal to two right angles, (iii) the exterior angle equal to the interior opposite angle.

(*b*) The opposite sides and angles of a parallelogram are equal.

Area.

The unit of area is the area of a square whose side is of unit length.

Prop. 15. The number of units of area in a rectangle is equal to the product of the number of units of length in one side multiplied by the number of units of length in the other.

Or, more shortly : Area of rectangle = length × breadth.

Prop. 16. The area of a triangle = $\frac{1}{2}$ base × altitude.

The Concurrencies of the Triangle.

Prop. 17. The lines bisecting the sides of a triangle at right angles are concurrent (i. e. meet at a point).

The point in which they meet is the centre of the circle passing through the three vertices and is called the **circumcentre.**

Prop. 18. The lines drawn from the vertices to bisect the opposite sides are concurrent.

These lines are called the **medians** and the point of concurrency is called the **centroid.**

Prop. 19. (*a*) The lines bisecting the angles are concurrent.

The point of concurrency is the centre of the circle that touches all the sides, and is called the **incentre.**

(*b*) If two of the sides be produced, the lines bisecting the exterior angles so formed and the line bisecting the interior angle contained by the produced sides are concurrent.

The point of concurrency is the centre of the circle that touches the two sides when produced and the third side (not produced); it is called an **e-centre.**

Prop. 20. The perpendiculars let fall from the vertices on the opposite sides are concurrent.

The point of concurrency is called **orthocentre.**

The Circle.

Prop. 21. The straight line passing through the centre, at right angles to a chord, bisects the chord.

Prop. 22. (*a*) The angle at the centre of a circle is twice the angle at the circumference on the same arc. (*b*) Angles in the same segment are equal. (*c*) The opposite angles of a quadrilateral inscribed in a circle are together equal to two right angles.

Prop. 23. The tangent at any point is at right angles to the radius drawn to that point.

Prop. 24. (*a*) Two tangents can be drawn to a circle from any external point. (*b*) The parts of these tangents between the external point and the points of contact are equal. (*c*) The line joining the external point to the centre bisects the angle between the tangents.

Prop. 25. The ratio of the circumference of any circle to its diameter is the same for all circles.

This ratio is denoted by the symbol π; its value is 3·1416 correct to five significant figures.

Prop. 26. The area of a circle equals the area of the rectangle contained by the radius and a straight line equal to half the circumference.

This is usually expressed in the formula : Area $= \pi r^2$.

GRAPHS *

Geometrical. If two straight lines are drawn in a plane, the position of any point in the plane can be determined by means of its distances from those lines.

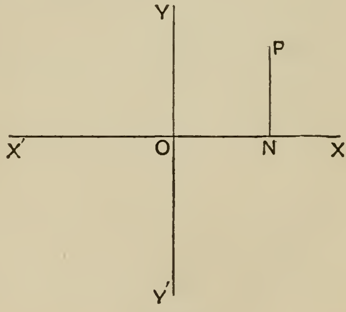


Fig. 1.

It is usual to draw one of the lines horizontal and the other perpendicular to it. The customary notation is shown in Fig. 1.

$X'OX$ is called the axis of x;

$Y'OY$ is called the axis of y;

O is called the **origin**;

ON is called the **abscissa** of the point P;

NP is called the **ordinate** of the point P.

The abscissa and ordinate are called the **co-ordinates** † of the point P.

* For a fuller treatment of Graphs see *School Algebra* published by the Clarendon Press.

† These co-ordinates are called Cartesian co-ordinates because they were first used by the French mathematician, Descartes.

The abscissa is said to be positive if drawn to the right, negative if drawn to the left. Similarly, the ordinate is positive if drawn upwards from N, negative if drawn downwards. The number of units of length in ON, preceded by the proper sign, is usually denoted by x, and the number of units of length in NP, preceded by the proper sign, is denoted by y. In each case the sign + is often omitted.

Thus, in Fig. 2, the co-ordinates of A are $x = -4$, $y = 2$, of C, $x = 4$, $y = 3$, of L, $x = 0$, $y = -7$.

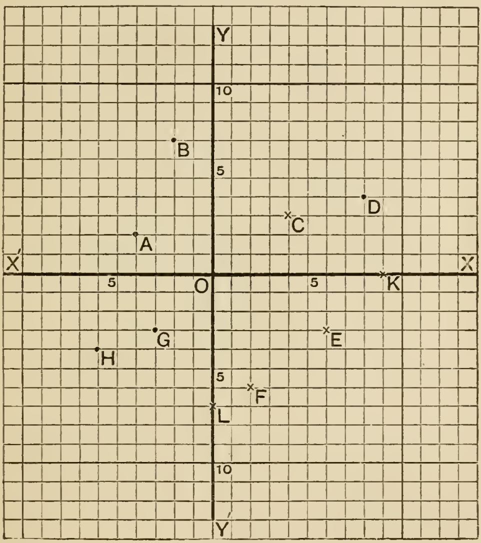


Fig. 2.

Very often a point is described by writing the values of the co-ordinates in brackets; e.g. the point H might be described as the point $(-6, -4)$.

Exercise. Write down the co-ordinates of all the points in Fig. 2.

Graphs of Statistics. The magnitude of any quantity may be represented by a straight line which contains as many units of length as the quantity contains units of its own kind.

If two quantities are changing their values at the same time, the simultaneous values may be represented in the same figure by taking horizontal lengths to represent one magnitude and vertical lengths to represent the other.

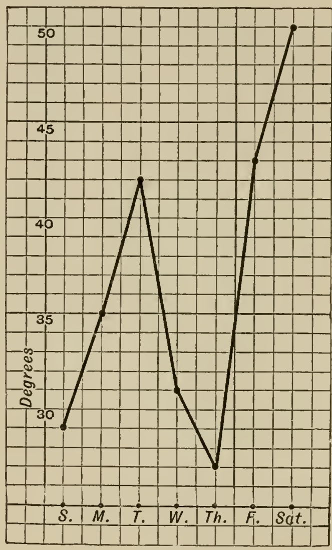


Fig. 3.

In Fig. 3 the changing quantities are time and temperature; and the dots show that at noon on Sunday the temperature was 29°, on Monday the temperature was 35, &c. In fact the figure conveys the same information as the following table :—

	Sun.	Mon.	Tues.	Wed.	Thurs.	Fri.	Sat.
Temp.	29°	35°	42°	31°	27°	43°	50°

If there is no information about intermediate temperatures, the points are joined by a series of straight lines. The figure now forms a graph.

In describing such a graph we should say that the abscissae represent time and the ordinates temperature.

Graphs of functions. If two quantities x and y are such that a change of value in the one causes a change of value in the other, then either of them is said to be a **function** of the other.

This is expressed thus: $y = f(x)$, or $x = f(y)$ where $f(x)$ means a function of x. A graph can be drawn in which the abscissae are proportional to the values of x and the ordinates to the values of y. This graph is called the graph of the function $f(x)$ or of the equation $y = f(x)$. This may be more easily understood by considering a few algebraical functions.

Example I. Draw the graph when $y = \frac{3}{5}x - \frac{2}{5}$.

(Choose values of x which will make y a whole number.)

$$y = \tfrac{1}{5}(3x - 2)$$

x	-6	-1	4	9
$3x-2$	-20	-5	10	25
y	-4	-1	2	5

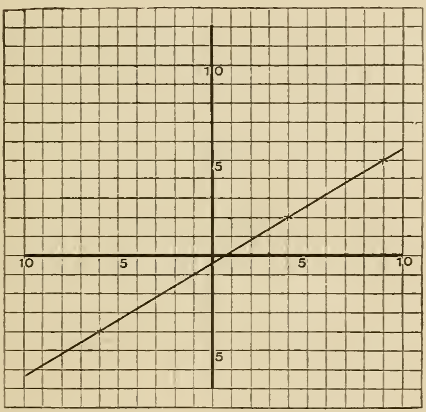


Fig. 4.

When the points corresponding to these values of x and y are plotted, it is found that they lie on the straight line shown in Fig. 4. It is also found

(i) That any simultaneous values of x and y connected by the given equation are the co-ordinates of some point on this straight line;

(ii) That the co-ordinates of any point on the straight line satisfy the equation.

It is found by experience (and can be proved from the geometrical propositions on proportion) that, when x and y are connected by an equation of the first degree, the graph is always a straight line.

Example II. In the same figure draw the graphs of

$$y = x^2 - 3x + 2 \text{ and } x = 2y^2 + 3.$$

Neither of these equations is of the first degree, therefore neither of the graphs is a straight line. At least six points must be found on each.

$$y = x^2 - 3x + 2. \tag{i}$$

x	-3	-2	-1	0	1	2	3	4	5
x^2	9	4	1	0	1	4	9	16	25
$-3x$	9	6	3	0	-3	-6	-9	-12	-15
y	20	12	6	2	0	0	2	6	12

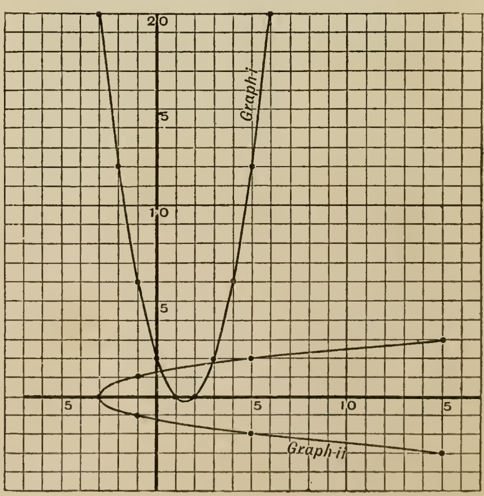


Fig. 5.

$$x = 2y^2 - 3. \qquad \text{(ii)}$$

y	-3	-2	-1	0	1	2	3	4
$2y^2$	18	8	2	0	2	8	18	32
x	15	5	-1	-3	-1	5	15	29

The co-ordinates of every point on graph i satisfy the first equation, and of every point on graph ii satisfy the second equation. Hence the co-ordinates of any points which are on both graphs, that is, the co-ordinates of the points of intersection, satisfy both equations.

Fig. 5 shows, therefore, that the values $x = \cdot 3$, $y = 1\cdot 3$, and $x = 2\cdot 8$, $y = 1\cdot 6$, are the solutions of the two equations.

This **graphical method of solving equations** is very useful, but is, of course, only approximate. If more accurate answers are required, the graphs must be drawn on a larger scale in the neighbourhood of their points of intersection.

LOGARITHMS *

Fractional and Negative Indices. It is shown in Algebra that

$$x^{\frac{p}{q}} = \sqrt[q]{x^p}, \qquad x^0 = 1, \qquad x^{-m} = \frac{1}{x^m},$$

where p and q are any positive integers and x is any positive quantity, integral or fractional.

A fractional index may be expressed as a decimal; thus such expressions as $4^{\cdot 36}$, $10^{\cdot 001}$ have a definite value. This value could in theory be found by reducing the decimal to a vulgar fraction and then replacing the power with a fractional index by a root, e.g.

$$10^{\cdot 301} = 10^{\frac{301}{1000}} = \sqrt[1000]{10^{301}}.$$

This is obviously not practical. The value can be found by a graphical method which is easy but only approximate.

Draw the graph of $x = 10^y$.

* For a fuller treatment of Indices and Logarithms see *School Algebra*, Chapters XXI and XXII.

y	0	·5	·25	·125	·75	·625	·875	1
x	1	3·16	1·76	1·33	5·62	4·26	7·49	10

The values of x are obtained as follows:—

$10^0 = 1$ by definition given above.

$10^{\cdot 5} = 10^{\frac{1}{2}} = \sqrt{10}$.

$10^{\cdot 25} = 10^{\frac{1}{4}} = \sqrt{\sqrt{10}}$; similarly $10^{\cdot 125} = \sqrt{10^{\cdot 25}}$.

$10^{\cdot 75} = 10^{\cdot 5 + \cdot 25} = 10^{\cdot 5} \times 10^{\cdot 25}$, &c.

The graph is shown on a small scale in Fig. 6.

It is seen that $10^{\cdot 30}$ is almost exactly 2; any other power of 10 can be found approximately from this graph when the index is between 0 and 10.

Definition of a logarithm. The logarithm of a number to a given base is the **index** of the power to which the base must be raised to equal the number. Thus $3^2 = 9$, therefore the logarithm of 9 to base 3 equals 2; this is written $\log_3 9 = 2$.

In dealing with numbers the base is 10. In the remainder of this chapter it is assumed that the base is always 10, so that log 731 means logarithm of 731 to base 10.

The equation $x = 10^y$ may be written $y = \log x$.

Hence Fig. 6 provides an approximate means of finding the logarithm of any number between 1 and 10.

Characteristic and mantissa. Consider a number, such as 4378. It means

$$4 \times 10^3 + 3 \times 10^2 + 7 \times 10 + 8.$$

Also a decimal number, such as ·0376, means

$$\frac{0}{10} + \frac{3}{10^2} + \frac{7}{10^3} + \frac{6}{10^4}.$$

If we use negative indices, this may be written

$$\cdot 0376 = 3 \times 10^{-2} + 7 \times 10^{-3} + 6 \times 10^{-4}.$$

Similarly

$$537 \cdot 13 = 5 \times 10^2 + 3 \times 10^1 + 7 \times 10^0 + 1 \times 10^{-1} + 3 \times 10^{-2}.$$

It follows that

$$4378 > 10^3 \text{ but } < 10^4, \therefore \log 4378 = 3 + \text{a decimal};$$
$$\cdot 0376 > 10^{-2} \text{ but } < 10^{-1}, \therefore \log \cdot 0376 = -2 + \text{a decimal};$$
$$537 \cdot 13 > 10^2 \text{ but } < 10^3, \therefore \log 537 \cdot 13 = 2 + \text{a decimal};$$
$$4 \cdot 37 > 10^0 \text{ but } < 10^1, \therefore \log 4 \cdot 37 = 0 + \text{a decimal}.$$

We now see (i) that the logarithm of any number consists of an integer (which may be positive, zero, or negative) and a positive

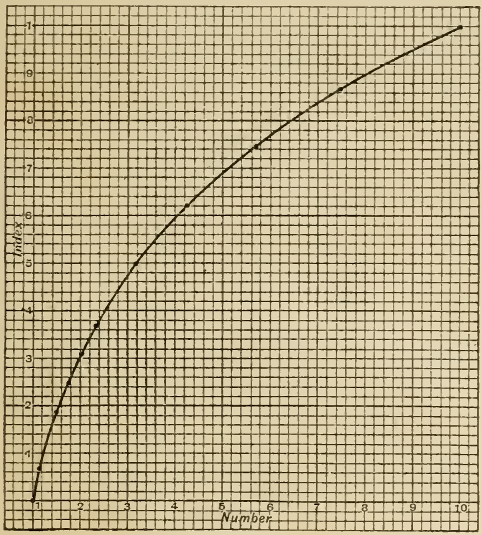


Fig. 6.

decimal, (ii) that the integer is the index of the highest power of 10 contained in the number.

The integral part of a logarithm is called the **characteristic**; the decimal part the **mantissa**.

Index	5	4	3	2	1	0	−1	−2	−3	−4	−5	−6
Number	3	7	8	9	0	0						
,,			4	3	7	6·	5	2				
,,						3·	4	6	7	8		
,,						0·	0	0	7	3	4	8
,,			2	0	0	0·	0	0	0			

Consideration of the preceding table shows that the characteristic (i. e. the highest index) may always be found by the following rule: Count from the unit place to the first significant figure (i. e. the first figure which is not 0), the unit place being counted as nothing. The characteristic is positive or zero if the number is greater than one, negative if it is less than one.

The mantissa is independent of the position of the decimal point. An example will make this clear.

Given that log 4·376 = ·6411, find log 4376 and log ·004376.

$$4376 = 1000 \times 4{\cdot}376$$

$$\text{but } 4{\cdot}376 = 10^{\cdot 6411} \text{ since } \log 4{\cdot}376 = {\cdot}6411.$$

$$\therefore\quad 4376 = 10^3 \times 10^{\cdot 6411} = 10^{3\cdot 6411}, \text{ i.e. } \log 4376 = 3{\cdot}6411$$

$$\cdot 004376 = \tfrac{1}{1000} \times 4{\cdot}376 = 10^{-3} \times 10^{\cdot 6411} = 10^{-3+\cdot 6411}$$

$$\therefore\quad \log {\cdot}004376 = -3 + {\cdot}6411.$$

The negative sign of a characteristic is always placed on top and the + before the decimal is omitted. Thus log ·004376 = $\bar{3}{\cdot}6411$.

To find the logarithm of any number.

(*a*) Four-figure tables. The mantissa is found from tables, of which a specimen is given below.

LOGARITHMS.

	0	1	2	3	4	5	6	7	8	9	1	2	3	4	5	6	7	8
51	7076	7084	7093	7101	7110	7118	7126	7135	7143	7152	1	2	3	3	4	5	6	7
52	7160	7168	7177	7185	7193	7202	7210	7218	7226	7235	1	2	2	3	4	5	6	7
53	7243	7251	7259	7267	7275	7284	7292	7300	7308	7316	1	2	2	3	4	5	6	6
54	7324	7332	7340	7348	7356	7364	7372	7380	7388	7396	1	2	2	3	4	5	6	6
55	7404	7412	7419	7427	7435	7443	7451	7459	7466	7474	1	2	2	3	4	5	5	6

Consider the logarithms of 5467 and ·05467.

By counting, the characteristic of log 5467 is found to be 3, and that of log ·05467 to be $\bar{2}$.

Both logarithms have the same mantissa. Look for 54 in the extreme left-hand column. In the same line with 54 and under 6 we find 7372; this is the mantissa of log 546. Under the 7 in the small columns to the right, and in a line with 54, we find 6; this must be added to the last digit of the mantissa already found. Hence the mantissa is ·7378.

Therefore log 5467 = 3·7378 and log ·05467 = $\bar{2}{\cdot}7378$.

(*b*) Five-figure tables. Find the logarithm of 346·73.

Proceeding as with four-figure tables, we find that the mantissa of log 346 is ·53908. Under 7 in the side columns, we find 88; this must be added to the last two digits already found. For a 3 in the fourth place we should add 38, for a 3 in the fifth place we add, therefore, $\frac{1}{10} \times 38$, i. e. 4 to nearest integer. Hence

$$\log 3467{\cdot}3 = 2{\cdot}53908 + 88 + 4 = 2{\cdot}54000.$$

To find the number corresponding to any logarithm.

Method I. Reverse the process for finding a logarithm. Suppose the logarithm is 3·7271.

Look in the logarithms for the mantissa nearest to 7271, but less than it. We find 7267, level with 53 and under 3; the first three figures of the number are 533. This leaves $7271 - 7267 = 4$; in the right-hand columns 4 is found under 5. Hence the first four figures are 5335.

The characteristic is 3, therefore the left-hand digit 5 represents 5×10^3; hence the number is 5335.

The number is called the **antilogarithm** of the logarithm.

Method II. If tables of antilogarithms are available, they are used in the same way as logarithm tables.

ANTILOGARITHMS.

0	1	2	3	4	5	6	7	8	9	1	2	3	4	5	6	7	8	9
5012	5023	5035	5047	5058	5070	5082	5093	5105	5117	1	2	4	5	6	7	8	9	11
5129	5140	5152	5164	5176	5188	5200	5212	5224	5236	1	2	4	5	6	7	8	10	11
5248	5260	5272	5284	5297	5309	5321	5333	5346	5358	1	2	4	5	6	7	9	10	11
5370	5383	5395	5408	5420	5433	5445	5458	5470	5483	1	3	4	5	6	8	9	10	11
5495	5508	5521	5534	5546	5559	5572	5585	5598	5610	1	3	4	5	6	8	9	10	12
5623	5636	5649	5662	5675	5689	5702	5715	5728	5741	1	3	4	5	7	8	9	10	12

Look for ·72 in the left-hand column; level with ·72 and under 7 we find 5333; in the small columns we find 1 under 1. Hence the first four figures are 5334.

The characteristic is 3; as before, the number is 5334.

Note.—The two methods give results differing by 1 in the last figure; this shows that the number is between the two results. On using five-figure tables, it is found that the antilogarithm of 3·7271 is 5334·5.

Use of Logarithms.

By the definition of logarithm $a = 10^{\log a}$, $b = 10^{\log b}$,

$$\therefore\ ab = 10^{\log a} \times 10^{\log b} = 10^{\log a + \log b}.$$

Hence $\qquad \log (ab) = \log a + \log b.$

Similarly $\qquad \log a/b = \log a - \log b,$

$$\log a^m = m \log a,$$

$$\log \sqrt[m]{a} = \frac{1}{m} \log a.$$

Thus, instead of

multiplying, we may use logarithms and add ;			
dividing,	,,	,,	subtract ;
raising to a power,	,,	,,	multiply ;
taking a root,	,,	,,	divide.

Note.—There is no process with logarithms to correspond with addition or subtraction with ordinary numbers.

Example I. Find the value of $\dfrac{516{\cdot}5 \times {\cdot}852}{36500}$.

$$\begin{aligned} \text{log of fraction} &= \log 516{\cdot}5 + \log {\cdot}852 - \log 36500 \\ &= 2{\cdot}7130 \\ &+ \bar{1}{\cdot}9304 - 4{\cdot}5623 \\ &= 2{\cdot}6434 \\ &- 4{\cdot}5623 \\ &= \bar{2}{\cdot}0811 \\ \therefore\quad \text{fraction} &= {\cdot}01205. \end{aligned}$$

Example II. Find the cube root of ·1765.

$$\begin{aligned} \text{log of cube root} &= \tfrac{1}{3} \log {\cdot}1765 \\ &= \tfrac{1}{3} \text{ of } \bar{1}{\cdot}2467 \\ &= \tfrac{1}{3} (\bar{3} + 2{\cdot}2467) \\ &= \bar{1}{\cdot}7489. \end{aligned}$$

Notice carefully this method of division when the characteristic is negative.

Hence $\sqrt[3]{{\cdot}1765} = {\cdot}5610.$

Exercises. Find the value of

(1) $\sqrt{319{\cdot}2} \times 1{\cdot}756$. Ans. 31·37.

(2) $\cdot 03056 \times 0{\cdot}4105$. Ans. ·01254.

(3) $3{\cdot}142 \times (71{\cdot}43)^2$. Ans. 16030.

(4) $\frac{4}{3} \times 3{\cdot}142 \times (9{\cdot}67)^3$. Ans. 3787.

(5) $254{\cdot}3 \div 0{\cdot}09027$. Ans. 2817.

(6) $\dfrac{(\cdot 1136)^{\frac{3}{4}} \times \sqrt{81{\cdot}86}}{\frac{3}{4} \times \sqrt[3]{2000}}$. Ans. ·1874.

THE GREEK ALPHABET

Greek letters are used so frequently in Trigonometry and other branches of Mathematics that it is useful to have the complete alphabet for reference.

Name.	Small.	Capital.
alpha	α	A
beta	β	B
gamma	γ	Γ
delta	δ	Δ
epsilon	ϵ	E
zēta	ζ	Z
ēta	η	H
thēta	θ	Θ
iōta	ι	I
kappa	κ	K
lambda	λ	Λ
mu	μ	M
nu	ν	N
xi	ξ	Ξ
omicron	o	O
pi	π	Π
rho	ρ	P
sigma	σ	Σ
tau	τ	T
upsilon	υ	Υ
phi	ϕ	Φ
chi	χ	X
psi	ψ	Ψ
omega	ω	Ω

CHAPTER I

ANGLES AND THEIR MEASUREMENT

1. Any angle such as *BAC* (Fig. I) may be thought of as having been formed by rotating the line *AC* about the point *A* from the position of coincidence with *AB* to its final position *AC*.

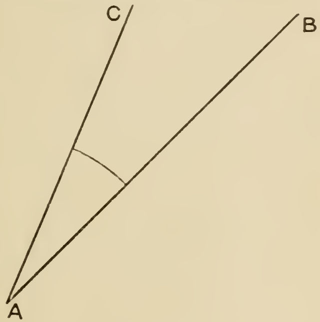


Fig. I.

This way of regarding an angle shows clearly the intimate connexion between angles and arcs of circles and this connexion leads to the usual method of measuring angles.

2. The Degree.* From very early times it has been the custom to divide the circumference of a circle into 360 equal parts or

* 'The current sexagesimal division of angles is derived from the Babylonians through the Greeks. The Babylonian unit angle was the angle of an equilateral triangle; following their usual practice this was divided into 60 equal parts or degrees, a degree was subdivided into 60 equal parts or minutes, and so on; it is said that 60 was assumed as the base of the system in order that the number of degrees corresponding to the circumference of a circle should be the same as the number of days in a year which it is alleged was taken, at any rate in practice, to be 360.' (From *A Short Account of the History of Mathematics*, by W. W. Rouse Ball.)

degrees, each degree into 60 parts or minutes,* each minute into 60 seconds.[1]

The angle at the centre of a circle, subtended by an arc of 1 degree, is taken as the unit angle, and it, too, is called a **degree**; it is divided into minutes and seconds in the same way as the arc degree.

The notation used is shown in the following example:— $47° 15' 37''$ is read 47 degrees 15 minutes 37 seconds.

If the line makes a complete rotation, thus returning to its original position, it has turned through an angle of 360°.

A right angle is produced by one-quarter of a complete rotation, and is, therefore, equal to 90°.

If two angles together equal a right angle, either of them is called the **complement** of the other. When the sum equals two right angles, either angle is the **supplement** of the other.

3. Positive and Negative Angles. In discussing the properties of a single angle it is usual to draw the initial line so that it is horizontal and to name it OA. If the rotating line moves in a direction opposite to that of the hands of a clock, the angle is said to be **positive**; if in the same direction as the hands of a clock, the angle is **negative**.

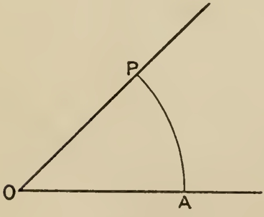


Fig. II.

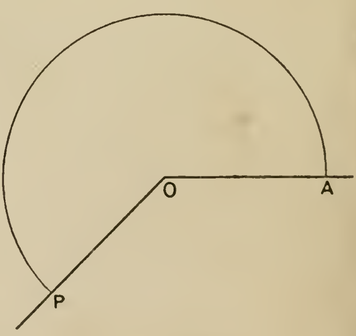


Fig. III.

* Minutes derived from the Latin *partes minutae*; seconds from the Latin *partes minutae secundae*.

In Fig. II the line OP has made $\frac{1}{8}$ of a complete turn, hence the angle $AOP = \frac{1}{8} \times 360° = 45°$; in Fig. III the angle AOP is reflex * and is equal to $\frac{5}{8} \times 360° = 225°$. If, in Fig. II, the line OP reached its position by turning in the negative direction it would have made $\frac{7}{8}$ of a complete turn so that the reflex angle AOP in Fig. II $= -315°$. Similarly in Fig. III the obtuse angle $AOP = -135°$.

4. Angles unlimited in size. In Fig. II the line OP might have made one, two, or any number of complete turns, either positive or negative, and then have moved on to its final position: hence the angle AOP may represent 405°, or 765°, or $-675°$. All possible values are included in the general formula

$$AOP = 360n + 45,$$

where n is any whole number, positive, zero, or negative.

Unless the problem under discussion allows the possibility of the angle being greater than 360°, it is always assumed that the angle is less than 360°.

5. The Grade. When the metric system was invented, the French Mathematicians introduced a new unit, the Grade, such that

100 grades = 1 right angle,
100 minutes = 1 grade,
100 seconds = 1 minute.

This system never came into general use, even in France, and now exists only in old-fashioned examination papers.

Examples I a.

1. Find the complement of each of the following angles: 32°, 47° 23′, 75° 13′ 14″, 68° 0′ 13″, 27° 42′ 18·6″.

2. Write down the supplements of 75°, 68° 14′, 115° 17′ 48″ 90°, 78° 24′ 36″.

3. The angles of a triangle are found to be 42° 13′ 17″, 73° 47′ 5″, 64° 0′ 38″. Is this correct?

4. Two angles of a triangle are 17° 43′, 92° 16′; calculate the third angle.

5. In a triangle ABC, $\frac{1}{2}(A+B) = 77° 29'$ and $\frac{1}{2}(A-B) = 16° 25'$; find all the angles.

* A reflex angle is an angle greater than two right angles, but less than four right angles.

6. Express in degrees, minutes, and seconds the angle of (*a*) a square, (*b*) a regular pentagon, (*c*) a regular heptagon.

7. Express each of the angles of question 6 in grades.

8. The magnitude of an angle may be expressed either as D degrees or G grades; find the equation connecting D and G.

9. Draw the angles A and $\frac{1}{3}A$ in each of the following cases: (*a*) $A = 54°$, (*b*) $A = 414°$, (*c*) $A = 774°$, (*d*) $A = 1134°$, (*e*) $234°$, (*f*) $-126°$.

10. Through what angles do the hour, minute, and second hands of a watch respectively turn between $12^h\,30'$ a.m. and $5^h\,3'$ a.m.?

6. The ratio of the length of the circumference of a circle to the length of its diameter is the same for all circles. This constant value is denoted by the Greek letter π (pronounced *pi*), so that if

the circumference $= c$ units of length,
and diameter $= d$ units of length,

then $$\frac{c}{d} = \pi.$$

The value of π can be found, correct to two or three significant figures, by actual measurement. By geometrical and trigonometrical calculations its value can be calculated to any desired number of places.

Correct to 5 significant figures, $\pi = 3{\cdot}1416$.

Correct to 6 significant figures, $\pi = 3{\cdot}14159$.

For mental calculations π may be taken as $3\frac{1}{7}$.

7. By using Prop. 5, p. 9, problems dealing with the lengths of circular arcs may often be solved.

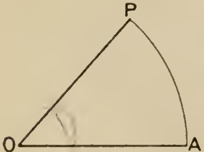


Fig. IV.

Example. *Find the length of an arc which subtends an angle of* 49° *at the centre of a circle whose radius is* 5 *feet.*

$$\frac{\text{arc } AP}{\text{semi-circumference}} = \frac{\text{angle } AOP}{2 \text{ right angles}},$$

i.e. $$\frac{\text{arc } AP}{5\pi} = \frac{49°}{180°}.$$

The calculation is easily completed.

Similarly $\dfrac{\text{area of sector } AOP}{\text{area of circle}} = \dfrac{\text{angle } AOP}{4 \text{ right angles}}$, (Prop. 26)

i. e. $\dfrac{\text{area of sector } AOP}{25\pi} = \dfrac{49}{360}$.

8. Circular measure. By the method of the last section it is easily shown that the length of an arc of a circle, radius r, subtending an angle $A°$ at the centre is $r\dfrac{A\pi}{180}$. In many other formulae the fraction $\dfrac{A\pi}{180}$ occurs in connexion with the angle $A°$. In theoretical work it has, therefore, been found convenient to use another unit angle, which simplifies formulae considerably.

The **radian** is the angle subtended at the centre of any circle by an arc equal in length to the radius.

Let $x°$ equal 1 radian

$$\frac{\text{arc equal to radius}}{\text{semi-circumference}} = \frac{\text{angle of 1 radian}}{2 \text{ right angles}} = \frac{x°}{180°},$$

i. e. $$\frac{r}{\pi r} = \frac{x°}{180°};$$

$\therefore$ $$x° = \frac{180}{\pi}.$$

Since π is the same for all circles, it follows that the radian is the same for all circles and may, therefore, be taken as a unit of measurement.

The number of radians in an angle is often called the **circular measure** of the angle. For this reason the symbol ^{c} is used to show that the angle is measured in radians, e. g. 2^c means 2 radians.

When the radian is the unit angle, it is customary to use Greek letters to denote the number of radians, and the symbol ^{c} is then often omitted. When capital English letters are used, it is usually understood that the angle is measured in degrees.

Examples I b.

1. How many times is an arc equal to the radius contained in the semi-circumference? Reduce 180°, 90°, 60°, 30° to radians. (Do not substitute for π.)

2. Show by simple geometry that the radian is less than 60°.

3. How many radians are there in 10°, 75°, 138°, respectively? Give the answers correct to 2 decimal places.

4. Express the angle of (i) an isosceles right-angled triangle, (ii) a regular nonagon, in circular measure. Give the answers in terms of π.

5. One angle of a triangle is $\frac{1}{6}\pi$, another is $\frac{1}{4}\pi$; what is the circular measure of the third angle?

6. Find the length of an arc of a circle which subtends an angle 78° at the centre, the radius being 18 feet.

7. An arc of length 5 feet subtends an angle of 132° at the centre; what is the radius of the circle?

8. Find the area of the sector of a circle if the radius is 12 feet and the angle 40°.

9. What time does the minute hand of a watch take to turn through (i) 3000°, (ii) 3000 grades, (iii) 3000 radians?

10. Fill in the missing values in the following table, which gives data about circular arcs.

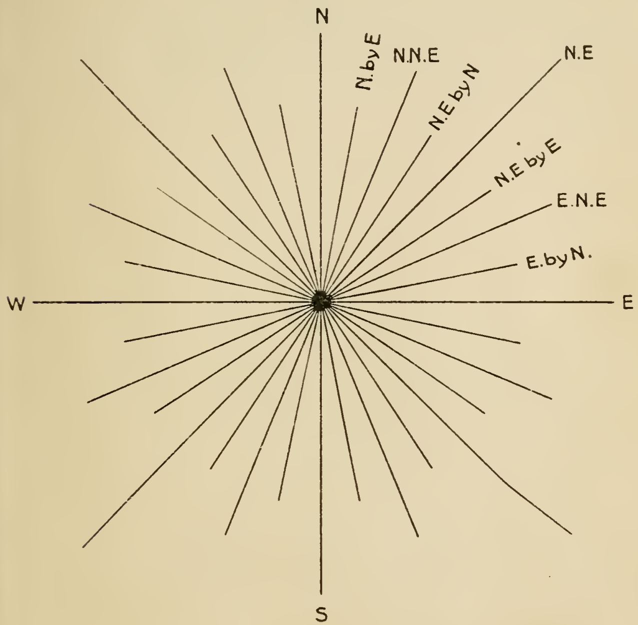
	Radius.	Angle.	Length.
(1)	5 inches	2 radians	
(2)	7·6 centimetres	74·6 grades	
(3)		314 degrees	413 feet
(4)	100 yards	radians	220 yards
(5)	320 metres	degrees	1 kilometre
(6)	yards	5 radians	half a mile

11. Express in radians the angle of a sector of a circle, being given that the radius is 7 inches and the area of the sector 100 sq. inches.

12. Show that the length of an arc subtending an angle θ^c at the centre of a circle, radius r, is $r\theta$. What is the area of the corresponding sector?

13. Find the circular measure of $1'$ and of $1''$, correct to 5 significant figures.

9. The points of the compass. The card of the Mariner's Compass is divided into four quadrants by two diameters pointing North and South, East and West respectively. These are the Cardinal Points. Two other diameters bisecting the angles between the previous diameters give four other points, viz. NE., NW., SW., SE. The eight angles so formed are bisected and



eight more points are thus obtained. These are named by combining the names of the points between which they lie, beginning with the cardinal point. Thus the point midway between E. and SE. is ESE. (East South-East).

The sixteen angles now formed are bisected so that the circumference is finally divided into thirty-two equal divisions. From their names the last sixteen points are called by-points. The point midway between N. and NNE. is called N. by E.; that midway between SW. and SSW. is called SW. by S., &c.

The angle between two consecutive points of the compass is also called a **point**, thus N. 2 points E. is the same as NNE.; WSW. $\frac{1}{2}$ W. means $\frac{1}{2}$ a point W. of WSW.

The ordinary degree is sometimes used in defining a direction, for instance ENE. can be referred to as $22\frac{1}{2}°$ N. of E. Similarly we may have 32° W. of N., 40° S. of W., &c.

10. Latitude and Longitude. The position of a point on a sphere can be defined by two angles, which may be compared with the abscissa and ordinate of plane geometry. These angles are easily understood by considering the special case of Longitude and Latitude.

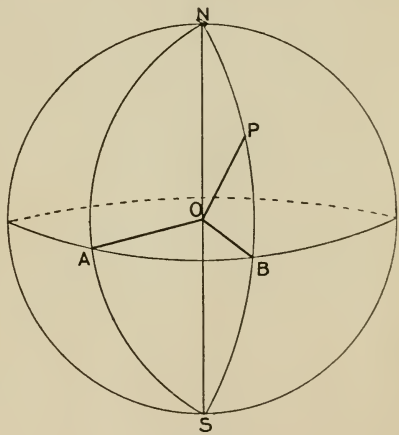


Fig. V.

In Fig. V the meridian through Greenwich cuts the equator at A; the meridian through P cuts at B. O is the centre of the Earth.

The **Longitude** of P is the angle AOB and may be either East or West of the Greenwich meridian.

The **Latitude** of P is the angle POB and may be either North or South of the Equator.

Note.—A geographical or nautical mile is the length of an arc of a meridian (or of the equator) subtending an angle of 1′ at the centre of the earth.

A ship travelling at the rate of 1 nautical mile per hour is said to have a speed of one knot.

11. Gradient. It is usual to estimate the inclination to the horizontal of a road or hill by the distance risen vertically for a certain horizontal distance. Thus a hill might be said to rise 3 in 5; this would mean that if a horizontal line were drawn through a point B on the hill to meet the vertical line through a lower point A at C, then AC/BC would equal $\frac{3}{5}$. The hill is said to have a **gradient** or **slope** of 3 in 5.

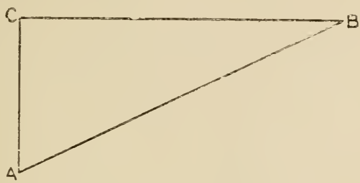


Fig. VI.

It is clear that in many cases it is easier to measure AB than BC; and some books take a gradient of 3 in 5 to mean a rise of 3 vertically for a distance 5 measured along the incline; so that in the figure AC/AB would be $\frac{3}{5}$. This latter interpretation of gradient is very common in books on Theoretical Mechanics.

If the inclination is small, it makes no practical difference which interpretation of gradient is taken.

It should be noticed that the angle is the same whatever units be used; that is whether we consider a rise of 3 inches in 5 inches, 3 furlongs in 5 furlongs, 3 miles in 5 miles. This follows from Prop. 12 c.

Examples I c.

1. Express in degrees the angle between

(*a*) NNE. and E. by N.;
(*b*) W. by S. and SE. by N.;
(*c*) ESE. and NE. by N.;
(*d*) NNW. and SSE.;
(*e*) N. by W. and SW.;
(*f*) S. 2 points W. and W. 2 points S.;
(*g*) 40° N. of W. and 30° E. of S.;
(*h*) NE. by E. and 1 point W. of N.;
(*i*) 30° S. of W. and ESE.;
(*k*) S. 2 points W. and W. 2 points N.

In the following questions take the radius of the Earth to be 4000 miles.

2. Two places on the Equator are 300 miles apart, find the difference of their Longitudes.

3. Quito (Longitude 79° W.) and Macapa (Longitude $51\frac{1}{2}°$ W.) are both on the Equator, find the distance between them. What time is it at Macapa when it is noon at Quito?

4. Find the distance between Poole (Lat. 50° 43′ N., Long. 1° 59′ W.) and Berwick (Lat. 55° 46′, Long. 1° 59′ W.).

5. Find the distance between Cape Breton Island (Lat. 45° 50′ N., Long. 60° W.) and the Falkland Isles (Lat. 51° 32′ S., Long. 60° W.).

Oral Questions.

1. What is a degree? How many degrees are there in the angle of a regular pentagon?

2. How big is each acute angle of an isosceles right-angled triangle?

3. One angle of a triangle is $A°$, another 30°, how big is the third angle?

4. What is meant by a negative angle? When screwing an ordinary screw in, is the turning in the positive or negative direction?

5. Does the earth rotate in the positive or negative direction? In which direction does the sun appear to move?

6. Do you usually draw a circle in the positive or negative direction?

7. The needle of a mariner's compass is deflected from its normal position through a positive angle $33\frac{3}{4}$ degrees, to what point of the compass does it then point?

8. Express the following angles in circular measure: 90°, 60°, 180°, 45°, 30°. (Give the answers in terms of π.)

9. What is the locus of all places having latitude 35° N.?

10. What is the locus of all places having longitude 15° W.?

11. It is noon at the same time at two different places, what do you know about their longitudes or latitudes?

12. Give the latitude and longitude of the N. pole.

Examples I.

1. In a triangle ABC, $A = 43° 15'$, $B = 67° 38'$, calculate the number of degrees in (i) the angle C, (ii) the angle subtended at

the centre of the circumcircle by the side BC, (iii) the angle subtended at the centre of the inscribed circle by the side BC.

2. Express in circular measure, correct to 3 significant figures, (*a*) the supplement of 1·37 radians, (*b*) 74°, (*c*) the angle of a regular octagon.

3. Define a radian and a grade. If an angle, containing D degrees, may be expressed as either θ radians or G grades, prove that $D/180 = \theta/\pi = G/200$.

4. The hands of a clock are coincident at noon, through what angle does the hour hand turn before they next coincide?

5. Prove that whatever be the radius of a circle the size of the angle at the centre, which subtends an arc equal to the radius, is constant. What is this angle called? Show, by a geometrical construction, that it is a little less than 60°.

6. A wheel of a cart is 4 feet in diameter, through what angle does it turn when the cart moves forward 10 feet?

7. Explain how to find the length of a circular arc being given the number of degrees in the angle subtended at the centre and the length of the radius.

8. Two places on the Equator differ in longitude by 37° 16′, find the distance between them, correct to three significant figures. (Radius = 4000 miles.)

9. Find the distance between a place, longitude 45° 17′ E., latitude 0°, and another place, longitude 38° 43′ W., latitude 0°.

10. Through what angle does the Earth turn between 9.30 a.m. and 4 p.m.?

11. When it is noon at Greenwich what time is it at (*a*) Calcutta (88° 15′ E.), (*b*) New York (74° W.), (*c*) Hawaii (156° W.)?

12. The co-ordinates of two points P and Q are (7, 8), (9, 11) respectively, find the gradient of the line PQ.

13. Draw an angle $AOP = 35°$, in OP take 3 points P, Q, R such that $OP = 1$ inch, $OQ = 1·7$ inch, $OR = 2·3$ inches. From P draw PH at right angles to OA, at Q draw QK at right angles to OQ, and from R let fall RL perpendicular to OA. Measure OH, OK, OL and calculate, to 3 decimal places, the ratios OH/OP, OQ/OK, OL/OR. Justify the result.

14. Explain what is meant by a radian, and find how many degrees and minutes it contains.

Express in degrees, and also in radians, the angle of a regular polygon of 100 sides.

15. An explorer reaches a latitude of $87° 28' 48''$. Find how many miles he is distant from the pole, assuming the earth to be a sphere whose circumference is 25000 miles.

16. Find the gradient of a straight line joining two points whose co-ordinates are (x', y') and (x'', y''). Hence find the equation of the straight line.

CHAPTER II

THE TRIGONOMETRICAL RATIOS

12. Definitions. Let OA the initial line be taken as axis of x, the axis of y being perpendicular to it at O; in the final

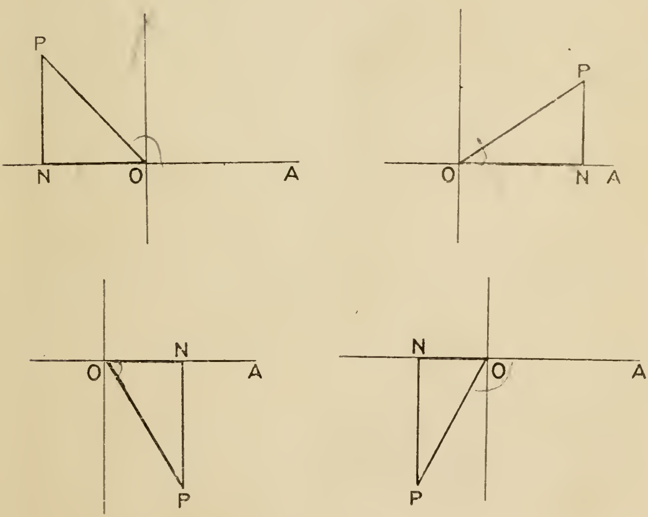


Fig. VII.

position of the rotating line take any point P. Figure VII shows four possible types of positions of OP.

Let fall PN perpendicular to OA or OA produced, so that ON is the abscissa of P and PN the ordinate. Then

$$\text{sine of AOP} = \frac{NP}{OP} = \frac{\text{ordinate}}{\text{radius}} = \frac{y}{r},$$

$$\text{cosine of AOP} = \frac{ON}{OP} = \frac{\text{abscissa}}{\text{radius}} = \frac{x}{r}.$$

$$\text{tangent of AOP} = \frac{NP}{ON} = \frac{\text{ordinate}}{\text{abscissa}} = \frac{y}{x}.$$

These are the most important ratios; the others are their reciprocals, viz.:

$$\text{cosecant of AOP} = \frac{OP}{NP} = \frac{\text{radius}}{\text{ordinate}} = \frac{r}{y},$$

$$\text{secant of AOP} = \frac{OP}{ON} = \frac{\text{radius}}{\text{abscissa}} = \frac{r}{x}.$$

$$\text{cotangent of AOP} = \frac{ON}{NP} = \frac{\text{abscissa}}{\text{ordinate}} = \frac{x}{y}.$$

The following abbreviations are usually used:

sin A instead of sine of AOP,
cos A ,, ,, cosine of AOP,
tan A ,, ,, tangent of AOP,
cosec A ,, ,, cosecant of AOP,
sec A ,, ,, secant of AOP,
cotan A ,, ,, cotangent of AOP.

Similarly, if AOP is measured in radians, sin θ, cos ϕ, cosec ψ, &c., are used.

13. Trigonometry was developed by Arabian and Greek astronomers who based their work on the circular arc and not on the angle. In the Middle Ages this early mathematical work was translated into Latin, and so the present names of the ratio were derived. The following section shows the reasons for these names.

14. Draw a circle with centre O cutting the initial line at A and the perpendicular to it at B.

Take a point P on the circumference of the circle.

Draw the tangent at A and produce OP to meet it at T.

Draw PN perpendicular to OA.

AT was called the **tangent** of the arc AP.

OT, which cuts the circle, was called the **secant** of the arc AP.

. NP was called the sine * of the arc AP.

Clearly the lengths of AT, OT, NP change when the radius OA changes, even if the angle AOP remain constant.

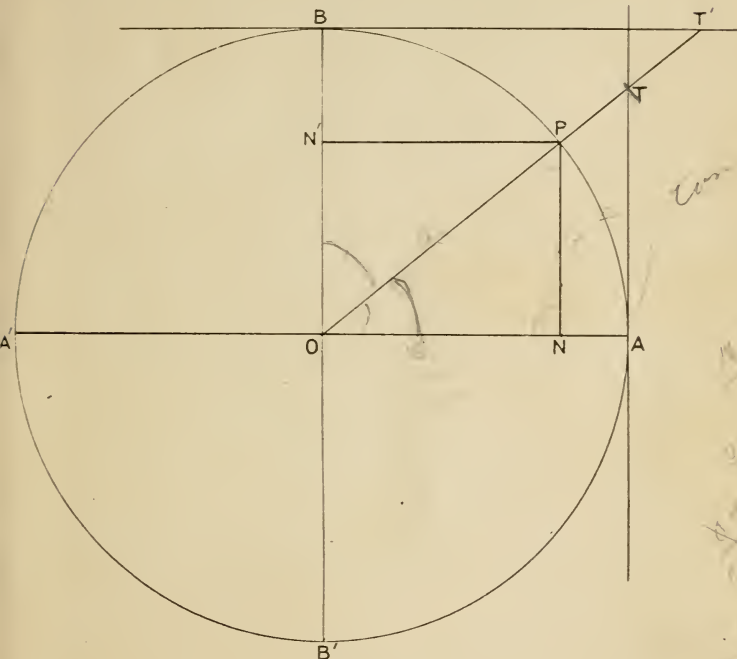


Fig. VIII.

But from similar triangles it is seen that, if the angle is constant, the ratios of AT, OT, NP to the radius are also constant. Hence, as Trigonometry developed, it was seen to be advisable to divide by

* The word 'sine' is derived from the Latin *sinus*. If in Fig. VIII PN be produced to meet the circumference at P', then PAP' resembles a bow (Latin *arcus*) of which PNP' is the string or chord (Latin *chorda*). To use the bow, the string is pulled till N touches the bosom (Latin *sinus*); hence PN is called the sine. NA is often called the sagitta of the arc.

the radius and to treat the subject as depending on the angle AOP rather than on the arc AP. Thus we have

$$\text{angle } AOP = \frac{\text{arc } AP}{\text{radius}} \text{ (when the angle is measured in radians),}$$

$$\sin AOP = \frac{\text{sine of arc } AP}{\text{radius}} = \frac{NP}{OP},$$

$$\tan AOP = \frac{\text{tangent of arc } AP}{\text{radius}} = \frac{AT}{OA} = \frac{NP}{ON},$$

$$\sec AOP = \frac{\text{secant of arc } AP}{\text{radius}} = \frac{OT}{OA} = \frac{OP}{ON}.$$

Now make a similar construction for the complementary arc BP. Then

sine of the complement of the angle AOP

$$= \frac{\text{sine of the complementary arc } BP}{\text{radius}}$$

$$= \frac{N'P}{OP}$$

$$= \frac{ON}{OP}.$$

'Sine of the complement of' was shortened into co-sine. Possibly 'complementary sine' was an intermediate stage. Similarly, co-tangent and cosecant were derived.

Since the values of the ratios depend on the values of the angle, the term **Trigonometrical Functions** is often used instead of Trigonometrical Ratios. Frequently the ratios are referred to as **Circular Functions.**

15. Ratios rarely used. In Fig. VIII

NA is called the versine (i. e. versed sine) of the arc AP.

$N'B$ is called the coversine of the arc AP.

AP (not joined in the figure) is the chord of the arc AP.

If we divide each of these by the radius we get the corresponding ratios of the angles AOP. These ratios are very rarely used. Another function that is now rarely used is the haversine, i. e. half the versed sine.

16. Projection Formulae. It is useful to remember that

ON (i. e. the projection of the radius on the initial line) $= r \cos \theta$

and NP (i. e. the projection of the radius on a line perpendicular to the initial line) $= r \sin \theta$.

17. Polar co-ordinates. The position of a point P is determined if the distance OP from a fixed point is known and also the angle this distance makes with a fixed line OX through O. The length is usually denoted by r and the angle by θ; these are the **polar co-ordinates** of P. In this connexion O is called the **pole.**

18. Graphs of Trigonometrical Functions.

The definitions of the last section lead to an easy method of drawing the graphs. On page 42 the sine graph (i.e. the graph of the equation $y = \sin x$) is given. It is obtained as follows:

Step 1. On the extreme left of the paper (which should be ruled in squares) draw a circle with its centre at the intersection of two lines. Take the horizontal radius CA as initial line.

Step 2. The perpendicular $B'CB$ gives the angles 90° and 270°.

The diagonals through C give the angles 45°, 135°, 225°, 315°.

By stepping off chords equal to the radius, starting from A, the angles 60°, 120°, 240°, &c., are obtained; and, by starting at B, the angles 30°, 150°, &c., are obtained.

Only the points P, P', . . ., on the circumference need be obtained as is shown in the third quadrant; the radii are not needed for drawing the sine graph.

Step 3. Take a point O as origin, some distance along the initial line, and, with a convenient scale, mark off abscissae to represent the angles 30°, 45°, 60°, &c., and, as far as space allows, mark off the negative abscissae.

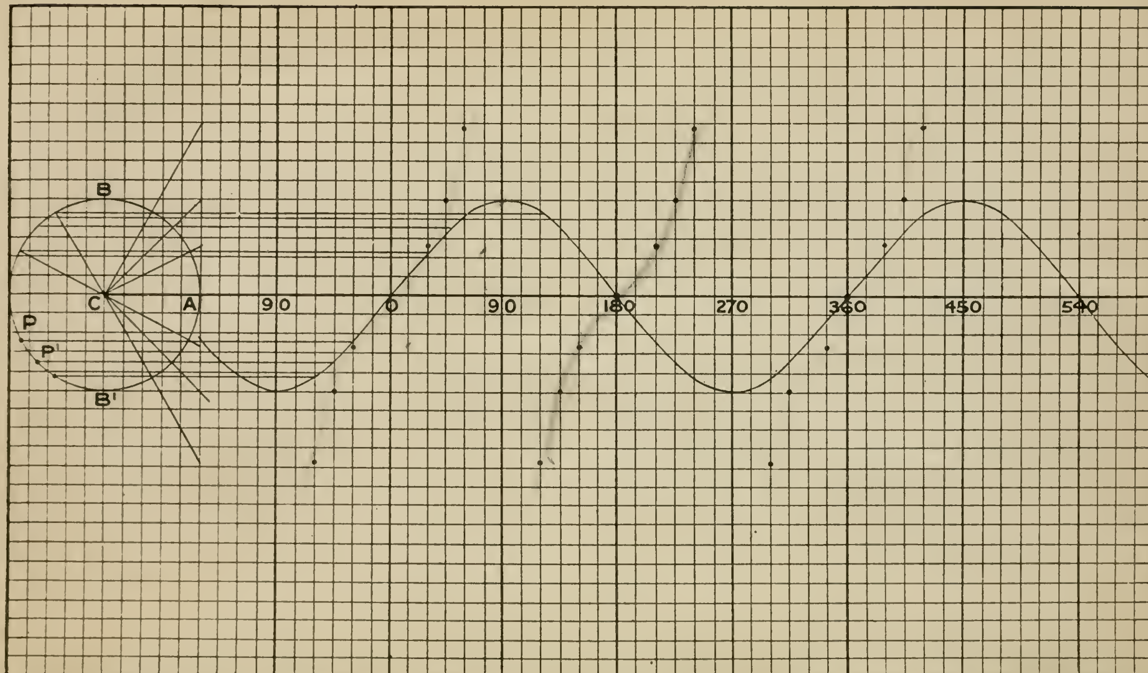
Step 4. Through the points on the circumference draw parallels to the initial line to cut the corresponding ordinates. These points of intersection are points on the graph.

The ordinates of this graph* are proportional to the sines; if we divide by the radius, the actual values of the sines are found.

The sine graph is shown on a larger scale in Fig. X.

19. The tangent graph. To obtain the ordinates for the tangent graph the radii must be produced to meet the tangent to

* The graph of the sine is a wavy or sinuous curve. The name sine is therefore appropriate, although it is improbable that the originators of the name ever drew the graph.



Scale:—for abscissae, side of small square = 15°; for ordinates, side = ·2.

Fig. IX.

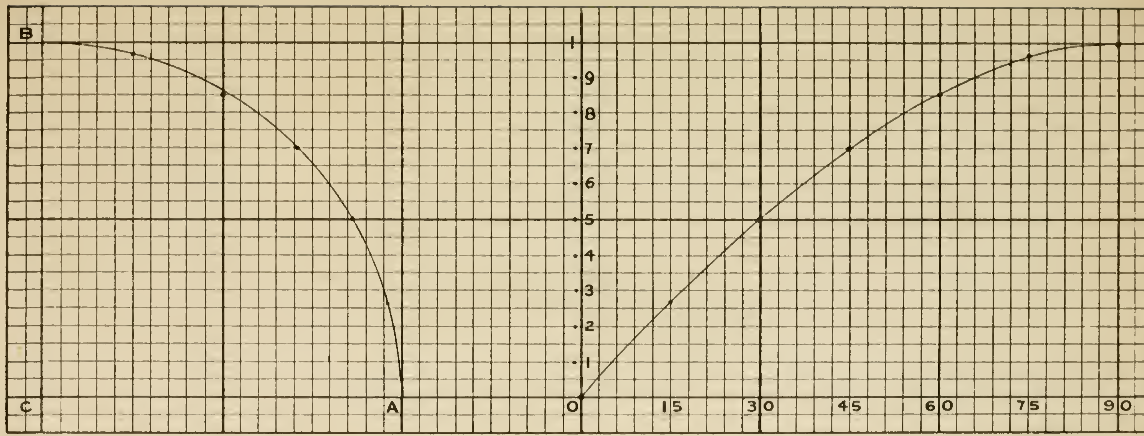


Fig. X.

the circle drawn at A. This is done in Fig. IX, and a few points of the graph are marked, but the graph is not drawn.

The cosine graph. Since the cosine of x is the sine of the complement of x, the student should be able to modify the method for the sine graph so as to obtain the cosine graph.

The secant graph. This is obtained by marking off along the respective ordinates the corresponding values of CT (see Fig. IX).

Examples II a.

(Answers should be given correct to 2 significant figures.)

By drawing to scale find the trigonometrical ratios of the following angles:

1. 30°.	2. 49°.	3. 79°.	4. 100°.
5. 78°.	6. 170°.	7. 250°.	8. 25°.
9. 300°.	10. 156°.	11. −80°.	12. 415°.

Find the ratios of the angle AOP when the coordinates of P are

13. (4, 3).	14. (4, −3).	15. (−4, −3).	16. (−4, 3).
17. (3, 2).	18. (−7, −3).	19. (−5, 4).	20. (63, −16).

The following graphs should be drawn carefully and kept for use:

21–26. A graph on a large scale for each function, for angles from 0° to 90°.

27–32. A graph on a smaller scale for each function for angles from −360° to +360°.

33. The blanks in the following table are to be filled with the sign (+ or −) of the respective ratios:

Angle	0°−90°	90°−180°	180°−270°	270°−360°
sine				
cosine				
tangent				

34. If the gradient of a hill, inclined at $A°$ to the horizon, is known, what trigonometrical ratio of the angle is known?

35. Construct an angle whose (i) tangent is 1·45, (ii) sine is ·75, (iii) cotangent is 1·45, (iv) secant is 2·7, (v) cosecant is 2·7, (vi) cosine is ·75. Measure each angle in degrees.

20. Powers of the Trigonometrical Functions. The square of $\sin A$ is written $\sin^2 A$; and a similar notation is used for other powers and ratios; thus, **in general,**

$$\sin^n A \text{ means } (\sin A)^n.$$

Inverse notation. There is one exception to the above statement. Suppose $\sin A = a$, then A is an angle whose sine is a. This is written $A = \sin^{-1} a$. Similarly, $\tan^{-1} a$ means an angle whose tangent is a; and so for the other ratios.

If we wish to express $\frac{1}{\sin A}$ as a power of $\sin A$, we must write $(\sin A)^{-1}$.

Note. Continental mathematicians denote the angle whose sin is x by arc sin x. This notation sometimes occurs in English books.

Example. *Determine, by drawing, the angle* $\sin^{-1} \frac{2}{3}$.

Step 1. Draw axes OA, OB.

Step 2. Draw circle centre O, radius 3 units.

Step 3. Along OB mark off OK equal 2 units.

Step 4. Through K draw a parallel to OA cutting circle at $P'OP$. Join OP, OP'.

We now have two angles AOP, AOP' each of which has its sine equal to $\frac{2}{3}$. AOP is 41°, AOP' 139°.

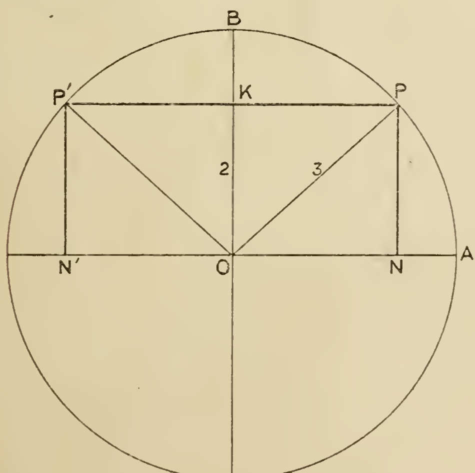


Fig. XI.

It is always understood that $\sin^{-1} a$ means the smallest positive angle that has the sine equal to a; and similarly for the other ratios.

Examples II.

1. Find, by drawing to scale, the sine, cosine, and tangent of 30°, 45°, 60°. Verify the results by calculation.

2. The sine of an acute angle is $\frac{3}{5}$; find the cosine, tangent, and secant.

3. The sine of an angle, not acute, is $\frac{3}{5}$; find the cosine and tangent.

4. The cosine of an angle is $\frac{5}{13}$; find the sine and tangent.

5. Draw as many angles as possible having the tangent equal to ·8.

6. Given that sin 63° = ·89 find cos 63° and cos 27°.

7. Find the value of $\sin^2 A + \cos^2 A$, it being known that

$$\sin A = \cdot 3907, \cos A = \cdot 9205.$$

Also find the values of $\tan A$ and $\sec A$.

8. Given $\tan \theta = \frac{12}{5}$, find $\cot \theta$ and $\sec \theta$.

9. Draw and measure an angle A such that (i) $\sin A = -\cdot 5$, (ii) $\cos A = -\cdot 5$, (iii) $\tan A = -\cdot 5$, (iv) $\sec A = -\cdot 5$.

10. Find the value of $\sec^2\theta - \tan^2\theta$, when $\sec\theta = 1\cdot 221$ and $\tan\theta = \cdot 7002$. Justify the answer by geometrical reasoning.

11. Are any of the following data inconsistent or impossible? Give reasons for your answers.

(a) $\sin A = \frac{5}{4}$; (b) $\sec A = \frac{5}{4}$;
(c) $\sin A = \frac{4}{5}$, $\cos A = \frac{3}{7}$; (d) $\sin A = \cdot 4$, $\cos A = \cdot 6$;
(e) $\sin A = \cdot 6$, $\cos A = \cdot 8$, $\tan A = \cdot 9$;
(f) $\sec A = \cdot 35$, $\tan A = 1\cdot 35$;
(g) $\tan A = 1$; (h) $\sin A = 1$; (i) $\operatorname{cosec} A = 1$.

12. Prove, by means of the definitions in § 12, that

$$\cos A = \sin (90 - A) \text{ and } \tan (\tfrac{1}{2}\pi - \theta) = \cot \theta.$$

13. Find, by drawing to scale, (a) sin 36° and sin 144°; (b) cos 42° and cos 138°; (c) cos 246° and cos 66°.

14. By means of graphs (or otherwise) test the following statements: (a) $\sin(180 - A) = \sin A$; (b) $\cos(180 + A) = -\cos A$; (c) $\sin(90 + A) = \sin A$.

15. By means of graphs find values for $\sin^{-1} \cdot 6$, $\tan^{-1} 2\cdot 5$ $\cos^{-1} \cdot 34$, $\cos^{-1} 1\cdot 5$.

16. Given $\sin 36° = \cdot 5878$, find $\cos 54°$, $\sin 144°$, $\sin 216°$, $\sin 324°$.

17. Given $\cos 53° = \cdot 6018$, find $\sin 37°$, $\cos 127°$, $\cos 233°$, $\cos 413°$, $\cos 307°$.

18. Prove that $\sin 117° = \cos 27°$.

19. Is it possible to find angles to satisfy the following equations? Give reasons.

(i) $\tan\theta = 1$; (ii) $\cos\theta = \dfrac{\cdot 934}{\cdot 866}$;

(iii) $\sin\theta + \cos\theta = 1$; (iv) $\sin^2\theta + \cos^2\theta = \frac{1}{2}$;

(v) $\sec\theta = 3\cdot 1416$; (vi) $\operatorname{cosec}\theta = \dfrac{1}{\sqrt{2}}$;

(vii) $\sin\theta = 0$; (viii) $\tan\theta = 100$;

(ix) $\cos\theta = 1$; (x) $\sec\theta = \cdot 78$.

20. Show that (i) sine and cosine cannot be numerically greater than 1; (ii) tangent and cotangent may be either greater or less than 1; (iii) secant and cosecant cannot be numerically less than 1. Why is the word numerically inserted?

21. Find all the trigonometrical functions of 0° and 90°.

22. (i) Show that the straight line whose equation is $y = mx$ makes an angle $\tan^{-1} m$ with the axis of x.

(ii) What is the tangent of the angle made with the axis of x by the straight line joining the two points (x_1, y_1) and (x_2, y_2)?

(iii) Show that the equation of the line joining the two points (x_1, y_1), (x_2, y_2) is $\dfrac{y - y_1}{y_1 - y_2} = \dfrac{x - x_1}{x_1 - x_2}$.

(iv) If the equation of a straight line is $y = mx + c$, give the geometrical meanings of m and c.

23. Show that, if x be any numerical quantity, positive or negative, an angle can be found whose tangent is equal to x.

Show what limitations in value, if any, exist in the case of each of the other trigonometrical ratios.

24. State concisely the changes in the sign and magnitude of $\sin A$ as A increases from 0° to 360°.

25. Define the cosine of an angle of any magnitude, explaining the conventions regarding the signs of the lines referred to in your definitions. Draw the graph of $\cos\theta$ from $\theta = 0$ to $\theta = \frac{1}{2}\pi$.

26. Define the sine of an angle and find by geometrical reasoning the values of $\sin 45°$, $\sin 90°$, $\sin 135°$.

27. Define the tangent and the versed sine of an angle; and find the greatest and least values which each can have.

28. With ruler and compasses construct an angle whose cosine is $\frac{1}{3}$; also an angle whose cosine is $-\frac{1}{3}$. Calculate the sine of the latter angle to three places of decimals.

29. ABC is a triangle in which AN is the perpendicular from A to BC. If $AB = 2·9$ inches, $AC = 2·5$ inches, $AN = 2$ inches, find the values of $\sin B$, $\cos C$, $\tan B$, $\operatorname{cosec} C$. Calculate the length of BC correct to one decimal place.

30. If A, B, C are the angles of a triangle, express $\sin \frac{1}{2}(A+B)$, $\cos \frac{1}{2}(A+B)$, $\tan \frac{1}{2}(A+B$ in terms of ratios of $\frac{1}{2}C$.

CHAPTER III

ELEMENTARY FORMULAE

21. Reciprocal Relations.

By definitions, $\sin A = \frac{y}{r}$, $\operatorname{cosec} A = \frac{r}{y}$;

$\therefore \quad \sin A \cdot \operatorname{cosec} A = 1$:

i. e. $\sin \mathbf{A} = \dfrac{1}{\operatorname{cosec} \mathbf{A}}$, $\operatorname{cosec} \mathbf{A} = \dfrac{1}{\sin \mathbf{A}}$.

In a similar way it can be proved that

$$\cos A \sec A = 1, \text{ \&c.}$$

$$\tan A \cot A = 1, \text{ \&c.}$$

22. Relations deduced from Pythagoras' Theorem (Prop. 13, p. 10).

In Fig. VII, § 12, we have in all cases

$$ON^2 + NP^2 = OP^2;$$

i. e. $x^2 + y^2 = r^2$.

Three sets of formulae are obtained by dividing in turn by r^2, x^2, y^2.

Divide by r^2, $\dfrac{x^2}{r^2} + \dfrac{y^2}{r^2} = 1$;

but $\dfrac{x}{r} = \cos A$, $\dfrac{y}{r} = \sin A$.

Substitute, $\cos^2 \mathbf{A} + \sin^2 \mathbf{A} = 1$.

The equivalent formulae must also be learnt, viz.:

$$\sin^2 A = 1 - \cos^2 A, \ \sin A = \pm \sqrt{1 - \cos^2 A};$$

$$\cos^2 A = 1 - \sin^2 A, \ \cos A = \pm \sqrt{1 - \sin^2 A}.$$

In a similar way the student should prove that

$$\tan^2 \mathbf{A} + 1 = \sec^2 \mathbf{A},$$

and $\cot^2 \mathbf{A} + 1 = \operatorname{cosec}^2 \mathbf{A}$.

23. Relation between sine, cosine, and tangent.

$$\tan A = \frac{y}{x}$$

$$= \frac{\frac{y}{r}}{\frac{x}{r}}$$

Substitute, $$\tan A = \frac{\sin A}{\cos A}.$$

In a similar way it is proved that

$$\cot A = \frac{\cos A}{\sin A}.$$

24. Identities. By means of the relations proved in the preceding sections, any expression containing trigonometrical functions can be put into a number of forms. It is a useful exercise to prove that two expressions, apparently different, are identical; such exercises serve to fix the relations in the memory and lead to facility in dealing with trigonometrical expressions.

Example. *Prove that* $\sec^2 A + \operatorname{cosec}^2 A \equiv \sec^2 A \operatorname{cosec}^2 A$.

[Express all ratios in terms of sine and cosine.]

Method I.

$$\text{L. H. S.} = \frac{1}{\cos^2 A} + \frac{1}{\sin^2 A} \qquad \text{by § 21}$$

$$= \frac{\sin^2 A + \cos^2 A}{\sin^2 A \cos^2 A}$$

$$= \frac{1}{\sin^2 A \cos^2 A} \qquad \text{using Formula of § 22}$$

$$= \frac{1}{\cos^2 A} \times \frac{1}{\sin^2 A}$$

$$= \sec^2 A \operatorname{cosec}^2 A. \qquad \text{by § 21}$$

Q. E. D.

Method II.

$$\sec^2 A + \operatorname{cosec}^2 A = \frac{1}{\cos^2 A} + \frac{1}{\sin^2 A} = \frac{\sin^2 A + \cos^2 A}{\sin^2 A \cos^2 A} = \frac{1}{\sin^2 A \cos^2 A},$$

$$\sec^2 A \operatorname{cosec}^2 A = \frac{1}{\cos^2 A} \times \frac{1}{\sin^2 A} = \frac{1}{\sin^2 A \cos^2 A};$$

$$\therefore \quad \sec^2 A + \operatorname{cosec}^2 A = \sec^2 A \operatorname{cosec}^2 A. \qquad \text{Q.E.D.}$$

Method III. This method is clumsy, and should be used only if Methods I and II have been tried unsuccessfully.

$$\sec^2 A + \operatorname{cosec}^2 A = \sec^2 A \operatorname{cosec}^2 A,$$

if $$\frac{1}{\cos^2 A} + \frac{1}{\sin^2 A} = \frac{1}{\cos^2 A} \cdot \frac{1}{\sin^2 A},$$

i. e. if $\sin^2 A + \cos^2 A = 1.$

But $\sin^2 A + \cos^2 A$ does equal 1 ;

$$\therefore \quad \sec^2 A + \operatorname{cosec}^2 A = \sec^2 A \operatorname{cosec}^2 A.$$

Note. The introductory 'if', or some similar conjunction, is vital to the logical statement of the work and must not be omitted.

25. Elimination. If two equations are satisfied by the same value of a single variable, there must be a relation connecting the constants of the equations; this is also the case when n equations are satisfied by the same values of $n-1$ variables. In order to find this relation we eliminate the variable or variables.

Example. *Eliminate θ from the equations $\sin\theta = a$, $\tan\theta = b$.*

By formulae $$\tan^2\theta = \frac{\sin^2\theta}{\cos^2\theta} = \frac{\sin^2\theta}{1-\sin^2\theta}.$$

Substitute $$b^2 = \frac{a^2}{1-a^2},$$

i. e. $$\frac{1}{a^2} - \frac{1}{b^2} = 1.$$

The result is called the *eliminant* of the original equations.

Examples III a.

1 If $\sin A = \frac{16}{65}$ use formulae to find the remaining ratios. Draw a figure to explain why some of the ratios may be either positive or negative.

2. Given that $\tan\theta = \frac{12}{5}$ find $\cot\theta$ and $\sin\theta$.

3. Find $\sec\theta$ in the following cases:

(i) $\cos\theta = \cdot 7921$; (ii) $\tan\theta = 1\cdot 352$; (iii) $\operatorname{cosec}\theta = 2\cdot 583$.

4. Show how all the ratios may be found when (i) the cosine, (ii) the tangent is known.

5. Prove the following identities :

(i) $\sin A\cot A + \cos A\tan A = \sin A + \cos A$:

(ii) $\tan A + \cot A = \sec A \operatorname{cosec} A$;

(iii) $\sin\theta\tan\theta + \cos\theta\cot\theta = \sec\theta + \operatorname{cosec}\theta - \sin\theta - \cos\theta$;

(iv) $\sec^2\theta - \operatorname{cosec}^2\theta = \tan^2\theta - \cot^2\theta$;

(v) $1 - 2\sin^2 A = 2\cos^2 A - 1$;

(vi) $\dfrac{\tan\alpha + \tan\beta}{1 - \tan\alpha\tan\beta} = \dfrac{\sin\alpha\cos\beta + \cos\alpha\sin\beta}{\cos\alpha\cos\beta - \sin\alpha\sin\beta}$;

(vii) $(1 - \tan^2 A) \div (1 + \tan^2 A) = \cos^2 A - \sin^2 A$;

(viii) $(\sin A + \cos A)^2 = 1 + 2\sin A\cos A$;

(ix) $\sin^3 A - \cos^3 A = (\sin A - \cos A)(1 + \sin A\cos A)$;

(x) $\sin A\cot A \operatorname{cosec} A + \cos A\tan A\sec A = \sec A\operatorname{cosec} A$;

(xi) $\tan X - \tan Y = (\sin X\cos Y - \cos X\sin Y) \div \cos X\cos Y$;

(xii) $(\tan A - \tan B) \div (\cot A - \cot B) = -\tan A\tan B$;

(xiii) $\cos^4\theta - \sin^4\theta = 2\cos^2\theta - 1$;

(xiv) $(3 - 4\sin^2 A) \div \cos^2 A = 3 - \tan^2 A$.

6. Prove that versin $A = 1 - \cos A$, coversin $A = 1 - \sin A$.

7. Show that the numerical value of $\sin^2 A \div (1 - \cos A)$ diminishes from 2 to 0 as A increases from 0° to 180°, and illustrate your answer by a diagram.

8. Which is greater, the acute angle whose cotangent is $\frac{4}{5}$, or the acute angle whose cosecant is $\frac{5}{4}$?

9. Prove that, if θ is an angle less than 180° for which $1 + \sin\theta = k\cos\theta$, then $\cos\theta = 2k \div (1 + k^2)$; and express $\tan\theta$ in terms of k.

10. Eliminate θ from the following :

(i) $\sin\theta = a$, $\cos\theta = b$;

(ii) $\sin\theta + \cos\theta = a$, $\sin\theta - \cos\theta = b$;

(iii) $\sec\theta - \tan\theta = a$, $\sec\theta + \tan\theta = b$;

(iv) $a \sin\theta + b\cos\theta = p$, $a\sin\theta - b\cos\theta = q$;

(v) $a\sin\theta + b\cos\theta = p$, $a'\sin\theta + b'\cos\theta = p'$.

11. If $a(1-\sin\theta) = b\cos\theta$, prove that $b(1+\sin\theta) = a\cos\theta$.

12. If $a(\sec\theta + 1) = b\tan\theta$, prove that $b(\sec\theta - 1) = a\tan\theta$.

13. If $x = a\cos\theta\cos\phi$, $y = a\cos\theta\sin\phi$, $z = a\sin\theta$, eliminate θ and ϕ.

26. Ratios of complementary angles.

Let $XOP = A°$ (Fig. XII) and $XOQ = 90 - A°$; make $OQ = OP$, and let fall QK, PN perpendicular to OX.

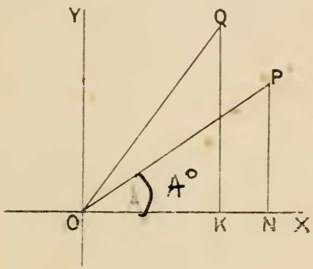


Fig. XII.

Then the triangles QOK, PON are congruent (Prop. 8 c); so that $KQ = ON$ and $OK = NP$.

$$\text{Hence}\quad \sin XOQ = \frac{KQ}{OQ} = \frac{ON}{OP} = \cos XOP.$$

i.e. $\mathbf{\sin(90 - A) = \cos A}$.

In a similar way it is proved that *

$\mathbf{\cos(90 - A) =}$

$\mathbf{\tan(90 - A) =}$

Compare these results with § 11.

What are the corresponding results when angles are measured in radians?

27. Ratios of supplementary angles.

Make $XOP = A°$, and $XOQ = 180 - A°$, so that $QOK = A°$.

* The student is expected to complete these formulae.

Make $OQ = OP$ and let fall the perpendiculars PN, QK.

Then the triangles QOK, PON are congruent (Prop. 8 *c*) so that $OK = ON$ (in magnitude) and $KQ = NP$. But OK and ON are of opposite sign.

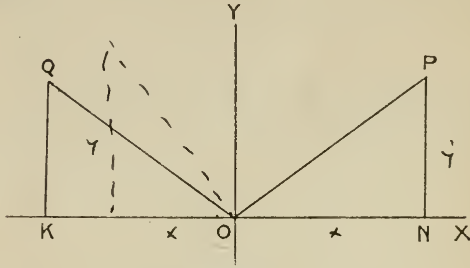


Fig. XIII.

Hence $\cos XOQ = \dfrac{(OK)}{OQ}$ * (where (OK) denotes the magnitude of OK with the proper sign prefixed)

$= \dfrac{-ON}{OP}$

$= -\cos XOP$,

i.e. $\cos(180 - A) = -\cos A$.

In a similar way it is proved that

$\sin(180 - A) =$

$\tan(180 - A) =$

28. Ratios of negative angles.

Make $XOP = +A°$ and $XOQ = -A°$.

Then $XOP = XOQ$ in magnitude.

Make $OQ = OP$.

Join PQ cutting OX at N.

Then in the triangles PON, QON

$OP = OQ$,
ON is common,
included angle NOP = included angle NOQ.

∴ $NP = NQ$ in magnitude,

and $ONP = ONQ$, so that PQ is perpendicular to OX.

* In writing it is usual to use the symbol $\overline{OK}$ to denote length preceded by correct sign; it is more convenient to print (OK).

$$\text{Hence} \quad \cos XOQ = \frac{(ON)}{OQ}$$
$$= \frac{(ON)}{OP}$$
$$= \cos XOP,$$

i.e. $\cos(-A) = \cos A.$

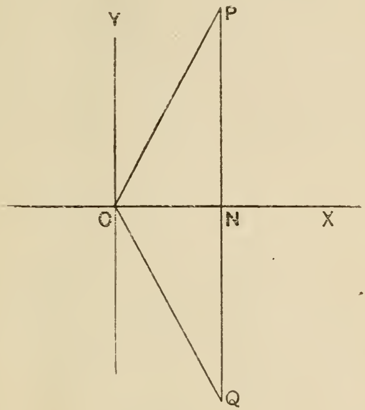


Fig. XIV.

In a similar way it is proved that

$\sin(-A) =$

$\tan(-A) =$

The student should also work out the ratio of $90 + A$, $180 + A$, $270 - A$, &c.

29. By means of the last three sections the ratios of any angle can be expressed in terms of the ratios of an acute angle not greater than 45°. For example

$$\cos 139^\circ = \cos(180^\circ - 41^\circ) = -\cos 41^\circ,$$
$$\cos 246^\circ = \cos(-114^\circ)$$
$$= \cos(114^\circ)$$
$$= \cos(180^\circ - 66^\circ)$$
$$= -\cos 66^\circ$$
$$= -\cos(90^\circ - 24^\circ)$$
$$= -\sin 24^\circ.$$

It is usually easy to work directly from the figure; thus in Fig. XV, where $XOP = 246°$ and $XOQ = 66°$,

$$\begin{aligned} \cos 246° &= \frac{(ON)}{OP} \\ &= -\frac{OK}{OQ} \\ &= -\cos 66° \\ &= -\sin 24°. \end{aligned}$$

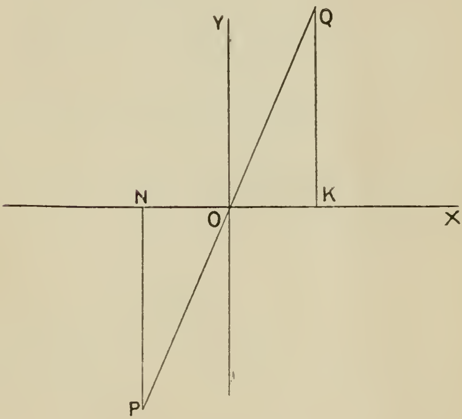


Fig. XV.

30. Ratios of 0° and 90°.

If $XOP = 0°$, P and N coincide; so that $NP = 0$, $ON = OP$.

Hence
$$\sin 0° = \frac{NP}{OP} = 0,$$
$$\cos 0° = \frac{ON}{OP} = 1,$$
$$\tan 0° = \frac{NP}{ON} = \frac{0}{ON} = 0.$$

If $XOP = 90°$, then PN falls along the y axis and N coincides with the origin O. In this case $NP = OP$ and $ON = 0$.

Hence
$$\sin 90^\circ = \frac{NP}{OP} = 1,$$
$$\cos 90^\circ = \frac{ON}{OP} = 0,$$
$$\tan 90^\circ = \frac{NP}{ON} = \frac{NP}{O} = \infty.*$$

31. Ratios of 30°, 45°, 60°.

Make XOP equal to 30°, Fig. XVI.

Let fall PN perpendicular to OX.

Make XOQ equal to 30° in magnitude, and produce PN to meet OQ in Q.

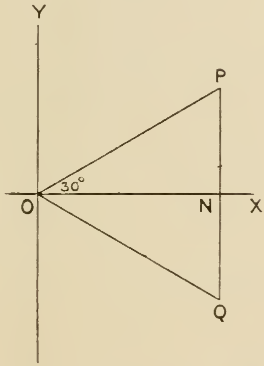


Fig. XVI.

Then, by Prop. 8 c, the triangles PON, QON are congruent. It follows that the triangle OPQ is equilateral.

* The symbol ∞ means 'infinity', i. e. a number greater than any number we can imagine.

Consider the value of $1/x$ as x gets smaller and smaller.

$$\frac{1}{\cdot 1} = 10, \quad \frac{1}{\cdot 001} = 1000, \quad \frac{1}{\cdot 000001} = 1000000.$$

As x diminishes, $1/x$ increases, and, by making x sufficiently small, we can make $1/x$ exceed any assigned value however great. This is expressed thus: when $x = 0$, $1/x = \infty$. Or more generally, if a is a constant, then $a/x = \infty$ when $x = 0$.

Hence $PN = \frac{1}{2}PQ$ since $PN = QN$

$= \frac{1}{2}OP$ since $PQ = OP$;

also $ON^2 = OP^2 - PN^2 = OP^2 - \frac{1}{4}OP^2 = \frac{3}{4}OP^2.$

$\therefore \quad ON = \frac{\sqrt{3}}{2}OP.$

Hence $\sin 30° = \frac{NP}{OP} = \frac{1}{2} = \cdot 5,$

$\cos 30° = \frac{ON}{OP} = \frac{\sqrt{3}}{2} = \cdot 866,$

$\tan 30° = \frac{NP}{ON} = \frac{1}{\sqrt{3}} = \cdot 577.$

Similarly $\sin 60° = \frac{\sqrt{3}}{2} = \cdot 866,$

$\cos 60° = \frac{1}{2} = \cdot 5,$

$\tan 60° = \sqrt{3} = 1 \cdot 732.$

As an exercise the student should find the values of the ratios of 45°.

$\sin 45° =$

$\cos 45° =$

$\tan 45° =$

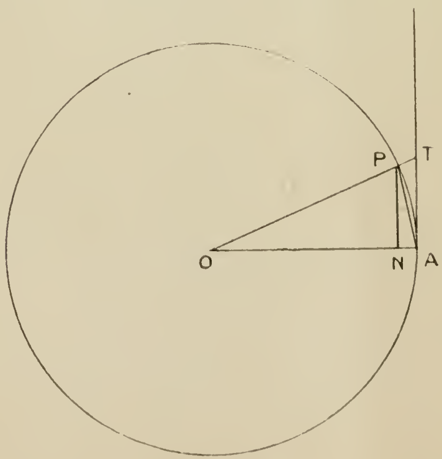


Fig. XVII.

32. The very small angle.

In Fig. XVII let the circular measure of the angle AOP be θ.

Then arc $AP = r\theta$, $NP = r \sin \theta$, $AT = r \tan \theta$.

Hence Area of triangle $AOP = \frac{1}{2} OA \, . \, NP$ (Prop. 16)

$= \frac{1}{2} r^2 \sin \theta$;

Area of sector $AOP = \frac{1}{2} r^2 \theta$; (§ 7)

Area of triangle $AOT = \frac{1}{2} OA \, . \, AT$

$= \frac{1}{2} r^2 \tan \theta$. (Prop. 16)

But, if AOP is an acute angle,

Triangle $AOP <$ sector $AOP <$ Triangle AOT,

i e. $\frac{1}{2} r^2 \sin \theta < \frac{1}{2} r^2 \theta < \frac{1}{2} r^2 \tan \theta$,

i. e. $\sin \theta < \theta < \tan \theta$.

This relation is true for any acute angle; if we multiply throughout by r we have

$r \sin \theta < r\theta < r \tan \theta$,

i. e. $NP <$ arc $AP < AT$.

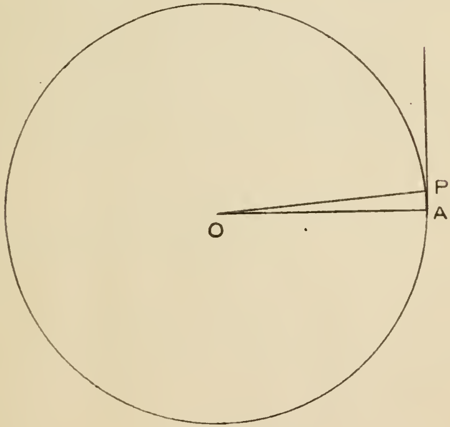


Fig. XVIII.

But, as the angle diminishes, these three lengths more and more nearly coincide ; and are practically indistinguishable when the angle is very small. This is shown in Fig. XVIII, which also shows that ON is indistinguishable from OA.

Hence, when θ is very small, there is very slight error in saying $NP =$ arc $AP = AT$, and $ON = OA$.

Substituting the trigonometrical values for the lengths of these lines, we have

$$\sin\theta = \theta = \tan\theta \quad \text{and} \quad \cos\theta = 1,$$

when θ, the circular measure of the angle, is small.

This may also be expressed thus: The limit of $\frac{\sin\theta}{\theta}$ or of $\frac{\tan\theta}{\theta}$, when θ is zero, is 1; or in symbols

$$\underset{\theta\to 0}{\mathrm{L}} \frac{\sin\theta}{\theta} = 1 \quad \text{and} \quad \underset{\theta\to 0}{\mathrm{L}} \frac{\tan\theta}{\theta} = 1.*$$

33. Error involved. Whatever be the value of θ, it has been shown that

$$\cos^2\theta + \sin^2\theta = 1.$$

Using the above approximations, we have

$$1 + \theta^2 = 1.$$

This last statement is true only when θ^2 is so small as to be negligible. Hence

If θ is so small that θ^2 may be neglected, we may say that

$$\sin\theta = \theta, \quad \cos\theta = 1, \quad \tan\theta = \theta.$$

It is shown in Higher Trigonometry that $\sin\theta = \theta$ gives correct results if $\frac{1}{6}\theta^3$ is negligible.

Example. *If accuracy is required to four decimal places, find the sine of* 1 *degree.*

$$1^\circ = \frac{1}{180}\pi \text{ radians} = \cdot 01745 \text{ radian}$$

$\cdot 01745^2 = \cdot 000295$. (We are not, therefore, justified in saying $\cos 1^\circ = 1$.)

$\cdot 01745^3 = \cdot 000005$. (This does not affect the first four places so we may use the approximation $\sin\theta = \theta$.)

Hence $\sin 1^\circ = \cdot 0175$, correct to four decimal places.

Examples III b.

1. Write down the sine, cosine, and tangent of
   (i) 150°, 240°, 330°, 840°;
   (ii) 60°, 300°, 135°, 225°;
   (iii) 180°, 270°, 405°, 210°.
2. Find the secant and cosecant of 60°, 45°, 120°, 225°.
3. Use the definitions of § 14 to find the ratios of $180 - A$.

* For explanation of the word 'limit' see *School Algebra*, Part ii, p. 446.

4. Correct, if necessary, the following statements :

$$\sin(180 - A) = \cos A;\ \cos(270 + A) = -\cos A;$$
$$\tan(180 + A) = \tan A;\ \sec(90 - A) = \sec A;$$
$$\cot(90 + A) = \cot A.$$

5. In a right-angled triangle the hypotenuse is 5 feet long and one of the angles is 60° ; find the lengths of the other two sides.

6. A ladder 25 feet long is leaning against a wall and is inclined 45° to the horizontal ; how far up the wall does it reach ?

7. Find sin 1′, correct to 3 significant figures.

8. Find sin 10′, cos 10′, tan 10′ correct to 5 decimal places.

9. What angle does a halfpenny (diameter 1 inch) subtend at the eye when at a distance of 10 feet ?

10. A post 25 feet high subtends an angle of 30′ at a certain point on the ground. How far from the post is the point ?

11. Find approximately the distance of a tower which is 51 feet high and subtends at the eye an angle $5\frac{3}{11}'$.

12. Prove that

$$\tan^2 60° - 2\tan^2 45° = \cot^2 30° - 2\sin^2 30° - \tfrac{3}{4}\operatorname{cosec}^2 45°.$$

13. Find approximately the number of minutes denoting the inclination to the horizon of an incline which rises $5\frac{1}{2}$ feet in 420 yards.

14. In any triangle show that

$$\cos(A+B) = -\cos C,\ \sin(B+C) = \sin A,\ \tan(B+C) = -\tan A.$$

Write down the other similar relations.

Oral Examples.

Fill in the right-hand sides of the following equalities:

1. (i) $\sin^2\theta =$
   (ii) $\sin 45° =$
   (iii) $\cos 135° =$
   (iv) $\tan\frac{1}{4}\pi =$
   (v) $\sin A \cot A$

2. (i) $\sec^2 A - \tan^2 A =$
   (ii) $\cos 60° =$
   (iii) $\tan\theta =$
   (iv) $\sin(180 - A) =$
   (v) $\sec(90 - B) =$

3. (i) $\cos^2 60° + \sin^2 60° =$
   (ii) $\operatorname{cosec}^2 C =$
   (iii) $\cot\frac{1}{6}\pi =$
   (iv) $\cos(180 - A) =$
   (v) $\cot^2\theta =$

4. (i) $\cos\theta\tan\theta =$
   (ii) $1 - \sin^2 x =$
   (iii) $\tan 210° =$
   (iv) $\cos^{-1}\frac{1}{2} =$
   (v) $\cos^2\frac{1}{7}\pi + \sin^2\frac{1}{7}\pi =$

5. (i) $1+\cot^2 A =$
(ii) $\sin\theta\cot\theta =$
(iii) $\sin(180+A) =$
(iv) $\cos^2 63^\circ+\sin^2 63^\circ =$
(v) $\tan 330^\circ =$

6. (i) $\cos^{-1}\dfrac{\sqrt{2}}{2} =$
(ii) $\cos 225^\circ =$
(iii) $\cos(90+A) =$
(iv) $\cos(-\theta) =$
(v) $\cos(180-B) =$

7. (i) $\tan\frac{1}{4}\pi =$
(ii) $\sec 60^\circ =$
(iii) $\sin^2\frac{1}{2}A+\cos^2\frac{1}{2}A =$
(iv) $\tan(\frac{3}{2}\pi+\theta) =$
(v) $\tan 135^\circ =$

8. (i) $\sin(360-A) =$
(ii) $\sin^{-1} 2 =$
(iii) $\sec^2\frac{1}{5}\pi-\tan^2\frac{1}{5}\pi =$
(iv) $\cos 0^\circ =$
(v) $\operatorname{cosec} 120^\circ =$

9. (i) $\tan 150^\circ =$
(ii) $\cos\theta/\sin\theta =$
(iii) $\cos 90^\circ =$
(iv) $\sec 240^\circ =$
(v) $\cot(180-A) =$

10. (i) $\sec 150^\circ =$
(ii) $\cos(360^\circ-A) =$
(iii) $\cos^{-1}\sqrt{3} =$
(iv) $\sin 77^\circ\cot 77^\circ =$
(v) $\tan\frac{7}{4}\pi =$

11. (i) $\tan 1200^\circ =$
(ii) $\tan(180^\circ+A) =$
(iii) $\tan\frac{1}{2}\pi =$
(iv) $\tan 15^\circ\cot 15^\circ =$
(v) $\tan^{-1}(-1) =$

12. (i) $\cos^2 23^\circ+\cos^2 67^\circ =$
(ii) $\cos(270^\circ+B) =$
(iii) $\sin^{-1}\cdot 4+\cos^{-1}\cdot 4 =$
(iv) $\sin(-\phi) =$
(v) $\sin 225^\circ =$

Examples III.

1. Prove, from first principles, that $\sin(90+A)=\cos A$, $\cos(180+A)=-\cos A$, $\tan(360-A)=-\tan A$.

2. Show that $\sin(180-A)=\sin A$, when A is (i) obtuse, (ii) between 180° and 270°, (iii) between 270° and 360°.

3. Show that $\cos(90-A)=\sin A$, when A is (i) obtuse, (ii) between 180° and 270°, (iii) between 270° and 360°.

4. Show that $\tan(180+A)=\tan A$, when A is (i) obtuse, (ii) between 180° and 270°, (iii) between 270°, and 360°.

5. Give 6 different solutions of each of the following equations:

(i) $\sin A=\frac{1}{2}$; (ii) $\sin A=\dfrac{\sqrt{2}}{2}$; (iii) $\cos\theta=\dfrac{\sqrt{3}}{2}$;

(iv) $\tan A=1$; (v) $\cos A=-\frac{1}{2}$; (vi) $\sin\theta=-\dfrac{\sqrt{3}}{2}$.

6. Show that all angles having the same sine as A are included in one or other of the forms: $180n+A$, if n is an even integer,

$180\,n - A$, if n is an odd integer; and that these are included in the single form $180\,n + (-1)^n A$ where n is any integer, positive or negative.

7. Show that all angles having the same cosine as A are included in the form $360\,n \pm A$, where n is any integer.

8. Show that all angles having the same tangent as A are included in the form $180\,n \pm A$, where n is any integer.

9. What do the forms of the three previous examples become when the angle is measured in radians?

10. If a small angle equals $A°$, what is the value of $\sin A$?

11. Show that using the approximation $\sin\theta = \theta$ is equivalent to regarding a circle as a polygon with a large number of sides.

12. What do the following equalities become when the angle θ is so small that θ^2 is negligible?

(i) $\sin 2\theta = 2\sin\theta\cos\theta$;

(ii) $\cos 2\theta = 1 - 2\sin^2\theta$;

(iii) $\sin(\theta + \phi) = \sin\theta\cos\phi + \cos\theta\sin\phi$;

(iv) $C = G\tan\theta$;

(v) $v^2 = 4\,gr\sin^2\frac{1}{2}\theta$.

13. Two strings are tied to two pegs A and B in the same horizontal line, and knotted together at C; when the strings are pulled tight, it is found that AC is 18 inches long and that the angles CAB, CBA are 30° and 60° respectively; how far apart are the pegs and how far is C from AB?

14. An inclined plane, length 4 feet, is inclined at 30° to the horizontal, what is the length of the base?

15. A pendulum is held so as to make an angle of 30° with the vertical, what is then the distance of the end of the pendulum from the vertical line through point of support?

16. Prove the following identities:

(i) $\cot^2 A\cos^2 A = \cot^2 A - \cos^2 A$;

(ii) $\sec^2 A - \sin^2 A = \tan^2 A + \cos^2 A$;

(iii) $\sin\theta(\operatorname{cosec}\theta - \sin\theta) = \cos^2\theta$;

(iv) $(\cos A + \operatorname{cosec} A)(\sin A + \sec A)$
$= 2 + \sin A\cos A + \sec A\operatorname{cosec} A$;

(v) $(\cos A + \sec A)(\sin A + \operatorname{cosec} A)$
$= \sin A\cos A + 2\sec A\operatorname{cosec} A$;

(vi) $\sec A - \sin A\tan A = \cos A$;

(vii) $(\sec A - \operatorname{cosec} A)(\sin A + \cos A) + \sec^2 A\cot A = 2\tan A$.

17. (i) If θ and ϕ differ by $\frac{1}{2}\pi$, prove that $\tan\theta\tan\phi = -1$;

(ii) Show that the lines whose equations are, respectively, $y = mx$ and $y = m'x$, are at right angles if $mm' = -1$;

(iii) Show that the graphs of the equations $ax+by+c=0$, $a'x+b'y+c'=0$ are at right angles if $aa'+bb'=0$, and are parallel if $a/a' = b/b'$.

18. If $\tan\theta = b/a$, find the value of $a\cos\theta + b\sin\theta$.

19. If $\tan^3\theta = b/a$, show that $a/\cos\theta + b/\sin\theta = (a^{\frac{2}{3}} + b^{\frac{2}{3}})^{\frac{3}{2}}$.

20. Give a general formula for all values of A which satisfy the equation $\cos A = -1$.

21. If $a\sin^2\theta + b\cos^2\theta = c$ and $a\cos^2\theta + b\sin^2\theta = d$, prove that $a+b=c+d$.

22. From the vertex C of an equilateral triangle ABC a perpendicular CD is let fall on AB; DC is produced to E so that CE equals CA, and AE is drawn. From the resulting figure find the sine, cosine, and tangent of 15° and 75°.

23. A is an angle between 180° and 270°, also $\cos A = -\frac{3}{5}$; find the value of $\operatorname{cosec} A + \tan A$.

24. Define the cosine of an angle of any magnitude and express the cosine of an angle between 180° and 270° in terms of each of the other trigonometrical ratios.

If $\cos\theta = -\frac{63}{65}$, find $\sin\theta$, $\sec\theta$, $\cot\theta$, and explain any double signs which occur in your answer.

25. Prove the following identities:

(i) $(\sin A\cos B + \cos A\sin B)^2 + (\cos A\cos B - \sin A\sin B)^2 = 1$;

(ii) $\sin^6\theta + \cos^6\theta = 1 - 3\sin^2\theta\cos^2\theta$;

(iii) $\cot A - \tan A = \sec A \operatorname{cosec} A\,(1 - 2\sin^2 A)$;

(iv) $(1-\sin A-\cos A)^2 = 2\,(1-\sin A)\,(1-\cos A)$;

(v) $(2\cos A - \sec A) \div (\cos A - \sin A) = 1 + \tan A$.

(vi) $(3\sin\theta\cos^2\theta - \sin^3\theta)^2 + (\cos^3\theta - 3\cos\theta\sin^2\theta)^2 = 1$;

(vii) $\sec^6\theta - \tan^6\theta = 1 + 3\tan^2\theta\sec^2\theta$;

(viii) $\operatorname{versin}(270° + A) \,.\, \operatorname{versin}(270° - A) = \cos^2 A$.

26. Prove that

$$\cos(180° - A) = -\cos A, \text{ and } \cos(90° + A) = -\sin A.$$

For what values of A is $\tan A = \sqrt{3}$ and $\sec A = -2$?

27. Solve for x the equations:

(i) $x^2 + 2x\sec\alpha + 1 = 0$;

(ii) $x^2 + 2x\cos\alpha = \sin^2\alpha$;

(iii) $x^2 + (\tan\alpha + \cot\alpha)x + 1 = 0$.

28. Prove that the number of seconds in an angle whose circular measure is unity is 206,265.

The moon subtends at the eye of an observer an angle of 30′, its distance is 240,000 miles, find its radius.

29. If $\tan^2\theta = \frac{5}{4}$, find versin θ, and explain the double result.

30. Eliminate θ from

(i) $a\tan\theta + b\cot\theta = c,\ a'\tan\theta + b'\cot\theta = c'$;

(ii) $a\tan\theta + b\sin\theta = c,\ a'\tan\theta + b'\sin\theta = c'$.

Revision Examples A.

1. Define the tangent of an angle. From your definition find tan 45° and tan 135°, and prove that $\tan(\frac{1}{2}\pi - \theta) = \cot\theta$.

2. A surveyor goes 10 chains in a direction 35° S. of E., then 7·8 chains 14° E. of S.; then 5·6 chains 10° N. of W. Find by drawing how far he is now from his starting-point.

3. Prove the relation $1 - \sin^2 A = \cos^2 A$ for the case where A lies between 90° and 180°.

Show that $(\sin A + \cos A)^4 = 1 + 4\sin A\cos A + 4\sin^2 A - 4\sin^4 A$.

4. The gradient of a railway is 1 in 270; find the inclination to the horizontal to the nearest second.

5. When the sun's altitude is 60°, find the length of the shadow cast by a vertical rod whose length is 10 feet.

6. Draw the graph of $\cos x$ between $x = 15°$ and $x = 135°$ without using tables.

---

7. Explain how to find the length of the arc of a circle of given radius, when the angle subtended at the centre is given in degrees.

A wheel, radius $4\frac{1}{2}$ feet, rolls along the ground; what horizontal distance does the centre travel when the wheel turns through 157°?

8. Why is the secant so called? Prove that the secant is the reciprocal of the cosine.

Given $\sec A = 2\frac{1}{2}$, find $\tan A$ and $\sin A$.

9. Show that the graph of the straight line $y = 2x - 5$ is inclined to the axis of x at an angle $\tan^{-1} 2$. Verify this by a careful drawing.

10. Trace the changes in $\sin\theta$ as θ changes from 0° to 360° and exhibit these changes by means of a graph.

11. Find the smallest angle which satisfies the equation

$$3 \cos \theta + 2 \sin^2 \theta = 0.$$

Give also four other solutions.

12. If $\sin A = \frac{3}{5}$, prove that $\sec A + 1/\cot A = 2$.

---

13. What is a radian? Prove the formula

$$\text{arc} = r \times \theta.$$

Show that if θ is small, $\sin \theta = \theta$ approximately.

14. Show that $\tan (180\, n + A) = \tan A$ where n is any integer.

If $\tan 3A = \sqrt{3}$, state three possible values for A that do not differ by 360°.

15. Find the value of the expression $\operatorname{cosec} A - \frac{5}{6} \cot A$, if $\sin A = \frac{63}{65}$ (i) when A is acute, (ii) when A is obtuse.

16. Prove the identity $2 \sin A \cos A = (2 \tan A) \div (1 + \tan^2 A)$.

17. In a triangle ABC, $C = 90°$, $AB = 15$, $\sin A = \cdot 37$; find the length of AC and BC.

18. Criticize the following statements:

(a) $\sin^2 \theta = 4$; (b) $\sin \theta \tan \theta = 1$;

(c) $\sin^{-1} (-\cdot 3) = 170°$; (d) $\sin \theta + \cos^2 \theta / \sin \theta = \tan \theta$.

---

19. Explain clearly what is meant by latitude.

A place has latitude 30° N., what is its distance from (i) the earth's axis, (ii) the Equator, measured along the surface? (Radius of earth = 4000 miles.)

20. Give a definition of cosine that applies to angles of any size. Prove that $\cos (180 - A) = -\cos A$.

If $\sin A = \frac{12}{13}$ and A is obtuse, find $\cos A$.

21. Prove that

$$(\cos A \cos B + \sin A \sin B)^2 + (\sin A \cos B - \cos A \sin B)^2 = 1.$$

22. Draw the graph of $y = \sec x$ from $x = 0°$ to $x = 180°$.

23. What is meant by the statement that $\tan 90° = \infty$?

Is $\sec 90°$ equal to $\tan 90°$? Give reasons.

24. Construct an angle A such that $\cos A = -\frac{2}{3}$ and $\tan A$ is positive.

---

25. Name the points of the compass between West and South.

How many degrees are there in the angle between SW. by S. and S. by E.?

26. Find the values of $\sin 45^\circ$, $\cos^{-1}\frac{1}{2}$, $\tan\frac{1}{4}\pi$.

27. Prove by means of a figure that

$$\sin^2 A + \tan^2 A = \sec^2 A - \cos^2 A.$$

Is this true when the angle is measured in radians? Give reasons.

28. Construct an angle such that its tangent $= \frac{3}{4}$ and its versine is greater than unity.

29. Prove the identity $\cos^2 A - \sin^2 A = (\cot^2 A - 1) \div \operatorname{cosec}^2 A$.

30. Find the value of $\tan\theta$ from the equation

$$3\tan^2\theta = 2\sqrt{3}\tan\theta - 1.$$

Hence find three different values of θ that satisfy the equation.

---

31. Write down six positive angles which have the same cosine as the angle α; and find the positive values of θ less than two right angles which satisfy the equation

$$\sin 4\theta = \cos 5\theta.$$

32. Show how to find by calculation the value of $\sin 30^\circ$ correct to four decimal places.

Verify, by substitution,

(i) $\sin 60^\circ = 2\sin 30^\circ\cos 30^\circ$;

(ii) $\sin 120^\circ - \sin 60^\circ = 2\cos 90^\circ\sin 30^\circ$;

(iii) $\cos 60^\circ - \cos 120^\circ = 2\sin 30^\circ\sin 90^\circ$.

33. Prove the identities:

(i) $\operatorname{cosec}^2 A - \operatorname{cotan}^2 A = 1$;

(ii) $\dfrac{5 + 13\sin\theta}{12 + 13\cos\theta} + \dfrac{12 - 13\cos\theta}{5 - 13\sin\theta} = 0.$

34. A steamer travels along the equator from longitude $37\frac{1}{2}^\circ$ W. to longitude $5^\circ\,30'$ E. in 4 days. What is the distance travelled in nautical miles? What was her average rate in knots?

35. What is meant by the chord of an angle?

For which angle is the chord equal to unity?

Explain how to draw an angle when a table of chords is given.

36. Express the following ratios as ratios of angles not greater than 45°:

$\sin 172^\circ$, $\cos 412^\circ$, $\tan 246^\circ$, $\sec 76^\circ$, $\operatorname{cosec} 147^\circ$, $\sec 236^\circ$, $\cot 138^\circ$, cosine 150°, $\sin 67^\circ$, $\tan 102^\circ$.

37. If the circumferences of the quadrants of two circles be divided similarly to the right angles they subtend, what would be the radius of a circle divided according to the French scale, in which the length of the arc of one grade would be equal to the length of the arc of one degree on a circle whose radius was 18 feet?

38. Point out which of the trigonometrical functions are never numerically less than unity, and which may be either less or greater than unity.

Express the numerical values of $\sin 135°$ and $\tan 150°$ with their proper signs.

39. If n be a positive whole number, show that the angles

$$(2n \,.\, 180° + A) \text{ and } \{(2n+1)180° - A\}$$

have the same sine as A.

Express these in a single formula.

40. Distinguish carefully between $(\sin A)^{-1}$ and $\sin^{-1} A$.

Show that $\cos^{-1} \frac{1}{2} + 2 \sin^{-1} \frac{1}{2} = 120°$.

41. Trace the changes in sign and magnitude of the expression $\cos x - \sin x$ as x increases from 0 to 2π. Illustrate your answer by a graph.

42. A church spire, whose height is known to be 45 feet, subtends an angle of $9'$ at the eye; find its distance approximately.

---

43. What is meant by $\tan^{-1} m$?

If $y = mx + c$ represents a straight line, state the geometrical interpretation of the coefficients m and c?

What is the angle between the lines whose equations are

$$y = x - 4, \; y = \sqrt{3}x + 2 \,?$$

44. Show that the equation of the line joining the points

$$(x_1, y_1), (x_2, y_2) \text{ is } (y - y_1) \div (y_1 - y_2) = (x - x_1) \div (x_1 - x_2).$$

45. Find the equation of a line passing through the origin and (i) parallel to, (ii) perpendicular to, the line whose equation is $y = mx + c$.

Deduce the conditions that the two lines whose equations are $ax + by + c = 0$, $a'x + b'y + c' = 0$, should be (i) parallel, (ii) perpendicular.

46. Find the equation of the line joining the origin to the point P whose co-ordinates are (x', y').

Find the equation of the line perpendicular to OP and passing through P.

Hence show that the equation of the tangent to a circle at the point x', y' is $xx' + yy' = r^2$, the equation of the circle being $x^2 + y^2 = r^2$.

47. If (r, θ) are the polar co-ordinates of a point, what locus is represented by

(i) $r = 3$, (ii) $\theta = \frac{1}{6}\pi$, (iii) $r \cos\theta = 5$, (iv) $r = 5 \cos\theta$?

48. If (x, y) are the Cartesian co-ordinates, (r, θ) the polar co-ordinates, of the same point, what relations connect them?

Express the equations of the previous example in Cartesian co-ordinates.

Express (i) $x^2 + y^2 - 4x + 5y = 7$, (ii) $3x + 4y = 5$ in polar co-ordinates.

49. The sum of two angles is 3 radians, their difference is 10 degrees. Find each angle in degrees, assuming that $43\pi = 135$.

50. A ring, 10 inches in diameter, is suspended from a point one foot above its centre by six equal strings attached to its circumference at equal intervals. Find the angle between two consecutive strings.

CHAPTER IV

USE OF TABLES

34. It has been shown in the previous chapters that the trigonometrical ratios of any angle may be found roughly by drawing to scale or by means of graphs. By methods which are explained in more advanced books on Trigonometry, the ratios can be calculated to any required degree of accuracy. There are many collections of tables published, containing not only the actual trigonometrical ratios (the natural functions as they are called) but also the logarithms of these ratios. These collections differ slightly in their arrangement, but the following general remarks apply to most of them.

35. Since any ratio of any angle is equal in magnitude to the same ratio of some angle less than 90°, it is necessary to tabulate the ratios only for angles between 0° and 90°. Thus

$$\sin 156^\circ = \sin(180^\circ - 24^\circ) = \sin 24^\circ,$$
$$\cos 215^\circ = \cos(180^\circ + 35^\circ) = -\cos 35^\circ.$$

But the tables may be made even shorter, for any function of an angle between 45° and 90° is equal to the complementary function of an angle less than 45°. Thus

$$\sin 76^\circ = \sin(90^\circ - 14^\circ) = \cos 14^\circ,$$
$$\tan 69^\circ = \tan(90^\circ - 21^\circ) = \cot 21^\circ.$$

This fact is used in two different ways. Some tables give all the ratios for angles from 0° to 45°; so that if, for instance, sin 72° is required, it must be looked up as cos 18°. Other tables give the values of sine, tangent, and secant for angles from 0° to 90°; in this case, cosine, cotangent, and cosecant must be looked for as the sine, tangent, and secant respectively of the complementary angle.

The slight mental work involved is avoided by giving each column a "footing" as well as a heading. Thus ·26892 is, in some tables, found on a page headed Natural Sines, on a level with 15° in the extreme left-hand column and under 36′, i.e. $\cdot 26892 = \sin 15^\circ 36'$.

But the same page has Natural Cosines at the bottom, ·26892 is on same level as 74° in the extreme right-hand column and above 24′, i.e. ·26892 = cos 74°24′.

A few minutes' inspection will make the arrangement of any set of tables quite clear.

36. Logarithmic Functions. Since the sine and cosine cannot be greater than unity, their logarithms cannot be greater than zero; hence these logarithms have a negative characteristic. In order to avoid difficulties of printing it has been the custom to add 10 to all these logarithms, and to the other logarithmic functions. The values thus tabulated are called **Tabular Logarithms** and are denoted in writing by L, thus $L \tan 75° = \log \tan 75° + 10$.

Some of the modern tables give the ordinary logarithms with the negative characteristics.

When tabular logarithms are used it is advisable to subtract 10 mentally and to work with the correct logarithm.

37. Interpolation. It is impossible to give the ratios for all angles. Four-figure tables usually give values for every 6′, seven-figure tables for every 1′. Intermediate values may, in some tables, be found from side columns giving the differences, as in the case of ordinary logarithms. If these side columns are not given, the method of **proportional parts** * must be used. This method is equivalent to assuming that the graph of the tabulated function may be treated as a straight line for portions lying between the points corresponding to two consecutive tabulated values. The practical use is easily followed from an example or two.

Example i. *Given that*

sin 28° 9′ = ·4717815, *and sin* 28° 10′ = ·4720380,

find sin 28° 9′ 43″.

$$\sin 28° 10' = \cdot 4720380,$$
$$\sin 28° \; 9' = \cdot 4717815.$$
$$\text{Increase for } 60'' = \cdot 0002565;$$
$$\therefore \quad \text{Increase for } 43'' = \tfrac{43}{60} \times \cdot 0002565$$
$$= \cdot 0001838;$$
$$\therefore \quad \sin 28° 9' 43'' = \cdot 4719653.$$

$$\begin{array}{r} \cdot 000004275 \times 43 \\ \hline \cdot 00017100 \\ 1282 \\ \hline \cdot 00018382 \end{array}$$

* For a fuller treatment see *School Algebra*, Part II, p. 376.

In practice the zeros are omitted as in the following example.

Example ii. *Given that log cos* $73° 15' = \bar{1}{\cdot}4058617$, *and log cos* $73° 16' = \bar{1}{\cdot}4053816$, *find the angle when the log cosine is* $\bar{1}{\cdot}4056348$.

Denote the angle by $73° 15' x''$.

$$\log\cos 73° 15' = \bar{1}{\cdot}4058617. \qquad \log\cos 73° 15' = \bar{1}{\cdot}4058617.$$

$$\log\cos 73° 15' x'' = \bar{1}{\cdot}4056348. \qquad \log\cos 73° 16' = \bar{1}{\cdot}4053816.$$

$$\text{Decrease for } x'' = \quad 2269. \qquad \text{Decrease for } 60'' = \quad 4801.$$

Hence $\frac{x}{60} = \frac{2269}{4801}$.

$\therefore \quad x = 28.$

$$\begin{array}{r} 2269 \\ 60 \\ \hline 4801)\overline{136140} \\ 4012 \\ 171 \end{array}$$

$\therefore$ required angle $= 73° 15' 28''$ to the nearest second.

Note. It is important to recollect that cosine, cotangent, cosecant, and their logarithms decrease as the angle increases; consequently proportional differences must be subtracted, not added.

If the graphs of the functions are carefully drawn, it is seen that in some parts they approach much more nearly to straight lines than in others. It follows that the method of proportional parts is more accurate for some angles than for others. For a complete discussion of Proportional parts see Nixon's *Elementary Plane Trigonometry* (Clarendon Press) or any advanced textbook.

Examples IV a.

Find, from tables, the natural function of the following angles, find the logarithm of the number found, and then look up the logarithmic function in the tables. There may be a slight discrepancy in the fourth decimal place.

1. $\sin 17° 15'$.	2. $\cos 73° 47'$.	3. $\tan 16° 39'$.
4. $\cos 23° 19'$.	5. $\sec 67° 15'$.	6. $\cot$an $44° 5'$.
7. $\tan 78° 53'$.	8. $\sin 83° 43'$.	9. $\cos 63° 28'$.
10. $\sin 156° 17'$.	11. $\tan 176° 16'$.	12. $\cot 100° 10'$.
13. $\cos 137° 42'$.	14. $\sin 126° 37'$.	15. $\tan 173° 14'$.

Explain carefully the difficulty that arises in connexion with some of the angles.

16. Find the Cartesian co-ordinates of a point whose polar co-ordinates are (i) 17, 16°; (ii) 25, 114°; (iii) 49, 227°.

Find the angles less than 180° which are determined by the following data:

17. $\sin\theta = \cdot 8732$.
18. $\cos A = \cdot 3469$.
19. $\sin B = \cdot 9340$.
20. $\log\tan A = \bar{1}\cdot 7932$.
21. $L\cos A = 9\cdot 7432$.
22. $\sec B = 2\cdot 5732$.
23. $\log\sin A = \bar{1}\cdot 3465$.
24. $L\tan A = 10\cdot 4385$.
25. $L\cot C = 10\cdot 7386$.

Find the sine, cosine, and tangent of the following angles, which are measured in radians:

26. $\frac{1}{12}\pi$. 27. $\frac{4}{13}\pi$. 28. $1\cdot 2$. 29. $\frac{5}{13}\pi$.

30. Verify that

$$\sin 112^\circ = \sin 70^\circ \cos 42' + \cos 70^\circ \sin 42'.$$

31. Find from the tables the values of

$$\sin \tfrac{1}{15}\pi \text{ and } \sin 27^\circ 18'/\cos 32^\circ 45'.$$

32. Employ the tables to verify the formula

$$\cot 24^\circ 45' - \cot 49^\circ 30' = \operatorname{cosec} 49^\circ 30'.$$

33. Find the values of $\cos 110^\circ$, $\cot 160^\circ$, $\sin 250^\circ$.

A quantity μ is such that $\mu = \sin i/\sin r$; complete the following tables:

	i	r	μ
34.	16°	12°	
35.	26° 18′		1·427.
36.		31° 52′	1·467.
37.		53° 49′	1·5.

38. Find the polar co-ordinates of points whose Cartesian co-ordinates are (i) (3, 7); (ii) (−3, 7); (iii) (−3, −7); (iv) (3, −7).

39. The angle of friction ϵ and the coefficient of friction μ are connected by the relation $\mu = \tan\epsilon$. Determine the missing quantity in the following cases:

ϵ	40° 15′		17° 39′	47° 8′		
μ		·67			·37	·50

40. In a circle of radius 17 find the lengths of chords subtending angles (i) 37°, (ii) 73°, (iii) 143° at the centre. What are the areas of the corresponding segments?

38. Graphs.

Example. *Draw the graph of*

$3 \sin(x+30°)-2\cos(x-30°)$ *from* $x=0°$ *to* $x=120°$.

In other words, draw the graph of

$$y=3\sin(x+30°)-2\cos(x-30°).$$

x	0	15°	30°	45°	60°	75°	90°	105°	120°
$\sin(x+30°)$.	·500	·707	·866	·966	1·000	·966	·866	·707	·500
$\cos(x-30°)$.	·866	·966	1·000	·966	·866	·707	·500	·259	0·000
$3\sin(x+30°)$	1·500	2·121	2·598		3·000	2·898	2·598	2·121	1·500
$2\cos(x-30°)$	1·732	1·932	2·000		1·732	1·414	1·000	·518	·000
y	−·232	·189	·598	·966	1·268	1·484	1·598	1·603	1·500

The graph is shown in Fig. XIX.

Use of Graph.

Interpolation. The value of the function can be found for any intermediate value of the angle. From the graph it is seen that $y=1·07$ when $x=50°$, and $y=1·54$ when $x=81°$. Calculation shows that the correct values are 1·075 and 1·542 respectively.

This is a useful method of testing the accuracy of a graph.

Maximum and Minimum. When, as x increases, y continually increases to a certain value and then decreases, that value is said to be a **maximum**; similarly, when y first decreases and then increases there is a **minimum** value. These maximum and minimum values are clearly shown on the graphs; the corresponding points are called **turning-points.**

From the graph the maximum value of

$$3\sin(x+30°)-2\cos(x-30°)$$

is found to be 1·62, and the corresponding angle is 99°.

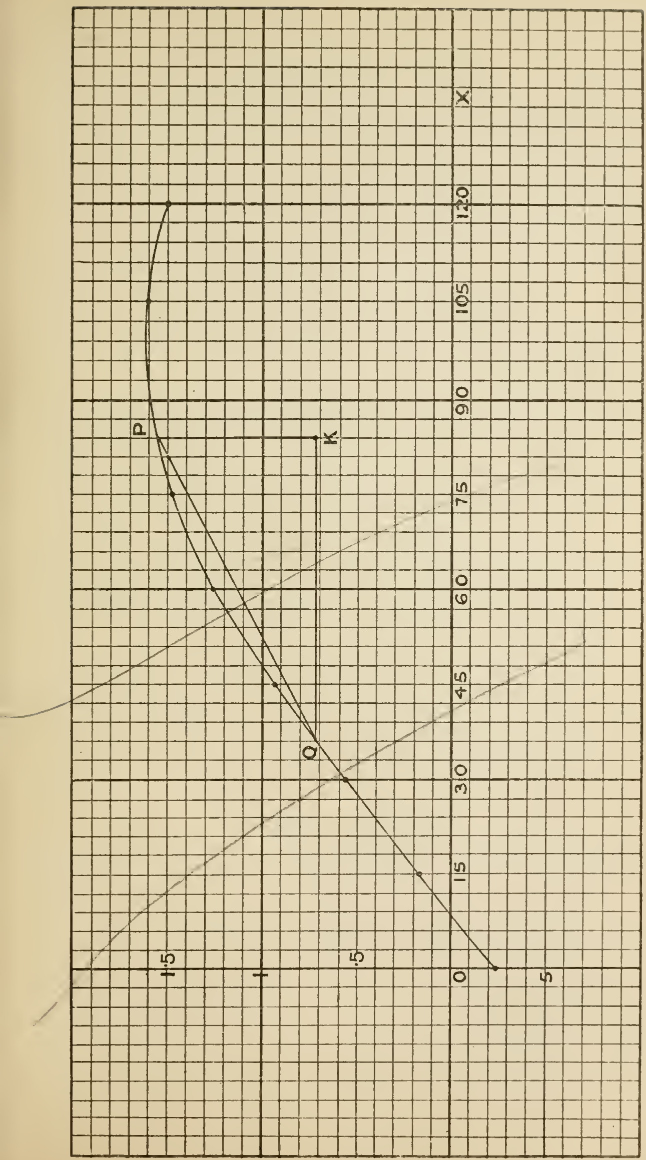


Fig. XIX.

Rate of change of the function. The graph shows that, when x changes from 15° to 30°, the increase in y is more than when x changes from 60° to 75°; consequently the curve is steeper between 15° and 30° than between 60° and 75°. Thus the rate at which y changes compared with x is shown by the steepness of the curve.

Join two points P and Q on the graph, and draw PK, QK parallel to the axes to meet in K. Then

$$\frac{\text{increase in } y}{\text{increase in } x} = \frac{KP}{QK} = \text{tangent of the angle } PQK$$

$$= \text{tangent of the angle } PQ \text{ makes with the axis of } x.$$

This is called the **slope** of the line PQ.

When Q approaches indefinitely near to P, the chord PQ becomes a tangent. Hence

The rate of increase of y at the point P is measured by the slope of the tangent at the point P.

Notice that the slope diminishes in the neighbourhood of a turning-point and is zero at the turning-point itself.

39. Solution of equations. By finding where two graphs intersect or where one graph intersects the axis of x or a line parallel to the axis, equations can be solved just as in Algebra.

Example. *Solve the equation*

$$3\sin(x+30°)-2\cos(x-30°) = 1{\cdot}5.$$

It is seen in Fig. XIX that the graph cuts the line whose equation is $y = 1{\cdot}5$ where $x = 77°$ and $x = 120°$. These are, therefore, the solutions within the range of the graph.

Examples IV b.

(The graphs should be verified in the way that the example of § 38 is verified.)

Draw the graphs and find the turning-points of:

1. $\sin\frac{1}{2}x$ from $x = 0°$ to $x = 90°$.
2. $\cos\frac{1}{2}x$ from $x = -90°$ to $x = 90°$.
3. $\sin\frac{1}{2}x+\cos\frac{1}{2}x$ from $x = 15°$ to $x = 135°$.
4. $\frac{1}{2}\tan(x-60°)$ from $x = 0°$ to $x = 90°$.
5. $\theta-\sin\theta$ from $\theta = 0$ to $\theta = \frac{1}{2}\pi$.
6. $\sec x-\tan x$ from 0° to 90°.

7. $\cos^2 \frac{1}{2} x + \sin^2 \frac{1}{2} x$ from 0° to 360°.

8. Draw the graph of $\sin x + \cos x$ between $x = 0$ and $x = 360°$. Solve $\sin x + \cos x = ·89$, and find the slope of the graph at the points corresponding to these values of x.

9. Draw the graph of $\cos x$ between the values of 0 and 2π for x. Show that an acute angle can be found to satisfy the equation $x = \cos x$.

10. Draw the graphs from $x = -1$ to $x = +1$ of (i) $\sin^{-1} x$, (ii) $\cos^{-1} x$, (iii) $\tan^{-1} x$. How are they related to the graphs of $\sin x$, $\cos x$, $\tan x$ respectively?

11. Draw the graphs whose polar equations are

(i) $r \sin\theta = 17$; (ii) $r = 10 \sin\theta$;
(iii) $r = 10 \cos\theta$; (iv) $\tan\theta = 2·45$.

12. Find from your tables the values of $\cos 2x$ for the values 0°, 10°, 20°, 30°, 40°, 50°, 60° of x.

Draw the graph of $\cos 2x - \cos x$ as x increases from 0° to 60°.

13. Find, by drawing graphs of $2 \sin A$ and $\sin 2A$, for what values of A, less than 90°, $2 \sin A - \sin 2A = 1$.

14. Find, by the aid of the tables, the values of $\sin x - \tan 2x$ for the values 0°, 10°, 20°. 45°, 60° of x.

Make a graph to give the values of $\sin x - \tan 2x$ from $x = 0$ to $x = 60°$.

15. Make a table giving the values of $\cos\theta$ at intervals of one-fifth of a radian from $\theta = 0$ to $\theta =$ two radians, taking the radian as 57° 30′.

From your table plot the graph of $\theta \cos\theta$; and hence find for what value of θ, between the limits 0 and 2, $\theta \cos\theta$ is greatest.

16. Plot the function $\frac{1}{2}\{\sin\theta + \sin 2(\theta + 20°)\}$ between $\theta = 0°$ and $\theta = 180°$, and find the maximum and minimum values of the function which occur within this range, and the corresponding values of θ.

17. Draw, in the same diagram, the graphs of $\sin x$ and $2 \cos x$ between $x = 0°$ and $x = 180°$. Show how to find from your diagram an angle whose tangent is 2.

18. Taking π as 3·1416 and using your tables, find the values of $\theta - \sin\theta$ when $\theta = \frac{1}{12}\pi$, $\frac{1}{6}\pi$, $\frac{1}{4}\pi$, $\frac{1}{3}\pi$, $\frac{5}{12}\pi$, and $\frac{1}{2}\pi$; and hence make a graph to give $\theta - \sin\theta$ from $\theta = 0$ to $\theta = \frac{1}{2}\pi$.

19. Draw the graph of (i) $\sin^{-1} x + \cos^{-1} x$; (ii) $\sin^{-1}(1/x)$.

Solution of Equations.

40. To solve a trigonometrical equation,

(i) express all the ratios involved in terms of one ratio,

(ii) find the value of this ratio by ordinary algebraical methods,

(iii) find the angle from the tables,

(iv) give the general solution.

Example i. *Solve* $2\sin x + 3\cos x = 2$.

Express in terms of sine,

$$\pm 3\sqrt{1-\sin^2 x} = 2 - 2\sin x.$$

Square $$9 - 9\sin^2 x = 4 - 8\sin x + 4\sin^2 x.$$

Transpose $$13\sin^2 x - 8\sin x - 5 = 0.$$

Factorize $$(13\sin x + 5)(\sin x - 1) = 0.$$

$$\therefore \quad \sin x = -\tfrac{5}{13} \text{ or } 1.$$

Substituting in the original equation, we find that:

(i) If $\sin x = -\frac{5}{13}$, $\cos x = \frac{12}{13} = \cdot 9231$.

Hence the bounding line is in the fourth quadrant.

From the tables it is found that $\cos 22^\circ 37' = \cdot 9231$.

Hence the smallest positive angle satisfying the equation is

$$360^\circ - 22^\circ 37' = 337^\circ 23'.$$

But we may add or subtract any multiple of 360° without altering the position of the bounding line; hence any angle satisfies the equation whose value is $360^\circ n + 337^\circ 23'$, where n is any integer positive or negative. This is the **general solution.**

(ii) If $\sin x = 1$, $\cos x = 0$.

The smallest solution is $x = 90^\circ$.

The general solution is $360^\circ n + 90^\circ$ or $(4n+1)90^\circ$.

Note. The same difficulty has arisen here that arises in Algebra when the original equation contains surds. After we have squared, the resulting equation is exactly the same as if we had started with the equation $2\sin x - 3\cos x = 2$. For this reason, after we found the value of $\sin\theta$, it was necessary to substitute in the original equation to find the corresponding value of $\cos x$.

Example ii. *Solve* $tan^2 A + 4\, sin^2 A = 5$.

Express in terms of tan A.

$$\tan^2 A + 4\frac{\tan^2 A}{1+\tan^2 A} = 5.$$

Multiply by $1+\tan^2 A$ $\qquad \tan^4 A = 5.$

Take logarithms $\qquad 4 \log \tan A = \cdot 6990.$

$$\therefore \quad \log \tan A = \cdot 1747.$$

Use tables $\qquad = \log \tan 56° 13'.$

Hence a solution is $\qquad A = 56° 13'.$

Consideration of the fundamental figure shows that the general solution is $\qquad A = 180\,n + 56° 13'.$

41. General Solutions.

General solutions can always be obtained by mentally considering the possible positions of the radius vector that give angles having the same function as some angle already found. This is what has been done in the two preceding examples. It is, however, useful to know the formulae that give these general solutions.

Find an expression for all angles that have a sine equal to sin α.

We have to solve $\qquad \sin\theta = \sin\alpha.$

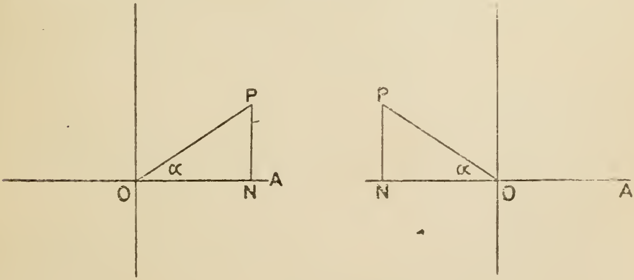


Fig. XIX a.

The bounding line may have either of the positions shown in Fig. XIX a.

Thus the line may revolve through $n\pi$, where n is even, and then go on α, or may revolve through $n\pi$, where n is odd, and then come back α. Hence $\theta = n\pi + \alpha$ if n is even,

or $\qquad n\pi - \alpha$ if n is odd.

These are included in the one formula

$$\theta = n\pi + (-1)^n \alpha.$$

If $\qquad \sin x = \sin A$, then $x = 180\,n + (-1)^n A$.

Exercises. In a similar way prove that

i. $\qquad \theta = 2n\pi \pm \alpha$, when $\cos\theta = \cos\alpha$;

$\qquad x = 360\,n \pm A$, when $\cos x = \cos A$.

ii. $\qquad \theta = n\pi + \alpha$, when $\tan\theta = \tan\alpha$;

$\qquad x = 180\,n + A$, when $\tan x = \tan A$.

Example. *Solve* $\sin 3\theta = \cos 5\theta$.

This is the same as

$$\sin 3\theta = \sin\left(\tfrac{1}{2}\pi - 5\theta\right);$$

$\therefore \qquad 3\theta = n\pi + (-1)^n\left(\tfrac{1}{2}\pi - 5\theta\right).$

If n is odd $\qquad 3\theta = n\pi - \tfrac{1}{2}\pi + 5\theta;$

$\therefore \qquad 2\theta = \tfrac{1}{2}\pi - n\pi.$

Put $n = -2p + 1$, then $\qquad \theta = p\pi - \tfrac{1}{4}\pi$, where p is any integer.

If n is even $\qquad 3\theta = n\pi + \left(\tfrac{1}{4}\pi - 5\theta\right);$

$\therefore \qquad 8\theta = n\pi + \tfrac{1}{2}\pi.$

Put $n = 2p$ $\qquad \theta = p\tfrac{1}{4}\pi + \tfrac{1}{16}\pi$, where p is any integer.

The complete solution is

$$\theta = p\pi - \tfrac{1}{4}\pi \text{ or } p\tfrac{1}{4}\pi + \tfrac{1}{16}\pi.$$

Examples IV c.

Solve:

1. $2\cos^2\theta = 3(1 - \sin\theta)$.
2. $\sin\theta + \cos\theta = 1$.
3. $\sin\phi + \cos\phi = \sqrt{2}$.
4. $12\tan^2 A - 13\tan A + 3 = 0$.
5. $2\cos^2 x - 1 = 1 - \sin^2 x$.
6. $\sin 3A = \sin 4A$.
7. $3\cot^4\theta - 10\cot^2\theta + 3 = 0$.
8. $2\sin A = \tan A$.
9. $\tan A + 3\cot A = 4$.
10. $\sin(x + A) = \cos(x - A)$.
11. $\tan^2 A + 4\sin^2 A = 6$.
12. $\sqrt{3}\tan^2\theta + 1 = (1 + \sqrt{3})\tan\theta$.
13. $\operatorname{cosec}\theta = \cot\theta + \sqrt{3}$.
14. $\cos(135° + A) + \sin(135° - A) = 0$.
15. $\cos^3 A - \cos A\sin A - \sin^3 A = 1$.
16. $\cos 3\theta + \sin\theta = 0$.
17. $3\tan^2 2\theta = 1$.

18. $\tan 2x = \tan 2/x$.

19. $2\sin^2\theta - 3\sin\theta - 2 = 0$.

20. $2\cos^2\theta + 3\cos\theta - 1 = 0$.

21. $\tan^2\theta + \sec^2\theta = 2$.

22. $1{\cdot}7\sin\theta - {\cdot}73 = 0$.

23. $3\sin\theta + 2\cos\theta = 2$.

24. $2\tan^2\theta + 7\tan\theta + 3 = 0$.

25. $\tan\theta - 2\cot\theta = 1{\cdot}7$.

42. Examples of the use of logarithms.

Example i. *Given that* $\dfrac{a}{\sin A} = \dfrac{b}{\sin B}$, *find* B *when* $a = 250,\ b = 240,\ A = 72^\circ\, 5'$.

We have $$\frac{\sin B}{b} = \frac{\sin A}{a},$$

i. e. $$\sin B = \frac{b \sin A}{a}.$$

Take logs.
$$\begin{aligned} \log\sin B &= \log b + \log\sin A - \log a \\ &= \quad 2{\cdot}3802 - 2{\cdot}3979 \\ &\quad + \bar{1}{\cdot}9784 \\ &= \quad 2{\cdot}3586 \\ &\quad - 2{\cdot}3979 \\ &= \quad \bar{1}{\cdot}9607; \end{aligned}$$

$$[\therefore\ \mathrm{L}\sin B = 9{\cdot}9607];$$

$$\therefore\ \sin B = \sin 66^\circ.$$

Hence $$B = 180\,n^\circ + (-1)^n\,66^\circ.$$

After a little practice the work may be arranged so that the logarithms are kept quite distinct from the remainder of the work. This same example is worked below to show the shorter method and the use of five-figure tables.

$$\frac{\sin B}{b} = \frac{\sin A}{a},$$
$$\sin B = 240\sin 72^\circ\, 5' \div 250$$
$$= \sin 65^\circ\, 59'.$$

Logarithms.

$$\begin{array}{r} 2{\cdot}38021 \\ +\bar{1}{\cdot}97841 \\ \hline 2{\cdot}35862 \\ -2{\cdot}39794 \\ \hline \bar{1}{\cdot}96068 \end{array}$$

Example ii. *The sides and angles of a triangle are connected by the relation* $\tan\frac{1}{2}(A - B) = \dfrac{a-}{a+b}\cot\frac{1}{2}C$: *find* A *and* B *when* $a = 242{\cdot}5,\ b = 164{\cdot}3,\ C = 54^\circ\, 36'$.

$$\tan\frac{A-B}{2} = \frac{78{\cdot}2}{406{\cdot}8}\cot 27^\circ 18'$$

$$\left[= \frac{78{\cdot}2}{406{\cdot}8}\tan 62^\circ 42'\right]$$

$$= \tan 20^\circ 26';$$

$$\therefore \quad \frac{A-B}{2} = 20^\circ 26',$$

by question $\frac{A+B}{2} = 62^\circ 42'$.

Hence $A = 83^\circ 8'$,
$B = 42^\circ 16'$.

Logarithms.

$$\begin{array}{r} 1{\cdot}89321 \\ + \ {\cdot}28723 \\ \hline 2{\cdot}18044 \\ -2{\cdot}60938 \\ \hline \bar{1}{\cdot}57106 \end{array}$$

The step in brackets is required if the tables do not give the cotangents. Since A and B are angles of a triangle, $\frac{1}{2}(A-B)$ cannot equal any of the angles $180\,n + 20^\circ 26'$ (except when $n=0$), so that there is no need to give the general solution.

Example iii. *If a, b, c are the sides of a triangle, $\tan \frac{1}{2}A = \sqrt{\frac{(s-b)(s-c)}{s(s-a)}}$ where s is half the sum of the sides. Find A when $a = 1762$, $b = 893$, $c = 1386$.*

$$s = (1762 + 893 + 1386) \div 2 = 4041 \div 2$$
$$= 2020{\cdot}5,$$
$$s-a = 258{\cdot}5,$$
$$s-b = 1127{\cdot}5,$$
$$s-c = 634{\cdot}5;$$
$$\therefore \quad \tan\frac{A}{2} = \sqrt{\frac{1127{\cdot}5 \times 634{\cdot}5}{2020{\cdot}5 \times 258{\cdot}5}};$$
$$\therefore \quad \frac{A}{2} = 49^\circ 29',$$
$$A = 98^\circ 58'.$$

Logarithms.

$$\begin{array}{r} 3{\cdot}05212 \\ 2{\cdot}80243 \\ +5{\cdot}85455 \\ \hline 3{\cdot}30546 \\ 2{\cdot}41246 \\ \hline -5{\cdot}71792 \\ 2)0{\cdot}13663 \\ \hline 0{\cdot}06831 \end{array}$$

Examples IV.

1. Use the tables to find the values of $\sin 52^\circ$, $\cos 140^\circ$, $\tan 220^\circ$, $\cos 340^\circ$, $\sin 340^\circ$.

2. Divide $\sin 52^\circ$ by $\cos 52^\circ$; verify your answer by finding the value of $\tan 52^\circ$ from the tables.

3. Write down by using tables the values of $\sin 140^\circ$, $\cos 160^\circ$, $\cos 220^\circ$, $\tan 320^\circ$.

4. Find the smallest positive value of θ which satisfies

$$\cos\theta = \sin\{(4n+3)\tfrac{1}{2}\pi + \alpha\}.$$

5. Find all the values of θ which satisfy the equation

$$4\cos\theta - 3\sec\theta = 2\tan\theta.$$

6. Find the inclination to the horizon of an incline which rises $5\frac{1}{2}$ feet in 420 yards.

7. Solve the equation $\tan^2\theta - (1+\sqrt{3})\tan\theta + \sqrt{3} = 0$.

8. Given that $\tan\frac{1}{2}C = \sqrt{(s-a)(s-b) \div s(s-c)}$, find C when $a = 32$, $b = 40$, $c = 66$.

9. Solve the equations $\cos(2x+3y) = \frac{1}{2}$, $\cos(3x+2y) = \sqrt{3}/2$.

10. Given $\log 2 = \cdot30103$ and $\log 3 = \cdot47712$, find (without the tables) $\text{L}\sin 60^\circ$ and $\text{L}\tan 30^\circ$.

11. Find the acute angle whose cosine equals its tangent.

12. The current C in a circuit, as determined by a tangent galvanometer, equals $G\tan\theta$, where G is a constant depending on the galvanometer only and θ is the deflexion of the needle. Determine the ratio of two currents which give deflexions of $27^\circ\ 14'$, $35^\circ\ 23'$ respectively.

13. The length of a degree of latitude in latitude ϕ is

$$(1111\cdot317 - 5\cdot688\cos\phi)\,10^4 \text{ centimetres.}$$

Find the length at London (latitude $51^\circ\ 31'$ N.) and Melbourne (latitude $37^\circ\ 50'$ S.).

14. The length of the seconds pendulum in centimetres, at a place whose latitude is λ, is $99\cdot3563 - \cdot2536\cos 2\lambda$. Find the length of the seconds pendulum at Paris (lat. $48^\circ\ 50'$ N.) and Calcutta (lat. $22^\circ\ 33'$ N.).

15. The acceleration of a falling body at a place whose latitude is λ, when measured in centimetres per second per second, is

$$980\cdot6056 - 2\cdot5028\cos 2\lambda.$$

Find the acceleration at Montreal (lat. $45^\circ\ 30'$ N.) and Cape Town (lat. $33^\circ\ 40'$ S.).

16. A quantity Δ is determined by the relation $\Delta = \frac{1}{2}ab\sin C$. Complete the following table:

	Δ	a	b	C
i.		17	43	$77^\circ\ 14'$
ii.	342·6	21·3	38·19	
iii.	984·2		43·82	$43^\circ\ 21'$.

17. Draw the graph of

$$\tan\theta - \theta \text{ from } \theta = 0 \text{ to } \theta = \frac{\pi}{2}.$$

Hence solve $\tan\theta = \theta + 3$.

18. Given that A and B, the angles of a triangle, are connected by the relation $a \sin B = b \sin A$, find B when $a = 181$, $b = 217$, $A = 34^\circ 15'$.

19. If $2R = a/\sin A$, find the value of A when $R = 179{\cdot}4$ and $a = 300$.

20. Verify that

$$\cos 146^\circ 43' - \cos 56^\circ 51' = -2 \sin 44^\circ 56' \sin 101^\circ 47'.$$

21. Find the length of (i) the chord, (ii) the arc, subtending an angle 70° at the centre of a circle of radius 25 cm. Find also the area of the segment.

22. Find the length of the side of a regular decagon (i) inscribed in, (ii) described about, a circle of radius 2·7 inches.

CHAPTER V

THE RIGHT-ANGLED TRIANGLE

43. In the previous chapters we have had to deal with only one angle at a time, and have been able to draw one of the lines containing that angle horizontal. In applications of Trigonometry we often have to deal with several angles in the same example, and the lines containing them are drawn in various directions; in such examples it would be difficult to apply the definitions of § 12. But it has been shown that the ratios of any angle can be expressed in terms of the ratios of an acute angle. In practice, therefore, it will often be found advisable to use the following definitions, which apply only to acute angles.

In a right-angled triangle an acute angle is contained by the hypotenuse and one of the other sides which is called the side **adjacent** to that angle. The remaining side is called the side **opposite**. Then in Fig. XX

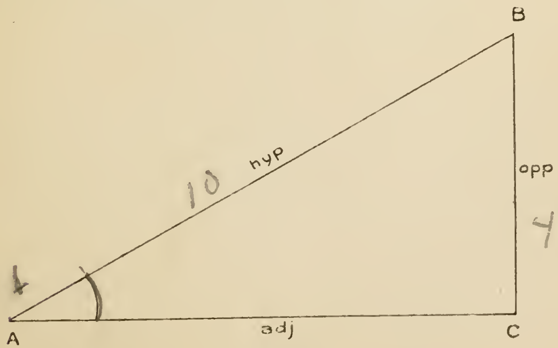


Fig. XX.

$$\sin BAC = \frac{\text{opposite}}{\text{hypotenuse}}; \quad \operatorname{cosec} BAC = \frac{\text{hypotenuse}}{\text{opposite}};$$
$$\cos BAC = \frac{\text{adjacent}}{\text{hypotenuse}}; \quad \sec BAC = \frac{\text{hypotenuse}}{\text{adjacent}};$$
$$\tan BAC = \frac{\text{opposite}}{\text{adjacent}}; \quad \cot BAC = \frac{\text{adjacent}}{\text{opposite}}.$$

These are clearly the same definitions as in § 12, the triangle *BAC* taking the place of the triangle *PON*; and the various formulae proved in Chap. III can be proved directly from the definitions of this section.

44. It is usual to denote the angles of any triangle *ABC* by the capital letters *A*, *B*, *C*; the lengths of the sides opposite the angles *A*, *B*, *C* are denoted by *a*, *b*, *c* respectively.

Hence, in a triangle *ABC*, right-angled at *C*,

$$\sin A = \frac{a}{c}, \text{ i.e. } a = c \sin A;$$
$$\cos A = \frac{b}{c}, \text{ i.e. } b = c \cos A;$$
$$\tan A = \frac{a}{b}, \text{ i.e. } a = b \tan A.$$

Examples V a.

1. Prove, from the definitions of § 43, that

(i) $\cos A = \sin B = \sin(90 - A)$;
(ii) $\sin A = \cos B = \cos(90 - A)$;
(iii) $\tan A = \cot B = \cot(90 - A)$.

2.

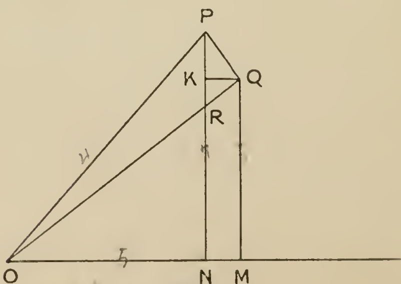


Fig. XXI.

In the figure PNO, QMO, QKP are right angles.

If $ON = 5$, $NP = 7$, $OM = 6$, $MQ = 5$, find the values of $\sin PON$, $\tan KPQ$, $\tan KQP$, $\sec QOM$, $\cos KQO$, $\operatorname{cosec} NRO$.

3. If, in Fig. XXI, $OP = 8$, $POQ = 30°$, $QON = 45°$, $PQO = 90°$, find the lengths of OQ, PQ, PK, QM, OM.

4. A circle is described on a horizontal diameter AB of length 10 inches; a point C is taken on the circumference, such that $BC = 7$, and CD is let fall at right angles to AB. Find the size of the angle BAC and the length of CD.

5. In a triangle, right-angled at C, a perpendicular is let fall from C to the hypotenuse; prove, by Trigonometry, that this perpendicular is a mean proportional between the sides containing the right angle.

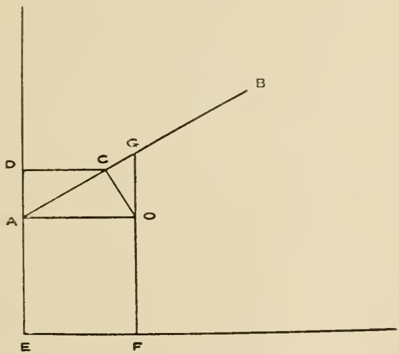


Fig. XXII.

In the above figure (which is not drawn to scale) AO is at right angles to DE, OC is at right angles to AG, OG is at right angles to AO and EF; also G is the middle point of AB.

Use this figure in the following examples.

6. If $AC = 10$, $CAD = 40°$, find, if possible, the lengths of all the other lines.

7. If $CD = 8$, $AB = 24$, find $\sin CAD$.

8. If $GF = 18$, $AE = 5$, $OC = 5$, find $\cos ACD$ and the length of AO.

9. If $AB = l$, $CAD = \theta$, find CD and AO.

10. If $CG = a$, $CGO = \theta$, find AD and AO.

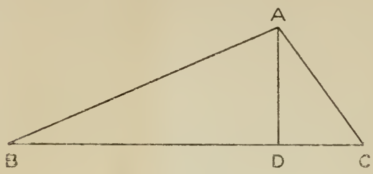


Fig. XXIII.

With the ordinary notation for the sides and angles of a triangle, find in the above figure:

11. The length of AD when $c = 70$, $B = 49°$.

12. The length of AD when $b = 42$, $C = 72°$.

13. The length of BD when $c = 76$, $B = 39°$.

14. The length of CD when $b = 114$, $C = 114°$.

(What geometrical fact does the negative sign in the result show?)

15. Prove that the area of the triangle $= \frac{1}{2}ab \sin C$; give the proof, also, when C is obtuse.

Solution of Right-angled Triangles.

45. The angles and sides of a triangle are sometimes called the six parts of a triangle. The determination of all the parts, when only some of the parts are known, is called **solving the triangle.** If the triangle is known to be right-angled, the triangle can be solved if one side and one other part are known.

Example i. *A man, standing* 100 *feet from the foot of a church steeple, finds that the angle* of elevation of the top*

* If a person is looking upwards, the angle his line of sight makes with the horizontal is the **angle of elevation**; similarly, if he is looking downwards, the angle his line of sight makes with the horizontal is the **angle of depression.**

is 50°. *If his eye is* $5\frac{1}{2}$ *feet from the ground, what is the height of the steeple?*

[The figure should be drawn neatly but need not be drawn to scale.]

In Fig. XXIV *AE* represents the steeple, *BC* the man; *CD* is drawn parallel to *BA*.

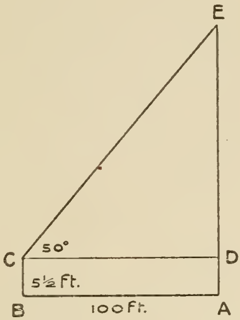


Fig. XXIV.

[**Mental.** In the right-angled triangle *CDE* we know that

$$CD = BA = 100 \text{ ft., angle } DCE = 50°,$$

and we wish to find *DE*.

$$\frac{\text{unknown side}}{\text{known side}} = \text{some ratio of known angle.}]$$

Here $$\frac{DE}{DC} = \tan DCE,$$

i.e. $$DE = 100 \tan 50° \text{ feet}$$
$$= 100 \times 1{\cdot}1918 \text{ feet}$$
$$= 119{\cdot}18 \text{ feet.}$$

Therefore height of steeple $= AD + DE = 124{\cdot}68$ feet.

Example ii. *The shadow, cast by the sun on a horizontal plane, of a vertical pole* 10 *feet high, is observed to be* 14 *feet long; find the altitude of the sun* (*i.e. the angle of elevation of the sun*).

In Fig. XXV *AB* represents the pole, *AC* the shadow; so that *CB* is the direction of one of the sun's rays.

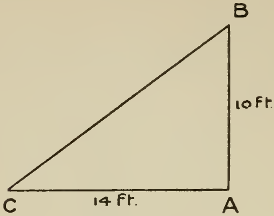


Fig. XXV.

[**Mental.** In the right-angled triangle *BAC* we know *BA* and *AC*, and wish to find the angle *ACB*.

ratio of known sides = some ratio of required angle.]

$$\frac{AB}{AC} = \tan ACB;$$

$$\therefore \quad \tan ACB = \tfrac{10}{14} = \cdot 71429.$$

$$\therefore \quad ACB = 35°32'.$$

$\tan 35°\ 30' = \cdot 71329.$

440 is diff. for 10′.

$\therefore$ 100 „ $\frac{100'}{44} = 2'.$

Sun's altitude = 35° 32′ to nearest minute.

Example iii. *A ship C is observed at the same time from two coastguard stations A and B,* 1459 *yards apart. The angle ABC is found to be* 90°, *and the angle BAC to be* 67° 14′, *what is the distance of the ship from station A?*

Fig. XXVI.

Here $\frac{AC}{AB} = \sec BAC;$

$$\therefore \quad \log AC = \log 1459 + \log \sec 67°14'$$
$$= 3\cdot 16406$$
$$+ \cdot 41111$$
$$+ 120$$
$$= 3\cdot 57637,$$

i.e. $AC = 3770\cdot 3.$

Distance of ship from *A* = 3770 yards to nearest yard.

* If the tables do not contain the secants, the working must be made to depend on the cosine.

$$\frac{AC}{AB} = \frac{1}{\cos BAC},$$

$$\log AC = \log 1459 - \log \cos 67^\circ 14'$$
$$= 3{\cdot}1641$$
$$- \bar{1}{\cdot}5877$$
$$= 3{\cdot}5764,$$

i.e. $AC = 3770.$

Example iv. *Two men, A and B,* 1370 *yards apart, observe an aeroplane C at the same instant and find the respective angles of elevation to be* 40° *and* 67°. *If the plane ABC is vertical, calculate the height of the aeroplane.*

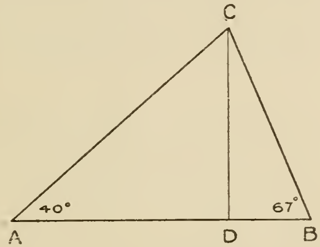


Fig. XXVII.

Let h feet be height of aeroplane.

From triangle ADC, $AD = h \cot 40°$.

From triangle BDC, $BD = h \cot 67°$;

but $AD + BD = AB$;

$\therefore \quad h \cot 40° + h \cot 67° = 137.$

$$\therefore \; h = \frac{1370}{\cot 40° + \cot 67°}$$

$$= \frac{1370}{1{\cdot}61622};$$

$\therefore \quad h = 847{\cdot}62;$

$\cot 40° = 1{\cdot}19175$

$\cot 67° = {\cdot}42447$

Logarithms

$3{\cdot}13672$

$-\ {\cdot}20852$

$2{\cdot}92820$

Height of aeroplane = 848 yards to nearest yard.

Examples V b.

1. The string of a kite is known to be 500 feet long, and it is observed to make an angle of 55° with the horizontal; find the height of the kite.

2. From the top of a cliff, 215 feet high, the angle of depression of a ship is observed to be 23° 20′; what is the distance of the ship from the foot of the cliff?

3. From a point 56 feet from the foot of a tree the angle of elevation of the top is 73°; find the height of the tree.

4. The top of a conical tent is 9 feet above the ground; the radius of the base is 5 feet; what is the inclination of the side of the tent to the horizontal?

5. The shadow thrown by a flagstaff is found to be $55\frac{1}{2}$ feet long when the sun's altitude is 53° 15′; what is the height of the flagstaff?

6. I know that a certain tower is 144 feet high. I find that its elevation observed from a certain point on the same level as the base of the tower is 37° 16′. Find the distance of that point from the base of the tower.

7. A sphere of radius 4 inches is suspended from a point A in a vertical wall so that it rests against the wall. The string is 11 inches long and is in the same straight line as a radius of the sphere. Find the inclination of the string to the vertical.

8. From the top of a cliff, 254 feet high, the angle of depression of a ship was found to be 9° 28′, and that of the edge of the sea 72° 40′; how far distant was the ship from the edge of the sea?

9. Two observers on the same side of a balloon and in the same vertical plane with it, a mile apart, find its angles of elevation to be 15° and 65° 30′ at the same moment. Find the height of the balloon.

10. From the top of a tower, 108 feet high, the angles of depression of the top and bottom of a vertical column are found to be 30° and 60° respectively. What is the height of the column?

11. A flagstaff, 30 feet high, is fixed in the centre of a circular tower 40 feet in diameter. From a point on the same horizontal plane as the foot of the tower the elevations of the top of the flagstaff and the top of the tower are observed to be 35° and 30° respectively. Find the height of the tower.

12. A river, the breadth of which is 200 feet, flows at the foot of a tower, which subtends an angle 25° 10′ at a point on the further bank exactly opposite. Find the height of the tower.

13. A person standing at the edge of a river finds that the elevation of the top of a tower on the edge of the opposite bank is 60°; on going back 30 feet he finds the elevation to be 45°; find the breadth of the river.

14. From the top of a tower, 50 feet high, the angle of depression of a man, walking towards the tower, is noticed to be 30°; a few moments after it was 45°. How far had the man walked between the two observations?

15. Two posts, 400 yards apart, at the sides of a straight road running E. and W., are observed to bear N. 20° E. and E. 20° N. respectively. Find the distance of the observer from the road.

16. Two points *A* and *B* and the foot *D* of a tower *CD* are in a horizontal straight line, and the angles of elevation of *C*, the top of the tower, as seen from *A* and *B* respectively, are 25° 46′ and 35° 25′. If the distance *AB* is 200 feet, find the height of the tower.

17. A vertical post casts a shadow 15 feet long when the altitude of the sun is 50°; calculate the length of the shadow when the altitude of the sun is 32°.

18. A vertical mast, having its base at *A*, is set up on a horizontal plane. *B* and *C* are points in the plane in a line with *A*, and such that the angular elevations of the top of the mast, when observed at these points, are respectively α and β. If $\tan \alpha = \frac{3}{4}$, $\tan \beta = \frac{2}{5}$ and the length of *BC* is 105 feet, find the height of the mast.

19. A man standing on a tower at a height of 80 feet from the ground observes that the angles of depression of two objects on a straight level road running close to the foot of the tower are 60° and 30°. If the objects are on the same side of the tower, how far are they apart?

20. *A*, *B*, *C* are three points in succession on a straight level road, and *P* is another point so situated that the angles *PAB*, *PBA*, *PCA* are respectively 90°, 60°, and 45°. If a man walks at a uniform rate from *A* to *B* in 25 seconds find, to the nearest second, how long it will take him, at the same rate, to walk from *B* to *C*.

21. A ray of light passes through a hole *A* in a graduated horizontal scale *AB* in a direction perpendicular to the scale and is reflected by a vertical mirror which is distant 30 inches from the scale and makes an angle $x°$ with the incident (i.e. approaching)

ray. After reflection the ray makes the same angle with the mirror as before and shines on the scale at a distance 8 inches from *A*. Find the value of x.

If the mirror now swings through an angle 1°, how far will the spot of light on the scale move?

Elementary Navigation.

{The student should revise §§ 9 and 10 dealing with latitude and longitude and the points of the compass.}

46. When a ship is sailing, the angle between its direction of sailing and the meridian the ship is crossing is called the **course.** If the course is constant, the ship is said to sail on a **rhumb-line.** The **distance** between two positions of the ship is then measured along the rhumb-line. The **difference of latitude** of two places is the arc of a meridian intercepted between the parallels of latitude passing through the two places. The **departure** between two meridians is the distance between the two meridians measured along a parallel of latitude; thus the departure between any two given meridians is not a constant but diminishes from the equator to the poles.*

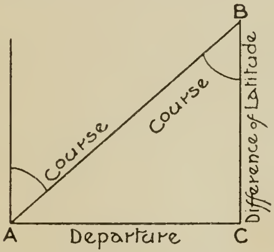


Fig. XXVIII.

47. A small portion of the earth's surface may be regarded as a plane; for distances small, compared with the earth's radius, we may therefore use the formulae of Plane Trigonometry.

Plane Sailing is the name given to that part of navigation which treats the surface of the earth as a plane. On this assumption the meridians become parallel straight lines, the rhumb-line becomes the hypotenuse of a right-angled triangle of which the departure is the side opposite to the course, and the difference of latitude is the side adjacent. Thus problems on Plane Sailing are merely examples in the solution of right-angled triangles.

* In navigation distances are usually measured in nautical miles; a nautical mile is the length of an arc of a meridian (or the equator) which subtends an angle of 1′ at the centre of the earth; thus a distance of 75 nautical miles is usually written 75′.

Examples V c.

(The distances are given in nautical miles.)

1. A ship sails SE. by S., a distance 81 miles; what is her departure and difference of latitude?

2. A ship sails N. 49° 41′ W., a distance 73 miles; what is the departure and difference of latitude?

3. A ship sails SSW. until its departure is 198 miles; what is the distance sailed and the difference of latitude?

4. If the course is $3\frac{1}{2}$ points W. of N., and the difference of latitude 149 miles, what is the distance?

5. A ship sails between North and West, making a difference of latitude $157\frac{1}{2}$ miles and departure 79 miles; what is the course?

6. A ship sails westward 247 miles along the equator from meridian 16° E.; what is now the longitude?

7. A ship sails 247 miles eastward along the parallel 40° N.; what is the change in longitude?

8. When a ship sails any distance (great or small) along a parallel of latitude, show that

difference of longitude in minutes = departure × secant of latitude.

9. A ship, from latitude 54° 22′ 10″ N., sails $195\frac{1}{2}$ miles $\frac{1}{4}$ of a point S. of SE.; what is now the latitude?

10. Leaving latitude 49° 37′ N., longitude 15° 22′ W., a ship sails SW. by W. 150 miles; find the new latitude and longitude.

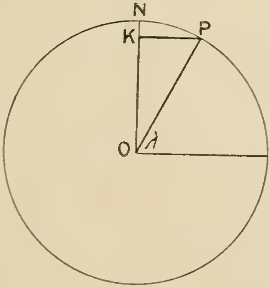


Fig. XXIX.

48. Parallel Sailing. If λ° is the latitude, then the radius of the parallel of latitude (KP in Fig. XXIX) is $\cos\lambda \times$ radius of the earth. If θ is the radian measure of the difference of longitude of two places on the same parallel, the length of the arc between them is $\theta \cos\lambda \times$ radius of the earth. The radius of the earth is $\frac{21600}{2\pi}$ nautical miles.

Hence $$\text{departure} = \frac{1600}{2\pi}\theta\cos\lambda.$$

When θ is reduced to minutes, this relation becomes

$$\text{departure} = \text{difference of longitude} \times \text{cosine of latitude}.$$

49. Middle Latitude Sailing. In Middle Latitude Sailing, the departure between two places, whose latitudes are λ and λ', is taken to be the departure between their meridians, measured at the latitude $\frac{1}{2}(\lambda+\lambda')$. On this assumption,

$$\text{departure} = \text{diff. of longitude} \times \cos\tfrac{1}{2}(\lambda+\lambda').$$

50. Traverse Sailing. If a ship sails on different courses, from A to B, from B to C, from C to D, &c., then, by the methods of Plane Sailing, the total changes in latitude and longitude can be worked out. This is called the method of **Traverse Sailing.** This method can only be used when the whole area traversed can be regarded as plane without introducing a great amount of error.

Example. *A ship left a position in which Oporto Light (lat.* 41° 9′ *N., long.* 8° 38′ *W.) bore W. by N.,* 15 *miles distant. Afterwards she sailed as under :*

Courses.	Distances.
N.W.	70′
S. by W. ½W.	55′
E.	35′
N.N.W.	42′
S.E.	51′.

Find her bearing and distance from the Light in her last position.

We have a series of right-angled triangles to solve, the hypotenuse and an acute angle being given in each case. In practical navigation special tables are used, called Traverse Tables.

The angle the hypotenuse makes with the meridian is taken in each case.

O to *A*.

Hypotenuse 15′, angle 7 points = 78° 45′.	1·1761
Diff. of latitude = 15′ × cos 78° 45′	$\bar{1}$·2902
≑ 2·926′ S.	·4663
Departure = 15′ × sin 78° 45′	1·1761
= 14·71′ E.	$\bar{1}$·9916
	1·1677

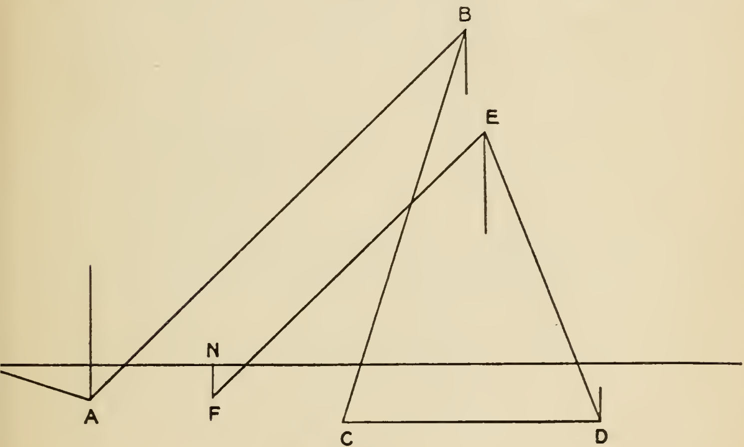


Fig. XXX.

Note that *O* bears W. by N. from *A*, but *A* bears E. by S. from *O*.

A to *B*.

Hypotenuse 70′, angle 45°.	·7071
Diff. of latitude = 70 × cos 45°	
= 49·497′ N.	
Departure = 49·497′ E.	

B to *C*.

	1·7404
Hypotenuse 55′, angle $1\frac{1}{2}$ points = 16° $52\frac{1}{2}$′.	$\bar{1}$·9809
Diff. of latitude = 55 cos 16° $52\frac{1}{2}$′	1·7213
= 52·64′ S.	1·7404
Departure = 55 sin 16° $52\frac{1}{2}$′	$\bar{1}$·4628
= 15·97′ W.	1·2032

The other triangles are worked in the same way.

Tabulate the results thus:

Direction.	Distance.	Diff. of Latitude.		Departure.	
		N.	S.	E.	W.
E. by S	15′		2·93	14·71	
NW.	70′	49·50		49·50	
S. by W. ½ W.	55′		52·64		15·97
E.	35′			35	
NNW.	42′	38·80			16·07
SE.	51′		36·06		36·06
		88·30	91·63	99·21	68·10
			88·30	68·10	
			3·33	31·11	

We see now that the final difference of latitude from the light is 3·33′ S., and departure 31·11 E.; so that we have to solve a right-angled triangle given the two sides.

In Fig. XXX $\tan FON = \dfrac{3·33}{31·11}$; $\quad$ ·5224

$1·4929$

$\therefore \quad FON = 6° 7'$. $\quad \bar{1}·0295$

$OF = \dfrac{31·11}{\cos FON}$ $\quad 1·4929$

$\bar{1}·9975$

$= 31·29$ $\quad 1·4954$

In her final position the ship bore 6° 7′ S. of E., 31·3 miles distant from the Light.

To find the longitude of the ship.

Latitude of O	41° 9′ N.
Diff. of latitude for F	3·3′ S.
Latitude of F	41° 5·7′ N.
Middle latitude	41° 7′.

Difference of longitude in minutes $= \dfrac{\text{departure}}{\text{cosine of middle latitude}}$

$= \dfrac{31·11}{\cos 41° 7'}$ $\quad 1·4929$

$-\bar{1}·8770$

$= 41·30'$. $\quad 1·6159$

Longitude of $F = 8° 38' - 41·30'$

$= 7° 57'$ W.

Examples V d.

1. Find the distance on the parallel between Cape Agulhas (lat. 34° 50′ S., long. 20° 1′ E.) and Monte Video (lat. 34° 50′ S., long. 56° 9′ W.).

2. A ship steamed at the rate of 12 knots from Albany (lat. 35° 3′ S., long. 118° 2′ E.) to Cape Catastrophe (lat. 35° 3′ S., long. 135° 58′ E.). How long did she take on the voyage?

3. A ship sailed from Port Elizabeth (lat. 34° 7′ S., long. 25° 40′ E.) SE. $\frac{1}{2}$ S., until her departure was 397′; find her final position.

4. Find the course and distance from Syracuse (lat. 37° 3′ N., long. 15° 15′ E.) to Fano (lat. 39° 52′ N., long. 19° 19′ E.).

5. A ship left a position from which Cape Clear (lat. 51° 26′ N., long. 9° 29′ W.) bore NE. by E. 12·5 miles distant and sailed South 150′ and then West 290 miles. Find the bearing and distance of Cape Clear from the ship in her last position.

6. Find, by Middle Latitude Sailing, the departure between two places whose positions are 13° S., 50° E. and 20° S., 60° E.

7. A ship sails from 50° N., 50° W. to latitude 48° N., the distance being 157′; find the new longitude.

8. Cape Ortegal (lat. 43° 45′ N., long. 7° 6′ W.) bore SW. $\frac{3}{4}$ W. 12 miles distant. Afterwards sailed as under:

True Courses.	Distances.
NNW. $\frac{1}{2}$ W.	70′
ESE.	85′
NNE. $\frac{3}{4}$ E.	101′
S.	50′
WSW.	92′

Find the final latitude and longitude.

9. A ship left the Texel (latitude 52° 58′ N.) and then sailed W. by N. 34′, S. by E. 45′, W. by S. 35′, SSE. 44′, WSW. $\frac{1}{2}$ W. 42′. Find the course and distance to Dungeness which lies 139′ West of the Texel in latitude 50° 55′ N.

10. A ship, latitude 17° 10′ N., is making for a harbour, latitude 13° 10′ N., and 180′ W. of the ship. She sails SW. by W. 27′, WSW. $\frac{1}{2}$ W. 30′, W. by S. 25′, W. by N. 18′, SSE. 32′, SSE. $\frac{3}{4}$ E., 27′, S. by E. 25′, S. 31′, SSE. 39′. Find the course and distance to the harbour.

11. A ship left a position in which Heligoland bore ENE. 12′, and then sailed NW. 24′, S. by W. 20′, NW. by W. 32′, S. by E. 36′, WNW. $\frac{1}{2}$ W. 42′, SSE. $\frac{1}{2}$ E. 16′, W. $\frac{3}{4}$ N. 45′. What is then the position of the ship? Heligoland lies 54° 12′ N., 7° 54′ E.

12. A ship sailed from Barcelona (41° 25′ N., 2° 10′ E.) SE. by E. $\frac{1}{2}$ E. until she reached latitude 36° 21′ N. What was then her longitude?

13. A ship left a position in which Sable Island (43° 24′ N., 65° 36′ W.) bore NW. $\frac{1}{2}$ W., distant 12 miles.

Afterwards sailed as under:

Courses.	Distances.
ESE.	72′
SW. $\frac{1}{4}$ W.	37′
NNE.	42′
E.	25′

Required the latitude and longitude reached.

51. The Double Angle.

In Fig. XXXI, the angle $BAC = A°$; on AB a semicircle is described with centre O, so that angle $BOC = 2A$.

Let fall CN perpendicular to AB.

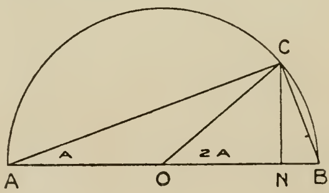


Fig. XXXI.

$$\cos 2A = \frac{ON}{OC}$$
$$= \frac{AN - AO}{OC}$$
$$= \frac{AN}{OC} - 1$$

$$= \frac{2\,AN}{} \cdot \frac{}{2\,OC} - 1$$

Fill in the vacant places with the hypotenuse of the triangle of which AN is a side.

$$= 2\frac{AN}{AC} \cdot \frac{AC}{AB} - 1$$

$$= 2\cos^2 A - 1.$$

Exercises. In a similar way prove

i. $\sin 2A = 2 \sin A \cos A.$

ii. $\cos 2A = 1 - 2\sin^2 A.$

Deduce

iii. $\cos 2A = \cos^2 A - \sin^2 A.$

iv. $\tan 2A = \dfrac{2 \tan A}{1 - \tan^2 A}.$

v. $\sin A = 2 \sin \frac{1}{2}A \cos \frac{1}{2}A$; $\cos A = \cos^2 \frac{1}{2}A - \sin^2 \frac{1}{2}A.$

vi. $2\cos^2 \frac{1}{2}A = 1 + \cos A.$

vii. $2 \sin^2 \frac{1}{2}A = 1 - \cos A.$

viii. $\tan \frac{1}{2}A = \sqrt{\dfrac{1-\cos A}{1+\cos A}} = \dfrac{\sin A}{1+\cos A} = \dfrac{1-\cos A}{\sin A}.$

ix. Prove the formulae for $\sin 2A$ and $\cos 2A$ when $2A$ is obtuse.

x. Do these proofs apply to angles of any size? If not, between what limits do they apply? Why is the ambiguous sign omitted in viii?

52. Geometrical questions may often be solved by using Trigonometry. For example:

If from a point outside a circle a secant and a tangent be drawn, the rectangle contained by the whole secant and the part outside the circle is equal to the square on the tangent.

In Fig. XXXII it is required to prove that rect. $PA \cdot PB =$ sq. on PT.

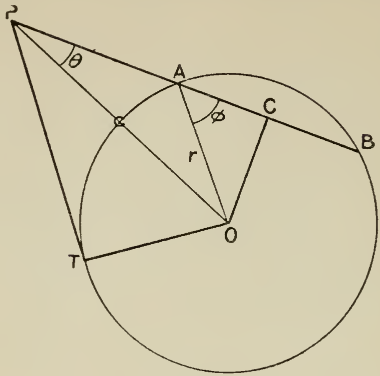


Fig. XXXII.

Let radius $= r$, $OP = c$, and angle $OPB = \theta$, angle $OAC = \phi$.

$$PA = PC - AC$$
$$= c\cos\theta - r\cos\phi.$$
$$PB = PC + CB$$
$$= PC + AC \quad \text{(Prop. 21)}$$
$$= c\cos\theta + r\cos\phi$$

$\therefore$ $$PA \cdot PB = c^2\cos^2\theta - r^2\cos^2\phi$$
$$= c^2 - r^2 - c^2\sin^2\theta + r^2\sin^2\phi.$$

But $$c\sin\theta = OC \text{ from triangle } OPC$$
$$= r\sin\phi \text{ from triangle } OAC.$$

Hence $$PA \cdot PB = c^2 - r^2$$
$$= OP^2 - OT^2$$
$$= PT^2 \text{ since } OTP \text{ is a right angle.}$$

53. Known results in Geometry are useful for proving Trigonometrical relations.

Show that, in any triangle,

$$\frac{\tan\frac{1}{2}(A-B)}{\tan\frac{1}{2}(A+B)} = \frac{a-b}{a+b}.$$

With centre C and radius CA (i.e. b), describe a circle cutting CB in E and CB produced in D.

Then $BE = a - b$, and $BD = a + b$.
Join AD and AE.
Through E draw EF parallel to DA and meeting AB at F.
Then the angle DCA at the centre $= 180 - C = A + B$.
So that the angle DEA at the circumference $= \frac{1}{2}(A + B)$.

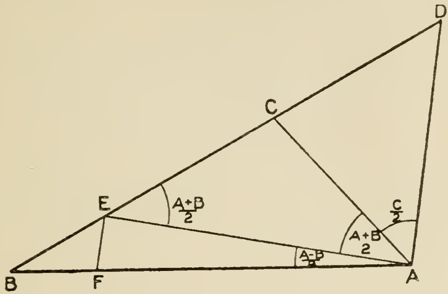


Fig. XXXIII.

Also the angle $BAE = BAC - EAC = A - \frac{1}{2}(A+B) = \frac{1}{2}(A-B)$.
Also the angle EAD, being in a semicircle, is a right angle.

$$\tan \tfrac{1}{2}(A+B) = \frac{AD}{AE},$$

$$\tan \tfrac{1}{2}(A-B) = \frac{EF}{AE}, \text{ since } AEF = EAD = \text{a right angle.}$$

Hence

$$\frac{\tan \frac{1}{2}(A-B)}{\tan \frac{1}{2}(A+B)} = \frac{EF}{AD}$$

$$= \frac{BE}{BD} \text{ since } EF \text{ is parallel to } AD. \text{ (Prop. } 12a.)$$

$$= \frac{a-b}{a+b}.$$

Corollary. $A + B + C = 180°$; $\quad\therefore\ \frac{1}{2}(A+B) = 90 - \frac{1}{2}C$.
Hence the above result may be written

$$\tan \tfrac{1}{2}(\mathrm{A} - \mathrm{B}) = \frac{\mathrm{a} - \mathrm{b}}{\mathrm{a} + \mathrm{b}} \cot \tfrac{1}{2}\mathrm{C}.$$

This formula will be used in a later chapter.

Examples V.

In the following examples:

A, B, C are the angles of a triangle ABC.

a, b, c are the sides, s = half the sum of the sides; R is the radius of the circumcircle.

r is the radius of the inscribed circle.

r_1 is the radius of the escribed circle touching the side BC.

Δ is the area of the triangle.

D, E, F are the middle points of the sides BC, CA, AB, respectively.

X, Y, Z are the feet of the perpendiculars let fall from A, B, C respectively on the opposite sides.

O is the centre of the circumcircle.

I is the centre of the inscribed circle.

K is the orthocentre.

1. Express in terms of the sides and angles the lengths of AX, BX, CX, AK, BK, CK.

2. Express the length of AD in terms of (i) a, b, C, (ii) a, b, B, (iii) a, b, c.

3. Show that $a/\sin A = b/\sin B = c/\sin C = 2R$. Deduce that $R = abc/4\Delta$.

4. Prove that $r(\cot \frac{1}{2}B + \cot \frac{1}{2}C) = a$. Write down the two similar formulae.

5. Prove that $r = \Delta/s$. (No trigonometry required.)

Deduce that $\tan \frac{1}{2}A = \Delta \div \{s(s-a)\}$.

6. Show that $BX = a - b\cos c$; hence prove that

$$c^2 = a^2 + b^2 - 2ab\cos c.$$

7. Prove that (i) $\Delta = \frac{1}{2}ab\sin C$, (ii) $\Delta = rs$, (iii) $\Delta = abc \div 4R$, (iv) $\Delta = \sqrt{s(s-a)(s-b)(s-c)}$.

8. Prove that

(i) $\sin \frac{1}{2}A = \sqrt{(s-b)(s-c) \div bc}$, (ii) $\cos \frac{1}{2}A = \sqrt{s(s-a) \div bc}$, (iii) $\tan \frac{1}{2}A = \sqrt{(s-b)(s-c) \div s(s-a)}$.

9. Show that the triangles ABC and AYZ are equiangular; hence prove that $YZ = a \cos A$.

10. Two tangents are drawn from a point P to a circle of radius 10 cm.; the tangents contain an angle of 43°. Find the lengths of the tangents and the distance of P from the centre.

11. A sheet of iron is shaped so that it can be rolled up to form a conical funnel 6 feet high with open circular ends 2 feet and 6 feet diameter respectively. Draw a plan of the sheet before rolling. What is the inclination of the edge of the funnel to the line joining the centres of the ends?

12. A circle rolls without slipping along a straight line: prove that the co-ordinates of a point fixed to the circumference are such that $x = a(\theta - \sin\theta)$, $y = a(1 - \cos\theta)$; the origin being taken at the point where the fixed point meets the straight line, and θ being the angle turned through by the circle.

13. One of the angles of a right-angled triangle is the acute angle whose sine is $\frac{2}{3}$, and the length of the shortest side of the triangle is 10 feet. Find the lengths of the other two sides.

14. A is the highest point of a sphere with centre O; a particle slides from a position P, where the angle $AOP = \theta$, to the position Q where the angle AOQ is ϕ. How much lower is Q than P and how much further from OA?

15. The time t of sliding from rest down a length s inclined at θ to the horizon is given by $s = \frac{1}{2} g t^2 \sin\theta$ where g is a constant. A circle is held with a diameter AB vertical; prove that the time of sliding along a chord from the highest point A to the circumference is the same whatever be the inclination of the chord, and that the time of sliding from the circumference along a chord to B is also independent of the inclination of the path.

16. A plane, inclined at 20° to the horizon, is placed with the line of greatest slope pointing north. A line is drawn on the plane, pointing NNE.; find the inclination of this line to the horizontal.

17. A man 6 feet high walks along a straight line which passes 3 feet from a lamp-post. If the light is 9 feet from the ground, find the length of the man's shadow when his distance from the point on his path nearest to the lamp is 10 feet. What is the locus traced out by the extremity of his shadow as he walks along the line?

18. If, in the previous question, there is a vertical wall parallel to the man's path and distant 2 feet from it on the side remote from the lamp, what is then the length of the shadow and the locus traced by its extremity?

19. Draw the graph of $\theta/\sin\theta$ from $\theta = 0$ to $\theta = \frac{1}{2}\pi$.

Use the graph to solve the following problem.

A string 30 inches long is tied to the ends of a cane 35 inches long, thus forcing the cane into a circular arc. Find the radius of the arc correct to the nearest inch.

20. Find the length of a strap which passes tightly round two pulleys of radii 2 feet and 3 feet, their centres being 6 feet apart.

CHAPTER VI

THE TRIANGLE

SEVERAL formulae connecting the sides and angles of a triangle have been proved in the examples of the preceding chapters. They are here gathered together for reference and proofs are given. **Care should be taken that the proof applies when the triangle is obtuse-angled; if it does not, a separate proof must be given.**

Relations between the sides and angles.

54. The angle formula. $A + B + C = 2$ **right angles.**

The sine formula. $$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C} (= 2R).$$

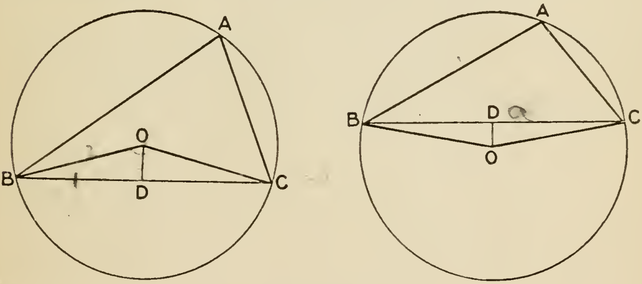


Fig. XXXIV.

Let O be the centre of the circumcircle, and D the middle point of BC.

Join OB, OC, OD.

Then, in the left-hand circle of Fig. XXXIV,

$$\text{angle } BOC = 2^{ce} \text{ angle } BAC$$
$$= 2A.$$

Triangles BOD and COD are congruent; (Prop. 8 *a*.)

$$\therefore \quad BOD = COD = A.$$

Also $$BD = \tfrac{1}{2} BC = \tfrac{1}{2} a.$$

In the right-angled triangle BOD,

$$BD = OB \sin BOD,$$

i.e. $\frac{1}{2} a = R \sin A$;

$$\therefore \quad \frac{a}{\sin A} = 2R.$$

In a similar way it may be proved that

$$\frac{b}{\sin B} = 2R \text{ and } \frac{c}{\sin C} = 2R.$$

Hence $$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C} = 2R.$$

Exercise. Supply the proof when the angle A is obtuse.

Note. In using this formula the following algebraic result is often useful :

If $\frac{a}{b} = \frac{c}{d} = \frac{e}{f}$, then each fraction equals $\frac{pa+qc+re}{pb+qd+rf}$.

55. The cosine formula $\cos A = \frac{b^2+c^2-a^2}{2bc}$, and its equivalent $a^2 = b^2+c^2-2bc\cos A$.

This can be proved very shortly by assuming Euclid II, 13 and 14; but it is better to base the proof on the theorem of Pythagoras.

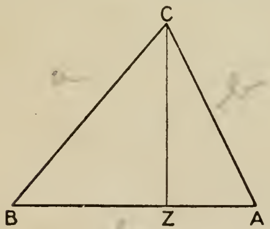


Fig. XXXV.

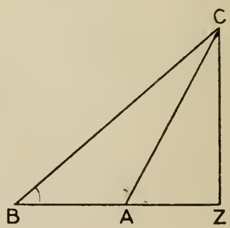


Fig. XXXV a.

Let CZ be the perpendicular from C on AB, Fig. XXXV.
Then $ZC = b \sin A$, $AZ = b \cos A$, and $BZ = c - b\cos A$.

$$BC^2 = BZ^2 + ZC^2,$$
$$a^2 = (c - b\cos A)^2 + (b \sin A)^2$$
$$= c^2 - 2bc\cos A + b^2\cos^2 A + b^2 \sin^2 A,$$

i.e. $$a^2 = b^2 + c^2 - 2bc\cos A,$$

or $$\cos A = \frac{b^2+c^2-a^2}{2bc}.$$

If A is obtuse, then in Fig. XXXV a,

$$ZC = b\sin(180 - A) = b\sin A,$$
$$AZ = b\cos(180 - A) = -b\cos A,$$
$$BZ = BA + AZ = C + (-b\cos A) = c - b\cos A.$$

The proof is now the same as before.

Exercise. Write down the corresponding formulae for $\cos B$ and $\cos C$.

56. The Projection formulae

$$\mathbf{c = b\cos A + a\cos B.}$$

In Fig. XXXV, BZ is the projection of BC on BA; and AZ is the projection of AC.

$$AB = AZ + BZ,$$

i.e. $$c = b\cos A + a\cos B.$$

Exercises. Supply the proof when A is obtuse.

Write down the other two corresponding formulae.

57. Area formulae

The symbol Δ is used to denote area of triangle.

(i) $\Delta = \frac{1}{2}$ **any side × perpendicular from opposite angle.** (Prop. 16.)

(ii) $\Delta = \frac{1}{2}\mathrm{AB} \times \mathrm{ZC} = \frac{1}{2}\mathrm{c}\,.\,\mathrm{b}\sin\mathrm{A} = \frac{1}{2}\mathrm{bc}\sin\mathrm{A}.$

(iii) $\Delta = \sqrt{\mathrm{s(s-a)(s-b)(s-c)}}.$

In Fig. XXXV, $BZ = a\cos B = a \times \dfrac{c^2 + a^2 - b^2}{2ca}.$

$$ZC^2 = a^2 - \frac{(c^2 + a^2 - b^2)^2}{(2c)^2};$$

$$\therefore\quad (2c\,.\,ZC)^2 = (2ac)^2 - (c^2 + a^2 - b^2)^2$$
$$= (a^2 + 2ac + c^2 - b^2)(b^2 - a^2 - 2ac + c^2)$$
$$= (a + b + c)(a - b + c)(a + b - c)(b + c - a).$$

Let $2s = a + b + c$, then $b + c - a = 2(s - a)$ &c.; so that

$$2c\,.\,ZC = \sqrt{2s\,.\,2(s-a)\,.\,2(s-b)\,.\,2(s-c)};$$

$$\therefore\quad \Delta = \tfrac{1}{2}\mathrm{AB\,.\,ZC}$$
$$= \sqrt{\mathrm{s(s-a)(s-b)(s-c)}}.$$

Exercise. Show that

$$16\Delta^2 = 2(b^2c^2 + c^2a^2 + a^2b^2) - (a^4 + b^4 + c^4).$$

58. From these formulae others may be deduced.

Example i. *To show that in any triangle*

$$\cos(A+B) = \cos A \cos B - \sin A \sin B.$$

From sine formula $\quad a \sin B - b \sin A = 0. \qquad$ (i)

From projection formula $\quad a \cos B + b \cos A = c. \qquad$ (ii)

Square and add, $\quad a^2 + b^2 + 2ab(\cos A \cos B - \sin A \sin B) = c^2.$

From cosine formula $\quad a^2 + b^2 - 2ab \cos C \qquad = c^2.$

It follows that

$$\cos C = -(\cos A \cos B - \sin A \sin B).$$

From the angle formula $C = 180 - (A+B)$,

i.e. $\qquad \cos C = -\cos(A+B).$

Hence $\qquad \cos(A+B) = \cos A \cos B - \sin A \sin B.$

Example ii. *In any triangle*

$$\sin(A-B) = \sin A \cos B - \cos A \sin B.$$

Multiply together equations (i) and (ii) above.

$$a^2 \sin B \cos B - b^2 \sin A \cos A - ab(\sin A \cos B - \cos A \sin B) = 0.$$

From Fig. XXXVI it is seen that

$a^2 \sin B \cos B = BZ \,.\, ZC = 2^{ce}$ triangle BZC,

and $\qquad b^2 \sin A \cos A = 2^{ce}$ triangle AZC

$\qquad\qquad = 2^{ce}$ triangle $A'ZC$,

($ZA' = ZA$, so that triangles CZA, CZA' are congruent).

$\therefore \quad a^2 \sin B \cos B - b^2 \sin A \cos A = 2^{ce}$ triangle BCA'

$\qquad = BC \,.\, CA' \sin BCA'$

$\qquad = ab \sin(A-B).$

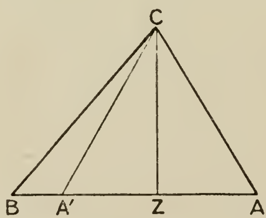


Fig. XXXVI.

Comparing this with the result above, we see that

$$\sin(A-B) = \sin A \cos B - \cos A \sin B.$$

This result can, however, be obtained more quickly.

For $$\frac{\sin BCA'}{A'B} = \frac{\sin CBA'}{CA'},$$

i.e. $$\frac{\sin(A-B)}{a\cos B - b\cos A} = \frac{\sin B}{b}.$$

Hence $$\sin(A-B) = \frac{a\sin B\cos B}{b} - \cos A\sin B$$
$$= \sin A\cos B - \cos A\sin B$$

since $a\sin B = b\sin A$

Example iii. *To show that the area of a quadrilateral inscribed in a circle is* $\sqrt{(s-a)(s-b)(s-c)(s-d)}$ *where* $s = \frac{1}{2}(a+b+c+d)$.

In Fig. XXXVII

Area of $ABCD$ = sum of triangles ABD and BCD
$$= \tfrac{1}{2}ad\sin A + \tfrac{1}{2}bc\sin(180-A)$$
$$= \tfrac{1}{2}(ad+bc)\sin A.$$

From triangle ABD,
$$BD^2 = a^2+d^2-2ad\cos A.$$

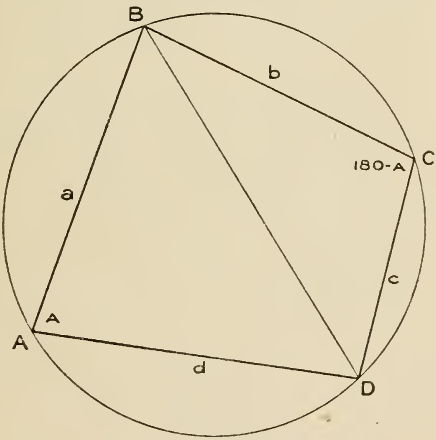


Fig. XXXVII.

From triangle BCD,
$$BD^2 = b^2+c^2-2bc\cos(180-A).$$

Hence $$a^2+d^2-2ad\cos A = b^2+c^2+2bc\cos A,$$

i.e. $$2(ad+bc)\cos A = a^2+d^2-(b^2+c^2);$$

$\therefore$ $$2(ad+bc)(1+\cos A) = (a+d)^2-(b-c)^2,$$

and $$2(ad+bc)(1-\cos A) = (b+c)^2-(a-d)^2.$$

Hence $4(ad+bc)^2(1-\cos^2 A)$

$$=(-a+b+c+d)(a-b+c+d)(a+b-c+d)(a+b+c-d),$$

i.e. $\{\frac{1}{2}(ad+bc)\sin A\}^2$

$$=\tfrac{1}{2}(-a+b+c+d)\,\tfrac{1}{2}(a-b+c+d)\,\tfrac{1}{2}(a+b-c+d)\,\tfrac{1}{2}(a+b+c-d)$$

$\therefore$ Area of $ABCD = \sqrt{(s-a)(s-b)(s-c)(s-d)}$.

Examples VI a.

1. From the three projection formulae deduce the three cosine formulae.

2. Prove that $\sin A = \sin B \cos C + \cos B \sin C$; and deduce that $\sin(B+C) = \sin B \cos C + \cos B \sin C$.

3. Prove that $\cos(A-B) = \cos A \cos B + \sin A \sin B$.

4. Show that $\Delta = \frac{1}{2}(b^2 \sin C \cos C + c^2 \sin B \cos B)$.

5. Show that $\Delta = \frac{1}{2}c^2\{\sin A \sin B \div \sin(A+B)\}$.

6. Prove that $\sin A + \sin B > \sin C$.

7. Prove that $\cot A + \cot B = c \operatorname{cosec} B \div a$.
What third expression are these equal to?

8. Show that
R (i.e. the radius of the circumcircle) $= s \div (\sin A + \sin B + \sin C)$.

9. Use the formula $\cos A = 1 - 2\sin^2 \frac{1}{2}A$ to prove that

$$\sin \tfrac{1}{2}A = \sqrt{(s-b)(s-c) \div bc}.$$

Write down the similar formulae for $\sin \frac{1}{2}B$ and $\sin \frac{1}{2}C$.

10. In a similar way to that suggested in the previous example, prove that $\cos \frac{1}{2}A = \sqrt{s(s-a) \div bc}$. Write down the formulae for $\cos \frac{1}{2}B$ and $\cos \frac{1}{2}C$. What is the formula for $\tan \frac{1}{2}A$?

11. Given $a = 17$, $b = 12$, $B = 37° 15'$, find A.

12. Given $a = 14$, $b = 13$, $c = 12$, find the greatest angle.

13. Given $a = 45$, $A = 45°$, $B = 60°$, find b.

14. Given $b = 17$, $c = 42$, $A = 72°$, find a.

15. Given $a = 176$, $b = 291$, $c = 352$, find all the angles.
(Choose a formula adapted for logarithms.)

16. Given $a = 7$, $b = 5$, $C = 49°$, find c.

17. Given $b = 9$, $c = 10$, $C = 57°$, find a.

18. By considering two forms for the area of an isosceles triangle, prove that $\sin A = 2 \sin \frac{1}{2}A \cos \frac{1}{2}A$.

19. Two sides of a triangle are 3 and 12 and the contained angle is $30°$; find the hypotenuse of an isosceles right-angled triangle of equal area.

20. Two adjacent sides of a parallelogram, 5 inches and 8 inches long respectively, include an angle of $60°$. Find the length of the two diagonals and the area of the figure.

21. If in a triangle $C = 60°$, prove that

$$1/(a+c)+1/(b+c) = 3/(a+b+c).$$

22. On a straight line AB, 4 inches long, describe a semicircle, and on the arc of the semicircle find points P, Q, R, S such that the areas of the triangles APB, AQB, ARB, ASB are 1 square inch, 2 square inches, 3 square inches, and 4 square inches respectively. If C is the centre of the circle, determine the sines of the angles ACP, ACQ, ACR, and ACS, and hence find, from the tables, the values of these angles.

23. If a quadrilateral can be inscribed in one circle and circumscribed about another, show that its area is $\sqrt{abcd}$, where a, b, c, d are the lengths of the sides.

The circles of the triangle.

59. It is shown in any Geometry textbook that

(i) the centre of the circumcircle is the point of concurrence of the perpendicular drawn at the middle points of the sides;

(ii) the centre of the inscribed circle is the point of concurrence of the three lines bisecting the three angles;

(iii) the centre of an escribed circle is the point of concurrence of the bisector of the opposite interior angle with the bisectors of the two adjacent exterior angles.

In Fig. XXXVIII, we have

$$AQ = AR, \qquad \text{(Prop. 24.)}$$
$$BP = BR,$$
$$CP = CQ;$$
$$\therefore \quad AQ+BP+CP = \tfrac{1}{2} \text{ sum of sides} = s.$$

Hence $\qquad AQ = s-a.$

Exercise. In a similar way, prove that

$$BP = \qquad CP' =$$
$$CQ = \qquad QQ' =$$
$$AQ' = \qquad PP' =$$

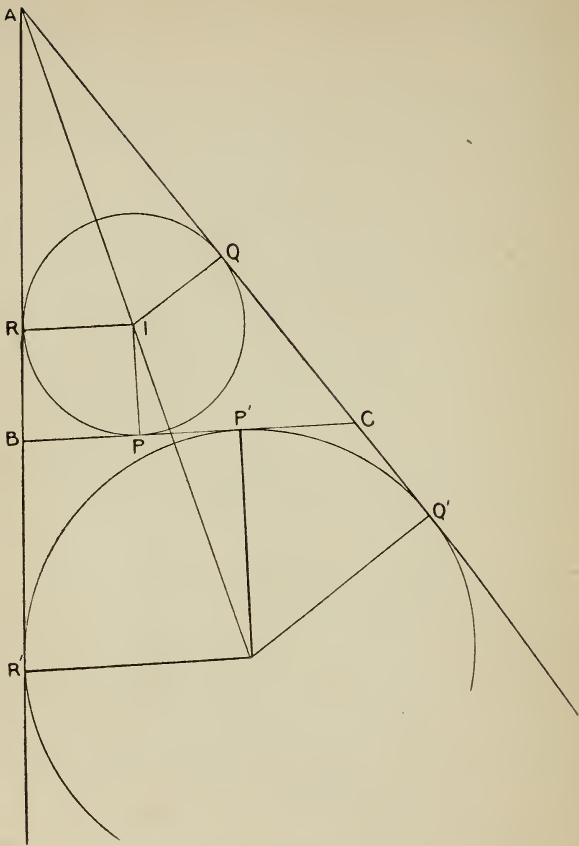


Fig. XXXVIII.

Examples VI b.

Prove the following formulae :

1. $R = a \div 2\sin A$.

2. $R = abc \div 4\Delta$.

3. $r = \Delta/s$. (Consider the sum of the triangles BIC, CIA, AIB.)

4. $r = a \div (\cot\frac{1}{2}B + \cot\frac{1}{2}C)$.

5. $r_1 = \Delta/(s-a)$.

6. $r_1 = a \div (\tan\frac{1}{2}B + \tan\frac{1}{2}C)$.

Using the above formulae, prove the following relations:

7. In a right-angled triangle $R+r=\frac{1}{2}(a+b)$.

8. $1/r_1+1/r_2+1/r_3=1/r$.

9. $1/r_2+1/r_3=2\div b\sin c$.

10. $r\,r_1\,r_2\,r_3=\Delta^2$.

11. $r\,r_1=(s-b)(s-c)$.

12. $(abc\div\sin A\sin B\sin C)^{\frac{1}{3}}$.

13. $2Rr=abc\div(a+b+c)$.

14. $4R\sin A\sin B\sin C=a\cos A+b\cos B+c\cos C$.

15. $\tan\frac{1}{2}A=\sqrt{(s-b)(s-c)\div s(s-a)}$.

16. $s^2=\Delta\cot\frac{1}{2}A\cot\frac{1}{2}B\cot\frac{1}{2}C$.

17. If ABC is a triangle such that $2b=a+c$, and p is the length of the perpendicular from B upon AC, show that $\tan\frac{1}{2}A$ and $\tan\frac{1}{2}C$ are equal to the roots of the equation

$$x^2-(b/p)\,x+\tfrac{1}{3}=0.$$

18. Show that the sum of the radii of the escribed circles of a triangle is equal to the radius of the inscribed circle together with four times the radius of the circumscribing circle.

19. Show that the area of the triangle formed by joining the centres of the escribed circles is

$$8R^2\cos\tfrac{1}{2}A\cos\tfrac{1}{2}B\cos\tfrac{1}{2}C.$$

20. The sides of a triangle are 3, 5, 6; find the radii of the inscribed and circumscribed circles.

21. In an isosceles triangle the base is 100 cm. and the perpendicular from the vertex is 70 cm.; find the radii of the inscribed and circumscribed circles.

22. A triangle is described with base $BC=5$ inches and angle $A=70°$. What is the radius of the circumcircle? Find the distance of the centre of the circumcircle from BC.

23. Find the radius of the circumcircle of the triangle ABC being given that $BC=7$, $CA=6$, and $C=60°$.

24. If $a=32$, $b=16$, $C=42°$, find R and r.

25. The area of a parallelogram having base 5·8 cm. and angle 123° is 37·7 sq. cm. Find the other sides and angles. Find the radii of the circles which pass through three of the corners of this parallelogram.

26. Two of the sides of a triangle are 7·5 cm. and 9·3 cm., the included angle is 37°. Find the radius of the circle which touches these sides produced and the third side.

Oral Revision Examples.

Complete the following identities and equations:

1. $\sin(270-A)=$
2. $\cos^2\theta=$
3. $2\tan A\cot A=$
4. If $\sin\theta=\frac{1}{2}$, $\theta=$
5. In any triangle $b^2=$
6. In any triangle $R=$
7. $\sin 2A=$
8. $\tan 225^\circ=$
9. If $\cos\theta=\frac{3}{5}$, $\tan\theta=$
10. Δ in terms of the sides $=$

---

11. $\tan^{-1}1=$
12. $\sec^2 A-1=$
13. length of arc $=$ radius $\times$
14. $\sin^2 B+\sin^2(90-B)=$
15. Definition of tangent.
16. In any triangle $\cos C=$
17. In any triangle $b\cos C+c\cos B=$
18. In any triangle $bc\sin A=$
19. In any triangle $r=$
20. $\tan\frac{1}{2}\pi=$

---

21. Definition of sine.
22. $\cos(360^\circ-B)=$
23. In any triangle $r_1=$
24. In any triangle $\cos A=$
25. What formula connects a, b, and B?
26. $\tan^{-1}(-\sqrt{3})=$
27. If $\cos\theta=-\frac{1}{2}$, $\theta=$
28. $\tan^2 73\frac{1}{2}^\circ+1=$
29. $abc=$
30. $37^\circ=$? radians.

---

31. $\cos^2(A-45^\circ)+\sin^2(A-45^\circ)=$
32. $b\sin C=$
33. Express R in terms of the sides.
34. If $\sin\theta=\sin\alpha$, then $\theta=$
35. $\cos\theta=$ (in terms of $\sin\frac{1}{2}\theta$).
36. Area of triangle $=$
37. $a^2+c^2-2ac\cos B=$
38. $\cos 1200^\circ=$
39. Maximum value of $2\sin\alpha\cos\alpha=$
40. $\cos^4\theta-\sin^4\theta=$ (in its simplest form).

---

41. $\Delta\div(s-a)=$
42. $\tan(180-B)=$
43. $a\cos C+c\cos A=$
44. $bc\sin A=$
45. $\sin^2(A+B)+\cos^2(A+B)=$
46. If $\cos x=\cos A$, then $x=$
47. In any triangle $\cos A=$
48. $\tan 60^\circ=$
49. $\cos 2\theta=$
50. How many radians $=A^\circ$?

Examples VI.

1. Prove that $(a \cos A - b \cos B) \div (a^2 - b^2) + \cos C/c = 0$.

2. Prove that $c^2 = (a+b)^2 \sin^2 \frac{1}{2} C + (a-b)^2 \cos^2 \frac{1}{2} C$.

3. In a triangle ABC the lines drawn from A and C, perpendicular to the opposite sides, intersect in O. If the angle A is acute, show that $OA = b \cos A/\sin B$.

Also draw a diagram in which A is an obtuse angle, and establish the corresponding expression for OA in that case.

4. Show that in any triangle the product of a side and the sines of the two adjacent angles is the same, whichever side be taken.

5. Find the area of a regular polygon of n sides circumscribed about a circle of radius r.

6. Regular polygons of 15 sides are inscribed in and circumscribed about a circle whose radius is one foot; show that the difference of their areas is nearly 20 square inches.

7. $ABCD$ are four points on a circle such that the angles BAC and BCA each equal θ. Show that $AD + CD = 2\,BD \cos \theta$.

8. If $2 \cos B = \sin A/\sin C$, prove that the triangle is isosceles.

9. If $\tan A / \tan B = \sin^2 A / \sin^2 B$, show that the triangle is isosceles or right-angled.

10. Express the sides of a triangle in terms of the angles and the semi-perimeter.

11. In a triangle ABC perpendiculars AD and BE are let fall on the opposite sides; prove that the radius of the circle circumscribing the triangle CDE equals $R \cos C$.

12. If in a triangle the median bisecting the base AB is perpendicular to the side AC, prove that $2 \tan A + \tan C = 0$.

13. If p and q are the lengths of the perpendiculars from A, B on any arbitrary line drawn through the vertex C of a triangle, prove that

$$a^2 p^2 + b^2 q^2 - 2ab\,pq \cos C = a^2 b^2 \sin^2 C.$$

14. An isosceles triangle, vertical angle 35°, is inscribed in a circle whose radius is 1·65 inches. Find the lengths of the sides.

15. Show that in any triangle

$$\frac{\cos A}{a} + \frac{\cos B}{b} + \frac{\cos C}{c} = \frac{a^2 + b^2 + c^2}{2abc}.$$

16. If R is the radius of the circumcircle of any triangle and x, y, z are the lengths of the perpendiculars let fall from its centre on the sides, prove that

$$R^3-(x^2+y^2+z^2)R-2xyz = 0.$$

17. The rectangular co-ordinates of the angular points of a triangle are (4, 5), (6, 7), (8, 6); determine the sum of the two smaller angles.

18. A rod AB, length $2a$, can turn about a hinge fixed to the wall at A; it is supported by a string BC, length l, fastened to a point C on the wall at a height h above A.

(i) If BC is horizontal, what is the inclination of the rod to the vertical?

(ii) If BC is horizontal, what is the inclination to the vertical of the line joining the hinge to the middle point of the string?

(iii) If the string and rod are inclined at θ and ϕ to the vertical respectively, prove that (i) $2a\sin\phi = l\sin\theta$, (ii) $l\cos\theta-2a\cos\phi=h$.

(iv) In the general case, what is the angle between the string and the rod? Give the answer in terms of h, a, θ or h, l, ϕ.

(v) In the general case, what is the inclination to the vertical of the line joining the hinge to the middle point of the string? Give the answer in terms of h, a, ϕ.

19. Three equal spheres of radius 7 centimetres are fixed in a horizontal plane so as to touch each other; a sphere of radius 6 cm. rests upon these three. Find the height of the centre of the fourth sphere above the horizontal plane, and the inclination to the vertical of the line joining the fourth centre to one of the lower centres.

20. Three equal rods of length 54 inches are fixed so as to form a tripod. If their feet are at the corners of an equilateral triangle, side 18 inches, find the inclination of each rod to the vertical.

21. In any triangle prove that the centroid trisects the line joining the circumcentre to the orthocentre.

22. Find the lengths of the sides of the pedal triangle of the triangle ABC. Find also the radii of the inscribed and circumscribed circles of that triangle.

(The pedal triangle is formed by joining the feet of the perpendiculars let fall from the vertices on the opposite sides.)

23. If $a = 5$ and $b = 4$, draw a graph to show the value of c as C varies from 0° to 180°. Hence find the value of c when $C = 40°$.

CHAPTER VII

SOLUTION OF TRIANGLES

60. It is known from Geometry that, if three parts of a triangle are given, the remaining parts can in some cases be found; and that, in other cases, relations between the missing parts may be found even though their exact values cannot be determined. When actual numbers are given, results can be obtained to a greater degree of accuracy by Trigonometrical methods than by drawing to scale. In all cases a formula is sought which shall contain the three given letters and one unknown letter.

61. Case I. Three angles given.

The angle formula shows that $A+B+C$ must be 180°. No formula contains the three angles and one side only; but from the sine formula, viz. $\dfrac{a}{\sin A}=\dfrac{b}{\sin B}=\dfrac{c}{\sin C}$, we can find the ratios of the sides.

62. Case II. Two angles and one side given.

The third angle can be found immediately since

$$A+B+C=180°.$$

Suppose a is the given side; and it is required to find b. The formula must contain a, b, and two of the angles; hence we use

$$\frac{b}{\sin B}=\frac{a}{\sin A}.$$

This is adapted for the use of logarithms as it involves no addition or subtraction. If the tables in use give the logarithms of the cosecant, it may be advisable to use the following logarithmic form $\qquad \log b=\log a+\log\sin B+\log\operatorname{cosec} A.$

63. Case III. One angle and the two sides containing the angle are given.

Suppose a, b, C are the given parts. Then the cosine formula $c^2=a^2+b^2-2ab\cos C$ enables us to determine c. When c is determined, the remaining angles can be found by the sine formula.

This method is of practical use only when the numbers involved are small; the cosine formula is not adapted for the use of logarithms. It is usual, therefore, to use the formula proved in §53,* viz.

$$\tan \tfrac{1}{2}(A-B) = \frac{a-b}{a+b}\cot \tfrac{1}{2} C.$$

This determines $\frac{1}{2}(A-B)$; also $\frac{1}{2}(A+B)$ equals the complement of $\frac{1}{2}C$; hence A and B are found by adding and subtracting.

The value of c is then calculated by the sine formula.

64. Case IV. One angle and the two sides not containing the angle are given.

Suppose a, b, A are given. Then we can determine c from the formula

$$a^2 = b^2 + c^2 - 2bc\cos A.$$

This is a quadratic equation to determine c, and it is seen that there is the possibility of two distinct values for c. This is also seen from the geometrical construction. On this account this case is usually known as the **Ambiguous Case.**

If there are two values of c, there will be two values for B and for C. This is seen independently if the sine formula is used (as it usually is, on account of its adaptability for logarithms):

$$\frac{\sin B}{b} = \frac{\sin A}{a}.$$

Suppose that this leads to the result

$$\sin B = \sin x.$$

Then

$$B = x \text{ or } 180 - x.$$

This shows that, if there are two solutions, those two solutions are supplementary. Hence one of the solutions will be obtuse. Preliminary geometrical considerations often show that there can be only one solution.

(i) If the given angle A is not acute, then B must be acute and the obtuse-angled solution must be rejected.

(ii) If $a > b$ or $= b$, then $A > B$ or $= B$; consequently B cannot be obtuse.

Exercises. When a, b, A are given, show (i) from the geo-

* Another proof is given on p. 103.

metrical solution, (ii) from the cosine formula, (iii) from the sine formula, that

(*a*) there is no solution, if $a < b \sin A$;
(*b*) there is one solution only, if $a = b \sin A$;
(*c*) there are two solutions, if $a > b \sin A$ but $< b$;
(*d*) there is one solution only, if $a > b$.

Point out the difference in nature of the one solution in (*b*) and (*d*).

65. Case V. Three sides given.

Here again the cosine formula may be used, if the numbers involved are not inconveniently large. For logarithmic calculation the formula for $\sin \frac{1}{2} A$, $\cos \frac{1}{2} A$, or $\tan \frac{1}{2} A$ is used. These half-angle formulae are derived from the cosine formula.

$$2 \sin^2 \tfrac{1}{2} A = 1 - \cos A \text{ * } \quad (\S\ 51)$$
$$= \frac{2bc - b^2 - c^2 + a^2}{2bc}$$
$$= \frac{a^2 - (b-c)^2}{2bc};$$
$$\therefore \quad \sin^2 \tfrac{1}{2} A = \frac{(a-b+c)(a+b-c)}{4bc};$$
$$\therefore \quad \mathbf{\sin \tfrac{1}{2} A = \sqrt{\frac{(s-b)(s-c)}{bc}}}.$$

Similarly
$$\mathbf{\cos \tfrac{1}{2} A = \sqrt{\frac{s(s-a)}{bc}}}.$$

Divide
$$\mathbf{\tan \tfrac{1}{2} A = \sqrt{\frac{(s-b)(s-c)}{s(s-a)}}}.$$

Of these three formulae it is best to use the tangent formula; for the logarithms used in finding $\tan \frac{1}{2} A$ are the same as those required for finding $\tan \frac{1}{2} B$ or $\tan \frac{1}{2} C$. If only one angle has to be found, it is indifferent which formula is used.

There is a simple geometrical proof for $\tan \frac{1}{2} A$.

* In old books on Trigonometry the 'haversine' was used for solving triangles, and the values of log haversine were tabulated in mathematical tables. The haversine equals half the versed sine; hence $\text{haversin } A = \frac{1}{2} \text{ versin } A = (1 - \cos A) \div 2 = \sin^2 \frac{1}{2} A$. The formula for solution of the triangle then becomes

$$\text{haversin } A = (s-b)(s-c) \div bc.$$

In Fig. XXXVIII I is the centre of the inscribed circle, E is the point of contact of the circle with AC.

Then
$$\tan\frac{A}{2} = \frac{IQ}{AQ}$$
$$= \frac{r}{s-a} \qquad (\S\ 59)$$
$$= \frac{\Delta}{s(s-a)}$$
$$= \sqrt{\frac{(s-b)(s-c)}{s(s-a)}}.$$

Examples VII a. (See p. 81 for arrangement of work.)

In the following triangles when

Case I.

1. $A = 79°\,20'$, $B = 64°\,10'$, find the ratios of the sides.

Case II.

2. $A = 58°\,12'$, $B = 64°\,33'$, $a = 385$, find b.
3. $A = 38°\,24'$, $C = 95°\,5'$, $c = 7{\cdot}832$, find a and b.
4. $B = 63°\,55'$, $C = 48°\,27'$, $c = 5{\cdot}75$, find b.

Case III.

5. $a = 409$, $b = 381$, $C = 58°\,12'$, find A and B.
6. $B = 23°\,46'$, $c = 9{\cdot}72$, $a = 8{\cdot}88$, find A and C.
7. $a = {\cdot}532$, $c = {\cdot}259$, $B = 39°\,33'$, find A and C.
8. $A = 73°\,15'$, $b = 7315$, $c = 8013$, find B and C.

Case IV.

9. $A = 38°\,14'$, $a = {\cdot}3\dot{3}$, $b = {\cdot}44$, find C.
10. $a = 409$, $b = 385$, $A = 64°\,32'$, find B and C.
11. $b = 6{\cdot}901$, $c = 5{\cdot}749$, $C = 48°\,27'$, find B.
12. $A = 73°\,15'$, $a = 7315$, $c = 8013$, find B and C.

Case V.

13. $a = 17$, $b = 13$, $c = 12$, find the least angle.
14. $a = 793$, $b = 937$, $c = 379$, find all the angles.
15. $s = 1410$, $a = 1437$, $b = 811$, find all the angles.
16. $s = 1437$, $a = 1410$, $b = 811$, find all the angles.

17. $a = 13$, $b = 7$, $C = 60°$, find A and B.
18. $a = 32$, $b = 40$, $c = 66$, find C.
19. $a = 250$, $b = 240$, $A = 72° 4'$, find B and C.
20. $a = 2b$, $C = 120°$, find A, B and the ratio of c to a.
21. $a = 36$, $b = 63$, $c = 81$, find the smallest angle.
22. $b = 5$, $c = 3$, $A = 42°$, find B and C.

Oral Examples.

State the formula to be used in the following cases:

1. Given a, b, C, find c.
2. Given a, b, C, find A and B.
3. Given b, c, C, find B.
4. Given b, c, C, find a.
5. Given c, a, C, find A.
6. Given c, a, C, find B.
7. Given c, A, B, find C.
8. Given c, a, A, find b.
9. Given a, b, B, find C.
10. Given a, B, A, find c.
11. Given a, b, c, find C.
12. Given A, B, C, find a.
13. Given A, C, b, find a.
14. Given a, b, B, find A.
15. Given a, c, B, find C.
16. Given c, A, B, find b.
17. Given a, b, c, find B.
18. Given b, c, A, find B.
19. What is the ambiguous case?
20. When a, c, A are given, what are the conditions that there should be no ambiguity?

Examples VII b.

Solve the following triangles:

1. $a = 5$, $b = 7$, $C = 30°$.
2. $b = 4$, $c = 3$, $C = 60°$.
3. $a = 65$, $b = 63$, $c = 16$.
4. $b = 8$, $c = 9$, $C = 45°$.
5. $a = 7$, $B = 120°$, $A = 45°$.
6. $a = 6$, $b = 7$, $c = 5$.
7. $b = 926{\cdot}7$, $A = 48° 24'$, $B = 31° 13'$.
8. $a = 407{\cdot}4$, $c = 115{\cdot}9$, $A = 127° 45'$.
9. $a = 1263$, $b = 1359$, $c = 1468$.
10. $a = 53{\cdot}94$, $b = 156{\cdot}5$, $C = 15° 13'$.
11. $b = 457{\cdot}2$, $c = 342{\cdot}6$, $A = 73° 45'$.
12. $a = 246{\cdot}7$, $b = 342{\cdot}5$, $B = 32° 17'$.
13. $c = 79{\cdot}48$, $A = 54° 16'$, $B = 85° 6'$.
14. $a = 7{\cdot}956$, $b = 10{\cdot}35$, $c = 9{\cdot}412$.
15. $b = 9463$, $c = 7590$, $C = 43° 47'$.
16. $a = 739$, $c = 937$, $B = 146° 12'$.

17. $c = 79{\cdot}5$, $A = 35^\circ 14'$, $C = 117^\circ 35'$.
18. $A = 89^\circ$, $B = 18^\circ 47'$, $C = 72^\circ 13'$.
19. $a = 87{\cdot}6$, $b = 57{\cdot}4$, $c = 46{\cdot}8$. 20. $a = 79$, $c = 97$, $\Delta = 2437$.
21. $A = 79^\circ$, $C = 97^\circ$, $R = 17{\cdot}2$. 22. $b = 73{\cdot}6$, $R = 57$, $a = 48{\cdot}9$.
23. $a^2 + b^2 = 841$, $\sin C = 1$, $\tan B = \frac{20}{21}$.
24. $A = 42^\circ 35'$, $a = 83$, $b = 74$.
25. $a = 2{\cdot}740$, $b = {\cdot}7401$, $C = 59^\circ 27'$.

Heights and Distances.

66. First a figure must be drawn, not necessarily to scale; the known lengths and angles should be indicated in the figure. It may be necessary to solve, or partly solve, more than one triangle before the required measurement is found. The scheme for working should be carefully thought out before the work is actually begun.

Example i. *Wishing to find the height of a house standing on the summit of a hill of uniform slope, I descended the hill for* 40 *feet, and then found the height subtended an angle of* $34^\circ 18'$. *On descending a further distance of* 60 *feet, I found the subtended angle to be* $19^\circ 15'$. *Find the height of the house.*

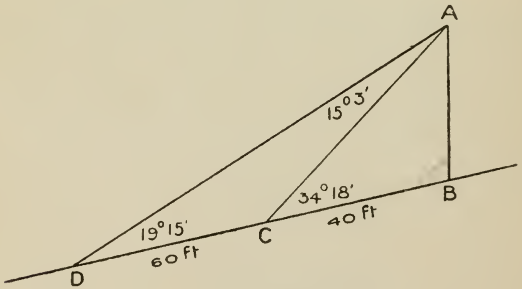


Fig. XXXIX.

Scheme.—In triangle *ADC* we know one side *CD* and all the angles; so *AC* can be found. Then in the triangle *ACB* two sides *AC*, *CB* are known, and the included angle, hence *AB* can be found.

From triangle ACD,

$$\frac{AC}{\sin ADC} = \frac{CD}{\sin DAC},$$

i.e. $$AC = 60 \sin 19^\circ 15' \operatorname{cosec} 15^\circ 3'$$
$$= 76{\cdot}182.$$

From triangle ABC,

$$\tan \tfrac{1}{2}(B-A) = \frac{b-a}{b+a} \cot \tfrac{1}{2} C$$
$$= \frac{36{\cdot}182}{116{\cdot}182} \cot 17^\circ 9';$$

$$\therefore \quad \tfrac{1}{2}(B-A) = 45^\circ 16',$$
$$\tfrac{1}{2}(B+A) = 72^\circ 51'.$$
$$\therefore \quad A = 27^\circ 35'.$$

Again, $$\frac{AB}{CB} = \frac{\sin ACB}{\sin CAB},$$

i.e. $$AB = \frac{40 \sin 34^\circ 18'}{\sin 27^\circ 35'}$$
$$= 48{\cdot}792.$$

Height of house = 48·8 feet.

Logarithms.
1·77815
$\bar{1}$·51811
·58559
1·88185
1·55849
+ ·51061
−2·06910
2·06514
·00396
1·60206
+$\bar{1}$·75091
1·35397
−$\bar{1}$·66562
1·68835

Example ii. *Wanting to know the height of a castle on a rock, I measured a base line of* 100 *yards, and at one extremity found the angle of elevation of the castle's top to be* 45° 15′, *and the angle subtended by the castle's height to be* 34° 30′; *also the angle subtended by the top of the castle and the other extremity of the base line was* 73° 14′. *At the other extremity the angle between the first extremity and the top of the castle was* 73° 18′. *Find the height of the castle.*

This requires a rough perspective figure of the whole, and subsidiary plane figures.

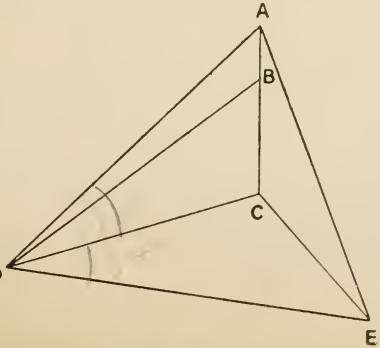


Fig. XL.

AB represents the castle.

C is the point in the same vertical as *AB*, and in the same horizontal plane as *DE*, the base line.

The following magnitudes are known:

DE = 100 yards.

ACD and *ACE* are each right angles.

ADC, *ADB* are known, therefore *BDC* is known.

ADE, *AED* are known.

Scheme. In triangle *ADE*, *DE* and the adjacent angles are known; hence *AD* can be found. *AB* can now be found from triangle *ABD*.

Fig. XLI.

From triangle *ADE*,

$$\frac{AD}{\sin 73^\circ 18'} = \frac{DE}{\sin 33^\circ 28'},$$

i.e. $$AD = \frac{100 \sin 73^\circ 18'}{\sin 33^\circ 28'}.$$

	2·
+	$\bar{1}$·9813
−	$\bar{1}$·7415
log *AD* =	2·2398

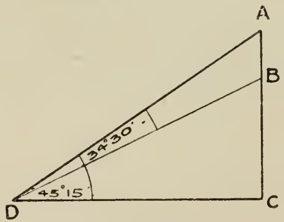


Fig. XLII.

From triangle *ABD*,

$$AB = \frac{AD \sin 34^\circ 30'}{\sin 79^\circ 15'}$$

Height of castle = 100 yards.

	2·2398
+	$\bar{1}$·7531
	1·9929
−	$\bar{1}$·9923
	2·0006

Example iii. *From the top of the Peak of Teneriffe the dip of the horizon is found to be* 1° 58′. *If the radius of the earth be* 4000 *miles, what is the height of the mountain?*

In Fig. XLIII C is the centre of the earth, AB is Teneriffe; BH is the tangent drawn from B to the earth's surface, so that H is the farthest point seen from B; in other words, H is on the horizon. The angle between BH and BD (the perpendicular to the vertical) is called the **dip of the horizon.**

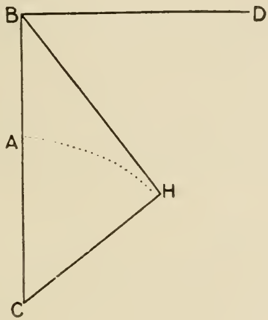


Fig. XLIII.

From triangle BCH,

$$\frac{BC}{CH} = \sec BCH,$$

$$\text{angle } BCH = \text{complement of } CBH = HBD;$$

$$\therefore \quad BC = 4000 \sec 1^\circ 58'$$
$$= 4000 \times 1{\cdot}00059$$
$$= 4002{\cdot}36 \text{ miles.}$$

$\therefore$ Height of mountain is 2·36 miles.

Note. Fig. XLIII is drawn much out of scale; for small heights BH and BD are practically identical. Even for mountains the dip is very small, as in this example; in fact, so small that we may use the approximation sine of dip = tan of dip = circular measure of dip.

If E be the other extremity of the diameter through B, we have, from § 52,

$$BA \,.\, BE = BH^2,$$

$$\text{i.e.}\quad h(2r+h) = d^2,$$

where r is radius of earth, h is height of place of observation, d is the distance of the horizon.

Hence $d = \sqrt{2rh + h^2}$;

$$\therefore\quad d = \sqrt{2rh}\left(1 + \frac{h}{2r}\right)^{\frac{1}{2}}$$

$$= \sqrt{2rh}\left(1 + \frac{h}{4r} - \frac{h^2}{32r^2}\dots\right) \text{ by the Binomial Theorem.}$$

So far the work is accurate; usually h/r is so small that it may be neglected. Hence for ordinary heights

$$\text{Distance of horizon} = \sqrt{2rh}.$$

Exercise. (i) In the formula just obtained r, h, and the distance are all expressed in the same units. By taking $r = 3960$ miles, prove that

Distance of horizon in miles

$$= \sqrt{\tfrac{3}{2} \times \text{height of place of observation in feet}}.$$

(ii) Show also that

$$\text{Dip in minutes} = \cdot 9784\sqrt{\text{height in feet}}.$$

Examples VII.

1. Standing at a horizontal distance 100 yards from the foot of a monument, a man observes the elevation of its top to be 25° 35′. Assuming the man's eye to be 5 feet from the ground, find the elevation of the top when the man stands 50 yards from the foot.

2. OA and OB are two straight roads intersecting at O and making with each other an angle of 35° 12′. A is a house 1572 yards from O, and B is a house 1129 yards from O. Find the direct distance between A and B.

3. A man observes the angles subtended by the base of a round tower at three points A, B, and C, in the same horizontal straight line with the centre of the circular base, to be 2α, 2β, 2γ respectively. Find the ratio of AB to BC, and find the diameter of the tower in terms of AC.

4. A man observes that the elevation of the top of a tower is $37° 40'$, and that the elevation of the top of a flagstaff on the tower is $43° 59'$; show that the height of the flagstaff is one-fourth of the height of the tower very nearly.

5. Having given that the least side of a triangle is 17·3 inches, and that two of the angles are $63° 20'$ and $72° 40'$, find the greatest side.

6. If two sides of a triangle are 7235 feet and 4635 feet respectively, and if the included angle is $78° 26'$, find the remaining angles of the triangle.

7. The base of a triangle being 7 feet, and the base angles $129° 23'$ and $38° 36'$, find the length of the shortest side.

8. Explain the ambiguous case of the solution of triangles. When a, b, A are given and the question is asked whether, from these data, two triangles, one triangle, or no triangle can be constructed, show that the question can be answered from a consideration of the roots of the equation

$$x^2 - 2\,bx \cos A + b^2 = a^2.$$

9. From each of two ships, a mile apart, the angle is observed which is subtended by the other ship and a beacon on shore; these angles are found to be $52° 25'$ and $75° 10'$ respectively. Find the distances of the beacon from each of the ships.

10. A ship sailing due north observes two lighthouses bearing respectively NE. and NNE. After the ship has sailed 20 miles the lighthouses are seen to be in a line due east. Find the distance in miles between the lighthouses.

11. The angles A, B, C of a triangle ABC are 40°, 60°, and 80° respectively, and CD is drawn from C to the base bisecting the angle ACB; if AB equals 100 inches, find the length of CD.

12. A man standing at a certain station on a straight sea-wall observes that the straight lines drawn from that station to two boats lying at anchor are each inclined at 45° to the direction of the wall, and when he walks 400 yards along the wall to another station he finds that the angles of inclination are 15° and 75° respectively. Find the distance between the boats and the perpendicular distance of each from the sea-wall.

13. From a house on one side of a street observations are made of the angle subtended by the height of the opposite house, first

from the level of the street, in which case the angle is $\tan^{-1}(3)$, and afterwards from two windows, one above the other, from each of which the angle is found to be $\tan^{-1}(-3)$. The height of the opposite house being 60 feet, find the height of each of the two windows above the street.

14. A segment of a circle stands on a chord AB 10 cm. long and contains an angle of 40°. A point C travels along the arc; for what value of the angle ABC is the chord CA three times the chord CB? Verify by drawing a graph showing the chord CA as a function of the chord CB.

15. If the sides of a triangle are 1011 and 525 feet, and the difference of the angles opposite to them is 24°, find (correct to the nearest degree) the smallest angle of the triangle.

16. A ladder is placed against the wall of a room and is inclined at an angle α to the floor. If the foot of the ladder slips outwards from the wall a distance of a feet, and the inclination of the ladder to the floor is then β, show that the distance which the top of the ladder will slide down the wall is $a \cot \frac{1}{2}(\alpha+\beta)$.

17. A man travelling due west along a straight road observes that when he is due south of a certain windmill the straight line drawn to a distant tower makes an angle of 30° with the direction of the road. A mile further on the bearings of the windmill and tower are NE. and NW. respectively. Find the distances of the tower from the windmill, and from the nearest point of the road.

18. A statue 10 feet high, standing on a column 100 feet high, subtends at the eye of an observer in the horizontal plane from which the column springs the same angle as a man 6 feet high standing at the foot of the column; find the distance of the observer from the column.

19. It is found that two points, each 10 feet from the earth's surface, cease to be visible from each other over a level plain at a distance of 8 miles; find the earth's diameter.

20. A plane, inclined at 33° to the horizontal, meets a horizontal plane in the line BC. From B a line BD is drawn on the inclined plane making an angle 27° with the horizontal plane. If BD is 18 inches long, find the height of D above the horizontal plane, and its distance from BC. Also find the angle BD makes with BC.

21. A lighthouse was observed from a ship to be N. 23° E.; after the ship had sailed due south for 3 miles, the same lighthouse bore N. 12° E. Find the distance of the lighthouse from the latter position of the ship.

22. Two streets meet at an acute angle; the one lies N. 51° W., and the other S. 48° W. The distance from the corner to a chemist's door in the first street is 315 yards; and the distance from the corner to a doctor's door in the other street is 406 yards. Find the length of a telephone wire going direct from the doctor's house to the chemist's.

23. From a vessel at anchor two rocks are observed to the westward, the one (A) bearing WNW., and the other (B) W. by S. from the vessel. From the chart it is found that A bears NNE. from B and is distant 645 yards from it. What are the distances of the rocks from the vessel?

24. Three objects A, B, and C forming a triangle are visible from a station D at which the sides subtend equal angles. Find AD, it being known that

$$AB = 12 \text{ miles}, \ AC = 6 \text{ miles}, \ CAB = 46° \, 34'.$$

25. A tower on the bank of a river, whose breadth is 100 feet, subtends angles $22\frac{1}{2}°$ and $67\frac{1}{2}°$ at two points A and B on the opposite bank of the river, whose distance apart is 600 feet, on a level with the base of the tower. Find the height of the tower.

*26. A, B, C are three given stations, so that the triangle ABC is completely known. Show how to determine, by means of angles measured at a fourth station P, the distances PA, PB, PC, the four stations being all in one plane, the case for consideration being that in which P is within the angle A, and the points P and A on opposite sides of BC.

If ABC is equilateral, and the angle BPC equals 60°, show that
$$2\cos(60° + BAP) + \cos(ABP - BPA) = 0.$$

27. A tower stands on the edge of a circular lake $ABCD$. The foot of the tower is at D, and the angles of elevation of the top of the tower from A, B, C, are α, β, γ respectively. If the angles BCA, BAC be each equal to θ, show that

$$\cot\alpha + \cot\gamma = 2\cot\beta\cos\theta.$$

* This example is best solved by using the formulae of §§ 83 and 84.

28. A mountain is observed from a place A to have elevation $15° 17'$ and to bear N. $24° 29'$ W. From another place B which is 2347 yards north of A its bearing is N. $37° 2'$ W. Deduce the elevation from B.

29. The extremity of the shadow of a flagstaff 6 feet high, standing on the top of a regular pyramid on a square base, just reaches a side of the base and is distant 56 feet and 8 feet from the extremities of that side. If the height of the pyramid be 34 feet, find the sun's altitude.

30. A man observes that when he has walked c feet up an inclined plane the angular depression of an object in the horizontal plane through the foot of the slope is α; and that, when he has walked a further distance of c feet, the angular depression of the object is β. Show that the inclination of the slope to the horizon is $\cot^{-1}(2\cot\beta - \cot\alpha)$; and determine the distance of the object observed from the foot of the slope.

31. A straight flagstaff, leaning due east, is found to subtend an angle α at a point in the plain upon which it stands, a yards west of the base. At a point b yards east of the base, the flagstaff subtends an angle β. Find at what angle it leans.

32. Four rods are loosely jointed at their extremities to form a parallelogram with sides 4 and 5 inches long. Two of the opposite corners are connected by an elastic string of length 7 inches. Find the angle between the string and the shorter side.

If the length of the other diagonal be diminished by 1 inch, what does the angle become?

*33. Three posts on the border of a lake are at known distances from each other, namely 63 yards, 44 yards, and 76 yards. At a boat on the lake it is found that the two posts, whose distance is 63 yards, subtend an angle $89° 15'$, and the two posts, whose distance is 76 yards, subtend an angle $130° 45'$. Find the distances of the boat from the three posts.

34. A base line AB is drawn 2 chains in length on a plane in the same horizontal plane as C the foot of a tree. The angles ABC, BAC are found to be $79° 56'$ and $78° 18'$ respectively; the angle of elevation of the top of the tree is found to be $19° 46'$ at A. Find the height of the tree to the nearest foot.

* This example is best solved by using the formulae of §§ 83 and 84.

35. A base line AB, 2527 links long, is measured on the sea-shore along the high water mark. C is a point where a distant rock meets the sea; the angles BAC, ABC are found to be $89° 15'$, $86° 21'$ respectively. The angle of elevation of the highest point of the rock, which is vertically above C, as observed at A, is $1° 48'$. Neglecting the curvature of the earth, find the height of the rock and its distance from A.

36. A hill slopes upwards towards the North at an inclination 14° to the horizontal. The sun is 15° W. of S., at an altitude of 47°; find the length of the shadow cast on the hill by a vertical post 39 feet high.

37. If, in the previous question, the post is perpendicular to the surface of the hill, what is the length of the shadow?

Revision Examples B.

1. Define the tangent of any angle, and prove from the definition that (i) $\tan(90+A) = -\cot A$; (ii) $\tan(180-A) = -\tan A$.
Express the other trigonometrical ratios in terms of the tangent.

2. Show by substitution that

$$\sin 45^\circ + \sin 30^\circ > \sin 60^\circ,$$

and

$$\cos 30^\circ - \cos 45^\circ < \cos 60^\circ.$$

3. Find the value of $\sin 45^\circ$ without using tables.
Solve the equation $4 \sin\theta \cos\theta + 1 = 2(\sin\theta + \cos\theta)$.
Give the general solutions.

4. A man walks directly across the deck of a ship, which is sailing due North at 4 miles an hour, in 12 seconds, and finds that he has moved in a direction 30° East of North. How wide is the deck?

5. Show that in any triangle ABC,
(i) $\sin A/a = \sin B/b = \sin C/c$;
(ii) $\sin C(a\cos B - b\cos A) = (a+b)(\sin A - \sin B)$.

6. Prove geometrically that

$$\cos 2A = 1 - 2\sin^2 A.$$

Hence find the value of $\sin 15^\circ$.

7. The angle of elevation of the top of a spire seen from A is 30°, and it is found that at a point B, $115\frac{1}{2}$ feet nearer the foot of the spire, it is 60°. Find the height of the spire to the nearest foot.

8. Plot a curve giving the sum of $4\sin\theta$ and $3\sin 2\theta$ from $\theta = 0^\circ$ to $\theta = 180^\circ$; and read off the angles at which the greatest and least values respectively of this sum occur.
Estimate the slope of the curve when $\theta = 90^\circ$ and when $\theta = 135^\circ$.

---

9. Define a radian. Express in degrees and minutes an angle of 1·36 radians.
Find the number of radians in the angle of a regular decagon.

10. Prove
(i) $\sin^2 A + \cos^2 A = 1$;
(ii) $\tan A \div (1 - \cot A) + \cot A \div (1 - \tan A) = \sec A \operatorname{cosec} A + 1$.

11. Draw the sine and cosine graphs, in the same figure, from $\theta = 10^\circ$ to $\theta = 20^\circ$.

From the graph find the angle which satisfies

$$\sin\theta + \cos\theta = 1{\cdot}2.$$

12. Find an expression which will include all angles having a given tangent. Write down the values of $\tan 225^\circ$, $\tan 780^\circ$, $\cot 1035^\circ$, $\cot 210^\circ$.

Construct an angle, having given the cotangent.

13. Find $a/\cos A + b/\cos B + c/\cos C$ in a form adapted to logarithmic calculation.

14. In any triangle prove that (i) $a = b\cos C + c\cos B$; (ii) $a(b\cos C - c\cos B) = b^2 - c^2$; (iii) $r\cos\frac{1}{2}A = a\sin\frac{1}{2}B\sin\frac{1}{2}C$.

15. If the sides of a parallelogram are a, b, and the angle between them ω, prove that the product of the diagonals is

$$\sqrt{a^4 - 2a^2b^2\cos\omega + b^4}.$$

16. A vessel is steaming towards the East at 10 miles an hour. The bearing of a lighthouse as seen from the vessel is 42° 24′ North of East at noon, and 25° 12′ East of North 25 minutes later. Find how far the vessel was from the lighthouse at noon, and find also at what time the bearing of the lighthouse will be due North.

17. Assuming that a circle may be treated as a regular polygon with an infinite number of sides, show that the ratio of the circumference of a circle to its diameter is constant.

What is the circular measure of the least angle whose sine is $\frac{1}{2}$, and what is the measure in degrees, &c., of the angle whose circular measure is ·15708?

18. Prove by a geometrical construction that

$$\cos 2A = \cos^2 A - \sin^2 A.$$

Solve the equation $\cos 2A = (\cos A + \sin A)^2$.

19. For what data will the solution of a triangle become ambiguous? Explain this.

Given $B = 30^\circ$, $c = 150$, $b = 50\sqrt{3}$, show that of the two triangles that satisfy the data one will be isosceles and the other right-angled. Find the third side in the greater of these triangles.

Would the solution be ambiguous if $B = 30^\circ$, $c = 150$, $b = 75$?

20. AB is a horizontal line whose length is 400 yards; from a point in the line between A and B a balloon ascends vertically,

and after a certain time its altitude is taken simultaneously from A and B; at A it is observed to be $64° 15'$; at B $48° 20'$; find the height of the balloon.

21. Find the radius of the circle circumscribing a triangle, in terms of its sides. If $c^2 = a^2+b^2$, show that this radius equals $\frac{1}{2}c$.

22. Define the trigonometrical ratios of A involved in the equation $\cot A + \tan A = \sec A \operatorname{cosec} A$; and establish its truth by a geometrical construction.

23. Prove that

$$\cos^{-1}\sqrt{(a-x)\div(a-b)} = \sin^{-1}\sqrt{(x-b)\div(a-b)}$$
$$= \cot^{-1}\sqrt{(a-x)\div(x-b)}.$$

24. Prove that $\sin\theta = \tan\theta \div \sqrt{1+\tan^2\theta}$.

Having given $\tan\theta = \frac{3}{4}$, find $\sin\theta$, $\cos\theta$, and $\operatorname{versin}\theta$.

---

25. If θ is an acute angle whose sine is $\frac{8}{17}$, calculate the value of $\tan\theta + \sec\theta$.

What would the value be if θ were obtuse?

26. What is the angle between the diagonal of a cube and one of the edges at its extremity?

27. Obtain an expression for all the angles which have a given tangent.

Find all the angles lying between $-360°$ and $+360°$ which satisfy the equation

$$\tan^2 x - \frac{2}{\sqrt{3}}\tan x - 1 = 0.$$

28. A circular wire of 3 inches radius is cut and then bent so as to lie along the circumference of a hoop whose radius is 4 feet. Find the angle which it subtends at the centre of the hoop.

29. A triangle ABC has angle $A = 34°$, $a = 11{\cdot}0$ cm., $c = 7{\cdot}8$ cm. Calculate the perpendicular from B on b, and the remaining angles and side of the triangle.

30. In a triangle $a = 74$, $b = 37$, $c = 97$; find the value of (i) $a\cos B + b\cos A$, (ii) $a\sin B - b\sin A$.

31. If ABC be a triangle, and θ an angle such that

$$\sin\theta = 2\sqrt{ab}\cos\tfrac{1}{2}C \div (a+b),$$

find c in terms of a, b, and θ.

If $a = 11$, $b = 25$, and $C = 106° 15\tfrac{1}{2}'$, find c.

32. Find the area of a regular quindecagon inscribed in a circle of one foot radius.

---

33. Find an expression for all angles having the same sine as the angle α.

Solve the equation $\sin(\alpha + x) + \sin(\beta + x) = 0$.

34. An angle α is determined by the equations $v^2 = 2gh$, $-b = tv \sin \alpha - \frac{1}{2}gt^2$, $tv \cos \alpha = a$. Show that

$$a^2 \tan^2 \alpha - 4ha \tan \alpha + a^2 - 4hb = 0.$$

35. Criticize the proposition that three measurements are sufficient and necessary to determine a triangle uniquely in shape and size.

36. A square house, measuring 30 feet each way, has a roof sloping up from all four walls at 35° to the horizontal. Find the area of the roof.

37. Draw up a table showing in three columns the values of $10 \sin \theta$, $10 \cos \theta$, and $8 \sin \theta + 6 \cos \theta$ for each 30° from 0° to 360°. From the table draw, in the same figure, the graphs of $y = 10 \sin \theta$ and $y = 8 \sin \theta + 6 \cos \theta$; and from the curves determine approximately a value of θ for which $\tan \theta = 3$.

38. Taking the earth as a sphere of radius 4000 miles, find the distance London travels in an hour in consequence of the rotation of the earth. (Latitude of London 51° 30′ N.)

39. *ABCD* is a quadrilateral in which *AB* and *DC* are parallel and 40 feet apart, and *AB* is 100 feet long. The angle *DAB* is 72° 30′, and the angle *CBA* is 38° 15′. Find the lengths of *AD*, *DC*, and *CB*, and the area of the quadrilateral.

40. State the local time at the following places when it is noon at Greenwich.

Cape Town 33° 56′ S., 18° 25′ E. Fiji 18° 0′ S., 178° 0′ E.
Edinburgh 55° 57′ N., 3° 10′ W. Singapore 1° 17′ N., 103° 50′ E.

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41. Define the cosine and the tangent of an angle, and show how to express the tangent in terms of the cosine.

Having given that $\cos A = \cdot 8$, and that A is less than 90°, find the value of $\tan A$; and by means of the tables find the value of A, both from its cosine and from its tangent.

42. Prove that, in any triangle ABC, $\sin B : \sin C = b : c$. In the triangle ABC the angle CAB is 50°, the angle ABC is 65°, and the side BC is 4 inches long. Find the length of the side AB.

43. Show how to find the height of a tree by means of a chain for measuring lengths and of an instrument for measuring angles.

44 Find an expression for all the angles which have (i) a given tangent, (ii) a given sine.

45. Explain how it is that, $\tan\theta$ being given, $\tan 2\theta$ is known; but that, $\sin\theta$ being given, $\sin 2\theta$ may have either of two values.

46. Prove that the area of a triangle is $\sqrt{s(s-a)(s-b)(s-c)}$. Show also that the area is $\frac{1}{2}c^2 \div (\cot A + \cot B)$.

47. Find the radius of the circumscribing circle of the triangle for which $A = 66° 30'$, $B = 11° 30'$, $c = 200$ feet.

48. A ship is sailing due East at a uniform rate: a man on a lighthouse observes that it is due South at 1 p.m. and 16° 30′ East of South at 1.20 p.m. In what direction will he see it at 2 p.m.?

CHAPTER VIII

PROJECTION. VECTORS

67. If from the extremities of a line PQ, of definite length, perpendiculars PK, QL are let fall on a line AB, which may be produced if necessary, then KL is called the **Projection** of PQ on the line AB.

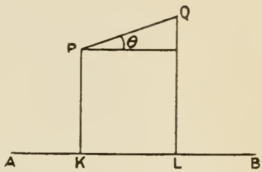


Fig. XLIV.

Projections are subject to the same convention of sign as are abscissae and ordinates. Thus, in the above figure, KL is positive, but LK is negative. It follows that the projection of PQ is not the same as the projection of QP, so that the order of the letters in naming a line is of great importance when we are dealing with projection. When the direction of the line is to be taken into account as well as its length, it is called a **directed length**; and we shall, in future, use the symbol (PQ) * to denote the directed length of the line from P to Q. The number of units of length in that line we shall continue to denote by the symbol PQ.

Thus, in Fig. XLIV, the projection of (PQ) is (KL),
and the projection of (QP) is (LK).

Note. When we speak of the sum of directed lengths in the same straight line, the algebraical sum is always meant. Geometrically this means that we require the directed length between the starting-point and final point, and not the length of the actual path traversed.

68. If the length of PQ is l, and if θ is the angle between PQ and the line AB, then

$$\text{projection of } (PQ) \text{ on } AB = l \cos \theta.$$

* This is usually written $\overline{PQ}$.

Some care is necessary in applying this formula; the safest plan is to keep l and θ both positive.

Consider, for instance, the projection of (QP) in Fig. XLV.

Imagine a line drawn from the initial point Q parallel to the line AB. Then it is seen that the angle between (QP) and AB is $\theta+\pi$, while the length QP is l.

Hence projection of (QP) on $AB = l\cos(\theta+\pi) = -l\cos\theta$.

Two other methods of treatment give the same result.

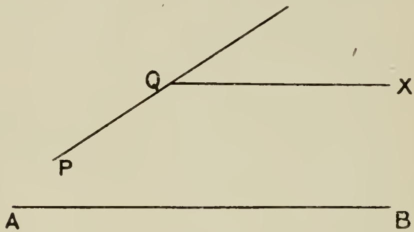


Fig. XLV.

In Fig. XLV the line QX is actually drawn parallel to AB; but it is usually sufficient to imagine it. Then we may take the angle between (QP) and QX to be the negative angle XQP, i.e. $-(\pi-\theta)$; the length QP is positive so that

$$\text{projection of } QP = l\cos(-\pi-\theta) = -l\cos\theta.$$

Or we may regard θ as being the angle between (QP) and QX; but this requires that the length of (QP) should be taken as $-l$, and so the projection of (QP) on $AB = -l\cos\theta$.

It will be found that, in all cases, $l\cos\theta$ gives both the magnitude and sign of the projection of (PQ) on AB. Similarly,

the projection of (PQ) on a line perpendicular to AB

$$= l\sin\theta.$$

69. Proposition A. *The sum of the projections on any* *line of two sides (AB), (BC) of a triangle is equal to the projection of the third side (AC).*

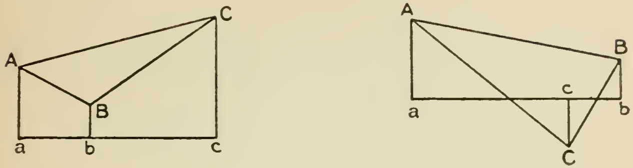


Fig. XLVI.

In either of the above figures (or in any other figure)

projection of (AB) + projection of $(BC) = (ab) + (bc)$

$= (ac)$

$=$ projection of (AC).

Proposition B. *The sum of the projections on any line of the three sides (AB), (BC), (CA) of a triangle is zero.*

Sum of projections of (AB), (BC), $(CA) = (ab) + (bc) + (ca)$.

Hence on the line of projection we start at the point a and finish at the same point, so that the distance between the initial and final points is zero. That is, the sum of the projections is zero.

Proposition C. *In any closed figure ABC ... HK, the sum of the projections of the sides (AB), (BC) ... (HK) equals the projection of (AK).*

Proposition D. *In any closed figure the sum of the projections of all the sides taken in order in the same direction is zero.*

Propositions C and D are proved exactly in the same way as Propositions A and B.

Example. *Prove that*

$$\cos A + \cos(120 + A) + \cos(120 - A) = 0.$$

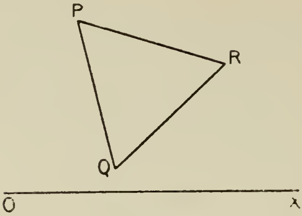


Fig. XLVII.

Draw an equilateral triangle PQR, side a units.

Draw a line OX inclined at an angle A to (QR).

Then (RP) is inclined at $A + 120$ degrees to OX; and (PQ) is inclined at $A + 240$ degrees.

Project on OX; then, by Proposition B,

$$a \cos A + a \cos(A + 120) + a \cos(A + 240) = 0;$$

but $\cos(A + 240) = \cos\{360 - (120 - A)\} = \cos(120 - A)$;

$$\therefore \quad \cos A + \cos(120 + A) + \cos(120 - A) = 0.$$

Examples VIII a.

(These examples should be verified by drawing a figure to scale.)

1. Show that the projection of a line on a line parallel to itself is equal to the projected line, and that the projection of a line on a line perpendicular to itself is zero.

2. A line of length r, making an angle θ with OX is projected on OX and at right angles to OX; calculate the lengths of the projections in the following cases:

(i) $r = 5,\ \theta = 60°$; (ii) $r = -5,\ \theta = 120°$;
(iii) $r = 5,\ \theta = 248°$; (iv) $r = 5,\ \theta = 300°$;
(v) $r = -5,\ \theta = 330°$.

3. Two rods AB, BC, of lengths 5 feet and 10 feet respectively, are joined together at an angle of 135°. The rods are fixed in

a vertical plane so that CB is inclined at 60° to the horizontal, and the angle ABC is beneath the rods; by projecting horizontally and vertically, find the inclination of the line AC to the horizontal.

4. By projecting a diagonal and two sides of a square on a line making an angle $A°$ with one of the sides, prove that

$$\cos(A+45°) = (\cos A - \sin A) \div \sqrt{2}.$$

Find a similar value for $\sin(A+45°)$.

5. PQR is a triangle right-angled at Q, having the angle at P equal to $A°$; PQ is inclined to OX at an angle $B°$.

Prove by projection that

$$\cos(A+B) = \cos A \cos B - \sin A \sin B,$$

and

$$\sin(A+B) = \sin A \cos B + \cos A \sin B.$$

70. If the projections of a line on two lines at right angles are given, the length and direction of the projected line can be found, but not its actual position.

Let r be the length of the line and θ the angle it makes with one of the lines of projection. Then $r\cos\theta$ and $r\sin\theta$ are known; suppose these values are x and y respectively, so that $r\cos\theta = x$ and $r\sin\theta = y$.

Then $r = \sqrt{x^2+y^2}$ and $\tan\theta = y/x$.

The projected line has therefore a definite length and a definite direction; it is the simplest example of a group of quantities called **vector quantities** or **vectors**.

71. A quantity which possesses a direction as well as magnitude is called a **vector**. Velocities and forces are examples of such quantities. The magnitude and direction can be represented by the length and direction of a directed straight line; hence the properties of a directed straight line that depend only on its length and direction represent properties common to all vectors.

72. Vector addition or Composition of Vectors.

A displacement from A to B followed by a displacement from B to C produces the same result as a single displacement from A to C.

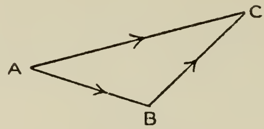


Fig. XLVIII.

Or we may regard the displacements as being simultaneous.

Suppose a point to start from A and move along AB, and while this point is moving, suppose the line AB to move parallel to itself, the point B moving to C while the point travels from A to B. The result of the two simultaneous displacements is that the point has travelled from A to C.

Hence the vector (AC) is called the resultant of the vectors (AB) and (BC).

Finding one quantity equivalent to two or more of the same kind is equivalent to the process of addition in Arithmetic.

If we use the sign + to denote this process, we have

$$(AC) = (AB) + (BC).$$

If P, Q, and R are the respective magnitudes of the vectors represented by (AB), (BC), and (AC), and if θ is the angle between the directions of (AB) and (BC) (in Fig. XLVIII the angle ABC is the supplement of θ);

then $$\mathbf{R}^2 = \mathbf{P}^2 + \mathbf{Q}^2 + 2\,\mathbf{PQ}\cos\theta.$$

Similarly, if a number of vectors are represented by the directed lengths (AB), (BC), $(CD) \ldots (HK)$, then their resultant is represented by the directed length (AK).

73. Resolution of vectors.

In Fig. XLVIII the vector (AC) may be replaced by the two vectors (AB) and (BC). Viewed in this light they are called the **components** of the vector (AC).

When we talk of the component of a vector in a given direction, and no mention is made of the direction of the other component, it is understood that the other component is at right angles to the first.

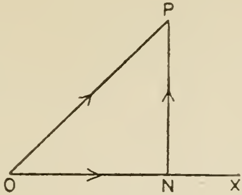


Fig. XLIX.

If (OP) in Fig. XLIX represents a vector of magnitude R inclined at an angle θ to OX, then its projection (ON) represents the component along OX, and the projection (NP) represents the component perpendicular to OX.

The vector is now said to be **resolved** along and perpendicular to OX.

Resolving along OX, we find that the component is $R\cos\theta$.

Resolving perpendicular to OX, we find that the component is $P\sin\theta$.

74. All the work of § 69 on projections can be applied to vectors and their components. For instance, Proposition C gives the following proposition:

The sum of the components of any number of vectors in a given direction is equal to the component of their resultant in that direction.

Examples VIII b.

[In the following examples the letters P, Q, R imply that the vectors are forces; the letters u, v, w imply that the vectors are velocities. When possible, figures should be drawn to scale to check the calculation.]

1. Find the resultant R in the following cases:

(i) $P = 17$, $Q = 13$, $\theta = 40^\circ$;
(ii) $P = 17$, $Q = 13$, $\theta = 140^\circ$;
(iii) $P = 114$, $Q = 75$, $\theta = 65^\circ$;
(iv) $P = 123$, $Q = 496$, $\theta = 117^\circ$.

2. Find P when Q the other vector, θ the angle between them, and R their resultant have the following values:

(i) $Q = 176,\ R = 249,\ \theta = 72°$;
(ii) $Q = 73,\ R = 193,\ \theta = 110°$;
(iii) $Q = 245,\ R = 92,\ \theta = 130°$;
(iv) $Q = 36,\ R = 84,\ \theta = 20°$.

3. Show that, if the resultant of three forces is zero, the sum of their components in any direction is zero.

4. Show that if three forces produce equilibrium (their resultant is, therefore, zero) they are parallel and proportional to the sides of a triangle.

5. A boat is being rowed due E. at a speed of 6 miles an hour; at the same time a current carries it due S. with a speed of 3 miles an hour; find the magnitude and direction of the actual velocity.

6. Find the resultant of velocities u and v inclined at an angle θ, when

(i) $u = 14,\ v = 16,\ \theta = 180°$;
(ii) $u = 14,\ v = 16,\ \theta = 65°$;
(iii) $u = 14,\ v = 16,\ \theta = 135°$.

7. Vectors of magnitudes 7, 8, 9 respectively are parallel to three consecutive sides of a regular hexagon. Find the sum of their components (i) parallel to, (ii) perpendicular to, the middle one of these sides. Hence find the magnitude and direction of their resultant.

8. Find the magnitude and direction of the resultant of four forces of magnitudes 5, 10, 15, 20 respectively, which act along the sides of a square.

9. A stream flows at the rate of 2 miles an hour. In what direction must a man swim in order that he may actually go straight across the river, his rate of swimming being 3 miles an hour?

10. A rod 5 feet long is hung by a string, attached to its two ends, over a smooth peg; it rests, at an angle of 20° to the horizontal, so that the two portions of the string are each inclined 35° to the vertical. Find the length of the string.

Projection on a Plane.

75. If from every point in a line, straight or curved, a perpendicular be let fall on a plane, the locus of the feet of the perpendiculars is called the projection of the line on the plane.

If from every point in the boundary of a surface a perpendicular be let fall on a plane, the area bounded by the locus of the feet of the perpendiculars is called the projection of the surface on the plane.

76. The angle between a straight line and its projection on a plane is called the angle between the straight line and the plane. It follows that the projection on a plane of a straight line of length l, making an angle α with the plane, is $l \cos \alpha$.

Any two planes, not parallel, intersect in a straight line. If from any point P in this line two perpendiculars PA, PB are drawn to it, one in each plane, then the angle APB measures the angle between the planes.

77. If any plane surface, of area A, is projected on a plane making an angle α with its own plane; then the area of the projection is $A \cos \alpha$.

*Step I. Consider a rectangle $ABCD$, having the side AB parallel to the plane of projection, and the side BC making an angle α with that plane; then α is the angle between the plane of the rectangle and the plane of projection.

Then, in Fig. L, $abcd$ is the projection of $ABCD$.

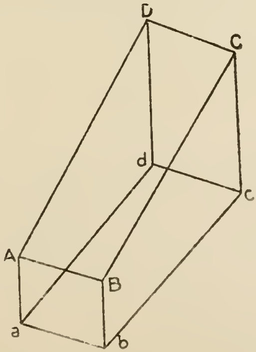


Fig. L.

* A slight knowledge of solid geometry is assumed in this proof.

Now Bb is perpendicular to the plane $abcd$, and therefore to the line ab;

$\therefore$ Bb is perpendicular to AB;

but BC is perpendicular to AB;

$\therefore$ AB is perpendicular to plane $BCcb$;

$\therefore$ ab is perpendicular to plane $BCcb$;

$\therefore$ ab is at right angles to bc,

i.e. $abcd$ is a rectangle.

Hence area of $abcd = ab \times bc$

$= AB \times BC \cos \alpha$

$=$ area of $ABCD \times \cos \alpha$.

Step II. Consider a plane area with curved or rectilinear boundary. In the plane of the figure draw any line PQ parallel to the plane of projection. Then in the area we can inscribe a number of rectangles having the short sides parallel to PQ and the longer sides perpendicular to PQ.

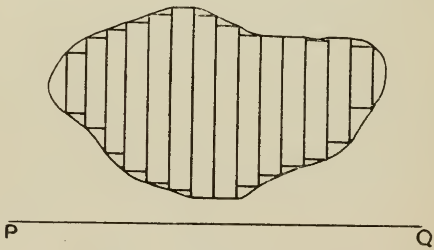


Fig. LI.

The sum of these rectangles is less than the original area, but may be made to differ from that area by as small a quantity as we please by making their width small enough; and then the sum of their projections will differ from the projection of the area by an even smaller quantity. Hence in the limit, when the width is indefinitely small, the sum of each set of rectangles will equal the area of the corresponding circumscribing figure.

But the sum of projections of rectangles $=$ sum of rectangles $\times \cos A$;

$\therefore$ the area of projected figure $=$ area of original figure $\times \cos A$.

Examples VIII c.

1. A pyramid $VABCD$ has a square base $ABCD$, side a, and the faces VAB, &c., are equilateral triangles. Find the length of the projection of VA on the base.

Verify that the sum of the areas of the projections of the four faces is equal to a^2.

2. A square house, whose side is 28 feet long, has a roof sloping up from all four walls at 40° to the horizontal, find the area of the roof.

3. Find, by projection, the area of the curved surface of a right circular cone, having height h, and semi-vertical angle 2α.

4. From a cone 6 feet high a smaller cone 2 feet high is cut off. If the radius of the base of the small cone is 1·6 feet, find the area of the curved surface of the remainder of the large cone.

Verify your answer by projecting this surface on the base.

5. A circle with radius a is projected into an ellipse with semi-axes a and b; show by projection that the area of the ellipse is πab.

6. The vertical angle of a conical tent is 67°, and the radius of the base is $5\frac{1}{2}$ feet; find (i) the slant height, (ii) the area of canvas used, (iii) the content of the tent.

7. A pyramid on a square base is such that each of the other faces is an isosceles right-angled triangle, find by projection the angle between a triangular face and the base.

Geometrical representation of imaginary quantities.

78. In Fig. LII OA is of length r.

By the usual convention a line OA drawn to the right is

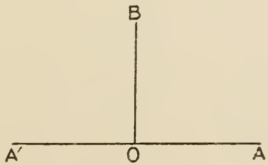


Fig. LII.

considered positive, so that (OA) represents $+r$. If now (OA) is turned through two right angles, it takes up the position (OA') and, by the usual convention, (OA') represents $-r$. Hence the geometrical operation of turning through two right angles represents the algebraical operation of multiplying by -1. Let us consider what the operation of turning through one right angle represents.

This is an operation which, if performed twice in succession, turns through two right angles, which represents multiplication by -1.

But the algebraical operation of multiplying by $\sqrt{-1}$, if performed twice in succession, multiplies by -1.

Hence it seems reasonable that the operation of turning a vector line through a right angle represents the algebraical operation of multiplying by $\sqrt{-1}$; that is, (OB) at right angles to (OA) represents $r \times \sqrt{-1}$, i.e. $\sqrt{-1}\, r$.

In future we shall denote $\sqrt{-1}$ by i.

79. With the interpretation of i suggested by the last section,

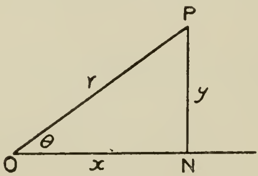


Fig. LIII.

$x + iy$ is represented by a vector line of length x followed by a vector line of length y at right angles to the first vector.

$$x + iy = (ON) + (NP) \qquad \text{(Fig. LIII.)}$$
$$= (OP). \qquad \text{(By vector addition, § 72.)}$$

Or, in words, $x + iy$ is represented by the vector (OP), that is by a vector line of length $\sqrt{x^2 + y^2}$, making an angle $\tan^{-1}\frac{y}{x}$ with the positive direction.

80. For our purposes this statement is more useful if reversed, viz.

$$(OP) = x + iy$$
$$= r\cos\theta + ir\sin\theta$$
$$= (\cos\theta + i\sin\theta)\, r.$$

Or, in words, the vector line of length r, in direction θ, represents the magnitude r multiplied by $\cos\theta + i\sin\theta$. This gives the important result that

turning through an angle θ represents multiplication by

$$\cos\theta + i\sin\theta.$$

Hence

turning twice in succession through θ represents multiplication by $\cos\theta + i\sin\theta$ repeated twice;

i.e. turning through 2θ represents multiplication by

$$(\cos\theta + i\sin\theta)^2;$$

but turning through 2θ represents multiplication by

$$(\cos 2\theta + i\sin 2\theta).$$

Hence the suggested interpretation of $\sqrt{-1}$ or i, leads to the identity $(\cos 2\theta + i\sin 2\theta) \equiv (\cos\theta + i\sin\theta)^2$.

If this is verified by algebraic multiplication and by the use of the ordinary formulae for $\cos 2\theta$ and $\sin 2\theta$, it will be found correct.

Carrying on the argument in the same way, we deduce that

$$(\cos n\theta + i\sin n\theta) \equiv (\cos\theta + i\sin\theta)^n,$$

where n is any positive integer.

Again, turning through a half θ is an operation which, if repeated, turns through θ, and, therefore, represents a multiplication which, if repeated, multiplies by $\cos\theta + i\sin\theta$;

i.e. $$(\cos\tfrac{1}{2}\theta + i\sin\tfrac{1}{2}\theta) = (\cos\theta + i\sin\theta)^{\frac{1}{2}}.$$

Similarly, $$\left(\cos\frac{\theta}{n} + i\sin\frac{\theta}{n}\right) = (\cos\theta + i\sin\theta)^{\frac{1}{n}};$$

and $$\left(\cos\frac{p}{q}\theta + i\sin\frac{p}{q}\theta\right) = (\cos\theta + i\sin\theta)^{\frac{p}{q}}.$$

Lastly, turning through $-\theta$ cancels turning through θ, and, therefore, represents an operation which cancels multiplication by $(\cos\theta + i\sin\theta)$;

i.e. $$\{\cos(-\theta) + i\sin(-\theta)\} = (\cos\theta + i\sin\theta)^{-1}.$$

Similarly,

$$\cos(-n\theta) + i\sin(-n\theta) = (\cos\theta + i\sin\theta)^{-n},$$

where n is any positive quantity.

We have now deduced from the geometrical interpretation of $\sqrt{-1}$ that

$$(\cos n\theta + i \sin n\theta) = (\cos\theta + i\sin\theta)^n$$

for all real values of n.

This is known as De Moivre's Theorem.

Example. *Use De Moivre's Theorem to find* $\sqrt[3]{1}$.

$$\cos 2n\pi = 1, \ \sin 2n\pi = 0,$$

where n is zero or any integer.

Hence $$1 = \cos 2n\pi + i\sin 2n\pi\,;$$

$$\therefore\ \sqrt[3]{1} = (\cos 2n\pi + i \sin 2n\pi)^{\frac{1}{3}}$$
$$= \cos\tfrac{2}{3}n\pi + i\sin\tfrac{2}{3}n\pi.$$

If $n = 0$, $\sqrt[3]{1} = \cos 0 + i \sin 0 = 1$.

If $n = 1$, $\sqrt[3]{1} = \cos\frac{2}{3}\pi + i\sin\frac{2}{3}\pi = \frac{1}{2}(-1 + i\sqrt{3})$.

If $n = 2$, $\sqrt[3]{1} = \cos\frac{4}{3}\pi + i\sin\frac{4}{3}\pi = \frac{1}{2}(-1 - i\sqrt{3})$.

If $n = 3$, $\sqrt[3]{1} = \cos 2\pi + i\sin 2\pi = 1$.

For other values of n it is seen that the three roots are repeated, Hence De Moivre's Theorem shows that there are three different cube roots of unity. They are, of course, the three roots of the equation $x^3 - 1 = 0$. The student should verify that the same roots are obtained by Algebra.

Examples VIII d.

Represent graphically and by imaginary quantities the following vectors:

1. (i) Magnitude, $r = 25$; direction $\theta = \alpha$ where $\alpha = \tan^{-1}\frac{4}{31}$.
   (ii) ,, $r = 25$; ,, $\theta = \pi - \alpha$;
   (iii) ,, $r = 25$; ,, $\theta = \pi + \alpha$;
   (iv) ,, $r = 25$; ,, $\theta = 2\pi - \alpha$;
   (v) ,, $r = 25$; ,, $\theta = -\alpha$.

2. Show graphically that
$$(x' + iy') + (x'' - iy'') = (x' + x'') + i(y' + y'').$$

3. Express the following in the form $\gamma(\cos\theta + i\sin\theta)$:
   (i) $3 + 4i$; (ii) $5 + 6i$; (iii) $7 - 8i$;
   (iv) $-5 - 12i$; (v) $-5 + 12i$; (vi) $8i$.

4. Interpret geometrically $(\cos\alpha + i\sin\alpha)(\cos\alpha - i\sin\alpha)r$; and justify your interpretation.

5. Show graphically that

$$(\cos\alpha + i\sin\alpha)(\cos\beta + i\sin\beta) = \cos(\alpha+\beta) + i\sin(\alpha+\beta).$$

6. Verify De Moivre's Theorem by calculation, when $n = 2, 3, -1, -2$.

7. Assuming De Moivre's Theorem, prove that $\sqrt[3]{-1}$ has three values, viz. -1, $\frac{1}{2}(1+\sqrt{-3})$ and $\frac{1}{2}(1-\sqrt{-3})$.

8. (*a*) Prove De Moivre's Theorem by induction when n is a positive integer.

(*b*) Deduce the proof when n is not a positive integer, by methods similar to those used for the Binomial Theorem in Algebra.*

Examples VIII.

1. A man walks one kilometre in a direction 16 degrees North of East; he then turns to the left, through an angle of 110 degrees, and walks one kilometre in the new direction. How far is he North and how far East of his starting-point?

2. Show that $a\cos\theta + b\sin\theta$ can be expressed in the form $r\cos(\theta-\alpha)$. Illustrate by a figure.

3. A number of rods are jointed together, and the two free ends are secured to two points A and B in the same horizontal line and distant c inches. If the length of the r^{th} rod is a_r, and its inclination to the horizontal is θ_r (all the angles being measured in the same sense), prove that (i) $\Sigma(a_r\cos\theta_r) = c$; (ii) $\Sigma(a_r\sin\theta_r) = 0$. (See § 89, Example ii.)

4. Prove by projection that

$$\sin(90+A) = \cos A \text{ and } \sin(270-A) = -\cos A.$$

5. In what respects can a vector quantity be represented by a straight line?

If three forces P, Q, R, acting at a point O, are such that $P/\sin QR = Q/\sin RP = R/\sin PQ$ (where $\sin PQ$ denotes the sine of the angle between P and Q), show that the three forces produce equilibrium.

6. A man walks one kilometre in a direction $A°$ North of East, one kilometre in a direction making 120° with the first direction, and one kilometre at an angle 240° with the first direction. Draw a figure showing that he has now returned to the starting-

* See *School Algebra*, pp. 407, 435, 465.

point; and by considering the distances he has gone to the East and North write down two trigonometrical identities concerning the sines and cosines of A, $120+A$, $240+A$.

7. Suggest a geometrical construction which may help to sum the series:

(i) $\cos\alpha+\cos(\alpha+\beta)+\cos(\alpha+2\beta)+\ldots n$ terms;

(ii) $\sin\alpha+\sin(\alpha+\beta)+\sin(\alpha+2\beta)+\ldots n$ terms.

Deduce that both these sums become zero if $n\beta=2\pi$.

8. A body which weighs 12 lb. is kept at rest by means of two cords, one being horizontal and the other inclined to the horizontal at an angle whose tangent is $\frac{3}{4}$; find the forces exerted by the cords.

9. A mine shaft is 1650 feet in length. It slopes downwards at an angle of 45° to the horizon for a certain part of its total length, say x feet, and at an angle of 35° for the rest of its length. If the total depth reached is 1000 feet, obtain an equation for x, and hence calculate x.

10. A man playing five holes of a golf course first walks 260 yards due East, then 140 yards 20° South of East, then 300 yards due South, then 200 yards 40° West of North, then 220 yards 30° West of South, thus arriving at the fifth hole. Find how far the fifth hole is from the first tee.

11. The perpendicular from the origin on a straight line equals p and makes an angle α with the axis of x; by projecting the co-ordinates of any point on the line show that the equation of the straight line may be put in the form $x\cos\alpha+y\sin\alpha=p$.

(This is known as the perpendicular form of the equation of a straight line.)

Hence find the length of the perpendiculars from the origin on the lines whose equations are (i) $3x+4y=7$; (ii) $5x-12y=2$; (iii) $x+2y=6$. Verify by drawing to scale.

12. The co-ordinates of a point referred to rectangular axes OX, OY are x, y; referred to two rectangular axes OV, OW through the same point O the co-ordinates are ξ, η. Prove by projection that $\xi=x\cos\alpha+y\sin\alpha$, where α is the angle between OX and OV.

Find three other similar relations connecting ξ, η, x, y.

CHAPTER IX

FORMULAE FOR (i) THE FUNCTIONS OF THE SUM OR DIFFERENCE OF TWO ANGLES, (ii) THE SUM OR DIFFERENCE OF THE FUNCTIONS OF TWO ANGLES, (iii) THE FUNCTIONS OF THE DOUBLE ANGLE AND THE HALF-ANGLE

81. To express cos (A + B) in terms of the sines and cosines of A and B.

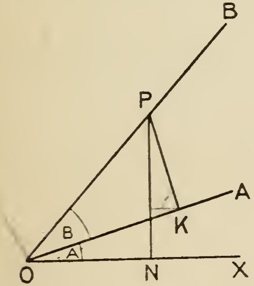


Fig. LIV.

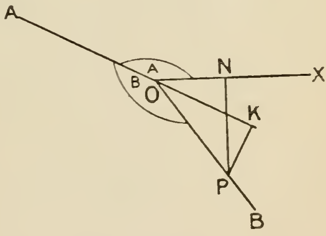


Fig. LV.

Let OX be the initial line; and let the revolving line first turn through the angle A to the position OA and then continue to turn through an additional angle B to the position OB. Then OB is the bounding line of the angle $A + B$. Along OB measure a length $OP = r$ units.

Project OP on the initial line, produced backwards if necessary also project OP on OA, produced backwards if necessary.

Figures LIV and LV show two of the many possible cases.

In all cases

the projection of (OP) on OX

$$= \text{sum of the projections of } (OK) \text{ and } (KP) \text{ on } OX;$$

i.e. $r\cos(A+B) = (OK)\cos A + (KP)\cos(A+90)$

$$= r\cos B\cos A + r\sin B(-\sin A).$$

Hence $\cos(A+B) = \cos A\cos B - \sin A\sin B.$*

Several proofs of this have already been given, but the earlier proofs have implied that A and B are together less than two right angles; this proof applies whatever be the values of A and B.

Exercises. (i) Deduce the formula for $\sin(A+B)$ by substituting $90-A$ in place of A, and $-B$ in place of B.

(ii) By similar substitutions deduce the formulae for $\cos(A-B)$ and $\sin(A-B)$.

(iii) By projecting perpendicular to OX, find the expanded form of $\sin(A+B)$.

(iv) Modify the construction so as to prove directly, by projection, the formula for $\sin(A-B)$ and $\cos(A-B)$.

(v) Complete the following formulae:

$$\cos(A+B) =$$
$$\cos(A-B) =$$
$$\sin(A+B) =$$
$$\sin(A-B) =$$

(vi) Learn these formulae in words, as:

cos sum = cos cos − sin sin.

———— = ————

———— = ————

———— = ————

* When Fig. LV is being used it must be recollected that (OK) is negative, and that its inclination to OX is XOA not XOK, see § 68. If (OK) is regarded as positive, its actual length is $-r\cos B$; but the angle is then XOK, the cosine of which is $-\cos A$. Whatever way it is taken, the projection of (OK) on OX is found to be $r\cos B\cos A$.

82. The following proof does not involve any knowledge of projection; its chief drawback is that it applies only to the case when $A+B$ is less than a right angle. It is easily modified to suit any other given case.

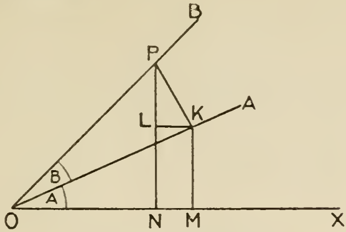


Fig. LVI.

Let $XOA = A$ and $AOB = B$;
then $XOB = A+B$, and in Fig. LVI is less than 90°.

Take a point P in the bounding line of $A+B$;
let fall PN perpendicular to OX;
,, PK ,, ,, OA;
,, KL ,, ,, PN and, therefore, parallel to OX;
,, KM ,, ,, OX.

$$\cos(A+B) = \frac{ON}{OP} = \frac{OM-MN}{OP}$$

$$\left[= \frac{OM}{}\frac{}{OP} - \frac{LK}{}\frac{}{OP}\right]$$

$$= \frac{OM}{OK}\frac{OK}{OP} - \frac{LK}{PK}\frac{PK}{OP}.$$

The spaces in the bracketed line (which does not appear in the completed work) are filled in with the hypotenuses of the triangles in which the respective numerators occur.

Now angle $LPK = 90° - LKP = LKO = A$; therefore

$$LK/PK = \sin A.$$

Hence $\cos(A+B) = \cos A \cos B - \sin A \sin B$.

From the same figure, prove that

$\sin(A+B) =$

To find the functions of $A-B$, the angle AOB must be made on the negative side of OA. The point P must be taken in the bounding line of $A-B$, and the construction and proof proceed as before. It is found that

$$\cos(A-B) =$$
$$\sin(A-B) =$$

Exercises. (i) Prove the four formulae when A and B are each less than 90° but $A+B$ is greater than 90°.

(Make the same construction as when $A+B$ is less than 90°, and pay careful attention to the signs of the lines.)

(ii) Prove the four formulae when A and B are each obtuse and together greater than 270°.

Examples IX a.

1. By using the formulae of § 81, verify that $\sin(90-A)=\cos A$, $\cos(90+A)=-\sin A$, $\sin(180-A)=\sin A$, $\cos(180-A)=-\cos A$, $\sin(270+A)=-\cos A$, $\cos(360-A)=\cos A$.

2. Express $\cos 70°$ in terms of the functions of (i) 40° and 30°; (ii) 45° and 25°; (iii) 95° and 15°; (iv) 35°.

3. Express $\sin 40°$ in terms of (i) 30° and 10°; (ii) 25° and 15°; (iii) 70° and 30°; (iv) 20°.

4. From the expansions of $\sin(A+B)$ and $\cos(A+B)$ deduce the expansion of $\tan(A+B)$ in terms of $\tan A$ and $\tan B$.

5. From the expansions of $\sin(A-B)$ and $\cos(A-B)$ deduce the expansion of $\tan(A-B)$ in terms of $\tan A$ and $\tan B$.

6. Verify that $\sin 0° = 0$ and $\cos 0° = 1$ by using the formulae for $A-B$.

7. Show that (i) $\sin(A+B)\cos B-\cos(A+B)\sin B=\sin B$;
(ii) $\cos(A+B)\cos B+\sin(A+B)\sin B=\cos B$.

8. From the formulae for $A+B$ deduce that

$$\sin 2A = 2\sin A\cos A \quad\text{and}\quad \cos 2A=\cos^2 A-\sin^2 A.$$

What is the value of $\tan 2A$?

9. Find the values of

(i) $\sin(A+B)+\sin(A-B)$; (ii) $\cos(A+B)+\cos(A-B)$;
(iii) $\sin(A+B)-\sin(A-B)$; (iv) $\cos(A+B)-\cos(A-B)$.

Account for the signs of (iii) and (iv) from first principles.

10. Prove that

$$\sin A = \sin \tfrac{1}{2}(A+B)\cos \tfrac{1}{2}(A-B) + \cos \tfrac{1}{2}(A+B)\sin \tfrac{1}{2}(A-B).$$

Prove similar results for $\cos A$, $\sin B$, and $\cos B$.

11. From the results of 10 deduce that

$$\sin A + \sin B = 2\sin \tfrac{1}{2}(A+B)\cos \tfrac{1}{2}(A-B),$$

and three similar results.

12. Prove that

(i) $\cos^2\theta + \cos^2\phi - 2\cos\theta\cos\phi\cos(\theta+\phi) = \sin^2(\theta+\phi)$;

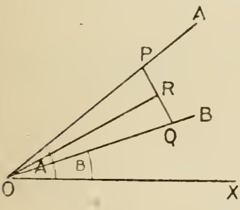
(ii) $\sin^2\theta + \cos^2\phi - 2\sin\theta\cos\phi\sin(\theta+\phi) = \cos^2(\theta+\phi)$.

83. Sums and differences of sines or cosines expressed as products.

These formulae are most easily derived from the formulae of § 81, as suggested in Examples IX *a*. They can be proved independently by projection.

Make the angle $XOA = A$ and the angle $XOB = B$.*

On OA and OB take lengths OP, OQ respectively, each equal to r units. Join PQ.



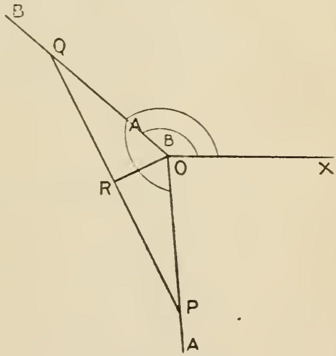


Fig. LVII. Fig. LVIII.

Bisect the angle QOP by a line cutting PQ at R.

Then the angles ROP, ROQ each equal $\frac{1}{2}(A-B)$; and the angle $XOR = \frac{1}{2}(A+B)$.

From congruent triangles $RP = RQ$, and PRO is a right angle.

The projection of (OP) = sum of projections of (OR) and (RP), and the projection of (OQ) = sum of projections of (OR) and (RQ).

* Notice the difference between this construction and the construction of § 81.

$\therefore$ projection of (OP) + projection of $(OQ) = 2^{ce}$ projection of (OR), since projections of (RP) and (RQ) are equal but opposite.

Projecting on a line perpendicular to OX, we have

$$r \sin A + r \sin B = 2^{ce} \text{ the projection of } (OR).$$

But (OR) is the projection of (OP) on the direction OR,

i.e. $$(OR) = r \cos \tfrac{1}{2}(A - B).$$

$\therefore$ $$r \sin A + r \sin B = 2r \cos \tfrac{1}{2}(A - B) \sin \tfrac{1}{2}(A + B).$$

Hence $$\sin A + \sin B = 2 \sin \tfrac{1}{2}(A + B) \cos \tfrac{1}{2}(A - B).$$

From the same figure, by projection on OX, we have

$$\cos A + \cos B = 2 \cos \tfrac{1}{2}(A + B) \cos \tfrac{1}{2}(A - B).$$

Again, projection of (OP) − projection of $(OQ) = 2^{ce}$ projection of (RP).

Hence $$\sin A - \sin B = 2 \cos \tfrac{1}{2}(A + B) \sin \tfrac{1}{2}(A - B),$$

$$\cos A - \cos B = -2 \sin \tfrac{1}{2}(A - B) \sin \tfrac{1}{2}(A + B).$$

The proofs apply to all cases whatever be the magnitudes of A and B.

The reason for the negative sign in the last of these formulae is obvious, for, if $A > B$, then $\cos A < \cos B$.

It is useful to learn the formulae in words, it being understood in all cases that the greater angle is put first.

sine + sine = 2 sine half sum cos half difference.

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84. Products of sines and cosines expressed as sums or differences.

In the formulae of the last section put

$$\tfrac{1}{2}(A + B) = X, \ \tfrac{1}{2}(A - B) = Y;$$

so that $A = X + Y$, $B = X - Y$.

Then $$\sin (X + Y) + \sin (X - Y) = 2 \sin X \cos Y,$$

i.e. $$2 \sin X \cos Y = \sin (X + Y) + \sin (X - Y).$$

Similar results are obtained from the other formulae. If we replace X by A and Y by B, the formulae become

$$2 \sin A \cos B = \sin (A + B) + \sin (A - B),$$
$$2 \cos A \sin B = \sin (A + B) - \sin (A - B),$$
$$2 \cos A \cos B = \cos (A + B) + \cos (A - B),$$
$$2 \sin A \sin B = \cos (A - B) - \cos (A + B).$$

These are more easily proved direct from the $A + B$ and $A - B$ formulae.

In using these formulae it is usual (but not necessary) to put the greater angle first; this shows why there are distinct formulae for $2 \sin A \cos B$ and $2 \cos A \sin B$.

Express the four formulae in words:

Twice sine cos = sin sum + sin difference.

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Examples IX b.

1. Apply the formulae of §§ 83, 84 to the following cases and verify from the tables:

(i) $A = 70°$, $B = 30°$; (iii) $A = 72°$, $B = 18°$;
(ii) $A = 110°$, $B = 75°$; (iv) $A = 78°$, $B = 46°$.

2. Prove, from the formula for $\sin A + \sin B$, that

$$\sin 2\theta = 2 \sin \theta \cos \theta,$$

and, in a similar way, show that

$$1 + \cos A = 2 \cos^2 \tfrac{1}{2} A, \quad 1 - \cos A = 2 \sin^2 \tfrac{1}{2} A.$$

3. Prove that

(i) $\sin A + \cos A = \sqrt{2} \cos (A - 45)$;

(ii) $\sin A - \cos B = -2 \sin \left(45 - \frac{A+B}{2}\right) \sin \left(45 - \frac{A-B}{2}\right)$;

(iii) $\cos A + \sin B = 2 \cos \left(45 - \frac{A-B}{2}\right) \cos \left(45 + \frac{A+B}{2}\right)$.

4. Prove that (i) $\sin 50° + \sin 130° = 2 \cos 40°$;
(ii) $\cos 50° - \cos 130° = 2 \sin 40°$.

Verify these by squaring and adding.

5. Prove that (i) $2 \cos 40° \sin 50° = 1 - \sin 10°$;
(ii) $2 \cos 40° \sin 40° = \sin 80°$;
(iii) $2 \sin 64° \sin 26° = \cos 38°$.

Verify this last result from the tables.

6. Fill in the right-hand side of the following:

(i) $\sin 70° + \sin 50° =$ (ii) $\cos 30° - \cos 110° =$
(iii) $2 \sin 75° \cos 10° =$ (iv) $\sin 37° + \cos 24° =$
(v) $2 \cos 84° \cos 72° =$ (vi) $\cos 79° - \cos 52° =$
(vii) $\sin 75° - \sin 116° =$ (viii) $2 \cos 80° \cos 35° =$
(ix) $\cos 24° - \sin 76° =$ (x) $2 \sin 17° \sin 48° =$

(xi) $2\cos 73^\circ \sin 15^\circ =$ (xii) $2\cos 14^\circ \cos 166^\circ =$

(xiii) $\cos \tfrac{1}{2}\pi - \cos \tfrac{1}{6}\pi =$ (xiv) $\sin \tfrac{1}{4}\pi + \cos \tfrac{1}{4}\pi =$

(xv) $2\sin 43^\circ \cos 47^\circ =$ (xvi) $2\cos 97^\circ \sin 46^\circ =$

(xvii) $\sin 81^\circ + \sin 10^\circ =$ (xviii) $\sin 49^\circ - \sin 53^\circ =$

(xix) $2\sin 79^\circ \sin 15^\circ =$ (xx) $\cos 43^\circ - \cos 216^\circ =$

7. Prove that $4\cos(75^\circ + A)\sin(75^\circ - A) = 1 - 2\sin 2A$.

Formulae for the double angle and half-angle.

85. It has already been shown in § 51 that

$$\sin 2A = 2\sin A\cos A;$$
$$\cos 2A = \cos^2 A - \sin^2 A$$
$$= 1 - 2\sin^2 A$$
$$= 2\cos^2 A - 1.$$

The proof there given assumed that $2A$ is less than 180°.

If we put A instead of B in the $A+B$ formulae the same results are obtained; thus they are true for all values of A.

The results can easily be proved independently by projection; the proofs are the same as in § 81, A taking the place of B.

86. From the last section, by putting $\tfrac{1}{2}A$ in place of A, we have

have $\cos^2 \tfrac{1}{2}A - \sin^2 \tfrac{1}{2}A = \cos A;$

also $\cos^2 \tfrac{1}{2}A + \sin^2 {}^{1}A = 1.$

Add $2\cos^2 \tfrac{1}{2}A = 1 + \cos A.$

Subtract $2\sin^2 \tfrac{1}{2}A = 1 - \cos A.$

Hence $\cos \tfrac{1}{2}A = \pm \tfrac{1}{2}(\sqrt{1+\cos A});$

$\sin \tfrac{1}{2}A = \pm \tfrac{1}{2}(\sqrt{1-\cos A}).$

If the value of A is given, there is no ambiguity of sign. If, for instance, $A = 140^\circ$, then $\tfrac{1}{2}A = 70^\circ$, and the sine and cosine are both positive; if $A = 264^\circ$, then $\tfrac{1}{2}A = 132^\circ$, and the sine is positive, the cosine negative.

If the value of $\cos A$ is given but not the value of A, the ambiguity cannot be removed. Suppose $\cos A = \tfrac{1}{2}$, then A may have any value of the form $360^\circ n \pm 60^\circ$. Hence $\tfrac{1}{2}A$ may have any value given by $180^\circ n \pm 30^\circ$. If we tabulate these values, we have

angle	cosine	sine
30°	$+\tfrac{1}{2}\sqrt{3}$	$+\tfrac{1}{2}$
150°	$-\tfrac{1}{2}\sqrt{3}$	$+\tfrac{1}{2}$
210°	$-\tfrac{1}{2}\sqrt{3}$	$-\tfrac{1}{2}$
330°	$+\tfrac{1}{2}\sqrt{3}$	$-\tfrac{1}{2}$

87. Tangent formulae.

From the sine and cosine formulae the following tangent formulae are derived; the proof of the first only is given:

$$\tan(A+B) = \frac{\sin(A+B)}{\cos(A+B)}$$

$$= \frac{\sin A \cos B + \cos A \sin B}{\cos A \cos B - \sin A \sin B}$$

$$= \frac{\tan A + \tan B}{1 - \tan A \tan B}.$$ (By dividing throughout by $\cos A \cos B$.)

Similarly,

$$\tan(A-B) =$$

$$\tan 2A =$$

$$\tan \tfrac{1}{2} A = \pm\sqrt{\frac{1-\cos A}{1+\cos A}}$$

$$= \pm\frac{1-\cos A}{\sin A} \text{ or } \pm\frac{\sin A}{1+\cos A}.$$ (By rationalizing.)

88. These may all be proved directly from the figures used for the sine and cosine formulae; e.g. in Fig. LVI.

$$\tan(A+B) = \frac{NP}{ON}$$

$$= \frac{MK+LP}{OM-LK}$$

$$= \frac{\frac{MK}{OM} + \frac{LP}{OM}}{1 - \frac{LK}{PK}\frac{PK}{OM}}.$$ (By dividing so as to make the first term in the denominator to be 1.)

The triangles LPK, OKM are similar;

$$\therefore \quad \frac{LP}{OM} = \frac{PK}{OK} = \tan B.$$

Hence $\tan(A+B) = \dfrac{\tan A + \tan B}{1 - \tan A \tan B}$.

Exercises. Prove that

(i) $\tan(A+B+C) = \dfrac{\tan A + \tan B + \tan C - \tan A \tan B \tan C}{1 - \tan B \tan C - \tan C \tan A - \tan A \tan B}$;

(ii) $\tan A + \tan B = \dfrac{\sin(A+B)}{\cos A \cos B}$;

(iii) $\tan A - \tan B = \dfrac{\sin(A-B)}{\cos A \cos B}$;

(iv) $\cot A + \cot B \quad = \dfrac{\sin(A+B)}{\sin A \sin B}$;

(v) $\cot A - \cot B \quad = -\dfrac{\sin(A-B)}{\sin A \sin A}$.

Example. *To prove that* $\tan^{-1}\frac{1}{99} + \tan^{-1}\frac{1}{239} = \tan^{-1}\frac{1}{70}$.

Let $\qquad A = \tan^{-1}\frac{1}{99}, \quad B = \tan^{-1}\frac{1}{239}$;

so that $\qquad \tan A = \frac{1}{99}, \quad \tan B = \frac{1}{239}$.

$$\tan(A+B) = \frac{\tan A + \tan B}{1 - \tan A \tan B}$$

$$= \frac{\frac{1}{99} + \frac{1}{239}}{1 - \frac{1}{99} \times \frac{1}{239}}$$

$$= \frac{239 + 99}{23900 - 240} = \frac{338}{23660}$$

$$= \tfrac{1}{70},$$

i.e. $\quad \tan^{-1}\frac{1}{99} + \tan^{-1}\frac{1}{239} = \tan^{-1}\frac{1}{70}$.

Examples IX c.

1. Prove that $\qquad (\sin A + \cos A)^2 = 1 + \sin 2A$,
and $\qquad (\sin A - \cos A)^2 = 1 - \sin 2A$.

2. Assuming the values of $\sin 45°$, $\cos 45°$, $\tan 45°$, deduce $\sin 90°$, $\cos 90°$, $\tan 90°$.

3. Find a formula for $\cot 2A$ in terms of $\cot A$.

4. Show that $\sin 3A = 3\sin A - 4\sin^3 A$. Explain how it is that there are three values of $\sin A$ when $\sin 3A$ is given.

5. Find the values of $\tan 22\frac{1}{2}°$, $\tan 67\frac{1}{2}°$, $\tan 157\frac{1}{2}°$.

6. Prove that $2\sin\frac{1}{2}A = \pm\sqrt{1+\sin A} \pm \sqrt{1-\sin A}$.
Find $\sin\frac{1}{2}A$ when $\sin A = \frac{1}{2}$. Illustrate by a figure.

7. Find $\cos\frac{1}{2}A$ when $\sin A = \frac{1}{2}$. Illustrate by a figure.

8. Prove that (i) $\sin 2A = 2\tan A \div (1 + \tan^2 A)$;
(ii) $\cos 2A = (1 - \tan^2 A) \div (1 + \tan^2 A)$;
(iii) $\tan 2A = 2\tan A \div (1 - \tan^2 A)$.

9. Prove that $\qquad \sin 3A = 3\sin A - 4\sin^3 A$;
$\cos 3A = 4\cos^3 A - 3\cos A$;
$\tan 3A = (3\tan A - \tan^3 A) \div (1 - 3\tan^2 A)$.

10. Show that

$$(\cos A+\sin A)^3+(\cos A-\sin A)^3 = 3\cos A-\cos 3A.$$

11 (*a*). Show that $\sin\frac{1}{2}A+\cos\frac{1}{2}A = \pm\sqrt{1+\sin A},$

and $\sin\frac{1}{2}A-\cos\frac{1}{2}A = \pm\sqrt{1-\sin A}.$

(*b*) Having given $4\sin 54° = \sqrt{5}+1$, apply the formulae in (*a*) to find $\sin 27°$ and $\cos 27°$, explaining how the ambiguities of sign are cleared up.

(*c*) Show that $8(\sin^2 42°-\cos^2 78°) = \sqrt{5}+1.$

12. Prove that

$$\tan(A+B+C) = \frac{\tan A+\tan B+\tan C-\tan A\tan B\tan C}{1-\tan B\tan C-\tan C\tan A-\tan A\tan B}.$$

Deduce the formula for $\tan 3A$.

What can be deduced if $A+B+C$ equals (i) 180°, (ii) 90°?

13. If $\tan A = \frac{13}{27}$ and $\tan B = \frac{7}{20}$, show that

$$A+B = (4n+1)\tfrac{1}{4}\pi.$$

14. Show that

$$\cos\theta+\cos 3\theta+\cos 5\theta+\cos 7\theta = 4\cos\theta\cos 2\theta\cos 4\theta.$$

15. Find all the solutions of the equation

$$\sin\theta\sin 3\theta = \sin 5\theta\sin 7\theta.$$

16. If $\tan A = \frac{1}{5}$, $\tan B = \frac{3}{7}$, $\tan C = \frac{5}{27}$, and each angle is acute, prove that $A+B+C = \frac{1}{4}\pi$.

17. If $\tan\theta = \tan\frac{1}{2}\alpha\tan\frac{1}{2}\beta$, show that

$$\tan 2\theta = (\sin\alpha\sin\beta)\div(\cos\alpha+\cos\beta).$$

18. (i) If $\theta = \tan^{-1}\frac{1}{3}$, find $\tan 2\theta$.

(ii) Show that $2\tan^{-1}\frac{1}{3}+\tan^{-1}\frac{1}{7} = \frac{1}{4}\pi$.

19. Prove that

$$\cos 2A-\cos 2B = 2(\cos^2 A-\cos^2 B) = 2(\sin^2 B-\sin^2 A).$$

20. Prove that

(i) $$\frac{1}{a+b\cos\theta} = \frac{\sec^2\frac{1}{2}\theta}{(a+b)+(a-b)\tan^2\frac{1}{2}\theta};$$

(ii) $$\frac{1}{a\cos\theta+b\sin\theta} = \frac{1+\tan^2\frac{1}{2}\theta}{a+2b\tan\frac{1}{2}\theta-a\tan^2\frac{1}{2}\theta}.$$

21. Solve the equations

(i) $x^2-\sqrt{2}\sin(\frac{1}{4}\pi+\alpha)x+\frac{1}{2}\sin 2\alpha = 0;$

(ii) $x^2-2\cot 2\beta\,.\,x-1 = 0.$

22. Solve for α and V the following equations:

$$2ag = V^2\sin 2\alpha,\quad 2bg = V^2\sin^2\alpha.$$

23. A hemispherical shell of radius 16 inches rests with its rim on a horizontal table; a rod is hinged to a vertical wall, 25 inches from the centre of the shell, at a point 5 feet above the table. The rod is in the same vertical plane as the hinge and centre of the shell, and touches the shell. Find its inclination to the vertical.

Oral Examples.

(*a*)
(i) $\sin(P-Q) =$
(ii) $\cos X + \cos Y =$
(iii) $\cos(90 - \frac{1}{2}\overline{A+B}) =$
(iv) $\sin 270° =$
(v) $2\sin\alpha\cos\beta =$
(vi) $\cos^2\theta - \sin^2\theta =$
(vii) $\tan(A-B) =$
(viii) $\sin B - \sin C =$
(ix) $\cos^2 45° - \sin^2 45° =$
(x) $\cos 2A =$

(*b*)
(i) $\cos(C+A) =$
(ii) $\sin B + \sin C =$
(iii) $2\sin\frac{1}{2}(B+C)\frac{1}{2}(B-C) =$
(iv) $\cos\theta + \cos\phi =$
(v) $2\cos^2\frac{1}{2}C - 1 =$
(vi) $\tan 2B =$
(vii) $\sin(180 - \overline{B+C}) =$
(viii) $\cos^2 75° + \sin^2 75° =$
(ix) $\cos(\frac{1}{2}\overline{A+B} + \frac{1}{2}\overline{A-B}) =$
(x) $\sin(360 - 2C) =$

(*c*)
(i) $\sin 2B =$
(ii) $\sin(P+Q) =$
(iii) $\cos(\pi - \overline{\alpha+\beta}) =$
(iv) $2\sin\frac{1}{2}C\cos\frac{1}{2}C =$
(v) $1 - 2\sin^2 B =$
(vi) $\cos^2\frac{1}{2}C - \sin^2\frac{1}{2}C =$
(vii) $\cos^2\frac{1}{2}C + \sin^2\frac{1}{2}C =$
(viii) $\tan(B-C) =$
(ix) $\cos C - \cos A =$
(x) $(\sin B + \cos B)^2 =$

(*d*)
(i) $\sin B\cos C - \sin B\sin C =$
(ii) $\cos(X-Y) =$
(iii) $\sin 3A =$
(iv) $\sin\frac{1}{2}B =$
(v) $\cos B + \cos C =$
(vi) $2\cos B\cos C =$
(vii) $\sin^2(B+C) + \cos^2(B+C) =$
(viii) $2\cos^2\frac{1}{2}\theta - 1 =$
(ix) $(\cos\frac{1}{2}A - \sin\frac{1}{2}A)^2 =$
(x) $\tan(90-C) =$

89. The preceding formulae lead to a number of useful identities in the cases where $A+B+C = 90°$ or $180°$. The method of dealing with these is shown in the following illustrative examples.

Example i. *In any triangle* $\tan \frac{1}{2}(B-C) = \frac{b-c}{b+c} \cot \frac{1}{2} A.$

[Here $\frac{b-c}{b+c}$ gives the clue to the proof.]

By the sine formula, $$\frac{b}{\sin B} = \frac{c}{\sin C},$$

i.e. $$\frac{\sin B}{\sin C} = \frac{b}{c},$$

Componendo et dividendo, $$\frac{\sin B - \sin C}{\sin B + \sin C} = \frac{b-c}{b+c};$$

$$\therefore \quad \frac{2 \cos \frac{1}{2}(B+C) \sin \frac{1}{2}(B-C)}{2 \sin \frac{1}{2}(B+C) \cos \frac{1}{2}(B-C)} = \frac{b-c}{b+c},$$

i.e. $$\frac{\tan \frac{1}{2}(B-C)}{\tan \frac{1}{2}(B+C)} = \frac{b-c}{b+c};$$

but $A+B+C = 180°$; $\therefore \frac{1}{2}A + \frac{1}{2}(B+C) = 90.$

Hence $\tan \frac{1}{2}(B+C) = \tan (90 - \frac{1}{2}A) = \cot \frac{1}{2}A.$

Substituting above, $$\tan \frac{1}{2}(B-C) = \frac{b-c}{b+c} \cot \frac{1}{2}A.$$

This formula has been proved geometrically in § 53; it is usually proved by the method given above.

Example ii. *In any triangle*

$$\cos A + \cos B + \cos C = 1 + 4 \sin \tfrac{1}{2} A \sin \tfrac{1}{2} B \sin \tfrac{1}{2} C.$$

L.H.S. $= \cos A + \cos B + \cos C$
$= 2 \cos \frac{1}{2}(A+B) \cos \frac{1}{2}(A-B) + 1 - 2 \sin^2 \frac{1}{2} C$
$= 2 \sin \frac{1}{2} C \cos \frac{1}{2}(A-B) + 1 - 2 \sin \frac{1}{2} C \cos \frac{1}{2}(A+B),$
since $\frac{1}{2}C = 90 - \frac{1}{2}(A+B)$
$= 1 + 2 \sin \frac{1}{2} C (\cos \frac{1}{2}\overline{A-B} - \cos \frac{1}{2}\overline{A+B})$
$= 1 + 4 \sin \frac{1}{2}A \sin \frac{1}{2}B \sin \frac{1}{2}C.$

The symbol $\Sigma \cos A$ is sometimes used to denote

$$\cos A + \cos B + \cos C;$$

and $\Pi \sin A$ to denote $\sin A \sin B \sin C$. The above result can be written: $$\Sigma \cos A = 1 + 4 \Pi \sin \tfrac{1}{2}A.$$

Example iii. *In any triangle* $\Sigma \cos^2 A = 1 - 2\,\Pi \cos A$.

(Questions involving the sum of the squares of sines or cosines are usually solved by expressing these squares in terms of the cosine of the double angle.)

$$2\,\Sigma \cos^2 A = 2\cos^2 A + 2\cos^2 B + 2\cos^2 C$$
$$= 1 + \cos 2A + 1 + \cos 2B + 2\cos^2 C. \quad \text{(Note that one angle is left unchanged.)}$$
$$\therefore\ \Sigma \cos^2 A = 1 + \cos(A+B)\cos(A-B) + \cos^2 C$$
$$= 1 - \cos C \cos(A-B) - \cos C \cos(A+B), \quad \text{since } C = 180 - (A+B)$$
$$= 1 - \cos C\,[\cos(A-B) - \cos(A+B)]$$
$$= 1 - 2\cos A \cos B \cos C.$$

Example iv. *Solve the equation*

$$\sin\theta + \sin 2\theta + \sin 3\theta + \sin 4\theta = 0.$$

Rearrange $\quad \sin\theta + \sin 4\theta + \sin 2\theta + \sin 3\theta = 0.$

Use formula for sum of two sines

$$2\sin\tfrac{5}{2}\theta\cos\tfrac{3}{2}\theta + 2\sin\tfrac{5}{2}\theta\cos\tfrac{1}{2}\theta = 0;$$

$\therefore$ either $\sin\frac{5}{2}\theta = 0$ or $\cos\frac{3}{2}\theta + \cos\frac{1}{2}\theta = 0$;

i.e. $\quad \frac{5}{2}\theta = n\pi$ or $2\cos\theta\cos\frac{1}{2}\theta = 0$;

i.e. $\quad \theta = \frac{2}{5}n\pi$, or $\cos\theta = 0$ or $\cos\frac{1}{2}\theta = 0$;

i.e. $\quad \theta = (2n+1)\frac{1}{2}\pi$ or $\frac{1}{2}\theta = (2n+1)\frac{1}{2}\pi$.

Hence the complete solution is

$$\theta = (2n+1)\pi,\ (2n+1)\tfrac{1}{2}\pi \text{ or } \tfrac{2}{5}n\pi.$$

Example v. *To prove that*

$$r = 4R\sin\tfrac{1}{2}A\,\sin\tfrac{1}{2}B\,\sin\tfrac{1}{2}C.$$

From the figure of § 59,

$$r\cot\tfrac{1}{2}B + r\cot\tfrac{1}{2}C = a,$$

i.e. $$r\left(\frac{\cos\frac{1}{2}B}{\sin\frac{1}{2}B} + \frac{\cos\frac{1}{2}C}{\sin\frac{1}{2}C}\right) = 2R\sin A,$$

i.e. $$r\,\frac{\sin\frac{1}{2}B\cos\frac{1}{2}C + \cos\frac{1}{2}B\sin\frac{1}{2}C}{\sin\frac{1}{2}B\sin\frac{1}{2}C} = 2R\sin A,$$

i.e. $$r\,\frac{\sin\frac{1}{2}(B+C)}{\sin\frac{1}{2}B\sin\frac{1}{2}C} = 2R\sin A,$$

i.e. $$r\,\frac{\cos\frac{1}{2}A}{\sin\frac{1}{2}B\sin\frac{1}{2}C} = 4R\sin\tfrac{1}{2}A\cos\tfrac{1}{2}A;$$

since $\frac{1}{2}(B+C) = 90 - \frac{1}{2}A$;

$$\therefore\ r = 4R\sin\tfrac{1}{2}A\sin\tfrac{1}{2}B\sin\tfrac{1}{2}C.$$

Exercise. Prove that $r_1 = 4R\sin\frac{1}{2}A\cos\frac{1}{2}B\cos\frac{1}{2}C$.

Example vi. *To show that the distance between the circum-centre and in-centre* $= \sqrt{(R^2 - 2Rr)}$.

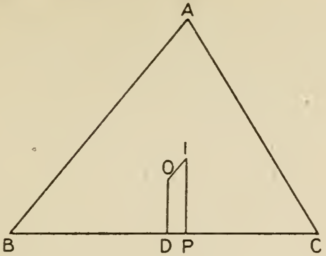

Fig. LIX.

In Fig. LIX, with the usual notation,

$$BD = R\sin A,\ DO = R\cos A,$$
$$BP = r\cot\tfrac{1}{2}B,\ PI = r.$$

$$\begin{aligned} OI^2 &= (BP - BD)^2 + (IP - OD)^2 \\ &= (r\cot\tfrac{1}{2}B - R\sin A)^2 + (r - R\cos A)^2 \\ &= R^2 - 2Rr(\sin A\cot\tfrac{1}{2}B + \cos A) + r^2(1 + \cot^2\tfrac{1}{2}B) \\ &= R^2 - 2Rr\frac{\sin(A + \frac{1}{2}B)}{\sin\frac{1}{2}B} + r^2\frac{1}{\sin^2\frac{1}{2}B} \\ &= R^2 - 2Rr\frac{\sin(A + \frac{1}{2}B)}{\sin\frac{1}{2}B} + 4Rr\frac{\sin\frac{1}{2}A\sin\frac{1}{2}B\sin\frac{1}{2}C}{\sin^2\frac{1}{2}B} \quad \text{(Substituting for } r.) \\ &= R^2 - 2Rr\frac{\cos\frac{1}{2}(A - C) - 2\sin\frac{1}{2}A\sin\frac{1}{2}C}{\sin\frac{1}{2}B}, \\ &\qquad\qquad \text{since } A + \tfrac{1}{2}B = 90 + \tfrac{1}{2}A - \tfrac{1}{2}C \\ &= R^2 - 2Rr\frac{\cos\frac{1}{2}(A + C)}{\sin\frac{1}{2}B}, \\ &= R^2 - 2Rr. \qquad \text{since } \tfrac{1}{2}(A + C) = 90 - \tfrac{1}{2}B. \end{aligned}$$

This is more shortly proved by Pure Geometry; but the method used here is a general method to find the lengths of lines connected with the triangle.

Example vii. *To prove that*

$$\sin A + \sin(A + B) + \sin(A + 2B) + \dots \text{ to } n \text{ terms}$$
$$= \frac{\sin\frac{1}{2}nB\sin(A + \frac{1}{2}\overline{n-1}B)}{\sin\frac{1}{2}B}.$$

Let $S = \sin A + \sin(A+B) + \ldots + \sin(A+\overline{n-1}B)$.

Multiply by $2\sin\frac{1}{2}B$.

Then $2\sin\frac{1}{2}B.S$

$= 2\sin A\sin\frac{1}{2}B + 2\sin(A+B)\sin\frac{1}{2}B + \ldots + 2\sin(A+n-1B)\sin\frac{1}{2}B$.

Use the formula for the product of two sines.

$$\begin{aligned}2\sin\tfrac{1}{2}B.S = {} & \cos(A-\tfrac{1}{2}B) - \cos(A+\tfrac{1}{2}B)\\ & + \cos(A+\tfrac{1}{2}B) - \cos(A+\tfrac{3}{2}B)\\ & + \ldots\ldots - \ldots\ldots\\ & \ldots\ldots\quad\ldots\ldots\\ & + \cos(A+\tfrac{1}{2}\overline{2n-3}B) - \cos(A+\tfrac{1}{2}\overline{2n-1}B)\\ = {} & \cos(A-\tfrac{1}{2}B) - \cos(A+\tfrac{1}{2}\overline{2n-1}B);\end{aligned}$$

$$\therefore \qquad S = \frac{\sin\frac{1}{2}nB\sin(A+\frac{1}{2}\overline{n-1}B)}{\sin\frac{1}{2}nB}.$$

Note. Compare this with the formula for the sum of n terms of an Arithmetic Progression. Notice that $A+\frac{1}{2}\overline{n-1}B$ = half the sum of the first angle (A) and the last angle ($A+\overline{n-1}B$).

Examples IX.

1. Prove the following identities:

(i) $\sin 3A = 4\sin A\sin(60°+A)\sin(60°-A)$;

(ii) $\sin 3A\sin^3 A + \cos 3A\cos^3 A = \cos^3 2A$;

(iii) $(1-2\sin^2 A)\div(1+\sin 2A) = (1-\tan A)\div(1+\tan A)$;

(iv) $\dfrac{\tan(45°+A)+\tan(45°-A)}{\tan(45°+A)-\tan(45°-A)} = \operatorname{cosec} 2A$;

(v) $\sin(y+z-x)+\sin(z+x-y)+\sin(x+y-z)-\sin(x+y+z)$
$= 4\sin x\sin y\sin z$;

(vi) $\cot\frac{1}{4}\theta - \cot\theta = \operatorname{cosec}\theta + \operatorname{cosec}\frac{1}{2}\theta$;

(vii) $\cos 4A + 2(\cos A+\sin A)^4 = 3+4\sin 2A$;

(viii) $\sin A+\sin B = \sin(A+B)+4\sin\frac{1}{2}A\sin\frac{1}{2}B\sin\frac{1}{2}(A+B)$;

(ix) $\sin A - 3\sin 3A + 3\sin 5A - \sin 7A = 8\sin^3 A\cos 4A$;

(x) $\cos(A+B+C)$
$= \cos A\cos B\cos C - \cos A\sin B\sin C - \sin A\cos B\sin C$
$- \sin A\sin B\cos C$;

(xi) $\cos\frac{1}{2}A(2\sin A - \sin 2A) = \sin^2\frac{1}{2}A(2\sin A+\sin 2A)$;

(xii) $\cos A+\cos B+\cos C+\cos(A+B+C)$
$= 4\cos\frac{1}{2}(A+B)\cos\frac{1}{2}(B+C)\cos\frac{1}{2}(C+A)$.

2. If A, B, C be the angles of a triangle, show that

(i) $\tan A + \tan B + \tan C = \tan A \cdot \tan B \cdot \tan C$;

(ii) $\sin 2A + \sin 2B + \sin 2C = 4 \sin A \sin B \sin C$;

(iii) $\sin^2 \frac{1}{2} A + \sin^2 \frac{1}{2} B + \sin^2 \frac{1}{2} C + 2 \sin \frac{1}{2} A \sin \frac{1}{2} B \sin \frac{1}{2} C = 1$;

(iv) $\sin A + \sin B + \sin C = 4 \cos \frac{1}{2} A \cos \frac{1}{2} B \cos \frac{1}{2} C$;

(v) $\cot A \cot B + \cot A \cot C + \cot B \cot C = 1$;

(vi) $\cot A + \cot B + \cot C$
$= \cot A \cot B \cot C + \operatorname{cosec} A \operatorname{cosec} B \operatorname{cosec} C$;

(vii) $\tan B \tan C + \tan C \tan A + \tan A \tan B$
$= 1 + \sec A \sec B \sec C$;

(viii) $\cos A \sin (B - C) + \cos B \sin (C - A) + \cos C \sin (A - B) = 0$;

(ix) $(\tan A + \tan B)(\tan A - \cot C) = \sec^2 A$;

(x) $\tan \frac{1}{2} B \tan \frac{1}{2} C + \tan \frac{1}{2} C \tan \frac{1}{2} A + \tan \frac{1}{2} A \tan \frac{1}{2} B = 1$.

3. Show geometrically that $\sin (A + B) = \sin A \cos B + \cos A \sin B$ when each of the angles A and B is between $\frac{1}{2}\pi$ and π, and $A + B$ is less than $\frac{3}{2}\pi$.

4. Solve the equation $\cos 3A + \cos 2A + \cos A = 0$.

5. Find all the values of θ which satisfy

(i) $\cos \theta + \cos 2\theta + \cos 3\theta + \cos 4\theta = 0$;

(ii) $\sin 3\theta + \sin 4\theta + \sin 5\theta = 0$.

6. Solve (i) $\sin (A + 30°) = 1 \div \sqrt{2}$;

(ii) $\sqrt{3} \sin A + \cos A = \sqrt{2}$;

(iii) $\sin A + \cos A = 1$;

(iv) $\sin A + \sqrt{3} \cos A = 2$;

(v) $\sqrt{2}(\cos 3x + \sin 3x) = 1$;

(vi) $a \cos \theta + b \sin \theta = c$ (put $a = r \cos \alpha$, $b = r \sin \alpha$).

7. Prove that (i) $2 \sin^{-1} \frac{1}{2}\sqrt{2} = 90°$;

(ii) $2 \tan^{-1} \frac{1}{2} = \tan^{-1} \frac{4}{3}$.

8. In any triangle show that
$$R(\sin 2A + \sin 2B + \sin 2C) = 2r(\sin A + \sin B + \sin C).$$

9. In any triangle show that
$$a^2 \cos 2B + b^2 \cos 2A = a^2 + b^2 - 4ab \sin A \sin B.$$

10. Prove the formula $(b + c) \tan \frac{1}{2}(B - C) = (b - c) \cot \frac{1}{2} A$.
Write down two corresponding formulae.

11. Using the fact that $3 \times 18° = 90° - 2 \times 18°$, find the values of $\sin 18°$ and $\cos 18°$.
Give a geometrical method for determining $\sin 18°$.

12. Simplify

(i) $\left(\dfrac{\sin 4A}{\sin A} - \dfrac{\cos 4A}{\cos A}\right) \div (\cot A + \cot 2A)$;

(ii) $\dfrac{\sin 5\theta - \sin 3\theta}{\cos 5\theta + \cos 3\theta} + \dfrac{2}{\sin 2\theta} + \dfrac{\sin 5\theta + \sin 3\theta}{\cos 5\theta - \cos 3\theta}$.

13. D, E, F are the feet of the perpendiculars from A, B, C on the opposite sides; P is the orthocentre. Prove that

(i) $AP = 2R\cos A$; (ii) $PD = 2R\cos B\cos C$;

(iii) perimeter of triangle $DEF = 4R\sin A\sin B\sin C$.

14. State the general formula for all angles having a given cosine.

Solve $\sin 3A + \sin 5A + \sin 7A = 0$.

15. Find $\sec(A+B)$ in terms of the secant and cosecant of A and B, and prove

$$\sec 105° = \sqrt{2}(1+\sqrt{3}).$$

16. Prove that

$\sin 18° = \frac{1}{4}(\sqrt{5}-1)$; and that $\sin^2 30° = \sin 18° \sin 54°$.

Show that in any circle the chord of an arc of 108° is equal to the sum of the chords of arcs of 36° and 60°.

17. Given $\cos A = \cdot 28$, determine the value of $\tan\frac{1}{2}A$, and explain fully the reason of the ambiguity which presents itself in your result.

18. Prove that

$$\cos^{-1}x + \cos^{-1}y = \sin^{-1}(x\sqrt{1-y^2} + y\sqrt{1-x^2}),$$

and solve the equation

$$\tan^{-1}\{(x+1)\div(x-1)\} + \tan^{-1}\{(x-1)\div x\} = \tan^{-1}(-7).$$

19. Express $\sin 3A \div (\sin 2A - \sin A)$ in terms of $\cos A$.

20. Prove the identities:

(i) $(1+\cos A)\tan^2\frac{1}{2}A = 1 - \cos A$;

(ii) $(\sec A + 2\sin A)(\operatorname{cosec} A - 2\cos A) = 2\cos 2A \cot 2A$.

21. In any triangle prove that $(b-c)\cos\frac{1}{2}A = a\sin\frac{1}{2}(B-C)$.

If $A = 80°$, $a = 10$, $b - c = 5$, find B and C.

22. Prove the identity $\cos 2x \sin 3x = \sin x \cos 4x + \cos x \sin 2x$.

23. Solve the equations $\cos 2\theta = \cos(\theta - a)$; $\cos 3\theta = \sin(\theta - \beta)$.

24. (i) If the equation of a straight line is put in the form $y = mx + c$, what is the geometrical interpretation of m?

(ii) Show how to find the angle between two lines whose equations are $y = mx + c$, $y = m'x + c'$.

(iii) Deduce that the lines are at right angles if $mm' = -1$; and parallel if $m - m' = 0$.

(iv) Prove that the lines whose equations are $ax + by + c = 0$, $a'x + b'y + c' = 0$, are perpendicular if $aa' + bb' = 0$, and parallel if $a/a' = b/b'$.

25. Find the angle between the lines whose equations are

(i) $3x - 4y = 5$, $4x - 2y = 7$;
(ii) $4x + 3y = 6$, $3x - 4y = 9$;
(iii) $2x - y = 3$, $4x + 5y = 1$;
(iv) $2x - y = 3$, $4x + 2y = 5$;
(v) $2x + 4y = 5$, $x + 2y = 3$.

In each case verify by drawing to scale.

26. Find the equations of the straight lines drawn through the point (3, 5), and respectively parallel and perpendicular to the line whose equation is $3x - 4y = 5$.

27. Find the equation of the straight line, parallel to the line whose equation is $x \cos \alpha + y \sin \alpha = p$, and passing through the point (x', y'). Deduce that the length of the perpendicular from (x', y') to the line $x \cos \alpha + y \sin \alpha = p$ is $x' \cos \alpha + y' \sin \alpha - p$.

28. Find in its simplest form the equation of the line joining the points $\{a \cos(\alpha + \beta),\ b \sin(\alpha + \beta)\}$, $\{a \cos(\alpha - \beta),\ b \sin(\alpha - \beta)\}$.

29. Prove that $\sin 55° \sin 15° - \sin 50° \sin 10° - \sin 65° \sin 5° = 0$.

30. Show that in any triangle

$$\frac{a^2 \sin(B - C)}{b + c} + \frac{b^2 \sin(C - A)}{c + a} + \frac{c^2 \sin(A - B)}{a + b} = 0.$$

31. If $2 \cos \theta = x + 1/x$ and $2 \cos \phi = y + 1/y$, prove that

$$2 \cos(\theta + \phi) = xy + 1/xy \text{ and } 2 \cos(\theta - \phi) = x/y + y/x.$$

32. If $\theta + \phi = 240°$, and $\operatorname{versin} \theta = 4 \operatorname{versin} \phi$, find the values of θ and ϕ.

33. Draw a curve to represent the variations in sign and magnitude of $(\sin \theta - \sqrt{3} \cos \theta) \div (\sqrt{3} \sin \theta + \cos \theta)$, from $\theta = 0$ to $\theta = \pi$.

34. If α and β are the roots of $a \sin \theta + b \cos \theta + c = 0$, prove that

$$\frac{\cos \frac{1}{2}(\alpha + \beta)}{b} = \frac{\cos \frac{1}{2}(\alpha - \beta)}{-c} = \frac{\sin \frac{1}{2}(\alpha + \beta)}{a}.$$

35. Eliminate θ and ϕ from

$$a \sin \theta + b \sin \phi = h, \qquad \text{(i)}$$
$$a \cos \theta - b \cos \phi = k, \qquad \text{(ii)}$$
$$\cos(\theta + \phi) = l. \qquad \text{(iii)}$$

36. Eliminate θ and ϕ when two equations are the same as (i) and (ii) in Ex. 35, and the third equation is (i) $\sin(\theta+\phi)=l$, (ii) $\tan(\theta+\phi)=l$.

37. Eliminate θ and ϕ from the equations

$$\frac{a}{b}=\frac{\cos(\phi+\alpha)}{\cos(\theta-\alpha)}=\frac{\sin\phi}{\sin\theta},\quad a\sin(\theta-\alpha)+b\sin(\phi+\alpha)=c.$$

38. Expand $\sin 5\theta$ in terms of $\sin\theta$, and $\cos 6\theta$ in terms of $\cos\theta$.

39. If $\sin B$ is the arithmetic mean between $\sin A$ and $\cos A$, prove that $\cos 2B=\cos^2(A+45°)$.

40. If $a\cos\theta+b\sin\theta=c$, show that

$$\theta=\tan^{-1}b/a+\cos^{-1}c/(\sqrt{a^2+b^2}).$$

41. Find the maximum and minimum values of

$$a\cos\theta+b\sin\theta=c.$$

Verify your answer when $a=3$, $b=5$, by drawing a graph.

42. Prove that

(i) $\sin\theta+\sin 2\theta+\sin 3\theta+\ldots$ to n terms

$$=\frac{\sin\frac{1}{2}n\theta\sin\frac{1}{2}\overline{n+1}\,\theta}{\sin\frac{1}{2}\theta};$$

(ii) $\cos A+\cos(A+B)+\cos(A+2B)+\ldots$ to n terms

$$=\frac{\sin\frac{1}{2}nB\cos(A+\frac{1}{2}\overline{n-1}\,B)}{\sin\frac{1}{2}B};$$

(iii) $\cos\alpha+\cos(\alpha\quad\beta)+\cos(\alpha+2\beta)+\ldots$ to n terms $=0$, if

$$n\beta=2\pi.$$

43. Find the sum of n terms in the following series:

(i) $\sin^2 A+\sin^2(A+B)+\sin^2(A+2B)+\ldots$;

(ii) $\cos^2 A+\cos^2(A+B)+\cos^2(A+2B)+\ldots$;

(iii) $\sin A\sin 2A+\sin 2A\sin 3A+\sin 3A\sin 4A+\ldots$.

Revision Examples C.

(All the following examples are taken from recent Examination Papers.)

1. Find, without reference to the tables, the values of (i) $\sin 45°$; (ii) $\cos 150°$; (iii) the tangent of the obtuse angle whose sine is $1/\sqrt{10}$.

2. Trace the graph of the function $\cos\theta + 2\sin\theta$ between the values 0 and 180° of θ, and determine from your figure the value of θ for which the function (i) is greatest, (ii) is decreasing most rapidly.

3. Express $\tan\theta$ in terms of $\sec\theta$.

Show that $(\sin\theta - \cos\theta)(\sec\theta + \operatorname{cosec}\theta) = \tan\theta - \cot\theta$.

4. Prove that the sines of the angles of a triangle are in the ratios of the sides opposite them.

5. Solve the equation $2\cos x + \sin x = 2$.

6. In a right-angled triangle ACB, C being the right angle, the angle A is 35°, the side AB is 10 inches; find the other sides.

7. If $\cos(A+B) = \cos A \cos B - \sin A \sin B$, calculate $\cos(A+B)$ when $A = 50°$ and $B = 50°$.

8. If α is measured in radians,

$$\sin\alpha = \alpha - \alpha^3/\underline{|3} + \alpha^5/\underline{|5} - \alpha^7/\underline{|7} + \ldots$$

where $\underline{|5}$ means $1 \times 2 \times 3 \times 4 \times 5$. Find $\sin\alpha$ correct to four significant figures when $\alpha = 0{\cdot}3$. What is the angle α in degrees?

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9. Define the tangent of an angle in such a way that your definition is true for all angles.

If θ be an acute angle, prove that $\cos(90+\theta) = -\sin\theta$.

10. Arrange in order of magnitude the angles

$$2\sin^{-1}{\cdot}51,\ \tfrac{1}{2}\cos^{-1}{\cdot}32,\ \tan^{-1}8{\cdot}9.$$

11. Draw the graph of $\cos x$ for values of x lying between 0° and 90°.

Use your figure to solve roughly the equation $x = 100\cos x°$, and verify your solution by the tables.

12. Given that $\sin 20° = 0·34$ and $\cos 20° = 0·94$, write down the values of $\sin 160° + \cos 160°$, of $\sin 250° + \cos 250°$, and of $\sin 340° + \cos 340°$.

13. In any triangle ABC, show that

(i) $c = a \cos B + b \cos A$;

(ii) $c^2 = a^2 + b^2 - 2ab \cos C$.

Find c when $a = 5$, $b = 6$, and $C = 155° 31'$, having given $\cos 24° 29' = 0·91$. Verify your result by a diagram drawn to scale.

14. Find to the nearest degree the angle subtended at a man's eye by a tower 50 feet high, when the man has stepped back 30 feet from the tower, assuming the height of his eye above the ground to be 5 feet 6 inches.

15. Write down a formula for $\sin \frac{1}{2}A$ in terms of the sides of the triangle ABC and explain the notation. How is the formula modified when $b = c$?

Given that the sides are 100, 200, 160 units in length, calculate the smallest angle.

16. A and B are two acute angles but $A + B$ is obtuse; prove that $\cos(A + B) = \cos A \cos B - \sin A \sin B$.

Solve completely $\cos x + \sin x = \cos \alpha - \sin \alpha$.

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17. Define the tangent of an angle, and show geometrically that

$$\tan A \tan(90° + A) + 1 = 0.$$

18. Draw a circle of diameter 1 inch. Draw a diameter AB and the tangent to the circle at B, divide either of the semi-circumferences between A and B into 8 equal parts, join A to the points of section, and produce the joining lines to meet the tangent at B. Measure the distances of the points so found from B, and use the results obtained for drawing the graph of $\tan A$ from $A = 0°$ to $A = 90°$.

19. Prove that the area of the triangle ABC is

$$\tfrac{1}{2} a^2 \sin B \sin C / \sin A.$$

Use this expression to find the area of the triangle when

$$a = 106·5 \text{ yards}, \; A = 56° 37', \; B = 75° 46'.$$

20. A person walking along a straight level road running due East and West observes that two objects P and Q are in a line bearing North-West, and after walking a further distance d he observes that P bears due North and that the direction of Q makes an angle A with the direction in which he is walking. Prove that the distance PQ is $d \cos A/\sin (A-45°)$. Find PQ when $d = 1372$ yards, and the angle $A = 56° 31'$.

21. (i) Show that $(\sin A + \cos A)^2 + (\sin A - \cos A)^2 = 2$.

(ii) Considering only values of A between 0° and 90°, find the value of A when $\sin A \cos A$ has its greatest value, and show that the same value of A gives the greatest value of $\sin A + \cos A$.

22. Let AD bisect the angle A of a triangle ABC, and let it meet BC in D; show that $BD \sin B = CD \sin C$.

Hence show that $BD . AC = DC . AB$.

23. (i) Show geometrically that

$$\sin (A+B) = \sin A \cos B + \cos A \sin B,$$

when A, B, and $A+B$ are each less than 90°.

(ii) By means of this formula, and in view of the restrictions under which it has been obtained, show that

$$\sin 464° = \sin 153° \cos 311° + \cos 153° \sin 311°.$$

24. Find $\tan \theta$ and x in terms of a and b from the equations

$$a \sin \theta + b \cos \theta = 3x,$$
$$a \cos \theta - 2b \sin \theta = 2x.$$

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25. An angle is made to increase gradually from 0° to 360°; state briefly how the values of its sine and of its cosine change during the increase of the angle.

26. Calculate the values of A between 0° and 360° for which $\tan A - 2 \cot A = 1$.

27. A and B are two milestones on a straight road running due East across a horizontal plane, C an object on the plane. The bearings of C as viewed from A and B are 35° North of East, and 55° North of West respectively. Find, to the nearest foot, (1) the distance of C from A, (2) the distance of C from the nearest point of the road.

28. Plot in relation to the same axis and origin the values of $\tan x$ and $2\sin x$ for the values $0°$, $12° 30'$, $37° 30'$, $50°$, $62° 30'$, $75°$ of x, draw the graphs of $\tan x$ and $2\sin x$, and find from them the values of x for which $\tan x = 2\sin x$. Give the general solution of the equation $\tan x = 2\sin x$.

29. Prove that $(\cos A + \sin A) \div (\cos A - \sin A) = \tan(A + 45°)$.

30. Prove for a triangle in which the angle B is obtuse the relation $\sin B/b = \sin C/c$, and deduce the relation

$$\tan \tfrac{1}{2}(B - C) = (b - c)/(b + c) \cot \tfrac{1}{2} A.$$

If $b = 27{\cdot}3$ yards, $c = 15{\cdot}8$ yards, $A = 48° 36'$, find B and C.

31. Prove that in a triangle $r = 4R \sin \frac{1}{2} A \sin \frac{1}{2} B \sin \frac{1}{2} C$. ABC is a triangle; $B'C'$ is drawn through A parallel to BC, $A'C'$ through C perpendicular to AC, and $A'B'$ through B perpendicular to AB. Prove that the area of the triangle $A'B'C'$ is

$$\tfrac{1}{2} a^2 \cos^2 (B - C) \div \cos B \cos C \sin A.$$

32. (i) Find $\sin A + \sin B$ in terms of functions of half the sum and of half the difference of the angles A and B.

(ii) If $A + B$ is between $90°$ and $180°$, find under what circumstances $\tan A + \tan B$ will be negative.

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33. Find to the nearest minute the angle of a regular polygon of 17 sides.

What angles less than $360°$ satisfy the equation

$$2\cos^2\theta + 11\sin\theta - 7 = 0?$$

34. Prove the identity

$$\frac{\tan^3 A}{1 + \tan^2 A} + \frac{\cot^3 A}{1 + \cot^2 A} = \frac{1 - 2\sin^2 A \cos^2 A}{\sin A \cos A}.$$

35. Assuming the formula $a^2 = b^2 + c^2 - 2bc\cos A$, establish a formula for $\tan \frac{1}{2} A$ in terms of the sides of the triangle, and find the greatest angle of the triangle whose sides are 13, 14, 15.

36. Prove that for any triangle ABC

$$a/\sin A = b/\sin B = c/\sin C.$$

If $B = 39° 17'$, $a = 4{\cdot}2$, and $b = 3{\cdot}5$, solve the triangle fully; draw a figure to illustrate your solution.

37. The angles of elevation of a vertical pole from two points on a horizontal line passing through its base and 6 feet apart are α and β; prove that the height of the pole is $b/(\cot\alpha - \cot\beta)$ feet.

38. From a point on a horizontal plane passing through the foot of a tower the angles of elevation of the top and bottom of a flagstaff 20 feet high, placed vertically at the summit of the tower, are $51{\cdot}2^\circ$ and $47{\cdot}3^\circ$. Find the height of the tower.

39. Prove that (i) $\sin(A+B) = \sin A \cos B + \cos A \sin B$;

(ii) $\cos 2A\,(1+\tan^2 A) = 1-\tan^2 A$.

Use (ii) to find the value of $\tan 15^\circ$.

40. Reduce the fraction $a \div (\cos^2 A - \sin^2 B)$ to a form suitable for logarithmic calculation, and perform the calculation when $a = 10$, $A = 29^\circ\, 55'$, and $B = 15^\circ\, 5'$.

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41. Prove that $\sin^2 A + \cos^2 A = 1$ for all values of A less than 180°.

A and B are each less than 180°, $\sin A = {\cdot}3900$, $\sin B = {\cdot}9208$, find four possible values of $A+B$.

42. Find from your tables the value to two decimal places of the expression $\sin\theta + \sin 2\theta$, when θ is 10°, 20°, 30°, ... 90°, and from these draw a graph of the expression on a suitable scale.

43. In a triangle ABC prove that

(i) $2bc\cos A = b^2 + c^2 - a^2$;

(ii) $\cos^2 A + \cos^2 B + \cos A \cos B = \frac{3}{4}$, if $C = 60^\circ$.

44. In a triangle $a = 12{\cdot}76$, $b = 10{\cdot}87$, $c = 8{\cdot}37$, find C.

45. Show how to find the distance between two visible but inaccessible objects.

46. In any triangle ABC show that four times the area equals $(a^2+b^2+c^2) \div (\text{cotan}\, A + \text{cotan}\, B + \text{cotan}\, C)$.

Show also that when C is a right angle this expression reduces to $2ab$.

47. Prove the identities:

(i) $1/\sin 2A = 1/\tan A - 1/\tan 2A = \tan A + 1/\tan 2A$;

(ii) $\tan 3A = \dfrac{\sin {}^2A + \sin 4A}{\cos 2A + \cos 4A}$.

48. What is the meaning of $\tan^{-1} x$?

Prove that $\tan^{-1} x + \tan^{-1} y = \tan^{-1}[(x+y) \div (1-xy)]$.

Prove that 45° is one value of $\tan^{-1}\frac{1}{2} + \tan^{-1}\frac{1}{4} + \tan^{-1}\frac{1}{13}$.

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49. Prove that (i) $\sec^2 A = 1 + \tan^2 A$;

(ii) $\text{cosec}\,\theta - \cot\theta = \tan\frac{1}{2}\theta$.

50. Construct an angle whose sine is 0·76. From your figure obtain the value of the cosine of the angle.

51. On squared paper draw graphs of $\tan\theta$ and $\cot\theta$ between $\theta = 10°$ and $\theta = 80°$. From the graph, or otherwise, find angles which satisfy the equation $\tan\theta + \cot\theta = 3$.

52. Let D be the point in which one of the escribed circles touches the side BC of a triangle ABC. If the sides a, b, c of the triangle are given, find expressions for the radius of that circle and for BD and CD.

53. A tree which grows at a point A on the north bank of a river is observed from the points B and C on the south bank. The distance BC is 200 metres, the angle ABC is $46° 30'$, and the angle ACB is $58° 20'$. Calculate the distance of A from the straight line BC.

54. Prove the formula $\sin\frac{1}{2}A = \sqrt{(s-b)(s-c) \div bc}$.

If, in a triangle ABC, $2b = a + c$, prove that

$$\sin\tfrac{1}{2}B = 2\sin\tfrac{1}{2}A\sin\tfrac{1}{2}C.$$

55. Find the angles B and C and the radius of the circumscribed circle of a triangle ABC in which $A = 32° 42'$, $a = 36$, $b = 44$.

56. State De Moivre's Theorem, and, assuming it for integral indices, prove it for fractional indices.

Write down all the values of $(\sqrt{-1})^{\frac{2}{5}}$.

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57. If A is an obtuse angle whose sine is $\frac{5}{13}$, find the values of $\cos A$ and $\tan A$.

58. (i) Show, by drawing graphs of the two expressions $\sin x$ and $\cos(x + 90°)$, that $\sin x = -\cos(x + 90°)$.

(ii) If $\sin x = \frac{1}{2}\sqrt{2}$, find a formula which gives all the values of x which satisfy the equation.

59. Prove that in a triangle

(i) $\tan\frac{1}{2}B = \sqrt{(s-c)(s-a) \div s(s-b)}$;

(ii) $b\cos B + c\cos C = a\cos(B - C)$.

60. If two sides of a triangle and the angle opposite one of them are given, show how to solve the triangle, and discuss by the aid of a figure all the cases that can arise.

One side of a triangle is 20 inches long, the opposite angle is $34° 42'$; another side is 30·41 inches. Find the sides and angles of the two possible triangles.

61. Assuming the formulae for the sine and cosine of half an angle of a triangle in terms of the sides, prove that

(i) $r = \sqrt{(s-a)(s-b)(s-c) \div s}$;

(ii) $R = a/2 \sin A$.

62. I observe the altitude of an airship to be 35°, and that of the sun, which is in the same vertical plane as my eye and the airship, to be 40°. The shadow of the airship falls on a tree on the same level as my eye and 500 feet in front of me. Find the height of the airship.

63. In any triangle prove that

$$\sin A - \sin B + \sin C = 4 \sin \tfrac{1}{2} A \cos \tfrac{1}{2} B \sin \tfrac{1}{2} C.$$

Assuming the formula for expanding $\tan(A+B)$, find expressions for $\tan 2A$ and $\tan 3A$ in terms of $\tan A$.

64. Make an angle AOC and bisect it by the line OB. From any point A in OA draw ABC perpendicular to OB, meeting OB, OC in the points B and C respectively, and draw AN perpendicular to OC. Use this figure to prove that

$$\text{(i) } \sin 2A < 2 \sin A; \text{ (ii) } \tan 2A > 2 \tan A.$$

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65. Prove that $\sin^2 A + \cos^2 A = 1$.

Having given that the sine of an angle is ·56, calculate its cosine.

66. Show how to construct an angle whose sine is ·6.

Find a value of x which satisfies the equation

$$4 \sin x + 3 \cos x = 1.$$

67. Given two sides of a triangle and the included angle, show how to find the remaining side and the other angles. Prove such formulae as you require.

If $a = 1097$ feet, $b = 781$ feet, $C = 31° 30'$, find c to the nearest foot.

68. A ship is sailing at the rate of 7 miles an hour. A man walks forward across the deck at the rate of 4 miles an hour

relative to the deck, in a direction inclined to the keel at an angle of 60°. Find the direction of his actual motion in space.

69. Prove the formula $\cos(A-B) = \cos A\cos B + \sin A\sin B$.

Show that if $xy = a^2+1$ then

$$\cot^{-1}(a+x)+\cot^{-1}(a+y) = \cot^{-1}a.$$

70. Find an expression for $\cos(\alpha+\beta+\gamma)$ in terms of sines and cosines of α, β, and γ.

Prove the identity

$$\cos\alpha\cos(\beta+\gamma)+\cos\beta\cos(\gamma+\alpha)+\cos\gamma\cos(\alpha+\beta)$$
$$= \cos(\alpha+\beta+\gamma)+2\cos\alpha\cos\beta\cos\gamma.$$

71. At what angle must forces of 4 dynes and 5 dynes act so that their resultant may be a force of 6 dynes?

72. If θ be the circular measure of an angle, prove that, as θ is indefinitely diminished, the ratios $\theta : \sin\theta$, $\theta : \tan\theta$ approach to the limit unity.

A man standing beside one milestone on a straight road observes that the foot of the next milestone is on a level with his eyes, and that its height subtends an angle of 2′ 55″. Find the approximate height of that milestone.

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73. Write down the values of $\sin 36°$ and $\cos 36°$ as given by your tables. Calculate the sum of the squares of these numbers to six decimal places, and explain why the result differs from unity.

74. Give definitions of the tangent and cotangent of an angle which is greater than 90° and less than 180°.

Prove that (i) $\tan(180-\theta) = -\tan\theta$;

(ii) $\tan(90+\theta) = -\cot\theta$.

75. In any triangle prove that $a/\sin A = b/\sin B = c/\sin C$.

If BC be 25 inches, and CA be 30 inches, and if the angle ABC be twice the angle CAB, find the angles of the triangle ABC, and show that the length of the third side is 11 inches.

76. P, Q, R are three villages. P lies 7 miles to the North-East of Q, and Q lies $11\frac{1}{4}$ miles to the North-West of R. Find the distance and bearing of P from R.

77. A point is moving with velocity 50 feet per second in a direction 60° North of East. Find the resolved parts of the velocity in directions East and North.

78. A man has before him on a level plane a conical hill of vertical angle 90°. Stationing himself at some distance from its foot he observes the angle of elevation α of an object which he knows to be half-way up to the summit. Show that the part of the hill above the object subtends at his eye an angle

$$\tan^{-1}\frac{\tan\alpha\,(1-\tan\alpha)}{1+\tan\alpha\,(1+2\tan\alpha)}.$$

79. The latitude of London is 51° N., and the radius of the Earth 4000 miles. How far is London from the Equator measured along the Earth's surface, and how far from the Earth's axis?

80. Prove that $\sin A+\sin B = 2\sin\frac{1}{2}(A+B)\cos\frac{1}{2}(A-B)$.

Show that $\sin 10^\circ+\sin 20^\circ+\sin 40^\circ+\sin 50^\circ = \sin 70^\circ+\sin 80^\circ$.

MISCELLANEOUS PROBLEMS

(The following examples are taken from recent Army Entrance and Civil Service Papers.)

1. I take measurements to determine the air space of a rectangular hall: length 18·4 metres, breadth 11·8 metres, inclination to floor of diagonal of side wall 31·8°, of diagonal of end wall 44°. Calculate the air space.

More measurements were taken than were necessary. Check the measurements by deducing one of them from the other three.

2. The ancient Greeks measured the latitude of a place by setting up a vertical rod and comparing its length with the length of its shadow. Supposing observations taken at mid-day at the equinox (when the sun is vertical at the equator) to give $\frac{5}{3}$ as the ratio of the rod to shadow at Alexandria, and $\frac{4}{3}$ as the ratio at Carthage, find the latitude of each place.

3. The following method of determining the horizontal distance PR, and the difference of level QR between two points P and Q, is often used. A rod with fixed marks A, B on it is held vertical at Q, and the elevations of these points, viz. $ACD = \alpha$, $BCD = \beta$, are read by a telescope and divided circle at C, the axis of the telescope being a distance $CP = a$ above the ground at P. If $QA = b$, and $AB = s$, write down expressions for PR and QR. Find PR and QR when $\alpha = 6^\circ 10'$, $\beta = 7^\circ 36'$, the values of a, b, and s being 5 feet, $2\frac{1}{2}$ feet, and 5 feet respectively.

4. Three balls, 5 cm. in diameter, lie on a floor in contact, and a fourth equal ball is placed on them. Find the height of the centre of the fourth ball above the plane of the other three centres. Find also the inclination to the vertical of any line that touches both the top ball and one of the lower balls.

5. The curved surface of a right circular cone whose semi-vertical angle is 45° is made by cutting out a sector from a circular sheet of copper, the diameter of the sheet being 56 cm. Determine the angle of the required sector.

6. If tangents be drawn to the inscribed circle of a triangle parallel to the sides of the triangle, show that the areas of the triangles cut off by these tangents are inversely proportional to the areas of the corresponding escribed circles.

7. A rod BC, of length 5·8 cm., rotates about B. Another rod CA, of length 8·6 cm., has one end C hinged to the first rod, while the other end A slides along the line BO. By drawing the rods in various positions, find how the length of BA varies as the angle B increases; and show BA as a function of angle B in a graph for one revolution of BC, showing the actual length of BA and representing 30° by 1 cm.

Write down an equation connecting the angle B and the lengths of the three sides of the triangle ABC. Solve the equation to find the length of BA when angle B is 35°.

8. The extreme range of the guns of a fort is 8000 metres. A ship, 14000 metres distant, sailing due East at 24 kilometres an hour, notices the bearings of the fort to be 20° 30′ North of East. Find, to the nearest minute, when the ship will first come within range of the guns.

9. The face of a building is 136 feet long. A photographer wants to take the building from a point at which the face subtends an angle of 37°, and for this purpose he starts off from one corner of the building in a direction making an angle of 127° with the face in question. Find by calculation the distance from the corner at which he must take the photograph. Calculate the area of ground in the triangular space between his position and the face of the building.

10. From the top of a telephone pole three wires radiate in a horizontal plane. One wire, A, exerts a tension of 100 lb. weight; the next, B, makes an angle of 90° with A and exerts a tension of 80 lb. weight; the third, C, makes an angle of 35° with B and an angle of 125° with A, and exerts a tension of 90 lb. weight. It is required to equilibrate the three tensions by means of a fourth wire. Find its direction and tension.

11. A man passing along a straight road measures the angle between the direction of his advance and a line drawn to a house on his left. At a certain moment the angle is 36° 21′. He walks on 1500 yards and finds that the angle between the same direction

and the line to the house is now 125° 36′. Find the distance of the house from the road.

12. Plot a curve giving the sum of $4 \sin \theta$ and $3 \sin 2\theta$ from $\theta = 0°$ to $\theta = 180°$, and read off the angles at which the greatest and the least values respectively of this sum occur. For the angle use 1 cm. to represent 10 degrees, and for $4 \sin \theta + 3 \sin 2\theta$ use 1 cm. to represent unity. Also estimate the slope of the curve when $\theta = 90°$ and when $\theta = 135°$.

13. A, B, and C are three buoys marking the corners of a triangular yacht racecourse round an island. The angles A, B, and C of the triangle ABC are found to be 75°, 63°, and 42° respectively. P is a flagstaff on the island, from which A and B can be seen, and the distances of P from A and B are found by a range-finder to be 650 yards and 535 yards respectively, and the angle APB to be 137°. Calculate the length of one lap of the course.

14. Draw an angle XOP of 30°, making OP 2″ long: through P draw PQ parallel to OX and in the same direction: produce XO to X', making $OX' = OP$, and join $X'P$: cut off $PQ = PX$. Join OQ and measure the angle XOQ carefully. Now denote XOQ by ϕ, XOP by θ, and OP by e, and write down an expression for the length PQ. Deduce an equation for θ and ϕ, and solve it for $\tan \phi$.

Use your tables to evaluate ϕ when $\theta = 30°$, and compare your result with the measured value. It is said that the given construction trisects an angle. What is the percentage error for 30°?

15. In running a survey the lengths of a series of lines are measured, and the angle each line makes with the direction of magnetic North is measured by a theodolite. The data booked are given in the table below:—

Line.	Length in feet.	Bearing.
AB	433	29° 15′
BC	521	89° 12′
CD	352	132° 38′
DE	417	233° 25′

The angles are measured clockwise from the magnetic North direction.

By an error the measured length of the closing line EA of the survey was not recorded, nor its bearing; from the data given in the table calculate these missing data.

16. AOB and COD are two straight roads crossing one another at an angle of 57°. A motor-car, travelling at the rate of 18 miles an hour along AOB, is 1500 yards from O, when a man, walking at the rate of three miles an hour along COD, is a quarter of a mile from O; car and man are both approaching O. Find graphically the motion of the car relative to the man. Hence find the least distance between the car and the man, and when they are at this distance from one another.

17. In a triangle $a = 10$ cm., $b = 7$ cm., one angle is 95°. There being no restriction as to which angle of the triangle is 95°, discuss how many distinct triangles can be made. Select any one case, and for this case calculate the remaining sides and angles.

18. X and Y are two fixed points in a straight line, P a point which so moves that $\cos PXY + \cos PYX = k$ (a constant). Prove the accuracy of the following construction for obtaining the locus of P: With X and Y as centres describe circles of radius XY/k. From any point N in XY draw NAB perpendicular to XY cutting the former circle in A and the latter in B. Draw XA and YB, intersecting in P. Then P is a point on the locus.

19. A candle, C, is placed on the floor at a distance r from a point O on a wall, and at the same level as the candle-flame, and the angle which OC makes with a perpendicular to the wall at O, is θ. The illumination received on the wall at O from the candle is known to be equal to $A \cos \theta / r^2$ where A is a constant. If the candle be moved about on the floor in such a way that this illumination remains constant, plot on a diagram the curve described by the candle-flame.

20. Two small islands are 5 miles apart, and there is known to be a rock distant 3 miles from each. A ship is in such a position that the islands subtend an angle of 66° at the ship. Calculate, to the nearest hundredth of a mile, her least possible distance from the rock.

21. Find by means of a graph two acute angles θ for which $5 \sin 2\theta = 3 \sin \theta + 2{\cdot}5$.

Find also the greatest value of $5 \sin 2\theta - 3 \sin \theta$ when θ is an acute angle, and the angle to which this value corresponds.

22. The elevation of an aeroplane which is flying horizontally on a fixed course at a height of 150 feet is taken at two instants

at an interval of 20 secs. At the first observation the elevation is 10° and the bearing is due North, and at the second the elevation is $6\frac{1}{2}$° and the bearing is N. 35° E. Find the course and speed of the aeroplane.

23. The strength of an electric current C is obtained from the formula $C = k \tan \theta$ where θ is the angle read off in degrees on an instrument, and k is a constant. If an observer makes an error of $\delta\theta$ in reading the angle θ, prove that the value of C thus obtained will be wrong by an amount equal to $\frac{1}{90}\pi\, C \operatorname{cosec} 2\theta\, \delta\theta$. Hence find the error per cent. in C produced by making a mistake of $\frac{1}{10}$ degree when θ is 60°.

What value of θ is likely to produce the smallest error in the value of C?

24. If P denote the pressure of wind in lb. per square foot on a plane surface at right angles to the direction of the wind, and p denote the normal pressure of wind in lb. per square foot on a plane surface inclined at an angle θ to the direction of the wind, the following formulae are used to determine the ratio $p : P$.

(i) $p/P = (\sin \theta)^{1 \cdot 84 \cos \theta - 1}$;

(ii) $p/P = 2 \sin \theta / (1 + \sin^2 \theta)$.

Compare the values of p/P given by these formulae for the values 10° and 50° of θ.

25. A man walks due W. from a point A up a straight path inclined at 10° to the horizon. After walking 2 miles he reaches B, and turns up another straight path to the NE., sloping 15° upwards. He reaches C after walking one mile from B. What is the distance in a straight line from C to A? What is the height of C above the level of A? Taking the face of the hill ABC as a plane surface, what is the greatest slope?

26. A flagstaff stands vertically on horizontal ground. Four ropes, each 56 feet long, are stretched from a point in the flagstaff, 50 feet above the ground, to four pegs in the ground, arranged at the corners of a square. Calculate the angle between two adjoining ropes.

27. Q is the centre of a circle of radius 10 cm., and QO is a radius. The seven points ABC... lie on the circumference and the angles OQA, OQB, OQC... have the values 10°, 20°, 30°...70°. Find by drawing or calculation the lengths of the chords OA, OB, OC..., and tabulate the results.

Draw a graph to give the length of chord of the circle in terms of the angle which it subtends at the centre (for angles up to 70°). Show the chord's actual size, and represent 4 degrees by 1 cm.

From your graph find the length of the chord which subtends an angle of 48°. Make a triangle having one side of this length, and the other two sides 10 cm. long, and therefore having an angle of 48°.

Check the accuracy of your drawing by measuring this angle.

28. A square made of jointed rods each 4 inches long is deformed into a rhombus having half the area of the square. Calculate the lengths of the diagonals of the resulting figure and check by drawing. If it is part of a lattice-work, the original height of which is 6 times the diagonal of one of these squares, find by calculation how much the height of the lattice-work could be increased if each square were reduced to half its area.

29. A straight rod AB, 3 feet 9 inches long, is held under water, A being 2 feet 6 inches and B 9 inches below the surface. Calculate (*a*) the distance below the surface of a point C on the stick which is 12 inches from A, (*b*) the angle which the stick makes with the surface of the water.

If a parallelogram is held under water, show that in every position the sum of the depths of the 4 corners is 4 times the depth of the point of intersection of the diagonals.

30. If a closed loop of thread is placed on a soap-film that covers a ring of wire, and the film within the loop is pierced, the film outside takes up as small an area as possible and thus pulls the thread at A into a circle. Calculate the diameter and the area of the circle formed by the thread if length of thread forming the loop is 6 cm.

If the ends BC of the thread are attached to the ring, and the film on one side of the thread is pierced, the thread again becomes a circular arc. If the thread BC is 6 cm. long, and the angle it subtends at the centre of the circle of which it forms an arc is 120°, calculate the length of the chord BC.

EXAMINATION PAPERS

OXFORD AND CAMBRIDGE SCHOOLS' EXAMINATION BOARD.

School Certificate, 1910.

1. Define the tangent of an angle.

Construct an acute angle whose sine is ·6, and find its cosine and cotangent.

2. Prove that $\cos(180-\alpha) = -\cos\alpha$.

Arrange the angles α, β, γ in order of magnitude, if

$$\sin\alpha = \cdot 8211,\ \cos\beta = \cdot 7738,\ \tan\gamma = -0\cdot 6104,$$

the angles being positive and each less than 180°.

3. What is the length of the shadow of a man, 5 feet 8 inches high, cast by the sun when its altitude is 55° 30′?

4. Draw the graph of $10+10\cos 2x$ for values of x between 0° and 60°. Find a value of x to satisfy the equation

$$x = 10+10\cos 2x^\circ.$$

[Take one-tenth of an inch as unit along both axes.]

5. Prove that in *any* triangle $\sin A/a = \sin B/b$.

If $A = 63°$, $B = 49°$, $a = 50$ inches, find b to the nearest tenth of an inch.

6. Prove that

(i) $$\frac{\cos\theta+\sin\theta}{\cos\theta-\sin\theta} + \frac{\cos\theta-\sin\theta}{\cos\theta+\sin\theta} = \frac{2}{1-2\sin^2\theta};$$

(ii) $(\sec\theta+\tan\theta)(\operatorname{cosec}\theta-\cot\theta) = (\operatorname{cosec}\theta+1)(\sec\theta-1)$.

7. If $2\sin\theta+5\cos\theta = 5$, prove that $\tan\theta = 0$ or $20/21$.

8. Prove that $\sin(A-B) = \sin A\cos B - \cos A\sin B$, where A and B are both acute angles and A is greater than B.

Prove that $$\frac{\sin 5A+\sin A}{\sin 3A-\sin A} = 1+2\cos 2A.$$

9. Show that in any triangle ABC

$$\frac{b+c}{a} = \frac{\cos\frac{1}{2}(B-C)}{\sin\frac{1}{2}A}.$$

If $b+c = 24\cdot 8$ cm., $a = 11\cdot 89$ cm., $A = 39°$, find B and C.

10. A lighthouse is observed from a ship which is steaming due N. to bear 62° W. of N.; after the ship has sailed 10 miles the lighthouse is observed to bear 40° W. of S. Calculate the distance of the ship from the lighthouse when it was nearest to it.

Higher Certificate, 1910.

Part I.

1. Give a definition of $\cos\theta$ that holds for all angles from 0° to 180°. Show that $\cos(180-\theta) = -\cos\theta$.

2. Show that $\sec^2\theta = 1+\tan^2\theta$.

Draw the graph of $1+\sin 3x°$, where x lies between 0° and 60°.

3. Construct an acute angle whose cotangent is 2, an obtuse angle whose sine is 3, and an obtuse angle whose secant is $-3{\cdot}5$. Measure these angles as accurately as you can with the protractor, and verify your results by means of tables.

4. (i) Verify that 30°, 45°, and 60° are solutions of the equation

$$\sin 3x+\cos 3x = 2\cos 2x.$$

(ii) Show that

$$(\operatorname{cosec} A+\sec A)^3+(\operatorname{cosec} A-\sec A)^3 = 2\operatorname{cosec}^3 A\,(3\sec^2 A-2).$$

5. Show that in an obtuse-angled triangle

$$\sin A/a = \sin B/b = \sin C/c.$$

A man observes that the angular elevation of the foot of a tower on a distant hillside is α, and that the angular elevation of the top of the tower is β, and he knows that the height of the tower is h feet. Show that his horizontal distance from the tower is $h\cos\alpha\cos\beta\operatorname{cosec}(\beta-\alpha)$.

Part II.

6. Draw the graph of $\cot x$ between the values -180 and $+180$ of x, taking the unit of x to be $\frac{1}{60}$ inch and the unit of y to be one inch.

Find an acute angle to satisfy the equation $x = 60\cot x°$.

7. Show that $\sin(A-B) = \sin A\cos B-\sin B\cos A$, taking A and B to be acute angles of which A is the greater.

If $\tan x = k\tan(A-x)$, show that

$$(k-1)\sin A = (k+1)\sin(2x-A).$$

Use this result and tables to solve the equation

$$\tan x = 2\tan(50°-x).$$

8. In the triangle in which $a = 72$ feet, $B = 40°$, and $C = 55°$, find c.

9. Find in terms of a, b, and c the radius of the circle escribed to the side BC of the triangle ABC.

If I_1 is the centre of this circle, show that

$$a\,AI_1^2 - b\,BI_1^2 - c\,CI_1^2 = abc.$$

10. AB is a diameter of a circle whose centre is O; on AB an equilateral triangle ABC is described, and a point D is taken in AB such that $7BD = 2AB$; CD is produced beyond D to meet the circle at E. Show that $\tan ADC = 7/\sqrt{3}$ and that $\sin OED = 3/\sqrt{52}$.

Hence, or otherwise, show that the error made in taking the arc BE to be one-seventh of the circumference of the circle is less than ·2 per cent.

Part III was beyond the scope of this book.

OXFORD LOCAL EXAMINATIONS.

Junior. 1910.

1. (i) Find the sine of $60°$;

(ii) If A is an acute angle, and $\cos A = \frac{4}{5}$, find the value of $4 \tan A + 5 \sin A$.

2. P and Q are points on a straight stretch of a river bank and R is a point on the other bank. If $\cot PQR = ·32$, $\cot QPR = ·43$, and the length of PQ is 15 yards, find the breadth of the river.

3. Draw the graph of $\sin(45° + 2x)$ between $x = 0$ and $180°$.

4. If A, B, $A - B$ are all positive acute angles, prove that

$$\cos(A - B) = \cos A \cos B + \sin A \sin B.$$

5. (i) A, B, C are the angles of a triangle; if $\tan A = \frac{1}{2}$ and $\tan B = \frac{1}{3}$, find the angle C.

(ii) Prove that $\dfrac{\cos 5A + \cos 3A}{\sin 5A - \sin 3A} = \cot A$.

6. Solve the equation $\cos 2\theta + \sin\theta = 1$.

7. Prove for any triangle that

(i) $a/\sin A = b/\sin B = c/\sin C$;

(ii) $(b+c)\cos A + (c+a)\cos B + (a+b)\cos C = a+b+c$.

8. Find the angles A and B of a triangle ABC in which $a = 13$, $b = 14$, $c = 15$, having given:

$$\log 2 = \cdot 3010,\ \log 7 = \cdot 8451,$$
$$\angle \tan 26^\circ 34' = 9\cdot 6990,$$
$$\angle \tan 29^\circ 44' = 9\cdot 7569.$$

Senior. 1910.

1. Find the tangent of 30°.

Using the values of $\tan 30^\circ$ and $\tan 45^\circ$, prove that

$$\tan 75^\circ = 2 + \sqrt{3}.$$

2. A man on a straight level road observed two objects P and Q (P being the nearer) in a horizontal straight line inclined to the direction of the road at an angle α. If $\tan\alpha = \cdot 75$, $PQ = 400$ yards, and the shortest distance of P from the road is 180 yards, what is the shortest distance of Q from the road?

3. Prove that $\cos 3A = 4\cos^3 A - 3\cos A$. Find $\sin 18^\circ$.

4. If ABC is a triangle in which $b = c = 5$ inches and $a = 8$ inches, find the values of $\tan A$ and $\tan B$.

5. Prove that

$$\cos^2 A + \cos^2 B = \sin^2(A+B) + 2\cos A\cos B\cos(A+B).$$

6. Prove that in any triangle $c = (a+b)\sin\theta$, where

$$\cos\theta = 2\sqrt{ab}\cos\tfrac{1}{2}C/(a+b).$$

In a triangle ABC, $a = 36$ feet, $b = 4$ feet, $C = 55^\circ$. Using the above formula, find the third side, having given

$$\log 6 = \cdot 7782,\quad \angle \cos 57^\circ 51' = 9\cdot 7261,$$
$$\angle \cos 27^\circ 30' = 9\cdot 9479,\quad \sin 57^\circ 51' = \cdot 8467.$$

7. Find the radius of the circle inscribed in the triangle ABC.

C is the centre of a circle of diameter d, and A, B are two points on the circumference of the circle. If l is the length of the chord AB and δ is the diameter of the circle which touches CA, CB and also the arc AB at its middle point, prove that $1/\delta = 1/d + 1/l$.

CAMBRIDGE LOCAL EXAMINATION.

JUNIOR. 1909.

1. Define the sine of an angle. What are the greatest and least values which the sine of an angle can have?

Prove that $\sin A = \cos A \times \tan A$, and that

$$\sin A \sin B \cot B = \cos A \cos B \tan A.$$

2. Construct an angle whose tangent is 1·45, and measure it with a protractor. Verify your results with the help of the tables.

3. Prove that

(i) $\sin A = \tan A/(\sqrt{1+\tan^2 A})$; (ii) $\cos(90°+A) = -\sin A$.

4. Find by drawing graphs of $\sin A$ and $\sin 2A$ for what value of A, less than 90°, $2\sin A - \sin 2A = 1$.

5. A vertical post casts a shadow 15 feet long when the altitude of the sun is 50°; calculate the length of the shadow when the altitude of the sun is 32°.

6. Prove that $\sin A + \sin B = 2\sin\frac{1}{2}(A+B)\cos\frac{1}{2}(A-B)$, and that $\tan 2A = 2\tan A/(1-\tan^2 A)$.

Show that $\sin A - 3\sin 3A + 3\sin 5A - \sin 7A = 8\sin^3 A\cos 4A$.

7. Prove that, in any triangle ABC, $a\cos B + b\cos A = c$.

Show also that $(\tan A + \tan B)(\tan A - \cot C) = \sec^2 A$.

8. Show how to solve a triangle when three sides are given.

Find the greatest angle of the triangle whose sides are 5·2 inches, 7·7 inches, and 9·1 inches.

SENIOR. 1909.

1. Show that the ratio of the circumference to the diameter of a circle is an invariable quantity.

Find to an inch the diameter of a wheel which makes 400 revolutions in rolling along a track one mile long.

2. Any positive proper fraction being given, show that there are two angles, one acute and the other obtuse, such that the sine of either is equal to this fraction.

If the fraction is $\frac{1}{3}$, use the tables to find the angles, and the cosine and tangent of each.

3. Find by aid of the tables the values of $\sin x - \tan 2x$ for the values 0°, 10°, 20°, 30°, 45°, 60° of x.

Make a graph to give the values of $\sin x - \tan 2x$ from $x = 0$ to $x = 60°$.

4. Show that $\sin A+\sin B = 2\sin\frac{1}{2}(A+B)\cos\frac{1}{2}(A-B)$.

Prove also that

(i) $\tan^2 A = (1-\cos 2A)\div(1+\cos 2A)$;

(ii) $\sin 55^\circ \sin 15^\circ - \sin 50^\circ \sin 10^\circ - \sin 65^\circ \sin 5^\circ = 0$.

5. Find the greatest angle of a triangle whose sides are 15, 21, 28 inches in length.

Show that in any triangle

$$\frac{a^2\sin(B-C)}{b+c}+\frac{b^2\sin(C-A)}{c+a}+\frac{c^2\sin(A-B)}{a+b}=0.$$

6. Find an expression for the radius of the inscribed circle of a given triangle.

Determine to one place of decimals the length of the radii of the inscribed circle, and of the escribed circle opposite the greatest angle of the triangle referred to in Question 5.

Questions 7 and 8 were outside the scope of this book.

(The two following questions may be taken instead of 7 and 8, but considerably lower marks will be assigned to them.)

A. Show that if A, B, C are the angles of a triangle,

$$\tan A+\tan B+\tan C = \tan A\tan B\tan C.$$

Show also that

$$\tan\tfrac{1}{2}B\tan\tfrac{1}{2}C+\tan\tfrac{1}{2}C\tan\tfrac{1}{2}A+\tan\tfrac{1}{2}A\tan\tfrac{1}{2}B = 1.$$

B. Solve the equation $a\cos\theta+b\sin\theta = c$.

Find all the solutions of $\sin\theta\sin 3\theta = \sin 5\theta\sin 7\theta$.

COLLEGE OF PRECEPTORS.

Christmas, 1910.

$1\frac{1}{2}$ Hours.

[Four-place tables of logarithms and of natural functions and square-ruled paper are provided. All diagrams should be drawn as accurately as possible.]

Part I.

1. Define a radian, and find its magnitude in degrees to two places of decimals ($\pi = \frac{22}{7}$).

If the angle of an equilateral triangle were taken to be the unit angle, what would be the measure of a radian to two places of decimals?

2. Define the sine and tangent of an acute angle. Prove that

$$\sin^2 A + \cos^2 A = 1.$$

If $\tan A = \frac{15}{112}$, find the value of $\cos A - 8 \sin A$.

3. Find, geometrically, $\tan 30°$.

If $A = 30°$, $B = 45°$, $C = 60°$, $D = 90°$, find the value of:

(i) $\sin A \cos B - \sin B \cos A$;
(ii) $(\tan^2 B - \operatorname{cosec}^2 A)/(\cot C + \cos D)$.

4. Use logarithms to find as nearly as possible the values of:

(i) $3{\cdot}142 \times {\cdot}9342/{\cdot}00532$; (ii) $\sqrt{562{\cdot}3/{\cdot}005984}$.

5. Solve, using the tables, the triangle in which $C = 90°$, $a = 654$, $A = 38° 45'$.

Part II.

6. Find all the positive values of θ, less than 360°, which satisfy the equations:

(i) $\cos^2\theta - \sin^2\theta = 0$;
(ii) $4 \sin^2\theta \cos^2\theta - \sin^2\theta = \frac{1}{2}$.

Which of the following statements are possible?

(i) $\tan\theta = -2$; (ii) $\sin\theta = \frac{3}{2}$.

7. Write down, without proof, the expansions of $\sin(A-B)$, $\cos(A-B)$.

Find the value of $\tan \overline{A-B}$ in terms of $\tan A$, $\tan B$.

If $\tan A = \frac{1}{3}$, $\tan B = \frac{24}{7}$, find $\tan(A+B)$.

8. Prove that, in a triangle, $a^2 = b^2 + c^2 - 2bc \cos A$ when the angle A is (i) acute, (ii) obtuse.

Deduce that $\tan \frac{1}{2} A = \sqrt{(s-b)(s-c) \div s(s-a)}$.

Find the greatest angle in the triangle whose sides are 256, 389, 401.

9. AB is a horizontal straight line. A vertical straight line is drawn from B upwards, and in it two points P, Q are taken, such that BQ is five times BP. If the angle BAP is 30°, calculate $\tan PAQ$.

LEAVING CERTIFICATE EXAMINATION (SCOTLAND). 1910.

1. Explain the circular measurement of angles.

Express 30°, 50°, 166° 40′ in radians.

Express ·0187 radian in degrees, minutes, and seconds, taking $\pi = 3{\cdot}1416$.

2. Taking a horizontal inch to represent 10° and 5 vertical inches to represent the unit of length, plot, with the help of your tables, the values of $\tan\theta$ when $\theta = 0, 10°, 20°, 30°, 40°, 50°$.

Plot also the values of $\sin\theta$ for the same angles, join both series of points by smooth curves, and thus find a graphic solution of the equation $5(\tan\theta - \sin\theta) = 1$.

3. State the relation which exists between the sine and cosine of any angle.

Use this relation to find, and express in a diagram, all the values of α, less than 180°, which satisfy the equation

$$5\sin\alpha + 6\cos^2\alpha = 7.$$

Either, 4 a. A man walked 5 miles due North and then walked 6 miles in a direction 27° East of North. Find by a figure drawn to scale how far he now is from his starting-point, and in what direction he should have originally started in order to go straight to his final position. Verify your results by calculation.

Or, 4 b. The sides of a parallelogram are 2 inches and 3 inches in length, and its area is $3\frac{2}{3}$ square inches. Find by a diagram the sizes of its angles and the length of its longer diagonal. Verify your results by calculation.

Either, 5 a. Draw a circle of radius 2 inches, and inscribe in it a triangle ABC, such that $\angle B = 34°$, $\angle C = 73°$.

Measure the lengths of the sides as nearly as possible.

Calculate with the help of the tables the lengths of the sides to the nearest hundredth of an inch, and thus test the correctness of your drawing.

Or, 5 b. State and prove the formula which gives $\tan(A+B)$ in terms of $\tan A$ and $\tan B$.

Apply this formula to find expressions for $\tan 2A$, $\tan 3A$, and $\tan 5A$ in terms of $\tan A$.

INTERMEDIATE EXAMINATION (IRELAND).

Middle Grade (Pass). 1910.

1. Prove that $\sin^2 A + \cos^2 A = 1$, where A is an obtuse angle.

2. Find the value of the expression $\operatorname{cosec} A - \frac{5}{6} \cot A$, if $\sin A = \frac{11}{61}$, when A is acute, and when A is obtuse.

3. Prove the identity $(1 - \tan^2 A) \div (2\cos^2 A - 1) = \sec^2 A$.

4. In a triangle $C = 90°$, $c = 65°$, $\tan A = \cdot 28$. Find a and b each to two decimal places.

5. In a triangle $a = 5\sqrt{3}$, $b = 11$, $C = 150°$. Find c and $\cos A$.

6. In a triangle $B = 45°$, $b = 20$, $c = 4$. Find $\sin C$, and prove that the perpendicular from A on BC divides BC into two segments one of which is seven times the other.

7. Prove that the length of the perpendicular from the vertex A of a triangle on the opposite side BC is equal to $a/(\cot B + \cot C)$, considering the cases when both angles are acute, when one is right, and when one is obtuse.

8. Find the angles between 0° and 360° which satisfy the equation $6 \sin\theta - 4 \operatorname{cosec}\theta + \cot\theta = 0$, being given $\cos 48° 11' 23'' = \frac{2}{3}$.

Middle Grade (Honours). 1910.

1. Show by a graph the values of $\operatorname{cosec} A$ for values of A between $-90°$ and $360°$.

2. If A is an angle in the first quadrant, prove that

$$\sin A + \cos A + \tan A + \cot A > \sec A + \operatorname{cosec} A.$$

3. Prove the identity

$$3(\sin A - \cos A)^4 + 6(\sin A + \cos A)^2 + 4(\sin^6 A + \cos^6 A) = 13.$$

4. The sides of a triangle are 37, 7, and 40. Find all the angles, being given that $\cos 69° 25' 48'' = \frac{13}{37}$.

5. In a triangle $a = \sqrt{5}$, $b = \sqrt{12}$, $C = 45°$. Find c, and prove that $\cot A = 2\sqrt{\frac{6}{5}} - 1$.

6. Find a solution between 180° and 270° of the equation

$$5(1 + \sin x) = -3\cos x,$$

being given $\cos 28° 4' 21'' = \frac{15}{17}$.

7. Prove by drawing a line through B, making an angle x with the side BC, or otherwise, that in a triangle ABC,

$$c \cos(B-x)+b\cos(C+x) = a\cos x.$$

P is a point on the hypotenuse AB of a right-angled triangle ABC. $AP = x$, $PB = y$, $PC = z$. Find $\cos CPB$ in terms of x, y, and z. Find the sides of the triangle when $x = 3-\sqrt{3}$, $y = \sqrt{3}+1$, $z = \sqrt{6}$.

SENIOR GRADE (PASS). 1910.

1. Find the distance from the earth to the moon, assuming that the moon's diameter, 2165 miles, subtends an angle of $31' 10''$ at the earth.

2. Prove that $\tan\frac{1}{2}A = (1-\cos A)/\sin A$.

Find $\tan 15°$ and $\tan 22\frac{1}{2}°$ without using the tables.

3. Find x if $\cos^{-1}x+\cot^{-1}2 = \frac{1}{4}\pi$.

4. Assuming the formulae for the sines of the sum and difference of two angles, prove that

$$\sin A-\sin B = 2\cos\tfrac{1}{2}(A+B)\sin\tfrac{1}{2}(A-B).$$

Find the corresponding expressions in factors for $\cos A-\cos B$.

5. Find the solutions between 0° and 360° of the equation

$$\cot 2x-3\tan x = 3.$$

6. In a triangle $a = 183$, $b = 247$, $C = 81° 40'$. Find A and B.

7. In a triangle $A = 54° 30'$, $B = 69° 20'$, $a = 341$. Find b and c.

8. Prove that in a triangle

$$a\cos B-b\cos A = (a^2-b^2)/c.$$

9. Prove that in a triangle $r\cot\frac{1}{2}A = s-a$, where r is the radius of the inscribed circle, and s the perimeter.

SENIOR GRADE (HONOURS). 1910.

1. An arc 40 feet in length is taken on a circle whose radius is 35 feet. Find, to the nearest inch, the length of the perpendicular from the centre on the chord of this arc.

2. Prove the identity

$$\cos 5A/\sin A+\sin 5A/\cos A = 2\operatorname{cosec} 2A-4\sin 2A.$$

3. If $\cos x+\cos y+\cos z+\cos x\cos y\cos z = 0$, prove that

$$\tan\tfrac{1}{2}x\tan\tfrac{1}{2}y\tan\tfrac{1}{2}z = \pm 1.$$

4. If $x = \cot^{-1}\sqrt{\cos y}-\tan^{-1}\sqrt{\cos y}$, prove that

$$y = 2\tan^{-1}\sqrt{\sin x}.$$

5. In a triangle $A = 35° 20'$, $a = 127$, $b = 194$. Find B, C, and c.

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INDEX

Five-figure Logarithmic and Trigonometrical Tables

ARRANGED BY

W. E. PATERSON, M.A., B.Sc.

MATHEMATICAL MASTER, MERCERS' SCHOOL

AUTHOR OF 'SCHOOL ALGEBRA,' 'ELEMENTARY TRIGONOMETRY'

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These five-figure tables are intended to give results correct to four figures; the fifth figure in the answer may be inaccurate.

The decimal point is printed before all the logarithms of numbers; it is hoped that this will obviate the common mistake of reading off logarithms instead of antilogarithms, and vice-versa.

The trigonometrical tables are arranged so that, at one opening of the tables, all the functions of an angle may be found on the left-hand page and their logarithms on the right-hand page; here again confusion is avoided. The characteristics of the logarithmic functions are the true characteristics; no useful purpose is served by increasing them by 10.

It should be noticed that, instead of dividing by a sine, one may multiply by the cosecant, &c., and, similarly, instead of subtracting the logarithm of a sine, one may add the logarithm of the cosecant, &c. In many cases this shortens calculation.

For quick reference the last page may be used, which gives the trigonometrical functions, to four figures only, for every whole degree up to 90° and the corresponding circular measure to five figures.

Logarithms of R for Compound Interest

R	log R	R	log R
1·0025	·00108438	1·0325	·01389006
1·005	·00216606	1·035	·01494035
1·0075	·00324505	1·0375	·01598811
1·01	·00432137	1·04	·01703334
1·0125	·00539503	1·0425	·01807606
1·015	·00646604	1·045	·01911629
1·0175	·00753442	1·0475	·02015403
1·02	·00860017	1·05	·02118930
1·0225	·00966332	1·0525	·02222210
1·025	·01072387	1·055	·02325246
1·0275	·01178183	1·0575	·02428038
1·03	·01283722	1·06	·02530587

Constants used in Mensuration and their Logarithms

	logarithm		logarithm
$\pi = 3{\cdot}14159265$	0·497150	$1 \div \pi = 0{\cdot}31830989$	$\bar{1}{\cdot}502850$
$\frac{1}{2}\pi = 1{\cdot}57079633$	0·196120	$1 \div 4\pi = 0{\cdot}07957747$	$\bar{2}{\cdot}900790$
$\frac{1}{6}\pi = 0{\cdot}52359878$	$\bar{1}{\cdot}718999$	$\sqrt[3]{6 \div \pi} = 1{\cdot}24070098$	0·093667
$\frac{4}{3}\pi = 4{\cdot}18879020$	0·622089	$\sqrt[3]{3 \div 4\pi} = 0{\cdot}62035049$	$\bar{1}{\cdot}792637$
$\sqrt{\pi} = 1{\cdot}77245385$	0·248575	$\sqrt{1 \div \pi} = 0{\cdot}56418958$	$\bar{1}{\cdot}751425$
$\pi^2 = 9{\cdot}86960440$	0·994300	$1 \div \pi^2 = 0{\cdot}10132118$	$\bar{1}{\cdot}005700$
$\sqrt[3]{\pi} = 1{\cdot}46459189$	0·165717	$\sqrt[3]{\pi^2} = 2{\cdot}14502940$	0·331433
$\pi/180 = 0{\cdot}01745329$	$\bar{2}{\cdot}241877$	$180/\pi = 57{\cdot}29577951$	1·758123

Naperian (or Natural) Logarithms

$e = 2{\cdot}7182182$ $\quad \log_{10} e = {\cdot}43429448$ $\quad \log_e 10 = 2{\cdot}30258509$

$\log_{10} N = \log_e N \times \log_{10} e.$ $\quad \log_e N = \log_{10} N \times \log_e 10$

LOGARITHMS OF NUMBERS

Mean Differences

	0	1	2	3	4	1	2	3	4	5	6	7	8	9
10	·00000	·00432	·00860	·01284	·01703	42	85	127	170	212	254	297	339	3
11	·04139	·04532	·04922	·05308	·05690	39	77	116	155	193	232	270	309	3
12	·07918	·08279	·08636	·08991	·09342	35	71	106	142	177	213	248	284	3
13	·11394	·11727	·12057	·12385	·12710	33	66	98	131	164	197	229	262	2
14	·14613	·14922	·15229	·15534	·15836	30	61	91	122	152	183	213	244	2
15	·17609	·17898	·18184	·18469	·18752	28	57	85	114	142	171	199	228	2
16	·20412	·20683	·20952	·21219	·21484	27	53	80	107	134	160	187	214	24
17	·23045	·23300	·23553	·23805	·24055	25	50	76	101	126	151	176	201	22
18	·25527	·25768	·26007	·26245	·26482	24	48	71	95	119	143	167	190	21
19	·27875	·28103	·28330	·28556	·28780	23	45	68	90	113	135	158	180	2
20	·30103	·30320	·30535	·30750	·30963	21	43	64	86	107	128	150	172	19
21	·32222	·32428	·32634	·32838	·33041	20	41	61	82	102	123	143	164	18
22	·34242	·34439	·34635	·34830	·35025	20	39	59	78	98	117	137	158	17
23	·36173	·36361	·36549	·36736	·36922	19	37	56	75	94	112	131	149	16
24	·38021	·38202	·38382	·38561	·38739	18	36	54	72	90	108	125	143	16
25	·39794	·39967	·40140	·40312	·40483	17	34	52	69	86	103	120	138	15
26	·41497	·41664	·41830	·41996	·42160	17	33	50	66	83	99	116	132	14
27	·43136	·43297	·43457	·43616	·43775	16	32	48	64	80	96	112	128	14
28	·44716	·44871	·45025	·45179	·45332	15	31	46	61	77	92	108	123	13
29	·46240	·46389	·46538	·46687	·46835	15	30	45	59	74	89	104	119	13
30	·47712	·47857	·48001	·48144	·48287	14	29	43	57	72	86	101	115	12
31	·49136	·49276	·49415	·49554	·49693	14	28	42	56	70	83	97	111	12
32	·50515	·50651	·50786	·50920	·51055	13	27	40	54	67	81	94	108	121
33	·51851	·51983	·52114	·52244	·52375	13	26	39	52	65	78	91	105	11
34	·53148	·53275	·53403	·53529	·53656	13	25	38	51	63	76	89	101	114
35	·54407	·54531	·54654	·54777	·54900	12	25	37	49	62	74	86	99	111
36	·55630	·55751	·55871	·55991	·56110	12	24	36	48	60	72	84	96	108
37	·56820	·56937	·57054	·57171	·57287	12	23	35	47	58	70	82	93	105
38	·57978	·58093	·58206	·58320	·58433	11	23	34	45	57	68	80	91	102
39	·59106	·59218	·59329	·59439	·59550	11	22	33	44	55	66	78	89	100
40	·60206	·60314	·60423	·60531	·60638	11	22	32	43	54	65	76	86	97
41	·61278	·61384	·61490	·61595	·61700	11	21	32	42	53	63	74	84	95
42	·62325	·62428	·62531	·62634	·62737	10	21	31	41	51	62	72	82	93
43	·63347	·63448	·63548	·63649	·63749	10	20	30	40	50	60	70	80	90
44	·64345	·64444	·64542	·64640	·64738	10	20	29	39	49	59	69	79	88
45	·65321	·65418	·65514	·65610	·65706	10	19	29	38	48	58	67	77	86
46	·66276	·66370	·66464	·66558	·66652	9	19	28	38	47	56	66	75	84
47	·67210	·67302	·67394	·67486	·67578	9	18	28	37	46	55	64	73	83
48	·68124	·68215	·68305	·68395	·68485	9	18	27	36	45	54	63	72	81
49	·69020	·69108	·69197	·69285	·69373	9	18	26	35	44	53	62	71	79
50	·69897	·69984	·70070	·70157	·70243	9	17	26	35	43	52	60	69	78
51	·70757	·70842	·70927	·71012	·71096	8	17	25	34	42	51	59	68	76
52	·71600	·71684	·71767	·71850	·71933	8	17	25	33	42	50	58	67	75
53	·72428	·72509	·72591	·72673	·72754	8	16	24	33	41	49	57	65	73
54	·73239	·73320	·73400	·73480	·73560	8	16	24	32	40	48	56	64	72

	5	6	7	8	9	1	2	3	4	5	6	7	8	9
10	**·02119**	**·02531**	**·02938**	**·03342**	**·03743**	**40**	**81**	**121**	**162**	**202**	**242**	**283**	**323**	**364**
11	·06070	·06446	·06819	·07188	·07555	37	74	111	148	185	222	259	296	333
12	·09691	·10037	·10380	·10721	·11059	34	68	102	137	170	204	238	272	307
13	·13033	·13354	·13672	·13988	·14301	32	63	95	126	158	190	221	253	284
14	·16137	·16435	·16732	·17026	·17319	29	59	88	118	147	177	206	236	265
15	**·19033**	**·19312**	**·19590**	**·19866**	**·20140**	**28**	**55**	**83**	**110**	**138**	**165**	**193**	**221**	**248**
16	·21748	·22011	·22272	·22531	·22789	26	52	78	104	130	156	182	208	233
17	·24304	·24551	·24797	·25042	·25285	24	49	73	98	123	147	171	196	220
18	·26717	·26951	·27184	·27416	·27646	23	46	70	93	116	139	162	185	208
19	·29003	·29226	·29447	·29667	·29885	22	44	66	88	110	132	154	176	198
20	**·31175**	**·31387**	**·31597**	**·31806**	**·32015**	**21**	**42**	**63**	**84**	**105**	**126**	**147**	**168**	**188**
21	·33244	·33445	·33646	·33846	·34044	20	40	60	80	100	120	140	160	180
22	·35218	·35411	·35603	·35793	·35984	19	38	57	77	96	115	134	153	172
23	·37107	·37291	·37475	·37658	·37840	18	37	55	73	91	110	128	146	165
24	·38917	·39094	·39270	·39445	·39620	18	35	53	70	88	105	123	140	158
25	**·40654**	**·40824**	**·40993**	**·41162**	**·41330**	**17**	**34**	**51**	**67**	**84**	**101**	**118**	**135**	**152**
26	·42325	·42488	·42651	·42813	·42975	16	32	49	65	81	97	114	130	146
27	·43933	·44091	·44248	·44404	·44560	16	31	47	63	78	94	110	125	141
28	·45484	·45637	·45788	·45939	·46090	15	30	45	61	76	91	106	121	136
29	·46982	·47129	·47276	·47422	·47567	14	29	44	58	73	87	102	117	131
30	**·48430**	**·48572**	**·48714**	**·48855**	**·48996**	**14**	**28**	**42**	**56**	**71**	**85**	**99**	**113**	**127**
31	·49831	·49969	·50106	·50243	·50379	14	27	41	55	68	82	96	109	123
32	·51188	·51322	·51455	·51587	·51720	13	27	40	53	66	80	93	106	119
33	·52504	·52634	·52763	·52892	·53020	13	26	39	51	64	77	90	103	116
34	·53782	·53908	·54033	·54158	·54283	13	25	38	50	63	75	88	100	113
35	**·55023**	**·55145**	**·55267**	**·55388**	**·55509**	**12**	**24**	**36**	**49**	**61**	**73**	**85**	**97**	**109**
36	·56229	·56348	·56467	·56585	·56703	12	24	35	47	59	71	83	95	106
37	·57403	·57519	·57634	·57749	·57864	12	23	35	46	58	69	81	92	104
38	·58546	·58659	·58771	·58883	·58995	11	22	34	45	56	67	78	90	101
39	·59660	·59770	·59879	·59988	·60097	11	22	33	44	55	66	76	87	98
40	**·60746**	**·60853**	**·60959**	**·61066**	**·61172**	**11**	**21**	**32**	**43**	**53**	**64**	**74**	**85**	**96**
41	·61805	·61909	·62014	·62118	·62221	10	21	31	42	52	62	73	83	94
42	·62839	·62941	·63043	·63144	·63246	10	20	30	41	51	61	71	81	91
43	·63849	·63949	·64048	·64147	·64246	10	20	30	40	50	60	70	79	89
44	·64836	·64933	·65031	·65128	·65225	10	19	29	39	49	58	68	78	87
45	**·65801**	**·65986**	**·65992**	**·66087**	**·66181**	**10**	**19**	**29**	**38**	**48**	**57**	**67**	**76**	**86**
46	·66745	·66839	·66932	·67025	·67117	9	19	28	37	47	56	65	74	84
47	·67669	·67761	·67852	·67943	·68034	9	18	27	36	46	55	64	73	82
48	·68574	·68664	·68753	·68842	·68931	9	18	27	36	45	54	62	71	80
49	·69461	·69548	·69636	·69723	·69810	9	17	26	35	44	52	61	70	78
50	**·70329**	**·70415**	**·70501**	**·70586**	**·70672**	**9**	**17**	**26**	**34**	**43**	**51**	**60**	**68**	**77**
51	·71181	·71265	·71349	·71433	·71517	8	17	25	34	42	50	59	67	75
52	·72016	·72099	·72181	·72263	·72346	8	16	25	33	41	49	58	66	74
53	·72835	·72916	·72997	·73078	·73159	8	16	24	32	40	48	57	65	73
54	·73640	·73719	·73799	·73878	·73957	8	16	24	32	40	48	55	63	71

Mean Differences

	0	1	2	3	4	1	2	3	4	5	6	7	8	9
55	**·74036**	**·74115**	**·74194**	**·74273**	**·74351**	**8**	**16**	**24**	**31**	**39**	**47**	**55**	**63**	**71**
56	·74819	·74896	·74974	·75051	·75128	8	15	23	31	39	46	54	62	69
57	·75587	·75664	·75740	·75815	·75891	8	15	23	30	38	46	53	61	68
58	·76343	·76418	·76492	·76567	·76641	7	15	22	30	37	45	52	60	67
59	·77085	·77159	·77232	·77305	·77379	7	15	22	29	37	44	51	59	66
60	**·77815**	**·77887**	**·77960**	**·78032**	**·78104**	**7**	**14**	**22**	**29**	**36**	**43**	**51**	**58**	**65**
61	·78533	·78604	·78675	·78746	·78817	7	14	21	28	36	43	50	57	64
62	·79239	·79309	·79379	·79449	·79518	7	14	21	28	35	42	49	56	63
63	·79934	·80003	·80072	·80140	·80209	7	14	21	27	34	41	48	55	62
64	·80618	·80686	·80754	·80821	·80889	7	14	20	27	34	41	47	54	61
65	**·81291**	**·81358**	**·81425**	**·81491**	**·81558**	**7**	**13**	**20**	**27**	**33**	**40**	**47**	**53**	**60**
66	·81954	·82020	·82086	·82151	·82217	7	13	20	26	33	39	46	52	59
67	·82607	·82672	·82737	·82802	·82866	6	13	19	26	32	39	45	52	58
68	·83251	·83315	·83378	·83442	·83506	6	13	19	25	32	38	45	51	57
69	·83885	·83948	·84011	·84073	·84136	6	13	19	25	31	38	44	50	56
70	**·84510**	**·84572**	**·84634**	**·84696**	**·84757**	**6**	**12**	**19**	**25**	**31**	**37**	**43**	**49**	**56**
71	·85126	·85187	·85248	·85309	·85370	6	12	18	24	31	37	43	49	55
72	·85733	·85794	·85854	·85914	·85974	6	12	18	24	30	36	42	48	54
73	·86332	·86392	·86451	·86510	·86570	6	12	18	24	30	36	42	48	53
74	·86923	·86982	·87040	·87099	·87157	6	12	18	23	29	35	41	47	53
75	**·87506**	**·87564**	**·87622**	**·87679**	**·87737**	**6**	**11**	**17**	**23**	**29**	**35**	**40**	**46**	**52**
76	·88081	·88138	·88195	·88252	·88309	6	11	17	23	29	34	40	46	51
77	·88649	·88705	·88762	·88818	·88874	6	11	17	22	28	34	39	45	51
78	·89209	·89265	·89321	·89376	·89432	6	11	17	22	28	33	39	44	50
79	·89763	·89818	·89873	·89927	·89982	5	11	16	22	27	33	38	44	49
80	**·90309**	**·90363**	**·90417**	**·90472**	**·90526**	**5**	**11**	**16**	**22**	**27**	**33**	**38**	**43**	**49**
81	·90849	·90902	·90956	·91009	·91062	5	11	16	21	27	32	37	43	48
82	·91381	·91434	·91487	·91540	·91593	5	11	16	21	27	32	37	42	48
83	·91908	·91960	·92012	·92065	·92117	5	10	16	21	26	31	37	42	47
84	·92428	·92480	·92531	·92583	·92634	5	10	15	21	26	31	36	41	46
85	**·92942**	**·92993**	**·93044**	**·93095**	**·93146**	**5**	**10**	**15**	**20**	**26**	**31**	**36**	**41**	**46**
86	·93450	·93500	·93551	·93601	·93651	5	10	15	20	25	30	35	40	45
87	·93952	·94002	·94052	·94101	·94151	5	10	15	20	25	30	35	40	45
88	·94448	·94498	·94547	·94596	·94645	5	10	15	20	25	30	34	39	44
89	·94939	·94988	·95036	·95085	·95134	5	10	15	19	24	29	34	39	44
90	**·95424**	**·95472**	**·95521**	**·95569**	**·95617**	**5**	**10**	**14**	**19**	**24**	**29**	**34**	**39**	**43**
91	·95904	·95952	·99599	·96047	·96095	5	10	14	19	24	29	33	38	43
92	·96379	·96426	·96473	·96520	·96567	5	9	14	19	24	28	33	38	42
93	·96848	·96895	·96942	·96988	·97035	5	9	14	19	23	28	33	37	42
94	·97313	·97359	·97405	·97451	·97497	5	9	14	18	23	28	32	37	41
95	**·97772**	**·97818**	**·97864**	**·97909**	**·97955**	**5**	**9**	**14**	**18**	**23**	**27**	**32**	**36**	**41**
96	·98227	·98272	·98318	·98363	·98408	5	9	14	18	23	27	32	36	41
97	·98677	·98722	·98767	·98811	·98856	4	9	13	18	22	27	31	36	40
98	·99123	·99167	·99211	·99255	·99300	4	9	13	18	22	27	31	35	40
99	·99564	·99607	·99651	·99695	·99739	4	9	13	17	22	26	31	35	39

	5	6	7	8	9	1	2	3	4	5	6	7	8	9
55	·74429	·74507	·74586	·74663	·74741	8	16	23	31	39	47	55	62	70
56	·75205	·75282	·75358	·75435	·75511	8	15	23	31	38	46	53	61	69
57	·75967	·76042	·76118	·76193	·76268	8	15	23	30	38	45	53	60	68
58	·76716	·76790	·76864	·76938	·77012	7	15	22	30	37	44	52	59	66
59	·77452	·77525	·77597	·77670	·77743	7	15	22	29	36	44	51	58	65
60	·78176	·78247	·78319	·78390	·78462	7	14	21	29	36	43	50	57	64
61	·78888	·78958	·79029	·79099	·79169	7	14	21	28	35	42	49	56	63
62	·79588	·79657	·79727	·79796	·79865	7	14	21	28	35	42	48	55	62
63	·80277	·80346	·80414	·80482	·80550	7	14	20	27	34	41	48	55	61
64	·80956	·81023	·81090	·81158	·81224	7	13	20	27	34	40	47	54	60
65	·81624	·81690	·81757	·81823	·81889	7	13	20	26	33	40	46	53	59
66	·82282	·82347	·82413	·82478	·82543	7	13	20	26	33	39	46	52	59
67	·82930	·82995	·83059	·83123	·83187	6	13	19	26	32	39	45	51	58
68	·83569	·83632	·83696	·83759	·83822	6	13	19	25	32	38	44	51	57
69	·84198	·84261	·84323	·84386	·84448	6	12	19	25	31	37	44	50	56
70	·84819	·84880	·84942	·85003	·85065	6	12	18	25	31	37	43	49	55
71	·85431	·85491	·85552	·85612	·85673	6	12	18	24	30	36	42	48	54
72	·86034	·86094	·86153	·86213	·86273	6	12	18	24	30	36	42	48	54
73	·86629	·86688	·86747	·86806	·86864	6	12	18	24	29	35	41	47	53
74	·87216	·87274	·87332	·87390	·87448	6	12	17	23	29	35	41	46	52
75	·87795	·87852	·87910	·87967	·88024	6	11	17	23	29	34	40	46	51
76	·88366	·88423	·88480	·88536	·88593	6	11	17	23	28	34	40	45	51
77	·88930	·88986	·89042	·89098	·89154	6	11	17	22	28	33	39	45	50
78	·89487	·89542	·89597	·89653	·89708	6	11	17	22	28	33	39	44	50
79	·90037	·90091	·90146	·90200	·90255	5	11	16	22	27	33	38	44	49
80	·90580	·90634	·90687	·90741	·90795	5	11	16	22	27	32	38	43	48
81	·91116	·91169	·91222	·91275	·91328	5	11	16	21	27	32	37	42	48
82	·91646	·91698	·91751	·91803	·91855	5	10	16	21	26	31	37	42	47
83	·92169	·92221	·92273	·92324	·92376	5	10	16	21	26	31	36	41	47
84	·92686	·92737	·92788	·92840	·92891	5	10	15	20	26	31	36	41	46
85	·93197	·93247	·93298	·93349	·93399	5	10	15	20	25	30	35	40	46
86	·93702	·93752	·93802	·93852	·93902	5	10	15	20	25	30	35	40	45
87	·94201	·94250	·94300	·94349	·94399	5	10	15	20	25	30	35	40	44
88	·94694	·94743	·94792	·94841	·94890	5	10	15	20	25	29	34	39	44
89	·95182	·95231	·95279	·95328	·95376	5	10	15	19	24	29	34	39	44
90	·95665	·95713	·95761	·95809	·95856	5	10	14	19	24	29	33	38	43
91	·96142	·96190	·96237	·96284	·96332	5	9	14	19	24	28	33	38	43
92	·96614	·96661	·96708	·96755	·96802	5	9	14	19	23	28	33	37	42
93	·97081	·97128	·97174	·97220	·97267	5	9	14	19	23	28	32	37	42
94	·97543	·97589	·97635	·97681	·97727	5	9	14	18	23	27	32	37	41
95	·98000	·98046	·98091	·98137	·98182	5	9	14	18	23	27	32	36	41
96	·98453	·98498	·98543	·98588	·98632	4	9	13	18	22	27	31	36	40
97	·98900	·98945	·98989	·99034	·99078	4	9	13	18	22	27	31	36	40
98	·99344	·99388	·99432	·99476	·99520	4	9	13	18	22	26	31	35	40
99	·99782	·99826	·99870	·99913	·99957	4	9	13	17	22	26	31	35	39

Mean Differences

	0	1	2	3	4	Mean Differences 1	2	3	4	5	6	7	8	9
·00	**10000**	**10023**	**10046**	**10069**	**10093**	**2**	**5**	**7**	**9**	**12**	**14**	**16**	**19**	**21**
·01	10233	10257	10280	10304	10328	2	5	7	9	12	14	17	19	21
·02	10471	10495	10520	10544	10568	2	5	7	10	12	15	17	20	22
·03	10715	10740	10765	10789	10814	2	5	7	10	12	15	17	20	22
·04	10965	10990	11015	11041	11066	3	5	8	10	13	15	18	20	23
·05	**11220**	**11246**	**11272**	**11298**	**11324**	**3**	**5**	**8**	**10**	**13**	**16**	**18**	**21**	**23**
·06	11482	11508	11535	11561	11588	3	5	8	11	13	16	18	21	24
·07	11749	11776	11803	11830	11858	3	5	8	11	14	16	19	22	24
·08	12023	12050	12078	12106	12134	3	6	8	11	14	17	19	22	25
·09	12303	12331	12359	12388	12417	3	6	9	11	14	17	20	23	26
·10	**12589**	**12618**	**12647**	**12677**	**12706**	**3**	**6**	**9**	**12**	**15**	**18**	**20**	**23**	**26**
·11	12882	12912	12942	12972	13002	3	6	9	12	15	18	20	24	27
·12	13183	13213	13243	13274	13305	3	6	9	12	15	18	21	24	27
·13	13490	13521	13552	13583	13614	3	6	9	12	16	19	22	25	28
·14	13804	13836	13868	13900	13932	3	6	10	13	16	19	22	26	29
·15	**14125**	**14158**	**14191**	**14223**	**14256**	**3**	**7**	**10**	**13**	**16**	**20**	**23**	**26**	**30**
·16	14454	14488	14521	14555	14588	3	7	10	13	17	20	24	27	30
·17	14791	14825	14859	14894	14928	3	7	10	14	17	21	24	27	31
·18	15136	15171	15205	15241	15276	4	7	11	14	18	21	25	28	32
·19	15488	15524	15560	15596	15631	4	7	11	14	18	22	25	29	32
·20	**15849**	**15885**	**15922**	**15959**	**15996**	**4**	**7**	**11**	**15**	**18**	**22**	**26**	**29**	**33**
·21	16218	16255	16293	16331	16368	4	8	11	15	19	23	26	30	34
·22	16596	16634	16672	16711	16749	4	8	11	15	19	23	27	31	34
·23	16982	17022	17061	17100	17140	4	8	12	16	20	24	28	32	35
·24	17378	17418	17458	17498	17539	4	8	12	16	20	24	28	32	36
·25	**17783**	**17824**	**17865**	**17906**	**17947**	**4**	**8**	**12**	**16**	**21**	**25**	**29**	**33**	**37**
·26	18197	18239	18281	18323	18365	4	8	13	17	21	25	30	34	38
·27	18621	18664	18707	18750	18793	4	9	13	17	22	26	30	34	39
·28	19055	19099	19143	19187	19231	4	9	13	18	22	26	31	35	40
·29	19498	19543	19588	19634	19679	5	9	14	18	23	27	32	36	41
·30	**19953**	**19999**	**20045**	**20091**	**20137**	**5**	**9**	**14**	**18**	**23**	**28**	**32**	**37**	**42**
·31	20417	20464	20512	20559	20606	5	9	14	19	24	28	33	38	43
·32	20893	20941	20989	21038	21086	5	10	15	19	24	29	34	39	44
·33	21380	21429	21478	21528	21577	5	10	15	20	25	30	35	40	44
·34	21878	21928	21979	22029	22080	5	10	15	20	25	30	35	40	46
·35	**22387**	**22439**	**22491**	**22542**	**22594**	**5**	**10**	**16**	**21**	**26**	**31**	**36**	**41**	**47**
·36	22909	22961	23014	23067	23121	5	11	16	21	27	32	37	42	48
·37	23442	23496	23550	23605	23659	5	11	16	22	27	33	38	44	49
·38	23988	24044	24099	24155	24210	6	11	17	22	28	33	39	44	50
·39	24547	24604	24660	24717	24774	6	11	17	23	28	34	40	45	51
·40	**25119**	**25177**	**25235**	**25293**	**25351**	**6**	**12**	**17**	**23**	**29**	**35**	**41**	**47**	**52**
·41	25704	25763	25823	25882	25942	6	12	18	24	30	36	42	48	54
·42	26303	26363	26424	26485	26546	6	12	18	24	30	36	43	49	55
·43	26915	26977	27040	27102	27164	6	12	19	25	31	37	44	50	56
·44	27542	27606	27669	27733	27797	6	13	19	26	32	38	45	51	57
·45	**28184**	**28249**	**28314**	**28379**	**28445**	**7**	**13**	**20**	**26**	**33**	**39**	**46**	**52**	**59**
·46	28840	28907	28973	29040	29107	7	13	20	27	33	40	47	53	60
·47	29512	29580	29648	29717	29785	7	14	21	27	34	41	48	55	62
·48	30200	30269	30339	30409	30479	7	14	21	28	35	42	49	56	63
·49	30903	30974	31046	31117	31189	7	14	21	29	36	43	50	57	64

	5	6	7	8	9	1	2	3	4	5	6	7	8	9
·00	10116	10139	10162	10186	10209	2	5	7	9	12	14	16	19	21
·01	10351	10375	10399	10423	10447	2	5	7	10	12	14	17	19	22
·02	10593	10617	10641	10666	10691	2	5	7	10	12	15	17	20	22
·03	10839	10864	10889	10914	10940	3	5	8	10	13	15	18	20	23
·04	11092	11117	11143	11169	11194	3	5	8	10	13	15	18	20	23
·05	11350	11376	11402	11429	11455	3	5	8	11	13	16	18	21	24
·06	11614	11641	11668	11695	11722	3	5	8	11	14	16	19	22	24
·07	11885	11912	11940	11967	11995	3	6	8	11	14	17	19	22	25
·08	12162	12190	12218	12246	12274	3	6	8	11	14	17	20	23	25
·09	12445	12474	12503	12531	12560	3	6	9	12	14	17	20	23	26
·10	12735	12764	12794	12823	12853	3	6	9	12	15	18	21	24	26
·11	13032	13062	13092	13122	13152	3	6	9	12	15	18	21	24	27
·12	13335	13366	13397	13428	13459	3	6	9	12	16	19	22	25	28
·13	13646	13677	13709	13740	13772	3	6	9	13	16	19	22	25	28
·14	13964	13996	14028	14060	14093	3	6	10	13	16	19	23	26	29
·15	14289	14322	14355	14388	14421	3	7	10	13	17	20	23	26	30
·16	14622	14655	14689	14723	14757	3	7	10	14	17	20	24	27	30
·17	14962	14997	15031	15066	15101	3	7	10	14	17	21	24	28	31
·18	15311	15346	15382	15417	15453	4	7	11	14	18	21	25	28	32
·19	15668	15704	15740	15776	15812	4	7	11	14	18	22	25	29	33
·20	16032	16069	16106	16144	16181	4	7	11	15	19	22	26	30	33
·21	16406	16444	16482	16520	16558	4	8	11	15	19	23	27	30	34
·22	16788	16827	16866	16904	16943	4	8	12	16	19	23	27	31	35
·23	17179	17219	17258	17298	17338	4	8	12	16	20	24	28	32	36
·24	17579	17620	17660	17701	17742	4	8	12	16	20	24	29	33	37
·25	17989	18030	18072	18113	18155	4	8	12	17	21	25	29	33	37
·26	18408	18450	18493	18535	18578	4	9	13	17	21	26	30	34	38
·27	18836	18880	18923	18967	19011	4	9	13	18	22	26	31	35	39
·28	19275	19320	19364	19409	19454	4	9	13	18	22	27	31	36	40
·29	19724	19770	19815	19861	19907	5	9	14	18	23	27	32	37	41
·30	20184	20230	20277	20324	20370	5	9	14	19	23	28	33	37	42
·31	20654	20701	20749	20797	20845	5	10	14	19	24	29	33	38	43
·32	21135	21184	21232	21281	21330	5	10	15	20	25	29	34	39	44
·33	21627	21677	21727	21777	21827	5	10	15	20	25	30	35	40	45
·34	22131	22182	22233	22284	22336	5	10	15	20	26	31	36	41	46
·35	22646	22699	22751	22803	22856	5	11	16	21	26	32	37	42	47
·36	23174	23227	23281	23335	23388	5	11	16	21	27	32	38	43	48
·37	23714	23768	23823	23878	23933	5	11	16	22	27	33	38	44	49
·38	24266	24322	24378	24434	24491	6	11	17	22	28	34	39	45	51
·39	24831	24889	24946	25003	25061	6	12	17	23	28	35	40	46	52
·40	25410	25468	25527	25586	25645	6	12	18	24	29	35	41	47	53
·41	26002	26062	26122	26182	26242	6	12	18	24	30	36	42	48	54
·42	26607	26669	26730	26792	26853	6	12	18	25	31	37	43	49	55
·43	27227	27290	27353	27416	27479	6	13	19	25	32	38	44	50	57
·44	27861	27925	27990	28054	28119	6	13	19	26	32	39	45	52	58
·45	28510	28576	28642	28708	28774	7	13	20	26	33	40	46	53	59
·46	29174	29242	29309	29376	29444	7	14	20	27	34	41	47	54	61
·47	29854	29923	29992	30061	30130	7	14	21	28	35	42	48	55	62
·48	30549	30620	30690	30761	30832	7	14	21	28	35	42	50	57	64
·49	31261	31333	31405	31477	31550	7	14	21	29	36	43	51	58	65

Mean Differences

	0	1	2	3	4	1	2	3	4	5	6	7	8	9
·50	31623	31696	31769	31842	31915	7	15	22	29	37	44	51	59	66
·51	32359	32434	32509	32584	32659	8	15	23	30	38	45	53	60	68
·52	33113	33189	33266	33343	33420	8	15	23	31	38	46	53	61	69
·53	33884	33963	34041	34119	34198	8	16	24	31	39	47	55	63	71
·54	34674	34754	34834	34914	34995	8	16	24	32	40	48	56	64	72
·55	35481	35563	35645	35727	35810	8	16	25	33	41	49	58	66	74
·56	36308	36392	36475	36559	36644	8	17	25	34	42	50	59	67	76
·57	37154	37239	37325	37411	37497	9	17	26	34	43	52	60	69	78
·58	38019	38107	38194	38282	38371	9	18	26	35	44	53	62	70	79
·59	38905	38994	39084	39174	39264	9	18	27	36	45	54	63	72	81
·60	39811	39902	39994	40087	40179	9	18	28	37	46	55	65	74	83
·61	40738	40832	40926	41020	41115	9	19	28	38	47	57	66	76	85
·62	41687	41783	41879	41976	42073	10	19	29	39	48	58	67	77	87
·63	42658	42756	42855	42954	43053	10	20	30	40	49	59	69	79	89
·64	43652	43752	43853	43954	44055	10	20	30	40	51	61	71	81	91
·65	44668	44771	44875	44978	45082	10	21	31	41	52	62	73	83	93
·66	45709	45814	45920	46026	46132	11	21	32	42	53	63	74	85	95
·67	46774	46881	46989	47098	47206	11	22	32	43	54	65	76	86	97
·68	47863	47973	48084	48195	48306	11	22	33	44	55	66	78	89	100
·69	48978	49091	49204	49317	49431	11	23	34	45	57	68	79	91	102
·70	50119	50234	50350	50466	50582	12	23	35	46	58	70	81	93	104
·71	51286	51404	51523	51642	51761	12	24	36	48	59	71	83	95	107
·72	52481	52602	52723	52845	52966	12	24	36	49	61	73	85	97	109
·73	53703	53827	53951	54075	54200	12	25	37	50	62	75	87	100	112
·74	54954	55081	55208	55335	55463	13	25	38	51	64	76	89	102	114
·75	56234	56364	56494	56624	56754	13	26	39	52	65	78	91	104	117
·76	57544	57677	57810	57943	58076	13	27	40	53	67	80	93	107	120
·77	58884	59020	59156	59293	59429	14	27	41	55	68	82	95	109	123
·78	60256	60395	60534	60674	60814	14	28	42	56	70	84	98	112	126
·79	61660	61802	61944	62087	62230	14	29	43	57	71	86	100	114	128
·80	63096	63241	63387	63533	63680	15	29	44	58	73	88	102	117	131
·81	64565	64714	64863	65013	65163	15	30	45	60	75	90	105	120	135
·82	66069	66222	66374	66527	66681	15	31	46	61	77	92	107	122	138
·83	67608	67764	67920	68077	68234	16	31	47	63	78	94	110	125	141
·84	69183	69343	69502	69663	69823	16	32	48	64	80	96	112	128	144
·85	70795	70958	71121	71285	71450	16	33	49	66	82	98	115	131	147
·86	72444	72611	72778	72946	73114	17	34	50	67	84	101	117	134	151
·87	74131	74302	74473	74645	74817	17	34	51	69	86	103	120	137	154
·88	75858	76033	76208	76384	76560	18	35	53	70	88	105	123	140	158
·89	77625	77804	77983	78163	78343	18	36	54	72	90	108	126	144	162
·90	79433	79616	79799	79983	80168	18	37	55	74	92	110	129	147	166
·91	81283	81470	81658	81846	82035	19	38	56	75	94	113	132	151	169
·92	83176	83368	83560	83753	83946	19	39	58	77	96	116	135	154	174
·93	85114	85310	85507	85704	85901	20	39	59	79	99	118	138	158	177
·94	87096	87297	87498	87700	87902	20	40	61	81	101	121	141	161	182
·95	89125	89331	89536	89743	89950	21	41	62	83	103	124	144	165	186
·96	91201	91411	91622	91833	92045	21	42	63	84	106	127	148	169	190
·97	93325	93541	93756	93972	94189	22	43	65	86	108	130	151	173	195
·98	95499	95719	95940	96161	96383	22	44	66	88	111	133	155	177	199
·99	97724	97949	98175	98401	98628	23	45	68	90	113	136	158	181	204

	5	6	7	8	9	1	2	3	4	5	6	7	8	9
·50	**31989**	**32063**	**32137**	**32211**	**32285**	**7**	**15**	**22**	**30**	**37**	**44**	**52**	**59**	**67**
·51	32734	32810	32885	32961	33037	8	15	23	30	38	45	53	61	68
·52	33497	33574	33651	33729	33806	8	15	23	31	39	46	54	62	70
·53	34277	34356	34435	34514	34594	8	16	24	32	40	48	55	63	71
·54	35075	35156	35237	35318	35400	8	16	24	32	41	49	57	65	73
·55	**35892**	**35975**	**36058**	**36141**	**36224**	**8**	**17**	**25**	**33**	**42**	**50**	**58**	**67**	**75**
·56	36728	36813	36898	36983	37068	9	17	26	34	43	51	60	68	77
·57	37584	37670	37757	37844	37931	9	17	26	35	44	52	61	70	78
·58	38459	38548	38637	38726	38815	9	18	27	36	45	54	62	71	80
·59	39355	39446	39537	39628	39719	9	18	27	36	46	55	64	73	82
·60	**40272**	**40365**	**40458**	**40551**	**40644**	**9**	**19**	**28**	**37**	**47**	**56**	**65**	**75**	**84**
·61	41210	41305	41400	41495	41591	10	19	29	38	48	57	67	76	86
·62	42170	42267	42364	42462	42560	10	20	29	39	49	59	68	78	88
·63	43152	43251	43351	43451	43551	10	20	30	40	50	60	70	80	90
·64	44157	44259	44361	44463	44566	10	20	31	41	51	61	72	82	92
·65	**45186**	**45290**	**45394**	**45499**	**45604**	**10**	**21**	**31**	**42**	**52**	**63**	**73**	**84**	**94**
·66	46238	46345	46452	46559	46666	11	21	32	43	54	64	75	86	96
·67	47315	47424	47534	47643	47753	11	22	33	44	55	66	77	88	99
·68	48417	48529	48641	48753	48865	11	22	34	45	56	67	79	90	101
·69	49545	49659	49774	49888	50003	11	23	34	46	57	69	80	92	103
·70	**50699**	**50816**	**50933**	**51051**	**51168**	**12**	**23**	**35**	**47**	**59**	**70**	**82**	**94**	**106**
·71	51880	52000	52119	52240	52360	12	24	36	48	60	72	84	96	108
·72	53088	53211	53333	53456	53580	12	25	37	49	62	74	86	98	111
·73	54325	54450	54576	54702	54828	13	25	38	50	63	75	88	100	113
·74	55590	55719	55847	55976	56105	13	26	39	52	64	77	90	103	116
·75	**56885**	**57016**	**57148**	**57280**	**57412**	**13**	**26**	**40**	**53**	**66**	**79**	**92**	**105**	**119**
·76	58210	58345	58479	58614	58749	13	27	40	54	67	81	94	108	121
·77	59566	59704	59841	59979	60117	14	28	41	55	69	82	97	110	124
·78	60954	61094	61235	61376	61518	14	28	42	56	71	85	99	113	127
·79	62373	62517	62661	62806	62951	14	29	43	58	72	87	101	116	130
·80	**63826**	**63973**	**64121**	**64269**	**64417**	**15**	**30**	**44**	**59**	**74**	**88**	**104**	**118**	**133**
·81	65313	65464	65615	65766	65917	15	30	45	60	76	91	106	121	136
·82	66834	66988	67143	67298	67453	15	31	46	62	77	93	108	124	139
·83	68391	68549	68707	68865	69024	16	32	48	63	79	95	111	127	143
·84	69984	70146	70307	70469	70632	16	32	49	65	81	97	114	130	146
·85	**71614**	**71779**	**71945**	**72111**	**72277**	**17**	**33**	**50**	**66**	**83**	**100**	**116**	**133**	**149**
·86	73282	73451	73621	73790	73961	17	34	51	68	84	102	119	136	153
·87	74989	75162	75336	75509	75683	17	35	52	70	87	104	122	139	156
·88	76736	76913	77090	77268	77446	18	36	53	71	89	107	124	142	160
·89	78524	78705	78886	79068	79250	18	36	55	73	91	109	127	145	164
·90	**80353**	**80538**	**80724**	**80910**	**81096**	**19**	**37**	**56**	**74**	**93**	**112**	**130**	**149**	**167**
·91	82224	82414	82604	82794	82985	19	38	57	76	95	114	133	152	171
·92	84140	84333	84528	84723	84918	19	39	58	78	97	117	136	156	175
·93	86099	86298	86497	86696	86896	20	40	60	80	100	120	140	160	179
·94	88105	88308	88512	88716	88920	20	41	61	82	102	122	143	163	184
·95	**90157**	**90365**	**90573**	**90782**	**90991**	**21**	**42**	**63**	**84**	**104**	**125**	**146**	**167**	**188**
·96	9225[illegible]	[illegible]470	92683	92897	93111	21	43	64	85	107	128	150	171	192
[illegible]	[illegible]	[illegible]24	94842	95060	95280	22	44	66	87	109	131	153	175	197
[illegible]	[illegible]	[illegible]	97051	97275	97499	22	45	67	90	112	134	157	179	201
[illegible]	[illegible]	[illegible]	99312	99541	99770	23	46	69	92	115	137	160	183	206

Mean Differences

NATURAL FUNCTIONS

Differences are given for every 10′. Intermediate values can be found by method of proportional parts; e. g.:—

To find tan 43° 56′ and cos 37° 34′

tan 43° 50′ =	·96008	cos 37° 30′ =	·79335
+ diff. for 6′ =	337	− diff. for 4′ =	− 71
∴ tan 43° 56′ =	·96345	∴ cos 37° 34′ =	·79264

When there is no entry in the difference column, the value of the function char too rapidly for correct interpolation by proportional parts. Greater accuracy is t obtained by expressing the function in terms of the sine and cosine.

To find tan 67° 23′

tan 67° 20′ = 2·39449 Diff. for 10′ = 1972

by proportional parts, diff. for 3′ = 592

This gives tan 67° 23′ = 2·40041. (The correct value is 2·40038.)

Subtract differences when dealing with co-functions

	sine	D	cosecant	D	tangent	D	cotangent	D	secant	D	cosine	D	
0°	**·00000**	**291**	**∞**		**·00000**	**291**	**∞**		**1·00000**	**000**	**1·00000**	**000**	**90**
10′	·00291	291	343·78		·00291	291	343·77		1·00000	002	1·00000	002	5
20′	·00582	291	171·89		·00582	291	171·89		1·00002	002	·99998	002	4
30′	·00873	291	114·59		·00873	291	114·59		1·00004	003	·99996	003	3
40′	·01164	290	85·946		·01164	291	85·940		1·00007	004	·99993	004	2
50′	·01454	291	68·757		·01455	291	68·750		1·00011	004	·99989	004	1
1°	**·01745**	**291**	**57·299**		**·01746**	**291**	**57·290**		**1·00015**	**005**	**·99985**	**006**	**89**
10′	·02036	291	49·114		·02037	291	49·104		1·00021	006	·99979	006	5
20′	·02327	291	42·976		·02328	291	42·964		1·00027	007	·99973	007	4
30′	·02618	290	38·202		·02619	291	38·188		1·00034	008	·99966	008	3
40′	·02908	291	34·382		·02910	291	34·368		1·00042	009	·99958	009	2
50′	·03199	291	31·258		·03201	291	31·242		1·00051	010	·99949	010	1
2°	**·03490**	**291**	**28·654**		**·03492**	**291**	**28·636**		**1·00061**	**011**	**·99939**	**010**	**88**
10′	·03781	290	26·451		·03783	292	26·432		1·00072	011	·99929	012	50
20′	·04071	291	24·562		·04075	291	24·542		1·00083	012	·99917	012	40
30′	·04362	291	22·926		·04366	292	22·904		1·00095	013	·99905	013	30
40′	·04653	290	21·494		·04658	291	21·470		1·00108	014	·99892	014	20
50′	·04943	291	20·230		·04949	292	20·206		1·00122	015	·99878	015	10
3°	**·05234**		**19·107**		**·05241**		**19·081**		**1·00137**		**·99863**		**87**
	cosine	D	secant	D	cotangent	D	tangent	D	cosecant	D	sine	D	

LOGARITHMIC FUNCTIONS

The values given here are the true logarithms; the characteristic is not increased by 10 as in many tables.

Differences are given for every 10′. Intermediate values can be found by the method of proportional parts.

The differences for the logarithm of a function and of the reciprocal of the function are the same in magnitude but opposite in sign.

When there is no entry in the difference column, the rate of change of the logarithm changes too rapidly for correct interpolation by proportional parts.

The following rules may be used when the angle is small :—

Log sine. Add $\bar{6}$·68557 to the log of the angle expressed in seconds and subtract $\frac{1}{3}$ of the log secant.

Log tan. Add $\bar{6}$·68557 to the log of the angle expressed in seconds and add $\frac{2}{3}$ of the log secant.

When the log sine is given, the angle is found in seconds by adding 5·31443 to the log sine and $\frac{1}{3}$ of the corresponding log secant (found in the ordinary way).

When the log tan is given, the angle is found in seconds by adding 5·31443 to the log tan and subtracting $\frac{2}{3}$ of the corresponding log secant (found in the ordinary way).

Subtract differences when dealing with co-functions

	log sin	D	log cosec	log tan	D	log cotan	log sec	D	log cos	
0°	— ∞		∞	— ∞		∞	**0·00000**		0·00000	**90°**
10′	$\bar{3}$·46373		2·53627	$\bar{3}$·46373		2·53627	0·00000	001	0·00000	50′
20′	$\bar{3}$·76475		2·23525	$\bar{3}$·76476		2·23524	0·00001	001	$\bar{1}$·99999	40′
30′	$\bar{3}$·94084		2·05916	$\bar{3}$·94086		2·05914	0·00002	001	$\bar{1}$·99998	30′
40′	$\bar{2}$·06578		1·93422	$\bar{2}$·06581		1·93419	0·00003	002	$\bar{1}$·99997	20′
50′	$\bar{2}$·16268		1·83732	$\bar{2}$·16273		1·83727	0·00005	002	$\bar{1}$·99995	10′
1°	**$\bar{2}$·24186**		**1·75814**	**$\bar{2}$·24192**		**1·75808**	**0·00007**	002	**$\bar{1}$·99993**	**89°**
10′	$\bar{2}$·30879		1·69121	$\bar{2}$·30888		1·69112	0·00009	003	$\bar{1}$·99991	50′
20′	$\bar{2}$·36678		1·63322	$\bar{2}$·36689		1·63311	0·00012	003	$\bar{1}$·99988	40′
30′	$\bar{2}$·41792		1·58208	$\bar{2}$·41807		1·58193	0·00015	003	$\bar{1}$·99985	30′
40′	$\bar{2}$·46366		1·53634	$\bar{2}$·46385		1·53615	0·00018	004	$\bar{1}$·99982	20′
50′	$\bar{2}$·50504		1·49496	$\bar{2}$·50527		1·49473	0·00022	004	$\bar{1}$·99978	10′
2°	**$\bar{2}$·54282**		**1·45718**	**$\bar{2}$·54308**		**1·45692**	**0·00026**	004	**$\bar{1}$·99974**	**88°**
10′	$\bar{2}$·57757		1·42243	$\bar{2}$·57788		1·42212	0·00031	005	$\bar{1}$·99969	50′
20′	$\bar{2}$·60973		1·39027	$\bar{2}$·61009		1·38991	0·00036	005	$\bar{1}$·99964	40′
30′	$\bar{2}$·63968		1·36032	$\bar{2}$·64009		1·35991	0·00041	006	$\bar{1}$·99959	30′
40′	$\bar{2}$·66769		1·33231	$\bar{2}$·66816		1·33184	0·00047	006	$\bar{1}$·99953	20′
50′	$\bar{2}$·69400		1·30600	$\bar{2}$·69453		1·30547	0·00053	006	$\bar{1}$·99947	10′
3°	**$\bar{2}$·71880**		**1·28120**	**$\bar{2}$·71940**		**1·28060**	**0·00060**		**$\bar{1}$·99940**	**87°**
	log cos	D	log sec	log cotan	D	log tan	log cosec	D	log sin	

	sine	D	cosecant	D	tangent	D	cotangent	D	secant	D	cosine	D	
3°	·05234	290	19·1073		·05241	292	19·0811		1·00137	016	·99863	016	87°
10′	·05524	290	18·1026		·05533	291	18·0750		1·00153	016	·99847	016	50′
20′	·05814	291	17·1984		·05824	292	17·1693		1·00169	018	·99831	018	40′
30′	·06105	290	16·3804		·06116	292	16·3499		1·00187	018	·99813	018	30′
40′	·06395	290	15·6368		·06408	292	15·6048		1·00205	019	·99795	019	20′
50′	·06685	291	14·9579		·06700	293	14·9244		1·00224	020	·99776	020	10′
4°	·06976	290	14·3356		·06993	292	14·3007		1·00244	021	·99756	020	86°
10′	·07266	290	13·7631		·07285	293	13·7267		1·00265	022	·99736	022	50′
20′	·07556	290	13·2347		·07578	292	13·1969		1·00287	022	·99714	022	40′
30′	·07846	290	12·7455		·07870	293	12·7062		1·00309	024	·99692	024	30′
40′	·08136	290	12·2913		·08163	293	12·2505		1·00333	024	·99668	024	20′
50′	·08426	290	11·8684		·08456	293	11·8262		1·00357	025	·99644	025	10′
5°	·08716	289	11·4737		·08749	293	11·4301		1·00382	026	·99619	025	85°
10′	·09005	290	11·1046		·09042	293	11·0594		1·00408	027	·99594	027	50′
20′	·09295	290	10·7585		·09335	294	10·7119		1·00435	028	·99567	027	40′
30′	·09585	289	10·4334		·09629	294	10·3854		1·00463	028	·99540	029	30′
40′	·09874	290	10·1275		·09923	293	10·0780		1·00491	030	·99511	029	20′
50′	·10164	289	9·83912		·10216	294	9·78817		1·00521	030	·99482	030	10′
6°	·10453	289	9·56677		·10510	295	9·51436		1·00551	031	·99452	031	84°
10′	·10742	289	9·30917		·10805	294	9·25530		1·00582	032	·99421	031	50′
20′	·11031	289	9·06515		·11099	295	9·00983		1·00614	033	·99390	033	40′
30′	·11320	289	8·83367		·11394	294	8·77689		1·00647	034	·99357	033	30′
40′	·11609	289	8·61379		·11688	295	8·55554		1·00681	034	·99324	034	20′
50′	·11898	289	8·40466		·11983	295	8·34496		1·00715	036	·99290	035	10′
7°	·12187	289	8·20551		·12278	296	8·14435		1·00751	037	·99255	036	83°
10′	·12476	288	8·01565		·12574	295	7·95302		1·00788	037	·99219	037	50′
20′	·12764	288	7·83443		·12869	296	7·77035		1·00825	038	·99182	038	40′
30′	·13052	289	7·66130		·13165	296	7·59575		1·00863	039	·99144	038	30′
40′	·13341	288	7·49571		·13461	297	7·42871		1·00902	040	·99106	039	20′
50′	·13629	288	7·33719		·13758	296	7·26873		1·00942	041	·99067	040	10′
8°	·13917	288	7·18530		·14054	297	7·11537		1·00983	041	·99027	041	82°
10′	·14205	288	7·03962		·14351	297	6·96823		1·01024	043	·98986	042	50′
20′	·14493	288	6·89979		·14648	297	6·82694		1·01067	044	·98944	042	40′
30′	·14781	288	6·76547		·14945	298	6·69116		1·01111	044	·98902	044	30′
40′	·15069	287	6·63633		·15243	297	6·56055		1·01155	045	·98858	044	20′
50′	·15356	287	6·51208		·15540	298	6·43484		1·01200	047	·98814	045	10′
9°	·15643	288	6·39245		·15838	299	6·31375		1·01247	047	·98769	046	81°
10′	·15931	287	6·27719		·16137	298	6·19703		1·01294	048	·98723	047	50′
20′	·16218	287	6·16607		·16435	299	6·08444		1·01342	049	·98676	047	40′
30′	·16505	287	6·05886		·16734	299	5·97576		1·01391	049	·98629	049	30′
40′	·16792	286	5·95536		·17033	300	5·87080		1·01440	051	·98580	049	20′
50′	·17078	287	5·85539		·17333	300	5·76937		1·01491	052	·98531	050	10′
10°	·17365		5·75877		·17633		5·67128		1·01543		·98481		80°
	cosine	D	secant	D	cotangent	D	tangent	D	cosecant	D	sine	D	

	log sin	D	log cosec	log tan	D	log cotan	log sec	D	log cos	
3°	$\bar{2}$·71880		1·28120	$\bar{2}$·71940		1·28060	0·00060	006	$\bar{1}$·99940	87°
10′	$\bar{2}$·74226		1·25774	$\bar{2}$·74292		1·25708	0·00066	008	$\bar{1}$·99934	50′
20′	$\bar{2}$·76541		1·23549	$\bar{2}$·76525		1·23475	0·00074	007	$\bar{1}$·99926	40′
30′	$\bar{2}$·78568		1·21432	$\bar{2}$·78649		1·21351	0·00081	008	$\bar{1}$·99919	30′
40′	$\bar{2}$·80585		1·19415	$\bar{2}$·80674		1·19326	0·00089	008	$\bar{1}$·99911	20′
50′	$\bar{2}$·82513		1·17487	$\bar{2}$·82610		1·17390	0·00097	009	$\bar{1}$·99903	10′
4°	$\bar{2}$·84358		1·15642	$\bar{2}$·84464		1·15536	0·00106	009	$\bar{1}$·99894	86°
10′	$\bar{2}$·86128		1·13872	$\bar{2}$·86243		1·13757	0·00115	009	$\bar{1}$·99885	50′
20′	$\bar{2}$·87829		1·12171	$\bar{2}$·87953		1·12047	0·00124	010	$\bar{1}$·99876	40′
30′	$\bar{2}$·89464		1·10536	$\bar{2}$·89598		1·10402	0·00134	010	$\bar{1}$·99866	30′
40′	$\bar{2}$·91040		1·08960	$\bar{2}$·91185		1·08815	0·00144	011	$\bar{1}$·99856	20′
50′	$\bar{2}$·92561		1·07439	$\bar{2}$·92716		1·07284	0·00155	011	$\bar{1}$·99845	10′
5°	$\bar{2}$·94030		1·05970	$\bar{2}$·94195		1·05805	0·00166	011	$\bar{1}$·99834	85°
10′	$\bar{2}$·95450		1·04550	$\bar{2}$·95627		1·04373	0·00177	011	$\bar{1}$·99823	50′
20′	$\bar{2}$·96825		1·03175	$\bar{2}$·97013		1·02987	0·00188	012	$\bar{1}$·99812	40′
30′	$\bar{2}$·98157		1·01843	$\bar{2}$·98358		1·01642	0·00200	013	$\bar{1}$·99800	30′
40′	$\bar{2}$·99450		1·00550	$\bar{2}$·99662		1·00338	0·00213	012	$\bar{1}$·99787	20′
50′	$\bar{1}$·00704		0·99296	$\bar{1}$·00930		0·99070	0·00225	014	$\bar{1}$·99775	10′
6°	$\bar{1}$·01923		0·98077	$\bar{1}$·02162		0·97838	0·00239	013	$\bar{1}$·99761	84°
10′	$\bar{1}$·03109		0·96891	$\bar{1}$·03361		0·96639	0·00252	014	$\bar{1}$·99748	50′
20′	$\bar{1}$·04262		0·95738	$\bar{1}$·04528		0·95472	0·00266	014	$\bar{1}$·99734	40′
30′	$\bar{1}$·05386		0·94614	$\bar{1}$·05666		0·94334	0·00280	015	$\bar{1}$·99720	30′
40′	$\bar{1}$·06481		0·93519	$\bar{1}$·06775		0·93225	0·00295	015	$\bar{1}$·99705	20′
50′	$\bar{1}$·07548		0·92452	$\bar{1}$·07858		0·92142	0·00310	015	$\bar{1}$·99690	10′
7°	$\bar{1}$·08589		0·91411	$\bar{1}$·08914		0·91086	0·00325	016	$\bar{1}$·99675	83°
10′	$\bar{1}$·09606	993	0·90394	$\bar{1}$·09947		0·90053	0·00341	016	$\bar{1}$·99659	50′
20′	$\bar{1}$·10599	971	0·89401	$\bar{1}$·10956	987	0·89044	0·00357	016	$\bar{1}$·99643	40′
30′	$\bar{1}$·11570	949	0·88430	$\bar{1}$·11943	966	0′88057	0·00373	017	$\bar{1}$·99627	30′
40′	$\bar{1}$·12519	928	0·87481	$\bar{1}$·12909	945	0·87091	0·00390	017	$\bar{1}$·99610	20′
50′	$\bar{1}$·13447	909	0·86553	$\bar{1}$·13854	926	0·86146	0·00407	018	$\bar{1}$·99593	10′
8°	$\bar{1}$·14356	889	0·85644	$\bar{1}$·14780	908	0·85220	0·00425	018	$\bar{1}$·99575	82°
10′	$\bar{1}$·15245	871	0·84755	$\bar{1}$·15688	889	0·84312	0·00443	018	$\bar{1}$·99557	50′
20′	$\bar{1}$·16116	854	0·83884	$\bar{1}$·16577	873	0:83423	0·00461	019	$\bar{1}$·99539	40′
30′	$\bar{1}$·16970	837	0·83030	$\bar{1}$·17450	856	0·82550	0·00480	019	$\bar{1}$·99520	30′
40′	$\bar{1}$·17807	821	0·82193	$\bar{1}$·18306	840	0·81694	0·00499	019	$\bar{1}$·99501	20′
50′	$\bar{1}$·18628	805	0·81372	$\bar{1}$·19146	825	0·80854	0·00518	020	$\bar{1}$·99482	10′
9°	$\bar{1}$·19433	790	0·80567	$\bar{1}$·19971	811	0·80029	0·00538	020	$\bar{1}$·99462	81°
10′	$\bar{1}$·20223	776	0·79777	$\bar{1}$·20782	796	0·79218	0·00558	021	$\bar{1}$·99442	50′
20′	$\bar{1}$·20999	762	0·79001	$\bar{1}$·21578	783	0·78422	0·00579	021	$\bar{1}$·99421	40′
30′	$\bar{1}$·21761	748	0·78239	$\bar{1}$·22361	769	0·77639	0·00600	021	$\bar{1}$·99400	30′
40′	$\bar{1}$·22509	735	0·77491	$\bar{1}$·23130	757	0·76870	0·00621	022	$\bar{1}$·99379	20′
50′	$\bar{1}$·23244	723	0·76756	$\bar{1}$·23887	745	0·76113	0·00643	022	$\bar{1}$·99357	10′
10°	$\bar{1}$·23967		0·76033	$\bar{1}$·24632		0·75368	0·00665		$\bar{1}$·99335	80°
	log cos	D	log sec	log cotan	D	log tan	log cosec	D	log sin	

	sine	D	cosecant	D	tangent	D	cotangent	D	secant	D	cosine	D	
10°	**·17365**	**286**	**5·75877**		**·17633**	**300**	**5·67128**		**1·01543**	**052**	**·98481**	**51**	**80°**
10′	·17651	286	5·66533		·17933	300	5·57638		1·01595	054	·98430	52	50′
20′	·17937	287	5·57493		·18233	301	5·48451		1·01649	054	·98378	53	40′
30′	·18224	285	5·48740		·18534	301	5·39552		1·01703	055	·98325	53	30′
40′	·18509	286	5·40263		·18835	301	5·30928		1·01758	057	·98272	54	20′
50′	·18795	286	5·32049		·19136	302	5·22566		1·01815	057	·98218	55	10′
11°	**·19081**	**285**	**5·24084**		**·19438**	**302**	**5·14455**		**1·01872**	**058**	**·98163**	**56**	**79°**
10′	·19366	286	5·16359		·19740	302	5·06584		1·01930	059	·98107	57	50′
20′	·19652	285	5·08863		·20042	303	4·98940		1·01989	060	·98050	58	40′
30′	·19937	285	5·01585		·20345	303	4·91516		1·02049	061	·97992	58	30′
40′	·20222	285	4·94517		·20648	304	4·84300		1·02110	061	·97934	59	20′
50′	·20507	284	4·87649		·20952	304	4·77286		1·02171	063	·); 875	60	10′
12°	**·20791**	**285**	**4·80973**		**·21256**	**304**	**4·70463**		**1·02234**	**064**	**·97815**	**61**	**78°**
10′	·21076	284	4·74482		·21560	304	4·63825		1·02298	064	·97754	62	50′
20′	·21360	284	4·68167		·21864	305	4·57363		1·02362	066	·97692	62	40′
30′	·21644	284	4·62023		·22169	306	4·51071		1·02428	066	·97630	64	30′
40′	·21928	284	4·56041		·22475	306	4·44942		1·02494	068	·97566	64	20′
50′	·22212	283	4·50216		·22781	306	4·38969		1·02562	068	·97502	65	10′
13°	**·22495**	**283**	**4·44541**		**·23087**	**306**	**4·33148**		**1·02630**	**070**	**·97437**	**66**	**77°**
10′	·22778	284	4·39012		·23393	307	4·27471		1·02700	070	·97371	67	50′
20′	·23062	283	4·33622		·23700	308	4·21933		1·02770	071	·97304	67	40′
30′	·23345	282	4·28366		·24008	308	4·16530		1·02841	073	·97237	68	30′
40′	·23627	283	4·23239		·24316	308	4·11256		1·02914	073	·97169	69	20′
50′	·23910	282	4·18238		·24624	309	4·06107		1·02987	074	·97100	70	10′
14°	**·24192**	**282**	**4·13357**		**·24933**	**309**	**4·01078**		**1·03061**	**076**	**·97030**	**71**	**76°**
10′	·24474	282	4·08591		·25242	310	3·96165		1·03137	076	·96959	72	50′
20′	·24756	282	4·03938		·25552	310	3·91364		1·03213	077	·96887	72	40′
30′	·25038	282	3·99393		·25862	310	3·86671		1·03290	078	·96815	73	30′
40′	·25320	281	3·94952		·26172	311	3·82083		1·03368	079	·96742	75	20′
50′	·25601	281	3·90613		·26483	312	3·77595		1·03447	081	96667	74	10′
15°	**·25882**	**281**	**3·86370**		**·26795**	**312**	**3·73205**		**1·03528**	**081**	**·96593**	**76**	**75°**
10′	·26163	280	3·82223		·27107	312	3·68909		1·03609	082	·96517	77	50′
20′	·26443	281	3·78166		·27419	313	3·64705		1·03691	083	·96440	77	40′
30′	·26724	280	3·74198		·27732	314	3·60588		1·03774	084	·96363	78	30′
40′	·27004	280	3·70315		·28046	314	3·56557		1·03858	086	·96285	79	20′
50′	·27284	280	3·66515		·28360	315	3·52609		1·03944	086	·96206	80	10′
16°	**·27564**	**279**	**3·62796**		**·28675**	**315**	**3·48741**		**1·04030**	**087**	**·96126**	**80**	**74°**
10′	·27843	280	3·59154		·28990	315	3·44951		1·04117	089	·96046	82	50′
20′	·28123	279	3·55587		·29305	316	3·41236		1·04206	089	·95964	82	40′
30′	·28402	278	3·52094		·29621	317	3·37594		1·04295	090	·95882	83	30′
40′	·28680	279	3·48671		·29938	317	3·34023		1·04385	092	·95799	84	20′
50′	·28959	278	3·45317		·30255	318	3·30521		1·04477	092	·95715	85	10′
17°	**·29237**		**3·42030**		**·30573**		**3·27085**		**1·04569**		**·95630**	**85**	**73°**
	cosine	D	secant	D	cotangent	D	tangent	D	cosecant	D	sine	D	

	log sin	D	log cosec	log tan	D	log cotan	log sec	D	log cos	
10°	1̄·23967	711	0·76033	1̄·24632	733	0·75368	0·00665	022	1̄·99335	**80°**
10′	1̄·24677	698	0·75323	1̄·25365	721	0·74635	0·00687	023	1̄·99313	50′
20′	1̄·25376	687	0·74624	1̄·26086	711	0·73914	0·00710	023	1̄·99290	40′
30′	1̄·26063	676	0·73937	1̄·26797	699	0·73203	0·00733	024	1̄·99267	30′
40′	1̄·26739	666	0·73261	1̄·27496	690	0·72504	0·00757	024	1̄·99243	20′
50′	1̄·27405	655	0·72595	1̄·28186	679	0·71814	0·00781	024	1̄·99219	10′
11°	**1̄·28060**	**645**	**0·71940**	**1̄·28865**	**670**	**0·71135**	**0·00805**	**025**	**1̄·99195**	**79°**
10′	1̄·28705	635	0·71295	1̄·29535	660	0·70465	0·00830	025	1̄·99170	50′
20′	1̄·29340	625	0·70660	1̄·30195	651	0·69805	0·00855	026	1̄·99145	40′
30′	1̄·29966	616	0·70034	1̄·30846	643	0·69154	0·00881	026	1̄·99119	30′
40′	1̄·30582	607	0·69418	1̄·31489	633	0·68511	0·00907	026	1̄·99093	20′
50′	1̄·31189	599	0·68811	1̄·32122	625	0·67878	0·00933	027	1̄·99067	10′
12°	**1̄·31788**	**590**	**0·68212**	**1̄·32747**	**618**	**0·67253**	**0·00960**	**027**	**1̄·99040**	**78°**
10′	1̄·32378	582	0·67622	1̄·33365	609	0·66635	0·00987	027	1̄·99013	50′
20′	1̄·32960	574	0·67040	1̄·33974	601	0·66026	0·01014	028	1̄·98986	40′
30′	1̄·33534	566	0·66466	1̄·34576	595	0·65424	0·01042	028	1̄·98958	30′
40′	1̄·34100	558	0·65900	1̄·35170	587	0·64830	0·01070	029	1̄·98930	20′
50′	1̄·34658	551	0·65342	1̄·35757	579	0·64243	0·01099	029	1̄·98901	10′
13°	**1̄·35209**	**543**	**0·64791**	**1̄·36336**	**573**	**0·63664**	**0·01128**	**029**	**1̄·98872**	**77°**
10′	1̄·35752	537	0·64248	1̄·36909	567	0·63091	0·01157	030	1̄·98843	50′
20′	1̄·36289	529	0·63711	1̄·37476	559	0·62524	0·01187	030	1̄·98813	40′
30′	1̄·36819	522	0·63181	1̄·38035	554	0·61965	0·01217	030	1̄·98783	30′
40′	1̄·37341	517	0·62659	1̄·38589	547	0·61411	0·01247	031	1̄·98753	20′
50′	1̄·37858	510	0·62142	1̄·39136	541	0·60864	0·01278	032	1̄·98722	10′
14°	**1̄·38368**	**503**	**0·61632**	**1̄·39677**	**535**	**0·60323**	**0·01310**	**031**	**1̄·98690**	**76°**
10′	1̄·38871	498	0·61129	1̄·40212	530	0·59788	0·01341	032	1̄·98659	50′
20′	1̄·39369	491	0·60631	1̄·40742	524	0·59258	0·01373	033	1̄·98627	40′
30′	1̄·39860	486	0·60140	1̄·41266	518	0·58734	0·01406	033	1̄·98594	30′
40′	1̄·40346	479	0·59654	1̄·41784	513	0·58216	0·01439	033	1̄·98561	20′
50′	1̄·40825	475	0·59175	1̄·42297	508	0·57703	0·01472	034	1̄·98528	10′
15°	**1̄·41300**	**468**	**0·58700**	**1̄·42805**	**503**	**0·57195**	**0·01506**	**034**	**1̄·98494**	**75°**
10′	1̄·41768	464	0·58232	1̄·43308	498	0·56692	0·01540	034	1̄·98460	50′
20′	1̄·42232	458	0·57768	1̄·43806	493	0·56194	0·01574	035	1̄·98426	40′
30′	1̄·42690	453	0·57310	1̄·44299	488	0·55701	0·01609	035	1̄·98391	30′
40′	1̄·43143	448	0·56857	1̄·44787	484	0·55213	0·01644	036	1̄·98356	20′
50′	1̄·43591	443	0·56409	1̄·45271	479	0·54729	0·01680	036	1̄·98320	10′
16°	**1̄·44034**	**438**	**0·55966**	**1̄·45750**	**474**	**0·54250**	**0·01716**	**036**	**1̄·98284**	**74°**
10′	1̄·44472	433	0·55528	1̄·46224	470	0·53776	0·01752	037	1̄·98248	50′
20′	1̄·44905	429	0·55095	1̄·46694	466	0·53306	0·01789	037	1̄·98211	40′
30′	1̄·45334	424	0·54666	1̄·47160	462	0·52840	0·01826	038	1̄·98174	30′
40′	1̄·45758	420	0·54242	1̄·47622	458	0·52378	0·01864	038	1̄·98136	20′
50′	1̄·46178	416	0·53822	1̄·48080	454	0·51920	0·01902	038	1̄·98098	10′
17°	**1̄·46594**		**0·53406**	**1̄·48534**		**0·51466**	**0·01940**		**1̄·98060**	**73°**
	log cos	D	log sec	log cotan	D	log tan	log cosec	D	log sin	

73°

	sine	D	cosecant	D	tangent	D	cotangent	D	secant	D	cosine	D	
17°	·29237	278	3·42030		·30573	318	3·27085		1·04569	094	·95630	085	7
10′	·29515	278	3·38808		·30891	319	3·23714		1·04663	094	·95545	086	5
20′	·29793	278	3·35649		·31210	320	3·20406		1·04757	096	·95459	087	4
30′	·30071	277	3·32551		·31530	320	3·17159		1·04853	097	·95372	088	3
40′	·30348	277	3·29512		·31850	321	3·13972		1·04950	097	·95284	089	2
50′	·30625	277	3·26531		·32171	321	3·10842		1·05047	099	·95195	089	1
18°	·30902	276	3·23607		·32492	322	3·07768		1·05146	100	·95106	091	7
10′	·31178	276	3·20737		·32814	322	3·04749		1·05246	101	·95015	091	5
20′	·31454	276	3·17920		·33136	324	3·01783		1·05347	102	·94924	092	4
30′	·31730	276	3·15155		·33460	323	2·98869		1·05449	103	·94832	092	3
40′	·32006	276	3·12440		·33783	325	2·96004		1·05552	105	·94740	094	2
50′	·32282	275	3·09774		·34108	325	2·93189		1·05657	105	·94646	094	1
19°	·32557	275	3·07155		·34433	325	2·90421		1·05762	107	·94552	095	7
10′	·32832	274	3·04584		·34758	327	2·87700		1·05869	107	·94457	096	5
20′	·33106	275	3·02057		·35085	327	2·85023		1·05976	109	·94361	097	4
30′	·33381	274	3·99574		·35412	328	2·82391		1·06085	110	·94264	097	3
40′	·33655	274	2·97135		·35740	328	2·79802		1·06195	111	·94167	099	2
50′	·33929	273	2·94737		·36068	329	2·77254		1·06306	112	·94068	099	1
20°	·34202	273	2·92380		·36397	330	2·74748		1·06418	113	·93969	100	70
10′	·34475	273	2·90063		·36727	330	2·72281		1·06531	114	·93869	100	5
20′	·34748	273	2·87785		·37057	331	2·69853		1·06645	116	·93769	102	4
30′	·35021	272	2·85545		·37388	332	2·67462		1·06761	117	·93667	102	3
40′	·35293	272	2·83342		·37720	333	2·65109		1·06878	117	·93565	103	2
50′	·35565	272	2·81175		·38053	333	2·62791		1·06995	120	·93462	104	1
21°	·35837	271	2·79043		·38386	335	2·60509		1·07115	120	·93358	105	69
10′	·36108	271	2·76945		·38721	334	2·58261		1·07235	121	·93253	105	50
20′	·36379	271	2·74881		·39055	336	2·56046		1·07356	123	·93148	106	40
30′	·36650	271	2·72850		·39391	336	2·53865		1·07479	123	·93042	107	30
40′	·36921	270	2·70851		·39727	338	2·51715		1·07602	125	·92935	108	20
50′	·37191	270	2·68884		·40065	338	2·49597		1·07727	126	·92827	109	10
22°	·37461	269	2·66947		·40403	338	2·47509		1·07853	128	·92718	109	68
10′	·37730	269	2·65040		·40741	340	2·45451		1·07981	128	·92609	110	50
20′	·37999	269	2·63162		·41081	340	2·43422		1·08109	130	·92499	111	40
30′	·38268	269	2·61313		·41421	342	2·41421		1·08239	131	·92388	112	30
40′	·38537	268	2·59491		·41763	342	2·39449		1·08370	133	·92276	112	20
50′	·38805	268	2·57698		·42105	342	2·37504		1·08503	133	·92164	114	10
23°	·39073	268	2·55930		·42447	344	2·35585		1·08636	135	·92050	114	67
10′	·39341	267	2·54190		·42791	345	2·33693		1·08771	136	·91936	114	50
20′	·39608	267	2·52474		·43136	345	2·31826		1·08907	137	·91822	116	40′
30′	·39875	267	2·50784		·43481	347	2·29984		1·09044	139	·91706	116	30′
40′	·40142	266	2·49119		·43828	347	2·28167		1·09183	140	·91590	118	20′
50′	·40408	266	2·47477		·44175	348	2·26374		1·09323	141	·91472	117	10′
24°	·40674		2·45859		·44523		2·24604		1·09464		·91355		66°
	cosine	D	secant	D	cotangent	D	tangent	D	cosecant	D	sine	D	

	log sin	D	log cosec	log tan	D	log cotan	log sec	D	log cos	
17°	$\bar{1}$·46594	401	0·53406	$\bar{1}$·48534	450	0·51466	0·01940	039	$\bar{1}$·98060	**73°**
10′	$\bar{1}$·47005	406	0·52995	$\bar{1}$·48984	446	0·51016	0·01979	039	$\bar{1}$·98021	50′
20′	$\bar{1}$·47411	403	0·52589	$\bar{1}$·49430	442	0·50570	0·02018	040	$\bar{1}$·97982	40′
30′	$\bar{1}$·47814	399	0·52186	$\bar{1}$·49872	439	0·50128	0·02058	040	$\bar{1}$·97942	30′
40′	$\bar{1}$·48213	394	0·51787	$\bar{1}$·50311	435	0·49689	0·02098	041	$\bar{1}$·97902	20′
50′	$\bar{1}$·48607	391	0·51393	$\bar{1}$·50746	432	0·49254	0·02139	040	$\bar{1}$·97861	10′
18°	$\bar{1}$·48998	387	0·51002	$\bar{1}$·51178	428	0·48822	0·02179	042	$\bar{1}$·97821	**72°**
10′	$\bar{1}$·49385	383	0·50615	$\bar{1}$·51606	425	0·48394	0·02221	041	$\bar{1}$·97779	50′
20′	$\bar{1}$·49768	380	0·50232	$\bar{1}$·52031	421	0·47969	0·02262	042	$\bar{1}$·97738	40′
30′	$\bar{1}$·50148	375	0·49852	$\bar{1}$·52452	418	0·47548	0·02304	043	$\bar{1}$·97696	30′
40′	$\bar{1}$·50523	373	0·49477	$\bar{1}$·52870	415	0·47130	0·02347	043	$\bar{1}$·97653	20′
50′	$\bar{1}$·50896	368	0·49104	$\bar{1}$·53285	412	0·46715	0·02390	043	$\bar{1}$·97610	10′
19°	$\bar{1}$·51264	365	0·48736	$\bar{1}$·53697	409	0·46303	0·02433	044	$\bar{1}$·97567	**71°**
10′	$\bar{1}$·51629	362	0·48371	$\bar{1}$·54106	406	0·45894	0·02477	044	$\bar{1}$·97523	50′
20′	$\bar{1}$·51991	359	0·48009	$\bar{1}$·54512	403	0·45488	0·02521	044	$\bar{1}$·97479	40′
30′	$\bar{1}$·52350	355	0·47650	$\bar{1}$·54915	400	0·45085	0·02565	045	$\bar{1}$·97435	30′
40′	$\bar{1}$·52705	352	0·47295	$\bar{1}$·55315	397	0·44685	0·02610	046	$\bar{1}$·97390	20′
50′	$\bar{1}$·53057	348	0·46944	$\bar{1}$·55712	395	0·44288	0·02656	045	$\bar{1}$·97344	10′
20°	$\bar{1}$·53405	346	0·46595	$\bar{1}$·56107	391	0·43893	0·02701	047	$\bar{1}$·97299	**70°**
10′	$\bar{1}$·53751	342	0·46249	$\bar{1}$·56498	389	0·43502	0·02748	046	$\bar{1}$·97252	50′
20′	$\bar{1}$·54093	340	0·45907	$\bar{1}$·56887	387	0·43113	0·02794	047	$\bar{1}$·97206	40′
30′	$\bar{1}$·54433	336	0·45567	$\bar{1}$·57274	384	0·42726	0·02841	048	$\bar{1}$·97159	30′
40′	$\bar{1}$·54769	333	0·45231	$\bar{1}$·57658	381	0·42342	0·02889	048	$\bar{1}$·97111	20′
50′	$\bar{1}$·55102	331	0·44898	$\bar{1}$·58039	379	0·41961	0·02937	048	$\bar{1}$·97063	10′
21°	$\bar{1}$·55433	328	0·44567	$\bar{1}$·58418	376	0·41582	0·02985	049	$\bar{1}$·97015	**69°**
10′	$\bar{1}$·55761	324	0·44239	$\bar{1}$·58794	374	0·41206	0·03034	049	$\bar{1}$·96966	50′
20′	$\bar{1}$·56085	323	0·43915	$\bar{1}$·59168	372	0·40832	0·03083	049	$\bar{1}$·96917	40′
30′	$\bar{1}$·56408	319	0·43592	$\bar{1}$·59540	369	0·40460	0·03132	050	$\bar{1}$·96868	30′
40′	$\bar{1}$·56727	317	0·43273	$\bar{1}$·59909	367	0·40091	0·03182	051	$\bar{1}$·96818	20′
50′	$\bar{1}$·57044	314	0·42956	$\bar{1}$·60276	365	0·39724	0·03233	050	$\bar{1}$·96767	10′
22°	$\bar{1}$·57358	311	0·42642	$\bar{1}$·60641	363	0·39359	0·03283	052	$\bar{1}$·96717	**68°**
10′	$\bar{1}$·57669	309	0·42331	$\bar{1}$·61004	360	0·38996	0·03335	051	$\bar{1}$·96665	50′
20′	$\bar{1}$·57978	306	0·42022	$\bar{1}$·61364	358	0·38636	0·03386	052	$\bar{1}$·96614	40′
30′	$\bar{1}$·58284	304	0·41716	$\bar{1}$·61722	357	0·38278	0·03438	053	$\bar{1}$·96562	30′
40′	$\bar{1}$·58588	301	0·41412	$\bar{1}$·62079	354	0·37921	0·03491	053	$\bar{1}$·96509	20′
50′	$\bar{1}$·58889	299	0·41111	$\bar{1}$·62433	352	0·37567	0·03544	053	$\bar{1}$·96456	10′
23°	$\bar{1}$·59188	296	0·40812	$\bar{1}$·62785	350	0·37215	0·03597	054	$\bar{1}$·96403	**67°**
10′	$\bar{1}$·59484	294	0·40516	$\bar{1}$·63135	349	0·36865	0·03651	055	$\bar{1}$·96349	50′
20′	$\bar{1}$·59778	292	0·40222	$\bar{1}$·63484	346	0·36516	0·03706	054	$\bar{1}$·96294	40′
30′	$\bar{1}$·60070	289	0·39930	$\bar{1}$·63830	345	0·36170	0·03760	055	$\bar{1}$·96240	30′
40′	$\bar{1}$·60359	287	0·39641	$\bar{1}$·64175	342	0·35825	0·03815	056	$\bar{1}$·96185	20′
50′	$\bar{1}$·60646	285	0·39354	$\bar{1}$·64517	341	0·35483	0·03871	056	$\bar{1}$·96129	10′
24°	$\bar{1}$·60931		0·39069	$\bar{1}$·64858		0·35142	0·03927		$\bar{1}$·96073	**66°**
	log cos	D	log sec	log cotan	D	log tan	log cosec	D	log sin	

	sine	D	cosecant	D	tangent	D	cotangent	D	secant	D	cosine	D
24°	·40674	265	2·45859		·44523	349	2·24604		1·09464	142	·91355	119
10′	·40939	265	2·44264		·44872	350	2·22857		1·09606	144	·91236	120
20′	·41204	265	2·42692		·45222	351	2·21132		1·09750	145	·91116	120
30′	·41469	265	2·41142		·45573	351	2·19430		1·09895	146	·90996	121
40′	·41734	264	2·39614		·45924	353	2·17749		1·10041	148	·90875	122
50′	·41998	264	2·38107		·46277	354	2·16090		1·10189	149	·90753	122
25°	·42262	263	2·36620		·46631	354	2·14451		1·10338	150	·90631	124
10′	·42525	263	2·35154		·46985	356	2·12832		1·10488	152	·90507	124
20′	·42788	263	2·33708		·47341	357	2·11233		1·10640	153	·90383	124
30′	·43051	262	2·32282		·47698	357	2·09654		1·10793	154	·90259	126
40′	·43313	262	2·30875		·48055	359	2·08094		1·10947	156	·90133	126
50′	·43575	262	2·29487		·48414	359	2·06553		1·11103	157	·90007	128
26°	·43837	261	2·28117		·48773	361	2·05030		1·11260	159	·89879	127
10′	·44098	261	2·26766		·49134	361	2·03526		1·11419	160	·89752	129
20′	·44359	261	2·25432		·49495	363	2·02039		1·11579	161	·89623	130
30′	·44620	260	2·24116		·49858	364	2·00569		1·11740	163	·89493	130
40′	·44880	260	2·22817		·50222	365	1·99116		1·11903	164	·89363	131
50′	·45140	259	2·21535		·50587	366	1·97681		1·12067	166	·89232	131
27°	·45399	259	2·20269		·50953	367	1·96261		1·12233	167	·89101	133
10′	·45658	259	2·19019		·51320	368	1·94858		1·12400	168	·88968	133
20′	·45917	258	2·17786		·51688	369	1·93470		1·12568	170	·88835	134
30′	·46175	258	2·16568		·52057	370	1·92098		1·12738	172	·88701	135
40′	·46433	257	2·15366		·52427	371	1·90741		1·12910	173	·88566	135
50′	·46690	257	2·14178		·52798	373	1·89400		1·13083	174	·88431	136
28°	·46947	257	2·13005		·53171	374	1·88073		1·13257	176	·88295	137
10′	·47204	256	2·11847		·53545	375	1·86760		1·13433	177	·88158	138
20′	·47460	256	2·10704		·53920	376	1·85462		1·13610	179	·88020	138
30′	·47716	255	2·09574		·54296	377	1·84177		1·13789	181	·87882	139
40′	·47971	255	2·08458		·54673	378	1·82906		1·13970	182	·87743	140
50′	·48226	255	2·07356		·55051	380	1·81649		1·14152	183	·87603	141
29°	·48481	254	2·06267		·55431	381	1·80405		1·14335	186	·87462	141
10′	·48735	254	2·05191		·55812	382	1·79174		1·14521	186	·87321	143
20′	·48989	253	2·04128		·56194	383	1·77955		1·14707	189	·87178	142
30′	·49242	253	2·03077		·56577	385	1·76749		1·14896	189	·87036	144
40′	·49495	253	2·02039		·56962	386	1·75556		1·15085	192	·86892	144
50′	·49748	252	2·01014		·57348	387	1·74375		1·15277	193	·86748	145
30°	·50000	252	2·00000		·57735	389	1·73205		1·15470	195	·86603	146
10′	·50252	251	1·98998	990	·58124	389	1·72047		1·15665	196	·86457	147
20′	·50503	251	1·98008	979	·58513	392	1·70901		1·15861	198	·86310	147
30′	·50754	250	1·97029	967	·58905	392	1·69766		1·16059	200	·86163	148
40′	·51004	250	1·96062	956	·59297	394	1·68643		1·16259	201	·86015	149
50′	·51254	250	1·95106	946	·59691	395	1·67530		1·16460	203	·85866	149
31°	·51504		1·94160		·60086		1·66428		1·16663		·85717	
	cosine	D	secant	D	cotangent	D	tangent	D	cosecant	D	sine	D

	log sin	D	log cosec	log tan	D	log cotan	log sec	D	log cos	
24°	$\bar{1}$·60931	283	0·39069	$\bar{1}$·64858	339	0·35142	0·03927	056	$\bar{1}$·96073	**66°**
10′	$\bar{1}$·61214	280	0·38786	$\bar{1}$·65197	338	0·34803	0·03983	057	$\bar{1}$·96017	50′
20′	$\bar{1}$·61494	279	0·38506	$\bar{1}$·65535	335	0·34465	0·04040	058	$\bar{1}$·95960	40′
30′	$\bar{1}$·61773	276	0·38227	$\bar{1}$·65870	334	0·34130	0·04098	057	$\bar{1}$·95902	30′
40′	$\bar{1}$·62049	274	0·37951	$\bar{1}$·66204	333	0·33796	0·04156	059	$\bar{1}$·95845	20′
50′	$\bar{1}$·62323	272	0·37677	$\bar{1}$·66537	330	0·33463	0·04214	058	$\bar{1}$·95786	10′
25°	**$\bar{1}$·62595**	270	**0·37405**	**$\bar{1}$·66867**	329	**0·33133**	**0·04272**	060	**$\bar{1}$·95728**	**65°**
10′	$\bar{1}$·62865	268	0·37135	$\bar{1}$·67196	328	0·32804	0·04332	059	$\bar{1}$·95668	50′
20′	$\bar{1}$·63133	265	0·36867	$\bar{1}$·67524	326	0·32476	0·04391	060	$\bar{1}$·95609	40′
30′	$\bar{1}$·63398	264	0·36602	$\bar{1}$·67850	324	0·32150	0·04451	061	$\bar{1}$·95549	30′
40′	$\bar{1}$·63662	262	0·36338	$\bar{1}$·68174	323	0·31826	0·04512	061	$\bar{1}$·95488	20′
50′	$\bar{1}$·63924	260	0·36076	$\bar{1}$·68497	321	0·31503	0·04573	061	$\bar{1}$·95427	10′
26°	**$\bar{1}$·64184**	258	**0·35816**	**$\bar{1}$·68818**	320	**0·31182**	**0·04634**	062	**$\bar{1}$·95366**	**64°**
10′	$\bar{1}$·64442	256	0·35558	$\bar{1}$·69138	319	0·30862	0·04696	062	$\bar{1}$·95304	50′
20′	$\bar{1}$·64698	255	0·35302	$\bar{1}$·69457	317	0·30543	0·04758	063	$\bar{1}$·95242	40′
30′	$\bar{1}$·64953	252	0·35047	$\bar{1}$·69774	315	0·30226	0·04821	063	$\bar{1}$·95179	30′
40′	$\bar{1}$·65205	251	0·34795	$\bar{1}$·70089	315	0·29911	0·04884	064	$\bar{1}$·95116	20′
50′	$\bar{1}$·65456	249	0·34544	$\bar{1}$·70404	313	0·29596	0·04948	064	$\bar{1}$·95052	10′
27°	**$\bar{1}$·65705**	247	**0·34295**	**$\bar{1}$·70717**	311	**0·29283**	**0·05012**	065	**$\bar{1}$·94988**	**63°**
10′	$\bar{1}$·65952	245	0·34048	$\bar{1}$·71028	311	0·28972	0·05077	065	$\bar{1}$·94923	50′
20′	$\bar{1}$·66197	244	0·33803	$\bar{1}$·71339	309	0·28661	0·05142	065	$\bar{1}$·94858	40′
30′	$\bar{1}$·66441	241	0·33559	$\bar{1}$·71648	307	0·28352	0·05207	066	$\bar{1}$·94793	30′
40′	$\bar{1}$·66682	241	0·33318	$\bar{1}$·71955	307	0·28045	0·05273	067	$\bar{1}$·94727	20′
50′	$\bar{1}$·66923	238	0·33078	$\bar{1}$·72262	305	0·27738	0·05340	067	$\bar{1}$·94660	10′
28°	**$\bar{1}$·67161**	237	**0·32839**	**$\bar{1}$·72567**	305	**0·27433**	**0·05407**	067	**$\bar{1}$·94593**	**62°**
10′	$\bar{1}$·67398	235	0·32602	$\bar{1}$·72872	303	0·27128	0·05474	068	$\bar{1}$·94526	50′
20′	$\bar{1}$·67633	233	0·32367	$\bar{1}$·73175	301	0·26825	0·05542	068	$\bar{1}$·94458	40′
30′	$\bar{1}$·67866	232	0·32134	$\bar{1}$·73476	301	0·26524	0·05610	069	$\bar{1}$·94390	30′
40′	$\bar{1}$ 68098	230	0·31902	$\bar{1}$·73777	300	0·26223	0·05679	069	$\bar{1}$·94321	20′
50′	$\bar{1}$·68328	229	0·31672	$\bar{1}$·74077	298	0·25923	0·05748	070	$\bar{1}$·94252	10′
29°	**$\bar{1}$·68557**	227	**0·31443**	**$\bar{1}$·74375**	298	**0·25625**	**0·05818**	070	**$\bar{1}$·94182**	**61°**
10′	$\bar{1}$·68784	226	0·31216	$\bar{1}$·74673	296	0·25327	0·05888	071	$\bar{1}$·94112	50′
20′	$\bar{1}$·69010	224	0·30990	$\bar{1}$·74969	295	0·25031	0·05959	071	$\bar{1}$·94041	40′
30′	$\bar{1}$·69234	222	0·30766	$\bar{1}$·75264	294	0·24736	0·06030	072	$\bar{1}$·93970	30′
40′	$\bar{1}$·69456	221	0·30544	$\bar{1}$·75558	294	0·24442	0·06102	072	$\bar{1}$·93898	20′
50′	$\bar{1}$·69677	220	0·30323	$\bar{1}$·75852	292	0·24148	0·06174	073	$\bar{1}$·93826	10′
30°	**$\bar{1}$·69897**	218	**0·30103**	**$\bar{1}$·76144**	291	**0·23856**	**0·06247**	073	**$\bar{1}$·93753**	**60°**
10′	$\bar{1}$·70115	217	0·29885	$\bar{1}$·76435	290	0·23565	0·06320	074	$\bar{1}$·93680	50′
20′	$\bar{1}$·70332	215	0·29668	$\bar{1}$·76726	290	0·23275	0·06394	074	$\bar{1}$·93606	40′
30′	$\bar{1}$·70547	214	0·29453	$\bar{1}$·77015	288	0·22985	0·06468	075	$\bar{1}$·93532	30′
40′	$\bar{1}$·70761	212	0·29239	$\bar{1}$·77303	288	0·22697	0·06543	075	$\bar{1}$·93457	20′
50′	$\bar{1}$·70973	211	0·29027	$\bar{1}$·77591	286	0·22409	0·06618	075	$\bar{1}$·93382	10′
31°	**$\bar{1}$·71184**		**0·28816**	**$\bar{1}$·77877**		**0·22123**	**0·06693**		**$\bar{1}$·93307**	**59°**
	log cos	D	log sec	log cotan	D	log tan	log cosec	D	log sin	

59°

	sine	D	cosecant	D	tangent	D	cotangent	D	secant	D	cosine	D
31°	·51504	249	1·94160	934	·60086	397	1·66428		1·16663	205	·85717	150
10′	·51753	249	1·93226	924	·60483	398	1·65337		1·16868	207	·85567	151
20′	·52002	248	1·92302	914	·60881	399	1·64256		1·17075	208	·85416	152
30′	·52250	248	1·91388	903	·61280	401	1·63185		1·17283	210	·85264	152
40′	·52498	247	1·90485	894	·61681	402	1·62125		1·17493	211	·85112	153
50′	·52745	247	1·89591	883	·62083	404	1·61074		1·17704	214	·84959	154
32°	·52992	246	1·88708	874	·62487	405	1·60033		1·17918	215	·84805	155
10′	·53238	246	1·87834	864	·62892	407	1·59002		1·18133	217	·84650	155
20′	·53484	246	1·86970	854	·63299	408	1·57981		1·18350	219	·84495	156
30′	·53730	245	1·86116	845	·63707	410	1·56969		1·18569	221	·84339	157
40′	·53975	245	1·85271	836	·64117	411	1·55966	994	1·18790	222	·84182	157
50′	·54220	244	1·84435	827	·64528	413	1·54972	985	1·19012	224	·84025	158
33°	·54464	244	1·83608	818	·64941	414	1·53987	977	1·19236	227	·83867	159
10′	·54708	243	1·82790	809	·65355	416	1·53010	967	1·19463	228	·83708	159
20′	·54951	243	1·81981	801	·65771	418	1·52043	959	1·19691	229	·83549	160
30′	·55194	242	1·81180	792	·66189	419	1·51084	951	1·19920	232	·83389	161
40′	·55436	242	1·80388	784	·66608	420	1·50133	943	1·20152	234	·83228	162
50′	·55678	241	1·79604	775	·67028	423	1·49190	934	1·20386	236	·83066	162
34°	·55919	241	1·78829	767	·67451	424	1·48256	926	1·20622	237	·82904	163
10′	·56160	241	1·78062	759	·67875	426	1·47330	919	1·20859	240	·82741	164
20′	·56401	240	1·77303	751	·68301	427	1·46411	910	1·21099	242	·82577	164
30′	·56641	239	1·76552	744	·68728	429	1·45501	903	1·21341	243	·82413	165
40′	·56880	239	1·75808	735	·69157	431	1·44598	895	1·21584	246	·82248	166
50′	·57119	239	1·75073	728	·69588	433	1·43703	888	1·21830	247	·82082	167
35°	·57358	238	1·74345	721	·70021	434	1·42815	881	1·22077	250	·81915	167
10′	·57596	237	1·73624	713	·70455	436	1·41934	873	1·22327	252	·81748	168
20′	·57833	237	1·72911	706	·70891	438	1·41061	866	1·22579	254	·81580	168
30′	·58070	237	1·72205	699	·71329	440	1·40195	859	1·22833	256	·81412	170
40′	·58307	236	1·71506	691	·71769	442	1·39336	852	1·23089	258	·81242	170
50′	·58543	236	1·70815	685	·72211	443	1·38484	846	1·23347	260	·81072	170
36°	·58779	235	1·70130	678	·72654	446	1·37638	838	1·23607	262	·80902	172
10′	·59014	234	1·69452	670	·73100	447	1·36800	832	1·23869	265	·80730	172
20′	·59248	234	1·68782	665	·73547	449	1·35968	826	1·24134	266	·80558	172
30′	·59482	234	1·68117	657	·73996	451	1·35142	819	1·24400	269	·80386	174
40′	·59716	233	1·67460	651	·74447	453	1·34323	812	1·24669	271	·80212	174
50′	·59949	233	1·66809	645	·74900	455	1·33511	807	1·24940	274	·80038	174
37°	·60182	232	1·66164	638	·75355	457	1·32704	800	1·25214	275	·79864	176
10′	·60414	231	1·65526	632	·75812	460	1·31904	794	1·25489	278	·79688	176
20′	·60645	231	1·64894	626	·76272	461	1·31110	787	1·25767	280	·79512	177
30′	·60876	231	1·64268	620	·76733	463	1·30323	782	1·26047	283	·79335	177
40′	·61107	230	1·63648	613	·77196	465	1·29541	777	1·26330	285	·79158	178
50′	·61337	229	1·63035	608	·77661	468	1·28764	770	1·26615	287	·78980	179
38°	·61566		1·62427		·78129		1·27994		1·26902		·78801	
	cosine	D	secant	D	cotangent	D	tangent	D	cosecant	D	sine	D

	log sin	D	log cosec	log tan	D	log cotan	log sec	D	log cos	
31°	**$\bar{1}.71184$**	**209**	**0.28816**	**$\bar{1}.77877$**	**286**	**0.22123**	**0.06693**	**077**	**$\bar{1}.93307$**	**59°**
10′	$\bar{1}.71393$	209	0.28607	$\bar{1}.78163$	285	0.21837	0.06770	076	$\bar{1}.93230$	50′
20′	$\bar{1}.71602$	207	0.28398	$\bar{1}.78448$	284	0.21552	0.06846	077	$\bar{1}.93154$	40′
30′	$\bar{1}.71809$	205	0.28191	$\bar{1}.78732$	283	0.21268	0.06923	078	$\bar{1}.93077$	30′
40′	$\bar{1}.72014$	204	0.27986	$\bar{1}.79015$	282	0.20985	0.07001	078	$\bar{1}.92999$	20′
50′	$\bar{1}.72218$	203	0.27782	$\bar{1}.79297$	282	0.20703	0.07079	079	$\bar{1}.92921$	10′
32°	**$\bar{1}.72421$**	**201**	**0.27579**	**$\bar{1}.79579$**	**281**	**0.20421**	**0.07158**	**079**	**$\bar{1}.92842$**	**58°**
10′	$\bar{1}.72622$	201	0.27378	$\bar{1}.79860$	280	0.20140	0.07237	080	$\bar{1}.92763$	50′
20′	$\bar{1}.72823$	199	0.27177	$\bar{1}.80140$	279	0.19860	0.07317	080	$\bar{1}.92683$	40′
30′	$\bar{1}.73022$	197	0.26978	$\bar{1}.80419$	278	0.19581	0.07397	081	$\bar{1}.92603$	30′
40′	$\bar{1}.73219$	197	0.26781	$\bar{1}.80697$	278	0.19303	0.07478	081	$\bar{1}.92522$	20′
50′	$\bar{1}.73416$	195	0.26584	$\bar{1}.80975$	277	0.19025	0.07559	082	$\bar{1}.92441$	10′
33°	**$\bar{1}.73611$**	**194**	**0.26389**	**$\bar{1}.81252$**	**276**	**0.18748**	**0.07641**	**082**	**$\bar{1}.92359$**	**57°**
10′	$\bar{1}.73805$	192	0.26195	$\bar{1}.81528$	275	0.18472	0.07723	083	$\bar{1}.92277$	50′
20′	$\bar{1}.73997$	192	0.26003	$\bar{1}.81803$	275	0.18197	0.07806	083	$\bar{1}.92194$	40′
30′	$\bar{1}.74189$	190	0.25811	$\bar{1}.82078$	274	0.17922	0.07889	084	$\bar{1}.92111$	30′
40′	$\bar{1}.74379$	189	0.25621	$\bar{1}.82352$	274	0.17648	0.07973	085	$\bar{1}.92027$	20′
50′	$\bar{1}.74568$	188	0.25432	$\bar{1}.82626$	273	0.17374	0.08058	085	$\bar{1}.91942$	10′
34°	**$\bar{1}.74756$**	**187**	**0.25244**	**$\bar{1}.82899$**	**272**	**0.17101**	**0.08143**	**085**	**$\bar{1}.91857$**	**56°**
10′	$\bar{1}.74943$	185	0.25057	$\bar{1}.83171$	271	0.16829	0.08228	086	$\bar{1}.91772$	50′
20′	$\bar{1}.75128$	185	0.24872	$\bar{1}.83442$	271	0.16558	0.08314	087	$\bar{1}.91686$	40′
30′	$\bar{1}.75313$	183	0.24687	$\bar{1}.83713$	271	0.16287	0.08401	087	$\bar{1}.91599$	30′
40′	$\bar{1}.75496$	182	0.24504	$\bar{1}.83984$	270	0.16016	0.08488	087	$\bar{1}.91512$	20′
50′	$\bar{1}.75678$	181	0.24322	$\bar{1}.84254$	269	0.15746	0.08575	089	$\bar{1}.91425$	10′
35°	**$\bar{1}.75859$**	**180**	**0.24141**	**$\bar{1}.84523$**	**268**	**0.15477**	**0.08664**	**088**	**$\bar{1}.91336$**	**55°**
10′	$\bar{1}.76039$	179	0.23961	$\bar{1}.84791$	268	0.15209	0.08752	090	$\bar{1}.91248$	50′
20′	$\bar{1}.76218$	177	0.23782	$\bar{1}.85059$	268	0.14941	0.08842	089	$\bar{1}.91158$	40′
30′	$\bar{1}.76395$	177	0.23605	$\bar{1}.85327$	267	0.14673	0.08931	091	$\bar{1}.91069$	30′
40′	$\bar{1}.76572$	175	0.23428	$\bar{1}.85594$	266	0.14406	0.09022	091	$\bar{1}.90978$	20′
50′	$\bar{1}.76747$	175	0.23253	$\bar{1}.85860$	266	0.14140	0.09113	091	$\bar{1}.90887$	10′
36°	**$\bar{1}.76922$**	**173**	**0.23078**	**$\bar{1}.86126$**	**266**	**0.13874**	**0.09204**	**092**	**$\bar{1}.90796$**	**54°**
10′	$\bar{1}.77095$	173	0.22905	$\bar{1}.86392$	264	0.13608	0.09296	093	$\bar{1}.90704$	50′
20′	$\bar{1}.77268$	171	0.22732	$\bar{1}.86656$	265	0.13344	0.09389	093	$\bar{1}.90611$	40′
30′	$\bar{1}.77439$	170	0.22561	$\bar{1}.86921$	264	0.13079	0.09482	094	$\bar{1}.90518$	30′
40′	$\bar{1}.77609$	169	0.22391	$\bar{1}.87185$	263	0.12815	0.09576	094	$\bar{1}.90424$	20′
50′	$\bar{1}.77778$	168	0.22222	$\bar{1}.87448$	263	0.12552	0.09670	095	$\bar{1}.90330$	10′
37°	**$\bar{1}.77946$**	**167**	**0.22054**	**$\bar{1}.87711$**	**263**	**0.12289**	**0.09765**	**096**	**$\bar{1}.90235$**	**53°**
10′	$\bar{1}.78113$	167	0.21887	$\bar{1}.87974$	262	0.12026	0.09861	096	$\bar{1}.90139$	50′
20′	$\bar{1}.78280$	165	0.21720	$\bar{1}.88236$	262	0.11764	0.09957	096	$\bar{1}.90043$	40′
30′	$\bar{1}.78445$	164	0.21555	$\bar{1}.88498$	261	0.11502	0.10053	098	$\bar{1}.89947$	30′
40′	$\bar{1}.78609$	163	0.21391	$\bar{1}.88759$	261	0.11241	0.10151	097	$\bar{1}.89849$	20′
50′	$\bar{1}.78772$	162	0.21228	$\bar{1}.89020$	261	0.10980	0.10248	099	$\bar{1}.89752$	10′
38°	**$\bar{1}.78934$**		**0.21066**	**$\bar{1}.89281$**		**0.10719**	**0.10347**		**$\bar{1}.89653$**	**52°**
	log cos	D	log sec	log cotan		log tan	log cosec	D	log sin	

52°

	sine	D	cosecant	D	tangent	D	cotangent	D	secant	D	cosine	D
38°	·61566	229	1·62427	602	·78129	469	1·27994	764	1·26902	289	·78801	179
10′	·61795	229	1·61825	596	·78598	472	1·27230	759	1·27191	292	·78622	180
20′	·62024	227	1·61229	590	·79070	474	1·26471	754	1·27483	295	·78442	181
30′	·62251	228	1·60639	585	·79544	476	1·25717	748	1·27778	297	·78261	182
40′	·62479	227	1·60054	579	·80020	478	1·24969	742	1·28075	299	·78079	182
50′	·62706	226	1·59475	573	·80498	480	1·24227	737	1·28374	302	·77897	182
39°	·62932	226	1·58902	569	·80978	483	1·23490	732	1·28676	304	·77715	184
10′	·63158	225	1·58333	562	·81461	485	1·22758	727	1·28980	307	·77531	184
20′	·63383	225	1·57771	558	·81946	488	1·22031	721	1·29287	310	·77347	185
30′	·63608	224	1·57213	552	·82434	489	1·21310	717	1·29597	312	·77162	185
40′	·63832	224	1·56661	547	·82923	492	1·20593	711	1·29909	314	·76977	186
50′	·64056	223	1·56114	542	·83415	495	1·19882	707	1·30223	318	·76791	187
40°	·64279	222	1·55572	536	·83910	497	1·19175	701	1·30541	320	·76604	187
10′	·64501	222	1·55036	532	·84407	499	1·18474	697	1·30861	322	·76417	188
20′	·64723	222	1·54504	527	·84906	502	1·17777	692	1·31183	326	·76229	188
30′	·64945	221	1·53977	522	·85408	504	1·17085	687	1·31509	328	·76041	190
40′	·65166	220	1·53455	517	·85912	507	1·16398	683	1·31837	331	·75851	190
50′	·65386	220	1·52938	513	·86419	510	1·15715	678	1·32168	333	·75661	190
41°	·65606	219	1·52425	507	·86929	512	1·15037	674	1·32501	337	·75471	191
10′	·65825	219	1·51918	503	·87441	514	1·14363	669	1·32838	339	·75280	192
20′	·66044	218	1·51415	499	·87955	518	1·13694	665	1·33177	342	·75088	192
30′	·66262	218	1·50916	494	·88473	519	1·13029	660	1·33519	345	·74896	193
40′	·66480	217	1·50422	489	·88992	523	1·12369	656	1·33864	348	·74703	194
50′	·66697	216	1·49933	485	·89515	525	1·11713	652	1·34212	351	·74509	195
42°	·66913	216	1·49448	481	·90040	529	1·11061	647	1·34563	354	·74314	194
10′	·67129	215	1·48967	476	·90569	530	1·10414	644	1·34917	357	·74120	196
20′	·67344	215	1·48491	472	·91099	534	1·09770	639	1·35274	360	·73924	196
30′	·67559	214	1·48019	468	·91633	537	1·09131	635	1·35634	363	·73728	197
40′	·67773	214	1·47551	464	·92170	539	1·08496	632	1·35997	366	·73531	198
50′	·67987	213	1·47087	459	·92709	543	1·07864	627	1·36363	370	·73333	198
43°	·68200	212	1·46628	455	·93252	545	1·07237	624	1·36733	372	·73135	198
10′	·68412	212	1·46173	452	·93797	548	1·06613	619	1·37105	376	·72937	200
20′	·68624	211.	1·45721	447	·94345	551	1·05994	616	1·37481	379	·72737	200
30′	·68835	211	1·45274	443	·94896	555	1·05378	612	1·37860	382	·72537	200
40′	·69046	210	1·44831	440	·95451	557	1·04766	608	1·38242	386	·72337	201
50′	·69256	210	1·44391	435	·96008	561	1·04158	605	1·38628	388	·72136	202
44°	·69466	206	1·43956	432	·96569	564	1·03553	601	1·39016	393	·71934	202
10′	·69675	208	1·43524	428	·97133	567	1·02952	597	1·39409	395	·71732	203
20′	·69883	208	1·43096	424	·97700	570	1·02355	594	1·39804	399	·71529	204
30′	·70091	207	1·42672	421	·98270	573	1·01761	591	1·40203	403	·71325	204
40′	·70298	207	1·42251	416	·98843	577	1·01170	587	1·40606	406	·71121	205
50′	·70505	206	1·41835	414	·99420	580	1·00583	583	1·41012	409	·70916	205
45°	·70711		1·41421		1·00000		1·00000		1·41421		·70711	
	cosine	D	secant	D	cotangent	D	tangent	D	cosecant	D	sine	D

	log sin	D	log cosec	log tan	D	log cotan	log sec	D	log cos	
38°	**1̄·78934**	**161**	**0·21066**	**1̄·89281**	**260**	**0·10719**	**0·10347**	**099**	**1̄·89653**	**52°**
10′	1̄·79095	161	0·20905	1̄·89541	260	0·10459	0·10446	099	1̄·89554	50′
20′	1̄·79256	159	0·20744	1̄·89801	260	0·10199	0·10545	101	1̄·89455	40′
30′	1̄·79415	158	0·20585	1̄·90061	259	0·09939	0·10646	100	1̄·89354	30′
40′	1̄·79573	158	0·20427	1̄·90320	258	0·09680	0·10746	102	1̄·89254	20′
50′	1̄·79731	156	0·20269	1̄·90578	259	0·09422	0·10848	102	1̄·89152	10′
39°	**1̄·79887**	**156**	**0·20113**	**1̄·90837**	**258**	**0·09163**	**0·10950**	**102**	**1̄·89050**	**51°**
10′	1̄·80043	154	0·19957	1̄·91095	258	0·08905	0·11052	104	1̄·88948	50′
20′	1̄·80197	154	0·19803	1̄·91353	257	0·08647	0·11156	103	1̄·88844	40′
30′	1̄·80351	153	0·19649	1̄·91610	258	0·08390	0·11259	105	1̄·88741	30′
40′	1̄·80504	152	0·19496	1̄·91868	257	0·08132	0·11364	105	1̄·88636	20′
50′	1̄·80656	151	0·19344	1̄·92125	256	0·07875	0·11469	106	1̄·88531	10′
40°	**1̄·80807**	**150**	**0·19193**	**1̄·92381**	**257**	**0·07619**	**0·11575**	**106**	**1̄·88425**	**50°**
10′	1̄·80957	149	0·19043	1̄·92638	256	0·07362	0·11681	107	1̄·88319	50′
20′	1̄·81106	148	0·18894	1̄·92894	256	0·07106	0·11788	107	1̄·88212	40′
30′	1̄·81254	148	0·18746	1̄·93150	256	0·06850	0·11895	109	1̄·88105	30′
40′	1̄·81402	147	0·18598	1̄·93406	255	0·06594	0·12004	109	1̄·87996	20′
50′	1̄·81549	145	0·18451	1̄·93661	255	0·06339	0·12113	109	1̄·87887	10′
41°	**1̄·81694**	**145**	**0·18306**	**1̄·93916**	**255**	**0·06084**	**0·12222**	**110**	**1̄·87778**	**49°**
10′	1̄·81839	144	0·18161	1̄·94171	255	0·05829	0·12332	111	1̄·87668	50′
20′	1̄·81983	143	0·18017	1̄·94426	255	0·05574	0·12443	111	1̄·87557	40′
30′	1̄·82126	143	0·17874	1̄·94681	254	0·05319	0·12554	112	1̄·87446	30′
40′	1̄·82269	141	0·17731	1̄·94935	255	0·05065	0·12666	113	1̄·87334	20′
50′	1̄·82410	141	0·17590	1̄·95190	254	0·04810	0·12779	114	1̄·87221	10′
42°	**1̄·82551**	**140**	**0·17449**	**1̄·95444**	**254**	**0·04556**	**0·12893**	**114**	**1̄·87107**	**48°**
10′	1̄·82691	139	0·17309	1̄·95698	254	0·04302	0·13007	114	1̄·86993	50′
20′	1̄·82830	138	0·17170	1̄·95952	253	0·04048	0·13121	116	1̄·86879	40′
30′	1̄·82968	138	0·17032	1̄·96205	254	0·03795	0·13237	116	1̄·86763	30′
40′	1̄·83106	136	0·16894	1̄·96459	253	0·03541	0·13353	117	1̄·86647	20′
50′	1̄·83242	136	0·16758	1̄·96712	254	0·03288	0·13470	117	1̄·86530	10′
43°	**1̄·83378**	**135**	**0·16622**	**1̄·96966**	**253**	**0·03034**	**0·13587**	**118**	**1̄·86413**	**47°**
10′	1̄·83513	135	0·16487	1̄·97219	253	0·02781	0·13705	119	1̄·86295	50′
20′	1̄·83648	133	0·16352	1̄·97472	253	0·02528	0·13824	120	1̄·86176	40′
30′	1̄·83781	133	0·16219	1̄·97725	253	0·02275	0·13944	120	1̄·86056	30′
40′	1̄·83914	132	0·16086	1̄·97978	253	0·02022	0·14064	121	1̄·85936	20′
50′	1̄·84046	131	0·15954	1̄·98231	253	0·01769	0·14185	122	1̄·85815	10′
44°	**1̄·84177**	**131**	**0·15823**	**1̄·98484**	**253**	**0·01516**	**0·14307**	**122**	**1̄·85693**	**46°**
10′	1̄·84308	129	0·15692	1̄·98737	252	0·01263	0·14429	123	1̄·85571	50′
20′	1̄·84437	129	0·15563	1̄·98989	253	0·01011	0·14552	124	1̄·85448	40′
30′	1̄·84566	128	0·15434	1̄·99242	253	0·00758	0·14676	124	1̄·85324	30′
40′	1̄·84694	128	0·15306	1̄·99495	252	0·00505	0·14800	126	1̄·85200	20′
50′	1̄·84822	127	0·15178	1̄·99747	253	0·00253	0·14926	126	1̄·85074	10′
45°	**1̄·84949**		**0·15052**	**0·00000**		**0·00000**	**0·15052**		**1̄·84949**	**45°**
	log cos	D	log sec	log cotan	D	log tan	log cosec	D	log sin	

45°

FOUR-FIGURE TRIGONOMETRICAL TABLES

Radians	Degrees	Sine	Cosec.	Tangent	Cotan.	Secant	Cosine		
·00000	0	·0000	∞	·0000	∞	1·0000	1·0000	90	1·57080
·01745	1	·0175	57·2986	·0175	57·2899	1·0002	·9998	89	1·55334
·03491	2	·0349	28·6537	·0349	28·6362	1·0006	·9994	88	1·53589
·05236	3	·0523	19·1073	·0524	19·0811	1·0014	·9986	87	1·51844
·06981	4	·0698	14·3356	·0699	14·3006	1·0024	·9976	86	1·50[illegible]98
·08727	5	·0872	11·4737	·0875	11·4301	1·0038	·9962	85	1·48[illegible]
·10472	6	·1045	9·5668	·1051	9·5144	1·0055	·9945	84	1·45[illegible]
·12217	7	·1219	8·2055	·1228	8·1443	1·0075	·9925	83	1·448[illegible]
·13963	8	·1392	7·1853	·1405	7·1154	1·0098	·9903	82	1·43117
·15708	9	·1564	6·3925	·1584	6·3138	1·0125	·9877	81	1·41372
·17453	10	·1736	5·7588	·1763	5·6713	1·0154	·9848	80	1·39626
·19199	11	·1908	5·2408	·1944	5·1446	1·0187	·9816	79	1·37881
·20944	12	·2079	4·8097	·2126	4·7046	1·0223	·97[illegible]1	78	1·36135
·22689	13	·2250	4·4454	·2309	4·3315	1·0263	·97[illegible]	77	1·34390
·24435	14	·2419	4·1336	·2493	4·0108	1·0306	·9703	76	1·3264[illegible]
·26180	15	·2588	3·8637	·2679	3·7321	1·0353	·965[illegible]	75	1·309[illegible]
·27925	16	·2756	3·6280	·2867	3·4874	1·0403	·961[illegible]	[illegible]4	1·291[illegible]
·29671	17	·2924	3·4203	·3057	3·2709	1·0457	·9563	73	1·274[illegible]
·31416	18	·3090	3·2361	·3249	3·0777	1·0515	·9511	72	1·[illegible]56[illegible]
·33161	19	·3256	3·0716	·3443	2·9042	1·0576	·9455	71	1·23918
·34907	20	·3420	2·9238	·3640	2·7475	1·0642	·9397	70	1·22173
·36652	21	·3584	2·7904	·3839	2·6051	1·0711	·9336	69	1·20428
·38397	22	·3746	2·6695	·4040	2·4751	1·0785	·9272	68	1·18682
·40143	23	·3907	2·5593	·4245	2·3559	1·0864	·9205	67	1·16937
·41888	24	·4067	2·4586	·4452	2·2460	1·0946	·9135	66	1·15192
·43633	25	·4226	2·3662	·4663	2·1445	1·1034	·9063	65	1·13446
·45379	26	·4384	2·2812	·4877	2·0503	1·1126	·8988	64	1·11701
·47124	27	·4540	2·2027	·5095	1·9626	1·1223	·8910	63	1·09956
·48869	28	·4695	2·1301	·5317	1·8807	1·1326	·8829	62	1·08210
·50615	29	·4848	2·0627	·5543	1·8040	1·1434	·8746	61	1·06465
·52360	30	·5000	2·0000	·5774	1·7321	1·1547	·8660	60	1·04720
·54105	31	·5150	1·9416	·6009	1·6643	1·1666	·8572	59	1·02974
·55851	32	·5299	1·8871	·6249	1·6003	1·1792	·8480	58	1·01229
·57596	33	·5446	1·8361	·6494	1·5399	1·1924	·8387	57	·99484
·59341	34	·5592	1·7883	·6745	1·4826	1·2[illegible]2	·8290	56	·97738
·61087	35	·5736	1·7434	·7002	1·4281	1·2208	·8192	55	·95993
·62832	36	·5878	1·7013	·7265	1·3764	1·2361	·8090	54	·94248
·64577	37	·6018	1·6616	·7536	1·3270	1·2521	·7986	53	·92502
·66323	38	·6157	1·6243	·7813	1·2799	1·2690	·7880	52	·90757
·68068	39	·6293	1·5890	·8098	1·2349	1·2868	·7771	51	·8901[illegible]
·69813	40	·6428	1·5557	·8391	1·1918	1·3054	·7660	50	·872[illegible]
·71559	41	·6561	1·5243	·8693	1·1504	1·3250	·7547	49	·85[illegible]
·73304	42	·6691	1·4945	·9004	1·1106	1·3456	·7431	48	·83776
·75049	43	·6820	1·4663	·9325	1·0724	1·3673	·7314	47	·82030
·76794	44	·6947	1·4396	·9657	1·0355	1·3902	·7193	46	·80285
·78540	45	·7071	1·4142	1·0000	1·0000	1·4142	·7071	45	·78540
		Cosine	Secant	Cotan.	Tangent	Cosec.	Sine	Degrees	Radians

[illegible]

[illegible]

$N = \frac{+17}{[illegible]}$

[illegible]

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