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ELEMENTS

OF

THEORETICAL MECHANICS;

BEING THE SUBSTANCE

OF A

COURSE OF LECTURES

ON

STATICS AND DYNAMICS.

BY

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PREFACE

It is the Author's official duty to give a course of lectures annually, during a session of six months, upon all the leading branches of [illegible] usually [illegible] headed under the title "[illegible] Philosophy." In doing this, he has hitherto followed no particular textbook, but has given his students the opportunity of taking very full and correct notes, and has endeavoured, as far as was in his power, to make [illegible] introductory to some of the [illegible] discussed. As the [illegible] subjects as are [illegible] rather a wide field [illegible] of discussing the [illegible] and Dynamics [illegible] other subjects [illegible] cultivation of a [illegible]

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PREFACE.

It is the Author's official duty to give a course of lectures annually, during a session of six months, upon all the leading branches of science usually comprehended under the title " Natural Philosophy." In doing this, he has hitherto followed no particular text-book, but has given his students the opportunity of taking very full and correct notes, and has endeavoured, as far as was in his power, to make the course introductory to some of the best treatises on the subjects discussed. As the lectures, though restricted to such subjects as are adapted to an elementary class, embrace rather a wide field, experience has shown the necessity of shortening the time hitherto devoted to Statics and Dynamics, that less injustice might be done to various other subjects, attention to which was demanded in the cultivation of a science so rapidly progressive. The assistance now given to his students in these departments of the course, induces the Author to hope that this may be accomplished without the sacrifice of any thing that seems due to the importance of departments so fundamental, and of such general utility. In the view of even extending the student's acquaintance with them, a few propositions not previously given in the lectures have been introduced, to assist those who, in respect of taste and acquirements, come best prepared for such studies, in making an easy transition to richer sources of information. The selection of some propositions from the late Dr. Robison's publications, in this and other parts of the course, will enable such to peruse with pleasure, and without much interruption, the treatises from which they are taken, by which their information will be much extended beyond the limited field of mere elementary tuition. Poisson's method of treating the general doctrines of Statics has been kept in view, without servilely following his demonstrations; and, after the conclusion of the course, a considerable part of his excellent *Traité de Mécanique*, may be perused without any difficulty. The student of mechanics is not now, however, under the necessity of learning a foreign language, to become acquainted with the approved methods of physico-mathematical research. After finishing the course here prescribed, and that of astronomy containing some investigations that are supplementary to it, he will find ample employment, for some time to come, and much immediate improvement, as well as attractive inducements to prosecute his researches, in the perusal of several works that have been lately presented to him from the Cambridge press; particularly of Mr. Whewell's *Mechanics* and *Dynamics*; the translation of Venturoli's *Mechanics*, by Mr. Cresswell; the *Mathematical Tracts*, by Mr. Airy, and Mr. Woodhouse's *Astronomy*.

ELEMENTS

OF

THEORETICAL MECHANICS.

STATICS.

1. When any number of forces, that, acting separately, would produce motion in a body, are applied at the same time to that body, and so that no motion ensues in consequence of their application, they are said to balance one another, and the body, as affected by them, is said to be *in equilibrio*.

2. Forces thus exerted are called pressures; and the science which treats of the laws of pressure in relation to the equilibrium of solid bodies is denominated Statics, from a Greek word signifying to *weigh*.

3. Pressures are of various kinds; but, viewed abstractly, as the objects of Statics, differ in only two respects, *quantity* and *direction*.

4. The quantity of a pressure is estimated by reference to some standard force of the same kind, for instance a certain weight. One pressure is said to be equal to another, when, being applied in an opposite direction, it would balance it; or, when each of the two, being separately applied in the same direction, would balance the same third force. Two such

pressures, when their actions conspire, constitute a double pressure; three a triple pressure; and so on.

5. When a pressure is balanced, not by a force tending to produce sensible motion in the opposite direction, but by the resistance merely of what we regard as an immoveable obstacle, that resistance is called reaction, and is measured by the force which it destroys.

6. The direction of a pressure is that in which it would produce motion, if unbalanced.

7. In Statical investigations the directions of pressures are represented by straight lines, and their proportional quantities by lines or numbers.

8. The *Equivalent* or *Resultant* of two or more pressures is that single force which would require the same force or forces as they do to produce an equilibrium.

9. The subordinate laws of Statics may be derived, by reasoning strictly demonstrative, from a few very obvious and general ones, which, as referable to the principle of the sufficient reason, or to familiar and universal experience, we shall state as *physical axioms.*

10. *Ax.* 1. If two equal pressures be applied to the same point, in directions making an angle, their resultant will be a force directed towards the same parts, and will bisect that angle.

11. *Ax.* 2. The resultant of forces applied to a point will not be affected by the application or removal of forces, under the influence of which, considered separately, that point would be in equilibrio.

12. *Ax.* 3. If two equal forces, acting towards the same parts, be applied perpendicularly to the extremities of an inflexible physical straight line, they will be balanced by a force equal to their sum applied to the middle point of the line, in an opposite direction: or their resultant is a parallel force equal to their sum, and bisecting the distance between them.

Corollaries to the Axioms.

13. *Cor.* 1. If any two pressures be applied to the same point in directions making an angle, that of their resultant must be intermediate.

If the resultant of P and Q, acting on the point A, (Fig. 1.) in the directions AP, AQ, be not between these directions, let it be without the angle, and on the side of P. Let two other forces $P'=P$, and $Q'=Q$, be applied to the same point A, so that P' is opposite to P, $\angle P'AQ' = \angle PAQ$, and AQ' and AQ on the same side of PP'. Let AR be the direction of R the resultant of P and Q; make $\angle P'AR' = \angle PAR$, and so that AR and AR' are on the same side of PP'; then shall AR' be the direction of the resultant R' of P' and Q'; also $R'=R$; hence the resultant of the four forces P, Q, P', Q', being that of R and R', will be in the direction AX, bisecting $\angle RAR'$: but as P and P' are equal and opposite, the resultant of the same four forces will be that of Q and Q', (by *Ax.* 2.) and therefore will be in the direction of AX', bisecting QAQ'; and these conclusions are inconsistent. A similar absurdity would follow the supposition, that the resultant R is without the angle PAQ on the side of Q, or coincides with AP or AQ.

14. *Cor.* 2. If two forces be equimultiples of two others that act in the same directions respectively; whatever multiple each of the former two is of its part, the same multiple shall the resultant of the former two be of the resultant of the latter, and their directions shall be the same, (11.) and (8.) If a, b and r be in equilibrio, or their resultant be $=0$, the resultant of ma, mb and mr in the same directions, will be $= 0$ (11.); and the equivalent of ma and mb will be equal and opposite to mr, while that of a and b is equal and opposite to r. (8.)

15. *Cor.* 3. If when two forces applied to the same point are increased or diminished together, the direction of the resultant remain invariable, the resultant of the greater forces shall be the greater.

For the resultant of the original forces, and that of their increments, must be in the same direction; otherwise that of the final resultant would (by *Cor.* 1.) be different from that of the first, which is contrary to the supposition: and the two resultants being in the same direction, the final resultant must be equal to their sum.

16. *Cor.* 4. If the angles formed by the directions of two constituent forces and their resultant be constant, while the constituent forces are increased or diminished in any the same ratio, the resultant shall be increased or diminished in that ratio.

Let a and b be any two constituent forces, and their resultant r. By *Cor.* 2. $\frac{a}{m}$ and $\frac{b}{m}$, acting in the directions of a and b, will have the resultant $\frac{r}{m}$ in the direction of r. Now let A and B be any other forces in the ratio of a to b, acting in the directions of a and b respectively, and R their resultant. If $A = \frac{na}{m}$, and consequently $B = \frac{nb}{m}$, $R = \frac{nr}{m}$, (by *Cor.* 2.) and consequently, according as $A \gtrless \frac{na}{m}$, and $B \gtrless \frac{nb}{m}$, R will be $\gtrless \frac{nr}{m}$, (*Cor.* 3.)

$$\therefore A : a = R : r. \quad \text{(Eucl. Book v. def. 5.)}$$

17. *Prop. I. Theorem.* Two pressures, represented by the adjoining sides of a parallelogram, are equivalent to one represented by the diagonal which passes through the point, at which the sides meet.

Lemma. If the equivalent of two pressures, which are in a constant ratio to each other, and are represented by the adjoining sides of a rectangle, be always represented in direction by the diagonal passing through the point at which they meet, it shall be represented by the same in quantity also.

Let $ABCD$ (Fig. 2.) be a rectangle, and AC the diagonal passing through the point to which the forces are applied; draw EAF perpendicular to AC, and let fall the perpendiculars *BF*, *BH*, *DG*, *DE*; then shall *ABCD*, *AFBH*, and *AEDG*, be similar rectangles; and if in each of these the resultant of the pressures represented in direction and quantity by the sides, be represented in direction by the diagonal, it shall be represented by the same in quantity also. For if the resultant of *AH* and *AF* be represented by $AK > AB$, take *AL*, *AM*, *AN*, so that $AB : AK = AD : AL = AC : AM = AM : AN$. Then *AE* and *AG* are equivalent to *AL*, *AB* and *AD* to *AM*, and *AK* and *AL* to *AN*; (16.) that is, *AF*, *AH*, *AE*, and *AG*, are together equivalent to *AN*. But *AE* and *AF* are equal and opposite; hence *AH* and *AG* are equivalent to *AN*, (11.) that is, to a force exceeding their sum in the same direction, which is absurd.

In the same way it may be proved, that the resultant of the forces *AH* and *AF* cannot be proportionally represented by a line $< AB$. It is therefore represented by *AB*, and consequently the resultant of *AB* and *AD* is represented by *AC*, (16.)

We now proceed to the demonstration of the proposition.

Case 1. Let *ABDC* (Fig. 3.) be any square; and let the sides *AB* and *CD* be produced indefinitely towards *B* and *D*; draw the diagonal *AD*; in *CD* produced take $DF = AD$; join *AF*; take $FH = AF$; join *AH*; and so on: then complete the rectangles *ACFE*, *ACHG*, &c. It is obvious that *AD*, *AF*, *AH*, &c. bisect the angles *BAC*, *BAD*, *BAF*, &c. respectively. Hence the resultant of *AB* and *AC*, which are equal, must always be represented in direction by *AD*, (10.) and therefore by the same in quantity also, (*Lem.*) The resultant of *AE* and *AC*, being the same with that of *AC*, *AB*, and *BE*, or of *AD* and *BE*, which are equal, will always be represented in direction, and therefore in quantity also by *AF*. And thus may the proposition be proved with respect to any rectangle, whose diagonal makes with one of the sides an angle found by the continued bisection of a right angle.

Case 2. Let *ABCD*, and *ABEL*, (Fig. 4.) be two rectangles, and let *BAC* and *CAE* be any two angles of which, con-

For the resultant of the original forces, [illegible]
increments, must be in the same direction, [illegible]
the final resultant would (by *Cor.* 1.) be [illegible]
of the first, which is contrary to the supp[illegible]
resultants being in the same direc[illegible]
must be equal to their sum. [illegible]

16. *Cor.* 4. If the angles formed by [illegible] AH. [illegible]
constituent forces and their resulta[illegible] are [illegible]
constituent forces are increased or [illegible] to AC [illegible] AF
ratio, the resultant shall be incre[illegible] are equivalent to AE
ratio. $FAH = \angle CAB$

Let a and b be any two con[illegible] the proposition has
ant r. By *Cor.* 2. $\frac{a}{m}$ and $\frac{b}{m}$, [illegible] AC are equivalent to
[illegible] AK; that is, by hy-
b, will have the resultant [illegible] m [illegible] series of those found
A and B be any other forces [illegible] Case 1; **BAC being taken**
the directions of a and b respec[illegible] of which the proposition
$A = \frac{na}{m}$, and consequ[illegible] $CAE = a$, it appears, by
[illegible] that it is true with respect

and consequently, accordi[illegible] be an angle incommen-
be $\frac{nr}{m}$, (*Cor.* 3.) [illegible] being still a rectangle.
[illegible]) in relation to BAE the
$\therefore A$: [illegible], and of BAG the multi-

17. *Prop. I.* [illegible] between these, $= a$, may
adjoining sides of [illegible] angle. In DA produced
presented by th[illegible] AB and AF are equivalent to
which the sides [illegible] (by Case 2.) Hence the equi-

Lemma. If [illegible] equivalent of AE and FD, and
contai[illegible] the equivalent of AB and
[illegible] and AK, or of AG and AK,
[illegible] But AC is the only direc-
[illegible] the hypothesis interme-
[illegible] therefore the diagonal AC, in
which the angle BAC is the
[illegible] represents the direction of

the resultant of *AB* and *AD*. Consequently it represents it in quantity also, (*Lem.*)

The proposition then is true of all rectangles whatever.

Case 4. Let *ABDC* (Fig. 6 and 7.) be any oblique-angled parallelogram.

Perpendiculars *AE*, *CG*, *DF* being let fall, as in the figure, *AB* and *AC* are equivalent to *AB*, *AG* and *AE*; that is to *AF* and *AE*; that is to *AD*; by what has been previously demonstrated. The proposition is therefore true universally, as enunciated.

18. It is very common to speak of a force acting at *A* (Fig. 6.) as represented by a line *BD* in which the point *A* is not situated. The meaning then is, that the force is represented in quantity by that line, and in direction by a line drawn from *A* parallel to it. In conformity to this language, *Prop. I.* may be enunciated thus:—

"The resultant of any two forces acting at one point, and represented by two sides of a triangle, is itself represented by the remaining side."

19. *Cor.* 1. If three forces, applied to one point, be represented by two sides of a parallelogram, and a line equal to the diagonal drawn through their point of intersection, but in the opposite direction; or by the three sides of a triangle taken in consecutive order; these three forces shall be in equilibrio.

20. *Cor.* 2. Two forces applied to the same point in directions making an angle, and their resultant, or any three such forces which are in equilibrio, are proportional to the sides of any triangle whose three sides are parallel, or perpendicular, or similarly inclined to their respective directions.

Let *ABDC* (Fig. 8.) be a parallelogram whose two sides *AB*, *AC* and diagonal *AD* represent the two constituent forces and their resultant *R*. These same forces may consequently be represented by the sides of the triangle *ABD*, viz. *AB*, *BD* and *AD*, respectively: and as any triangle whose sides are parallel to *AB*, *BD* and *AD* is similar to *ABD*, and the parallel sides are homologous, the same forces may be represented by the sides of this other triangle which are parallel to their respective directions.

Again, let ABC (Fig. 9.) be a triangle whose sides represent three forces, related as above described, and let abc be another triangle whose sides ab, bc, ca meet AB, BC, CA perpendicularly in d, e and f. $Adaf$ is a four-sided figure, and d and f being right angles, dAf and daf are together equal to two right angles, that is to $daf + dac$. Hence $\angle bac = \angle BAC$. In the same manner it may be proved that $\angle abc = \angle ABC$, or $\angle acb = ACB$. Consequently the triangles are similar, and the perpendicular sides are homologous; or the forces are represented in quantity by the sides of abc, which are perpendicular to their respective directions.

Let da, fc and eb now revolve equally, so as to make equal angles with dA, fC and eB respectively; the one of the angles Ada, Afa becoming as much greater than a right angle as the other is less, their sum will still be $=$ two right angles as before: and so of the rest: $\therefore$ abc is still similar to ABC, and the latter part of the corollary is manifest.

21. *Cor.* 3. Each of three forces related as constituents and resultant, and not in the same straight line, or each of three such forces which are applied to one point and are in equilibrio, may be represented in quantity by the sine of the angle contained by the directions of the other two: and any two of them are reciprocally as the perpendiculars drawn to their directions from the same point in the direction of the third force.

For if AB or P (Fig. 8.) be represented by the sine of $\angle BDA$ or of $\angle DAQ$, or its supplement $D'AQ$; then, since the sides of a plane triangle are as the sines of its opposite angles, AC or BD, that is Q, will be proportionally represented by the sine of DAP or of its supplement $D'AP$, and AD or AD', that is R, by the sine of DBA or of its supplement BAC. Moreover, since $P : Q =$ Sine QAR : Sine PAR, $P : Q = DF : DE$, these lines DF and DE, drawn from any point *D* in the direction *AR* perpendicular to *AQ* and *AP*, being the sines of *QAR* and *PAR* to the radius *AD*: and in the same manner may the other analogies above indicated be deduced.

22. *Cor.* 4. Let the forces which are represented by the lines *AB* and *AC*, or *AB* and *BD*, (Fig. 6 and 7.) be denoted by the numerical symbols a, b; and let the angle *BAC*, or its equal *FBD*, be denoted by θ; Then since $AD^2 = AB^2 + BD^2 \pm 2\,AB.BF = AB^2 + BD^2 + 2\,AB.BD \cos.\,\theta$, the analytical expression of the resultant r will be

$$\sqrt{a^2 + b^2 + 2\,a\,b \cos.\,\theta},$$

and if α and β be the angles which the direction of r makes with those of a and b respectively,

$$\text{Sin.}\ \alpha = \frac{b}{r}\ \text{Sin.}\ \theta,\quad \text{Sin.}\ \beta = \frac{a}{r}\ \text{Sin.}\ \theta.\quad (21.)$$

23. *Cor.* 5. Any number of forces acting at one point may be reduced to one resultant.

Let three forces acting at *A* (Fig. 10.) be represented by the lines *AB*, *AC* and *AD*; from *B* and *C* draw lines parallel to *AC* and *AB*, meeting in *E*; then *AE* represents a force equivalent to *AB* and *AC*. From *E* and *D* draw lines parallel to *AD* and *AE* meeting in *F*; then shall *AF* be the resultant of *AE* and *AD*, or of *AB*, *AC*, and *AD*; and by the same mode of proceeding may any number of forces be compounded. The graphical process will be simplified by omitting some parts of it which are unnecessary in practice. Thus all that is essential to the composition here proposed is to draw *BE* equal and parallel to *AC*, which determines the point *E*; and *EF* equal and parallel to *AD*, which determines *F*. *A* and *F* being joined, *AF* will represent the resultant sought. This has suggested the following elegant Statical theorem:—" If forces acting at a point *A* be represented by all the sides of a polygon but one, taken in consecutive order, as *AB*, *BE*, *EF*, *FD*, the resultant of the whole will be represented by the remaining side *AD*." Or it may have occurred as an obvious extension of the observation, (18.) The resultant of *AB* and *BE* is *AE*; that of *AE* and *EF* is *AF*; and that of *AF* and *FD* is *AD*. Hence too, "If the forces be represented by *all* the sides of the polygon taken in order, the point *A* will be in equilibrio."

24. *Cor.* 6. The forces considered in the last corollary may be either all in one plane or not in one plane. In the latter case the lines *AB*, *AC*, and *AD*, which represent them, are three adjacent linear sides of a parallelopiped; and the line *AF*, which represents their resultant, is manifestly the diagonal of the solid which passes through the point at which the forces are applied. The order in which the forces are compounded is immaterial: we may regard *AB* and *AC* as equivalent to *AE*, and *AE* and *AD* to *AF*; or *AD* and *AB* as equivalent to *AG*, and *AG* and *AC* to *AF*, &c.

25. *Cor.* 7. When it is necessary that a pressure *R*, represented by the given line *AD*, (Fig. 8.) should be applied at *A*, but, by reason of some obstacle, we cannot employ for that purpose a single force $= R$, the end may be accomplished by two forces, *P* and *Q*, applied in such directions *AP* and *AQ*, as may be convenient. To find these forces and their directions is an indeterminate problem: but if, of the quantities of the forces to be employed and their directions, any two be assigned, the other two may be found. Thus if, when the directions *AP*, *AQ* are given, that is, the angles which they make with *AR* given in position, we draw *DB* and *DC* parallel to *AQ* and *AP* respectively, *AB* will represent the force *P*, and *AC* the force *Q*. If *AC* be given, representing *Q* both in direction and quantity, join *CD*; and *AB*, drawn equal and parallel to *CD*, will represent *P* in the same respects. If the direction of *Q* be given and the quantity of *P*, let $R : P = AD : M$; take *D* for centre and *M* for radius, and describe an arch intersecting *AQ*, when possible, in a point *C*, or in two points *C*, *C'*; then *AC* or *AC'* will represent the magnitude of *Q*, and a parallel to *DC* or *DC'* drawn through *A*, the direction of *P*. So if the quantities are given, and the directions to be determined, the problem is manifestly reducible to the construction of a triangle whose three sides are given, and subject to a similar limitation. For

ulation, which should be employed when we have given in the 1st *Case*, one side the angles; in the 2d, two sides *AD*, *AC*, angle: in the 3d, two sides *AD*, *DC*, with one angle opposite to the latter; and in the 4th, the three sides.

The operation here described is termed the *resolution of forces*; and is of the greatest utility in the investigation of Statical principles, as well as in their practical application.

26. *Cor.* 8. When we resolve the force *AD* (Fig. 8.) into *AC* and *AB*, or *AC* and *CD*, and the latter is not at right angles to *AQ*, it may be further resolved into *CF* and *FD*, of which the former is in the direction of that line, and the latter perpendicular to it. The only resolution that gives the just effect of *AD* reduced to the direction *AQ* is that which is obtained by drawing the perpendicular *DF*. Then, as the force *FD* neither conspires with *AF* nor opposes it, the analysis is complete, and *AF* represents the whole effect of *AD* in the direction *AQ*. Hence, if a force *R* acting in any direction, be reduced to, or estimated in, another direction, making an angle θ with the former, the analytical expression of the force so reduced is R cos. θ.

27. *Cor.* 9. If any number of constituent forces be reduced to the same direction, the sum of the forces so reduced is equal to the resultant estimated in that direction.

Let *AB*, *BC*, *CD*, (Fig. 11.) represent the forces, and *AD* their resultant: let *AM* be the direction to which the forces are to be reduced; draw *BE*, *CF* and *DG* perpendicular to *AM*, *BH* perpendicular to *CF*, and *CK* perpendicular to *DG*. Then *AE*, *BH*, *CK*, or *AE*, *EF*, *FG* are the representatives of the constituent forces reduced to the direction *AM*, and *AG* is that of the resultant *AD* (23.) reduced to the same. If the constituent forces are not all in one plane, *FC*, *GD*, &c. to which *BH*, *CK*, &c. are drawn perpendicular, may be supposed to represent planes to which *AM* is perpendicular.

28. *Cor.* 10. If any number of forces applied to one point

be in equilibrio, the equilibrium will exist, in whatever common direction the balanced forces are estimated.

29. *Cor.* 11. If any number of forces applied to one point be reduced by projection to the same plane, the projection of the resultant shall be the resultant of the projections of the constituent forces: and if any number of forces be applied to one point, and in equilibrio, their projections on the same plane will also represent a system in equilibrio.

For the forces to be projected being represented by the sides of a polygon, whether in the same plane or not, their projections will form a polygon.

30. *Cor.* 12. In the composition of forces, when the number of them is considerable, the most convenient way is that which proceeds by a previous resolution of each into its constituents, in the direction of rectangular axes of co-ordinates, given in position.

Case 1. Let AP, AP', AP'', &c. (Fig. 12.) represent any number of forces applied to the point A and all acting in the same plane. Draw through A, and in that plane, any two straight lines, AX and AY at right angles to each other, and resolve the forces AP, AP', AP'', &c. into AB, AB', AB'', &c. whose sum let be AH, in the direction of the line AX; and AC, AC', AC'', &c. whose sum let be AK, in the direction of the line AY; complete the parallelogram $AHRK$, and join AR; then shall AR represent, in direction and quantity, the resultant of all the forces. Let the forces expressed numerically be denoted by the symbols P, P', P'', &c. and the angles which their directions make with AX by α, α', α'', &c. respectively: and let AH be denoted by X, AK by Y, and AR by R; also let a denote the angle RAX. Then

$$X = P \cos.\ \alpha + P' \cos.\ \alpha' + P'' \cos.\ \alpha'', \&c. = \int P \cos.\ \alpha,$$

$$Y = P \sin.\ \alpha + P' \sin.\ \alpha' + P'' \sin.\ \alpha'', \&c. = \int P \sin.\ \alpha,$$

$$R = \sqrt{X^2 + Y^2} \text{ and } \cos.\ a = \frac{X}{R} = \frac{X}{\sqrt{X^2 + Y^2}}$$

The process is a little simplified, by taking one of the axes of the co-ordinates, AX or AY, in the direction of one of the forces.

Case 2. Let the forces, one of which is represented by AP, (Fig. 13.) have their directions in different planes. Draw through A three straight lines AX, AY, AZ which may be considered as three adjacent linear sides of a rectangular parallelopiped; or such that each of them may be at right angles to the plane of the other two. From P let fall PB perpendicular to one of these planes, suppose XAY, and join AB. PB will of course be at right angles to AB, and AP is resolved into AB in the plane XAY, and $AE = BP$ perpendicular to it, or in the direction AZ. Now draw BC perpendicular to AX, and BD perpendicular to AY; and AB may be considered as resolved into AC and AD in the directions AX and AY. Thus, AP is resolved into AC, AD and AE, in the directions of the three axes AX, AY and AZ respectively. Let each of the forces be thus resolved, and let AF, AH and AK denote their sums in these respective directions: then, by the 1*st Case,* find AG the resultant of AF and AH, in the plane XAY; and AR the resultant of AG and AK in the plane GAZ; AR will be the resultant of the whole.

With a view to calculation, let P, P', P'', &c. and R denote, as before, the forces and their resultant; $\alpha, \alpha', \alpha''$, &c. and a, the angles which their respective directions make with AX; β, β', β'', &c. and b, those which they make with AY; and $\gamma, \gamma', \gamma''$, &c. and c, those which they make with AZ: Also let X, Y, Z, denote the sums of the forces estimated in the directions AX, AY, AZ respectively. Then as

$$AR^2 = AG^2 + AK^2 = AF^2 + AH^2 + AK^2,$$

we shall have

$$R = \sqrt{X^2 + Y^2 + Z^2}.$$

Moreover, if RH and RF be joined, it will be obvious that $AF = AR \cos. RAF$, and $AH = AR \cos. RAH$; for the planes RGF, RGH are at right angles to the plane XAY; and AF, AH in that plane are perpendicular to the common sections GF, GH, and therefore at right angles to the planes RGF, RGH, and to RF and RH respectively.

$$\text{Hence } \cos. a = \frac{X}{R}, \ \cos. b = \frac{Y}{R}, \ \cos. c = \frac{Z}{R}.$$

Now $X = \int P \cos. \alpha$, $Y = \int P \cos. \beta$, $Z = \int P \cos. \gamma$, and are all given; therefore R, and the angles a, b, and c, which determine its position, are given.

In taking the products $P \cos. \alpha$, &c. the cosines are to be considered as positive in the first and last quadrants, and negative in the second and third; and by the sums of the forces estimated in a particular direction, we always understand such as are found by addition in the algebraic sense.

That the position of the resultant is determined by the angles a, b, and c, may be illustrated thus: Let x, y, z, and g, (Fig. 14.) denote the points in which the axes AX, AY, and AZ, and the line AG meet the surface of a sphere, whose centre is A. The arches xy, xz, yz, and zg, are quadrants, and the angles at g right angles; and if xR, or xR', the measure of a, and yR, or yR' the measure of b, be given, they limit the point where the direction of the resultant meets the spherical surface to one of two positions R, R', both in the arch zgz', and the one as much above g as the other is below it. The angle c, or its measure zR, or zR', therefore, being also given, determines which of the two points R, R' is to be chosen, and this, with the given point A, determines the position of the line sought.

31. *Prop. II. Theorem.* If three parallel forces be in equilibrio by the intervention of an inflexible physical straight line, to which their directions are perpendicular, any two of them will be inversely proportional to their distances from the point at which the third is applied.

This is generally called the principle of the lever, and was first demonstrated, so far as we know, by the celebrated Archimedes. Of the two principles expressly assumed by him, (αἰτήματα,) the first is, that equal weights suspended at equal distances from the fulcrum of a lever will balance each other; and taken along with another, which is tacitly assumed in the course of his demonstration, that the fulcrum in that case supports the sum of the weights, is the same with our 3d

Ax. (12.) The second is, that if equal weights be suspended at unequal distances from the fulcrum, the more distant will preponderate. This, however, may be deduced from the former. Let C (Fig. 15.) be the fulcrum, M any weight or force perpendicularly applied, and D and E two points in the lever on the same side of the fulcrum, of which E is the more distant; M at E shall balance more than M at D. Bisect CD in B; in DC produced towards C take $CA=CB$; and let $BE' = BE$. Then 2 M at B will balance 2 M at A, (12.) But 2 M at B is equivalent to M at C and M at D, (12.) and the force, whose direction passes through the fulcrum, can have no effect upon the rotation; therefore M at D balances 2 M at A. Again, 2 M at B is equivalent to M at E and M at E'; therefore M at E balances 2 M at A, together with M at E'.

The demonstration of the proposition for weights or pressures of any kind that are commensurable, proceeds thus: Let EF (Fig. 16.) be a lever, whose fulcrum is C, and let the weights P and Q be applied at E and F, so that $P : Q = CF : CE$, there shall be an equilibrium. Produce EF towards both E and F, making $EA = CF$, and $FB = CE$, and the whole CB consequently $= CA$; take $ED = EA$; then AD is bisected in E; as is also DB in F; for CF being $= ED$, because each of them is $= EA$, to each add CD, and $FD = CE = FB$.

Now suppose the weight $P+Q$ to be uniformly distributed over the whole line AB, so that equal parts of the weight may rest on any equal parts of the line, which is supposed to be horizontal, there will be an equilibrium, (12.); for every particle resting on CA is balanced by an equal one at the same distance on the other side, (12, 11.) Also, by the same axioms, all the matter on AD may be collected at E, and all that which is upon DB at F, without destroying the equilibrium; for the resultant of every two equal weights equidistant from E passes through that point; and so it may be affirmed of the other point F. Now

$$AD : DB = \tfrac{1}{2}AD : \tfrac{1}{2}DB = CF : CE = P : Q,$$

$$\text{and comp. } AB : AD = P + Q : P,$$
$$\text{or } AB : DB = P + Q : Q;$$

and as $P + Q$ was uniformly applied to the whole line AB, the parts on AD and DB must have been P and Q respectively. It appears then, that when $P : Q = CF : CE$, there will be an equilibrium; and by the second principle above mentioned and demonstrated, if Q be applied to any point in the line CB nearer to C, or more remote, the equilibrium will be destroyed, so that P cannot balance Q, unless $P : Q = CF : CE$.

Let $E'F'$ (Fig. 17.) represent the lever EF reversed upon the same fulcrum C, and let a weight P be applied both at E and at E', and Q at both F and F'. There will be an equilibrium, (11.) and there can be no reason why the pressure at C, arising from the one pair P, Q, should differ from that arising from the other, so that one-half of the whole must arise from each. But the whole pressure at C is manifestly $2P + 2Q$, (12.) therefore that arising from P and Q, applied to E and F, must be $= P + Q$. Let this be called S. Instead of the fulcrum at C then we may substitute a force $S = P + Q$, acting in a direction opposite to that of P and Q;

$$\text{and since } P : Q = CF : CE,$$
$$\text{comp. } S : P = FE : FC,$$
$$\text{and } S : Q = EF : EC.$$

Note. We may consider P as equal and opposite to the resultant of S and Q, so that when two unequal forces S and Q act perpendicularly on the arms of a straight lever, and in opposite directions, the resultant is equal to their difference, and acts towards the same parts as the greater. Also the resultant and the less of the two forces are on opposite sides of the greater.

A very simple demonstration of the property of the lever in the case of commensurable weights, is given by the celebrated Maclaurin, (see his account of Newton's Disc. p. 150.) Another, that seems very plain, may be as follows:

Let *C*, (Fig. 18.) be the fulcrum of the lever *AF*, and let $AC = CB = BD = DE$ &c. 2 *P* at *B* balances 2 *P* at *A*, and the pressure on the fulcrum = 4 *P*. But 2 *P* at *B* is equivalent to *P* at *C* and *P* at *D*. Let the forces be so applied, and we must still have a pressure = 4 *P* acting upwards at *C*. But this admits of simplification, for *P* acting downwards at *C* and an equal part of the force acting upwards may be removed without subverting the equilibrium, (11.) Hence, *P* at *D* balances 2 *P* at *A*, and the pressure at *C* is 3 *P*. Again, 2 *P* at *B* balancing 2 *P* at *A* as before, and the fulcrum being pressed with the force 4 *P*; let 2 *P* at *B* be resolved into *P* at *A* and *P* at *E*; then *P* at *E* balances 3 *P* at *A*, and the pressure at *C* is their sum. Now, if the proposition be true of mAC, and of nAC, it shall be true of their sum. Let $BC = mAC$, and $DC = nAC$, (Fig. 19.) Also take $DE = BC$, so that $EC = (m + n)AC$, and bisect *CE* or *BD* in *O*. *P* at *B* and *P* at *D* balance $(m + n)P$ at *A*, and the pressure on *C* is $(m + n + 2)P$, by Hypoth. and (11.) But *P* at *B* and *P* at *D* are equivalent to *P* at *C* and *P* at *E*, (12.) Taking away *P* acting downwards at *C*, and as much of the force there acting upwards, we have *P* at *E* in equilibrio with $(m + n)P$ at *A* and $(m + n + 1)P$ acting in the opposite direction at *C*. Lastly, if *CF* (Fig. 18.) $= mAC$ and $CG = nAC$ or nBC, *P* at *A* will be balanced by $\frac{1}{m} P$ at *F*, and *P* at *B* by $\frac{1}{n} P$ at *G*, and *P* at *A* balancing the equal force at *B*, let these be removed, and we shall have $\frac{1}{m} P$ and $\frac{1}{n} P$ in equilibrio (11.), the former at the distance mAC, and the other at *n* times that distance. Whence, universally, when the arms of the lever are commensurable, the perpendicular forces in equilibrio are reciprocally as their distances.

Let *CA* and *CB* (Fig. 20.) be now supposed incommensurable, and let *P* applied perpendicularly at *A*, and *Q* at *B* be in equilibrio,

$$P : Q = CB : CA.$$

For, if possible, let $P : Q = CD : CA$, CD being $< CB$. Let Cd be $> CD$, but $< CB$ and commensurable with AC, and let N at d balance P at A.

$$\begin{aligned} \text{Then } P : Q &= CD : CA, \text{ by Hyp.} \\ P : N &= Cd : CA, \text{ by part 1.} \\ \text{and } Cd : CA &> CD : CA \\ \therefore\ P : N &> P : Q \end{aligned}$$

and $Q > N$, which is absurd; for by Archimedes's second principle N at B would overbalance P at A, much more would Q do so if greater than N.

In the same way it may be proved that P cannot be to Q as CD', any line greater than CB, to CA.

Otherwise thus: If $CB = \frac{n}{m} CA$, $P = \frac{n}{m} Q$ by part 1; if CB be $> \frac{n}{m} CA$, P is $> \frac{n}{m} Q$, for it is $\frac{n}{m}$ times the force which balances itself at a less distance; and if CB be $< \frac{n}{m} CA$, P is $< \frac{n}{m} Q$, for it is $\frac{n}{m}$ times the force which balances itself at a greater distance: that is,

$$P \gtreqless \frac{n}{m} Q, \text{ according as } CB \gtreqless \frac{n}{m} CA,$$

Whence $P : Q = CB : CA$, (Eucl. B. v. def. 5.)

32. *Cor.* 1. The rectangles under any two of the lines representing the forces and their respective distances from the point at which the third is applied are equal to one another: or, if the ratio of the forces and that of the distances be expressed by numbers, the corresponding products are equal.

$$\begin{aligned} &P.\ CE = Q.\ CF \text{ (Fig. 16.)} \\ &S.\ FC = P.\ FE \\ &S.\ EC = Q.\ EF. \end{aligned}$$

33. *Cor.* 2. If the arms of a lever make an angle with each other at the fulcrum, and two forces, applied perpendicularly to the arms, be reciprocally as their distances from the fulcrum, these forces shall be in equilibrio.

Let ACB (Fig. 21.) be a crooked lever having two arms AC, BC, which make an angle at C, where the fulcrum is supposed to be placed : and let P and Q be two forces represented by AP and BQ, acting perpendicularly upon the arms AC, BC, and so related to each other that $P : Q = BC : AC$, or $P . AC = Q . BC$; these forces shall be in equilibrio. Produce BC to A', making $CA' = CA$, and let CA' represent a physical line rigidly connected with the lever ACB. Suppose a force Q', represented by BQ', equal and opposite to Q, to be applied at B, and a force $P' = P$, to be applied at A' perpendicularly to the arm CA', and tending to turn it in the direction $A'P'$: the lever will manifestly be in equilibrio, for Q' will balance Q, and P' will balance P; there being nothing to determine that either should prevail. But since $P' = P$, and $Q' = Q$, and $P : Q = BC : AC$ by Hyp. $P' : Q' = BC : AC = BC : A'C$, and P' may be considered as in equilibrio with Q'. Hence P and Q must be in equilibrio, (11.)

34. *Cor.* 3. If two forces acting in the same plane be applied to the arms of a lever in any directions whatever, and be in equilibrio, they shall be reciprocally as the perpendiculars drawn from the fulcrum to their respective directions; and conversely.

If in the straight or angular lever ACB, (Fig. 22 and 23.) P and Q, represented by AM and BN respectively, be in equilibrio, and CF be drawn perpendicular to AM, and CG to BN; $P : Q = CG : CF$. Draw the perpendiculars MD, NE as in the figures. Then AM may be resolved into AD and DM, and BN into BE and EN, of which AD and BE have no effect in producing rotation. The equilibrium in that respect is maintained by DM and EN, acting at A and B perpendicularly to the arms of the lever, so that we must have $DM : EN = BC : AC$, (31.) Hence, and by similar triangles, the following serieses of quantities will be proportionals in an inverse order:

$$AM : DM : EN : BN,$$
$$CG : BC : AC : CF,$$

and $AM : BN = CG : CF$, (Ex. æq.)
or $P.CF = Q.CG$.

Otherwise thus,

$MD.AC = EN.BC$. (32.)
i.e. $AC.P.\sin. A = BC.Q.\sin. B$
or $P.CF = Q.CG$.

Conversely if $AM : BN = CG : CF$,

$$\left.\begin{array}{l} MD : AM : BN : EN. \\ CB : CG : CF : AC. \end{array}\right\} \text{In prop. pert.}$$

$MD : EN = CB : AC$. (Ex. æq.)

and there is an equilibrium. (31.)

Note. The rectangles under the lines representing the forces and the perpendicular distances of the directions in which they act from a fulcrum or axis, or the corresponding products when the forces and the distances are represented by numbers, may be adopted as measures of momentum or rotative energy, and are themselves generally denominated *momenta*.

35. *Cor.* 4. When two forces in the same plane are applied to the arms of a lever as before represented, and are in equilibrio, the pressure on the fulcrum is the same in quantity and direction as if both forces were immediately applied to that point, in directions parallel to the lines in which they are actually exerted.

This has been taken for granted as a part of our third axiom, or has been already proved, when the lever is straight and the forces perpendicular to it: it remains to be proved of oblique forces applied to the straight lever, and of any forces applied to the angular lever.

The forces AD, BE (Fig. 22, 23, 24.) are propagated to the fulcrum, which, when the lever is straight, as in (Fig. 22, 23.), sustains also the sum of the perpendicular pressures DM, EN, as has been assumed, or proved: and AD, DM thus transmitted recompose a pressure equal and parallel to AM, while BE and EN produce one equal and parallel to BN.

Now, let ACB (Fig. 25.) be an angular lever, A and B forces perpendicular to the arms, and in equilibrio. In BC

and AC produced make $CB' = CB$ and $CA' = CA$; let the levers BB' and AA' be inflexibly connected at C, and let $A' = A$ and $B' = B$, all acting as represented by the perpendicular lines in the figure. A' and B' will also be in equilibrio, and their resultant will be in the same line with that of A and B; for the one will make the same angles with CA' and CB' that the other does with CA and CB respectively. This is very obvious when A' and B' act in the opposite to their present directions. But these two forces, and those equal and opposite to them, would destroy one another, and so consequently would their resultants do, which must of course be also equal and opposite. As the resultants of the two pairs then are manifestly equal, and in the present case conspire in direction, the pressure arising from A and B alone is the half of the resultant of all the four. But the equivalent of B and its equal B' is $CD = 2\ B$ parallel to B; and that of A and A' is $CE = 2\ A$, parallel to A, (12.) and CF the resultant of CD and CE represents the whole pressure on the fulcrum. A' and B' being removed, the resultant, retaining, as has been proved, the same direction must be reduced to $\frac{1}{2}\ CF$, which is equivalent to $\frac{1}{2}\ CE$, equal and parallel to A, and $\frac{1}{2}\ CD$, equal and parallel to B, whence, as in the cases formerly discussed, the truth of the corollary is manifest.

36. *Cor.* 5. When three forces in the same plane, and not parallel, are in equilibrio by the intervention of any lever, their directions meet in one point, and the forces are such as would be in equilibrio if applied to one point, their directions preserving the same relative position.

Let the forces P and Q be represented by AP and BQ, (Fig. 26.) which meet when produced in F. Draw from the fulcrum CK equal and parallel to AP, and CM equal and parallel to BQ, complete the parallelogram $CKLM$, and CL (35.) shall represent the pressure on the fulcrum, a force R equal and opposite to which will hold P and Q in equilibrio. The direction of this force meets CK and CM by construction, and therefore PA and QB which are parallel to them. Now, it must meet them both in F their point of intersection; for, if not, let it meet the one in

G, and the other in *H*. Draw the perpendiculars *CE*, *CD*, *LO*, *LN*.

$$\text{Then } P : Q = CD : CE, \text{ (24.)}$$
$$P : Q = LN : LO, \text{ (21.)}$$
$$\therefore LN : LO = CD : CE,$$
$$\text{Alt. } LN : CD = LO : CE,$$

And, therefore, by similar triangles,

$$LC : CG = LC : CH,$$
$$\text{or } CG = HC, \text{ which is absurd.}$$

This demonstration applies equally when the lever is angular at *C*.

P, *Q*, and *R* then are directed to one point *F*; and, being represented by three sides of a triangle *CK*, *KL*, and *LC*, are such as would be in equilibrio, if actually applied to one point.

Let *ACBD* (Fig. 27.) be a lever with two inflexions, one at the fulcrum *C*, and the other at *B*. Take *FG* to represent the pressure on the fulcrum, or the opposite force *R*, which may replace the resistance of the fulcrum, when it is removed, and be in equilibrio with *P* and *Q* represented by *FH* and *FE* sides of the parallelogram *FHGE*. Draw *HK* and *HL* parallel to *FC* and *FD*. *S* and *T*, represented by *FK* and *FL*, are, by what was proved in the former case, equivalent to *P*, so that the forces *S*, *T*, *Q*, and *R*, are in equilibrio, *P* being removed. But *R* and *T* are directly opposed to each other; therefore *T* and its equal, that part of *R* which is represented by *LF*, being removed, *S*, *Q* and $R' = R - T$ are in equilibrio. The directions of these three balanced forces still meet in *F*, and being represented by *LH*, *HG*, *GL* the sides of a triangle, are such as would be in equilibrio if actually applied, as stated in the corollary, to one point. We may then apply to three forces in equilibrio by the intervention of a lever the 2d and 3d corollaries to *Prop. I.* (20, 21.)

The student will moreover remark, that any one of the forces *S*, *Q*, *R'* may be considered as supplying the place of a fulcrum to the other two, so situated that its resistance may be in the direction of that force. Thus may the demonstration be extended to a lever of any number of angles, and consequently to bodies of any shape used as levers: for the fulcrum may be considered as connected with the point of application of each force by an angular lever of the same number of sides.

37. *Cor.* 6. Two forces that are equal and opposite will, by the intervention of a solid body of any shape, destroy each other, though they be not applied at the same point.

For if the body be regarded as a lever having a fixed point or fulcrum, there will be no rotatory motion (34), and no pressure on the fulcrum (35), therefore no motion of translation, the fulcrum being removed.

38. *Cor.* 7. A force in respect of both its momentum or tendency to rotation, and the pressure exerted at the fulcrum, may be conceived as applied at any point in the line of its direction, provided that point be inflexibly connected with any point of the lever.

This also evidently follows from the 3d and 4th *Cor.* (34, 35.)

39. *Cor.* 8. Whatever forces are applied to a lever, and in whatever directions in the same plane, if the sum of the moments of the forces which tend to turn it in one direction be equal to the sum of the moments of those which tend to turn it in the opposite direction, there will be an equilibrium; and conversely.

Let B, D, E, F, and G act perpendicularly on the lever DG whose fulcrum is C (*Fig.* 26.), and in the directions represented in the figure: if

$$B.BC + D.DC = E.EC + F.FC + G.GC,$$

there shall be an equilibrium. Let $Ac = AC$, and let $A = A$ be such, that $A.AC =$ each of these sums of moments. A may be divided into two parts b and d, such that $b.AC = B.BC$, and $d.AC = D.DC$: b will balance B, and d will balance D, or A will balance B and D. In like manner, A must be divisible into three parts which we shall call e, f, and g, such that

$$e.AC = E.EC, f.AC = F.FC, \text{ and } g.AC = G.GC \therefore$$

e, *f*, and *g* will balance *E*, *F*, and *G*; that is, *A* will balance *E*, *F*, and *G*. Now let *A* and *A'*, which must be in equilibrio, be removed, and the remaining forces will be in equilibrio, (11.) The demonstration of the converse is obvious.

We may suppose any number of levers in the same plane united at the fulcrum *C*, or an angular lever of any number of arms, and any forces in that plane applied to each in any directions. The demonstration would be exactly similar to that just given.

40. *Cor.* 9. If any forces whatever in the same plane applied to a lever be in equilibrio, forces represented by their orthogonal projections on any plane shall be in equilibrio, when applied, as represented in the projection, to a lever, represented by the orthogonal projection of the former upon the same plane.

Let there be two forces *P* and *Q*, represented by *BP* and *AQ*, (Fig. 29.) on the one side of the fulcrum, and one upon the other represented by *DS*. Let fall the perpendiculars *CE*, *CF*, *CG*, as in the figure, and join *P* and *C*, *Q* and *C*, *S* and *C*.

$$PB.CF + QA.CE = SD.CG, \text{ (39.)}$$

and the halves of these are equal; that is,

$$\triangle PCB + \triangle QCA = \triangle SCD.$$

There will, consequently, be an equation between the corresponding areas in the projection, and between the doubles of the same, or the measures of the projected momenta.

41. *Cor.* 10. If any number of given parallel forces be applied to given points of a solid inflexible body, a point may be found through which their resultant, when possible, will always pass, whatever be the position of a straight line to which their directions are parallel.

Case 1. Let there be two forces *P* and *Q* applied at the points *A* and *B*, (Fig. 30.) and acting towards the same parts in any parallel directions, and let the point *C* be taken in the line *AB*, so that $Q : P = AC : BC$, or

$$Q + P : Q = AB : AC;$$

through *C* draw *DCE* perpendicular to *AP* or *BQ*; the triangles ACD, BCE will be similar; and since $Q : P = AC : BC$, $Q : P = CD : CE$, and P and Q will be in equilibrio if the point C be fixed, (34.) and the point C is subjected to a pressure equal to their sum, (35.) which is their resultant.

Case 2. Let the forces P and Q (Fig. 31.) act towards opposite parts in parallel directions. If they be equal, and their directions be not in the same line, they cannot have any resultant; for supposing them to have one passing through any point of the line joining the points of application, a fulcrum being placed there capable of sustaining it P and Q would be in equilibrio, which is manifestly impossible; for they will either conspire in turning the line in the same direction, or they will tend to turn it in opposite directions with unequal momenta. Let them then be unequal, and Q the greater. In AB, produced towards B the point at which the greater is applied, take a point C so that

$$Q : P = AC : BC, \text{ or}$$
$$Q - P : Q = AB : AC;$$

through C draw CED perpendicular to AP, or BQ. Then $Q : P = CD : CE$, and if a force $R = Q - P$ be applied at C in the direction CR, parallel to AP or QB, the three forces P, Q, and R will be in equilibrio; for Q and P are reciprocally as their distances from C, and $Q - P = R$, or $Q = P + R$. The resultant of Q and P is therefore $R' = Q - P$ acting at C towards the same parts as Q.

Suppose P very nearly equal to Q; then $Q - P$ is very small; the ratio $Q : Q - P$, or its equal $AC : AB$ is very great; and as P approaches indefinitely near to Q in value, AC becomes indefinitely great in comparison with the finite line AB; that is, there is no lever AC of any assignable length, to which a third force being applied could maintain an equilibrium with P and Q. In this way we see, as before, that the problem is in that case impossible. If a third force, such as that now described, could be found, one equal and opposite to it would be the resultant sought.

Case 3. Let A, B, C, &c. be points of a solid body, that is, points any way inflexibly connected with each other, and let forces denoted by the same letters be applied to them respectively, all parallel to a line $O x$ given in position, (Fig. 32.)

First, Suppose all the forces to act towards the same parts. In AB find D, so that $A : B = BD : AD$. The resultant of A and $B = A + B$, which call D, shall pass through D, and be parallel to $O x$: join the points C, D, and in CD find E, so that $C : D = ED : EC$; the resultant of C and $D = D + C = A + B + C =$ the resultant of the whole, shall pass through E parallel to $O x$. Now let the same forces, applied to the same points, act all in lines parallel to another straight line given in position, as $O y$; it may be shown, in the same words, that the resultant, which is as before the sum of the forces, passes through E parallel to $O y$. The investigation is similar for any number of forces; and the point E thus found is called the *Centre of parallel forces.*

Secondly, Suppose that there are also parallel forces acting towards the opposite parts, a point E', which may be the same with E, or different, will be found by a similar process, such that their resultant shall always pass through it. Call this resultant R', and that of the first mentioned forces R. If the points E and E' are the same, R and R' are directly opposed to one another, and the final resultant is their difference. If, as will generally be the case, E and E' are different, a point S may be found in EE', produced towards E or E' according as R or R' is the greater, through which the resultant of the whole, that is of R and R', shall always pass, or which will be the centre of the whole, when these partial resultants are unequal. If they are equal, no single resultant can be found, nor is there in this case any centre of parallel forces.

42. The centre of parallel forces obviously remains the same, though the forces should be increased or diminished, provided they be all increased or all diminished in the same ratio.

43. It is clearly implied in what has been said of this centre, that if a force be always applied to it equal to the algebraic sum of the forces taken with a contrary sign and paral-

lel to their directions, or if it is preserved fixed by an immoveable obstacle, the body will be in equilibrio, whatever be the relative position of a line to which the forces are parallel.

44. *Cor.* 11. Forces which, in respect of a tendency to rotation, would be in equilibrio when applied in the same plane to the same point, will be in equilibrio when applied in parallel planes to the same axis.

It will be sufficient to state the proof in the simplest case, when the parallel planes are perpendicular to the axis.

Let CA and CB (Fig. 33.) represent two inflexible lines, connected perpendicularly with the same axis represented in projection by the point C; and let P acting by the lever CA, and Q acting by CB, in the directions AP and BQ, which are in planes perpendicular to the axis, be in equilibrio, the ratio of P to Q shall be the same whether these planes be different or not. In the line CA take $CD = CB$, and let two forces S and S', each equal to Q, be applied at D, as in the figure, in opposite directions, parallel to BQ or AP. This will not alter the statical condition of the system. But S will balance Q, for they are in circumstances precisely similar, whether CB and CD be in the same plane, or in parallel planes. P will therefore balance Q, if it will balance $S' = Q$ in its own plane of momentum.

45. *Prop. III. Theorem.* The momentum of the resultant of any number of forces acting in one plane, referred to a fixed point in that plane, is equal to the sum of the momenta of its constituent forces referred to the same; and that whether the directions of the forces be parallel, or inclined to each other.

Case 1. Let there be any number of parallel forces A, B, D, (Fig. 34.) acting in one plane, and from the point O, with reference to which the momenta are to be estimated, let there be drawn the straight line OE perpendicular to their directions, and meeting them in A, B, and D. This line being considered as a lever, let C be that point at which a fulcrum being placed, the forces A, B, and D would balance each other. The pressure on C will be the resultant, or the sum of the forces, estimated with their proper signs; and

$$A.AC + B.BC = D.D\,C;\ \text{that is,}$$

$$A(OC - AO) + B(OC - BO) = D(DO - OC);$$

or by transposition,

$$(A + B + D)OC = A.AO + B.BO + D.DO,$$

If the point O were situated between A and B, we should have $AC = OC + AO$, and the term $A.AO$ in the second member of the equation would become negative. This may be technically represented, by considering the distance AO in that case as negative. In like manner, whatever be the position of O in the line OE, the distances on opposite sides of it are, in taking the sum of the momenta with reference to it, to be estimated with opposite signs. Also in stating the equation of equilibrium as above, forces that act in opposite directions are to be taken with contrary signs.

Case 2. Let $ABDE$ (Fig. 35 and 36.) be a parallelogram, AB and AE representing two constituent forces, and AD their resultant; and from any point C in the plane of the figure draw CG, CF, CH perpendicular to the directions AD, AB, AE; join A, C, and on AC as a diameter, and in the same plane, describe a circle, which will pass through the points F, G, H, because the angles subtended by AC at these points are all right angles; join FH and GH. Then since

$$\angle GCH = \angle GAH = \angle ADB,\ \text{and}\ \angle CHG = \angle CAG,$$

the triangle CGH is similar to the triangle ADK. In like manner it may be shown that the triangle HCF is similar to the triangle ABK; whence

$$AD : DK = CH : CG,\ \text{or}\ AD.CG = DK.CH,$$

$$\text{and}\ AB : BK = CH : CF,\ \text{or}\ AB.CF = BK.CH;$$

and by adding equals to equals when C is within the angle BAE or its vertical angle, and taking equals from equals when it is without, we shall have

$$AD.CG \pm AB.CF = BD.CH = AE.CH,$$

$$\text{or}\ AD.CG = AE.CH \mp AB.CF,$$

a conclusion which may be expressed as in the enunciation of the proposition, if the momenta of the forces tending to produce rotation in opposite directions be marked with opposite signs.

It is obvious that the forces *AE* and *AB* being each resolved into two constituent forces, and so on indefinitely, the proposition will still be true.

46. *Cor.* 1. This proposition, in so far as it relates to parallel forces, suggests a ready method of finding their centre, or the point through which their resultant always passes. We have only to suppose the forces, while applied at the same points, to act perpendicularly on the line, and divide the sum of the momenta, referred to any given point in the same, by the sum of the forces, the distances and forces being both estimated as formerly with characteristic signs; the quotient will be the distance of the centre of parallel forces from the point in reference to which the momenta are calculated, and its sign will indicate the direction in which it is to be taken.

If the sum of the momenta with respect to any point be $= 0$, that point itself is the centre, and conversely.

Note. It is usual among writers on Statics, in treating of that part of the subject on which we are now entering, to designate by the term *momenta*, the products of parallel forces into their distances from a point, line or plane, without any immediate reference to them as measures of rotative energy; or, in other words, without supposing, as above, that the given parallel directions, when oblique, are changed into others perpendicular to the lines which measure the above-mentioned distances.

47. *Cor.* 2. If any number of parallel forces be applied to points which are all situated in the same plane, the sum of the momenta, with reference to any straight line in that plane, shall be equal to the momentum of their resultant with reference to the same.

Let the parallel forces *A*, *B*, (Fig. 37.) be applied to the points of the same name, in any plane passing through *A*, *B* let *MN* be a straight line to which are drawn the perpendiculars $A\,a$, $B\,b$, and from *C* the centre of parallel forces $C\,c$. If *AB* be parallel to *MN*, $A\,a = B\,b = C\,c$, and it is obvious that $A.Aa + B.Bb = (A+B)\,Cc$. If AB meets MN in O, then $A.AO + B.BO = (A+B)\,CO$, by Case 1., and $A.Aa + B.Bb = (A+B)\,Cc$, all the terms being reduced in the same ratio, viz. rad : sin. BON.

If points of application lie on opposite sides of O as B and A

$$B.BO - A.AO = (A+B)CO,$$
$$\text{and } B.Bb - A.Aa = (A+B)Cc.$$

This demonstration applies to any number of points of application which are in the same straight line. If A, B and D (Fig. 38.) be not in the same straight line, find C the centre of A and B, and E the centre of A, B and D, and draw Aa, Bb, Cc, Dd, Ee, all perpendicular to MN. Then by the first step in this case,

$$A.Aa + B.Bb = C.Cc,$$
$$\text{and } C.Cc + D.Dd = E.Ee;$$

that is, $A.Aa + B.Bb + D.Dd = (A+B+D)Ee$.

48. This suggests a convenient method of finding the centre of any number of given parallel forces applied to given points in one plane. Draw in that plane two rectangular axes of co-ordinates Ox, Oy, (Fig. 39.) Let P, P', P'', &c. be the forces, and R their resultant; that is, their algebraic sum. Let the perpendicular distances of P, P', P'', &c. and R from Oy, or the ordinates parallel to Ox be denoted by x, x', x'', &c. and $x_{,}$, and the co-ordinates of the same parallel to Oy, by y, y', y'', &c. and $y_{,}$, ∴ then

$$Rx_{,} = Px + P'x' + P''x'', \text{ \&c. or } x_{,} = \frac{\int Px}{R},$$

$$\text{and } Ry_{,} = Py + P'y' + P''y'', \text{ \&c. or } y_{,} = \frac{\int Py}{R}.$$

In the lines Ox and Oy take OA and OB, such that their numerical measures are equal to $x_{,}$ and $y_{,}$ respectively, and draw through A and B, AF parallel to Oy and BD parallel to Ox; the point G, where these lines intersect, is obviously the centre of parallel forces sought.

If the sum of the momenta with respect to any line be $= 0$, the centre is in that line; and conversely.

49. *Cor.* 3. If any number of parallel forces be applied to points in any positions whatever, the momenta of these forces, with reference to any plane, shall be equal to the momentum of their resultant with reference to the same.

Let A, B and D (Fig. 40.) represent the forces, and the points to which they are applied; and, without respect to their being, when of this number, necessarily in one plane, let $MNPO$ represent a different plane, to which are drawn from A, B and D the perpendiculars Aa, Bb, Dd; and from C the centre of the parallel forces A and B, and E the centre of the whole, perpendiculars Cc and Ee. The points a, c, b, are manifestly in one straight line, as are also the points c, e, d. (Playf. Eucl. Prop. 17. and def. 2. B. 2. Suppl.) Hence by *Cor.* 2,

$$A.Aa + B.Bb = C.Cc,$$

$$\text{and } C.Cc + D.Dd = E.Ee;$$

$$\text{that is, } A.Aa + B.Bb + D.Dd = (A + B + D)Ee;$$

and in the same way may the proof be extended to any number of forces.

50. This suggests a convenient method of finding the centre of any given parallel forces applied to any given points.

Assume three rectangular axes of co-ordinates AX, AY, AZ, (Fig. 13.) Let P, P', P'', &c. be the forces, and R their resultant. Let the perpendicular distances of P, P', P'', &c. and R, from the plane ZAY, be denoted by x, x', x'', &c. and $x_{,}$, the perpendicular distances of the same from the plane ZAX, by y, y', y'', &c. and $y_{,}$, and their perpendicular distances from the plane XAY by z, z', z'', &c. and $z_{,}$.

$$\text{Then } Rx_{,} = Px + P'x' + P''x'' \text{ \&c. or } x_{,} = \frac{\int Px}{R}$$

$$Ry_{,} = Py + P'y' + P''y'' \text{ \&c. or } y_{,} = \frac{\int Py}{R}$$

$$\text{and } Rz_{,} = Pz + P'z' + P''z'' \text{ \&c. or } z_{,} = \frac{\int Pz}{R}$$

The numbers $x_{,}$, $y_{,}$ and $z_{,}$, attention being paid to their signs, will give the position of three planes parallel to ZAY, ZAX, and XAY respectively, in each of which the centre of parallel forces must lie. It will, therefore, be the point of their intersection which is given in position.

If the sum of the momenta with respect to any plane be $= 0$, the centre is in that plane, and conversely.

51. *Cor.* 4. It is obvious, from the preceding investigations, that when the forces are given, and the points of their application, there is only one point which can be the centre of parallel forces.

52. *Cor.* 5. As in the demonstration of the 2d *Case* of this proposition the resultant may be considered as applied at any point in the line of its direction, let x and y be the co-ordinates of any point in that direction, parallel to axes of co-ordinates $A\,x$ and $A\,y$ respectively, and x' and y' the corresponding co-ordinates of any other point in the same; and let X and Y denote the constituent forces of the resultant R, the former parallel to $A\,x$, and the latter to $A\,y$; then shall

$$X\,y' - Y\,x' = X\,y - Y\,x,$$

each of these sums of momenta being equal to the momentum of the resultant.

53. *Cor.* 6. When the sum of the momenta of the constituent forces with relation to any point is $= 0$, the resultant passes through that point; and conversely. into

c. 69) 54. *Prop. IV. Theorem.* A momentum of rotation in any plane, with reference to a fixed point in the same, may be resolved into equivalent momenta with reference to that point, and in the planes of three rectangular axes passing through it, the momentum in each being represented by the cosine of the angle which it makes with the first mentioned plane, and that in the first mentioned plane itself by the radius.

From O, the fixed point, (Fig. 41.) draw a perpendicular (not here represented) to the direction of the force, and conceive O as the centre of a sphere, of which that perpendicular is the radius, and the intersections of whose superficies with the plane of the momentum to be resolved, and those of the three rectangular axes, are the circles to which belong the arch EF, and the quadrantal arches AB, BC, AC. The force P, whose momentum is to be resolved, may be considered as applied *tangentially* at any point in the circumference EF; for of two forces P', P'' each $= P$, and applied at that point tangentially, and in opposite directions, in the plane of momentum, one P'' may be considered as balancing either P' or P. Conceive then the force $= P$ to be applied at G, and resolved into the tangential forces A in the plane BC, and Q in the plane AGD. Then

$$A = P \cos. EGB = P \cos. \alpha,$$
$$Q = P \cos. EGD = P \sin. \alpha.$$

But Q may be considered as applied *tangentially* at A, and there resolved into B and C in the planes AC, AB, making with AGD angles measured by GC and GB.

Hence $B = Q \cos. GC = P \sin. \alpha \cos. GC = P \cos. \beta$,

$$C = Q \cos. GB = P \sin. \alpha \cos. GB = P \cos. \gamma;$$

and all the forces here considered acting by levers of equal length, and being therefore as their momenta, if R, X, Y, Z, denote the momenta of P, A, B, C respectively.

$$X = \qquad R \cos. \alpha,$$
$$Y = R \sin. \alpha \cos. GC = R \cos. \beta,$$
$$Z = R \sin. \alpha \cos. GB = R \cos. \gamma.$$

55. *Cor.* 1. $X^2 + Y^2 + Z^2 = R^2 (\cos.^2 \alpha + \sin.^2 \alpha (\cos.^2 GC + \cos.^2 GB)) = R^2 (\cos.^2 \alpha + \sin.^2 \alpha) = R^2$.

56. *Cor.* 2. A momentum of rotation about the diagonal of any parallelopiped, and represented by that diagonal, is equivalent to momenta of rotation about the three linear sides which meet in one of its extremities, and proportionally represented by these sides respectively.

For the three linear sides may represent the three rectangular axes OA, OB, OC, (Fig. 41.) which will make with the axis of EF angles equal to α, β and γ respectively; and any part of the latter axis being taken as the diagonal of the parallelopiped, and denoted by R, the three linear sides will be denoted by $R \cos. \alpha$, $R \cos. \beta$, $R \cos. \gamma$.

57. *Cor.* 3. When the momenta of all the forces applied to a solid body, are resolved into their equivalent momenta

of rotation about three rectangular axes, the sum of the squares of these equivalent momenta is the same, whatever be the positions of the said axes, provided their point of common intersection be always the same: also if the sums be taken of the momenta reduced to each of the rectangular axes, the sum of the squares of these sums is constant in the like circumstances.

The first part is obvious from *Cor.* 1. for $\int X^2 + \int Y^2 + \int Z^2 = \int R^2$, which is constant, the momenta to be resolved being given: the second, from this consideration, that the resultant of the same momenta must be always the same, by whatever legitimate resolution and recomposition, *i. e.* substitution of exact equivalents, it may be found. If a more detailed proof of the latter part be sought, it may be found by the following well known proposition in Spherics.

Let ABC (Fig. 42.) be a spherical triangle, whose sides AB, BC, AC, are quadrantal arches, and let K and K' be any two points on the surface of the sphere, connected with each other by the arch θ, and with A, B and C by the arches α, α'; β, β', &c. as in the figure.

$\text{Cos.}\,\alpha \cos.\,\alpha' + \cos.\,\beta \cos.\,\beta' + \cos.\,\gamma \cos.\,\gamma' = \cos.\,\theta$; For $\text{Cos.}\,\theta = \cos.\,\alpha \cos.\,\alpha' + \sin.\,\alpha \sin.\,\alpha' \cos.\,KAK'$, (by Sp. Trig.) and the arches AK, AK' being produced to D and D', $\sin.\,\alpha = \cos.\,KD$, $\sin.\,\alpha' = \cos.\,K'D'$, $\cos.\,KAK' = \cos.\,(BD' - BD) = \cos.\,BD \cos.\,BD' + \sin.\,BD \sin.\,BD' = \cos.\,BD \cos.\,BD' + \cos.\,CD \cos.\,CD'$, also $\cos.\,BD \cos.\,KD = \cos.\,\beta$, &c. whence, by substitution, the truth of the lemma will easily appear. Now

$$(\int X)^2 = X^2 + X'^2, \text{ \&c. } + 2XX', \text{ \&c.}$$
$$(\int Y)^2 = Y^2 + Y'^2, \text{ \&c. } + 2YY', \text{ \&c.}$$
$$(\int Z)^2 = Z^2 + Z'^2, \text{ \&c. } + 2ZZ', \text{ \&c.}$$

and as $XX' + YY' + ZZ' = RR' \cos.\,\alpha \cos.\,\alpha' + RR' \cos.\,\beta \cos.\,\beta' + RR' \cos.\,\gamma \cos.\,\gamma' = RR' \cos.\,\theta$,

it is evident that $(\int X)^2 + (\int Y)^2 + (\int Z)^2$ is $= \int R^2 +$ twice the sum of the products, found by multiplying together the numerical measures of each two of the momenta RR' &c.

and the cosine of the angle between their axes; and, the momenta to be resolved and the positions of their axes being given, the second member of the last equation is constant.

58. *Cor.* 4. Momenta may be resolved or compounded by a method perfectly analogous to that, by which forces applied directly to one point are resolved or compounded.

If the momenta be given and the directions of their axes, the rectangular axes being assumed given in position, $\int X$, $\int Y$ and $\int Z$ will be known,

$$(\int X)^2 + (\int Y)^2 + (\int Z)^2 = R^2,$$

R being now the finally resulting momentum; and if α, β and γ be the angles which its axis makes with those of X, Y and Z respectively.

$$\text{Cos. } \alpha = \frac{X}{R}, \text{ cos. } \beta = \frac{Y}{R}, \text{ cos. } \gamma = \frac{Z}{R},$$

or the resulting momentum is given in magnitude, and its axis in position.

59. *Cor.* 5. A momentum R in any plane, reduced to another plane passing through the same centre of momenta, and making the angle θ with the former, is, with respect to that centre, $= R$ cos. θ.

60. *Cor.* 6. If three momenta referred to the same point be in equilibrio, their planes intersect in the same line, and each momentum is represented by the sine of the angle contained between the planes of the other two.

61. *Prop. V. Problem.* To find the conditions of equilibrium in the case of parallel forces, applied to a solid body unconnected with any fixed point or fulcrum.

That such forces may be in equilibrio, it is plain that the resultants of those which have contrary signs must be equal and directly opposed to each other. These resultants being equal, if their momenta, with reference to any two intersecting planes, parallel to their directions, be equal, considered as if they had the same sign, their perpendicular distances from these planes will be the same; (50.) or each of them will be in each of the same two planes parallel to these, and there-

[illegible] section. But the [illegible] the momenta of its co[illegible] ons of equilibrium will be, [illegible] forces be = 0. 2. Thus the [illegible] with reference to two intersecti[illegible] be = 0.

[illegible] xed point or fulcrum, the [illegible] its resistance will supply [illegible] condition, or will [illegible] provided [illegible] when possible, [illegible] which the sum of the [illegible] therefore, requisite [illegible] = 0 with reference [illegible] fixed point parallel [illegible] straight line [illegible] parallel forces [illegible]

[illegible] find the resultant [illegible] given points, [illegible]

[illegible] angle = [illegible] through their points of application [illegible] their components parallel to AX [illegible] which suppose to be Y and which it passes, viz. that whose co-ordinates are $x_,$ and $y_,$, and the angle which its direction makes with a line given in position. It is, therefore, in every respect determined, due attention being paid to the signs of the quantities, except in the case of the forces being reducible to two which are equal in magnitude, with contrary signs, and not directly opposed to each other; and in this case, as formerly observed, (41.) no single resultant can be found.

64. *Prop. VII. Prob.* To find the conditions of equilibrium when forces, not all parallel to the same straight line, but all acting in directions coinciding with the same plane, are applied to a body unconnected with any fixed point or fulcrum.

There will evidently be an equilibrium, if one of the forces P be equal and directly opposed to the resultant R' of all the rest; otherwise not. Assume two rectangular axes of co-ordinates XAX', YAY', and let α be the angle which the direction of P makes with AX; α', α'', &c. those which P', P'', &c. make with AX'.

Then $P \cos. \alpha = R' \cos. \alpha = P' \cos. \alpha' + P'' \cos. \alpha''$, &c.

$P \sin. \alpha = R' \sin. \alpha = P' \sin. \alpha' + P'' \sin. \alpha''$, &c.

or if, as is usual, we refer the directions of P', P'', &c. as well as P to the same positive axis AX, the opposite of AX', we must add to each of the angles α', α'', &c. 180°, or, which is equivalent, prefix to each of the terms involving them the negative sign.

Then $\int P \cos. \alpha = P \cos. \alpha + P' \cos. \alpha'$, &c. $= 0$ (A)

$\int P \sin. \alpha = P \sin. \alpha + P' \sin. \alpha'$, &c. $= 0$ (B)

These equations (A) and (B), however, though necessarily resulting from the equality and direct opposition of the forces P and R', would also result from their equality and parallelism, and therefore are not sufficient conversely to determine the coincidence of their directions. Since they are in the same plane, and opposite to each other, their directions will coincide if they be on the same side of A, and at the same perpendicular distance from it; that is, if their momenta with respect to that point be equal and opposite, or the sum of the momenta of all the forces with reference to the same $= 0$; whence (45.)

[illegible] intersection. But the [illegible] moments [illegible] equilibrium [illegible] resistances [illegible] point of [illegible] resistance [illegible]

[illegible]

$$(1.)\ Xy - Yx = Xy_1 - Yx_2 = \int Py \cos.\ \alpha - \int Px \cos.\ \beta = L$$
$$(2.)\ Zx - Xz = Zx_3 - Xz_1 = \int Px \cos.\ \gamma - \int Pz \cos.\ \alpha = M$$
$$(3.)\ Yz - Zy = Yz_2 - Zy_3 = \int Pz \cos.\ \beta - \int Py \cos.\ \gamma = N$$

Here we may assign any given value to one of the co-ordinates x, y, z, suppose to x, and the two equations in which it is involved will determine the other two y and z. But as all the three equations hold when a resultant is possible, the values found must be such as will, by substitution, satisfy the third, or give $LZ + MY + NX = 0$. If this be not the case, we shall conclude that the forces cannot be reduced to one resultant.

When by the substitution mentioned the third equation is satisfied, we have

$$R = \sqrt{X^2 + Y^2 + Z^2}$$
$$\text{Cos. } a = \frac{X}{R},\ \cos.\ b = \frac{Y}{R},\ \cos.\ c = \frac{Z}{R}.$$

Thus the magnitude of R, the co-ordinates x, y, z, and the angles a, b, c, are all determined.

67. *Prop. IX. Prob.* To determine the conditions of equilibrium when any forces whatever are applied to any points of a solid body.

There will evidently be an equilibrium, if one of the forces P be equal and directly opposed to the resultant R of all the rest; otherwise not. Adopting the previous notation we find, as in *Prop. VII.*

$(1.) \int P \cos.\ \alpha = 0$, $(2.) \int P \cos.\ \beta = 0$, $(3.) \int P \cos.\ \gamma = 0$.
But these would result from the equality and parallelism of P and R. That their directions may coincide, those of their projections upon the three rectangular planes of the axes must coincide. Therefore, as in *Prop. VII.* we must have

$$(4.) \int Py \cos.\ \alpha - \int Px \cos.\ \beta = 0,$$
$$(5.) \int Px \cos.\ \gamma - \int Pz \cos.\ \alpha = 0,$$
$$(6.) \int Pz \cos.\ \beta - \int Py \cos.\ \gamma = 0.$$

68. *Cor.* If there be a fixed point or fulcrum, then what-

ever be the forces applied their resultant will be balanced by the resistance of that point, provided it pass through it; and this is the only condition of equilibrium in that case. Now the resultant will pass through it provided its projections upon the planes of the axes pass through it when it is taken for the origin of the co-ordinates; that is, provided (53.)

$$(1.)\ Xy_1 - Yx_2 = 0,$$
$$(2.)\ Zx_3 - Xz_1 = 0,$$
$$(3.)\ Yz_2 - Zy_3 = 0;$$

conditions which are identical with equations (4.) (5.) and (6.) above.

69. *Prop. X. Prob.* To determine the conditions of equilibrium when forces are applied to the angular points of a flexible polygon.

Let *ABCD*, (Fig. 43.) be the polygon fixed at *A* and *D*, and stretched by the forces *P* and *Q* applied at the angular points; the angles which the direction of *P* makes with the sides *AB* and *BC* being α, α'; and those which the direction of *Q* makes with *BC* and *CD* being β and β' respectively. Let *P* and *Q* be resolved into their constituent forces *E* and *F*, *G* and *H* in the directions of the sides produced. An equilibrium being established, the opposite forces applied to any side must be equal to each other. Therefore, since (21.)

$$P : E = \sin.\ (\alpha + \alpha') : \sin.\ \alpha$$
$$\text{and } Q : G = \sin.\ (\beta + \beta') : \sin.\ \beta'$$
$$\frac{P \sin.\ \alpha}{\sin.\ (\alpha + \alpha')} = E = G = \frac{Q \sin.\ \beta'}{\sin.\ (\beta + \beta')}$$

70. *Cor.* 1. If all the forces applied to the polygon, including the resistances of the supports at *A* and *B*, were applied, in parallel directions, to any one point, they would be in equilibrio.

For, after resolution as before, each would be destroyed by one equal and opposite.

71. *Cor.* 2. If all the forces, from one of the extremities *A* to any angular point *C*, including the resistance of *A*, were

applied at C, their resultant would be CH, the tension of the next following side CD.

72. *Cor.* 3. If the forces are parallel, they and all the sides of the polygon are in one plane, and $\acute{\alpha}$ is the supplement of β, whence $\sin.\ \acute{\alpha} = \sin.\ \beta$

$$\text{and } \frac{P \sin.\ \alpha \sin.\ \acute{\alpha}}{\sin.\ (\alpha + \acute{\alpha})} = \frac{Q \sin.\ \beta \sin.\ \beta'}{\sin.\ (\beta + \beta')}$$

$$\text{or } \frac{P}{\cot.\ \alpha + \cot.\ \alpha'} = \frac{Q}{\cot.\ \beta + \cot.\ \beta'}$$

as will appear by expressing the sines of the angles $\alpha + \alpha'$ and $\beta + \beta'$ in terms of the sines and cosines of their segments.

73. *Cor.* 4. $E \sin.\ \alpha'$ or $G \sin.\ \beta$ expresses the tension estimated in a direction perpendicular to the applied forces when parallel, which is thus proved (72.) to be constant.

74. *Cor.* 5. If the forces be parallel, a straight line drawn through their centre parallel to their directions, and the extreme sides of the polygon produced, shall meet in the same point.

Let the sides AB and DC produced meet in K, and draw KL parallel to BP or CQ meeting BC in L: then the forces in equilibrio at B and those at C may be represented by the sides of the triangles BLK and CLK respectively (20.) and

$$\left.\begin{array}{l} P : E \text{ or } G : Q \\ LC : \ LK \ \ : LB \end{array}\right\} \text{in prop. pert.}$$

$$\therefore\ P : Q = LC : LB \quad \text{(Eucl. v. 23.)}$$

or L is the centre of the parallel forces P and Q. If there be more sides of the polygon than are represented in this figure, we may conceive the sum of P and Q as applied at K, in the same directions as before; then BC being removed and the contiguous sides extended to K, these will have the same tensions as before, the equilibrium will remain undisturbed, and the distance of the centre of parallel forces from any given axis of co-ordinates parallel to BP, CQ, &c. will also remain without alteration (48.) Thus the proof may be easily extended to any number of sides.

75. *Cor.* 6. If any of the forces as P be applied by means of a ring or in any way admitting of its sliding freely along the flexible line, it is, further, a condition of equilibrium that the direction of the force shall bisect the angle of the polygon through which it passes.

Let MBM' bisect the angle ABC (Fig. 44.); there being no reason from the data why the connexion of the physical point B of the polygon with the sides AB and BC should be different, the resultant of the tensions will bisect the angle, or coincide with BM'. If the direction of the force P then be not opposite to this, there cannot be an equilibrium: for if BP represent P, draw PN and PM parallel to BM' and BC respectively; and, while BM is destroyed by the resultant of the equal tensions, BN will cause the ring to slide towards C.

76. *Cor.* 7. The tension of the contiguous sides is in this case $= \dfrac{P \sin. \alpha}{\sin. 2\,\alpha}$ (69.) $= \dfrac{P \sin. \alpha}{2 \sin. \alpha \cos. \alpha} = \dfrac{P}{2 \cos. \alpha} = \frac{1}{2} P$ sec. α.

77. *Cor.* 8. If the sides of the flexible or funicular polygon be conceived as multiplied indefinitely while forces are applied to each physical point, the limit of the polygon is what is called a funicular curve; and if the forces applied to the elements of the curve $F d s$, $F' d s$, &c. be resolved into $P d s$, $P' d s$, &c. parallel to the axis AX, and $Q d s$, $Q' d s$, &c. parallel to AY, these latter forces being conceived as substituted for the former, which will make no change in the statical condition of the curve, we may then consider those applied between the origin and any point C, or $\int P ds$ and $\int Q ds$, as transferred, in their respective directions, to that point, the tangential force at which will be that of which they are the co-ordinate constituents, (71.) If we consider as positive the force T with which the following element of the curve is drawn tangentially towards the origin of the co-ordinates, the elementary forces $P d s$, $Q d s$, &c. which act in the direction of the positive co-ordinates are to be taken with the negative sign, because such increments produce decrements of T. We

may refer *all* the forces to the directions of the positive co-ordinates, that is, mark all those with the negative sign which act in the opposite directions, and then prefix the negative sign to the integrals $\int P\,ds$ $\int Q\,ds$, when considered as constituents of T. We shall then have by similar triangles

$$dx : dy = \int P\,ds : \int Q\,ds$$
$$dx : ds = -\int P\,ds : T$$
$$dy : ds = -\int Q\,ds : T$$
$$dx^2 + dy^2\ (= ds^2) : ds^2 = (\int P\,ds)^2 + (\int Q\,ds)^2 : T^2$$
$$\text{or } T^2 = (\int P\,ds)^2 + (\int Q\,ds)^2$$

But, if we have no forces acting in the direction of the positive co-ordinates, our simplest way is to consider the actual directions of the forces as positive, and to take both the integrals with the positive sign.

To exemplify the application of these analogies, conceive the line *MN* (Fig. 53.) to be horizontal, and a fine chain or perfectly flexible physical line *MAN*, of uniform density, to be attached to the points *M*,*N*, and to form itself into a curve by its own weight. Suppose *A*, the lowest point, to be taken as the origin of the co-ordinates; let the weight of the unit of length of the chain be taken as the unit of weight; and, in reference to the same unit, let k express the constant horizontal tension, or the tension at *A*. Then taking the actual directions of the forces as positive

$$\int P ds = s,\ \int Q\,ds = k$$
$$dx : ds = s : \sqrt{s^2 + k^2}$$
$$dx = \frac{s\,ds}{\sqrt{s^2 + k^2}}$$

and the fluent corrected so as to give $x = 0$ when $s = 0$ is

$$x = \sqrt{s^2 + k^2} - k$$
$$dy : ds = k : \sqrt{s^2 + k^2}$$
$$dy = \frac{k\,ds}{\sqrt{s^2 + k^2}}$$

which being integrated so that $y = 0$ when $s = 0$ gives

$$y = k \log. \frac{\sqrt{s^2 + k^2} + s}{k}$$

or by substitution of the value of s from the preceding equation, so that y may be expressed as a function of x

$$y = k \log. \frac{x + k + \sqrt{x^2 + 2\,k\,x}}{k}$$

This curve is called the *Catenary.*

Suppose equal weights suspended from the points of the chain at equal and very small horizontal distances, as is nearly the case in forming the roadway of a suspension bridge: the sum of the weights applied from the origin to the termination of any ordinate y may then be denoted by y, and the application being treated as uninterrupted, and the weight of the chains and suspending rods being neglected, we have

$$d\,x : d\,y = y : k$$

$$k\,d\,x = y\,d\,y$$

$$\text{and } 2\,k\,x = y^2$$

so that as the case occurs in practice the figure of the chains will be a polygon approximating to the common Parabola.

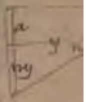
k in the former example is that length of the chain, and in the present that length of the roadway whose weight is equal to the horizontal tension.

If the depth of the inverted arch be h and half its span b, $2\,k\,h = b^2$ or $k = \frac{b}{4\,h}\ 2\,b$ so that the horizontal tension is $\frac{b}{4\,h}$ times the weight of the whole roadway. k being found, we have $\frac{\sqrt{k^2 + y^2}}{2\,b}$ for the tension at any point whose ordinate is y, and $\frac{\sqrt{k^2 + b^2}}{2\,b}$ for the same at either abutment or point of support, expressed, in both cases, as a multiple of the weight of the roadway. The latter may also be thus expressed by $\frac{\sqrt{b^2 + 4\,h^2}}{4\,h}$, and the angle of what we may call

$\frac{y}{k}$ $\qquad \frac{b}{} \times \frac{2h}{b^2}$

the dip, which gives the direction of this strain, is that whose Tang.$=\frac{dx}{dy}=\frac{2h}{b}$ at the pier. $=\frac{y}{k}=\frac{b}{\frac{k^2}{2h}}=\frac{2h}{b}$

If the weight for the unit of horizontal length at the lowest point of the curve be represented by a line *a*, and if it increase towards either side, as the breadth of a trapezoid the less of whose parallel sides is *a* and the inclination of the other two is the angle whose tangent $= n$; the fluxion of the weight will be $(a + ny)dy$, and $\int P ds$ for the arch whose ordinate is *y* will be $ay + \frac{1}{2}ny^2$. Hence $dx : dy = ay + \frac{1}{2}ny^2 : k^2$. Here we take k^2 to represent the horizontal tension, because in our present notation the unit of weight is that which corresponds to a unit of surface.



$$k^2 x = \tfrac{1}{2} a y^2 + \tfrac{1}{6} n y^3 = \tfrac{1}{6} (3a + ny) y^2$$

$$k^2 = \frac{3a + nb}{6h} b^2$$

or the horizontal stress is $\frac{3a + nb}{6h}$ times the weight of a prism of the given materials the area of whose end is b^2 and whose length is that of a transverse section of the roadway.

The tangent of the depression of the curve at the pier $= \frac{dx}{dy}$ for that point$=\frac{3h}{b} \times \frac{2a + nb}{3a + nb}$ and the horizontal stress multiplied by the secant of this angle gives the tension of the curve, or the tangential force, at the pier.

Of the Centre of Gravity.

78. In all our ordinary applications of the principles of Statics gravity may be regarded as constant, and as acting in parallel directions. Hence there must be in every body possessing weight a point through which the resultant of the gravitations of its individual particles will pass, whatever be the positions of any given points in it with relation to the constant vertical direction of this force. That point, the

[illegible] determinated the cen- [illegible], which its position is known, [illegible] cation into most of our me- [illegible] us to state the conditions of [illegible] masses of matter with the [illegible] can express those connected [illegible] as many single particles.

[illegible] of the Centre of Gravity.

[illegible] ported at its centre of gravity by a [illegible] weight it will remain in equilibrio [illegible] (13.)

[illegible] conditions of equilibrium, when a weight [illegible] concerned, we may always consider the [illegible] acting at the centre of gravity: [illegible] the same with the actual [illegible]

[illegible] supported at one [illegible] above or ver- [illegible] in equilibrio, (62.) [illegible] be indifferent to [illegible] point of support, [illegible] position will altoge- [illegible] difficulty of sup- [illegible] vertically [illegible] be destroyed. [illegible] latter has [illegible]

[illegible] verti- [illegible] within the [illegible] the point [illegible] the [illegible]

83. A body, whose centre of gravity is high, is more easily overset than one having the same base and whose centre is lower; that is, it will be overset by a smaller inclination, and, therefore, from accident more frequently.

Let G and G' be the centres of the rectangular parallelopipeds LC, AC, (Fig. 45.) and let a force, applied with the same mechanical advantage, tend to overset each of them, by turning it round a horizontal axis passing through C. To make them just ready to fall by their own weight the former requires to be turned through the angle GCB; the latter only through $G'CB$. Thus, a carriage loaded with wool, hay, straw, or the like, will be overset upon a transverse slope, across which one loaded with the same weight of stone, iron, or lead might be drawn with safety. On this principle is explained the danger of loading a carriage much above, or making it top-heavy as it is called: and of the passengers starting up, as they are apt to do, from sudden alarm, when it is ready to overset.

84. The statical properties of the centre of gravity account for many particulars in the motions and attitudes of animals, to which they are habituated by early training, and which are therefore formed or assumed with such rapidity as often to seem instinctive.

Geometrical Properties of the Centre of Gravity.

85. The sum of the products of the particles of a body into the squares of their distances from a plane parallel to another given in position is a minimum when the first mentioned plane passes through the centre of gravity.

Let MN, $M'N'$ (Fig. 46.) represent two parallel planes of which $M'N'$ passes through the centre of gravity of any body, G; and let A denote a particle of the body. Draw ABC and GD perpendicular to the parallel planes. Then

$$AC^2 = (AB \pm DG)^2$$

$$\text{or } AC^2 = AB^2 + DG^2 \pm 2\,AB.DG$$

$$A \,.\, AC^2 = A \,.\, AB^2 + A \,.\, DG^2 \pm 2\, DG \,.\, A \,.\, AB$$
$$\int A \,.\, AC^2 = \int A \,.\, AB^2 + m \,.\, DG^2 + 2\, DG \int \pm A \,.\, AB$$

where $m = \int A$, or the whole mass, the number of equal material particles.

$$\text{Now} \int \pm A \,.\, AB = 0 \text{ (50.)}$$

Therefore $\int A \,.\, AC^2 = \int A \,.\, AB^2 + m \,.\, DG^2$ and $\int A \,.\, AC^2$ will be least when $DG = 0$, that is, when $M'N'$ coincides with MN.

86. The sum of the products of the particles of a body into the squares of their distances from an axis parallel to a straight line given in position, is a minimum when that axis passes through the centre of gravity.

Let the plane of the fig. (Fig. 47.) represent that which passes through any particle A perpendicular to the axes, which meet it in C and G; the latter being a point in that which passes through the centre of gravity: and let MN, $M'N'$ represent parallel planes, perpendicular to that of the figure and to the plane XY, which, as passing through the axes, must also be perpendicular to the same. Join AC, AG, and draw the perpendiculars AB, ADE. Then

$$\int A \,.\, AC^2 = \int A \,.\, AB^2 + \int A \,.\, AE^2$$

and $\int A \,.\, AG^2 = \int A \,.\, AB^2 + \int A \,.\, AD^2$, consequently

$$\int A \,.\, AC^2 = \int A \,.\, AG^2 + \int A \,.\, AE^2 - \int A \,.\, AD^2$$
$$= \int A \,.\, AG^2 + m \,.\, GC^2 \quad \text{(85.)}$$

87. The sum of the products of the particles of a body into the squares of their distances from a given point is a minimum when that point is the centre of gravity.

Let G (Fig. 47.) now represent the centre of gravity, and C any other point. Let A represent the orthogonal projection of the position P of any particle upon a given plane passing through CG to which MN, $M'N'$ are perpendiculars as before. The point P, which is not represented, but supposed to be above or below A in a line perpendicular to the plane of the figure, being joined to C and G, we shall have

$$\int A \,.\, PC^2 = \int A \,.\, AC^2 + \int A \,.\, PA^2,$$
$$\int A \,.\, PG^2 = \int A \,.\, AG^2 + \int A \,.\, PA^2.$$

Hence the excess of $\int A \,.\, PC^2$ above $\int A \,.\, PG^2$ is equal to the excess of $\int A \,.\, AC^2$ above $\int A \,.\, AG^2$. But the latter excess was found to be $m \,.\, GC^2$, (86.) and the truth of the proposition is manifest.

2

88. The centre of gravity of a body is its centre of position.

Let the body consist of m equal particles A, B, C, &c. (Fig. 48.) whose perpendicular distances from any plane MN are Aa, Bb, Cc, &c. and let Gg be the perpendicular distance of the centre of gravity G from the same. By (50.)

or 49

$$Gg = \frac{A.Aa + B.Bb + C.Cc, \&c.}{A + B + C, \&c.} = \frac{A(Aa + Bb + Cc, \&c.)}{mA}$$

$$= \frac{Aa + Bb + Cc, \&c.}{m}.$$ So that the distance of G from any plane whatever is the mean or average distance of all the particles of the body from the same; and the position of the body is properly denoted by that of its centre of gravity, which is the meaning of the proposition.

Investigation of the Position of the Centre of Gravity in Bodies of various Figures.

In this investigation, when we speak of a line or of a surface, we mean a physical line, or physical surface, of uniform density unless it be otherwise expressed, and having as little of the dimensions which are excluded from the geometrical conception of a line and surface as the limits to the actual subdivision of matter may admit of.

89. The centre of gravity of a right line, or single row of particles connected by the force of cohesion, is evidently the point of bisection; for that is a point through which the resultant of the weights of every two equidistant particles must pass in all positions of the line.

90. If there be any number of lines connected as forming the perimeter of a polygon, or otherwise; suppose all the matter in each to be collected in its middle point or centre of

gravity, then the centre of the whole may be found by the method of momenta, (48, 49.)

The quantity of matter in each side is in this process to be expressed by its length, which will be proportional to the mass, the density being uniform. In other words, we take for the *unit of mass* the mass of the *unit of length.* We may also proceed thus: Let the figure be $ABCD$, (Fig. 49.) bisect the sides in E, F, H, K; join two of these points of bisection as E, F, and divide EF in L, so that $AB : BC = FL : EL$; join L, H, and divide LH in M, so that $AB + BC : CD = HM : LM$; join M, K, and find in the line which joins them the point G, so that $AB + BC + CD : AD = KG : MG$. Then, if there be no more sides, G is the centre of gravity of the given perimeter. If there are more sides, proceed as before.

91. Let ABC (Fig. 50.) be a triangular surface; and from A, one of the angular points, draw AD bisecting in D the opposite side BC. It shall also bisect all straight lines as FK, parallel to the base, and terminated by the other sides of the triangle; For

$BD : FH = AD : AH = DC : HK$, or $BD : DC = FH : HK$.

Now the whole triangular physical plane may be conceived as made up of physical lines, as FK, *i. e.* trapezoids of very small breadth, of each of which the centre is in the point of bisection; that is, in AD: the centre of the whole is therefore in AD. Also, if CE be drawn bisecting in E the side AB, the centre of the whole must for a similar reason be in CE. It is therefore in G, the point of intersection. Now let E,D be joined; ED will be parallel to AC (Eucl. vi. 2.) and BDE, BCA are similar triangles, as also EDG, GAC

$$\therefore DG : GA = ED : AC = BD : BC = 1 : 2$$

or, by composition,

$$DG : DA = 1 : 3.$$

92. *Cor.* If any parallel forces, applied to a straight line, with equal and indefinitely small intervals, be directly as the distances of their points of application from the one extremity, their centre is one-third of the length of the line distant from the other.

93. The centre of gravity of a parallelogram is the point in which the diagonals intersect one another. For in that point they also bisect one another, and by (92.) it is manifest that the centre is in each of them.

94. To find the centre of any polygon we may draw diagonals from one of the angular points, dividing the surface into triangles, find the centres of the triangles, (91.) and the centre of the whole by the method of momenta, (48.) or by the other method above described, (90.) representing the quantity of matter in each of the triangles by its area, or taking the mass of the unit of surface for the unit of mass.

For regular polygons a shorter method will readily occur. If the number of sides be even, draw two diagonals joining opposite angles; if odd, draw from two of the angular points perpendiculars to the opposite sides; in either case the point of intersection of the lines so drawn will be the centre.

95. In prisms and cylinders the centre of gravity bisects the straight line joining the centres of gravity of their opposite ends. For a line so situated will pass through the centres of the laminæ parallel to the ends into which the solids may be conceived as divided; and, the matter of each equal lamina being conceived as concentrated in the physical point of intersection, the line referred to will be of uniform density, and its centre of gravity will be its middle point, (89.)

96. In pyramids and cones, join the vertex and the centre of gravity of the base, and divide the intercepted line so that the segment towards the base may be one-fourth part of the whole; the point in which it is so divided is the centre.

Let *VABC* be a triangular pyramid, of which *V* is the vertex. Let *VBD*, *VCE* be two planes whose intersections with the base *BD* and *CE* bisect the sides *AC*, *AB* in *D* and *E*. It may be easily demonstrated that their intersections with the plane of a section parallel to the base *abc* bisect *ac*, *ab* in *d* and *e*, (Playf. Sup. ii. 14. and Eucl. vi. 4.) and, consequently, their common section *VG* passes through the centres of gravity of the base and all the physical planes parallel to it, of which the solid may be considered as composed. The centre of gravity, then, of the whole solid is in this line.

Take $EF = \frac{1}{3} EV$ and join C,F. The centre of the whole will also be in CF, and therefore in the point H where it intersects VG, and if G,F be joined

$$GH : HV = GF : VC = EG : EC = 1 : 3$$

$$\text{or by comp. } GH : GV = 1 : 4.$$

Again, let there be a quadrangular pyramid, whose vertex is V, (Fig. 52.) and whose base is divided into two triangles A and B, the centres of gravity of which are represented by G and F respectively. Join G, F, and let GH be to HF as B to A; H will be the centre of gravity of the base. Let $g\,h\,f$ be the intersection of the plane VGF with a plane parallel to the base, and whose altitude above it is one-fourth part of the altitude of the pyramid. The centres of gravity of the two triangular pyramids whose bases are A and B will be g and f by what has been just demonstrated and (Playf. Sup. ii. 16.), and $hf : gh = HF : GH = A : B$; that is, since the pyramids have the same altitude, as the pyramid whose base is A to the pyramid whose base is B, (Playf. Sup. iii. 15. *Cor.* 2.) and h will be the centre of gravity of the whole quadrangular pyramid, (46.) Now $H\,h = \frac{1}{4}\,HV$ (Playf. Sup. ii. 16.)

In the same way may the demonstration, obviously, be extended to all other pyramids, and to cones considered as pyramids the number of whose sides is indefinitely great.

25 97. *Cor.* If any parallel forces, applied to a straight line with equal and indefinitely small intervals, be directly as the squares of the distances of their points of application from the one extremity, their centre is one-fourth of the length of the line distant from the other.

98. Every physical solid, bounded by plane surfaces, may be divided into pyramids, and its centre of gravity may be found by the process now described (96.) and the method of momenta, these being taken with reference to the planes of three rectangular axes, (50.)

The investigation of a general rule to find the centre of gravity of curve lines or curve surfaces, and of planes or solids bounded by them respectively, in whole or in part, requires an application of the higher geometry.

99. Let s be the sum of the momenta estimated with reference to any point, straight line, or plane, up to a certain distance x; s' the sum up to any greater distance X; also let m and m' denote the quantities of matter between the given point, line, or plane and another point or parallel line or plane at the distances x and X respectively. The increment of the sum of the momenta, being the sum of the products of every particle in the mass $m'—m$ into its distance, will always be greater than if all the distances were equal to the least, but less than if they were all equal to the greatest; that is,

$$s'-s > (m'-m)x \text{ but } < (m'-m)X$$

$$\text{or } \frac{s'-s}{m'-m} > x \text{ but } < X.$$

But, if the cotemporary increments be continually diminished, X approaches to x as a limit; therefore x is also the limit of $\frac{s'-s}{m'-m}$ which is always, as we have just seen, intermediate between X and x, that is, $\frac{ds}{dm} = x$ or $ds = x\,dm$ and $s = \int x\,dm$; whence, the distance to the centre of gravity, $D = \frac{\int x\,dm}{m}$.

If its distance be found in this way from the planes of three rectangular axes, its place will be determined, (50.) If the particles of matter are all in one physical plane, we need only two axes of momenta: and if they be symmetrically arranged with respect to any straight line, so that the parts on each side are perfectly equal and similar, we need only one.

100. The general formula now investigated may be conveniently accommodated to cases of the latter description, as follows. Let MAN (Fig. 53.) represent a curve line whose axis of abscissæ is AX, and whose ordinates DE are all bisected by that axis; the particles of the curve at the extremity of each ordinate will have their centre of gravity in the

tance from C to the centre of gravity of the whole will be $\frac{r' c'}{z'} = \frac{\frac{2}{3} r . \frac{2}{3} c}{\frac{2}{3} z} = \frac{\frac{2}{3} r c}{z}$.

109. The centre of gravity of a segment of an ellipse is the same as that of the corresponding segment of the circle described upon either axis, the base of the segment being parallel to the other axis; for, the corresponding ordinates of the two curves being in a constant ratio, the numerator and denominator in the value of D will be changed in the same ratio by the substitution of the one for the other.

110. Let EAD represent a segment of a spherical shell,

$$d z : d x = r : y \text{ or } y d z = r d x$$

$$\text{and (103.) } D = \frac{\int r x d x}{\int r d x} = \tfrac{1}{2} x$$

111. Convex surface of a right cone. Here $x : y$ is a constant ratio, as is also $d z : d x$

$$\text{or } y = m x$$
$$d z = n d x$$

$$\text{Hence (103.) } D = \frac{\int x^2 d x}{\int x d x} = \tfrac{2}{3} x = \tfrac{2}{3} a$$

for the whole cone whose axis $= a$.

112. For the centre of gravity of the cone's solidity investigated in this way, let $y = m x$; then (104.) $D = \frac{\int x^3 d x}{\int x^2 d x} = \frac{3}{4} x = \frac{3}{4} a$ for the whole cone, agreeing with the deduction by a former method, (96.)

113. Let DAE be a paraboloid, the equation of whose generating curve is $y^2 = a x$; $D = \frac{2}{3} x$.

114. Let DAE represent a spherical segment whose radius $= r$,

$$D = \frac{8 r - 3 x}{12 r - 4 x} x = \tfrac{5}{8} r$$

when the segment is a hemisphere.

115. The same is the value of D for a segment of an ellipsoid if r be the semiaxis of revolution.

116. Let $CEAD$ represent a spherical sector, the sagitta of whose terminating segment is x. By conceiving the solid to be divided into an indefinite number of very slender cones or pyramids, and proceeding in a way analogous to the second method, (108.) we shall find the distance from the centre of the sphere $= \frac{3}{4} r - \frac{1}{2} \cdot \frac{3}{4} x$ (110.) $= \frac{1}{8} (6 r - 3 x)$ or $D = \frac{1}{8} (2 r + 3 x)$.

Geometrical Properties of the Centre of Gravity demonstrated by the Fluxional Calculus.

117. Let MAN (Fig. 54.) be any line, straight or curved, x, y the co-ordinates, and Y the distance of the centre of gravity from the line BC, $Y = \frac{\int y\, d z}{z}$ $\therefore 2 \pi Y z = 2 \pi \int y\, d z = \int 2 \pi y\, d z$. Now $2 \pi y\, d z$ is well known to be the fluxion of the surface of revolution described by the line MAN about BC as an axis. Therefore,

Any surface of revolution is equal to a rectangle under the generating curve and the path described by its centre of gravity.

118. Suppose now Y to denote the distance from BC to the centre of gravity of the surface $MBFN = S$,

$Y = \frac{\frac{1}{2} \int y^2\, d x}{S}$ or $2 \pi Y S = \int \pi y^2\, d x =$ the solid of revolution described by the area S, and $2 \pi Y$ is the path of its centre of gravity in making a complete revolution. Hence,

The solid generated by the revolution of a plane surface is equal to that whose measure is the product of the surface itself into the path described by its centre of gravity.

It is obvious that, if the revolution of the line (117.) or surface (118.) be incomplete, the surface or solid described will be as the angle of revolution; for when, in addition to the other data, the angles are equal, every determining circumstance is the same.

119. There are many cases in which, from the irregularity of the figures of bodies, and their irregular or unknown density, we cannot find the position of the centre of gravity by either of the methods that have been described. We may, however, consider an irregular body as divided into small parts which may be accounted regular, and, from a near approximation to the centres of the separate parts, find also nearly the position of the common centre of the whole. The same thing may also be often done more conveniently by experiment, thus:

120. Suspend a body by a thread, or wire, which its weight is sufficient to stretch, attached to two different points of it successively, and mark the direction of the line of suspension in both cases: the intersection of the two directions will determine the centre of gravity.

It is often sufficient for the purpose in view to know its distance from either extremity of a body. This may be found by balancing it on the edge of a prism. The centre of gravity is then vertically above the edge. Or, if its weight w be known, we may allow one end to rest upon a prism or axis, and balance the body by a weight p made to act upwards at a given distance a from the fulcrum by means of a balance or a pulley: then, if x be the distance from the fulcrum to the centre of gravity, $w x = p a$ and $x = \frac{p}{w} a$.

Of Machinery.

121. Having now treated, at as much length as we can allot to that department, of the general principles of statics, we are next to proceed to some of the most useful and interesting applications of them; and first, to the investigation of the conditions of equilibrium in machines composed entirely of solid matter. All machinery of this description, however

complicated, may be reduced to a few simple constituent elements, which are usually denominated the Mechanic Powers. Their names are the lever, the pulley, the wheel and axle, the inclined plane, the wedge, and the screw.

122. Our object in the employment of any one of these, or any combination of them, is some advantageous modification of two opposed forces, by means of others supplied by the machine itself, or rather its solid supports. Of the two opposed forces, the action of either of which may be conceived as, in its modified state, directly opposed to the other, the one is usually called the *power*, and the other the *resistance*. In the statical consideration of machinery our object is to ascertain what pressure is exerted by either at the point where the other is applied, or, in other words, what must be the ratio of two forces that, by the intervention of a given machine, they may balance each other. When the resistance is given, if we calculate, by this ratio, the force which it opposes directly to the power, we then know what unbalanced power is left to overcome other resistances, and to produce motion and useful effect. In a few cases our object is, simply, to produce an equilibrium; but, in general, it is to produce a modified motion. To calculate the effect of such motion in given circumstances belongs to the science of Dynamics: but, for the reason above stated, the statical consideration of machinery must precede the dynamical. We shall first, then, trace the conditions of equilibrium in the mechanic powers considered separately, and afterwards show how to calculate the statical effect of given combinations of them. In the course of this investigation we shall find that the analysis might be carried farther, and that all the six above mentioned are reducible to two,—the *lever* and the *inclined plane*.

1. *The Lever.*

123. The lever is, in theory, considered as an inflexible rod or bar moveable about a fulcrum or point of support, so

that a power applied to the one extremity may balance a resistance applied to the other.

124. The conditions of equilibrium in this power have been already explained in the general doctrine of parallel forces. The most convenient statement or formula for the student's recollection seems to be that, in every variety of the machine, there will be an equilibrium when the products of the power and the resistance into the perpendiculars drawn from the fulcrum to their respective directions are equal; or when

$$Pa = Qb$$

a being the perpendicular distance from the fulcrum to the direction of P, and b that which is drawn to the direction of Q.

125. If we take into view the weight of the lever itself, we may consider it as collected in its centre of gravity. Let the distance of this point from the fulcrum be c, and W the weight of the instrument; then $Pa = Qb + Wc$, when the centre is on the same side of the fulcrum with Q; and $Pa = Qb - Wc$, when it is on the other side.

126. Levers are usually divided into three kinds. The *first* is when the fulcrum is between the power and the resistance; the *second*, when the resistance is between the power and the fulcrum; the *third*, when the power is between the resistance and the fulcrum. An obvious example of the first kind is the iron bar commonly used for raising stones. To the second kind may be referred the oars of a boat, the resistance of the water serving as a momentary fulcrum, in each position of the impelling oar, and the resistance which the boat meets with in passing through the water being the obstacle to be overcome. We find the same kind of lever combined with the wedge in a kind of knife having a long handle and a joint at the farther end, immediately beyond the edge. So far as this instrument operates, in cutting, by multiplication of pressure, it is a lever; so far as its energy depends on its edge merely it belongs to the wedge. To the third kind of lever may be referred spring shears and tongs. When a man rears a ladder, by placing one end of it against

a wall, it is first a lever of the second kind, and then of the third. A drawbridge consists of two levers, one of the first and one of the second kind united.

127. A machine is said to give a *mechanical advantage* when it enables a power to balance a resistance greater than itself. We shall use the phrase in that sense, although the real purpose of a machine, or, in other words, the real advantage which it presents, may be a diminution of power, or the knowledge of the equality of two powers, or of the ratio they bear to each other. The first kind of lever then may give, to what we call the power, either a mechanical advantage or the contrary. In the second, when the directions of the forces are parallel, it has always a mechanical advantage, and in the third a mechanical disadvantage.

128. If we regard the lever as without weight, the farther the power is from the fulcrum, *cæteris paribus*, the greater is the energy with which it acts, and that in the exact ratio of its perpendicular distance; but, if we consider the case of an actual lever, which must be a heavy body, its weight will alter that ratio. In the first kind the weight will generally conspire with the power, and will aid it the more the greater the length of the arm by which it acts; for the weight of that arm will be increased, *cæteris paribus*, by lengthening it, and the centre of its gravity will be farther removed from the fulcrum, so that, on both accounts, the momentum of the weight Wc will be increased. In the second kind, which we shall here suppose of uniform thickness and density, the extension of the arm by which the power acts will, as before, augment its energy; but the increase of the weight of the instrument will always more and more counteract this effect; and there is a limit to the advantageous increase of the length, or a particular length which in each case gives the greatest practical advantage to the power.

129. Let P and Q be as before the power and the resistance, a and b the arms by which they act, c the distance from the fulcrum to the centre of gravity of the unloaded lever, W its whole weight, and g its specific gravity, which

we may here take as the weight of the unit of length, the other dimensions being supposed to be constant. Then $W = g a$ and $c = \frac{1}{2} a$. Hence $P a - \frac{1}{2} g a^2 = Q b$ or $P = \frac{1}{2} g a + \frac{Q b}{a}$. For a given value of Q and b then, P admits of a minimum, which may be found by the usual fluxionary method, or by this consideration, that, in every case when the product of the numbers representing two quantities is constant, their sum is a minimum when they are equal. Thus in the present case $\frac{1}{2} g a = \frac{Q b}{a}$ and $a = \sqrt{\frac{2Qb}{g}}$, or $\sqrt{2 Q b g}$ is the minimum value of P. For the principle here employed the student may be referred to Eucl. B. ii. prop. 5. from which it appears that a given rectangle is formed by the two segments of the least line, or two lines whose sum is a minimum, when these segments or lines are taken equal. Or it may be more directly proved thus. Let $m = \frac{1}{2}$ the sum and $n = \frac{1}{2}$ the difference of two numbers, and let $(m + n).(m - n) = a^2$ or $m^2 = a^2 + n^2$; m and $2m$ will be least when $n = 0$.

130. By a lever the proportional strength of two persons may be ascertained with tolerable accuracy; and a burden carried by them conjointly may be equalised, or proportioned to their respective capacities of bearing a load. Thus let P and Q (Fig. 55.) draw the bar AB upwards against a fulcrum C, and slide the bar along till they find that position of it by which, when exerting their utmost force, neither can prevail, the strength of P is to that of Q as BC to AC. Suppose again AB to represent a pole or a hand-barrow, carried by P and Q, and a weight W to be laid upon it, it ought to be so laid at C that $BC : AC = P$'s strength : Q's strength. So, if C represent a ring connected with the beam of a plough, and which may be slid into different grooves or notches in the bar, the draught may be equitably divided between two horses of unequal strength.

131. To the lever are referred two instruments in very common use, the balance and the steel-yard. Of the first of

these we shall treat in some detail, as it is an instrument of the greatest use in various departments of experimental philosophy, and in the business of ordinary life. The balance is a lever of the first kind. It consists of a beam turning upon edges of tempered steel, a little rounded and resting upon planes or cylindrical grooves of the same or some other hard and polished substance. The two arms measured from these edges to the points at which the scales are appended are equal, and the whole is so adjusted that the equality between a weight and its counterpoise shall be indicated by the position of the beam being horizontal, or, in other words, by making the figure such that the straight line joining the points of suspension shall be perpendicular to the plane of the centre of gravity of the beam and its axis of revolution. In the direction of a straight line drawn from the centre of gravity at right angles to the line joining the points of suspension is a slender projecting pointed rod of metal called the *index* or *tongue* of the balance. When that line is horizontal the tongue will be vertical, and conversely; and the vertical position of the tongue is easily ascertained by its coincidence with the pendulous fork called the *cheeks*, which supports the beam, or other means furnished by the artist in the original construction and adjustment of the instrument. It is of no consequence to the accuracy of the instrument, unless it be as causing an unnecessary addition to the friction of the axis, though the two arms should be unequally heavy, provided they are of equal length; only should the one be heavier than the other, it must be counterpoised before we proceed to weigh.

132. The two principal requisites in the construction of a good balance are, that it possess *delicacy* or *sensibility* and *stability*. The former implies that a small excess of weight on either side shall be indicated by a sensible preponderance, or deviation of the index from the vertical line, and the latter that there is a force sufficient to render the state of equilibrium *stable* when it is once attained, or which shall restore [illegible] may here [illegible] in [illegible]

[illegible]

in the present case [illegible]

is the minimum value [illegible] ed the student may [illegible] which it appears that [illegible] segments of the leas[illegible] mum, when these [illegible] may be more dire[illegible] $n = \frac{1}{2}$ the differe[illegible] a' or $m' = a$ [illegible]

130. By a [illegible] may be ascerta[illegible] carried by the[illegible] to their respe[illegible] and Q, Fig. [illegible] C, and [illegible] which, [illegible] th[illegible]

[illegible] B in [illegible] cen- [illegible] the [illegible] repre- [illegible] small [illegible] and, there [illegible] OF, and

$$+ M \text{ [illegible]} = \text{[illegible]} . OF.$$

[illegible] perpendicular to it, and [illegible] angle of in- [illegible] H.

$$\text{[illegible]}\, q . AC . \text{[illegible]} - q . OC . \varphi$$

$$\text{[illegible]}\, H - CK$$

$$\text{[illegible]} = q . AC . \text{[illegible]}$$

$$\frac{q . AC}{\text{[illegible]}\, OC + M . OG}.$$

[illegible] tangent of the inclination, [illegible]eater the greater the length [illegible], the less its load and the [illegible]

[illegible] be in equilibrio, the beam [illegible], the momentum of the force [illegible] horizontal position, or the [illegible] $(H . OC + M . OG)\, \varphi$. To [illegible] quantity should be as great [illegible] sensibility of the instrument, [illegible] as small as possible. It [illegible] quantities in the extreme at

[illegible], as they sometimes [illegible] being still below [illegible] by $M . OG -$ [illegible] may be a good one; for greater ones it will be sluggish; and for greater still it will be indifferent to any position, or will overset by the slightest inclination according as $W.OC$ is equal to $M.OG$ or exceeds it.

137. If G falls above the axis and C below it, the measure of the stability is $(W.OK - M.OG)\phi$. Hence it may be a C
good balance for great weights, but will be sluggish for those that are less, and when $W.OC \angle M.OG$ it is useless.

138. A balance in which both C and G coincide with O, or both fall above it, is useless. If either of them fall above it, it cannot be generally useful. The best forms are when one of these two points coincides with the axis and the other falls below it; and, of these, the one or the other will be preferred according to the object in view.

139. Let G coincide with O, and C fall below it. The stability is now expressed by $W.OC.\phi$, or for any given inclination by $W.OC$, which increases as the imposed weights increase, and, therefore, as the friction of the axis, the force which it has to overcome, increases. The instrument in this much
form is consequently ~~fit only~~ for the coarser processes of fitte
weighing, or the estimation of great weights. It does not, however, so sensibly indicate a given excess of weight when the load is great as when it is small. The tangent of inclination being $= \dfrac{q.AC}{(W+q)\,OC}$ it will indicate with equal sensibility differences proportional to the load which it carries.

140. Now, let C coincide with O, and G fall below it. In this form the stability is constant with a given inclination, and is expressed by $M.GO$. But the friction of the axis increases with an increase of weight, and the restoring force may be at last counterbalanced. It is accordingly most used in delicate experiments, or in weighing small commodities. The less GO is, the less will be the stability and the greater the sensibility; but it will take the longer time to weigh any thing accurately, that is to say, with all the precision that the ations will be slower.

$$= \frac{q . AO}{M . OG}.$$

arately may be accele-
ht line or arm attached
pare the excursions of
. If these appear une-
ht in one of the scales,
til the balance settle in
ends; and thus continue to
pear equal. If the instru-
sure that the counterpoise is
e may proceed in a different
loaded beam to settle in the
ends, and observing by the in-
n. By the last equation $OG =$

. Thus, if we find the weight of

ance, measure half its length be-
n = AO, and observe the inclina-
eight q, we shall find OG, and
the last equation. The process

constant quantity, which call

ve the angle of inclination we
ghts, and know at once what to
process may be accelerated by
e difference of two succeeding
posite sides will be very nearly
en the beam comes to rest.

es, they must be enclosed in
gitation by the air. Some of
, when loaded with several
part of the weight or less.

been here delivered on the
tance of a paper *De Bilan.*

by Euler, *Com. Petrop.* tom. x. See also Biot's *Trait. de Phys.* and Nicholson's *Chem.*

143. *False Balance.*—A false balance is one whose arms are of unequal length, used as if they were equal, for the purpose of giving an undue advantage to him who employs it. The one arm may be longer than the other, and compensation may be made by the thickness, or by the difference of the scales, so that it shall appear perfectly in equilibrio when not loaded. If the weights are then put into the scale appended to the shorter arm, less than an equal weight of goods in the other scale will balance them, and the purchaser will be defrauded. If the arms are as 21 : 20 the seller will gain five per cent. upon all the goods sold, from this cause alone. In a balance which is just if properly used, the cords may be entangled about one end of the beam so as virtually to shorten it and produce the same effect. This, however, is too obvious to escape the notice of any but the most ignorant.

144. The equality of the arms, or the justness of a balance, is easily ascertained by first counterpoising a weight exactly and then making the weight and counterpoise exchange places, care being taken to have the beam and scales in equilibrio before the process of weighing is begun, or to make the scales exchange places, if moveable, along with the weights which they respectively contain. After this transference the greater weight must necessarily be appended to the longer arm, and consequently will, if *sensibly* greater, preponderate.

145. The true weight of a body may be easily discovered by a balance with unequal arms. Let x be the true weight, at present unknown. Weigh the body in each scale, and let P be its counterpoise when in the scale F, and Q when in the scale E, (Fig. 16.)

$$P : x = FC : EC$$

$$x : Q = FC : EC$$

$$\therefore\ P : x = x : Q,$$

and the true weight is the geometrical mean of the two counterpoises. If this experiment be carefully made, the ratio of

ghing will
$= \sqrt{P}\sqrt{Q}$.
we should
of the nomi-
difference in the
actual mean may be

a false balance,
scale, and an exact
ce in the other; re-
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restored. Let the
evident that w and w'
or, more generally,
the present case, that
for they then balance
circumstances. The
attention to which is
is required, is, that a
qual to that of its own
ospheric pressure, as is
es; and thus bodies of
ts; and the difference is
of the atmosphere.
of weighing is now pre-
required, as the most ac-
fact a false one, for no
of the same length with
ere be any balance with-
mode of weighing, how-
are as far as possible
ed are placed in the same

ed to gain a little stabi-
bility in a balance, some
small sliding piece at-

to the index, the motion of which up or down raises or depresses the centre of gravity.

149. In all cases, the instrument is aided in overcoming the friction, or rendered more sensible, by communicating to it a slight tremulous motion. as by gently striking the table, or still better by holding a piece of hard wood or the like, in contact with some part of its stand, and drawing over it the teeth of a fine saw. or notched edge of any kind. The following causes probably contribute to this effect. If there be a preponderance on one side while the beam, in consequence of friction, refuses to quit the horizontal position, the centre of gravity of the whole must be on that side of a vertical plane passing through the axis on which is the greater weight; and the small succussions communicated to the axis will tend to make the whole, while momentarily detached, revolve about that centre. Besides, the beam and its weights descend *with an impulse*, by which the excess of weight may overcome a resistance that would balance its pressure. We can much more easily *strike* an axe or chisel into a resisting cleft than we can *press* it into it. The simple process of driving and pulling out a nail affords another familiar example.

150. We may just observe, in conclusion of what relates to the common balance, that in philosophical experiments we seldom require to know the absolute so much as the relative or proportional weights of bodies; and consequently it is of little importance whether the arms be exactly equal or not, provided we always put the things to be weighed into the same scale. This will still give us truly the *ratios* of their weights.

151. *The Steelyard.*—This is a lever of the first kind, whose arms are unequal, and in using which the body to be weighed is suspended from the shorter arm at a fixed distance from the fulcrum, and counterpoised by a moveable weight, which slides upon the longer arm, and consequently in weighing different quantities of matter, is placed at different distances from the fulcrum. Let a constant weight be suspended from the shorter arm to balance the longer. If we then divide

the longer arm into parts of equal length with that by which the weight P acts, and number them 1, 2, 3, 4, &c. beginning from the axis, Q at the distance 1, 2, 3, &c. will balance Q, $2Q$, $3Q$, &c. respectively; or P will be the same multiple of Q, that the arm by which Q acts is of that by which P acts.

152. The steelyard has this advantage over the common balance in estimating great weights, that the load upon the fulcrum is less, which diminishes the friction. It has, however, this disadvantage, that the longer arm is apt to bend while the shorter does not, at least sensibly; whereas the arms of the balance being made as equal and similar as possible, are likely to suffer flexure in the same degree.

153. A kind of steelyard is said to be employed in some of the northern kingdoms, as Denmark and Sweden, in which both weights are applied at fixed points, and the fulcrum is moveable.

AB, (Fig. 57.) is a smooth cylindrical beam, with a constant weight A at the one end, and a hook or scale B at the other, to which the goods to be weighed are appended. Let Q be the weight of the goods, and P that of the whole beam with its hook or scale, whose common centre of gravity suppose to be E. The position of equilibrium is found by sliding the beam through a ring C by which it is suspended; then $BC : EC = P : Q$, and P being known, we get Q in terms of it. The graduation of EB may be performed experimentally with known weights, or by geometrical construction thus.

Let EB, (Fig. 58.) be the line to be divided, draw from E and B any two parallel lines EH, BD on opposite sides of EB; let $BD = EF = FG = GH$, &c. and draw DF, DG, DH, &c. intersecting EB in the points 1, 2, 3, &c. When the beam is balanced and the ring is at 1, $Q = P$; when it is at 2, $Q = 2P$, and so on; for $E1 : B1 = FE : BD$ or in a ratio of equality; $E2 : B2 = EG : BD = 2 : 1$; $E3 : B3 = EH : BD = 3 : 1$, &c. The lines BE, $B1$, $B2$, $B3$, &c. are in harmonic progression or proportional to the reciprocals of a series of numbers in arithmetical progression. The same divisions being transferred to the other half of the line will indicate what part Q is of P. This is a very imperfect instrument, and unsuitable where accuracy is required; but it gives with despatch such an approximation to the true weight as may be sufficient in the sale or barter of cheap commodities.

154. *Bent-lever Balance*, (Fig. 59.) This is also a species of steelyard. AG represents a quadrantal arch attached to the stand BH. BF is a rod moveable round an axis B, to which may be attached a small wheel or pulley. The rod BF is loaded with a weight, and their joint centre of gravity we suppose to be E. Of this weight the momentum will vary with its position being always $= W.BL$; where W is the weight that may be regarded as concentrated at E. Thus a weight Q suspended at D may be balanced provided its momentum $Q.BC$ do not exceed $W.BA'$. The graduation is most conveniently performed by suspending known weights at D, as 1, 2, 3, &c. ounces or pounds, and marking with the numbers 1, 2, 3, &c. respectively, the points at which the index stands upon the arch. Whatever body we afterwards suspend at D, if the index stand at 3, we are sure that it is 3 oz. or 3 lbs. or in general 3 of such units of weight as were originally employed in the graduation.

155. Employing this method of graduation we may estimate weights also by their effect in dilating a spring. We have thus what is called a spring steelyard. It does not, however, class with the mechanic powers, and has nothing in common with the ordinary steelyard but the purpose to which it is applied. The common steelyard may be employed, as formerly explained, in comparing the force of certain mechanical agents. The spring steelyard too may be employed as a dynamometer, either in the way of dilatation or of compression, and that in cases where the common steelyard is inapplicable. Thus by interposing one or more of them between a horse and his draught we may ascertain which of two carriages of different construction is drawn at

a given rate with the most facility. In like manner, if two boats are attached to a vessel under sail by ropes into which two well graduated instruments of this kind are introduced, we shall learn which of the two meets with the less resistance from the water, and the absolute measure of the resistance in both cases.

2. *The Pulley.*

156. The pulley is a thin cylinder with a channel or groove cut around its edge about which a rope passes, so that a power applied to the one extremity may balance or overcome a resistance applied to the other.

157. The pulley is always moveable round its axis, when not in a state of equilibrium; but it is said to be fixed or moveable according as its axis is fixed or moveable.

158. In the fixed pulley, (Fig. 60.) there is an equilibrium when the power and the resistance are equal. It is manifestly a lever of equal arms, and as $AC = BC$, P must be $= Q$, so that no mechanical advantage is gained in the sense in which we employ that phrase. It enables us, however, to change a direct resistance into a more advantageous one. Thus by a rope laid over a fixed pulley, (Fig. 61.) a horse walking horizontally may be made to draw vertically as in raising water or the like; and a man without leaving his place may elevate a weight, not equal to his own, to any proposed height.

159. In the single moveable pulley whose strings or cords are parallel, (Fig. 62.) the condition of equilibrium is that the power be to the resistance as 1 to 2. The horizontal diameter AB is manifestly a lever kept in a state of equilibrium by three parallel forces G, D, or the tensions of the strings GA, DB, and the weight Q applied at the middle between them. The source of the advantage here is, that the fixed point G supports the half of the weight, or serves,

by means of the rope GA, as a fulcrum, to prevent A from descending, and ACB is a lever of the second kind. In the weight Q must be understood as included that of the pulley to which it is appended. For conveniency, the rope BD is made to pass over a fixed pulley E; and as we disregard the effect of friction, in our elementary theory, and suppose no force to be spent in the flexure of the rope, though it is otherwise in practice, we conceive B to be drawn upwards with the same force as P is drawn downwards, and all the parts of the rope $PFDBAG$ as in a state of equal tension.

160. A combination of pulleys in this form, (Fig. 63.) or any equivalent and more commodious one, having a block containing two or more moveable pulleys, and a fixed block containing a corresponding number of others, round which the same rope passes, is called a *tackle*. Here, the whole of the rope being supposed as before to have equal tension, the five parts a, b, c, d, e may be conceived as supporting equal weights. Now e supports P, therefore a, b, c, d support $4\,P$ and $P : Q = 1 : 4$; Q being understood to include the weight of the lower block. The number of pulleys in each block might be augmented. The addition of another to each would give an advantage as $6 : 1$, and in general $P : Q = 1 :$ the number of strings by which the lower block is supported.

161. Other forms, on the same principle and giving the same energy to the power, are when the pulleys in each block turn on axes which are vertical to each other; or, which is still more convenient, when those of each block turn on the same axis. The rope, in passing over one of each set alternately, must then have a slight obliquity, which is a disadvantage, as the power will not act with its whole energy; but the loss of power from this cause is trifling unless the pulleys be numerous or the blocks very near to each other. If the pulleys in this form be numerous, the axis must be of considerable length, and the one end will be drawn considerably below the level of the other, which occasions a great waste of

power. To avoid this, Mr. Smeaton, the celebrated engineer, made an ingenious and powerful combination by the following disposition, which is represented and described in the *Lond. Phil. Trans.* vol. 47. An idea of it may be communicated by the following very rude and simple sketch, (Fig. 64.)

There are two blocks as usual, a fixed and a moveable one. In each block are two parallel axes, the one beneath the other, *AB* and *CD* in the upper; *EF* and *GH* in the lower. On each axis are five pulleys: those on the two extreme axes all equal, and of a larger diameter than those on the axes *CD* and *EF* which are also equal to each other. *K* being a fixed point in the upper block, the cord is attached to it; and, descending, we may suppose, on the side next to us, passes over the pulley in the middle of the lower axis marked 1, then, going up behind, comes over that marked 2, and so on alternately in the consecutive order of the figures or numbers 3, 4, &c. The advantage gained is as 1 : 20.

162. What is called White's pulley contains, properly speaking, only one fixed and one moveable pulley, which by a particular contrivance are made equivalent to blocks with several. Each is cut into circular grooves of different diameters, and having their centres in a common axis. A cord attached to the fixed point *K*, (Fig. 65.) connected with the upper block, passes, as in the figure, over the groove marked 1, then over that marked 2, and so on. Such a one as the figure represents would increase the energy of the power in the ratio of 1 : 6. One more groove added to each would make it 1 : 8, and universally it will be 1 : *n*, *n* being the number of ropes by which the lower pulley is suspended. If the weight
2 be raised one inch, 1 inch of the rope will pass over the pulley 1; 2 inches over the pulley 2, for, by the ascent of the weight, each of the ropes, as *a*, *b* is shortened 1 inch; 3 over the pulley 3; and so on. It will be of great advantage, then, to make the radii of the grooves 1, 2, 3, &c. in the propor-
3, 5, tion of these numbers, and those of the other block in the ratio of 2, 4, 6, &c.; for then the circumference of each

one piece with the rest of the handle [illegible] one or more persons may apply their strength [illegible] *CD* or *FG* now stands in place of the [illegible] in our analogy expressing the [illegible] practice two buckets are used [illegible] while the other is raised [illegible] tages. There is a great saving of [illegible] in raising the full bucket by the [illegible] and its rope.

167. A section of the capstan [illegible] and the momentum of the power [illegible] of levers, [illegible] the number and strength [illegible] who apply their force at them and their distance [illegible] axis.

168. A very great force may be [illegible] on a cylinder in a plane perpendicular [illegible] that acts on the same [illegible] sent the radius of a cylinder [illegible] surface; let the plane of *OA* and [illegible] axis, and let *AC* equal to *AB* [illegible] would balance the [illegible] on the radius *OA*. [illegible] balance *AB* act in the same tangent [illegible] *AE*, its value must be [illegible] to be a right angle. Now [illegible] crease indefinitely as [illegible] the direction *AE* is parallel with [illegible] this principle [illegible] ous parallel [illegible] *M*, *N* and we [illegible] several times round, having made them [illegible] possible, *M* and *N* will be [illegible] great force. Each of the [illegible] considered as a [illegible] twisted.

The carpenters make use of this method of [illegible] small flexible saw

169. In the common construction of the wheel and axle, if we attempt to increase the power of the machine to a great degree, we must either make the diameter of the wheel very large, which would be inconvenient and clumsy, or diminish that of the axle, and so render it weak and unable to support a great burden. There is, however, a very simple and ingeniously contrived windlass which becomes stronger the more it multiplies the power. Its two ends are, as in the common wheel and axle, cylinders of unequal diameter, and the difference between them consists merely in this, that the same rope passes round both cylinders in the one we are now to explain, and is coiled round them in opposite directions, passing under a single moveable pulley beneath, so that while it is taken up by the one, it is given off by the other.

Let $AC = R$, $BC = r$, (Fig. 72.) and first let an equilibrium be maintained by a power P acting in the direction of the tangent at A; then as the two parts of the string AF, BG will be equally stretched, we may consider $\frac{1}{2} Q$ as acting at A, while the other half acts at B. This latter aids the power, as it tends to turn the machine in the same direction. Hence

$$PR = \tfrac{1}{2} QR - \tfrac{1}{2} Qr = \tfrac{1}{2} Q (R - r),$$

$$\text{or } P \times 4 R = Q (2 R - 2 r),$$

$$\text{that is, } P : Q = D - d : 2 D;$$

the power is to the weight, including of course that of the pulley, as the difference of the diameters of the two cylinders to twice that of the greater. Thus, suppose the diameters are 19 and 20 inches respectively, $P : Q = 1 : 40$, a great increase of energy. We are here supposing P to act at the circumference of the greater cylinder. If we employ a lever to turn it, as in the common windlass, and if the power be thus applied at ten times the former distance, a tenth part of its former value will suffice, and $P : Q = \frac{1}{10} : 40 = 1 : 400$. But, without using a lever, the energy may be augmented to this extent, and in fact indefinitely, by making the less cylinder more nearly equal to the greater, and thus at the same time adding to the strength of the machine instead of diminishing it, as in the common construction. Let $D = 20$ inches as before, and $d = 19.9$, or in tenths of an inch 200 and 199 respectively.

$$P : Q \text{ still} = 1 : 400.$$

The thickness of the rope is here disregarded. By using another rope attached to a larger wheel, and giving it the advantage of a compound or double axle like the whole machine now described, it is evident that we may have an increase of power corresponding to that obtained by the insertion of a lever. The principle of this contrivance is the making the resistance partly conspire with the power that is to balance it. It is a very ingenious thought, and has been adopted also, as we shall see, in the construction of the screw.

170. In the employment of the wheel and axle, or any of its modifications, as a mechanic power, it is possible that the force applied may be constant, while the resistance is variable, or that, conversely, the force may be variable while the resistance is constant. In either case the energy of the agent may be so modified as to produce an equable action, or one nearly such.

171. If a man works at a common windlass to draw water from a deep pit, by two buckets with ropes coiled in different directions, the weight of the empty bucket and its rope will, as formerly mentioned, assist him in drawing up the full bucket; but the weight of the rope being often in such cases very considerable, the exertion to which he is exposed at different stages of the operation will be very unequal. He will be most assisted when he least needs it, and least when he requires it most. The advantages of equable action will be better considered afterwards; at present we shall confine our attention to a simple method of producing it.

Suppose then, that instead of the common windlass, we use one of a barrel shape, or resembling two truncated cones placed base to base as in the figure, and that the ropes are fixed to the smaller ends, so that each bucket as its rope is coiled up approaches the middle and acts by a longer radius, the full bucket as it is drawn up acts with a greater mechanical advantage in proportion as the weight of loose rope on that side is diminished; and on the other hand, the radius by which the empty bucket assists our action becomes smaller in proportion as the weight of its uncoiling rope increases.

Let R = the rad. at the middle,
r = the rad. at each end,
b = the weight of each bucket,
w = that of the water each contains,
c = that of each rope,
f = the force applied at the distance a from the axis.

To equalise the momenta of the opposed forces at the beginning and end of the ascent of the bucket, which will render them with sufficient accuracy in this case equal throughout, we have

1. When Q is beginning to ascend,

$$fa + bR = (b + w + c)\, r,$$

2. When Q is at the top,

$$fa + (b + c)\, r = (b + w)\, R,$$

By transposition of fa, and addition,

$$(2b + w)\, R = (2b + w + 2c)\, r,$$

$$\therefore R = a \cdot \frac{f}{w} \cdot \frac{2b + w + 2c}{2b + w + c},$$

$$r = a \cdot \frac{f}{w} \cdot \frac{2b + w}{2b + w + c}.$$

Let d = the diameter of the rope, n = the number of coils from D to G, l = its length: the spires of the rope being regarded approximatively as circles, will increase in arithmetical progression, and the extremes will be $2\pi R$, $2\pi r$, the half sum of which multiplied by $n = n\pi\,(R + r) = l$.

$$\text{Hence } n = \frac{l}{\pi\,(R + r)},\quad DG = nd = \frac{ld}{\pi\,(R + r)},$$

$$\text{and } HE = \sqrt{\left(\frac{ld}{\pi(R + r)}\right)^2 - (R - r)^2}$$

Boss. Mec. § 212.

172. In the mechanism of the fusee of a watch again, we have an adjustment, the object of which is to equalise the effect of a power of variable intensity. *c d b* (Fig. 74.) represents the barrel which is moveable and connected by the main-spring with the fixed arbor *a*. The main-spring joins the barrel at *d*. A chain is coiled round the barrel and also round the fusee *e f g*. When the fusee is turned round in winding up the watch by applying the key at *o*, the chain is evolved from the barrel, by whose revolution, thus produced, the spring is bent. When the key is removed, the recoil of the spring carries round the wheels by its action on the fusee, which takes hold of one of them by a catch on its return, and at the same time serves as a maintaining power to the balance spring which regulates the motion. Now a spring has always most force when most bent, if not injured by straining its elasticity too much, which is here avoided. It acts with the greatest absolute force then when the watch is newly wound up, relaxes its energy by the angular motion of recoil, and when it is well constructed has a force proportional to its remaining angular distance from the position of quiescence. 2.4

To equalise this variable action the fusee is made of a tapering shape like a truncated cone, (Fig. 75.) When the timepiece is just wound up the chain is near the top of the fusee, and the power acts in the line *e m* a tangent to the circle *e f g*; after some time the spring, now weaker, acts in the line *q n* a tangent to a circular section of longer radius, and these radii may be so adjusted to the force of the spring that its momentum shall be constant. It is of importance, though we cannot perhaps practically reach this limit, to approach it as nearly as possible in the construction of a machine for the accurate measurement of time, to which an equable motion is essential. Let us try then to find the nature of that line, whether straight or curved, by the revolution of which round an axis the surface of a fusee should be formed.

173. Let F and f be the forces of the spring at two dif-

ferent angular distances, in the ratio $a : x$, from the position of quiescence, r and y the radii of the fusee by which the chain acts in these two cases respectively: then $Fr = fy$. But $F : f = a : x$. Hence $ar = xy$, which is the equation of the equilateral hyperbola referred to the assymptotes as the axes of co-ordinates.

4. *Inclined Plane.*

174. If a body of any form whatever touch a plane only in one point and be solicited by only one force, it is necessary to an equilibrium, 1. That the direction of the force be perpendicular to the plane; 2. That it pass through the point in which the body touches the plane. If it be not perpendicular to the plane, it may be resolved into two, one of which shall be perpendicular and the other parallel to it. And as we suppose the plane to be perfectly smooth, there is nothing to resist the parallel force.

175. If a body touch a plane in several points and be still solicited by only one force, then in order that it may remain in equilibrio the force must be perpendicular to the plane, and must either pass through one of the points which touch the plane, or be resolvable into two or more forces parallel to itself, each passing through one of these points. The same thing is to be understood of the resultant of several forces acting in any directions.

176. *Cor.* 1. If the single force by which the body is solicited be that of gravity, the plane must be horizontal. As we cannot change the direction of gravity and make the force perpendicular to the plane, we must make the plane perpendicular to the direction of the force.

177. *Cor.* 2. If a body be solicited by only two forces, they must be in the same plane, their resultant must be perpendicular to the plane on which the body rests, and consequently the plane of the forces will be perpendicular to the same. Let a plane pass through, or contain in it, the direc-

tion of one of the forces and the point of application of the other. If the second do not act in this plane, it may be resolved into two, one in the plane and one perpendicular to it, if it be not itself perpendicular. Supposing then the two forces in the plane to have a resultant, it cannot be destroyed by the remaining force. The rest of the corollary is obvious.

178. *Cor.* 3. If one of these forces be gravity, the plane of the forces must be vertical, and must pass through the centre of gravity; for the resultant of the body's gravity is in a vertical line passing through that centre.

179. As we are here to investigate the conditions of equilibrium produced by a body's weight and the other forces by means of an inclined plane, we have only occasion to represent a vertical section of the body passing through the centre of gravity and perpendicular to that plane.

180. Let ABC then (Fig. 74) represent a section of the inclined plane as described, MNH a body sustained by three forces in equilibrio, its own weight Q, and a power P acting at G in the lines GQ, GP, and the resistance of the plane in the line HG perpendicular to itself. Draw AD perpendicular to GP, AE perpendicular to CA, and produce BC to D; and observe that AC is called the length of the plane, AB its height, and BC its base; call the resistance of the plane R. Then since the sides of the triangle ACD are perpendicular to the directions of the three sustaining forces,

$$P : Q : R = AD : CD : AC.$$

181. If the direction of the power be parallel to the base, DA, always by construction perpendicular to the line, will also be perpendicular to the base, and will coincide with AB, and ABC will now be the triangle whose sides are proportional to the forces, or

$$P : Q : R = AB : BC : AC,$$

and the power is to the weight as the height to the base.

182. If GP be parallel to the length AC, AD will coincide with AE, and

$$P \cdot Q : R = AE : CE : AC.$$

or referring to the homologous sides of the triangle ABC,

$$P : Q : R = AB : AC : BC,$$

and the power is to the weight as the height to the length.

183. The power is to the weight then, as the height to the base, or as the height to the length, according as it acts parallel to the base or to the length respectively.

184. We shall now find what direction gives to the power the greatest advantage in balancing Q. Returning to the general representatives of the forces, the sides of the triangle ACD, we have

$$P : Q = AD : CD = \sin.C : \sin.CAD,$$

or $P = \frac{Q \sin.C}{\sin.CAD}$. Now for an inclined plane of given elevation, and a given weight Q, the numerator is constant, and P will be least when $\sin.CAD$ is greatest; that is, when CAD is a right angle, or AD coincides with AE, which is the case when the direction of the power's action is parallel to the length of the plane. This then is the most advantageous line of action; it requires the least power to balance a given weight.

185. It may be useful to the student as a simple exercise in the application of his analytical formulæ to investigate the above results as follows:

Let i = angle ACB = angle HGK,
and θ = angle LGP, GL being parallel to AC.

Then $Q \cos.i$ = pressure on the plane,
$Q \sin.i$ = pressure down the plane,
$P \sin.\theta$ = pressure on or from the plane,
$P \cos.\theta$ = pressure up the plane.

$$1.\ P \cos.\theta = Q \sin.i, \text{ or } P = \frac{Q \sin.i}{\cos.\theta}.$$

$$2.\ Q \cos.i \pm P \sin.\theta = R,$$

$$i.\ e.\ Q \cos.i \pm \frac{Q \sin.i \sin.\theta}{\cos.\theta} = R,$$

$$\text{or } \frac{Q \cos.(i \mp \theta)}{\cos.\theta} = R.$$

If parallel to the plane, $\iota = 0$; $\cos.\iota = 1$;

$$P = Q\sin.i\,;\ \ R = Q\cos.i.$$

If GP be parallel to the base, $\iota = i$.

$$P = Q\,\mathrm{tang}.i\,;\ \ R = \frac{Q\cos.(i-\iota)}{\cos.\iota} = \frac{Q}{\cos.i} = Q\sec.i$$

186. *Cor.* The equilibrium of bodies upon curve surfaces may be determined by the same principles, if we consider the surface as represented at any point by the plane which touches it there.

5. *Wedge.*

187. The wedge is a triangular prism used for the purpose of cleaving bodies, and sometimes for compression, and for the raising of weights.

188. We here regard the sides as perfectly smooth, so that the only effective resistance will be in directions perpendicular to the sides, and the conditions of equilibrium are very easily determined.

Let the power P be applied perpendicularly to AB the back of the wedge ABD, Fig. 77. and let the resistances as in cleaving a piece of wood be in the directions QF, RF. These three forces being supposed to be in equilibrium, their directions must meet in one point as O, and their quantities will be as the three sides of a triangle perpendicular to their respective directions. Now the sides of the triangular section of the wedge itself are perpendicular to the directions, and $P : Q : R = AB : BD : AD$, and $P : Q + R = AB : BD + AD$.

This demonstration applies whatever be the species of the triangular section. If it be an isosceles triangle, which it generally is, $P : Q + R = AC : AD$, or the power is to the sum of the resistances as half the back of the wedge to either side. Let S express the sum of the resistances, and ι half the angle of the edge. $P = S\sin.\iota$; whence it is evident, that the

or referring to the homologous [illegible] is [illegible] ba
$P : Q : R = AB$ [illegible]
and the power is to the weig[illegible] parall
183. The power is to th[illegible] R'' [illegible] D, or
the base, or as the height [illegible] resolv[illegible] ane in
parallel to the base or to [illegible] ar to CD and BC, which
184. We shall now fi[illegible] he same with R' and R'', for
the greatest advantag[illegible]
neral representati[illegible]
ACD, we have

$$P : [illegible] = AD : CD,$$

$$[illegible] R = AD : CD,$$

$$[illegible] R = AC : AD,$$

$$[illegible] R = AC : CD,$$

or $P = \frac{Q \sin [illegible]}{\sin [illegible]}$ [illegible] ing the sum of the resistances paral-
vation, and [illegible] wedge. This gives its energy in rais-
P will be less [illegible] pressing them, when by some resisting
is a right an[illegible] are destroyed, and the motion when
when the [illegible] perfect constrained to be parallel to the
length of [illegible]

[illegible] the inclined plane, we supposed it to
[illegible] suppose its base to be perfectly smooth,
[illegible] a perfectly smooth horizontal plane, the
[illegible] direction HG, (Fig. 78, 79.) cannot exist
[illegible] applied to the upright side AB. The
[illegible] to a vertical and a horizontal force.
[illegible] resisted by the support of the base; but
[illegible] the horizontal one unless we supply
[illegible], or a fixed obstacle. In Fig. 78. let
[illegible] obstacle unconnected with the inclined
[illegible] surface, which is supposed perpendicu-
[illegible] whose centre of gravity is G and whose
[illegible] F, presses at K; this plane will
[illegible] direction KGD parallel to the in-
[illegible] resolvable into GH and HF, whereof
[illegible] resistance of MN, just as it would
[illegible] force Q acting parallel to the plane.
[illegible] GH, which again may be resolved
[illegible] horizontally. Hence the pressure

[illegible] on the plane $= HO : GH = AB : AC$. [illegible] nder this point of view becomes a [illegible] ower P prevails, or exceeds what [illegible] the proportion now stated, will [illegible] elevate it by causing it to move [illegible] or parallel to and guided by [illegible] surface MN is represented as [illegible] will resist in a direction parallel to the base. [illegible] resolved into GE and GD in the lines of the [illegible] resistances. GE is destroyed by MN, and the pressure on the plane AC, represented by GD, may be resolved into GF vertically and FD horizontally. Hence the horizontal force or that on the back of the wedge must be to the vertical force $= FD : FG = AB : BC$. If P now prevails, the motion must be parallel to AB, guided by MN.

191. All this coincides with the theory of the wedge as previously given. Observe particularly, *with a view to a future reference*, that the tendency of the heavy body to separate the two planes MN and AB, creates a pressure tending to move AB backwards, or parallel to CB; and that the vertical pressure is to its horizontal tendency as the base of the plane to its height. Observe at the same time, that if the plane MN were produced to meet the inclined plane and rigidly connected with it, so that they should constitute one body, no such force as P would be required. The pressure exerted on MN would then counteract the equal and opposite pressure on AB.

192. It is generally the *force of percussion* that is applied as a power to the back of the wedge; but its energy in this case will depend on the same principles as in the case of pressure.

193. In cleaving elastic woods, as dry fir or ash, longitudinally, the cleft generally reaches to some distance beyond the edge; and thus, in extending the fibres, combines the energy of the lever with that of the instrument under consideration.

194. To the principle of the wedge are referable all edged

tools and instruments with a sharp point, as swords, axes, knives, lancets, planes, chisels, saws, files, nails, &c.

195. The edge of a saw consists of a number of teeth, which are all wedges of small depth, and cannot sink far into the wood which it is to cut. It thus takes the fibres in small parcels, and breaks them in succession, just as we break a bundle of rods more easily one by one, than by taking them all together. All sharp instruments whatever cut most easily when used on the principle of the saw, and in fact they are all saws though not so called. The finest edge has a vast number of small inequalities, which are not visible unless the eye is assisted by a microscope. These serve the same purpose as the teeth of the saw, and accordingly all such instruments cut more easily when drawn or pushed in a *slanting* direction, than when pressed or struck in a direction *perpendicular* to the edge. In planing such woods as give considerable resistance, the joiner uses a plane in which the edge is not perpendicular to the sides, but oblique. The curvature of the sabre seems to be given to it with a view to this effect. While *it* gives a deep cut by reason of the obliquity of the edge to the line of percussion, a sword with a straight edge and little obliquity, urged with no greater force, and meeting with more resisting obstacles at once, would have its velocity sooner reduced to the same degreè, or destroyed; and the effect would be a wound *less deep*, with more *contusion*.

196. On the principle just explained and illustrated depends the theory of sharpening all edged tools or instruments. The finest hone having its surface full of small cavities and protuberances will always communicate corresponding inequalities to the edge, and make it equivalent to a fine saw. Hence, in sharpening, it ought to be so applied to the instrument, or the instrument ought to be so applied to it as to lay the points of these small teeth to which we have alluded in the direction in which the instrument is to be drawn or pushed in using it. This is always done in sharpening a scythe, the teeth, as we have called them, on the edge of

which, are visible, being formed by a piece of free-stone or wood covered with fine sand. In a sickle, the same thing is still more distinctly visible. It is often formed with a very rough edge.

197. A file is a sort of saw with a very broad edge, and a number of teeth placed collaterally as well as longitudinally. If, instead of forming a saw of hard steel with teeth, we make it of soft iron, and strew some hard gritty powder under its edge, the powder, from the softness of the metal, will be partly imbedded in it, and thus form a sort of file. The marble cutters use a saw of this kind.

6. *The Screw.*

198. This instrument may be conceived as formed by a spiral thread or groove, cut round the surface of a convex or of a concave cylinder, and every where making the same angle with lines parallel to the axis; so that if the surface of the cylinder with the spiral threads upon it were unfolded, becoming a plane, the spiral threads would form straight inclined planes parallel to each other.

199. The screw, as a mechanic power, consists of two parts which, agreeably to the definition now given, are severally called screws, the *exterior* and the *interior*; the latter being so formed, that it may be considered as a mould in which the former is cast. The exterior has a spiral protuberance formed on the outer surface of a convex cylinder, the interior a corresponding spiral groove on the inner surface of a concave one. The part of the machine which contains the interior screw, is in fact exterior in situation, and is called sometimes the *box* or *nut*. The one of these screws is fixed, and the other moveable: and in giving the theory of the instrument, it is of no consequence which of the two we consider as fixed. The nut being screwed on at the top and pressed vertically by a weight, the under side of the protuberance of the nut will be pressed down upon the upper surface of the

protuberance of the convex screw, and we may suppose it equally distributed so far as the contact extends. This weight may be raised by turning the nut backwards, while the rest remains fixed; or by turning the exterior screw forwards, while the nut is prevented from rotation. In the former case, the process is like that of drawing a body up an inclined plane; in the latter, it is like our using the same inclined plane as a wedge, and pressing it under the body. The screw may, in this point of view, be regarded as a wedge with a *circular base*, pressed under a body with a *circular motion* communicated by the energy of a *lever*.

200. Let *ABCD*, (Fig. 80.) represent a convex cylinder, and let a number of equal and similar right-angled triangles, whose bases are exactly equal to the circumference of a circular section of the solid, be lapped round it, so as to have all the sides homologous to *EF* in the figure coincident with the same line *BC* parallel to the axis, and one beginning where another ends, the hypotenuses, if the triangles be physical surfaces, will form the continued spiral protuberance. Now let a particle of the nut be pressed down vertically in the direction db on the point b of the spiral, which we may consider as an elementary portion of the inclined plane, and similar to the whole *EGF*. The direction of the base at b is that of bf a tangent to a circular section of the nut, whose circumference passes through the point of application. Let f be the pressure in the direction bf which balances the vertical pressure q on the small inclined plane, f must be to q as the height to the base, or $FE : FG$. Now FE is the distance of two contiguous threads of the screw parallel to the axis, or what is called the *step* of the screw; let this be h, and let c denote the circumference of the circle, which is $= FG$; then

$$f : q = h : c.$$

But the point of application b is within the nut, and we cannot apply the force f at b, in the direction bf, *immediately*; but we can insert a lever represented by the radius ob produced, and to its extremity apply a force p less than f, which shall excite a pressure $= f$ at the distance $o\,b$ from the common axis; p must just be that force which acting the opposite way would balance f; and if C be the circumference to the radius $o\,p$, $p : f = c : C$.

Compounding this analogy with the last we have $p : q = h : C$. Now let q' be another portion of the weight resting on a point of the spiral, either at the same distance from the axis or at a different distance, the circumference passing through which is c', and let f' be the tangential force and p' the force which at the distance $o\,p$ would balance q'; f' and c' will disappear, as f and c in deducing the last analogy, and we shall have

$$p' : q' = h : C \text{ and so on,}$$
$$\therefore\ p + p', \text{\&c.} : q + q', \text{\&c.} = h : C$$
$$\text{or } P : Q = h : C.$$

The whole power is to the whole resistance, including that of the nut itself, as the step of the screw to the circumference of the circle described by the power. Hence, the finer the screw, or the more nearly contiguous its threads, the greater is the energy gained, with a lever of the same length.

201. If, with any lever of a given convenient length, we attempt to increase very much the energy of this power, we must make the distance of the contiguous threads very small, and consequently the threads themselves very weak and unable to withstand the resistance proposed. A particular construction of the screw has, however, been invented, which gives very great energy without making the threads fine. It is on the same principle as that of the double cylinder employed as a capstan, formerly described; namely, the making the resistance in part to balance itself or to conspire with the power.

Let *GL* (Fig. 81.) be a convex screw turned by a lever *GN*, and working through the solid beam *AB* which is supported by *AC*,*BD* and serves as a fixed nut; and when turned in the direction *MNO*, let it work into a hollow screw in a heavy mass of matter, or one from which is suspended a heavy mass, *EF*, which is prevented from rotation: and lastly, let the upper and lower parts *GK* and *KL* have a different number of threads in the inch, the one 10 and the other 11 for instance. This is the circumstance on which its peculiar energy depends. If the upper part is the finer screw, it will raise the body *EF*, and if the lower is the finer it will depress it with very great force; so that, like the common screw, it may be used either for raising a great weight or for strong compression. We investigate the energy, from first principles, thus. Let h be the step of the lower part and h' of the upper, and Q the weight of *EF*. As this weight cannot turn round, it will tend to turn the screw backward, (191.) exciting a force P at N in the direction *NM;* and this force P is just what, acting in the direction *NO*, will balance the weight. *So far* the case is the same as before, and

$$P : Q = h : C.$$

EF here acts like the loaded nut in the former case. But the weight Q at the same time tends to draw the convex screw vertically through the upper or fixed nut *AB*, and this tends to make it turn in the opposite direction to that formerly supposed, or to make the end of the lever move towards *O*. The force P' thus excited at N is of course that which, acting in an opposite direction, would balance that tendency and

$$P' : Q = h' : C.$$

This will be easily conceived by reference to Fig. 80, where b may be considered as a small portion of a protuberant thread of the convex screw drawn in the direction $d\,b$ against the spiral protuberance of the interior one, tending to move down the slope and restrained as before by a force at p, exciting at b the tangential pressure f analogous to *FD*, (Fig. 79.) Or we may view the matter very simply in this light: when the convex screw is drawn downwards against the fixed nut above, the reaction has the same effect as if the fixed nut were unfixed and drawn with the given force vertically upwards. The nuts are therefore virtually drawn with equal force, viz. the weight Q, in opposite directions, and tend to turn the convex screw opposite ways. Then, as the pressures P and P' at N are excited in opposite directions, the one conspires with the force that is opposed to the other;

nd, if we suppose P the greater, $P - P'$ is the only force equired to sustain the weight. Comparing the two analogies investigated we find the consequent terms the same in both, and

$$P - P' : Q = h - h' : C,$$

or the power is to the weight as the difference between the steps of the two parts of the screw is to the circumference of the circle described by the power. The last analogy is deduced from the two preceding in the same way as Prop. 24. B. v. of Playf. Eucl. is demonstrated, by substituting the word "division" instead of "composition." To show the enormous multiplication of power by this machine, without making the screws at all fine, let us suppose the lower part to have 10 and the upper part 11 threads in the inch,

$$P - P' : Q = 1 : 110\,C,$$

C being expressed in inches. If the radius corresponding to C be merely 20 inches, 10

$$P - P' : Q = 1 : 6911\tfrac{1}{2} \text{ nearly}.$$

The instrument constructed as now mentioned might evidently be used to compress any body placed above EF and confined between it and a resisting plane; but, if the body to be compressed is to be below EF, we must have the upper part of the screw the coarser, with 10 threads in the inch, for instance, while the lower has 11.

The instrument now described and explained is called Hunter's Screw, from the name of the inventor.

202. The screw, even when exciting a very great force in compression, will generally retain its position, in consequence of the friction, which is great in this instrument. Hence it is much used in the formation of presses of all kinds. We see, in the case of a smith's vice, the advantage gained even by a coarse screw in this way.

203. It is also very useful when minute and accurate divisions are to be made, as in micrometers for telescopes, and 2 the division of scales of various kinds. For this purpose it is fitted by the steadiness of its motion and by the regularity of its threads. A convex screw is formed by turning a cylinder through a concave one, and, if the threads were not very exactly equidistant in the hollow screw or nut, as well as in the convex one, when formed, the one would not pass through the other. Suppose then we take a screw with 50 threads in the inch, every complete revolution will move the end backwards or forwards $\frac{1}{50}$ inch; and, if a ruler be fixed to the end, we may thus draw any number of equidistant lines, at this interval. But we are not limited even to this in the use of the same screw, for a circle may be fitted to it to measure the angular motion of an index attached to the end of the screw, and thus we can turn it through the $\frac{1}{100}$ or the $\frac{1}{300}$ of a revolution. Each turn through $\frac{1}{100}$ part will evidently move the dividing scale through $\frac{1}{5000}$ inch in the case supposed. A more minute subdivision will be obtained by using a *finer screw*, or the *same* with a more *minute subdivision* of the circle. When we come to the science of Optics, we shall see, in the explanation of the sextant, that a circle of no very great radius may be subdivided into 129,600 equal parts, all distinctly visible by the aid of a small microscope. Accordingly, if the index of our dividing screw were of the same length with that of a sextant, 7 or 10 inches, and fitted with the same artificial helps, we might turn it through $\frac{1}{129600}$ of an inch, and the ruler would move forward or backward $\frac{1}{50}$ of this.

I do not mean to suggest that such minute subdivision of a scale is necessary, or to be actually attempted in practice. The divisions are to be such as a good artist can lay down with accuracy and distinctness; and, when an estimate of greater precision is required than the scale immediately furnishes, enlarged images of objects may be formed on optical principles, which being measured by the scale, and reduced in proportion to the magnifying power employed, we shall obtain the dimensions of the objects themselves, so as to leave the possible error extremely small. A very minute subdivision may be made by Hunter's screw, without a graduated arch, and furnished merely with an index to mark the complete revolution. Let a part of the screw which passes through a fixed nut have 50 threads in the inch, and let the farther end, which turns into a moveable nut which may slide along a groove without rotation, have 51; a turn of the screw will, in so far as depends upon the motion through the fixed nut, carry forward the sliding nut, or a ruler attached to it, $\frac{1}{50}$ inch; but, in so far as depends upon the motion into the sliding nut, it will draw it backwards $\frac{1}{51}$ inch. It will therefore move forwards $\frac{1}{50} - \frac{1}{51} = \frac{1}{2550}$ inch.

204. Reviewing the conditions of equilibrium in all the mechanic powers, we discover by induction the observance of a general law which it is of importance to notice and explain, as, being equally applicable to all their combinations, it affords in the more complicated cases the simplest practical means of estimating the energy of a machine. The rule is this: If a momentary addition be made to the power or the resistance by some extrinsic force, so that the machine may be put in motion in a state of what may be called dynamical equilibrium, and the velocities of the points of application of the two balanced forces reduced to the directions in which they act be called the velocities of these forces respectively, the balanced forces themselves shall be reciprocally as their velocities. Reference is here made by anticipation to a principle of Dynamics, (298.) that a motion or velocity, represented by the diagonal of a parallelogram, may be considered as compounded of motions or velocities represented by the two sides which meet in one of its extremities; and that, consequently, a velocity V in any direction, reduced to another which makes with it an angle θ, is $V\cos.\theta$. In the application of the principle to cases where the direction in which either point of application moves is perpetually changing, we must take the motion in its nascent state, or suppose it indefinitely small.

205. Before proceeding to the investigation of the law, we must recollect that if a body, whose parts are inflexibly connected, revolve about an axis, the absolute velocities of the different particles will be as their distances from the axis.

They describe similar arches in any given time, and these are as their radii.

206. When the lever (Fig. 22, 23.) begins to revolve, G and F describe arches of circles to which GN, FM are tangents, and all the points of the same straight line which is moved in the direction of its length will move with the same velocity; so that V and v, the velocities of P and Q, will be the same as those of F and G, and these are as the radii CF, CG, or $CG : CF = v : V$. But $P : Q = CG : CF$ (34.) $\therefore$ $P : Q = v : V$.

If the forces be applied perpendicularly to the arms themselves, these arms of course are to be taken instead of CF and CG.

207. In the fixed pulley (Fig. 60.) the power and the resistance when in equilibrio are equal; and so, it is obvious, are their velocities when put in motion.

208. In any system of moveable pulleys, (Fig. 62, &c.) when the same rope passes round the whole, if Q be elevated one inch, and the number of strings supporting the lower block be n, each of them being shortened one inch, P will descend n inches: that is, $v : V = 1 : n$. But $P : Q = 1 : n$ (160.) $\therefore$ $P : Q = v : V$.

209. In the pulley (Fig. 66.) where the directions of the power and the resistance are inclined to each other, we may consider AB as a lever whose momentary fulcrum is B, and on which P and Q act by the perpendicular arms BF and BD. Hence $v : V = BD : BF =$ sin. BQD : sin. $BQA =$ sin. CBA : sin. $ACB = AC : AB = P : Q$. (163.)

210. In the wheel and axle (Fig. 67.) $P : Q = AC : CB$ (164.) $= v : V$ (205.)

211. Suppose the capstan (Fig. 72.) to make one revolution, the power being applied to the circumference of the larger cylinder, the rope will be wound up to the length C, and evolved to the length c; C and c denoting the circumferences of the larger and the smaller cylinder, respectively. [illegible] will therefore be taken up to the extent $C - c$, and, each [illegible] equally shortened, Q will rise $\frac{1}{2}(C - c)$ while C

will be the motion of *P*. That is, $v : V = \frac{1}{2}(C - c) : C = \frac{1}{2}(R - r) : R = D - d : 2\,D$, which is the ratio of *P* to *Q*, (169.)

212. In the case of the inclined plane, let the power act with uniform velocity parallel to the length and continue the motion till the resisting body is dragged to the top. The line of action of the resistance, here supposed to be a weight, is vertical; and, in the application of the law, it is in that line that we must estimate its velocity. Now it rises through the height of the plane *BA*, (Fig. 76.) while the power, retaining its direction, passes over a space equal to the length *CA*. Hence $v : V = AB : AC = P : Q$, (182.)

213. If the power act parallel to the base, it must pass over a space equal to the base, while the weight rises through the height; and $v : V = AB : BC = P : Q$, (181.)

214. If it begin to act at *C* in any other direction as *fl*, (Fig. 82.) pushing the body in that oblique direction till it reach the top, when the direction of the power will be *FL*, parallel to *fl*, draw *AD* perpendicular to *FL* or *fl*, meeting the latter in *K*; then, while the ascent of *Q* is *BA*, the actual motion of the point of *P*'s application is *CA*; but, estimated in the direction of *P*'s action, it is *CK*. Hence $v : V = AB : CK = AD : CD = P : Q$, (180.)

215. If *ABC* (Fig. 78.) be regarded as a wedge, and a force *P* applied perpendicularly to the back balance a resistance *Q* acting in a direction parallel to *MN*, let the wedge be made to move through *BC* in the direction of its length, so that the point *H*, where the resistance is applied, may rise from *C* to *S* determined by drawing *BS* perpendicular to *AC*: $v : V = BS : BC = AB : AC = P : Q$, (188.)

If the resisting body *Q* be constrained to rise, as in Fig. 79. in a direction parallel to *AB*, $v : V = AB : BC = P : Q$, (189.)

216. In the common screw, while the point to which the power is applied describes the circle whose circumference was denoted by *C*, (200.) the resistance moves through *h* the distance of two adjoining threads parallel to the axis; and,

small the motion, the ratio is constant; ∴ $v : V =$
$P : Q$.

In Hunter's screw, while the power makes one com-
revolution, the motion of the resistance will be $h - k'$,
difference of the two steps; and, however small the an-
motion of the lever be, the same ratio is maintained.
if P' denote the power applied, the $P - P'$ of our for-
alogy, (201.)

$$P : Q = h - k' : C = v : V.$$

In all of these cases, (206—217.) if we consider the
as positive or negative according as it conspires or
direction with the force in whose line of action it is
ated, we may express the result in this form, $PV +$
0.

Observing the law to hold in such a variety of in-
considerably dissimilar, we are naturally led to sus-
that it is included in one yet more general, and which
be considered as expressing a condition of equilibrium
ally; and it may, in fact, be generalized as follows:
case of equilibrium, whatever be the number and
of the forces applied, if an indefinitely small
communicated to the system, provided that the con-
when it is flexible, be inextensible and remain
and if the nascent velocities of the points of applica-
forces P, P', &c. estimated in the directions of
and considered as positive or negative according
or not with the forces in direction, be de-
&c. respectively,

$$+ P'p', \&c. = 0, \text{ or } \int Pp = 0.\text{''}$$

is extremely easy in the case of rigid
points to which the forces are applied
configuration. The motion of such
progressive, all the points describing
directions, or rotatory, so that they
gular motion about the same axis, or
two, that is rotatory about a pro-

If the motion communicated be simply a progressive one, or what is called a motion of translation, let c, which in this case may be of any magnitude, be the space which each of the points of application describes in the same direction. Then if we assume the axis of the co-ordinates AX parallel to this, $\int P \cos.\alpha = 0$ (67.)

$$\text{That is, } P \cos.\alpha + P' \cos.\alpha', \text{ \&c.} = 0,$$

$$\therefore Pc \cos.\alpha + P'c \cos.\alpha', \text{ \&c.} = 0.$$

$$\text{But } c \cos.\alpha = p, \; c \cos.\alpha' = p', \text{ \&c.}$$

$$\therefore Pp + P'p' + P''p'', \text{ \&c.} = 0.$$

If the motion be a rotatory one, the forces being in equilibrio, there will be no tendency arising from them to revolve about the axis of the motion extrinsically communicated whatever it may be; and if AX (Fig. 13.) be taken for that axis of motion, and we project the forces P, P', &c. upon ZAY, or planes parallel to it, with which they make the angles θ, θ', &c. respectively; the forces $P \cos.\theta$, $P' \cos.\theta'$, &c. their projections, will also be in equilibrio, as is evident from (67. Eq. 4, 5, and 6.) or from the consideration, that the forces as estimated parallel to AX can have no effect upon the equilibrium with respect to it. The forces $P \cos.\theta$, $P' \cos.\theta'$, &c. then being in equilibrio, if we let fall perpendiculars r, r', &c. from the axis AX upon their directions, and consider the momenta as positive, which conspire with the motion communicated, and those as negative which oppose it, we shall have, (39, 44.)

$$Pr \cos.\theta + P'r' \cos.\theta', \text{ \&c.} = 0.$$

Let c now denote the angle of rotation, or rather the length of the arch which measures it at the unit of distance, which, in order to comprehend every case, we must for any finite distance conceive as indefinitely small. Then

$$Pcr \cos.\theta + P'cr' \cos.\theta', \text{ \&c.} = 0.$$

But cr, cr', &c. being the absolute velocities of rotation of the points where the perpendiculars terminate, $cr \cos.\theta$, $cr' \cos.\theta'$, &c. will be these velocities estimated in the directions of P, P', &c. respectively; that is, $cr \cos.\theta = p$, $cr' \cos.\theta' = p'$, &c.

217. I. H … hen p depends on a
from a rotatory one,
combined, and will
ns of the same form.
ay take the funicular
which the forces are
on, each point may not
translation or the same
another point, and it be-
the connecting forces or
for each is the same at each

for the force applied to an
stand for the tensions of the
let p, p', p'', &c. denote as be-
gular point estimated in the di-
ng forces. The forces applied to
quilibrio when reduced to the di-
point begins to move, (28.) con-
he angles which their lines of

$$\cos. \alpha'', \&c. = 0;$$

motion of that point,

$$+ P''c \cos. \alpha'', \&c. = 0,$$

$$p'', \&c. = 0;$$

application,

$$p'', \&c.) = 0.$$

d is constant, and it always
whatever longitudinal motion
ngs to the whole, and is the
extremity reduced to the di-
on in that direction result-
ar motion is = 0. As each
panied with one equal to it-
d the velocities p' are in the
ing in direction with its cor-

responding tension, and the other being opposite to that with which it corresponds, $\int(P'p'+P''p''$, &c.) is made up of quantities which, taken two and two, are equal with opposite signs; it is therefore $=0$, and $\int Pp=0$.

The law here stated and explained, is what is called the "Principle of virtual velocities."

Of Compound Machines.

220. Under this head we shall only describe a few of the simpler combinations, which will be sufficient to illustrate the way of calculating the statical effect or energy in any case. In the further prosecution of the subject, the student ought to seek opportunities of examining actual machinery, or of inspecting models, and reading descriptions illustrated by plates. He may begin with such of the latter as he will find of easy access, as in *Fergusson's Lectures*, edited by Dr. Brewster, Imison's *School of Arts*, Emerson's *Mechanics*, Gregory's *Mech.* vol. ii. and any of the *Encyclopædias*.

221. Compound machines may consist of the same mechanic power repeated a certain number of times, or of different powers. To the first of these classes belong the compound lever or steelyard, the compound pulley, the compound of wheels and axles: to the latter, the lever with the screw, already described, the lever with the screw and wheel and axle, the same with the addition of a pulley, as exhibited in a part of the class apparatus, the pulley or the wheel and axle with the inclined plane, &c. In all such combinations, the energy of each preceding is the power applied to the next following in the order of the transmission of motion or pressure, and the same uniform rule may be employed in calculating the energy in every such case: "the ratio of the increase of energy is that which is compounded of the ratios of increase belonging to the powers in combination taken separately.". In other words, let the quotient of the resistance divided by the power in each of the elementary parts be

called the measure of its energy, or, for the sake of abbreviation, its energy; then "the energy of the machine is the product of the separate energies of its parts."

222. *Compound lever*, (Fig. 83.) Let the pressures at B and E be denoted by these letters; then

$$P : B = b : a,$$
$$B : E = b' : a',$$
$$E : Q = b'' : a'',$$
$$\therefore P : Q = b\,b'b'' : a\,a'a'',$$
$$\text{or } \frac{Q}{P} = \frac{a}{b} \times \frac{a'}{b'} \times \frac{a''}{b''}.$$

It is evident that the velocity will decrease throughout the series, when the machine is put in motion, exactly as the pressure increases.

? 223. A more convenient form of the compound lever or steelyard, is that which is sketched in Fig. 84.

AD is a lever of the second kind, (126.) whose fulcrum B is supported by the cheeks BK, and the weight is suspended at C. The $\frac{BC}{BD}$ part of this weight, or an equivalent pressure of any kind, acting upwards at D will support it. This is supplied by the energy of a lever of the first kind EG, whose axis is sustained at F, and whose arms are FG, FE; the point E being vertically over D in the state of equilibrium, and connected with it by a chain, whatever pressure upwards is wanted at D or E, may be excited by the $\frac{EF}{FG}$ part of itself acting downwards at G. And thus

$$P = \frac{BC}{BD} \times \frac{EF}{FG}\,Q.$$

If each of the longer arms be ten times the corresponding shorter, $P = \frac{1}{100}\,Q$, and a simple steelyard having the weight suspended at the same distance from the fulcrum, if required to possess the same energy, must be ten times as long, and be subject to greater inequality of flexure.

224. *Steelyard for weighing loaded carriages*, (Fig. 85.) In the form here sketched, AD represents a lever of which

the [illegible] is [illegible] [illegible] is [illegible] [illegible] equal arms, whose [illegible] *FH*, *LL*, *DE* are [illegible] then $= EI -$ [illegible] on the [illegible] section being one [illegible] transverse [illegible] plate [illegible] stage is [illegible] two on each [illegible] small revolution about the [illegible] parallel to the horizon. [illegible] the same, because the [illegible] being each equal and opposite [illegible] be equidistant from the middle of the [illegible] to equalise the pressure on the two [illegible] though this is not essential to the just performance of the machine. With whatever force *L* is pressed downwards, *E* and *B* are pressed upwards, and the effect upon [illegible] where the counterpoise is applied will be the same as if [illegible] with a simple [illegible] and the whole weight appended at *L*.

225. A very simple weighing machine might be formed on the principle of the double cylinder capstan. Let the fulcrum of a steelyard be placed a little nearer to *B* than to *A*, (Fig. 86.) let the supports of the platform cover *a* and *b* be equidistant from *O* the middle of *AB*, and let a small weight *P* be suspended from *F* to balance that of the carriage *Q*.

Then $P.CF = \frac{1}{2} Q(AC - BC)$, or

$$P = \frac{AC - BC}{2\,CF} Q.$$

The energy is therefore the same with that of an ordinary lever, of which the longer arm is *CF*, and the shorter *CO*, half the difference of *AC* and *CB*.

If $AC = 10$ inches, $BC = 9$, and $FC = 20$; *P* will be $\frac{1}{40}$ *Q*, and the half of a hundred weight will balance a ton.

If $BC = 10$ inches and $AC = 9$, *O* will be on the side of *C* next to *F*, and the lever becoming of the second kind, may

be supported at F by one of the first, as in the fir
of the compound steelyard, (223.) If this second
as before an energy 10, we shall have $P = \frac{1}{100} Q$,
will balance a ton.

226. *Compound Pulley*, (Fig. 87.) Here qo is a
to change the direction of the power, whose energ
by the first moveable one lk; the energy of this l
of supporting a weight, is applied as a power to
doubles it again, and so on. With three moveal
therefore, in this form, $P : Q = 1 : 8$, and with
leys, $P : Q = 1 : 2^n$. The ratio of the velocities v
conveniently traced thus. Let the lowest with
rise one inch; each branch of its rope will be sh
inch; hence the next pulley will rise two inches;
of its rope will be shortened two inches, and t
rise four inches; each side of its rope will be s
inches, and P will descend eight inches; theref
and the resistance are reciprocally as their velo

227. Conceive the fixed pulley to be remo
act upwards by the part of the rope lo; if
two fixed obstacles, Q would be drawn upward
the force $8 P$, and M downwards with the for
maining unit to balance Q being supplied by
ceive now the figure to be inverted, and M to
which are attached the strings of all the pul
a fixed body acting as a support; $P : M =$
vestigate this in the usual form, and to shew
specting the velocity is observed, may be p
temporary exercise.

228. *Spanish Barton*, (Fig. 88.) D is a
fixed point; $ABCEGFP$ a rope passin
able pulleys BC and EF, while the latter
rope $GHKD$ passing round the fixed pull
BC. The ropes being supposed parall
$= 1 : 4$, and $V : v. = 4 : 1$.

229. *Combination of Wheels.*—The power
circumference of the first wheel, whose axle

the axle of this the third, and so on; the resistance being applied to the last axle. Let R, R', R'' be the radii of the wheels, r, r', r'' those of the axles, and it is demonstrated as in (222.) that $\frac{Q}{P}=\frac{R}{r}\times\frac{R'}{r'}\times\frac{R''}{r''}$.

230. As the surfaces are never perfectly smooth, we may conceive each axle to move the following wheel by friction merely. But in this case they would soon, by the continued attrition, diminish each others diameter, and require a new adjustment of the axes to bring them into close contact. They may be connected, when at some distance, by bands or belts passing round them; and when these become relaxed, they can, with very little inconvenience, be adjusted anew by tightening them. In general, the connexion is by teeth. The axle or small wheel is in this case called a pinion, and its teeth, leaves. In large wooden wheels, the teeth are generally made each of a separate piece of wood, and are in that case called cogs. A cog-wheel often drives a pinion of a particular form, composed of two thin cylindrical discs joined by a number of small parallel and equidistant cylinders. This piece of mechanism, which may be seen in the common flour-mill, is called a lantern or trundle, and its parallel cylinders uniting the two ends, serving instead of teeth or leaves, are called rounds or spindles.

2

231. A cog-wheel whose cogs or teeth are perpendicular to its plane, and drive a lantern, may be conveniently employed to change the axis of a rotatory motion. See the representation of the flour-mill in Imison's *School of Arts*, vol. i. pl. 3.

232. The axis of a motion of rotation may be changed to one making with it any given angle by *bevelled wheels*, whose teeth may be considered as belonging to a fluted cone, as those of common wheels may be said to belong to fluted cylinders. Thus, let it be proposed to change a motion round the axis AB (Fig. 89.) into one round the axis AC, making with it the given angle BAC. If no other condition be proposed, the problem is indeterminate, for the thing may be done in an infinite variety of ways. But, if the angular velocities of rotation about AC and AB are to be in a given ratio $m : n$, take $AD : AF = m : n$; complete the parallelogram $ADEF$; join AE; in AE take any point G, the position of which in that line will depend on the proposed diameters of the wheels: draw the perpendiculars GH, GL; and the cones formed by the revolutions of the triangles AGH, AGL will be the bases on which are to be formed the bevelled teeth. For the angular velocities will be inversely as the corresponding radii: therefore, if C and B denote the velocities of rotation about AC and AB respectively,

$$C : B = GL : GH = \sin. GAL : \sin. GAH = AD : AF = m : n$$

as was proposed.

233. Wheels of the same diameter are sometimes applied to each other without pinions, and then they are employed merely for the transmission of motion or power.

234. When one wheel is employed to drive another, the action should be equable, without those sudden variations of pressure and momentum in the connecting parts which, giving an advantage at one time to the power and at another time to the resistance, occasion sudden alternations of acceleration and retardation in the performance of work, and strain the parts more than would be done by a constant and moderate pressure. If the machine be subject to variations of strain, it must evidently be made strong enough to withstand the greatest; yet more work is not performed than would result from a more equal pressure throughout and an equable velocity of an intermediate degree, while more of the power is expended in moving the machine itself than, with a better construction, would be found necessary. It is therefore an important problem to construct the teeth of wheels so that the energy with which one acts upon another may be constant when the power applied is so. This may be done by making the teeth epicycloidal, (See Camus on the Teeth of Wheels.) A better form, however, and of much easier execution, is recommended by the late Professor Robison, (See his Art. *Mach.* § 36.) Let afg (Fig. 90.) be a thin curved

disc having a thread lapped round it in the direction *gfe* and terminating at *a*. Connect a pencil with the extremity *a* of this thread and evolve it from the curve, keeping it tight so that it may always be a tangent to the arch as represented at *e* and *f*: the curve thus described is termed the involute of *afg*; *be*, *cf*, &c. may be regarded as momentary radii; and indefinitely small portions of the curve at *b*, *c*, &c. may be considered as belonging to circles of different curvature determined by these radii. In other words, the radii *be*, *cf*, &c. are always at right angles to the curve at the points where they meet it, that is, to the tangents of the curve at those points: or a straight line, drawn perpendicular to the involute at any point will, if produced, meet the generating curve or evolute, and be a tangent to it.

As it is on this property that the application of the process of evolution to the formation of the teeth of wheels depends, it may perhaps be desirable to demonstrate it more strictly; and in doing so we cannot adopt a simpler or more perspicuous method than that of the elegant geometer Huyghens, the author of the theory of this process.

235. Let ACF (Fig. 91.) be the evolute, ABE the involute, CB and FE the radii of evolution at C and at some point F more distant from the origin. Produce BC to G and let FE meet in D the line drawn perpendicular to CB at B; the point D shall be without the curve. For GD is $> GB$ $\therefore$ $FD > FG + GB$; much more is $FD >$ arch $FC + CB$, that is, $> FC + CA$, that is, $> FE$.

Again, let EF (Fig. 92.) be the radius of the involute at F a point nearer to A than C is; draw the chord FC, join C,E and let CE meet BD in D: then $CD > CB$, that is, $> CFA$, that is, $>$ arch $CF + FE$: much more is it $>$ chord $CF + FE$ which again are $> CE$: consequently $CD > CE$, and D, on this side also, is without the curve.

236. Now, to apply this to the purpose in view, let the surfaces of the tooth *abc* (Fig. 93.) have their curvature determined by the evolution of the circle representing the wheel *AEF*, and let the curvature of *dg* and and *eg* be that of the involute of *BGH.* If the sides touch or have a common tangent at *n*, the perpendicular to this tangent is the line in which the pressure and reaction are exerted, and, when produced, it will be a tangent to each of the wheels, (235.) to the one at *A*, and to the other at *B;* and the momentum of both pressure and resistance will be the same as if the wheels were in contact, the one impelling the other by friction merely, and the action of each applied perpendicularly to the radius of the other at its extremity.

237. If the wheel and axle or the pulley be combined with the inclined plane, the energy of either will be multiplied by the cosecant or the cotangent of the plane's inclination according as the line of traction is parallel to the length or to the base.

238. A very powerful combination for raising weights may be formed by the screw and a wheel and axle, the wheel having teeth adapted to the step of the screw. Suppose the screw to be formed as in figure 94. and that *q* represents a tooth of the wheel. While the cylinder, which is confined to a motion of rotation, is turned in the direction *sx*, the end of the spiral *a* insinuates itself behind, or to the left of the tooth *q;* and the rest of it, as it comes round, being always more and more to the right, will cause the tooth to move in that direction till it is detached at the other extremity of the spiral. Every time the end *a* comes round it catches a new tooth, and thus there may be several teeth impelled at once; but it is evident that there cannot be many unless the screw and the teeth be both very fine, or the wheel very large. The screw thus employed is called the perpetual screw. The power is easily calculated. Using our former notation we express the power of the screw by $\frac{C}{h}$ and that of the wheel by $\frac{R}{r}$; consequently the two combined multiply the pressure $\frac{C.R}{hr}$ times.

Statical Principles applied to Architectural Structure.

239. The general principles of Statics enable us to understand and to calculate the strains to which the different parts of some of our most important mechanical structures are subject, and discover to the intelligent and scientific artist the most advantageous way of disposing his materials, so as to combine security with economy. Their application in this way we shall illustrate in some observations upon the construction of frames of carpentry, of which the formation of roofs, as one of the most important, may serve as an example: and, for the better understanding of this subject, it will be proper to premise some consideration of the mechanism of resistance to transverse strains, the most dangerous of those to which timber and other materials employed in the framing of such structures are exposed.

Transverse Strain.

240. Let $ACBD$ (Fig. 95.) be a rectangular beam or joist, of uniform texture, one of whose ends MC is firmly inserted into a wall, or otherwise fixed. We may consider it as composed of equal parallel fibres, as AD, CB, mq, &c. equal in strength and in close apposition to each other. The force with which it resists being pulled asunder in the direction of its length, is called its *absolute strength;* and if a weight p, suspended at some point B, be gradually increased till it is as great as the beam can bear without fracture, this is called the measure of its *relative strength.* It is proposed, then, having given the dimensions of the beam and its absolute strength, for which we must have recourse to experiment, to find, by calculation, on the hypothesis as to texture above mentioned, its relative strength.

241. A beam of wood and a rod of metal, though they be naturally straight, may be bent or incurvated with a tendency

to recover their original form. Their parts are therefore distensible and elastic; and it is observable in springs of all kinds that, within certain limits at least, their elastic force is proportional to their tension or dilatation. Considering, then, *ACB* as an angular lever, whose fulcrum is *C*, and in which the momentum of *p* is balanced by the contractile force of the distended fibres, which, like so many elastic threads, connecting the solids on opposite sides of the section *AC*, tend to prevent their separation, we naturally conceive the distension, and consequently the contractile force to be as the distance from *C*. Whenever the part *A a* then of the fibre *AD*, and the corresponding parts of the uppermost fibres through the whole breadth, are distended to the greatest degree that they can bear, the least addition to the weight *p* will cause the fibres of the whole section to give way. For when the collateral fibres of the upper horizontal plate or physical plane give way, those of the next inferior lamina will be distended to the utmost, and will of course give way also, because the momentum of resistance is now less than before, and they are less supported than the uppermost fibres were when exerting the same force. The weight *p*, it is obvious, will tend to produce fracture at any other vertical section also, as *EF*; but the beam being supposed of uniform texture and strength throughout, it must first fail at *AC*, the section against which *p* acts with the greatest momentum.

242. There is another hypothesis, that of Galilæo, respecting these contractile forces of the fibres, according to which, instead of their being as the distances from the axis of fracture, they are considered as all equal.

We shall give the investigation of the relative strength on each of these hypotheses, and first on that of Galilæo as the most simple.

243. Let *b* be the breadth of the section, *d* its depth, *l* the length of the beam from *C* to *B*, and *f* the absolute strength of the section for each unit of surface, or the force with which a rod of the same materials whose section is the square of the unit of breadth or length resists the being directly pulled

asunder; the absolute strength of the whole section will be fbd, and as this force is composed of parallel forces equably distributed, their centre of action will be the same with the centre of gravity of the surface AC considered as uniformly dense. Its height above the axis C, therefore, $= \frac{d}{2}$, and the momentum is $\frac{fbd^2}{2} = pl$, or $p = \frac{fbd^2}{2l}$.

244. On the other hypothesis (241.) the amount of force simultaneously exerted is less, but the diminution is partly compensated by an increase of the distance between the axis and the resultant. Let the rectangular physical surface AD (Fig. 96.) be considered as composed of equal physical lines, as AB, EG, &c. and the triangular surface ABC of AB, EF, and others which are always as the distance CE. The force of each of the uppermost filaments when the beam is just ready to break is f, the breadth of the fibre being taken as the unit of linear dimension, and may be represented by the same line AB on either hypothesis. But on Galilæo's the forces being all equal to this, will be represented by the sum of the equal lines composing the surface AD, while, on the other, decreasing as the distance CE, they will be represented by the sum of the decreasing lines as EF composing the triangular surface, which is $= \frac{1}{2} AD$. As the sum then is numerically represented by fbd on the hypothesis of equal distension, it will on the other be expressed by $\frac{fbd}{2}$. But as the forces are now as the distance from C, one of the extremities of the line denoted by d, the distance of their resultant from that point, or from the axis of fracture which it represents, will be $\frac{2}{3} d$ (92.) Hence the momentum is $\frac{fbd^2}{3} = pl$, and $p = \frac{fbd^2}{3l}$.

Otherwise, and more concisely, thus: The force of a fibre at the distance x from $C = \frac{fx}{d}$; the number of fibres exerting

1st, That whose section is the inscribed square, (Fig. 99.) is among those of given length that which contains the greatest quantity of matter: for $AD.AB$ is a maximum when $\sqrt{DE}.\sqrt{BE}$ is a maximum, *i.e.* when $DE.BE$ is a maximum, which is, when $DE = EB$, (Eucl. ii. 5.) or when $DA = AB$.

2dly, The strongest section is when $AD^2.AB$, or $DB.DE.\sqrt{DB}.\sqrt{BE}$ is greatest; that is, when $DE.\sqrt{BE}$, or $DE^2.BE$, or $\frac{1}{4}DE^2.BE$ is a maximum, which is, when $DE = 2BE$, or $DB.DE = 2DB.BE$, or $AD^2 = 2AB^2$. We may derive this conclusion from Eucl. ii. 5. by which it is evident that a rectangular parallelopiped, whose three adjoining linear sides are the three segments of a given straight line, is a maximum when these segments are all equal.

If we denote BD by d and AB by x, AD^2 will be $= d^2 - x^2$,

the measure of the area $= x\sqrt{d^2 - x^2}$,
the relative strength is $\div\, x(d^2 - x^2)$,

and by stating the fluxions of these expressions as $= 0$, or instead of the fluxion of the former that of its square as $= 0$, we shall obtain more concisely the same results.

249. *Cor.* 4. A hollow cylindrical tube is stronger than a solid cylindrical rod of equal length, of similar materials, and containing the same quantity of matter.

Let AF, (Fig. 100.) be the exterior diameter, BE the interior, and C the centre; draw BD a tangent to EHB at B meeting FGA in D, and join CD. The area of the annular section is $\pi.AC^2 - \pi.CB^2 = \pi.(CD^2 - CB^2) = \pi.BD^2$; so that BD is the radius of a circle equal in area to the ring. On the hypothesis of equal distension, the sum or resultant of the forces will in each case be the same, and its distance from the axis of fracture will in the case of the ring be FC, and in the case of the solid rod BD, so that the momenta resisting fracture will be as FC to BD.

On the hypothesis of distension proportional to the distance, let f, as before, be the force of one fibre at the distance d, $\frac{fx}{d}$ will be the force, and $\frac{fx^2}{d}$ the momentum of each fibre at the distance x, and the sum of the momenta will be for the same kind of matter $\div \int \frac{x^2}{d}$. Now, it will be proved in Dynamics, that $\int x^2$ for a circle whose radius is R, supposing x to be referred to a diameter, is $\frac{1}{4}\pi R^4$; and consequently if the distances be measured from a tangent, it will be $\frac{1}{4}\pi R^4 + \pi R^4$, (86, at the end.) So for a circle whose radius is r, if x be referred to a line whose distance from the centre is R, it will be $\frac{1}{4}\pi r^4 + \pi r^2 R^2$ (86.) Hence if the radius FC be denoted by R, BC by r, and BD by $k = \sqrt{R^2 - r^2}$, the strength of the ring will be to that of the solid rod in the following ratio:

$$\frac{\pi R^4 + \frac{1}{4}\pi R^4 - \pi R^2 r^2 - \frac{1}{4}\pi r^4}{2R} : \frac{\frac{5}{4}\pi k^4}{2k}$$

$$= \frac{R^2(R^2 - r^2) + \frac{1}{4}(R^2 + r^2)(R^2 - r^2)}{2R} : \frac{\frac{5}{4}k^2.(R^2 - r^2)}{2k}$$

$$= R + \frac{1}{5}\frac{r^2}{R} : k$$

$= \frac{6}{5}\, FC : BD$ nearly, when FC and BC are nearly equal; so that on this hypothesis the advantage is still more decidedly in favour of the ring.

250. We thus see it as an instance of wise adjustment, that the bones of animals and the quills of the feathered tribes are made hollow, by which they are at once strong and light.

251. Let the beam now rest on two props A and B, (Fig. 101.) and let a weight W be suspended at any point D, if the weight is just not sufficient to break the beam, there will be an equilibrium, and it will be in the same circumstances as if supported at A and B by two weights equal to the pressures at these points, and drawing upwards by means of strings passing over fixed pulleys. If the weight be gradually increased till the beam break, the fracture will be at D, where the weight is suspended, as we shall find immediately, and will begin on the under side at d: for the effect is the same as if the end AD were firmly fixed, and a force equal to the reaction at B drawing it upwards, or the end BD firmly fixed, and one equal to the reaction at A drawing AD upwards. Now if AB be taken to represent the whole

[illegible] pressures at *B* and [illegible] the strain at any point [illegible] simplest to employ the re- [illegible] lever: for if we employ [illegible] we must subtract the [illegible] the calculation rather more [illegible]. The demonstration [illegible] now be very easy.

[illegible] *BC*

[illegible] *3*

[illegible] *DB*

[illegible] *3*

[illegible] *AB*.

[illegible] at any point from a given [illegible] proportional to the rect- [illegible]: the strain from any load [illegible]: the strains at different [illegible] are as the rectangles under [illegible] of all occasioned by the [illegible] a given beam is when it is [illegible] then the same as if the [illegible], and ¼ *W* were suspended [illegible] the same as if half the beam [illegible] were suspended from its extre- [illegible] projecting beam carry at its ex- [illegible] bear *four* times as much on [illegible] supported.

[illegible] from a weight *W* uniform- [illegible] from the weight *W* of the [illegible] half as much as if the whole

[illegible] *B*, if unbalanced, would be [illegible] the weight of *BC*, or that [illegible] which may be considered as [illegible] the point of bisection, *E*. This weight is $W\cdot\frac{BC}{AB}$, and $EC = \frac{1}{2} BC$. Therefore the unbalanced momentum producing the strain, is $\frac{1}{2} W.BC - \frac{1}{2} W\cdot\frac{BC.BC}{AB} = \frac{1}{2} W\cdot\frac{AC.BC}{AB}$.

257. The strain arising from a pressure uniformly distributed is, like the former, greatest at the middle section, and is there $= \frac{1}{8} W.AB$, so that if a projecting beam will just carry without fracture a certain weight, it will carry *eight* times as much uniformly distributed over its length when both extremities are supported.

258. If the beam AB, resting on two props A, B, and projecting beyond them, have its ends fixed so that they cannot rise, it will bear twice as much in the middle as before. For being of uniform texture, it cannot now break at the middle without a simultaneous fracture at A and B. The strain at the middle we found to be $= \frac{1}{2} W.\frac{1}{2} AB$ (254.) That is, as before explained (255.) if it require W to break it in the middle when the ends are free, each half projecting from a wall would require $\frac{1}{2} W$. Suppose now the beam to be cut through at W, it will require $\frac{1}{2} W$ suspended at the extremity of each half to produce fracture at A and B; the sum of these is W; and it will require another W to break it in the middle as before when the cohesion of the fibres there is restored.

259. The transverse strain from a weight upon a beam of given length is diminished, when it is placed obliquely to the horizon, in the ratio of rad. : cos. Elev.

If BC, (Fig. 102.) represent the weight of any point of the beam, it may be resolved into BD and DC, the first producing a thrust upon the abutment, the second a transverse strain, and $BC : DC =$ rad. : cos. $BCD =$ rad. : cos. A.

260. *Cor.* The strain on rafters for roofs of the same width is as their length, when the covering is of the same weight per square foot.

Let W be the weight of covering of the same kind which would belong to AC in a flat roof of the same span. This is of course a constant quantity; $\frac{W}{\cos.A} = W.\sec.A$ is the load belonging to AB; $W.\sec.A.\cos.A = W$ is its transverse pressure, and $\frac{1}{8}\, W.L$ its momentum in producing a strain, (257.) which varies as L.

261. As a general corollary from the propositions now given, it may be remarked, that what succeeds very well with a model will often fail when tried without due precaution on a large scale. To illustrate this remark, which is of great importance to the engineer and the artist, suppose the homologous lines in a model of a machine or structure of any kind, as a roof, arch, bridge, or the like, and in the machine or structure itself to be as $1 : n$, and let the three dimensions of length, breadth, and depth in the model be denoted by the symbols already employed l, b, d. The materials being of the same kind, the weight or stress of all similar parts or loads, similarly placed, will obviously be increased in the larger work in the ratio $1 : n^3$. But the *absolute strength*, or that which resists a direct pull, being as the area of the section, will be increased only in the ratio $1 : n^2$: for $b\,d$ is to $n\,b \times n\,d$ in that ratio.

The *relative strength* resisting transverse fracture will also be increased in the latter ratio: for that of the model will be to that of the machine $= \frac{b\,d^2}{l} : \frac{n\,b \times n^2 d^2}{n\,l} = 1 : n^2$.

262. There isth us a limit to the increase of size in any structure composed of given materials. Let W be the greatest weight which one of the beams, or more generally one of the parts of a model, or any other structure which it is proposed to increase similarly, can bear; and let w be the stress or weight which it actually sustains: the stress that it must actually sustain in the enlarged structure, when every linear dimension is increased in the same ratio is $n^3 w$; but the greatest stress whether absolute or relative that it can bear is $n^2 W$: consequently $n^3 w$ must not exceed $n^2 W$, or we must have n not $> \frac{W}{w}$.

263. If it is wished to increase the smaller structure beyond this limit, all the dimensions cannot be increased similarly; we must alter the rctio for at least one of them. The depth of beams is the dimension which, generally speaking, and without a limitation imposed by some specialty in the circumstances, it will be advantageous to increase in the higher ratio; because a given addition to the strength in resisting transverse strains will in that case be made with the least expenditure of materials. Suppose then that it is wished to preserve the ratio of increase $1 : n$, but that a particular beam whose greatest strength is W would in that case be too weak to sustain the pressure $n^3 w$ to which it will be exposed: let its depth be increased in the ratio $1 : x$, while the breadth and length are each increased in the ratio $1 : n$. The strength will now be $\frac{nb \times x^2 d^2}{nl} = x^2 \cdot \frac{bd^2}{l}$, that is x^2 times what it was in the model; and it will now sustain $x^2 W$ and no more, consequently $x^2 W$ must be not $\angle$ $n^3 w$, or

$$x \text{ not } \angle\ n \,.\, \sqrt{\frac{nw}{W}} \,.\ (\text{Ventur. § 660.})$$

The same propositions and the general principle of the composition and resolution of pressure suggest some other important maxims. Such are the following:

264. *Avoid as much as possible transverse strains.* This is done, when we cannot supersede the forces producing them, by converting them into longitudinal pulls or thrusts. We shall see instances of this afterwards in explaining the construction of roofs, and wooden bridges. In the mean time we shall illustrate the maxim by reference to the manner in which the late ingenious Mr. Watt formed, at one time, the working beams of his powerful steam engines. *AB* (Fig. 103.) is the main beam, *C* the gudgeon or axis placed above it, *CD* and *CE* are strutts or braces, abutting at *C*, and whose ends *D* and *E* are connected with each other and with the ends of the beam, *A* and *B*, from which the piston and pump rods are suspended by iron rods *DE*, *AD*, *EB*. The pressures arising from the weights at *A* and *B*, which would produ

a very great transverse strain at the axis upon a simple beam *AB*, are resolvable into longitudinal strains in the directions *DA*, *AC*, and *EB*, *BC*; *DA*, and *EB* being stretched, and the the ends of the beam compressed. The force in the direction *EB* may be resolved into one stretching *DE* and compressing *EC*; and the corresponding one on the other side may be similarly resolved. Every strain here is a pull or a thrust, and there is no transverse strain at all. The distribution of the pressure will be illustrated by reference to Fig. 104. which requires no explanation.

265. *Avoid loading much a very obtuse angle, especially if the beams meeting there abut on parts naturally weak or ill supported.* Let the angle $= 2a$, and let the direction of the pressure W bisect it; then if P be the longitudinal thrust in the direction of either beam $P : W = \sin. a : \sin. 2a$, or $P = W \cdot \frac{\sin. a}{\sin. 2a} = \frac{W}{2 \cos. a}$, and if the angle be very obtuse $\sin. 2a$ will become very small, approaching to zero as a limit, while $\sin. a$ approaches to unity, or $2 \cos. a$ will approach to zero as a becomes more and more nearly a right angle. Hence a finite weight W may produce a thrust exceeding any force assigned on the rafters which form the angle, and these will either be crippled, or injure and perhaps overset their supports.

Let *AC* (Fig. 105, 106.) be a beam projecting obliquely from a wall or other fixed support *AG*, and supporting a weight at *C*. Let this be represented by *CD*, supposed equal in the two figures. Let *BC* be a brace or strutt supporting *AC*; resolve *CD* into *CE*, *CF*, as represented in the figures, and it will be evident that the disposition in Fig. 105. is by much the more advantageous. In the case represented by Fig. 106. there is a strong distension of *AC* and compression of *CB*. When *BCF* is a right angle the distension of *AC* will be moderate, and the compression of the brace represented by $EC = DF$ the least possible for a given value of the appended weight and a given position of *AC*.

266. *If a part is to be cut out of a beam for the insertion of*

another, let it be done on the side that becomes concave by any strain to which it may be exposed.

Suppose *AD* (Fig. 107, 108.) to be a projecting beam, with a weight *p* suspended from the end *BD*; it is plain that when the cavity is filled up by the inserted beam, of which *FEHG* is a section, the part *Ffg G* has more momentum to resist fracture in the situation denoted by Fig. 107. than in that indicated by the other. The sum of the forces called into action is greater, and the resultant at the same time is more distant from the axis of fracture. If these were beams supported at both ends like joists, and loaded in the middle, the second would be the stronger; the axis of fracture being then on the upper side.

267. *If a beam that has a weak part, as when two are formed into one by scarfing, is to be strengthened there by the addition of another, let the addition be made on the side that becomes convex when strained.*

Let *AB* (Fig. 109, 110.) represent the scarfed beam, and *CD* the piece added fixed to the former by iron straps or otherwise, so as to strengthen it for opposing some transverse strain. If the beam project, and a weight be hung on at *B*, the part *CD* is in the first case strained so as to exert more nearly its utmost force than the weakened beam *AB*, which therefore receives the most aid that it can from the additional piece. In the second case, the contrary happens: the weaker beam is exposed to the greater strain, and *CD* does not support it advantageously. The case, as before, is reversed if these are beams supported at the ends and loaded at the middle.

268. *Avoid an unnecessary load occasioned by giving to some parts an useless strength.*

The possibility of this we shall illustrate by a few problems.

If *AB* (Fig. 95.) represent a projecting rectangular beam, loaded at *B*, and if the section *AC* be strong enough to resist fracture, the sections *EF*, *GH*, &c. must be unnecessarily strong; for the momentum of *p* to produce fracture is always less in proportion as the lever by which it acts is shorter. If

p were just sufficient to break the beam at AC, it would not be sufficient to break it at EF, and still less at KH. One or both of the dimensions of breadth and depth then, may be gradually diminished from C towards B. By this process of reduction materials may be saved that would otherwise be uselessly expended. The beam is even relatively strengthened by the reduction. There is a strain on AC from the weight of the beam itself, which becomes of course less when every section is reduced to the same relative strength. Let AB now represent a square beam supported at both ends and loaded between them; a similar reduction may be made with safety and advantage. For instance, if the load be uniformly distributed, the strain will be every where as the rectangle under the segments of the length. Now this rectangle (Eucl. ii. 5.) gradually decreases from the middle, where it is a maximum, towards each end. The strength of the section then should be proportional to this rectangle; to make it greater is useless, and even hurtful.

269. *Prob. I.* To make a beam equally strong throughout to resist a transverse strain occasioned by a given weight applied at the extremity.

On each of our hypotheses respecting the cohesive forces, $fbd^2 \doteqdot pl$, and here p is given and also f as we refer always to the same beam whose texture is supposed to be uniform. Hence $bd^2 \doteqdot l$. Suppose, moreover, that the depth is constant, then $b \doteqdot l$, and if the sides be vertical planes, the horizontal sections will be triangles. The beam in short is a triangular prism or wedge with the base at the wall, the edge at DB, and the triangular ends above and below. But if the breadth be constant while the depth varies, $d^2 \doteqdot l$; the beam may have an indefinite variety of shapes, but if either the upper or under surface be a horizontal plane, the other must have the curvature of the common parabola. If the sections are to have neither breadth nor depth constant, but in a constant ratio to each other, so that $b = md$; d^3 is $\doteqdot md^3 \doteqdot bd^2 \doteqdot l$; and the curvature of one of the sides if the other be a plane will be that of the cubical parabola.

270. *Prob. II.* To make a projecting beam equally strong to resist the stress arising from a weight uniformly distributed over its length.

Suppose the sides to be parallel vertical plains; the vertical section in the direction of the length must be a triangle; for the weight of uniform covering of equal breadth between any given section and the extremity must be as the length of that part, and the distance to its centre of gravity is also as that length being always equal to its half. Hence $b\,d^2 \doteqdot l^2$, and as b is constant by supposition, $d^2 \doteqdot l^2$ and $d \doteqdot l$.

271. *Prob. III.* To make a beam of given breadth supported at both ends, equally strong throughout to resist the stress from a weight uniformly distributed.

b being given, d^2 must be every where as the rectangle under the segments, (256.) and the depths must be the ordinates of an ellipse of which the length of the beam is an axis.

Of Roofs.

272. The simplest form of a roof is that of two equal rafters AB, AC (Fig. 111.) abutting against each other at A the vertex, and resting on the walls at B and C. If the weight of each rafter and of the covering which it supports, as lead, slates, tiles, &c. be supposed to be uniformly distributed, it will press with the half of its weight vertically at B or C, and with the other half at A. Hence the load which may be conceived to rest on the angle A is the half of that of the whole portion of the roof supported by AB and AC, including the weight of the rafters themselves. Complete the rhombus $ABGC$, and let the weight at A be represented by the diagonal AG. It may be resolved into longitudinal thrusts AB, AC, and each wall will be pressed outwards with a force represented by FB or $FC = \frac{1}{2} A \operatorname{tang}. m$; A denoting the load at the angle of the same name. The vertical pressure at each of the points B, C, resulting from the pressure at A, that is, from the resolution of the longitudinal thrusts of the rafters

is *AF* or ½ *A*, so that altogether the vertical pressure at each of these points is *A*, and the two together = the whole weight of the roof. This, however, supposes the horizontal thrusts to be withstood, so that those in the directions of the rafters may take effect.

273. This simple form may suffice when the rafters at *B* and *C* are to rest on the ground, and to form a shed merely, or when they are to rest on any immoveable obstacles in which their ends are so firmly fixed as to prevent them from sliding. But they could not safely be made to rest, so secured, on the tops of walls, on which they would act with a momentum proportioned to the height of the walls and to the horizontal thrust, tending to overset them. In the case of very flat roofs, composed as here of abutting rafters, the strain would be enormous. This horizontal thrust is in such cases counteracted by what is called the tie-beam *BC*, which connects the lower extremities of the rafters, and then the whole frame of the roof rests on the walls like a simple joist. *BC* being in a state of distension, its place as a tie merely might be supplied by a rope, chain, or any thing flexible, if of sufficient strength, and, as to sense, inextensible. It is, however, generally a beam or solid parallelopiped of wood to support a floor, and may thus be exposed to a considerable transverse as well as longitudinal strain. This is ingeniously counteracted by the introduction of what is called a king-post *AF*, acting as a tie, and not resting on *BC* and supporting *A*, as the term *post* seems to indicate. It is dependent from *A* where the rafters are joggled into it, or abut against it obliquely as represented in Fig. 112, the upper end being a truncated wedge like the key-stone of an arch. An inextensible chain might perform the office of the king-post. This addition then, it will be obvious, converts the transverse strain on the middle of the tie-beam into a longitudinal stretching of the king-post, *that* of course like any other vertical pressure at *A*, into longitudinal thrusts of the rafters, and these again finally into a stretching of the tie-beam in the direction of its length which it bears much better than a cross strain.

The rafters are also exposed to a transverse strain proportional to their load, their length, and the cosine of their elevation. This is counteracted by what are called strutts or braces *FD*, *FE*, abutting on the king-post at *F*, as may be represented by Fig. 112. inverted. The stress on these beams is a thrust, and by their composition they produce a resultant stretching *AF*, and finally the tie-beam, as before.

274. When there are more rafters than two, they form what may be called a polygonal roof. The common appellation is a kirb-roof. The parts ought evidently to be joined together in a state of equilibrium; for, though it will be an unstable one, and will require ties and braces, these will, in that case, have nothing to do but to resist an incipient change of form; whereas, if the parts are not in equilibrio at first, there will be an unnecessary and often violent strain on these subsidiary parts.

275. If the beams and their loads be uniform in structure and distribution, the vertical pressure at each angle will be half the sum of the weights of the two rafters there mutually abutting, with their proportional shares of the load of covering; and if the loads be otherwise disposed in any given manner, they may be resolved (31, 34.) into their equivalent vertical pressures at the angles. The conditions of equilibrium for the polygonal roof are then the same as for the funicular polygon loaded with weights, a particular case of *Prop.* X. *Cor.* 3. (72.) We may suppose that polygon loaded with weights to be inverted, and the sides to become stiff with moveable joints: then if *A*, *B*, *C*, &c. denote the vertical pressures at the angles of the same names, and the segments of these angles made by the verticals be α, α'; β, β'; γ, γ', respectively $\frac{A}{\cot.\alpha + \cot.\alpha'} = \frac{B}{\cot.\beta + \cot.\beta'} = \frac{C}{\cot.\gamma + \cot.\gamma'}$, &c. equations which not only give the ratios of the pressures at the different angles as dependent on the positions of the rafters, but express the equality of the horizontal thrust at all the angles and abutments, (73.)

276. Let *DACE* (Fig. 113.) be a quadrangular roof, in which *AD* and *CE* are equal, and similarly inclined to the

horizon, and *AC* of any length parallel to it. Also let *A* and *D* denote the weights pressing vertically at the points *A*, *D*.

The horizontal stress on the supports, or on the tie-beam, is here $\frac{A}{\text{cot.} n + 0} = A$ tang. *n*, for *GAC* is a right angle, and its cotangent $= 0$.

The vertical stress at *D*, arising from *A*, is = horizontal stress × cot. $n = A$ tang. *n* cot. $n = A$, and the whole vertical stress at $D = A + D$.

This roof, in its simplest state, is not secured against accidental inequality of pressure. Should an undue load be imposed on $\angle A$, as, by the wind blowing strongly upon it, a fall of snow partially drifted, or the like, that angle would be depressed, and the equilibrium would cease. This is prevented by what are called queen-posts, represented by *AG* and *CH*, which are beams attached at *G* and *H* to the tie-beam, and, on the joggles of which the oblique rafters abut, as formerly represented in the case of the king-post, while the horizontal or truss beam *AC* is mortised into the sides of the heads next to itself. When the transverse strain on the oblique rafters is great, it may be opposed by the insertion of braces *GL*, *HM*, abutting on the lower extremity of the queen-posts. A good support to these braces is obtained by the insertion of a beam *NO*, in contact with the upper surface of the tie-beam, and extending between *G* and *H*, so as to prevent their approximation. This piece *NO* is called a *straining sill.* If the truss beam *AC* require support to counteract a transverse strain, it may be given by another straining sill *PQ*, extending like a smaller truss-beam below it, on which the braces *NP*, *OQ* abut at *P* and *Q*, while their other extremities rest on joggles of the queen-posts at *N* and *O*. With this addition, the straining sill *NO* becomes less necessary.

277. Let *DABCE*, (Fig. 113.) be a pentagonal roof, having $BA = BC$, and $AD = CE$, the parts on each side of *BK* being perfectly symmetrical.

Here *GAB* is the supplement of *m*, and cot. $GAB = -$cot. *m*, so that (275.)

$$\frac{B}{2\,\text{cot.}\,m} = \frac{A}{\text{cot.}\,n - \text{cot.}\,m}$$

$$\text{Hence } B\,\text{cot.}\,n - B\,\text{cot.}\,m = 2\,A\,\text{cot.}\,m.$$

$$B\,\text{cot.}\,n = (2\,A + B)\,\text{cot.}\,m.$$

$$\text{or } B\,\text{tang.}\,m = (2\,A + B)\,\text{tang.}\,n.$$

Suppose now that the rafters and their loads are all equal: then $B = A$, and tang. $m = 3$ tang. n.

In this case, if the span and height proposed be given, the roof may be easily constructed. Let half the width $DK = b$, the height $BK = h$, and let x be the length of each rafter. Then

1st, x (sin. m + sin. n) $= b$.

2d, x (cos. m + cos. n) $= h$.

3d, tang. $m = 3$ tang. n, or $\frac{\text{sin.}\,m}{\text{cos.}\,m} = \frac{3\,\text{sin.}\,n}{\text{cos.}\,n}$, or sin. m cos. n $= 3$ sin. n cos. m.

By Eq. 1. and 2. $\frac{b}{h} = \frac{\text{sin.}\,m + \text{sin.}\,n}{\text{cos.}\,m + \text{cos.}\,n} = \text{tang.}\,\frac{1}{2}\,(m + n.)$

By Eq. 3. $\frac{1}{2}$ sin. $(m + n) + \frac{1}{2}$ sin. $(m - n) = \frac{3}{2}$ sin. $(m + n)$ $- \frac{3}{2}$ sin. $(m - n)$ or sin. $(m - n) = \frac{1}{2}$ sin. $(m + n)$.

Thus $m + n$ and $m - n$ are found; thence are easily derived m and n, and, by Eq. 1, or 2, x.

278. To prevent the equilibrium from being subverted by one of the angles A,C yielding outwards, they must be connected by a tie-beam AC. The structure then consists of the triangular and quadrangular roofs combined.

Let the weight of the tie-beam $= W$; B produces at A the pressure outwards $\frac{1}{2}\,B$ tang. m, and the vertical pressure $\frac{1}{2}\,B$. To this vertical force is to be added A, the pressure resulting from AD and AB, with their loads at the point of the same name, and also $\frac{1}{2}\,W$, so that the whole vertical pressure at A is $A + \frac{1}{2}\,(B + W)$. Hence the horizontal thrust outwards at D is $= (A + \frac{1}{2}\,(B + W))$ tang. n, (276.) By the reaction from D towards A, a force equal to this is exerted inwards at A, and an equal one at C, compressing the tie-beam. This counteracts the outward thrusts at A and C, arising from B and $= \frac{1}{2}\,B$ tang. m, as before stated, so that the longitudinal strain of the tie-beam is $\frac{1}{2}\,B$ tang. $m - (A + \frac{1}{2}\,(B + W))$ tang. n, and will operate as a distension or compression according as the algebraic sum of these terms is positive or negative.

The vertical pressure at each abutment D and E is $A + D + \frac{1}{2}(B + W)$. (See Vent. § 602.)

279. An equilibrated roof of this or any number of sides, may be constructed experimentally by threads with weights suspended, so as to give it a polygonal form. To construct the last mentioned roof in this way, take a string of any convenient length $DABCE$, and divide it into four equal parts, suspending any equal weights from the three points of division A, B, C, and fixing the ends D and E in the same horizontal line. Then if DK be not to KB in the given ratio of $b : h$, lengthen or shorten the interval DE till these two lines be in that ratio. Suppose now the thread, becoming rigid, to be inverted, so that the pulls are converted into thrusts, it will still be in equilibrio, and will exhibit the proposed roof in miniature. The two figures will be similar polygons: therefore, as half the span of the threads to one of the divisions $= \frac{1}{4}$ of the string, or as twice the span of the polygonal string to its length, so will b, half the proposed span of the roof, be to x, one of the sides of the roof; and the angles will be the same for the one as for the other.

Of Wooden Bridges.

280. The principles that explain the structure of roofs may be further illustrated by an examination of the construction of wooden bridges of various forms.

The simplest form of a bridge of this sort is when one or two beams or planks, as AB, (Fig. 114.) are laid from side to side of a brook or narrow ravine. But if the length of the beam is to be considerable, and more especially if the bridge of which it may represent a section, is to be subjected to any considerable stress, as by cattle or carriages passing along it, there will be a transverse strain to be counteracted. This may be done in various ways. If there are any means of firm support at a considerable height above *A* and *B*, as at *P* and *Q*, it may be done by chains connecting these supports with the middle point *D*. Their connexion with this point in particular is to be preferred, because it is there that the beam is exposed to the most violent strain, (255, 257.); and if *P* and *Q* are but little elevated above *A* and *B*, the chains will be subjected to a severe strain from the obtuse angle at which they meet, (265.) *AB* may also be supported from beneath by braces meeting at *D*, and abutting on the banks below *A* and *B*, if the banks have considerable depth; but if they have not this, the braces will meet at an angle too obtuse, and a violent thrust will be occasioned, which may cripple them, or derange the equilibrium, by the yielding of their abutments. A more generally convenient way, therefore, will be to employ the frame or truss represented in the figure, and constructed like a common triangular roof. *AC* and *BC* are rafters abutting at *C*, on the joggles of a king-post *CD*, which, by the sagging of *AB*, will be strained as a tie. It may have a broad stirrup at *D*, to embrace *AB*, instead of being mortised into it, which would weaken the beam. Or the plane of *AC*, *CB*, which contains also *CD*, may be a little to one side of the parallel vertical plane in which *AB* lies, and then *CD* may support one end of a transverse horizontal beam, reaching from side to side, and supported by a similar framing at the other.

Another form, which requires less elevation of the king-post, may be that represented in Fig. 115.

The bridge, when supported in the middle in either of these ways, will bear six times the load that it could safely be subjected to without such aid. Dr. Robison says *ten* times the load; but I suspect some oversight in his estimate.

If the load is to be so great that even the halves *AD* and *DB* cannot safely be exposed to their shares of it, we may truss these parts in the same way as, according to the first method, the middle is supported; and this will produce a form not inelegant and more pleasing to the eye than that of the simpler structure. The sketch (Fig. 116.) represents it.

Here the addition is that of the rafters *AG*, *DG* abutting at *A* and *D* and supporting *H*, the middle of *AD*, by a new king-post *GH*, and a similar truss on the other side *DKB*. The abutment of the rafters *GD* and *KD* on *D* produces a resultant stretching *CD*, which protects *AB* there from any transverse strain from this source.

Another advantageous way of supporting *AB* is that exhibited in Fig. 117. The introduction of the straining beam *DE* enables us to give a less oblique position to the braces *CD*, *FE*, than if they were to meet. If more support is wanted, it may be given by an addition of two other braces, with a shorter straining beam immediately below *DE*.

For a river, piers may be erected at convenient intervals, and the construction now described, or some of the former, repeated from pier to pier; and for a wide ravine or river, a number of firmly built trapezoidal frames of carpentry may be made to support each other by mutual abutment, like the stones of an arch. Bold and elegant structures of this kind have been sometimes made of iron.

Of Arches.

281. An arch is composed of truncated wedges, as *BFF'B'*, *BFGC*, &c. (Fig. 118.) which, considered as composed each of a transverse course of stones extending from side to side, are named *voussoirs*. The uppermost, or that at the crown of the arch *BFF'B'*, is called the key-stone, and the surfaces *BF*, *CG*, &c. where the voussoirs meet, are called the joints. These wedges, though they were perfectly smooth, might be made to preserve, by their mutual pressure, a state of equilibrium, each endeavouring to insinuate itself between the adjoining two. Their weights acting in vertical lines through the centres of gravity, may be resolved into forces perpendicular to the sides, and the perpendicular forces meeting at the joints must be equal. The ratio of the weight of any part of the arch, then, to the pressures which it occasions at the

joints forming its terminations, may be found by the following simple geometrical construction.

282. Let OA be a vertical line bisecting the parallel sides of the end of the key-stone; from any point O in that line draw Ob, Oc, Od, &c. parallel to the joints BF, CG, DH, &c. and meeting a horizontal line NN' in b, c, d, &c. Then will the triangles Oab, Obc, Ocd, &c. have their sides perpendicular to the directions of the balanced forces, and represent them in magnitude, (20.) We may, for the greater simplicity of investigation, suppose the key-stone bisected by a vertical joint AA', Oa will then represent the horizontal thrust at AA', and which is common to all the joints, ab the weight of $AA'FB$, half the key-stone, and Ob the pressure perpendicular to BF. In like manner, bc, Ob, and Oc will represent the weight of the voussoir $BCGF$, and the pressures on the joints BF and CG arising from its weight, and that on the same scale as before, because the two pressures of which Ob is made the representative are equal, as a condition of equilibrium. It is plain that, with respect to the weight and the pressures on the terminal joints, we may consider any part of the arch, for instance the half, $AEKA'$ as a single wedge; and, in this case ae represents the weight, Oa the horizontal thrust at either extremity, and Oe the perpendicular pressure on the abutment LM, the knowledge of which is necessary to the secure construction of the pier. These qualities are more accurately determined in practice by an easy trigonometrical calculation founded on this construction. Let β be the angle of the key-stone, or the inclination of the sides BF, $B'F'$, and let w be the weight of its half $ABFA'$; then

Tang. $\frac{1}{2}\beta$: rad. $= w$: horizontal thrust, which call A, and let the angle which any joint makes with the vertical be called θ; the weight of the arch from the crown to that joint will be $= A$ tang. θ, and the pressure perpendicular to the joint $=$ A sec. θ.

283. In the theory of equilibration, it is usual to conceive the series of truncated wedges as indefinitely thin, so that

their upper and lower extremities, forming polygons of an indefinite number of sides, may be regarded as curves. That which bounds the lower extremities is called the *Intrados*, and the outer one the *Extrados*. The chief problem is, having given the proposed intrados and the forces to which the arch is to be subjected, in quantity and direction, to find the extrados.

284. The voussoirs being supposed to be all formed of the same materials, their weights will be as their solid contents; that is, since they are prisms of the same length, as the areas of their ends. Let *AN* and *FM* (Fig. 119.) be portions of the extrados and intrados, and *CBED* an indefinitely thin wedge whose sides meet in a horizontal line, represented in projection by the point *O*. Let $\angle BOA = \theta$, and consequently $\angle BOC = d\theta$. Also, let $OB = R$, $OE = r$, $BE = v$, the values of the two latter for the crown of the arch being l and k respectively. The weight being $\div$ area $BEDC$ will be $\div OBC - OED \div \frac{1}{2} OB^2 d\theta - \frac{1}{2} OE^2 d\theta \div ((r+v)^2 - r^2) d\theta$. But by (282.) it is obvious that the weight of the elementary voussoir is $\div d.\ \text{tang.}\ \theta \div \frac{d\theta}{\cos.^2\theta}$, whence $(r+v)^2 - r^2 \div \frac{1}{\cos.^2\theta}$ and $= \frac{c^2}{\cos.^2\theta}$, c^2 being a constant quantity to be afterwards determined. We have therefore

$$R^2 = (r+v)^2 = \frac{c^2}{\cos.^2\theta} + r^2.$$

$$R = \sqrt{\frac{c^2}{\cos.^2\theta} + r^2}.$$

$$\text{and } v = R - r.$$

At the crown of the arch $\theta = 0$ and $\cos.^2\theta = 1$, therefore $c^2 = (l+k)^2 - l^2 = k^2 + 2kl$.

285. As an easy example of the application of our formula, let the intrados be a horizontal straight line as *MN*, (Fig. 120.) the lines representing the joints converging to some point as *O* below it.

Let $AD = k$, $OD = l$, $OE = r$, and $OB = R$. Then $c^2 = k^2 + 2\,k\,l$, $\cos.^2\theta = \frac{l^2}{r^2}$, and

$$R = \sqrt{\frac{r^2}{l^2}\,(k^2 + 2k\,l) + r^2} = \sqrt{\frac{r^2\,.\,(l+k)^2}{l^2}} = \frac{r\,.\,(l+k)}{l},$$

$$\text{or } l : l + k = r : R,$$

and the locus of the point B is evidently the straight line AM' parallel to MN.

To enter into the more complex calculations necessary when the intrados is a curve line, would be irksome to the greater number of students in a first course, and would exhaust much time that might be, in such circumstances, more advantageously employed. In addition to what is contained in the treatises recommended in the Preface, those who wish to prosecute farther the doctrine of Arches, and to extend the elementary views that have been given respecting some of the preceding topics, will consult with advantage the article "BRIDGE," with the additions to the former article "CARPENTRY," by Dr. T. Young, in the last *Supplement* to the *Encyclopædia Britannica*, Vol. II.

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ELEMENTS

OF

THEORETICAL MECHANICS.

DYNAMICS.

286. THE name of this science is derived from the Greek word signifying *force;* but, forces being known to us only as characterised by the motions which suggest them, our real object in the prosecution of it is the discovery and classification of the phenomena of sensible motion. The phenomena of nature, even when considered merely as phenomena of motion, are, in their complex state, distinct and varied, to an extent that no language could express and no memory retain; but, when they have been successfully analysed, such resemblances are detected among the principles, or least complex assemblages, that they can be arranged under a few general heads; and, as resemblances detected amidst variety, especially when extensively prevalent, are naturally, and even necessarily, ascribed to some presiding influence, and considered as the result of regulation, the generalized expressions of what observation and experiment have discovered in the composition and consecutive order of past events, considered as indications of superintending intelligence, and declarations of what is yet to be, are, in the language of Philosophy, denominated Laws of Nature. Those that fall to be particularly considered in this branch of mechanical philosophy, are the laws of Motion.

287. The most general of these, as laid down by Sir Isaac Newton in his *Principia*, are three:

1. "Every body continues, when at rest, in a state of rest, and, when in motion, in a state of uniform and rectilinear motion, unless it be affected by some force impressed."

2. "Change of motion is always proportional to the motive force impressed, and is made in the direction of that force."

3. "There is always a reaction equal and contrary to action; or the actions of bodies are mutual, equal, and opposite."

288. Though these laws are few and simple, they are of most extensive application, reaching from the phenomena that we regard as the most familiar, to that sublime elevation of science from which the disciple of Newton contemplates with admiration, in the order of mechanical connection and harmonious dependence, the magnificent machinery of the solar system. It is of the highest importance then that we should carefully study them, and endeavour to acquire a clear and accurate conception of their import.

289. *Explanation of the first.*—If two bodies be loosely connected, and so situated that the one supports the other; and if, by a sudden impulse, the supporting body be put in motion, the other will be left behind; and if, on the other hand, they be both in motion with a common velocity, and the undermost be suddenly stopped, the uppermost will go on in the same direction in which it previously moved. Thus, if a vessel of water be suspended by a rope, and we strike it towards the one side, the fluid, from its slight connection with the vessel, will tend to remain in its place, and will be spilled over that side on which the blow was struck; and if, while the whole is in motion, the vessel be suddenly stopped, it will be spilled over the side towards which the motion is directed. Thus also, when a person stands upright in a boat which is suddenly pushed off from the shore, he feels a tendency to fall over the stern, and he falls towards the prow when the boat in motion with considerable velocity strikes unexpected-

ly against a bank. An instance of the operation of this law is exhibited whenever one body is detached from another while in motion. The detached body is always projected with the velocity that was common to the two at the instant of separation. A fall is almost unavoidable in springing to the ground from a carriage, when dragged by horses at full speed. The danger arises from the tendency of the rest of the body to continue the motion when the feet are stopped by the ground, and would be much less could a person in such circumstances alight on a sheet of smooth ice of some extent. The whole body would then continue in motion, and would be *gradually* brought to rest, by the want of perfect smoothness in the surface with which it was in contact. If a person standing on a ship's deck, throw a ball vertically upwards into the air, and keep his hand in the same relative position, the ball will fall back into the hand, whether the ship be at rest or in motion, provided that in the latter case her motion be uniform and rectilinear. It is owing to this same law that we can throw a stone from the hand, or discharge it from a sling. We hold the stone fast, moving the hand briskly, till we have communicated to it and to the stone together a certain common velocity; we then stop the hand, and the stone continues its motion in the direction of a tangent to the curve which it was describing at the instant of separation; and we can never in this way give to the stone a greater velocity than we can give to that part of the hand in which it is held. But we can communicate a much greater velocity, by means of a sling. When, by continued pressure, we have brought the hand, and the part of the sling which is held by it, into a state of as rapid motion as possible, the stone in the loop at the farther extremity must have a motion still more rapid, as describing a curve at the extremity of a greater radius; and this motion, be what it may, will always be that of the stone in the first moment of separation, when the string breaks or we let go one of its extremities.

290. These familiar phenomena plainly indicate a tendency in bodies to continue in their present state, whether it be of motion or of rest, the property which we term the *inertia* of matter.

291. The proposition under our consideration, however, expresses more than the familiar and ordinary observation of such phenomena is sufficient to prove. It is indeed very difficult to prove it, in all its generality, to one's entire satisfaction, in the first stages of his progress in this study. There is perhaps no moving body whose state as such is not affected by circumstances involving the consideration of all the three laws of motion. Bodies have a mutual influence on each others mechanical condition, and there is no such thing as permanency of motion in both direction and velocity, for any sensible time, actually observed. How is it then that we are led to the conception and belief of this first law? It will not do, as it should seem, to say, with a late highly respectable author, that by the constitution of our nature we are taught to consider every deviation from uniform rectilinear motion as indicating the agency and measuring the intensity of a changing force. This would be merely a conventional law of philosophical nomenclature, to be observed in a sort of hypothetical delineation of phenomena. It would not express the fact or contingent truth that matter *is* inert, and that every change in the condition of a body with respect to motion has a reference to something extrinsic, and would cease were every other portion of matter annihilated. To this latter statement, however, as the basis of something more than the mere demonstration of relations between certain conventional terms, we are led by a careful and refined comparison of phenomena.

292. In the *first* place, we observe a given motion to be *more* retarded in certain circumstances than in others. This suggests to us the idea of *obstacles* to motion, which being in a still greater degree removed, the velocity would approach nearer to permanency and uniformity. Thus if we attempt to make a body slide along the ground, we scarcely succeed; but if we project it along a smoother surface as that of a sheet of ice covered slightly with snow, the motion will continue

for a considerable time. This suggests to us the probability that if the surface were made smoother still, as by sweeping away the snow, the diminution of rate would be less, which on trial is found to be the case. In prosecuting such experiments through various stages of less and more perfect artificial preparation, it would occur to us as probable that, if we could make the surface perfectly smooth and remove entirely the resistance of the air, which is found to be another obstacle, the rate of motion would suffer no change. This condition, however, we cannot experimentally fulfil. One of the nearest practicable approaches to its virtual fulfilment would be to drop a perfectly spherical ball from a balloon, which has been for some time in free motion, and to observe whether it strikes the ground exactly under it. So a motion of rotation communicated to a wheel and axle, continues for a very short time, without some extrinsic maintaining power, if the axle or its supports be rough; longer, if they be smooth; longer still, if we give the wheel a mechanical advantage against the friction, by supporting it on friction wheels; and longest of all, if we remove as far as possible the resistance of the air, by communicating motion to an axle, thus circumstanced, in the exhausted receiver of an airpump. But as we cannot do more than *diminish* the friction and *rarefy* the resisting medium in which the motion must take place, it is always destroyed sooner or later, and the extent of our legitimate inference and conviction is that the law is *probable.*

293. *Secondly,* Although there is never presented to our observation a motion that we are assured is either perfectly rectilinear or perfectly uniform, we have in all the celestial motions instances of permanency, and in these as well as in cases with which we are more familiar, we discover, though not immediately, yet indirectly and by analysis, the tendency both to uniformity of rate and sameness of direction. The planetary motions of rotation are strictly constant. In the path of the moon's revolution round the earth, we perceive a constant deflection towards the earth's centre. If, abstracting the consideration of some minute inequalities, we analyse

this curvilinear motion, we shall distinguish in it two tendencies, the one directed towards the centre of the earth, and the other in the direction of the tangent to the lunar orbit, the former varying according to a determinate law: in other words, a tendency regulated by this precise law combined with one, the effect of which separately would be an uniform and rectilinear motion in the direction of the tangent at any given point, will, if we disregard some minuter and less perceptible tendencies towards other bodies of the system, account for or completely represent the moon's actual motion round the earth at every other point of her path. Were the moon's *independent* motion not uniform and rectilinear, but having some unknown tendency to curvature, without respect to the position of our planet, we should in the process of analysis above mentioned estimate the amount of the central deflections by reference to a wrong standard, and thus in reality taking them at random, we should scarcely find them to be comprehended under any precise rule, and should as soon expect to draw the numbers from a lottery wheel in the order of some determinate progression. In the same manner, we find all the motions of inanimate matter that have yet been analysed to be such as may be synthetically reproduced by the combination of a certain number, one of which is always in its tendency uniform and rectilinear, while the rest are severally referable to the positions of some other body or bodies, and regulated by determinate laws. When our minds shall by habit be more familiarized to physical in analysis, and inductive ~~re~~ference, we shall be in a condition to pronounce as a legitimate result that the law is *certain*.

294. *Explanation of the second.*—A change of motion in a given body may refer either to the velocity, or to the direction, or to both together. If a body be so situated that when free to move it acquires the velocity v, in a certain direction, in the time t'', and if the experiment be repeated, with this and this *only* variation of circumstance, that the body is *projected* in the direction of the previous motion, or in its oppo-

site, there will be produced in the one case an acceleration, in the other a retardation of the projectile motion to the amount of the velocity formerly generated from rest in an equal time, and this independent of the particular value of the initial or projectile velocity itself. The velocity generated from rest, the acceleration, or the retardation indifferently, may be assumed as indicating the agency and measuring the intensity of the force; but *one* instance of the change, supposing that such could *with perfect accuracy* be observed, is sufficient for this purpose; and what the second law of motion, as here applied, declares, is the fact above stated, that the change in question is altogether independent of the particular velocity of the given body which it modifies. To say that the change of velocity is as the force, is the same thing in effect as to say that the change produced in the unit of time is the same with that other change which was *assumed as the measure* of the force. But before we can understand fully the application of this second law, we must understand what is called the composition of motion, and what is meant, in the cases where that consideration is involved, by change of motion.

295. Let a straight line AG (Fig. 121.) move always parallel to one given in position, so that its extremity A shall describe the line AC uniformly in a certain time T; and in that same time let a point describe uniformly the line AG; this point partaking of both motions shall describe uniformly the diagonal AK.

In the time t, let the line AG have reached the position BH, and let the point which describes AG have passed over AD; complete the parallelogram $ABED$, and at the end of the time t the point will be at E; for BE represents AD in its new position.

$$\text{Now } AC : AB = T : t = AG : AD,$$
$$\text{alt. } AC : AG = AB : AD,$$

so that the parallelograms AK, AE are similar and similarly situated, having also a common angle, and therefore are about the same diameter, (Eucl. vi. 26.) that is, the point E

is always found in the diagonal AK. Its motion is, moreover, uniform when the constituent motions are uniform;

$$\text{for } AK : AE = AC : AB = T : t,$$

or the space described is always as the time, and equal spaces are described in any equal times.

296. When the change upon a motion is, as far as our senses can distinguish, abrupt and instantaneous, its measure is always understood to be that motion which, entering as now described into composition with the former motion, produces the new one; that is to say, when the motion AC is changed into AK, the change of motion is AG; and on the authority of that law which we are now explaining, we may add that the changing force is that which would produce the motion AG upon a body previously at rest. This is not a necessary or intuitive truth, or a law of human thought. It is perfectly conceivable, and involves no intuitive absurdity to say, that the same relative situation which would produce the motion AG upon a body previously at rest may not change the motion AC into the motion AK, that is, may not produce upon a body already moving with any given velocity a change of motion in the sense above explained, the same with that which it produces upon any other given motion of the same body, or with the entire motion which it produces upon that or an equal body at rest. It is matter of observation, however, that it does this, and we affirm it as a physical law, on the authority of universal experience.

297. Hence there is a composition and resolution of motion and of moving forces analogous to the composition and resolution of pressures formerly considered.

298. From this a train of corollaries may be deduced similar to those of *Prop. I.* Statics, and among them this important one already anticipated, (204.) "if V be the velocity of a body moving in any direction, its velocity estimated in any other direction, making an angle θ with the former, will be $V \cos. \theta$."

299. It may be convenient to introduce here the following

proposition also as a corollary to the doctrine of the composition of motion.

If *A* and *B* (Fig. 122.) be two bodies of which one at least, *A*, is in motion, a spectator in *A*, insensible of his own change of place, will ascribe to *B* a relative motion compounded of the real motion of that body, if any, and one parallel, equal and opposite to that of *A*.

Selecting for illustration and proof the more complex case, let us suppose that *A* and *B* are both in motion, the former describing *AC* and the latter *BD*, uniformly, in the same time; join *CD*, and complete the parallelogram *ACDE*; the spectator at *C* will view *B* as at *D*; but, supposing himself to be at *A*, he will refer it to the point *E*, for the distances *AE* and *CD* are equal, and these lines will appear directed to the same point at such a distance that *AE* would subtend at it no sensible angle. By taking any two intermediate cotemporary positions, it is easily shewn that *B* appears to describe the line *BE*, and that uniformly. But the motion *BE* is compounded of the motions *BD* and *DE*, of which the latter is equal and parallel to *AC*.

In like manner, a spectator in *B*, when at *D*, and observing *A* at *C*, will, if insensible of his own change of place, refer the position of that body to *F*, *BF* being equal and parallel to *DC*; and he will ascribe to *A* the relative motion *AF*, compounded of *AC* and *CF*, of which the latter is parallel, equal and opposite to his own motion *BD*.

300. It is obvious, too, that *BE* and *AF* are equal and parallel, or that the apparent relative motions are parallel, equal and opposite, whatever the real motions may be.

301. In many cases, some of them very important ones, it is difficult to distinguish between real motions and such as are only apparent; this, however, must be accomplished before we can consider our analysis of such motions as complete, and assign to the constituent motions their proper places in our classification of phenomena, or indicate the forces concerned in the production of the changes contemplated. In its accomplishment we are aided by an enlarged acquaintance with

the established analogies of nature, by the knowledge of existing motive forces, and by the observation of concomitant circumstances known to be the appropriate accompaniments and characteristics of real motion, as distinct from that which is merely apparent.

302. The change of motion which is stated to be proportional to the motive force impressed, may, when our attention is confined to the same body, and when there is no alteration of direction, be considered as merely an increment or decrement of velocity; but, in its most unrestricted sense, it is understood, with reference to what is called *quantity of motion*, as measured by the velocity and the mass jointly. Here a question occurs, which it requires some reflection to answer with satisfactory precision, What is meant by mass, or quantity of matter, and how are we to ascertain the ratio of one mass to another? What, in short, are we to adopt as the measure of quantity of matter?

303. Those masses of matter are considered as equal, which the same force, the weight of the same body for instance, accelerates to the same degree in a given time; or those may be considered as equal, which, in like circumstances, produce equal accelerations in the same body. Two aggregates of the former description, we may state, as possessing *equal inertia;* two of the latter as having *equal motive gravitation;* and, as those which have equal motive gravitation in the same region of the earth, are uniformly found to balance each other in like circumstances, we may state them as possessing equal weight. If, by the method first mentioned, we determine a number of units of equal inertia, we shall find their equality to be established by the other tests also; and, the correspondence being once strictly ascertained, and no exception discovered, we adopt *weight* as the most convenient measure in all experimental researches, it being of easiest determination, while in the case of those distant bodies which are to us subjects of observation merely, and beyond the reach of our ordinary tests, we assume as the indication of equality of mass the equality of absolute attractive power, or

of acceleration produced in a particle situated at a given distance.

304. In many of our future researches, it is necessary to consider mass and weight not absolutely, but in relation to volume. The mass of the unit of volume is called the *density* of a body, and the weight of the same, its *specific gravity.* Hence, if Q be the whole quantity of matter in any body, W its weight, D its density, G its specific gravity, and V its volume,

$$Q = DV,\ D = \frac{Q}{V},\ V = \frac{Q}{D};$$

$$W = GV,\ G = \frac{W}{V},\ V = \frac{W}{G};$$

$$\text{also } Q \div W$$

$$\text{and } D \div G.$$

305. It is received as a result of observation and experiment, and meant by Newton to be included in the second law of motion, that the quantity of motion produced in a given time by any unbalanced pressure, is proportional to that pressure, and is the same, the pressure being given, whatever be the quantity of matter on which it is exerted. If we determine the equality of inertia, or of mass, by the equality of acceleration produced by the same weight, and then state it as a law that the same weight produces, in the same mass, the same acceleration, we may seem, so far as this extends, to be stating a merely tautological proposition. On reflection, however, we perceive, even in this restricted application of the law, an important generalization. The equality of mass is supposed to be determined by the equality of the change of velocity produced by some determinate force, upon a body having previously a determinate mechanical condition, and in a determinate time, while the proposition refers to any constant force, any previous state with respect to motion, and any given time. The law itself, in its most unrestricted sense, as applied to the production or extinction of motion by pressure, affirms that, if a number of units of equal mass be as-

certained, as above described, and any the same force employed to generate motion, the force that, acting for a given time, would generate, in the unit of mass, the velocity, or change of velocity v, will, in a mass compounded of two such units, produce the velocity or change of velocity $\frac{1}{2}v$; in a mass composed of three, $\frac{1}{3}v$; and, universally, in a mass $\gtreqless \frac{m}{n}$ units, a velocity or change of velocity $\lesseqgtr \frac{n}{m}v$; the change being always the same whatever be the previous condition upon which it is superinduced, provided always that the body be free, or that the resistance to the change arise from its inertia merely.

When of the motions entering into composition two are in different directions, and one of them variable while the other is uniform, or both are variable but according to different laws, the resultant of the two is a curvilinear motion. In all such cases, indeed in all cases whatever in which this law is to be applied, the place of the body at the end of any assigned time may be found by supposing it to obey, for an equal time, and in any order, each of the forces impressed, successively.

306. *Explanation of the third.*—When the mechanical condition of one body is altered by the influence of another, it is always observed that the influence is mutual. If *A* communicates a velocity or change of velocity to *B*, and both bodies are unconstrained, there is always a velocity or change of velocity communicated to *A*. These changes are seldom equal. The action and reaction which, in the enunciation of the law, are stated as equal, are measured not by velocity merely, but by quantity of motion as above defined, (302.) The quantity of motion lost or gained by *A* is always equal to that gained or lost respectively by *B*; so that the whole quantity of motion belonging to the two bodies together, estimated in any given direction, remains unaffected by their mutual attraction, repulsion, or collision. This law serves, as is observed by Maclaurin, to render the first law more general, and to extend it to any number of bodies; for as by the first law a

body perseveres in its state of rest, or of uniform rectilinear motion, till some external influence affect it; so it follows from this law that "the sum of the motions of any number of bodies, estimated in a given direction, perseveres the same in their mutual actions and collisions, till some external influence disturb them."

307. Having thus stated and illustrated the most general phenomena of motion, we proceed to describe the subordinate resemblances that characterise its different species, so far as they can with propriety be considered in an elementary course. These resemblances, consisting in certain relations of quantities contemplated as measurable, may be demonstratively deduced from the general laws as exhibited in combination, or in various circumstances of assigned specific modification; and, when so deduced, and tabulated, will present convenient means of reference in the classification of natural phenomena under the various denominations of force.

Of Uniform Motion.

308. A motion is said to be uniform when equal spaces are described by the moving body in any equal parts of the time during which it continues. It has place whenever a body in motion is affected by no unbalanced force; but in relation to us it is, like the subject of a definition in pure geometry, rather a conception of the mind, or ideal standard of reference, than a reality, the existence of which we can in any case with certainty recognise.

309. We adopt as the measure of *velocity* in uniform motion the space described in that portion of time which is taken as the unit, generally one second.

310. With regard to the term *force*, we may observe that it is not unfrequent in philosophical discussion to adopt the phraseology of common language, but generally in a restricted or more definite sense. In common language, for instance, we speak of greater force being required to put in motion two pounds of matter than to put in motion one, with the

same velocity; and we say that it requires more to give the velocity 10 feet per second, than to give the velocity 9, when the mass is the same. But we say this rather laxly perhaps, and without any very distinct reference to a measure of force. In reference to our own efforts, we may mean what appears to us a greater exertion continued for the same time, or an equal or even less exertion continued for a longer time; and we cannot pronounce with certainty that the exertions are in the one case as 2 : 1, or in the other as 10 : 9. In physical investigations, whenever the magnitude of a motive force is concerned, we adopt, after Newton, as its definite measure, *the quantity of motion produced or expended in a given time.*

Men of science have occasionally used the term force with a laxity of signification very remote from the analytic precision which distinguishes the philosophy of Newton, and in the language of the Leibnitzian school, it is employed with all the latitude of the word cause, or its correlative, effect. Force, they affirm, is to be measured by its effect. Now, referring to a body moving with a given velocity, as a standard, and considering that the same, with a double velocity, will rise in opposition to gravity four times as high, will penetrate a given substance of uniform texture to a quadruple depth, or will, by a certain arrangement, bend four times as many springs of equal strength to the same degree, and viewing these as effects, they state this body as possessing a quadruple force. Here no account is taken of the time that elapses in the accomplishment of the effect, as is done in the Cartesian measure adopted by Newton and his followers. It may be very proper to have a term to designate the whole of what is accomplished, without reference to time, in a particular department of mechanical agency, by the expenditure of a given motion or mechanical power, whether it be simply space described or work performed. But we are always apt to be misled in our reasonings by the misty and ill-defined language that deals in terms of uncertain comprehension, and in the analysis of mechanical action, that of a stream of water of given dimensions and velocity, for instance, we must have a term to denote the quantity of motion produced or expended in a given time, whether that is to be traced in its sawing effect, its grinding effect, its corn-thrashing, or its cotton-spinning effect.

In cases of impact, where the production of motion seems to depend on a repulsive force of unknown but variable intensity, according to the specific nature of the bodies concerned and the successive degrees of approximation, we seldom consider the progressive generation and extinction of motion, and we treat it as if it were, what it is to sense, instantaneous. Our measure of motive force impressed is, in this case, the quantity of motion expended; and, on the authority of the third law of motion, we may state an absolute equation between this measure of force, and that of the motion which it produces. But, if in speaking of the force the reference be to any pressure by which a certain velocity, now continued uniformly, has been produced, we can only state a proportional equation between such force and the resulting quantity of motion.

311. Let V denote the velocity of an uniform motion.
S the space described.
T the time of the motion's continuance.
Q the quantity of matter moved.
F the motive force.

And the relations of all the quantities concerned may be expressed as in the following table:

$$S = VT = \frac{FT}{Q}.$$

$$V = \frac{S}{T} = \frac{F}{Q}.$$

$$T = \frac{S}{V} = \frac{SQ}{F}.$$

$$F = QV = \frac{QS}{T}.$$

$$Q = \frac{F}{V} = \frac{FT}{S}.$$

The space described in the unit of time multiplied by the number of these units T must be the whole space; whence $S = VT$; and F being $= QV$, as above explained, the remaining equations are obviously deducible from these by division and substitution.

312. The same are often stated as proportional equations, as $S \doteqdot VT$, &c. and translated thus, "the space is as the velocity and the time jointly;" or "the spaces are in the ratio which is compounded of the ratios of the velocities and of the times;" $S : s = VT : v\,t$ being, at full length, the expression of an analogy for which $S \doteqdot VT$ is a convenient abbreviation.

313. *Cor.* If $S \doteqdot V$, $\frac{S}{V}$ is constant, $\therefore T$ is constant; that is, if, in the comparison of any two uniform motions, the spaces described be as the velocities, the times of the description are equal.

Of uniformly varied Motion, or Motion depending on the Agency of a Constant Force.

314. A motion uniformly varied is one that receives equal increments or equal decrements of velocity in any equal times during its continuance; and the rate of the variation, that is, the velocity produced or destroyed in a given time, commonly taken one second, is assumed as the measure of what we are to call accelerating and retarding force. Let this be denoted by φ; and, in a case of acceleration, let V be the final velocity acquired in the number of seconds T: then

$$1.\quad V = \varphi\, T,$$

or the final velocity is equal to the velocity acquired in one second multiplied by the number of seconds employed in its acquisition.

315. If V denote initial velocity, and φ retarding force, the same equation still holds, T being the time of V's extinction.

316. 2. $S = \frac{1}{2} VT$,

or the space described during the acquisition or extinction of any velocity by a constant force, or by an uniform variation of the motion, is the half of that which would be described in the same time with the final or initial velocity, respectively, continued uniform.

In the demonstration of this property of uniformly varied motion, it will be best to confine our attention at first to one of the two cases, suppose that of acceleration. Let the whole time of the motion then be considered as divided into a certain number of equal parts, for instance seconds, and let s be the increment of velocity received in the course of each second. If we suppose this increment to be received at the *beginning* of each second, the spaces so described will be all *greater* than those which are described in the corresponding times with an equable and continuous acceleration; and if we suppose them to be all received at the *end* of each second, the spaces so described will be all *less* than the true. If the number of seconds in the whole time be t, the final velocity will, on each supposition, be ts, and the space which would be uniformly described in the same time with this velocity will be t^2s. If S be the space described from rest by the continuous acceleration, we shall, therefore, have

$$S : t^2s \angle s + 2s + 3s - - - - + ts : t^2s,$$

$$\text{but} > 0 + s + 2s - - - - + (t - 1)s : t^2s,$$

$$\text{or } S : t^2s \angle (t + 1)\frac{st}{2} : t^2s, \text{but} > (t - 1)\frac{st}{2} : t^2s,$$

or by reduction,

$$S : t^2s \angle 1 + \frac{1}{t} : 2 \text{ but } > 1 - \frac{1}{t} : 2.$$

Now, as the change of velocity is equable and continuous, we may apply the same process of reasoning, however small, and, consequently, however numerous be the equal times into which the whole time is subdivided: and as t the number of the moments indefinitely increases, $\frac{1}{t}$ approaches to 0 as its

limit, so that 1 : 2 [illegible] equations (320.)
$\frac{1}{t}$: 2, and $> 1 -$ [illegible] described
[illegible] the squares of
[illegible] in extinguish-

In applying the [illegible] ares, or as the
tarded motion, we may [illegible] n.
locity to take place [illegible] gravity in each
ments, and secondly, [illegible] of seconds du-
of these suppositions [illegible] not only denote
described $ts_{,} + (t - 1)$ [illegible] rest, but an in-
and the latter a series [illegible] pressed velocity c;
and the reasoning will [illegible]

317. By substituting [illegible]
$S = \frac{1}{2} VT$, the values of [illegible]
have two others of [illegible] eration that in our
[illegible] may conceive S to be
[illegible] accumulated increment
[illegible] viously at rest or in mo-

318. According [illegible] already the velocity c
[illegible] is required to find
produced or destroyed [illegible] ally the velocity v.
[illegible] and s' that due to
note a motive force, [illegible]

$\frac{V}{T}$ its equal φ,

And if we multiply [illegible]
4th by Q, and for Q [illegible]
[illegible] through which
[illegible] so that its ini-

319. From these [illegible] pealed to as a
lowing Table, exhib[illegible] forces are often
the quantities conc[illegible] Thus if a force
[illegible] feet every se-

u, it may be said to be twice that of gravity. In general, $\varphi = mg$, then our equations in (314, 316, 317.) may stand :

$$1.\ v = mgt,$$
$$2.\ s = \tfrac{1}{2} vt,$$
$$3.\ s = \tfrac{1}{2} mgt^2,$$
$$4.\ s = \frac{v^2}{2mg}.$$

the force is to be determined and as a multiple of gravity, is the quantity sought.

326. Before proceeding farther in the science of motion, shall exhibit an illustration and proof of the laws already plained. For this purpose we employ a very ingeniously trived machine, invented by the late Mr. Atwood, the ode of using which will be understood by the following concise description.

Let *ADB* (Fig. 123.) represent a wheel or pulley, the fric- n of whose axis is as much as possible diminished; to each d of a very fine silk line passing over this pulley, let a eight *Q* be attached: the two weights *Q* will of course be equilibrio. Let now an additional weight *P* be attached one end of the line. Motion will then commence, and ying out of view for a moment the inertia of the pulley, the eight of *P* will not have to put in motion merely the mass , as in the case of an unimpeded simple fall of *P* to the ound; for it cannot at present descend without causing its n mass and that of 2 *Q* to move with the same velocity. ow, by what has preceded, the masses of bodies are as their eights, and the accelerative forces are directly as the moving forces and inversely as the quantities of matter. Hence e accelerative force in the circumstances supposed will be the whole accelerative force of gravity $= \frac{P}{P+2Q} : \frac{P}{P}$ $= \frac{P}{P+2Q} : 1$; that is, the accelerative force in our experiment will be $\frac{P}{P+2Q}$ as a multiple of gravity. But the inertia of the wheel is not in practice to be disregarded. If its mass were all at the circumference, so as to be made to move with the same velocity as the descending and ascending masses, we should only have to add its mass to the denominator of our fraction $\frac{P}{P+2Q}$. The descending weight, however, acts with a mechanical advantage against those parts of the pulley that lie nearer to the centre than the circumference is, and they oppose a less resistance than if they were all placed in that line. By an experiment to be afterwards described under the head of rotatory motion, or indeed by one of the kind to be made immediately, we can ascertain what mass placed at the circumference would oppose an equivalent resistance to the moving force. Let this mass be w. Then the accelerative force as a multiple, or rather a submultiple of gravity, will by the same steps as before be found to be $\frac{P}{P+2Q+w}$. The machine is so constructed by the maker that, as in the one originally used by Mr. Atwood himself, w shall be $= 2\frac{3}{4}$ oz. or $\frac{11}{4}$ oz. Troy. To avoid troublesome computations in the experiments $\frac{1}{4}$ oz. Troy is chosen as the unit of weight, of which consequently all the rest employed are expressed as multiples or submultiples. This unit of weight is called m, and the other weights are $4\,m$, $2\,m$, $\frac{1}{2}\,m$, $\frac{1}{4}\,m$, $\frac{1}{8}\,m$. Each of the boxes or supports is itself $= 6\,m$, and the equivalent of the inertia of the pulley and its friction wheels $= 11\,m$. Most of the weights used are circular, a few of them are rods, for a special purpose to be afterwards mentioned.

Suppose then that we include $4\frac{3}{4}$ oz. or $19\,m$ in each box; this with the box itself will make on each side $25\,m$, to which if we add the inertia of the pulley and wheels, the whole will be $61\,m$: add now a circular weight m to each box; and the mass to be moved is $63\,m$ perfectly in equilibrio and (setting aside the small remaining friction) movable by the least weight, just as if the same force were applied to communicate motion to the mass $63\,m$ existing in free space and void of gravity. Let the weight added be *m*; and the accelerative force φ will be $= \frac{m}{64m} = \frac{1}{64}$ of gravity. Taking this for *m* in our last series of equations, we have by the 3d $s=3t^2$ in inches, so that the spaces described in 1, 2, and 3 seconds will be 3, 12, 27 inches respectively. We take $\frac{1}{2}g$ here as $= 192$ inches to avoid a minute fraction, which in so small a descent could not be appreciated. If we now take the circular weight *m* from *B* and put it upon *A*, the mass moved will still be 64 *m*, but the preponderating weight or moving force will be 3 *m*, and $\varphi = \frac{3}{64}$ of gravity. To make the moving force 2 *m* while the mass is still the same, let all the circular weights *m* be removed; the mass is then reduced to 61 *m*; add $\frac{1}{2}$ *m* to each side, it is now 62 *m* in equilibrio; then add 2 *m* to the box *A* and the mass is 64 *m* as before; but the value of φ is $\frac{2}{64} = \frac{1}{32}$ of gravity.

The spaces described are measured by a scale of more than 64 inches, with a small stage to receive the box *A* in its descent.

The time is estimated by the beats of a pendulum which swings seconds.

The mode of measuring the velocity is very ingenious. We cannot take as its measure the space actually described in a given time following the instant at which it is wished to note it; the space actually described being partly due to the further acceleration, which is continuous. If we could remove the motive force leaving the remaining mass in equilibrio, there would be no cause of further acceleration or retardation; by the first law of motion the mass would proceed uniformly, and then the velocity which had been acquired would be measured by the space described per second. Now this may be done as follows: Let the mass 63 *m* be in equilibrio as before, but instead of adding a circular weight *m* add one in the form of a rod, two or three inches long: allow the box to descend with a continual acceleration a certain number of inches according to the velocity wanted, and then let the rod m be intercepted by a *circular* stage in the form of a ring through which the box passes. After this interception the motion will be uniform if a few grains have been previously added to balance the retarding power of the remaining friction.

The following construction of an experiment gives a greater range of illustration, the motion being slower:

$$\begin{array}{llll}
\quad A & \quad B & & \\
\text{Itself } 6m & \text{Itself } 6m & & \\
\text{add } 36\tfrac{1}{4}m & \text{add } 36\tfrac{1}{4}m & & \\
\quad + \tfrac{1}{2}m & & & \\
\overline{\quad 42\tfrac{3}{4}m} \quad + & \overline{\quad 42\tfrac{1}{4}m} & = 85m & \\
& \text{Inertia of wheels} & = 11m & \\
& & \overline{\quad 96m} &
\end{array}$$

$$F = \tfrac{1}{2}m,\ \varphi = \frac{\tfrac{1}{2}m}{96m} = \frac{1}{192}.$$

Times.	Spaces.
1	1 inch.
2	4
3	9
4	16
5	25
6	36
7	49
8	64

To shew the velocity acquired, remove the circular weight $\frac{1}{2}m$, and use the flat rod $\frac{1}{2}m$ in place of it. Let it be intercepted when the *bottom* of the box comes to 16 inches on the scale, or at the end of the 4th second. The mass having described with uniform acceleration 16 inches in four seconds is now according to the theory in a state, according to which it would describe 32 inches uniformly in that time, or, which is the same thing, 8 inches in each second. Therefore at the end of

5″ it will strike the square stage at	24 inches.
6	32
7	40
8	48
9	56
10	64

Experiment on Retardation.

Let 18 ½ m be placed in A and 19 ½ m in B. Then

A with its load $= 24\frac{1}{2}m$,
B with its load $= 25\frac{1}{2}m$.

The sum is $50m$, which with the inertia of the wheels is $61m$, and the preponderance is m on the side of B. But let two rods each $= m$ be added to A, then the mass is $63m$, and the preponderance is m on the side of A, $\varphi = \frac{1}{63}g$, and by descending till the bottom of the box A reaches 26.44 inches, a velocity of 18 inches per second will be acquired. The circular stage being so placed as to intercept the two rods, will leave the mass $61m$, and a preponderance m on the side of B, so that we shall have the mass $61m$ projected with a velocity of 18 inches per second in opposition to a retarding force $= \frac{1}{61}g$, and the space described during the extinction of this motion ought to be 25.6 inches. The bottom of the box will therefore descend to about 52 inches before its motion be destroyed.

In making the calculations for this experiment, we take for $\frac{1}{2}g$ 193 inches, which is more accurately its true value than 192, because we cannot avoid fractions by using the approximate number.

Of the Motion of Bodies upon Inclined Planes.

327. In descending down a smooth inclined plane, the force by which a body is impelled being for each particular

$g : P :: l : h$

$\therefore\ g = \frac{2h}{l}$

$r : F = l : h$

$\therefore\ \frac{F}{W} = l$

with b… accelerati… or down… motion… have

elevation a determinate part of the whole weight, may be regarded as constant. Hence the laws of uniform acceleration and retardation are applicable to motions so modified, and from the application some interesting theorems are deduced. Let h be the height, and l the length of an inclined plane; then $\frac{hg}{l}$ is the accelerative force, or it is sg, s being the sine of the plane's elevation. In applying our general formulæ, therefore, $v = mgt$, &c. to this class of motions, we have only to substitute s or $\frac{h}{l}$ for m, (325.)

328. The velocity acquired by a body in falling down an inclined plane is as the square root of the perpendicular height, and is equal to that which would be acquired by falling vertically through that height,

$$v^2 = 2mgs = \frac{2hgl}{l} = 2gh,$$

$$\text{and } v = \sqrt{2gh}, \text{ or } v \div \sqrt{h}.$$

329. The time of a body's descent down an inclined plane varies directly as the length, and inversely as the square root of the perpendicular height,

$$t^2 = \frac{2s}{mg} = \frac{2l}{\frac{h}{l}g} = \frac{2l^2}{gh},$$

$$\text{or } t = \frac{l}{\sqrt{h}} \times \sqrt{\frac{2}{g}}, \text{ and } t \div \frac{l}{\sqrt{h}}.$$

330. *Cor.* The times of descent along inclined planes of the same vertical height are as their lengths.

331. The spaces described in the same time by a free descent in a vertical direction, and along an inclined plane are as the length to the height. Let S be the first of these spaces and s the second.

$$S : s = \tfrac{1}{2}gt^2 : \tfrac{1}{2}mgt^2 = 1 : m = l : h.$$

332. *Cor.* 1. If AC (Fig. 124.) be an inclined plane, of which BC is the base and AB the vertical height, and we

draw BD perpendicular to AC, a body will fall from A to D in the same time in which it would fall vertically to B; for

$$S : s = l : h = AC : AB = AB : AD.$$

333. *Cor.* 2. If a circle (Fig. 125.) be situated in a vertical plane and chords be drawn from either extremity of the vertical diameter, the velocities acquired by falling down these chords will be proportional to their lengths, and the times of descent through any of the chords and through the vertical diameter will be equal.

By (328.) $v^2 \div BD \div AB.BD \div BC^2$, and $v \div BC$.

By (332.) since ACB is a right angle, the time of descent down AC is the same with the time of descent down AB, and the same may be affirmed of the descent down CB, which is equal to that of the descent down BC when the figure is inverted.

On this property of the circle may be founded several elegant theorems or problems, some of which are usually prescribed as exercises.

334. If a body fall down a series of inclined planes, and no velocity be lost at the transitions from the superior to the inferior, the final velocity will be that which is due to the vertical height of the whole.

The velocity acquired by falling through BE (Fig. 126.) = that acquired by falling through CE, which has the same vertical height; (328.) and as, by supposition, no velocity is lost in passing from BE to EF, the velocity acquired by falling through BE, EF, will be the same with that acquired by falling through CE, EF, that is, through CF. For a similar reason that acquired by falling through CF, FG, or BE, EF, FG, will be equal to that acquired by falling through DG, or to that which is due to the vertical height DH, (328.)

335. If the planes BE, EF, &c. be so numerous, and their inclinations to each other so small, that their vertical section $BEFG$ may be considered as a curve, the velocity acquired by falling through any part of it will be that which is due to the vertical distance between its two extremities.

To prove this, we must shew that the velocity lost by the continued change of direction is evanescent.

Let *AB*, *BC* (Fig. 127.) be two inclined planes making a finite angle *ABD*, whose versed sine to the rad. $1 = s$. Take *AB* to represent the velocity v with which the body comes to *B*. This may be considered as equivalent to *AE* and *EB*, of which the former is destroyed by the collision, and *EB* remains in the direction of *BC*. Consequently the velocity lost v' will be represented by $AB - BE = ED$, which is the versed sine of *ABD* to the rad. *BD;* and $v' : v = s : 1$, or $v' = sv$. If the angle become indefinitely small, s vanishes, and $v' = 0$.

Again, let *CBF* (Fig. 128.) be any given angle, and let *CBE* be the $\frac{1}{n}$ part of it; the sum of the versed sines of all the n equal parts $= \frac{n.(\text{chord } EC)^2}{D} < \frac{n.(\text{arc } EC)^2}{D}$ that is $< \frac{n^2.\text{arc}^2}{nD}$ that is $< \frac{CF^2}{nD}$. But $\frac{CF^2}{D}$ is constant, and hence n may be taken so great that $\frac{CF^2}{nD}$ shall be less than any assigned quantity. The sum of the versed sines then of the angles whose number is n, or $\int s = 0$, and consequently $V\int s = 0$ even supposing V to be the greatest finite velocity of all.

Let *AC* and *BC* (Fig. 129.) be tangents to any portion of a curve, at its extremities, meeting in *C*, and let *AC* be produced to *D;* subdivide the angle *BCD* into equal parts *BCE*, *ECF*, &c. indefinitely, and conceive successive chords of the curve to be drawn parallel to *CE*, *CF*, &c. beginning from *B*. By falling down the polygon formed by these chords or down the curve which is its limit, no finite velocity will be lost; and the proposition is demonstrated.

336. In falling down curve surfaces however the force derived from gravity will not be constant, as upon an inclined plane of given elevation; and before we can exhibit the results of a motion so regulated, we must treat of acceleration and retardation depending on the agency of variable forces.

Of Accelerated and Retarded Motions depending on the Agency of Variable Forces.

337. Let s and s' denote any two successive values of the space described, t and t' the corresponding times, v and v' the velocities at the beginning and end of the increment of time; then, for accelerated motion,

$$s'-s > v\,(t'-t), \text{ but } < v'\,(t'-t),$$

$$\text{or } \frac{s'-s}{t'-t} > v, \text{ but } < v'.$$

But as the cotemporary increments $s'-s$ and $t'-t$ diminish, v' approaches to v as its limit; therefore $\frac{s'-s}{t'-t}$, whose value we have just seen is always intermediate between v and v', must have v also as its limit; or $\frac{ds}{dt} \doteq v$; whence

$$ds = v.dt$$

$$dt = \frac{ds}{v}.$$

For retarded motions the same investigation will suffice, if the symbols $>$ and $<$ be interchanged.

338. Let the accelerative forces at the beginning and end of the increment of time be φ and φ', and let v and v' be the corresponding velocities; then, if the accelerative force be increasing,

$$v'-v > \varphi\,(t'-t), \text{ but } < \varphi'\,(t'-t);$$

$$\text{or } \frac{v'-v}{t'-t} > \varphi, \text{ but } < \varphi';$$

but as the increment of time decreases φ is the limit of φ', and therefore of $\frac{v'-v}{t'-t}$;

$$\text{or } \frac{dv}{dt} \doteq \varphi$$

$$dv = \varphi.dt$$

$$dt \doteq \frac{dv}{\varphi}$$

339. $\frac{dv}{\varphi} = dt = \frac{ds}{v}$, (338, 337.)

$$\text{or } v\,dv = \varphi\,ds,$$

$$\text{whence also } \varphi = \frac{v}{ds} dv = \frac{1}{dt}$$

$$\text{or } \varphi = \frac{dds.dt - ddt.ds}{dt^3} = \frac{dds}{dt^2} \text{ whe}$$

340. Our first fluxional equation, ds the first equation under the head of u $v = \varphi t$, furnishes a very concise demo or $s = \frac{1}{2} v t$; for

$$ds = v.dt = \varphi t.dt, \text{ and } s =$$

341. As the investigations to which tions are adapted, are, by some dis among others by Newton, in his *Pr* aid of a geometrical representation, ing farther, give the elements of th

Lemma. (*Prin.*

Let *FS* (Fig. 130.) be any cur the rectangles *AG*, *BM*, &c. and circumscribed, as in the figure, th If the bases *AB*, *BC*, &c. be all vided by continual bisection, and made at each, the sum of the inte tially exterior rectangles will e near area *AFSE* as a limit.

For the difference of the ab *OP*, &c. $= DE.(ST+QP$, &c.) diminution of its breadth become or the sums of the two sets of each other, and consequently by any assigned difference.

It is not necessary to suppo we have done, that the bases a Fig. 131

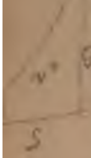
is not less than any of the other bases, the difference of the sums above-mentioned will be obviously not $> DE.(ST + QP$, &c.) that is, not $> US$.

342. Let the abscissæ of a curve AB, AC, &c. (Fig. 131.) represent the times, while the ordinates AE, BG, &c. represent the velocities at the instants denoted by the points of the line of abscissæ where they terminate; the areas corresponding to any portions of that line, as their bases, will proportionally represent the increments of the space described in the times which these portions denote.

In uniform motion $s = v\,t$, $s' = v\,t'$, or $s' - s = v\,(t' - t)$; that is, the increments of the space are equal to the velocities into the increments of the time; or if, in any number of cases compared, we represent the velocities and the times proportionally each by lines, the increments of the space will be represented proportionally by the rectangles under the lines denoting the velocities and those denoting the increments of the times. Suppose then the velocity AE to remain constant during the small time AB, then to become BG, and to remain constant for the succeeding small time BC, and so on: the increments of the space between any two points of time as A and B, B and D, will be represented by the corresponding rectangle AF, or sum of rectangles $BH + CL$. If we now suppose the moments of time during which the several velocities continue constant to be continually diminished, the increase or diminution of velocity will become more and more nearly continuous, and the rectangles between any two ordinates will approach to the corresponding area, bounded on one side by the curve as a limit; whence the truth of the proposition is obvious.

343. If the abscissæ represent the times as before, and the ordinates the successive values of the accelerative force, the areas will represent the increments of the velocity.

For if $v = \varphi\,t$ and $v' = \varphi\,t'$, φ being constant, $v' - v = \varphi\,(t' - t)$. Therefore the increments of the velocity when the accelerative force is constant, may be proportionally represented by rectangles under the lines representing the accelerative forces and the increments of the time. Suppose then the accelerative forces denoted by *AE*, *BG*, *CK*, to continue constant during the times *AB*, *BC*, *CD*, respectively, and, by considering the times during which each value of ϕ remains unchanged to be continually diminished, we find, as in the demonstration of the last proposition, the result stated in the present.



344. If the line of abscissæ represent the space through which a body moves, and the ordinates the values of the accelerative force at the different points of that space, the areas will represent the square of the velocity generated if the motion begins from a state of rest, or universally, the increment of the square of the velocity.

If $v^2 = 2\phi s$ and $v'^2 = 2\phi s'$, ϕ being supposed constant, $v'^2 - v^2 = 2\phi(s' - s)$ or $v'^2 - v^2 \overset{\cdot}{\underset{\cdot}{=}} \phi(s' - s)$, that is, while the accelerative force remains constant, the increment of the square of the velocity is proportional to a rectangle under the lines representing the accelerative force and the increment of the space. Thus, if the accelerative forces *AE*, *BG*, &c. remain constant through the spaces *AB*, *BC*, &c. respectively, the rectangles *AF*, *BH*, &c. will represent the successive increments of the square of the velocity, and the sum of *AF* and *BH*, the whole increment of the same in passing through the space *AC*. The rest of the demonstration proceeds as before, (342.)

345. *Cor.* 1. If a body be subjected to the same accelerative or retarding forces at the same points of the space described, the square of the velocity will receive the same increment or decrement in passing through that space, whatever the previous velocity may have been.

Cor. 2. A given decrement of the square of the velocity may be the result of a very small diminution of the velocity itself, and consequently attended with a very small loss of quantity of motion; but the loss of force as measured by quantity of motion, is always equal to the force impressed. Hence we may understand how a musket ball may be discharged through a screen or a thin board without oversetting it.

Cor. 3. If the ordinates in Fig. 131. as constructed with reference to this proposition be each quadrupled, the area, and the square of the extinguished velocity, which it may represent, will each be quadrupled. Of course the velocity itself which is destroyed in passing over the same space will only be doubled: In this way is easily explained, in conformity with the Newtonian measure of force, one of the most striking of the facts adduced by the adherents of Leibnitz in support of their measure. (See 310.)

We now proceed to exemplify the equations above investigated, and the theorems just demonstrated, in their application to one of the most important classes of mechanical phenomena, that of motions depending on the agency of an accelerating or retarding force which is as the distance from a given fixed point.

346. Let a body begin to move from A towards C (Fig. 132.) under the influence of a single force whose intensity is directly as the distance from C, and,

Let $AC = a$,

$AB = x$ the space through which the body has moved,

$CB = a - x$,

$f =$ the value of the force at the unit of distance.

Then $f(a-x) =$ its value at the dist. $a - x$, and as $ds = dx$,

$$v.dv = f(a\,dx - x\,dx)$$

$$\frac{v^2}{2} = f\left(ax - \frac{x^2}{2}\right)$$

$$\text{or } v = \sqrt{f}\sqrt{2ax - x^2}.$$

Cor. The square of the velocity acquired in moving from A to C is the half of what it would have been had the force which acted upon the body at A continued constant.

For $v^2 = f(2ax - x^2) = fa^2$ when $x = a, = fa.a$; now fa is the intensity of the force at A, or at the distance a. Denote this by the symbol mg: then $v^2 = mga$. But had the force mg been constant from A to C, we should have had $v^2 = 2mga$, (325.)

$$-\frac{x}{a} = \frac{\frac{dx}{a}}{\sqrt{\frac{2x}{a} - \frac{x^2}{a^2}}} = \frac{\frac{[illegible]x}{a}}{\sqrt{\frac{2ax - x^2}{a^2}}} = \frac{[illegible]}{\frac{1}{a}\sqrt{2ax - x^2}}$$

347. Having found the value of x, we have $dt = \frac{dx}{\sqrt{f}\cdot\sqrt{2ax-x^2}}$, (337.)

Let A be the arch whose versed sine $= \frac{x}{a}$ to the radius 1;

$$t = \frac{1}{\sqrt{f}} A = \tfrac{1}{2}\pi \frac{1}{\sqrt{f}} \text{ when } x = a;$$

and, as this expression for t is independent of a, the distance from C at which the motion begins, the time is the same whatever be the distance.

348. In AC produced towards C take $CA' = CA$, and the body if free to pass the point C will continue its motion by its inertia, *(first law of motion)*; and its velocity being destroyed according to the same rate by which it was produced, it will pass from C to A' in the same time in which it moved from A to C. At the point A' its velocity acquired at C is just extinguished, and it will return from A' to A, spending in each of these transitions from side to side, which are called oscillations, the time $\pi \frac{1}{\sqrt{f}}$.

If there be any slight cause obstructing the alternate excursions, the oscillations will at last cease, and C will be sensibly the point of final quiescence.

349. We shall arrive at the same conclusions by a geometrical representation thus:

Let AB (Fig. 133.) represent the accelerative force at the point A where the motion commences. Join BC and draw BO parallel to AC, and if V be the velocity acquired in moving from A to C by the constant force AB and v the velocity acquired in passing over the same space when the force is AB at first, but varies as the distance from C: $V^2 : v^2 = AO : ABC = 2 : 1$, (344.) Now, if $AC = a$, and φ represent AB, $V^2 = 2\varphi a$, and therefore $v^2 = \varphi a$.

The velocity at any point E is proportionally represented by DE, the sine of the arch whose versed sine is the space described: For $r^2 \doteq ABHE, \doteq ABC - EHC, \doteq AC^2 - EC^2, \doteq DC^2 - EC^2 \doteq ED^2$, and $r \doteq ED$.

350. *Cor.* On this scale the final velocity will be represented by the radius *FC*.

351. The space *AC* may be considered as described by a succession of increasing velocities, each continuing constant for a small space, if we consider these spaces as indefinitely diminished.

The time of describing *Dm* with the velocity *FC* or *DC* is the same with that of describing *Dd* or *Ee* with the velocity *DE*. For $Dm : DC = Dd : DE$, by similar triangles, or

$$\frac{Dm}{DC} = \frac{Dd}{DE}$$

But *Dm* and *Dd* are spaces, and *DC* and *DE* are proportional to the velocities with which they are described; and, since the quotients of the spaces divided by the velocities are equal, the times are equal. 312.

352. *Cor.* The time of describing *AC* with the variable velocity is the same with that of describing the quadrantal arch *AF* with the constant velocity *FC*, or $\sqrt{ga}$, that is,

$$t = \frac{\frac{1}{2}\pi a}{\sqrt{ga}} = \frac{1}{2}\pi \sqrt{\frac{a}{g}}, \text{ as before.}$$

For *g* is in our present notation the value of the force at the distance *a*, and *f* was its value at the distance 1; and the force being as the distance, $\frac{a}{g} = \frac{1}{f}$.

353. This investigation is applicable to a cycloidal motion, if the tangential accelerating force be always as the arch intercepted between the point where the body is and the position of quiescence.

354. The force of a well constructed spring varying according to this law, it is applied as the regulator of motion in a watch.

355. Let *LCM* (Fig. 134) be a cycloid, *ALC* the half of the generating circle, *AC* the axis, *EDF* an ordinate cutting

317. Having D and C,
parallel to
F, and let FK
into FH perpen-
rection, or in that of
f its force estimated
f considered as pres-
may represent also
direction and down the
FK : FG = FE : EB =

therefore $f \doteq CD \doteq$

where the tangential
a body describing a cy-
refore be performed in equal
expressed, will be isochronous.
always parallel to the chord
D, the tangent at M will
The whole force is there tan-
the force at the distance

complete oscillation is $\pi\sqrt{\frac{CM}{g}}$,

the writers on the cycloid, that
an equal cycloid. If we there-
place the two cycloidal cheeks
represented in Fig. 135. by MC,
equal to the base of the cy-
the same thing, the cheeks hav-
at M, a flexible pendulum of
describe the cycloid CPC′, and,

cycloid described on each side of
small, it will not sensibly deviate
with the radius L. Hence, in
vibrations will be sensibly iso-

depends the application of clocks.

a simple pendulum vibrating solve the weight represented in Fig. 134, (355.) and draw the CA;

$: = CB : BD$, (by sim. triang.)

$\frac{B}{CB} \div BD.$

ative force is not here as the arch, but as f there be any number of arches the greatest ry small, they are nearly in the ratio of their of the forces, whence may be derived the same before.

required to find the time of vibration in a finite r arch, as 2 AB. Let the body, which is here con- red as a particle of matter, at the extremity of a straight $CB, = l$, without weight or inertia, or as forming what is simple pendulum, be at the point H of its ascent ough AB; let $AD = h$, the height due to the velocity at the west point, and $AH = z$. The remaining velocity when the is at H, will be that which is due to a vertical descent gh DK, and $\therefore = \sqrt{2g} . \sqrt{DK} = \sqrt{2g} . \sqrt{AD \quad AK}$ $= \sqrt{2g} . \sqrt{h - l \text{ ver. sin. } z} = \sqrt{2g} . \sqrt{h - 2l . \sin.^2 \frac{1}{2} z}$,

nd $dt = \dfrac{l.dz}{\sqrt{2g}.\sqrt{h - 2l.\sin.^2 \frac{1}{2}z}} = \dfrac{\sqrt{l}}{\sqrt{g}} . \dfrac{\frac{1}{2} dz}{\sqrt{\frac{h}{2l} - \sin.^2 \frac{1}{2} z}}$,

if we take $c^2 = \frac{h}{2l}$ and $c.\sin. \varphi = \sin. \frac{1}{2} z$, we shall have,

$$dt = \sqrt{\frac{l}{g}} . \frac{d\varphi}{\sqrt{1 - c^2 . \sin.^2 \varphi}}$$

$$= \sqrt{\frac{l}{g}} . d\varphi \left(1 + \tfrac{1}{2} c^2 . \sin.^2 \varphi + \tfrac{3}{8} c^4 . \sin.^4 \varphi + \tfrac{[illegible]}{16} [illegible] \right)$$

at the even powers of sin. φ [illegible]

ries composed each of a constant term and [illegible]

the circle in *D*, *CD* the chord of the arcl
FE a tangent to the cycloid at *F*, and t
DC. Let a body be on the cycloidal cu
represent its weight. This may be reso'
dicular to the curve, and *FG* in its di
the tangent. Let *F* be the weight, an
in the tangential direction; as *F* an
sures have to move the same mass,
the accelerative forces in the vertical
curve respectively. Now $F : f =$

$DC : CB = AC : CD$, or $f = \frac{F.CF}{AC}$

$2\,CD \doteq$ arch $CF \doteq$ distance fro
force vanishes. The oscillations [illegible] c^6, &c.)
cloid by its weight alone, will th
times, (347.) or, as it is usually [illegible] sin.^{2} $\frac{1}{2} z = c^2 =$

356. As the tangent at *F* is a [illegible] $z = AB$, the semi-
joining *C* to the correspondin
be parallel to *AC*, or vertical. [illegible] $2c$ is not very small,
gential, and $f = F = g$. He [illegible] readily obtained from
Exer. de Cal. Int.

$1 = \frac{g}{CM}$, and the time of a [illegible] ndefinitely small and $t =$
(348.)

357. It is demonstrated [illegible] with sufficient approxima-
the involute of the cycloid
fore divide our cycloid a [illegible] practice, the pendulum
MC, *LC* in the position [illegible] rate through a large angle.
MC, *CC* being hori [illegible] on in a cycloidal or inde-
cloid, or, which [illegible] that of falling through half
ing a common vertical [illegible] circumference of a circle to
the length $L = MC$

$t = \pi \sqrt{\frac{L}{g}}$ [illegible] and if $s = \frac{1}{2} l$

358. [illegible]
the [illegible] $\sqrt{\frac{l}{g}} = \pi : 1.$
from [illegible]
[illegible] circular arcs a [illegible] on through any given angle

[illegible], [illegible] the
[illegible]

[illegible] of the vibrations
[illegible] pendulums whose times
[illegible] $= n$ or N [illegible] $=$
[illegible] and $l =$ [illegible]
[illegible] minutes then if n be $=$
$= \frac{140900 \, . \, l}{L}$ and l [illegible]

[illegible] equations we may correct the length
[illegible] it does not swing seconds exactly,

[illegible] length may also be found more di-

$$N^2 : n^2 = l : L$$

$$N^2 - n^2 : N^2 = \Delta l : l$$

$$\Delta L = \frac{l(N+n)\,(N-n)}{NN},$$

[illegible] is to be applied with its proper sign to l, to

[illegible] 1. It has been found, that while a pendulum is [illegible] any parallel of latitude towards the equator, the [illegible] its vibration becomes longer, independently of the ef[illegible] temperature; and since $t \div \frac{\sqrt{l}}{\sqrt{g}}$, and increase in the [illegible] though l is constant, it must be [illegible] of g. The accelerative force of [illegible] diminishes as we proceed from [illegible] latitude. If l is constant, [illegible] always [illegible]

[illegible] $\sqrt{\frac{l}{g}}$ is constant and $g \div l$

363. *Cor.* 5. Since $t = \pi\sqrt{\frac{l}{g}}$, $t^2 = \frac{\pi^2 l}{g}$, or $g = \frac{\pi^2 l}{t^2}$, and if t be $= 1''$, l is 39.139 English inches in the latitude of London, and $g = 386.2862$ inches $=$ the measure of gravity in that latitude. It is the velocity produced by it in $1''$ when the fall is unresisted: and $\frac{1}{2}g = 193.1431$ inches $= 16\frac{1}{12}$ feet, omitting the fraction of an inch, is the actual fall in that time. This is the most accurate way of determining the accelerative force of gravity. From the isochronism of the pendulum, if an error of $\frac{1}{1000}$th part be committed in the estimation of the length of one oscillation, it will be easily appreciated, amounting to an unit in 1000 oscillations. This ought to be remarked as one of the means of insuring accuracy as far as possible in observation or experimental research. Instead of observing a simple difference or error, we observe a given multiple of it: and, this being estimated with all the precision that our senses and our instruments admit of, we reduce the simple error in the ratio in which we can increase the multiple. Great use is made of this principle in astronomy.

364. *Cor.* 6. The pendulum may be employed advantageously to prove that in any given place the accelerative force of gravity is the same in all matter. We can shew, by means of a common experiment with the airpump, that bodies, of whatever matter composed, descend with equal velocity in vacuo. There might, however, be minute differences not to be detected in this way. It is proved, with a precision much more satisfactory, by taking pendulums of the same length, and composed of different materials, which, in vacuo, are always found to perform their oscillations through equal arches in equal times. In such experiments, it is obvious that we cannot employ what is called a simple pendulum, the existence of which is ideal merely, but must use one composed throughout of inert matter, possessing also weight variously distributed through its whole length. Any compound pendulum, however, can always, by calculation or experiment, be reduced with sufficient approximation to an equivalent simple one, as will be proved under the head of *Rotation*;

and, when the principle of this reduction is explained, we can employ, in such observations as have been alluded to, a fine wire with a solid or hollow ball, as may appear convenient, at its extremity. In the last experiment mentioned, it will be very convenient to employ the same hollow ball, filled successively with various kinds of matter.

365. *Cor.* 7. The pendulum may also be used with advantage as a standard of linear measure. All our ordinary standards may be injured or lost; but, so long as the constitution of the terrestrial globe remains unchanged, the length of a pendulum that makes a vibration in a given time, as 1″, will remain invariable in the same latitude, and subject to the same correction for difference of latitude. If its length be found by astronomical observation, and be divided into 39 139 equal parts, 1000 of these will give the length of the English inch, 12000 of them a foot, and so on. It is, however, a matter of some delicacy to find the proper length of the seconds pendulum. The best method of doing it is that recently invented by Captain Kater, and will be described under the head of Rotation.

366. The time of an oscillation varying with the length, it must evidently depend on the variations of temperature to which the pendulum is exposed. A variation of 0.000909 inch, which we may take as $\frac{1}{1000}$ inch, in the length, corresponds to a variation of 1″ per day in the rate. From the known expansion of a bar of steel, we find that in the going of a clock regulated by a pendulum whose rod is of that metal, there will be a variation of 1″ per day for every change of temperature amounting to 4° of Fahrenheit's thermometer. As clocks that shall go with extreme precision are necessary in the highly improved state of modern astronomy, no observation that is not made with great accuracy being now of almost any consequence, various contrivances have been employed to prevent the irregularity occasioned by changes of temperature.

367. Of one of these the principle will be easily understood by a short description. *AC* and *BD*, (Fig. 137.) are two steel

rods, connected above and below by two cross bars, of the same or any other metal. On the lower bar rest two other rods, of a more expansible metal, as zinc or some of its alloys. These are connected at the upper ends by a bar *GH*, from which depends the pendulum rod of steel *KP*, passing freely through an aperture in the middle of the lower bar at *O*, and carrying the ball *P*. Suppose *AB* to retain sensibly its position, the expansion of the two outer steel rods will lower the bar *CD*, and thereby tend to lower as much the bar *GH*, and the ball *P*, which will also sink still lower by the expansion of the rod *KP*. But the simultaneous expansion of the rods *EG*, *FH*, will tend to raise the bar *GH* and the ball *P*; and, if the expansion of *GE* or *HF* be equal to the sum of the expansions of *AC* and *KP*, the ball of the pendulum will neither rise nor fall. It does not, however, follow that the equivalent simple pendulum is neither lengthened nor shortened. There is a new distribution of the matter of the pendulum which, as we shall afterwards see, may affect this a little, but the remaining irregularity will now be very small, and may still be diminished if the instrument be so constructed as to admit of adjustment by trial. This is called Harrison's pendulum, from the name of the inventor, a London artist. It is also known by the name of the gridiron pendulum, an appellation derived from its shape. It has the name, now common to it with others, of compensation pendulum, from the principle of its construction. The same principle has been applied in various forms. Let *CA* (Fig. 138.) be a steel rod supporting a hollow cylinder partly filled with mercury, *BEFD*; and let *CO* be the length of the equivalent simple pendulum. The expansion of the rod and cylinder tends to lower the point *O*, but the expansion of the mercury raises it. By taking out or adding mercury, the quantity may be found experimentally, which being employed, there shall be no difference in the rate of going when the clock is exposed to great artificial inequalities of temperature, and then it may be used with the greater confidence through the smaller range of the common atmospherical changes. This is called Graham's compensation pendulum. The inventor was a celebrated clockmaker in London, distinguished by some other very ingenious improvements in his art.

368. Since a variation of $\frac{1}{1000}$ inch nearly corresponds to a change of rate = 1″ per day, it is easy to regulate a clock by observing the number of threads in an inch of the screw below the pendulum ball. Suppose they are 25; then to correct a variation of 10″ per day, which will require the pendulum to be lengthened or shortened $\frac{10}{1000}$ inch, = $\frac{1}{100}$ inch, = $\frac{1}{4} \times \frac{1}{25}$ inch, the nut supporting the ball may be turned, for a first and very near approximation when the rod is light and the ball heavy, $\frac{1}{4}$ of a revolution. By a proportion founded on the observed reduction of error which this produces, we may find, with all requisite accuracy, what corrective adjustment is further to be employed.

369. To the balances of the finer kinds of watches, called chronometers, the principle of compensation is also applied. The more the balance is expanded by heat the spring has the less power to turn it, and its vibrations become slower. When it is contracted by cold, again, all the parts are brought nearer to the axis, and the accelerative forces at all corresponding points of the range of oscillation become greater. The forces are still as the distances from the point of quiescence, and the vibrations are still isochronous, but they are not synchronous, as it is sometimes termed, with those of a balance unaffected by temperature. The compensation balance is often constructed thus. The rim is divided into three equal parts as in Fig. 139. each connected with the axis by a separate radius, as *ab*, *ac*, *ad*. The three divisions of the rim are perfectly similar, and each composed of two plates of metal closely united or soldered together, the inner of steel, the outermost of brass. At *e*, *f* and *g*, near to the detached extremities of the three divisions of the rim, are fixed three small knobs which screw into them, and which may have their positions shifted a little for the sake of adjustment. Suppose now the balance to be expanded by heat: The matter in the radii and in the parts

of the rim adjoining to them will be carried outwards; and, in so far as depends on this, the watch will tend to go slower. But the exterior metal of the compound plate is more expansible than the interior. This incurvates the rim, and throws the parts towards *e*, *f* and *g* nearer to the axis; and, in so far as depends on this cause, the watch will go faster. Should the compensation want much of being complete, we may substitute for *e*, *f* and *g* three other screws with larger knobs, and if it wants but little we may remove those which we have a little nearer to the extremities to which they are adjacent, which will cause them to approach nearer still to the axis, by a given increase of temperature. By cold, the radii and the adjacent parts of the rim will be contracted, or brought nearer to the axis; but the brass contracting more than the steel will diminish the curvature of the rim, and carry the detached ends and their screws outwards. A watch of this kind may be considered as perfect if it preserves an uniform *rate*; that is, if it always gains or always loses the same number of seconds per day; for when the rate is found, allowance can be made for it in calculation. It is however convenient that the rate should be small, and to accomplish this object, there may be three other small screws at *b*, *c* and *d*, on the outside of the rim. When the motion is too quick we unscrew these a little, and equally, so as to preserve the equipoise of the wheel. When it is too slow we screw them down. The reason of this may be understood from what has been already stated, and will be still better comprehended when we have considered the subject of rotatory motion. The weight of these compensation balances is considerable, and, when exposed to concussion, is apt to break the pivots, so that they do not answer well in pocket watches.

370. Before we leave this subject, it may not be amiss to show the way in which a pendulum or a balance spring is applied to the regulation of clock or watch movements.

The toothed wheel, of which *D* (Fig. 140.) is the centre, called the *swing wheel* or *pallet wheel*, is the last in a train of wheelwork driven by the weight or maintaining power. The crooked lever *BAC*, with which the pendulum rod *AP* is connected, so as to have the same angular motion, is termed the crutch. Its arms terminate in two oblique faces at *B* and *C*, called the pallets, on which the teeth of the swing wheel drop alternately. When the pendulum ball *P* is moving to the left, carrying with it the arm *AB*, a tooth escapes from *B*, and one drops on *C*; on the return of the pendulum towards the right this tooth escapes from *C*, and another drops on *B*, and thus a tooth escapes from the crutch at every second vibration of the pendulum. There are thirty teeth in the swing wheel, and consequently it turns once round for every sixty vibrations of the pendulum, that is, in one minute. On the axis of this wheel, projecting a little through the dial plate, the seconds hand is fixed. Knowing the weight which puts the train of wheelwork in motion, and the radii of the different wheels and their axes or pinions, we can find what force is exerted at the extremity of a tooth of the swing wheel in the direction of its motion. Let this be *F*. Draw *EG* perpendicular to the surface of the pallet, and let fall upon it the perpendiculars *DE*, *AG*: let the radius of the swing wheel from *D* to the extremity of a tooth be called *R*, and let *F* be the pressure exerted at *E* or at *G*, considered as having a rigid connection with the pendulum, and F'' the tangential pressure at *P*. This is the maintaining power which prevents the gradual diminution of the oscillations that would result from the resistance of the air and other obstructions. It may be estimated as follows: $F.R = F'.DE$ or $F = \frac{F.R}{DE}$; $F.AG = F''.AP$, or $F'' = \frac{F'.AG}{AP}, = \frac{F.R}{AP} \cdot \frac{AG}{DE}$; where F, R and AP are constant.

When a tooth of the swing wheel escapes from *C*, to which it has been for some time adding motion, the one that drops on *B* is, by the obliquity of the pallet, during the remainder of the pendulum's excursion to the right, pushed a little backwards. The same thing happens in the opposite vibration. When a tooth escapes from *B*, one moves forward through a small space to drop on C, and, by the remainder of C's motion to the left, is forced backwards through a smaller space. Hence this escapement, which is pretty generally used in common house-clocks, is called the *recoiling 'scapement.* It may be known by the motion of the seconds hand being alternately forwards and then, through a smaller space, backwards. When this escapement is used, we have seen that a pressure resulting from the weight is applied to the pendulum, in the way of acceleration or retardation, throughout the whole extent of its vibration. Now, unless this additional force follow the same law as the cycloidal modification of gravity, that is, be as the distance of the pendulum from the lowest point, or nearly as the sines of its small deviations from the vertical, which it is not, the isochronism of the oscillations will be disturbed. Hence the recoiling 'scapement, though quite sufficient for ordinary purposes, is unfit to be employed when great precision in the measurement of time is of importance.

371. Could the motion be maintained by an instantaneous impulse communicated to the pendulum ball, at the lowest point D, (Fig. 141.) the isochronism would not be disturbed. The ball having descended from A, would commence its ascent through DA' with the same velocity as if it had descended not from A but from some higher point B, and if the impulse restore the velocity which would be lost from the resistance of the air, &c. in descending from B to D, and communicate about as much more, it will ascend to the corresponding point B'.

372. To approach as near as possible to this state of things, is the object of what is called the *dead 'scapement*, or *dead beat escapement*, the invention of the ingenious Mr. Graham.

BAE, (Fig. 142.) represents the crutch, CD and GF the pallets, which are oblique planes as before, BC and EG cylindrical surfaces, whose common axis is a straight line passing through A perpendicular to the plane of oscillation. When a tooth escapes now from the pallet on the right, one drops on C, and during the remainder of that excursion of the pendulum it occasions very little obstruction to the motion, only that which arises from the viscidity of the oil, and a very slight friction as it slides from *C* towards *B*, along the hard and well polished cylindrical surface. On the return of the pendulum, it can give it no impulse so long as it slides from *B* to *C*; for the pressure, being always perpendicular to the surface on which it is exerted, is directed to the axis *A*. The only impulse during this oscillation is communicated by a pressure exerted for a short time, near the middle of the vibration, as the tooth slides along the oblique plane *CD*. Another then drops on *G*, slides from *G* towards *E*, and back to *G*, along the concave cylindrical surface, and finally communicates its impulse in sliding from *G* to *F*, near the middle of the following vibration. It is from each tooth remaining still, or without any angular recoil, during the time of its sliding along the convex or concave cylindrical surface, that this escapement has got its name. The seconds hand of course has no recoil.

373. Let *ABD* (Fig. 143.) represent the balance of a watch, and *c* the termination of its axis or verge, *ca* and *cb* the pallets, making an angle of about 95°. These are small planes, situated as in Fig. 144, and are represented in Fig. 143, as seen when we look to the verge in the direction of its length, or as projected on a plane perpendicular to the verge. When the plane of the balance is horizontal, the axis of the wheel which communicates to it the maintaining power is also horizontal, or the plane of its motion is vertical. *BD* represents the projection of a part of its rim on the upper side, with its teeth inclining to the right. On the lower part of the rim they will of course slope towards the left, when seen from above, and *efg* represents one of them in that situation. In the figure the tooth *h* of the crown wheel has impelled the pallet *ca* to the right, and is just ready to escape. As soon as it has escaped, the tooth *efg* drops on the lower pallet *cb*, but, the balance continuing its excursion for some time in the direction *BDA*, the pallet *cb* will draw the tooth *e* a little backward. When the balance stops, by the influence of the

spring, and the pressure of the tooth *e*, it immediately begins to return, and the tooth *e* exerts its maintaining power by pressing the pallet *cb* round till it escape from it, when the tooth *d* will drop on *ca*, and the former motion will be repeated. When the recoiling 'scapement is thus applied to a watch, whose balance must be light, that it may not break the fine pivots on which it turns, the inequalities of the maintaining power, and its deviation from the law that regulates the action of the balance-spring, occasion much greater variations in the rate than the corresponding inequalities do when the same sort of 'scapement is applied to a clock. The want of momentum in the balance, arising from the smallness of its mass, and its proximity to the axis, is partly compensated by giving it a considerable velocity, and by an imitation of, or an improvement on Graham's invention. In what are called detached 'scapements the balance is subjected to the action of the mainspring during a small part only of the vibration; and this is found to be a great improvement.

Of Collision, or of the Laws of Impulse as a Motive Force.

374. In treating of this subject, bodies may be divided into hard, soft, and elastic.

A perfectly hard body is conceived as one which suffers no compression, or change of form, by any force applied to it.

A perfectly soft body is one, the parts of which, on the application of any force, however small, suffer compression; and, taking a new arrangement, exert no force to regain their original position.

A perfectly elastic body is one whose parts suffer compression or dilatation, and exert the same force to recover their former state as they exerted in opposition to the change in them.

These definitions may be considered as marking the ideal limits of certain observed varieties in the constitution of bodies, no one being actually found which is either perfectly hard, perfectly soft, or perfectly elastic. In some bodies, however, as glass, ivory, and hard-tempered steel, the degree of elasticity is very great, approaching nearly to that which is defined as perfect. In other bodies, as lead or clay, it is, in the ordinary ways of trial, scarcely sensible.

275. All bodies appear to possess a repulsive force, the sphere of which extends to a very small and generally imperceptible distance around them, but which, increasing very rapidly as the distance decreases, is sufficient to prevent absolute or geometrical contact. When *A*, moving faster than *B*, overtakes it, or appears to do so, this mutual repulsion begins to act, and continues to take something from *A*'s velocity, and add something to *B*'s, till the two velocities are equal. The effect of *A*'s greater velocity before this equality is attained, is merely to keep the two bodies within the sphere of each other's influence, so as to prolong the action of the repulsive force; and, if the bodies are either perfectly hard, or perfectly soft, they will suffer no sensible change after the common velocity is attained. But if they are perfectly elastic, they will, during the change just described, compress each other, and the external particles of either body will be repelled inwards towards those next to them, until the elastic or repulsive force which these exert to prevent a nearer approximation is in equilibrio with the repulsion exerted against the other body. Thus two spherical balls of ivory will have, in their state of nearest approximation, not a point but a surface of apparent contact, the base of a spherical segment in each. Two balls of clay will also flatten each other, in the same way; but, the clay having no sensible tendency to recover its form, there is no cause preserving the action of the repulsive force; whereas, in elastic bodies, there is such a cause, the elasticity of the compressed interior particles in each body repelling the exterior particles towards the other; and if the elasticity be considered as perfect, there will be the same force mutually exerted, and the same changes made on the velocities of the two bodies, respectively, during the recoil as during the compression.

It follows, from what has now been observed, that it will be sufficient to consider bodies as divided into two classes, *elastic* and *unelastic*; and we begin with the latter, as presenting the simpler problem, the solution of which moreover naturally conducts us to the solution of that which is presented by the other. In what follows the bodies are supposed to be spherical.

Collision of Unelastic Bodies.

376. Suppose the body A moving with the velocity a to overtake B whose velocity is b, and that it is required to find the common velocity after impact. Let this be called x: then,

$a-x=$ velocity lost by A,

$x-b=$ velocity gained by B,

$A(a-x)=$ quantity of motion lost by A,

$B(x-b)=$ quantity of motion gained by B.

$A(a-x)=B(x-b)$, by third law of motion,

$$\text{whence } x=\frac{Aa+Bb}{A+B}.$$

Cor. Velocity lost by $A=\dfrac{B(a-b)}{A+B}$;

velocity gained by $B=\dfrac{A(a-b)}{A+B}$.

The changes of velocity, then, depend solely on the *relative velocity*, or velocity of approach, and on the *ratio* of the masses.

377. Suppose A to meet B, and to have the greater quantity of motion.

The velocities being denoted as before,

$a-x=$ velocity lost by A,

$b+x=$ velocity gained by B in the direction of A's motion,

$A(a-x)=B(b+x)$, by third law of motion, and

$$x=\frac{Aa-Bb}{A+B}.$$

[illegible] determined by the relative [illegible] bodies.

[illegible] by the use of a double

$$\text{Velocity lost by } A = \frac{B\,(a \mp b)}{A+B};$$

$$\text{velocity gained by } B = \frac{A\,(a \mp b)}{A+B}.$$

A third case, when B is at rest and struck by A, requires no separate solution, as we may accommodate the above equations to that case by supposing $\pm b = 0$.

Collision of Perfectly Elastic Bodies.

378. The result in this case is easily found by recollecting what was stated at the outset, that the change of velocity in each body is doubled by the recoil.

$$A\text{'s velocity after impact} = a - \frac{2\,B\,(a \mp b)}{A+B}.$$

$$B\text{'s velocity after impact} = \frac{2\,A\,(a \mp b)}{A+B} \pm b.$$

379. *Cor.* 1. The relative velocity remains unchanged in respect of quantity, but its sign is changed by the impact; in other words, the velocity of approach before the impact is equal to the velocity of recess after it. If a' and b' be the velocities of A and B, respectively, after the recoil, $a \mp b = b' - a'$.

This may be proved by subtracting A's velocity from B's, as determined above, or still more simply, by this consideration, that since, during the congress, by a subtraction from A's velocity and an addition to B's, the relative velocity is destroyed, or the actual velocities made equal, the equal mutual action during the recoil, taking as much more from A, and adding as much more to B, will reproduce the same difference as before, but in a contrary order.

380. *Cor.* 2. In the case of perfectly elastic bodies, not only the products of the quantities of matter into the velocities, estimated in the same direction, remain, as in all other cases of collision, unchanged; but also the products of the same into the *squares* of the velocities, for,

$$A\,(a - a') = B\,(b' \mp b), \text{ by third law of motion,}$$

$$\text{and } a + a' = b' \pm b, \textit{ Cor. } 1.$$

$$\therefore A\,(a^2 - a'^2) = B\,(b'^2 - b^2),$$

$$\text{or } A\,a^2 + B\,b^2 = A\,a'^2 + B\,b'^2.$$

381. *Cor.* 3. If A strike B at rest, or if $b = 0$,

$$A\text{'s velocity after impact} = \frac{A - B}{A + B}\,a;$$

$$B\text{'s velocity after impact} = \frac{2\,A}{A + B}\,a.$$

Thus, if $A = 2\,B$,

$$A\text{'s velocity after impact} = \frac{1}{3}\,a;$$

$$B\text{'s velocity after impact} = \frac{4}{3}\,a.$$

And if $B = 2\,A$,

$$A\text{'s velocity after impact} = -\frac{1}{3}\,a;$$

$$B\text{'s velocity after impact} = \frac{2}{3}\,a;$$

that is, A is reflected backwards with $\frac{1}{3}$ of its previous velocity, and B is progressive with $\frac{2}{3}$ of the same.

It is obvious from the formula that in this case the striking body, if the less of the two, must be always reflected.

382. *Cor.* 4. If A and B be equal, and A strike B at rest, A will remain at rest after collision, and B will proceed with

secting FG in C and join CB, A must be impelled in the line AC. The demonstration is obvious.

388. *Prob.* 2. Let A and B (Fig. 147.) be two equal spherical balls, and let A, moving in the direction and with the velocity AG, strike B obliquely; it is required to find the motion of each after impact. With the centre B and radius $BG =$ the sum of the radii, *i. e.* in this case $2\,BL$, describe a circle, and let the direction of A's motion meet its circumference in G. Take AG to represent A's velocity and resolve it into AH, HG perpendicular and parallel, respectively, to the tangent plane DLF. Then, 1*st*, If the bodies are perfectly elastic, the velocity AH will be destroyed, and B will proceed in the line BC, with a velocity represented by AH, and A will move in the direction GK, with a velocity represented by GK, $=$ HG. 2*dly*, Let the bodies be unelastic; B will still take the direction BC, but now with the velocity $\frac{1}{2}AH$. A will retain the velocity $\frac{1}{2}AH$ in that direction; and, if we make as before $GK = HG$, and draw KE perpendicular to it, and $= \frac{1}{2}AH$, the path and velocity of A after impact will be represented by GE.

If both the spheres be in motion when they meet, we require the solution of the following preliminary problem.

389. *Prob.* 3. Having given the radii of two spheres moving so that their centres are always found in the same plane, their velocities, the directions of their motions, and two contemporary positions of their centres, to find their position at the moment of impact, and that of the plane which is their common tangent at that instant.

Let the spheres be A and B, (Fig. 148.), let AD be described by A and BG by B in the same time, if moving uninterruptedly; complete the parallelogram $ABCD$, join CG, and let it be cut if possible in a point H by a line $DH =$ the sum of the radii of A and B. Complete the parallelogram $DHFE$;

$$\text{Then } BC : BG = FH : FG, \text{ or}$$
$$AD : BG = ED : FG,$$
$$\text{and } AD : BG = AE : BF, \text{ Eucl. v. 19.}$$

$\left(\frac{2a}{a+b}\right)^2 v$, and that of the last, $\left(\frac{2a}{a+b}\right)^{m-1} v$, in which, if n be the ratio of the progression, may be reduced to this form $\left(\frac{2}{1+n}\right)^{m-1} v$; and, as the bodies may be thus represented, $a, na, n^2a, \ldots n^{m-1}a$, the quantity of motion of the last will be $=\left(\frac{2n}{1+n}\right)^{m-1} av$. From this a curious consequence is deduced. If the progression be a decreasing one, the velocity will increase, and the quantity of motion will decrease, both without limit; and, if the progression be an increasing one, the velocity will decrease, and the quantity of motion increase without limit.

2

386. *Cor.* 9. A perfectly elastic ball is reflected from an immovable plane so as to make the angle of reflection equal to the angle of incidence.

1. Let it be considered as a physical point or single particle; and resolve the velocity of incidence *AB* (Fig. 145.) into *AD*, *DB*. When the particle strikes the plane *DE* at *B*, the velocity $DB = BE$ continues undiminished, and *AD* or *FB* is destroyed, and, as far as our senses can distinguish, *instantly* reproduced in the direction *BF*, and the velocities *BE*, *BF* will, by composition, produce $BC = AB$, making the angle $CBF =$ the angle *ABF*.

2. If *B* be a ball whose radius is sensible, let *D′E′* be the reflecting plane, and, having drawn *DE* parallel to it, at a distance = radius of *B*, and on the side on which the body approaches it, let *AB* meet this line in *B*, and proceed as before.

387. *Prob.* 1. Let *A* and *B* (Fig. 146.) be two spherical balls, of which *A* is perfectly elastic, and *F′G′* a plane surface considered as immovable, it is required to find in what direction *A* must be impelled so as after reflection from the plane to strike *B*. Let *FG* be parallel to the plane *F′G′*; at a distance = the radius of *A*, draw *BD* perpendicular to *FG*, and produce it to *E*, making $DE = BD$. Draw *AE* inter-

secting *FG* in *C* and join *CB*, *A* must be impelled in the line *AC*. The demonstration is obvious.

388. *Prob.* 2. Let *A* and *B* (Fig. 147.) be two equal spherical balls, and let *A*, moving in the direction and with the velocity *AG*, strike *B* obliquely; it is required to find the motion of each after impact. With the centre *B* and radius $BG =$ the sum of the radii, *i. e.* in this case $2\,BL$, describe a circle, and let the direction of *A*'s motion meet its circumference in *G*. Take *AG* to represent *A*'s velocity and resolve it into *AH*, *HG* perpendicular and parallel, respectively, to the tangent plane *DLF*. Then, 1*st*, If the bodies are perfectly elastic, the velocity *AH* will be destroyed, and *B* will proceed in the line *BC*, with a velocity represented by *AH*, and *A* will move in the direction *GK*, with a velocity represented by $GK, = HG$. 2*dly*, Let the bodies be unelastic; *B* will still take the direction *BC*, but now with the velocity $\frac{1}{2}\,AH$. *A* will retain the velocity $\frac{1}{2}\,AH$ in that direction; and, if we make as before $GK = HG$, and draw *KE* perpendicular to it, and $= \frac{1}{2}\,AH$, the path and velocity of *A* after impact will be represented by *GE*.

If both the spheres be in motion when they meet, we require the solution of the following preliminary problem. J. K

389. *Prob.* 3. Having given the radii of two spheres moving so that their centres are always found in the same plane, their velocities, the directions of their motions, and two cotemporary positions of their centres, to find their position at the moment of impact, and that of the plane which is their common tangent at that instant.

Let the spheres be *A* and *B*, (Fig. 148.) let *AD* be described by *A* and *BG* by *B* in the same time, if moving uninterruptedly; complete the parallelogram *ABCD*, join *CG*, and let it be cut if possible in a point *H* by a line $DH =$ the sum of the radii of *A* and *B*. Complete the parallelogram *DHFE*;

Then $BC : BG = FH : FG$, or

$AD : BG = ED : FG$,

and $AD : BG = AE : BF$, Eucl. v. 19.

Therefore since AE and BF are in the ratio of the velocities, they are spaces described in the same time, (313.) and E and F will be cotemporary positions of the centres of the spheres; but EF is $= DH =$ the sum of the radii, and the spheres must touch each other when in this situation. Take $EO =$ the radius of the sphere A, and through O let a plane pass at right angles to EF, this will be the tangent plane required.

This elegant solution is to be fouud in the commentary on Newton's *Principia.* Wood has also given it in his *Mechanics.* To find the motions of the spheres after impact, we must find, as in this problem, the tangent plane common to both at the instant of the impact; resolve the velocity of A into a perpendicular to that plane, and a' parallel to it, and the velocity of B in like manner into b and b', the former perpendicular, the latter parallel to the same: then, by the formulæ already given for bodies elastic or unelastic, as the case may require, find by the quantities of matter, and the velocities a, b, what will be the velocity of each body after impact, and in what direction; these compounded with a' and b', which remain undiminished, will give the new motions.

390. In treating of impulse we have hitherto considered only the simplest cases, in all of which, the line of effective collision passes through the centre of gravity. But even when the motion of the one body is directed towards the other's centre of gravity, the impulse may not be communicated in that direction. It will not be so directed if the perpendicular to the common tangent plane at the point of impact do not pass through the centre. Thus CE (Fig. 149.) being the tangent plane, and A's velocity, AD, being resolved into AE, ED, the body B will be struck with what is lost by A of the quantity of motion $A.AE$, and in the direction DH, not passing through the centre of gravity G. The result is now less simple than in the former case; and, before it can be explained, we must demonstrate some dynamical properties of the centre of gravity, and the laws of rotatory motion.

Of the Motion of the Centre of Gravity.

391. If *C* (Fig. 150.) be the centre of gravity of the masses *A* and *B*, they are kept in equilibrio, as has been formerly demonstrated, by a pressure equal to the sum of their weights acting in the parallel and opposite direction *CD*, (43.) This force then applied to the centre of gravity, is so distributed as to oppose an equal and opposite pressure to the gravitation of every equal particle; for the equal tendency of each to descend is just counteracted. Were the weights of *A* and *B* suddenly annihilated, the inertia of each equal particle would oppose the same resistance to motion, and the whole would begin to move in the directions in which they are pressed, that is, in parallel directions, and with the same progressive velocity.

If we consider impulse as a case of pressure arising, as formerly described from the action of repulsive forces, it will follow that when a body is struck so that the whole efficient force is directed to the centre of gravity, every particle will have an equal progressive motion, so that there will be no tendency to rotation.

Or, if we consider impulse as a motive force *sui generis*, we may still demonstrate the same thing, by the aid of a dynamical axiom analogous to the third in Statics: "If two equal quantities of motion be applied perpendicularly to the extremities of an inflexible physical straight line, they will be balanced by a quantity of motion equal to their sum, applied in a contrary direction, at the middle point between them." Taking this for granted as a contingent truth, agreeable to universal experience, we can demonstrate a series of consequences relating to the lever and its modifications, as affected by impulsive forces, or the quantities of motion that measure them, exactly similar to those proved in statics in relation to pressures; and thus we show that there is a centre of parallel impulses, as well as of parallel pressures, and the properties of the centre of gravity in the one case are analogous to those which obtain in the other. Conceive a body to be descending so that every particle has the same progressive velocity, in parallel directions, and that a quantity of motion equal to its own is impressed in a vertical line passing through the centre of gravity, the velocity of each particle will be annihilated, and the force which accomplishes this would have impressed upon them all equal and opposite velocities, had those supposed not previously existed.

392. If any uniform motions whatever in parallel directions be impressed upon a number of unconnected bodies, their common centre of gravity, or of position, as it should rather in this case be named, will either remain at rest, or its motion will be *uniform*, *rectilinear*, and parallel to that in which the motive forces are applied.

Let the parallel lines *HD*, *IE*, and *LC*, (Fig. 151.) whether lying in the same plane or not, be perpendicular to a plane whose projection is *RS*; and let the bodies *A*, *B*, *C*, whose centre of gravity is *G*, have such motions impressed on them that they describe the spaces *AD*, *BE*, *CF*, uniformly in the same time; draw *GK* perpendicular to *RS*, and therefore parallel to the directions of the impressed motions; the centre of gravity shall always be in this line. For let a plane pass through *KG*, and, taking *KM* any other line in that plane, draw *IN* parallel to it, and the planes *MKG*, *NIB* will be parallel. In the same manner planes may be conceived to pass through *HD* and *LC* parallel to the plane *MKG*. While the bodies describe *AD*, *BE*, *CF*, they remain in those parallel planes, and their perpendicular distances from the plane *MKG* will be constant, as also the momenta or products of the masses into those distances. Now the sum of these momenta $= 0$, when the bodies are in the positions *A*, *B*, *C*; for the plane to which they are referred passes through the centre of gravity; and as the sum continues $= 0$, the centre must always be found in that plane, (50.) In the same way it may be proved to continue in any other plane passing through *KG*; it must, therefore, be always in *KG* the line of common section.

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o m-site, will not affect the state of the centre of gravity; that is, if any forces whatever be impressed upon any parts whatever of a *system of connected bodies*, the motion of the centre will be the same as if the bodies were free, and the same as if these forces were directly applied to itself.

398. *Cor.* 6. In eccentric impact there is combined with this progressive motion a motion of rotation about the centre of gravity.

G (Fig. 152.) being the centre, and GOQ a right angle, let OQ represent the force impressed F. Bisect OQ in R, so that OR and RQ may each represent $\frac{1}{2}F$: take $GB = GO$, and suppose the forces BN, BP, which are each $= OR$, to be applied at B, in opposite directions; then, as BN, BP balance each other, the body will have the same motions whether the single force OQ be impressed, or the four equal forces OR, RQ, BN, BP be applied to it simultaneously. Now the effect of BP and OR, whose resultant is $=$ their sum, $= F$, and passes through G, will be to produce an uniform progressive velocity in the whole body, while BN and RQ, with a momentum $= BN.BG + RQ.GO, = F.GO$, conspire in producing a rotation about G the middle point between them.

Note.—This reasoning will apply to pressure as well as impulse.

399. *Cor.* 7. Though the eccentric impact produces both a progressive and a rotatory motion, the latter adds nothing to the quantity of motion estimated in any given direction.

Let FE and CD (Fig. 153.) be two planes passing through the centre of gravity, and let their intersection be the axis of rotation: let GA be a straight line perpendicular to the axis, AC perpendicular to GA, and AB to GC. Then if v be the angular velocity, $GA.v$ is the absolute velocity of the particle A, and $A.GA.v$ its quantity of motion in the direction AC; this is equivalent to $A.GA.v.\cos.\ BAC = v.A.GA.\cos.\ BGA = v.A.GB$ parallel to FE, and $A.GA.v.\sin.\ BAC = v.A.GA.\sin.\ BGA = v.A.BA$ parallel to GC. Therefore $\int A.GA.v$ or $v\int A.GA$ is equivalent to $v\int A.GB$ and $v\int A.BA$, in the directions of two rectangular axes, and $\int A.GB = 0, \int A.BA = 0$, (50. end); whence the truth of the Corollary is manifest.

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$c\,p$, &c. and those on
, $C\,c\,p'$, &c.
e distance d to produce

:

$$\frac{C\,c^2p'}{d} = F,$$

r^2

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ative force at that distance.
ed in terms of the weight
gives the accelerative force
ivity.

$$\frac{Aa^2v}{d}, + \frac{Bb^2v}{d} + \frac{Cc^2v}{d}, \&c. = F;$$

$$\text{whence } v = \frac{Fd}{Aa^2 + Bb^2 + Cc^2}, \&c. = \frac{Fd}{\int prr},$$

p denoting any particle indeterminately, and r its distance from the axis.

402. *Cor.* 1. Thus we see that a body resists the communication of angular motion not merely in proportion to its inertia p, but also in proportion to the square of its distance from the axis: for the resistance to angular motion may be considered as proportional to the force required at the unit of distance to impress a given velocity there; and this force, we have just seen, varies as $\int prr$. F at the distance d is equivalent to Fd at the distance 1, and

$$Fd \doteqdot \int prr \text{ when } v \text{ is given.}$$

Note.—$\int prr$ is called, though not perhaps with very great propriety, the *momentum of inertia*.

403. *Cor.* 2. When Fd is given, v is inversely as $\int prr$, and will be constant when that is constant, and greatest when that is least.

404. *Cor.* 3. A point may be assigned such that if all the matter of a body m were there concentrated, a given force impressed at a given distance from the axis would produce the same angular velocity as it does with the matter in its natural form.

We have only to suppose $mk^2 = \int prr$,

$$\text{or } k = \sqrt{\frac{\int prr}{m}};$$

and any point at the distance k from the axis will have the property mentioned. This point is called the *Centre of Gyration*, and we may call the distance k the *Radius of Gyration.*

405. *Cor.* 4. It is often convenient in calculation to assign the mass m' of equivalent resistance to be placed at a given distance from the axis, for instance the distance d at which a force is applied: then

$$m'd^2 = \int p\,r\,r = m\,k^2$$

$$m' = \frac{\int p\,r\,r}{d^2} = \frac{m\,k^2}{d^2}.$$

Note.—F may then be considered as impressed upon m' in the way of direct impact, and the velocity at the distance d, $= \frac{F}{m'}$, will be known; hence, too, the angular velocity is given, and the absolute velocity at any assigned distance.

406. *Prop. II. Prob.* Let there be a system of bodies A, B, C, &c. as in *Prop. I*, and let a pressure F be applied in a direction to which d is the perpendicular distance; it is proposed to find the accelerative force at the point of application.

From the connection of the parts the velocity communicated in a given time to any particle, or the measure of its accelerative force, must be as the distance from the axis. But the accelerative force is also as the pressure applied to the unit of mass. Let p' then be this pressure at the distance 1 from the axis:

At the distances a, b, c, &c. the pressures on units of mass are ap', bp', cp', &c. and those on the masses A, B, C, &c. $A\,ap'$, $B\,bp'$, $C\,cp'$, &c.

The corresponding pressures at the distance d to produce these will be $\frac{Aa^2p'}{d}$, $\frac{Bb^2p'}{d}$, $\frac{Cc^2p'}{d}$;

$$\therefore \frac{A\,a^2p'}{d} + \frac{B\,b^2p'}{d} + \frac{C\,c^2p'}{d} = F,$$

$$\text{and } p' = \frac{Fd}{\int p\,r\,r},$$

$$\text{or } pd = \frac{Fd^2}{\int p\,r\,r}.$$

Now pd, being the pressure on the unit of mass at the distance d, is a measure of the accelerative force at that distance.

Note.—If the pressure be assigned in terms of the weight of the unit of mass, this formula gives the accelerative force as a multiple or submultiple of gravity.

407. *Cor.* 1. $\frac{Fd}{\int prr}\,g$ is the angular accelerative force.

408. *Cor.* 2. If $m' = \frac{\int prr}{d^2}$, $\frac{Fg}{m'}$ is the accelerative force at the distance d.

409. When a body is connected with a fixed axis, if a force impressed be such as to make it strike that axis, the reaction will affect the motion of the centre of gravity. If, therefore, in any case, the motion of this centre be both in quantity and direction the same as if the body were free, it is a proof that no force is, in the nascent state of the motion, exerted by the axis, and that the body so struck would begin to turn round the same geometrical axis in free space. A line in that situation is called the *Axis of Spontaneous Rotation.*

410. *Prop. III. Prob.* To find the axis of spontaneous rotation.

Let a force F be impressed at O, (Fig. 154.) so as to make the body whose centre of gravity is G to revolve about an axis passing through C. This force may be impressed in a direction perpendicular to CO, and the incipient motion of the centre of gravity Gg will then be in the same direction, whether the body be considered as free or as connected with the axis. Also the mass of equivalent inertia being $m' = \frac{\int prr}{CO^2}$, (405.) the velocity of O will be $\frac{F}{m'} = \frac{F.CO^2}{\int prr}$, and that of G $= \frac{F.CO.CG}{\int prr}$. If m be the mass of the body, $\frac{F}{m}$ is the velocity which the centre of gravity would have were the body free, and CO, when possible, will be determined by the following equation,

$$\frac{F.CO.CG}{\int prr} = \frac{F}{m},$$

$$\text{or } CO = \frac{\int prr}{m.CG}.$$

411. *Cor.* 1. Let $\int p r' r'$ represent the momentum of inertia for an axis parallel to the former and passing through G.

$$\text{Then (86.) } CO = \frac{\int p r' r' + m.CG^2}{m.CG}, = \frac{\int p r' r'}{m.CG} + CG,$$

$$\text{and } GO = \frac{\int p r' r'}{m.CG}, \text{ or } CG = \frac{\int p r' r'}{m.GO}.$$

Now, for the same axis $\frac{\int p r' r'}{m}$ is a constant quantity, and $CG \div \frac{1}{GO}$; so that when GO is indefinitely small CG is indefinitely great, and if O coincide with G, there is no axis of spontaneous conversion. This is agreeable to what has been previously demonstrated otherwise, that when the line of effective impulse passes through the centre of gravity, the whole mass has an equal progressive motion, without rotation, (391.)

412. *Cor.* 2. It is evident (397.) that the rotation about an axis passing through C determined as above is momentary. The continued rotation is about an axis passing through the centre of gravity, and $CG = \frac{\int p r' r'}{m.GO}$ is the distance at which the absolute velocity of rotation is equal to the progressive motion. Every point of the axis passing through C is affected by two equal and opposite velocities, and therefore momentarily quiescent.

413. *Cor.* 3. If an unelastic body whose mass is M, moving with the velocity V, strike in the direction FO (Fig. 154.) a body whose centre of gravity G and centre of spontaneous rotation C are given, the motions produced may be determined thus:

The velocity common to the body and the point struck after impact will be

$$\frac{MV}{\frac{\int prr}{CO^2} + M}, = \frac{MV.CO^2}{\int prr + M.CO^2}, = \frac{MV.CO^2}{m.CG.CO + M.CO^2} =$$

$\frac{MV.CO}{m.CG + M.CO}$. This being the velocity with which O will begin to revolve about C, and the velocities being as the distances from the axis, that of the centre of gravity G will be $=\frac{MV.CG}{m.CG+M.CO}$. This, which is equal to the progressive velocity, or to the velocity of C's actual revolution about G, being divided by $2\pi.CG$, the circumference of the circle described by C, will give $\frac{\frac{1}{2\pi}MV}{(M+m)CG+M.GO}$ for the angular velocity in revolutions and parts of a revolution per second.

414. There is a certain point in every body revolving about an axis, to which if a fixed obstacle be presented, the body having struck it shall remain in equilibrio, or without a tendency to revolve about it in any direction. This point is called the *Centre of Percussion.*

415. *Prop. IV. Prob.* To find the distance from the axis of motion to the centre of percussion.

Let C (Fig. 154.) represent the axis, and O the centre to be found. If the angular velocity be v, the absolute velocity of A in the direction AB, perpendicular to CA, is $AC.v$, and its quantity of motion $A.AC.v$. This considered as applied at B and estimated perpendicular to CO is $A.AC.v.\sin.B = A.AC.v.\sin.CAD = A.CD.v$. ∴ The momentum with reference to an obstacle at O considered as an edge or line parallel to the axis is $A.CD.OB.v = A.CD.v\,(CO - CB)$ $= A.CD.v\left(CO - \frac{AC^2}{CD}\right)$; and, by the conditions of the problem, the sum of these momenta $= 0$: that is, $\int A.CD.CO$, or $CO\int A.CD = \int A.CA^2$. Whence $CO = \frac{\int A.CA^2}{\int A.CD} = \frac{\int prr}{m.CG}$.

If the revolving body be a straight line as CR, (Fig. 155.) or any surface or solid of uniform density constructed symmetrically with respect to that line, for example, a rectangle bisected by it, an ellipse or ellipsoid of which it is an axis, &c. the point O, determined by the distance CO found as above, will be the centre of percussion. In other cases, the sum of such momenta as $HB.v.BD$, that is, $v\int HB.BD$ will not be equal on each side, and there will be a tendency to rotation about CR, or the centre of percussion will not be O, but some other point O' at the same distance from the axis.

416. In the case of any body revolving about a fixed axis by its weight alone, there may be found a point at which all the matter of the body being collected, and there also acted on by its weight alone, the angular accelerative force shall, at any given deviation from the vertical, be the same as before. This point is called the *Centre of Oscillation.* Could the matter be actually concentrated, as supposed, it would evidently constitute a simple pendulum, oscillating through any given angle in the same time as the body to which it refers does in its natural form. The distance from the axis to the centre of oscillation of a pendulous body, we shall occasionally call its equivalent simple pendulum.

417. *Prop. V. Prob.* To find the centre of oscillation of a body turning round a fixed axis.

Let w be the weight of the body, m its mass, G (Fig. 154.) the centre of gravity, O the centre of oscillation to be found, and φ the sine of deviation of the line CG from the vertical. In the actual form of the body the weight or motive force may be considered as acting at the centre of gravity, and the angular accelerative force as a multiple of gravity is $\frac{\varphi\, w.CG}{\int prr}$, (407.) Again, when the matter is all collected at O, the angular accelerative force is, by the same rule, $\frac{\varphi\, w.CO}{m.CO^2} = \frac{\varphi\, w}{m.CO}$.

Hence, by the conditions of the problem, $\frac{\varphi\, w.CG}{\int prr} = \frac{\varphi\, w}{m.CO}$, and $CO = \frac{\int prr}{m.CG}$.

418. *Cor.* 1. $GO = \frac{\int pr'r'}{m.CG}$, (411.)

419. *Cor.* 2. It appears by the last corollary that the distance from G to the centre of oscillation is equal to $\frac{\int pr'r'}{m}$

divided by the distance from G to the axis. Let then an axis parallel to the former pass through O, and the distance from G to the centre of oscillation will now be $\frac{\int p\,r'r'}{m.GO}$. But $CG = \frac{\int p\,r'r'}{m.GO}$, (418.) Two parallel axes then passing through C and O may be called conjugate axes of oscillation. To each as an axis of motion the centre of oscillation is in the other. This property has lately been ingeniously employed by Captain Kater to find the length of the equivalent simple pendulum, and thence, by observing the number of its vibrations in a given time, the length of the pendulum that swings seconds in the same place.

420. *Cor.* 3. For the same body, and axis parallel to a line given in position, $\frac{\int p\,r'r'}{m}$ is constant, and CG varies inversely as GO. Therefore while CG is constant GO is constant, and consequently their sum CO remains of the same length. That is, if a circle be described from the centre G at the distance GC in a plane perpendicular to a line given in position, the body will oscillate in the same time through whatever point of its circumference the axis of oscillation pass, provided it be always parallel to that line.

421. *Cor.* 4. As the rectangle under CG and GO is constant, their sum CO and the corresponding time of vibration will be a minimum with reference to the same set of parallel axes when they are equal, or when $CG^2 = \frac{\int p\,r'r'}{m}$.

In every other case there will be *four* points in the line CGO, the oscillations about parallel axes passing through which will be performed in the same time. In finding the length of the equivalent simple pendulum by Kater's method, care must be taken that those selected shall be conjugate to each other.

422. *Cor.* 5. The distance from the axis of suspension to the centre of oscillation may be found experimentally, where extreme precision is not required, by taking a small spheri-

cal ball suspended by a fine wire or thread, and lengthening or shortening it till it vibrate in the same time as the body, and for the greater exactness applying a small correction, to be afterwards explained (434.); or, considering the length of the seconds pendulum as ascertained, we may cause the body to vibrate, and count the small oscillations as long as they are perceptible; then supposing N to be the number per minute, the distance to the centre of oscillation, or length of the simple pendulum whose vibrations would be synchronous, is $L = \frac{140900.4}{N^2}$, (360.)

423. *Cor.* 6. Hence $\int prr$ or $\int pr'r'$ may be found experimentally. m as measured by weight may be found by a balance, CG as in (120.), and CO by one of the methods just described; then

$$m.CG.CO = \int prr;$$
$$m.CG.GO = \int pr'r'.$$

424. *Cor.* 7. Let k be the radius of gyration; that also may be found experimentally; for $k^2 = \frac{\int prr}{m}$, (404.) $= CG.CO$.

425. *Cor.* 8. When a pendulum has completed the semiarch of vibration, so that the centre of oscillation is at the lowest point, its velocity, or that of any other point in a line drawn through it parallel to the axis, is that which is due to a fall through the versed sine of the arch described; a property which does not belong to any point at a greater or less distance.

The first part is obvious by a comparison of the motion with that of the extremity of the equivalent simple pendulum; the second by recollecting that the velocities of different points are as the radii of the arches which they describe, that is, as the versed sines of the semiarches of vibration, while those due to falls through the versed sines are in the subduplicate ratio of the same, (320.)

426. *Cor.* 9. Let a body be composed of several parts whose masses m, m', m'', &c. are known, as also the distances to their

respective centres of gravity h, h', h'', &c. and the distances to their centres of oscillation l, l', l'', &c. and let L be the distance from the axis to the centre of oscillation of the whole;

$$L = \frac{mhl + m'h'l' + m''h''l', \&c.}{mh + m'h' + m''h'', \&c.}, \text{ (417.) (423.) (49.)}$$

427. *Cor.* 10. Let m, h and l denote as before, and let a small sliding piece μ be attached to the pendulum rod, the variable distances from the axis to whose centres of gravity and oscillation may be always accounted equal, and each $= \lambda$;

$$L = \frac{mhl + \mu\lambda^2}{mh + \mu\lambda}.$$

Hence L will vary with the position of μ, that is, with the variation of λ, and the sliding piece may be employed to regulate with precision the time of vibration. Considering L and λ as variable, we derive this fluxional equation,

$$dL = \frac{\mu^2\lambda^2 + 2m\mu h\lambda - m\mu hl}{(mh + \mu\lambda)^2}.d\lambda.$$

When the numerator of the coefficient is positive (or $\mu\lambda > \sqrt{m^2h^2 + m\mu hl} - mh$, that is, λ a little greater than $\frac{1}{2}l$,) the signs of $d\lambda$ and dL will be the same, when it is negative they will be opposite.

For a detailed and perspicuous statement of the effect in different positions as deduced from these premises, see Whewell's *Dynamics*, § 92.

We shall now show how to find by calculation $\int prr$ in some of the cases of most frequent occurrence, on the supposition of uniform density.

428. Let s' represent the increment of the required sum between two distances from the axis of which r is the less and R the greater, and m' the increment of the mass,

$$s' \text{ is } > r^2m', \text{ but } < R^2m',$$

$$\text{or } \frac{s'}{m'} \text{ is } > r^2 \text{ but } < R^2;$$

but r^2 is the limit of R^2, and consequently of the quantity always intermediate $\frac{s'}{m'}$, while the time of the cotemporary increments is continually diminished: and $\frac{ds}{dm}$ is another expression of the same limit, $\therefore \frac{ds}{dm} = r^2$, $ds = r^2 dm$,

$$\text{and } s = \int prr = \int r^2 dm.$$

429. Let the length of the physical straight line CA (Fig. 156.) $= a$, $CB = x$; the mass of the unit of length being taken as the unit of mass, a and x may denote quantity of matter as well as length, and $dm = dx$. Hence $s = \int x^2 dx$ $= \frac{1}{3}x^3 = \frac{1}{3}a.a^2$, when $x = a, = \frac{1}{3}ma^2$, m being our general symbol for the mass of the body, so that for any very small cylinder or prism we may conceive the effect of inertia to be the same as if one third of the mass of the body were placed at the extremity, and the rest annihilated.

Cor. Let $ABED$ be a rectangle whose length $AD = a$ and breadth $AB = b$, and let CF, parallel to AD, bisect AB in C. Then

(*a.*) If the rectangle revolve about the axis AB,

$$s = \tfrac{1}{3}m.a^2.$$

(*b.*) If it revolve about the axis CF,

$$s = \tfrac{1}{12}m.b^2.$$

(*c.*) If it revolve about an axis perpendicular to its plane, passing through its centre of gravity G,

$$s = \tfrac{1}{12}m(a^2 + b^2).$$

Note.—This will also serve for a parallelopiped whose mass is m, and whose section perpendicular to the axis of revolution is the rectangle $ABED$.

(*d.*) If it revolve about an axis perpendicular to its plane, and passing through C,

$$s = \tfrac{1}{3}m(a^2 + \tfrac{1}{4}b^2).$$

(*e.*) If it revolve about an axis perpendicular to its plane and passing through one of the angular points,

$$s = \tfrac{1}{3}m(a^2 + b^2).$$

Note.—The last two formulæ will equally serve for a parallelopiped whose section is the given rectangle revolving about the same axes.

430. Let *ADE* (Fig. 158.) represent a physical circle, or thin cylindrical lamina, revolving about an axis passing through its centre and perpendicular to its plane. Let r be the radius of this circle, and let x be the variable radius of the dilating concentric circle *XFG*, by whose circumference we may conceive the area of the given circle to be generated. m for this circle $= \pi x^2$, $dm = 2\pi x\,dx$, and $s = \int 2\pi x^3 dx$,

$$= \tfrac{1}{2}\pi x^4, = \tfrac{1}{2}\pi r^4, \text{ when } x = r;$$

$$\text{or } s = \tfrac{1}{2}\pi r^2.r^2 = \tfrac{1}{2} m r^2;$$

so that the resistance to angular motion is the same as if one half of the mass were placed at the circumference and the rest annihilated.

Cor. 1. The result will be the same for any solid revolving cylinder of uniform density, whatever be its length.

Cor. 2. For a hollow cylinder the exterior radius of whose section $= r$, and the interior $= k$,

$$s = \tfrac{1}{2}\pi r^4 - \tfrac{1}{2}\pi k^4, = \tfrac{1}{2}\pi(r^4 - k^4),$$

$$= \tfrac{1}{2}\pi(r^2 - k^2).(r^2 + k^2) = \tfrac{1}{2} m (r^2 + k^2).$$

Cor. 3. From the result above stated, we have an easy method of finding s in reference to a diameter of the circle. Let *AB* and *DE* (Fig. 159.) be any two diameters of the circle at right angles to each other; let *CH* represent the distance of any particle from the centre, and draw *HF*, *HG* perpendicular to *AB* and *DE*; then $CH^2 = HF^2 + HG^2$, and $\int prr = \int p.CH^2, = \int p.HF^2 + \int p.HG^2, = 2\int p.HF^2$; and since $\int p.CH^2 = \tfrac{1}{2}\pi r^4, \int p.HF^2$, or its equal $\int p.HG^2 = \tfrac{1}{4}\pi r^4 = \tfrac{1}{4} m r^2$.

Cor. 4. From this result, again, we have an easy determination of $\int prr$ with reference to any axis in the plane of the circle or parallel to it, as *MN*, at a given distance d from the centre; for by the last corollary and the property of the centre of gravity (86.) it must be

$$\tfrac{1}{4}\pi r^4 + m d^2 = m(\tfrac{1}{4} r^2 + d^2);$$

and if *MN* be a tangent to the circle, it is $\tfrac{5}{4}\pi r^4$.

Cor. 5. Let *ABDE* (Fig. 160.) be a cylinder revolving about an axis *CF* passing through its centre of gravity *G*;

let $FB = a$, and $LK = b$, and $c^3 =$ the area of a section as KM. Also let the variable line $GL = x$. Then, with reference to KM as an axis of rotation, for the area of the section $s = \frac{1}{4} c^2 b^2$, (*Cor.* 3.) and therefore with reference to the parallel axis CGH it will be $\frac{1}{4} c^2 b^2 + c^2 x^2$, and for a very thin lamina whose thickness is dx, it will be $\frac{1}{4} c^2 b^2 dx + c^2 x^2 dx$, the integral of which, when $x = a$, is $\frac{1}{4} c^2 b^2 a + \frac{1}{3} c^2 a^3$, or $\frac{1}{2} m (\frac{1}{4} b^2 + \frac{1}{3} a^2.)$ To this adding its equal for the other half of the cylinder, or as we sometimes express it, integrating between the limits $x = a$ and $x = -a$, we have for the whole cylinder

$$s = m (\tfrac{1}{3} a^2 + \tfrac{1}{4} b^2) = \tfrac{1}{12} m (4 a^2 + 3 b^2).$$

431. Some of these deductions may be simply and elegantly enough verified by suspending cylinders and parallelopipeds of given materials and dimensions, and causing them to vibrate horizontally by the torsion of wires. It is easy to construct our experiments so that the ratio of the times of their oscillations, as deduced from the above data, shall be known.

432. Let $ADBE$ (Fig. 159.) now represent a sphere, and AB, DE, the projections of two of its great circles, at right angles to each other : $\int p r r$ with reference to their intersection as an axis $= \int p . HF^2 + \int p . HG^2 = 2 \int p . HF^2$ as before, these sums being now taken with reference to the planes of which AB and DE are the projections.

Let $ADBE$ (Fig. 161.) be a sphere, of which AB, DE are great circles as before, draw FG perpendicular to the radius CE, and join CF. Then CG being denoted by x, and FG by y, $dm = \pi y^2 dx$, and s, for the matter on one side of the plane AB, $= \int \pi y^2 x^2 dx$, $= \pi \int (r^2 - x^2) x^2 dx$, $= \frac{2}{15} \pi r^5$, when $x = r$. Hence for the whole sphere in reference to this plane it is $\frac{4}{15} \pi r^5$, and in reference to the axis coinciding with the intersection of the two planes

$$s = \tfrac{8}{15} \pi r^5 = \tfrac{2}{5} . \tfrac{4}{3} \pi r^3 . r^2 = \tfrac{2}{5} m r^2,$$

or it is equivalent to two-fifths of the whole mass accumulated at the extremities of its equatorial radii.

433. Let *ADBG* (Fig. 162.) be an oblate spheroid, generated by the revolution of the semi-ellipse *ADB* about the less axis *AB;* let *CE* or $CA = r$, and $CD = a$. *KIH* being supposed parallel to *CD*,

$$KI : KH, = r : a,$$

and, the sphere and spheroid being conceived as composed of corresponding circular laminæ whose radii are all as $r : a$, and $\int p\,r\,r$ for such laminæ being as the fourth powers of their radii, (430.)

$$r^4 : a^4 = \tfrac{8}{15}\pi r^5 : s, \text{ for the axis } AB \text{ of the spheroid.}$$

$$\therefore s = \tfrac{8}{15}\pi a^4 r = \tfrac{2}{5} m a^2.$$

It is obvious from what was proved in the investigation for the sphere (432.) that one half of this, or $\frac{4}{15}\pi a^4 r$, will be $= \int p \,.\, HF^2$ or $= \int p \,.\, HG^2$ in reference to one of the elliptical sections passing through the centre of the spheroid, that is in reference to *ADBG* in any one of its positions. Let us next find $\int p \,.\, HF^2$ or $\int p \,.\, HG^2$ in reference to the equator. If $CK = x$, and $HK = y$,

$$r^2 : a^2 = r^2 - x^2 : y^2 = \frac{a^2}{r^2}(r^2 - x^2),$$

and as y^2 for the sphere would be $= r^2 - x^2$ and $\frac{2}{15}\pi r^5 = \int p \,.\, HF^2$ for the same on each side of a great circle, it will be here $\frac{a^2}{r^2} \cdot \frac{2}{15}\pi r^5, = \frac{2}{15}\pi a^2 r^3$; and for both sides or for the whole spheroid $\frac{4}{15}\pi a^2 r^3$.

Cor. When the spheroid revolves or librates round an equatorial diameter, the value of s, in reference to that diameter, will be $\frac{4}{15}\pi a^4 r + \frac{4}{15}\pi a^2 r^3, = \frac{4}{15}\pi a^2 r (a^2 + r^2), = \frac{1}{5} m (a^2 + r^2)$, equivalent to two-fifths of the mass, nearly, placed at a distance which is a geometrical mean between the two semi-axes, when their difference is small.

434. To find the centres of gyration, oscillation, &c. is very simple when $\int p\,r\,r$ has been investigated. Thus suppose k the distance to the centre of gyration;

For a very slender rod or physical line, $k^2 = \frac{\int p r r}{m}, = \frac{\frac{1}{3} m a^2}{m}, = \frac{1}{3} a^2$; and $k = \frac{4}{7} a$, nearly

For a circle or cylinder, $k^2 = \frac{\frac{1}{2} m r^2}{m}, = \frac{1}{2} r^2$; and $k = \frac{5}{7} r$, nearly.

For a sphere, $k^2 = \frac{\frac{2}{5} m r^2}{m}, = \frac{2}{5} r^2$; and $k = \frac{7}{11} r$, nearly.

The distance to the centre of oscillation of a very slender rod, or of a rectangular surface revolving about a side, $= \frac{\int p r r}{m . CG} = \frac{\frac{1}{3} m a^2}{m . \frac{1}{2} a}, = \frac{2}{3} a$.

To find the distance from the axis to the centre of oscillation of a spherical ball, of uniform density, suspended by a fine wire or thread whose quantity of matter may be disregarded, let *C*, (Fig. 163.) represent the axis of oscillation, projected on the plane of the figure, to which we suppose it perpendicular, and let *G* represent the projection of a line parallel to the former passing through the centre of the ball. $\int p r' r'$, in reference to *G* as an axis of rotation, we have found to be $\frac{2}{5} m r^2$, (432.) Therefore if *CG* be denoted by *l*, and *O* be the place of the centre sought, $GO = \frac{\int p r' r'}{m . CG}, = \frac{\frac{2}{5} m r^2}{m l}, = \frac{2}{5} \frac{r^2}{l}$.

If the pendulum be very long compared with the radius of the ball, we may consider *O* as coincident with *G*.

Composition of Rotatory Motion.

435. A motion of rotation about the diagonal of any parallelogram, with an angular velocity represented by that diagonal, may be considered as compounded of motions of rotation round the sides which meet in one of its extremities,

with angular velocities represented by those sides respectively.

Let the rotation about CY (Fig. 164.) tend to depress, and that about CZ to elevate a point R, situated in the plane of these axes, and within the angle YCZ; take CH and CD, as in the figure, to represent the angular velocities of rotation about CY and CZ, or the lengths of the arches at the unit of distance described in a certain given time, complete the parallelogram $CDEH$, draw the diagonal CE, and produce it to M; the rotations about CY and CZ shall by composition constitute a rotation about CM, with an angular velocity represented by CE.

1. If R' be any point in the line CM, and $R'O'$, $R'B'$ be drawn perpendicular to CY, CZ, it will be depressed with the absolute velocity $R'O'.CH$ and elevated with the velocity $B'R'.CD$, and these being equal, by (45, *Case* 2d.) the whole line CM will be at rest.

2. If R be a point in the plane of the axes, and not in the line CM, draw RO, RB, RA perpendicular to CY, CZ and CM, respectively. The motion of R being perpendicular to RO and to RB, and therefore to the plane of the axes, will, consequently, be perpendicular to RA, and its velocity will be $RO.CH \backsim BR.CD = AR.CE$, when R is within the angle ZCY; but if it be without that angle the velocity will be $RO.CH + BR.CD = AR.CE$, (45, *Case* 2d.) Hence it is obvious that any particle whatever in the plane of the axes, and not in the line CM, is in a state of rotation about that line, with the common angular velocity represented by CE.

3. Let F be any point not in the plane of the axes; draw FR perpendicular to that plane, and, having drawn RO, RB, RA as before, join FO, FB, FA. The planes FOR, FBR, FAR are all perpendicular to the plane of the axes, (Eucl. xi. 18.) and CO, CB, CA, which are in that plane, and perpendicular to the common sections RO, RB, RA, are perpendicular to the planes FOR, FBR, FAR, and therefore to FO, FB, FA, respectively. The velocity of F, as revolving about CY in a circle of which the radius is FO, is $FO.CH$, in the direction FS a tangent to that circle, and, as revolving about CZ,

its velocity is $BF.CD$, in the direction $S'F$ a tangent to the circle whose radius is BF. The velocity $FO.CH$ in the direction FS is equivalent to velocities in the direction of FR and of a line parallel to RS, of which the former will be $FO.CH.\cos. SFR =$ (since OFS is a right angle,) $FO.CH.\cos. FOR = RO.CH$; and by this and other similar resolutions we have as follows,

$$FO.CH \text{ equivalent to } \begin{cases} RO.CH \text{ in direction } FR, \\ FR.CH \text{ parallel to } RS; \end{cases}$$

$$BF.CD \text{ equivalent to } \begin{cases} BR.CD \text{ in direction } RF, \\ FR.CD \text{ parallel to } RB; \end{cases}$$

and recompounding in a different order we have $RO.CH \mp BR.CD = RA.EC$ in the line FR; and as Ra or AR produced makes the angles SRa, BRa equal to HCE and DCE respectively, $FR.CH$ parallel to RS and $FR.CD$ parallel to RB are equivalent to $FR.CE$ parallel to AR.

Lastly. The velocities at F, $RA.EC$ in the line FR, and $FR.CE$ parallel to Ra, will produce by composition $FA.CE$ perpendicular to AF; so that F is in a state of rotation about A, with the absolute velocity $FA.CE$, and consequently with the angular velocity CE.

To avoid the necessity of drawing another figure, or of making this one too complex, we may demonstrate the last step thus: Let the resulting velocity $= r$, the angle which the direction of the resulting motion makes with $FR = \theta$, and $\angle FAR = a$;

$$\text{Then } RA.EC = \cos. a.FA.EC$$

$$FR.EC = \sin. a.FA.EC$$

$$r^2 = FA^2.EC^2 (\cos.^2 a + \sin.^2 a) = FA^2.EC^2$$

$$r = FA.EC;$$

$$\cos.\theta = \frac{RA.EC}{r}, = \frac{\cos. a.FA.EC}{FA.EC}, = \cos. a;$$

$$\therefore \theta = a;$$

and by adding $\angle AFR$ to each it will be obvious that the resulting motion is at right angles to AF.

436. *Cor.* 1. A motion of rotation about the diagonal of any parallelopiped, with an angular velocity represented by that diagonal, may be considered as compounded of motions of rotation about the linear sides which meet in one of its extremities, with angular velocities represented by those sides, respectively.

437. *Cor.* 2. In the composition of two motions of rotation, the angular velocities of revolution about any two of the axes are reciprocally as the sines of the angles which these axes make with the third.

438. *Cor.* 3. In the resolution of a given motion of rotation, of the positions of the new axes and the angular velocities of revolution about them any two being given the other two may be found.

We shall now illustrate the utility of the principles explained under the head of Rotation by some important applications of them.

Theory of the Ballistic Pendulum.

439. This is the name given to a machine, invented by the ingenious Mr. Robins, for measuring the velocity of military projectiles.

It is composed of a large square block of wood, *OPRQ*, (Fig. 165.) suspended by a strong iron rod, on an axis *MCN*, about which, supported on knife edges, like the beam of a balance, it may oscillate freely and with very little friction. Let a ball be discharged so as to strike the block of wood in a direction perpendicular to the radius, let the distances from *C* to *O* the centre of oscillation, and to *I* the point of impact, be given, and the angle through which the pendulum rises; we can then find the velocity *V* with which the point *I* begins to move. Instead of observing the angle directly, Mr. Robins measured, by means of a piece of tape *AB* drawn, with a resistance merely sufficient to keep it sensibly straight, through a slit on a fixed support at *B*, the chord of the arch described at the given distance *CB*. *AB* being known by this, or any preferable means, the calculation may proceed as follows. (See 425.)

$$CB : CO = AB : DO = \frac{AB.CO}{CB}; \text{ (Fig. 166.)}$$

$$EO = \frac{DO^2}{2CO} = \frac{AB^2.CO}{2CB^2};$$

$$\text{velocity of } O = \sqrt{2g.EO} = \frac{AB}{CB}\sqrt{g.CO};$$

where both g and CO are to be in feet if the velocity is to be expressed in feet.

$$\text{Hence velocity of } I = V, = \frac{CI}{CO}.\frac{AB}{CB}\sqrt{g.CO}.$$

Again, if we know the weight of the pendulum m, as representing its mass, and CG the distance from the axis to the centre of gravity, we can find the mass of equivalent inertia which may be conceived as substituted at I. This will be

$$M = \frac{m.CG.CO}{CI^2}. \quad (405, 423.)$$

Let B represent the weight of the ball, and x its velocity. By the doctrine of rotation the problem is now reduced to this simple statement: there being given the velocity V communicated to a given mass M, existing in free space, by a ball of given mass B, which lodges in it, it is proposed to find the velocity of the ball before impact. By third law of motion,

$$Bx = (M + B)V, \text{ and } x = \left(\frac{M}{B} + 1\right)V,$$

$$\text{or } x = \left(\frac{m.CG.CO}{B.CI^2} + 1\right)\frac{CI}{CO}.\frac{AB}{CB}\sqrt{g.CO}.$$

Theory of the Torsion Balance.

440. If a cylinder be suspended by a fine metallic wire, and the wire be a little twisted, it will spontaneously untwist itself, and the oscillations will be isochronous. Hence the force with which the wire resists torsion is as the angle of

this principle has been employed with great succes [. Coulomb of the French Academy, for the accurat ation of very minute forces, particularly to ascertair .w of electrical and magnetical action.

The resuit of M. Coulomb's investigations in relation to the principle and construction of his balance is, that if F be the force of tor on, A the angle of torsion, D the diameter of the wire, a its length, $F \doteqdot \frac{AD^4}{L}$, or $F = \frac{m\,AD^4}{L}$; where m is a cor u r ior wires the same material and texture, but ien se are lifferent.

It ma ere t ie investigation of this for ime very simple instance of e by experiment.

ts.

Steel Wire.	Gr.	Brass Wire.	Gr.
No. 12. weight of 6 feet .	5	No. 12. weight of 6 feet .	5
7.	14	7.	18.5
1.	56	1.	66

I. *Iron Wire.*

Length of wire = 9 inches.

Cylinder of lead = 19 lines in diameter, and 6½ lines in height; weight = ½ lb.

Experiment 1. No. 12. 20 oscillations in 120″
2. . 7. 42
3. . 1. wire not stretched.

Cylinder of lead—height 26 lines—diameter as before—weight = 2 lbs.

4. No. 12. . . . . 242″
5. . 7. 85
6. . 1. 28

II. *Brass Wire.*

Cylinder as in Experiments 1st, 2d, and 3d.

7. No. 12 . . . . 230″
8. . 7. 57
9. . 1. . . . . . .

Cylinder as in Experiments 4th, 5th, and 6th.

10. No. 12 . . . . 442″
11. . 7. 110
12. . 1. 32

Brass wire No. 7. Cylinder = 2 lbs.; length of wire = 36 inches.

13. . . 20 oscillations in 222″.

Now let n be the force of torsion, as measured by the weight in pounds which at the extremity of the cylinder would keep it in equilibrio, at the distance of 1 line from the position of quiescence;

$\int p r r = \frac{1}{2} m r^2$, equivalent to $\frac{1}{2} m$ at the circumference of the cylinder, or at the distance at which we conceive n to be applied. If φ be the accelerative tangential force at the extremity of the radius,

$$\varphi = \frac{n g}{\frac{1}{2} m}, = \frac{2 n g}{m}, \text{ (406.)}$$

and if t be the time of one complete oscillation,

$$t = \pi \sqrt{\frac{1}{\varphi}}, \text{ (348.)} = \pi \sqrt{\frac{m}{2 g n}};$$

$$\text{and } n = \frac{\pi^2 m}{2 g t^2}, \doteqdot \frac{m}{t^2}.$$

Here $g = 4348.224$ being the measure of the accelerative force of gravity expressed in lines or 12th parts of a French inch; for by the investigation of the equation $t = \pi \sqrt{\frac{1}{\varphi}}$, 1 and φ are in the same denomination of measure.

torsion; and this principle has been emplo success by M. Coulomb of the French Acade curate estimation of very minute forces, pa certain the law of electrical and magnetical ac

The result of M. Coulomb's investigations i principle and construction of his balance is, force of torsion, A the angle of torsion, D the wire, and L its length, $F \div \frac{AD^4}{L}$, or $F = \frac{m}{}$ is a constant number for wires of the same m ture, but different when these are different.

It may be useful to give here the investig mula, as a good and at the same time very s the scientific interrogation of nature by exp

Experiments.

Steel Wire.	Gr.	B
No. 12. weight of 6 feet .	5	No. 12.
7.	14	7.
1.	56	1.

I. *Iron Wire.*

Length of wire = 9 inches.

Cylinder of lead = 19 lines in di height; weight = ½ lb.

Experiment 1. No. 12. 20
2. . 7. .
3. . 1. wir

Cylinder of lead—height 26 l weight = 2 lbs.

4. No. 1
5.
6.

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of which, is the sum
$= \frac{1}{4}\pi f r^4$ when $x = r$, a

to a familiar standard,
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being 19 lines, the circu
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$= \frac{1}{389025}$ lb. = the force whic right angles to a radius of the cylinder at its extremity, balances, in the given circumstances, a torsion of 1°. One pound, French measure, is 7560 grains English, and the force above investigated will be $\frac{1}{51 \cdot 438}$ grain.

By doubling the length of the wire, the force balancing the torsion of 1° would be about $\frac{1}{103}$ grain. But balances far exceeding this in delicacy, and which are used for detecting very minute degrees of electrical excitation, are formed by suspending a fine needle by a fibre of silk as it comes from the silkworm.

Biot in his *Traité de Physique*, tome ii. p. 351, mentions an experiment of Coulomb, in which a small circular plate, five lines in diameter, and weighing 8½ grains, was suspended by a silk fibre four inches long, and made one oscillation in 45″. From these data, it is easily found, that the force requisite at the distance of one line from the axis to balance the torsion of 57°.295, &c. or the angle whose intercepted arch is equal to the radius, must have been only $\frac{1}{35857}$ grain. In perusing what Biot or Coulomb has written on this subject, the student will observe, that if n, as in our investigation, be the force at the extremity of radius r, when the arch $=$ one line, it will be $n\,r$ for arch $= r$ lines to radius r, or arch of one line to radius 1, and the force equivalent to this (or the force giving the same momentum) at the distance $1 = n\,r^2$.

441. In illustration of what was observed in (431.) let n be the force of torsion at the unit of distance for the angle whose measure is the radius, and let the body suspended be a cylinder whose mass is m and whose radius $= a$, the line of suspension being the axis produced. The mass of equivalent inertia at the distance 1 is $\frac{1}{2}\,m\,a^2$; hence the accelerative force at that distance, as a multiple of gravity, is $\frac{n}{\frac{1}{2}\,m\,a^2} = \frac{2\,n}{m\,a^2}$, and if t be the time of an oscillation $t = \pi\sqrt{\frac{m\,a^2}{2\,g\,n}}$.

In the same manner for a very slender rod, whether cylindrical or prismatical, whose length is $2a$, and which is suspended by the middle, we find $t = \pi \sqrt{\frac{m a^2}{3 g n}}$.

In either case, we may from known values of m, a and t, find the value of n and then, retaining the same wire, or otherwise constructing the experiments so that n may remain the same, we may employ different values of m and a, and compare the calculated with the observed values of t.

In either case, considered separately, if m and n be given, that is, if we employ the same, or perfectly similar and equal wires and append equal cylinders of different radii, or equal rods of different lengths, $t \doteqdot a$, or the number of vibrations in a given time will be in the one case inversely as the radius, in the other inversely as the length.

If the body suspended be a cylindrical rod of considerable thickness, the mass of equivalent inertia at the distance 1 is $\frac{1}{3} m (a^2 + \frac{3}{4} b^2)$ (430.); for it is always $\frac{\int p r r}{d^2}$ at the distance d, and here $d = 1$. Hence $t = \pi \sqrt{\frac{m(a^2 + \frac{3}{4} b^2)}{3 g n}}$.

In the same manner for a parallelopiped whose length $= 2a$ and breadth $= 2b$, $\int p r r = \frac{1}{3} m (a^2 + b^2)$ (429.c.), and $t = \pi \sqrt{\frac{m (a^2 + b^2)}{3 g n}}$.

Of the Accelerative Force of Cylinders and Spheres rolling down Inclined Planes.

442. Before the invention of Atwood's machine, experiments on the laws of uniform acceleration were often made by means of inclined planes. By diminishing at pleasure the plane's elevation, the experimenter could reduce the space described in one, two, or three seconds to a measurable quantity; but, with a small elevation, all motion is prevented by friction, unless the descending body be made to roll, for which

purpose it is made cylindrical or spherical. In such experiments, though there is then no sensible retardation from friction, the spaces described in given times are considerably different from what we should expect, taking into view merely the measure of gravity as modified by the slope. The reason of this we are now in a condition to explain. When a cylinder rolls down an inclined plane without sliding, the velocity of rotation at its surface is equal to the progressive velocity of the axis; and a part of the pressure in the direction of the plane's length is employed in producing this motion of rotation. Let *ADB* (Fig. 167.) represent the body rolling down the inclined plane *EG*, m the numerical measure of its mass and weight, and $s = \frac{EF}{EG}$, = the sine of the plane's elevation.

If the body be a cylinder, the motive force may be considered as divided into two parts, the one of which has to accelerate the mass m down the plane, and the other has to do what is equivalent to an equal acceleration of the mass $\frac{1}{2}m$, the equivalent of the inertia when referred to the circumference. The accelerations being equal, the parts of the whole motive force employed in them must be as the masses, and therefore as 2 : 1; or $\frac{2}{3}sm$ will be the motive force employed in producing the motion of translation and $\frac{1}{3}sm$ that which produces the rotatory motion about *C*.

Or we may arrive at the same result perhaps more perspicuously thus: By the composition of the progressive and the rotatory motion, the cylinder is in a state of rotation about the horizontal axis represented by A; $\int prr$, for an axis passing through C, is $\frac{1}{2}mr^2$, and for the axis A it is $\frac{1}{2}mr^2+mr^2=\frac{3}{2}mr^2$, (86.) and this is equivalent to $\frac{3}{2}m$ at C (405.) where the motive force sm is applied. Hence the accelerative force at C is $\frac{sm}{\frac{3}{2}m} = \frac{2}{3}s$ as a multiple of gravity.

For a sphere it will be $\frac{sm}{m+\frac{2}{5}m} = \frac{5}{7}s$, (432.) (86.)

The force in each case being constant, the spaces will be as the squares of the times, (317.); but for a cylinder their absolute values will be $\frac{2}{3}$, and for a sphere $\frac{5}{7}$ of what they would have been without friction or rotation.

To produce a rotation without sliding when the plane has considerable elevation, or even when the descent is vertical, a thread may be lapped round the middle section, and the body suffered to fall by unrolling.

Let *ABD* (Fig. 168.) be a cylinder, or, for the greater generality, a cylindrical groove cut upon a body of any shape and dimensions, having the middle of its geometrical axis coincident with the body's centre of gravity, and let it unroll vertically by means of the thread $A\,q$ lapped round it, and in the first place fixed at q. Let m denote the mass and weight of the body and t the tension of the thread. (By 398, *Note.*) it appears that the descent of this body will be vertical, and that the tension t is propagated, in its proper direction, to the centre C, so that the acceleration of C, as a multiple of the natural acceleration of gravity, will be $1 - \frac{t}{m}$. This will also be the acceleration of A round C, another expression for which is $\frac{t}{\frac{m\,k^2}{a^2}} = \frac{t\,a^2}{m\,k^2}$, k being the radius of gyration, and a the radius of the cylinder. Hence

$$1 - \frac{t}{m} = \frac{t\,a^2}{m\,k^2}, \text{ or } t = \frac{m\,k^2}{a^2 + k^2};$$

and the acceleration of $C = 1 - \frac{t}{m} = \frac{a^2}{a^2 + k^2}$.

When *ABD* is a cylinder $k^2 = \frac{1}{2}\,a^2$, (434.) Hence the measure of the accelerative force is in that case $\frac{a^2}{a^2 + \frac{1}{2}\,a^2} = \frac{2}{3}$, gravity being denoted by unity.

Suppose now that the thread instead of being fixed at q passes round a small fixed pulley, whose inertia may be disregarded, and has a weight p suspended as in the figure. t being the tension of the thread, the acceleration of C downwards, and of A about C from this cause, will be, as before,

$1-\frac{t}{m}$: also the acceleration of p downwards, and of A round C from this cause, will be $1-\frac{t}{p}$. Therefore,

$$1-\frac{t}{m}+1-\frac{t}{p}=\frac{t a^2}{m k^2},$$

$$\text{or } t=\frac{2 p m k^2}{p a^2+(m+p) k^2};$$

$$\text{acceleration of } C=1-\frac{t}{m}, =\frac{p a^2+(m-p) k^2}{p a^2+(m+p) k^2};$$

$$\text{acceleration of } p=1-\frac{t}{p}, =\frac{p a^2-(m-p) k^2}{p a^2+(m+p) k^2}.$$

Of Rotative Machinery in a State of Constant Acceleration.

443. Let ADB represent a fixed pulley, or cylinder, free to move round the axis C, and let there be two weights p and w suspended by a line passing round it as in (Fig. 169.) of which p exceeds w.

If the inertia of the machine be disregarded, as well as the friction and other resistances, the accelerative force will be $\frac{p-w}{p+w} g$.

If the weight of the pulley or cylinder be m, then, taking into account the inertia, we shall have the accelerative force $=\frac{(p-w) g}{p+w+\frac{1}{2} m}$.

If C represent an axis passing through the centre of gravity of a body of irregular shape, or whose density is not uniform, and if m be the weight representing its mass, k the distance from the axis to the centre of gyration, and a the radius of a groove ADB round which the rope passes, the accelerative force will be $\frac{(p-w) a^2 g}{(p+w) a^2+m k^2}$.

If the mass of equivalent inertia $\frac{m k^2}{a^2}$ be unknown, it may be found experimentally thus: Suppose it to be x. Then the accelerative force is $\frac{(p-w)g}{p+w+x}$. Allow the weight to descend and observe the space s described in t'',

$$\frac{(p-w)\frac{1}{2}gt^2}{p+w+x}=s,$$

$$\text{and } x=\frac{(p-w)gt^2}{2s}-(p+w).$$

If there be no weight w,

$$x=\frac{pgt^2}{2s}-p.$$

Suppose $p=30$ grains Troy weight,

$t=3''$,

$s=38\frac{1}{2}$ inches;

$x=1323$ grains $=2\frac{3}{4}$ oz. nearly.

This is the experiment referred to in the description of Atwood's machine, (326.)

Note.—The friction arising from the weight of the pulley ought to be balanced.

Let *FGH* (Fig. 170.) represent a wheel and axle; let $AC=a$, the mass of the machine, represented by a weight, $=m$, the distance to the centre of gyration $=k$, and let there be various grooves to which by a thread lapped round them the power p may be applied, to raise the weight w appended to a line which is lapped round the axle; a, p, w, m and k being given, it is proposed to find the radius of the groove x to which p may be applied most advantageously, that is, so as to produce the greatest acceleration in w's motion of ascent in a given time.

The part of p which is balanced by w is $\frac{wa}{x}$, and the moving force at $D=p-\frac{wa}{x}$,

$$\text{accelerative force at } D=\frac{\left(p-\frac{wa}{x}\right)g}{p+\frac{mk^2}{x^2}+\frac{wa^2}{x^2}}=\frac{(px^2-wax)g}{px^2+mk^2+wa^2}.$$

The accelerative force at *A*, or the velocity acquired by the ascending weight in the unit of time, $= \frac{(p a x - w a^2) g}{p x^2 + m k^2 + w a^2}$.

That this may be a maximum, we must have $\frac{px - wa}{px^2 + mk^2 + wa^2}$, or for the sake of abbreviation $\frac{p x - d}{p x^2 + f}$, a maximum.

$$\therefore p\,dx\,(p x^2 + f) = 2 p x\,dx\,(p x - d),$$
$$p^2 x^2 - 2 d p x = p f,$$
$$p^2 x^2 - 2 d p x + d^2 = d^2 + p f,$$
$$p x - d = \sqrt{d d + p f},$$
$$\text{and } x = \frac{d + \sqrt{d^2 + p f}}{p} = \frac{a w + \sqrt{a^2 w^2 + p m k^2 + p w a^2}}{p}.$$

If the inertia of the wheel be disregarded,

$$x = \frac{a}{p}\left(w + \sqrt{w^2 + p w}\right).$$

If the body whose weight is *w*, attached to the same axle, is to be dragged along a horizontal plane without friction, the term containing *w* in the numerator of the value of the accelerative force disappears, and $x = \sqrt{\left(\frac{m k^2 + w a^2}{p}\right)}$.

Let *AHB* (Fig. 171.) represent a movable pulley, to which is attached the weight *w*, and let the weight of the pulley itself be *m*; let a force, represented by a weight *p*, act upwards at *k*, and let it be required to find the accelerative force of ascent at *k*.

We may consider the pulley as in a state of momentary rotation, about *A*; and the point *B* having a motion compounded of the velocity of the pulley's ascent and the equal velocity of rotation, will have the velocity of *k*'s ascent. $\int p r r$ with respect to an axis at *A* is $\frac{1}{2} m r^2 + m r^2 = \frac{3}{2} m r^2 = \frac{3}{8} m d^2$, where $r = AC$, and $d = AB$. The resistance of the pulley then is equivalent to that of $\frac{3}{8} m$ at *B*, or at *k*. Again, the inertia of *w* at *C* is equivalent to $\frac{1}{4} w d^2$, or $\frac{1}{4} w$ at *B* or *k*. Hence the accelerative force of *k*'s ascent $= \frac{p - \frac{1}{2}(w + m)}{p + \frac{3}{8} m + \frac{1}{4} w}$.

If the string pass over a fixed pulley, whose weight is also m, its inertia is equivalent to $\frac{1}{2}m = \frac{3}{8}m$ at F, and the accelerative force of p's descent will now be $\frac{p - \frac{1}{2}(w+m)}{p + \frac{7}{8}m + \frac{1}{4}w}$ as a multiple of gravity.

Let AE and HB (Fig. 172.) represent two wheels, of which the former turns the latter by the pinion whose radius is CF, and p applied at A raises a weight w attached to the axle DG. Let S and S' be the centres of gyration of the two wheels with their axles, and let

$$\begin{array}{lll} CA = a, & DB = a_{,}, & \frac{b}{a} = n, \\ CF = b, & DG = b_{,} & \frac{b_{,}}{a_{,}} = n_{,}, \\ CS = k, & DS' = k_{,}, & \end{array}$$

while the masses of the wheels are denoted by m and m.

The inertia of the wheel and axle HB together with that of the mass w, is equivalent to $\frac{m_{,}k_{,}^2}{a_{,}^2} + \frac{w\,b_{,}^2}{a_{,}^2} = \frac{m_{,}k_{,}^2}{a_{,}^2} + w\,n_{,}^2$, at F in the circumference of that wheel or of the axle which impels it with the same absolute velocity. Again, this inertia at F, together with that of the wheel and axle AE, is equivalent to $\frac{m\,k^2}{a^2} + \left(\frac{m_{,}k_{,}^2}{a_{,}^2} + w\,n_{,}^2\right)\frac{b^2}{a^2}$ at A, and the moving force there is $p - \frac{w\,b\,b_{,}}{a\,a_{,}} = p - w\,n\,n_{,}$. Therefore the accelerating force at A is

$$\frac{p - w\,n\,n_{,}}{p + w\,n^2 n_{,}^2 + \frac{m\,k^2}{a^2} + \frac{m_{,}n^2 k_{,}^2}{a_{,}^2}}.$$

This multiplied by $\frac{b}{a} \times \frac{b_{,}}{a_{,}} = n\,n_{,}$ gives the accelerative force at G.

444. We shall conclude this part of our subject with a short explanation of the regulating power of what is called a *Fly*.

Equability of motion is of great importance in most kinds of machinery. To maintain a motion already uniform, requires only a pressure sufficient to *balance* the resistance which accompanies it; but to restore any part of the acceleration that may have been lost requires the exertion of a greater force, or such as shall *overbalance* it. An irregular desultory motion strains the machinery, and the fabric that supports it, as well as the agent when animal power is employed, more than one that is equable, and the strength of the machine and the agent must be adapted to the greatest strains that occur, whilst at the same time more work is not performed than if the motion were at a mean rate and uniform. To regulate a motion which would otherwise be subject to variation or reciprocation, as in the applications of the steam engine, a *fly* is used. It is a heavy mass of matter, so shaped as to balance itself, connected with the machinery, and turning round the same axis with a part of it. Its effect depends on the momentum of inertia, denoted by the symbol $\int prr$. Its momentum, when revolving with the angular velocity v, is $v\int prr$; and, if v' be the increment of angular velocity, the increment of momentum is $v'\int prr$. Consequently, if this be a constant quantity, $v' \div \frac{1}{\int prr}$. When the mass or the diameter then is considerable, and still more when both are so, it may acquire a great momentum with but little increase of angular velocity, or lose a considerable momentum with little diminution of that velocity. It thus becomes a receptacle for the surplus energy of the power when it acts with most intensity, or when the resistance is least, and preserves it for future demand. If, by a diminution of resistance, or an increase of power, the machinery would otherwise be considerably accelerated, the motive force is in a great measure expended upon the fly, in which it generates a proportional momentum with little increase of velocity; and when the resistance is increased and the moving power or its momentum diminished, and the machinery would be very sensibly retarded, the momentum accumulated in the fly continues the motion with little diminution of its own velocity, at least if the interval of reciprocation, or of unequal resistance, be short.

When a more powerful regulator is wanted, we should increase the momentum of inertia of the fly by enlarging the diameter rather than the mass, because we thus produce the same effect with less weight, consequently with less transverse strain upon the axle and supports, and less friction.

The nearer the axis of the fly is to the position where the angular velocity is greatest the greater will be its dominion or regulating power, *cæteris paribus.*

All rotative machinery, particularly that whose parts are massy and of large diameter, will obviously act as a fly in regulating its own motion.

445. The principles that have been explained in relation to the motion of machines are still insufficient to give us a complete knowledge of the actual performance which is to be expected from them. Large and complex machinery is never sensibly accelerated for any considerable time, but soon arrives at a state of apparent uniformity. The chief causes of this seem to be, that with an increase of velocity there is generally an increase of resistance, and at the same time, in some of the most common mechanical agents, a diminution of effective power. The effect of the former of these causes may be illustrated by observing what happens when a boat is dragged along a canal. It displaces with a certain velocity a portion of the water. Let its velocity be now doubled; it must displace twice as much water in a given time as before, and with twice the velocity; so that from the inertia of the water alone there will be a quadruple resistance and retardation; and, universally, the retardation from this cause alone may be expected to be as, or nearly as, the square of the velocity. Besides this there is something analogous to friction in overcoming the viscidity of the water; and, what is of greater consequence, the water in front exerts its whole

pressure on the prow, while that which is behind has to *follow* the boat, and exerts upon it a less force. Suppose a man then, who drags the boat to be capable of increasing his velocity, still exerting a constant pressure, this pressure will at last be balanced by the continually increasing resistance. The retarding and the accelerating forces will be equal, and the motion which is the result of the previous accelerations will become uniform.

In fact, however, the man who drags the boat and accelerates his motion is incapable of exerting the supposed constant pressure, and there is a certain velocity which would expend his whole energy in simply continuing his own motion. This must accelerate the attainment of the state of uniformity; and a similar cause operates whenever the mechanical agent employed is an animal, or a current of water or air.

The resistance occasioned by the inertia of the air itself, though a rare fluid, and not sensibly diminishing a very slow motion, may produce an angular velocity sensibly uniform. When a clock begins to strike, the descending weight would soon accelerate the motion of the wheels to a considerable degree, in consequence of which the strokes would not occur at equal intervals, and, which is of more consequence, the momentum acquired would be such as to expose the parts to a violent strain, and destroy the wheelwork by the shock, when the motion is suddenly stopped, were there not means employed to prevent the acceleration from passing a certain limit. Upon the axis most remote from the weight, in the train of wheelwork which forms the striking part, is a small plate of metal, the leaves of which make about 49 revolutions for every stroke, and the resistance which it meets with from the air in this state of rapid rotation, is sufficient to balance the acceleration of the descending weight, against which it acts with great mechanical advantage, so that the motion already acquired is simply continued. Similar to this is the resistance you meet with in turning a corn-fan, which you find it impossible to accelerate beyond a certain degree. Here, however, both the causes above mentioned operate, which is not the case where the moving power is a weight. It will farther illustrate the subject if we consider what would happen in the case of a clock striking a number of times successively, without the employment of the fly, and on the supposition that the parts are strong enough to withstand a considerable force. A part of the impetus of the descending weight is expended at every stroke in lifting the hammer against the force of the spring which makes it strike. Suppose this expenditure constant and to be less than the force acquired by the descending weight in the interval between the strokes. There will of course be an acceleration; the strokes will occur more frequently, and there will be less intermediate acceleration of the weight, till at last it will be sensibly equal to what is required for raising the hammer. There will then be a small acceleration still *visible* during each interval, but destroyed at the end of it, and none *audible*.

To the causes producing uniformity already mentioned, may be added the effect of friction. Though the mere increase of velocity in a machine does not, according to the best experiments, sensibly alter the friction, which is in each case a determinate part of the pressure, yet whenever the increase of velocity is accompanied by an increase of resistance, the friction must increase in the same ratio.

446. This circumstance of the speedy attainment of uniformity in the motion of machinery performing work leads to a very important principle, which tends greatly to simplify our investigations respecting its most advantageous performance. As long as the power or force actually impressed, as it would be measured by the dilatation or compression of an interposed spring, exceeds the statical counterpoise of the resistance estimated in the same manner, the machine must accelerate; and, consequently, when it arrives at a state of uniform motion, the power and the resistance then actually exerted upon it effectively, are such as would, by its interven-

tion, balance each other when at rest. Now it has been formerly proved, that powers which so balance each other are reciprocally proportional to the velocities with which they *begin* to move in their respective lines of action, and, in the case of rotative machines, *continue* to move when put in motion. Hence, if the power and the resistance be estimated by weights, their quantities of motion will be equal. But the performance of the machine is properly estimated by the quantity of motion of the resistance, or the product of the resistance into the velocity with which it is overcome. It will therefore be measured by the equal product of the effective power into its velocity or that of its point of application; and, when this is a maximum, the performance of the machine will be a maximum. What is now, then, chiefly wanted is to ascertain experimentally with respect to our ordinary mechanical agents as men, horses, wind, streams of water, &c. according to what law their effective energy varies with the variation of velocity. If this be discovered, we shall know what velocity of each agent gives the product of that velocity into the corresponding pressure a maximum; and then also we may be assured that we shall have the effect of the machine the greatest possible.

447. Let v be the actual velocity of one of the above-mentioned mechanical agents, and let a be that which puts an end to all effective pressure. The greater v is, or the less $a-v$, the less is the effective pressure which it can exert. Suppose the effective pressure then actually exerted to be p, and that exerted when $v=0$, and which may be considered as wholly effective, P; and that

$$P : p = a^n : (a-v)^n, \text{ or } p = \frac{P}{a^n}(a-v)^n.$$

The measure of the performance will be $pv = \frac{P}{a^n}(a-v)^n . v$, and this will be a maximum when $v = \frac{a}{n+1}$; and consequently the maximum effect $= Pa . \frac{n^n}{(n+1)^{n+1}}$.

In the case of a stream of water impelling the floatboard of a millwheel, v is the velocity of the floatboard, and a is that of the stream, and it has been supposed that the impulse of the stream is as the square of the relative velocity, that is, $\div (a - v)^2$. In the case of men employed as mechanical agents, Dr. Robison found the effective pressure to vary according to the same law. Here then $n = 2$. The velocity which gives the greatest effect is $\frac{1}{3} a$, in the case first mentioned $\frac{1}{3}$ of the velocity of the stream, and the greatest effect itself $\frac{4}{27} P a$, that is, $\frac{4}{27}$ of what it would be if we could combine the two extremes, the pressure that excludes all velocity with the velocity that excludes all pressure. In the case of horses, he found n to be more nearly 1.7, or the maximum performance about $\frac{10}{59} P a$.

448. M. Coulomb estimates the energy of men, as mechanical agents, in a different way, of which the following may serve as a specimen.

A man is supposed by him to weigh at a medium 70 chilogrammes, each chilogramme being about $2\frac{2}{10}$ lb. avoirdupois; and, in carrying a weight up a declivity, his most advantageous action, or that which gives the product of the weight carried into the velocity the greatest, is when he is unloaded or supports merely his *own* weight. It is then 205 chilogrammes raised one chilometre in a day, which is supposing him to ascend 2.928 chilometres.

N. B.—The chilometre is, as the name indicates, 1000 metres and = 3281 English feet nearly.

When he is loaded with 68 chilogrammes, the measure of his energy is found to be

109 chilogrammes raised one chilometre;

so that the load 68 diminishes the energy by 96 chilogrammes raised one chilometre.

Coulomb considers the diminution in the effect thus measured as proportional to the load, so that if P be the load $68 : 96 = P : 1.41\ P =$ the diminution of the maximum of performance occasioned by the load P. Hence if h be the height ascended through,

$$(70+P)h = 205 - 1.41\,P,$$

$$h = \frac{205 - 1.41\,P}{70+P},$$

and $Ph = \frac{P\,(205 - 1.41\,P)}{70+P}$ = the measure of the useful effect. This is a maximum when $P = 53$ chilogrammes, and the effect is then equal to 56 chilogrammes carried one chilometre in vertical height. The work, it is supposed, may be continued from seven to eight hours *per diem*.

We shall find the maximum effect here not to differ much from that which would be assigned by the other formula, (447.) on the supposition that $n = 2$.

The velocity which excludes all additional weight here being 2.928 chilometres *per diem*, if we combine this with a force equal to that which a man can exert in raising a weight from the ground while he stands still, and which, by a number of trials with Regnier's dynamometer, may be estimated at 130 chilogrammes, we shall find the product to be 380.64; that is, 380.64 chilogrammes raised one chilometre in a day, and $\frac{4}{27}$ of this is about 56 chilogrammes raised one chilometre. The formula itself, however, it must be observed, gives a different value of the pressure which extinguishes the velocity, and which is obtained by supposing the useful effect = 0, or $205 = 1.41P$.

When a man of 70 chilogrammes travels horizontally unloaded, his action of that kind is a maximum, and he can continue his travel about 50 chilometres *per diem*, for several successive days. This is 3500 chilogrammes carried one chilometre.

When his load is 58 chilogrammes additional, the measure of the action is found to be 2000, and the reduction, of course, 1500. Supposing, as before, that the reduction is proportional to the load P,

$$58 : 1500 = P : 25.86\,P.$$

Let d be the distance travelled with the load P,

$$(70+P)\,d = 3500 - 25.86\,P,$$

$$\text{whence } Pd = \frac{P\,(3500 - 25.86\,P)}{70+P}.$$

When this is a maximum $P = 49.8$ nearly, and the maximum effect will be about 919 or 920 chilogrammes carried horizontally one chilometre.

Of Friction.

449. It is impossible to obtain by machinery the useful effect which is sought without submitting to a partial, and that often considerable expenditure of power, from resistances the overcoming of which makes no part of our primary object, and which, consequently, it must be our aim to diminish as much as possible. One of the most considerable of these, which are sometimes denominated passive forces, is friction, or the resistance arising from the asperities of the surfaces that slide or roll upon each other. Several valuable series of experiments have been made with a view to ascertain the laws of this resistance as opposing the commencement of motion, and retarding or preventing the acceleration of motion already commenced, but by none on so large a scale, or so successfully, as by the justly celebrated philosopher last quoted, Coulomb. His memoir on the subject obtained a prize from the French *Academie des Sciences* in 1781, and has been printed separately at Paris in 1809. A very full abstract of it is given by Prony in his *Architecture Hydraulique*, and a short summary by Venturoli in his *Elements of Practical Mechanics.*

A measure of the force of friction in given circumstances may be obtained in different ways. Coulomb used sledges resting upon a horizontal surface, and observed what weight it was necessary to suspend from a string, connected with the sledge and passing horizontally over a pulley, in order to commence motion. With the same apparatus he could observe what weight was sufficient to continue, with uniformity, a slow motion already begun. This weight of course would be a measure of the friction accompanying the particular velocity which was communicated. If the weight of traction overbalanced the friction of the body in motion, it might be learned from the circumstances of the motion whether the accelerative force was constant or not; and, if it appeared to be constant, the space described in a given time would afford a measure of its value. If the spaces described be as the squares of the times, it is a proof of the constancy of the accelerative force. Coulomb divided the whole space described into two equal parts, which we shall denote by the symbols L, N, L being that which was first in order, and observed, by a pendulum swinging half seconds, the time of describing each half. Suppose t to be the time of describing L, and T that of describing $L + N$, or $T - t$ that of describing N, he would have in a case of uniform acceleration,

$$T^2 : t^2 = 2 : 1 \text{ or } T : t = \sqrt{2} : 1,$$

$$\text{or } T - t : t = \sqrt{2} - 1 : 1 = 41 : 100.$$

And thus, if the time of describing N was about $\frac{41}{100}$ of that of describing L, he considered the accelerative force, which was the excess of gravity above the friction, as constant. But the former may in all such experiments be regarded as constant, therefore so in this case must be the latter. Then to find the measure of the friction, w being the weight that measures the force of traction, W that of the sledge and its load, m the weight of the pulley as measuring its inertia, and F the friction, $\frac{w - F}{w + W + \frac{1}{2} m}$, or, for the sake of abbreviation, $\frac{w - F}{M}$. Therefore $s = \frac{1}{2} g t^2 \frac{w - F}{M}$, and

$$F = w - \frac{2 s M}{g t^2}.$$

Let $\frac{F}{W} = f$, and f is what we call the coefficient of friction.

In Coulomb's experiments as s is expressed in French measure, g must be taken on the same scale or $= 30$, being the number of French feet in the measure of the accelerative force of terrestrial gravity.

The inclined plane has also been used to find the value of f, and affords a very ready means of doing it. Let w be the weight of a body which rests on an inclined plane, raise the plane slowly till you find the elevation θ at which the body just begins to slide. Then w sin. θ is the force tending to make it slide, w cos. θ is the pressure exerted perpendicularly on the plane, and fw cos. θ the friction. Hence fw cos. $\theta =$ w sin. θ and $f = \frac{\sin.\ \theta}{\cos.\ \theta} = \text{tang.}\ \theta$.

In this way, too, we may find by the rate of acceleration as compared with that of the relative gravity, the friction of the body when in motion.

The following are some of the most useful and interesting results which have been discovered in one or other of these ways.

450. Friction, in like circumstances, is proportional to the pressure, and if this be W, it will be $= fW$: but there are many circumstances a variation in which is accompanied by a change in f the coefficient.

451. Friction is generally greater as measured by the force requisite to begin a motion than as measured by that which is sufficient to maintain one already produced. The case of metals rubbing on metals, however, presents an exception to this, the friction being in both cases sensibly the same.

452. The force necessary to overcome the friction of a body at rest increases in many cases with a continuance of the contact.

In the case of wood resting on wood, or metals on metals, the time of attaining the maximum is very short, a few seconds, or at most a minute or two; but when wood rests on metals, it is sometimes four or five days.

453. The coefficient of friction varies with the change of the substance or the polish of the surfaces in contact, and is very much diminished when a layer of some unctuous substance is interposed.

454. It depends very little on the breadth of the surfaces,

if one of them be not so narrow as to form a point or edge sufficient to penetrate the texture of the other, and to produce a sensible abrasion. When the surface is of considerable breadth, and the pressure small, there appears to be an increase in the value of *f*. This is thought to arise from a force of adhesion making a part of the resistance usually ascribed to friction, and which being constant for all pressures to which the same surface is exposed will most affect the ratio of the resistance to the pressure when the latter is least.

We shall here add a specimen of Coulomb's results.

455. *In commencing motion, when wood is drawn over wood, both surfaces being polished, but without unctuosity, and the fibres of both surfaces in the same direction.*

For oak resting on oak . . . $f = 0.43$
oak . . . fir . . . 0.65
fir . . . fir . . . 0.56
elm . . . elm . . . 0.47.

When the fibres cross each other at right angles the friction is considerably reduced. In that case,

For oak resting on oak . . . $f = 0.26$

For iron resting on oak . . . $f = 0.20$
brass . . . oak . . . 0.18
iron . . . iron . . . 0.28
iron . . . brass . . . 0.26.

Friction of Bodies in Motion, sliding.

For oak sliding on oak . . . $f = 0.105$
oak . . . fir . . . 0.158
fir . . . fir . . . 0.167
elm . . . elm . . . 0.100,

and no sensible dependence on the velocity.

For oak sliding on iron . . . $f = 0.08$
or 0.16,

the former of these values being found when the velocity was insensible, the latter when it was one foot per second: so that in the friction of wood on metals there is an increase with an increase of velocity.

When a coating of some consistent unctuous substance is interposed.

For oak sliding on	oak . . .	$f = 0.035$
iron . . .	oak . . .	0.028
brass . . .	oak . . .	0.021

Friction of Bodies in Motion, rolling.

When a cylinder rolls along a plane, f is, *cæteris paribus*, inversely as the diameter; but is in all such cases extremely small and scarcely worthy of consideration.

When a cylinder of lignum vitæ of 6 inches diameter rolls on a plane of oak, $f = 0.006$; and when on a plane of elm, 0.01.

The friction of axes is also found to be nearly proportional to the pressure, and does not depend sensibly on the velocity. When a polished iron axle turns in a box of brass, $f = 0.164$; but if it is covered with a coating of tallow, it is 0.09. When the axis and the box are of wood, the coefficient is still less. Here f denotes the coefficient of the friction reduced to the axis: or as it would appear if the power applied to balance it had no mechanical advantage against it.

456. The momentum of the friction of an axis may be diminished by supporting it on friction wheels.

Let *ANK*, *BML* (Fig. 173.) be two wheels whose axes are equal, having the radii *DR*,*ES*, with *D*,*E* in the same horizontal line; and let there be two others, exactly equal and similar in every respect, so placed that the axle *C* of a pulley *PQT* may rest upon the four horizontally. Having joined *C*,*D* and *C*,*E*, complete the parallelogram *CDGE*. Suppose

W to be the weight of the pulley PQT with its load, and f to be, as above, the coefficient of friction; the value of the friction, if the axis of the pulley rested on a horizontal plane, would be fW. Let this be represented by CG; it may be resolved into CD, CE, pressing the wheels in the lines of the centres at A and B. The pressure, and consequently the friction which is proportional to it, are therefore, by this transference to the axes of the wheels, increased in the ratio $CG : CD + CE = CF : CD = AH : AD$. But the points of the axle of the pulley which are in contact with the wheels obtain a mechanical advantage against it in the ratio of $AD : DR$. Hence the resistance to the motion of the axle is, *ex æquali*, diminished in the ratio $AH : DR$, which, if the centres are near, is nearly $= AD : DR$.

457. The effect of friction in the case of sliding bodies, as sledges, may be diminished by making the line of traction oblique to the horizon; and the line which gives the greatest advantage in this respect is that which makes with the horizontal plane the angle whose tangent is the coefficient of friction.

Let W (Fig. 174.) be the weight of the body, and let w be a weight equal to the force of traction in the line BA, which makes with the horizontal plane MN the angle x. The force w is partly employed in diminishing the downward pressure of W, and partly in drawing it horizontally. The diminution of pressure is $w \sin. x$, the remaining weight $W - w \sin. x$, and the friction $= f(W - w \sin. x)$, the force of horizontal traction which is supposed to be just ready to overcome the friction is $w \cos. x$. We have therefore

$$w \cos. x = f(W - w \sin. x) \text{ or}$$

$$w = \frac{fW}{\cos. x + f \sin. x},$$

and, fW being regarded as constant, w will be least when $\cos. x + f \sin. x$ is greatest, that is, when,

$$- dx \sin. x + f dx \cos. x = 0 \text{ or } f = \frac{\sin. x}{\cos. x} = \text{tang. } x.$$

Thus for oak resting on oak, to commence motion x should be 23° 16′; but when motion is once begun, it should be only 6°.

458. If a given force is to be overcome by means of a rope coiled round a cylinder, and if the length of the rope in contact with the cylinder be made to vary in arithmetical progression, the force required to produce the state bordering on motion will vary in geometrical progression.

Let the arch of a circular section of the cylinder embraced by the rope be AD, (Fig. 175.) and let it be stretched by the forces W and Q. If Q be $= W$, they will, independently of friction, balance each other. Construct the figure as formerly for oblique forces applied to the pulley, and if R be the resultant of Q and W, $W : R = AC : AD$, or $R = \frac{W.AD}{AC} = \frac{Wa}{r}$ when the arch is indefinitely small, a being the arch and r the radius. But in this case R is the force with which an elementary part of the rope is pressed against the cylinder; consequently the friction thence arising will be $\frac{Wfa}{r}$ which suppose $= nW$. In the state bordering on motion then Q must be $= W + nW = (1 + n) W$. Suppose now the contact of the rope and cylinder to extend to E, so that $DE = AD = a$, which though here represented as large must be conceived as indefinitely small, and that a force P is applied tangentially at E, sufficient to produce the tension Q, or $(1 + n) W$ at D, P must, for a similar reason, be $= (1 + n) Q = (1 + n)^2 W$; and by pursuing the same mode of reasoning the truth of the proposition will be obvious.

Cor. Let the arithmetical progression of the arches be

$$0, a, 2a, 3a ---- ma = A,$$

and the corresponding geometrical progression of the tensions,

$l(n+x) = l\,n + \frac{x}{a} - \frac{x^2}{2a^2} + \frac{x^3}{3a^3} - \&c$

$l(1+n) = 0 + \frac{n}{1} - \frac{n^2}{2} + \&c$

$$W,\ b\,W,\ b^2W,\ b^3W - - - - b^mW = P$$

$$\frac{A}{a} = m, = \frac{1}{\log. b}.\log.\frac{P}{W}, \text{ or } \log.\frac{P}{W} = \frac{A\log. b}{a};$$

$$\text{but } \log. b = \log.(1+n) = \log.\left(1+\frac{fa}{r}\right) = \frac{fa}{r}$$

very nearly for the Neperian logarithms, because a is indefinitely small. Hence

$\log.\frac{P}{W} = \frac{A\log. b}{a} = \frac{Af}{r}$: or if e be the radix of the Neperian system, $\frac{P}{W} = e^{\frac{Af}{r}}$ and $P = We^{\frac{Af}{r}}$.

If A be expressed in terms of the circumference to the radius r, or by $2\pi r q$, $P = We^{2\pi fq}$.

Suppose $f = \frac{1}{3}$, and q to be expounded successively by 1, 2, 3, 4, 8; P will be equal to

8.121 W
65.943 W
535.492 W
4348.473 W
and 18,909,217.000 W, respectively;

so that when the rope passes 8 times round the cylinder the force W will not yield to one less than nearly 19 millions of times itself.

It is most convenient to employ in this calculation the common logarithms. Let the symbol of Neperian or hyperbolic logarithm be l and of common logarithm L. Then since

$$l.\frac{P}{W} = 2\pi fq.$$

$$L.\frac{P}{W} = 2\pi fq \times 0.43429448, \&c.$$

$$\text{or } L.\frac{P}{W} = 0.909584\,q \text{ when } f = \tfrac{1}{3}.$$

That a prodigious advantage is thus gained in opposing a small force to a great one is well known to many who are un-

[illegible] 2.71828

acquainted with the principles by which its amount is ascertained. The seaman, for instance, often secures his ropes in this way.

In our calculation the slight disadvantage under which one of the forces may act from the spiral form of the coils, (168.) is disregarded, as well as the force required for the flexure of the rope.

459. Friction, while to a certain extent it prevents or destroys motion, is favourable to the maintenance of equilibrium. In the use of the wedge or the screw for compression it is generally sufficient to secure the advantage already gained, without the continued exertion of active power. Without it nails, screws and bolts would be useless, and no machine or permanent structure could be formed of parts once disjoined, no animal on a horizontal plane could exert any force but in a direction exactly vertical, nor could a body be transported from one place to another unless by the action of a fluid, or a tendency downwards never to be reversed.

460. Ropes of any considerable thickness occasion, by their rigidity, a further diminution of power. When in the use of the pulley or of the wheel and axle the power prevails, and the weight begins to rise, the curvature of the part of the rope applied to the instrument, combined with its stiffness, throws the vertical line of the power's action as it were below the pulley or cylinder, or a little nearer to the axis than the extremity of the horizontal diameter, and the other side of the rope is made to project a little outwards, so as virtually to increase the horizontal lever by which the weight acts. The resistance arising from this cause is found to depend on various circumstances, not easily subjected to calculation; for instance, upon the newness of the rope, or the degree in which it has been used, on its texture, and the degrees of torsion employed in its formation; but, *cæteris paribus*, it is found to be inversely as the radius of the cylinder to which it is applied, in the direct sesquiplicate ratio nearly of its own radius or diameter, and directly as a sum composed of two terms, the one constant and the other varying as the tension.

Of the Parabolic Theory of Projectiles.

461. *Prop. I. Theorem.* **If a body be projected in a direction not perpendicular to the horizon, and under the future influence of no force but that of gravity, its path will be sensibly a parabola.**

The elevation of a projectile at the highest point of its flight is so small compared with the earth's radius, and its range so small a part of the earth's circumference, that we may here, as in our statical investigations, consider gravity both as a constant force and as acting in parallel directions. On this supposition the path of a projectile, *in vacuo*, when not vertical, will be accurately a parabola.

Let t, T be the times of describing AB, AC, (Fig. 176.) with the uniform projectile velocity V, and of descending from rest through AE, AF by the force of gravity; complete the parallelograms AG, AH; then

$$AE : AF = t^2 : T^2 \text{ (317.)}$$

$$AB^2 : AC^2 = t^2 : T^2 \text{ (311.)}$$

$$\therefore AE : AF = AB^2 : AC^2 = EG^2 : FH^2,$$

and the *locus* of the points G, H, &c. is a parabola of which AL is a diameter, A its vertex, and EG a semiordinate. But, by the second law of motion, the projectile will at the end of the times t, T, &c. be found in the points G, H, &c. Whence the proposition is manifest.

462. *Cor.* 1. The direction of the projectile motion is a tangent to the curve, because it passes through the vertex of a diameter, and is parallel to its ordinates.

463. *Cor.* 2. The velocity in any point of the curve is that which is due to a fall from rest through a space equal to a fourth part of the *parameter*, or *latus rectum*, belonging to the diameter which passes through that point.

Let AF be the height due to the tangential velocity at any point A, which may be the point of projection. Then AC or

$FH = 2\,AF$, (316.) and, P being taken to denote the parameter,

$$P.AF = FH^2 = 4\,AF^2;$$
$$\text{or } AF = \tfrac{1}{4}P = AL,$$

LR being the directrix.

464. *Cor.* 3. The velocity is least at the vertex of the axis, and equal at equal distances on each side of it.

465. *Cor.* 4. If the projections be made in different directions with the velocity acquired by falling through LA, and if a circle be described from the centre A, with the radius AL, its circumference will be the *locus* of all the foci.

466. *Cor.* 5. The time of describing any arch, as AG, is the same with that of falling vertically through the abscissa AE, or its equal BG, and is therefore $= \sqrt{\frac{2\,BG}{g}}$, (320, Eq. 3.)

467. *Cor.* 6. $P = \frac{2V^2}{g}$. For $\frac{1}{4}P$ is the height due to V, (463.) and therefore $= \frac{V^2}{2g}$, (320, Eq. 4.)

468. *Prop. II. Problem.* There being given the position of the point from which the projection is made, that of an object in the same horizontal plane, and the velocity of projection, it is required to describe the parabola in which the projectile must move to hit that object.

The velocity being given the parameter is known, (467.) Let AB (Fig. 177.) be this parameter placed at right angles to the horizontal plane AM in which is situated the object F which is to be hit. On AB, and in the plane BAF, describe the semicircle BHA, of which let the centre be C; draw FED perpendicular to the horizon, and, if possible, cutting or touching the semicircle. If it cut it in E, it will also, when produced, meet it in another point D. Join A,E and A,D, and let a parabola be described of which BAN may be a diameter, A the vertex of that diameter, BA its parameter, and to which AE or AD may be a tangent in A; this parabola shall pass through F.

Draw FL parallel to AE or to AD, according as the former or the latter is chosen as the direction of the impulse; it will be in the direction of an ordinate to the diameter AN. Join BE or BD. Then, taking for demonstration the case when AE is the line of projection, we have by the similar triangles BAE, AEK, (EK being perpendicular to AB,) $BA.AK = AE^2$, or since $AK = EF = AL$ and $AE = LF$

$$BA.AL\ (= P.AL) = FL^2,$$

wherefore F is in the parabola; and by *Prop. I.* and its *Cor.* 1 and 2, it is evident that if a body be projected with the given velocity in the direction AE or AD it will describe this parabola.

AF is called the horizontal range or amplitude, FAE or FAD the elevation, and $\frac{1}{4}AB$ the impetus.

469. *Cor.* 1. If F coincide with G, the point where the vertical tangent of the semicircle meets the horizontal plane, the problem admits of only one solution, and it is evident that the corresponding direction AH, or an elevation of 45°, is that which gives the greatest range with a given velocity.

470. *Cor.* 2. Let AF or $EK = A$, and EAF the angle of elevation $= E$. Then EBA also $= E$, and $ECA = 2\,E$;

$$\text{Hence } AF = KE, = \tfrac{1}{2} P \sin. 2\,E,$$

$$\text{or } 1.\ A = \frac{V^2 \sin. 2\,E}{g};$$

$$2.\ \sin. 2E = \frac{Ag}{V^2};$$

$$3.\ V = \sqrt{\frac{Ag}{\sin. 2E}}.$$

471. *Cor.* 3. Since the vertical velocity $= V \sin. E$, the time of destroying it will be $\frac{V.\sin. E}{g}$, (320, Eq. 1.) and the time of destroying and reproducing it, or the whole time of flight, $= \frac{2\,V.\sin. E}{g}$.

Otherwise thus, the time of flight is that of describing AE uniformly with the velocity of projection, and consequently

The parameter is known as before. Let AB (Fig. 179.) be a vertical straight line $= P$, and describe on it a segment of a circle which may contain an angle $= FAL$, the supplement of the zenith distance of the object. Of this circle let C be the centre. Draw FED parallel to AB and, when possible, cutting the circle in E and D, or touching it as GH: join A, E and A, D, and describe a parabola, as in the preceding problem; it shall pass through F. Join BE, and draw FL parallel to AE,

$$\angle BEA = \angle FAL \text{ by construction, } = \angle AFE,$$
$$\text{and } \angle BAE = \angle AEF.$$
$$\therefore \triangle BAE \text{ is similar to } \triangle AEF,$$
$$\text{and } BA : AE = AE : EF,$$
$$\text{whence } BA.EF = AE^2,$$
$$\text{or } P.AL = LF^2.$$

$\therefore F$ is in the parabola; and by *Prop. I.* and its *Cor.* 1. and 2. the projectile will describe this parabola.

476. *Cor.* 1. When F coincides with G, the point where the vertical tangent to the circular segment AHB meets the tangent to the same at the point of projection, the problem admits of only one solution.

477. *Cor.* 2. Let $\angle BAM = Z$, $\angle BAE = z$, and $\angle EAM = E$. Then

$$AB : AE = \sin. Z : \sin. E,$$
$$\text{and } AE : AF = \sin. Z : \sin. z,$$
$$\therefore AB : AF = \sin.^2 Z : \sin. E . \sin. z,$$
$$\text{or } \frac{2V^2}{g} : A = \sin.^2 Z : \sin. E . \sin. z,$$
$$\text{whence 1. } A = \frac{V^2}{g} \cdot \frac{2 \sin. E . \sin. z}{\sin.^2 Z}.$$
$$2.\ V^2 = \frac{A g \sin.^2 Z}{2 \sin. E . \sin. z}.$$
$$3.\ 2 \sin. E . \sin. z = \frac{A g \sin.^2 Z}{V^2}.$$

But $2 \sin. E . \sin. z = \cos. (E - z) - \cos. (E + z) = \cos. (E - z)$

$-\cos. Z$. Hence $E-z$ and $E+z$ are both known when A, Z and V are given, and therefore E may be found.

478. *Cor.* 3. The velocity of projection V may be resolved into a velocity $\frac{V.\sin.z}{\sin.Z}$ in the direction AF and a velocity $\frac{V.\sin.E}{\sin.Z}$ in the vertical direction. In the half of the time the space described in the direction AF will be $\frac{1}{2}AF$, or the body will then be at the vertex of the diameter to which AF is an ordinate. Its motion, being in the direction of the tangent, will be parallel to AF, so that the velocity $\frac{V.\sin.E}{\sin.Z}$ has just been destroyed. The time of destroying this velocity is $\frac{V.\sin.E}{g.\sin.Z}$, and consequently the time of flight is $\frac{2V.\sin.E}{g\sin.Z}$.

Otherwise thus; the time of flight is $=\frac{AE}{V}$, or that of describing AE with the velocity of projection, and for AE may be substituted its value $\frac{P.\sin.E}{\sin.Z}=\frac{2V^2}{g}\cdot\frac{\sin.E}{\sin.Z}$.

479. *Cor.* 4. When the projectile is at the vertex of the diameter to which AF is an ordinate, and consequently is moving parallel to AF, it must be at its greatest height above that line; but in that position it bisects the subtangent which is $=\frac{1}{2}EF$. Hence the greatest height $H=\frac{1}{4}EF=\frac{\frac{1}{4}P.\sin.^2E}{\sin.^2Z}$ $=\frac{V^2}{2g}\cdot\frac{\sin.^2E}{\sin.^2Z}$.

That the value of EF is here truly assigned is manifest thus: The triangles EAF and EBF were proved to be similar, (475.) $\angle EBA=\angle EAF=E$, and $\angle BEA=\angle EFA=Z$.

$$\begin{aligned}\text{Hence } AB : AE &= \sin.Z : \sin.E,\\ AE : EF &= \sin.Z : \sin.E,\\ \therefore AB : EF &= \sin.^2Z : \sin.^2E.\end{aligned}$$

480. *Cor.* 5. Of the two directions that with the same im-

petus give equal ranges upon the inclined plane, the one is as much above the line bisecting the zenith distance of the plane as the other is below it.

This is evident by inspection of the figure, where the arches *HD* and *HE* must be equal; or by considering the value of *A* in *Cor.* 2. where since E and z make a given sum Z, we may, when they are unequal, change E into z and z into E, and thus have two different divisions of Z the products of whose sines shall be equal.

481. *Cor.* 6. With a given velocity of projection the greatest range upon a given plane *AF* will be when the direction of the impulse bisects its zenith distance.

Since $\angle EAF = \angle EBA$, *AF* is a tangent to the circle, as is also *GH*. Therefore $GA = GH$, and $\angle HAG = \angle GHA = \angle HAB$. Now *AH* is the direction which gives the amplitude *AG*.

The same conclusion may be derived from *Cor.* 2. by considering that since for each plane $E + z$ is a given sum, $\sin. E. \sin. z$ will be a maximum when $E = z = \frac{1}{2} Z$. This may be proved, for exercise, analytically, or by a very simple geometrical demonstration.

482. *Cor.* 7. The greatest range A' upon the plane *AF* $= \frac{1}{4} P. \sec.^2 \frac{1}{2} Z$.

$$\text{For } A' = \frac{P. \sin.^2 \frac{1}{2} Z}{\sin.^2 Z} = \frac{P. \sin.^2 \frac{1}{2} Z}{4 \sin.^2 \frac{1}{2} Z \cos.^2 \frac{1}{2} Z}.$$

Or thus: if we draw a line *GO* perpendicular to *AH*, $AK . \sec. \frac{1}{2} Z$, or $\frac{1}{2} P. \sec. \frac{1}{2} Z = AH$, and $AO. \sec. \frac{1}{2} Z$, or $\frac{1}{2} AH. \sec. \frac{1}{2} Z = AG$.

483. *Cor.* 8. While the velocity of projection is constant, but the directions different, the points of greatest distance that can be struck are all in a parabola given in position.

For the parameter being constant, (467.) the points *H* are all situated in the same line *CK*, which is drawn from the centre bisecting the parameter as a vertical chord of the circle, and *GH*, perpendicular to *CH*, is always $= GA$. Hence if a parabola be described with the focus *A*, and directrix *CH*, it will be the *locus* of all the points *G*, or the limit of amplitudes for planes of all different elevations or depressions.

As the foci of both parabolas are in the same line GA, and GH is perpendicular to the directrix of each, the straight line bisecting the angle AGH will be a common tangent of both. Hence the two curves touch each other in G; and it is obvious that they do not meet in any other point, for to reach any point of the limiting curve, the direction of the primitive impulse must always bisect its zenith distance. The direct geometrical proof of this last statement is very simple, but would require an additional construction.

This elegant Dynamical theorem we owe to Torricelli, the disciple of Galilæo.

484. The formulæ investigated as corollaries to the two preceding problems may be easily found without reference to the parabola.

That a projectile discharged from A (Fig. 180.) may strike F on the same horizontal plane, the direction AB must be such that it would reach the point B by the projectile impulse in the same time in which it would fall vertically through BF. Suppose $\angle BAF$ to be denoted by E, and AF by A: $AB = \frac{A}{\cos. E}$, the time of describing AB uniformly with the velocity $V = \frac{A}{V \cos. E}$, and the square of this time $= \frac{A^2}{V^2 \cos.^2 E}$. Again, $BF = \frac{A. \sin. E}{\cos. E}$, and the square of the time of describing it with uniform acceleration $= \frac{2A.\sin.E}{g.\cos.E}$, (320, Eq. 3.)

$$\therefore \frac{A^2}{V^2 \cos.^2 E} = \frac{2\,A.\sin.E}{g.\cos.E},$$

$$\text{and } A = \frac{2\,V^2.\sin.E.\cos.E}{g}, = \frac{V^2.\sin.2\,E}{g}.$$

If B is the object to be hit, we must make the discharge in the direction AC, determined by this condition that the body shall describe CB by the force of gravity in the time in which it would describe AC uniformly with the motion of projection. If we now denote *AB* by *A*, and the angles *B*, ~~*D*~~ and *BAC* by *Z*, *z*, and *E*, respectively;

$$AC = \frac{A.\sin. Z}{\sin. z}, \quad BC = \frac{A.\sin. E}{\sin. z},$$

$$\frac{A^2.\sin.^2 Z}{V^2 \sin.^2 z} = \frac{2\,A.\sin. E}{g \sin. z}, = T.$$

$$\text{Whence } A = \frac{2\,V^2}{g}.\frac{\sin. E.\sin. z}{\sin.^2 Z}.$$

The relations of the same quantities may be otherwise investigated and stated in a form somewhat different, thus, (Fig. 181.)

Let $AK = a$, $FK = b$, the co-ordinates of *F* the point to be struck; $\angle DAM = e$, $\frac{v^2}{2g} = H$, the height due to the velocity of projection; $t =$ the time of describing $AB = x$ horizontally, or $BC = y$ vertically,

$$v \cos. e = \text{horizontal velocity},$$

$$x = t v \cos. e,\ t^2 = \frac{x^2}{v^2 \cos.^2 e}, = \frac{x^2}{2 g H \cos.^2 e};$$

$$y = x \text{ tang. } e - \tfrac{1}{2} g t^2 \text{ (323.)}$$

$$\text{or } y = x \text{ tang.} e - \frac{x^2}{4 H \cos.^2 e}.$$

For $\frac{1}{\cos.^2 e}$ substitute $1 + \text{tang.}^2 e\ (= \sec.^2 e)$,

$$\text{and you have tang. } e = \frac{2 H \pm \sqrt{4 H^2 - 4 H y - x^2}}{x}.$$

By substituting *a* and *b* for *x* and *y*, *e* will be found.

The double sign indicates in general two values of *e*, either of which will answer. But if $4 H^2 - 4 Hb = a^2$, there is only one solution.

If the point *F* be in the same horizontal plane with the point of projection, $4 Hb = 0$, and then if $a^2 = 4 H^2$ or $a = 2 H$, the problem admits of only one solution.

When in this case *a* is $> 2 H$ or in general $a^2 > 4 H^2 -$

$4\, H b$, the value of tang. e becomes imaginary, and the problem impossible.

485. The author of the parabolic theory of projectiles was Galilæo, the founder of the mechanical philosophy, who published it in his *Dialogues on Motion* in the year 1638. As one of the first examples of physico-mathematical science, it claims attention in every academical course of Dynamics; and, were no force concerned in the modification of projectile motion but that of gravity, the deductions of this theory might be safely confided in as applicable in practice. But military projectiles must move through the air, and cannot do so without displacing a portion of it. To this they must communicate certain quantities of motion, and in doing so must themselves lose equal quantities. This occasions a very sensible retardation to bodies specifically light, and exposing a considerable surface, even when discharged with very moderate velocities. A ball of cork cannot be thrown to any great distance. A sharp-edged flat plate of stone or metal may be thrown to some distance while it proceeds edge foremost; but if it happens to turn the flat side foremost, it descends in a path much incurvated, and sometimes almost perpendicularly. Philosophers were long aware of the existence of this force before they were disposed to allow it much effect in retarding the flight of military projectiles, which are always composed of matter which has great specific gravity, as lead or iron. The effect being then confessedly small in the case of slow motions, they did not sufficiently attend to its very great increase when the velocity is such as that with which military projectiles are discharged. Did the resistance depend on the *inertia* of the air alone, it would be as, or nearly as, the square of the velocity: for a ball of given diameter whose velocity is doubled or tripled would remove double or triple the mass of air formerly displaced, and that with a velocity the double or triple respectively of its former velocity, so that its loss of quantity of motion and therefore of velocity, the body being the same, would be in the one case four times, in the other nine times what it was at first. But in addition to the resist-

ance from inertia, there is an unbalanced pressure on the anterior surface, which in the case of a velocity of 1300 or 1400 feet per second will in the lower regions of the atmosphere amount to between 14 and 15 lbs. avoirdupois on the square inch of the hemispherical base. In the use of cannon and musketry, then, the parabolic theory of projectiles is of little or no value, but as leading the way to a more accurate knowledge of the effect of the air's resistance, of which it may assist us in obtaining a measure.

The reason that artillerists were so long in detecting the true cause of the deficiency of this theory, as a foundation of practice, was the want of an independent method of estimating with tolerable precision the initial velocity communicated by a given charge of powder. We have formerly proved that if a ball is discharged at an angle of elevation E through an unresisting medium, $V = \sqrt{\frac{Ag}{\sin. 2E}}$, A being the amplitude upon a horizontal plane; consequently E being given and the amplitude measured, it is easy to find what must have been the initial velocity, if the body has moved through such a medium. But, since the air resists the motion, the amplitude will be thereby diminished, and will become equal to one corresponding to a smaller velocity *in vacuo.* A velocity inferior to the true one will of course be deduced by this method. Could the artillerist have accurately estimated the velocity of projection by any method independent of the parabolic theory, he would soon have perceived that the amplitude was much inferior to what, on the principles of that theory, it ought to be; and would thus have been led to appreciate more justly the amount of the air's resistance: but it is evidently impossible that he should detect any inconsistency between the initial velocity and the actual range, so long as the amount of the one was inferred from the extent of the other.

486. About the year 1742 Mr. Robins, an ingenious English mathematician, laid the foundation of a complete revolution in the science of gunnery, by the discovery of two methods of estimating the velocity of shot, which are both de-

serving of our particular notice. The first was by finding experimentally, as accurately as he could, the initial force of newly inflamed gunpowder filling a space of given capacity, and the law according to which its pressure decreases by expansion, and thence calculating the velocity which it must communicate to a given body impelled by it through a given space. The second, and most perfect, and, on different accounts, the most useful, was by a machine of his own contrivance, already described, the ballistic pendulum, (439.)

487. The elastic force of fired gunpowder according to Mr. Robins's experiments with an airpump, and subsequent calculations, is, at the first moment of explosion, equal to 1000 times the pressure of the atmosphere, when the barometer stands at nearly 30 inches; and the elasticity of the generated fluid varies as its density, or inversely as the space occupied.

Mr. Robins in calculating, from these *data*, the velocity of shot, employs the geometrical method of representation explained in treating of variable forces, (344.), proceeding thus:

Let AC (Fig. 182.) represent the length of a musket barrel, AB that of the chamber occupied by the powder; take BD to represent the weight of the ball, and, making BD to BG as the weight of the ball to the initial force of the powder, let LGM be the curve whose ordinates to the line of abscissæ AC represent the accelerative force, or which is here the same thing, since the mass impelled is constant, the pressure of the powder on the ball at the different points of the line BC. Then if DE be drawn parallel to BC, and v and V represent the velocities communicated to the ball, in passing over a space $= BC$, by its own weight and by the elastic force of the inflamed powder, respectively

$$BCED : BCMG = v^2 : V^2;$$

now $v^2 = 2g.BC$ (320, Eq. 4.) and is known, therefore V^2 will be also given, if we can find the ratio of $BCED$ to $BCMG$. But the force of the powder being inversely as the *space* occupied, wlll be, in a cylindrical barrel, inversely as the *length* occupied. Therefore $NQ \div \frac{1}{AN}$, or $NQ : BG = AB : AN$, and the curve is an hyperbola, whose assymptotes are *AH* and *AC*. The ratio of the spaces above mentioned then is given; for we know the ratios *BC* : *AB* and *BD* : *BG*, therefore the ratio compounded of the two, viz. *BCED* : *ABGF*; also the ratio $ABGF : BCMG = 1 : \text{hyp. log.} \frac{AC}{AB}$, whence V^2 and *V* are given.

488. As an exemplification of our analytical formulæ, we may also solve the problem thus:

Let a body be impelled from *B* to *C*, (Fig. 183.) by a force which is inversely as the distance from *A*, and let it be proposed to find the velocity communicated, on the supposition that the force at the distance *b* is *f* and that $AC = a$ $AB = b$ and *f* are given.

Let $AD = x$; the force at *D* will be $\frac{bf}{x}$, and $v\,dv = \frac{bf\,dx}{x}$, (339.)

$$\text{Hence, } v^2 = 2\,bf.\log. x + C,$$
$$0 = 2\,bf.\log. b + C,$$
$$\text{and } v^2 = 2\,bf.(\log. x - \log. b),$$
$$\text{or } v = \sqrt{2\,bf.\log.\frac{a}{b}}, \text{ when } x = a.$$

To apply this more particularly to the case under consideration, let us adopt the following symbols, used by Dr. Hutton in his *Course of Mathematics.*

$d =$ diameter of the ball in inches,

$c = 0.7854$ area of a circle to diameter 1,

$m = 230$ oz. avoird. = pressure of air on 1 square inch,

$n : 1$ the ratio of the initial force of fired gunpowder to the pressure m,

$w =$ the weight of the ball, } in avoird.

$p =$ half that of the powder. } ounces.

This latter element is introduced into the calculation of the accelerative force, on the principle that, in impelling the ball the powder must move its own mass with the half of the sam velocity as estimated by that of its centre of gravity, whic exhausts the same force as if it had to move half its own mass with a velocity always equal to that of the ball.

$$\text{Hence } f = \frac{g m n c d^2}{p+w},$$

$$\text{and } v = \sqrt{\frac{2 g b m n c d^2}{p+w} L . \frac{a}{b}}.$$

Hence L denotes Neperian or hyperbolic logarithm. If the common logarithms are used, we must introduce the factor 2.302585. If v is to be in feet, $g = 32\frac{1}{6}$, and b must be in the same denomination, because in the formula $r\,d\,v = \frac{b f d x}{x}$ dx and consequently x and b must be in measure of the same denomination with v.

489. This calculation, when we take $n = 1000$, which is below its true value, is sufficient to show the great excess of the initial velocity of shot above what had been contemplated, and the total insufficiency of any theory in which the resistance of the air is disregarded, as a guide in artillery practice. But the number n in our formula is not to be ascertained with much approach to accuracy in the way employed by Mr. Robins, and his second method of measuring the velocity of a ball is greatly preferable to this one. When v is ascertained by the pendulum, the formula above investigated will give the value of n; and in this way Dr. Hutton generally found it to be at least $\frac{3}{2}$ of the value asssigned by Mr. Robins.

490. In order to put to the test the received opinion respecting the resistance of the air, Mr. Robins charged a musket three times successively with the same allotment of powder, and with a leaden ball $\frac{3}{4}$ of an inch in diameter, and about $\frac{1}{12}$ lb. avoirdupois in weight, and discharged it into a ballistic pendulum placed successively at the distances 25, 75 and 125 feet. The velocities with which it reached the pendulum in these situations were found to be 1670, 1550, and 1425 feet per second, respectively; so that in passing through 50

feet of air with the mean rate of 1610 feet, the loss of velocity was 120 feet per second. The time in which this velocity was lost, or that of passing through 50 feet of air, must have been about $\frac{1}{32}$ of a second; and in this time a retarding force equal to the ball's weight would have destroyed only one foot per second of the velocity. The resistance of the air, then, to this ball moving with the velocity above stated, a velocity often equalled or exceeded in practice, must have been 120 times its weight, or equal to about 10 lbs. avoirdupois. Thus in the parabolic theory the deflecting force neglected greatly exceeds that taken into account.

491. Hence it follows that the curve actually described will differ greatly from the parabola. In describing the latter, the horizontal velocity is constant and the same with that at the vertex, and the ascending and descending branches are in every respect equal and similar. But in moving through the air, the horizontal velocity is continually diminished by the resistance of the medium, and the vertical velocity due to impulse and gravity diminished both in ascending and descending; the descending branch of the curve will, therefore, supposing it to terminate at the extremity of the horizontal amplitude, be shorter and more incurvated towards the axis than the ascending branch, the vertex will be nearest to the farther extremity of the range, the motion will be slowest not at the highest point D (Fig. 184.) but a little beyond it, as at E, where the horizontal velocity at D, already much reduced below its original value, is still farther diminished by the air's resistance, and not yet compounded with any considerable velocity of descent. The point of greatest curvature too, will for the same reason not be at the summit, but a little beyond it, as at F, and nearer to it than E. Without the resistance of the air the greatest range upon a horizontal plane would be with an elevation of 45° and $= \frac{V^2}{g}$. A musket or cannon ball, a piece of cork, a feather, and the finest cobweb, projected at that elevation, with a velocity of 1600 feet per second, would range to the very same distance, about 15 miles, where-

as we know that in moving through the air the lighter bodies mentioned would reach to an exceedingly small distance, the musket ball not much above half a mile, and a 24 lb. cannon ball to little more than two miles. The reason why the cannon ball ranges farther with the same projection than the musket bullet is, that the resistance to the former, though much greater than that presented to the latter, is a less *proportional* part of its weight. The extent of the range depends not so immediately on the resistance as on the part of it that falls to the share of each equal particle of the moving body, when equally distributed, that is, on the retarding force, the measure of which is $\frac{F}{Q}$, F being the resistance, and Q the quantity of matter. If balls of different diameters be cast of the same metal, or be of equal density, the resistance will be nearly as the surface, or as the square of the diameter, while the quantity of matter will be as its cube. That is, Retardation $\doteq \frac{F}{Q} \doteq \frac{d^2}{d^3} \doteq \frac{1}{d}$, or the retardation will be inversely as the ball's diameter. If the balls are of different matter, let s denote specific gravity, or the weight of the unit of volume: then, Retardation $\doteq \frac{d^2}{sd^3} \doteq \frac{1}{sd} \doteq \frac{1}{s}$ when d is given. The specific gravity of cast iron being to that of cork as 7425 to 240, if two balls of these materials and of equal diameter were projected with equal velocity, the ball of cork would in passing through the same small space lose more than 30 times as much velocity as the ball of iron.

In practice the angle of elevation giving the greatest range is found to depend on various circumstances. It may be nearly 45° for the heaviest shot and smallest velocities; but for the smallest shot discharged with the greatest velocity it is not above 30°, and it may be, according to varying circumstances, any one between these limits. According to Borda's calculation, by an approximative formula, we have for a 24 lb. ball the following table of angles giving the greatest range corresponding to the initial velocities opposite to them.

[illegible]

[illegible] will be deflected [illegible] side with [illegible] will thus be continually deflected to the [illegible] A similar effect must result from irregularity [illegible] presented to the air's

That [illegible] Mr. Robins does [illegible] two very simple and beautiful [illegible] In the [illegible] he took a light wooden globe, and [illegible] by a double string of convenient length. The [illegible] then twisted in a given direction, and the globe [illegible] like a pendulum. its oscillations continued [illegible] vertical plane till the string, by untwisting, be-

Velocity in French feet per second.	Angle giving the greatest range.
300	42° 10′
600	36 30
1000	33 —
1200	31 40
1500	30 10
1800	28 50
2000	28 10.

The investigation of this formula, and Euler's method of calculating the path of a projectile in a medium of which the resistance is as the square of the velocity, with a demonstration of some things above stated in relation to the curves thus described, we shall give in an Appendix.

492. There is another effect of the air's resistance discovered by Mr. Robins, which very much embarrasses the practice of the artillerist. A bullet is not only deflected vertically from the line of projection, and from the parabolic path due to the action of gravity, but it is also frequently deflected to the right or left of the vertical plane of projection. If much windage is allowed, the ball may strike obliquely against the side of the barrel, or the bore of the cannon, in different parts, rebounding from side to side, and the last rebound from some point very near to the muzzle may occasion the projection to be in a vertical plane making an angle with the axis of the piece. This may and probably does often occasion a very considerable deviation from the expected course. But the motion, so far as we yet perceive, would still be in a vertical plane, and if two shot were discharged from the same point, through parallel screens, and affected only in this way, the horizontal distances of the perforations would be as the distances of the screens from the station where the shot were discharged. The irregularity, however, discovered by Mr. Robins, though perhaps blended with this, was of a different kind. He found a *curvilinear* deflection to the right or left of the vertical plane of projection, which of

course indicated the *continued* action of a disturbing force. This was incontestably proved in the following way.

J.28 *M*, *N*, *P* (Fig. 185.) being three parallel screens of fine paper, which could present no sensible resistance, and still less inequality of resistance to the ball, were set up at known distances from each other, and from the point *A* where the bullet was discharged. If the motions of the balls then were confined to the same vertical planes, whose projections on the plane of the horizon may be represented by *AF*, *AG*, it is obvious that the distances of the perforations, reduced also to the horizontal plane, and represented by *BC*, *DE*, *FG*, would be as their distances from *A*; or if the point of discharge *A* were supposed to be a little different, still if we take *BC* from each of the other distances *DE*, *FG*, the remainders *DH* and *FK* would be to each other as the known distances *BD*, *BF*. But if the distances as *Bb*, *DE*, *Fg* were not according to this law, then the path *AbEg* was curvilinear. This was uniformly found to be more or less the case. Mr. Robins accounts for it ingeniously by a motion of rotation, that, he supposes, will generally be communicated to the ball by rubbing on the side of the barrel. On one side of the ball the velocity of rotation will conspire with the progressive velocity, and the actual velocity will be their sum; on the opposite side it will be their difference. The side of the ball on which the velocity happens to be greatest will meet with most resistance. It will be deflected as if moving all with equal velocity it met on that side with a denser fluid, and will thus be continually deflected to the opposite side. A similar effect must result from irregularity in the curvature of the ball on the side presented to the air's resistance. That the cause assigned by Mr. Robins does operate, he demonstrated by two very simple and beautiful experiments. In the one he took a light wooden globe, and suspended it by a double string of convenient length. The string being then twisted in a given direction, and the globe made to oscillate like a pendulum, its oscillations continued in the same vertical plane till the string, by untwisting, began to produce in it a considerable velocity of rotation. It then began to deflect to that side of the original vertical plane, which according to the theory could be predicted from the known direction of the motion; and that this could not be owing to the untwisting of the string, operating otherwise in some unknown way, was proved by the continuance of the deflection towards the same side during the whole time of rapid rotation, though in the latter half of that time the string, having untwisted itself, was by the motion acquired twisted in a contrary direction. In the second experiment he took a gun barrel, and bent three or four inches of it towards the muzzle to the one side, which we shall call the left; as in Fig. 186. This he charged with a ball of considerable windage, and, foreseeing that the tendency to continue its motion through the barrel in the right line determined by the straight part *AC* would occasion it to rub against the concave interior surface at *C*, and thus, retarding by friction that side of the ball, give it a motion from left to right about a vertical axis, he predicted that, though thrown at first to the left of the line of aim by the curvature from *BC* to *D*, it would be finally deflected to the right of that line; and his expectation was verified.

493. To whichsoever of the causes assigned this deflection, which makes the path a line of double curvature, be chiefly due, a remedy, in the case of musketry, is found in the use of what is called a rifled barrel. The inside of a common barrel is a smooth hollow cylinder, but the rifled barrel has its interior surface cut by a number of spiral channels or grooves, proceeding from the breech to the muzzle, and making about one turn in the whole length. If they made only half a turn or even less, it would occasion less resistance to the impulse, and the sensible effect would probably be the same. The ball is made a little larger than what would have fitted the bore before the spiral channels were cut, and is driven home to the powder by a mallet and strong iron rammer, or the musket may be charged at the breech by unscrewing the barrel. Upon the discharge, the ball

takes the direction of the spiral threads, and thus acquires a rapid motion of rotation round an axis coinciding with the line of projection. In this case, there can be no inequality of resistance from rotation so long as the axis of rotation sensibly coincides with the direction of the progressive motion, and the effect of small irregularities in the surface will be equally distributed. The motion cannot deviate far to one side till it will be corrected by an equal deviation to the other, in consequence of the part which occasions the irregularity having made half a revolution. This is illustrated by the similar effect of the feathers of an arrow, when put on with a slight obliquity to the axis, or direction of its length. The strong relative current of air produced by the rapid flight of the arrow, meeting the feathers obliquely, will give by resolution a force tending to turn the arrow about its axis, and prevent it, when truly aimed, from ever deviating more from the right course than what corresponds to the time of half a revolution.

Dr. Charles Hutton, late professor of mathematics in the Royal Military Academy at Woolwich, has very ably prosecuted and extended the plan begun by Mr. Robins. His *Tracts*, containing the detail of his experiments on gunnery, and many deductions from them, may be studied with great advantage by the young artillerist, as containing much important information; and by those who wish to improve their habits of philosophical induction, as containing an excellent specimen of experimental research. The second volume of his *Course of Mathematics*, may also be perused with advantage by such as wish for ampler details concerning the practice of gunnery than would be suitable in a general course.

Of the Central Forces of Bodies Revolving in Circular Orbits.

494. When by a constant acceleration a body describes DC (Fig. 187.) in a certain time, and when by a constant retardation the motion BD is changed into BC in that time, a

force is equally indicated, which, being opposed, would occasion pressure equivalent to a weight that may, by principles already explained, be inferred from the change of motion. The same thing is true when the motion *DC* generated or prevented, is a deflection, (Fig. 188.) *AC* being an element of a curve, *AD* its tangent, and *DC* perpendicular to the curve, if at the end of the element of time dt after the body has been at *A* it be found not at *D* but at *C*, a force may be inferred which, operating as a pressure, would be the same with the weight that would cause the same body to fall through a space equal to *DC* in the same time. If, in the time in which a body falls from *D* to *C*, a body of the same inertia and void of gravity were drawn by an interposed spring through the same space, so as to have always the same velocity at any given point, the spring would be dilated to the same degree as it would be when simply supporting the body's weight. In like manner, were this same body, projected in the opposite direction, to be suddenly caught by a spring when it is at *C*, and were the motion *CD*, by this spring acting with constant energy, prevented in the same time, the spring would be still dilated or compressed to the same degree as it would be by the weight above defined. Now such a motion is prevented in a body virtually possessing it when it describes the curve *AC*, and the indefinitely small motion *AD* is changed into *AC*. If a body is whirled round at the extremity of a spring, the spring is always stretched; and, when the other extremity is fixed, the tension may be considered as a central force, and is denominated centripetal or centrifugal just as we happen to view it. As dilating the spring it is centrifugal; considered as a contractile force it is centripetal.

The centrifugal tendency arising from a motion of rotation, may be further familiarly illustrated by fixing a spherical body of soft consistence, as wax, putty, or clay, upon an axis, which, on our thereby whirling it round, will change its shape and become oblate; by agitating water with a circular motion in a vessel, when that which is towards the side will be made to stand above the level of the central portion; or by setting a vessel filled with water upon the inside of the rim of a wheel, and turning it round with such rapidity that the fluid shall not be spilled, even when at the highest point.

495. In circular orbits the co-existent centripetal and centrifugal forces are always equal; for were either to prevail, the body would approach nearer to the centre or recede from it.

496. In such orbits also the tangential velocity is uniform; for the centripetal force, being always at right angles to the direction of the motion, can neither accelerate nor retard it.

497. *Theorem.* If a particle of matter describe the circumference of a circle in consequence of continual deflection by a centripetal force, the measure of that force is the square of the velocity divided by the radius.

When the body is at *A* (Fig. 189.) its motion is in the direction of the tangent *AF*, and its velocity in the direction *AC* is nothing; and when it is at *B*, its motion is in the direction *FB*, parallel to *AC*, and its velocity parallel to *AF* is extinguished. The tangential velocity then at *B*, and which is constant, (496.) may be considered as generated by a force acting from *H* towards *L*, while the line *HL* is carried from the position *AC* to the position *FB*. Now if the centripetal force at *A* be denoted by *AC*, being constant it will at *G* be denoted by *GC*, and this may be resolved into *GK* and *GL*, whereof the latter alone accelerates the motion in the direction *HL*, or perpendicular to *AF*. The motion of *G* then, in this direction, is produced by a force which is always as the body's distance from *L*, and if v be the velocity produced when it reaches that point or comes by the curvilinear motion to *B*, r the radius, and f the measure of the accelerative centripetal force at *A*, $v^2 = fr$, (349.) and

$$f = \frac{v^2}{r}.$$

It will easily appear more particularly by (349, 350.) that if the final or tangential velocity be represented by a line equal to the radius, the velocity generated at *G*, in the direction *GL*, will be represented in magnitude by *GK*; and of the velocity at *A* that which remains at *G*, after the retardation occasioned by the force always represented by *GK*, will be denoted by *GL*; also that the velocities *GK* and *GL*, in the respective directions *GL* and *GM*, are equivalent to a velocity represented in magnitude by the radius, and in direction by the tangent at *G*. As *GK* and *KC* are equivalent to *GC*, the last conclusion will be obvious if we suppose the lines *GK*, *GC* and *GL* to revolve, each through a right angle, and so that *GL* may be in the position *GM*.

498. To illustrate to the student the method of using the fluxionary calculus in such investigations, we shall apply it to this easy case, already analysed and understood.

Let *A x* and *A y* be taken for axes of co-ordinates, *AH* being denoted by x and *HG* by y; and let the central force be denoted by f; then, (339.) dt being supposed constant,

$$(A.)\quad \frac{ddx}{dt^2} = -\frac{f.KG}{GC} = -\frac{fx}{r}.$$

$$(B.)\quad \frac{ddy}{dt^2} = \frac{f.GL}{GC} = \frac{f}{r}(r-y).$$

$$\text{by } (A.)\quad \frac{dx\,ddx}{dt^2} = -\frac{fx\,dx}{r},$$

$$\frac{dx^2}{dt^2} = C - \frac{fx^2}{r} = \frac{f}{r}(r^2-x^2),$$

because $\frac{dx}{dt}$ represents the velocity in the direction parallel to *A x*, and is $=0$, when $x=r$.

$$\text{by } (B.)\quad \frac{dy\,ddy}{dt^2} = \frac{f}{r}(r-y)\,dy,$$

$$\frac{dy^2}{dt^2} = \frac{f}{r}(2ry-y^2) = \frac{fx^2}{r}$$

$$\therefore \frac{dx^2+dy^2}{dt^2} = \frac{ds^2}{dt^2} = \frac{f}{r}(r^2-x^2+x^2) = fr,$$

or $v^2=fr$, as before,

$\frac{ds}{dt}$ being the velocity in the curve.

499. *Cor.* 1. If t be the periodic time, or the time of a complete revolution, $v = \frac{2\pi r}{t}$, and $f = \frac{4\pi^2 r}{t^2}$, (497.)

$$\text{or } f \doteqdot \frac{r}{t^2}.$$

500. *Cor.* 2. If ω be the angular velocity, the velocity in the curve will be $r\omega$, and, by (497.)

$$f = r\omega^2.$$

501. *Cor.* 3. If the periodic times are equal, the centripetal forces are as the radii, (499.)

If the velocities are equal, the forces are reciprocally as the radii, (497.)

If the radii are equal, the forces are in the duplicate ratio of the velocities, (497.)

If the forces are equal, the velocities and the periodic times are in the subduplicate ratio of the radii, (497, 499.) or the angular velocities in the inverse subduplicate ratio of the same, (500.)

If the squares of the periodic times be as the cubes of the radii, the forces are inversely as the squares of the radii or distances, (499.)

502. *Cor.* 4. Since $v^2 = fr$, $v^2 = 2f.\frac{1}{2}r$, which compared with the equation $v^2 = 2\varphi s$, (317.) proves that the velocity in a circular orbit is that which would be acquired by falling from rest through half the radius of that circle, in consequence of the action of the centripetal force continued constant.

503. *Cor.* 5. Since $f = \frac{4\pi^2 r}{t^2}$ (499.) $\frac{f}{g} = \frac{4\pi^2 r}{g t^2}$; or, if f' denote the central force as a multiple of gravity,

$$f' = 1.2268 \frac{r}{t^2},$$

where r is to be expressed in feet and t in seconds.

Thus if a ball of one ounce be whirled round in a sling which, when doubled, is two feet long, so as to make two revolutions in a second, the centrifugal force by which the string is stretched will be 0.8144 ounces: or it is 9.8 &c. times the ball's weight, whatever that may be.

N. B.—The ball is here considered as a point, or as having in every part the same velocity of rotation. Also f in the preceding formulæ is of the nature of an accelerative force, and is stated as having a ratio to g, or as divisible by g, which is the measure of the accelerative power of gravity. But this will occasion no difficulty, when it is recollected that the acceleration is as the weight or pressure when the mass acted on is given. f is an abstract number.

The following conclusion, important in some future parts of the course, may be derived from the above formula.

If the earth be a sphere, whose radius is 3965 miles, revolving in 23^h 56' 4", the portion of a body's natural weight, which, at the equator, is lost, or balanced by the centrifugal force, is $\frac{1}{289}$ part, or $\frac{1}{288}$ of the apparent weight.

504. *Cor.* 6. Since $f = \frac{4\pi^2 r}{t^2}$, $t = 2\pi\sqrt{\frac{r}{f}}$, or the time of an entire revolution is to the time of falling, by the centripetal force remaining constant, through half the radius, as twice the circumference of a circle to the diameter, or it is twice the time of the cycloidal oscillation of a pendulum whose length is equal to the radius.

505. *Cor.* 7. If a ball suspended by a string oscillate through a whole semicircle, the string will be stretched at the lowest point with three times the ball's weight. For, if v be the velocity at the lowest point, $v^2 = 2gr$, and $f = \frac{v^2}{r} = \frac{2gr}{r} = 2g$. So that the centrifugal pressure will be that which would produce in the mass of the ball twice the natural acceleration of gravity; it is therefore twice the ball's weight; and to this must be added its actual weight.

506. *Cor.* 8. If a ball attached to a string revolve in a vertical circle, the string must be able to sustain at least six times the ball's weight.

507. *Cor.* 9. From what has been demonstrated above, we

may find the time in which a conical pendulum will make a revolution.

Let AD (Fig. 190.) represent this pendulum, D being considered as a single particle at the extremity of a line void of weight and inertia AD, and describing a circle $DEBF$, while the point A to which the line is attached remains fixed. Let $AC = a$, $CD = r$, and let the acceleration of gravity be denoted by unity. While D revolves, it is under the influence of three balanced forces, its weight acting parallel to AC, the centrifugal force acting in the direction CD, and the tension of the string in the direction DA. The balanced forces are therefore as the sides of this triangle which denote their directions,

$$1 : \frac{4\pi^2 r}{g t^2} = a : r,$$

$$\text{or } t = 2\pi \sqrt{\frac{a}{g}},$$

and the time of a semicircular oscillation is equal to that in which a simple pendulum whose length is AC would make a complete vibration through a cycloidal arch.

508. *Cor.* 10. It being assumed that when a perfect fluid is in equilibrio the resultant of the pressures applied to any particle in the surface must be perpendicular to that surface, we may determine, by this proposition, the velocity of circulation in different parts of a vortex that the surface of revolution which it assumes may be given; and conversely.

Let AHD (Fig. 191.) be a section of the surface by a plane containing the axis KH, AC a tangent to the section at any point A, whose co-ordinates are x and y, AB and BC parallel to the axes of the co-ordinates; a particle at A is under the influence of three balanced forces, its gravity, its centrifugal tendency, and the reaction of the rest of the fluid, to which BC, AB and AC are perpendicular, respectively. Now

$$AB : BC = dx : dy.$$

$$\text{Therefore } dx : dy = \frac{v^2}{y} : g.$$

If AC be a straight line as in Fig. 192. that is, if the surface be conical, $dx : dy$ is a constant ratio, and, g being constant, $\frac{v^2}{y}$ or $y\omega^2$ is also constant, and the square of the angular velocity in different parts proportional inversely, or the square of the periodic time proportional directly to their distances from the axis.

$$\text{Since } dx : dy = y\omega^2 : g,$$

$$\text{or } dx = \frac{\omega^2}{g} y\,dy,$$

if ω be constant, or the whole fluid circulate in the same time, $y^2 = \frac{2g}{\omega^2}x$, and the surface is a paraboloid.

to it; also let SP meet Dd in H, then $PH = AC$, $=$ the mean distance, $= D$, and supposing $R =$ the actual distance or radius vector, and $P =$ the periodic time, we have

$$SY^2 : SP^2 = PM^2 : PH^2 = PM^2 : AC^2 = BC^2 : CD^2,$$

$$\text{or } SY^2 = \frac{SP^2.BC^2}{CD^2},$$

$$\text{also } PV = \frac{2.CD^2}{AC} \text{ (by Conics.)}$$

$$\therefore \frac{1}{SY^2.PV} \div \frac{AC}{SP^2.BC^2}.$$

But if A be the area described in the unit of time

$$= \frac{\pi AC.BC}{P}$$

$$F \div \frac{V^2}{PV} \div \frac{A^2}{SY^2.PV} \div \frac{A^2.AC}{SP^2.BC^2} \div \frac{AC^3}{P^2.SP^2} \div \frac{D^3}{P^2.R^2}.$$

Now in the same ellipse D and P are both constant, and $F \div \frac{1}{R^2}$; also in the different ellipses, though neither D nor P may be the same, $D^3 \div P^2$, and $\frac{D^3}{P^2}$ is constant by hypoth.

$$\text{therefore } F \text{ is still} \div \frac{1}{R^2},$$

$$\text{and } FR^2 = \frac{mD^3}{R^2} \text{ a constant quantity.}$$

Now FR^2 is the value or measure of the centripetal force at the unit of distance.

Prop. V. If two bodies in free space be connected by mutual attraction, and an impulse be communicated to either of them while they are subjected to no other extrinsic influence, they will both revolve about their common centre of gravity describing similar curves, the radius vector in each of which will describe equal areas in equal times, while the centre will move uniformly forward in a straight line.

The demonstration will be easy after perusing §§ 392, 397, 398, from which also it is obvious that if equal and opposite is denoted by the area BNb; or as the triangles [illegible] the same may be also proportionally represented by [illegible] BN^2 respectively. Also if $ND = \frac{2}{3} AN$, and NC [illegible] D and C will be the centres of these two sets of parallel and we may regard the point C as the fulcrum of a le[illegible] which the perpendicular forces at D and H balance other. If the forces at C and D be called C and D, re[illegible]tively, $Q = D - C$, and Q will be represented, on the [illegible] adopted, by $AN^2 - BN^2$. Hence as $CD = \frac{2}{3}(BN^2 +$ [illegible]

$$AN^2 - BN^2)\ CH = \frac{2}{3} AN^2.AB,$$

$$\text{or } 2\,AB.NO.CH = \frac{2}{3} AN^2.AB,$$

$$\text{and } NO.CH = \frac{1}{3} AN^2;$$

$$\therefore NO.HO = \tfrac{1}{3}(AN^2 - 3\,NO.CO),$$

$$= \tfrac{1}{3}\left(AO^2 + NO(3.\overline{AO - CO} - \overline{AO - NO})\right), = \tfrac{1}{3} AO^2,$$

$$= \tfrac{1}{12} AB^2; \text{ for } 3\,(AO - CO) = 3\,BC = BN, = AO - NO.$$

$$\therefore NO = \frac{AB^2}{12.HO}.$$

Dr. T. Young's *Nat. Phil.* vol. ii. § 320.

2. Let $AEHF$ (Fig. 194.) represent a vertical section of an elastic beam from the neutral line upwards, while it is inflected downwards by an appended weight. Let AD be a very small arch of the exterior curve, conceived as coincident with its circle of curvature at A, or rather at the middle point, and let AC, DC be radii of this circle; draw GB parallel to FA; then BD is the distension of AD, bd of ad, and so on. Now the elastic force of the fibre or physical line ABD will be as the distension of the immediately consecutive particles, or as the whole distension divided by the number of equal intervals, as $\frac{BD}{AD} = \frac{GB}{AC} \div \frac{1}{AC}$ for the same beam or lamina, and as the whole force of the section while BG is constant must, for different degrees of flexure, be in a constant ratio to that of the extreme fibre, and the position of the centre of action is also constant, being always $\frac{2}{3} GB$ from G, the momentum of elasticity will vary as the force of the extreme fibre, or as that of any other whose situation is given, and will be

$\div\frac{1}{R}$ or $=\frac{E}{R}$, R being the radius of curvature and E a quantity constant while the beam is the same, but varying with a change of elasticity, or of the dimensions of the section.

3. If w be a weight equivalent to the force by which a beam is stretched or compressed, and Δl denote the distension or compression for the length l, w will for the same beam be proportional to $\frac{\Delta l}{l}$; or $w=m.\frac{\Delta l}{l}$.

In this case m is the *weight* of what has been named by Dr. Young the *modulus* of elasticity.

The weight w is proportional to that length of the same beam of which it is the weight, and if it be put for this length in the above equation, m will be the *height* of the modulus. The weight of the modulus we shall in what follows denote by M.

If m be the height of the modulus, b the breadth, and d the depth of a transverse section of a rectangular beam, all in feet, and s the specific gravity, water being the standard; then, as 62.406 lbs. is the weight of a cubic foot of water, the weight of the modulus will be $m\,b\,d\,s \times 62.406$, and the weight required to stretch or compress a beam $\frac{1}{n}$ part of its length will be $\frac{62.406.mbds}{n}$ lbs. avoird.

A table of the *moduli* for different substances, with a collateral one of specific gravities, will be found in Dr. Young's 2d vol. p. 509.

4. Referring again to Fig. 193, let us denote by f the contractile force of an unit of surface throughout which the distension is uniform and equal to that of the fibre at A. f may denote the force of that single fibre if we take its thickness for the linear unit. Then $\frac{1}{2}fb.AN=$ the contractile force; and $\frac{1}{2}fb.AN \times \frac{2}{3}AB = \frac{1}{3}fbd.AN =$ the momentum of this force in reference to the centre of repulsion as a fulcrum, $= Q.CH$.

is denoted by the area BN [illegible]
the same may be also proj[illegible]
BN^2 respectively. Also if [illegible]
D and C will be the [illegible]
and we may regard the [illegible] $= \frac{M.AB}{12.Q.HO}$
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[illegible] the axis, and one in the
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[illegible] $K = HL.OH$, each being $=$
$\therefore NO.HO$ [illegible] HK; in other words $Fa =$
[illegible]

$$\frac{M.AB^2}{12.F.a}.$$

[illegible] HF makes with the axis increase
[illegible] right angle, OH becomes continu-
[illegible] becomes OL the length of the
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[illegible] point of the axis is $\div \frac{1}{Fa}$, a being
[illegible] point to the direction of the force,

and as $\frac{E}{R}$ was assumed equal to the momentum of elasticity, or that of the external force which balances it, $E = R.Fa = \frac{M.d^2}{12}$, d being $= AB$ the depth of the beam.

8. Now let AB, (Fig. 195.) be an elastic beam or lamina, of given dimensions and elasticity, slightly bent by a weight suspended from it at B, and let it be required to find a near approximation to the figure which it will assume when in equilibrio.

Let BF be assumed as the axis of abscissæ, and B for the origin. The momentum with which P tends to turn any part of the lamina commencing at B round a point E, and which is opposed by the sum of the momenta of elasticity belonging to the transverse section at that point, is $P.E\,G = Py$. Therefore

$$\frac{E}{R} = Py.$$

The radius of curvature is inversely as y, and consequently diminishes as the ordinate increases, and is least at the vertex A. Supposing ds constant, we have two expressions for the radius of curvature, $\frac{ds.dy}{ddx}$ and $\frac{ds.dx}{ddy}$, of which we take the former with the negative sign, because while the ordinates increase and dx, dy, ds are positive $\frac{dx}{ds}$ diminishes. Hence

$$\frac{Eddx}{ds.dy} = -Py.$$

But, as we suppose the flexure very small, we may consider dy as $= ds$, and therefore as also constant, or

$$\frac{Eddx}{dy} = -Pydy \text{ and}$$

$$\frac{E.dx}{dy} = C - \tfrac{1}{2}Py^2.$$

The horizontal position of the beam at A being a tangent to the curve, $\frac{dx}{dy}$ there vanishes, or $0 = C - \frac{1}{2}Pb^2$, AF being denoted by b, and

$$\frac{E.dx}{dy} = \tfrac{1}{2} P (b^2 - y^2),$$

multiplying by dy and integrating again, we have

$$Ex = \tfrac{1}{2} P (b^2 y - \tfrac{1}{3} y^3)$$

or $x = \frac{1}{6} . \frac{P}{E} (3 b^2 y - y^3)$ which, as y vanishes with x, requires no correction.

9. If a be the abscissa for the point A, we shall have

$$a = \frac{P b^3}{3 E}.$$

Note.— a taken of this value will also be the depth of flexure for an elastic bar whose length $l = 2 b$, loaded at the middle with a weight $= 2 P$; or if P be the load in the middle, which will produce a pressure $= \frac{1}{2} P$ upwards at each end,

$a = \frac{P b^3}{6 E} = \frac{P l^3}{4 M d^2} = \frac{1}{4 m s} . \frac{P l^3}{b d^3}$. See Tredgold's *Carp.* § 80.

10. Had the force P been in the direction of the ordinate or parallel to the axis, and applied as before at the origin of the co-ordinates, we should have had

$$\frac{E}{R} = Px, \text{ or } \frac{E ddx}{dy^2} = -Px, \text{ nearly;}$$

$$\text{Therefore } \frac{E dx . ddx}{dy^2} = -Px.dx;$$

that this, since we regard dy^2 as constant,

$$\frac{E.dx^2}{dy^2} = C - Px^2;$$

$\frac{dx}{dy}$ being $= 0$ when $y = b$ or when $x = a$, the correct integration gives here

$$\frac{E.dx^2}{dy^2} = P (a^2 - x^2),$$

$$\text{or } dy = \frac{\sqrt{E}}{\sqrt{P}} . \frac{dx}{\sqrt{a^2 - x^2}},$$

$$\text{and } y = \frac{\sqrt{E}}{\sqrt{P}} \cdot \text{arc}\left(\text{sin.} = \frac{x}{a}\right),$$

$$\text{or } x = a \,\text{sin.}\left(y \cdot \frac{\sqrt{P}}{\sqrt{E}}\right).$$

The same equations will serve if the beam or lamina have the length $l = 2h$, and be supported by a proper abutment at the end opposite to B, or be inflected by equal and opposite forces P applied to the ends.

11. As when $x = a$, $y = b$, we have, by the last but one of the above equations, $b = \frac{1}{2}\pi \frac{\sqrt{E}}{\sqrt{P}}$, or $2b = \pi \sqrt{\frac{E}{P}}$. Therefore unless l exceed this quantity, that is, unless $\frac{l}{d}$ exceed $0.9069 . \sqrt{\frac{M}{P}}$ there can be no inflexion.

12. If ABD (Fig. 196.) be an elastic beam bent by two equal and opposite longitudinal forces P, P, so that the depth of inflexion is BC, and if we suppose EF and $E'F'$, equal and parallel to each other and equidistant from BC, to represent rigid lines connected with the axis, we may consider the two longitudinal forces as acting upon the segment of the beam EE' by the levers EF and $E'F'$ instead of EA and $E'D$, the momentum applied to every section from E to E' will be the same as before, and the curvature will of course remain unchanged. In this way we may learn the depth of inflexion BG produced by equal and opposite longitudinal forces applied to the extremities of a straight prismatic beam at a given distance from the axis. See Young's Nat. Phil. vol. ii. sect. 324.

Let $BC = d$, $EF = GC = a$, BG or u, or u must be found by the equation $x = d \,\text{sin.}\left(y \cdot \frac{\sqrt{P}}{\sqrt{E}}\right)$,

$$\text{which gives } a = d \,\text{sin}\left(\tfrac{1}{2}l \cdot \frac{\sqrt{P}}{\sqrt{E}}\right).$$

$$\text{and } a = a' \sin.\left(AF.\frac{\sqrt{P}}{\sqrt{E}}\right).$$

$$\text{Hence } a'-a : a = \sin.\left(AC.\frac{\sqrt{P}}{\sqrt{E}}\right) - \sin.\left(AF.\frac{\sqrt{P}}{\sqrt{E}}\right) : \sin.\left(A\right.$$

$$\text{or } a'-a = a\left(\frac{1}{\sin. AF.\frac{\sqrt{P}}{\sqrt{E}}} - 1\right)$$

$$= a\left(\frac{1}{\sin.\frac{1}{2}\pi.\frac{AF}{AC}} - 1\right)$$

$$= a\left(\frac{1}{\cos.\frac{1}{2}\pi.\frac{CF}{AC}} - 1\right).$$

$$\text{But as } \sin.\left(AC.\frac{\sqrt{P}}{\sqrt{E}}\right) = 1,\ AC.\frac{\sqrt{P}}{\sqrt{E}} = \frac{1}{2}\pi,$$

$$\text{and } \frac{1}{2}\pi.\frac{CF}{AC} = CF.\frac{\sqrt{P}}{\sqrt{E}} = CF.\sqrt{\frac{12P}{Md^3}}$$

$$= \frac{2CF}{d}\sqrt{\frac{3P}{M}} = \frac{l}{d}\sqrt{\frac{3P}{M}}. \text{ Therefore}$$

$$BG = a' - a = a\left(\sec.\left(\frac{l}{d}\sqrt{\frac{3P}{M}}\right) - 1\right).$$

13. If an elastic bar as AF, (Fig. 195.) be slightly be its own weight, this may be represented by s, the arc as sured from the origin of the co-ordinates, or approxim by y, and the lever by which it acts is $\frac{1}{2}y$. Hence

$$\frac{E.ddx}{ds.dy} = -y \times \frac{1}{2}y \text{ or very nearly}$$

$$\frac{E.ddx}{dy} = -\frac{1}{2}y^2\,dy,$$

$$\frac{E.dx}{dy} = \frac{1}{6}(b^3 - y^3), \text{ as corrected on the prin}$$

that $\frac{dx}{dy} = 0$ when $y = b$.

By integrating again,

$$24\,E\,x = 4\,b^3 y - y^4.$$

The depth of flexure $a = \frac{b^4}{8E}, = \frac{3}{2}\,\frac{b^4}{M d^2}, = \frac{3}{2}\,b.\frac{b^2}{d^2}.\frac{b}{M}.$ In the numerator of the last factor b must according to the notation adopted denote a weight, viz. that of the beam, if M be the weight of the modulus, or it may be conveniently taken in its primary signification of the length of the beam if M be the height of the modulus.

APPENDIX I.

TO

DYNAMICS.

1. Let a spherical body descend from rest by its gravity considered as a constant force, the measure of whose acceleration *in vacuo* is g, through a medium whose resistance varies as the square of the velocity, and the weight of a cubic unit of which is $\frac{1}{n}$ of that of the same unit of the solid, and suppose it required to determine the relations of the space described, the final velocity and the time of the descent.

It is known, from the principles of Hydrostatics and Pneumatics, that the body loses the $\frac{1}{n}$ part of its weight by immersion in the fluid; and therefore the force left to overcome its inertia, without taking into view the resistance of the fluid, is $\left(1-\frac{1}{n}\right)$ times its natural weight, or the accelerative force is not g, but $\frac{n-1}{n}g$. This for heavy bodies, such as balls of iron or lead, descending through the air, we may without sensible error consider as equal to g ; but when n is not a large number we must in all applications of the formulæ about to be investigated assign to g its corrected value as the measure of relative gravity.

The resistance of the fluid being supposed proportional to v^2, its retarding force as applied to the same body will vary in the same ratio, and may be supposed $= k\,v^2$, k being constant. The effective accelerating force will then be $g—kv^2$. The value of k is deduced from the Newtonian theory on the principle that a cylinder moving through a fluid in the direction of its axis is resisted by a force equal to the weight of a column of the fluid whose base is the end of the cylinder, and whose length is h or $\frac{v^2}{2g}$, the height due to its velocity; and that the resistance to a sphere of equal radius is the half of this. Now the area of the base of the cylinder is $\frac{\pi d^2}{4}$; therefore taking the linear measures in feet, and 1 to denote the weight of a cubic foot of the fluid, $\frac{\pi d^2 h}{4} =$ the resistance to the cylinder, $\frac{\pi d^2 h}{8} =$ the resistance to the sphere, and $\frac{\pi d^2 h}{8 \times \frac{1}{6} \pi d^3 n} =$ the force retarding the same, as a multiple of gravity. By reduction this becomes $\frac{3}{8nd}v^2$, or $\frac{3}{8nd}$ is the theoretical value of k. It is better, however, in practice to take its value as derived from direct experiment. For motions not exceeding 200 feet per second, the resistance of the air may be expressed by $0.00007015\, d^2 v^2$ in avoirdupois ounces, and k by $\frac{0.00746844}{s\,d}$, where v is in feet, d in inches, and s is the specific gravity of the ball, that of water being denoted by 1.

$$d\,s = \frac{v\,d\,v}{g - k\,v^2} \quad (339.)$$

$$ds = -\frac{1}{2k} \times \frac{-2kvdv}{g - kv^2}$$

$$s = \frac{1}{2k}\left(C - L.(g - kv^2)\right),$$

and, supposing $v = 0$ when $s = 0$,

$$C = L.g, \text{ and } s = \frac{1}{2k} L. \frac{g}{g - kv^2}.$$

Hence also $\frac{g}{g - kv^2} = e^{2ks}$, e being the base of the Neperian or hyperbolic logarithms,

$$\text{and } v = \sqrt{\frac{g}{k}} \sqrt{1 - e^{-2ks}}$$

$$dt = \frac{dv}{g - kv^2} \text{ (338.)}$$

$$= \frac{1}{2\sqrt{kg}} \left(\frac{dv\sqrt{k}}{\sqrt{g} + v\sqrt{k}} - \frac{-dv\sqrt{k}}{\sqrt{g} - v\sqrt{k}} \right),$$

which being integrated on the supposition that $v = 0$, when $t = 0$ gives $t = \frac{1}{2\sqrt{kg}}\left(L.(\sqrt{g} + v\sqrt{k}) - L.(\sqrt{g} - v\sqrt{k})\right)$

$$\text{or } t = \frac{1}{2\sqrt{kg}} L. \frac{\sqrt{g} + v\sqrt{k}}{\sqrt{g} - v\sqrt{k}}.$$

Were V such a value of the velocity that $k V^2 = g$, the acceleration would cease, and the descent become uniform. But as $v\sqrt{k}$ approaches to the value g the number $\frac{\sqrt{g} + v\sqrt{k}}{\sqrt{g} - v\sqrt{k}}$ and its logarithm increase indefinitely; which shows that in the case considered, or when the body descends from rest, V is a limiting value of the velocity, never actually attained. This also appears very clearly from the value of $v = \sqrt{\frac{g}{k}} \sqrt{- \frac{1}{e^{2ks}}} = \sqrt{\frac{g}{k}} \sqrt{1 - \frac{1}{e^{\frac{3s}{4nd}}}}$; unless n be very great, the last term of the second factor soon becomes very small, and $v = \sqrt{\frac{g}{k}}$ nearly. This limiting value of v is called the terminal velocity.

2. Let the sphere be projected vertically upwards with th velocity u, and let it be proposed to determine the height t which it will rise, and the time that will elapse before the velocity be extinguished.

Here the motion is retarded both by gravity and by th resistance of the air, or by the force $g+kv^2$, and

$$ds = \frac{-v\,dv}{g+kv^2},$$

$$\therefore s = \frac{1}{2k}\left(C - L.(g+kv^2),\right)$$

and as, when $s=0$, $v=u$,

$$C = L.(g+ku^2), \text{ and } s = \frac{1}{2k} L. \frac{g+ku^2}{g+kv^2}.$$

$$dt = \frac{-dv}{g+kv^2} = -\frac{1}{\sqrt{kg}} \cdot \frac{dv\sqrt{kg}}{(\sqrt{g})^2 - (v\sqrt{k})^2}$$

$$t = \frac{1}{\sqrt{kg}} . \left(C - \text{arc}\left(\text{tang.} = \frac{v\sqrt{k}}{\sqrt{g}}\right)\right),$$

and as when $t=0$, $v=u$,

$$C \text{ is manifestly} = \text{arc}\left(\text{tang.} = \frac{u\sqrt{k}}{\sqrt{g}}\right).$$

When the initial velocity is great, as in the case of military projectiles, a much more accurate solution is obtained by supposing the resistance to be expressed by the sum of two terms, of which the one varies as v and the other as v^2. In this case, the retarding force, when the body is projected upwards may be represented by $g+hv+kv^2$. The student will find this case elegantly and perspicuously treated, as usual, by Mr. Whewell, *Dyn.* p. 180, or he may consult Dr. Hutton's *Tracts*, vol. iii. p. 233, near to which he will find the data furnished by experiment for the determination of h and k, (§ 33.)

3. Let us now investigate Borda's formula for the motion of a ball projected obliquely through the air, as was promised in § 491. In doing this we shall, for the student's easier reference, adopt the author's notation: but, for the greater variety,

we shall investigate a little differently the general formula, and then supply the investigation of the particular and approximative one.

Let R be the retarding force directly opposed to the ball, and let the horizontal and vertical co-ordinates be x and y respectively, $d\,s$ the element of the curve, and $d\,x$ constant. Then (339.)

$$(A.) \qquad -\frac{d\,x.d\,d\,t}{d\,t^3} = -R\frac{d\,x}{d\,s}.$$

$$(B.) \qquad \frac{d\,d\,y}{d\,t^2} - \frac{d\,y.d\,d\,t}{d\,t^3} = -g - R\,\frac{d\,y}{d\,s}.$$

Multiply the two members of Eq. (*A.*) by $\frac{d\,y}{d\,x}$ and subtract the resulting equation from (*B.*), then

$$(C.) \qquad d\,d\,y = -g\,d\,t^2.$$

$$(D.) \qquad d^3\,y = -2\,g.d\,t.d\,d\,t, \text{ by Eq. } (C.)$$

Substitute for $d\,d\,t$ and then for $d\,t^4$ their values from (*A.*) and (*C.*)

$$(E.) \qquad -2\,R.d\,d\,y^2 = g\,.\,d\,s.d^3\,y.$$

Now suppose $R \div v^2$ or $2\,a\,R = v^2$, if $2\,a\,g = V^2$, V will be what is called the terminal velocity and a is the height due to it.

Since $2\,a\,R = v^2 = \frac{d\,s^2}{d\,t^2}$, or $2\,R = \frac{d\,s^2}{a\,d\,t^2}$, by substitution in Eq. (*E.*) from this Eq. and from (*C.*) we find,

$$(F.) \qquad d\,s.d\,d\,y = a\,d^3\,y.$$

Let v be the velocity of the projection, h the height due to it, e the angle of elevation, n its secant, c the base of the hyp. log. At the beginning of the motion $d\,s = n\,d\,x$, and for a tolerable approximation we may suppose this to hold throughout for elevations below 45°, or

$$(G.) \qquad n\,d\,x\,.\,d\,d\,y = a\,d^3y,$$

$$\frac{d^3\,y}{d^2\,y} = \frac{n\,d\,x}{a},$$

$$L.d^2y = \frac{n\,x}{a} + L.B\,d\,x^2,$$

$$L.d^2\,y = L.c^{\frac{n\,x}{a}} + L.B\,d\,x^2,$$

H.) $$\text{or } d^2 y = B\,c^{\frac{nx}{a}}\,dx^2, \text{ that is,}$$

$$d^2 y = \frac{a}{n}\,B\,c^{\frac{nx}{a}}\,\frac{n}{a} dx.dx.$$

(K.) $$dy = \frac{a}{n}\,B\,c^{\frac{nx}{a}}\,dx + D.dx, \text{ that is,}$$

$$dy = \frac{a^2}{n^2}\,B\,c^{\frac{nx}{a}}\,\frac{n}{a}\,dx + D.dx,$$

$$y = \frac{a^2}{n^2} B\,c^{\frac{nx}{a}} + Dx + C,$$

and y being $= 0$ when $x = 0$,

$$y = \frac{a^2}{n^2} B\,c^{\frac{nx}{a}} + Dx - \frac{a^2}{n^2}\,B, \text{ or}$$

$$y = Dx + \frac{a^2}{n^2} B\left(c^{\frac{nx}{a}} - 1\right)$$

By (*H.*) $$Bc^{\frac{nx}{a}} = \frac{d^2 y}{dx^2} = -g\frac{dt^2}{dx^2} \text{by } (C.)$$

but $\frac{dx^2}{dt^2} = v^2 \cos.^2 e$ when $x = 0$; hence

$$B = -\frac{g}{v^2 \cos.^2 e} = -\frac{1}{2\,h \cos.^2 e}.$$

By Eq. (*K.*) $\frac{dx}{dy} = \frac{a}{n}\,Bc^{\frac{nx}{a}} + D$; but $\frac{dy}{dx} = \text{tang.}\,e$ at the commencement of the motion, or when $x = 0$, and consequently $c^{\frac{nx}{a}} = 1$. Therefore $D = \text{tang.}\,e + \frac{a}{2nh\cos.^2 e}$, and, as $n = \text{sec.}\ e = \frac{1}{\cos.\ e}$, we have finally

(*L.*) $$y = x\left(\text{tang.}\,e + \frac{a}{2\,h \cos.\ e}\right) - \frac{a^2}{2h}\left(c^{\frac{x}{a \cos.\,e}} - 1\right).$$

This is the equation employed by Borda to find the elevation for the greatest range.

For this purpose suppose $y = 0$ without x vanishing: then,

(*M.*) $$x = \frac{\frac{a^2}{2\,h}\left(c^{\frac{x}{a}\,\text{sec.}\,e} - 1\right)}{\text{tang.}\ e + \frac{a}{2h}\ \text{sec.}\ e}.$$

Fig. 65.

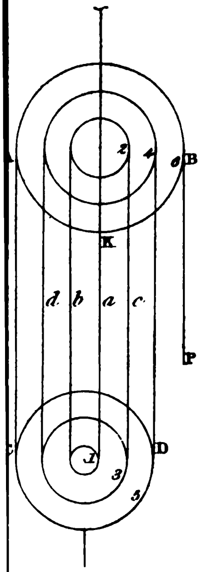


Fig. 66.

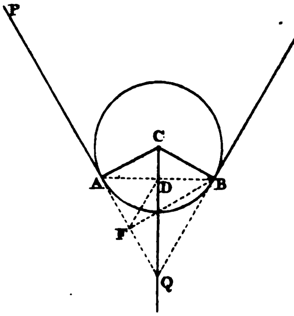


Fig. 67.

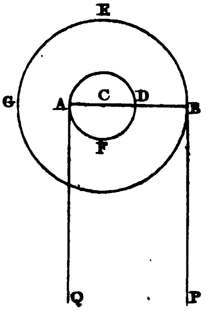


Fig. 69.

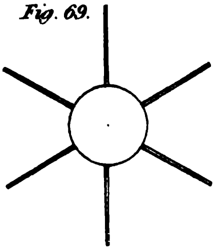


Fig. 72.

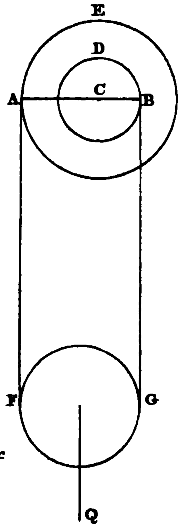
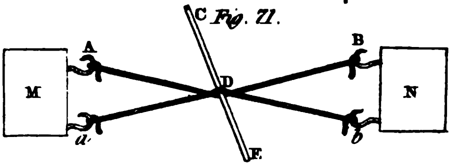


Fig. 71.



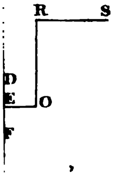


Fig. 74.

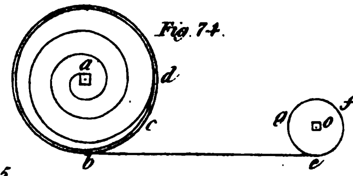


Fig. 75.

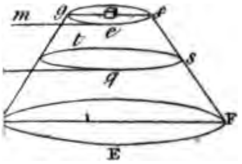


Fig. 78.

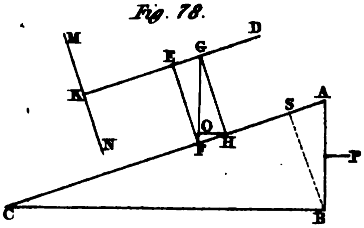


Fig. 77.

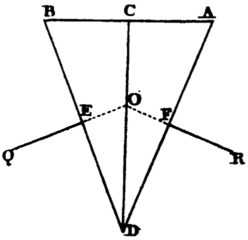
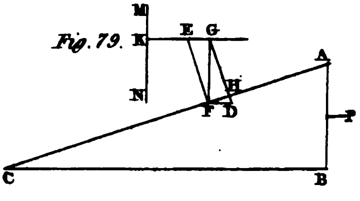


Fig. 79.



McNab & Smellie Sc.

$$\text{and } s' - s = \frac{1}{2k} L. \frac{n + P'}{n + P}.$$

Thus find the lengths of the curve in small portions between the limits of depression θ and θ' whose tangents are p and p' corresponding to P and P', and let the limits be always taken with so near an approximation that to the extent of $s' - s$ the ratios $\Delta s : \Delta x$ and $\Delta s : \Delta y$ may be sensibly constant: then

$$x' - x = (s' - s) \cos. \theta;$$
$$y' - y = (s' - s) \sin. \theta;$$

or, for the greater accuracy, instead of $\cos. \theta$ we may take $\frac{1}{2} (\cos. \theta + \cos. \theta') = \cos. \frac{1}{2} (\theta' + \theta) . \cos. \frac{1}{2} (\theta' - \theta)$; and, in like manner, for $\sin. \theta$, $\sin. \frac{1}{2} (\theta' + \theta) . \cos. \frac{1}{2} (\theta' - \theta)$.

Let v be the velocity of the projectile when the angle of depression is θ; $v^2 = \left(\frac{dx}{dt}\right)^2 \sec.^2 \theta$ or by $(C.) = \frac{A^2}{e^{2ks}} \cdot (1 + p^2)$

$$= \frac{A^2 n (1 + p^2)}{n + P} = \frac{g}{2k} \times \frac{1 + p^2}{n + P}.$$

$$\text{In like manner } v'^2 = \frac{g}{2k} \times \frac{1 + p'^2}{n + P'},$$

and if u be the mean of these two values $t' - t = \frac{s' - s}{u}$ very nearly.

The above investigation is easily accommodated to the ascending branch by conceiving the motion at the vertex to be reversed, and the retarding force of the air converted into an accelerating force. All the terms of Eq. (*A.*) and (*B.*) will then be positive; Eq. (*C.*) will become $\frac{dt}{dx} = \frac{e^{-ks}}{A}$, and finally,

$$dP = \frac{g e^{-2ks}}{A^2} \text{ or } -dP = \frac{g}{2kA^2} e^{-2ks} \times -2kds,$$

$$n - P = \frac{g e^{-2ks}}{2kA^2},$$

$$\frac{n - P}{n} = e^{-2ks},$$

$$-s = \frac{1}{2k} L. \frac{n - P}{n},$$

$$\text{or } s = \frac{1}{2k} L. \frac{n}{n - P}$$

To find when v or v^2 is a *minimum*, put the fluxion o $\frac{1 + p^2}{n + P} = 0$, or $2p\,dp\,(n + P) = dP\,(1 + p^2)$; that is, $2p(n+P) = (1 + p^2)\sqrt{1 + p^2}$. As p is very small near to the vertex, if we substitute for P its value, before given, and reduce, we shall find $P = p + \frac{1}{6}p^3$ very nearly, and $2pn =$ $1 - \frac{1}{2}p^2 + \frac{1}{24}p^4$,

$$\text{or } p = \frac{1}{2n} - \frac{p^2}{4n} = \frac{1}{2n} - \frac{1}{16n^3} \text{ very nearly;}$$

and if n be considerable we may suppose $p = \frac{1}{2n}$.

The expression for the radius of curvature when dx is assumed constant is $\frac{ds^3}{-dx\,ddy} = r$. Now as $dy = p\,dx$, $ddy = dp\,dx$, and $ds^2 = dx^2\,(1 + p^2)$, we shall have

$$r = \frac{dx}{dp}(1 + p^2)^{\frac{3}{2}}, = \frac{A^2}{g\,e^{2ks}}\,(1 + p^2)^{\frac{3}{2}}, = \frac{1}{2k}\frac{(1 + p^2)^{\frac{3}{2}}}{n + P}.$$

To find the minimum value of this radius we have, rejecting dp and $(1 + p^2)^{\frac{1}{2}}$, as common to both sides of the equation, $3p\,(n + P) = (1 + p^2)^{\frac{3}{2}}$.

$$\text{whence } 3pn = 1 - \tfrac{3}{2}p^2 \text{ nearly,}$$

or $p = \frac{1}{3n} - \frac{1}{18n^3}$, that is, when n is considerable, $p = \frac{1}{3n}$.

The point corresponding to this value of p is therefor nearer to the vertex than that where the velocity is least.

The number n in the application of the preceding formul may be found from the velocity of projection and the angle o elevation being given. Let the tangent of elevation be q and let Q be what P becomes when $p = q$, $v^2 = \frac{g}{2k} \cdot \frac{1 + q^2}{n - Q}$

We take Q with the negative sign because q refers to a poi in the ascending branch of the curve.

$$\text{Hence } n = \frac{g}{2\,k} \times \frac{1 + q^2}{v^2} + Q,$$

and $\frac{1 + q^2}{v^2} = \frac{1}{v^2 \cos.^2 e} = \frac{1}{2\,g\,h \cos.^2 e}$ if h be the *impetus.*

If φ be the angle whose tangent is p, the quantity P will be most easily found by this equation.

$$2\,P = \text{tang. } \varphi \text{ sec. } \varphi + L \,.\, \text{tang. } (45 + \tfrac{1}{2}\,\varphi),$$

which may be thus derived. In the value of P formerly expressed, $p = \text{tang. } \varphi$, $\sqrt{1 + p^2} = \text{sec. } \varphi$, and $p + \sqrt{1 + p^2} = \text{tang. } \varphi + \text{sec } \varphi =$

$$\frac{\sin.\ \varphi}{\cos.\ \varphi} + \frac{1}{\cos.\ \varphi} = \frac{1 + \sin.\ \varphi}{\cos.\ \varphi} =$$

$$\frac{\sin.\ 90 + \sin.\ \varphi}{\sin.(90 + \varphi)} = \frac{2 \sin.\ (45 + \frac{1}{2}\,\varphi) \cos.\ (45 - \frac{1}{2}\,\varphi)}{2 \sin.\ (45 + \frac{1}{2}\,\varphi) \cos.\ (45 + \frac{1}{2}\,\varphi)} =$$

$$\text{tang. } (45 + \tfrac{1}{2}\,\varphi), \text{ because } \frac{\cos.\ (45 - \frac{1}{2}\,\varphi)}{\sin.\ (45 + \frac{1}{2}\,\varphi)} = 1.$$

If the common logarithmic tangent be employed in this calculation, it must be taken to the radius 1, and multiplied by the hyperbolic logarithm of 10.

APPENDIX II.

ON

DYNAMICS.

Containing some Elementary Propositions in Physical Astronomy, to assist the Student in following Newton's Analysis of Kepler's Laws, and his consequent Induction of the Law of Universal Gravitation.

Prop. I. LET *PZ* (Fig. 197.) be a curve described by a body projected under the influence of a centripetal force always directed to the point *S*, *PY* the tangent to the curve and to *PVH* its circle of curvature at *P*, draw *SY* perpendicular to the tangent, let *SP* meet the circle of curvature in *V*, and, *PH* being the diameter of that circle drawn through *P*, join *HV*; then if *F* be the centripetal force, and *V* the tangential velocity at *P*,

$$F \doteqdot \frac{V^2}{PV}.$$

For, *F* being the force in the direction *PS*, $\frac{F.SY}{SP} = \frac{F.PV}{PH} =$

$\frac{F.PV}{2R} = \frac{V^2}{R}$ (497.) where $R =$ the radius of curvature. Hence
$F = \frac{2V^2}{PV} \div \frac{V^2}{PV}$.

Prop. II. Let the force be a function of the distance, and and therefore always the same when the distance is the same; if, in describing different trajectories, the velocity be the same at a given distance from the centre, it shall be the same at any other given distance.

Let $SP = \varrho$, the fluxion of the curve $= + ds$, $\angle SPY = \theta$, v the tangential velocity at P. Then $v\,dv = f \cos.\theta.ds$ $= f.ds \cos.\theta = -f d\varrho$; $\therefore$ the increment of the square of the velocity is a function of the distance, and the same between the same limits of distance, independent of the species of the curve.

Prop. III. If the data be as in propositions *I.* and *II.* the radius vector, a straight line joining the projectile, considered as a point, and the centre of force, describes equal areas in any equal times.

Let $SY = p$, $SC = c$, $CP = r$, and $SP = \varrho$ as before.

$$\varrho^2 = c^2 + r^2 + 2rp,$$

$$\text{and } \varrho.d\varrho = r.dp,$$

r being regarded as constant, becanse the body may be considered as momentarily describing the circle of curvature.

But $\frac{fp}{\varrho} = \frac{v^2}{r}$ (*Prop. I.*) or $f = \frac{v^2\varrho}{pr}$, and $v\,dv = -f d\varrho$ $= -\frac{v^2\varrho.d\varrho}{pr} = -\frac{v^2r.dp}{pr} = -\frac{v^2dp}{p}$, and $p\,dv + v\,dp = 0$,

$$\text{or } pv = \text{constant},$$

and pv is the fluxion of the area.

Otherwise thus.

See Maclaurin's *Fluxions*, § 467.

Let PF (Fig. 198.) represent the fluxion of the curve at P, let S be a fixed point, and PK the measure of the deflective accelerative force at P, whose direction is supposed to

APPENDIX

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Prop. I. LET *PZ* (Fig. 197.) be
projected under the influence
directed to the point *S*, *PY* th ellipse about anoth
PVH its circle of curvature at deflective force is inve
the tangent, let *SP* meet th and if among any num
PH being the diameter of thares of the periodic time
HV; then if *F* be the centres, the force that connec
velocity at *P*, ntre is the same at a giv
whole extent of the syst

the diameter conjugate
For, *F* being the force in the major axis, and
angent at *P*, and *SY* perpen

in H, then $PH = AC = $ the

osing $R =$ the actual distance

odic time. we have

$AC = BC^2 : CD^2$.

unit of time

$$C \div \frac{AC^3}{P^2 . SP^2} \div \frac{D^3}{P^2 . R^2}$$

and P are both constant, and

nt ellipses, though neither D nor

P, and $\frac{D^3}{P^2}$ is constant by hypoth.

F is still $\div \frac{1}{R^2}$,

$= \frac{m D^3}{R^2}$ a constant quantity.

value or measure of the centripetal force at

nce.

If two bodies in free space be connected by mu- n. and an impulse be communicated to either of they are subjected to no other extrinsic influence, both revolve about their common centre of gravity similar curves, the radius vector in each of which ribe equal areas in equal times, while the centre will niformly forward in a straight line.

demonstration will be easy after perusing §§ 392, 397, from which also it is obvious that if equal and opposite be PO at the instant for which the fluxions are supposing the construction to be obvious from th will be the fluxion of AC,

$$\text{or } PG = d.AC = -d.PD,$$
$$PE = d.CP,$$
$$CG = d.\text{area } ACP,$$
$$DE = d.\text{area } APDS,$$
$$PH = d.PG,$$
$$PL = d.PE, \text{ and negative as here}$$
$$2 \text{ area } ASP = ACP + APDS, \text{ and}$$
$$d.(ACP+APDS) = CG+DE.$$

But $d.CG = PC\,d.PG + PG\,d.PC = CH + GE,$

And $d.DE = PD d.PE + PE\,d.PD = DL + GE,$

Hence the second fluxion of the area or $d.($ $-DL = CK - DK = 0$, when PK coinci with PS, and consequently the first fluxion o constant.

Cor. When PK is on the side of SP, body is moving, $CK - DK$ is positive, an area is increasing; when it is on the ot is negative, and the description of th whence the converse of the proposition areas described in any equal times ab or the fluxion of the area be const directed to that centre."

Prop. IV. If a body describe an el situated in one of the foci, the deflec the square of the distance; and i bodies thus revolving the squares o the cubes of the mean distances, th of them with the common centre tance, and varies through the who cording to the same law.

Let DCd (Fig. 199.) be the which passes through P, Aa the minor axis, PY the tangent

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[illegible] Juan

[illegible]

[illegible]

[illegible] de [illegible]