EUCLID'S

ELEMENTS OF GEOMETRY

BOOKS I—IV, VI AND XI.

Euclides G. B. Smith

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ELEMENTS OF GEOMETRY

BOOKS I—IV, VI AND XI

EDITED FOR THE USE OF SCHOOLS

BY

CHARLES SMITH, M.A.,

MASTER OF SIDNEY SUSSEX COLLEGE, CAMBRIDGE

AND

SOPHIE BRYANT, D.Sc.,

HEAD MISTRESS OF THE NORTH LONDON COLLEGIATE SCHOOL FOR GIRLS.

PREFACE.

IN the following School Edition of Euclid's Elements of Geometry the subject is treated in Euclid's order and manner, but with no special regard for the exact words of Simson's translation, which appears to have been scrupulously followed by many English editors.

Further explanation has been given whenever this appeared to be necessary or desirable, and we have not hesitated occasionally to give proofs different from those of Euclid. In Book I. we have, for example, discarded altogether Euclid's incomplete proof of Prop. 24, which would, we imagine, be now accepted as satisfactory by few examiners; and we have made Prop. 22 logically complete by shewing that, under the given conditions, the two circles will necessarily intersect. It may be interesting to remark that, with the additional axiom which must be explicitly or implicitly assumed (and which is indeed implied though not expressed in I. 1), I. 22 may be taken immediately after I. 3. No alternative proofs of I. 5 and I. 6 have been given, because the experience both of teachers and examiners appears to shew that the average beginner finds Euclid's proofs easier to understand—at any rate easier to reproduce—than the alternatives which have been suggested.

The changes we have made in Book II. are more considerable than in Book I. We have substituted for Euclid's proofs of Props. 9 and 10 the proof in which the equality to be established is shewn directly from the diagram. The advantage of this for educational purposes need not be dwelt upon; but as this proof is somewhat long, alternative proofs of these propositions are given, as also of II. 8, which do not require the construction of the different squares and rectangles. These alternative proofs being logically sound and strictly geometrical, may be given in examinations, except when a proof by means of a diagram is definitely asked for. Proofs of II. 12 and II. 13 are also given in which the equality that has to be established is proved at once from the diagram.

These proofs might be given immediately after I. 47, as in Lardner's Euclid, where, so far as we can discover, these interesting and instructive extensions of I. 47 first appear.

The second book of Euclid is often found to be difficult and distasteful to beginners; we hope, however, that the changes we have ventured to introduce, and the explanations and additions we have given, will make the study of this important section of Geometry more attractive and more valuable.

In Books III. and IV., although the propositions have been taken in Euclid's order, Euclid's methods have not always been followed.

"The contact of circles," says De Morgan, "is in a state of confusion in Euclid: there is a positive assumption," the more objectionable because it is implied but not definitely expressed, "that a circle which touches another internally is entirely within and that a circle which touches another externally is entirely without the other circle." This judgment of De Morgan, from which few would dissent, is a sufficient justification of a departure from Euclid's method of treating the contact of circles.

Propositions 26, 27, 28 and 29 are proved by superposition. This is the more instructive method; and it has the additional advantage that as each of the propositions can be proved quite independently of the others, it is not necessary to remember the order in which they happen to have been placed.

Of Euclid's two alternative proofs of III. 9 and III. 10 we have in both cases given the first proof; whereas Simson, with what seems to us to be singular want of judgment, gave only Euclid's second and inferior proof.

Book IV. only contains the solutions of the problems (1) to inscribe a regular polygon in and to circumscribe a regular polygon about a given circle, when this can be done by a geometrical construction, and (2) to inscribe a circle in and circumscribe a circle about a regular polygon. The plan of the whole book is however somewhat concealed by the fact that in the case of the triangle it is not necessary that the figure should be regular. We have slightly altered Euclid's constructions so as to bring into greater prominence the fact that the solution of the first problem depends only upon the

possibility of dividing four right angles into the requisite
number of equal parts.

The Theory of Proportion as treated by Euclid is very
difficult, and the Vth Book of the Elements is now rarely
read by students, or even by teachers; we think, however,
that a more geometrical treatment of the subject presents no
great difficulty.   After making a few simple deductions from
Euclid's definition of proportion, we have proved all the
fundamental theorems required in Plane Geometry, and which
therefore are all that we are concerned with, directly from the
theorem that rectangles of equal altitude are to one another
as their bases, and its converse.   These proofs have the great
advantage of being independent of one another, and we
believe that they will be found both easy and interesting;
and we are confirmed in this belief by the opinions of all who
have read the proof sheets.   The English words *alternately*
and *inversely* are used instead of '*alternando*' and '*invertendo*,'
and the unnecessary and misleading terms '*componendo*' and
'*dividendo*,' and some others, are omitted.

A few additional propositions have been given in the text;
and in a note following each book we have grouped together
the other important theorems with which the student should
make himself familiar on a second reading.   The Appendix
to Book VI. is especially important.   Most of the examples
which are given under the different propositions are very
simple, and could be solved by an average well-taught student
quite early in his study of Geometry.

The greater part of the examples have been selected from
Mathematical Journals and Examination papers; but many
are original.

Abbreviations and symbols for words have been used at
an early stage, because the different steps of the reasoning are
more easily followed when contractions are employed; and,
when students are themselves required to write out proofs, it
is of the utmost importance that each of the different steps
of the proof should be made to stand out as clearly as possible
by the use of abbreviations.

De Morgan remarked a generation ago that 'many
teachers think it meritorious to insist upon their pupils
remembering the *very words* of Simson'; and we are afraid
that even at the present day there are some teachers who

still hold that view.   It should on the contrary be considered distinctly meritorious to depart from the exact words of text-book or teacher, provided that a sound proof is given.

References to previous propositions &c. are now given in the margins of most editions of Euclid ; and it may be well to point out that these references are given for the *convenience of the learner*, to shew him where to look for some knowledge he may have forgotten.   Examiners stand in no need of such help, and by no means wish to impose on the memory of students the heavy and useless burden of learning how to give accurately numbered references to the different propositions of Euclid.   On the other hand, in the reports of examiners there is not unfrequently a complaint that candidates who possess this knowledge, or more often perhaps who only think they possess it, sometimes give a mere reference by number to a preceding proposition when some concise reason for a statement is really required.   Euclid himself never referred to previous propositions by name or number ; such words as ' it has been demonstrated ' being used, without further specification.

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CHARLES SMITH.
SOPHIE BRYANT.

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EUCLID.

Very little is known about the life of Euclid, the author of the *Elements*, except that he was born about 330 B.C. and died about 275 B.C., and that he was the first and one of the most famous mathematicians of the University of Alexandria, where he taught for many years.

Euclid's *Elements* consists of thirteen books.   The first four and the sixth are on plane geometry ; the fifth is on the theory of proportion, and applies to magnitude in general ; the seventh, eighth, and ninth are on arithmetic ; the tenth on incommensurable magnitudes ; the eleventh and twelfth on the elements of solid geometry ; and the thirteenth on the regular solids and miscellaneous propositions in plane and solid geometry.

EUCLID'S ELEMENTS OF GEOMETRY.

BOOK I.

DEFINITIONS.

1. *A* **point** *has position but no magnitude.*

2. *A* **line** *has position and length but no breadth or thickness.*

The extremities of a line are points, and the intersection of two lines is a point.

3. *A* **surface** *has position, length and breadth, but no thickness.*

Clearer ideas of the nature of a surface, a line and a point will be obtained by reversing the order in which they are considered.

Any object, a cricket ball for example, takes up a certain amount of space, and the *surface* of the ball is the boundary between the space occupied by the ball and the surrounding air. The surface is not a thin layer of leather next to the air any more than it is a thin layer of air next to the leather; it is the boundary between the two, and *it has no thickness whatever.* It will be noticed that the terms length and breadth are not very appropriate to the surface, or any part of the surface, of the ball.

Again, the cricket ball might have a black patch upon it, and the boundary of this patch is a *line.* The line is not a thin strip of the black surface next to the red any more than it is a thin strip of the red surface next to the black; it is the boundary between the two and *it has no breadth whatever.* There may be two lines on a surface which intersect one another, and if this be the case a *point* will be common to the two lines, and this point will have no magnitude whatever.

Thus, instead of Euclid's definitions, the following may be given.

A **surface** *is the boundary of a portion of space. A surface has no thickness whatever.*

A **line** *is the boundary of a portion of a surface. A line has no breadth or thickness but length only.*

We say that a line has been drawn on paper when a *very narrow* portion of the surface has been discoloured, but this discoloured portion of the paper would not be visible if it were entirely without width. It must be always remembered that geometrical lines have absolutely no width, although this is by no means true of the strokes made on paper to represent lines, and to enable us to reason about them. A point is represented on paper by the intersection of two lines, or by a small dot; and however small the mark put on paper to represent a point may be, it must have some magnitude or it would not be visible; the geometrical point which is represented by the mark has, however, no magnitude whatever.

A point is denoted by a single letter of the alphabet placed near it.

A line is denoted by two or more letters of the alphabet which denote points on the line.

4. *A* **straight line** *is one which lies evenly between its extreme points.*

A clearer idea of the nature of a straight line is obtained by considering how to test whether a line drawn on paper is or is not straight. If we apply to the line the straight edge of a flat ruler, and if the edge can be made to coincide with the line throughout its whole length, then the line must be straight, but not otherwise. If, however, we do not know that the edge itself is straight, but if it is found that the line will everywhere coincide with the edge, and will continue to do so when the edge is moved along the line, and if moreover the line will still coincide with the edge when the ruler is turned about the edge, then both the edge and the line must be straight lines.

5. *A* **plane surface,** *or a* **plane**, *is a surface such that the straight line joining any two points on the surface will lie entirely on the surface.*

Thus a plane surface is one on which a straight edge will lie throughout its entire length in any position whatever.

There are surfaces which are not plane on which a straight line will lie in *certain positions*, for example, a straight line may be made to coincide in certain positions with the surface of a circular cylinder—the surface of an ordinary pencil is made to be as nearly as possible of this shape—but a straight line will not lie along the cylinder in *all positions*.

The above definition of a plane is the one now usually given; Euclid, however, gave the following definition:

A plane surface is a surface which lies evenly between straight lines on it.

6. *A* **plane angle** *is the inclination to one another of two straight lines which meet together but are not in the same straight line.*

The point where the two straight lines meet is called the **vertex** of the angle, and the lines themselves are sometimes called the **arms** of the angle.

When only two straight lines meet at a point A, the angle formed by the lines may be called the angle A.

When, however, more than two straight lines meet in a point A, so that there is more than one angle whose vertex is A, each angle must be described by three letters, the outside letters denoting points one on each of the lines bounding the angle and the middle letter denoting the vertex of the angle. Thus the angle between the straight lines AB and AC is called the angle BAC, and the angle between the straight lines AC and AD is called the angle CAD, also the angle between the lines AB
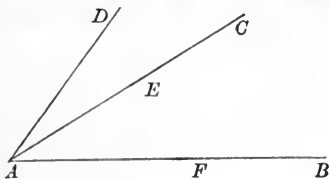

and AD is called the angle BAD. It should be noticed that the angle BAD is equal to the *sum* of the angles BAC and CAD, and the angle BAC is equal to the difference of the angles BAD and CAD.

It must be carefully noticed that the angle between two straight lines does not depend on the lengths of the bounding lines. Thus the same angle may be called BAC, FAE, EAB or CAF.

Two angles which have a common vertex and are on opposite sides of a common bounding line, are called *adjacent angles*. Thus the angles BAC and DAC are adjacent angles.

7. *When one straight line standing on another straight line makes the adjacent angles equal, each of these angles is a* **right angle,** *and the straight line which stands on the other is called a* **perpendicular** *to it.*

Thus, if the adjacent angles ACD and BCD are equal, and ACB is a straight line, each is a *right angle*.
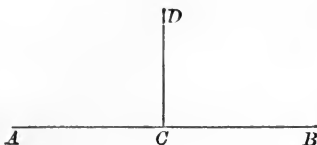
It will be proved later on that when two straight lines cross one another, so as to form four angles at their point of intersection, then if one of these angles is a right angle they will all four be right angles. Hence each of the lines is perpendicular to the other, and the lines are said to be *perpendicular*, or *at right angles*, to one another.

8. *An angle greater than a right angle is called an* **obtuse angle**.

9. *An angle less than a right angle is called an* **acute angle**.
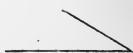
10. *Any portion of a plane surface bounded by one or more lines is called a* **plane figure**, *and if the figure is entirely bounded by straight lines it is called a* **plane rectilineal figure**. [See note on page 13.]

The straight lines which form the boundary of a rectilineal figure are called its **sides**.
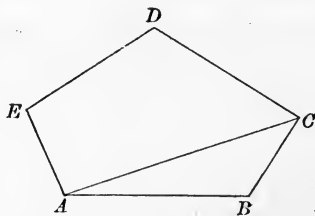
The sum of the lengths of the straight or curved lines which form the boundary of a plane figure is called its **perimeter**.

If the sides of a rectilineal figure are all equal it is said to be **equilateral**.

If the angles of a rectilineal figure are all equal it is said to be **equiangular**.

A figure is described by putting letters at different points along its boundaries, one letter being placed at each angular point, if there are any.

Thus the whole figure *ABCDE* is divided by the straight line *AC* into the two parts *ABC* and *ACDE*.


The perimeter of the figure *ABCDE* is the sum of the lengths of its five sides *AB*, *BC*, *CD*, *DE* and *EA*.

11. *A* **circle** *is a plane figure bounded by one line, called the* **circumference**, *and is such that all straight lines drawn from a certain point within it, called the* **centre**, *to the circumference are equal to one another.*

A straight line drawn from the centre of a circle to the circumference is called a **radius** *of the circle.*

Thus, by definition, all radii of the same circle are equal to one another.

A straight line drawn through the centre of a circle and terminated both ways by the circumference is called a **diameter** *of the circle.*
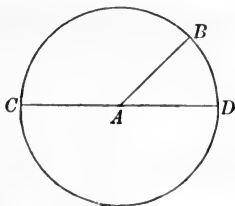
Although, by the above definition, a circle is the *figure* enclosed by its circumference, the circumference itself is often called the circle when no ambiguity would arise. For example, two *circles* are said to cut one another when the circumferences intersect; the two circles, strictly speaking, have in this case a certain area in common.

12. *A plane figure bounded by three straight lines is called a* **triangle**.
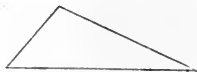
13. *Any plane figure bounded by four straight lines is called a* **quadrilateral**.

A straight line joining two opposite angular points of a quadrilateral is called a **diagonal** of the quadrilateral.
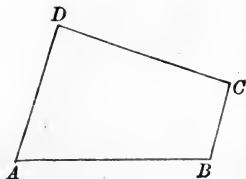
14. *A plane figure bounded by more than four straight lines is called a* **polygon**.

It will be seen at once that a three-sided figure (triangle) has *three* angles, and that a quadrilateral has *four* angles; and after a little consideration it will be seen that **any rectilineal figure has as many angles as sides.**

15. *An* **equilateral triangle** *is a triangle whose three sides are equal.*

16. *An* **isosceles triangle** *is a triangle which has two sides equal.*

17. *A* **scalene triangle** *is a triangle which has three unequal sides.*

18. *A* **right angled triangle** *is a triangle one of whose angles is a right angle.*
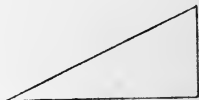
In a right angled triangle the side which is opposite to the right angle is called the **hypotenuse.**

19. *An* **obtuse angled triangle** *is a triangle one of whose angles is an obtuse angle.*
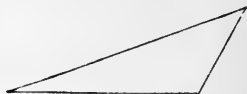
20. *An* **acute angled triangle** *is a triangle all of whose angles are acute.*

It will be proved later on that every triangle has at least *two* acute angles, so that no triangle can have two right angles or two obtuse angles or one right angle and one obtuse angle.

21. **Parallel straight lines** *are straight lines in a plane which do not meet however far they are produced in either direction.*

22. *A* **parallelogram** *is a quadrilateral whose opposite sides are parallel.*

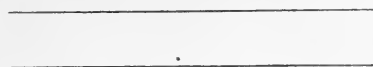**23.** *A* **rectangle** *is a quadrilateral whose opposite sides are parallel and one of whose angles is a right angle.*

24. *A* **rhombus** *is a quadrilateral all of whose sides are equal.*

25. *A* **square** *is a quadrilateral all whose sides are equal and one of whose angles is a right angle.*

Thus a square is a right angled rhombus.

POSTULATES.

In order to draw the diagrams required for the study of geometry it is necessary to assume the possibility of performing certain simple operations; and these operations which we take for granted that we can perform should be as few in number and as simple in character as possible.

A geometrical construction which we take for granted that we can perform is called a **postulate**.

The postulates assumed by Euclid are three in number :

Post. i. *A straight line may be drawn from any one point to any other point.*

Post. ii. *A terminated straight line may be produced to any length in that straight line.*

Post. iii. *A circle may be described with any given point as centre and with any given line from that point as radius.*

These postulates require the use of a *straight ruler* and a *pair of compasses.* It must however be borne in mind that the ruler is not supposed to be divided or graduated in any way, so that it cannot be used to draw a line of any proposed length ; nor are we supposed, in the third postulate, to be able to draw a circle with any given point for centre and with a radius equal to a given straight line but which is not drawn from the given centre.

Axioms. *

It is necessary to assume the truth of certain elementary facts of geometry ; and these elementary facts which we assume to be true, and on which the whole science of geometry is to be built up, should be as few in number and as simple and obvious as possible.

A geometrical truth which is taken for granted without proof is called an **axiom**, *or a* **Postulate**.

The following are Euclid's axioms :

Ax. i. *Things which are equal to the same thing are equal to one another.*

Ax. ii. *If equals be added to equals, the wholes are equal.*

Ax. iii. *If equals be taken from equals, the remainders are equal.*

Ax. iv. *If equals be added to unequals, the wholes are unequal.*

Ax. v. *If equals be taken from unequals, the remainders are unequal.*

Ax. vi. *Things which are the doubles of the same thing, or of equal things, are equal.*

Ax. vii. *Things which are halves of the same thing, or of equal things, are equal.*

Ax. viii. *The whole is greater than a part.*

Ax. ix. *Any figure, or diagram, can be transferred from one position to another without change of shape or size.*
Magnitudes which can be made to coincide are equal.

This axiom, the first part of which was not definitely enunciated by Euclid, gives the meaning and supplies a test of the equality of geometrical magnitudes.

To test, for example, the equality of two angles.
Suppose one of the angles to be taken up and put down again without change but so that its vertex coincides with the vertex of the other, and one of its arms falls on an arm of the other, and so that the remaining arms are both on the same side of those which were made to coincide ; then the remaining arms will also coincide if the angles are equal, but not otherwise.

The placing one geometrical magnitude upon another is called **superposition**, and the one magnitude is said to be **applied** to the other.

Ax. x. *Two straight lines cannot enclose a space.*

The axiom with reference to straight lines may be stated in the following forms :

If two straight lines have two points in common they will coincide throughout.

If two straight lines coincide for any portion of their lengths, they will coincide throughout.

Ax. xi. *All right angles are equal.*

It is not necessary to assume that all right angles are equal; for this can, and therefore should, be *proved*.

Ax. xii. *If a straight line meet two other straight lines, so as to make the two interior angles on one side of it together less than two right angles, the other straight lines will meet if continually produced on the side on which are the angles which are less than two right angles.*

The consideration of this axiom should be deferred to a later stage when the need for it arises.

The first eight axioms are true of magnitudes of all kinds, and not merely of geometrical magnitudes. They were called '*common notions*' by Euclid, and are now sometimes called **General Axioms**, the others being called **Geometrical Axioms**.

With reference to the first seven axioms it should be noticed that when restricted to plane geometrical magnitudes, and it is only geometrical magnitudes with which we are concerned, they can all be *proved* by superposition ; this will be seen by considering any definite kinds of magnitude, straight lines or angles, for example.

Axiom viii is only an indirect definition of a *part*.

It should also be noticed that Axioms i to vii are not all *independent* of each other, for example Axiom vi is only a particular case of Axiom ii.

Moreover the axioms are not arranged in proper order, for Ax. ix, which gives the meaning and supplies a test of the equality of magnitudes, should precede all the others.

In addition to the axioms definitely enunciated by Euclid there are some others which he tacitly assumed : these will be pointed out when it is necessary to use them. For example, see Propositions 1, 2, 12 and 22 of Book I.

Other axioms of the same type as (i) to (vii) frequently occur, for example the axiom 'If one magnitude be greater than a second, and the second be greater than a third, then will the first be greater than the third.' The truth of one of these axioms will be at once obvious in any particular case, and a formal enunciation of such an axiom is no more necessary in Geometry than in Arithmetic.

Each of the different propositions which are considered by Euclid proposes to effect some geometrical construction or to prove some geometrical truth.

A **problem** is a geometrical construction which is to be performed.

To *solve* a problem is to shew how the required construction can be effected by means of the postulates and other constructions which we have previously shewn how to perform.

A **theorem** is the statement of some geometrical truth.

The **hypothesis** of a theorem is that which is supposed to be true, and the **conclusion** is that which, it is asserted, necessarily follows from the hypothesis.

To *prove* a theorem is to employ the fundamental axioms and other theorems which have been already proved, to shew that the conclusion necessarily follows from the hypothesis.

The following symbols and abbreviations will be used. These should be adopted in all written work, not only to save time in writing, but more especially because the different steps of the reasoning stand out more clearly when contractions are used, and consequently the argument is more easily understood and reproduced.

The first eight propositions will, however, be printed in full.

$\therefore$ for *therefore* or *hence*,	$\odot$ for *circle*,
$\because$ ... *because* or *since*,	$\perp^r$ ... *perpendicular*,
$>$ ... *is greater than*,	rt. $\angle$ ... *right angle*,
$<$ ... *is less than*,	$\triangle$ ... *triangle*,
$=$ ... *is equal to*,	$\parallel$ ... *is parallel to*,
$\angle$ ... *angle*,	$\parallel^m$ ... *parallelogram*,
$\angle^s$ ... *angles*,	par^m. ... *parallelogram*,
st. ... *straight*,	post. ... *postulate*,
const. ... *construction*,	def. ... *definition*,
hyp. ... *hypothesis*,	ax. ... *axiom*,
pt. ... *point*,	rect. ... *rectangle*,
alt. ... *alternate*,	perp. ... *perpendicular*,
sq. ... *square*,	adj. ... *adjacent*.

Note. In the best Greek manuscripts which have come down to us, Axioms I to IX are called ' *Common Notions*,' and x, xi and xii are given as Postulates vi, iv, v respectively.

[See Peyrard's Edition of Euclid's Elements in Greek, Latin and French.] We have not, however, thought it necessary to alter the arrangement as given by Simson.

The first English edition of Euclid's Elements was that of Billingsley published in 1570. This edition contains a translation of all the thirteen books with many notes and additions.

PROPOSITION I. PROBLEM.

To describe an equilateral triangle upon a given finite straight line.

Let *AB* be the given straight line.

It is required to describe an equilateral triangle upon AB.
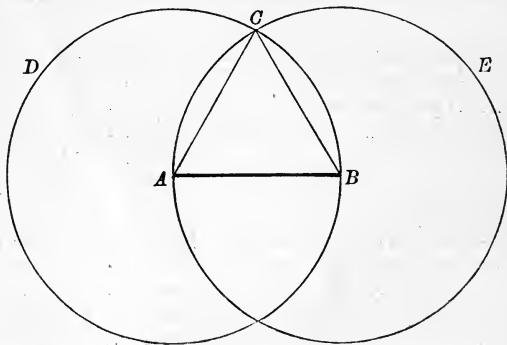
Const. With *A* as centre and *AB* as radius describe the circle *BCD*. [*Post.* iii.

With *B* as centre and *BA* as radius describe the circle *ACE*. [*Post.* iii.

These circles must intersect.

Let *C* be a point of intersection.

Draw the straight lines *CA*, *CB*; [*Post.* i.

then *ABC* is an equilateral triangle constructed as required.

Proof. Because *A* is the centre of the circle *BCD*,

$$AC \text{ is equal to } AB. \qquad [\textit{Def. } 11.$$

Also because *B* is the centre of the circle *ACE*,

$$BC \text{ is equal to } BA. \qquad [\textit{Def. } 11.$$

And, since *AC* and *BC* are both equal to *AB*, the three lines *AC*, *BC* and *AB* are all equal, so that *ABC is an equilateral triangle*, and it is *described on the given straight line AB*.

It is now usual to describe each proposition as a Problem or Theorem, as the case may be ; the distinction between problems and theorems was not, however, so marked by Euclid.

The statement of what has to be done, or of what has to be proved, is called the **enunciation** of the proposition.

The enunciation is repeated with special reference to a particular diagram, and this is called the **particular enunciation**. Then follows the **construction**, namely, the directions for drawing such lines and circles as may be required to solve the problem or to enable us to prove the theorem ; and lastly the **proof** that the construction given does really effect what is required in the case of a problem, or that the theorem enunciated is really true.

A **finite** straight line is a straight line with fixed ends; when the ends of a straight line are not fixed it is called an **indefinite** straight line, or a straight line of *unlimited length*.

The two circles (that is, the two circumferences) will cut in *two* points, one on each side of the line *AB*; thus two equilateral triangles can be drawn, one on each side of the given line.

It should be noticed that no proof that the circles will intersect is offered. Thus Euclid tacitly assumes as an axiom that *if two circles be such that the centre of either is on the circumference of the other, then the two circumferences will intersect.*

Ex. 1. Produce a given finite line *AB* both ways, and find points *C*, *D* on *AB* produced such that *BC* and *CD* may each be equal to *AB*; find also points *E*, *F* on *BA* produced such that *AE* and *EF* may each be equal to *BA*.

Ex. 2. Produce the given finite straight line *AB* and find on the line so produced a point *C* such that *AC* is equal to four times *AB*.

Ex. 3. Describe on a given straight line *AB* an isosceles triangle each of whose equal sides is the double of *AB*.

Ex. 4. On a given straight line describe an isosceles triangle each of whose equal sides is four times the given line.

Ex. 5. If *F* be the other point of intersection of the circles in the figure* to Prop. I, and the lines *AF*, *BF* be drawn, shew that the figure *ACBF* is a rhombus.

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* Sometimes any collection of lines and points is called a *figure*. The student will have no difficulty in seeing when the word figure is used only for a picture or diagram, and when it has the meaning given on page 4.

PROPOSITION II. PROBLEM.

From a given point to draw a straight line equal to a given straight line.

Let A be the given point, and BC the given straight line.

It is required to draw from the point A a straight line equal in length to BC.
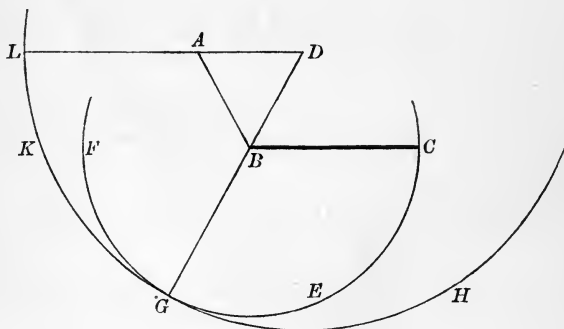
Const. From A to B draw the straight line AB. [*Post.* i.

On AB describe the equilateral triangle ABD. [I. 1.

With centre B and radius BC describe the circle CEF.
[*Post.* iii.

Produce DB until it meets the circle CEF in the point G.
[*Post.* ii.

With centre D and radius DG describe the circle HGK.
[*Post.* iii.

Produce DA until it meets the circle HGK in the point L.
[*Post.* ii.

Then the line AL is drawn as required.

Proof. Because D is the centre of the circle $HGKL$,

$$DL \text{ is equal to } DG. \qquad [Def. 11.$$

But the lines DA and DB are equal since they are sides of the equilateral triangle ABD.

Hence, taking DA from DL and DB from DG, the remainder AL must be equal to the remainder BG.

Again, B is the centre of the circle CEG, therefore BC is equal to BG.

But BG has been proved to be equal to AL.

Hence *AL is equal to the given line BC, and it is drawn from the given point A.*

It should be noticed that it is taken for granted that any straight line through the centre of a circle will, if produced far enough, cut the circle. The most general case of this assumption is

Any straight line drawn through a point within a circle (or any other closed figure) will, if produced far enough both ways, cut the circle in two points.

For if a straight line be drawn in any direction from a point *within* a circle, a point on the line must sooner or later be *outside* the circle, since the circle is limited in all directions, and it is at once seen to be impossible to join a point within a circle (or any other *closed* figure) to a point outside the circle by any line straight or curved which does not cross the boundary.

The point A may be joined to *either* end of the given line, and the equilateral triangle can be drawn on *either* side of the joining line. Moreover DB in the diagram will cut the circle CEF in another point, X suppose; and if a circle be described with D as centre and DX as radius, and AD be produced *in the direction from A to D* to cut this circle in Y, then AY will be equal to BC, as is easily seen. Thus by varying Euclid's construction we can in general draw *eight* straight lines from A equal to BC.

Ex. 1. Draw a diagram for the case in which A is on the line BC.

Ex. 2. On a given base describe an isosceles triangle each of whose equal sides is equal to a given straight line.
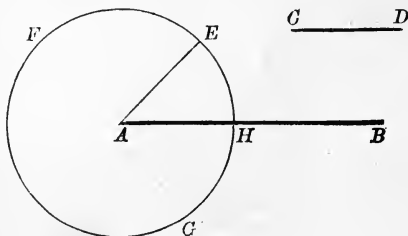
Ex. 3. From a given point draw a straight line equal in length to twice a given straight line.

PROPOSITION III. PROBLEM.

*From the greater of two given straight lines to cut off a part
equal to the less.*

Let *AB* and *CD* be the given straight lines, of which *AB*
is the greater.

It is required to cut off from AB a part equal to CD.

Const. From *A* draw the straight line *AE* equal to *CD*.
[I. 2.

With centre *A* and radius *AE* describe the circle *EFG*.
[*Post.* iii.

Then this circle must cut *AB*, since *AB* is greater than *AE*.

Let the circle cut *AB* in the point *H*, then *AH* is the
part required.

Proof. Because *A* is the centre of the circle *EFGH*,

AH is equal to *AE*.   [*Def.* 11.

But *AE* was made equal to *CD*.

Therefore *AH* is equal to *CD*, and it is cut off from *AB*.

Propositions II. and III. are only necessary because the compasses allowed to be used by Postulate iii. are supposed to close of themselves when they are taken up from the paper.

N.B. The references to the definitions, axioms, postulates and preceding propositions which are put in the margin are given for the convenience of the learner who may not be sufficiently familiar with what has already been proved; these numbered references must not, however, be considered as forming part of the proof. References were not in fact given by Euclid.

Sometimes the letters Q.E.D. are placed at the end of a Theorem, and the letters Q.E.F. at the end of a Problem. These are the initial letters of *quod erat demonstrandum* (which was to be proved) and *quod erat faciendum* (which was to be done), respectively.

Ex. 1. Produce the smaller of two given straight lines so that the whole line so produced may be equal to the greater.

Ex. 2. Draw a straight line equal to the sum of two given straight lines.

Ex. 3. Draw a straight line equal to the difference of two given straight lines.

Ex. 4. Draw a straight line equal to the sum of three given straight lines.

Sometimes one particular side of a triangle is distinguished by being called the **base**: the opposite angular point is then called the **vertex**.

If two sides of a triangle are equal, the third side of the triangle is generally called the base of the isosceles triangle.

One theorem is said to be a **Corollary** of a second when the truth of the first becomes obvious as soon as the truth of the second is established.

PROPOSITION IV. Theorem.

If, in two triangles, two sides and the included angle of the one be respectively equal to two sides and the included angle of the other; then will the base and the remaining angles of the one be respectively equal to the base and remaining angles of the other, those angles being equal which are opposite to equal sides; also the two triangles will be equal.

Let *ABC* and *DEF* be two triangles in which the sides *AB*, *AC* are equal respectively to the sides *DE* and *DF*, and the included angle *BAC* is equal to the included angle *EDF*; then *it is required to prove that BC is equal to EF, that the angles ABC, ACB are equal respectively to DEF, DFE, and that the triangles are equal.*
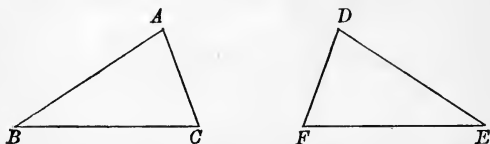
Proof. Suppose the triangle *ABC* to be taken up and applied to the triangle *DEF*, so that the point *A* may be on the point *D*, and the line *AB* on the line *DE*, the triangle *ABC* being turned over if necessary so that it may be on the same side of *DE* as the triangle *DFE*.

Then, because the point *A* coincides with the point *D*, and the line *AB* lies along *DE*, and *AB* is equal to *DE*,

therefore the point *B* must coincide with the point *E*.

Again, because *AB* coincides with *DE*, and the angle *BAC* is equal to the angle *EDF*,

the line *AC* must fall on the line *DF*.

And, since the point *A* coincides with the point *D*, and the line *AC* lies along *DF*, and *AC* is equal to *DF*,

therefore the point *C* must coincide with the point *F*.

And, since it has been proved that the point B coincides with the point E, and the point C coincides with the point F,

the line CB must coincide with the line FE,   [*Ax.* x.

and CB must be equal to FE.

Moreover, the triangle ABC altogether coincides with the triangle DEF, and is equal to it; also the angle ABC coincides and is equal to the angle DEF, and the angle ACB coincides and is equal to the angle DFE.

Figures which can be made to coincide by superposition are said to be **congruent**. Congruent figures are **equal in all respects**; for since they can be made to coincide by superposition, each side of one figure must be equal to the corresponding side of the other, and each angle of one figure must be equal to the corresponding angle of the other, the two figures must also be equal in area.

It should be noticed that the third form of Axiom x. is assumed in the above proof, for B would not necessarily coincide with E if AB could fall along DE for a certain distance and then diverge from it.

It may or may not be necessary to turn one triangle over in order to make it coincide with the other. The figures ABC and DEF are so drawn that it would in this case be necessary to turn ABC over.

The following theorems can also be proved by superposition.

Ex. 1. In the triangles ABC and DEF, the side BC is equal to the side EF, and the angles ABC, ACB are equal respectively to the angles DEF, DFE. Shew that the triangles are equal in all respects.

Ex. 2. In the quadrilaterals $ABCD$ and $EFGH$, the sides AB, BC, CD are equal respectively to the sides EF, FG, GH, and the angles ABC, BCD are equal respectively to the angles EFG, FGH. Shew that the quadrilaterals are equal in all respects.

PROPOSITION V. Theorem.

The angles at the base of an isosceles triangle are equal to each other; and if the equal sides be produced, the angles on the other side of the base will also be equal to each other.

Let ABC be an isosceles triangle in which the sides AB and AC are equal, and let AB and AC be produced to X and Y respectively.

It is required to prove that the angles ABC and ACB are equal, and also that the angles CBX and BCY are equal.
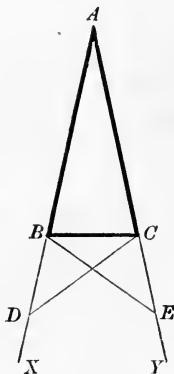
Const.   In BX take any point D, and from AY cut off a part AE equal to AD.   [I. 3.

Join DC and BE.

Proof.   In the triangles DAC and EAB,

$\qquad\qquad DA$ is equal to EA,   [*Const.*

$\qquad\qquad CA$ is equal to BA,   [*hyp.*

and the included angle DAC is the same as the included angle EAB.

Therefore the triangles *DAC* and *EAB* are equal in all respects, so that   [I. 4.

$$DC \text{ is equal to } EB,$$

the angle *DCA* is equal to the angle *EBA*,

and the angle *ADC* is equal to the angle *AEB*.

Again, because the whole *AD* is equal to the whole *AE*,   [*Const.*

of which the parts *AB* and *AC* are equal,   [*hyp.*

therefore the remainder *BD* is equal to the remainder *CE*.

Then, in the triangles *BDC* and *CEB*, we have proved that

$$BD \text{ is equal to } CE,$$

$$DC \text{ is equal to } EB,$$

and that the included angle *BDC* is equal to the included angle *CED*.

Hence the triangles *BDC* and *CEB* are equal in all respects, so that   [I. 4.

the angle *DBC* is equal to the angle *ECB*, and the angle *DCB* is equal to the angle *EBC*.

And, since it has been proved that

the angle *DCA* is equal to the angle *EBA*, and the angle *DCB* is equal to the angle *EBC*,

therefore, taking equals from equals, the remaining angle *BCA* is equal to the remaining angle *CBA*.

Thus we have proved, as was required, that *the angles ABC and ACB are equal* and also that *the angles DBC and ECB are equal.*

Corollary.   *An equilateral triangle is also equiangular.*

Ex. 1.   Prove that the angles *DBE* and *DCE* are equal.

Ex. 2.   Prove that, if *DE* be drawn, the triangles *DBE* and *DCE* are equal in all respects.

Ex. 3.   If two isosceles triangles are on the same base and on the same side of it, one triangle must be entirely within the other.

Ex. 4.   Prove that the opposite angles of a rhombus are equal to one another.

Ex. 5.   Two isosceles triangles ACB, ADB are on the same base AB; shew that the triangles ACD, BCD are equal in all respects.

Ex. 6.   Points D, E, F are taken on the sides AB, BC, CA respectively of an equilateral triangle ABC, so that AD, BE and CF are all equal; shew that the triangles FAD, DBE, ECF are equal in all respects, and that the triangle DEF is equilateral.

PROPOSITION VI.   Theorem.

If two angles of a triangle be equal to each other, the sides which are opposite to the equal angles will be equal to each other.

Let ABC be a triangle in which the angles BCA and CBA are equal to each other; *it is required to prove that the sides AB and AC, which are opposite to the equal angles are equal to each other.*
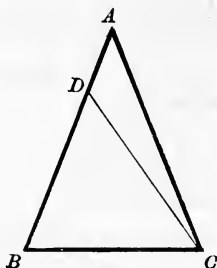
Proof.   For, if the two sides are not equal, one of them must be greater than the other.

Suppose then that BA is greater than CA, and cut off from BA a part BD equal to CA.   [I. 3.

Join DC.   [*Post.* i.

Then, in the triangles DBC and ACB

$\qquad\qquad DB$ is equal to AC [*Const.*

$\qquad\qquad BC$ is equal to CB

and the included angle DBC is equal to the included angle ACB.   [*hyp.*

Hence the triangles DBC and ACB are equal in all respects; but this is impossible, for one triangle is manifestly smaller than the other.

It is therefore impossible that one of the sides AB, AC should be greater than the other. *The two sides AB, AC must therefore be equal to each other.*

Corollary. *An equiangular triangle is also equilateral.*

Converse Theorems. Two theorems are said to be *converse theorems* when the hypothesis of each is the conclusion of the other.

Propositions V. and VI. are converse theorems. This will be seen at once when they are enunciated as under :

If two sides of a triangle are equal, the opposite angles will be equal.

If two angles of a triangle are equal, the opposite sides will be equal.

It must be carefully noticed that the converse of a true theorem is by no means always true.

For example, it is true that *all men are mortal*, but it is not true that *all mortals are men*. Again, it is true that *a right-angled triangle has two acute angles*, but it is not always true that *a triangle with two acute angles is right-angled*.

To prove a converse theorem in geometry, an **indirect method** is generally adopted ; that is, the theorem is proved to be true by shewing that it is impossible for it to be false. In Prop. VI., for example, the sides are proved to be equal by shewing that it is impossible for them to be unequal.

The proof of a theorem by shewing that the supposition that the theorem is not true leads to an absurdity, is sometimes called a **Reductio ad absurdum.**

Ex. 1. If, in the figure to Prop. V., the lines BE and CD intersect in F ; prove that BF is equal to CF, and that the triangles BFD and CFE are equal in all respects.

Ex. 2. If, in the figure to Prop. V., the lines BE and CD intersect in F ; prove that the straight line AF will bisect the angle BAC and also the angle BFC.

Ex. 3. Shew that either diagonal of a rhombus bisects each of the angles through which it passes.

PROPOSITION VII. Theorem.

On the same base and on the same side of it, there cannot be two triangles which have their sides terminated in one extremity of the base equal to one another and also those terminated in the other extremity of the base equal to one another.

If possible let *ABC* and *ABD* be two triangles on the same base *AB* and on the same side of it such that *CA* is equal to *DA* and *also CB* equal to *DB*.

When two triangles are on the same base and on the same side of it (I.) the vertex of each triangle is without the other triangle, or (II.) the vertex of one triangle is within the other, or else (III.) the vertex of one triangle is on a side of the other.

Thus there are three and only three cases to be considered.
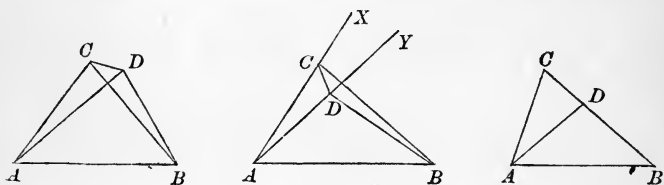
Case I. When the vertex of each triangle is without the other. Join *CD*.

Then, because *AC* is equal to *AD*,   [*hyp.*
the angle *ACD* is equal to the angle *ADC*.   [I. 5.

But the angle *ADC* is less than the angle *BDC* ;

therefore the angle *ACD* is less than the angle *BDC*.

Therefore the angle *BCD*, which is a part of the angle *ACD*, is less than the angle *BDC*.

Again, because *BD* is equal to *BC*,   [*hyp.*
the angle *BCD* is equal to the angle *BDC*.   [I. 5

Thus the angle *BCD* is both equal to and less than the angle *BDC*, which is impossible.

Therefore, in this case, *BC* cannot be equal to *BD* at the same time that *AC* is equal to *AD*.

Case II. When the vertex of the triangle ADB is within the triangle ACB.

Join CD, and produce AC and AD to X and Y respectively.

Then, because AC is equal to AD,   [*hyp.*

the angle XCD is equal to the angle YDC.   [I. 5.

But the angle BDC is greater than the angle YDC ;

therefore the angle BDC is greater than the angle XCD ;

and the angle XCD is greater than the angle BCD ;

therefore the angle BDC is greater than the angle BCD.

Again, because BC is equal to BD,   [*hyp.*

the angle BDC is equal to the angle BCD.   [I. 5.

Thus the angle BDC is both equal to and greater than the angle BCD, which is impossible.

Therefore, in this case, BC cannot be equal to BD at the same time that AC is equal to AD.

Case III. needs no demonstration, for it is obvious that BD and BC are unequal.

It has therefore been proved that *in no case can CA be equal to DA and at the same time CB equal to DB.*

Ex. There cannot be two equilateral triangles on the same base and on the same side of it.

In the Greek only one case is given. Both are given by Billingsley.

PROPOSITION VIII. Theorem.

If in two triangles the three sides of the one are respectively equal to the three sides of the other, the triangles will be equal in all respects, those angles being equal which are opposite to equal sides.
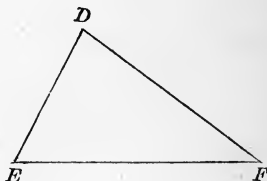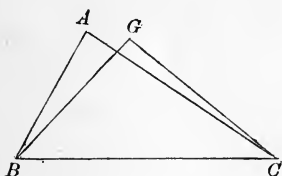
Let *ABC* and *DEF* be two triangles such that *BC* is equal to *EF*, *CA* equal to *FD* and *AB* equal to *DE*; *it is required to prove that the triangles are equal in all respects.*

Proof. For, if the triangle *DEF* be applied to the triangle *ABC*, so that the point *E* may be on *B* and the straight line *EF* on *BC*, the two triangles being on the same side of *BC*; then will the point *F* fall on the point *C*, since *EF* is equal to *BC*.

And when *EF* coincides with *BC* the sides *ED* and *FD* must coincide respectively with the sides *BA* and *CA*; for, if they took up any other position such as *BG* and *CG*, there would be two triangles *ABC*, *GBC* on the same base and on the same side of it having *GB* equal to *AB* and also *GC* equal to *AC*, which we know is impossible.   [I. 7.

Hence, when *E* coincides with *B* and *F* with *C*, *ED* will coincide with *BA* and *FD* with *CA*; thus *the two triangles will altogether coincide, and are therefore equal in all respects.*

ALTERNATIVE PROOF.   (Philo's Proof.)

The following proof, which has the great advantage of being independent of I. 7, is often given.

Let ABC and DEF be two triangles having the sides AB, BC, CA of the one respectively equal to the sides DE, EF, FD of the other; then will the triangles be equal in all respects.

Let the triangle DEF be applied to the triangle ABC so that EF coincides with BC, with the sides terminating in B equal as also those terminating in C, and so that the triangles DEF and ABC may be on *opposite* sides of the line BC.

Let GBC represent the triangle so applied, so that G corresponds to D.

Join AG. Then AG may (i) cut BC or (ii) fall outside BC or (iii) may pass through B or C.
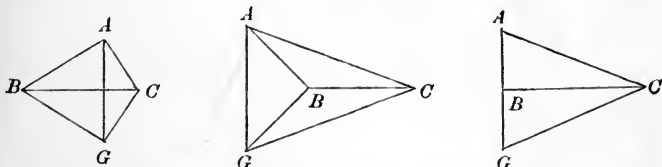
Then, since BA is equal to BG, the angles BAG and BGA are equal. Also, since CA is equal to CG, the angles CAG and CGA are equal.

Hence, taking in the first case the sum and in the second case the difference of these equals, it follows that the angle BAC is equal to the angle BGC.

Thus, in every case, we have the two sides BA, AC and the included angle BAC equal respectively to the two sides BG, GC and the included angle BGC; whence it follows, from I. 4, that the triangles BAC and BGC are equal in all respects, and therefore the triangles BAC and DEF are equal in all respects.

Ex. 1.   Shew that equilateral triangles on equal bases are equal in all respects.

Ex. 2.   Shew that a diagonal of a rhombus divides it into two triangles equal in all respects.

Ex. 3.   In the quadrilateral $ABCD$ the sides AB and AD are equal and the sides CB and CD are equal; shew that the angles ABC and ADC are equal and that AC bisects each of the angles BAD and BCD.

Ex. 4.   If two isosceles triangles be on the same base, the line joining their vertices, produced if necessary, will bisect the vertical angles of the isosceles triangles and will also bisect the common base.

PROPOSITION IX. PROBLEM.

To bisect a given angle, that is, to divide it into two equal angles.

Let *BAC* be the given angle, *it is required to divide it into two equal angles.*
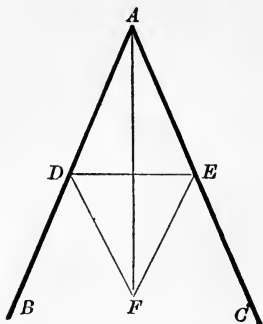
Const. In *AB* take any point *D*, and from *AC* cut off a part *AE* equal to *AD.* Join *DE.*

On the side of *DE* remote from *A* describe the equilateral triangle *DFE*, and join *AF.*

Then the straight line AF will bisect the angle BAC.

Proof. In the triangles *ADF* and *AEF*,

$$\because \begin{cases} AD = AE, & \text{[Const.} \\ AF = AF, \\ \text{and } DF = EF; & \text{[Const.} \end{cases}$$

∴ the △ˢ *ADF* and *AEF* are equal in all respects, and in particular [I. 8.

$$\angle DAF = \angle EAF.$$

Therefore *the angle BAC is bisected by AF.*

The equilateral triangle *DFE* is constructed on the side of *DE remote* from *A*, for the construction would otherwise fail if *DAE* happened to be itself an equilateral triangle. If, however, *DAE* is not an equilateral triangle, *DFE* can be on either side of *DE.*

PROPOSITION X. Problem.

To bisect a given finite straight line.

Let *AB* be the given straight line. *It is required to bisect AB.*
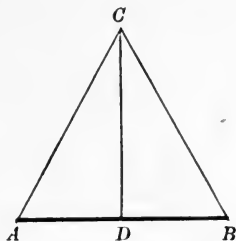
Const. Upon *AB* describe the equilateral triangle *ABC*. [I. 1.

Bisect the angle *ACB* by the st. line *CD* which meets *AB* in the point *D*. [I. 9.

Then AB will be bisected in D.

Proof. In the triangles *ACD*, *BCD*,

$$\because \ \begin{cases} AC = BC, & [Const. \\ CD = CD, & \\ \text{and } \angle ACD = \angle BCD; & [Const. \end{cases}$$

∴ the △ˢ *ACD* and *BCD* are equal in all respects, and in particular [I. 4.

$$AD = DB.$$

Hence *AB* is bisected in *D*.

Ex. 1. Shew that, in the figure to Prop. IX., *AF* bisects the angle *DFE*.

Ex. 2. Divide a given angle into four equal parts, and also into eight equal parts.

Ex. 3. Shew that, in the figure to Prop. X., *CD* is perpendicular to *AB*.

Ex. 4. Divide a straight line into four equal parts, also into eight equal parts.

PROPOSITION XI. Problem.

To draw a straight line at right angles to a given straight line, from a given point upon it.

Let AB be the given straight line and C a given point on it.

It is required to draw from C a straight line at right angles to AB.
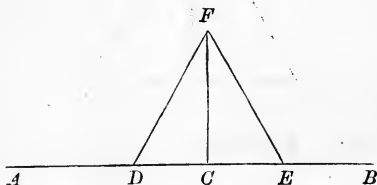
Const. In CA take any point D, and from CB cut off a part CE equal to CD.

On DE describe the equilateral triangle DEF, and join FC.

Then FC will be the line required.

Proof. In the triangles DCF, ECF,

$$\therefore \left\{ \begin{array}{ll} DC = CE, & [Const. \\ CF = CF, & \\ \text{and } DF = EF; & [Const. \end{array} \right.$$

$\therefore$ the triangles DCF, ECF are equal in all respects, and in particular $\qquad\qquad$ [I. 8.

$$\angle DCF = \angle ECF,$$
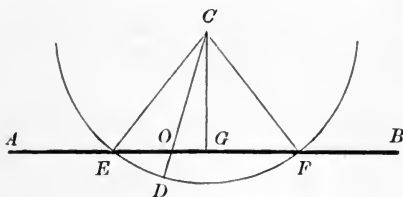
and they are adjacent angles.

Hence, by definition, each is a right angle, and FC *is at rt. $\angle^s$ to AB.*

PROPOSITION XII. Problem.

To draw a straight line perpendicular to a given straight line of unlimited length from a given point without it.

Let AB be the given straight line, and C the given point without it.

It is required to draw from C a straight line perpendicular to AB.

Const. Take any point D on the side of AB remote from C, and with centre C and radius CD describe a circle. This circle will cut the line AB, produced if necessary, in two points E and F. Bisect EF in G, and join CG.

Then CG is the line required.

Join CE and CF.

Proof. Then, in the triangles EGC, FGC,

$$\because \quad \left\{ \begin{array}{l} EG = GF. \qquad\qquad\qquad [Const. \\ CG = CG, \\ \text{and } CE = CF, \text{ being radii of a circle}; \qquad [Const. \end{array} \right.$$

$\therefore$ the triangles EGC and FGC are equal in all respects, and in particular $\qquad\qquad\qquad\qquad\qquad\qquad\qquad$ [I. 8.

$$\angle EGC = \angle FGC,$$

and these are adjacent angles.

Hence, by definition, CG is at rt. $\angle^s$ to AB.

The point D is taken on the side of the line AB opposite to that on which C is situated, in order to ensure that the circle will cut the line AB. The line AB must be of *unlimited* length, so that it may be produced if necessary so as to cut the circle.

It is *assumed* that the circle will in this case cut the line AB, and the nature of this assumption requires examination. Since C and D are on *opposite* sides of AB, CD will cut AB in some point O between C and D; and since CO is less than CD, the point O will be *within* the circle; and any straight line drawn through a point within a circle will, if produced far enough both ways, cut the circle in two points. [See p. 15.]

Ex. 1. Shew that the diagonals of a rhombus bisect each other and are at right angles.

Ex. 2. Prove that if any two isosceles triangles are on the same base, the line joining their vertices, produced if necessary, will bisect the base and will be at right angles to the base.

Ex. 3. Shew that the three straight lines joining the angular points of an equilateral triangle to the middle points of the opposite sides are equal to one another.

Ex. 4. Shew that in any isosceles triangle the bisector of the vertical angle is perpendicular to the base and bisects the base.

Ex. 5. Shew that, if the line joining an angular point of a triangle to the middle point of the opposite side be perpendicular to that side, the triangle must be isosceles.

Ex. 6. Shew that, if a line be drawn bisecting a second line at right angles, any point on the first line will be equally distant from the ends of the second.

Ex. 7. Construct a rhombus, having given the length of a side and one diagonal.

Ex. 8. Construct a right-angled triangle, having given the length of the hypotenuse and of one side.

Ex. 9. If the triangle ABC be turned over about its side AB, shew that the line joining the two positions of C will be perpendicular to AB.

Ex. 10. In the quadrilateral $ABCD$, $AB = AD$ and $\angle ABC = \angle ADC$; shew that $BC = CD$.

Ex. 11. Prove by superposition that, if all the sides of one quadrilateral be equal respectively to the sides of another quadrilateral and if also one pair of corresponding angles be equal, then will the quadrilaterals be congruent. [See page 19.]

PROPOSITION A. Theorem.

All right angles are equal.

Let the st. line AB standing on CD make the adjacent angles CBA, ABD equal to one another; then, by definition, each of these angles is a right angle.

Also let EF standing on GH make the adjacent angles GFE, EFH equal to one another, so that each of these angles is a right angle.
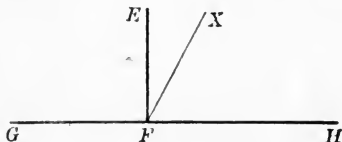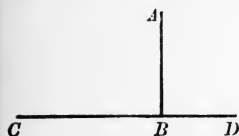
It is required to prove that either of the angles CBA or ABD is equal to either of the angles GFE or EFH.

Let the st. line CBD be applied to the st. line GFH so that the pt. B may be on the pt. F, and AB and EF on the same side of GFH. Then we have to prove that AB will coincide with EF.

For, if BA does not coincide with FE, but falls in some other position FX;

then since $\qquad \angle CBA = \angle ABD$,

$\therefore \qquad \angle GFX = \angle XFH.$

But, since FE is perp. to GH,

$$\angle GFE = \angle EFH.$$

And $\qquad \angle GFX$ is greater than $\angle GFE$;

$\therefore \qquad \angle GFX$ is greater than $\angle EFH.$

Also $\qquad \angle EFH$ is greater than $\angle XFH$;

$\therefore \qquad \angle GFX$ is greater than $\angle XFH.$

But it is impossible that $\angle GFX$ should be both equal to and greater than $\angle XFH.$

Hence the line BA must coincide with FE, and therefore $\angle CBA = \angle GFE.$

PROPOSITION XIII. Theorem.

The angles which one straight line makes with another, on one side of it, are either two right angles or are together equal to two right angles.

Let *AB* be a st. line making the ∠ˢ *ABC*, *ABD* with the st. line *CBD* and on one side of it; then *it is required to prove that the angles ABC, ABD are either right angles or are together equal to two right angles.*
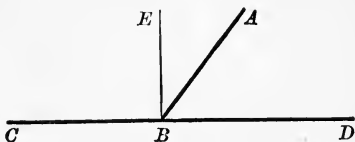
If ∠ *ABC* = ∠ *ABD*, each is a rt. ∠, by definition.

But, if ∠ *ABC* is not equal to ∠ *ABD*, draw from the point *B* the st. line *BE* perp. to *CD*.   [I. 11.

Then the two ∠ˢ *CBE* and *EBD* are right angles, and ∠ *EBD* is equal to ∠ *EBA* and ∠ *ABD* together.

∴ ∠ˢ *CBE, EBA* and *ABD* are together equal to two rt. ∠ˢ.

But ∠ˢ *CBE* and *EBA* make up the ∠ *CBA*.

Hence *the ∠ˢ CBA and ABD are together equal to two rt. ∠ˢ.*

Cor. I. *If two straight lines cut one another the four angles at their point of intersection are together equal to four right angles.*

Cor. II. *If any number of straight lines meet at a point all the angles between successive lines are together equal to four right angles.*

Ex. 1. If one of the angles which two intersecting lines make with one another is a right angle, they will all four be right angles.

Ex. 2. If *AB* and *CBD* are any two straight lines, the bisectors of the angles *CBA* and *ABD* will be at right angles.

Def. Two angles are said to be **supplementary** when their sum is two right angles, and either angle is called the **supplement** of the other.

Two angles are said to be **complementary** when their sum is a right angle, and either angle is called the **complement** of the other.

Thus the angles *CBA* and *ABD* are supplementary, and the angles *EBA* and *ABD* are complementary.

PROPOSITION XIV. Theorem.

If at a point in a straight line two other straight lines on opposite sides of it make the adjacent angles together equal to two right angles, these two straight lines must be in one and the same straight line.

At the point *B* in the st. line *AB*, let the two st. lines *CB*, *DB* on opposite sides of *AB* make the adjacent ∠ˢ *ABC*, *ABD* together equal to two rt. ∠ˢ; then *it is required to prove that CB and BD are in the same st. line.*
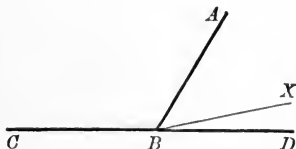
For, if *BD* be not in the same st. line as *CB*, produce *CB* beyond *B* and let *BX* be the produced part.

Then, *CBX* is a st. line,

∴ ∠ˢ *ABC* and *ABX* are together equal to two rt. ∠ˢ.

[I. 13.

But, by hypothesis,

∠ˢ *ABC* and *ABD* are together equal to two rt. ∠ˢ.

Hence sum of ∠ˢ *ABC* and *ABX* = sum of ∠ˢ *ABC* and *ABD*. [Ax. xi. or Prop. A.

Take away the common ∠ *ABC* from these equals ;

then $\angle ABX = \angle ABD$,

and this is impossible unless *BX* coincides with *BD*.

Hence *BD* and *CB* must be in the same st. line.

PROPOSITION XV. Theorem.

If two straight lines cut one another, the vertically opposite angles will be equal.

Let the st. lines AB and CD cut one another at the point E; then *it is required to prove that* $\angle AED = \angle CEB$ *and* $\angle AEC = \angle BED$.

Since CE meets AEB,

∴ $\angle AEC$ and $\angle CEB$ together = two rt. $\angle$ s.   [I. 13.

And, since BE meets CED,

∴ $\angle CEB$ and $\angle BED$ together = two rt. $\angle$ s.   [I. 13.

Hence $\angle AEC$ and $\angle CEB = \angle CEB$ and $\angle BED$.   [Ax. xi.

Take away the common $\angle CEB$ from each of these equals; then

$$\angle AEC = \angle BED.$$

And in the same way it can be proved that

$$\angle CEB = \angle AED.$$

Ex. 1. A line ABC is met in the point B by the two straight lines DB and BF which are on opposite sides of ABC, and the angles ABD and CBF are equal. Shew that DBF is a straight line.

Ex. 2. Shew that, in the figure to Prop. xv, the bisectors of the angles AEC and DEB are in the same straight line.

Ex. 3. The straight lines AB and CD bisect each other and are at right angles; shew that $ACBD$ is a rhombus.

Ex. 4. Four straight lines OA, OB, OC and OD meet in the point O, and the angles AOB, COD are equal and the angles BOC, DOA are also equal; shew that AOC and BOD are straight lines.

PROPOSITION XVI. Theorem.

If one side of a triangle be produced, the exterior angle will be greater than either of the interior opposite angles.

Let the side *BC* of the △ *ABC* be produced to *D*; then *it is required to prove that the exterior angle ACD is greater than either of the interior and opposite angles BAC and ABC.*
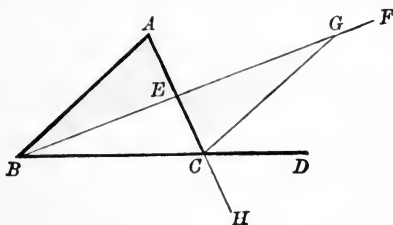
Const. Bisect *AC* in the point *E*. [I. 10.

Join *BE* and produce it to *F*, and from *EF* cut off *EG* = *BE*. Then join *GC*.

Proof. In the △ˢ *AEB* and *CEG*,

$$\because \left\{ \begin{array}{ll} AE = EC, & [\textit{Const.} \\ BE = EG, & [\textit{Const.} \\ \text{and} \angle AEB = \text{vertically opp.} \angle CEG. & [\text{I. 15.} \end{array} \right.$$

$$\therefore \angle BAE = \angle ECG. \qquad [\text{I. 4.}$$

But $\qquad \angle ACD > \angle ECG;$

$$\therefore \angle ACD > \angle BAC.$$

Also, by producing *AC* to *H* and bisecting *BC*, it can be proved in a similar manner that

$$\angle BCH > \angle ABC.$$

But $\qquad \angle BCH = \text{vertically opp.} \angle ACD;$

$$\therefore \angle ACD > \angle ABC.$$

Thus $\angle ACD$ is greater than either $\angle BAC$ or $\angle ABC.$

Ex. 1. Is an exterior angle of a triangle greater than the interior adjacent angle?

Ex. 2. Every triangle has at least two acute angles.

Ex. 3. Each of the base angles of an isosceles triangle is an acute angle.

Ex. 4. Shew that, if the sides of a triangle are produced, any two exterior angles are together greater than two right angles.

Ex. 5. Shew that only one perpendicular can be drawn to a straight line from any given point without it.

PROPOSITION XVII. Theorem.

Any two angles of a triangle are together less than two right angles.

Let ABC be a triangle; then *it is required to prove that any two of its angles are together less than two right angles.*
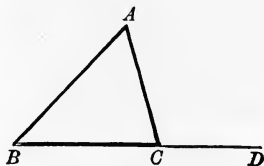
Produce BC to D.

Then ext. $\angle ACD >$ int. opp. $\angle ABC$.   [I. 16.

To each of these unequals add $\angle ACB$; then

sum of $\angle^s ACD$ and $ACB >$ sum of $\angle^s ABC$ and ACB.

But $\angle^s ACD$ and ACB are together equal to two rt. $\angle^s$.

[I. 13.

∴. $\angle^s ABC$ and ACB are together less than two rt. $\angle^s$.

In the same manner it can be shewn that $\angle^s ABC$ and BAC, and also that $\angle^s BCA$ and BAC are together less than two rt. $\angle^s$.

This proposition may be enunciated in the following form:

If a straight line intersects two other straight lines which meet in a point, the three lines not all passing through the same point, the two interior angles which it makes with those straight lines are together less than two right angles.

It will now be seen that Axiom xii is the Converse of Proposition XVII.

PROPOSITION XVIII. Theorem.

If, in any triangle, one of two sides be greater than the other, the angle which is opposite to the greater side will be greater than the angle which is opposite to the smaller.

In the triangle *ABC* let *AC* be greater than *AB*; then *it is required to prove that* ∠*ABC* *is greater than* ∠*ACB*.
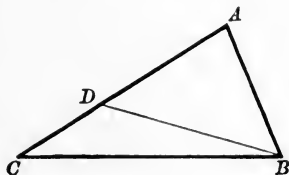
From *AC* cut off *AD* equal to *AB*, and join *BD*.

Then   ∵ *AB* = *AD*,

         ∴ ∠*ADB* = ∠*ABD*.   [I. 5.

But   exterior ∠*ADB* > int. opp. ∠*DCB*;   [I. 16.

         ∴ ∠*ABD* > ∠*DCB*.

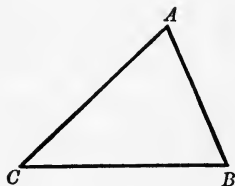But   ∠*ABC* > ∠*ABD*;

         ∴ ∠*ABC* > ∠*ACB*.

Simson's enunciation was as follows:

The greater side of any triangle is opposite to the greater angle.

PROPOSITION XIX. Theorem.

If, in any triangle, one of two angles be greater than the other, the side which is opposite to the greater angle is greater than the side which is opposite to the smaller.

In the $\triangle ABC$ let $\angle ABC$ be greater than $\angle ACB$; then *it is required to prove that AC is greater than AB.*

For, if AC be not greater than AB, it must either be equal to AB or less than AB.

Now AC cannot be equal to AB, for then $\angle ABC$ would $= \angle ACB$, which is not the case.

And again AC cannot be less than AB, for then $\angle ABC$ would, by the preceding proposition, be less than $\angle ACB$, which is not the case.

Hence AC must be greater than AB.

Simson's enunciation was as follows:

*The greater angle of any triangle is opposite to the greater side.*

Simson's enunciations of Propositions XVIII. and XIX., which are unfortunately still sometimes given, are a fruitful source of error, for no one can tell from these enunciations what is supposed to be known and what is to be proved. It must therefore be remembered that *what is supposed to be known is put first* in these enunciations.

PROPOSITION XX. Theorem.

Any two sides of a triangle are together greater than the third side.

Let *ABC* be a triangle. *It is required to prove that any two of its sides are together greater than the third side.*
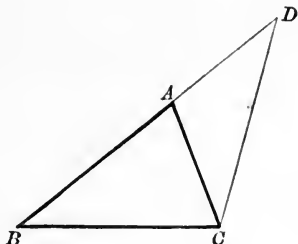
Produce *BA* to *D*, making *AD = AC*; and join *CD*.

Then, $\quad\quad\quad\quad\quad \because AD = AC,$ $\quad\quad\quad\quad\quad$ [*Const.*

$\quad\quad\quad\quad \therefore \angle ACD = \angle ADC.$ $\quad\quad\quad\quad\quad$ [I. 5.

But $\quad\quad\quad\quad\quad \angle BCD > \angle ACD;$

$\quad\quad\quad\quad \therefore \angle BCD > \angle ADC.$

Therefore, in the $\triangle BCD$,

the side *BD* opp. $\angle BCD >$ the side *BC* opp. $\angle BDC.$ [I. 19.

But *BD* is made up of *BA* and *AD*, and *AD = AC*;

$\quad\quad\quad\quad \therefore$ sum of *BA* and *AC > BC*.

In the same way it can be proved that the sum of *BC* and *CA* is greater than *AB*, and that the sum of *CB* and *BA* is greater than *AC*.

Ex. 1. The difference of two sides of a triangle is less than the third side.

Ex. 2. Any three sides of a quadrilateral are together greater than the fourth side.

Ex. 3. The sum of the sides of a quadrilateral is greater than the sum of its diagonals.

PROPOSITION XXI. Theorem.

If from the ends of one side of a triangle two straight lines be drawn to any point within the triangle, these straight lines will together be less than the sum of the other two sides of the triangle, but they will contain a greater angle.

From the ends B, C of the side BC of the $\triangle ABC$ let the two st. lines BD, CD be drawn to any point D within the $\triangle$; then *it is required to prove that the sum of BD and DC is less than the sum of BA and AC, and that $\angle BDC > \angle BAC$.*
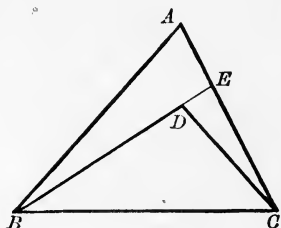
Produce BD to meet AC in the point E.

Then   the sum of BA and $AE > BE$.   [I. 20.

Add EC to each of these unequals, then

     the sum of BA, AE and $EC > BE$ and EC,

i.e. BA and $AC > BE$ and EC.

Again, the sum of DE and $EC > DC$.

Add BD to each of these unequals, then

     the sum of BD, DE and $EC > BD$ and DC,

i.e. BE and $EC > BD$ and DC.

But BA and $AC > BE$ and EC;

       $\therefore BA$ and $AC > BD$ and DC.

Again,   ext. $\angle BDC >$ int. opp. $\angle DEC$,   [I. 16.

and   ext. $\angle DEC >$ int. opp. $\angle BAE$;   [I. 16.

       $\therefore \angle BDC > \angle BAE$.

The proposition may be generalised as follows:

If there are two convex rectilineal figures on the same base, one of which is entirely within the other, the perimeter of the inner figure will be the smaller.

[A **convex rectilineal figure** is such that the production of any one of its sides would lie entirely outside the figure.]

For example, let the figures be $ABCD$ and $AEFD$, and let AE produced cut BC in X, and EF produced cut CD in Y.
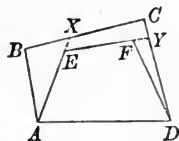
Then, sum of AB and $BX > AX$;

∴ perimeter of $ABCD >$ perimeter of $AXCD$.

Again, sum of EX, XC and $CY > EY$;

∴ perimeter of $AXCD >$ perimeter of $AEYD$.

And, sum of FY and $YD > FD$;

∴ perimeter of $AEYD >$ perimeter of $AEFD$.

Hence perimeter of $ABCD >$ perimeter of $AEFD$.

Ex. 1. Shew that in a right-angled triangle the side opposite to the right angle is the greatest side.

Ex. 2. In an obtuse-angled triangle the greatest side is that which is opposite to the obtuse angle.

Ex. 3. Shew that the perpendicular from a given point on a given straight line is shorter than any other straight line drawn from the given point to the given straight line.

Ex. 4. Shew that any straight line drawn from the vertex of a triangle to a point in the base is less than the longer of the two sides, or than either side if they are equal.

Ex. 5. In the quadrilateral $ABCD$, AB is the greatest side and CD is the least; shew that $\angle BCD$ is greater than $\angle DAB$, and $\angle CDA$ greater than $\angle ABC$.

Ex. 6. If O be any point within the $\triangle ABC$, the sum of the lines OA, OB, OC will be less than the sum but will be greater than half the sum of the sides of the triangle.

Ex. 7. Shew that the sum of the diagonals of a convex quadrilateral is greater than the sum of either pair of opposite sides, and also greater than half the sum of the sides.

Ex. 8. $ABCD$ is a convex quadrilateral and O is any point within it; shew that the sum of OA, OB, OC, OD is greater than the sum of AC and BD except when O is at the point of intersection of AC and BD.

Ex. 9. The sum of the distances of any point within a rectilineal figure from the angular points is greater than half the sum of the sides.

PROPOSITION XXII. Problem.

To construct a triangle the sides of which shall be respectively equal to three given straight lines, provided that any two of these lines are together greater than the third.

Let *A, B, C* be any three straight lines any two of which are together greater than the third. Then *it is required to construct a triangle the sides of which are respectively equal to the lines A, B, and C.*

We may suppose that a line which is not less than either of the others is called *A*.
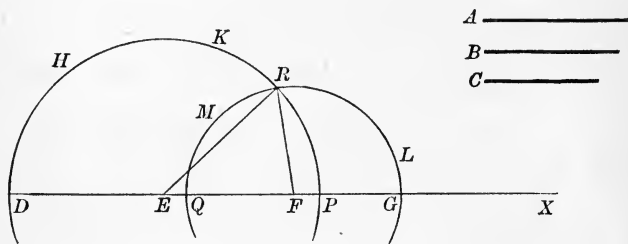
Const. Draw any straight line *DX*, and from it cut off *DE* equal to *A* ; and then, in the same direction, *EF* equal to *B* and *FG* equal to *C*.

With centre *E* and radius *ED* describe the circle *DHK* cutting the line *DX* in *P*. Then, since, by hypothesis, *DE* or *EP* is not less than *EF* but is less than *EF* and *FG* together, the point *P* must be between *F* and *G*.

Again, with centre *F* and radius *FG* describe the circle *GLM* cutting the line *DX* in *Q*. Then, since by hypothesis *FG* is less than *FE* and *ED* together, the point *Q* must lie between *F* and *D*.

Since $FD > FQ$, the point D must be *without* the circle GLM.

And, since $FP < FG$, the point P must be *within* the circle GLM.

But when there are two points on the circumference of one circle one of which is within and the other without another circle, it is obvious that the two circles must cut one another.

Hence the circles DHK and GLM will cut one another.

Let R be one of the points of intersection, and join RE and RF. Then EFR will be the triangle required.

For radius $ER = $ radius $ED = $ line A,

and radius $FR = $ radius $FG = $ line C,

also $EF = $ line B.

Thus the sides of the triangle EFR are equal respectively to the three given straight lines.

The axiom which must be here assumed is the following :—

'*Any line which joins a point within a circle (or any other closed figure) to a point without the figure must cut the boundary of the figure.*'

It will be seen that the whole difficulty in the above proposition lies in the proof that, *with the given conditions*, the circles drawn must cut one another.

The proposition may be enunciated in the following form, which will be seen to be the converse of Proposition XX. :—

It is always possible to construct a triangle whose sides are respectively equal to three given straight lines, provided that any two of these lines are together greater than the third.

Euclid gives the condition that any two of the lines are together greater than the third in the enunciation of the problem, but he makes no attempt to shew the necessity and sufficiency of this condition, the condition is not indeed mentioned except in the enunciation. All that Euclid does is therefore to shew how to make a triangle whose sides are respectively equal to three straight lines, *assuming that it is always possible to make such a triangle.*

PROPOSITION XXIII. Problem.

At a given point in a given straight line to make an angle equal to a given angle.

Let AB be the given st. line and A the given point in it, and let C be the given angle; then *it is required to make an angle equal to the angle C, with its vertex at A and having AB for one of its arms.*
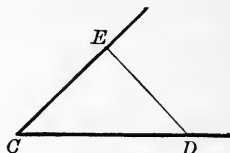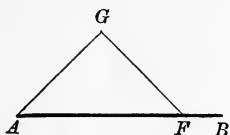
Const. Take any two points D, E one on each of the lines bounding the angle C, and join DE.

From AB cut off $AF = CD$, and make the $\triangle AFG$ having its sides AF, FG, GA equal respectively to CD, DE, EC.

[I. 22.

Then, in the $\triangle^s AFG$, CDE,

$$\because \begin{cases} AF = CD, \\ GA = EC, \\ \text{and } FG = DE; \end{cases}$$

$$\therefore \angle FAG = \angle DCE. \qquad [\text{I. 8.}$$

Thus $\angle FAG$ *is the angle required.*

PROPOSITION XXIV. Theorem.

If two sides of one triangle be equal respectively to two sides of another, and if the included angles be unequal, the triangle which has the greater included angle will have the greater base.

Let ABC, DEF be two triangles having the sides AB, BC equal respectively to the sides DE, EF and $\angle ABC$ greater than $\angle DEF$; then *it is required to prove that the base AC is greater than the base DF.*
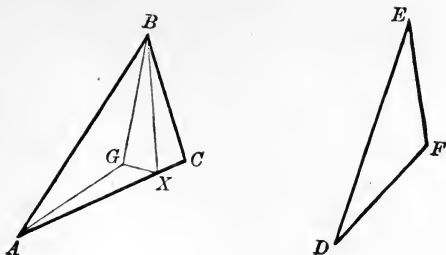
At the point B in the st. line AB make

$$\angle ABG = \angle DEF,$$

and $BG = EF$.   Join AG.

In the $\triangle^s\ ABG,\ DEF$,

$$\because \left\{ \begin{array}{ll} AB = DE, & [Hyp. \\ BG = EF, & [Const. \\ \text{and } \angle ABG = \angle DEF; & [Const. \end{array} \right.$$

$$\therefore AG = DF. \qquad [\text{I. 4.}$$

Now if the point G is on the line AC it is obvious that $AG < AC$. But, if G be not on AC, bisect the angle CBG by the line BX which meets AC in X.   Join GX.

Then, in the $\triangle^s\ GBX,\ CBX$,

$$\because \left\{ \begin{array}{ll} BG = BC, & [Const. \\ BX = BX, & \\ \text{and } \angle GBX = \angle CBX; & [Const. \end{array} \right.$$

$$\therefore GX = XC. \qquad [\text{I. 4.}$$

$$\therefore GX \text{ and } XA = CX \text{ and } XA$$

$$= CA.$$

But GX and XA are greater than AG;   [I. 20.

$$\therefore CA > AG.$$

The above proof is now usually given instead of Euclid's, which is defective. The defect in Euclid's proof was pointed out by Billingsley (1570), who completed the proof. Simson refers to a rigid proof by Campanus, but does not give one.

PROPOSITION XXV. Theorem.

If two sides of one triangle be equal respectively to two sides of another, and if the bases be unequal, the triangle which has the greater base will have the greater included angle.

Let ABC, DEF be two triangles having the sides AB, BC equal respectively to the sides DE, EF and the base AC greater than the base DF; then *it is required to prove that* $\angle ABC$ is greater than $\angle DEF$.
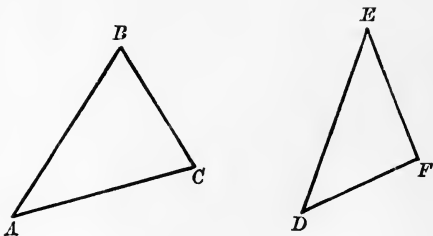
For, if $\angle ABC$ be not greater than $\angle DEF$, it must be either equal to or less than $\angle DEF$.

But, since $BA = DE$ and $BC = EF$, the $\angle ABC$ cannot be equal to $\angle DEF$, for then by I. 4, the base AC would be equal to the base DF, which is not true.

Nor can $\angle ABC$ be less than $\angle DEF$, for then by the preceding proposition AC would be less than DF, which is not true.

Hence $\angle ABC$ must be greater than $\angle DEF$.

Ex. 1. Make an angle equal to the sum of two given angles.

Ex. 2. Make an angle equal to the difference of two given angles.

**Ex. 3.** Construct a triangle having given the base and the two angles at the base.

Ex. 4. Construct a triangle having given the lengths of two sides and the angle contained by those sides.

PROPOSITION XXVI. Theorem.

If in two triangles, two angles of the one be equal respectively to two angles of the other, and if also the sides adjacent to the equal angles, or the sides opposite to a pair of equal angles, be equal, then the two triangles will be equal in all respects, those sides being equal which are opposite to equal angles.

In the triangles ABC, DEF, if $\angle ABC = \angle DEF$, and $\angle ACB = \angle DFE$, and if also either $BC = EF$ or $AB = DE$; then *it is required to prove that the triangles are equal in all respects.*

Case I. When the sides BC and EF adjacent to the equal angles are equal.
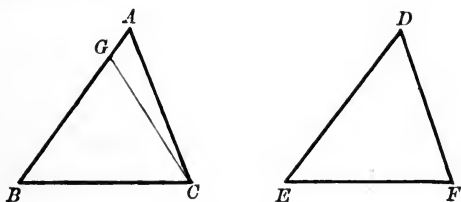
If BA be not equal to ED, one of them must be the greater. If possible, let BA be the greater, and cut off from BA the part BG equal to ED. Join GC.

Then in the $\triangle^s$ GBC, DEF,

$\because$ $\begin{cases} & BG = ED, & \text{[Const.} \\ & BC = EF, & \text{[Hyp.} \\ \text{and included } & \angle GBC = \text{included } \angle DEF; & \text{[Hyp.} \end{cases}$

$\therefore \angle BCG = \angle EFD.$ [I. 4.

But $\angle EFD = \angle BCA$;   [*Hyp.*

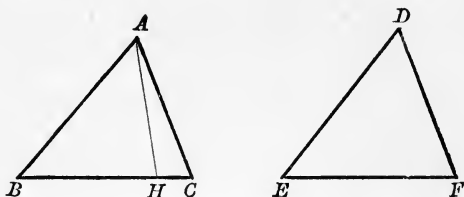$\therefore \angle BCG = \angle BCA,$

the part equal to the whole, which is impossible.

Hence ED cannot be unequal to BA, i.e. $ED = BA$.

And then, since $BA = ED$, $BC = EF$, and $\angle ABC = \angle DEF$;

$\therefore \triangle^s ABC$, DEF are equal in all respects.

CASE II. When the sides *AB*, *DE* opposite to a pair of equal angles are equal.

If *BC* be not equal to *EF*, one of them must be the greater. If possible, let *BC* be the greater, and cut off the part *BH* equal to *EF*. Join *AH*.

Then, in the △ˢ *ABH, DEF*,

$$\because \begin{cases} AB = DE, & [Hyp. \\ BH = EF, & [Const. \\ \text{and included} \angle ABH = \text{included} \angle DEF; & [Hyp. \end{cases}$$

$$\therefore \angle AHB = \angle DFE. \qquad [\text{I. } 4.$$

But $\qquad \angle DFE = \angle ACB; \qquad [Hyp.$

$$\therefore \angle AHB = \angle ACB,$$

that is, an exterior angle of the triangle *AHC* equal to an interior opposite angle, and this is impossible.   [I. 16.

Hence *EF* cannot be unequal to *BC*, i.e. *EF = BC*.

And then $\qquad \because \begin{cases} BC = EF, \\ AB = DE, \\ \text{and} \angle ABC = \angle DEF; \end{cases}$

$\therefore \triangle$ˢ *ABC, DEF are equal in all respects.*

PROPOSITION B. Theorem.

If two sides of one triangle be respectively equal to two sides of another triangle, and if the angles opposite to one pair of equal sides be also equal, then will the angles opposite the other pair of equal sides be either equal or supplementary; and, in the former case, the triangles will be equal in all respects.

In the triangles ABC, DEF let $AB = DE$, $AC = DF$, and $\angle ABC = \angle DEF$; then *it is required to prove that the angles ACB and DFE are either equal or supplementary.*
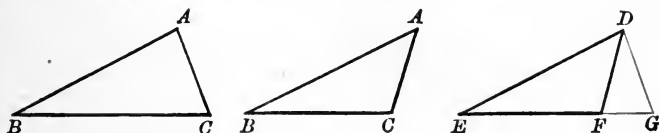
For, if $\angle BAC = \angle EDF$,

then, since $AB = DE$, $AC = DF$ and $\angle BAC = \angle EDF$;

$\therefore$ $\triangle^s$ ABC and DEF are congruent, and $\angle ACB = \angle DFE$.

But, if $\angle BAC$ be not equal to $\angle EDF$, on the same side of ED that F is, make

$$\angle EDG = \angle BAC;$$

and let EF, produced if necessary, meet DG in G.

Then, in the $\triangle^s$ DEG, ABC

$$\because \begin{cases} ED = BA, & [Hyp. \\ \angle EDG = \angle BAC, & [Const. \\ \angle DEG = \angle ABC; & [Hyp. \end{cases}$$

$\therefore$ $DG = AC$, and $\angle DGE = \angle ACB$. [I. 26.

Also, since $AC = DF$, and $DG = AC$, $DG = DF$;

$\therefore$ $\angle DFG = \angle DGF = \angle ACB$.

But $\angle DFE$ is supplementary to $\angle DFG$; [I. 13.

$\therefore$ $\angle DFE$ is supplementary to $\angle ACB$.

4—2

Cor. *If two sides of one triangle are equal respectively to two sides of another and the angles opposite to one pair of equal sides are also equal, the two triangles are equal in all respects when the angles opposite the other pair of equal sides are both acute, or both obtuse, or when one of these angles is a right angle.*

For one of two supplementary angles is greater and the other is less than two right angles, or else both are right angles.

Ex. 1. The bisectors of the equal angles ABC, ACB of the isosceles triangle ABC meet the sides AC, AB respectively in the points E, F; shew that $BE = CF$.

Ex. 2. The hypotenuse and one side of one right-angled triangle are respectively equal to the hypotenuse and one side of another right-angled triangle. Shew that the two triangles are equal in all respects.

Ex. 3. The bisector of the angle BAC of any triangle meets the opposite side in the point D, and $BA > CA$. Shew that $BD > CD$.

Ex. 4. D is any point on the side BC of the triangle ABC; shew that the perimeter of the triangle is greater than twice AD.

Ex. 5. Shew that, if the line bisecting an angle of a triangle passes through the middle point of the opposite side, the triangle must be isosceles.

Ex. 6. From any point on the bisector of an angle perpendiculars are drawn to the lines containing the angle; shew that these perpendiculars are equal.

Ex. 7. Shew that, if the perpendiculars drawn from a point on two intersecting straight lines are equal, the point must be on one or other of the lines which bisect the angles between the intersecting lines.

Ex. 8. Shew that, if the perpendiculars from two of the angular points of a triangle on the opposite sides are equal, the triangle must be isosceles.

When a straight line intersects two other straight lines it makes with them eight angles to which particular names are given.
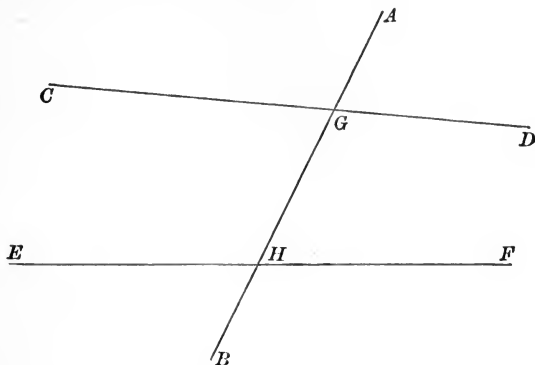
Thus, in the figure,

AGD, AGC, EHB, BHF are called **exterior** angles.

HGD, HGC, GHE, GHF are called **interior** angles.

CGH and *GHF*, also *DGH* and *GHE*, are called

alternate angles.

Also *AGD* and *GHF* are called respectively an *exterior angle* and the *interior and opposite angle on the same side of the line A B.*

PROPOSITION XXVII. Theorem.

If a straight line cutting two other straight lines make the alternate angles equal, these two straight lines will be parallel.
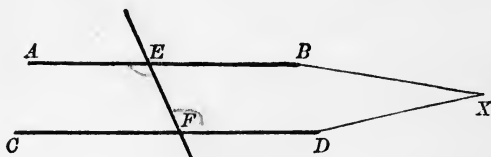
Let the straight line EF cut the two straight lines AB, CD in the points E, F respectively, and let the alternate angles AEF and EFD be equal; then *it is required to prove that AB and CD are parallel.*

For, if AB and CD are not parallel, they will meet if produced far enough. Let them be produced towards B and D, and if possible let them meet in the point X.

Then EXF would be a triangle and the exterior $\angle AEF$ would be equal to the interior and opposite $\angle EFX$, which we know is impossible.

It is therefore impossible that AB and CD should meet when produced towards B and D; and it can be proved in a similar manner that it is impossible that they should meet when produced towards A and C.

$$\therefore AB \text{ is} \parallel \text{to } CD.$$

PROPOSITION XXVIII. Theorem.

If a straight line cut two other straight lines and make an exterior angle equal to the interior and opposite angle on the same side of the line, or make the two interior angles on the same side of the line together equal to two right angles, the two straight lines will be parallel.

Let the st. line *EF* cut the two st. lines *AB*, *CD* in the points *G*, *H* respectively, and make the exterior angle *EGB* equal to the interior and opposite angle *GHD* on the same side of *EF*, or make the two interior angles *BGH*, *GHD* on the same side of *EF* equal to two right angles; then *it is required to prove that AB and CD are parallel.*
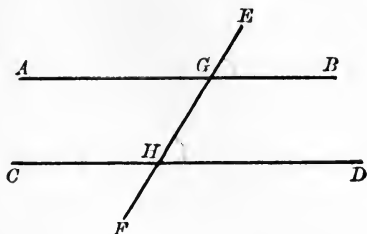
First, suppose that $\angle EGB = \angle GHD$.

Since $\qquad\qquad \angle EGB =$ vertically opp. $\angle AGH$, [I. 15.

$\qquad \therefore \angle AGH =$ alt. $\angle GHD$;

$\qquad\qquad \therefore AB$ is $\parallel$ to CD. $\qquad\qquad$ [I. 27.

Next, suppose that $\angle^s BGH$ and GHD together equal two rt. $\angle^s$.

Since $\angle AGH$ and $\angle BGH$ together $=$ two rt. $\angle^s$,   [I. 13.

Sum of $\angle^s AGH$ and $BGH =$ sum of $\angle^s BGH$ and GHD;

$\qquad\qquad \therefore \angle AGH = \angle GHD$,

and these are alternate angles.

$\qquad\qquad \therefore AB$ is $\parallel$ to CD. $\qquad\qquad$ [I. 27.

Ex. 1.   Prove Prop. xxviii. in the same manner as Prop. xxvii.

Ex. 2.   Shew that, if two straight lines are perpendicular to the same straight line, they are parallel to one another.

Ex. 3.   Shew that every rhombus is a parallelogram.

PROPOSITION XXIX. Theorem.

If a straight line cut two parallel straight lines, it will make alternate angles equal, each exterior angle equal to the interior opposite angle on the same side of the straight line, and the two interior angles on the same side of the line together equal to two right angles.

Let the st. line EF cut the two parallel lines AB, CD in the points G, H respectively; then *it is required to prove that the alternate $\angle^s$ AGH, GHD are equal, the ext. $\angle$ EGB = int. opp. $\angle$ GHD, and that the two int. $\angle^s$ BGH, GHD are together equal to two rt. $\angle^s$.*
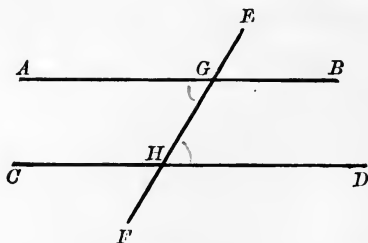
For, if $\angle AGH$ be not equal to $\angle GHD$, one of them must be the greater; if possible, let $\angle AGH$ be greater than $\angle GHD$.

Add $\angle HGB$ to each of these unequals,

then $\angle AGH$ and $\angle HGB > \angle GHD$ and $\angle HGB$.

But $\angle AGH$ and $\angle HGB =$ two rt. $\angle^s$;

∴ $\angle GHD$ and $\angle HGB <$ two rt. $\angle^s$.

Hence, by Axiom XII.*, the lines AB and CD will meet if produced towards B and D; but this is impossible since the lines are parallel.

∴ $\angle^s$ AGH and GHD cannot be unequal,

i.e. $\angle AGH =$ alt. $\angle GHD$.

Then,   ∵ $\angle AGH = \angle GHD$,

and $\angle AGH =$ vertically opp. $\angle EGB$;

∴ $\angle EGB = \angle GHD$.

* See Note, page 58.

Again, $\quad\quad\quad \because \angle GHD = \angle AGH,$

adding $\angle BGH$ to each,

$\quad\quad \angle GHD$ and $\angle BGH = \angle AGH$ and $\angle BGH.$

But $\quad\quad \angle^s AGH$ and $BGH =$ two rt. $\angle^s$;

$\quad\quad \therefore \angle^s GHD$ and $BGH =$ two rt. $\angle^s.$

PROPOSITION XXX. Theorem.

Straight lines which are parallel to the same straight line are parallel to one another.

Let each of the st. lines AB and CD be parallel to EF; then *it is required to prove that AB is parallel to CD.*
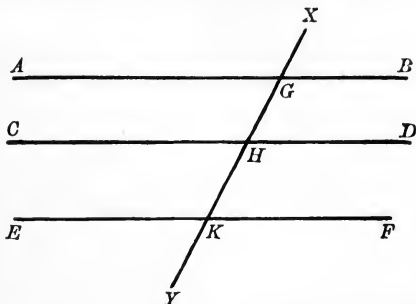
Draw some st. line XY which will cut AB, CD, EF in the points G, H, K respectively.

Then $\quad\quad\quad \because AB$ is $\parallel$ to EF,

$\quad\quad \angle AGH =$ alternate $\angle HKF.$ [I. 29.

And $\quad\quad\quad \because CD$ is $\parallel$ to EF,

$\quad\quad \angle CHK =$ alternate $\angle HKF.$ [I. 29.

Hence exterior $\angle CHK =$ interior opp. $\angle AGH$;

$\quad\quad \therefore AB$ is $\parallel$ to $CD.$ [I. 28.

Cor. *If a straight line intersect one of two parallel straight lines it will intersect the other.*

Ex. 1. If a straight line is parallel to one of two parallel straight lines it is parallel to the other.

Ex. 2. If a straight line is perpendicular to one of two parallel straight lines it is perpendicular to the other.

NOTE ON AXIOM XII.

As the definition of parallel straight lines simply describes them by the negative property that they never meet, it is necessary that some positive property of parallel lines should be assumed as an axiom. The axiom assumed by Euclid has been objected to on the ground that it is not self-evident, and it must in fact be allowed that it is not more self-evident than Proposition XVII., of which it is the converse. Many attempts have been made to found the theory of parallel lines upon an axiom more nearly self-evident than Euclid's Axiom XII., but the only substitute for Euclid's Axiom which appears to really diminish the difficulty is the following :

'*Two straight lines which intersect one another cannot both be parallel to the same straight line.*'

The above axiom is generally called Playfair's Axiom, although Professor Playfair was not the first to suggest it.

It would be a good exercise for the student to prove Prop. XXIX. by assuming Playfair's Axiom instead of Euclid's Axiom XII.

PROPOSITION XXXI. PROBLEM.

To draw a straight line through a given point parallel to a given straight line.

Let *A* be the given point and *BC* the given st. line; then *it is required to draw through the pt. A a st. line parallel to BC.*
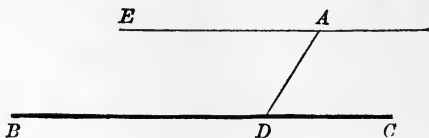
In *BC* take any point *D*, and join *AD*. Then at the point *A* in the st. line *AD* make ∠ *DAE* equal to ∠ *ADC*, these ∠ˢ being on opposite sides of *DA*.   [I. 23.

Then   ∠ *EAD* = alt. ∠ *ADC* ;   [*Const.*

∴ *EA* is parallel to *BC*,   [I. 27.

and *EA* is drawn through the given point *A*.

PROPOSITION XXXII. Theorem.

If a side of a triangle be produced, the exterior angle will be equal to the sum of the two interior opposite angles.

The three interior angles of any triangle are equal to two right angles.

Let the side BC of the $\triangle ABC$ be produced to D; then it is required to prove (a) that $\angle ACD =$ sum of $\angle^s ABC$ and CAB, and (β) that the three angles ABC, CAB and BCA are together equal to two rt. $\angle^s$.
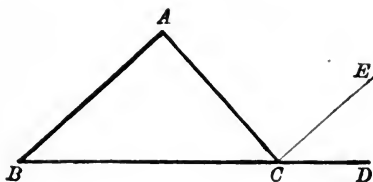
Through C draw $CE \parallel$ to AB.   [I. 31.

Then $\because AB \parallel$ to CE;

     $\therefore \angle ACE =$ alt. $\angle BAC$,   [I. 29.

and   ext. $\angle ECD =$ int. opp. $\angle ABC$.   [I. 29.

     $\therefore \angle^s ACE$ and $ECD = \angle^s BAC$ and ABC.

But $\angle^s ACE$ and ECD together make up $\angle ACD$.

     $\therefore$ exterior $\angle ACD = \angle^s BAC$ and ABC.

Now add $\angle ACB$ to each of these equals, then

     $\angle^s ACD$ and $ACB = \angle^s BAC$, ABC and ACB.

But $\angle^s ACD$ and ACB together $=$ two rt. $\angle^s$;   [I. 13.

     $\therefore \angle^s BAC$, ABC and ACB together $=$ two rt. $\angle^s$.

Cor. I. *All the interior angles of any rectilineal figure together with four right angles are equal to twice as many right angles as the figure has sides.*

For any rect. figure *ABCDE* can be divided into as many △ˢ as there are sides by drawing st. lines from any point *O* within it to each of its angular points.
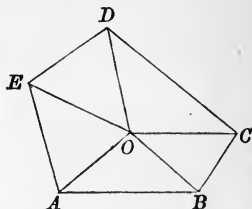
Then the sum of the interior angles of all these triangles will be equal to twice as many right angles as the figure has sides.

But the interior angles of these triangles make up the interior angles of *ABCDE* together with the angles at the point *O*, and the angles at *O* are together equal to four right angles.

Thus the interior angles of the figure together with four right angles are equal to twice as many rt. ∠ˢ as the figure has sides.

For example, *the sum of the interior angles of any quadrilateral is equal to four right angles.*

Cor. II. * *All the exterior angles of any convex rectilineal figure are together equal to four right angles.*

A **convex** rectilineal figure is one which would not be cut by the production of any of its sides.
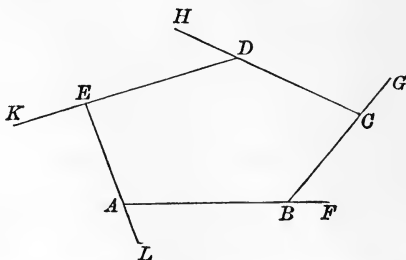
Every int. ∠ of a *convex* rectilineal figure together with its corresponding ext. ∠ are equal to two rt. ∠ˢ.

Hence all the interior angles together with all the exterior angles are equal to twice as many right angles as the figure has sides.

Hence, by Cor. I., the exterior angles are together equal to four right angles.

* Cor. I. and Cor. II. were given in Billingsley's edition (1570).

The quadrilateral *BDCA* on page 42 is an example of a figure which is not convex. An angle such as *BDC* in this figure, which is greater than two right angles, is called a **re-entrant** angle.

Ex. 1. Prove that the sum of the three interior angles of a triangle is equal to two right angles, by drawing a line through one angular point parallel to the opposite side.

Ex. 2. Shew that, if two angles of one triangle are equal respectively to two angles of another triangle, their third angles will also be equal.

Ex. 3. Shew that, if one of the angles of a triangle is equal to the sum of the other two, the triangle is right-angled.

Ex. 4. Shew that, if one of the angles of a triangle is greater than the sum of the other two, the triangle is obtuse-angled.

Ex. 5. If in a triangle every angle is less than the sum of the other two, the triangle is acute-angled.

Ex. 6. The angles of all equilateral triangles are equal.

Ex. 7. Divide a right angle into three equal parts.

Ex. 8. Make an angle equal to one-sixth of a right angle.

Ex. 9. Equilateral triangles *BCD*, *CAE*, *ABF* are described on the sides of any triangle *ABC*, the equilateral triangles being all outside the triangle *ABC*; shew that *AD*, *BE*, *CF* are all equal.

Ex. 10. Shew that if two isosceles triangles have equal vertical angles, the base angles will also be equal.

Ex. 11. Shew that the sum of the exterior angles of a convex quadrilateral is equal to the sum of the interior angles.

Ex. 12. Shew that the sum of the interior angles of any convex hexagon is double the sum of the exterior angles.

Ex. 13. Shew that each of the angles of a regular* pentagon is six-fifths of a right angle.

Ex. 14. Shew that each of the angles of a regular hexagon is four-thirds of a right angle.

Ex. 15. Each of the angles of a polygon is three-halves of a right angle. How many sides has the polygon?

Ex. 16. The side *BA* of the triangle *ABC* is produced to *D*, and the line bisecting the angle *CAD* is parallel to *BC*. Shew that the triangle is isosceles.

---

* A polygon which is both equilateral and equiangular is called a **regular** polygon.

PROPOSITION XXXIII. Theorem.

The straight lines which join the extremities of two equal and parallel straight lines, towards the same parts, are themselves equal and parallel.

Let AB and CD be two equal and parallel st. lines and let them be joined towards the same parts by the st. lines AC and BD; then *it is required to prove that AC and BD are themselves equal and parallel.*
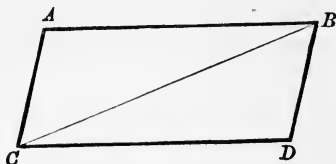
Join BC.

$$\because \; AB \text{ is} \parallel \text{to } CD,$$
$$\therefore \angle ABC = \text{alt. } \angle BCD. \qquad [\text{I. 29.}$$

Then in $\triangle^s$ ABC, DCB

$$\because \begin{cases} AB = CD, & [Hyp. \\ BC = CB, \\ \text{and included } \angle ABC = \text{included } \angle DCB; \end{cases}$$
$$\therefore AC = BD. \qquad [\text{I. 4.}$$

Also $\qquad \angle ACB = \angle CBD,$

and these are alternate angles; $\quad \therefore AC$ is $\parallel$ to BD. $\quad [\text{I. 27.}$

Ex. 1. The straight lines which join the extremities of two equal and parallel straight lines, towards opposite parts, are bisected at their points of intersection.

Ex. 2. If each of two equal and parallel straight lines be divided into the same number of equal parts, the lines joining corresponding points of division will all be parallel.

Ex. 3. The perpendicular distances of the points A, B from the straight line LM are equal, shew that LM is either parallel to AB or bisects AB.

PROPOSITION XXXIV. Theorem.

The opposite sides and angles of a parallelogram are equal, and each diagonal bisects the parallelogram.

Let $ABCD$ be a par^m. of which BD is a diagonal; *it is required to prove that opposite sides are equal, that opposite angles are equal and that a diagonal bisects the figure.*
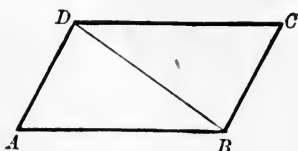
Since AB is ∥ to DC,

$$\angle ABD = \text{alt.} \angle BDC.$$ [I. 29.

Since AD is ∥ to BC,

$$\angle ADB = \text{alt.} \angle DBC.$$ [I. 29.

Then, in the $\triangle^s$ ABD, CDB

∵ $\left\{ \begin{array}{l} \angle ABD = \angle BDC, \\ \angle ADB = \angle DBC, \\ \text{and the side } DB \text{ adjacent to the equal angles is common;} \end{array} \right.$

$$\therefore AB = DC,$$ [I. 26.

$$AD = BC,$$

$$\angle DAB = \angle BCD,$$

and $$\triangle ABD = \triangle BCD.$$

Similarly, if AC be drawn, it can be proved that

$$\angle ABC = \angle CDA,$$

and that AC bisects the figure.

Ex. 1. Shew that a quadrilateral is a parallelogram,

   (i)   if one pair of opposite sides are equal and parallel,

   (ii)   if pairs of opposite sides are equal,

   (iii)   if pairs of opposite angles are equal,

   (iv)   if the diagonals bisect each other.

Ex. 2. Shew that the diagonals of a parallelogram bisect each other.

Ex. 3. If one angle of a parallelogram is a right angle, they are all right angles.

Ex. 4. Shew that a square, or any other rhombus, is a parallelogram, and that its diagonals are at right angles.

Ex. 5. Shew that all the angles of a square are right angles.

Ex. 6. Shew that the diagonals of a square, or of any other rhombus, are at right angles.

Ex. 7. Shew that the diagonals of a rectangle are equal.

Ex. 8. Shew that, if the diagonals of a parallelogram are equal, it must be a rectangle.

Ex. 9. Shew that, if the diagonals of a parallelogram are equal and perpendicular, the parallelogram must be a square.

Ex. 10. From any two points on a straight line perpendiculars are drawn on a parallel straight line; shew that these perpendiculars are equal.

Ex. 11. Shew that any line through *O*, the point of intersection of the diagonals of a parallelogram, is cut by either pair of opposite sides in two points equidistant from *O*.

Ex. 12. Shew that any straight line through the point of intersection of the diagonals of a parallelogram will divide the parallelogram into two equal parts.

Ex. 13. Shew how to find a point which is at given perpendicular distances from two given intersecting straight lines.

PROPOSITION XXXV. Theorem.

Parallelograms on the same base and between the same parallels are equal.

Let the ∥^{ms} *ABCD*, *EBCF* be on the same base and between the same parallels *AF*, *BC*; then *it is required to prove that these* ∥^{ms} *are equal.*
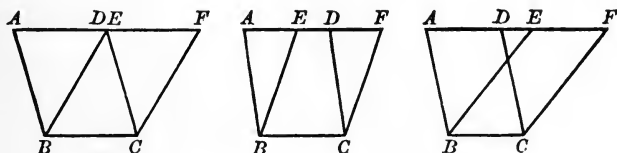
∵ *ABCD* is a ∥^{m}, and opposite sides of a ∥^{m} are equal;

$$\therefore AB = DC$$

∵ *AB* is ∥ *CD*, ∠*BAE* = ∠*CDF*,   [I. 29.

∵ *BE* is ∥ *CF*, ∠*AEB* = ∠*DFC*.   [I. 29.

Hence in   △^{s} *AEB*, *DFC*,

∵ {
$$AB = DC,$$
$$\angle BAE = \angle CDF,$$
and ∠*AEB* = ∠*DFC*.

$$\therefore \triangle AEB = \triangle DFC.$$

∴ *ABCFA* diminished by △*AEB*

= *ABCFA* diminished by △*DFC*;

$$\therefore \parallel^m EBCF = \parallel^m ABCD.$$

The two parallelograms although equal in area are not equal in all respects, unless by chance the angles *BCD* and *EBC* are equal. It is easily proved that two parallelograms are equal in all respects when, and only when, two adjacent sides and the included angle of the one are equal to two adjacent sides and the included angle of the other.

This is the first example of two figures being proved to be equal in area without being equal in all respects.

In the case of two parallelograms on the same base and between the same parallels it is easy to subdivide one of the figures into parts which, when properly fitted together, will coincide with the other figure. For set off distances *DG*, *GH* along *AF*, each equal to *AD*, until the last point of subdivision of *AF* comes within *EF*. Then draw lines through *G*, *H* parallel to *AB*, as in the figure.
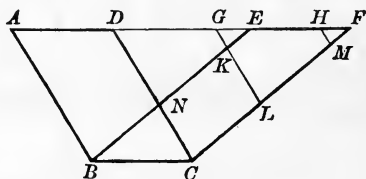
Then it is easily proved that △*HFM* is equal in all respects to △*GEK*.

Then again the figure *LGHM* can be proved to be equal in all respects to the figure *NDGK*; and the figure *CDGL* to the figure *BADN*.

Hence, remove the triangle *HFM* to the position *GEK*, and then remove the figure *LGHM* to the position *NDGK*, and finally remove the figure *CDGL* to the position *BADN*; we shall then have moved the parts into which the parallelogram *BEFC* is divided so as to make them coincide with the parallelogram *BADC*.

Ex. 1. Make a rhombus equal to a given parallelogram and having each of its sides equal to the longer side of the parallelogram.

Ex. 2. Make a rectangle equal to a given parallelogram and having one of its sides equal to a side of the parallelogram.

Ex. 3. Shew that, if the lengths of the sides of a parallelogram be given, the area will be greatest when it is a rectangle.

. PROPOSITION XXXVI. Theorem.

Parallelograms on equal bases and between the same parallels are equal.

Let $ABCD$, $EFGH$ be two ‖ᵐˢ on equal bases BC, FG and between the same parallels BG and AH; then *it is required to prove that $ABCD$ is equal (in area) to $EFGH$.*
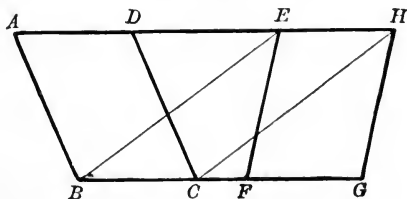
Join BE and CH.

Then $$BC = FG, \qquad [hyp.$$

and since opposite sides of a ‖ᵐ are equal, $FG = EH$;

$$\therefore \ BC \text{ and } EH \text{ are equal and } \|.$$

Hence $\qquad BCHE$ is a ‖ᵐ. $\qquad$ [I. 33.

Then the ‖ᵐˢ $ABCD$ and $EBCH$ are on the same base BC and between the same ‖ˢ;

$$\therefore \ \|^{\mathrm{m}} ABCD = \|^{\mathrm{m}} EBCH. \qquad [\text{I. 35.}$$

Also the ‖ᵐˢ $EBCH$ and $EFGH$ are on the same base EH and between the same ‖ˢ;

$$\therefore \ \|^{\mathrm{m}} EBCH = \|^{\mathrm{m}} EFGH. \qquad [\text{I. 35.}$$

Hence $\qquad \|^{\mathrm{m}} ABCD = \|^{\mathrm{m}} EFGH.$

The **altitude** of a parallelogram, with reference to a particular side as base, is the length of the perpendicular drawn to the base from any point on the opposite side.

It is easily seen that ‖ᵐˢ which are between the same ‖ˢ have equal altitudes, and that ‖ᵐˢ which have equal altitudes can be so placed as to be between the same parallels.

Ex. 1. Parallelograms on equal bases and with equal altitudes are equal.

Ex. 2. Equal parallelograms on equal bases must be between the same parallels, or must have the same altitude.

Ex. 3. Equal parallelograms which have the same altitude must be on equal bases.

Ex. 4. Divide a given parallelogram into four equal parallelograms.

PROPOSITION XXXVII. Theorem.

Triangles on the same base and between the same parallels are equal.

Let the $\triangle^s$ *ABC*, *ADC* be on the same base *AC*, and between the same $\parallel^s$ *AC* and *BD*; then *it is required to prove that* $\triangle$ *ABC* = $\triangle$ *ADC*.
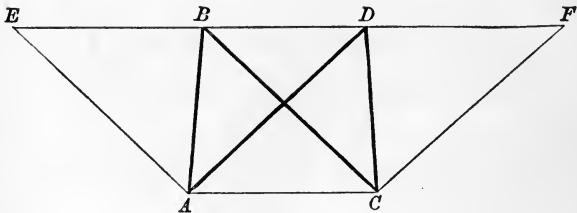
Join *BD* and produce it indefinitely both ways. Through *A* draw *AE* $\parallel$ to *CB*, and *meeting BD in E*. Also through *C* draw *CF* $\parallel$ to *AD* meeting *BD* in *F*.

Then *EACB* and *DACF* are $\parallel^{ms}$ on the same base *AC* and between the same $\parallel^s$.

Hence $\parallel^m$ *EACB* = $\parallel^m$ *DACF*. [I. 35.

But $\triangle$ *ABC* is half $\parallel^m$ *EACB*, [I. 34.

and $\triangle$ *ADC* is half $\parallel^m$ *DACF*. [I. 34.

$\therefore \triangle$ *ABC* = $\triangle$ *ADC*.

PROPOSITION XXXVIII. Theorem.

Triangles on equal bases and between the same parallels are equal.

Let ADC and DEF be two $\triangle$s on equal bases and between the same parallels BF and AD; then *it is required to prove that these $\triangle$s are equal.*
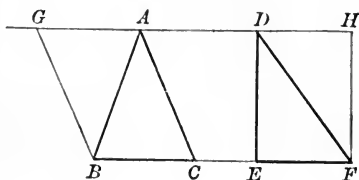
Join AD and produce it indefinitely both ways. Through B draw $BG \parallel$ to AC *and meeting DA in G.* Also through F draw $FH \parallel$ to ED and meeting AD in H.

Then by const. the figures $GBCA$ and $DEFH$ are $\parallel^{\text{ms}}$; and they are on equal bases and between the same $\parallel$s.

Hence $\parallel^{\text{m}} GBCA = \parallel^{\text{m}} DEFH.$ [I. 36.

But $\triangle ABC$ is half $\parallel^{\text{m}} GBCA,$ [I. 34.

and $\triangle DEF$ is half $\parallel^{\text{m}} DEFH$;

      $\therefore \triangle ABC = \triangle DEF.$

The **altitude** of a triangle, with reference to any particular side as base, is the perpendicular drawn to the base from the opposite angular point.

It is easily seen that $\triangle$s which are between the same parallels have equal altitudes, and that $\triangle$s which have equal altitudes can be so placed as to be between the same parallels.

Ex. 1. Shew that equal △ˢ which are between the same parallels are on equal bases.

Ex. 2. Of two triangles between the same parallels that which has the greater base has the greater area.

Ex. 3. If the base of a triangle be divided into any number of equal parts and the points of division be joined to the vertex, the triangle will thereby be divided into the same number of equal triangles.

Ex. 4. Two triangles are between the same parallels and one triangle is double the other; shew that the base of one triangle must be double the base of the other.

Ex. 5. Two triangles have a common vertex and their bases are in the same straight line; shew that if the base of one triangle be three times the base of the other, then the area of the first triangle will be three times that of the second.

Ex. 6. If two sides of a triangle are given, the area will be greatest when these sides are at right angles.

Ex. 7. If D, E are the middle points of the sides AB, AC of the triangle ABC, and DE be drawn, the triangle ADE will be one-quarter of the triangle ABC.

Ex. 8. If D, E be the middle points of the sides AB, AC of the triangle ABC and if BE, CD intersect at F, the triangle BFC will be equal to the quadrilateral $ADFE$.

Ex. 9. Shew that, if the line CD is bisected by AB, the triangles ACB and ADB are equal in area.

Ex. 10. Two triangles of equal area are on the same base and on opposite sides of it; prove that the straight line joining their vertices is bisected by the base, produced if necessary.

Ex. 11. Shew that, if D be the middle point of the base BC of the △ ABC, and P be any point on AD, the △ˢ APB and APC will be equal.

Ex. 12. Shew that, if the △ˢ APB and APC are equal, the point P must be on the line joining A to the middle point of BC.

Ex. 13. Shew that the diagonals of a parallelogram divide the figure into four equal triangles.

Ex. 14. Shew that, if two triangles have two sides of the one equal respectively to two sides of the other, and if the included angles are supplementary, the triangles will be equal in area.

PROPOSITION XXXIX. Theorem.

Equal triangles on the same base, and on the same side of it, are between the same parallels.

Let ABC, DBC be equal $\triangle$s on the same base BC and on the same side of it; *it is required to prove that AD is parallel to BC.*
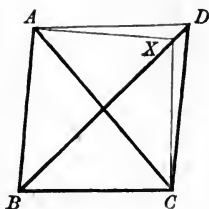
Join AD. Draw $AX \parallel$ to BC *and meeting BD, produced if necessary*, in the point X. Join XC.

Then ABC, XBC are $\triangle$s on the same base and between the same $\parallel$s.

$$\therefore \triangle ABC = \triangle XBC. \qquad [\text{I. 37.}$$

But, by hyp., $\triangle ABC = \triangle DBC.$

$$\therefore \triangle XBC = \triangle DBC,$$

which is impossible, unless X coincides with D, so that AD must itself be $\parallel$ to BC.

PROPOSITION XL. Theorem.

Equal triangles on equal bases in the same straight line and on the same side of it are between the same parallels.

Let ABC, DEF be equal $\triangle^s$ on equal bases BC, EF in the same straight line BF, and on the same side of BF; then *it is required to prove that AD is parallel to BF.*
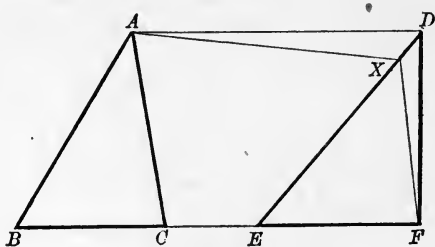
Join AD. Draw AX parallel to BF and meeting ED, produced if necessary in the point X. Join XF.

Then ABC, XEF are $\triangle^s$ on equal bases and between the same $\|^s$.

$$\therefore \triangle ABC = \triangle XEF. \qquad \text{[I. 38.}$$

But, by hyp., $\qquad \triangle ABC = \triangle DEF.$

$$\therefore \triangle XEF = \triangle DEF,$$

which is impossible, unless X coincides with D, so that AD must itself be parallel to BC.

Ex. 1. Shew that equal triangles on equal bases have the same altitude.

Ex. 2. Shew that equal triangles which have equal altitudes have equal bases.

Ex. 3. The straight line joining the middle points of two sides of a triangle is parallel to the third side and equal to half of it.

Ex. 4. If D, E, F be the middle points of the sides BC, CA, AB of the triangle ABC, shew that the straight lines DE, EF, FD divide the triangle ABC into four other triangles equal in all respects.

Ex. 5. If a quadrilateral be bisected by each of its diagonals, it must be a parallelogram.

Ex. 6. Shew that if one diagonal of a quadrilateral bisect the quadrilateral it will bisect the other diagonal.

Ex. 7. A quadrilateral is divided into four triangles by its diagonals; shew that, if two adjacent triangles be equal, the other two triangles will also be equal.

Ex. 8. A quadrilateral is divided into four triangles by its diagonals; shew that, if two opposite triangles are equal, one pair of opposite sides will be parallel.

PROPOSITION XLI. THEOREM.

If a parallelogram and a triangle be on the same base and between the same parallels, the parallelogram will be double the triangle.

Let the $\parallel^m$ $ABCD$, and the $\triangle$ EBC be on the same base and between the same parallels BC and AE; then *it is required to prove that $\parallel^m$ $ABCD$ is double $\triangle$ EBC.*
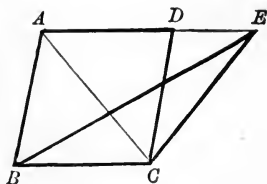
Join AC. Then the $\triangle^s$ ABC, EBC are on the same base and between the same parallels;

$$\therefore \triangle ABC = \triangle EBC. \qquad [\text{I. 37.}$$

But $\parallel^m$ $ABCD$ is double $\triangle$ ABC, for any $\parallel^m$ is bisected by a diagonal.

$$\therefore \parallel^m ABCD \text{ is double } \triangle EBC.$$

A parallelogram is often described by naming two opposite angular points instead of all the four angular points in order.

Thus the $\parallel^m$ $ABCD$ may be called the $\parallel^m$ AC, or the $\parallel^m$ BD.

Ex. 1. If O be any point within the parallelogram $ABCD$, the sum of the triangles OAB and OCD will be equal to half the parallelogram $ABCD$.

Ex. 2. Through the ends of each diagonal of a quadrilateral lines are drawn parallel to the other diagonal; shew that the area of the parallelogram so formed is double the area of the quadrilateral.

PROPOSITION XLII. Problem.

To describe a parallelogram equal to a given triangle, and having one of its angles equal to a given angle.

Let ABC be the given $\triangle$, and D the given $\angle$. *It is required to make a* $\parallel^m$ *equal in area to the* $\triangle$ *ABC and having one of its angles equal to the angle D.*
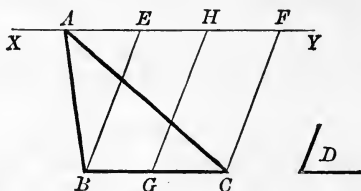
Through A draw a line XY parallel to BC. At the point B in the st. line CB make an angle CBE equal to $\angle D$, and let BE meet XY in the point E. Draw CF parallel to BE meeting XY in F.

Then $\triangle ABC$ is half $\parallel^m EBCF$, $\because$ they have the same base and are between the same parallels.

Bisect BC in G, and draw GH parallel to BE and meeting XY in H.

Then $\parallel^m EBGH$ is also half $\parallel^m EBCF$.

Hence $\parallel^m EBGH = \triangle ABC$, and $\angle EBG = \angle D$.

PROPOSITION XLIII. Theorem.

The complements of the parallelograms, which are about the diagonals of any parallelogram are equal.

Let $ABCD$ be a $\parallel^m$, and AC one of its diagonals, let EH, GF be $\parallel^{ms}$ about AC (that is $\parallel^{ms}$ one diagonal of which is along AC); then *it is required to prove that the* $\parallel^{ms}$ *BK, KD, which make up the figure ABCD* (and which are therefore called the complements of the $\parallel^{ms}$ EH and GF), *are equal to one another.*
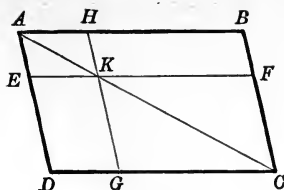
Since every ‖^m is bisected by its diagonal,

$$\triangle\ ABC = \triangle\ ADC,$$

$$\triangle\ AHK = \triangle\ AEK,$$

and $$\triangle\ KFC = \triangle\ KGC.$$

From $\triangle\ ABC$ take the sum of the $\triangle$^s AHK and KFC, and from the $\triangle\ ADC$ take the sum of the $\triangle$^s AEK and KGC; then the remainders will be equal.

Hence   ‖^m HF = ‖^m KD.

It should be noticed that the converse of this theorem is true, namely :—

'*If any* ‖^m *be divided into four* ‖^{ms} *by two lines* ‖ *respectively to two adjacent sides; then if two opposite* ‖^{ms} *be equal, the other two* ‖^{ms} *will be about a diagonal of the given* ‖^m.'

Let GE, EH, HF, FG be the four ‖^{ms} into which ‖^m AC is divided by lines ‖ to adjacent sides, and let ‖^m EG = ‖^m FH. Then we have to prove that AKC is a st. line.

For, if AKC is not a st. line, let AK cut DC in the point X. Draw XY ‖ to AD and cutting AB in Y. Then the complements GE and KY are equal. Hence ‖^m KY = ‖^m KB, which is impossible. Hence AKC must be a straight line.

Ex. 1. Shew that each of the parallelograms about the diagonal of a rhombus, is a rhombus.

Ex. 2. Shew that each of the parallelograms about the diagonal of a rectangle, is a rectangle.

Ex. 3. Shew that the parallelograms DH and AF are equal.

Ex. 4. Shew that the parallelograms EC and GB are equal.

Ex. 5. Shew that, if *KD* and *KB* be joined, the triangles *AKD* and *AKB* will be equal.

Ex. 6. Shew that the triangles *DEB, DHB* are equal.

Ex. 7. Shew that the triangles *DGB, DFB* are equal.

Ex. 8. Shew that the straight lines *EH, DB, GF* are parallel.

PROPOSITION XLIV. PROBLEM.

To a given straight line apply a parallelogram which shall be equal to a given triangle, and have one of its angles equal to a given angle.

Let *AB* be the given st. line, *C* the given ∠ and *PQR* the given triangle; then *it is required to make a ‖^m equal to △ PQR, having AB for one of its sides, and having one of its angles equal to ∠ C.*
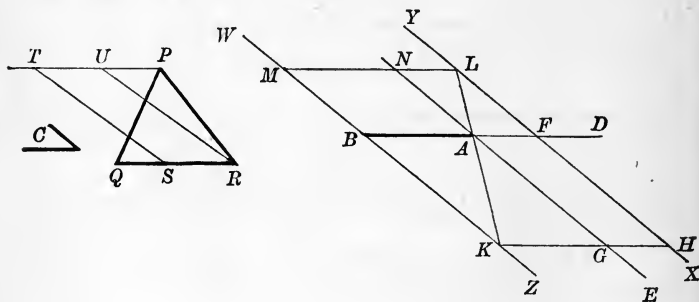
Make ‖^m *TSRU* equal to △ *PQR* and with ∠ *SRU* = ∠ *C*.
[I. 42.

Produce *BA* to *D*. Make ∠ *DAE* = ∠ *C*. From *AD* cut off *AF* = *RS*, and from *AE* cut off *AG* = *RU*.

Through *F* and *B* draw *XY, ZW* parallel to *AE*. Through *G* draw *KGH* ‖ to *BAD, cutting XY in H* and *ZW in K*.

Join *KA*, then *KA* cannot be ‖ to *XY*, since it meets *AG* which is parallel to *XY*. Produce *KA* to meet *XY* in *L*. Lastly, through *L* draw *LNM* ‖ to *AB*, and *meeting EA* produced in *N* and *ZW in M*.

Then *AM* is the ‖^m required.

For, by construction, AM and AH are the complements of the ‖$^{\text{ms}}$ about the diagonal KL of the ‖$^{\text{m}}$ $KMLH$.

Hence   ‖$^{\text{m}}$ $AM =$ ‖$^{\text{m}}$ AH.   [I. 43.

But ‖$^{\text{m}}$ AH was made equal in all respects to the ‖$^{\text{m}}$ $TSRU$, and by construction ‖$^{\text{m}}$ $TSRU = \triangle PQR$.

Hence   ‖$^{\text{m}}$ $AM = \triangle PQR$.

Thus ‖$^{\text{m}}$ AM is the ‖$^{\text{m}}$ required, for it is equal to the given $\triangle$, and has AB for one of its sides and the angle BAN is equal to the given $\angle C$.

Euclid proves at some length that KA will meet XY, but offers no proof that KG will meet XY, although this equally needs proof. Both alike are included in Prop. xxx. **Cor.** Other cases occur in Props. xxxvii., xxxviii., xxxix. and xl.

The problem is sometimes enunciated in the following form:

‘*Construct a parallelogram equal to a given parallelogram and having one of its sides of given length.*’

It will be seen that the problem is solved by drawing a ‖$^{\text{m}}$ such that a given ‖$^{\text{m}}$ $FAGH$ may be one of its complements and that the other complement may have one of its sides of given length.

It would be well for the student first to consider the simpler case— ‘To make a rectangle equal to a given rectangle and having one side of given length.’

The following example leads to another construction which may be given instead of that in the text:

Construct a triangle equal to a given triangle, and having one of its sides of given length.

Let ABC be the given triangle. Take a point X on BC, or BC produced, such that BX is equal to the given length. Join XA, and through C draw the line $CY \parallel XA$ and meeting BA, produced if necessary, in the point Y. Join XY.

Then since $CY \parallel XA$, $\triangle YAX = \triangle CAX$. Add $\triangle AXB$ to each. Then $\triangle YXB = \triangle CAB$, and the side BX is of the required length.

PROPOSITION XLV. Problem.

To describe a parallelogram equal to a given rectilineal figure, and having an angle equal to a given angle.

Let *ABCDE* be the given rectilineal figure, and *F* the given angle; *it is required to make a ‖^m equal to the figure ABCDE and having one of its ∠^{s} equal to ∠ F.*

Divide the figure into △^{s} by joining angular points.
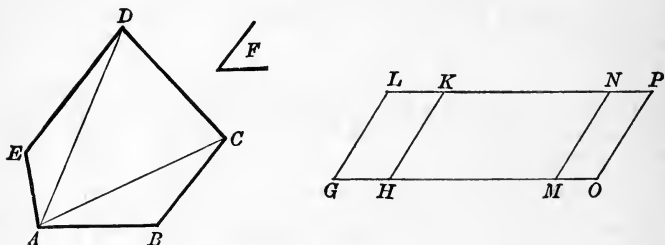
Make the ‖^m *GHKL* equal to △ *ABC* and having

$$∠ HGL = ∠ F.$$ [I. 42.

To the line *HK* apply a ‖^m *HKNM* equal to △ *ACD* and having ∠ *MHK* = ∠ *F*. [I. 44.

To the line *MN* apply ‖^m *MNPO* equal to △ *ADE* and having ∠ *OMN* = ∠ *F*.

Proceed in this way until a ‖^m has been made equal to each of the △^{s} into which the given figure was divided.

Then   ∵ *LG* is ‖ to *KH*,

sum of ∠^{s} *LGH* and *KHG* = two rt. ∠^{s}.   [I. 29.

But, by construction,

$$∠ LGH = ∠ F = ∠ KHM;$$

∴ sum of ∠^{s} *KHM* and *KHG* = two rt. ∠^{s}

Hence   *GHM* is a st. line.   [I. 14.

Then, since   *KN* is ‖ to *GHM*,

and   *LK* is ‖ to *GHM*;

∴ *LKN* is also a st. line.

Similarly it can be proved that *HMO* and *KNP* are st. lines, and so for any number of ∥^{ms}.

Hence the figure *LO* is a ∥^{m}, and it is made up of parts which by construction are equal respectively to the different △^{s} into which the given figure was divided.

The ∥^{m} *LO* is therefore equal to the given figure, and ∠*OGL* = ∠*F*.

Ex. 1. Make a rectangle equal to the sum of two given rectangles.

Ex. 2. Make a rectangle equal to the difference of two given rectangles.

PROPOSITION C. PROBLEM.

To describe a triangle equal to a given rectilineal figure.*

Let *ABCDE* be the given rectilineal figure; then *it is required to make a triangle equal to the figure ABCDE.*
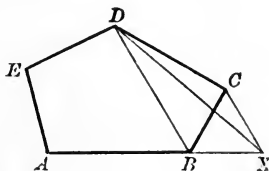
Join *BD*, and through *C* draw the line *CX* parallel to *BD* and meeting *AB* produced in *X*.

Then, ∵ *CX* is ∥ to *BD*,

△*BXD* = △*BCD*. [I. 37.

To each add the figure *BDEA*.

Then figure *XDEA* = figure *BCDEA*.

We have therefore made a figure *XDEA* equal in area to, but having one side fewer than, the given figure *BCDEA*.

* This is given in T. Simpson's Geometry, 1780.

This process may now be applied to the figure $AXDE$, and may be repeated as often as may be necessary until at last a figure of only three sides will be made whose area is equal to that of the given rectilineal figure.

Euclid's method of making a parallelogram equal to a given rectilineal figure by means of Propositions XLII., XLIV. and XLV. would be practically very long and tedious, and it would be much shorter and simpler first to make a triangle equal to the given figure by the method of Prop. C, and then to use Euclid's Prop. XLII.

PROPOSITION XLVI. Theorem.

On a given straight line to describe a square.

Let AB be the given st. line ; then *it is required to describe a square on* AB.
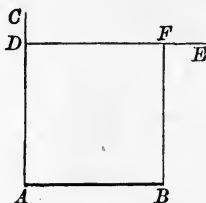
From A draw $AC \perp^r$ to AB, and cut off AD equal to AB. Through D draw $DE \parallel$ to AB, and through B draw $BF \parallel$ to AB and meeting DE in the point F. Then $ABFD$ is the square required.

For, by construction, $ABFD$ is a $\parallel^m$, and its opp. sides are ∴ equal.

Hence $AB = DF$ and $BF = AD$.

But, by construction, $AD = AB$;

$$\therefore AB = AD = DF = FB.$$

Also the angle BAD is a rt. $\angle^e$, by construction.

The figure $ABFD$ is therefore a square, for all its sides are equal and one of its angles is a right angle.

Cor. *If two adjacent sides of a rectangle are equal, it is a square.*

Euclid's definition of a square is that it is 'a four sided figure which has all its sides equal and *all* its angles right angles.'

Now if the four sides of a quadrilateral are all equal, and a diagonal be drawn, it follows at once [I. 8] that the two triangles into which the figure is divided are equal in all respects, and therefore [I. 27] opposite sides are parallel, so that *every rhombus is a parallelogram.* And it follows at once from I. 29 that all the angles of a parallelogram are right angles if any one angle is a right angle.

Euclid's definition is therefore objectionable since it contains more than is necessary. The same objection would apply to the definition of a triangle as 'a three sided figure which has three angular points.'

Ex. 1. Prove that two squares are equal in all respects when their sides are equal.

Ex. 2. Prove that the sides of two equal squares must be equal.

Ex. 3. Describe a rhombus one of whose sides is a given straight line and one of whose angles is equal to a given angle.

Ex. 4. Describe a square which will have a given straight line for one of its diagonals.

Ex. 5. Shew that the diagonals of a square are equal and are at right angles to one another.

Ex. 6. Shew that the parallelograms about the diagonals of a square, are squares.

Ex. 7. Construct a square which has one angular point at a given point and two others on a given straight line.

Ex. 8. Construct a rectangle having given the lengths of two adjacent sides.

PROPOSITION XLVII. Theorem.

The square described on the hypotenuse of a right-angled triangle is equal to the sum of the squares described on the other two sides.

Let ABC be a right-angled triangle, BAC being the right angle; then *it is required to prove that the square described on BC is equal to the sum of the squares described on CA and AB.*
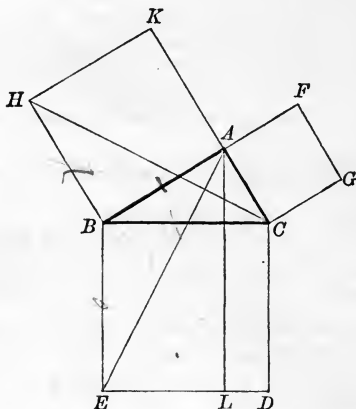
On BC, CA, AB describe the squares BD, CF, AH.

Through A draw $AL \parallel$ to BE or CD and meeting ED in L.

Join AE and HC.

$\because \angle^s BAC$ and BAK are right angles,

$\therefore CAK$ is a st. line.   [I. 14.

Also $\because \angle^s CAB$ and CAF are right angles,

BAF is a st. line.

Now the rt. $\angle HBA =$ rt. $\angle EBC$.

Add $\angle ABC$ to each of these equals;

then   whole $\angle HBC =$ whole $\angle EBA$.

Then in $\triangle^s$ *HBC, EBA,*

$$\because \begin{cases} & HB = BA, \qquad\qquad [Const.\\ & BC = BE \qquad\qquad [Const.\\ \text{and}\ \angle\,HBC = \angle\,EBA. \end{cases}$$

Hence $\qquad\qquad \triangle HBC = \triangle EBA.$

But sq. *BK* is double $\triangle HBC$, since they are on the same base and between the same parallels.

And, for a similar reason, $\|^m$ *BL* is double $\triangle ABE$.

Hence $\qquad\qquad$ sq. $BK = \|^m\ BL.$

Similarly by joining *AD* and *BG* it can be proved that

$$\text{sq. } CF = \|^m\ CL.$$

But $\|^{ms}$ *BL* and *CL* together make up the whole square *BD*.

Hence

$$\text{sq. } BD = \text{sum of the squares } BK \text{ and } CF,$$

i.e. square on *BC* = sum of squares on *CA* and *AB*.

Ex. 1. Shew that the square on a diagonal of a square is double the original square.

Ex. 2. Find a straight line the square on which is (1) double, and (2) treble a given square.

Ex. 3. Shew how to find a square which is equal to the sum of any number of given squares.

Ex. 4. Find a line the square on which is equal to the difference of two given squares.

Ex. 5. Shew that, in the figure to I. 47,

(i) $\qquad\qquad\qquad BF = CK.$

(ii) *HAG* is a straight line.

(iii) *BK* and *CF* are parallel to the bisector of the angle *BAC*.

(iv) if *CH* cut *AB* in *X*, and *BG* cut *AC* in *Y*; then will

$$\triangle\ CXK = \triangle\ CBA = \triangle\ BYF.$$

(v) $\qquad\qquad\qquad AX = AY.$ $\qquad\qquad$ [From (i) and (iv).]

(vi) *AE* is perpendicular to *CH*, and *AD* to *BG*.

The above proposition, which is supposed to have been discovered by Pythagoras, is of very great importance.

Many ways of shewing how two squares can be cut up into pieces and put together again so as to form a single square have been invented. The simplest method is the following.

Let the two squares be *ABCD* and *AEFG*, and let them be placed side by side, as in the figure, with the angular point *A* common, the side *AE* lying along *AD*, the sides *GA* and *AB* being therefore in the same straight line, since the angles *GAE* and *BAE* are right angles.
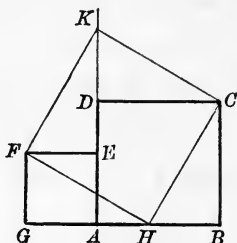
Cut off *BH* equal to *GA*. Also produce *AD* to *K* making *DK* equal to *AE*.

Join *FH, HC, CK, KF*.

Then, since $GA = HB$, $GH = AB$.

Also, since $DK = AE$, $EK = AD$.

Hence the four right-angled triangles *FGH, HBC, CDK* and *KEF* are equal in all respects, for in each case the sides containing the right angle are equal respectively to *AB* and *GA*. And, since these four triangles are equal in all respects,

$$FH = HC = CK = KF.$$

Also $\angle BCH = \angle DCK.$

Hence $\angle HCK =$ sum of $\angle^s$ *HCD* and *DCK*

$=$ sum of $\angle^s$ *HCD* and *HCB*.

Thus $\angle HCK$ is a right angle.

The figure *FHCK* is therefore a square.

If then the triangle *FGH* be moved to the position *FEK*, and the triangle *HBC* to the position *KDC*, the parts into which the squares *ABCD* and *AEFG* are divided by the straight lines *FH* and *HC* will make up the square *FHCK*, which is the square on the hypotenuse of a right-angled triangle of which the sides are equal to *AB* and *AG* respectively.

PROPOSITION XLVIII. Theorem.

If the square on one side of a triangle be equal to the sum of the squares on the other two sides, the angle contained by those two sides is a right angle.

Let ABC be a $\triangle$ such that sq. on BC = sum of squares on CA and AB; then *it is required to prove that $\angle BAC$ is a right angle.*
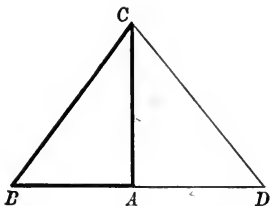
Draw $AD \perp^{r}$ to AC, and make $AD = AB$.

Join DC.

Then, $\because \angle CAD$ is a rt. $\angle$,

sq. on CD = sum of squares on DA and AC [I. 47.

          = sum of squares on BA and AC [*Const.*

          = square on BC.   [*Hyp.*

Hence $CD = BC$.

Then, in the $\triangle^{s}$ BAC, DAC

$$\because \begin{cases} BA = AD, \\ AC = AC \\ \text{and } BC = DC. \end{cases}$$

$$\therefore \angle BAC = \angle DAC. \qquad [\text{I. } 8.$$

But $\angle DAC$ is a rt. $\angle$, by construction.

$\therefore \angle BAC$ is also a rt. $\angle$.

ADDITIONAL PROPOSITIONS.

There are certain properties of straight lines, triangles and parallelograms which are not definitely proved by Euclid but which are both interesting and important. Some of these have already been given as examples, but it will be convenient to collect together the most important of the theorems and problems which the student should know, and which may be quoted as known results, in addition to those included in Book I. of Euclid's Elements.

We know that every point on the circumference of a certain circle is at a fixed distance from the centre of the circle, and that no point not on the circumference is at that distance from the centre. This is expressed by saying that *the locus of a point whose distance from a given point is equal to a fixed length is a circle whose centre is at the fixed point.*

Again it is known that if *ABC* be any triangle, the triangle whose vertex is at any point on the line through *A* parallel to *BC* is equal in area to the triangle *ABC*; and it is also known that if any triangle is equal in area to the triangle *ABC*, and is on the same base *BC* and on the same side of it, its vertex must be on the line through *A* parallel to *BC*. This is expressed by saying that *the locus of the vertices of equal triangles on the same base and on the same side of it is a straight line parallel to the common base.*

If the equal triangles are not necessarily on the same side of the common base, it is easily seen that the locus of their vertices will be a pair of straight lines parallel to the common base and on opposite sides of it.

Def. If an unlimited number of points satisfy a given condition, and if all these points lie on a certain line (or lines), and if also every point on this line (or lines) satisfies the given condition; then the line (or lines) is called the **locus** of the points which satisfy the given condition.

I. *The locus of a point which is equidistant from two given points is a straight line.*

Let *A*, *B* be the two given points. Join *AB* and bisect it in *C*.

Then, since *CA = CB*, *C* is one point on the required locus.

Let *D* be any other point such that *DA = DB*. Join *DC*.

Then the three sides of the triangle *ACD* are equal respectively to the three sides of the triangle *BCD*;

$$\therefore \ \angle DCA = \angle DCB,$$

and therefore, by definition, each of these angles is a right angle.
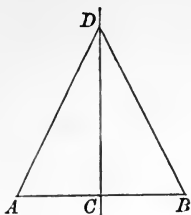
Hence every point which is equidistant from A and B lies on the line which is $\perp^r$ to AB and passes through its middle point.

Moreover every point on the line through C $\perp^r$ to AB is equidistant from A and B.

For, if P be any point on this line, AC, CP and $\angle ACP$ are equal respectively to BC, CP and $\angle BCP$; $\therefore AP = BP$.

Thus *the locus of a point which is equidistant from two given points is the straight line which bisects and is perpendicular to the line joining the given points.*

This may also be enunciated in the following form:

The locus of the vertices of all isosceles triangles on a given base is the straight line which bisects the given base and is perpendicular to it.

Ex. Find a point which is equidistant from three given points.

From the above, the locus of points equidistant from the first two of the given points is a straight line, and the locus of points equidistant from the second and third of the given points is another straight line. Hence the point which is equidistant from the three given points is the point of intersection of these loci. It should be noticed that there is no point equidistant from three given points which are on the same straight line.

II. *The locus of a point whose perpendicular distances from two given straight lines are equal is the pair of straight lines which bisect the angles between the given straight lines.*

Let ABC, DBE be the given straight lines.

Draw BX, BY bisecting the angles CBE, EBA respectively.

[It should be noticed that the bisectors of the angles CBE, ABD are in the same straight line which is perpendicular to the line bisecting the angles EBA, DBC.]
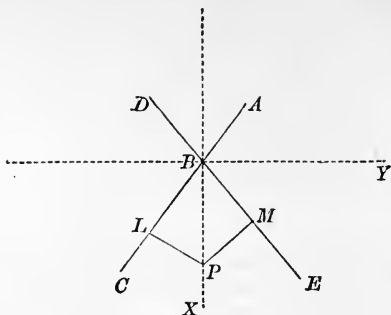
Let P be any point on BX, and draw the $\perp^{rs}$ PL, PM to the lines BC, BE respectively.

Then, in the triangles BPL, BPM

$$\because \begin{cases} & \angle PBL = \quad \angle PBM & \text{[Hyp.} \\ & \text{rt. } \angle BLP = \text{rt. } \angle BMP \\ \text{and} & BP, \text{ opposite the rt. } \angle^s, \text{ is common} \end{cases}$$

$$\therefore PL = PM. \qquad\qquad \text{[I. 26.}$$

Thus the perpendiculars from any point on BX to the lines BC and BE are equal.

Again, let P be any point within the compartment CBE such that the $\perp^{rs}$ from P on BC and BE are equal.

Then, in the triangles PLB, PMB, the two sides PL, PB are equal respectively to the two sides PM, PB and the angles opposite to PB in the two triangles are *right angles*;

$$\therefore \angle PBL = \angle PBM. \qquad\qquad \text{[Prop. B.}$$

Hence, if the perpendiculars from P on the given lines are equal to one another, P must be on the bisector of one of the angles between the given lines.

This proves the theorem.

Ex. Find a point which is equidistant from three given lines. How many such points are there?

III. *The locus of a point whose perpendicular distance from a given straight line is equal to a given length is a pair of straight lines parallel to the given line and on opposite sides of it.*

This follows at once from I. 33 and I. 34, since by I. 28 straight lines which are perpendicular to the same straight line are parallel to one another.

IV. *A quadrilateral is a parallelogram* (i) *when both pairs of opposite sides are equal,* (ii) *when both pairs of opposite angles are equal, and* (iii) *when the diagonals bisect each other.*

Let $ABCD$ be a quadrilateral whose diagonals AC and BD meet in O.
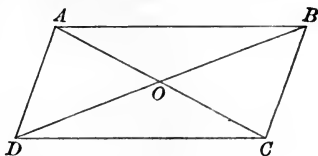
(i) Let $AB = CD$ and $BC = DA$.

Then the three sides of $\triangle ABD$ are equal respectively to the three sides of $\triangle CDB$; hence the angles opp. to equal sides are equal;

$$\therefore \ \angle ADB = \angle CBD, \text{ and } \angle ABD = \angle BDC,$$

and these are pairs of alternate angles.

Hence AD is $\parallel$ to BC, and AB is $\parallel$ to DC, so that $ABCD$ is a $\parallel^{\text{m}}$.

(ii) Let $\angle DAB = \angle BCD$ and $\angle ABC = \angle CDA$.

Then the sum of the angles DAB and ABC is equal to the sum of the angles BCD and CDA.

Hence the sum of the angles DAB and ABC is half the sum of all the angles of the quadrilateral.

But we know that the sum of the interior angles of any quadrilateral is equal to four right angles.

Hence sum of $\angle^s$ DAB and ABC = two rt. $\angle^s$;

$$\therefore \ AD \text{ is } \parallel \text{ to } BC. \qquad \text{[I. 28.}$$

Similarly sum of $\angle^s$ BAD and ADC = two rt. $\angle^s$;

$$\therefore \ AB \text{ is } \parallel \text{ to } CD. \qquad \text{[I. 28.}$$

Hence the figure $ABCD$ is a $\parallel^{\text{m}}$.

(iii) Let $AO = OC$ and $BO = OD$.

Then, in the $\triangle^s$ AOB and COD,

$$AO = OC, BO = OD, \text{ and } \angle AOB = \text{vert. opp. } \angle COD;$$

$$\therefore \ \angle ABO = \angle CDO,$$

and these are alternate angles.

Hence AB is $\parallel$ to CD.   [I. 27.

Similarly by considering the $\triangle^s$ AOD, BOC it can be proved that

$$AD \text{ is } \parallel \text{ to } BC.$$

Hence the figure $ABCD$ is a parallelogram.

Ex. i. If E is the middle point of the side CA of the triangle ABC, the sum of BA and BC will be greater than twice BE.

Produce BE to G, making $EG = BE$. Join CG and GA.

[See figure to Prop. xvi.]

Then, since BG and AC bisect one another at E, $BCGA$ is a $\|^m$, and $CG = BA$. But the sum of BC and CG is greater than BG.

Hence sum of BA and BC is greater than BG, that is greater than twice BE.

Ex. ii. In a right-angled triangle the distance of the middle point of the hypotenuse from the right angle is equal to half the hypotenuse.

Let D be the middle point of the hypotenuse BC of the right $\angle^d$ $\triangle ABC$. Produce AD to E so that $DE = AD$. Join EB and EC. Then $ABEC$ is a rectangle, and its diagonals BC and AE are equal. ∴ &c.

V. *The diagonals of a parallelogram bisect each other, and the point of intersection of the diagonals of a parallelogram is the middle point of any straight line drawn through it and terminated by a pair of parallel sides; also every such line bisects the area of the parallelogram.*

Let $ABCD$ be a $\|^m$ whose diagonals intersect in the point O.

Then, in the $\triangle^s$ AOB, COD,

$$AB = CD, \quad \angle ABO = \text{alt.} \ \angle CDO, \quad \angle BAO = \text{alt.} \ \angle DCO;$$
$$\therefore AO = OC \text{ and } BO = OD.$$
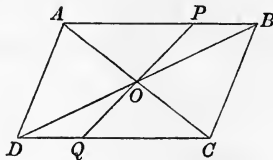
Now let POQ be any line drawn through O and meeting the parallel sides AB, CD in P, Q respectively.

Then in the $\triangle^s$ BOP, DOQ

$$BO = OD, \quad \angle BOP = \text{vert. opp.} \ \angle DOQ, \text{ and } \angle PBO = \text{alt.} \ \angle ODQ;$$
$$\therefore PO = OQ, \text{ and } \triangle BPO = \triangle DQO.$$

To each of these equal triangles add the figure $BOQC$.

Then figure $BPQC = \triangle BDC = \text{half } \|^m BADC$.

Thus any line through the intersection of the diagonals of a parallelogram bisects the area of the parallelogram.

Since the portion of any line, drawn through the intersection of the diagonals of a parallelogram, intercepted by the boundary of the figure, is bisected at the intersection of the diagonals, this point is often called the **centre** of the parallelogram.

VI. *The line joining the middle points of any two sides of a triangle is parallel to, and equal to half, the third side;* and conversely, *the line through the middle point of one side of a triangle parallel to a second side will bisect the remaining side of the triangle.*

Let E, F be the middle points of the sides AC, AB respectively of the $\triangle ABC$. Join EF, BE and CF.
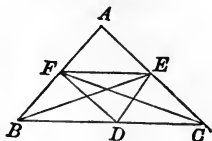
Then, since $\qquad\qquad BF = FA$

$$\triangle BFC = \triangle AFC; \qquad\qquad \text{[I. 38.}$$

$$\therefore \triangle BFC = \text{half } \triangle ABC.$$

Similarly $\qquad\qquad \triangle BEC = \text{half } \triangle ABC.$

Hence $\qquad\qquad \triangle BFC = \triangle BEC,$

and $\qquad\qquad \therefore FE \text{ is } \parallel \text{ to } BC.$ $\qquad\qquad$ [I. 39.

Now let D be the middle point of BC, and join DE, DF.

Then, it can be proved in the same manner that DE is $\parallel$ to AB and $DF \parallel$ to AC.

Hence $BFED$ and $DFEC$ are $\parallel^{ms}$, whence it follows that

$$BD = FE \text{ and } DC = FE.$$

$$\text{Thus } FE \text{ is equal to half } BC.$$

Conversely, let F be the middle point of BA, and let FE be drawn $\parallel$ to BC to meet AC in E.

Then, since FE is $\parallel$ to BC

$$\triangle BFC = \triangle BEC.$$

But, since $BF = FA$, $\triangle BFC = \text{half } \triangle BAC.$

Hence $\qquad\qquad \triangle BEC = \text{half } \triangle BAC,$

and therefore E is the middle point of AC.

Thus *the line drawn through the middle point of one side of a triangle parallel to a second side will bisect the remaining side of the triangle.*

It is easily proved that the four triangles AEF, BFD, CDE, and DEF are equal in all respects, so that *the three lines joining the middle points of the sides of a triangle in pairs will divide the original triangle into four congruent triangles.*

VII. *The four middle points of the sides of any quadrilateral are the angular points of a parallelogram.*

Let P, Q, R, S be the middle points of the sides AB, BC, CD, DA respectively of the quadrilateral $ABCD$.

Join AC, BD, PQ, QR, RS, SP.
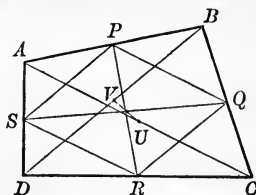
Then, since $AP = PB$ and $CQ = QB$; we know that

$\qquad PQ$ is $\parallel$ to AC and $PQ =$ half AC. $\qquad$ **[VI.]**

Similarly $\qquad RS$ is $\parallel$ to AC and $RS =$ half AC.

Hence $PQ = RS$ and PQ is $\parallel$ to RS.

$\qquad \qquad \therefore PQRS$ *is a parallelogram.*

Now let U, V be the middle points of the diagonals AC, BD respectively.

Then, since P, V are the middle points of AB, DB respectively,

$\qquad PV$ is $\parallel$ to AD and $PV =$ half AD.

Similarly $\qquad UR$ is $\parallel$ to AD and $UR =$ half AD.

Hence $\qquad \qquad PVRU$ *is a parallelogram.*

So also $\qquad \qquad SVQU$ *is a parallelogram.*

And, since the diagonals of a $\parallel^{m}$ bisect each other, *the middle points of PR, QS and UV coincide.*

VIII. *In the triangle ABC, if P be any point on the line joining A to the middle point of BC, then will $\triangle APB = \triangle APC$; and conversely if $\triangle APB = \triangle APC$, the line AP, produced if necessary, will bisect BC.*

First let D be the middle point of BC, and let P be any point on DA. Join PB, PC.

Then $\qquad \because BD = DC$

$\qquad \qquad \triangle ADB = \triangle ADC$

and $\qquad \triangle PDB = \triangle PDC$.

Hence, taking equals from equals,

$\qquad \triangle APB = \triangle APC$.
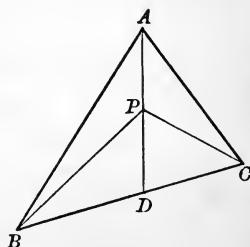
Conversely, let $\triangle APB = \triangle APC$.

Join P to the middle point of BC.

Then we have to prove that APD is a straight line.

Since $BD = DC$, $\triangle BDP = \triangle CDP$.

And, by hyp., $\triangle APB = \triangle APC$.

Hence the sum of the $\triangle$s APB and BPD is equal to half $\triangle ABC$.

But the *straight line AD* bisects $\triangle ABC$.

Hence the $\triangle$s APB and BPD are together equal to the $\triangle$ ADB, which is impossible unless the point P is on the straight line AD.

It can be proved in the same manner that, if the triangles APB and APC are in the ratio of any two whole numbers, then will AP, produced if necessary, cut BC in a point D such that BD and DC are in the ratio of the same two whole numbers. For example, if $\triangle APB$ is three-fourths of $\triangle APC$, then will BD be three-fourths of DC; and conversely, if BD be three-fourths of DC, and P be any point on AD, then will $\triangle APB$ be three-fourths of $\triangle APC$.

Ex. 1. Points D, E are taken on the sides BC, CA respectively of the $\triangle ABC$, so that $2BD = DC$ and $CE = EA$. Shew that, if the lines AD, BE intersect in the point P, $BP = PE$.

Since $BD =$ half DC, $\triangle APB =$ half $\triangle APC$.

And, since $CE = EA$, $\triangle APE =$ half $\triangle APC$.

Hence $\triangle APB = \triangle APE$, and therefore $BP = PE$.

Ex. 2. Points D, E are taken on the sides BC, CA respectively of the $\triangle ABC$, so that $3BD = DC$ and $2CE = EA$. Shew that, if AD, BE intersect in P, $2BP = PE$.

Def. A line drawn from an angular point of a triangle to the middle point of the opposite side is called a **median** of the triangle.

IX. *The three medians of a triangle meet in a point, and their common point is a point of trisection of each median.*
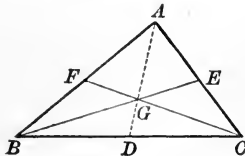
Let D, E, F be the middle points of the sides BC, CA, AB respectively of the $\triangle ABC$. Join BE, CF and let them intersect in G. Join AG and GD. Then we have to prove that AGD is a straight line.
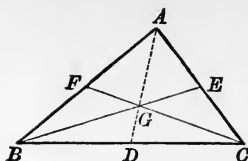
Since $\qquad BF = FA$, $\triangle BFC = \triangle AFC$

and also $\qquad\qquad \triangle BFG = \triangle AFG$.

Hence, taking equals from equals, $\triangle BGC = \triangle AGC$.

Since $CE = EA$, it can be proved in the same manner that

$$\triangle BGC = \triangle BGA.$$

Hence $\qquad\qquad \triangle BGA = \triangle CGA$.

But, since $\quad BD = DC$, $\triangle BGD = \triangle CGD$.

Hence sum of $\quad \triangle^s BGA, BGD = $ sum of $\triangle^s CGA, CGD$.

Hence sum of $\quad \triangle^s BGA, BGD = $ half $\triangle ABC$.

Now, if AGD is not a straight line, draw the straight line AD.

Then, since $BD = DC$, $\triangle ADB = $ half $\triangle ABC$,

and we have proved that the sum of $\triangle^s AGB, BGD$ is half $\triangle ABC$.

Hence the sum of $\triangle^s AGB$ and BGD is equal to $\triangle ADB$, and this is impossible unless the point G is on the straight line AD.

Hence *the three medians AD, BE, CF meet in a point.*

Again, we have proved that

$$\triangle BGA = \triangle CGA.$$

And, since $CE = EA$, $\triangle CGA = $ twice $\triangle EGA$.

$$\therefore \triangle BGA = \text{twice } \triangle AGE ;$$

$$\therefore BG = 2GE.$$

Similarly $CG = 2GF$ and $AG = 2GD$.

If GD be produced to H so that $DH = GD$; then, since BC and CH bisect each other, $GCHB$ is a $\parallel^m$ and $CH = BG$. It is then easily seen that in the $\triangle GCH$ the sides are parallel to the medians of ABC, and that each side is two-thirds of the corresponding median.

Def. The point of intersection of the medians of a triangle is called the **centroid.**

Ex. If the three lines joining a point within a triangle to the angular points divide the triangle into three equal parts, the point must be the centroid of the triangle.

X. *A median of a triangle will bisect any straight line parallel to the side bisected by that median and terminated by the other two sides.*

Let AD be the median bisecting the side BC of the $\triangle ABC$, and let EKF be any line parallel to BC and meeting AB, AD, AC (produced either way if necessary) in the points E, K, F respectively. Join ED, FD.
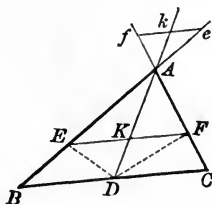
Then, since $BD = DC$, $\triangle ABD = \triangle ADC$.

And, since $BD = DC$, and EF is ∥ to BC, $\triangle BED = \triangle CFD$.

Hence, taking equals from equals, $\triangle AED = \triangle AFD$.

Now, if EK be not equal to KF, and X be taken on KF, or KF produced, such that $EK = KX$.

Then $\because EK = KX$, $\triangle AKE = \triangle AKX$.

and also $\triangle DKE = \triangle DKX$.

Hence $\triangle AED = \triangle AXD$.

But $\triangle AED = \triangle AFD$.

                                          $\therefore \triangle AFD = \triangle AXD$,

and this is impossible, for FX cannot be parallel to AD.   [I. 39.

Hence KF cannot be unequal to KE.

In precisely the same manner can be proved the following proposition, of which the above is a particular case :—

If two equal triangles are on equal bases in the same straight line, the sides of the two triangles will intercept equal lengths from any straight line drawn parallel to their bases.

XI. *If two equal triangles are on the same base and on opposite sides of it, the line joining their vertices will be bisected by the base or the base produced.*

Let ABC, DBC be the equal triangles, and let the base, produced if necessary, cut AD in O.

Take a point X on OD, or OD produced, such that $AO = OX$.

Then it can be easily proved that $\triangle BCX = \triangle BCA$.

Hence $\triangle BCX = \triangle BCD$, and therefore X must coincide with D, for XD cannot be parallel to BC.

XII.   *To divide a given finite straight line into any number of equal parts.*

Let AB be the given finite straight line and let it be required to divide it into five equal parts.
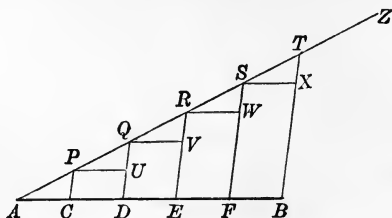
Draw any straight line AZ through one extremity of the given line, and set off along AZ equal lengths AP, PQ, QR, RS, ST, as many in number as there are to be equal parts in AB.

Join TB, and through P, Q, R, S draw lines parallel to TB and cutting AB in the points C, D, E, F respectively.

Then AB will be divided into five equal parts at the points C, D, E, F.

Draw lines through P, Q, R, S parallel to AB and meeting QD, RE, SF and TB respectively in the points U, V, W, X.

Since PC, QD, RE, &c. are parallel, the angles APC, PQU, QRV, &c. are equal.

Also, since AB, PU, QV, &c. are parallel, the angles PAC, QPU, RQV, &c. are equal.

And the sides AP, PQ, QR, RS and ST were made all equal.

Hence the triangles PAC, QPU, RQV, &c. are equal in all respects.

Hence $AC = PU = QV = RW = SX$.

But, since PD, QE, RF, SB are $\|^{\text{ms}}$, $PU = CD$, $QV = DE$, $RW = EF$ and $SX = FB$.

Hence $AC = CD = DE = EF = FB$.

XIII.   *Find a point on a given straight line the sum of whose distances from two given points is the least possible.*

Let AB be the given straight line, and C, D the given points.

Then, if the two given points are on opposite sides of AB, and if the line joining them cut AB, produced if necessary, in the point P, the sum of the lines joining P to C and D is obviously less than the sum of the lines joining any other point on AB to C and D.

If, however, C and D are on the same side of AB [since (by I.) the distance of any point on AB from D is equal to its distance from the point E which is such that DE is $\perp^r$ to AB and is bisected by it], draw $DL \perp^r$ to AB and produce it to E so that $LE = DL$.

Then, if CE cut AB in Q, Q is the point required.
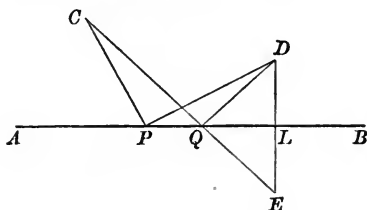
For, since the distance of any point on AB from D is equal to its distance from E, the sum of the distances of any point on AB from C and D is equal to the sum of its distances from C and E, and this latter sum is least for the point Q. Hence the sum of QC and QD is less than the sum of PC and PD, where P is any other point on AB.

Since $\angle DQL = \angle LQE =$ vert. opp. $\angle CQA$, the point Q, the sum of whose distances from C and D is the least possible, is such that *the lines CQ and DQ make equal angles with AB.*

XIV. *Of all triangles on a given base and of given area the isosceles triangle has the least perimeter.*

Let AB be the common base of the triangles. Then, since all the triangles have the same area, their vertices must all lie on a line, CD suppose, parallel to AB.
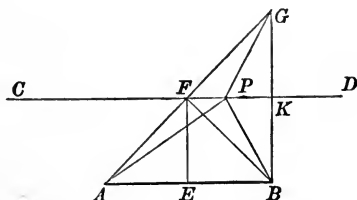
Bisect AB in E, and through E draw the line $EF \perp^r$ to AB and meeting CD in the point F. Join AF, FB.

Then AFB is the isosceles triangle which is on the base AB and has the given area.

Draw $BK \perp$ to CD, and produce BK and AF to meet in G.
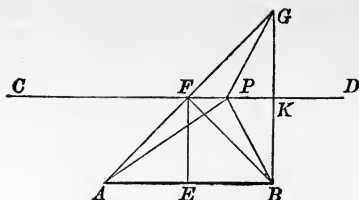
Then, since FK is parallel to AB,

$$\angle GFK = \angle FAB \text{ and } \angle KFB = \angle FBA.$$

Also, since $FB = FA$; $\angle FAB = \angle FBA$,

$$\therefore \angle GFK = \angle BFK.$$

Hence, in the triangles GFK, BFK

$\angle GFK = \angle BFK$, rt. $\angle GKF =$ rt. $\angle BKF$ and FK is common;

$$\therefore GF = BF ;$$

$$\therefore \text{ sum of } AF \text{ and } FB = AG.$$

Now let PAB be any other of the triangles, and join PG.

Then, in the triangles PKG, PKB,

$GK = KB$, PK is common and rt. $\angle PKG =$ rt. $\angle PKB$;

$$\therefore PG = PB.$$

Hence the sum of AP and PB is equal to the sum of AP and PG, and the sum of AP and PG is greater than AG, *i.e.* greater than the sum of AF and FB.

Hence the sum of AP and BP is greater than the sum of AF and FB.

Thus *the perimeter of the isosceles triangle AFB is less than the perimeter of any other triangle APB which is on the same base and is of equal area.*

XV. *Of all triangles on a given base and of given perimeter the isosceles triangle has the greatest area.*

Let ACB be the isosceles triangle on the base AB which has the given perimeter.

Draw through C the line XY parallel to AB.
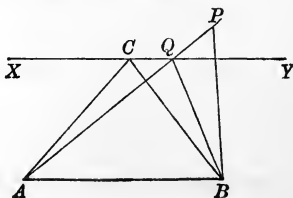
Then the area of any triangle on the base AB is greater than, equal to, or less than the triangle ACB according as its vertex is on the opposite side of XY, on XY, or on the same side of XY as AB.

Now, let PAB be any triangle whose vertex is on the side of XY opposite to AB. Let AP cut XY in Q, and join QB.

The sum of AP and BP is greater than the sum of AQ and QB. But, by the preceding prop., the sum of AQ and QB is greater than the sum of AC and CB.

Hence every triangle on the base AB whose area is equal to or greater than that of the triangle ACB has a greater perimeter, and therefore the area of any triangle with the same perimeter as the $\triangle ACB$ must be less than the area of the $\triangle ACB$.

We have already proved that the three medians of a triangle meet in a point.

Def. When three or more straight lines meet in a point, they are said to be **concurrent**.

The following other cases of concurrence are of importance.

XVI. *The three lines which bisect the angles of a triangle are concurrent.*

Let the bisectors of the angles ABC, ACB of the $\triangle ABC$ meet in the point O.
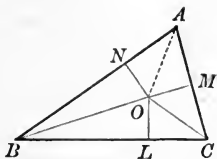
Join OA, then we have to prove that OA bisects the angle BAC.

Draw the lines OL, OM, ON perpendicular to BC, CA, AB respectively.

Then, since BO bisects the angle ABC, we know that $OL = ON$. [II.

And, since CO bisects the angle ACB, we know that $OL = OM$. [II.

Hence $OM = ON$, whence it follows [II.] that OA bisects the angle BAC.

If the sides of a triangle be produced it can be proved in the same manner that the bisector of one of the interior angles of a triangle and the bisectors of the other two exterior angles will meet in a point.

XVII. *The three lines drawn through the middle points of the sides of a triangle perpendicular to those sides are concurrent.*

Let D, E, F be the middle points of the sides BC, CA, AB of the $\triangle ABC$.
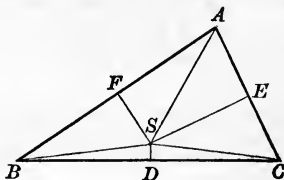
Let the lines through E, F perpendicular to CA, AB respectively meet in S, and join SD.

Then we have to prove that SD is $\perp^r$ to AB.

Join AS, BS, CS.

Then in $\triangle^s$ AES, CES, $AE = EC$, ES is common, and

rt. $\angle AES =$ rt. $\angle CES$.

Hence $\qquad\qquad AS = CS$.

Similarly $\qquad\qquad AS = BS$; and $\therefore BS = CS$.

Then, in $\triangle^s$ BDS, CDS,

$$BD = DC,\ SD \text{ is common, and } BS = CS ;$$
$$\therefore \angle BDS = \angle CDS,$$

and $\qquad\qquad \therefore SD$ is $\perp^r$ to BC.

The point S is equidistant from the three points A, B, C.

XVIII. *The three lines drawn through the angular points of a triangle perpendicular to the opposite side are concurrent.*

Let AD, BE, CF be drawn from the points A, B, C $\perp^r$ to BC, CA, AB respectively; then it is required to prove that AD, BE, CF are concurrent.
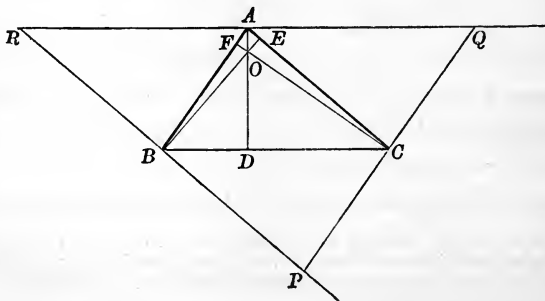
Through A, B, C draw lines *QAR, RBP, PCQ* parallel to *BC*, *CA*, *AB* respectively forming the triangle *PQR*.

Then, since *ABCQ* and *ACBR* are ‖ᵐˢ *BC* = *AQ* and *BC* = *RA*.

Hence A is the middle point of *RQ*, and similarly it can be proved that B and C are the middle points of *RP* and *PQ* respectively.

Also, since *RQ* is ‖ to *BC*, and *AD* is ⊥ʳ to *BC*, *AD* is also ⊥ʳ to *QR*. So also *BE* is ⊥ʳ to *RP* and *CF* ⊥ʳ to *PQ*.

But we know (by the previous proposition) that the lines drawn through the middle points of the sides of a triangle ⊥ʳ to those sides respectively, will meet in a point.

Hence *AD*, *BE* and *CF* are concurrent.

Def. The lines drawn from the angular points of a triangle perpendicular to the opposite sides are called the **perpendiculars** of the triangle, and the point of concurrence of the perpendiculars is called the **orthocentre** of the triangle.

MISCELLANEOUS THEOREMS AND PROBLEMS.

No general rules can be given which will enable a student to prove a new theorem. When the different propositions established by Euclid in Book I. have been thoroughly mastered, together with those marked I. to XVIII., the student must rely upon his own resources. The following additional exercises may, however, be suggestive.

It will be noticed that in order to solve a problem it is generally best to begin by supposing that what is required to be done is already done; and then, by an examination of the diagram, we try to find out in what way the required construction depends upon, and can be effected by means of, other constructions which we have previously shewn how to perform.

In order to prove a new theorem it is also often desirable to begin by supposing that what has to be proved is really true, and then proceed to consider what is *necessary* and *sufficient* to ensure the truth of this assumption. We thus shew, it may be in a series of steps, how the theorem which we require to prove follows necessarily from some theorem which is already known.

This is the reverse of the course adopted in all cases for purposes of demonstration by Euclid, who begins with certain known theorems or known constructions, and proceeds to shew how these known theorems necessarily lead to that which it is required to prove, or how, by means of certain known constructions, the new construction can be effected.

The 'putting together' of known results in order to obtain some new theorem, or some new construction, is called **Synthesis**.

The 'taking to pieces' of a proposed theorem or construction in order to see from what known truths it necessarily follows, or by means of what known constructions it can be effected, is called **Analysis**.

Ex. 1. *Draw a line parallel to the base BC of the triangle ABC and cutting the sides AB, AC in the points F, E respectively so that FE may be equal to the sum of BF and CE.*
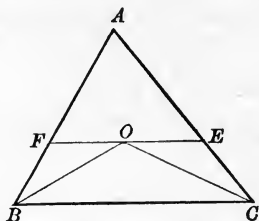
[Suppose the line *FE* to be drawn as required.

Then it naturally suggests itself to take the point *O* on *FE* such that *FO = BF*, and therefore *OE = EC*.

Then, since *BF = FO*, if *BO* be joined, ∠ *FOB* = ∠ *FBO*.

But, since *FOE* is ∥ to *BC*, ∠ *FOB* = ∠ *OBC*.

Hence *O* must be *on the line bisecting the angle ABC*.

Similarly *O* must be *on the line bisecting the angle ACB*.

Hence the required line can be drawn by the following construction*.]

Const. Bisect the angles *ABC, ACB* by the lines *BO* and *CO*, and through *O*, the point of intersection of these bisectors, draw a line parallel to *BC* and cutting *AB, AC* in *F, E* respectively.

Then, by construction,   ∠ *FBO* = ∠ *OBC*,

and,   ∵ *BC* ∥ to *FO*,

                              ∠ *OBC* = ∠ *FOB*;

                           ∴ ∠ *FBO* = ∠ *FOB*;

                           ∴   *FO* =   *FB*.

Similarly *OE = EC*, and ∴ *FE = BF + CE*.

Ex. 2. *Draw through any point P within the angle XOY a straight line cutting the lines OX, OY in the points Q, R respectively so that QP = PR.*

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* In future the 'analysis' will be put in brackets, as in this case.

The student will find it necessary, in all except the very simplest cases, to make an Analysis; this need not, however, be written out for an examiner in addition to the Synthesis.

[Suppose the line QPR to be drawn as required.

Then, if OP be joined, and OP be produced to S so that $PS = OP$, the lines OS and QR bisect each other in P.

Hence $ORSP$ is a $\|^m$. [IV.

Hence the required line can be drawn as follows :]
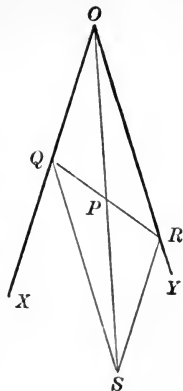
Const. Join OP, and produce it to S so that $PS = OP$.

Draw SR parallel to OX and meeting OY in R.

Draw SQ parallel to OY and meeting OX in Q.

Then, since $SROQ$ is a $\|^m$, its diagonals bisect each other; $\therefore RQ$ passes through the middle point of OS, *i.e.* through P, and RQ is bisected in P.

Let P be without the angle XOY, and let it be required to draw a line PQR cutting OX, OY in Q, R respectively so that $PQ = QR$.

[Suppose the line PQR drawn as required.

Take the point S on OX so that $QS = OQ$.

Then, since PR and OS bisect each other in Q, $PORS$ is a $\|^m$. Hence PS is $\|$ to OY and $SR \|$ to PO. The required line can therefore be drawn by the following construction.]
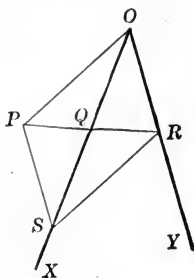
Const. Draw $PS \|$ to OY and meeting OX in S.

Draw $SR \|$ to PO and meeting OY in R.

Then $PORS$ is a $\|^m$, and $\therefore PR$ will be bisected by OS.

Ex. 3. *Bisect a triangle by a straight line drawn through a given point in one of its sides.*

Let P be the given point in the side BC.
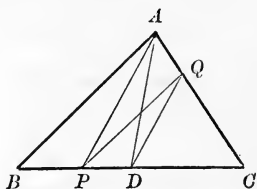
[Suppose the line PQ to be the required line through P which bisects the area of the triangle, so that $\triangle PQC =$ half $\triangle ABC$.

Bisect BC in D and join AD.

Then $\qquad\qquad \triangle ADC =$ half $\triangle ABC = \triangle PQC$.

$\qquad\qquad \therefore \triangle ADC = \triangle PQC$.

Take away the $\triangle QDC$ from each of these equals; then

$$\triangle ADQ = \triangle PDQ,$$

and $\therefore DQ$ is $\parallel$ to AP.

Hence PQ is found by the following construction.]

Const. Bisect BC in D, and draw DQ parallel to AP to cut AC in Q. Join PQ.

Then $\qquad\qquad \because AP$ is $\parallel$ to DQ

$\qquad\qquad\qquad \triangle PQD = \triangle AQD$.

Add $\triangle DQC$ to each of these equals;

then $\qquad\qquad \triangle PQC = \triangle ADC$,

and $\qquad\qquad \triangle ADC =$ half $\triangle ABC$, since $CD =$ half CB;

$\qquad\qquad \therefore \triangle PQC =$ half $\triangle ABC$.

Ex. 4. *Bisect a quadrilateral by a straight line through one of its angular points.*

Let C be the angular point through which the line is to be drawn to bisect the quadrilateral.

Produce DA, and draw $BL \parallel$ to AC to meet DA in L. Then $\triangle ABC = \triangle LAC$, and $\therefore \triangle CLD =$ quad. $ABCD$.

Now bisect LD in P and join CP. Then CP will bisect the quadrilateral.

For, since $\qquad\qquad DP =$ half DL

$\qquad\qquad \triangle CDP =$ half of $\triangle CDL$

$\qquad\qquad\qquad\quad =$ half of quad. $CDAB$.
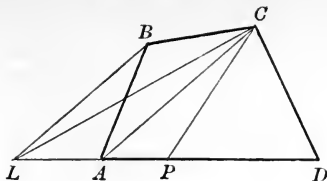
It will be seen that the above construction would fail if LA were greater than AD, that is if $\triangle ALC$, or $\triangle ABC$, were greater than $\triangle ADC$. In that case it would be necessary to draw a line through D parallel to AC to meet BA produced in M. Then the quadrilateral would be bisected by drawing a line from C to the middle point of BM.

Ex. 5. *Bisect a quadrilateral by a straight line drawn through a given point in one of its sides.*
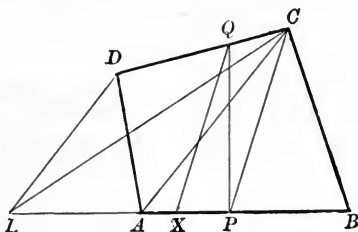
Let P be the given point in the side AB; then it is required to draw a straight line through P which will bisect the quadrilateral $ABCD$.

Draw DL parallel to CA to meet BA produced in L. Join CL.

Then $\triangle CLB =$ quad. $ABCD$.   [See Ex. 4.

Bisect BL in X, then $\triangle CXB =$ half quad. $ABCD$.

Now draw through X a line $\parallel$ to PC to meet CD in Q. Then PQ will bisect the quadrilateral.

For,   since XQ is $\parallel$ to PC, $\triangle CQP = \triangle CXP$;

$\therefore$, adding $\triangle CPB$ to each, fig. $CQPB = \triangle CXB$

$$= \text{half quad. } ABCD.$$

[If BA be greater than AL and BP be greater than BX, the line through $X \parallel$ to CP must be drawn to cut BC in R, and then the line PR will bisect the quadrilateral.]

Ex. 6. *Draw a straight line equal and parallel to a given straight line and having its extremities on two given straight lines.*
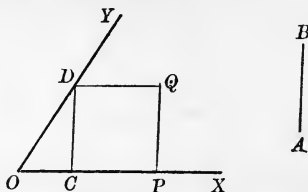
[Suppose *CD* the line required equal and parallel to the given line *AB* and with its ends on the given lines *OX*, *OY*.

Through any point *P* on *OX* draw a line ∥ to *AB*, and cut off a length *PQ*=*AB*.

Then ∵ *DC* and *PQ* are equal and parallel, *DQ* is ∥ to *OX*.

Hence *DC* can be drawn as follows:]

Const. Take any point *P* on *OX* and draw the line *PQ* equal and parallel to *AB*.

Through *Q* draw *QD* ∥ to *OX* to cut *OY* in *D*. Through *D* draw *DC* ∥ to *QP* to cut *OX* in *C*.

Then, since *DP* is a ∥ᵐ, *DC* is equal and ∥ to *QP*, and therefore equal and ∥ to *BA*; *DC* is therefore the line required.

Ex. 7. *Find the locus of a point the sum of whose perpendicular distances from two given intersecting straight lines may be equal to a given length.*
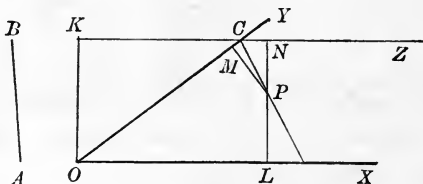
Let *P* be any point such that, if *PL*, *PM* be the ⊥ʳˢ on *OX* and *OY*, the sum of *PL* and *PM* may be equal to *AB*.

Draw *OK* ⊥ʳ to *OX* and make *OK*=*AB*.

Draw through *K* the line *KZ* ∥ to *OX* and cutting *OY* in the point *C*.

Produce *LP* to cut *KZ* in the point *N*.

Then, since *OLNK* is a ∥ᵐ,

$$LN=OK=AB; \text{ and } LN=LP+PM;$$
$$\therefore PN=PM.$$

Thus the perpendiculars from *P* to the fixed lines *OC*, *CZ* are equal, and therefore *P* must be on the line through *C* bisecting the angle *OCZ*.

Ex. 8. *On the sides of any triangle ABC, and external to the triangle the squares BCDE, CAFG, ABHK are described, X, Y, Z being the centres of these squares, and the parallelograms FAKL, EBHM, DCGN are completed. Ba, Bc are the perpendiculars from B on the lines AE, CH respectively, and O is the point of intersection of AE and CH. Prove the following properties of the figure :*

(i) $\triangle FAK = \triangle HBE = \triangle DCG = \triangle ABC$.

(ii) *LA, MB, NC* are $\perp^r$ to *BC, CA, AB* respectively.

(iii) *LA, MB, NC* meet in a point.

(iv) The medians of the $\triangle ABC$ through *A, B, C* are $\perp^r$ to *FK, HE, DG* respectively.

(v) *BLAE, CLAD,* &c. are $\parallel^{ms}$.

(vi) If *AV* be $\perp^r$ to *BC*, and if *BC* be produced to cut *HM, GN* in *S, T* respectively, $SB = CT = AV$; also $GT = CV$ and $SH = BV$.

(vii) *HC* and *AE* intersect at right angles at *O*.

(viii) *CH, BG, LA* meet in a point.

(ix) *BO* bisects $\angle HOE$.
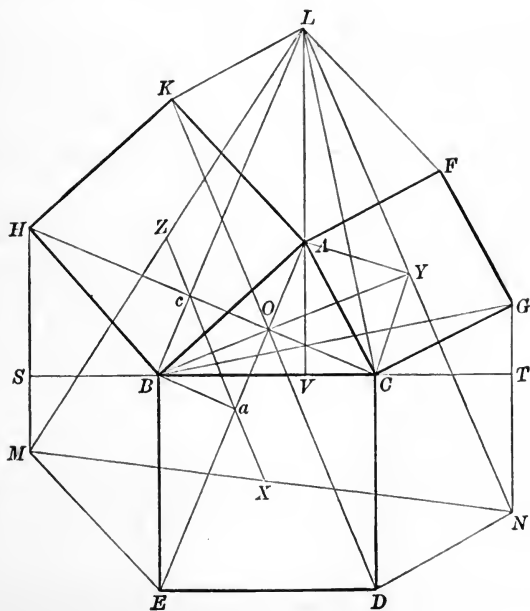
   (x)   *BOY* is a straight line.

   (xi) *KOD* is a straight line.

  (xii) *BaOc* is a square.

 (xiii) *XacZ* is a straight line $\perp^r$ to *BOY*.

 (xiv) *AX, BY, CZ* meet in a point.

  (xv) *X, Y, Z* are the middle points of *MN, NL, LM* respectively.

[The student is recommended to attempt to prove these theorems for himself, and also to see in what way they are modified when the triangle is right angled.]

   (i) Since *BAK* and *CAF* are rt. $\angle$ˢ, and all the angles at a point are together equal to four rt. $\angle$ˢ, $\angle$ˢ *BAC* and *FAK* are supplementary. But $\angle$ˢ *LFA* and *FAK* are supplementary.

Hence $\angle AFL = \angle CAB$; also $AF = AC$ and $FL = AK = AB$.

Hence $\triangle$ˢ *FAL* and *CAB* are congruent.

Then $\qquad\qquad \triangle FAK = \triangle FAL = \triangle CAB,$

and similarly $\qquad \triangle HBE = \triangle CAB = \triangle DCG.$

  (ii) It has just been proved that $\triangle$ˢ *FAL* and *ACB* are congruent, so that $\angle FAL = \angle ACB$. Let *LA* produced cut *BC* in *V*; then, since $\angle CAF$ is rt. $\angle$, $\angle FAL$ and $\angle CAV =$ rt. $\angle$; $\therefore \angle ACV$ and $\angle CAV =$ rt. $\angle$.

Hence *LA* is $\perp^r$ to *BC*. Similarly *MB* $\perp^r$ to *CA* and *NC* $\perp^r$ to *AB*.

 (iii) Since *LA, MB, NC* are perpendicular to the opposite sides *BC, CA, AB* respectively, they will meet in a point.　　　　　　[XVIII.

 (iv) By completing the ‖ᵐ *BACP*, it can be proved exactly as in (ii) that the diagonal *AP*, which will we know bisect *BC*, is $\perp^r$ to *FK*.

  (v) We have proved in (ii) that *LA* is parallel to *BE*, and in (i) that $AL = BC = BE$. Hence *BLAE* is a ‖ᵐ. Hence also *ALHM* and *ALCD* are ‖ᵐˢ, &c. Thus *BL* is equal and ‖ to *EA*, *CL* equal and ‖ to *AD*, &c.

 (vi) Since $\angle ACV$ and $\angle GCT$ are complementary, $\angle ACV = \angle CGT$, $\angle CAV = \angle GCT$. Also $CG = CA$.

Hence $\triangle$ˢ *CGT* and *ACV* are congruent.

Hence $TG = CV$ and $CT = AV$.

Similarly $SH = BV$ and $BS = AV$.

 (vii) Let *AE* cut *BC* in *U*. Then $\triangle$ˢ *EBA* and *CBH* are congruent, since $EB = CB$, $BA = BH$ and $\angle EBA = \angle CBH$. Thus $\angle BCH = \angle BEA$; also $\angle CUO = \angle BUE$. Hence sum of $\angle$ˢ *BCO* and *CUO* = sum of $\angle$ˢ *BEA* and *BUE* = rt. $\angle$.

Hence $\angle COE$ is a right angle.

Thus *CH* is $\perp^r$ to *AE*. Similarly *BG* is $\perp^r$ to *AD*, &c.

(viii) *CH* is $\perp^r$ to *AE* by (vii). Hence *CH* is $\perp^r$ to *BL* by (v). So also *BG* is $\perp^r$ to *CL*. And, from (ii), *LA* is $\perp^r$ to *BC*.

Thus *AL, BG, CH* are the $\perp^{rs}$ of the $\triangle$ *LBC*; they are $\therefore$ concurrent.

(ix) If *Bc, Ba* be $\perp^r$ to *CH, AE* respectively, the $\triangle^s$ *BCc* and *BEa* are equal in all respects; $\therefore$ *Bc* = *Ba*. And, since the $\perp^{rs}$ from *B* on the lines *OH*, *OE* are equal, *BO* bisects the angle *HOE*.

(x) The diagonals of a square are $\perp^r$; $\therefore \angle AYC$ = rt. $\angle^s = \angle COA$. But the four angles of the quad. *YAOC* are together equal to four rt. $\angle^s$. Hence the $\angle^s$ *YAO, YCO* are supplementary, and if *Ya, Yγ* be the perpendiculars on *OA, OC* respectively, $\angle YAa = \angle YC\gamma$; also $\angle AaY = \angle C\gamma Y$, and *YA* = *YC*. Hence the $\perp^{rs}$ *Ya, Yγ* are equal, and therefore *Y* is on the bisector of $\angle AOC$.

Hence *BOY* is a straight line.

(xi) Since $\angle AOH$ = rt. $\angle = \angle HKA$, the angles *KAO* and *KHO* are supplementary, and *KA* = *KH*.

Hence, as in (x), *KO* bisects $\angle AOH$. Similarly *DO* bisects the vertically opposite $\angle COE$.

Hence *KOD* is a straight line.

(xii) Since *BcOa* is a rectangle, and the adjacent sides *Ba, Bc* are equal, *BcOa* is a square, and $\therefore$ *ca* bisects *OA* at rt. $\angle^s$.

(xiii) Since *EOC* is a rt. $\angle$, and *X* is the middle pt. of *CE*, *XO* = *XC* = *XB*. Similarly *ZO* = *ZB*. Hence *X, Z* are vertices of isosceles $\triangle^s$ on the base *BO*. Hence *XZ* bisects *BO* at rt. $\angle^s$. But, by (xii), *ca* bisects *BO* at rt. $\angle^s$. Hence *XacZ* is a straight line.

(xiv) Since *YOB* is $\perp^r$ to *XZ*, and similarly *XA* $\perp^r$ *YZ* and *ZC* $\perp^r$ *XY*, the lines *AX, BY, CZ* meet in the orthocentre of the $\triangle XYZ$.

(xv) *YF* = *YA, FL* = *AB*. Also $\angle AFL = \angle CAB$, and $\angle YFA = \angle YAC$; $\therefore \angle YFL = \angle YAB$.

Hence $\triangle^s$ *YFL* and *YAB* are congruent, so that *YL* = *YB* and $\angle LYF = \angle BYA$. Hence sum of $\angle^s$ *LYF* and *AYL* = sum of $\angle^s$ *BYA* and *AYL*, so that $\angle BYL$ is a right angle.

Similarly $\angle BYN$ is a rt. $\angle$, and *YN* = *YB*.

Hence *Y* is the middle point of *LN*.

So also *X, Z* are the middle points of *MN* and *LM* respectively. Also *AX* = *YZ* = half *MN*, *BY* = *ZX* = half *NL*, and *CZ* = *XY* = half *LM*.

Otherwise thus :—

From (i) and (ii) *LA* is equal and $\perp^r$ to *BC*. Hence *LA* is equal and $\parallel$ to *HM*. Hence *LAMH* is a $\parallel^m$, and therefore the middle point of *AH*, *i.e.* the point *Z*, is the middle point of *LM*. Similarly *X* is the middle point of *MN*, and *Y* is the middle point of *NL*.

MISCELLANEOUS EXERCISES.

1. Shew that, if the line bisecting an angle of a triangle passes through the middle point of the opposite side, the triangle must be isosceles.

2. Shew that, if the line bisecting an angle of a triangle be perpendicular to the opposite side, the triangle must be isosceles.

3. On a given straight line find a point which is equidistant from two given points.

4. On a given straight line find a point which is equidistant from two given straight lines.

5. On a given base describe an isosceles triangle equal in area to a given triangle.

6. In an equiangular polygon each exterior angle is one-tenth of a right angle. How many sides has the figure?

7. Each of the angles of a polygon is nine-fifths of a right angle. How many sides has the polygon?

8. What is the least number of triangles into which a plane figure of n sides can be divided?

9. Shew how to find a point which is at given perpendicular distances from two given intersecting straight lines.

10. Draw a straight line which will bisect each of two given parallelograms.

11. Shew that no convex polygon can have more than three of its exterior angles obtuse, or more than three of its interior angles acute.

12. Shew that the straight lines which bisect two opposite angles of a parallelogram are either coincident or parallel.

13. Shew that the three distances of any point within an equilateral triangle from the angular points are such that the sum of any two is greater than the third.

14. On a given base describe an isosceles triangle of given perimeter.

15. Draw through a given point a straight line making equal angles with two given straight lines.

16. Two quadrilaterals have equal angles and two adjacent sides of the one are equal respectively to the two corresponding adjacent sides of the other; shew that the quadrilaterals are equal in all respects.

17. Two quadrilaterals have equal angles and two opposite sides of the one are equal respectively to the two corresponding opposite sides of the other; shew that the two quadrilaterals are equal in all respects unless the other opposite sides of each quadrilateral are parallel.

18. *ABC*, *ACB* are the equal angles of the isosceles triangle *ABC*, and the bisectors of these angles meet the opposite sides in the points *X*, *Y* respectively. Shew that, if *XY* be drawn, the straight lines *BY*, *YX*, *XC* will all be equal.

19. *D*, *E*, *F* are the middle points of the sides *BC*, *CA*, *AB* of the triangle *ABC*. Shew that, if *BA* be greater than *CA*, *BE* will be greater than *CF*.

20. Shew that the sum of any two of the medians of a triangle are together greater than the third median.

21. Construct a triangle whose medians are equal to three given straight lines the sum of any two of which is greater than the third.

22. Shew that the sum of the medians of a triangle is greater than three-fourths of the perimeter of the triangle.

23. Shew that, if the line joining the middle points of two opposite sides of a quadrilateral bisect the quadrilateral, these opposite sides must be parallel.

24. A line parallel to the diagonal *BD* of the parallelogram *ABCD* cuts the sides *BC*, *CD* in the points *P*, *Q* respectively; shew that the triangles *ABP* and *ADQ* are equal in area.

25. The sides *BC*, *CA*, *AB* of a triangle are produced to *D*, *E*, *F* respectively so that *CD*=*BC*, *AE*=*CA* and *BF*=*AB*. Shew that the area of the triangle *DEF* is seven times that of the triangle *ABC*.

26. The triangle *ABC* is three times the triangle *A'BC*; shew that, if *AA'*, produced if necessary, cut *BC* in *D*, *A'D* will be equal to one-third of *AD*.

27. Shew that, if two parallelograms have a common diagonal, their other angular points are at the corners of another parallelogram.

28. Construct a parallelogram whose diagonals and one side are given in length.

29. Four points lie in a plane, and no one of the points is within the triangle having the other three for angular points. Find the point in the plane the sum of whose distances from the four given points is the least possible.

30. *ABC* is an equilateral triangle, *BC* is produced to *D* making *CD*=*BC*, and *AB* is produced to *E* making *BE*=2*AB*; shew that *ED*=2*AD*.

31. *D* is the middle point of the side *BC* of the triangle *ABC*, and any other line is drawn through *D* cutting the sides *AB*, *AC*, produced if necessary, in the points *P*, *Q*. Shew that the triangle *APQ* is greater than the triangle *ABC*.

32. Shew that, if two of the medians of a triangle are equal, the triangle must be isosceles.

33. Shew that, if two of the perpendiculars of a triangle are equal, the triangle must be isosceles.

34. D is the middle point of the side BC of the triangle ABC. Shew that, if the angle BAC is acute, AD is greater than DC.

35. E, F are the middle points of the sides AB, CD of the parallelogram ABCD. Shew that the lines ED, BF will trisect the diagonal AC.

36. Shew that the sum of the lengths of the perpendiculars, drawn to the sides of an equilateral triangle from any point within it, is constant.

37. Through the extremities of each diagonal of a quadrilateral lines are drawn parallel to the other diagonal. Shew that the area of the parallelogram so formed is double that of the original quadrilateral.

38. Shew that, if ABCD be a parallelogram and O be any point on the diagonal AC, then will △AOB = △AOD. Shew also that, if the △AOB = △AOD, then will O be on AC.

39. The sides AB, CD of the quadrilateral ABCD are parallel; E, F are the middle points of AD, BC respectively, and the straight line EF cuts the diagonals AC, BD in the points X, Y respectively. Shew that EF is parallel to AB or CD, that EF is equal to half the sum of AB and CD, and that XY is equal to half the difference of AB and CD.

40. The four feet of the perpendiculars let fall from one angular point of a triangle on the internal and external bisectors of the other two angles will all lie on a straight line which passes through the middle points of two of the sides.

41. ABCD is a square, and a line AXY is drawn through A cutting DC in X and BC produced in Y. Shew that the sum of AX and AY is greater than twice AC.

42. Construct a right-angled triangle, having given the length of the hypotenuse and the difference of the other two sides.

43. Construct a triangle having given one side, the angle opposite to that side, and the sum of the other two sides.

44. Construct a triangle having given one side, the angle opposite to that side, and the difference of the other two sides.

45. Construct a triangle equiangular to a given triangle and having a given perimeter.

46. Divide a straight line into two parts the square on one of which may be double the square on the other.

47. Construct a triangle having given the lengths of two sides and the corresponding median.

48. Construct a triangle having given two of the sides and the area.

49. Find a point on the base of a triangle such that the difference of the perpendiculars from it on the sides may be equal to a given length.

50. Find a point on the base of a triangle such that the sum of the perpendiculars from it on the sides may be equal to a given length.

51. Divide a given straight line into two parts such that the difference of the squares on the two parts may be equal to a given square.

52. Divide a given straight line into two parts such that the sum of the squares on the two parts may be equal to a given square.

53. Divide a triangle into three equal parts by lines drawn through a given point in one of its sides.

54. Divide a triangle into four equal parts by lines drawn through a given point in one of its sides.

55. Divide a parallelogram into four equal parts by lines drawn through a given point in one of its sides.

56. Construct a right-angled triangle having given one of the sides containing the right angle and the difference between the hypotenuse and the other side.

57. Bb, Cc are the perpendiculars drawn from the points B, C respectively on the internal bisector of the angle BAC. Shew that $2 \triangle CAb = 2 \triangle ABc = \triangle ABC$.

58. Find the condition that must exist in order that it may be possible to fold the four corners of a quadrilateral piece of paper flat down on the paper so that the four angular points meet in a point, and the paper is everywhere doubled.

59. Any two points D, E are taken on the sides AB, AC respectively of the triangle ABC, and F is the point of intersection of BE and CD. Shew that the sum of FD and FE is less than the sum of AD and AE.

60. Any parallelograms $ABDE$, $ACFG$ are described externally on the sides AB, AC of any triangle ABC. Shew that, if DE and FG be produced to meet in L, and BM, CN be drawn each equal and parallel to AL, the parallelogram $BMNC$ will be equal to the sum of the parallelograms $ABDE$ and $ACFG$.

61. In the quadrilateral $ABCD$ the sides AB and CD are parallel and are together equal to BC; shew that the bisectors of the angles ABC and BCD intersect on AD.

62. Points A, B, C are taken on three parallel straight lines; BC, CA, AB, produced if necessary, meet the lines through A, B, C respectively in the points D, E, F. Prove that the triangles AEF, BFD, CDE are all equal.

63. Shew that, in the figure to I. 47, the line joining the centre of the square $BCDE$ to the point A, will bisect the angle BAC.

64. Prove that, if the diagonals of a quadrilateral intersect at right angles, the sum of the squares on one pair of opposite sides is equal to the sum of the squares on the other pair of opposite sides.

65. Through the angular points of a triangle are drawn three parallel straight lines; shew that the area of the triangle formed by joining the points in which each parallel meets the side, produced if necessary, of the original triangle opposite to the angular point through which it is drawn, is equal to twice that of the original triangle.

66. *ABCD* is a quadrilateral in which the angles *ABC* and *BCD* are equal; shew that the angle *BAD* is greater than, equal to, or less than, the angle *CDA*, according as *CD* is greater than, equal to, or less than, *AB*.

67. Shew that, if the middle points of three of the sides of a quadrilateral be three given points, the middle point of the remaining side will be one or other of three other fixed points.

68. Shew that, if one pair of opposite sides of a quadrilateral are equal, the middle points of the other two sides and the middle points of the diagonals are at the angular points of a rhombus.

69. *AB*, *CD* are two given finite straight lines, find the locus of a point *P* which is such that the triangle *APB* is equal to the triangle *CPD*.

70. *AB*, *CD*, *EF* are any three given finite straight lines which are not all parallel. Find a point *O* such that the three triangles *AOB*, *COD*, *EOF* are all equal.

71. The angular points of one parallelogram are on the sides of another; shew that the two parallelograms have the same centre.

72. *ABCD* is a square and any point *E* is taken in *AB*, and in *BC*, *CD*, *DA* respectively points *F*, *G*, *H* are taken so that each of the lines *BF*, *CG*, *DH* is equal to *AE*. Shew that *EFGH* is a square.

73. Through the point of intersection of the diagonals of a square any two perpendicular lines are drawn meeting the sides in order in the points *P*, *Q*, *R*, *S* and the sides produced in the points *P'*, *Q'*, *R'*, *S'*. Shew that *PQRS* and *P'Q'R'S'* are squares.

74. Find the square of least area whose angular points are respectively on the four sides of a given square.

75. The four angular points of a rectangle with unequal sides are respectively on the four sides of a given square; shew that the sides of the rectangle are parallel to the diagonals of the square, and that the perimeter of the rectangle is equal to the sum of the diagonals of the square. Shew also that the area of the rectangle is less than one quarter of the area of the square.

76. *ABCDE* is a regular pentagon, and *AB*, *DC* are produced to meet in *F*. Shew that *CF = CA = BF*.

77. *ABCD* is a parallelogram having the side *AD* double of *AB*; the side *AB* is produced both ways to *E* and *F* till each produced part is equal to *AB*, and lines are drawn from *C* and *D* to *E* and *F* respectively so as to cross within the parallelogram; shew that they will meet at right angles.

78. Points *D*, *E*, *F* are taken on the sides *BC*, *CA*, *AB* respectively of the triangle *ABC*, such that *BD*=2*DC*, *CE*=2*EA* and *AF*=2*FB*. Shew that the triangle *DEF* is one-third of the triangle *ABC*.

79. Points *E*, *F* are taken on the sides *CA*, *AB* of the triangle *ABC* such that *AE*=2*EC* and *BF*=2*FA*, and the lines *BE* and *CF* intersect in *O*; shew that *BO*=6*OE*.

80. Points *F*, *D* are taken on the sides *AB*, *BC* respectively of the triangle *ABC*, so that *AF* is the fourth part of *AB*, and *CD* the third part of *CB*, and *AD*, *CF* intersect in *O*; prove that *AD* is bisected in *O*.

81. A point *D* is taken on the side *BC* of the triangle *ABC* such that *CD*=5*DB*, and *O* is the middle point of the line *AD*. Shew that, if *BO* be produced to cut *AC* in *E*, *CE*=6*AE* and 7*EO*=5*OB*.

82. From any point *P* on the side *BC* of the triangle *ABC*, lines *PQ*, *PR* are drawn parallel to *AB*, *AC* respectively and meeting *AC*, *AB* respectively in the points *Q*, *R*. Shew that the parallelogram *PQAR* is greatest when *P* is the middle point of *BC*.

83. Shew that the sum of the areas of the complements of the parallelograms about the diagonal of a given parallelogram cannot be greater than half the area of the parallelogram.

84. In a given triangle inscribe a parallelogram equal to half the triangle, so that one side of the parallelogram may be in the same straight line with one side of the triangle and one angular point of the parallelogram at a given point on that side.

85. Prove that, if *O* be any point in the plane of a parallelogram *ABCD* and the parallelograms *OAEB*, *OBFC*, *OCGD*, *ODHA* be completed, then will *EFGH* be a parallelogram whose area is double that of the parallelogram *ABCD*.

86. A quadrilateral is divided into four equal triangles by lines joining its angular points to a point within it; prove that no such point exists unless one of its diagonals be bisected by the other.

87. In the figure to I. 47, *HK* and *CF* are produced to meet in *M*, and *EB*, *DC* are produced to meet *HM*, *GM* respectively in *P*, *Q*; shew that *HPQC* is a square, and that *MAL* is a straight line. Shew also that, if *HB*, *KC* be produced so as to meet the line through *E* parallel to *BA* in the points *R*, *S* respectively, then will *BRSA* be a square.

88. Shew that the four internal bisectors of the angles of any parallelogram are the sides of a rectangle whose diagonals are parallel to the sides of the original parallelogram and equal to the difference between them.

89. The external angles of a parallelogram are bisected: prove that the figure formed by the four bisectors is a rectangle, the sum of whose diagonals is equal to the perimeter of the parallelogram.

90. Through a given point O draw two equal and perpendicular lines having their extremities respectively on two given straight lines.

91. Describe a square whose angular points shall lie on the sides, or the sides produced of a given parallelogram.

92. Through a given point O draw two equal lines inclined at a given angle and whose extremities are respectively on two given straight lines.

93. Describe an equilateral triangle having one of its vertices at a given point on one side of a given triangle and having its other vertices respectively on the other two sides, produced if necessary.

94. If three parallelograms are described having their sides parallel to two given straight lines and having for diagonals the sides of a given triangle, the other three diagonals will meet in a point.

95. In a given triangle inscribe a square.

96. In a given triangle inscribe a rectangle the difference of whose adjacent sides is equal to a given length.

97. DE is drawn parallel to the base BC of the triangle ABC and meets the sides AB, AC in the points D, E respectively. Shew that, if BE and CD meet in K, AK will bisect the lines DE and BC.

98. On the sides of a parallelogram as hypotenuses right-angled isosceles triangles are described external to the parallelogram. Shew that the vertices of the triangles are at the angular points of a square.

99. On the sides AB, BC of the parallelogram $ABCD$ equilateral triangles ABP, BCQ are described exterior to the parallelogram; shew that the triangle PQD is equilateral.

100. Equilateral triangles are described on the four sides of a parallelogram external to the parallelogram, prove that their vertices are at the angular points of a parallelogram, which is a rhombus when the original parallelogram is a rectangle, and a rectangle when the original parallelogram is a rhombus.

BOOK II.

DEFINITIONS.

1. A rectangle is said to be *contained by* any two of its adjacent sides.

Since, by definition, a rectangle is a parallelogram with one of its angles a right angle, it follows at once from Euclid I. 29, that *all* its angles are right angles. It is then easily seen by superposition that any two rectangles are equal in all respects if two adjacent sides of one rectangle are equal respectively to two adjacent sides of the other.

The construction of a rectangle which is to have two of its adjacent sides equal respectively to two given straight lines, can be effected in a similar manner to the construction of a square on a given straight line. We may therefore speak of the rectangle contained by two given straight lines, which are not adjacent sides of any rectangle actually drawn, meaning thereby any rectangle, two of whose adjacent sides are equal respectively to the two given straight lines.

The abbreviation '*rect. AB, BC*' will be used for '*the rectangle contained by AB and CD.*'

It is easily seen that the rectangle contained by two equal straight lines is equal to the square on either, or to the square on any straight line equal to either of the given lines.

2. If C be any point on the straight line AB, and if D be any point on AB produced, the straight line is said to be divided *internally* into the two *segments* AC and CB, and *externally* into the two *segments* AD and DB, the points C and D being called the *points of section*.

Book II. deals with certain cases of the equality of squares and rectangles. These must be proved by purely geometrical methods; and those proofs are to be preferred in which the equality is demonstrated directly by means of a figure.

The measurement of areas in relation to the measurement of the lines containing them has no place in a purely geometrical treatment of areas. This subject is, however, dealt with in a separate note at the end.

PROPOSITION I. Theorem.

If there be two straight lines, one of which is divided into any number of parts, the rectangle contained by the two straight lines is equal to the sum of the rectangles contained by the undivided line and the several parts of the divided line.

Let AB and CD be the two straight lines, and let AB be divided into any number of parts AE, EF, FB. Then it is required to prove that the rect. AB, CD is equal to the sum of rect. AE, CD, rect. EF, CD and rect. FB, CD.
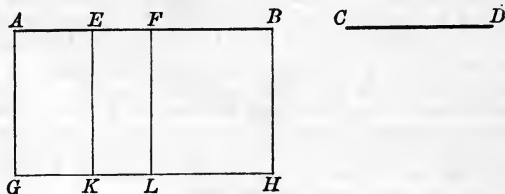
From A draw AG at right angles to AB, and make $AG=CD$.

Through G and B draw lines parallel to AB and AG respectively and intersecting in H, thus completing the rectangle $AGHB$.

Through E and F draw lines parallel to AG meeting GH in the points K, L respectively.

Then all the figures AH, AK, EL, FH are rectangles.

Since $AG=CD$,

figure AH is the rect. AB, CD.

Since opposite sides of ‖ᵐˢ are equal,

$EK=FL=AG=CD$; and therefore

figure AK is the rect. AE, CD ;

figure EL is the rect. EF, CD ;

and   figure FH is the rect. FB, CD.

But figure AH is equal to the sum of the figures AK, EL and FH.

Hence the rectangle contained by AB and CD is equal to the sum of the rectangles contained by AE and CD, EF and CD, and FB and CD.

Conversely. *The sum of the rectangles contained by any straight line and two or more other straight lines is equal to the rectangle contained by the first straight line and a straight line which is equal to the sum of the other straight lines.*

Precisely similar reasoning can be applied to prove that the rectangle contained by two straight lines both of which are divided into parts is equal to the sum of the rectangles contained by every pair of parts, one being taken from each of the two lines in all possible ways.
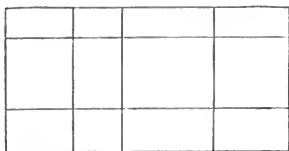
The student should identify the rectangles contained by each pair of segments in the above figure, and in other figures which he should draw for himself.

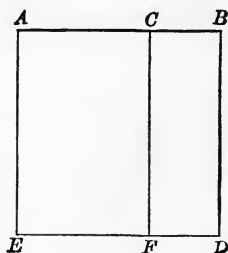## PROPOSITION II. Theorem.

If a straight line be divided into two parts, the square on the straight line is equal to the sum of the rectangles contained by the whole line and each of the parts.

Let the straight line AB be divided into two parts at the point C; then *it is required to prove that the square on the whole line is equal to the sum of the rect. AB, AC and rect. AB, CB.*

On AB describe the square $ABDE$.

Draw CF parallel to AE to meet DE in F.

Then the figures AD, AF, CD are rectangles.

By construction $AE = BD = AB$.

Hence figure AF is the rect. AB, AC, and figure CD is the rect. AB, CB.

But figure AD = the sum of the figures AF and CD;

∴ the square on AB is equal to the sum of the rectangles contained by AB and AC and by AB and CB.

Ex. If a straight line be divided into any number of parts the square on the line is equal to the sum of the rectangles contained by the line and each of the parts.

PROPOSITION III. Theorem.

If a straight line be divided into any two parts the rectangle contained by the whole straight line and one of the parts is equal to the square on that part together with the rectangle contained by the two parts.

Let the straight line AB be divided into any two parts at the point C. Then, *it is required to prove that rect. AB, AC is equal to the sum of the sq. on AC and the rect. AC, CB.*
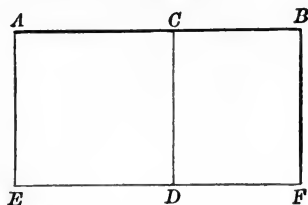
On AC describe the square $ACDE$.

Produce ED to meet BF drawn $\parallel$ to CD, or AE, in the point F, thus completing the rect. $ABFE$.

Since AE, CD, BF are opposite sides of rectangles, $AE = CD = BF$; and, by construction, $AE = AC$.

Hence   figure AF is the rect. AB, AC;

figure CF is the rect. CB, AC;

also   figure AD is the square on AC.

But figure AF is the sum of AD and CF.

$\therefore$ rect. AB, AC = the sum of the square on AC and the rect. CB, AC.

It should be noticed that Propositions II. and III. are special cases of Prop. I.

PROPOSITION IV. Theorem.

If a straight line be divided into two parts the square on the whole line is equal to the sum of the squares on the two parts together with twice the rectangle contained by the parts.

Let the line AB be divided into two parts at C. Then *it is required to prove that the square on AB is equal to the sum of the squares on AC, CB and twice the rect. AC, CB.*

On AB describe the square $ABDE$. Through C draw CF ∥ to BD to meet DE in F.

From BD cut off $BG = BC$. Through G draw GHK ∥ to BC and meeting CF in H and AE in K.
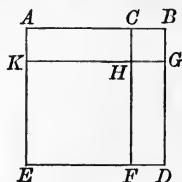
Then the figures AH, CG, HD and KF are all rectangles by construction.

By construction, $BD = BA$ and $BG = BC$; ∴ $GD = AC$.

Since opposite sides of rectangles are equal,

∴ $KE = GD = AC$, $EF = AC$ and $GH = CB$.

Figure CG is square on BC; for $BG = BC$.

Figure AH is rect. AC, CB; for $CH = BG = BC$.

Figure KF is equal to the square on AC; for

$$KE = GD = AC, \text{ and } EF = AC.$$

Also figure HD is equal to rect. AC, CB; for $HG = CB$ and $GD = AC$.

But figure AD is made up of the figures CG, KF, AH and HD.

Hence square on $AB =$ the sum of square on CB, square on AC and twice rect. AC, CB.

The reasoning of this proposition may be extended to the case of a line divided into any number of parts. Thus it may be proved that

'*If a straight line be divided into any number of parts the square on the whole line is equal to the sum of the squares on the different parts together with twice the rectangles contained by the parts two and two in all possible ways.*'

It should be noticed that the above theorem is a particular case of the generalisation of Prop. I. stated on page 119.

Prop. IV. may be enunciated as follows:—

'*The square on the sum of two straight lines is equal to the sum of the squares on the lines and twice the rectangle contained by them.*'

Every proposition about divided lines is equivalent to some proposition about the sums or differences of lines, and the student should practise himself in variation of statement. The difficulty which learners may find in doing this arises from paying too much attention and effort of memory to the form of words instead of concentrating the intelligence on the geometrical figure.

[It is recommended that Prop. VII. should be taken immediately after Prop. IV.]

Ex. 1. The square on any straight line is equal to four times the square on half the line.

Ex. 2. The square on any straight line is equal to nine times the square on one-third of the line.

Ex. 3. Divide a given square into 4, or 9, or 16 equal squares.

**Ex. 4.** Divide an equilateral triangle into 4, or 9, or 16 equal equilateral triangles.

Ex. 5. In the figure to II. 4 shew that *EHB* is a straight line.

Ex. 6. Shew that, if *KF* and *CG* be joined, the rect. *KF, CG* is twice the rect. *AC, CB.*

PROPOSITION V. Theorem.

If a straight line be divided into two equal parts and also into two unequal parts, the rectangle contained by the unequal parts and the square on the line between the points of section are together equal to the square on half the line.

Let the straight line *AB* be bisected at *C* and divided unequally at *D*. Then, *it is required to prove that rect. AD, DB and square on CD are together equal to the square on CB or AC.*
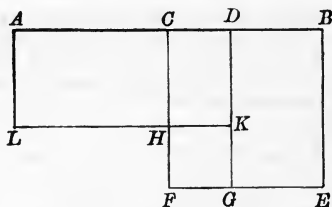
On *CB* describe the square *CBEF*.

Draw *DG* ∥ to *CF* cutting *FE* in *G*, and cut off *DK = DB*.

Through *K* draw a line ∥ to *AB* cutting *CF* in *H* and the line through *A* ∥ to *CF* in *L*.

When all the quadrilaterals are rectangles, and opposite sides of rectangles are equal.

Hence *DG = BE = BC* and *DK = BD*; ∴ *KG = CD*.

Hence fig. *AK* is rect. *AD, DB*; for *DK = DB*.

fig. *AH* is rect. *AC, DB*; for *CH = DK = DB.*

and   fig. *DE* is rect. *AC, DB*; for *BE = BC = AC.*

Hence   fig. *AH* = fig. *DE.*

Also fig. *HG* is equal to square on *CD*; for

$$KH = CD = KG.$$

Then sum of rect. *AD*, *DB* and sq. on *CD*

$$= \text{sum of } AK \text{ and } HG$$

$$= \text{sum of } AH,\ CK \text{ and } HG$$

$$= \text{sum of } DE,\ CK \text{ and } HG$$

$$= \text{fig. } CE$$

$$= \text{square on } CB.$$

Ex. 1. If a given straight line be divided into any two parts, the rectangle contained by the parts is greatest when the parts are equal.

Ex. 2. If a given straight line be divided into any two parts the sum of the squares on the parts is least when the parts are equal.

Ex. 3. Shew that, if the perimeter of a rectangle is given, the area is greatest when it is a square.

Ex. 4. The square on the sum of two unequal lines is greater than four times their rectangle.

Ex. 5. Prove that the square on any straight line drawn from the vertex of an isosceles triangle to a point on the base, is less than the square on a side of the triangle by the rectangle contained by the segments of the base.

Ex. 6. A line is drawn from the right angle of a right-angled triangle perpendicular to the opposite side. Shew that the square on the perpendicular is equal to the rectangle contained by the segments of the hypotenuse.

PROPOSITION VI.

If a straight line be bisected and produced to any point, the rectangle contained by the whole line thus produced, and the part of it produced, together with the square on half the bisected line, is equal to the square on the straight line which is made up of the half and the part produced.
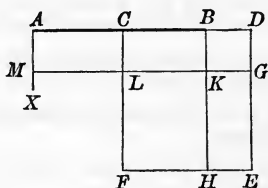
Let the straight line AB be bisected at C and produced to D. Then *it is required to prove that* rect. AD, DB and sq. on CB are together equal to sq. on CD.

On CD describe the square $CDEF$. Draw $AX \parallel$ to CF, and BH parallel to CF meeting FE in H. From DE cut off $DG = DB$, and through G draw $GKLM \parallel$ to AD and meeting BH, CF, AX in K, L, M respectively.

Then all the quadrilaterals are rectangles, and opposite sides of rectangles are equal.

Hence $AM = CL = BK = DG$, and DG was made equal to DB.

Hence figure AG is rect. AD, DB; and figure AL is rect. AC, BD.

Also, since $DE = DC$ and $DG = DB$; $\therefore$ $GE = BC = CA$.

Hence figure KE is rect. AC, BD; for $KG = BD$.

$$\therefore \text{ rect. } KE = \text{rect. } AL.$$

And, since $LK = CB$ and $LF = GE = CB$;

$$\therefore \text{ figure } LH \text{ is equal to sq. on } CB.$$

Now sq. on CD = fig. CE

$\quad\quad\quad\quad$ = sum of figures CG, KE, LH

$\quad\quad\quad\quad$ = sum of figures CG, AL, LH

$\quad\quad\quad\quad$ = sum of figures AG, LH

$\quad\quad\quad\quad$ = sum of rect. AD, DB and sq. on CB.

It is important to notice that Prop. V. and Prop. VI. are both included under the following enunciation :—

'*If a straight line be bisected and be also divided (internally or externally) into two unequal segments, the rectangle contained by the unequal segments is equal to the difference of the squares on half the line and on the line between the points of section.*'

This Proposition and also Proposition V. may be enunciated in terms of the sum and difference of two straight lines.

If we take AC and CD as the straight lines, then AD is their sum and DB is their difference. Hence the proposition is equivalent to the following :—

The rectangle contained by the sum and difference of two straight lines is equal to the difference of the squares on the lines.

Again, if we take AD and BD as the lines, CD is half their sum and CB is half their difference. Hence the proposition is equivalent to the following :—

The rectangle contained by two straight lines is equal to the difference of the squares on half their sum and half their difference.

PROPOSITION VII. Theorem.

If a straight line be divided into any two parts the sum of the squares on the whole line and one of the parts is equal to twice the rectangle contained by the whole line and that part together with the square on the other part.

Let the straight line AB be divided into any two parts in the point C. Then *it is required to prove that the sum of the squares on AB and BC is equal to twice rect. AB, BC and sq. on AC.*
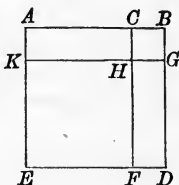
On AB describe the square $ABDE$. Through C draw CF parallel to AE meeting ED in F.

From BD cut off BG equal to BC, and through G draw a line || to AB cutting CF in H and AE in K.

Then all the quadrilaterals are rectangles, and since $AB = BD$ and $BG = BC$,

∴ AG is rect. AB, BC, and CD is also rect. AB, BC.

Hence AG and CD are together twice rect. AB, BC.

Also CG is the square on BC, for $BG = BC$.

And, since $BD = BA$ and $BG = BC$; ∴ $GD = CA$.

But opposite sides of rectangles are equal;

∴ $EF = AC$, and $KE = GD = CA$.

Hence KF is equal to the square on AC.

Now the two squares AD and CG are equal to the sum of the figures KF, AG and CD.

Hence the sum of the squares on AB and BC is equal to the square on AC together with twice the rectangle AB, BC.

Since AC is the difference between AB and BC, the above proposition can be enunciated in the following more interesting form :—
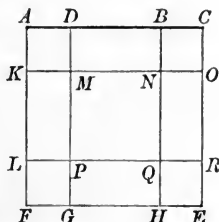
'*The square on the difference of any two straight lines is less than the sum of the squares on the lines by twice the rectangle contained by them.*'

Ex. 1. Shew that the sum of the squares on two straight lines is never less than twice the rectangle contained by them.

Ex. 2. Shew that the sum of the squares on two straight lines is never less than half the square on the sum of the lines.

PROPOSITION VIII. Theorem.

The square on the sum of two straight lines exceeds the square on their difference by four times the rectangle contained by the lines.

Let AB and BC be the two given straight lines, placed so that ABC is a straight line.

Cut off from AB a part AD equal to BC.

Then AC is the sum of the given lines and DB is the difference. *It is required to prove that sq. on AC exceeds the square on DB by four times the rect. AB, BC.*

On AC describe the square $ACEF$. Through D, B draw lines parallel to AF meeting FE in G, H respectively.
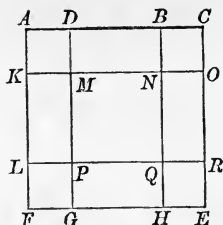
From AF cut off $AK = AD$ and $FL = AD$.

Draw through K, L lines parallel to AC cutting DG, BH, CE in M, N, O and P, Q, R respectively.

Then all the quadrilaterals in the diagram are rectangles.

Since $AF = AC$, $AK = AD$ and $LF = AD = BC$;

$$\therefore KL = DB, \text{ and } AL = AB = KF.$$

But opposite sides of rectangles are equal,

$$\therefore MP = KL = DB, \text{ and } MN = DB.$$

Hence figure PN is equal to the sq. on DB.

Now figure FC is equal to the sum of the figures PN, DO, OH, HL and LD.

Figure DO is rect. AB, BC,

for $DC = AB$ and $CO = AK = BC$.

Figure OH is rect. AB, BC,

for $OE = KF = AB$ and $NO = BC$.

Figure HL is rect. AB, BC, for $HF = AB$ and $FL = BC$.

Figure LD is rect. AB, BC, for $AL = AB$ and $AD = BC$.

Hence the square on AC is equal to the square on DB together with four times the rectangle AB, BC.

The enunciation of this proposition given above is more interesting than that given by Euclid, which is as follows:—

If a straight line be divided into any two parts, four times the rectangle contained by the whole line and one of the parts, together with the square on the other part, is equal to the square on the line made up of the whole and the first part.

ALTERNATIVE PROOF.

Let *AB*, *BD* be the two given straight lines, *ABD* being a st. line. Cut off from *AB* (which is supposed to be the greater) a part *BC* equal to *BD*.

Then *AD* is the sum of the lines and *AC* is their difference.

[Or with Euclid's Enunciation :—

Let *AB* be a straight line divided into two parts at the point *C*. Produce *AB* to *D*, making *BD* = *BC*. Then *AD* is the line made up of *AB* and *BC*.]
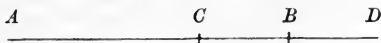
Then, by II. 4,

Sq. on *AD* = sum of sq. on *AB*, sq. on *BD* and twice rect. *AB*, *BD*.

But, since *BC* = *BD*,

Sq. on *BC* = sq. on *BD*, and rect. *AB*, *BD* = rect. *AB*, *BC*.

∴ sq. on *AD* = sum of sq. on *AB*, sq. on *BC* and twice rect. *AB*, *BC*.

Again, by II. 7,

Sum of sq. on *AB* and sq. on *BC* = sq. on *AC* and twice rect. *AB*, *BC*.

Hence sq. on *AD* = sq. on *AC* and four times rect. *AB*, *BC*.

PROPOSITIONS IX. AND X.

If a straight line be bisected and be also divided, internally or externally, into two unequal parts, the sum of the squares on the unequal parts will be equal to twice the square on half the line and twice the square on the line between the points of section.
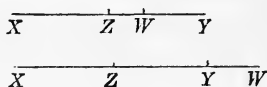
Let XY be a straight line bisected at Z and divided into two unequal parts at W.

$$\overline{X \qquad\qquad Z\ \ W \qquad Y\ \ }$$

$$\overline{X \qquad\qquad Z \qquad\qquad Y\ \ W}$$

Then *it is required to prove that the sum of sq. on XW and sq. on WY is equal to twice sq. on XZ and twice sq. on ZW.*

Now XW is equal to the sum of XZ and ZW, and WY is equal to the difference of XZ (or ZY) and ZW.

Hence both cases are included in the following enunciation, in which form the theorem will be proved :—

The sq. on the sum of two straight lines and the sq. on their difference are together equal to twice the sum of the squares on the given lines.

Let AB and BC be the two given straight lines, placed so that ABC is a straight line.

Cut off from AB a part AD equal to BC.

Then AC is the sum of the given lines and DB is their difference. *It is required to prove that the sum of the squares on AC and DB is double the sum of the squares on AB and BC.*
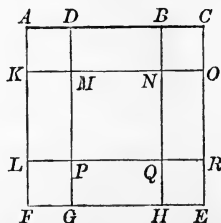
On AC describe the square $ACEF$. Through D, B draw lines || to AF meeting FE in G, H respectively.

From AF cut off $AK = AD$ and $FL = AD$.

Draw through K, L lines parallel to AC cutting DG, BH, CE in M, N, O and P, Q, R respectively.

Then all the quadrilaterals in the diagram are rectangles.

Now sum of figures FC and PN

$\qquad$ = sum of figures LB, GO, FP and NC.

Since $AF = AC$, $AK = AD$ and $LF = AD = BC$;

$\qquad \therefore KL = DB$ and $KF = AL = AB = DC$.

But opposite sides of a rectangle are equal,

$\qquad \therefore MP = KL = DB$, and $MN = DB$.

Hence figure PN is equal to the sq. on DB.

Figure LB is sq. on AB, for $AL = AB$.

Figure GO is sq. on AB, for $MG = KF = AB$

$\qquad\qquad$ and $MO = DC = AB$.

Figure FP is sq. on BC, for $LP = AD = BC$
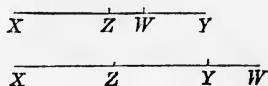
$\qquad\qquad$ and $LF = AD = BC$.

Figure NC is square on BC, for $CO = AK = AD = BC$.

Also FC is square on AC.

Hence sum of squares on AC and BD is equal to twice sum of squares of AB and BC.

Alternative Proof.

The following proof may be given :—

$$\overline{X \qquad Z \quad W \quad Y}$$

$$\overline{X \qquad Z \qquad Y \quad W}$$

By Euclid II. 4,

Sq. on XW = sum of sq. on XZ, sq. on ZW, and twice rect. XZ, ZW.

Also by Euclid II. 7,

Sq. on WY and twice rect. ZY, ZW = sum of sq. on ZY and sq. on ZW.

Hence, as $ZY = XZ$,

Sq. on WY and twice rect. XZ, ZW = sum of sq. on XZ and sq. on ZW.

Hence, by addition,

Sq. on XW, sq. on WY and twice rect. XZ, ZW = twice sq. on XZ, twice sq. on ZW and twice rect. XZ, ZW.

Take away twice rect. XZ, ZW, which is common; then

Sum of squares on XW and WY = twice sum of squares on XZ and ZW.

PROPOSITION A. Theorem.

The difference of the squares on any two straight lines is equal to the rectangle contained by the sum and the difference of the lines.

[We have already shewn [see page 127] that this theorem is included in Prop. V. or Prop. VI. On account, however, of the importance of the theorem, an independent proof is given.]

Let AB, AC be the two straight lines. On AB, AC describe squares $ABDE$, $ACFG$, both squares being on the same side of the line ACB. The side AG of the smaller square will be along the side AE of the larger, since $\angle^s BAE$ and CAG are right angles.
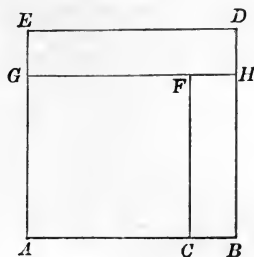
Produce GF to cut BD in H.

Then, since $AE = AB$ and $AG = AC$,

$$GE = CB = \text{difference of lines } AB, AC.$$

Since opp. sides of a rectangle are equal, $BH = AG = AC$, and $ED = AB$.

Hence FB is rect. AC, CB, and EH is rect. AB, CB.

Now the difference of the squares on AB and AC

$=$ the sum of EH and FH

$=$ the sum of rect. AC, CB and rect. AB, CB

$=$ rect. contained by CB and the sum of AC and AB [II. 1.

$=$ rect. contained by the sum and the difference of AB and AC.

PROPOSITION B. Problem.

To produce a given straight line so that the rectangle contained by the whole line so produced and the part produced may be equal to a given square.

Let AB be the given straight line, and let CD be a side of the given square. Then, *it is required to produce AB to some point E such that rect. AE, EB may be equal to the square on CD.*
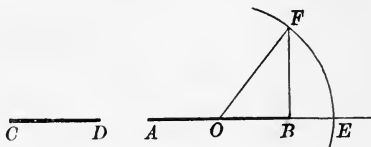
[Suppose that E is the point required. Then, by II. 6, if O is the middle point of AB,

sq. on OE = sum of rect. AE, EB and square on OB

= sum of squares on CD and OB.

Hence, if BF be drawn perp. to AB and equal to CD, and OF be joined, OF will be equal to the required line OE.

Hence the following construction*.]

Draw BF perp. to AB, making $BF = CD$.

Bisect AB in O, and join OF.

With O as centre and OF as radius describe a circle cutting AB produced in E. Then rect. AE, EB will be equal to the square on CD.

For sq. on OE = rect. AE, EB and sq. on OB.

And sq. on OE = sq. on OF = sum of squares on BF and OB.

Hence rect. AE, EB and sq. on OB = sq. on BF and sq. on OB.

Take away the common sq. on OB.

Then rect. $AE \cdot EB$ = sq. on BF = sq. on CD.

* See note on page 102.

PROPOSITION XI. Problem.

To divide a given straight line into two parts so that the rectangle contained by the whole line and one of the parts may be equal to the square on the other part.

Let AB be the given straight line. On AB describe the square $ABCD$.

Bisect AD in E, and join BE.

Produce EA to F so that $EF = EB$.
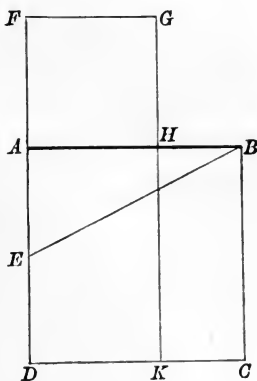
On AF describe the square $AFGH$; then H will fall on AB, since $\angle^s\ FAH$ and FAB are rt. $\angle^s$, and AB will be divided in H so that rect. AB, BH is equal to sq. on AH.

Produce GH to meet CD in K.

Then ∵ DA is bisected in E and produced to F,

∴ rect. DF, FA and sq. on AE = sq. on EF [II. 6.

= sq. on EB (since $EF = EB$)

= sum of squares on AB and AE.   [I. 47.

From these equals take the square on AE; then

rect. DF, FA = sq. on AB.
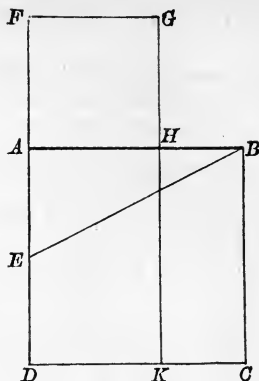
But figure *FK* is rect. *DF*, *FA*, since *FG = FA*.

Hence   figure *FK* = sq. *AC*.

From these equals take away the fig. *AK* which is common ; then

$$\text{fig. } FH = \text{fig. } HC.$$

But fig. *FH* is a square, and is ∴ the sq. on *AH*, and fig. *HC* is equal to rect. *AB*, *HB*, for *BC = AB*.

Hence   sq. on *AH* = rect. *AB*, *BH*.

Def. When a straight line is divided into two parts so that the rectangle contained by the whole line and one of the parts is equal to the square on the other part, the line is said to be divided in '**medial section.**' The line is also said to be divided in '**extreme and mean ratio,**' for in this case, as will be seen in Book vi, the ratio of the whole line to one part is equal to the ratio of that part to the other.

The analysis [see page 101] of this problem will shew how the above construction could be invented, and will enable the student to solve other analogous problems.

Analysis. Suppose that *AB* is divided in the required manner at the point *H*. Construct *AFGH*, the square on *AH*, and also the rectangle *HB*, *BA*, *these being put on opposite sides of the line AB*, as in the figure.

Then it is natural to complete the square *AB*, and as *FK* is equal to *AC*, we see that *DA* is to be produced to *F* so that the rectangle contained by the whole line produced and the part produced may be equal to the square on *AB*. Thus the problem is reduced to a particular case of that considered in Prop. B.

Ex. 1. If in the diagram to Prop. XI, CB and FG are produced to meet in R, shew that DHR is a straight line.

Ex. 2. Shew that, if the lines GB, FC and AK be drawn, they will all be parallel.

Ex. 3. If FC cut AB, HK in P, Q respectively; then $FP = QC$.

Ex. 4. $\|^m GC = \|^m FB = \|^m AK$.

Ex. 5. The lines KF and HC are parallel.

Ex. 6. If KF cut AH in the point X, $HX = BH$.

Ex. 7. FB is $\perp^r$ to DH.

Ex. 8. If DH and EB intersect in O, AO is parallel to FB, and perpendicular to DH.

Ex. 9. If BA be produced to Z so that $AZ = HA$,

$$\text{rect. } BZ \cdot AZ = AB^2. \qquad \text{[Euclid XIII. 5.]}$$

Ex. 10. The sum of the squares on AB and BH is three times the square on AH. [Euclid XIII. 4.]

Ex. 11. The square on the sum of AB and BH is five times the square on AH.

Ex. 12. The difference of the squares on AH and HB is equal to the rectangle AH, HB.

Ex. 13. If, in the figure to Euclid II. 11, a point L be taken on ED produced such that $EL = EB$, and if a square $ALMN$ be described so that the squares AC and AM are on opposite sides of ADL; shew that the line BA will be divided externally at N so that sq. on AN is equal to rectangle BA, BN.

Ex. 14. If X be taken on HA such that $HX = HB$, then square on HX is equal to the rectangle HA, AX.

[This result is important. It shews that if a straight line be divided in medial section, and if the lesser segment be cut off from the greater, this latter is thereby divided in medial section. And this process can be continued; whence it follows that AB and AH are incommensurable. (Euclid XIII. 6.)]

Ex. 15. DF is divided in medial section at A.

Ex. 16. DR is divided in medial section at H.

Ex. 17. GX is parallel to DH.

Ex. 18. Divide a straight line into two parts such that the sum of the squares on the whole line and one part may be equal to three times the square on the other part.

Def. The **projection** of a terminated straight line on any other straight line is the length intercepted between the feet of the perpendiculars from the ends of the terminated line on the other.

Thus, if *AL*, *BM* be the perpendiculars from *A* and *B* on the line *XY*, then *LM* is the projection of *AB* on the line *XY*.
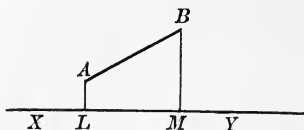
Also, if *AD* be the perpendicular from *A* on the line *BC*, produced if necessary; then *BD* is the projection of *BA* on *BC*.
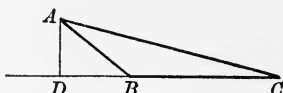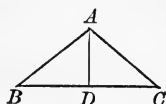
It is easily seen that the projections of a finite line on any two parallel lines are equal; and also that the projections on any straight line of two equal and parallel straight lines are equal.

PROPOSITION XII. Theorem.

In an obtuse-angled triangle, the square on the side opposite to the obtuse angle is equal to the sum of the squares on the other two sides together with twice the rectangle contained by either of these sides and the projection upon it of the other.

Let *ABC* be a triangle having the obtuse angle *BAC*. Draw *CD* ⊥ʳ to *BA* produced, then *AD* is the projection of *AC* on *BA*.

It is required to prove that sq. on BC exceeds the sum of the squares on BA, AC by twice the rect. BA, AD.
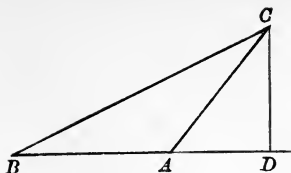
Then, by II. 4,

sq. on BD = sum of sq. on BA, square on AD and twice rect. BA, AD.

To each of these equals add the square on CD.

Then, sum of squares on BD and DC

= sum of squares on BA, AD and DC together with twice rect. BA, AD.

But, since ADC is a right angle,

sum of squares on BD and DC is equal to sq. on BC.

Also sum of squares on AD and DC is equal to sq. on AC.

Hence the square on BC is equal to the sum of the squares on BA and AC together with twice rect. BA, AD.

Euclid's enunciation of this theorem is :

In obtuse-angled triangles, if a perpendicular be drawn from either of the acute angles to the opposite side produced, the square on the side subtending the obtuse angle, is greater than the squares on the sides containing the obtuse angle, by twice the rectangle contained by the side upon which, when produced, the perpendicular falls, and the straight line intercepted without the triangle between the perpendicular and the obtuse angle.

ALTERNATIVE PROOF.

The following alternative proof which is strictly analogous to I. 47, and which shews directly the equality of the figures, is much more instructive than Euclid's proof*.

Let *ABC* be a triangle having the obtuse angle *BAC*. On the sides *BC*, *CA*, *AB* describe the squares *BCDE*, *CAFG*, *ABHK*, all the squares being external to the triangle.
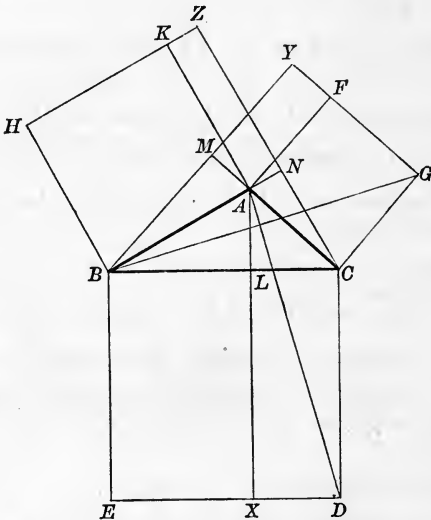
Draw *AL* ⊥ʳ to *BC*, and produce it to meet *DE* in *X*.

Draw *BM* ⊥ʳ to *CA* produced, and let *BM* and *GF* produced meet in *Y*.

Draw *CN* ⊥ʳ to *BA* produced, and let *CN* and *HK* produced meet in *Z*.

Then, since *AM* is the projection of *BA* on *AC*, *it is required to prove that sq. on BC exceeds the sum of the squares on CA and AB by twice rect. CA, AM.*

Join *AD* and *BG*.

---

* This interesting extension of I. 47 is given in Lardner's Euclid. *London*, 1828, but cannot be traced earlier.

Add $\angle\ BCA$ to each of the right $\angle^s\ BCD$ and ACG; then

$$\angle\ BCG = \angle\ DCA.$$

Also $\qquad\qquad BC = DC$ and $CG = CA.$

Hence $\qquad\qquad \triangle\ BCG = \triangle\ DCA.$

But $\qquad$ rect. CY is double $\triangle\ BCD$,

because they are on the same base and between the same parallels,

and similarly rect. CX is double $\triangle\ CAD$.

Hence $\qquad\qquad$ rect. $CX = $ rect. CY.

Similarly $\qquad\qquad$ rect. $BX = $ rect. BZ,

these being respectively the doubles of the equal $\triangle^s\ ABE$, HBC.

And $\qquad\qquad$ rect. $AY = $ rect. AZ,

these being respectively the doubles of the equal $\triangle^s\ BAF$, KAC.

Hence $\qquad$ sq. $BD = $ sum of rect. BX and rect. CX.

$\qquad\qquad\qquad = $ sum of rect. CY and rect. BZ.

$\qquad\qquad\qquad = $ sum of rectangles CF, BK, AY and AZ.

But $\qquad\qquad CF$ is square on CA,

$\qquad\qquad\qquad BK$ is square on AB,

sum of AY and $AZ = 2AY = 2$ rect. CA, AM, for $AF = CA$.

Hence sq. on $AB = $ square on CA, square on AB and twice rect. CA, AM.

PROPOSITION XIII. Theorem.

In any triangle the square on the side opposite to an acute angle is less than the sum of the squares on the sides containing the acute angle by twice the rectangle contained by either of those sides and the projection upon it of the other side.

In the triangle ABC let BCA be an acute angle.

Draw $BD \perp^r$ to CA, or CA produced; then CD is the projection of CB on CA.

Then, *it is required to prove that the sq. on AB is less than the sum of the squares on BC and CA by twice rect. AC, CD.*
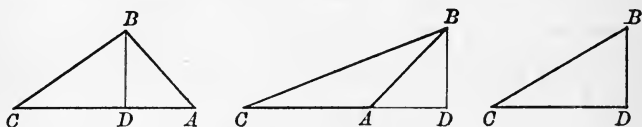
Then, whether D falls on CA or CA produced, by II. 7

sq. on CA and sq. on CD

$\qquad$ = sq. on AD and twice rect. AC, CD.

To each of these equals add the sq. on BD.

Then sq. on CA, sq. on CD and sq. on BD

$\qquad$ = sq. on AD, sq. on BD and twice rect. AC, CD.

But, since BD is $\perp^r$ to CA,

$\qquad$ sq. on CD and sq. on BD = sq. on CB,

also $\qquad$ sq. on AD and sq. on BD = sq. on AB.

Hence sq. on CA and sq. on CB

$\qquad$ = sq. on AB and twice rect. AC, CD;

i.e. sq. on AB is less than sum of squares on CA and CB by twice rect. AC, CD.

When BAC is a right angle, CA is the projection of CB on CA, and we have to prove that sq. on AB is less than sum of squares on CA and CB by twice the square on CA, which follows at once from I. 47.

ALTERNATIVE PROOF.

This proposition can be proved in a manner analogous to the proof of I. 47, as in the Alternative proof of Prop. XII.

Let ABC be an acute angle of the triangle ABC; then the figure being constructed as on p. 142, we first prove as before that

$$\text{rect. } AY = \text{rect. } AZ,$$
$$\text{rect. } BZ = \text{rect. } BX,$$
and $$\text{rect. } CX = \text{rect. } CY.$$
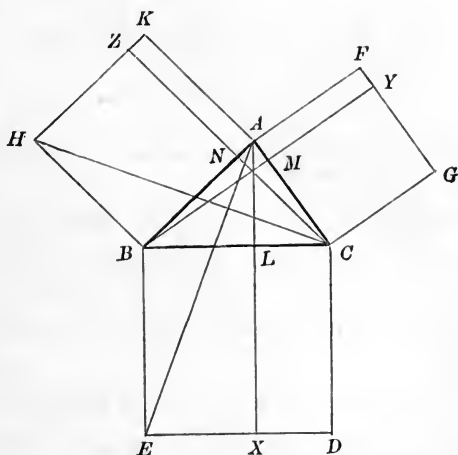
Hence square CF is less than the sum of the squares CE and BK by the sum of rect. BX and rect. BZ, that is by twice the rect. BX.

But BX is rect. BC, BL, and BL is the projection of BA on BC.

Hence *sq. on AC is less than the sum of the squares on AB and BC by twice the rectangle contained by BC and the projection of AB upon BC.*

The student should go through the proof when the triangle ABC is obtuse-angled, as in the figure on p. 142.

PROPOSITION XIV. PROBLEM.

To describe a square equal to a given rectilineal figure.

Let X be the given rectilineal figure.

Construct a rectangle $ABCD$ equal to the figure X. [I. 45.
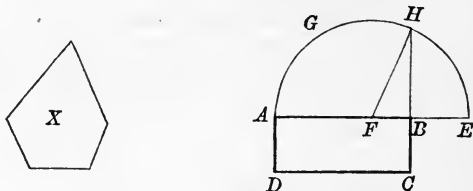
[If by chance $BC = BA$, the square is already constructed. But, if not]

Produce AB to E, making $BE = BC$. Bisect AE in F.

With centre F and radius FE describe the circle EGA, and produce CB to meet the circumference in H.

Then the square on BH will be equal to the given figure.

Join HF.

Then, AE is bisected at F and divided unequally at B;

$\therefore$ sq. on FE = rect. AB, BE and sq. on FB.

But $FE = FH$;

$\therefore$ sq. on FE = sq. on FH = sq. on HB and sq. on FB.

Hence sq. on HB and sq. on FB

$\qquad$ = rect. AB, BE and sq. on FB.

Take away the common square on FB; then

$\qquad$ sq. on HB = rect. AB, BE = rect. AC.

But rect. AC was made equal to the given figure X.

Hence $\qquad$ sq. on HB is equal to figure X.

It should be noticed that the above is the last step in the solution of the important problem to find the side of a *square* which is equal in area to any given rectilineal figure, the previous steps being given in Propositions 42, 44 and 45 of Book I—or in Prop. 42 and Prop. C.

By '*squaring a figure*' is meant the drawing of a square whose area is equal to that of the given figure.

Ex. 1.　Describe a rt. ∠ᵉᵈ △ equal to a given rectilineal figure and such that one of its sides containing the right angle is double of the other.

Ex. 2.　Describe an isosceles right-angled triangle equal to a given rectilineal figure.

Ex. 3.　Having given one side of a rectangle equal to a given square, find the other side.

Ex. 4.　Point out the succession of steps by which Euclid ' squares' any rectilineal figure.

Ex. 5.　Describe a rectangle equal to a given square, and having the sum of two of its adjacent sides equal to a given st. line.

Ex. 6.　Construct a rectangle equal to a given square and having the difference of two adjacent sides equal to a given st. line.

Ex. 7.　Shew that of all rectangles of equal area, the square has the smallest perimeter.

NOTE I.

In Pure Geometry no attempt is made to estimate the area of squares and rectangles in relation to the lengths of their sides. In those special cases, however, in which the lengths of adjacent sides of a rectangle can be expressed in terms of some common unit length; then, if the square on that unit length be taken as the unit of area, it is proved in Arithmetic[1] that the number of units of area in the rectangle is equal to the product of the number of units of length in two of its adjacent sides.

Lines that have a common measure are said to be **commensurable**, and lines which have no common measure are said to be **incommensurable**.

Pairs of lines which are incommensurable occur in the simplest geometrical figures. For example, the square on a diagonal of a square is twice the square on a side, so that the ratio of a diagonal to a side is $\sqrt{2}$ to 1, and we know that $\sqrt{2}$ cannot be expressed as a Vulgar Fraction, and therefore no line which is contained an exact number of times in the side of a square can be contained an exact number of times in the diagonal, so that a side and a diagonal of a square are incommensurable.

It is not necessary to give other cases of lines which are incommensurable; the student must, however, constantly bear in mind that there is no security, and in fact little probability, that the lines in any figure are commensurable.

Now, *if we assume that the different lines*, referred to in the cases considered in the Propositions 1 to 10 of Book II. of Euclid's Elements, *are commensurable*, the geometrical proofs lead at once to certain algebraical identities; these algebraical formulae are thus established, with the limitation, however, that the letters therein refer only to commensurable numbers. Conversely the geometrical truths are established by the algebraical proofs, *but only for commensurable lines*.

Algebraical Formulae

analogous to, and deduced from, euclid ii. 1—10.

Prop. I. On the supposition that the different parts of the divided line are commensurable and contain a, b, c, &c. units of length, and that the undivided line contains x units of length, the proposition proves that

$$(a + b + c + \dots\dots)\, x = ax + bx + cx + \dots.$$

The student should write for himself the result when the second line is also divided. [See C. Smith's *Elementary Algebra*, Art. 46.]

Prop. II. On the supposition that the two parts of the divided line are commensurable, and contain a and b units of length respectively, we have

$$(a + b)^2 = (a + b)\, a + (a + b)\, b.$$

Prop. III. We have in this case

$$a\,(a + b) = a^2 + ab.$$

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[1] See C. Smith's *Arithmetic*, page 161.

Prop. IV. On the supposition that the two parts of the divided line are commensurable, and contain a and b units of length respectively, we have

$$(a+b)^2 = a^2 + b^2 + 2ab.$$

Prop. V. On the supposition that the parts of the divided line are commensurable, and that AC, or CB, contains a units and that CD contains b units, we have from

rect. $AD \cdot DB +$ sq. on $CD =$ sq. on CB,

the algebraical identity

$$(a+b)(a-b) + b^2 = a^2.$$

Prop. VI. On the supposition that the parts of the divided line are commensurable, and that AC, or CB, contains a units, and that CD contains b units, we have from

rect. $AD \cdot BD +$ sq. on $CB =$ sq. on CD,

the algebraical identity

$$(a+b)(b-a) + a^2 = b^2.$$

Prop. VII. On the supposition that the two parts of the line are commensurable, and contain a, b units respectively, we have from

sq. on $AB +$ sq. on $BC =$ sq. on $AC + 2$ rect. $AB \cdot BC$,

the algebraical identity

$$(a+b)^2 + b^2 = a^2 + 2(a+b)b.$$

Or, if AB contain a units and BC contain b units, then

$$a^2 + b^2 = (a-b)^2 + 2ab.$$

Prop. VIII., IX. and X. On the supposition that the two lines are commensurable, and contain a and b units respectively, we have from VIII. the identity

$$(a+b)^2 = (a-b)^2 + 4ab.$$

Also from IX. or X., the identity

$$(a+b)^2 + (a-b)^2 = 2a^2 + 2b^2.$$

Prop. A. On the supposition that the sides of the squares are commensurable, and contain a, b units respectively, we have

$$a^2 - b^2 = (a+b)(a-b).$$

Prop. B. If the given straight line contain a units, and the side of the given square contain b units; then the problem gives the geometrical solution of the quadratic equation

$$(a+x)x = b^2.$$

NOTE II.

It should be noticed that after Prop. 1, Book II., has been proved by means of a diagram, it is possible to deduce from it all the remaining propositions included in Euclid II., 1 to 10, without reference to any figure in which the different squares and rectangles are actually constructed, and that this can be done by strictly geometrical methods. This procedure, though logically sound, would be far inferior to the method adopted in the text, where in each case the equality which it is desired to establish is shewn directly by means of a figure. It would, however, be a useful exercise for the student to make the deductions in this manner.

N.B. The symbol AB^2 is often used for shortness instead of '*the square on* AB,' and the symbol $AB.BC$ in the place of '*the rect.* AB, BC.' In all examinations these symbols, and also the signs $+$ and $-$, are now allowed to be used in writing out any theorems or problems which are not given in Euclid's text. The symbols AB^2 and $AB.BC$ may not, however, be used in writing out the propositions given by Euclid.

The reason for the distinction is that it is thought that no one who is able to do deductions is likely to imagine that these symbols could have the same meanings as the algebraical symbols a^2 and $a.b$, or that they are in any way connected with the numerical measures of rectangles.

The use of these symbols ought never to be allowed at any time until it is clear that AB^2 and $AB.BC$ are used by the student simply as the shortest way of writing '*the square on* AB' and '*the rectangle contained by* AB *and* BC' respectively.

MISCELLANEOUS PROBLEMS AND THEOREMS.

I. *Find two lines, having given their sum and their difference.*
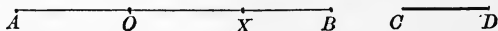
Let AB be the given sum of the lines and CD their given difference.

From BA cut off $BX = CD$, and bisect AX in O; then AO and OB are the lines required.

For the sum of OB and OA is AB; and since $OX = AO$, the difference of OB and OA is equal to XB, which was made equal to CD.

II. *Find two lines, having given their sum and the area of the rectangle contained by them.*

Let AB be the given sum, and let the rectangle contained by them be equal to the square on CD.

[If the area is given equal to a certain rectilineal figure, the square whose area is equal to that of the given figure can be constructed by II. 14.]

Bisect AB in O.
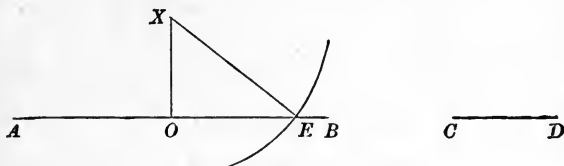
[Now, if AE and EB be the required lines, we know that

$$AE \cdot EB + OE^2 = OB^2;$$

$$\therefore \; CD^2 + OE^2 = OB^2.$$

Hence, if OX **be drawn** $\perp^r$ **to** AOB **and such that** $OX = CD$, **we have**

$$OX^2 + OE^2 = OB^2;$$

$$\therefore \; XE^2 = OB^2 \text{ (I. 47) and } XE = OB.$$

Hence E can be found by the following construction.]

Draw $OX \perp^r$ to AOB, and take $OX = CD$.

With X as centre and radius equal to OB, describe a circle cutting AB in E. Join XE.

Then, since $AO = OB$, $AE \cdot EB + OE^2 = OB^2$ [II. 5.

$$= XE^2 \quad \text{[const.}$$

$$= OX^2 + OE^2. \quad \text{[I. 47.}$$

Hence $AE \cdot EB = OX^2$

$$= CD^2. \quad \text{[const.}$$

Thus the lines AE, EB are such that their sum is AB, and the rectangle contained by them is equal to the square on CD.

    III.   *Find two straight lines, having given their difference and the area of the rectangle contained by them.*

This is easily seen to be Prop. B.

IV. *Find two straight lines, having given their sum and the difference of the squares described on them.*

Let AB be the given sum of the lines.

We know [Prop. A] that the difference of the squares on any two straight lines is equal to the rectangle contained by their sum and difference.

Hence if we apply to AB a rectangle $ABCD$ equal to the given difference of the squares on the lines, BC will be equal to the difference of the lines. And now that the sum and the difference of the required lines are known, the lines can be found as in I.

V. *Find two straight lines, having given their difference and the difference of the squares described on them.*

VI. *The sum of the squares on any two sides of a triangle is equal to twice the square on half the third side and twice the square on the median that bisects the third side.*
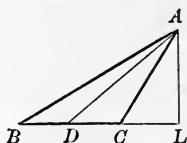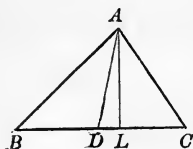
Let D be the middle point of the side BC of the $\triangle ABC$. Join AD. Then *it is required to prove that*

$$AB^2 + AC^2 = 2DC^2 + 2AD^2.$$

Then, if AD is $\perp^{r}$ to BC, the theorem follows at once from I. 47.

But, if AD be not $\perp^{r}$ to BC, draw $AL \perp^{r}$ to BC, produced if necessary. Then one of the angles ADC, ADB must be obtuse and the other acute. Let ADB be the obtuse angle, as in the figures. Then, by II. 12 and II. 13,

$$AB^2 = BD^2 + AD^2 + 2BD \cdot DL;$$

and $$AC^2 = DC^2 + AD^2 - 2DC \cdot DL.$$

But, since $$BD = DC, \ BD^2 = DC^2,$$

and $$BD \cdot DL = DC \cdot DL.$$

Hence, by addition,

$$AB^2 + AC^2 = 2DC^2 + 2AD^2.$$

Ex. 1. *Find the locus of a point which moves so that the sum of the squares on the lines joining it to two given points is constant.*

If A, B be the two given points and O be the middle point of AB; then if P be any point on the locus, $PA^2 + PB^2 = 2AO^2 + 2PO^2$. Hence, if $PA^2 + PB^2$ is constant, PO^2 is constant, so that P is at a fixed distance from O. The locus required is therefore a circle whose centre is the middle point of the line joining the two given points.

VII. *The sum of the squares on the sides of any quadrilateral exceeds the sum of the squares on the diagonals by four times the square on the line joining the middle points of the diagonals.*
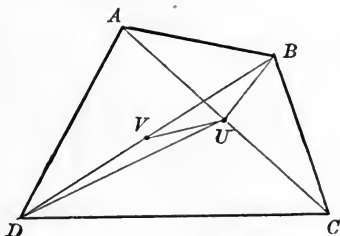
Let $ABCD$ be any quadrilateral, and let U, V be the middle points of its diagonals AC, BD respectively.

Then, by VI., since U is the middle point of AC,

$$AB^2 + BC^2 = 2BU^2 + 2AU^2,$$

and $$AD^2 + DC^2 = 2DU^2 + 2AU^2.$$

$$\therefore AB^2 + BC^2 + CD^2 + DA^2 = 2BU^2 + 2DU^2 + 4AU^2.$$

Again, since V is the middle point of BD,

$$2BU^2 + 2DU^2 = 4UV^2 + 4BV^2.$$

But $$4AU^2 = AC^2 \text{ and } 4BV^2 = BD^2.$$

Hence $$AB^2 + BC^2 + CD^2 + DA^2 = AC^2 + BD^2 + 4UV^2.$$

Cor. I. *The sum of the squares on the sides of a parallelogram is equal to the sum of the squares on the diagonals.*

For the diagonals of a parallelogram bisect each other, and therefore UV is zero.

Cor. II. *If the sum of the squares on the sides of a quadrilateral is equal to the sum of the squares on the diagonals, the quadrilateral must be a parallelogram.*

For the points U and V must coincide, and therefore the diagonals of the quadrilateral must bisect each other, whence it follows that the quadrilateral must be a parallelogram.

VIII.   *The difference of the squares on two sides of any triangle is equal to twice the rectangle contained by the base and the projection on the base of the corresponding median.*

Let D be the middle point of the side BC of the triangle ABC, and let AL be perpendicular to the base, produced if necessary.   Then, if AB be supposed to be greater than AC, *it is required to prove that* $AB^2 - AC^2 = 2BC \cdot DL$.
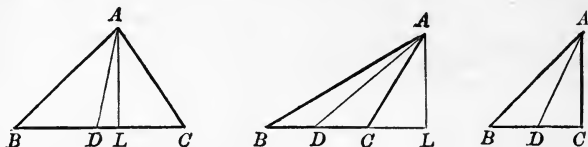
Since $\angle\ ALB$ is a rt. $\angle$,

$$AB^2 = BL^2 + AL^2 \text{ and } AC^2 = CL^2 + AL^2;$$
$$\therefore\ AB^2 - AC^2 = BL^2 - CL^2.$$

But the difference of two squares is equal to the rectangle contained by their sum and difference [II. A].

Now, in fig. I. [or in fig. III. where L and C coincide]

$$BL + LC = BC, \text{ and } BL - LC = 2DL;$$

and in fig. II., $BL + CL = 2DL \text{ and } BL - CL = BC$;

$\therefore$ in all cases, $BL^2 - CL^2 = 2BC \cdot DL.$

Hence $AB^2 - AC^2 = 2BC \cdot DL.$

IX.   *Divide a given straight line into two parts so that the square on one part may be double the rectangle contained by the whole line and the other part.*

Let AB be the given straight line.

[Suppose that AB is divided at C so that $AC^2 = 2AB \cdot BC$.   Construct $ACDE$ the square on AC, and also $CBFG$ the rectangle $AB \cdot BC$, placing them on opposite sides of AB, as in the figure.

Then, since EC is by supposition equal to twice CF, it is natural to produce BF and CG to H, K so that $FH = BF$ and $GK = CG$. Then sq. $EC =$ fig. CH. Hence, if we complete the rectangle $ABHL$ and the square $ABFM$, we shall have rect. $EK =$ rect. AH. But rect. $AH = 2$ sq. $AF =$ sq. on MB. Hence the line LA is produced to E so that $LE \cdot AE = MB^2$, and $\therefore\ ME^2 - MA^2 = MB^2$, or $ME^2 = MB^2 + MA^2$. We have therefore the following construction.[1]]

On AB describe the square $ABFM$.
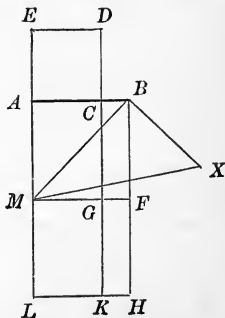
[1] See p. 102.

Join MB, and draw $BX \perp^{r}$ to MB and such that $BX = BA = MA$.

Join MX.

Produce MA to E making ME equal to MX. On AE describe the square $AEDC$, then C will be on AB, and AB will be divided at C in the manner required.

Produce AM, BF to L, H respectively so that $ML = MA$ and $FH = BF$. Join LH and draw $CGK \parallel$ to AL and cutting MF, LH in G, K respectively.

Then, since M is the middle point of LA,

$$LE \cdot AE + MA^2 = ME^2 \qquad \text{[II. 6.}$$
$$= MX^2 \qquad \text{[const.}$$
$$= MB^2 + BX^2 \qquad \text{[I. 47.}$$
$$= MB^2 + MA^2 \qquad \text{[const.}$$

Hence $\qquad LE \cdot AE = MB^2 = 2AB^2;$

$$\therefore \text{ rect. } EK = \text{rect. } AH.$$

Take away the common rectangle AK.

Then $\qquad$ sq. $EC = \text{rect. } CH = 2 \text{ rect. } CF.$

Hence $\qquad\qquad AC^2 = 2AB \cdot BC.$

MISCELLANEOUS EXERCISES.

1. Divide a given straight line into two parts such that the rectangle contained by the parts may be the greatest possible.

2. Shew that, if the perimeter of a rectangle be given, its area will be greatest when it is a square.

3. Shew that the greatest right-angled triangle, which has the sum of the sides containing the right angle equal to a given straight line, is isosceles.

4. Divide a given straight line so that the sum of the squares on the two parts may be the least possible.

5. Prove that, of all right-angled parallelograms having the same perimeter, the square has the shortest diagonals.

6. Given that the sum of the squares on two lines is equal to one given square, and that the difference of the squares on the lines is equal to another given square; find the lines.

7. Prove Euler's Theorem that, if any four points A, B, C, D be taken in order on a straight line, then will

rect. AB, CD + rect. AD, BC = rect. AC, BD.

8. Find a line the square on which is one-eighth the square on a given line.

9. Divide a straight line into two parts such that the rectangle contained by the parts may be equal to one-eighth of the square on the given straight line.

10. Divide a given straight line, internally or externally as the case may be, into two parts the difference of the squares on which is equal to a given square.

11. From one angle of a triangle a perpendicular is drawn on the opposite side, and the square on the perpendicular is equal to the rectangle contained by the segments of the opposite side. Shew that the triangle must be right angled.

12. If O be the orthocentre [see page 101] of the triangle ABC, shew that the sum on the squares of BC and OA is equal to the sum of the squares on CA and OB and also to the sum of the squares on AB and OC.

13. Shew that the locus of a point which moves so that the difference of the squares of its distances from two fixed points is constant, is a pair of parallel straight lines.

14. D is any point on the base BC of an equilateral triangle ABC; shew that the square on AD exceeds the sum of the squares on CD and DB by the rect. CD, DB.

15. Shew that the sum of the squares on the lines joining any point to two opposite vertices of a rectangle is equal to the sum of the squares on the lines joining the same point to the other two opposite vertices.

16. The sum of the squares on the four lines joining any point to the four corners of a square is equal to four times the square on the line drawn to the centre of the square together with the square on one of the diagonals of the square.

17. $ABCD$ is any quadrilateral and F, G, H, K are the middle points of AC, BD, AD, BC respectively. Shew that the sum of the squares on AB and CD is equal to twice the sum of the squares on FG and HK.

18. In any quadrilateral the sum of the squares on the diagonals is equal to twice the sum of the squares on the lines joining the middle points of opposite sides.

19. Shew that, if two sides of a quadrilateral are parallel, the squares on the diagonals are together equal to the squares on the two sides which are not parallel and twice the rectangle contained by the sides which are parallel.

20. If squares $ABDE$, $ACFG$ be described outwards on the sides AB, AC of the triangle ABC; shew that the sum of the squares on EG and BC is double the sum of the squares on AB and AC.

21. Shew that, if squares be described on the sides of any triangle and adjacent corners of the squares be joined so as to form a hexagonal figure, the sum of the squares on the sides of the hexagon is equal to four times the sum of the squares on the sides of the original triangle.

22. A point is taken within a rectangle, and straight lines are drawn from it to the angular points of the rectangle, and others perpendicular to the sides. Prove that the sum of the squares on the former is double the sum of the squares on the latter, and that these sums are least when the point is the centre of the rectangle.

23. The line AB is bisected in C and produced to D so that the square on CD is equal to the sum of the squares on AB and BC; shew that the rectangle AD, BD is equal to the square on AB.

24. Shew that three times the difference of the squares on the lines drawn from the vertex of a triangle to the points of trisection of the base is equal to the difference of the squares on the two sides of the triangle.

25. Points D, E are taken on the base BC of the triangle ABC such that $BD = DE = EC$; shew that the sum of the squares on AB and AC is equal to the sum of the squares on AD and AE together with four times the square on DE.

26. The squares on the straight lines drawn from the right angle to the points of trisection of the hypotenuse of a right-angled triangle are together equal to five times the square on the line between the points of trisection.

27. Inscribe a square within the greater of two given squares such that its area may be the mean of the areas of the two given squares.

28. Divide a straight line into two parts such that the rectangle contained by them may be equal to the square on their difference.

29. Find the locus of a point P which moves in the plane of the triangle ABC so that twice the square on PA is equal to the sum of the squares on PB and PC.

30. A, B are two given points, and CD a given straight line not perpendicular to the line joining AB. Find the point P on the line CD, produced if necessary, such that the difference of the squares on PA and PB may be equal to twice the square on AB.

31. Shew that, if the sum of the squares on two opposite sides of a quadrilateral is equal to the sum of the squares on the other two opposite sides, the diagonals of the quadrilateral must be at right angles.

32. AB is divided into two parts at C, and D, E are the middle points of AC and CB respectively. Shew that the square on AE together with three times the square on EB are equal to the square on BD together with three times the square on DA.

33. Find the locus of a point which moves so that the sum of the squares on the lines joining it to four fixed points is constant, and find the position of the point when this sum is least.

34. Shew that the sum of the squares on the distances of the middle point of either of the diagonals of a quadrilateral from the four angular points is equal to half the sum of the squares on the sides.

35. The sum of the squares on the medians of a triangle is equal to three-fourths of the sum of the squares on the sides.

36. If G is the centroid of the triangle ABC, the sum of the squares on the sides of the triangle is three times the sum of the squares on the lines GA, GB, GC.

37. Find two straight lines having given any *two* of the following:

(i) their sum, (ii) their difference, (iii) the rectangle contained by them, (iv) the sum of the squares on the lines, (v) the difference of the squares on the lines.

[Taking the above five quantities two together in all possible ways we shall have *ten* different problems. One of the ten problems, namely, the case when we have given the rectangle contained by the lines and the difference of their squares, cannot be solved without the aid of Book III.]

38. Produce a given straight line so that the square on the whole line produced may be double the square on the part produced.

39. Shew that the area of any square inscribed in a given square is greater than that of any inscribed rectangle whose sides are unequal.

40. Shew that the perimeter of any square inscribed in a given square is greater than that of any inscribed rectangle whose sides are unequal.

41. Divide a given straight line into two parts so that the rectangle contained by the whole line and one of the parts may be equal to the rectangle contained by the other part and a given straight line.

42. Divide a given straight line into two parts so that the rectangle contained by one segment and one given straight line may be equal to the rectangle contained by the other segment and another given straight line.

43. Divide a given straight line into two parts such that the rectangle contained by the whole line and one of the parts may be four times the square on the other part.

44. Divide a given straight line into two parts such that the rectangle contained by the whole line and one part may be one-fourth the square on the other part.

45. Divide a given straight line into two parts such that the square on one part may exceed the rectangle contained by the whole line and the other part by a given square.

46. Divide a given straight line into two parts so that the sum of the squares on the whole line and one part may be equal to five times the square on the other part.

47. Produce a given straight line so that the sum of the squares on the whole line so produced and the part produced may be three times the square on the given line.

48. Divide a given straight line into two parts so that the rectangle contained by one segment and a given straight line may be equal to the square on the other segment.

49. Shew that, if the area of a quadrilateral be given, the perimeter will be least when it is a square.

50. Shew that, if the perimeter of a quadrilateral be given, the area will be greatest when it is a square.
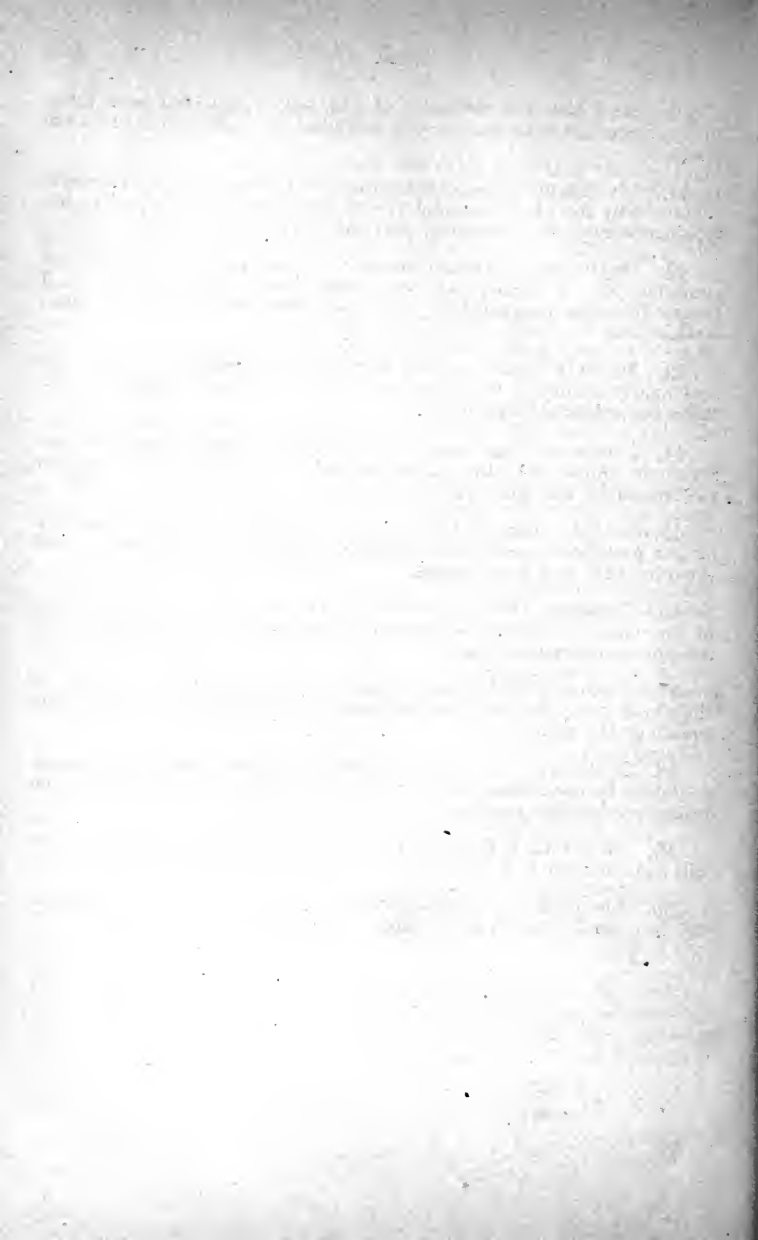
BOOK III.

DEFINITIONS.

1. *A* **circle** *is a plane figure bounded by one line, called the* **circumference,** *and is such that all straight lines drawn from a certain point within it, called the* **centre,** *to the circumference are equal to one another.*

2. *A straight line drawn from the centre of a circle to the circumference is called a* **radius.**

3. *A straight line drawn through the centre of a circle and terminated both ways by the circumference is called a* **diameter** *of the circle.*

Although, by the above definition, a circle is the *figure* enclosed by its circumference, the circumference itself is often called the circle when no ambiguity would arise.

The following simple properties of a circle, which are not, however, directly proved by Euclid, are of importance, and follow at once from the definition. Some of these properties are required, and have already been considered, in Book I.

(i)   A circle is a *closed* figure.

(ii)   The centre of a circle is *within* the figure.

It will be seen that (i) and (ii) are implied in the definition of a circle.

(iii)   Any straight line drawn through a point within a circle will, if produced sufficiently far in both directions, cut the circumference in *two* points.   [See note on I. 2.]

(iv)   A point is within or without a circle according as its distance from the centre is less or greater than the radius.

(v)   All diameters of a circle are equal, and each is bisected at the centre.

For the length of any diameter is clearly twice the length of a radius.

(vi)   Two circles which have equal diameters, or equal radii, are equal.

This is given as a definition by Euclid, but it is really a theorem which is easily proved by superposition.   For, if one circle be applied to the other so that their centres coincide; then, since the radii are equal, every point on the circumference of one circle will coincide with a point on the circumference of the other.   Thus the circles altogether coincide.

(vii)   A circle is bisected by any diameter, and each portion is therefore called a **semi-circle.**

For, if one of the two portions into which a circle is divided by a diameter be applied to the other, so that the common diameters coincide, every point on the circumference of one portion will, since all radii of a circle are equal, coincide with a point on the circumference of the other, so that the two portions will then altogether coincide.

(viii)   Two circles which have the same centre cannot intersect.

For, any point on the circumference of the circle which has the smaller radius is at a distance from the centre of the larger circle which is less than the radius of that circle.   Hence every point on the circumference of the smaller circle must be *within* the larger.

(ix)   Equal circles have equal radii.

(x)   A circle can only have *one* centre.

For, if the two points O, O' could both be centres of a circle, and if A, B were the extremities of the diameter through O and O'; then O and O' would both bisect AB, which is impossible.

4. *Any part of the circumference of a circle is called* **an arc.**

5. *Any straight line joining two points on the circumference of a circle is called a* **chord.** *The straight line joining the extremities of any arc is called* **the chord of the arc.**
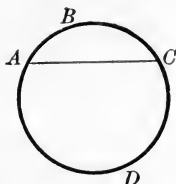
Thus the whole circumference of the circle *ABCD* is divided at the points *A* and *C* into the two *arcs ABC, ADC* ; and the straight line *AC* is the chord of each of these arcs.

6. *A* **segment** *of a circle is the figure contained by a chord and either of the two arcs into which it divides the circumference.*

Thus the chord *AC*, in the figure above, divides the circle *ABCD* into the two *segments ABC* and *ADC*.

7. *Circles which have the same centre are said to be* **concentric.**

8. *A straight line is said to* **touch a circle** *when it meets the circle but does not* **cut** *it at the point of meeting. The straight line is called a* **tangent** *to the circle, and the point which is common to the straight line and the circle is called* **the point of contact** *of the tangent.*
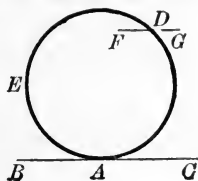
In passing along the line *FDG* from *F* to *G* we pass at *D* from one side of the arc to the other, so that the line *cuts* the circle at *D*.

The line *BAC*, however, touches the circle *ADE* at the point *A*, because the line and the circle have the point *A* in common and the line does not *cut* the circle at *A*.

9. **Circles are said to touch one another** *when they meet, but do not* **cut** *one another at the point of meeting.*
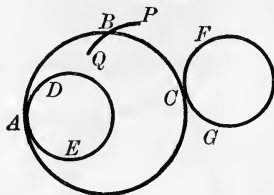
The circle of which *PBQ* is an arc does not touch the circle *ABC*, for of the two points *P*, *Q* near the point *B*, and on opposite sides of it, one is without and the other is within the circle *ABC*, so that the circles *PBQ* and *ABC* cut one another at the point *B*.

The circle *DAE* touches the circle *ABC*, for they have the point *A* in common, and any two points *D*, *E* on one circle near the point *A*, and on opposite sides of it, are *both within* the circle *ABC*, so that the two circles do not *cut* at the point *A*. So also the circle *FCG* touches the circle *ABC*, for the point *C* is common, and any two points *F*, *G* on one circle near the point *C*, and on opposite sides of it, are *both outside* the circle *ABC*, so that the two circles do not *cut* at *C*.

In the diagram the circle *DAE* touches the circle *ABC* **internally,** and the circle *ABC* touches the circle *DAE* **externally ;** also each of the circles *ABC*, *FCG* touches the other **externally.**

It will be proved later on that, if one circle touch another internally, *every* point of the first circle, except the point of contact itself, will be within the other circle ; it will also be proved that if two circles touch each other externally, *every* point of either circle, except the point of contact, will be outside the other circle. [See Prop. XIII.]

10. *The length of the perpendicular drawn from a point to a straight line is called* **the distance** *of the point from the straight line.*

Thus two chords of a circle are said to be *equally distant* from the centre of the circle when the perpendiculars drawn to the chords from the centre are equal ; also when the perpendiculars from the centre on two chords are unequal, the chord on which the ⊥ʳ is the greater is farther from the centre than the other chord.

11. *The* **angle in a segment** *of a circle is the angle contained by the two straight lines drawn from any point of the bounding arc to the two extremities of its chord.*
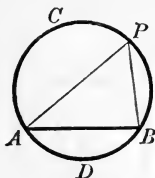
Thus the angle *APB* is an angle in the segment *ACB*.

The angle *APB* is also sometimes said to *stand on* the arc *ADB*.

12. *A* **sector of a circle** *is the figure bounded by two radii and the arc of a circle intercepted between them.*
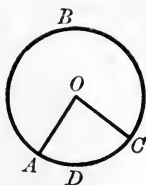
Thus the figure *AOCD* is a *sector* of the circle *ABCD*, *OA* and *OC* being radii of the circle.

13. *Segments of circles which contain equal angles are said to be* **similar.**

PROPOSITION I. Problem.

To find the centre of a given circle.

Let *ABC* be the given circle. *It is required to find its centre.*
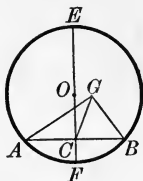
Take any two points *A*, *B* on the circumference. Join *AB*, and bisect it in *C*. Through *C* draw a line ⊥^{r} to *AB*, and produce this line both ways to meet the circumference in the points *E*, *F*. Bisect *EF* at *O*.

Then the point *O* is the required centre.

For, if possible, let some point *G*, which is not on *EF*, be the centre. Join *AG*, *BG* and *CG*.

Then, in the △^{s} *ACG*, *BCG*

$$\because \left\{ \begin{array}{ll} AC = CB, & [Const. \\ CG = CG, & \\ \text{and } GA = GB. & [Hyp. \end{array} \right.$$

$$\therefore \angle ACG = \text{adjacent } \angle BCG. \quad [\text{I. 8.}$$

$$\therefore \angle ACG \text{ is a right angle.}$$

But ∠*ACE* is a right angle.

$$\therefore \angle ACG = \angle ACE, \text{ which is impossible.}$$

It is therefore impossible for any point not on the line *EF* to be the centre; the centre must therefore be on the line *EF*.

But, if the centre is on the line *EF*, it must be at *O*, the middle point of *EF*, for the distances of *E* and *F* from every other point on *EF* are unequal.

Hence the point *O* is the centre.

Cor. *A line which bisects any chord of a circle and is at right angles to it will pass through the centre.*

PROPOSITION II. THEOREM.

If any two points be taken on the circumference of a circle, the straight line which joins them will be entirely within the circle.

Let ABC be a circle and A, B any two points on the circumference. Then *it is required to prove that every point on the line AB, between A and B, is within the circle.*
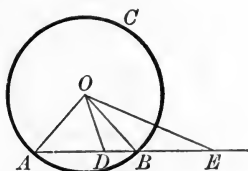
Find O the centre of the circle.   [III. 1.

In AB take *any* point D, and join OD.

Then the exterior $\angle ADO >$ int. opp. $\angle DBO$.   [I. 16.

But, since $OA = OB$, $\angle ABO = \angle OAB$.   [I. 5.

Hence $\angle ADO > \angle OAD$.

But the greater side of a triangle is opposite to the greater angle ;

$$\therefore \ OA > OD \ ;$$ [I. 19.

and since the distance of the point D from the centre is less than the radius of the circle, the point D must be *within* the circle.

Thus *any* point on the line AB between A and B is within the circle.

Cor. I. *If AB or BA be produced, every point on the line produced is without the circle.*

For, if E be *any* point on AB produced, and OE be joined, the ext. $\angle OBA >$ int. opp. $\angle OEB$; but $\angle OAB = \angle OBA$; $\therefore \angle OAB > \angle OEA$; $\therefore OE > OA$. Hence E is without the circle.

Cor. II. *A straight line cannot cut a circle in more than two points.*

PROPOSITION III. Theorem.

A straight line drawn from the centre of a circle to bisect any chord which does not pass through the centre, will cut it at right angles; and conversely, a straight line drawn through the centre of a circle perpendicular to any chord will bisect that chord.
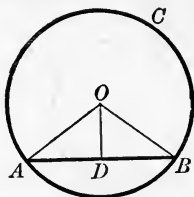
Let O be the centre of the circle ABC, and D the middle point of any chord AB, which does not pass through O. Join OD. Then, *it is required to prove that OD is $\perp^r$ to AB.*

Join OA, OB.

Then, in the $\triangle^s$ ODA, ODB

$$\therefore \begin{cases} AD = DB, \\ OD = OD, \\ \text{and radius } OA = \text{radius } OB; \end{cases} \qquad [Hyp.$$

$$\therefore \; \angle ODA = \text{adjacent } \angle ODB;$$

$$\therefore \; OD \text{ is } \perp^r \text{ to } AB.$$

Now let OD be drawn from the centre O perpendicular to any chord AB. Then, *it is required to prove that $AD = DB$.*

In the triangles ODA, ODB

$$\therefore \begin{cases} \angle ODA = \angle ODB, \\ \angle OAD = \angle OBD, \text{ since } OA = OB, \\ \text{and } OA = OB, \text{ these equal sides being opposite} \\ \qquad \text{to equal angles}; \end{cases} \qquad [Hyp.$$

$$\therefore \; AD = DB. \qquad\qquad [\text{I. } 26.$$

It follows from Props. I. and III. that a straight line

    (i) passes through the centre of a circle,

    (ii) is at right angles to a chord of the circle,

and   (iii) bisects that chord,

provided that it satisfies *any two* of these conditions.

Ex. 1. The locus [see p. 86] of the middle points of all parallel chords of a circle is the diameter perpendicular to the chords.

Ex. 2. The locus of the centres of all the circles which pass through two given points is a straight line.

Ex. 3. Through any point O within a circle draw a chord which will be bisected in O.

Ex. 4. Shew that the line joining the middle points of any two parallel chords of a circle passes through the centre of the circle.

Ex. 5. Shew that the line joining the middle points of any two parallel chords of a circle is perpendicular to the chords.

Ex. 6. Shew that, if the line joining the middle points of two chords of a circle be perpendicular to one of the chords, it will also be perpendicular to the other.

Ex. 7. The line joining the middle points of two chords of a circle passes through the centre; shew that the chords must be parallel.

Ex. 8. AB, AC are equal chords of a circle; shew that they make equal angles with the radius through A.

Ex. 9. The chords AB, AC of a circle make equal angles with the radius through A. Shew that $AB = AC$.

Ex. 10. Through two given points A, B describe a circle whose diameter will be equal to a given straight line which is not less than the straight line AB.

Ex. 11. Shew that, if two chords of a circle be equal, they will subtend equal angles at the centre of the circle. Prove also the converse theorem.

Ex. 12. Shew that, if two chords of a circle be unequal, the greater chord will subtend the greater angle at the centre.

PROPOSITION IV. Theorem.

Two chords of a circle, which do not both pass through the centre, cannot bisect each other.

Let *ABCD* be a circle, and *AC*, *BD* any two chords intersecting in the point *E* which is not the centre of the circle. Then, *it is required to prove that E cannot be the middle point both of AC and of BD.*
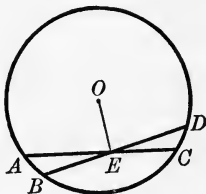
For, if one of the chords pass through the centre, that chord cannot be bisected in the point *E* which is not the centre, since the centre is the middle point of every diameter of a circle.

And, if neither of the chords pass through the centre, find the centre, *O* suppose, and join *OE*.

Then, if both *AC* and *BD* were bisected in *E*, the line *OE* through the centre of the circle would, by the preceding proposition, be perpendicular both to *AC* and to *BD*; and this is impossible.

It is therefore impossible for the chords *AB* and *CD* to bisect each other.

Ex. 1. Shew that the diagonals of any parallelogram inscribed in a circle (that is, which has its vertices on the circumference of the circle) intersect in the centre.

Ex. 2. Shew that any parallelogram inscribed in a circle is a rectangle.

Ex. 3. Shew that a rhombus inscribed in a circle is a square.

Ex. 4. Shew that, if *AB*, *CD* be any two diameters of a circle; then *A*, *B*, *C*, *D* will be at the angular points of a rectangle.

PROPOSITION V. Theorem.

Two circles which have a common point cannot have the same centre.

Let the two circles *BAC*, *DAE* have the point *A* in common. Then, *it is required to prove that the circles cannot have a common centre.*
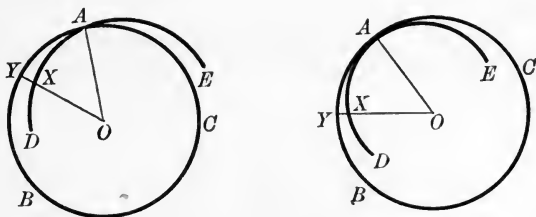
For, let *O* be the centre of the circle *BAC*. Join *OA*.

Then, whether the circles cut or touch one another at the point *A*, unless they altogether coincide with one another, it must be possible to draw some line through *O* which will meet the circles in different points. Let *OXY* be such a line meeting the circle *BAC* in *Y* and the circle *DAE* in *X*. Then, since *O* is the centre of the ⊙ *BAC*,

$$OY = OA.$$

But *OY* is not equal to *OX*;

$$\therefore OA \text{ is not equal to } OX.$$

Hence *O* is not the centre of the circle *DXAE*.

Euclid divides this very simple proposition into two, which are enunciated thus:

Prop. V. *If two circles intersect, they cannot have the same centre.*

Prop. VI. *If one circle touches another internally they cannot have the same centre.*

Cor. *Two concentric circles cannot have a common point.*

PROPOSITION VII. Theorem.

If from any point within a circle which is not the centre, straight lines be drawn to the circumference, the greatest is that which passes through the centre, and the remainder of that diameter is the least; and of any two other such lines the greater is that which is the nearer to the greatest; also from the same point there can only be drawn two equal straight lines to the circumference, one being on each side of the diameter through the point.

Let *ABC* be a circle whose centre is *O*, and let *P* be any point within it. Through *P* draw the diameter *LOPM*. Then, *it is required to prove*
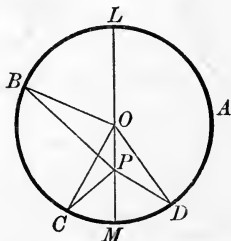
(i)   that *PL*, in which the centre lies, is the longest line which can be drawn from *P* to the circumference ;

(ii)   that *PM*, the remainder of the diameter, is the least ;

(iii)   that of any two other straight lines drawn from *P* to the circumference, the longer is that which makes the smaller angle with *PL* ; and

(iv)   that only two equal straight lines can be drawn from *P* to the circumference, and that they must be on opposite sides of the diameter *LOPM*.

Let *PB* be any straight line from *P* to a point in the circumference.   Join *OB*.

Then *PO* and *OB* together are > *PB*.

But the radii *OB* and *OL* are equal;

∴   *PO* and *OL* are > *PB*, that is *PL* > *PB*.

Thus, *PL is greater than any other straight line from P to the circumference.*

Again, PB and PO together are $> OB$, and $OB = OM$;

$$\therefore PB \text{ and } PO > OM,$$

that is, PB and $PO > PM$ and PO.

Take away PO, which is common; then $PB > PM$.

Thus, *PM is less than any other straight line from P to the circumference.*

Now let PB and PC be any two lines from P such that PB makes a smaller angle with PL than PC does;
then the point B must be between L and C on the arc, so that

$$\angle POB > \angle POC.$$

Hence, in the $\triangle^s$ POB, POC

radius $OB =$ radius OC, $PO = PO$,

and included $\angle POB >$ included $\angle POC$;

$$\therefore PB > PC. \hspace{2cm} \text{[I. 24.}$$

Thus, *of any two straight lines from P to the circumference, that is the greater which makes the smaller angle with the longest line PL.*

This proves that no two lines from P to the circumference can be equal which are on the same side of the diameter LPM. If, however, any two radii OC, OD are on opposite sides of LOM but make equal angles with it, and if PC, PD be joined; then in the $\triangle^s$ POC, POD

radius $OC =$ radius OD, $PO = PO$,

and included $\angle POC =$ included $\angle POD$;

$$\therefore PC = PD, \text{ and } \angle CPO = \angle DPO.$$

Thus, *there is one other, and only one other, straight line from P to the circumference which is equal to any line PC, and these equal lines make equal angles with the diameter through P and are on opposite sides of it.*

The particular case should be noticed when P is on the circumference, and when therefore the least distance from P of a point on the circumference is zero.

PROPOSITION VIII. Theorem.

If from any point without a circle straight lines be drawn to the circumference, the greatest is that which passes through the centre, and the least is that which when produced passes through the centre, and of any two others, that which subtends the greater angle at the centre is the greater; also from the same point there can only be drawn two equal straight lines to the circumference.

Let ABC be any circle whose centre is O, and let P be any point without the circle. Let the straight line PO cut the circle in M, and PO produced cut the circle in L.
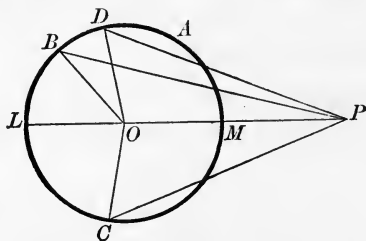
Then, *it is required to prove*

(i)   that PL is the longest straight line which can be drawn from P to the circumference;

(ii)   that PM is the shortest;

(iii)   that of any two other straight lines drawn from P to the circumference that is the greater which subtends the greater angle at the centre; and

(iv)   that only two equal straight lines can be drawn from P to the circumference.

Let PB be any straight line from P to a point in the circumference. Join OB.

Then PO and OB together are $> PB$.

But the radii OB and OL are equal;

$\therefore$ PO and $OL > PB$, that is $PL > PB$.

Thus, *PL is greater than any other straight line from P to the circumference.*

Again, PB and BO are together $> OP$; and $OB = OM$;

$$\therefore\ PB \text{ and } OM > OP,$$

that is, $\qquad\qquad PB \text{ and } OM > OM \text{ and } MP.$

Take away OM, which is common,

then $\qquad\qquad\qquad PB > PM.$

Thus, *PM is less than any other straight line from P to the circumference.*

Now let PB and PD be any two straight lines from P to the circumference, and let $\angle POB$ be greater than $\angle POD$.

Then, in the $\triangle^s$ POB, POD

$$\left\{ \begin{array}{c} \text{radius } BO = \text{radius } DO, \\ OP = OP, \\ \text{and included } \angle BOP > \text{included } \angle DOP; \end{array} \right.$$

$$\therefore\ PB > PD. \qquad\qquad \text{[I. 24.}$$

Thus, *of any two straight lines from P to the circumference, that is the greater which subtends the greater angle at the centre.*

This proves that no two lines from P to the circumference can be equal which are on the same side of the line PO. If, however, any two radii OD, OC are on opposite sides of PO but make equal angles with it, and if PD and PC be joined; then, in the $\triangle^s$ POD, POC

$$\left\{ \begin{array}{c} \text{radius } OD = \text{radius } OC, \\ OP = OP, \\ \text{and included } \angle POD = \text{included } \angle POC; \end{array} \right.$$

$$\therefore\ PD = PC.$$

Thus, *there is one other, and only one other, straight line from P to the circumference which is equal to any such line PD, and these equal lines are on opposite sides of PO and subtend equal angles at the centre.*

PROPOSITION IX. Theorem.

If from a point within a circle more than two equal straight lines can be drawn to the circumference, that point must be the centre of the circle.

Let ABC be a circle, and let the three lines OA, OB, OC, drawn to the circumference from a point O within the circle, be equal; then, *it is required to prove that the point O is the centre of the circle.*
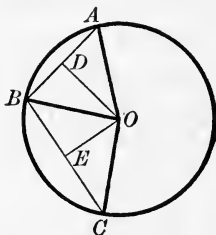
Join AB, BC and bisect them in the points D, E respectively. Join OD and OE.

Then, in the $\triangle^s$ ODA, ODB

$$\left\{ \begin{array}{l} AD = BD, \\ DO = DO, \\ \text{and radius } OA = \text{radius } OB; \end{array} \right. \qquad [\textit{Const.}$$

$$\therefore \angle ODA = \text{adjacent } \angle ODB;$$

$$\therefore OD \text{ is } \perp^r \text{ to } AB.$$

And, since DO is $\perp^r$ to AB and bisects AB, the centre of the circle must be in the line DO. [III. 1 Cor.

Similarly the centre of the circle must be in the line EO.

Hence the centre must be at O, the only point which is common to the lines DO and EO*.

* Euclid gave two proofs of Prop. IX., of which the above is the first.

PROPOSITION X. Theorem.

One circle cannot cut another in more than two points.

If it be possible let the circles ABC, DBC cut one another in the three points B, C, E.
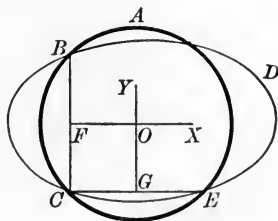
Join BC, CE and bisect them in the points F, G respectively. Through F draw $FX \perp^r$ to BC, and through G draw $GY \perp^r$ to CE. Then the lines FX and GY will intersect each other since the lines to which they are at right angles intersect each other. Let FX, GY intersect in O.

Then BC, a chord of each circle, is bisected at right angles by FX;

∴ the centre of each circle is in the line FX. [III. 1. Cor.

For the same reason, the centre of each circle must be in the line GY.

Hence O, the point of intersection of FX and GY, must be the centre of both circles.

But it has been proved that two circles which have a common point cannot be concentric*. [III. 5.

Hence *two circles cannot have more than two common points.*

Cor. *Two circles cannot have a common arc.*

---

* Euclid gave two proofs of Prop. X. of which the above is the first.

The line joining the centres of two circles will bisect their common chord at right angles.

For two circles cannot cut one another in more than two points [III. 10], and the line which is perpendicular to their common chord and bisects that chord must pass through both centres [III. 1. Cor.].
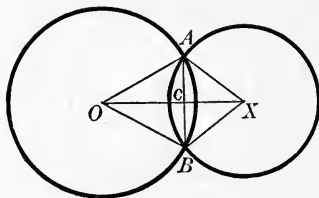
The following proof may be given.

Let A, B be the points of intersection of two circles whose centres are O, X respectively.

Join OA, OB, XA, XB, AB and OX, and let C be the point of intersection of AB and OX, produced if necessary.

Then, in the $\triangle^s$ OAX, OBX

$$OA = OB, \ AX = BX \text{ and } OX \text{ is common};$$

$$\therefore \angle AOX = \angle BOX.$$

Then, in the $\triangle^s$ COA, COB

$$OA = OB, \ OC \text{ is common, and } \angle AOC = \angle BOC;$$

$$\therefore AC = CB \text{ and } \angle OCA = \text{adj. } \angle OCB.$$

Thus OX bisects AB and is at right angles to it.

The following problem is of importance:

To describe a circle whose circumference will pass through three given points not on the same straight line.

Let A, B, C be the three given points.
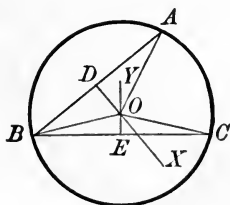
Join AB, BC and bisect them in D, E respectively.

Draw $DX \perp^r$ to AB and $EY \perp^r$ to CB. Then the lines DX and EY will intersect since the lines to which they are $\perp^r$ intersect. Let O be the point of intersection of DX, EY.

Join OA, OB, OC.

Then, in the $\triangle^s$ ADO and BDO

$AD = BD$, DO is common and included angles ADO and BDO are equal, being rt. $\angle^s$.

Hence $\qquad\qquad AO = BO.$

Similarly $BO = CO$, so that $AO = BO = CO$.

Hence, a circle described with centre O and radius AO will pass through the three points A, B, C, and therefore will be the circle required.

Ex. Find a point which is equidistant from three given points.

PROPOSITIONS XI. AND XII. THEOREMS.

*If one circle touch another, internally or externally, the
straight line joining their centres, produced if necessary, will
pass through the point of contact.*

Let O be the centre of the circle ABC, and let X be the
centre of any other circle which passes through the point A.

Then, *it is required to prove that if the two circles touch
one another at the point A, the point A must be on the line OX,
or OX produced.*
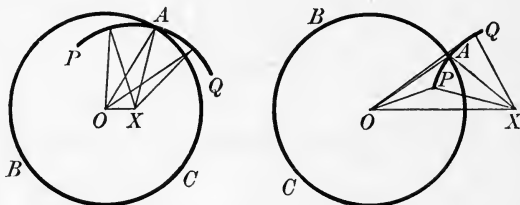
For, if the point A be not on the line OX, or OX produced,
we can take two points P, Q on the circle whose centre is X
near the point A but on opposite sides of it, P, Q being both
on the same side of OX. Join XP, XA, XQ and OP, OA, OQ.

Then $\angle OXP < \angle OXA$ and $\angle OXA < \angle OXQ$.

Hence, in the $\triangle^s$ OXP, OXA radius $XP =$ radius XA,
XO is common, and $\angle OXP < \angle OXA$;

$$\therefore \ OP < OA.$$

So also, since $\angle OXA < OXQ$, $OA < OQ$.

Now, since $OP <$ radius OA, P is *within* the $\odot$ ABC;
and, since $OQ >$ radius OA, Q is *without* the $\odot$ ABC.

Hence, if A, the common point of the two circles, is not on
the straight line OX, or OX produced, the arc PAQ of the
circle whose centre is X will *cut* the circle ABC, which is
contrary to the hypothesis that the two circles *touch* at A.

Hence, if two circles *touch* each other, the line joining their centres, produced if necessary, must pass through their common point.

Conversely. *If two circles have a common point which is on the line joining their centres, or on this line produced, the two circles will touch one another at that point.*

Let O, X be the centres of two circles which have a common point A on the line OX or OX produced.
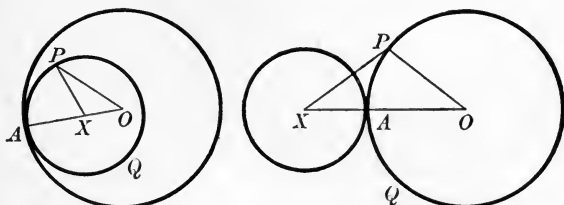
First, let the common point be in OX produced, as in the figure on the left.

Take any point P on the circle whose centre is X, and join PX, PO.

Then the sum of PX and $XO > PO$.

But rad. $PX =$ rad. AX;

$$\therefore \text{ the sum of } AX \text{ and } XO > PO,$$

that is, $AO > PO$.

Hence P must be *within* the circle whose centre is O. And, as every point on the circle centre X, except the point A, is *within* the $\odot$ centre O, the two circles cannot *cut* at A, and must therefore touch one another.

In the second figure, if P is any point on the $\odot$ centre O,

$$XP \text{ and } PO > XO$$
$$> XA \text{ and } AO ;$$
$$\therefore XP > XA, \text{ since } PO = AO.$$

Hence P is *without* the circle centre X. And, as every point, except A, of the circle centre O is *without* the circle whose centre is X, the two circles cannot *cut* at A, and must therefore touch one another.

PROPOSITION XIII. Theorem.

Two circles cannot touch each other in more than one point.

Let the circle *PAQ* touch the circle *ABC* at the point *A*. Then, *it is required to prove that the two circles will have no other common points.*
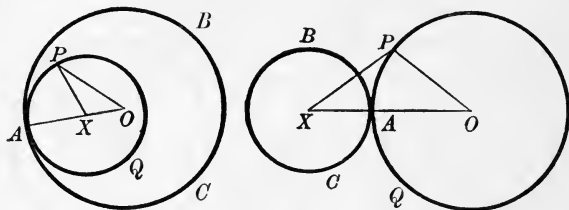
It has been proved that when two circles touch each other, the line joining their centres, produced if necessary, will pass through the point of contact.

Let *O*, *X* be the centres of the circles *ABC*, *PAQ* respectively.

First, let the point of contact be in *OX* produced.

Let *P* be any point on the circle *PAQ*, which in the figure is the circle of smaller radius.   Join *PX* and *PO*.

Then *PO* < *PX* and *XO* together.

But radius *AX* = radius *PX* ;

$$\therefore \; PO < AX \text{ and } XO \; ; \text{ i.e. } PO < AO.$$

And, as the distance of *P* from the centre of the circle *ABC* is less than the radius of that circle, *P* must be *within* the circle *ABC*.

Thus, *if two circles touch each other internally, every point on the smaller circle, except the point of contact, is within the larger,* so that the two circles have no other points in common, and therefore cannot touch (or cut) one another again.

Next, let the point of contact be in OX.

Let P be any point on the circle PAQ.   Join PX, PO.

Then OP and PX are together $> OX$;

i.e. OP and $PX > OA$ and AX.

But radius $PO =$ radius AO; $\therefore XP > XA$.

And, as the distance of P from the centre of the circle ABC is greater than the radius of that circle, P must be *without* the circle ABC.

Thus, *if two circles touch each other externally, every point on one of the circles, except the point of contact, is outside the other circle*, so that the two circles have no other points in common, and therefore cannot touch (or cut) one another again.

Ex. 1. If two circles touch each other, the distance between their centres is equal to the sum or to the difference of the radii.

Ex. 2. If two circles cut one another the distance between their centres must be less than the sum and greater than the difference of the radii.

Ex. 3. Find the locus of the centre of a circle which touches a given circle at a given point.

Ex. 4. A circle of given radius touches a given circle, shew that its centre lies on one or other of two concentric circles.

Ex. 5. Find the two points, one on each of two circles external to one another, which are farthest apart, and the two points which are nearest together.

Ex. 6. With a given point as centre describe a circle to touch a given circle. How many such circles can be drawn?

Ex. 7. Describe a circle of given radius to touch two given circles.

Ex. 8. Draw a circle of given radius to touch a given circle and pass through a given point.

Ex. 9. PP', QQ' are parallel diameters of two circles which touch one another. Shew that either PQ and $P'Q'$ or PQ' and $P'Q$ will pass through the point of contact.

PROPOSITION XIV. Theorem.

*Equal chords in a circle are equally distant from the centre;
and conversely, chords which are equally distant from the centre
are equal.*

Let AB, CD be equal chords in a circle whose centre is O.
Then, *it is required to prove that they are equally distant
from O.*
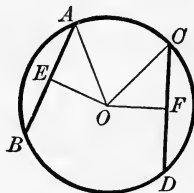
Draw OE, OF perpendicular to AB, CD respectively.
Join OA, OC.

Then, $\because$ OE is drawn from the centre $\perp^r$ to AB,

$\therefore$ $AE = EB$, and $\therefore$ $AE =$ half AB.

Similarly $CF =$ half CD.

But, by hypothesis, $AB = CD$; $\therefore$ $AE = CF$;

$\therefore$ sq. on $AE =$ sq. on CF.

The radius $OA =$ radius OC;

$\therefore$ sq. on $OA =$ sq. on OC.

But, since OEA is a rt. $\angle^e$,

sq. on $OA =$ sq. on AE with sq. on OE.

So also   sq. on $OC =$ sq. on CF with sq. on OF.

Hence

sq. on AE with sq. on $OE =$ sq. on CF with sq. on OF.

And   sq. on $AE =$ sq. on CF;

$\therefore$ sq. on $OE =$ sq. on OF,

and $\therefore$ $OE = OF$.

Thus, *if the chords AB and CD are equal, they are equally
distant from the centre.*

Conversely, let the chords AB and CD be equally distant from the centre; then, *it is required to prove that the chords are equal.*

The same construction being made, we have proved that

sq. on AE with sq. on OE = sq. on CF with sq. on OF.

But   sq. on OE = sq. on OF; since, by hyp., $OE = OF$.

$$\therefore \text{ sq. on } AE = \text{sq. on } CF;$$

and $\therefore$ $AE = CF$.

But it has been proved that AE = half AB, and that CF = half CD.

Hence $AB = CD$.

Thus, *if the chords AB and CD are equally distant from the centre, they are equal in length.*

Ex. 1.   Find the locus of the middle points of equal chords of a circle.

Ex. 2.   Through a point O within a circle two chords are drawn equally inclined to the diameter through O; shew that these chords are equal.

Ex. 3.   Two equal chords AB, CD of a circle intersect in O; shew that AO is equal to CO or to DO.

Ex. 4.   Two chords AB, CD of a circle intersect in O, and $AO = CO$; shew that $BO = DO$.

Ex. 5.   Two parallel chords of a circle are cut by a diameter in two points equally distant from the centre; shew that the chords are equal.

Ex. 6.   Through two points on a diameter of a circle equally distant from the centre two parallel chords are drawn; shew that the extremities of these chords are at the angular points of a parallelogram.

Ex. 7.   Draw a chord of a given circle equal to one given chord and parallel to another.

PROPOSITION XV. Theorem.

A diameter is the greatest chord of a circle ; and, of others, a chord which is nearer the centre is greater than one more remote ; and, conversely, of any two chords the greater is that which is nearer to the centre.

First, let AB be any diameter of a circle whose centre is O, and let CD be any other chord. Then, *it is required to prove that $AB > CD$.*
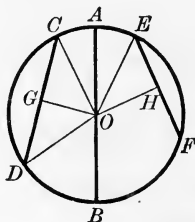
Join OC, OD.

Then, since all the radii of a circle are equal ;

$$OA = OB = OC = OD.$$

Hence sum of OC and OD = sum of OA and $OB = AB$.

But sum of $\qquad OC$ and $OD > CD$;

$$\therefore\ AB > CD.$$

Next, let EF be any chord further from the centre than CD. Then, *it is required to prove that $CD > EF$.*

Draw OG, $OH \perp^{r}$ to CD, EF respectively, and join OE.

Then, since OG and OH are the $\perp^{rs}$ from the centre on CD, EF respectively, $CG = GD$ and $EH = HF$; \qquad [III. 3.

$$\therefore\ CG = \text{half } CD \text{ and } EH = \text{half } EF.$$

Since $\angle OGC$ is a right $\angle^e$,

$$\text{sq. on } OC = \text{sq. on } OG \text{ with sq. on } GC. \qquad [\text{I. 47.}$$

So also, sq. on $OE = $ sq. on OH with sq. on EH. $\qquad [\text{I. 47.}$

But rad. $OC = $ rad. OE; $\therefore$ sq. on $OC = $ sq. on OE.

Hence

sq. on OG with sq. on $GC = $ sq. on OH with sq. on EH.

But sq. on $OG < $ sq. on OH, since $OG < OH$.

Hence sq. on $GC > $ sq. on EH,

and $\therefore$ $GC > EH$.

Hence $CD > EF$.

Thus, *if CD is nearer the centre than EF, then $CD > EF$.*

Conversely, let CD be $> EF$; then, *it is required to prove that CD is nearer the centre than EF.*

The same construction being made, we have proved that

$$CG = \text{half } CD \text{ and } EH = \text{half } EF.$$

But, by hypothesis, $CD > EF$;

$\therefore CG > EH$ and sq. on $CG > $ sq. on EH.

Also, as before

sq. on OG with sq. on $CG = $ sq. on OH with sq. on EH.

But sq. on $CG > $ sq. on EH.

Hence sq. on $OG < $ sq. on OH;

$\therefore OG < OH$.

Thus, *if $CD > EF$, it is nearer the centre than EF.*

Cor. *The longest chord of a circle which can be drawn through a given point within it is the diameter through the point, and the shortest chord is that which is perpendicular to that diameter.*

Ex. 1. Through a given point within a circle draw the shortest possible chord.

Ex. 2. If the shortest chords which can be drawn through the points O, O' within a circle are equal, prove that O, O' are equally distant from the centre of the circle.

PROPOSITION XVI. Theorem.

The straight line drawn through any point on a circle perpendicular to the radius through that point is a tangent to the circle, and every other straight line through that point will cut the circle.

Let A be any point on a circle whose centre is O, and let BAC be the st. line through $A \perp^{r}$ to the radius AO; then, *it is required to prove that BAC touches the circle and that any other straight line through A will cut the circle.*
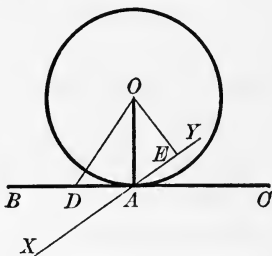
Take any point D on the line BAC, and join OD.

Then, in the $\triangle OAD$, since $\angle OAD$ is a rt. $\angle^{e}$,

$$\angle ODA \text{ is } < \text{ a rt. } \angle^{e}.$$

Hence $OD >$ radius OA.   [I. 19.

And, since the distance of any point D on the line BAC from the centre of the circle is greater than its radius, every point on the line BAC, except the point A, is *without* the circle.

$\therefore BAC$ does not cut the circle, i.e. *it is a tangent to the circle.*

Now, let XAY be any other straight line through the point A.

Draw $OE \perp^{r}$ to XAY.

Then, since $\angle OEA$ is a rt. $\angle^{e}$, $\angle OAE$ is $<$ a rt. $\angle^{e}$;

$$\therefore OE < \text{ radius } OA.$$

Hence the point E must be *within* the circle, and therefore the st. line XAY *cuts* the circle at the point A.

Ex. 1. If two circles touch one another at any point, the two circles have the same tangent line at that point.

Ex. 2. Shew that the tangents to a circle at the two extremities of any diameter are parallel.

Ex. 3. Shew that if two tangents to a circle are parallel, their points of contact are extremities of a diameter.

Ex. 4. Draw a tangent to a circle which will be parallel to a given straight line.

Ex. 5. Draw a tangent to a given circle perpendicular to a given straight line.

Ex. 6. Draw a circle to touch a given line at a given point and to pass through another given point.

Ex. 7. Shew that the locus of the middle points of all chords of a circle which are of given length is a concentric circle.

Ex. 8. Shew that all chords of a given circle which are of given length will touch a concentric circle.

Ex. 9. Shew that a straight line will cut, touch, or lie entirely outside a circle, according as its distance from the centre is less than, equal to, or greater than the radius of the circle.

Ex. 10. Two circles are concentric; shew that all chords of the outer which touch the inner are equal in length.

PROPOSITION XVII. Problem.

Draw a straight line from a given point so as to touch a given circle.

Case I. When the given point is within the given circle, the problem is impossible, for every straight line drawn from a point within a circle must *cut* the circle.

Case II. When the given point is on the circumference of the given circle.

Join the given point to the centre, and draw a straight line through the given point perpendicular to this radius. Then, by the preceding proposition, this perpendicular is the tangent required.

Case III. When the given point is without the given circle.

Let *A* be the given point, and *O* the centre of the given circle *BCD*.

Join *OA* cutting the circle *BCD* in the point *E*.
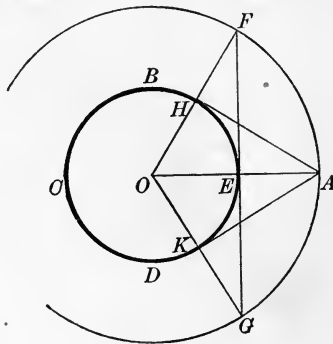
With *O* as centre and *OA* as radius describe a circle *FAG*.

Through *E* draw a st. line $\perp^r$ to *OA* and produce it to cut the circle *FAG* in the points *F*, *G*.

Join *OF*, *OG* cutting the circle *BCD* in the points *H*, *K* respectively.

Join *AH*, *AK*. Then *AH* and *AK* will be the required tangents from *A* to the circle *BCD*.

For, in the △ˢ *OEF*, *OHA*,

$$\begin{cases} \text{radius } OE = \text{radius } OH, \\ \text{radius } OF = \text{radius } OA, \\ \text{and included } \angle AOF \text{ is common;} \end{cases}$$

$$\therefore \ \angle OEF = \angle OHF.$$

But, by construction, ∠ *OEF* is a rt. ∠ᵉ;

$$\therefore \ \angle OHF \text{ is a right angle.}$$

But a st. line drawn from the extremity of a radius of a circle and at rt. ∠ˢ to it, is a tangent to the circle. [III. 16.

Thus *AH* is a tangent to the circle *BCD*.

So also *AK* is a tangent to the circle *BCD*.

Cor. *The two tangents drawn to a circle from any external point are equal in length.*

For, since *OHA* and *OKA* are rt. ∠ˢ,

sq. on *OA* = sq. on *OH* with sq. on *AH*,

and sq. on *OA* = sq. on *OK* with sq. on *AK*.

Hence

sq. on *OH* with sq. on *AH* = sq. on *OK* with sq. on *AK*.

But sq. on *OH* = sq. on *OK*, since radius *OH* = radius *OK*.

$$\therefore \text{ sq. on } AH = \text{sq. on } AK; \ \therefore \ AH = AK.$$

Then, since the three sides of the △ *AHO* are equal respectively to the three sides of the △ *AKO*, these triangles are equal in all respects, so that ∠ *AOH* = ∠ *AOK* and ∠ *HAO* = ∠ *KAO*.

Thus *two tangents can be drawn to a circle from any external point and these two tangents are equal in length; also the two tangents subtend equal angles at the centre, and the line joining the external point to the centre bisects the angle between the tangents.*

PROPOSITION XVIII. Theorem.

The straight line drawn from the centre to the point of contact of any tangent to a circle is perpendicular to that tangent.

Let the straight line DAE touch the circle ABC, whose centre is O in the point A. Then, *it is required to prove that $\angle OAE$ is a right angle.*
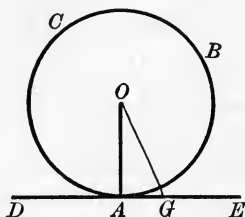
For, if possible, let OA be not perpendicular to DAE.

Draw $OG \perp^r$ to DAE.

Then, since $\angle OGA$ is a rt. $\angle^e$, $\angle OAG$ is less than a rt. $\angle^e$;

$$\therefore \angle OAG < \angle OGA;$$
$$\therefore OG < OA.$$

Hence G would be *within* the circle, and therefore the line DAE would *cut* the circle.

Thus, if DAE is a tangent, it must be $\perp^r$ to OA.

PROPOSITION XIX. Theorem.

The straight line drawn from the point of contact of a tangent to a circle perpendicular to that tangent passes through the centre of the circle.

It has been proved that a tangent is $\perp^r$ to the radius through its point of contact, that is to say, the line joining the centre to the point of contact is $\perp^r$ to the tangent, and there is only *one* perpendicular to a straight line at a given point; this $\perp^r$ must therefore pass through the centre.

Ex. 1. From any point on the outer of two concentric circles tangents are drawn to the inner; shew that these tangents are of constant length.

Ex. 2. Find the locus of a point from which the tangents drawn to a given circle are of given length.

Ex. 3. Find, when possible, a point on a given straight line such that the tangents from it to a given circle may be of given length.

Ex. 4. Shew that the two tangents at the extremities of any chord of a circle make equal angles with the chord.

Ex. 5. Shew that, if A, B are any two points on a circle, the perpendicular from A on the tangent at B is equal to the perpendicular from B on the tangent at A.

Ex. 6. Find the locus of the centres of circles which touch two parallel straight lines.

Ex. 7. Shew that the locus of the centres of circles which touch two given intersecting straight lines is two straight lines.

Ex. 8. Find the centre of a circle which touches three given straight lines which are not all parallel and which do not meet in a point. How many such circles can be drawn?

Ex. 9. Describe a circle of given radius to touch two given intersecting straight lines.

Ex. 10. Shew that a circle can be drawn to touch the sides of any rhombus.

Ex. 11. Construct a rhombus, having given its angles and the radius of its inscribed circle.

Ex. 12. Shew that, if the four sides of a quadrilateral touch a circle, the sum of one pair of opposite sides is equal to the sum of the other pair.

Ex. 13. Shew that, if the six sides of a hexagon touch a circle, the sum of three alternate sides is equal to the sum of the other three alternate sides. Shew also that the corresponding theorem is true for any polygon of an even number of sides all of which touch the same circle.

Ex. 14. Shew that, if the sides of a quadrilateral touch a circle, the sum of the angles which one pair of opposite sides subtends at the centre is equal to two right angles.

Ex. 15. TA, TB are two fixed tangents to a circle whose centre is O, and any other tangent to the circles cuts TA, TB respectively in the points P, Q. Prove that PQ subtends a constant angle at the centre of the circle, and that the perimeter of the triangle TPQ is constant.

NOTE. Before considering the next proposition it is neces-
sary to extend the meaning given to an angle. Hitherto it
has been understood that an angle was less than two right
angles. Euclid never definitely frees himself from this restric-
tion, although, as will be seen, angles of indefinite magnitude
are necessarily introduced in the proof of Euclid VI. 33.

The following definition of an angle should now be sub-
stituted for that previously given.

Def. *If a straight line turn about the fixed extremity O,
starting from the initial position OA, when it is in any position
OB it is said to have described the angle AOB.*
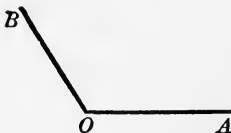
As there is no limit to the *amount of turning* of the line,
an angle can be of any magnitude whatever. It will, how-
ever, be sufficient for our present purposes to consider angles
not greater than four right angles, that is angles described in
the first complete revolution of the moving line.

An angle greater than two right angles is distinguished in
Geometry by the mark ⌢ placed over it.

Thus any two straight lines *AO, CO*, which meet in a
point, form two angles, one of which is greater than and the
other is less than two right angles;
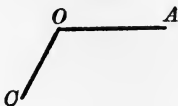
of these angles ∠ $\widehat{AOC}$ stands for that which is greater than two
right angles, and ∠ *AOC* for that which is less than two right
angles with the same bounding lines, so that ∠ $\widehat{AOC}$ and
∠ *AOC* are together equal to four right angles.

PROPOSITION XX. THEOREM.

An angle at the centre of a circle is double an angle at the circumference which stands on the same arc.

In the circle $ABCD$ let $\angle ABC$ at the circumference and $\angle AOC$ at the centre, O, stand on the same arc ADC; then, *it is required to prove that $\angle AOC$ is double $\angle ABC$.*
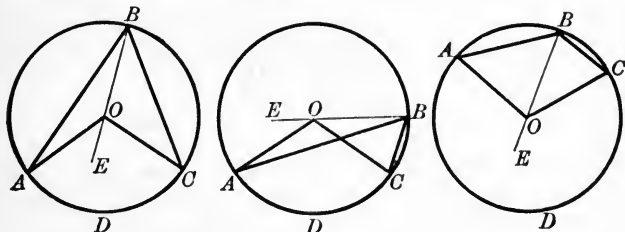
Join BO and produce it to E.

Then, since $AO = BO$, $\angle OBA = \angle OAB$.

But ext. $\angle AOE = $ sum of $\angle^s ABO$ and OAB;

$$\therefore \quad \angle AOE = \text{twice } \angle ABO.$$

So also $\qquad \angle EOC = \text{twice } \angle OBC.$

Hence, in figure 1,

sum of $\angle^s AOE$ and $EOC = $ twice sum of $\angle^s ABO$ and OBC;

i.e. $\qquad \angle AOC = \text{twice } \angle ABC.$

And, in figure 2,

difference of $\angle^s EOC$ and $EOA = $ twice difference of

$$\angle^s OBC \text{ and } OBA;$$

i.e. $\qquad \angle AOC = \text{twice } \angle ABC.$

Also, in figure 3,

sum of $\angle^s AOE$, $EOC = $ twice sum of $\angle^s ABO$, OBC;

i.e. $\qquad \angle \widehat{AOC} = \text{twice } \angle ABC.$

Hence, in all cases, *the angle at the centre is double the angle at the circumference which stands on the same arc.*

PROPOSITION XXI. Theorem.

The angles in the same segment of a circle are equal.
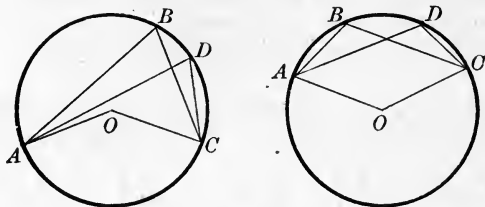
Let the angles ABC, ADC be any two angles in the segment $ABDC$ of the circle $ABDC$; then, *it is required to prove that* $\angle ABC = \angle ADC$.

Let O be the centre of the circle. Join OA, OC.

Then the $\angle AOC$ at the centre and the $\angle ABC$ at the circumference stand on the same arc AEC;

$$\therefore \angle AOC = \text{twice } \angle ABC; \qquad \text{[III. 20}$$

or, in figure 2, $\qquad \angle \widehat{AOC} = \text{twice } \angle ABC.$

Similarly $\qquad \angle AOC = \text{twice } \angle ADC;$

or, in figure 2, $\qquad \angle \widehat{AOC} = \text{twice } \angle ADC.$

Hence $\qquad \angle ABC = \angle ADC.$

Conversely. *The locus of a point at which a given straight line subtends a constant angle is two arcs of circles through the extremities of the given line.*

Let AB be the given straight line, and let C be any point such that the angle ACB is equal to the constant angle; also let P be any other point such that $\angle APB = \angle ACB$, P being on the same side of AB as the point C.

Describe a circle through the points A, C, B.

Then we have to prove that this circle will pass through the point P.

For, if the circle do not pass through P it will cut BP, or BP produced, in some point Q. Join QA.

Then, the angles AQB, ACB are in the same segment $AQCB$ of a circle;

$$\therefore \angle AQB = \angle ACB.$$

But we know that $\angle APB = \angle ACB$.

Hence $\angle AQB = \angle APB$; but this is impossible, for one of these angles is an exterior angle of the triangle APQ, of which the other is an interior opposite angle.

Hence the circle through A, C, B must pass through P.

Similarly all the points on the other side of AB at which AB subtends the constant angle are on another arc of a circle through A and B.

PROPOSITION XXII. Theorem.

The opposite angles of a quadrilateral inscribed in a circle are together equal to two right angles.

Let $ABCD$ be a quadrilateral inscribed in the circle $ABCD$; then, *it is required to prove that the sum of any two opposite angles is equal to two right angles.*
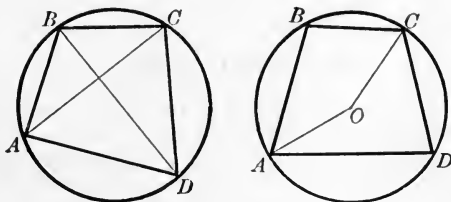
Draw the diagonals AC and BD.

Then the $\angle^s$ CAD, CBD are in the same segment;

$$\therefore \angle CAD = \angle CBD. \qquad \text{[III. 21}$$

Also the angles BAC, BDC are in the same segment;

$$\therefore \angle BAC = \angle BDC.$$

Hence

sum of $\angle^s$ BAC and $CAD =$ sum of $\angle^s$ BDC, and CBD,

i.e. $\qquad \angle BAD =$ sum of $\angle^s$ BDC and CBD.

To each of these equals add the $\angle BCD$;

then

sum of $\angle^s$ BAD and $BCD =$ sum of $\angle^s$ BDC, CBD and BCD.

But the sum of the three angles BDC, CBD and BCD of the $\triangle BCD$ is equal to two right angles.

Hence the sum of $\angle^s$ BAD and BCD is equal to two right angles.

In the same manner it can be proved that the sum of $\angle^s$ ABC and ADC is equal to two right angles.

Cor. I. *If one of the sides of a quadrilateral inscribed in a circle be produced, the exterior angle will be equal to the opposite angle of the quadrilateral.*

Cor. II. *The two segments into which a circle is divided by any chord contain supplementary angles.*

Alternative Proof.

Join two opposite angular points A and C to O, the centre of the circle. [See figure on the preceding page.]

Then $\angle AOC$ at the centre and $\angle ADC$ at the circumference stand on the same arc ABC.

$$\therefore \angle ADC = \text{half } \angle AOC.$$

Also $\angle \widehat{AOC}$ at the centre and $\angle ABC$ at the circumference stand on the same arc ADC.

$$\therefore \angle ABC = \text{half } \angle \widehat{AOC}.$$

Hence sum of $\angle^s$ ADC and ABC

$$= \text{half sum of } \angle^s \ AOC \text{ and } \widehat{AOC}$$
$$= \text{half four right angles}$$
$$= \text{two right angles.}$$

Conversely. *If two opposite angles of a quadrilateral be together equal to two right angles, a circle can be described about the quadrilateral.*

Let the sum of the angles *ABC*, *ADC* of the quadrilateral *ABCD* be equal to two right angles. Describe a circle through the three points *A*, *B*, *C*; then we have to prove that this circle passes through the point *D*.

For, if the ⊙ˢ did not pass through *D*, it would cut *CD*, or *CD* produced, in some point *X*. Join *AX*.

Then, since *A*, *B*, *C*, *X* lie on a circle,

Sum of ∠ˢ *ABC* and *AXC* = two right angles.

But, by hyp., sum of ∠ˢ *ABC* and *ADC* = two right angles.

Hence sum of ∠ˢ *ABC* and *AXC* = sum of ∠ˢ *ABC* and *ADC*.

Hence ∠ *AXC* = ∠ *ADC*; but this is impossible, for one of these angles is an exterior angle of the triangle *ADX* of which the other is an interior opposite angle.

Hence the circle through *A*, *B*, *C* must also pass through *D*.

Def. A quadrilateral which is such that a circle can be described through its four angular points is called a **cyclic quadrilateral**.

Ex. 1. Shew that, if *ABC* be any segment of a circle, and *P* be a point on the same side of *AC* as the segment; then will the angle *APC* be less or greater than the angle *ABC* according as *P* is outside or inside the segment *ABC*.

Ex. 2. A triangle is inscribed in a circle; shew that the angles in the three segments exterior to the triangle are together equal to four right angles.

Ex. 3. An equilateral triangle is inscribed in a circle, shew that the angle subtended by one of the sides at any point of the circle is twice the angle subtended at that point by either of the other two sides.

Ex. 4. A square is inscribed in a circle; shew that at any point of the circumference one side subtends an angle three times the angle subtended by any one of the other sides.

Ex. 5. A regular hexagon is inscribed in a circle; shew that at any point on the circumference one side subtends an angle which is equal to five times the angle subtended by one of the other sides.

Ex. 6. *AB*, *CD* are chords of a circle which intersect in *O*. Shew that the triangles *OAC*, *ODB* are equiangular, and that the triangles *OAD*, *OCB* are also equiangular.

Def. *Segments of circles which contain equal angles are said to be* **similar**.

PROPOSITION XXIII. Theorem.

On the same straight line, and on the same side of it, there cannot be two similar segments of circles not coinciding with one another.

If it be possible, upon the same straight line *AB*, and on the same side of it, let there be the two similar segments of circles *ACB*, *ADB*, which do not coincide with one another.
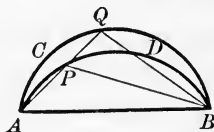
Then, since the arcs *ACB*, *ADB* do not coincide, it is possible to draw a st. line *APQ* cutting them in different points *P*, *Q* respectively.

Join *PB*, *QB*.

Then, since the segments are similar, by definition

$$\angle APB = \angle AQB.$$

But it is impossible that *APB*, an exterior angle of the $\triangle PQB$, should be equal to an interior opposite angle *PQB*.

It is therefore impossible that the segments *ADB*, *ACB*, which are not coincident, should be similar.

PROPOSITION XXIV. Theorem.

Similar segments of circles on equal straight lines, are equal to one another.

Let *ABC*, *DEF* be two similar segments of circles upon the equal straight lines *AC*, *DF*.

Then, *it is required to prove that the segments ABC, DEF are equal to one another.*
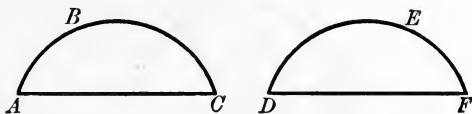
For, if the segment *DEF* be applied to the segment *ABC* so that the point *D* falls upon the point *A* and the st. line *DF* upon the st. line *AC*, the arcs *DEF* and *ABC* being upon the same side of *AC*.

Then, since *DF = AC*, the point *F* will fall on the point *C*.

There will then be two similar segments on the same st. line *AC* and on the same side of it; and therefore, by the preceding proposition, the two segments must altogether coincide.

$$\therefore \text{ segment } DEF = \text{ segment } ABC.$$

It should be noticed that this proposition is equivalent to the following:

If two segments of circles on equal straight lines contain equal angles, the two circles of which the segments are parts are equal.

From Prop. xx. it also follows that

If in two circles equal chords subtend equal or supplementary angles at the circumference, the circles must be equal.

Ex. 1. Prove that, if *AB*, *AC* are equal sides of the isosceles triangle *ABC*, and *D* be any point on *BC*, or *BC* produced, the circles *ABD* and *ADC* will be equal.

PROPOSITION XXV. THEOREM.

An arc of a circle being given: complete the circle.

Let *ABC* be a given arc of a circle. *It is required to find the centre and complete the circle.*
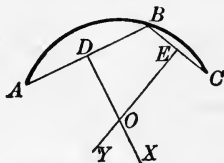
Take any point *B* in the arc, and join *AB*, *CB*.

Bisect *AB* and *BC*, and let *D*, *E* be their middle points respectively.

Through the points *D*, *E* draw the st. lines *DX*, *DY* respectively ⊥ʳ to *AB* and *BC*.

Since *AB*, *CB* intersect, *DX* and *EY* must also intersect. Let *O* be the point of intersection.

Then, since *DX* bisects *AB* and is ⊥ʳ to it, the centre of the ⊙ is in *DX*.   [III. 1 Cor.

So also the centre of the ⊙ is in *EY*.

Hence *O*, the only point which is common to *DX* and *EY*, must be the centre of the circle.

The circle of which *ABC* is an arc can now be completed, for it is the circle with *O* as centre and *OA* as radius.

PROPOSITION XXVI. Theorem.

In equal circles the arcs which subtend equal angles at the centres, or at the circumferences, are equal.

Let *ABCD*, *EFGH* be equal circles, and let the $\angle^s$ *AOC*, *EKG* at the centres, and therefore the angles *ABC*, *EFG* at the circumferences, be equal; then, *it is required to prove that the arcs ADC, EHG are equal.*
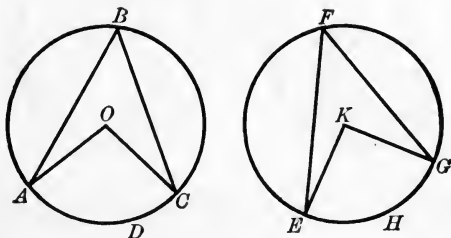
For let the $\odot$ *EFGH* be applied to the $\odot$ *ABCD* so that the centre *K* is on the centre *O*, and the radius *KE* on the radius *OA*. Then, since the radii of the circles are equal, the point *E* will fall on the point *A*, and the whole circumferences *EFGH*, *ABCD* will coincide.

And, since *EK* coincides with *AO*, and $\angle EKG = \angle AOC$, *KG* will lie on *OC*, and the point *G* will coincide with the point *C*, since *KG = OC*.

Hence the arc *EHG* coincides with, and is therefore equal to the arc *ADC*.

Thus, *in equal circles the arcs which subtend equal angles at the centres, or at the circumferences, are equal.*

Cor. *In equal circles chords which subtend equal angles at the centres, or at the circumferences, are equal.*

PROPOSITIONS XXVII. and XXIX. Theorems.

*In equal circles the angles at the centres, or at the circum-
ferences, which stand on equal arcs, are equal. Also, the chords
of equal arcs are equal.*

Let *ABCD*, *EFGH* be equal circles, and let the arcs *ADC*,
EHG be equal; then, *it is required to prove that the angles
AOC, EKG, which are subtended at the centres by the equal
arcs, are equal, and also that the chords AC and EG are equal.*
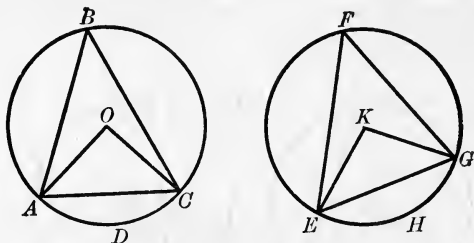
For let the ⊙ *EFGH* be applied to the ⊙ *ABCD* so that
the centre *K* is on the centre *O*, and the radius *KE* on the
radius *OA*. Then, since the radii of the circles are equal, the
point *E* will fall on the point *A*, and the whole circumferences
EFGH, *ABCD* will coincide.

And, since the arc *EHG* lies on the arc *ADC*, the point *E*
being on the point *A*, and the arcs are equal, their other
extremities must also coincide.

Thus the point *G* falls upon the point *C*.

Hence, as *G* is on *C* and *K* on *O*, the radius *KG* coincides
with the radius *OC*, and therefore ∠ *EKG* = ∠ *AOC*.

Thus, *in equal circles, the angles at the centres* (and
therefore also the angles at the circumferences which are the
halves of the angles at the centres), *which stand on equal arcs,
are equal.*

Also, since *E* is on *A* and *G* on *C*, the chord *EG* coincides
with the chord *AC*, and is therefore equal to it.

Thus, *in equal circles, the chords which join the extremities
of equal arcs are equal.*

PROPOSITION XXVIII. Theorem.

In equal circles, if two chords be equal, the arcs they cut off from the circumferences are equal.

In the equal circles *ABCD, EFGH*, whose centres are *O, K* respectively, let the chords *AC* and *EG* be equal; then, *it is required to prove that the arcs ADC, EHG and also the arcs ABC, EFG cut off by these chords are equal.*
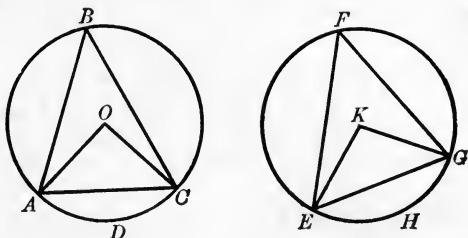
Join *OA, OC, KE, KG.*

Then, in the △ˢ *AOC, EKG*, since the radii of the circles are equal

$$AO = EK, \ OC = KG; \text{ also, by hyp., } AC = EG;$$
$$\therefore \angle AOC = \angle EKG.$$

Now let the ⊙ *EFGH* be applied to the ⊙ *ABCD*, so that the centre *K* is on the centre *O*, and the radius *KE* on the radius *OA*.

Then, since the radii of the circles are equal, the point *E* will fall on the point *A*, and the whole circumferences *EFGH, ABCD* will coincide.

And, since *KE* coincides with *OA*, and ∠ *EKG* = ∠ *AOC*, *KG* will lie on *OC*, and the point *G* will coincide with the point *C*.

Hence the arc *EHG* coincides with, and is therefore equal to the arc *ADC*. So also the arcs *ABC* and *EFG* coincide and are equal.

Thus, *in equal circles, equal chords cut the circumferences into parts which are respectively equal.*

NOTE. The enunciations of Propositions XXVI. to XXIX., as given by Euclid, refer only to *equal* circles; the student will, however, have no difficulty in seeing that all these enunciations may be given in the form 'In equal circles, *or in the same circle*, &c.'

These Theorems are in fact more important when considered as properties of a single circle, than when they are considered only as properties of equal circles.

Thus in a circle, or in equal circles, if any one of the pairs *arcs, chords, angles subtended at centre*, are equal, they will all be equal.

The following proposition is of importance:

If equal chords of two circles subtend equal angles at the centres of the circles (or subtend equal or supplementary angles at two points on the circumferences), the two circles must be equal.

For, let the chords AB and CD of two circles subtend equal angles at their centres O, K respectively.

Then, since $\angle AOB = \angle CKD$,

the sum of $\angle^s OAB$, OBA = sum of $\angle^s OCD$, ODC.

But, since $OA = OB$ and $KC = KD$, $\angle OAB = \angle OBA$ and $\angle KCD = \angle KDC$. Hence these four angles are all equal.

Therefore, in the $\triangle^s ABO$, CDK, we have $AB = CD$,

$$\angle OAB = \angle KCD \text{ and } \angle OBA = \angle KDC;$$

$\therefore OA = OC$. [See also Prop. XXIV.]

Ex. 1. Two triangles are inscribed in a circle and two of the sides of the one are respectively parallel to two sides of the other; shew that the third sides are equal.

Ex. 2. A triangle is inscribed in a given circle and one of its angles is of constant magnitude; shew that the opposite side touches a fixed circle.

Ex. 3. Two given circles intersect in the points A, B. Any line is drawn through A cutting the circles again in the points P, Q respectively. Shew that the angle PBQ is constant.

Ex. 4. Two given circles intersect in the points A, B. Through any point P on one of the circles lines PAX, PBY are drawn cutting the other circle in X, Y respectively. Shew that the straight line XY is of constant length.

PROPOSITION XXX. Problem.

To bisect a given arc of a circle.

Let ABC be the given arc. *It is required to bisect it.*
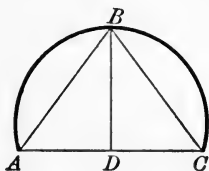
Join AC, and bisect it in D.

Through D draw a line $\perp^r$ to AC and let it cut the circumference in the point B. Then the arc ABC will be bisected in B.

Join AB, CB.

Then, in the $\triangle^s$ ADB, CDB

$$AD = DC$$
$$DB = DB$$
and rt. $\angle ADB =$ rt. $\angle CDB$ and these are the included angles;

$$\therefore AB = BC,$$

and since the chords AB, BC are equal, they cut the whole circumference into arcs which are equal, the greater arc equal to the greater and the less arc equal to the less.   [III. 29.

Hence the arcs AB and BC are equal, for they are both less than the semi-circumference, since BD produced is a diameter.

Ex. 1.   Divide a given arc of a circle into four equal parts.

Ex. 2.   Divide a given arc of a circle into eight equal parts.

Ex. 3.   Divide the circumference of a circle into twelve equal parts.

PROPOSITION XXXI. Theorem.

The angle in a semi-circle is a right angle, the angle in a segment greater than a semi-circle is less than a right angle, and the angle in a segment less than a semi-circle is greater than a right angle.

Let $ABCD$ be a circle of which O is the centre and AOD is a diameter, and let any chord DB be drawn dividing the circle into the segments BAD, BCD of which the segment
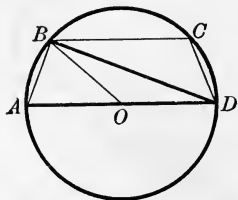
BAD is greater and the segment BCD is less than a semi-circle.

Take any point C in the arc BCD, and join CB, CD, AB. Then, *it is required to prove that* $\angle ABD$ *is a rt.* $\angle$, *that* $\angle BAD$ *is less than a rt.* $\angle$, *and that* $\angle BCD$ *is* $>$ *a rt.* $\angle$.

Join OB.

Then, since $OA = OB = OD$,

$\angle OBA = \angle OAB$ and $\angle OBD = \angle ODB$.

Hence sum of $\angle^s$ OBA and OBD is equal to the sum of $\angle^s$ OAB and ODB.

That is, $\angle ABD =$ sum of $\angle^s$ BAD and BDA.

Hence $\angle ABD =$ half the sum of the angles of the $\triangle BAD$
$=$ half two right angles.

Thus, *the angle ABD in a semi-circle is a right angle.*

Then, since $\angle ABD$ is a rt. $\angle$, $\angle BAD$ must be less than a rt. $\angle$.

Thus, *the angle BAD in the segment BAD, which is greater than a semi-circle, is less than a rt.* $\angle$.

Again, since $ABCD$ is a quadrilateral inscribed in a circle, the sum of the opposite angles BAD and BCD is equal to two rt. $\angle$ˢ.

But of these $\angle BAD$ has been proved to be less than one rt. $\angle$;

$$\therefore \angle BCD \text{ is greater than a rt. } \angle.$$

Thus, *the angle BCD in the segment BCD, which is less than a semi-circle, is greater than a rt. $\angle$.*

Ex. 1. Shew that every diameter of a circle subtends a right angle at any point on the circumference.

Ex. 2. Shew that, if the chords AB, CD of a circle intersect at O within the circle, the angle AOC will be equal to the sum of the angles at the circumference which stand on the arcs AC and BD.

Ex. 3. AB and CD are two perpendicular chords of a circle; shew that the sum of the arcs AC and BD is equal to half the circumference of the circles.

Ex. 4. Two circles intersect in the points A, B and a straight line PAQ is drawn through A cutting the circles in P, Q respectively. Shew that, if the circles are equal the chords PB and QB will be equal; and conversely that, if $PB = QB$, the circles must be equal.

Ex. 5. Two circles intersect in AB and through A two straight lines PAQ, RAS are drawn cutting one circle in P, R respectively and the other circle in Q, S. Shew that, if the chord PR is equal to the chord QS, the circles must be equal; and conversely that, if the circles are equal the chords PR and QS will be equal.

Ex. 6. Chords of a circle whose centre is O pass through a fixed point P; shew that the locus of the middle point of the chord is the circle whose diameter is OP.

Ex. 7. Describe a rectangle, having given one of its diagonals in magnitude and position and also a point through which the other diagonal passes.

Ex. 8. Two circles touch one another at the point A and are touched by a straight line in the points B, C respectively. Shew that the circle whose diameter is BC passes through A. Shew also that if the lines BA, CA be produced to cut the circles again in C', B' respectively, the lines BB' and CC' will be diameters.

PROPOSITION XXXII. Theorem.

If a straight line touch a circle, and through the point of contact a chord of the circle be drawn, the angles which this chord makes with the tangent are equal to the angles in the alternate segments of the circle.

Let the st. line XAY touch the circle $ABCD$ in the point A, and let the chord AC be drawn through A. Then, *it is required to prove that the angles XAC, YAC are equal to the angles in the alternate segments CDA, ABC respectively.*
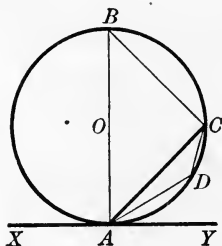
From A draw a st. line $\perp^r$ to XAY meeting the circle again in the point B.

Then, since AB is $\perp^r$ to the tangent XAY and passes through its point of contact, AB must be a diameter of the circle. [III. 19.

Join BC. Also take any point D on the arc CDA and join DC, DA.

Then, since AB is a diameter,

$$\angle BCA \text{ is a rt. } \angle. \qquad \text{[III. 31.}$$

Hence the sum of $\angle^s BAC$, $ABC = $ a rt. $\angle$.

But the sum of $\angle^s BAC$, $CAY = $ a rt. $\angle$.

Hence the sum of $\angle^s BAC$, $CAY = $ sum of $\angle^s BAC$, ABC;

$$\therefore \angle CAY = \angle ABC,$$

which is the angle in the alternate segment.

And, since $ABCD$ is a quadrilateral inscribed in a circle,

$$\text{sum of } \angle^s\ ABC,\ ADC = 2 \text{ rt. } \angle^s,$$

and $$\text{sum of } \angle^s\ CAY,\ CAX = 2 \text{ rt. } \angle^s.$$

Hence sum of $\angle^s\ CAY,\ CAX = $ sum of $\angle^s\ ABC,\ ADC.$

But it has been proved that $\angle\,CAY = \angle\,ABC$;

$$\therefore\ \angle\,CAX = \angle\,ADC,$$

which is the angle in the alternate segment.

CONVERSELY. *If a straight line, drawn through one extremity of a chord of a circle, make with the chord an angle equal to the angle in the alternate segment of the circle, the straight line will touch the circle.*

Ex. 1. Two circles touch internally or externally at the point A, and through A two straight lines are drawn cutting one circle in P, R respectively and the other circle in Q, S. Shew that PR is parallel to QS, and that the tangents at P, Q, and also the tangents at R, S are parallel.

Ex. 2. Shew that, if an equilateral triangle be inscribed in a circle, the tangents at its angular points will form another equilateral triangle.

Ex. 3. Shew that, if D is a point on the base AB of the $\triangle\,ABC$ such that $\angle\,CDB = \angle\,ACB$, then will BC touch the circle through the points A, C, D.

Ex. 4. $ABCD$ is a parallelogram whose diagonals AC, BD intersect in the point O. Shew that, if the circles AOB, COD be drawn, they will touch one another at O.

Ex. 5. Two circles intersect in the points A, B and through these points any two lines PAQ, RBS are drawn cutting one circle in P, R respectively and the other circle in Q, S. Shew that PR is parallel to QS.

Ex. 6. A is a common point of two circles, and through A two straight lines are drawn cutting one circle in P, R respectively and the other circle in Q, S. Shew that, if PR is parallel to QS the two circles must touch at the point A.

PROPOSITION XXXIII. Problem.

Upon a given straight line describe a segment of a circle which will contain an angle equal to a given angle.

Let AB be the given straight line, and X the given angle. *It is required to describe on AB a segment of a circle containing an angle equal to the angle X.*

First, let the angle X be a rt. $\angle$.
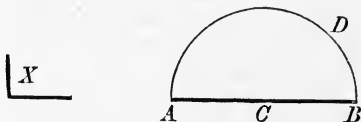
Bisect AB in C, and with C as centre and CA or CB as radius describe a semi-circle ADB. Then, since the angle in a semi-circle is a right angle, this semi-circle will be the segment required.

But, if $\angle X$ be not a rt. angle.

At the point A make the angle BAD equal to the given angle X, and through A draw $AE \perp^r$ to DA.

Bisect AB in C, and draw through C the line FCG perpendicular to AB, and let FG cut AE in O.
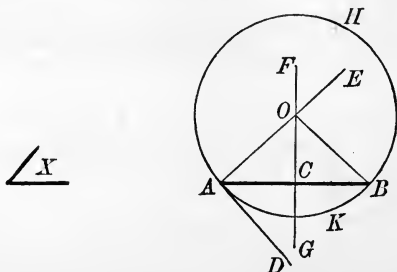
Join *OB*.

Then, in the △ˢ *ACO, BCO*,

AC = BC, CO is common, and rt. ∠ *ACO* = rt. ∠ *BCO* ;

$$\therefore AO = BO.$$

Hence the circle described with *O* as centre and *OA* as radius will pass through *B*. Let this be the circle *AHBK*.

Then, *AD* is, by construction, ⊥ʳ to rad. *AO* ;

$$\therefore AD \text{ touches the circle } AHBK.$$

Hence ∠ *DAB* is equal to the angle in the alternate segment *AHB*.

But ∠ *DAB* = given angle *X*.

Hence we have described on *AB* a segment *AHB* containing an angle equal to the given angle *X*.

Ex. 1. Construct a triangle having given one side, the angle opposite to that side and the sum of the other two sides.

Let *AB* be the given side, *X* the given angle and *YZ* a straight line equal to the sum of the other two sides.

[Suppose that *ACB* is the required triangle. Since the sum of *AC* and *CB* is equal to *YZ*, produce *AC* to *D* making *CD = CB* ; then (1) *AD = YZ*. And, since *CD = CB*, ∠ *CDB* = ∠ *CBD* ; ∴ ∠ *ACB* = sum of ∠ˢ *CDB* and *CBD* = twice ∠ *CDB*. Thus ∠ *ADB* = half ∠ *ACB* = half given angle *X*. Hence (2) *D* is on a segment of a circle of which *AB* is the chord and which contains an angle equal to half the given angle. The conditions (1) and (2) determine the point *D* :—hence we have the following construction.]

On *AB* describe a segment of a ⊙ containing an angle equal to half the given angle *X*. Then, with *A* as centre and radius equal to *YZ* describe a ⊙ cutting the former ⊙ in a point *D*. Join *AD*. From ∠ *DBA* cut off ∠ *DBK* = ∠ *ADB*, and let *BK* cut *AD* in the point *C*. Then, since ∠ *DBC* = ∠ *BDC*, *CD = CB*, and therefore sum of *AC* and *CB = AD* = given st. line *YZ*. Also

∠ *ACB* = sum of equal ∠ˢ *CDB* and *CBD*

= twice ∠ *CDB* = given angle *X*.

Hence △ *XCB* is the triangle required.

Ex. 2. Construct a triangle having given one side, the opposite angle, and the length of the perpendicular drawn on the given side from the opposite angular point.

Ex. 3. Construct a triangle having given one side, the opposite angle, and the length of the line drawn from the opposite angular point to the middle point of the base.

Ex. 4. Construct a triangle having given one side, the opposite angle, and the difference of the other two sides.

Ex. 5. Find, when possible, a point on a given straight line at which another given finite straight line subtends an angle equal to a given angle.

Ex. 6. Find a point on the circumference of a given circle at which the line joining two fixed points subtends an angle equal to a given angle.

Ex. 7. Construct a right-angled triangle having given the lengths of the hypotenuse and one of the other sides.

Ex. 8. A is a fixed point on a given circle and P is any other point on the circumference, AP is produced to Q so that $PQ = AP$. Shew that the locus of Q is a circle.

Ex. 9. A, B are two fixed points on a circle and P is any other point on the circle. On AP, or AP produced, a point Q is taken so that $PQ = PB$; shew that Q lies on one or other of two fixed circles through A and B.

Ex. 10. The straight line AB of given length moves so that its extremities A, B lie on the two fixed lines OX, OY respectively. Shew that the locus of the centre of the circle through the points O, A, B is a circle.

Ex. 11. Two circles intersect in the points A, B, and through A, B parallel straight lines PAQ, RBS are drawn cutting one circle in P, R respectively and the other circle in Q, S. Shew that $PQSR$ is a parallelogram.

Ex. 12. Shew that, of all triangles on the same base and with equal vertical angles, the isosceles triangle has the greatest area.

Ex. 13. $ABCD$ is a quadrilateral in a circle, and the angles DAB, ABC are equal; prove that CD is parallel to AB.

Ex. 14. Shew that the lines bisecting the interior angles of any quadrilateral form a cyclic quadrilateral.

Ex. 15. Shew that the lines bisecting the exterior angles of any quadrilateral form a cyclic quadrilateral.

Ex. 16. Prove that the lines bisecting any angle of a quadrilateral inscribed in a circle and the line bisecting the opposite exterior angle intersect on the circle.

Ex. 17. AB and CD are parallel chords of a circle, and the chords AC, AD, BC, BD are drawn. Shew that $AC = BD$ and $AD = BC$.

Ex. 18. The circle A goes through the centre of the circle B; shew that the tangents to B at their points of intersection will meet on the circumference of A.

PROPOSITION XXXIV.

From a given circle to cut off a segment, which will contain an angle equal to a given angle.

Let ABC be the given circle and X the given angle. *It is required to cut off from the circle ABC a segment which will contain an angle equal to the angle X.*
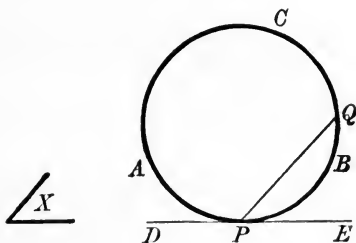
Take any point P on the circle, and draw DPE the tangent at P.   [III. 17.

At P make $\angle EPQ = \angle X$, PQ cutting the circle ABC in Q.

Then, since PE is a tangent to the circle, and PQ a chord through its point of contact,

$\angle EPQ =$ the angle in alternate segment $PACQ$.

But $\angle EPQ = \angle X.$

Hence the angle in the segment $PACQ = \angle X$, so that the segment $PACQ$ is the segment required.

PROPOSITION XXXV. Theorem.

If any number of chords of a circle pass through a fixed point within it, the rectangles contained by their segments are all equal.

Let the point through which the chords pass be the centre of the circle; then it is obvious that the rectangle contained by the segments of any chord is equal to the square of the radius.

Next, let the chords all pass through a point P which is not the centre.

Draw the diameter AB through P, and let O be the centre.
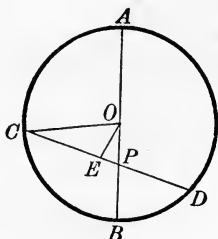
Then, since AB is bisected in O,

　　　　rect. AP, PB with sq. on $OP =$ sq. on OA,　　　[II. 5.

so that rect. AP, PB is equal to the difference of the squares on the radius and on the distance OP.

Now let CD be any other chord through P.

Draw $OE \perp^r$ to CD, and join OC.

Then, since OE passes through the centre of the $\odot$ and is $\perp^r$ to CD, it bisects CD.

Hence rect. CP, PD with sq. on $EP =$ sq. on CE.　　　[II. 5.

Add the sq. on OE to each of these equals; then

rect. CP, PD with sq. on EP and sq. on OE

$= $ sq. on CE and sq. on OE.

But, since OE is $\perp^r$ to DC,

sq. on EP and sq. on $OE = $ sq. on OP,   [I. 47.

also   sq. on EC and sq. on $OE = $ sq. on OC.

Hence rect. CP, PD with sq. on $OP = $ sq. on OC,

so that rect. CP, PD is equal to the difference of the squares on the radius and on the distance OP.

Thus, *the rectangles contained by the segments of all chords of a $\odot$ which pass through a point P within it are equal to one another, and equal to the difference of the squares on the radius of the circle and the distance of P from its centre.*

CONVERSELY. *If two straight lines intersect in a point and the rectangles contained by their segments are equal, a circle can be drawn through their four extremities.*

For, let the lines AB, CD intersect in O so that rect. AO, $OB = $ rect. CO, OD. Draw a circle through A, B, C; and, if possible, let this circle cut CD, produced if necessary, in the point X. Then, by the preceding prop., rect. AO, $OB = $ rect. CO, OX. Hence rect. CO, $OD = $ rect. CO, OX, which is impossible unless X coincides with D.

Ex. 1. Two circles intersect in the points A, B, and any line is drawn cutting AB in the point O, one of the circles in P, Q and the other in R, S. Shew that the rectangles PO, OQ and RO, OS are equal.

Ex. 2. Two circles intersect in the points A, B; and, through any point O on the common chord AB, two lines are drawn one cutting one of the circles in the points P, Q and the other cutting the other circle in the points R, S. Shew that a circle will pass through the points P, Q, R, S.

PROPOSITION XXXVI. Theorem.

If from any point without a circle two straight lines be drawn, one of which cuts the circle and the other touches it; the rectangle contained by the whole line which cuts the circle and the part of it without the circle will be equal to the square on the tangent.

Let P be any point without the circle ABC and let PAC, PB be two lines through P, of which PAC cuts the $\odot$ in the points A, C, and PB touches the circle at the point B. Then, *it is required to prove that rect. PA, $PC =$ sq. on PB.*
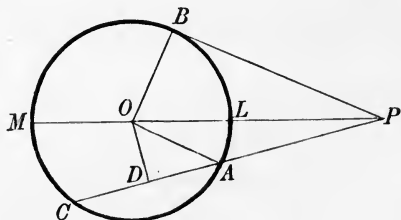
Let O be the centre of the circle. Draw $OD \perp^r$ to PAC. Join OP, OA and OB.

Then, since OD passes through the centre and is $\perp^r$ to AC, it will bisect AC.

And, since CA is bisected at D,

rect. PA, PC with sq. on $AD =$ sq. on PD.   [II. 6.

Add the sq. on OD to each of these equals; then

rect. PA, PC with sq. on AD and sq. on OD

$=$ sq. on PD and sq. on OD.

But, since OD is $\perp^r$ to PAC,

sq. on AD and sq. on $OD =$ sq. on OA,   [I. 47.

also   sq. on PD and sq. on $OD =$ sq. on OP.

Hence

> rect. *PA*, *PC* with sq. on radius *OA* = sq. on *OP*.

If the line which cuts the circle passes through the centre *O* and cuts the circle in the points *L*, *M*.

Then, since *LM* is bisected in *O*,

> rect. *PL*, *PM* with sq. on radius *OL* = sq. on *OP*. [II. 6.

Thus, *the rectangle contained by the segments of any chord through P is equal to the difference of the squares on OP and on a radius of the circle.*

Again, since *PB* touches the ⊙ at the point *B*, *PB* is ⊥ʳ to the radius *BO*.

> ∴ sq. on *PB* with sq. on radius *BO* = sq. on *OP*.

Thus, *the square on the tangent from P to the ⊙ is equal to the difference of the squares on OP and on a radius of the circle.*

Hence, *the rectangle contained by the segments of any chord of a circle which passes through an external point is equal to the square on the tangent drawn to the circle from that point.*

CONVERSELY. *If two straight lines be both produced to meet in a point and if the rectangles contained by their segments are equal, a circle can be drawn through their four extremities.*

Ex. 1. *AB*, *CD*, *EF* are the common chords of three circles which cut one another in pairs; shew that the lines *AB*, *CD*, *EF* are all parallel, or that they will (being produced if necessary) meet in a point.

First, suppose that two of the chords are parallel. Then, since the line joining the centres of two circles is at right angles to their common chord, it is easily seen that, if two of the common chords of three circles taken in pairs be parallel, the three centres of the circles are on a straight line perp. to these ∥ chords; and the third common chd. must also be ⊥ to the line of centres, and must ∴ be ∥ to the other two common chds.

Next, let *CD* and *EF*, two of the common chords be not parallel.

Let *O* be the point of int. of *CD* and *EF*, produced if necessary. Join *OA*, then we have to shew that *OA*, or *OA* prod. will pass through the pt. *B*.

For, if OA do not pass through B it will cut the two circles through A, B in different pts., X, Y suppose. Then, since A, X, C, D are on the circle $ABCD$, rect. $OA \cdot OX =$ rect. $OC \cdot OD$; and, since A, Y, E, F are on the circle $ABEF$, rect. $OA \cdot OY =$ rect. $OE \cdot OF$. But, since C, D, E, F are on a circle, rect. $OC \cdot OD =$ rect. $OE \cdot OF$. Hence rect. $OA \cdot OX =$ rect. $OA \cdot OY$; the pts. X, Y must $\therefore$ coincide, and OA must pass through B the only other point which is common to the circles AEF and CDA.

Ex. 2. Two circles touch one another at the point O and are cut by any other circle in the points A, B and C, D respectively; shew that AB, CD and the common tangent at O will meet in a point or be all parallel.

Ex. 3. Each of three given circles touches the other two; shew that the common tangents at the three points of contact will meet in a point or be all parallel.

Ex. 4. Any number of circles are drawn through two fixed points A, B, and from any fixed point on the line AB produced tangents are drawn to the circles; shew the locus of the points of contact of these tangents is a circle.

PROPOSITION XXXVII. THEOREM.

If from a point without a circle there be drawn two straight lines, one of which cuts the circle in two points and the other meets it; and if the rectangle contained by the whole line which cuts the circle and the part of it without the circle be equal to the square on the line which meets the circle; then the line which meets the circle must be a tangent.

From a point P without the circle ABC, let the two lines PAC, PB be drawn, of which PAC cuts the circle in the points A, C and PB meets the circle in B; then, *it is required to prove that, if the rect. PA, PC be equal to the square on PB, then PB will touch the circle at the point B.*
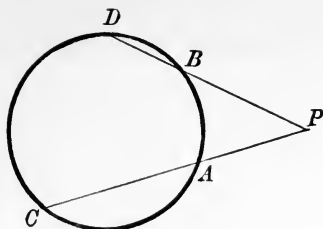
For, if *PB* were not a tangent, it would cut the circle at *B*, and would therefore, if produced, cut the circle again at some point *D*.

But, in that case, by the preceding proposition

$$\text{rect. } PA, PC = \text{rect. } PB, PD.$$

But $\qquad$ rect. $PA, PC = \text{sq. on } PB.$

Hence $\qquad$ sq. on $PB = \text{rect. } PB, PD,$

which is impossible.

Hence *PB* cannot *cut* the circle.

SECANTS AND TANGENTS.

Def. *A line which cuts a circle in two points is called a* **secant.**

If any secant be drawn through a fixed point *P* on a circle, this secant can be turned about the point *P* so that its second point of intersection with the circle will move up to and ultimately coincide with the point *P* itself.

Def. *The limiting position of a secant of a circle when its two points of intersection coincide is called a* **tangent** *to the circle.*
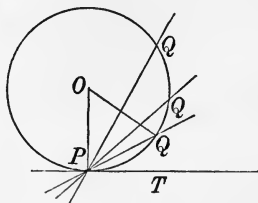
The figure shews different positions of the secant *PQ*, its ultimate position, when *Q* coincides with *P*, being the tangent *PT*.

If in any position of the secant *PQ* the radii *OP*, *OQ* be drawn, since *OPQ* is an isosceles triangle, ∠ *OPQ* differs from a rt. ∠ by half ∠ *POQ*; hence when *PQ*, and therefore also ∠ *POQ*, vanishes, the ∠ *OPT*=rt. ∠. Thus the new definition leads to the same result as the old.

Since the tangent to a circle is a secant whose points of intersection are coincident, the properties of tangents are merely particular cases of the properties of secants, and it is most important that the student should make himself familiar with the method of deducing the properties of tangents from those of secants.

Thus Props. XVI., XVIII., XIX. are only particular cases of Prop. I., Cor., and Prop. III.; Prop. XXXII. is a particular case of Prop. XXI.; and Prop. XXXVI., Exs. 2 and 3 are particular cases of Ex. 1.

In like manner the properties of touching circles follow from those of intersecting circles. For example, Props. XI. and XII. are particular cases of the theorem (page 178) that the line joining the centres of two circles bisects their common chord. Also Prop. XIII. follows from Prop. X.

ADDITIONAL PROPOSITIONS.

In addition to the properties of circles proved by Euclid, the following are important. Most of these have already been given, but it will be convenient to collect together all those additional propositions with which the student must make himself familiar.

I. *The locus of the middle points of all equal chords of a circle is a concentric circle.* [From III. 14.

II. *All straight lines, whose perpendicular distances from a given point are equal to one another, touch a fixed circle of which that point is the centre.* [From III. 16.

III. *All chords of a circle which are of equal length touch a fixed concentric circle.* [From I. and II.

IV. *If two circles cut each other, the line joining their centres bisects their common chord.* [Page 178.

V. *Equal chords of a circle subtend equal angles at the centre, and conversely chords of a circle which subtend equal angles at the centre are equal.* [Page 206.

VI. *The two tangents which can be drawn to a circle from any external point are equal in length and make equal angles with the line joining the external point to the centre of the circle.* [Page 191.

VII. *If a straight line subtend equal angles at two points on the same side of it, a circle will pass through these two points and the two extremities of the line.* [Converse of III. 21.

VIII. *If a straight line subtend supplementary angles at two points on opposite sides of it, a circle will pass through these two points and the two extremities of the line.* [Converse of III. 22.

IX. *Chords of a circle which subtend equal or supplementary angles at points on the circumference are equal, and therefore* [III] *touch the same concentric circle.*

X. *If in two circles equal chords subtend equal or supplementary angles at points on the circumference, the circles must be equal.* [Page 206.

XI. *If a straight line, drawn through one extremity of a chord of a circle, make with the chord an angle equal to the angle in the alternate segment of the circle, the straight line will touch the circle.* [Converse of III. 32.

XII. *If two straight lines intersect in a point and if the rectangles contained by their segments be equal, a circle will pass through their four extremities.* [Converse of III. 35.

XIII. *If two straight lines be both produced to meet in a point and if the rectangles contained by their segments be equal, their four extremities are concyclic.* [Converse of III. 36.

MISCELLANEOUS THEOREMS AND PROBLEMS.

1. *Divide a given straight line so that the rectangle contained by the two parts may be equal to a given square.*

Let AB be the given st. line, and let the sq. on XY be the given sq.
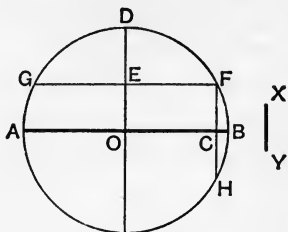
Bisect AB in O, and draw a $\odot$ with centre O and rad. OA, or OB.

[Then, if C is the reqd. pt., and FCH the chd. of the $\odot$ $\perp^r$ to AB; rect. AC, CB = rect. FC, CH = sq. on FC, since FH is bisected by the $\perp^r$ diam^r AB; $\therefore$ FC = given line XY. Hence we have the following construction :—]

Draw the rad. OD $\perp^r$ to AB, and from OD cut off $OE = XY$. Through E draw the chord FG $\perp^r$ OD and $\therefore$ ∥ to AB. Through F draw the chd. FH $\perp^r$ to AB and cutting AB in C. Then AB will be divided at C in the manner required.

For, since FH is $\perp^r$ to the diam^r AB, it will be bisected in C. Also rect. AC, CB = rect. FC, CH = sq. on FC. But $OCFE$ is a ∥^{m}; $\therefore$ CF = OE, and $OE = XY$. Hence rect. AC, CB = sq. on XY.

2. *Produce a given straight line so that the rectangle contained by the whole line so produced and the part produced may be equal to a given square.*

Let AB be the given st. line and let the square on XY be the given sq. Bisect AB in O, and draw a $\odot$ $APBQ$ with centre O and rad. OA, or OB.
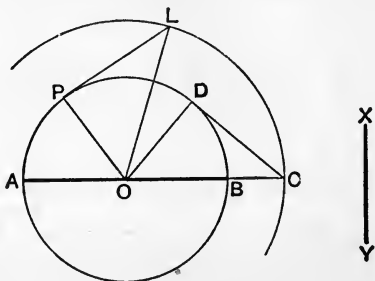
[Then, if C be the pt. on AB such that AC, $BC=XY^2$, and if the tangent CD be drawn to the $\odot$; then $CD^2=CB \cdot CA=XY^2$, and $\therefore CD=XY$. Now *all points from wh. the tangents drawn to a given $\odot$ are of given length lie on a concentric $\odot$*; hence we have the following construction:—]

Take any pt. P on the $\odot$ *APBQ*, and draw the tangent PL, making $PL=XY$. With centre O and radius OL des. a $\odot^e$ cutting AB produced in C. Then C is the point reqd.

From C draw CD touching the $\odot$ *APBQ* in D. Join DO, LO and PO. Then, since PL and DC are tangents, $\angle$ ° OPL and ODC are rt. $\angle$ °; $\therefore$ $OL^2=OP^2+PL^2$ and $OC^2=OD^2+DC^2$. But by const. $OL=OC$ and $OP=OD$, whence it follows that $DC=PL$, and $PL=XY$. Hence $XY^2=DC^2=CA \cdot CB$. Thus C is the pt. required.

[See also Prop. B, page 136.

3. *The four extremities of any two parallel lines which are both bisected at right angles by the same st. line, lie on a circle.*

Let AB, CD be any two parallel st. lines whose middle points are U, V respectively, and let UV be at right angles to AB and CD; then *it is required to prove that a circle will pass through the four points A, B, D, C.*
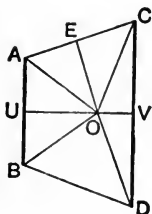
Join AC and BD. Through E the middle point of AC draw a line $\perp^r$ to AC and cutting UV in O. Join AO, BO, CO, DO.

Then, since AC is bisected at rt. $\angle$ ° by EO, any point on EO is equally distant from A and C. Thus $AO=OC$. For the same reason $OC=OD$ and $OB=OA$. Hence $OA=OB=OC=OD$, and therefore the circle whose centre is O and radius OA will pass through all the four points.

Or thus. It is sufficient to prove that a pair of opposite angles are supplementary. Now, if $AUVC$ be folded over along the line UV, UA will fall on UB, since rt. $\angle$ $VUA=$ rt. $\angle VUB$; and A will fall on B, since $UA=UB$. Similarly C will fall on D, and therefore $UACV$ will coincide with $UBDV$. Hence $\angle ACV = \angle BDV$. But, since AB and CD are $\parallel$, the angles A and C are supplementary; hence the angles A and D are supplementary.

4. *To draw the common tangents of two given circles.*

Let A and B be the centres of the given circles, of wh. the $\odot$ whose centre is A is the greater.
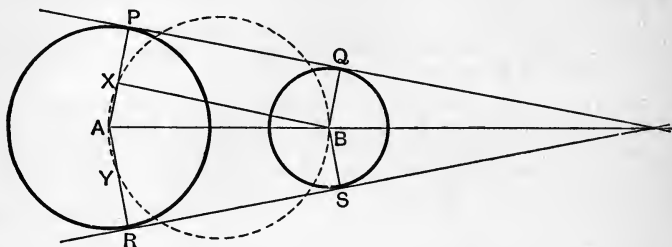
[Suppose that PQ has been drawn to touch the circles at P, Q respectively on the same side of AB. Then, if AP and BQ be joined, the angles APQ and BQB are rt. $\angle$s, since PQ touches both circles. Draw $BX \parallel$ to PQ to cut AP in X; then $PQBX$ is a $\parallel$m, and $\therefore XP = BQ$. Hence AX *is equal to the difference of the radii of the circles*; and, since $\angle AXB$ is a rt. $\angle$, X *is on the circle whose diameter is* AB. Hence the construction is as follows :—]

On AB as diam^r describe a $\odot$, and in this $\odot$ place a chord AX equal to the difference of the radii of the given circles. Draw the radius AXP through X, and draw the radius $BQ \parallel$ to AP and on the same side of AB. Join PQ, then PQ will touch both $\odot$s.

For, since AX = difference of AP and BQ, $XP = BQ$, and XP is also $\parallel$ to BQ; $\therefore XPQB$ is a $\parallel$m.

But, since AB is the diam. of the $\odot$ AXB, AXB is a rt. $\angle$, and $\therefore$ all the angles of $XPQB$ are rt. $\angle$s; and, since the angles APQ, BQP are right angles, it follows that PQ touches both the given circles.

A chord AX of the circle whose diameter is AB can always be drawn equal to the difference of the radii of the given circles unless this difference is greater than AB, in which case one circle lies entirely within the other. When one chord AX can be drawn, another chord AY can also be drawn on the other side of AB such that $\angle BAY = \angle BAX$, and $AY = AX$.

A common tangent to two circles whose points of contact are on the same side of the line joining their centres is called a **direct common tangent**.

Thus two given circles, neither of which is entirely within the other, have two direct common tangents; and it is easily seen that these common tangents make equal angles with the line joining the centres of the circles and cut this line produced in the same point.

Next suppose that PQ has been drawn to touch the circles at P, Q respectively on opposite sides of AB.
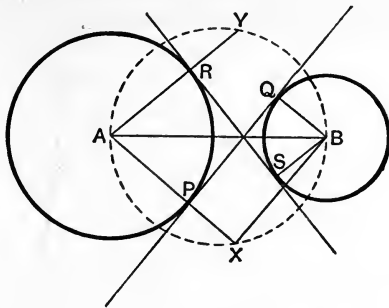
Then, if AP and BQ be joined, the angles APQ, BQP are rt. $\angle$ˢ, since PQ touches both circles. Draw $BX \parallel$ to PQ to cut AP produced in X; then $PQBX$ is a $\parallel$ᵐ, and $\therefore PX = QB$. Hence AX *is equal to the sum of the radii of the circles*; and since AXB is a rt. $\angle$, X *is on the circle whose diameter is AB.*

Hence AX can be drawn, and then PQ in the same manner as for the direct common tangents. A chord AX of the circle whose diameter is AB can always be drawn equal to the sum of the radii of the given circles unless this sum is greater than AB, in which case the two circles will cut one another, or one ⊙ will be entirely within the other; and when one such chord can be drawn, another chord equally inclined to AB can also be drawn.

A common tangent to two circles whose points of contact are on opposite sides of the line joining their centres is called a **transverse common tangent.**

Thus two given circles which do not cut one another, and neither of which is entirely within the other, have two transverse common tangents, which make equal angles with the line joining their centres and which cut this line in the same point.

The student should consider the special cases when the two given circles touch one another.

5. *The three 'perpendiculars' of a triangle meet in a point.*

Let BE, CF be the perpendiculars from B, C upon the opposite sides CA, AB respectively of the triangle ABC, or on those sides produced; and let BE, CF, produced if necessary, meet in O. Then *it is required to prove that if OA be joined and produced to cut BC at D, AD will be $\perp$ʳ to BC.*

Join FE.

Then, since the ∠ˢ *OFA, OEA* are rt. ∠ˢ, their sum is equal to two rt. ∠ˢ;

∴ *O, F, A, E* are cyclic   [III. 22, Converse.

Hence ∠ *OFE* = ∠ *OAE* in the same segment.
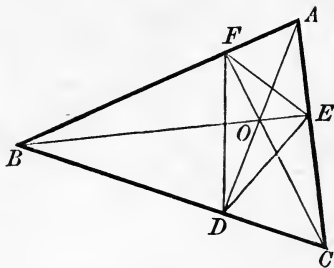
Again, since rt. ∠ *BFC* = rt. ∠ *BEC*,

B, F, E, C are cyclic   [III. 21, Converse.

Hence ∠ *CFE* = ∠ *CBE* in the same segment.

Hence ∠ *OAE* = ∠ *CBE.*

To each add ∠ *ACB*;

then   ∠ *OAE* and ∠ *ACB* = ∠ *CBE* and ∠ *ACB.*

But   ∠ *CBE* and ∠ *ACB* = rt. ∠, since ∠ *BEC* = rt. ∠;

∴ ∠ *OAE* and ∠ *ACB* = rt. ∠,

and therefore   ∠ *ADC* = rt. ∠.

The case of an obtuse-angled triangle requires no separate examination. For when △ *ABC* is acute-angled, as in the figure, the △ *AOB* is obtuse-angled, and *AE, BD, OF* are the 'perpendiculars' of the △ *AOB*, and these meet in the point *C*.

It should be noticed that, if *O* be the orthocentre of the △ *ABC*, then *A, B, C* will be the orthocentres of the △ˢ *BOC, COA, AOB* respectively, the feet of the perpendiculars for all four triangles being the points *D, E, F*.

The △ *DEF*, whose angular points are the feet of the perpendiculars of the △ *ABC* is called the **pedal triangle** of the △ *ABC*.

It should also be noticed that, since the angles *BOC* and *BAC* are supplementary, the ⊙ˢ *BOC* and *BAC* are equal; and similarly each of the ⊙ˢ *COA, AOB* is equal to the ⊙ *ABC*.

Thus, *if O is the orthocentre of the triangle ABC, the four circles BOC, COA, AOB, ABC are all equal.*

Conversely, *if ABC be any triangle and O a point such that the circles BOC, COA, AOB are equal, then each is equal to the circle ABC and O is the orthocentre of the triangle ABC.*

It is of importance to notice that the sides of the $\triangle DEF$ make equal angles with the sides of the $\triangle ABC$ on which they meet.

For, since A, E, D, B are cyclic,

$$\angle EDC = \angle BAC.$$

And, since A, F, D, C are cyclic,

$$\angle FDB = \angle BAC.$$

Hence $\qquad \angle BDF = \angle CDE = \angle BAC.$

Similarly $\qquad \angle DEC = \angle AEF = \angle CBA,$

and $\qquad \angle AFE = \angle BFD = \angle ACB.$

Now, if XYZ be the triangle of minimum perimeter whose angular points are on the sides BC, CA, AB respectively of an *acute-angled* triangle ABC, it follows from XIII. page 96 that any two sides of the triangle XYZ must make equal angles with the side of the $\triangle ABC$ on which they meet. For, if YX, ZX were not equally inclined to BC, we could by keeping Y and Z fixed and taking the point X' on BC which is such that YX', ZX' make equal angles with BC, obtain a $\triangle X'YZ$ whose perimeter is less than that of XYZ. Now we have proved that the sides of the $\triangle DEF$ make equal angles with the sides of the $\triangle ABC$ on which they meet, and it is easy to prove that no other such triangle can be inscribed in ABC.

Hence *the pedal triangle of an acute-angled triangle is the inscribed triangle of minimum perimeter.*

In the case of an obtuse-angled triangle the inscribed triangle of minimum perimeter has the foot of the $\perp^r$ drawn from the obtuse angle for one angular point and the two others are indefinitely near the obtuse angle.

6. *If from any point on the circumference of a circle, perpendiculars be drawn to the sides of an inscribed triangle, the three feet of the perpendiculars lie on a st. line.*

From any pt. O on the $\odot ABC$ draw $OD, OE, OF \perp^r$ to BC, CA, AB. Join DF and FE; then we have to prove that DF and FE are in the same st. line. To prove that DFE is a st. line, we have only to prove that $\angle DFB = \angle EFA$.
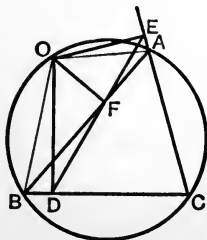
Join *OB*, *OA*.

Now, since ∠ˢ *ODB* and *OFB* are rt. ∠ˢ, *O*, *D*, *B*, *F* are cyclic.

Hence ∠ *DFB* = ∠ *DOB*

         = complement of ∠ *OBD*

         = complement of ∠ *OAE*   [III. 22, Cor.

         = ∠ *EOA*

         = ∠ *EFA*, since *E*, *A*, *O*, *F* are cyclic.

     ∴ *DF* is in the same st. line as *FE*.

Conversely. *From any point O the* ⊥ˢ *OD, OE, OF are drawn to the sides BC, CA, AB of the triangle ABC; then, if the three points D, E, F lie on a st. line, the point O must be on the circle through A, B and C.*

With the above figure

                   ∠ *DFB* = vertically opp. ∠ *EFA*.

But   ∠ *DFB* = ∠ *DOB*, since *D*, *B*, *F*, *O* are cyclic

                     = complement of ∠ *OBD*.

And   ∠ *EFA* = ∠ *EOA*, since *E*, *F*, *O*, *A* are cyclic

                     = complement of ∠ *OAE*.

Thus complement of ∠ *OBD* = complement of ∠ *OAE* ;

            ∴ ∠ *OBD* = ∠ *OAE*,

whence it follows that *O*, *B*, *C*, *D* lie on a circle, so that *O* must be on the circle through *A*, *B* and *C*.

The line on which the feet of the perpendiculars lie is called the **Pedal Line** of *O* with respect to the triangle*.

  Ex. *The four circles circumscribing the four triangles formed by four given straight lines, no two of which are parallel, have a common point of intersection.*

Let *O* be the second pt. of intersection of the ⊙ˢ *ABF* and *FDE*, and

---

* This line is sometimes called the 'Simson Line.' The theorem was not, however, discovered by Simson but by Wallace.

let OL, OM, ON, OP be the $\perp^s$ from O on the st. lines ABC, AFD, BFE, CDE respectively.
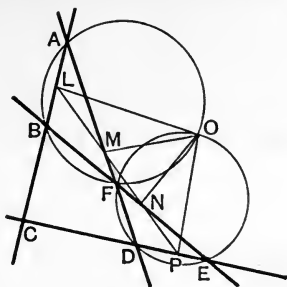
Then, since O is on the $\odot ABF$, L, M, N are on a st. line.

And, since O is on the $\odot FDE$, M, N, P are on a st. line.

Hence L, M, N, P are all on a st. line.

Then, since L, M, P are on a st. line, O is on the $\odot ACD$.

And, since L, N, P are on a st. line, O is on the $\odot BCE$.

Hence O is on all four of the $\odot^s$ ABF, FDE, ACD and BCE.

Or thus. Since O is on the $\odot BAF$,

$$\angle BOF = \angle BAF.$$

And, since O is on the $\odot FDE$,

$$\angle FOE = \text{supplement of } \angle FDE = \angle CDA.$$

Hence $\angle BOE = \text{sum of } \angle^s\ CAD$ and CDA ;

$$\therefore \angle BOE \text{ and } \angle BCE = 2 \text{ rt. } \angle^s ;$$

$$\therefore O \text{ is on the } \odot BCE.$$

Similarly O is on the $\odot CDA$.

7. *If two pairs of opposite sides of a hexagon inscribed in a circle be parallel, the third pair will also be parallel.*
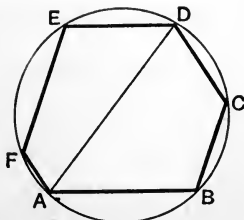
Let *ABCDEF* be a hexagon inscribed in a circle, and let *AB* be ‖ to *DE*, and *BC* ‖ to *EF*; then it is required to prove that *CD* is ‖ to *AF*.

Join *DA*.

Then, *CD* is ‖ to *FA*, if ∠ *CDA* = ∠ *FAD*.

But   ∠ *CDA* = supplement of ∠ *ABC*,

and   ∠ *FAD* = supplement of ∠ *DEF*.

We have ∴ to prove that ∠ *ABC* = ∠ *DEF*.

Join *EB*.

Then, since *AB* is ‖ to *ED*, ∠ *ABE* = ∠ *BED*.

And, since *BC* is ‖ to *EF*,   ∠ *CBE* = ∠ *BEF*.

Hence, by addition,   ∠ *ABC* = ∠ *FED*.

A corresponding theorem is true for a decagon, or for any polygon inscribed in a circle and having an *odd* number of pairs of sides.

8. *In any hexagon inscribed in a circle, the sum of three alternate angles is equal to the sum of the other three alternate angles.*

Let *ABCDEF* be the hexagon inscribed in a ⊙. Join *AD*.
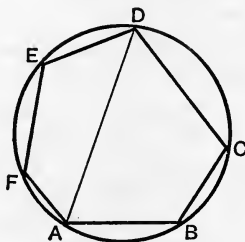
Then the sum of the angles *ABC* and *CDA* is equal to half the angles of the quad. *ABCD*.

Also the sum of the angles *ADE* and *EFA* is equal to half the angles of the quad. *ADEF*.

Hence, by addition, the sum of the angles *B*, *D*, *F* of the hexagon is equal to half the sum of all the angles of the hexagon.

A corresponding theorem is true for an octagon inscribed in a circle, or for any polygon of an even number of sides.

9. *Through A, B, the points of intersection of two circles, the parallel lines PAQ, RBS are drawn which cut one circle in the points P, R respectively and the other circle in the points Q, S respectively. Prove that PQSR is a parallelogram, and that the lines PQ and RS are greatest when they are parallel to the line joining the centres of the circles.*

Let X, Y be the centres of the circles.

Draw $XK \perp^r$ to PA, and produce KX to meet RB in the pt. L. Then, since RB is $\parallel$ to PA, and KXL is $\perp^r$ to PA, it will also be $\perp^r$ to RB.

Hence PA is bisected at K and RB at L.

So also, draw through Y a line MYN to meet AQ, BS at rt. $\angle^s$ at M, N respectively.

Then AQ is bisected at M and BS at N.

Hence $PQ = 2KM$ and $RS = 2LN$.

But by construction $KMNL$ is a $\parallel^m$, and $\therefore KM = LN$.

Hence $PQ = RS$, and PQ is $\parallel$ to RS;

$$\therefore PQSR \text{ is a parallelogram.}$$
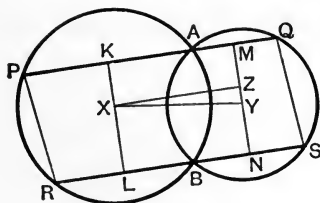
If PQ is parallel to XY, $KXYM$ is a $\parallel^m$, and $\therefore 2XY = 2KM = PQ$.

But, if PQ be not $\parallel$ to XY, draw $XZ \parallel$ to PQ to meet YM in Z. Then, since YZM is $\perp^r$ to PAQ, and XZ is $\parallel$ to PAQ, YZ must be $\perp^r$ to XZ.

Hence $\angle XYZ < \angle XZY$, and $\therefore XZ < XY$.

But $XZ = KM = \frac{1}{2}PQ$; $\therefore PQ < 2XY$.

Thus *PQ is greatest when it is parallel to XY, and its greatest length is twice XY.*

To draw through A the line PAQ of *given length*, we have only to place in the $\odot$ whose diam. is XY a chord equal to half the given length, and draw PAQ parallel to this chord.

The following proof that $PQSR$ is a $\parallel^m$ should be noticed:

Since $AQSB$ is a quad. in a $\odot$,

$$\angle AQS = \text{supplement of } \angle ABS.$$

And, since *PRBA* is a quad. in a ⊙, ∠ *ABS* = ∠ *APR*.

Hence ∠ˢ *AQS* and *APR* are supplementary, and ∴ *QS* and *PR* are parallel.

If the st. line *BQ* be drawn, since *AQ* is ‖ to *BS*, ∠ *AQB* = ∠ *QBS*; ∴ the chords *AB*, *QS* subtending these equal angles are equal.

Thus, *QS* = *PR* = *AB*, for all positions of the parallel lines *PAQ*, *RBS*.

10. There are many interesting and instructive problems in which a circle is required to be drawn so as to satisfy three given conditions. Some cases of this problem can be solved at this stage; but other cases, including that of the construction of a circle so as to touch three given circles, must be deferred as a knowledge of some of the theorems proved in Book vi. is required for their solution. The construction of a circle to pass through three given points, and of a circle to touch three given straight lines are given in Euclid, Book iv.

When in these problems the Analysis only is given, the proofs will present no difficulties.

(i) *Draw a circle through two given points so as to touch a given st. line.*

It is required to draw a ⊙ through the pts. *A*, *B* so as to touch the st. line *CD*.
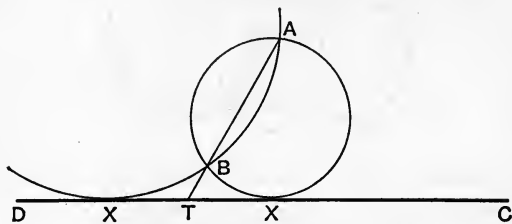
Suppose the ⊙ drawn as required touching *CD* in the pt. *X*.. Join *AB* and produce it to cut *CD* in *T*. Then TX^2 = rect. *TB . TA*. Hence we have only to find the side of a sq. equal to the known rect. *TB . TA*, and set off along *CD*, on either side of *T*, *TX* equal to the side of this square. Then the ⊙ through *A*, *B*, *X* will be the circle required.
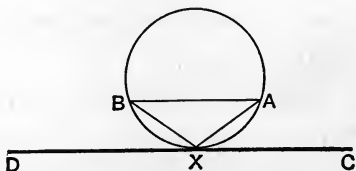
If AB is ∥ to CD; then, since CD touches the ⊙ at X, ∠ AXC = ∠ ABX. But, since AB ∥ to CD, ∠ AXC = ∠ BAX. Hence ∠ ABX = ∠ BAX, and $AX = BX$. Thus X is on the line which bisects AB at rt. ∠ˢ.

[It should be noticed that there are *two* solutions of the problem when AB is produced to cut CD, *one* solution if AB is ∥ to CD, and that the problem is impossible if CD cuts AB between A and B.]

(ii) *Draw a circle through two given points to touch a given circle.*

Let A, B be the given points and CDE the given ⊙. Through A, B draw any circle cutting the given ⊙ in the points C, D. Join AB and CD and produce them to meet in T. From T draw TX to touch the given ⊙ in X. Then the ⊙ through A, B and X will be the circle required.
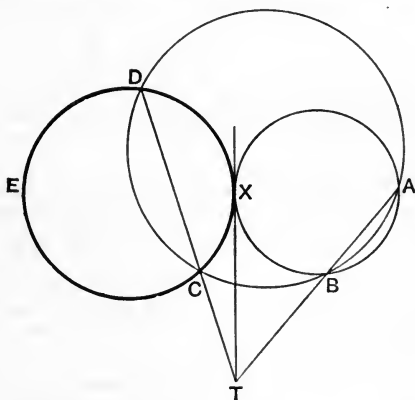
For, since TX touches the given ⊙, $TX^2 = TC \cdot TD$.

And, since A, B, C, D lie on a ⊙, $TC \cdot TD = TA \cdot TB$.

Hence $TX^2 = TA \cdot TB$, wherefore TX touches the ⊙ ABX at X; and since TX touches both ⊙ˢ at the point X, the ⊙ˢ touch one another.

When the point is determined as above, *two* tangents can be drawn from it to the given circles and there are two corresponding circles through A and B, one of which touches the given ⊙ externally and the other touches it internally.

When the line CD is parallel to AB, let X, Y be the points of contact of the tangents to the given ⊙ which are ∥ to AB; then it is easily seen that the ⊙ˢ ABX, ABY will be the required circles.

(iii) *Draw a circle to touch a given circle, to pass through a given point and to have a given tangent at that point.*

This is a particular case of the preceding problem, when the two given points are coincident.

Or thus. Let O be the centre of the given $\odot$, AB the given st. line and C the given point on AB.
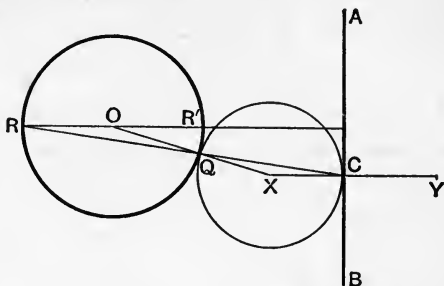
Suppose that a circle centre X has been drawn to touch AB at C and the given $\odot$ at Q.

Then XC is $\perp^r$ to AB; also XQO is a st. line and $XQ=XC$; $\therefore XO=XC$ plus the radius of the given circle. We therefore produce XC to Y so that $CY=QO$. Then $XO=XY$, so that X is equidistant from O and Y, and is $\therefore$ on the line which bisects OY at rt. $\angle^s$. X is also on the line through C $\perp^r$ to AB; hence the position of X is determined, and the $\odot$ can be constructed.

If Y be taken on the side of AB on which O lies, the same construction will give the centre of the circle which touches AB at C and which is touched by the given circle internally.

Or thus. The required circle being supposed to be drawn to touch the given $\odot$ in Q. Join CQ and produce it to cut the given $\odot$ again in R, and join OR.

Then, since $RO=OQ$ and $XQ=XC$,

$$\angle ORQ = \angle OQR = \text{vert.} \angle XQC = \angle XCQ.$$

Hence OR is $\parallel$ to XC, and $\therefore \perp^r$ to AB.

Hence the construction:

Draw from O a radius OR $\perp^r$ to AB, and join RC cutting the given $\odot$ in Q. Produce OQ to cut the line through C $\perp^r$ to AB in X. Then X is the centre of a circle which touches AB in C and the given circle in Q.

If R' be the other extremity of the diameter ROR', and CR' cut the given $\odot$ again in Q', a $\odot$ can be drawn to touch AB at C and the given $\odot$ internally at Q'.

(iv) *Draw a circle to touch two given st. lines and to pass through a given point.*

Suppose that a ⊙ whose centre is X has been drawn to pass through the given pt. P and to touch the lines OA, OB at Q, R respectively.
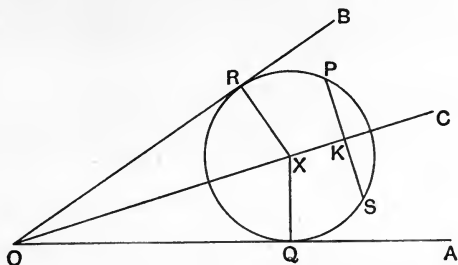
Then, since the ⊙ PQR touches OQ, OR its centre must lie on the st. line OC which bisects the ∠ AOB. And if the chd. PS of the ⊙ be drawn ⊥ to OC it will be bisected by OC; ∴ S can be found by drawing PK ⊥ to OC and producing it to S so that $KS = PK$. Two points P and S on the required ⊙ are now known, and the problem is thus reduced to (i).

(v) *Draw a circle to touch two given st. lines and a given circle.*

Suppose that a ⊙ whose centre is X has been drawn to touch the given st. lines OA, OB at P, Q respectively; and the given ⊙, whose centre is C, externally at R.

Then XRC is a st. line, and XC exceeds XQ and XP by the rad. RC.
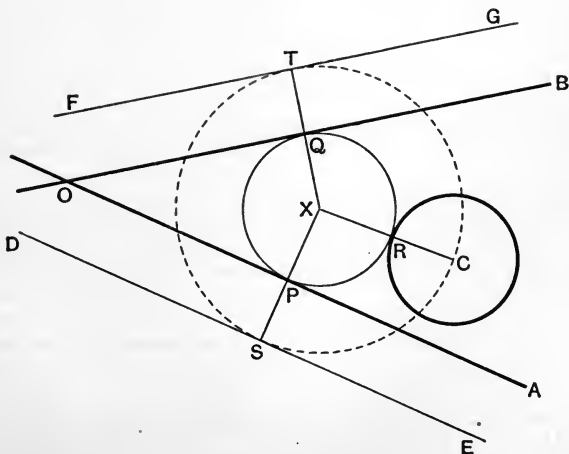
Hence, if XP, XQ be produced to S and T so that $PS=QT=RC$; and if lines ESD, FTG be drawn through S, T parallel respectively to OA and OB, the circle whose centre is X and rad. XC will touch ED and FG. Hence we have first to draw the lines ED, FG and then describe a ⊙ to touch ED, FG and pass through C [see iv.]; then, a concentric ⊙ will touch OA, OB and the given circle.

Two circles can be drawn through the point C to touch the st. lines ED, FG, and there are two corresponding circles wh. will touch the lines AO, BO and the given ⊙ externally; and, if the given ⊙ cuts neither of the lines OA, OB, two other circles can be drawn to touch OA, OB and the given circle *internally*.

11. *Produce a given st. line AB both ways and find two points X, Y, one on each of the produced parts, such that the rectangles XA . AY and XB . BY may be equal respectively to given squares.*

Let the given squares be the squares on CD and EF, and draw through A, B lines PAQ, RBS ⊥ʳ to AB and such that $PA=AQ=CD$ and $RB=BS=EF$.
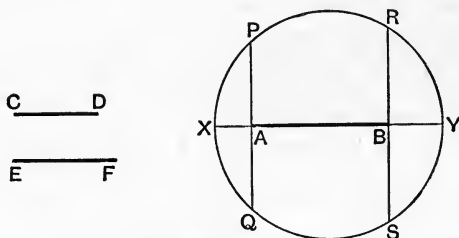
[Now it is known (Ex. 3) that P, Q, R, S lie on a circle; and if we suppose that X, Y are the points required, so that XA . $AY=CD^2=PA$. AQ and XB . $BY=EF^2=RB$. BS, we see that X, Y are on the circle through P, Q, R, S.]

Draw a circle through P, Q, R, S and let this circle cut the line AB produced in the points X, Y; then these are the points required. For, since XY, PQ, RS are chords of a circle, XA . $AY=PA$. AQ and XB . $BY=RB$. BS. Hence, by construction XA . $AY=CD^2$ and XB . $BY=EF^2$.

12. *Draw a circle so as to bisect the circumferences of each of three given circles.*

Suppose that the circle $PQRSTU$ is drawn as required so as to bisect the circumferences of the three given circles APQ, BRS, CTW, whose centres are D, E, F respectively.

Then, since the three circumferences are bisected, PQ, RS, TU are diameters.
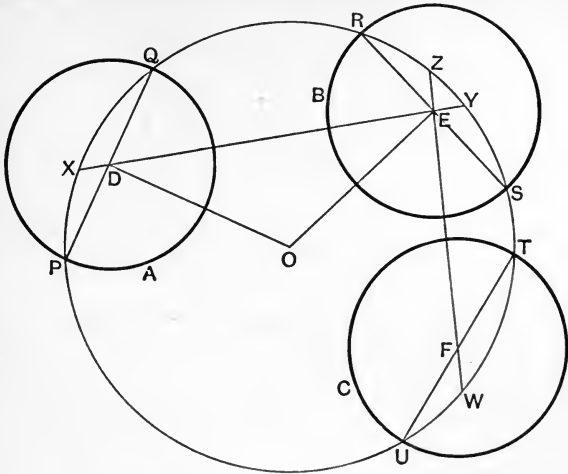
Now if DE be joined and produced to meet the $\odot$ $PQRSTU$ in X, Y,

rect. $XD \cdot DY = $ rect. $QD \cdot DP = $ known square,

and   rect. $XE \cdot EY = $ rect. $RE \cdot ES = $ known square.

Hence, by the preceding problem, the two points X, Y where the required circle will cut the line DE can be found; and we can find in a similar manner the two points Z, W in which the required circle will cut the line FE. The required circle can therefore be constructed, since four points on its circumference can be found.

Or thus. Let O be the centre of the required circle of which the diameters PDQ, RES and TFU are the chords of intersection with the given circles.

Then, since the chords of intersection are bisected at D, E, F respectively, OD is $\perp^r$ to PDQ, $OE \perp^r$ to RES and $OF \perp^r$ to TFU.

Hence $OD^2 + DQ^2 = OQ^2 = OR^2 = OE^2 + ER^2$.

Hence the difference of the squares on OD and OE is equal to the difference of the squares on the radii RE and DQ.

Hence [viii. p. 154] O lies on a fixed straight line $\perp^r$ to DE.

Similarly O lies on a fixed straight line $\perp^r$ to EF.

By drawing these st. lines the position of O is determined, and the radius of the required circle can at once be found.

Ex. 1. *Draw a circle through a given point so as to bisect the circumferences of two given circles.*

13. *If two of the sides of a triangle equal in all respects to a given triangle pass respectively through two given points, the third side will touch a fixed circle.*
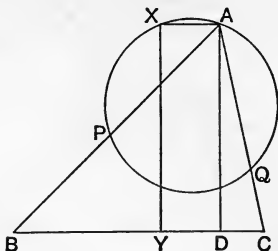
Let the two sides AB, AC pass respectively through the given points P, Q.

Then, since the angle PAQ is const., the pt. A will be on a fixed $\odot$ through P and Q.

Through A draw the chord AX of the $\odot$ PAQ parallel to BC.

Then $\angle PAX = \angle ABC = $ const. $\angle$;

$\therefore$ arc PX is const., so that X is a fixed point.

Draw AD, $XY \perp^r$ to BC. Then $AXYD$ is a $\parallel^m$.

Hence $XY = AD = $ const.

Hence BC always touches a fixed circle whose centre is the fixed point X and whose radius is equal to the perp. distance of A from BC.

14. *If two of the sides of a triangle equal in all respects to a given triangle touch respectively two given circles, the remaining side will also touch a fixed circle.* BOBILLIER'S THEOREM.
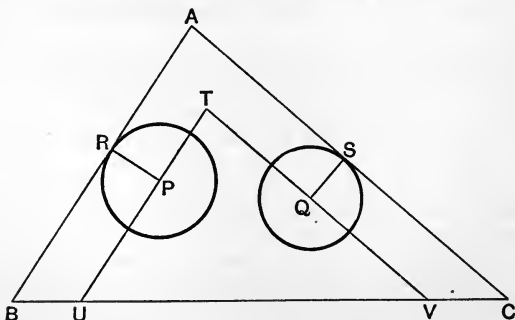
Let the sides AB, AC touch the given $\odot^s$ whose centres are P, Q in the points R, S. Through P, Q draw lines TPU, TQV parallel to BA, CA respectively intersecting at T and cutting BC in U, V respectively.

Then since UPT, VQT are $\parallel$ to BA, CA respectively and are at given $\perp^r$ distances from these lines, the $\triangle$ UTV is of constant shape and size. Hence, by the preceding theorem, UV touches a fixed circle.

15. *Draw three straight lines through three given points so as to make a triangle equal in all respects to a given triangle.*

Let P, Q, R be the three given points.

Suppose that ABC is a $\triangle$ whose sides pass through P, Q, R respectively and which is equiangular to the given $\triangle$ $A'B'C'$.
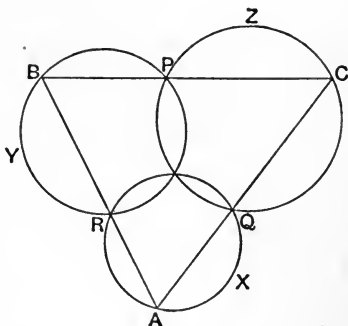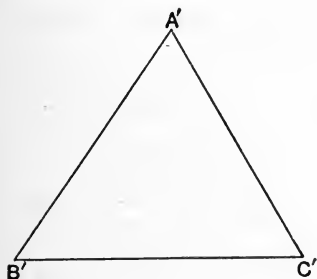
Then, since $\angle QAR = \angle B'A'C'$, A must lie on a fixed $\odot$ QXR through Q and R. So also B must lie on a fixed $\odot$ RYP through R and P, and C must lie on a fixed $\odot$ PZQ through P and Q.

[It is easy to see that these three $\odot^s$ intersect in a point.]

Now, if *any* line BPC be drawn through P cutting the $\odot^s$ RYP, PZQ in B, C respectively, and BR, CQ be produced to meet in A; then, since $\angle RBP = \angle C'B'A'$ and $\angle PCQ = \angle A'C'B'$, $\angle QAR$ must be $=$ to $\angle B'A'C'$, so that A must lie on the $\odot$ QXR.

Thus an infinite number of $\triangle^s$ can be drawn whose sides pass through P, Q, R respectively and which are equiangular to the given $\triangle A'B'C'$.

Now BPC is of greatest length when BC is $\parallel$ to the line joining the centres of the $\odot^s$ RYP, PZQ, and this greatest length is twice the distance between the centres [see 9, page 233]. And, provided $B'C'$ is not greater than twice the distance between the centres of the $\odot^s$ RYP, QZC, a line can be drawn through P to cut the circles in B, C respectively so that $BC = B'C'$ [see 9, page 233]. The $\triangle$ ABC will then be the $\triangle$ required.

MISCELLANEOUS EXERCISES.

1. Draw a circle with a given radius to touch two given circles.

2. Find the shortest and greatest straight lines whose extremities are one on each of two given non-intersecting circles.

3. Find a point without a given circle such that the angle between the tangents drawn from that point to the given circle may be equal to a given angle.

4. One circle is entirely within another; draw the greatest and least chords of the outer which touch the inner circle.

5. AA', BB', CC' are parallel chords of a circle; shew that the triangles ABC, $A'B'C'$ are equal in all respects.

6. Shew that if two circles cut one another in the points A, B and AC, AD are the diameters through A, the line CD will pass through B.

7. Find a point in the diameter produced of a given circle, from which the tangent drawn to the circle will be of given length.

8. Construct a triangle having given the base, the vertical angle, and the length of the line drawn from the vertex to the middle point of the base.

9. Divide a circle into two segments the angle in one of which is twice the angle in the other.

10. Divide a circle into two segments the angle in one of which is five times the angle in the other.

11. Find the complete locus of a point at which the equal sides of a given isosceles triangle subtend equal angles.

12. AB is the diameter of a semi-circle and P, Q, R, ..., K any number of points on the circumference taken in order from A. Shew that the square on AB is greater than the sum of the squares on AP, PQ, QR, ... KB.

13. Describe a circle of given radius, with its centre on one given circle and touching another given circle.

14. Draw a circle touching two given circles and having its centre on a given diameter of one of those circles.

15. In a given circle inscribe a triangle so that two of the sides may pass respectively through two fixed points and the third side may be of given length.

16. Shew that if two of the sides of a triangle inscribed in a circle are parallel respectively to two given straight lines, the third side will touch a fixed circle.

17. In a given circle inscribe a triangle so that one side may pass through a given point and the other two may be parallel to given straight lines.

18. Shew that, if any quadrilateral be inscribed in a circle, a quadrilateral with the same sides taken in any other order can be inscribed in the same circle.

19. In a given circle inscribe a quadrilateral having two opposite sides equal respectively to two given straight lines and the other two sides equal to one another.

20. In a given circle inscribe a quadrilateral so that two opposite sides may be equal respectively to two given straight lines and that the sum of the other two sides may be equal to a third given straight line.

21. AB, AC are two chords of a circle, and BD is drawn parallel to the tangent at A to meet AC in D; shew that the circle BCD will touch AB.

22. Through a point O within a given circle draw a chord AOB such that the difference between AO and OB may be equal to a given straight line.

23. Shew that, of all triangles which have the same base and equal vertical angles, the isosceles triangle has the greatest area and the greatest perimeter.

24. Shew that, if AC and BD are parallel chords of a circle and if O is the point of intersection of AB and CD, the two circles OAC, OBD will touch one another.

25. Find a chord of a given circle which is of given length and which subtends a right angle at a given point.

26. Through two given points describe a circle so that it may intercept a given length on a given straight line.

27. Through one of the points of intersection of two given circles two straight lines are drawn cutting one of the circles in P, Q and the other in P', Q'. Shew that the angle between PQ and $P'Q'$ is constant.

28. Two circles touch one another internally at O, and a line is drawn cutting one circle at P, P' and the other in Q, Q'; shew that PQ and $P'Q'$ subtend equal angles at O.

29. One circle touches another internally at the point O, and the tangent to the inner circle at any point P cuts the outer circle in the points Q, R; shew that OP bisects the angle QOR.

30. A, B are two fixed points on a circle and C, D the extremities of a chord of constant length; shew that the point of intersection of AC and BD is on one or other of two fixed circles.

31. AB is a chord and AC an equal length on the tangent at A to a circle ABM; BC, produced if necessary, cuts the circle again in D, and M is the middle point of the arc cut off by AB and on the side opposite to that on which C lies; shew that $ACDM$ is a parallelogram.

32. AB, CD are any two parallel diameters of two circles and AC cuts the circles again in the points P, Q respectively. Shew that the tangents at P and Q are parallel.

33. Construct a triangle having given the base, the vertical angle and the area. Hence, or otherwise, find two points, P, Q on the given straight lines AB, AC respectively so that the triangle APQ may be of given area and PQ of given length.

34. Equilateral triangles BCA', CAB', ABC' are described on three sides of the triangle ABC, the equilateral triangles being all three on the same side of their bases as the triangle ABC, or all three on the opposite side. Shew that, in either case, the lines AA', BB', CC', will meet in a point.

35. The tangent at A to one circle is parallel to the chord BC of another circle; AB, AC cut the first circle in D, E and the second circle in F, G; shew that DE is parallel to FG.

36. Two parallel chords are drawn in a circle; shew that the four straight lines joining their extremities will all touch each of two circles whose centres are on the given circle.

37. O, O' are the centres of two circles which touch one another at A, and B is the middle point of OO'. Through a point P on the tangent at A a line is drawn perpendicular to PB, shew that the two circles intercept equal chords on this straight line.

38. AB and CD are parallel chords of a circle and E is the middle point of AB. Shew that, if DE meet the circle again in P, PA, PB are tangents to the circles CAE, CBE, respectively.

39. O is the middle point of the chord AB of a circle and PQ is any chord through O. Shew that AB produced cuts the tangents at P and Q in points equidistant from the centre.

40. Shew that the locus of the middle points of all the chords of a circle which pass through a given point is a circle.

41. Two circles intersect in the points A, B, and C, D are the points of contact of a common tangent; shew that CD subtends supplementary angles at A and B.

42. The diagonals AC, BD of the parallelogram $ABCD$ intersect in O; shew that the circles AOB, COD touch each other.

43. Through one of the points of intersection of two given circles draw a chord of either circle such that its middle point is on the other circle.

44. Shew that, if from any point on a given arc of a circle perpendiculars be let fall on the radii to its extremities, the line joining the feet of these perpendiculars will be of constant length.

45. Construct a triangle having given the base, the difference of the sides and the difference of the base angles.

46. Four fixed points A, B, C, D are taken on a circle, and two other circles are drawn to touch each other, one circle passing through A and B and the other through C and D. Shew that the locus of the point of contact is a circle.

47. Construct a triangle having given the base, the sum of the sides and the difference of the angles at the base.

48. Shew that, if the chords of a circle which bisect two of the angles of an inscribed triangle be equal, the triangle must be isosceles or the third angle equal to an angle of an equilateral triangle.

49. Draw when possible a line cutting two given concentric circles so that the chord intercepted by one circle may be double the chord intercepted by the other.

50. AB, AC are equal chords of a circle, and AP, AQ any two other chords which cut BC in the points R, S respectively. Shew that P, Q, S, R are cyclic.

51. O is any point on the circumference of the circle circumscribing the triangle ABC, and OA', OB', OC' are chords of the circle perpendicular respectively to BC, CA, AB. Shew that the triangles ABC, $A'B'C'$ are equal in all respects.

52. A, B are two fixed points within a circle. Describe a circle through A and B and cutting the given circle in D and E so that the lines DA, EB may intersect on the given circle.

53. ABC is any triangle inscribed in a circle, and AP, BQ are chords of the circle parallel to BC, CA respectively; shew that PQ is parallel to the tangent at C.

54. The bisectors of the angles of the triangle ABC inscribed in a circle meet in a point O and cut the circle again in the points A', B', C' respectively; shew that O is the orthocentre of the triangle $A'B'C'$.

55. A, B are the points of intersection of two given circles, and any other circle through A cuts the given circles again in C, D respectively. Shew that, if any line through B cut the circles ACB, ADB in E, F respectively, the lines CE, FD will intersect on the circle CAD.

56. Through A, one of the points of intersection of two given circles, draw a line PAQ cutting the circles in P, Q respectively, so that the difference between AP and AQ may be equal to a given line.

57. Through one of the points of intersection of two given circles a line is drawn cutting the circles again in P, Q. Shew that the locus of the middle point of PQ is a circle.

58. Shew that, if a circle A passes through the centre of a circle B, the tangents to B at their points of intersection will meet on the circle A.

59. A, B, C are any three points on a circle and the tangents at B, C meet in O; shew that, if a chord of the circle be drawn through O parallel to BA, it will be bisected by AC.

60. Two circles touch one another externally in the point P, and a straight line touches the circles in the points A, B respectively; shew that the circle whose diameter is AB passes through P and touches the line joining the centres of the circles.

61. Shew that, if P be any point on the circle circumscribing the equilateral triangle ABC, one of the lines PA, PB, PC is equal to the sum of the other two.

62. Shew that, if a quadrilateral be inscribed in a circle and all but one of its sides be drawn parallel respectively to three given straight lines, the remaining side will be parallel to a fixed straight line.

Shew that the corresponding theorem is true for an inscribed hexagon, and for any inscribed polygon of an even number of sides.

63. Shew that if a pentagon be inscribed in a circle and all but one of the sides be drawn in given directions, the remaining side will be of given length. Shew also that the theorem is true for any polygon of an odd number of sides.

64. Two circles cut one another at right angles, and tangents are drawn to one of the circles from any point on the other; prove that the middle point of the chord of contact of these tangents is on the second circle.

65. Two circles cut each other at right angles at A, B; P is any point on one of the circles and PA, PB cut the other circle in the points Q, R respectively. Shew that QR is a diameter.

66. Any point P is taken on a given segment of a circle described on the line AB, and perpendiculars AG, BH are dropped on BP, AP respectively; shew that GH is of constant length and touches a fixed circle whose centre is the middle point of AB.

67. AD, BE, CF are the perpendiculars of the triangle ABC; prove that the feet of the perpendiculars from D on AB, AC, BE, CF are on a straight line.

68. Through a point on a circle any three chords are drawn ; prove that the circles described on these chords as diameters will intersect again in three points on a straight line.

69. Inscribe a triangle in a given circle so that two of its sides may pass respectively through two given points and that the third side may be parallel to the line joining the given points.

70. If A, B, C, D be four points on a circle, and if AB, CD produced meet in F, and AD, BC produced meet in G, the lines which bisect the angles F and G are perpendicular to one another, and parallel to the bisectors of the angles between AC and BD.

71. If a straight line be equally inclined to the diagonals of a quadrilateral inscribed in a circle, it will be equally inclined to either pair of opposite sides.

72. Shew that, if from the middle point of each side of a quadrilateral inscribed in a circle a perpendicular be drawn to the opposite side, these four perpendiculars will meet in a point.

73. Three circles BCO, CAO, ABO meet in a point O, and from any point D on the circle BCO the lines DB, DC are drawn to cut the circles ABO, CAO in F, E respectively ; shew that EAF is a straight line. Shew also that the lines joining the centres of the three circles form a triangle equiangular to the triangle DEF.

74. Shew that the bases of all triangles with a common angle and the same perimeter touch a circle.

75. Construct a triangle having given the perimeter, one angle and the length of the perpendicular drawn from that angular point to the opposite side.

76. On a given circle find two points which shall be at once concyclic with two given points A, B and also with two other given points C, D.

77. Through the centre of the circle circumscribing the triangle ABC lines are drawn parallel to AB, AC meeting the tangents at B, C respectively in E, F. Shew that EF touches the circle.

78. P, Q are points one on each of two concentric circles, and the tangents at P, Q meet in T ; shew that, if the line joining T to the centre of the circles bisects PQ, the tangents at P and Q must be at right angles.

79. From any point T the tangents TP, TP' are drawn to a circle and the tangents TQ, TQ' are drawn to a concentric circle. Shew that PQ, PQ' make equal angles with the tangent at P.

Shew also that the lines $PQ, PQ', P'Q, P'Q'$ touch two circles whose centres are respectively T and the centre of the given circles.

80. *A* and *B* are any two points one on each of two given circles. Draw through *A* and *B* two equal and parallel chords.

81. A point *P* on one given circle is joined to a point *Q* on another given circle; shew that there is one and only one other line *RS* which is equal and parallel to *PQ* and which has its extremities one on each of the given circles.

82. Through a point *P* draw a straight line cutting two given straight lines in the points *A*, *B* respectively such that the rectangle *AP*, *PB* may be equal to a given square.

83. Shew that, if one circle can be drawn cutting three given circles so that the three chords of intersection are parallel respectively to three given straight lines, then an infinite number of such circles can be drawn.

84. Draw a circle to cut three given circles so that two of the chords of intersection may be parallel respectively to given straight lines, and that the third may pass through a given point.

85. Draw a circle through two given points to cut a given circle so that the common chord may be of given length.

86. Describe a circle bisecting the circumferences of two given circles and cutting a third given circle so that the common chord may be of given length.

87. Shew that the locus of the centre of a circle, which cuts two given circles so that each of the chords of contact may pass through a fixed point, is a straight line.

88. Describe a circle so that each of its chords of intersection with three given circles shall pass through a fixed point.

89. Shew that, if a point *O* be taken within the parallelogram *ABCD* such that the angles *OBA*, *ODA* are equal, the circles *AOB*, *BOC*, *COD* and *DOA* will all be equal.

90. On the sides *BC*, *CA*, *AB* any three points *D*, *E*, *F* respectively are taken; shew that the three circles *EAF*, *FBD*, *DCE* intersect in a point *O*. Shew also that if the triangle *DEF* is equiangular to a given triangle, the point *O* will be fixed.

91. Four given points in a plane are joined two and two by three pairs of straight lines. Shew that, if the bisectors of the angles between any one of these pairs be parallel to the bisectors of the angles between either of the other pairs, the four given points must be cyclic.

92. Describe an isosceles triangle, equiangular to a given triangle, so that the extremities of its base may be on a given circle, and that the two sides may pass respectively through two given points.

93. Construct a square such that two of its sides may pass respectively through two given points and that the other two may meet at a third given point.

94. Describe a square whose sides pass through the four given points P, Q, R, S respectively.

95. Describe a square whose four sides touch respectively four given circles.

96. ABC is an equilateral triangle inscribed in a circle, and P is any point on the circumference ; shew (i) that the sum of the perpendiculars from A, B, C on the tangent at P is constant, and (ii) that the sum of the squares on PA, PB, PC is constant.

97. The sides of a triangle with given angles pass respectively through three given points ; shew that the loci of the in-centre, of the circum-centre, of the centroid, and of the orthocentre are circles.

98. Shew that, if the sides of a triangle with given angles pass respectively through three given points, every line invariably connected with it passes through a fixed point.

99. If the sides of a moving angle of constant magnitude touch each a given circle, every line dividing the angle into parts of constant magnitude will touch a fixed circle whose centre is on a circle which passes through the centres of the given circles.

100. Shew that if one angular point of a polygon of given species be fixed, and if one of the sides not passing through the given angular point touches a fixed circle, then will all the other sides touch fixed circles.

BOOK IV.

DEFINITIONS.

1. WHEN each of the angular points of one rectilineal figure lies on a side of a second rectilineal figure, the first figure is said to be **inscribed in** the second.

2. When each of the sides of one rectilineal figure passes through an angular point of a second rectilineal figure, the first figure is said to be **described about,** or **circumscribed to**, the second.
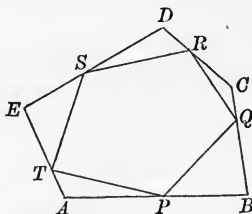
Thus the figure *PQRST* is inscribed in the figure *ABCDE*, and the figure *ABCDE* is described about the figure *PQRST*.

3. When each of the angular points of a rectilinear figure is on the circumference of a circle, the figure is said to be **inscribed in the circle,** and the circle is said to be **circumscribed to the rectilineal figure.**

4. When each of the sides of a rectilineal figure touches the circumference of a circle, the figure is said to be **circumscribed to the circle,** and the circle is said to be **inscribed in the figure.**

5. A straight line is said to be placed in a circle when the extremities of the line are on the circumference of the circle.

PROPOSITION I.   Problem.

In a given circle to place a chord equal to a given straight line which is not greater than the diameter of the circle.

Let ABC be the given circle and XY the given straight line; then *it is required to draw a chord of the circle equal to XY.*
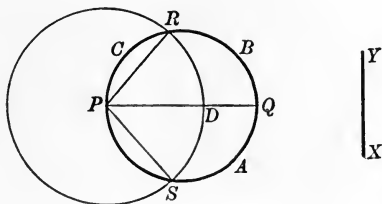
Take any point P on the circle and draw the diameter PQ.

Then, if $PQ = XY$, what was required is done.

But, if $XY < PQ$, cut off from PQ the line $PD = XY$.

With P as centre and PD as radius describe a circle.

Then this circle will cut the given circle in two points, R, S suppose.

Join PR, PS; then each of the chords PR, PS will be equal to XY.

For radius PR = radius PS = radius PD,

and $PD = XY$, by construction.

Hence $PR = PS = XY$.

Ex. 1.   Draw a chord of a given circle equal to one given straight line and parallel to another.

Ex. 2.   Through any given point draw a chord of a circle equal to a given straight line.

PROPOSITION II.

In a given circle inscribe a triangle equiangular to a given triangle.

Let ABC be the given $\odot$, and DEF the given $\triangle$; then *it is required to inscribe in the $\odot$ ABC a $\triangle$ equiangular to the $\triangle$ DEF.*
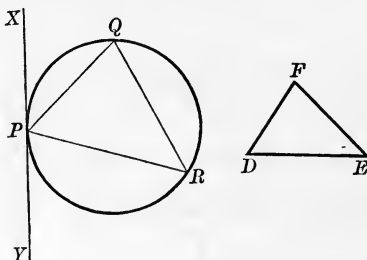
Draw the tangent XPY at any point P on the circle.

Make $\angle XPQ = \angle DEF$, PQ cutting the $\odot$ at Q.

Also, make $\angle YPR = \angle EFD$, PR cutting the $\odot$ at R.

Join QR.   *Then $\triangle PQR$ is the req^d $\triangle$.*

For, since XPY is the tangent to the $\odot$ at the pt. P,

$\qquad\qquad \angle XPQ = \angle PRQ$ in the alt. segment,

and $\qquad\qquad \angle YPR = \angle PQR$ in the alt. segment.

Thus $\qquad\qquad \angle PRQ = \angle XPQ = \angle DEF$,   [Const.

and $\qquad\qquad \angle PQR = \angle YPR = \angle EFD$;   [Const.

$\qquad \therefore$ remaining $\angle QPR =$ remaining $\angle FDE$.

Hence $\triangle$ PQR is equiangular to the $\triangle$ DEF and it is inscribed in the given circle.

Or thus:

Find O, the centre of the given circle, and draw any radius OP.

Make $\angle POQ =$ twice $\angle DEF$, and $\angle POR =$ twice $\angle EFD$.

Join PQ, QR, RP.   Then PQR is the $\triangle$ required.

For $\qquad\qquad \angle PRQ =$ half $\angle POQ = \angle DEF$,

and $\qquad\qquad \angle PQR =$ half $\angle POR = \angle DFE$.

Hence remaining $\angle RPQ =$ remaining $\angle FDE$.

PROPOSITION III. Problem.

About a given circle to describe a triangle equiangular to a given triangle.

Let ABC be the given $\odot$, and DEF the given $\triangle$; then *it is required to describe about ABC a $\triangle$ equiangular to the $\triangle DEF$.*
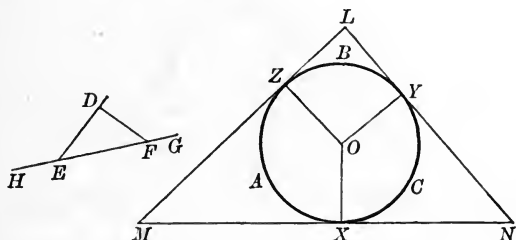
Produce EF both ways to the points G, H.

Take O the centre of the $\odot$ ABC, and draw any radius OX.

At O make the $\angle$s XOY, XOZ equal to the $\angle$s DEH, DFG respectively, X, Z being on the $\odot$.

Draw the tangents at X, Y, Z and produce them to meet in pairs in the points L, M, N.

Then LMN is the triangle required.

For, all the $\angle$s of the quad. $OXNY$ are equal to 4 rt. $\angle$s.

But, since MN and NL touch the $\odot$ at X, Y respectively,

$$\angle OXN = \angle OYN = \text{rt. } \angle^e.$$

Hence remaining angles XOY and $XNY =$ two right $\angle$s.

But $\qquad \angle^s DEH$ and $DEF =$ two right $\angle$s,

$\therefore \angle^s XOY$ and $XNY = \angle^s DEH$ and DEF,

but $\angle XOY = \angle DEH$; $\therefore \angle XNY = \angle DEF$.

Similarly $\qquad \angle NML = \angle DFE$.

But the three angles of any $\triangle$ are equal to two rt. $\angle$s;

$\therefore$ the remaining $\angle MLN =$ remaining $\angle EDF$.

$\therefore \triangle LMN$ is equiangular to $\triangle DEF$ and it is described about the $\odot$ ABC.

PROPOSITION IV. Problem.

To inscribe a circle in a given triangle.

Let ABC be the given $\triangle$; then *it is required to inscribe a $\odot$ in the $\triangle ABC$.*
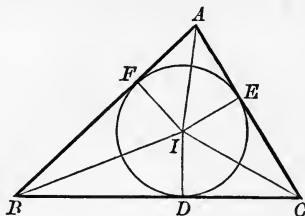
Bisect the angles ABC, ACD and let the bisectors meet in I.

From I draw ID, IE, IF $\perp^{\mathrm{r}}$ to BC, CA, AB respectively.

Then, in the $\triangle^{s}$ IBD, IBF

$$\because \begin{cases} \angle IBD = \angle IBF, \\ \text{rt. } \angle IDB = \text{rt. } \angle IFB, \\ \text{and} \quad IB \text{ opposite to equal angles is common}; \end{cases}$$

$$\therefore ID = IF.$$

Similarly $\qquad ID = IE.$

Hence $\qquad ID = IE = IF,$

and therefore the $\odot$ described with centre I and radius ID will pass through the three points D, E, F, and it will touch the three sides of the $\triangle ABC$ since these sides are $\perp$ to the radii ID, IE, IF respectively, and pass through their extremities.

Cor. I. *The three bisectors of the angles of a triangle meet in a point.*

Join AI. Then, since $\angle AFI = \angle AEI = \text{rt. } \angle$,

sq. on AF and sq. on $FI =$ sq. on $AI =$ sq. on AE and sq. on EI.

But $\qquad FI = EI$, and $\therefore$ sq. on $FI =$ sq. on EI.

Hence $\qquad$ sq. on $AF =$ sq. on AE ; and $\therefore AF = AE$.

Then, since AI, AF, FI are equal respectively to AI, AE, EI,

$$\angle FAI = \angle EAI.$$

Cor. II. *If a circle touch two given straight lines its centre must be on one or other of the two straight lines which bisect the angles between the given straight lines.*

Def. The circle *DEF* is called the **inscribed circle** of the triangle *ABC*, and its centre is called the **centre of the inscribed circle** or the **in-centre**.

Ex. 1. In a given circle inscribe an equilateral triangle.

Ex. 2. Inscribe a right-angled isosceles triangle in a given circle.

Ex. 3. Shew that, if an equilateral triangle be inscribed in a circle and the tangents at the angular points be drawn, the triangle so formed will also be equilateral.

Ex. 4. Shew that, if an equilateral triangle be circumscribed to a circle, and the points of contact of its sides be joined, the triangle so formed will be equilateral.

Ex. 5. Shew that all equilateral triangles inscribed in a given circle are equal.

Ex. 6. Shew that all equilateral triangles circumscribed to a given circle are equal.

Ex. 7. Shew that a circle inscribed in an equilateral triangle touches each side at its middle point.

Ex. 8. An equilateral triangle is inscribed in a circle, and the tangents to the circle at its angular points are drawn; shew that the triangle formed by the tangents is four times the original triangle.

Ex. 9. Shew that, in any triangle, each side subtends an obtuse angle at the in-centre.

Ex. 10. Find the centre of a circle which cuts off equal lengths from the sides of a triangle.

Ex. 11. Shew that, if the triangle formed by joining the points of contact of the circle inscribed in the triangle *ABC* be equiangular to the triangle *ABC*, both triangles must be equilateral.

Ex. 12. Construct a triangle, having given the centres of three of the circles which touch its sides.

Ex. 13. Shew that the triangle whose vertices are the centres of the three escribed circles of any triangle is acute-angled.

Proposition IV. is a particular case of the problem to describe a circle to touch three given straight lines.

Let the st. lines $PBCK$, $QCAL$, $RABM$ be the given st. lines.

Bisect the $\angle^s$ MBC, BCQ by lines meeting at I_1.

Then, since I_1B bisects $\angle MBC$, the $\perp^{rs}$ from I_1 on MBA and BC are equal; and, since I_1C bisects $\angle BCQ$, the $\perp^{rs}$ from I_1 on BC and ACQ are equal.

Hence a circle whose centre is I_1 will touch BC and the productions of AB and AC.

Since the $\perp^{rs}$ from I_1 on ABM and ACQ are equal, it follows (as in Cor. I.) that I_1A bisects the angle BAC, and therefore CII_1 is a straight line.
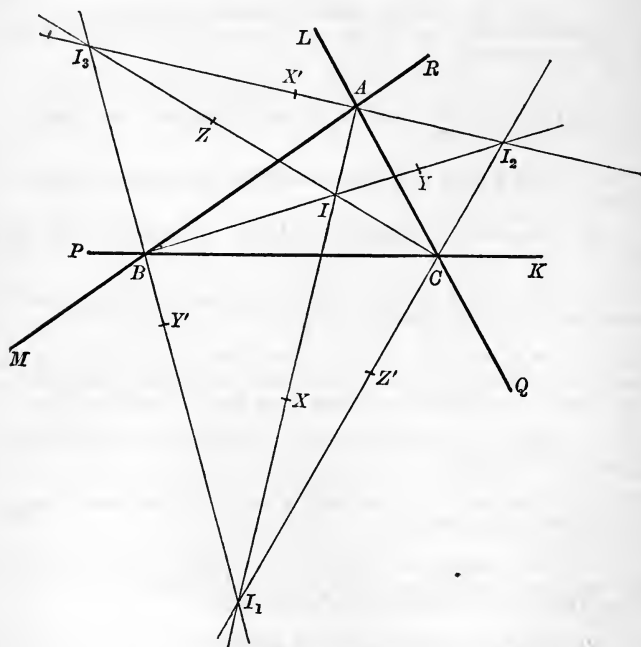
Similarly, if I_2 is the point of intersection of lines bisecting $\angle^s$ RAC, ACK, a $\odot$ whose centre is I_2 will touch AC and the productions of BA, BC.

Also, if I_3 is the pt. of intersection of the bisectors of $\angle^s$ LAB, ABP, a $\odot$ whose centre is I_3 will touch AB and the productions of CA, CB.

Thus, when three straight lines intersect in three points, there are *four* circles which touch the st. lines; one of the circles touching the sides of the △ formed by the lines and each of the others touching one side of that △ and the prolongation of the remaining sides.

Def. *A circle which touches one side of a given triangle and the other two sides produced is called an* **escribed circle** *of the triangle.*

It is important to notice that *the line joining the centres of any two of the four circles*, produced if necessary, *will pass through an angular point of the* △.

Also, since the two bisectors of the angles between two st. lines are at rt. $\angle$ ⁸, it follows that *the line joining any two centres is* $\perp$ʳ *to the line joining the other two centres*, so that each centre is the orthocentre of the △ whose angular points are the three other centres.

Thus the △ ABC is the pedal triangle of the triangle whose vertices are the centres of any three of the four circles which touch its sides.

Since $\angle$ ⁸ IBI_1 and ICI_1 are rt. $\angle$ ⁸, the ⊙ whose diameter is II_1 passes through B and C.

Hence X, the middle point of II_1, is the centre of the ⊙ $BICI_1$ and $\therefore \angle BXC = 2 \angle BI_1C$.

But
$$\angle BI_1C = \angle BI_1I + \angle II_1C$$
$$= \angle BCI + \angle IBC \text{ since } I, B, I_1, C \text{ are cyclic}$$
$$= \tfrac{1}{2} \angle BCA + \tfrac{1}{2} \angle ABC.$$

Hence $\qquad \angle BXC = \angle BCA + \angle ABC;$

$\therefore \angle BXC + \angle BAC = 2$ rt. $\angle$ ⁸, and therefore X is on the ⊙ ABC.

And it can be proved in a similar manner that the circum-circle of ABC passes through all the points X, X', Y, Y', Z, Z' which are the middle points of the lines II_1, I_2I_3, II_2, I_3I_1, II_3, I_1I_2. [See page 279.]

Again, since $XB = XC$, the line through X $\perp$ʳ to BC will bisect BC and will $\therefore$ be a diameter of the ⊙ ABC. But XX' is the diameter through X, since $\angle XAX'$ is a rt. $\angle$. Hence the diameter XX' cuts BC at rt. $\angle$ ⁸, in D suppose.

Then, since X' is the middle point of I_2I_3, sum of $\perp$ ʳˢ from I_2, I_3 on BC is equal to $2X'D$.

And, since X is the middle point of II_3, I and I_3 being on opposite sides of BC,

difference of $\perp$ ʳˢ from I_1, I on BC is equal to $2XD$.

Hence, if r, r_1, r_2, r_3 be the radii of the ⊙ ⁸ whose centres are I, I_1, I_2, I_3 respectively and R the radius of the circum-circle of the △, we have
$$r_1 + r_2 + r_3 - r = 2X'X = 4R.$$

PROPOSITION V. Problem.

To describe a circle about a given triangle.

Let ABC be the given $\triangle$; then *it is required to describe a circle about it.*
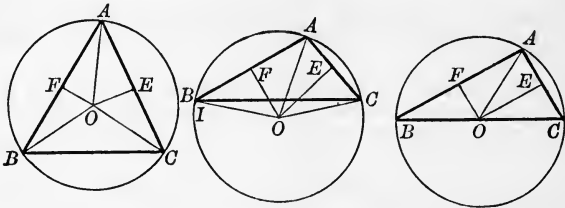
Bisect the sides AC, AB in the points E, F respectively.

Through E, F draw lines $\perp^r$ to AC, AB respectively and meeting in the point O.

Join AO; also, if the point O is not on BC, join BO and CO.

Then, in the $\triangle^s AFO$, BFO

$$\because \left\{ \begin{array}{l} AF = BF, \\ FO = FO, \\ \text{and rt.} \angle AFO = \text{rt.} \angle BFO ; \end{array} \right.$$

$$\therefore AO = BO.$$

Similarly $\qquad AO = CO.$

$$\therefore AO = BO = CO.$$

Hence the circle described with O as centre and OA as radius will pass through all three points A, B, C and will therefore be the circle required.

Def. The $\odot ABC$ is called the **circumscribed circle** of the $\triangle ABC$, and its centre is called the **centre of the circumscribed circle** or the **circum-centre**.

Cor. *The three lines drawn perpendicular to the three sides of a triangle and through their middle points will meet in a point.*

It will be obvious that the above is the same problem as that considered on p. 179.

Ex. 1. Shew that, in an equilateral triangle, the in-centre coincides with the circum-centre.

Ex. 2. Shew that, if the in-centre and the circum-centre of a triangle coincide, the triangle must be equilateral.

Ex. 3. Shew that the radius of the circumscribing circle of an equilateral triangle is double the radius of the inscribed circle.

PROPOSITION VI. PROBLEM.

To inscribe a square in a given circle.

Let *ABC* be the given circle; then *it is required to inscribe a square in the circle.*
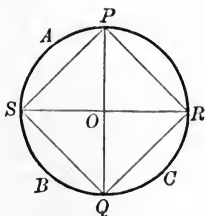
Draw any diameter *POQ* and the perpendicular diameter *ROS*. Join *PS, SQ, QR, RP*.

Then *PSQR* is the square required.

For, since the chords *PS, SQ, QR, RP* all subtend equal angles at the centre,

$$\therefore PS = SQ = QR = RP.$$

And $\angle PRQ$ is a right angle, since *POQ* is a diameter.

$\therefore$ *PSQR* is a square, and it is inscribed in the given circle.

PROPOSITION VII. PROBLEM.

To describe a square about a given circle.

Let *ABC* be the given circle; then *it is required to describe a square about it.*
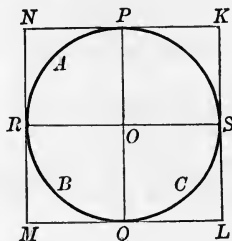
Draw any diameter *POQ* and the perpendicular diameter *ROS.*

Draw tangents to the circle at the points *P, S, Q, R* and let the tangents be produced to meet in the points *K, L, M, N.* Then *KLMN is the square required.*

The tangents at *P* and *Q* are ⊥^{r} to the diameter *POQ,*
and   *ROS* is also ⊥^{r} to *POQ* ;

∴ *NPK, ROS, MQL* are all ∥.

So also   *KSL, POQ, NRM* are all ∥,

and are ⊥^{r} to *NPK, ROS* and *MQL.*

Hence all the quadrilaterals in the figure are rectangles.

But the opposite sides of rectangles are equal;

∴ *NK = ML = RS =* diameter of circle,
and   *KL = NM = PQ =* diameter of circle.

Hence the sides of *KLMN* are all equal, and its angles are rt. ∠^{s}, so that it is a square.

Ex. 1. Shew that the square inscribed in a given circle is half the circumscribed square.

Ex. 2. Shew that the square inscribed in a given circle is equal to twice the square on the radius.

Ex. 3. Shew that every parallelogram inscribed in a circle is a rectangle.

Ex. 4. Shew that every parallelogram circumscribed to a circle is a rhombus.

Ex. 5. In a given circle inscribe a quadrilateral equiangular to a given quadrilateral in which the sum of one pair of opposite angles is two right angles.

Ex. 5. About a given circle describe a quadrilateral equiangular to a given quadrilateral.

PROPOSITION VIII. Problem.

To inscribe a circle in a given square.

Let *ABCD* be a square; then *it is required to describe a ⊙ in it.*

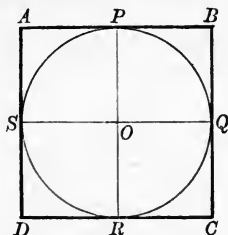Bisect the sides *AB, BC, CD, DA* in the points *P, Q, R, S* respectively. Join *PR, QS* and let them meet in *O*.

Then, since *AB = DC*, their halves are equal;

∴ *AP = DR*, and *AP* is also ∥ to *DR*.

Hence   *PR* is ∥ to *AD* or *BC*.

Similarly   *QS* is ∥ to *AB* or *CD*.

Hence *ASOP* is a ∥ᵐ; but opposite sides of ∥ᵐˢ are equal;

∴ *SO = AP* = half *AB*.

Similarly *PO, QO, RO* are each equal to half a side of the square, so that   *PO = QO = RO = SO*.

Hence the ⊙ described with centre *O* and radius *OP* will pass through the four points *P, Q, R, S*.

And, since *PR* is ∥ to *AD*, ∠ *OPB* = alt. ∠ *BAD* = rt. ∠.

Hence *AB* touches the ⊙ whose centre is *O* and radius *OP*.

Similarly all the other sides of the square touch the ⊙ *PQRS*.

Hence the circle *PQRS* is inscribed in the given square.

It is of importance to notice that *a circle can be inscribed in any convex quadrilateral in which the sum of one pair of opposite sides is equal to the sum of the other pair.*

Let *ABCD* be the quad. such that *AB + CD = BC + DA*.

Draw a circle to touch the three sides *AB, BC, CD* in *P, Q, R*.

Then, if *AD* does not touch this circle, draw from *A* the tangent *AX* and produce it so as to cut *CD* (produced, if necessary) in the point *Y*.

Then, since the two tangents drawn to a circle from any point are equal,

AP = AX, BP = BQ, CR = CQ, YR = YX.

Hence   *AB + CY = BC + AY.*

But   *AB + CD = BC + AD*;   [Hyp.

∴ *CY ∼ CD = YD*, which is impossible.

Hence *AD* must touch the ⊙ which touches *AB, BC* and *CD*.

PROPOSITION IX. Problem.

To circumscribe a circle to a given square.

Let *ABCD* be the given square; then *it is required to circumscribe a circle about it.*
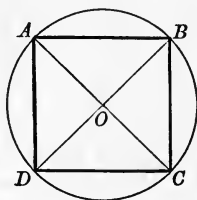
Draw the diagonals *AC*, *BD*, and let *O* be their point of intersection.

Then, since $AB = AD$, $\angle ABD = \angle ADB$.

But, since $\angle DAB$ is a rt. $\angle$, $\angle^s$ *ABD* and *ADB* are together equal to a rt. $\angle$.

Hence $\angle ABD = \angle ADB = $ half a rt. $\angle$.

So also $\angle BAC = \angle BCA = $ half a rt. $\angle$,

$\angle DAC = \angle DCA = $ half a rt. $\angle$,

and $\angle DBC = \angle BDC = $ half a rt. $\angle$.

And, since $\angle ABO = $ half a rt. $\angle = \angle OAB$; $\therefore AO = BO$.

Similarly $BO = CO = DO$.

Hence the circle described with centre *O* and radius *OA* will pass through the four points *A*, *B*, *C*, *D* and will be the ⊙ required.

PROPOSITION X. Problem.

To construct an isosceles triangle having each of the angles at the base double the vertical angle.
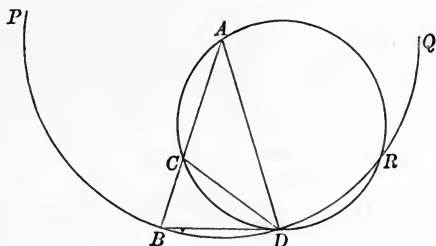
Take any st. line AB, and divide it at the point C so that the rect. AB, BC = sq. on AC.

With A as centre and AB as radius describe the circle PBQ. In the circle PBQ place the chord BD equal to AC. Join AD. Then *ABD is the triangle required.*

Join CD, and about the $\triangle ACD$ describe the circle ACD.

Then rect. BA, BC = sq. on AC [*Const.*

                     = sq. on BD.   [*Const.*

Hence BD touches the $\odot ACD$ at the point D;

     $\therefore \angle BDC = \angle CAD$ in the alt. segment.

To each add $\angle CDA$;

then whole $\angle BDA$ = sum of $\angle$s CAD and CDA.

But $\angle BCD$ = sum of $\angle$s CAD and CDA.

Hence $\angle BCD = \angle BDA$.

But, since $AD = AB$, $\angle BDA = \angle ABD$;

        $\therefore \angle BCD = \angle CBD$;

     $\therefore BD = CD$, and $BD = CA$;

   $\therefore CD = CA$, and $\therefore \angle CAD = \angle CDA$.

But it has been proved that $\angle BDC = \angle CAD$.

Hence   whole $\angle BDA$ = twice $\angle BAD$.

But $\angle ABD = \angle BDA$, since $AD = AB$.

Thus each of the $\angle$s ABD, BDA is double the $\angle BAD$, so that ABD is constructed as required.

NOTE. The student may have no difficulty in the above proposition, but will doubtless feel that it is unsatisfactory because there is nothing to shew what suggested the construction adopted. It will, however, appear by the following Analysis that Euclid's construction suggests itself in a perfectly straightforward manner.

ANALYSIS. Suppose that the $\triangle ABD$ is such that

$$\angle ABD = \angle ADB = 2 \angle BAD.$$

Then the given relation suggests at once the bisection of the $\angle BDA$ by the st. line DC.

Then $\angle BDC = \frac{1}{2} \angle BDA = \angle BAD.$

Hence BD will *touch* at D the circle circumscribing the $\triangle ACD$;

$$\therefore \text{ sq. on } BD = \text{rect. } BC, BA.$$

But, since $\angle CDA = \angle CAD$, $CD = CA$.

And $\angle BCD = $ sum of $\angle$s CAD and CDA

$$= 2 \angle BAD = \angle ABD$$

$$\therefore BD = CD = CA.$$

Hence   sq. on CA = rect. BC, CA,

which at once suggests Euclid's construction.

The division of a right angle into equal parts.

(i) Since any angle can be bisected, and the halves bisected again, and so on without limit, *a right angle can be divided into* 2, 4, 8, 16, 32,... *equal parts.*

(ii) Since the angles of an equilateral triangle are all equal and are together equal to two right angles, each of the angles is two-thirds of a right angle, and by bisecting any one of the angles we obtain an angle equal to *one-third* of a right angle.

Hence, by (i), *a right angle can be divided into* 3, 6, 12, 24,... *equal parts.*

(iii)   If an isosceles triangle be constructed each of whose base angles is double the vertical angle, the vertical angle will be one-fifth of two right angles ; and if the vertical angle be bisected we obtain an angle equal to *one-fifth* of a right angle.

Hence, by (i), *a right angle can be divided into* 5, 10, 20, 40,... *equal parts.*

(iv)   The difference between an angle of one-third of a right angle and an angle of one-fifth of a right angle is an angle of two-fifteenths of a right angle, and by bisecting this angle we obtain an angle equal to *one-fifteenth* of a right angle.

Hence, by (i), *a right angle can be divided into* 15, 30, 60, 120,... *equal parts.*

(v)   Gauss proved that a right angle can by a geometrical construction be divided into any number of equal parts provided that the number is prime and of the form $2^n + 1$. Numbers of this form are 3, 5, 17, 257,.... The construction for the division of a right angle into 17 equal parts is, however, extremely complicated.

It is very important to remember that it is impossible to divide a right angle into a number of equal parts, unless that number is included in one of the sets given above—it is impossible, for example, to divide a right angle into 7 equal parts or into 9 equal parts by a geometrical construction. This of course does not mean that a construction cannot be found which will *very approximately* effect the required subdivision, which is all that is required in geometrical drawing.

Ex. 1.   Shew that, if the circles ACD and PBQ intersect again in the point R, and AR be joined, $\angle ADR = \angle ARD = 2 \angle DAR$.

Ex. 2.   Shew that in the isosceles $\triangle ACD$ each of the angles at the base is one-third the vertical angle.

Ex. 3.   Shew that, if the tangent at A to the circle ACD cut DB produced in Y, then $BY = BA$.

Ex. 4.   Shew that, if the circles ACD, PBQ intersect again in the point R, CR will be parallel to BD, and will bisect the angle ARD.

Ex. 5.   Shew that $CR = AB$.

Ex. 6.   Construct an isosceles triangle whose vertical angle is three times each of the base angles.

Ex. 7.   Construct an isosceles triangle whose vertical angle is two-ninths of each of the base angles.

Ex. 8.   Construct an isosceles triangle each of whose base angles is three-fourths of the vertical angle.

PROPOSITION XI. Problem.

To inscribe a regular pentagon in a given circle.

Let *O* be the centre of the given circle; *it is required to inscribe a regular pentagon in the circle.*

Since all the sides of the regular pentagon to be inscribed in the circle are to be equal, they will all subtend equal angles at the centre. But the angles at the centre are together equal to four rt. $\angle$ s. Hence each side must subtend at the centre an angle equal to one-fifth of four rt. $\angle$ s.

Construct an isosceles $\triangle$ *XYZ* having each of the $\angle$ s *XYZ, XZY* double the $\angle$ *YXZ*; then $\angle$ *XYZ* = two-fifths of two rt. $\angle$ s = one-fifth of four rt. $\angle$ s.
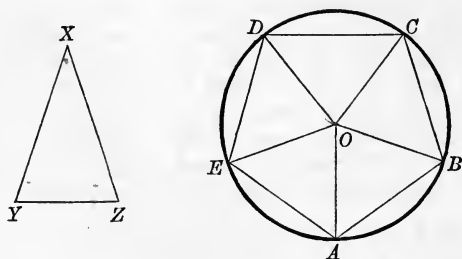
Draw any radius *OA*, and also the radii *OB, OC, OD, OE* so that $\angle$ *AOB* = $\angle$ *BOC* = $\angle$ *COD* = $\angle$ *DOE* = $\angle$ *XYZ*.

Then, since $\angle$ *XYZ* = one-fifth of four rt. $\angle$ s, and all the angles at *O* are together equal to four rt. $\angle$ s, the remaining $\angle$ *EOA* = one-fifth of four rt. $\angle$ s.

Join *AB, BC, CD, DE, EA.*

Then *ABCDE* is the required regular pentagon.

For, all the radii of the ⊙ are equal, and the angles *AOB*, *BOC*, *COD*, *DOE*, *EOA* are all equal,

∴ the isosceles △ˢ *AOB*, *BOC*, *COD*, *DOE*, *EOA* are equal in all respects.

Hence their bases are all equal, so that the pentagon is equilateral; also their base angles are all equal, whence it follows that the angles of the pentagon are the doubles of the base angles, and are all equal.

Hence *ABCDE* is a regular pentagon, and it is inscribed in the given circle.

NOTE. It will be seen at once that the above method is applicable to all polygons provided that the necessary sub-multiple of four rt. ∠ˢ can be found by a geometrical construction. The cases in which this can be done are enumerated in the note to the previous proposition.

1. If the diagonals of a regular pentagon be drawn, as in the figure below, prove that the pentagon *PQRST* is regular.
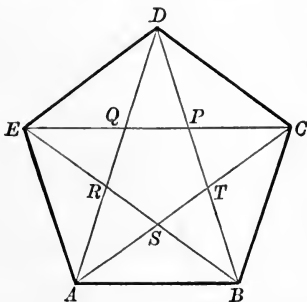
2. Shew that the square on *AS* is equal to the rectangle *AT*, *ST*. [Euclid XIII. 8.]

3. Shew that *ABPE* is a rhombus.

4. Shew that the ⊙ *EPB* touches *DE* and *CB*.

5. Shew that, if an equilateral pentagon have any three of its angles equal, it will be equiangular. [Euclid XIII. 7.]

6. Shew that, if *AB* be the side of a regular decagon inscribed in a circle, and if *AB* be produced to *D*, *BD* being equal to the side of a regular hexagon inscribed in the same circle, then will $BD^2 = AD \cdot AB$. [Euclid XIII. 9.]

7. Shew that, if AB, BC, CD be three consecutive sides of a regular decagon inscribed in a circle, then will AD be equal to the sum of AB and the radius of the circle.

8. Shew that the square on the side of a regular pentagon is equal to the sum of the squares on the sides of a regular hexagon and a regular decagon inscribed in the same circle.   [Euclid xiii. 10.]

PROPOSITION XII.   Problem.

To describe a regular pentagon about a given circle.

Let O be the centre of the given circle.   *It is required to describe a regular pentagon about the circle.*
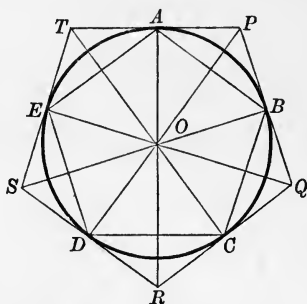
Construct an isosceles $\triangle XYZ$ having each of the $\angle^s$ XYZ, XZY double $\angle YXZ$. Then $\angle XYZ =$ two-fifths of two rt. $\angle^s =$ one-fifth of four rt. $\angle^s$.

Draw any radius OA, and also the radii OB, OC, OD, OE such that

$$\angle AOB = \angle BOC = \angle COD = \angle DOE = \angle XYZ.$$

Then, since $\angle XYZ$ is one-fifth of four rt. $\angle^s$, and all the angles at O are together equal to four rt. $\angle^s$, the remaining $\angle EOA =$ one-fifth of four rt. $\angle^s$.

Draw the tangents to the circle at the points A, B, C, D, E and produce each tangent to meet the consecutive one so as to form the pentagon $PQRST$.

Then *the pentagon $PQRST$ will be the figure required.*

Join OP, OQ, OR, OS, OT.

Then we know that *if two tangents be drawn from any external point to a circle the tangents are equal in length and subtend equal angles at the centre,* and also that *the line joining the external point to the centre bisects the angle between the tangents.* [III. 17, Cor.]

Hence $PA = PB$, $QB = QC$, $RC = RD$, &c.;

also $\angle POA = \angle POB$, $\angle QOB = \angle QOC$, &c.;

and $\angle APO = $ half $\angle APB$, $\angle BQO = $ half $\angle BQC$, &c.

But, by construction, $\angle AOB = \angle BOC$; ∴ their halves are equal, i.e. $\angle POB = \angle BOQ$.

And, in the $\triangle^s$ *POB, QOB*

∵ $\begin{cases} \angle POB = \angle QOB \\ \text{rt. } \angle OBP = \text{rt. } \angle OBQ \\ \text{and } OB, \text{ adjacent to equal angles, is common} \end{cases}$

∴ $PB = QB$ and $\angle OPB = \angle OQB$.

Hence $PQ = $ twice PB.

Similarly all the sides of the pentagon $PQRST$ can be shewn to be bisected at their points of contact.

Hence $PT = 2PA = 2PB = PQ$.

Similarly $PQ = QR$, $QR = RS$ and $ST = TP$.

Thus the figure $PQRST$ is equilateral.

Also, since $\angle OPB = \angle OQB$, their doubles are equal,

i.e. $\angle TPQ = \angle PQR$.

Similarly $\angle PQR = \angle QRS = \angle RST$.

Thus the angles of the pentagon $PQRST$ are all equal.

Hence $PQRST$ is a regular pentagon and it is described about the given circle.

NOTE. It will be seen that the above method is applicable to all polygons provided that the necessary sub-multiple of four rt. $\angle^s$ can be found by a geometrical construction.

PROPOSITION XIII. Problem.

To inscribe a circle in a given regular pentagon.

Let *ABCDE* be the given regular pentagon ; then *it is required to inscribe a circle in the pentagon.*
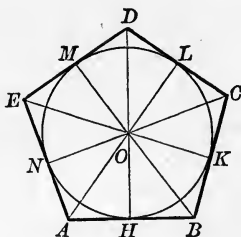
Bisect the angles *EAB, ABC* by st. lines meeting in the pt. *O.* Join *OC, OD, OE.*

In the △ˢ *OAE, OAB*

$$\therefore \begin{cases} AE = AB & \text{[Hyp.} \\ OA = OA \\ \text{and } \angle OAE = \angle OAB & \text{[Const.} \end{cases}$$

$$\therefore \angle OEA = \angle OBA.$$

But $\angle OBA = \frac{1}{2} \angle ABC$, and $\angle ABC = \angle AED$ (hyp.),

$$\therefore \angle OEA = \frac{1}{2} \angle AED,$$

so that *OE* bisects ∠ *AED.*

And then, since *OA* bisects ∠ *BAE* and *OE* bisects ∠ *AED,* it can be proved in the same manner that *OD* bisects ∠ *EDC,* and then that *OC* bisects ∠ *DCB.*

Hence *the angles of the pentagon are bisected by the straight lines AO, BO, CO, DO; EO respectively.*

Now draw from *O* the lines *OH, OK, OL, OM, ON* ⊥ʳ respectively to *AB, BC, CD, DE, EA.*

Then, in the △ˢ *OBH, OBK*

$$\therefore \begin{cases} \angle HBO = \angle KBO \\ \text{rt. } \angle OHB = \text{rt. } \angle OKB \\ \text{and } OB, \text{ opp. to equal angles, is common} \end{cases}$$

$$\therefore OH = OK.$$

In the same way it may be proved that

$$OK = OL, \ OL = OM, \ OM = ON \text{ and } ON = OH.$$

Hence OH, OK, OL, OM, ON are all equal, and $\therefore$ a $\odot$ with centre O and radius OA will pass through the five points H, K, L, M, N; and, since the $\angle^s$ at H, K, L, M, N are rt. $\angle^s$, this circle will touch all the sides of the given pentagon, and is therefore the circle required.

PROPOSITION XIV. Problem.

To describe a circle about a given regular pentagon.

Let $ABCDE$ be the given regular pentagon; then *it is required to describe a circle about it.*
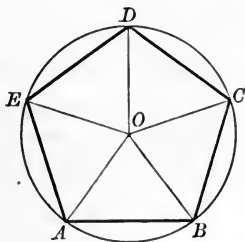
Bisect the $\angle^s$ EAB, ABC by st. lines meeting in the pt. O. Join OC, OD, OE.

In the $\triangle^s$ OAE, OAB

$$\because \begin{cases} AE = AB & \text{[Hyp.} \\ OA = OA \\ \text{and } \angle OAE = \angle OAB & \text{[Const.} \end{cases}$$

$$\therefore \angle OEA = \angle OBA.$$

But $\angle OBA = \frac{1}{2} \angle ABC$, and $\angle ABC = \angle AED$ (hyp.);

$\therefore \angle OEA = \frac{1}{2} \angle AED$, so that OE bisects $\angle AED$.

And then, since OA bisects $\angle BAE$ and OE bisects $\angle AED$, it can be proved in a similar manner that OD bisects $\angle EDC$, and then that OC bisects $\angle DCB$.

Hence *the angles of the pentagon are bisected by the straight lines OA, OB, OC, OD, OE respectively.*

Since the angles of the pentagon are all equal, and are bisected by the st. lines OA, OB, OC, OD, OE it follows that

$$\angle OAB = \angle OBA, \ \angle OBC = \angle OCB, \ \angle OCD = \angle ODC,$$

and
$$\angle ODE = \angle OED ;$$

$$\therefore \ OA = OB, \ OB = OC, \ OC = OD \text{ and } OD = OE.$$

Hence a circle whose centre is O and radius OA will pass through the five points A, B, C, D, E and will therefore be the circle required.

NOTE. It will easily be seen that the constructions given in Props. XIII and XIV do not depend on the number of the sides of the pentagon. Thus we have the following theorem:—

If two of the angles of any regular polygon be bisected by st. lines which meet in the point O, the lines joining O to the other vertices will bisect the remaining angles of the polygon, a circle whose centre is O will touch all the sides of the polygon, and another circle whose centre is O will pass through all the vertices of the polygon.

PROPOSITION XV. PROBLEM.

To inscribe a regular hexagon in a given circle.

Let O be the centre of the given circle; then *it is required to inscribe a regular hexagon in the circle.*

Since all the sides of the regular hexagon to be inscribed in the circle are to be equal, they will all subtend equal angles at the centre, and each side must $\therefore$ subtend at the centre an angle equal to one-sixth of four rt. $\angle$ˢ.

Construct an equilateral triangle XYZ; then, since the angles are all equal, each angle must be one-third of two rt. $\angle$ˢ or one-sixth of four rt. $\angle$ˢ.
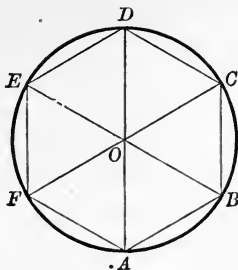
Draw any radius *OA*, and draw the radii *OB*, *OC*, *OD*, *OE*, *OF* so that

$$\angle AOB = \angle BOC = \angle COD = \angle DOE = \angle EOF = \angle XYZ.$$

Then, since $\angle XYZ =$ one-sixth of four rt. $\angle$ ˢ, and all the angles at *O* are together equal to four rt. $\angle$ ˢ, the remaining $\angle FOA$ is also one-sixth of four rt. $\angle$ ˢ.

Join *AB*, *BC*, *CD*, *DE*, *EF*.

Then *ABCDEF is the required regular hexagon.*

For all the radii of the circle are equal, and the angles *AOB*, *BOC*, *COD*, &c. are all equal;

∴ the isosceles $\triangle$ ˢ *AOB*, *BOC*, *COD*, &c. are equal in all respects.

Hence their bases are all equal, so that the hexagon is equilateral; also their base angles are all equal, whence it follows that the angles of the hexagon are all equal.

Hence *ABCDEF* is a regular hexagon and it is inscribed in the given circle.

Ex. 1. Shew that the side of a regular hexagon inscribed in a circle is equal to the radius of the circle.

Ex. 2. One regular hexagon is inscribed in a given circle and another is circumscribed about the circle; shew that the area of one hexagon is three-fourths the area of the other.

Ex. 3. An equilateral triangle and a regular hexagon are inscribed in the same circle; shew that the area of the triangle is half that of the hexagon.

Ex. 4. An equilateral triangle and a regular hexagon are circumscribed about a given circle; shew that the area of the hexagon is two-thirds of that of the triangle.

PROPOSITION XVI. Problem.

To inscribe a regular quindecagon in a given circle.

Since all the sides of the regular quindecagon to be inscribed in the circle are to be equal, they will all subtend equal angles at the centre, and each side must therefore subtend at the centre an angle equal to one-fifteenth of four rt. $\angle$ s.

Construct a triangle ABC having each of the $\angle$ s ABC, ACB double the angle BAC; then $\angle BAC$ is equal to one-fifth of two rt. $\angle$ s or one-tenth of four rt. $\angle$ s. On AC construct the equilateral $\triangle$ CAD, then $\angle CAD$ is one-third of two rt. $\angle$ s, or one-sixth of four rt. $\angle$ s.

Hence $\angle BAD$ is the difference between one-sixth and one-tenth of four rt. $\angle$ s and is therefore one-fifteenth of four rt. $\angle$ s.
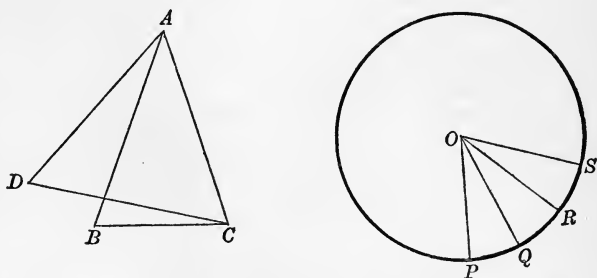
Now draw any radius OP, and draw the radii OQ, OR, OS,... such that $\angle POQ = \angle QOR = \angle ROS = ... = \angle BAD$.

Then, since $\angle BAD$ is one-fifteenth of four rt. $\angle$ s, there will be fifteen radii in all drawn from the point O, the angle between consecutive radii being the same throughout.

Join PQ, QR, RS, ...

Then $PQRS...$ *is the required regular quindecagon.*

For all the radii of the circle are equal, and the angles POQ, QOR, ROS,... are all equal; $\therefore$ the isosceles $\triangle$ s POQ, QOR, ROS,... are equal in all respects.

Hence their bases are all equal, so that the quindecagon is equilateral; also their base angles are all equal, whence it follows that the angles of the quindecagon are all equal.

Hence $PQRS...$ is the required regular quindecagon.

MISCELLANEOUS THEOREMS.

1. *An equilateral polygon inscribed in a circle is also equiangular.*

For join the centre O to each of the angular points A, B, C, D, ...
Then, since the sides of the polygon are all equal, they will subtend equal
angles at the centre; and ∴ the isosceles $\triangle^s$ AOB, BOC, COD, ... are
equal in all respects. Hence their base angles are equal, and the angles
of the polygon are ∴ the doubles of these base angles and are equal.

2. *An equiangular polygon inscribed in a circle has its alternate sides
equal.*

For, if AB, BC, CD, DE, ... be consecutive sides of the equiangular
polygon, and AC, BD be joined.
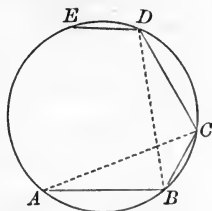
Then, in the $\triangle^s$ BAC, BDC

  $\angle ABC = \angle BCD$ (hyp.), $\angle BAC = \angle BDC$ in the same segment,

and BC opp. to equal angles is common;

$$\therefore AB = CD.$$

Similarly $\hspace{3cm} BC = DE.$

3. *An equiangular polygon with an odd number of sides inscribed in
a $\odot$ is also equilateral.*

This follows at once from the preceding.

4. *To describe a regular pentagon on a given straight line.*

Let AB be the given st. line.
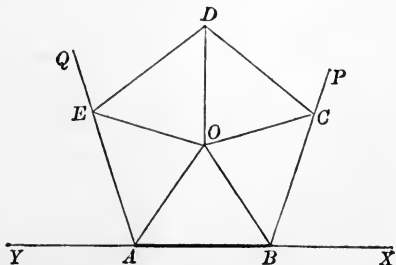
Since the pentagon is to be a regular pentagon, all its angles are to be equal, and therefore if its sides be produced all its exterior angles will be equal.

Hence each exterior angle will be one-fifth of four rt. $\angle$ˢ.

Produce therefore AB both ways to X, Y respectively, and make $\angle XBP = \angle YAQ =$ one-fifth of four rt. $\angle$ˢ. [This is equal to one of the base angles of an isosceles $\triangle$ each of whose base angles is double the vertical $\angle$.]

Bisect $\angle$ˢ ABP, BAQ by st. lines meeting in O.

Then, since $\angle ABP = \angle BAQ$, $\angle ABO = \angle BAO$, and $\therefore AO = BO$.

Describe a $\odot$ with centre O and radius OB.

Then $\angle AOB =$ supplement of $\angle$ˢ OAB, OBA

                       $=$ supplement of $\angle ABC$

                       $= \angle XBC =$ one-fifth of four rt. $\angle$ˢ.

Hence AB is a side of a regular pentagon inscribed in the circle, and the construction can be completed.

Any regular polygon can be constructed in a similar manner provided the required submultiple of four rt. $\angle$ˢ can be found by a geometrical construction.

5. *To find, in terms of the sides, the lengths of the tangents drawn from the angular points of a triangle to its inscribed and escribed circles.*

Let a, b, c be the lengths of the sides BC, CA, AB of the $\triangle$ ABC, and let s be half the sum of the sides.
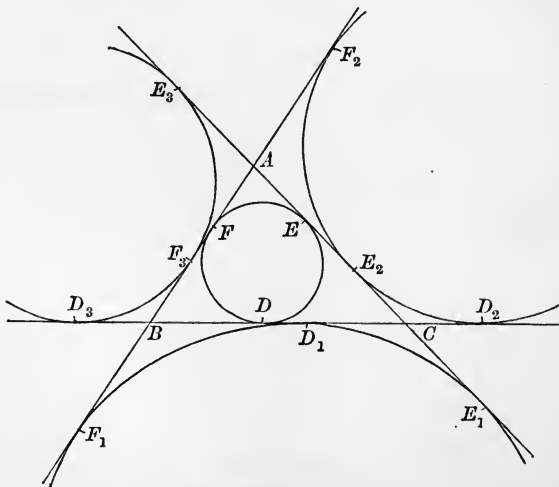
Since the two tangents drawn to a circle from any point are equal,

$$AF_1 + AE_1 = AB + BF_1 + AC + CE_1$$
$$= AB + BD_1 + AC + CD_1$$
$$= AB + BC + CA = 2s;$$
$$\therefore AF_1 = AE_1 = s \quad \dots\dots\dots\dots\dots\dots\dots\dots(i).$$

Similarly $\quad BD_2 = BF_2 = s,$ and $CD_3 = CE_3 = s.$

Again, $\quad 2AB + 2CD = AB + AF + FB + 2CD$
$$= AB + AE + BD + CE + CD$$
$$= AB + BC + CA = 2s;$$
$$\therefore CD = CE = s - c \dots\dots\dots\dots\dots\dots\dots(ii).$$

Similarly $\quad AE = AF = s - a,$ and $BD = BF = s - b.$

From (i) $\quad BD_1 = BF_1 = AF_1 - AB = s - c;$

$\therefore$ from (ii) $\qquad\qquad BD_1 = CD \dots\dots\dots\dots\dots\dots\dots(iii),$

so that D and D_1 are equidistant from the middle point of BC.

Similarly E, E_2 are equidistant from the middle point of CA, and F, F_3 are equidistant from the middle point of AB.

Also, since $CD_1 = BD = s - b,$ and $CD_2 = CE_2 = AE = s - a;$

$$\therefore D_1D_2 = s - b + s - a = c = AB.$$

Again, $\quad DD_1 = BD_1 - BD = CD - BD = (s - c) - (s - b) = b - c.$

6. *If O is the orthocentre of the $\triangle ABC$, and AL the chord of the circum-circle perpendicular to BC, OL will be bisected by BC.*
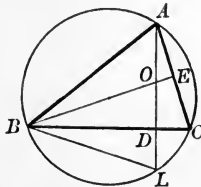
Let AD, BE be drawn perpendicular to BC, CA respectively, and let O be their point of intersection, then O is the orthocentre. And, if AD be produced to meet the circum-circle in L, we have to prove that $OD = DL$.

Now, since $\angle^s BDA$, BEA are equal, being rt. $\angle^s$, therefore points B, D, E, A are on a circle.

Hence $\qquad\qquad \angle DBE = \angle DAE.$

But ∠ *DAE, i.e.* ∠ *LAC* = ∠ *LBC,* in the same segment.

Hence in the △ˢ *BDO, BDL,*

$$∠ DBO = ∠ DBL, \text{ rt. } ∠ BDO = \text{rt. } ∠ BDL,$$

and *BD* adjacent to the equal angles is common ;

$$∴ OD = DL.$$

[The student should draw a figure for the case when the triangle is obtuse-angled.]

7. *The distance of the orthocentre from any vertex of a triangle is twice the perpendicular distance of the circum-centre from the opposite side of the triangle.*

Let *AD, BE, CF* be the three 'perpendiculars' of the △ *ABC,* meeting in the orthocentre *O,* and let *SA′, SB′, SC′* be the perpendiculars from the circum-centre on the sides *BC, CA, AB* respectively.
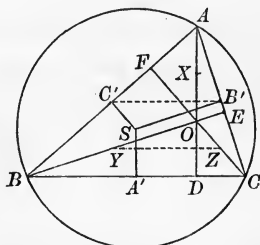
Then we know that *A′, B′, C′* are the middle points of the sides on which they lie.

Bisect *OA, OB, OC* in *X, Y, Z* respectively.

Join *YZ, B′C′.*

Then, since *BC′ = C′A* and *CB′ = B′A, B′C′* is ‖ to *BC* and equal to half *BC.*

And, since *OY = YB* and *OZ = ZC, YZ* is ‖ to *BC* and equal to half *BC.*

Hence *B′C′* and *YZ* are equal and parallel.

And, since the three sides of the △ *YOZ* are parallel respectively to the three sides of the △ *B′SC′,* the △ˢ are equiangular; and the corresponding sides *YZ* and *B′C′* are equal.

Hence △ˢ *YOZ, B′SC′* are equal in all respects, so that *OY = SB′* and *OZ = SC′.*

Hence *OB = 2SB′, OC = 2SC′,* and similarly *OA = 2SA′.*

8. The Nine-point Circle. *In any triangle the three middle points of the sides, the three feet of the perpendiculars drawn from the angular points on the sides, and the three middle points of the lines joining the orthocentre to the angular points all lie on a circle called the nine-point circle whose radius is half that of the circum-circle of the triangle.*

Let AD, BE, CF be the perpendiculars of the $\triangle ABC$ intersecting in the orthocentre O. Let S be the circum-centre, and SA', SB', SC' the $\perp^{rs}$ from S on the sides BC, CA, AB respectively; then A', B', C' are the middle points of the sides on which they lie. Let X, Y, Z be the middle points of OA, OB, OC respectively.

Then *we have to prove that* A', B', C', D, E, F, X, Y *and* Z *lie on a circle.*
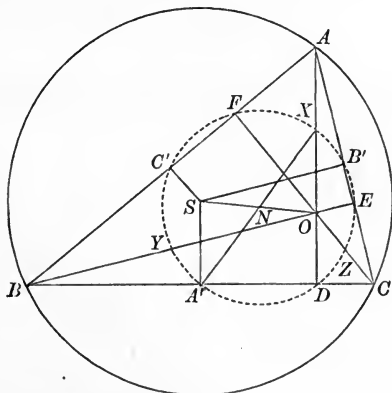
Join SA and $A'X$ and let $A'X$ cut OS in the point N.

Then, we know that $OA = 2A'S$;

$\therefore XA = A'S$ and is parallel to it.

Hence $A'X$ is parallel to SA, and $A'X = SA$.

And, in the $\triangle^s$ ONX, SNA'

$$\begin{cases} \angle OXN = \angle SA'N, \text{ since } OX \parallel SA', \\ \angle NOX = \angle NSA', \\ \text{and } OX = \tfrac{1}{2}OA = SA'; \end{cases}$$

$\therefore ON = NS$, and $NX = NA' = \tfrac{1}{2}A'X = \tfrac{1}{2}SA$.

Thus the circle whose centre is N, the middle point of SO, and whose radius is half the radius of the circum-circle will pass through A' and X, and therefore also through D, since $\angle XDA'$ is a rt. $\angle^e$.

And it can be proved in a similar manner, that the same circle will pass through B', Y and E and also through C', Z and F.

Thus *the nine-points* A', B', C', D, E, F, X, Y *and* Z *all lie on a circle whose radius is half that of the circum-circle and whose centre is the middle point of the line joining the circum-centre and orthocentre.*

9. *The centroid of any triangle is on the line joining the circum-centre and the orthocentre of the triangle.*

Let S be the circum-centre and O the orthocentre of the $\triangle ABC$. Join SO.

Let A' be the middle point of BC. Join AA' cutting SO in G.
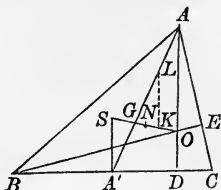
Then we know that SA' is $\perp^r$ to BC and that $AO = 2SA'$.

Bisect GA in L and GO in K, and join LK.

Then LK is $\parallel^l$ to AO and $LK = \frac{1}{2}OA$.

Hence LK is equal and parallel to SA'.

Hence the $\triangle^s SGA'$, KGL are equal in all respects, so that

$$A'G = GL \times \frac{1}{2}GA,$$

which proves that G is the centroid.

10. *The pedal line of any point on the circum-circle of a triangle bisects the line joining the point to the orthocentre.*

Let P be any point on the circum-circle of the $\triangle ABC$, whose orthocentre is O and circum-centre S.

Draw PL, PM the $\perp^{rs}$ from P on BC, CA respectively.

Produce PL to meet the circum-circle again on the point a, and join Aa.

Let LM and OA, produced if necessary, cut in X.

Then, since $\angle^s PLC$ and PMC are rt. $\angle^s$, P, L, M, C are cyclic ;

$$\therefore \angle PLM = \angle PCA = \angle PaA.$$

Hence LM is $\parallel$ to aA, and PL, OA are $\perp^r$ to BC and $\therefore$ are $\parallel$;

$$\therefore LaAX \text{ is a } \parallel^m \text{ and } La = XA.$$

Draw the $\perp^{rs} SA'$, SV on BC, Pa respectively.

Then, since V is the middle point of Pa,

$$PL - La = 2VL = 2SA' = OA ;$$
$$\therefore PL = OA + La = OA + XA = OX.$$

Hence PL is equal and $\parallel$ to OX, so that $PLOX$ is a $\parallel^m$, and $\therefore$ the diagonal OP is bisected by the diagonal LX.

Hence *the pedal line of any point on the circum-circle bisects the line joining the point to the orthocentre.*

Let p be the middle point of OP, through which, as we have just proved, the pedal line of P passes.

Join Np, where N is the centre of the nine-point circle.

Then, since $Op = pP$ and $ON = NS$, Np is $\parallel$ to SP and $Np = \frac{1}{2}SP$.

Hence p *is on the nine-point circle.*
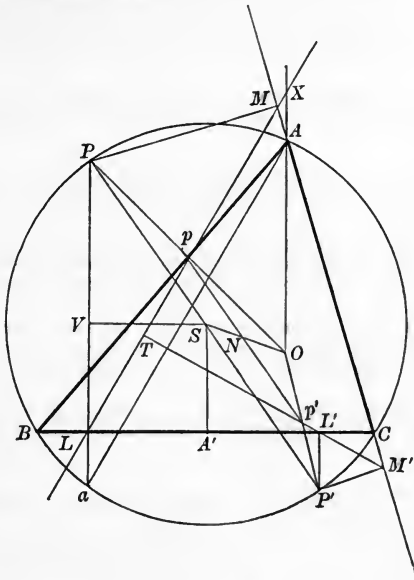
So also, if P' be the other extremity of the diameter PSP' of the circum-circle, the pedal line of P' will cut OP' in a point p' on the nine-point circle, and Np' will be parallel to SP', and therefore pp' is a diameter of the nine-point circle.

Now we have proved that the pedal line of P makes with BC an angle equal to the complement of the angle PCA or $PP'A$. The pedal line of P' will similarly make with BC an angle equal to the complement of the angle $P'PA$.

Hence *the pedal lines of P, P' the extremities of any diameter of the circum-circle are at right angles.*

And, since the pedal lines of P and P' are at rt. $\angle^s$ and pass respectively through the extremities of a diameter of the nine-point circle, their point of intersection must be on the nine-point circle, which is Steiner's Theorem:

STEINER'S THEOREM. *The pedal lines of the two extremities of any diameter of the circum-circle intersect at right angles on the nine-point circle.*

MISCELLANEOUS EXERCISES.

1. Construct a rhombus having given its angles and the radius of its inscribed circle.

2. Shew that, if an equilateral polygon be circumscribed to a circle, its alternate angles will be equal; and that, if the number of sides be odd, the polygon will be regular.

3. Shew that the greatest triangle inscribed in a given circle is equilateral, and that the greatest quadrilateral inscribed in the circle is a square.

4. Shew that the triangle formed by joining the points of contact of one of the circles which touch three given straight lines is equiangular to the triangle formed by joining the centres of the other three circles.

5. Shew that the line joining the feet of the perpendiculars from two angles of a triangle on the opposite sides is at right angles to the line joining the other angle to the centre of the circum-circle.

6. Divide a parallelogram into two quadrilaterals by a straight line so that a circle may be inscribed in each quadrilateral. When is the problem impossible?

7. Construct an isosceles triangle each of whose base angles is seven times the vertical angle.

8. Construct a triangle having one angle equal to three times and another equal to six times the third angle.

9. AB is the side of an equilateral triangle inscribed in a circle, and AC is the side of an inscribed square; shew that BC is the side of a regular polygon of twelve sides inscribed in the circle.

10. AB is the side of a regular pentagon inscribed in a circle, and AC the side of an inscribed hexagon; shew that BC is equal to the side of a regular polygon of thirty sides inscribed in the circle.

11. $PQRS$ is a cyclic quadrilateral and the opposite sides PQ, RS are cut by two straight lines in the points A, B and C, D respectively; shew that, if the four points A, B, C, D lie on a circle, the lines AC and BD will cut QR and SP in four points which lie on a circle, and will also cut PR and QS in four cyclic points.

12. Shew that, if any quadrilateral is divided into four triangles by its diagonals, the circum-centres of the four triangles are at the angular points of a parallelogram.

13. Shew that, if the base and vertical angle of a triangle be given the centres of the four circles which touch its sides will lie on one or other of two fixed circles through the extremities of the base.

14. Having given any three points A, B, C on a given circle, find a fourth point D on the circle such that a circle can be inscribed in the quadrilateral $ABCD$.

15. $ABCD$ is a cyclic quadrilateral, and the sides AB, CD meet in E and AD, BC meet in F. Shew that the circles ABF, DCF, BCE, ADE meet in a point on EF.

16. Construct a triangle having given the orthocentre, the circumcentre, and one angular point.

17. A given circle touches two given straight lines. Draw another tangent to the given circle so that the part of it intercepted between the given tangents may be of given length.

18. Construct a triangle having given one side and the radii of the in-circle and circum-circle.

19. Construct a triangle having given the inscribed circle, the position of one angular point, and the sum of the two sides which meet in that angular point.

20. AB, AC are two given straight lines and O is any point within the angle BAC. Shew how to draw through O a straight line BOC so that the in-circle of ABC may touch BC at O.

21. $ABCD$ is a quadrilateral described about a circle, and BD is joined. Shew that the circles inscribed in the triangles ABD, CBD will touch one another, and that a circle can be described to pass through the four points where these inscribed circles touch the sides of $ABCD$.

22. Shew that the four points of contact of the direct common tangents of two given circles which are external to each other, the four points of contact of the transverse common tangents, and the four points of intersection of common tangents which are not on the line joining the centres of the circle, lie on three concentric circles.

23. $ABCDE$ is a regular pentagon and P is any point on its circumcircle; shew (1) that the sum of the perpendiculars from A, B, C, D, E on the tangent at P is constant and (2) that the sum of the squares on PA, PB, PC, PD and PE is constant.

24. Shew that, if I be the in-centre of the triangle ABC, and AI, BI, CI be produced to meet the circum-circle of ABC in A', B', C' respectively, then I will be the orthocentre of the triangle $A'B'C'$.

25. Shew that, if circles be described to touch the sides, three and three, of any convex quadrilateral, either all internally, or one side externally and the two adjacent sides produced; then the centres of either system of circles will lie on a circle.

26. Shew that the centres of the four circum-circles of the four triangles formed by four straight lines will lie on a circle through their common point.

27. Having given the base and the vertical angle of a triangle, prove that the loci of the orthocentre, the nine-point centre and the centroid are all circles.

Prove also that the nine-point circle touches a fixed circle whose radius is equal to that of the circum-circle.

28. Shew that, if a line BC of constant length have its extremities on the two fixed straight lines AX, AY, the loci of the circum-centre and the orthocentre of the triangle ABC are circles.

29. Construct a triangle having given the vertical angle and the lengths of the two segments into which the base is divided by the point of contact of the inscribed circle.

30. Construct a triangle having given the length of one side, the difference of the other two sides, and the radius of the inscribed circle.

31. Construct a triangle having given the base, the vertical angle, and the length of the line cut off by the base from the bisector of the vertical.

32. Construct a triangle having given the length of the line from the vertical angle to the middle point of the base, the length of the bisector of the vertical angle cut off by the base and the difference of the angles at the base.

33. Having given two circles in magnitude and position, and a line given in position, draw two parallel tangents to the given circles which will intercept a given length on the given straight line.

34. From S the circum-centre of the triangle ABC, the perpendiculars SA', SB', SC' are drawn to the sides, and these perpendiculars are produced to X, Y, Z respectively so that $SA' = A'X$, $SB' = B'Y$, and $SC' = C'Z$. Shew that the triangles ABC, XYZ have the same nine-point circle.

35. Prove that the three perpendiculars to the sides of a triangle from the three internal points of contact of the three escribed circles will meet in a point.

36. Through a given point O draw a straight line cutting two given straight lines AX, AY in the points B, C respectively so that the perimeter of the triangle ABC may be equal to a given straight line; also draw through O the straight line which makes with OX, OY the triangle of minimum perimeter.

37. Shew that, if a quadrilateral be circumscribed to a circle, the orthocentres of the four triangles formed by two consecutive tangents and their chord of contact are at the angular points of a parallelogram.

38. Shew that, if $ABCD$ is a cyclic quadrilateral and the diagonals AC, BD be drawn, the orthocentres of the four triangles BCD, CDA, DAB, ABC are at the angular points of a quadrilateral equal in all respects to the given quadrilateral.

39. Shew that, if S be the centre and R the radius of the circum-circle of a triangle and I, I_1, I_2, I_3 be the centres of the circles which touch its sides, then

$$(1) \quad II_1{}^2 + I_2 I_3{}^2 = 16R^2,$$

$$(2) \quad SI^2 + SI_1{}^2 + SI_2{}^2 + SI_3{}^2 = 12R^2.$$

40. A triangle is divided into two others by a line from the vertex to the point of contact of the inscribed circle with the base. Shew that the in-circles of the two triangles so formed will touch one another.

41. A parallelogram is divided into two triangles by a diagonal; shew that the nine-point circles of these two triangles touch one another.

42. Through the middle point of each side of a cyclic quadrilateral a line is drawn perpendicular to the opposite side; shew that the four perpendiculars meet in a point.

43. Inscribe a triangle in a given circle so that the orthocentre may be at a given point, and that one of the sides may pass through another given point.

44. Shew that, in any triangle ABC the foot of the perpendicular from the orthocentre on the bisector of the angles BAC is on the diameter of the nine-point circle which passes through the middle point of BC.

45. Shew that, if a circle X pass through the centre of a circle Y, an infinite number of quadrilaterals can be inscribed in X whose sides, or sides produced, will touch Y.

46. ABC is any triangle and A', B', C' the middle points of its sides; P, Q, R are the feet of the perpendiculars from A', B', C' on $B'C'$, $C'A'$, $A'B'$ respectively, and P', Q', R' are the middle points of the sides of PQR; also, X, Y, Z are the feet of the perpendiculars of the triangle $P'Q'R'$. Shew (1) that the circle inscribed in PQR is concentric with the circum-circle of ABC, and (2) that the circum-circle of PQR is concentric with the in-circle of XYZ.

47. Shew that, if the radius of one of two circles is double the radius of the other, and the circles are not entirely external to one another, an infinite number of triangles can be constructed such that the given circles are respectively the circum-circle and the nine-point circle of the triangle.

48. Two given straight lines AX, AY are cut by a moving line in the points B, C respectively so that the sum of AB and AC is equal to a given straight line. Shew the loci of the circum-centre, the orthocentre, the nine-point centre, and the centroid of the triangle ABC, for different positions of BC, are all straight lines.

49. Shew that, if four points be taken on a circle, the four nine-point circles of the four triangles whose angular points are three of the four given points, will meet in a point.

50. Shew that, if four points be taken on a circle, the four pedal lines of each point with respect to the triangle formed by the other three will meet in a point which is the point of intersection of the nine-point circles of the four triangles.

BOOKS VI AND XI
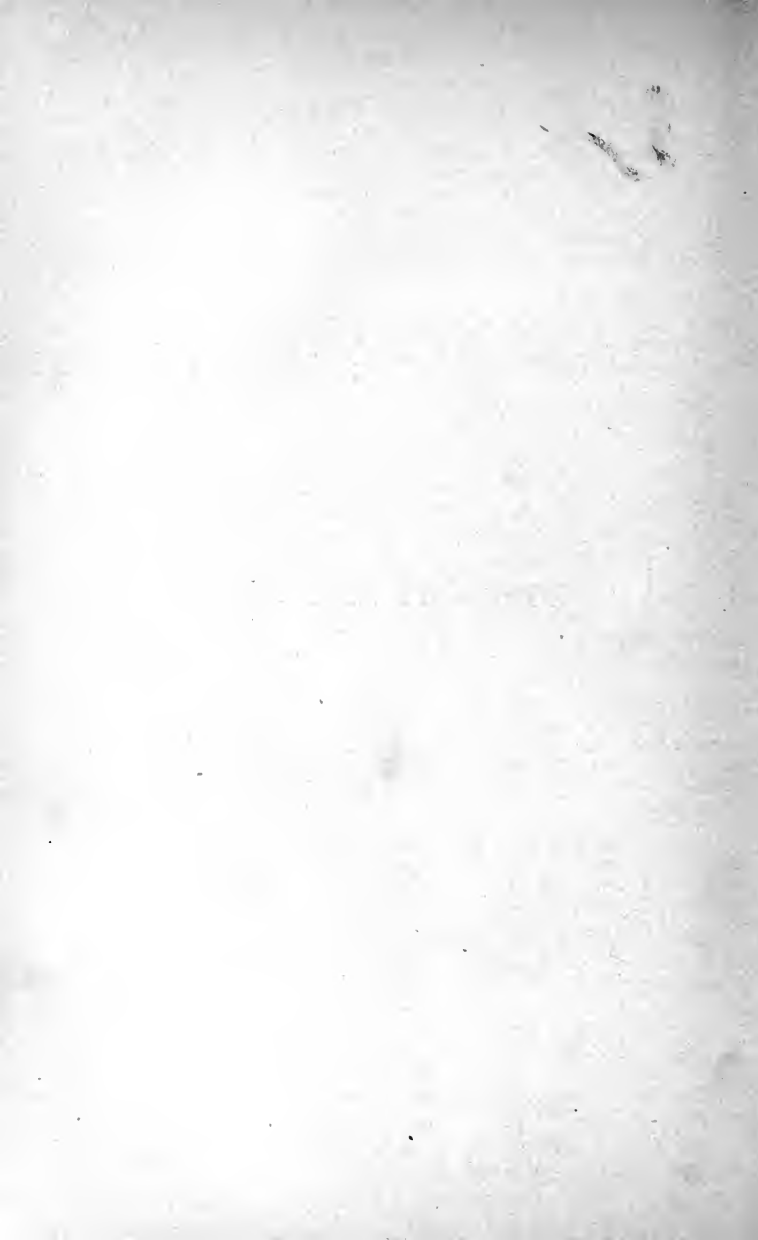
BOOK VI.

DEFINITIONS.

1. *If one magnitude be equal to another repeated twice, thrice, or any other whole number of times, the first magnitude is said to be a* **multiple** *of the second, and the second is said to be a* **sub-multiple**, *or a* **measure**, *of the first.*

2. *Two magnitudes of the same kind are said to be* **commensurable** *when they have a common measure, and to be* **incommensurable** *when they have no common measure.*

Magnitudes which are incommensurable are of frequent occurrence in Geometry; for example, a side and a diagonal of a square are incommensurable, the side of an equilateral triangle and the radius of its inscribed circle are incommensurable, and the area of an equilateral triangle is incommensurable with the area of a square described on one of its sides.

Capital letters A, B, C,... will generally be employed to denote magnitudes (not numerical representations of magnitudes but the magnitudes themselves), and multiples of magnitudes will be denoted by using numbers, or small letters to represent *whole numbers*.

Thus $2A$, $5B$, mA, nB, pC, ... represent multiples of the magnitudes A, B, C, ... Also any equimultiples of the magnitudes A and B will be represented by mA and mB, or by nA and nB, &c.

3. *The relation of two magnitudes of the same kind to one another in respect to relative greatness is called their* **ratio**.

The ratio of the two magnitudes A and B is denoted by $A : B$, which is read 'A to B.'

The first of the two magnitudes is sometimes called the **antecedent** and the second the **consequent**.

Hitherto two magnitudes have been compared only with respect to equality or inequality. Unequal magnitudes could be precisely compared if it were always possible to find a common measure of both; but this is, as we have seen, by no means always the case. But although a side and a diagonal of a square, to take an example of two incommensurable magnitudes, have no common measure, we can form an approximate idea of their relative lengths, that is of their ratio. For, if the side were divided into 10 equal parts it would be found that the diagonal contained more than 14 and less than 15 of these parts, and if the side were divided into 1000 equal parts it would be found that the diagonal contained more than 1414 and less than 1415 of such parts.

Euclid's definition that 'magnitudes are said to have a ratio to one another, when the less can be multiplied so as to exceed the greater,' is only an indirect way of stating that two magnitudes have a ratio when, and only when, they are of the *same kind.* Thus two straight lines have a ratio to one another, and so also have two areas or two angles; but we cannot compare an angle with an area, or a weight with a length.

'In definition 3 Euclid gives that sort of inexact notion of a ratio which defines it in the case of commensurable quantities, and gives some light on its general meaning. It stands here like the definition of a straight line, "that which lies evenly between its extreme points" prior to the common notion "two straight lines cannot enclose a space" which is the actual subsequent test of straightness.' *De Morgan.*

The exact definition of the equality of ratios is given in the following definition.

4. *The ratio of the first of four magnitudes to the second is said to be equal to the ratio of the third to the fourth, provided that whenever any equimultiples of the first and third are taken and also any equimultiples of the second and fourth, the multiple of the first is always greater than, equal to or less than the multiple of the second according as the multiple of the third is greater than, equal to or less than the multiple of the fourth.*

Thus, if A and B be the first and second of the magnitudes (which must be magnitudes of the same kind) and C and D be the third and fourth (which must also be magnitudes of the same kind, though not necessarily of the same kind as A and B); and if any equimultiples mA, mC of the first and third are taken, and also any equimultiples nB, nD of the second and fourth; then the ratio of A to B is equal to the ratio of C to D, provided that

$$mA > nB \text{ whenever } mC > nD,$$
$$mA = nB \text{ whenever } mC = nD,$$
and
$$mA < nB \text{ whenever } mC < nD,$$

i.e.
$$mA \gtreqless nB \text{ according as } mC \gtreqless nD,$$

whatever whole numbers m and n may be.

It should be noticed that two magnitudes are commensurable when some multiple of one is equal to a multiple of the other. For, if $mA = nB$; then, if A be divided into n equal parts, mn of those parts are contained in mA, and therefore also in nB, whence it follows that B contains m of the parts. Thus A and B are commensurable, the nth part of A being the same as the mth part of B.

5. *Four magnitudes are said to be* **in proportion**, *or to be* **proportionals**, *when the ratio of the first to the second is equal to the ratio of the third to the fourth.*

If the four magnitudes be denoted by the letters A, B, C, D; then A, B, C, D are in proportion if the ratio of A to B is equal to the ratio of C to D, which is written in the form

$$A : B = C : D,$$

and read '*A to B equals C to D.*'

The relation is sometimes written in the form

$$A : B :: C : D,$$

which is read '*A is to B as C is to D.*'

The first and fourth of four magnitudes in proportion are called the *extremes*, and the second and third are called the *means*.

In a proportion, the *antecedents* of the equal ratios, that is the first and third terms of the proportion, are sometimes said to be **homologous**; so also the *consequents*, namely the second and fourth terms, are said to be *homologous*.

6. *When magnitudes of the same kind are such that the ratio of the first to the second, of the second to the third, of the third to the fourth, and so on, are all equal, the magnitudes are said to be in* **continued proportion**.

When three magnitudes are in continued proportion, the second is called the **mean proportional** *between the first and third, and the third is called the* **third proportional** *to the first and second.*

Thus, if $\qquad\qquad A : B = B : C,$

B is a *mean proportional* between A and C, and C is the *third proportional* to A and B.

7.   *When there are any number of magnitudes of the same kind, the first is said to have to the last the* **ratio compounded** *of the ratio of the first to the second, of the ratio of the second to the third, and so on to the last.*

Thus, if there are three magnitudes *A*, *B*, *C* of the same kind, the ratio of *A* to *C* is compounded of the ratios of *A* to *B* and *B* to *C*.

A ratio which is compounded of two equal ratios is said to be the **duplicate** *of either of the equal ratios.   So also a ratio which is compounded of three equal ratios is said to be the* **triplicate** *of any one of those equal ratios.*

Thus, if $A : B = B : C$, the ratio $A : C$ is the *duplicate* of the ratio *A* to *B* or *B* to *C*.

8.   *Rectilineal figures which have the angles of the one taken in order equal respectively to the angles of the other taken in the same order, and in which the ratio of the side adjacent to two angles in one figure to the side adjacent to the equal angles in the other figure is the same for all the pairs of sides, are said to be* **similar**.

Thus the figures *ABCD*, *PQRS* are similar, if the angles *A*, *B*, *C*, *D* are equal to the angles *P*, *Q*, *R*, *S* respectively, and if also

$$AB : PQ = BC : QR = CD : RS = \&c.$$

N.B.   It must be carefully noted that when two figures *ABCD*..., *PQRS*... are said to be similar, it is always understood that *A* and *P*, *B* and *Q*, *C* and *R*, ..., are equal angles.

9.   *The* **altitude** *of a parallelogram, with reference to a particular side as base, is the length of the perpendicular drawn to the base from any point on the opposite side.*

It is easily seen that $\parallel^{ms}$ which are between the same $\parallel^{s}$ have equal altitudes, and that $\parallel^{ms}$ which have equal altitudes can be so placed as to be between the same parallels.

The **altitude** *of a triangle, with reference to any particular side as base, is the perpendicular drawn to the base from the opposite angular point.*

It is easily seen that $\triangle^{s}$ which are between the same parallels have equal altitudes, and that $\triangle^{s}$ which have equal altitudes can be so placed as to be between the same parallels.

THEORY OF PROPORTION.

PROPOSITION i.

If four magnitudes be proportional, they will be proportional when taken inversely.

Let A, B, C, D be the four magnitudes in proportion, so that

$$A : B = C : D.$$

Then, *it is required to prove that*

$$B : A = D : C.$$

Since A, B, C, D are in proportion, we know that for all integral values of m and n,

$$mA \gtreqless nB \text{ according as } mC \gtreqless nD,$$

i.e.
$$nB \lesseqgtr mA \text{ according as } nD \lesseqgtr mC.$$

Hence, by definition,

$$B : A = D : C.$$

~ PROPOSITION ii.

Ratios which are equal to the same ratio are equal to one another.

Let A, B; C, D; E, F be three pairs of magnitudes such that

$$A : B = C : D,$$
and
$$C : D = E : F;$$
then, *it is required to prove that*

$$A : B = E : F.$$

Of A, C, E take any equimultiples mA, mC, mE; and of B, D, F take any equimultiples nB, nD, nF.

Then, by hypothesis,

$$mA \gtreqless nB \text{ according as } mC \gtreqless nD,$$

and
$$mC \gtreqless nD \text{ according as } mE \gtreqless nF.$$

Hence
$$mA \gtreqless nB \text{ according as } mE \gtreqless nF,$$

and therefore, by definition, $\quad A : B = E : F.$

PROPOSITION iii.

Equal magnitudes have the same ratio to the same magnitude, or to equal magnitudes.

For, let $A = B$ and $C = D$; then it is obvious that $mA = mB$ and $nC = nD$.

Hence
$$mA \gtreqless nC \text{ as } mB \gtreqless nD;$$

$$\therefore \quad A : C = B : D;$$

or, if C is the same as D,

$$A : C = B : C.$$

PROPOSITION iv.

Magnitudes which have the same ratio to the same magnitude, or to equal magnitudes, must be equal.

Let A, B, C, D be four magnitudes in proportion such that $B = D$; then, *it is required to prove that* $A = C$.

For, if possible, let A exceed C by X; then, however small X may be, some multiple of X, mX suppose, will be greater than B. And, since A exceeds C by X, mA will exceed mC by mX, so that the difference between mA and mC will be greater than B, and therefore some multiple of B, nB suppose, will lie between mA and mC.

Hence $mA > nB$, but $mC < nB$.

But $B = D$, so that $nB = nD$.

Hence $mA > nB$, but $mC < nD$, which is impossible since $A : B = C : D$.

Hence, if $A : B = C : D$ and $B = D$, A must be equal to C.

PROPOSITION v.

Two magnitudes and any two of their equimultiples are in proportion.

Let A and B be any two magnitudes, and let pA, pB be any two of their equimultiples; then, *it is required to prove that*

$$pA : pB = A : B.$$

It is obvious that

$$p \cdot mA = p \cdot nB \text{ if } mA = nB,$$

$$p \cdot mA > p \cdot nB \text{ if } mA > nB,$$

and
$$p \cdot mA < p \cdot nB \text{ if } mA < nB.$$

But $p \cdot mA = m \cdot pA$, and $p \cdot nB = n \cdot pB$;

$$\therefore \quad m \cdot pA \gtreqless n \cdot pB \text{ according as } mA \gtreqless nB.$$

Hence, by definition,

$$pA : pB = A : B.$$

PROPOSITION vi.

If any number of ratios are equal, all the magnitudes being of the same kind, the ratio of the sum of all the antecedents to the sum of all the consequents is equal to the ratio of any one of the antecedents to the corresponding consequent.

Let the pairs of magnitudes be $A, B; C, D; E, F; \ldots$

Then $\qquad A : B = C : D = E : F = \ldots\ldots$

By definition of equal ratios, whatever whole numbers m and n may be,

$$\text{if } mA > nB,$$

$$\text{then will } mC > nD,$$

$$\text{also } nE > nF,$$

$$\ldots\ldots\ldots\ldots$$

Hence $\qquad m(A + C + E + \ldots) > n(B + D + F + \ldots).$

So also, if

$$mA = nB, \quad m(A + C + E + \ldots) = n(B + D + F + \ldots),$$

and if $\qquad mA < nB, \quad m(A + C + E + \ldots) < n(B + D + F + \ldots).$

Thus, for all values of m and n,

$$m(A + C + E + \ldots) \gtreqless n(B + D + F + \ldots) \text{ according as } mA \gtreqless nB.$$
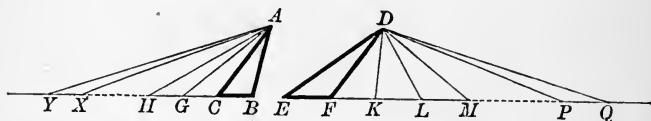
Hence, by definition,

$$A + C + E + \ldots : B + D + F + \ldots = A : B.$$

PROPOSITION I.

Triangles and parallelograms of equal altitudes are to one another as their bases.

First let the $\triangle^s$ *ABC*, *DEF* have the same altitude; then *it is required to prove that*

$$\triangle \ ABC : \triangle \ DEF = BC : EF.$$

Produce *BC*, and cut off any number of parts *CG*, *GH*, ..., *XY* each equal to *BC*, and join *AG*, *AH*, ..., *AX*, *AY*.

Also produce *EF*, and cut off any number of parts *FK*, *KL*, *LM*, ..., *PQ* each equal to *EF*, and join *DK*, *DL*, *DM*, ..., *DP*, *DQ*.

Then $\triangle ABC = \triangle ACG = \triangle AGH = ... = \triangle AXY$,

for all the $\triangle^s$ have the same altitude and

$$BC = CG = GH = ... = XY.$$

Hence $\triangle ABY$ is the same multiple of $\triangle ABC$ that *BY* is of *BC*.

Similarly $\triangle DEQ$ is the same multiple of $\triangle DEF$ that *EQ* is of *EF*.

Moreover, since the $\triangle^s$ *ABY* and *DEQ* have equal altitudes,

$$\triangle ABY > \triangle DEQ, \text{ if } BY > EQ,$$
$$\triangle ABY = \triangle DEQ, \text{ if } BY = EQ,$$

and $\qquad\quad \triangle ABY < \triangle DEQ, \text{ if } BY < EQ.$

Thus of four magnitudes, namely the $\triangle$ *ABC*, the $\triangle$ *DEF*, the base *BC* and the base *EF*, we have taken *any* equimultiples of the first and third, and also *any* equimultiples of the second and fourth; and we have shewn that the multiple of the first is always greater than, equal to or less

than the multiple of the second according as the multiple of the third is greater than, equal to or less than the multiple of the fourth. Hence, by definition,

$$\triangle ABC : \triangle DEF = BC : EF.$$

Next let $ABCC'$, $DEFF'$ be $\parallel^{ms}$ with equal altitudes. Then it can be proved in a precisely similar manner that

$$\parallel^m ABCC' : \parallel^m DEFF' = BC : EF.$$
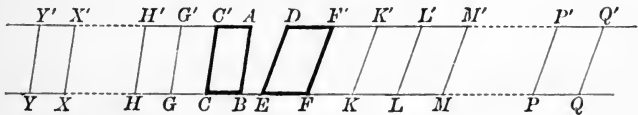
[Since $\parallel^m AC = 2 \triangle ABC$ and $\parallel^m DF = 2 \triangle DEF$, it follows from Prop. v. that $\parallel^m AC : \parallel^m DF = \triangle ABC : \triangle DEF$. Hence, by Prop. ii., if the theorem is true either for parallelograms or for triangles it is true for both.]

Conversely. *If two triangles or two parallelograms are to one another in the ratio of two sides one in each figure, their altitudes with reference to those sides are equal.*

Let $\parallel^m CBAC' : \parallel^m DEFF' = BC : EF$.

Then, if the altitudes of the $\parallel^{ms}$ with reference to their sides BC, EF respectively be not equal, construct the $\parallel^m EFST$ of the same altitude as $\parallel^m CBAC'$.

Then, since the $\parallel^{ms}$ $CBAC'$, $EFST$ have the same altitude,

$$\parallel^m CBAC' : \parallel^m EFST = BC : EF.$$

$\therefore$ $\parallel^m CBAC' : \parallel^m DEFF'$

$$= \parallel^m CBAC' : \parallel^m EFST. \quad \text{[Prop. ii.}$$

Hence [Prop. iv] $\parallel^m DEFF' = \parallel^m EFST$,

and therefore ST coincides with $F'D$.
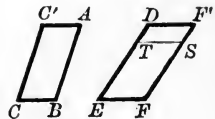
Cor. *Triangles and parallelograms on equal bases are to one another as their altitudes.*

Theory of Proportion continued.

We can now prove the remaining theorems in proportion which are required in Book VI.

It must first be noted that the only geometrical magnitudes which need be considered are *straight lines* and *rectilinear areas*; there is, however, one theorem involving *angles*, but this is proved directly from the definition of proportion.

It was shewn in Book I. how to construct a rectangle equal to any given rectilinear area, and also how to construct a rectangle equal to a given rectangle and having one of its sides of *given length*. It therefore follows from VI. 1 that two straight lines can be found whose ratio is equal to that of any two given rectilinear areas; also rectangles can be constructed whose ratio is equal to that of any two given straight lines.

PROPOSITION vii.

If four magnitudes of the same kind be proportionals, they will be proportionals when taken alternately.

Let P, Q, R, S be the four magnitudes of the same kind such that

$$P : Q = R : S;$$

then, *it is required to prove* that

$$P : R = Q : S.$$

All the four magnitudes must either be areas or straight lines.

First let all the magnitudes be areas.

Construct a rectangle *abcd* equal to the area P, and to *bc* apply the rectangle *bcef* equal to Q.
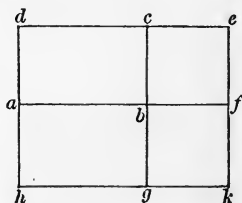
Also to *ab* and *bf* apply rectangles *ag*, *bk* equal to R and S respectively.

Then, since the rectangles *ac*, *be* have the same altitude, they are to one another as their bases.

Hence $\qquad P : Q = ab : bf.$

But $\qquad P : Q = R : S ;$

$$\therefore \quad R : S = ab : bf,$$

i.e. $\qquad$ rect. ag : rect. $bk = ab : bf.$

Hence [by Euclid VI. 1 converse] the rectangles ag and bk have the same altitude, so that k is on the line hg.

Hence the rectangles ac, ag have the same altitude, namely ab; also be, bk have the same altitude, namely bf.

$$\therefore \quad \text{rect. } ac : \text{rect. } ag = bc : bg,$$

and $\qquad$ rect. be = rect. $bk = bc : bg ;$

$$\therefore \quad \text{rect. } ac : \text{rect. } ag = \text{rect. } be : \text{rect. } bk ;$$

$$\therefore \quad P : R = Q : S.$$

Next let the magnitudes be the four straight lines AB, BC, CD, DE.

Construct the rectangles Ab, Bc, Cd, De with the same altitude.
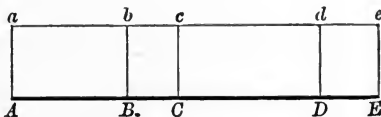
Then $\qquad Ab : Bc = AB : BC,$

and $\qquad Cd : De = CD : DE.$ $\qquad$ [VI. 1.

But $\qquad AB : BC = CD : DE,$ $\qquad$ [*Hyp.*

$$\therefore \quad Ab : Bc = Cd : De.$$ $\qquad$ [Prop. ii.

Hence by the first case

$$Ab : Cd = Bc : De,$$

and these rectangles have the same altitude,

$$\therefore \quad AB : CD = BC : DE.$$

PROPOSITION viii.

If there are six magnitudes such that the first is to the second as the fourth to the fifth and also the second to the third as the fifth to the sixth, then will the first be to the third as the fourth to the sixth.

Let the six magnitudes A, B, C, X, Y, Z be such that

$$A : B = X : Y \text{ and } B : C = Y : Z ;$$

then *it is required to prove* that

$$A : C = X : Z.$$

The three magnitudes A, B, C must be of the same kind and the three magnitudes X, Y, Z must also be of the same kind.

First suppose that all the magnitudes are areas.

Construct a rectangle $abcd$ equal to A; to bc apply the rectangle $bcef$ equal to B, and to ef apply the rectangle $efhg$ equal to C.
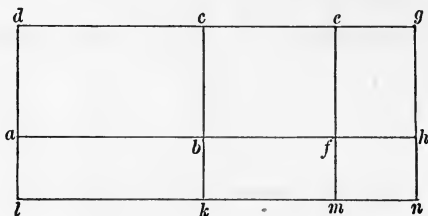
Also to ab, bf, fh apply rectangles ak, bm, fn equal respectively to X, Y, Z, as in the figure.

Then, since the rectangles ac, be, fg have equal altitudes, they are to one another as their bases.

Hence $\qquad ab : bf = \text{rect. } ac : \text{rect. } be$

$\qquad\qquad\qquad\quad = A : B \qquad\qquad\qquad\qquad$ [*Const.*

$\qquad\qquad\qquad\quad = X : Y \qquad\qquad\qquad\qquad$ [*Hyp.*

$\qquad\qquad\qquad\quad = \text{rect. } ak : \text{rect. } bm \qquad$ [*Const.*

Hence the rectangles ak and bm have the same altitude.

$\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad$ [VI. 1, *Converse.*

Similarly the rectangles bm and fn have the same altitude, so that the three rectangles ak, bm, fn have all the same altitude.

Hence
$$A : C = \text{rect. } ac : \text{rect. } fg$$
$$= ab : fh \qquad \text{[VI. 1.}$$
$$= \text{rect. } ak : \text{rect. } fn \qquad \text{[VI. 1.}$$
$$= X : Z.$$

Secondly, let the magnitudes A, B, C be straight lines and the magnitudes X, Y, Z be areas.

Let ab, bf, fh be equal to the straight lines A, B, C respectively, and to these lines apply rectangles ak, bm, fn equal to X, Y, Z respectively.

Then, as in the first case, these three rectangles must have the same altitude.

Hence
$$A : C = ab : fh$$
$$= \text{rect. } ak : \text{rect. } fn$$
$$= X : Z.$$

Thirdly, let all the magnitudes be straight lines.

Apply to the lines X, Y, Z rectangles P, Q, R of the same altitude.

Then
$$A : B = X : Y \qquad \text{[}Hyp.\text{}$$
and
$$X : Y = P : Q ; \qquad \text{[VI. 1.}$$
$$\therefore A : B = P : Q. \qquad \text{[Prop. ii.}$$

Similarly
$$B : C = Q : R.$$

Hence, by the second case,
$$A : C = P : R$$
$$= X : Z. \qquad \text{[VI. 1.}$$

Cor. If $A : B = X : Y$, *then the duplicate of the ratio $A : B$ is equal to the duplicate of the ratio $X : Y$.*

For, if $A : B = B : C$ and $X : Y = Y : Z$, and if also $A : B = X : Y$;

then will
$$B : C = Y : Z,$$
and $\therefore$ (Prop. vii.)
$$A : C = X : Z.$$

But, by def., $A : C$ is the duplicate ratio of $A : B$,

and $X : Z$,,   ,,   ,, $X : Y$.

PROPOSITION ix.

If the four magnitudes A, B, C, D are in proportion; then will

$$A + B : A \quad = C + D : C,$$
$$A + B : B \quad = C + D : D,$$
$$A - B : A \quad = C - D : C,$$

and
$$A + B : A - B = C + D : C - D.$$

First let all the magnitudes be areas.

Construct a rectangle *abcd* equal to *A*, and to *bc* apply the rectangle *bcef* equal to *B*.
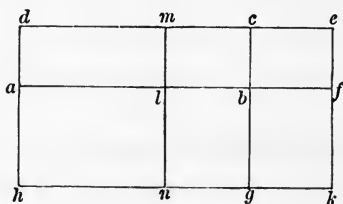
Also to *ab*, *bf* apply the rectangles *ag*, *bk* equal to *C* and *D* respectively.

Then, since the rectangles *ac*, *be* have equal altitudes *bc*, they are to one another as their bases.

Hence
$$ab : bf = \text{rect. } ac : \text{rect. } be$$
$$= A : B \qquad \qquad [\textit{Const.}$$
$$= C : D \qquad \qquad [\textit{Hyp.}$$
$$= \text{rect. } ag : \text{rect. } bk \qquad [\textit{Const.}$$

Hence, by the converse of VI. 1, the rectangles *ag*, *bk* have the same altitude, so that *k* is on the straight line *hg*.

Hence
$$A + B : A = \text{rect. } ae : \text{rect. } ac$$
$$= af : ab$$
$$= \text{rect. } ak : \text{rect. } ag$$
$$= C + D : C.$$

Similarly
$$A + B : B = C + D : D.$$

Now from ba cut off $bl = bf$, and through l draw mln parallel to dah meeting de, hk respectively in m, n.

Then rectangles am, an are clearly equal to $A - B$ and $C - D$ respectively.

Hence
$$A - B : A = \text{rect. } am : \text{rect. } ac$$
$$= al : ab$$
$$= \text{rect. } an : \text{rect. } ag$$
$$= C - D : C.$$

Also
$$A + B : A - B = \text{rect. } ae : \text{rect. } am$$
$$= af : al$$
$$= \text{rect. } ak : \text{rect. } an$$
$$= C + D : C - D.$$

Next let the magnitudes A, B be straight lines and the magnitudes C, D be areas.

Let ab, bf be equal to the straight lines A, B, and to these lines apply the rects. ag, bk equal to C, D respectively; then as before the rects. ag, bk have the same altitude. Also cut off from ba the line bl equal to bf.

Then
$$A + B : A = af : ab$$
$$= \text{rect. } ak : \text{rect. } ag$$
$$= C + D : C.$$

Similarly
$$A + B : B = C + D : D,$$
$$A - B : A = C - D : C,$$

and
$$A + B : A - B = C + D : C - D.$$

Lastly, let all the magnitudes be straight lines.

Apply to the lines C, D rectangles P, Q having the same altitude.

Then [VI. 1]
$$P : Q = C : D.$$

Hence, by the second case,
$$A + B : A = P + Q : Q.$$

Also
$$P + Q : Q = C + D : C;$$
$$\therefore A + B : A = C + D : C.$$

And the other results can be proved in a similar manner.

PROPOSITION II. Theorem.

A straight line parallel to one side of a triangle cuts the other two sides (or these sides produced) proportionally ; and, conversely, the straight line joining points which divide two sides of a △ (or both these sides produced) in the same ratio is parallel to the third side.
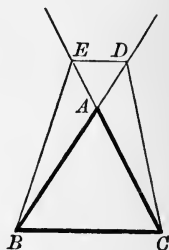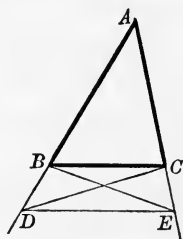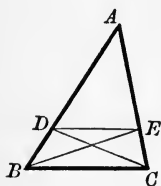
Let ABC be the given triangle, and let DE be any st. line ‖ to BC and cutting AB, AC, or these produced, in the points D, E respectively ; then *it is required to prove that*

$$AD : DB = AE : EC.$$

Join BE and CD.

Then, since DE is ‖ to BC, $\triangle BDE = \triangle CED$.

But equal magnitudes have the same ratio to the same magnitude ;

$$\therefore \ \triangle BDE : \triangle ADE = \triangle CED : \triangle ADE.$$

But $\qquad \triangle BDE : \triangle ADE = BD : DA,$ [VI. 1.

and $\qquad \triangle CED : \triangle ADE = CE : EA.$

But ratios which are equal to equal ratios are equal to one another ;

$$\therefore \ BD : DA = CE : EA.$$

Conversely. Let the points D and E be taken on AB, AC respectively (or on both these produced) such that $BD : DA = CE : EA$; then *it is required to prove that DE is $\parallel$ to BC.*

For $\qquad BD : DA = \triangle BDE : \triangle ADE,$ [VI. 1.

and $\qquad CE : EA = \triangle CED : \triangle AED.$

But, by hyp. $BD : DA = CE : EA$;

$\therefore \quad \triangle BDE : \triangle ADE = \triangle CED : \triangle AED.$

Hence $\qquad \triangle BDE = \triangle CED,$

and they are on the same base DE.

$$\therefore \quad DE \text{ is } \parallel \text{ to } BC.$$

Cor. It will be easily seen that

$$AB : BD = AC : CE,$$

[For $\triangle ABE = \triangle ACD$;

$\therefore \quad \triangle ABE : \triangle BDE = \triangle ACD : \triangle CED.$]

and $\qquad AB : AD = AC : AE.$

Ex. 1. Through any point O within the triangle ABC straight lines AOD, BOE, COF are drawn to meet the opposite sides of the triangle in D, E, F respectively. Shew that

$$\triangle AOB : \triangle AOC = BD : DC, \text{ &c.}$$

Ex. 2. Find a point O within the triangle ABC such that

$$\triangle BOC = \triangle COA = \triangle AOB.$$

Ex. 3. Shew that the three medians divide a triangle into six equal parts.

Ex. 4. Find a point O within the triangle ABC such that

$$\triangle BOC = 2 \triangle COA = 4 \triangle AOB.$$

Ex. 5. Shew that, if O be the point defined in Ex. 4, and AO cut BC in D,

$$4 \triangle AOB = 3 \triangle BOD \text{ and } 4OA = 3OD.$$

Ex. 6. On the sides BC, CA of the triangle ABC the points D, E are taken respectively such that $CD = 2BD$ and $CE = 2EA$. The lines AD, BE intersect at O and CO is produced to cut AB in K. Shew that $AK = KB$, $CO = 4OK$, and $2BO = 3OE$.

Ex. 7. D, E are points on the sides BC, CA respectively of the triangle ABC such that $BD = \frac{1}{2}DC$ and $CE = EA$; shew that AD bisects BE.

Ex. 8. From any point O on the diagonal AC of the quadrilateral $ABCD$ lines OX, OY are drawn parallel to AB, AD respectively so as to meet CB, CD respectively in X, Y. Shew that XY is parallel to BD.

Ex. 9. One straight line cuts three parallel straight lines in A, B, C respectively and another straight line cuts them in P, Q, R respectively; shew that

$$AB : BC = PQ : QR.$$

[For $AB : BC = \triangle AQB : \triangle CQB = \triangle PBQ : \triangle RBQ = PQ : QR.$]

Ex. 10. AB, CD are two parallel straight lines and P, Q are points on AB, CD respectively such that $AP : PB = CQ : QD$. Shew that AC, BD and PQ will meet in a point.

Ex. 11. ABC, PQR are two triangles such that PA, QB, RC meet in a point; shew that, if AB is parallel to PQ and BC parallel to QR, then will AC be parallel to PR.

Ex. 12. A line parallel to the side BC of the triangle ABC cuts AB, AC respectively in F, E, and BE, CF intersect at O; shew that AO will pass through the middle points of BC and FE.

[$\triangle BOC : \triangle COA = BF : FA = CE : EA = \triangle COB : \triangle BOA$;

$$\therefore \triangle COA = \triangle BOA, \text{ whence result.}]$$

Ex. 13. O is any point on the median AD of the triangle ABC, and BO, CO are produced to meet CA, BA respectively in E, F. Shew that EF is parallel to BC.

Ex. 14. E, F are the middle points of the sides AD, BC of the parallelogram $ABCD$; shew that BE, DF will trisect AC.

Ex. 15. E, F are the middle points of the sides DA, DC of the parallelogram $ABCD$; shew that BE, BF will trisect AC.

Ex. 16. D is any point on the side BC of the triangle ABC, and any line parallel to BC cuts AB, AD, AC in P, Q, R respectively; shew that

$$PQ : QR = BD : DC.$$

PROPOSITION III. THEOREM.

If an angle of a triangle be bisected by a straight line which cuts the base, the ratio of the segments of the base will be equal to the ratio of the other sides of the triangle; and conversely, if one side of a triangle be divided into segments whose ratio is equal to that of the adjacent sides of the triangle, the straight line joining the point of section to the opposite vertex will bisect the vertical angle.

Let BAC be a triangle, and let the bisector of the angle BAC cut BC in D; then *it is required to prove that*

$$BD : DC = BA : AC.$$
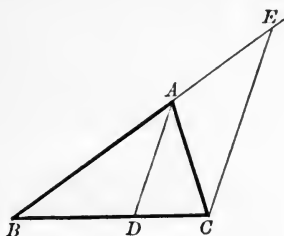
Draw $CE \parallel$ to DA meeting BA produced in E.

Then, since $AD \parallel CE$,

$\angle BAD =$ int. opp. $\angle AEC$ and $\angle DAC =$ alt. $\angle ACE$.

But $\qquad\qquad \angle BAD = \angle DAC,$ $\qquad\qquad$ [*Hyp.*

$\therefore \angle AEC = \angle ACE$, and $\therefore AE = AC$.

But, since AD is $\parallel$ to CE,

$$BD : DC = BA : AE, \qquad\qquad [\text{VI. 2.}$$
$$\therefore BD : DC = BA : AC.$$

Conversely. Let D be such that $BD : DC = BA : AC$; then *it is required to prove that DA will bisect $\angle BAC$.*

Through C draw $CE \parallel$ to DA meeting BA produced in E.

Then, since $DA \parallel$ to CE,

$$BD : DC = BA : AE. \qquad\qquad [\text{VI. 2.}$$

But, by hyp., $\qquad BD : DC = BA : AC.$

Hence $\qquad\qquad BA : AE = BA : AC,$

and $\qquad\qquad \therefore AE = AC$, and $\angle AEC = \angle ACE$.

But, since $AD \parallel CE$,

$\angle BAD =$ int. opp. $\angle AEC$ and $\angle DAC =$ alt. $\angle ACE$.

Hence $\qquad\qquad \angle BAD = \angle DAC.$

PROPOSITION III*. Theorem.

*If the exterior angle of a triangle, made by producing one of its sides, be bisected by a straight line which cuts the base, the ratio of the segments of the base will be equal to the ratio of the other sides of the triangle; and conversely, if one side of a triangle be divided externally into segments whose ratio is equal to that of the other sides of the triangle, the straight line drawn from the point of section to the vertex will bisect the exterior angle of the triangle.*

In the $\triangle BAC$ let BA be produced to D, and let the bisector of the $\angle CAD$ cut BC produced in E; then *it is required to prove that*

$$BE : CE = BA : AC.$$
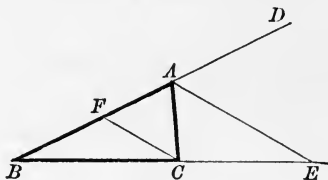
Through C draw $CF \parallel$ to AE cutting BA in F.

Then, $\because CF$ is parallel to EA,

$$BE : CE = BA : AF. \qquad \text{[VI. 2.}$$

And, since CF is $\parallel$ to EA,

$$\angle CAE = \text{alt.} \angle FCA,$$

and $\qquad\qquad \angle DAE = \text{int. opp.} \angle AFC.$

But by hyp. $\qquad \angle CAE = \angle DAE$;

$$\therefore \angle FCA = \angle AFC, \text{ and } \therefore AF = AC.$$

Hence $\qquad\qquad BA : AF = BA : AC$;

$$\therefore BE : CE = BA : AC.$$

Conversely. Let E be such that $BE : CE = BA : AC$; then *it is required to prove that EA will bisect the ext. $\angle CAD$.*

Through C draw $CF \parallel$ to EA cutting BA in F.

Then $\qquad BE : CE = BA : AF.$ [VI. 2.

But $\qquad BE : CE = BA : AC.$ [*Hyp.*

$\qquad \therefore \; BA : AF = BA : AC,$ and $\therefore \; AF = AC.$

Hence $\qquad \angle AFC = \angle ACF.$

But, since CF is $\parallel$ to EA,

$\qquad \angle AFC = \angle DAE$ and $\angle ACF = \angle CAE.$

Hence $\qquad \angle DAE = \angle CAE.$

This extension of Prop. III. was not given by Euclid. It was, however, quoted by Pappus as a known theorem.

Ex. 1. Shew that, in an isosceles triangle, the bisector of the external vertical angle is parallel to the base. Shew that this agrees with III*.

Ex. 2. The internal and external bisectors of the angle BAC cut the base BC in D, E respectively and the circle ABC in F, G respectively, and X is the middle point of BC. Shew that FG is the diameter of the circle ABC perpendicular to BC; shew also that FC touches the $\odot ACD$ and GC touches the $\odot ACE$, and that $FC^2 = FD \cdot FA$ and $GC^2 = GA \cdot GE$.

Ex. 3. ACB is a right angle, and the bisectors of the angle ACB cut AB in D, E. Shew that, if O be the middle point of AB, OC touches the circle DCE.

Ex. 4. Construct a triangle having given the base, the vertical angle, and the angle the bisector of the vertical angle makes with the base.

Ex. 5. Construct a triangle having given the base and the position of the line bisecting the vertical angle.

PROPOSITION IV. Theorem.

In equiangular triangles, the sides about any pair of equal angles are proportionals.

Let ABC, DEF be equiangular triangles, having the angles A, B, C equal respectively to the angles D, E, F; then *it is required to prove that the sides about a pair of equal angles are proportionals.*
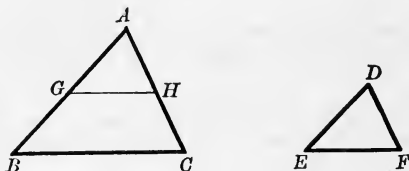
Apply the $\triangle EDF$ to the $\triangle ABC$ so that D falls on A and DE falls on AB, then DF will fall on AC since $\angle EDF = \angle BAC$, and E, F will fall at some points, G, H suppose, on AB, AC respectively.

Then, $\because$ $\angle AGH = \angle DEF$, and $\angle DEF = \angle ABC$, [*Hyp.*

$\angle AGH = \angle ABC$, and $\therefore$ GH is $\parallel$ to BC.

Hence $AB : AG = AC : AH$;   [VI. 2.

$\therefore$ alternately $AB : AC = AG : AH$ [VI. vii.

$= DE : DF$,   [*Const.*

so that the sides about the equal angles BAC and EDF are proportionals, and it can be proved in a similar manner that the sides about either of the other pairs of equal angles are proportionals.
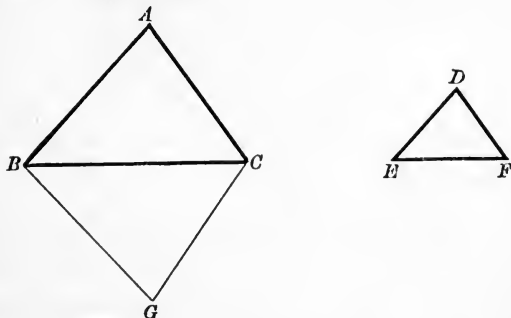
PROPOSITION V. Theorem.

If the sides of two triangles about each of their angles be proportionals, the triangles will be equiangular; and those angles will be equal which are opposite to homologous sides.

Let the triangles ABC, DEF have their sides proportional, so that

$$AB : BC = DE : EF,$$
$$BC : CA = EF : FD ;$$
$$\text{and} \therefore CA : AB = FD : DE ; \qquad \text{[VI. viii.}$$

then *it is required to prove that the triangles ABC and DEF are equiangular.*

At points B and C respectively make $\angle^s\ CBG$, BCG equal to $\angle^s\ DEF$, DFE respectively. Then will $\angle BGC = \angle EDF$.

And, since $\triangle^s\ BGC$ and EDF are equiangular,

$$BC : BG = FE : ED. \qquad \text{[VI. 4.}$$

But, by hyp.,

$$BC : BA = FE : ED ;$$
$$\therefore BC : BG = BC : BA ; \qquad \text{[VI. iv.}$$
$$\therefore BG = BA.$$

And, similarly, $CG = CA$.

Hence in the $\triangle^s\ BGC$, BAC

$$BA = BG, \ CA = CG \text{ and } BC \text{ is common ;}$$

$\therefore$ the two triangles are equiangular.

But the $\triangle^s\ BCG$ and EFD are equiangular ;

$\therefore$ the $\triangle^s\ BAC$ and EFD are equiangular.

PROPOSITION VI. Theorem.

If two triangles have one angle of the one equal to one angle of the other and the sides about these equal angles proportionals, the triangles will be similar.

In the triangles BAC, EDF, let $\angle BAC = \angle EDF$, and $BA : AC = ED : DF$; then *it is required to prove that the* $\triangle^s$ BAC, EDF *are similar.*
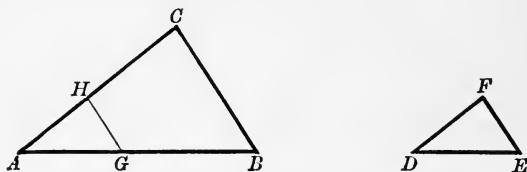
From AB, produced if necessary, cut off $AG = DE$; and from AC, produced if necessary, cut off $AH = DF$. Join GH.

Then, $\because$ $AG = DE$, $AH = DF$, and the included $\angle GAH =$ included $\angle EDF$, the $\triangle^s$ EDF, GAH are equal in all respects.

But $\qquad BA : AC = ED : DF$

$$= AG : AH;$$

$\therefore$ alternately $\qquad BA : AG = AC : AH.$

Hence GH is parallel to BC;

$$\therefore \angle ABC = \angle AGH = \angle DEF,$$

and $\qquad\qquad \angle ACB = \angle AHG = \angle DFE.$

$\therefore$ $\triangle^s$ BAC, EDF are equiangular, and are therefore similar. $\qquad\qquad\qquad\qquad\qquad\qquad$ [VI. 4.

Ex. 1. Shew that two isosceles triangles are similar if their vertical angles are equal.

Ex. 2. The length of the shadow of an upright stick 3 feet 6 inches long is 2 feet 10 inches, and at the same time the length of the shadow of a tree is 75 feet; what is the height of the tree?

Ex. 3. Shew that, if any two chords AB, CD of a circle intersect in the point O, the triangles AOC, BOD are similar.

Ex. 4. In two different circles the chords AB, CD subtend equal angles at the circumferences. Shew that the ratio of AB to CD is equal to the ratio of the radii of the circles.

Ex. 5. Equilateral triangles are inscribed in different circles; shew that their sides are in the ratio of the radii of the circles.

Ex. 6. Shew that, if the sides AB, AC of the triangle ABC be equal, and D be any point on the side BC, then will the circles ABD, ACD be equal.

Ex. 7. D is any point on the side BC of the triangle ABC; shew that the radii of the circles ADB, ADC are in the ratio of AB to AC.

PROPOSITION VII. THEOREM.

If two triangles have one angle of the one equal to one angle of the other, and the sides about one other angle in each proportional so that the sides opposite the equal angles are homologous, then will the third angles of the triangles be either equal or supplementary, and if they are equal the triangles will be similar.
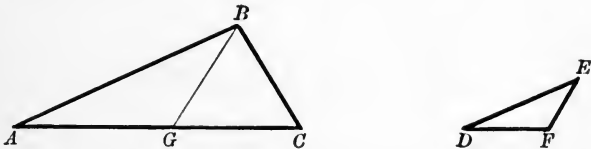
In the $\triangle^s$ BAC, EDF let $\angle BAC = \angle EDF$, and $AB : BC = DE : EF$, the sides BC, EF opposite to the equal angles being homologous; then *it is required to prove that the* $\angle^s$ BCA, EFD *are either equal or supplementary.*

If $\angle ABC = \angle DEF$, then will $\angle BCA = \angle EFD$, and the two $\triangle^s$ will be equiangular and therefore similar.

But, if $\angle ABC$ be not equal to $\angle DEF$, make $\angle ABG = \angle DEF$, BG cutting AC, produced if necessary, in the point G.

Then, $\because \angle BAG = \angle EDF$, and $\angle ABG = \angle DEF$, the remaining $\angle^s$ BGA, EFD will be equal and the $\triangle^s$ ABG, DEF will be equiangular.

Hence $\qquad AB : BG = DE : EF.$ $\qquad$ [VI. 4.

But $\qquad AB : BC = DE : EF;$

$\qquad \therefore AB : BG = AB : BC,$ and $\therefore BG = BC.$

$\qquad\qquad\qquad\qquad\qquad\qquad$ [VI. ii. and iv.

Hence $\qquad \angle BCA = \angle BGC$

$\qquad\qquad\qquad = $ supplement of $\angle BGA$

$\qquad\qquad\qquad = $ supplement of $\angle EFD.$

PROPOSITION VIII. THEOREM.

In a right-angled triangle the perpendicular drawn from the right angle to the base will divide the triangle into two parts which are similar to the whole and to each other.

Let BAC be a right-angled triangle, the $\angle BAC$ being the rt. $\angle$, and let AD be drawn $\perp$ to BC; then *it is required to prove that $\triangle^s\ DBA,\ DAC$ are similar to $\triangle\ ABC$ and to each other.*
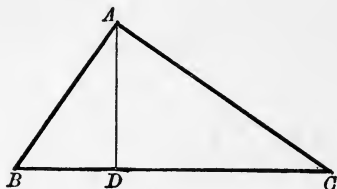
Since $\angle ADB$ is a rt. $\angle$, $\angle^s\ DBA$ and BAD are together equal to a rt. $\angle$.

But $\angle DAC$ and $\angle BAD$ together make up the rt. $\angle BAC$.

Hence

$$\angle^s\ DBA \text{ and } BAD = \angle^s\ DAC \text{ and } BAD;$$

$$\therefore \angle DBA = \angle DAC,$$

and similarly $\qquad \angle DAB = \angle DCA.$

Hence the three $\triangle^s\ DBA,\ DAC,\ ABC$ are equiangular, and are $\therefore$ similar.

Cor. *DA is a mean proportional between BD and DC.*

For, from the similar $\triangle^s\ BDA,\ ADC$, we have

$$BD : DA = DA : AC.$$

Also *CA is a mean proportional between CD and CB.*

For, from the similar triangles $ACB,\ DCA$,

$$BC : CA = CA : CD.$$

So also *AB is a mean proportional between BD and BC.*

Ex. 1. Two parallel tangents to a circle are cut by the tangent at the point P in the points K, L respectively. Shew that the radius of the circle is a mean proportional between PK and PL.

Ex. 2. Two circles touch externally at O, and P, Q are the points of contact of a common tangent which does not pass through O; shew that POQ is a right angle, and that PQ is a mean proportional between the diameters of the circles.

PROPOSITION IX. PROBLEM.

From a given straight line to cut off any assigned sub-multiple.

Let AB be the given straight line. Then *it is required to cut off any assigned sub-multiple of AB.*
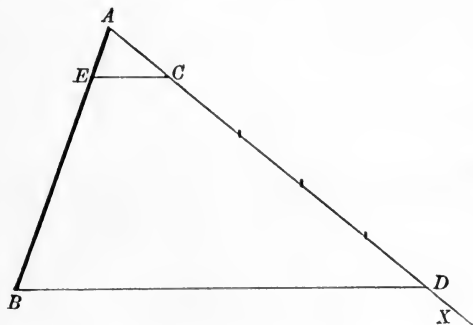
From A draw any indefinite straight line AX, and in AX take any point C.

Along AX set off lengths equal to AC, until a length AD is found which is the same multiple of AC that AB is of the required part.

Join BD, and through C draw a line $\parallel$ to DB so as to cut AB in the point E.

Then, since CE is $\parallel$ to DB,

$$AB : AE = AD : AC.$$

Hence AB is the same multiple of AE that AD is of AC.

$\therefore$ AE is the required sub-multiple of AB.

PROPOSITION X. Problem.

To divide a given straight line similarly to a given divided straight line.

Let AB be the given undivided straight line, and place AC the given straight line which is divided into any number of parts at the points E, F so that the two lines make any angle at the point A.
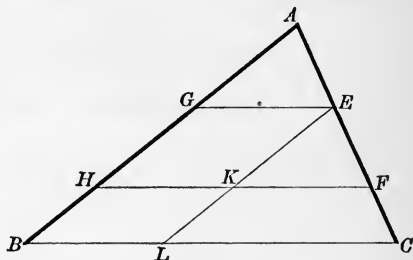
Join BC, and draw through E, F lines parallel to BC so as to cut AB in the points G, H respectively. Then AB will be divided in the required manner.

For, since EG is ∥ to FH,

$$AG : GH = AE : EF.$$

Let EKL be drawn parallel to AB so as to cut FH, CB in K, L respectively; then GK and HL are ∥ᵐˢ;

$$\therefore GH = EK \text{ and } HB = KL.$$

But, since FK is ∥ to CL,
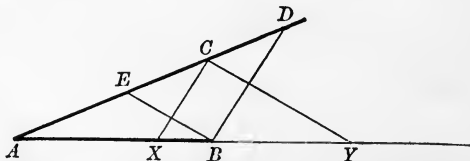
$$EK : KL = EF : FC;$$
$$\therefore GH : HB = EF : FC.$$

Ex. 1. Shew that the ratio of the perpendiculars from two given points A, B on any straight line which cuts AB in a fixed point is constant.

Ex. 2. *To divide a given straight line internally and externally in a given ratio.*

Let AB be the given straight line which it is required to divide in the ratio of $AC : CD$.

From CA cut off $CE = CD$. Join BD, BE and through C draw CX, CY parallel respectively to BD, BE and cutting AB, or AB produced, in X, Y. Then X, Y are the points required.

Ex. 3. Through two given points on a given circle draw two parallel chords which are in a given ratio.

Ex. 4. A, B are fixed points and P any other point on a given circle. Shew that if AP is produced to Q so that $PQ : PB$ is constant, the locus of Q is a circle through A, B.

Ex. 5. On two given straight lines OX, OY points A, B are taken respectively so that the sum of OA and OB is equal to a given length; shew that, if the parallelogram $OAQB$ be completed, the locus of Q is a straight line, and the locus of the middle point of AB is a straight line.

Ex. 6. ABC is an isosceles triangle, AB and AC being the equal sides, and any line is drawn cutting BC in D, CA in E and AB produced in F; shew that

$$CD : DB = CE : BF.$$

Ex. 7. Shew that, if a quadrilateral have two parallel sides one of which is double the other, the two diagonals intersect at a point of trisection.

Ex. 8. The straight lines AB, AC, AD meet in a point, and from any point P on AC the perpendiculars PE, PF are drawn to AB, AD respectively; shew that $PE : PF$ is constant for all positions of P on AC.

Ex. 9. Find a point O within the triangle ABC such that, if OD, OE, OF be the perpendiculars on the sides BC, CA, AB respectively, $OD : OE : OF$ may be equal to given ratios.

Ex. 10. Divide the triangle ABC into three triangles BOC, COA, AOB such that $\triangle BOC : \triangle COA : \triangle AOB$ may be equal to given ratios.

Ex. 11. Through the middle point of the side BC of the triangle ABC a straight line is drawn cutting the sides AB, AC respectively in the points X, Y; shew that $CY : YA = XB : XA$.

Ex. 12. Shew that, if D be the middle point of the side BC of the triangle ABC, and if any straight line be drawn through C, cutting AD in E and AB in F, then will $AE : ED = 2AF : FB$.

Ex. 13. Draw through a given point a straight line cutting the three given straight lines AX, AY, AZ in P, Q, R respectively so that $PQ : QR$ may be equal to a given ratio.

Ex. 14. Two given circles intersect in the points A, B and any other circle touches them both in the points P, Q respectively; shew that the tangents at P and Q meet on AB produced, and that $AP : BP = AQ : BQ$.

Ex. 15. AA', BB', CC' are the three diagonals of a complete quadrilateral, and O is the common point of the circum-circles of the triangles formed by the lines taken in threes; shew that BOC, $B'OC'$; COA, $C'OA'$ and AOB, $A'OB'$ are pairs of similar triangles, and that $OA \cdot OA' = OB \cdot OB' = OC \cdot OC'$. [Use VI. 16.]

Ex. 16. On the sides BC, CA, AB are taken the points D, E, F respectively such that $BD = 2DC$, $CE = 2EA$ and $AF = 2FB$. Also BE and CF meet in P, CF and AD meet in Q and AD and BE meet in R. Shew that $AR = RQ = 3QD$, and that $7 \triangle PQR = \triangle ABC$.

PROPOSITION XI.   Problem.

To find a third proportional to two given straight lines.

Let *AB* and *AC* be the two given straight lines.   *It is required to find a third proportional to AB and AC.*
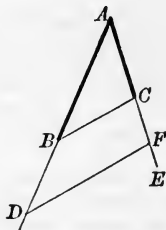
Produce *AB* and cut off *BD = AC*.

Join *BC*, and draw *DF* ∥ to *BC* so as to cut *AC* produced in *F*.

Then, since *BC* is ∥ to *DF*,

$$AB : BD = AC : CF.$$

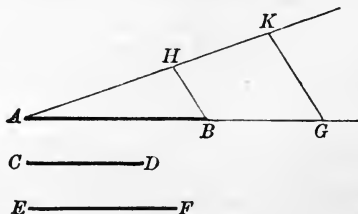But $$BD = AC ;$$

$$\therefore AB : AC = AC : CF.$$

Hence *CF* is the required third proportional.

PROPOSITION XII.   Problem.

To find a fourth proportional to three given straight lines.

Let *AB*, *CD*, *EF* be the three given straight lines.   *It is required to find a fourth proportional to AB, CD and EF.*

Produce AB and cut off $BG = CD$.

Draw any line AHK through A, and cut off $AH = EF$.

Join BH, and through G draw $GK \parallel$ to BH so as to cut the line AHK in the point K.

Then, since BH is $\parallel$ to GK,

$$AB : BG = AH : HK.$$

But $\qquad\qquad BG = CD \text{ and } AH = EF;$

$$\therefore AB : CD = EF : HK.$$

Hence HK is the required fourth proportional.

PROPOSITION XIII. PROBLEM.

To find a mean proportional between two given straight lines.
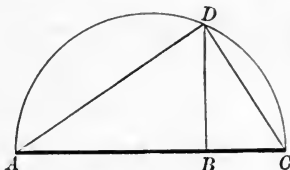
Place the given st. lines AB, BC in the same st. line.

On AC describe a semicircle, and through B draw a line $\perp^r AC$ so as to meet the circumference in the point D.

Then BD is the required mean proportional between AB and BC.

Join AD and DC.

Then, since ADC is an angle in a semicircle, it is a rt. $\angle$.

Hence sum of $\angle^s BAD$, $ADB =$ sum of $\angle^s CDB$, ADB;

$\therefore \angle BAD = \angle CDB$, and similarly $\angle ADB = \angle BCD$.

Hence the $\triangle^s BAD$, BDC are similar,

and $\qquad\qquad AB : BD = BD : BC.$

Thus BD is the required mean proportional between AB and BC.

Def. *If the ratio of a side of one polygon to a side of another be equal to the ratio of an adjacent side of the second to an adjacent side of the first, these four sides are said to be Reciprocally Proportional.*

PROPOSITION XIV. Theorem.

*Equal parallelograms, which have one angle of the one equal to one angle of the other, have their sides about the equal angles reciprocally proportional; and conversely, parallelograms which have one angle of the one equal to one angle of the other, and their sides about the equal angles reciprocally proportional, are equal to one another.*

Let AC, AF be equal $\|^{\text{ms}}$, which have the $\angle^{\text{s}}$ at A equal. Then *it is required to prove that the sides about the equal angles are reciprocally proportional*, that is that $DA : AG = EA : AB$.
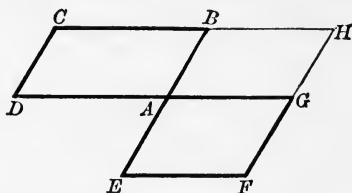
Let the sides DA, AG be placed in the same st. line; then EA and AB will also be in a st. line, since $\angle EAG = \angle DAB$.

Complete the $\|^{\text{m}}$ $BAGH$.

Then, since $\|^{\text{m}} CA = \|^{\text{m}} AF$,

$$\|^{\text{m}} CA : \|^{\text{m}} BG = \|^{\text{m}} AF : \|^{\text{m}} BG.$$

But, since $\|^{\text{ms}}$ CA, BG are between the same $\|^{\text{s}}$ DAG, CBH,

$$\|^{\text{m}} CA : \|^{\text{m}} BG = DA : AG.$$

Similarly $\qquad \|^{\text{m}} AF : \|^{\text{m}} BG = EA : AB.$

Hence $\qquad DA : AG = EA : AB.$

Next let the sides about the equal angles DAB, EAG of the $\|^{\text{ms}}$ AC, AF be reciprocally proportional, that is let

$$DA : AG = EA : AB.$$

The same construction being made, we have as above

$$DA : AG = \|^{\text{m}} CA : \|^{\text{m}} BG,$$

and $\qquad EA : AB = \|^{\text{m}} AF : \|^{\text{m}} BG.$

Hence $\qquad \|^{\text{m}} CA : \|^{\text{m}} BG = \|^{\text{m}} AF : \|^{\text{m}} BG,$

and therefore $\qquad \|^{\text{m}} CA = \|^{\text{m}} AF.$

PROPOSITION XV. Theorem.

Equal triangles which have one angle of the one equal to one angle of the other, have their sides about the equal angles reciprocally proportional; and conversely, triangles which have one angle of the one equal to one angle of the other, and their sides about the equal angles reciprocally proportional, are equal to one another.

Let ABC, DAE be equal $\triangle$s which have $\angle BAC = \angle DAE$. Then *it is required to prove that* $BA : AE = DA : AC$.

Place the $\triangle$s so that BA and AE are in the same st. line; then, since $\angle DAE = \angle BAC$, DA and AC will also be in a st. line.

Join BD.

Then, since $\triangle ABC = \triangle DAE$,

$$\triangle ABC : \triangle BAD = \triangle DAE : \triangle BAD.$$
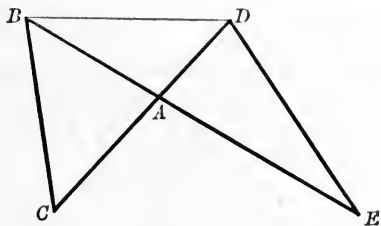
But $\triangle BAC : \triangle BAD = CA : AD$,   [VI. 1.

and $\triangle DAE : \triangle BAD = EA : AB$.

Hence $CA : AD = EA : AB$.   [VI. ii.

Next, let the sides about the equal angles BAC, DAE be reciprocally proportional, that is, let $CA : AD = EA : AB$.

Then, the same construction being made,

$$CA : AD = \triangle CAB : \triangle DAB,$$

and $$EA : AB = \triangle EAD : \triangle DAB.$$

Hence $\triangle CAB : \triangle DAB = \triangle EAD : \triangle DAB$,

and $\therefore \triangle CAB = \triangle EAD$.

PROPOSITION XVI. Theorem.

If four straight lines be proportionals, the rectangle contained by the extremes will be equal to the rectangle contained by the means; and conversely, if the rectangle contained by two straight lines be equal to the rectangle contained by two other straight lines, the four straight lines will be proportionals.

Let the four straight lines AB, CD, EF, GH be such that $AB : CD = EF : GH$; then *it is required to prove that the rectangle AB, GH is equal to the rectangle CD, EF.*
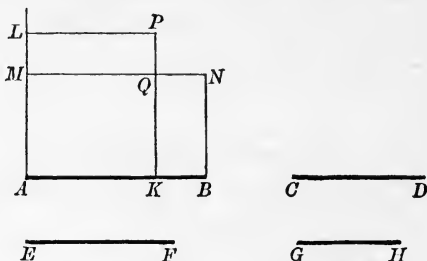
Let AB be greater than CD, and cut off from it $AK = CD$.

Through A draw a straight line perpendicular to AB, and on it set off AM, AL equal to GH, EF respectively, and complete the rectangles $BAMN$, $KALP$.

Then, since $AB : CD = EF : GH$,

and $AB > CD$, EF will be $> GH$.

Hence KP will cut MN, in Q suppose.

Then rect. AN : rect. $AQ = AB : AK$ [VI. 1.

$= AB : CD$,

and rect. AP : rect. $AQ = AL : AM$ [VI. 1.

$= EF : GH$.

But, by hyp., $AB : CD = EF : GH$;

∴ rect. AN : rect. AQ = rect. AP : rect. AQ

    [VI. ii.

∴ rect. AN = rect. AP,   [VI. iv.

that is, rect. AB, GH = rect. CD, EF.

Next, let rect. AB, GH = rect. CD, EF.

Then, the same construction being made,

$$\therefore \text{ rect. } AP = \text{rect. } AN,$$

rect. AP : rect. AQ = rect. AN : rect. AQ.

[VI. iii.

But rect. AP : rect. $AQ = AL : AM$ [VI. 1.

$$= EF : GH,$$

and rect. AN : rect. $AQ = AB : AK$ [VI. 1.

$$= AB : CD.$$

Hence $AB : CD = EF : GH.$ [VI. ii.

PROPOSITION XVII. Theorem.

If three straight lines be in continued proportion, the rectangle contained by the extremes is equal to the square on the mean; and conversely, if the rectangle contained by two straight lines is equal to the square of a third, the third straight line is a mean proportional to the other two.

Let the three straight lines AB, CD, GH be in continued proportion, so that $AB : CD = CD : GH$; then *it is required to prove that* rect. AB, GH = sq. on CD.
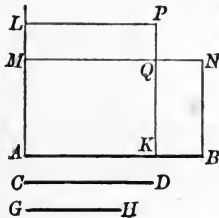
Let AB be greater than CD, then CD will be greater than GH.

Cut off $AK = CD$, and on AK describe the square $KALP$.

From AL cut off $AM = GH$, and complete the rectangle $AMNB$, and let PK cut MN in the point Q.

Then the proof is precisely the same as in the preceding proposition (of which this is a particular case), except that AP is now a square.

21—2

Ex. 1. The tangents at the ends of the diameter AB of a circle are cut by any other tangent in the points P, Q respectively; shew that $4AP \cdot BQ = AB^2$.

Ex. 2. P, Q are the points of contact of a common tangent of two circles which touch one another externally at O, and PL, QM are diameters of the circles. Shew that the triangles LPQ, PQM are similar and that $PQ^2 = PL \cdot QM$.

Ex. 3. Shew that, in the figure to [IV. 10],

$$\triangle ABD : \triangle ACD = \triangle ACD : \triangle BCD.$$

PROPOSITION XVIII. PROBLEM.

Upon a given straight line to describe a rectilineal figure similar and similarly situated to a given rectilinear figure.

Let $ABCDE$ be the given rectilinear figure and FG the given straight line. *It is required to describe on FG a rectilineal figure similar and similarly situated to $ABCDE$.*

Divide the figure $ABCDE$ into triangles by drawing lines from A to the other angular points.
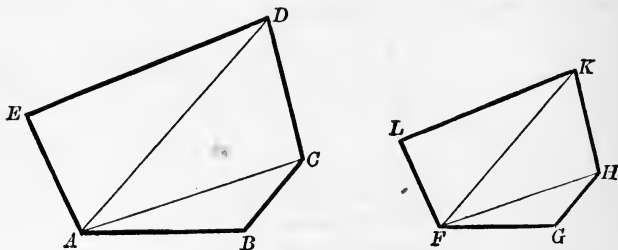
At F and G make the $\angle^s$ GFH, FGH equal respectively to the $\angle^s$ BAC, ABC.

Then at F and H make the $\angle^s$ HFK, FHK equal respectively to the $\angle^s$ CAD, ACD.

And at F and K make the $\angle^s$ KFL, FKL equal to the $\angle^s$ DAE, ADE.

Then $FGHKL$ will be the figure required.

For, since in each of the pairs of △s *FGH, ABC*; *FHK, ACD*; *FKL, ADE*; two angles of the one are by const. equal respectively to two angles of the other, the pairs of triangles must be equiangular.

Thus $\angle FGH = \angle ABC$,

$\qquad \angle GHK = $ sum of $\angle$s *FHG, FHK*

$\qquad\qquad = \ldots\ldots\ldots\ldots ACB, ACD = \angle BCD$,

$\qquad \angle HKL = $ sum of $\angle$s *HKF, FKL*

$\qquad\qquad = \ldots\ldots\ldots\ldots CDA, ADE = \angle CDE$,

$\qquad \angle KLF = \angle DEA$,

and $\qquad \angle GFL = $ sum of $\angle$s *GFH, HFK, KFL*

$\qquad\qquad = \ldots\ldots\ldots\ldots BAC, CAD, DAE = \angle BAE$.

Hence the figures *FGHKL, ABCDE* are *equiangular*.

Again, since △s *FGH, ABC* are equiangular, they are similar;

$$\therefore FG : AB = GH : BC = FH : AC.$$

And, since △s *FHK, ACD* are equiangular, they are similar;

$$\therefore FH : AC = HK : CD = FK : AD.$$

And, since △s *FKL, ADE* are equiangular, they are similar;

$$\therefore FK : AD = KL : DE = LF : EA.$$

Hence, in the figures *FGHKL, ABCDE*, the ratios *FG : AB, GH : BC, HK : CD, KL : DE* and *LF : EA* are all equal [and are also equal to the ratios *FH : AC, FK : AD*].

Hence the figures *FGHKL, ABCDE* are similar.

The same construction and proof would apply however many sides the given rectilinear figure might have.

N.B. It should be noticed that two polygons are similar when the ratios of all pairs of corresponding lines, *including diagonals as well as sides*, are equal. Also that two polygons are similar if all pairs of corresponding angles are equal, including the angles made by diagonals.

PROPOSITION XIX. Theorem.

Similar triangles are to one another in the duplicate ratio of their homologous sides.

Let the $\triangle^s$ ABC, DEF be similar, having the $\angle^s$ at A, B, C equal respectively to the $\angle^s$ at D, E, F, so that BC and EF are a pair of corresponding sides; then *it is required to prove that* $\triangle$ $ABC : \triangle$ DEF *is equal to the duplicate of the ratio* $AB : DE$.
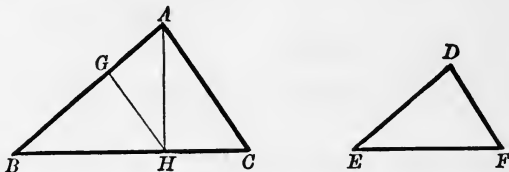
Let BC be $> EF$; then, since $BC : EF = AB : DE$, AB will be $> DE$.

From BC cut off $BH = EF$, and from BA cut off $BG = ED$. Join GH and HA.

Then in the $\triangle^s$ DEF, GBH,

DE, EF and included $\angle DEF$ are equal respectively to GB, BH and included $\angle GBH$.

Hence the $\triangle^s$ DEF, GBH are equal in all respects.

Hence $\angle BGH = \angle EDF = \angle BAC$;

         $\therefore$ GH is $\parallel$ to AC,

and $\therefore$ $BG : BA = BH : BC$.

Now $BG : BA = \triangle$ $BGH : \triangle BHA$,

and $BH : BC = \triangle$ $BHA : \triangle$ BAC.

    $\therefore$ $\triangle BGH : \triangle$ $BHA = \triangle$ $BHA : \triangle$ BAC.

Hence, by definition,

        $\triangle$ $BGH : \triangle$ $BAC = $ duplicate of $\triangle$ $BGH : \triangle BHA$

                       $ = $ duplicate of $BG : BA$

                       $ = $ duplicate of $ED : BA$.

Hence $\triangle$ $DEF : \triangle$ $ABC = $ duplicate of $ED : BA$.

PROPOSITION XIX*. Theorem.

Similar triangles are to one another as the squares on corresponding sides.

Let the △• *ABC*, *DEF* be similar, having the ∠• at *A, B, C* equal respectively to the ∠• at *D, E, F*.

Construct the squares *BK*, *EM* on the corresponding sides *BC, EF*.
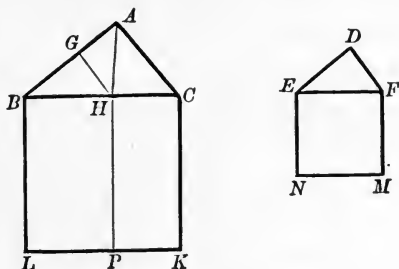
From *BC* cut off *BH=EF*, and from *BA* cut off *BG=ED*. Join *GH, AH*.

Through *H* draw *HP* ∥ to *BL* so as to cut *LK* in *P*.

Then, *BG, BH* and the included ∠ *GBH* are equal respectively to *DE, EF* and the included ∠ *DEF*.

Hence △• *GBH, DEF* are equal in all respects;

$$\therefore BG : BA = ED : BA = EF : BC = BH : BC.$$

Now $\triangle ABC : \triangle ABH = BC : BH,$

and   sq. *BK* : rect. *BP* = *BC* : *BH* ;

$$\therefore \triangle ABC : \triangle ABH = \text{sq. } BK : \text{rect. } BP.$$

And $\triangle ABH : \triangle GBH = AB : BG$;

$$\therefore \triangle ABH : \triangle DEF = BC : BH ;$$

But   rect. *BP* : sq. *EM* = *BL* : *EN* = *BC* : *BH* ;

$$\therefore \triangle ABH : \triangle DEF = \text{rect. } BP : \text{sq. } EM,$$

and $\triangle ABC : \triangle ABH = \text{sq. } BK : \text{rect. } BP$;

$$\therefore \triangle ABC : \triangle DEF = \text{sq. } BK : \text{sq. } EM.$$

PROPOSITION XX. Theorem.

A pair of similar polygons may be divided into the same number of similar triangles, having the same ratio to one another that the polygons have, and the ratio of the polygons is the duplicate of the ratio of homologous sides.

Let the polygons $ABCDE$, $FGHKL$ be similar, AB and FG, BC and GH, &c. being corresponding sides. Then *it is required to prove that the polygons can be divided into pairs of similar triangles whose ratio and also the ratio of the whole polygons is the duplicate of the ratio of AB to FG, or of BC to GH, &c.*
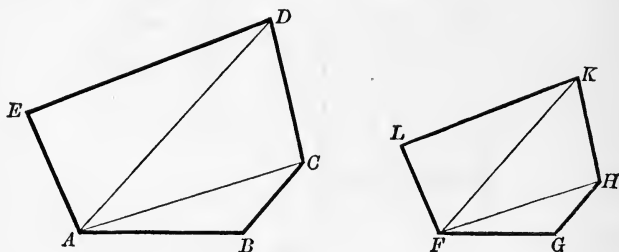
Join AC, AD, FH and FK.

Then, since the polygons are similar,

$$\angle ABC = \angle FGH \text{ and } AB : FG = BC : GH.$$

Hence the $\triangle^s$ ABC, FGH are similar;

$$\therefore \angle ACB = \angle FHG,$$

and $\qquad AC : FH = BC : GH.$

Again, since the polygons are similar, $\angle BCD = \angle GHK$, and we have proved that $\angle ACB = \angle FHG$;

$$\therefore \angle ACD = \angle FHK.$$

Also $\qquad BC : GH = CD : HK$;

and we have proved that $BC : GH = AC : FH$;

$$\therefore AC : FH = CD : HK,$$

i.e. the sides about the equal $\angle^s$ ACD and FHK are proportionals.

Hence the $\triangle^s$ ACD, FHK are similar.

It can clearly be proved in a similar manner that $\triangle^s ADE$, *FKL* are similar, and the proof would apply however many sides the polygons might have.

Now, since the ratio of similar $\triangle^s$ is the duplicate of the ratio of corresponding sides,

$$\triangle \, ABC : \triangle \, FGH = \text{duplicate of ratio } AB : FG,$$

and $\qquad \triangle \, ACD : \triangle \, FHK = \quad,, \qquad,, \qquad,, \quad CD : HK$

$$= \quad,, \qquad,, \qquad,, \quad AB : FG,$$

and $\qquad \triangle \, ADE : \triangle \, FKL = \quad,, \qquad,, \qquad,, \quad EA : LF$

$$= \quad,, \qquad,, \qquad,, \quad AB : FG.$$

Hence the ratio of each of the triangles into which *ABCDE* is divided to the corresponding triangle in *FGHKL* is the duplicate of the ratio $AB : FG$.

Hence (Prop. vi) the ratio of the sum of all the triangles which make up *ABCDE* to the sum of all the triangles which make up *FGHKL* is the duplicate of the ratio $AB : FG$, so that

fig. $ABCDE$: fig. $FGHKL = $ duplicate of ratio $AB : FG$.

Cor. I. *The ratio of the perimeters of similar polygons is equal to the ratio of any pair of corresponding sides.*

Cor. II. *Two similar polygons which are equal in area, are equal in all respects.*

For the polygons can be divided into the same number of similar triangles, which have the same ratios as the polygons ; and it is easily seen that, if similar triangles are equal in area, they are equal in all respects.

It should be noticed that we have only proved the theorem when the two figures are divided into triangles by lines drawn from corresponding vertices. It would be a good exercise for the student to prove that, if *any* point *X* be taken within *ABCDE* and the lines *XA*, *XB*, ... be drawn, a point *Y* can be found within the figure *FGHKL* such that if *YF*, *YG*, ... be drawn, the pairs of $\triangle^s AXB$ and *FYG*, *BXC* and *GYH*, &c. will be similar.

PROPOSITION XXI. Theorem.

Rectilinear figures, which are similar to the same rectilinear figure, are similar to one another.

Let each of the rectilinear figures *ABCD, EFGH* be similar to the figure *XYZW*; then *it is required to prove that the figures ABCD, EFGH are similar to one another.*
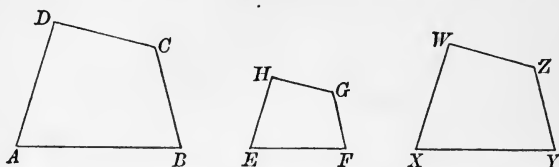
By definition, if two rectilinear figures are similar corresponding angles are equal, and all pairs of corresponding sides are in the same ratio.

Hence $\angle A = \angle X$, and $\angle X = \angle E$;

$\therefore \angle A = \angle E$, and similarly $\angle B = \angle F$, &c. (1)

Again $AB : XY = BC : YZ =$ &c.,

and $XY : EF = YZ : FG =$ &c.;

$\therefore AB : EF = BC : FG =$ &c. [VI. viii.] (2)

From (1) and (2) it follows that *ABCD* and *EFGH* are similar.

PROPOSITION XXII. Theorem.

If four straight lines are proportionals, and any similar and similarly situated rectilinear figures be described on the first and second and any similar and similarly situated rectilinear figures be also described on the third and fourth, then will the four figures be proportionals; and conversely, if four rectilinear figures so described be proportionals, the straight lines on which they are described will also be proportionals.

Let the st. lines *AB, CD, EF, GH* be such that

$$AB : CD = EF : GH,$$

and let the similar and similarly situated rectilinear figures *P, Q* be described on *AB, CD* respectively, and the similar and similarly situated figures *R, S* be described on *EF, GH* respectively; then *it is required to prove that P : Q = R : S,*

To AB, CD take a third proportional XY, and to EF, GH take a third proportional ZW.
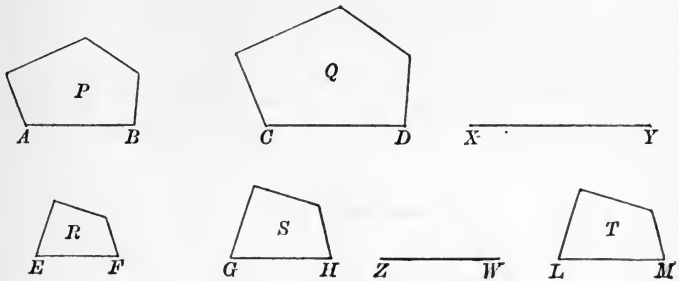
Then, since $AB : CD = EF : GH$,

and $CD : XY = GH : ZW$;

$\therefore AB : XY = EF : ZW.$ [VI. viii.

But $P : Q = $ duplicate of ratio $AB : CD$

$= AB : XY$,

and similarly $R : S = EF : ZW.$

Hence $P : Q = R : S.$

Conversely, let $P : Q = R : S.$

To AB, CD, EF take a fourth proportional LM, and on LM describe the figure T similar and similarly situated to the figure R.

Then, by the above, since

$$AB : CD = EF : LM;$$

$$\therefore P : Q = R : T.$$

But $P : Q = R : S;$

$\therefore R : T = R : S$, and $\therefore T = S.$

But the similar figures T and S which are equal in area, must be equal in all respects; [VI. 20 Cor. ii.

$$\therefore LM = GH.$$

But $AB : CD = EF : LM;$

$\therefore AB : CD = EF : GH.$

PROPOSITION XXIII. Theorem.

Equiangular parallelograms have to one another the ratio which is compounded of the ratios of their sides.

Let AC, AF be equiangular $\parallel^{ms}$, which have the $\angle^s$ at A equal. Then *it is required to prove that the ratio of the $\parallel^{ms}$ is equal to the ratio compounded of the ratios of their sides which contain the equal angles.*
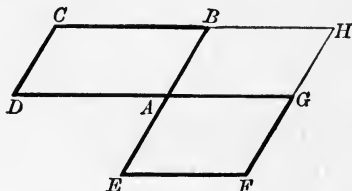
Let the sides DA, AG be placed in a straight line; then EA and AB will also be in a st. line, since $\angle EAG = \angle DAB$.

Complete the $\parallel^m BAGH$.

Then $\parallel^{ms} AC$, AH are between the same parallels DAG, CBH;

$$\therefore \parallel^m AC : \parallel^m AH = DA : AG.$$

Similarly $\qquad \parallel^m AH : \parallel^m AF = BA : AE.$

Hence $\parallel^m AC : \parallel^m AF =$ ratio compounded of $\parallel^m AC : \parallel^m AH$ and $\parallel^m AH : \parallel^m AF$

$\qquad = $ *ratio compounded of ratios equal to these ratios* *

$\qquad = $ ratios compounded of $DA : AG$ and $BA : AE$.

---

* Euclid's proof is slightly different from the above, but in both proofs it is *assumed* that the ratio compounded of any two given ratios is equal to the ratio compounded of any two other ratios which are equal respectively to the given ratios. In the Geometry of the Association for the Improvement of Geometrical Teaching this assumption is included in the *definition* of the compounding of ratios.

PROPOSITION XXIII*.

The ratio of equiangular parallelograms is equal to the ratio of the rectangles contained by their adjacent sides.

Let *ABCD*, *EFGH* be equiangular parallelograms which have the angles at *A* and *E* equal to one another.

Then *it is required to prove that*

$$\|^{\mathrm{m}} ABCD : \|^{\mathrm{m}} EFGH = \text{rect. } AB, AD : \text{rect. } EF, EH.$$
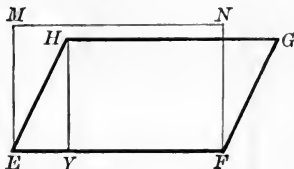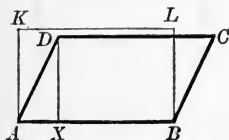
Draw *AK*, *EM* ⊥ʳ to *AB*, *EF* respectively, making *AK = AD* and *EM = EH*. Complete the rectangles *BAKL* and *FEMN*.

Draw *DX*, *HY* ⊥ʳ to *AB*, *EF* respectively.

Then, since ∠*DAX* = ∠*HEY* and ∠*AXD* = ∠*EYH*, the △ˢ *DAX* and *HEY* are similar;

$$\therefore DX : DA = HY : HE.$$

But $\|^{\mathrm{m}} DB : \|^{\mathrm{m}} KB = DX : KA = DX : DA,$ [VI. 1, Cor.

and $\|^{\mathrm{m}} HF : \|^{\mathrm{m}} MF = HY : ME = HY : HE.$

Hence $\|^{\mathrm{m}} DB : \|^{\mathrm{m}} KB = \|^{\mathrm{m}} HF : \|^{\mathrm{m}} MF;$

∴, alternately, $\|^{\mathrm{m}} DB : \|^{\mathrm{m}} HF = \text{rect. } KB : \text{rect. } MF.$

Note. Euclid's Prop. XXIII. conveys no clear idea, for Euclid offers no suggestion as to the nature of the operation by which ratios can be compounded. From the above form of the theorem we learn that the ratio compounded of the ratios of two pairs of straight lines is equal to the ratio of the rectangle contained by the antecedents to the rectangle contained by the consequents of the pairs of ratios.

PROPOSITION XXV. Problem.

To describe a rectilineal figure which shall be similar to one and equal to another given rectilineal figure.

Let *ABCD* and *EFGH* be the given rectilineal figures. Then *it is required to describe a rectilineal figure similar to ABCD and equal to EFGH.*
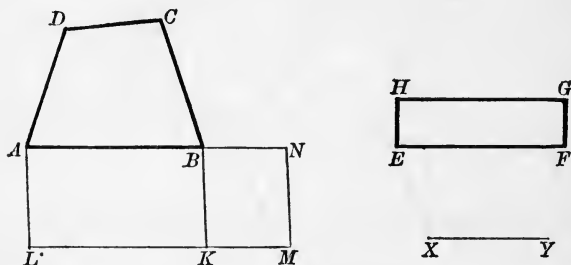
On *AB* describe the rectangle *ABKL* equal in area to the figure *ABCD*.

Also on *BK* describe the rectangle *BKMN* equal to *EFGH*, so that *AB*, *BN* may be in the same st. line.

Find *XY* the mean proportional between *AB* and *BN*.

Then the rect. figure described on *XY* similar to *ABCD* will be equal to *EFGH*, and will be the figure required.

For figure *ABCD* : sim^r. fig. on *XY* = duplicate of *AB* : *XY*

$$= AB : BN;$$

since $AB : XY = XY : BN.$

But *AB* : *BN* = rect. *AK* : rect. *BM*

$$= \text{figure } ABCD : \text{figure } EFGH. \qquad [Const.$$

Hence

fig. *ABCD* : sim^r. fig. on *XY* = fig. *ABCD* : fig. *EFGH*.

Hence figure on *XY* similar to figure *ABCD* is equal to the figure *EFGH*.

PROPOSITION XXIV. Theorem.

Parallelograms about the diagonal of any parallelogram are similar to the whole parallelogram and to one another.

Let $ABCD$ be a $\|^m$, and EH, FG $\|^{ms}$ about the diagonal AC. Then *it is required to prove that* $\|^{ms}$ *EH*, *FG*, *BD are all similar.*
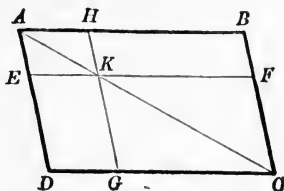
In the $\|^{ms}$ EH, DB,

$$\angle EAH = \angle DAB,$$

$$\angle AHK = \angle ABC, \text{ since } HKG \text{ and } BC \text{ are } \|,$$

and the opp. $\angle^s$ of $\|^{ms}$ are equal;

$\therefore$ $\|^{ms}$ EH, BD are equiangular.

Again, since HK is $\|$ to BC and EK to DC,

$$AH : AB = AK : AC$$

$$= AE : AD.$$

And, since opp. sides of $\|^{ms}$ are equal, it follows that the ratios of all pairs of corresponding sides of the $\|^{ms}$ EH, BD are equal.

Hence $\|^{ms}$ EH, BD are similar.

And it can be proved in the same manner that $\|^{ms}$ FG, BD are similar.

But rect. figures which are similar to the same figure are similar to one another, so that the three $\|^{ms}$ EH, **FG**, BD are all similar.

PROPOSITION XXVI. Theorem.

If two similar parallelograms have a common angle, and be similarly situated, their diagonals will coincide.

Let the ||^{ms} $ABCD$, $AHKE$ which have the $\angle$^s at A common be similar and similarly situated, so that AB and AH are corresponding sides. Then *it is required to prove that the diagonals AC and AK are coincident.*
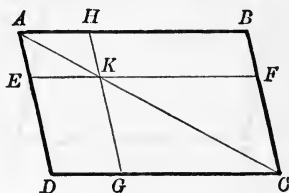
Since the ||^{ms} BD, HE are similar, the sides about the equal $\angle$^s ABC, AHK are proportionals.

Thus $\angle ABC = \angle AHK$, and $AB : AH = BC : HK$;

$\therefore \triangle$^s ABC, AHK are similar; and $\therefore \angle BAC = \angle BAK$,

so that the st. lines AC, AK coincide.

Cor. *If two similar polygons have a common angle, and be similarly situated, all their diagonals through the common angle will coincide.* [See figure on page 328.]

PROPOSITION XXX. Problem.

To cut a given straight line in extreme and mean ratio.

Let AB be the given st. line; then *it is required to divide it into two parts at the point C so that $AB : AC = AC : BC$.*

If $AB : AC = AC : BC$,

rect. AB, BC = sq. on AC.

Hence the problem is the same as that given in Book II., Proposition 11.

N.B. Propositions XXVII., XXVIII. and XXIX. are omitted. Of these XXVII. is unimportant, and XXVIII. and XXIX. have already been given on page 224.

PROPOSITION XXXI. Theorem.

In a right-angled triangle, the rectilinear figure described upon the side opposite to the right angle is equal to the sum of the similar and similarly situated figures described on the sides containing the right angle.

Let ABC be the right-angled $\triangle$, having the rt. $\angle BAC$, and let the three similar and similarly situated rectilinear figures X, Y, Z be described on the sides BC, CA, AB respectively; then *it is required to prove that the figure X is equal to the sum of the figures Y and Z.*
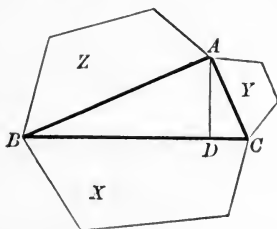
Draw $AD \perp^r$ to BC.

Then we know that $\triangle^s ABC$, DAC are similar, so that

$$BC : CA = CA : CD ;$$

$\therefore BC : CD =$ the duplicate of the ratio $BC : CA$.

But since the figures X and Y are similar and similarly situated,

$$\text{fig. } X : \text{fig. } Y = \text{the duplicate of the ratio } BC : CA$$
$$= BC : CD.$$

Similarly fig. X : fig. $Z = BC : BD$.

Hence fig. X : sum of figures Y and $Z = BC$: sum of CD and BD.

But BC is equal to the sum of CD and BD;

$\therefore$ fig. X is equal to the sum of figures Y and Z.

PROPOSITION XXXIII. Theorem.

In equal circles angles, whether at the centre or the cir-cumference, have the same ratio as the arcs on which they stand. Also the areas of sectors are in the same ratio as their angles.

Let *AB*, *LM* be arcs of equal circles and let *AOB*, *LSM* be two angles at their centres. Then *it is required to prove that* $\angle AOB : \angle LSM = arc\ AB : arc\ LM$, *and also that*

$$sector\ AOB : sector\ LSM = \angle AOB : \angle LSM.$$
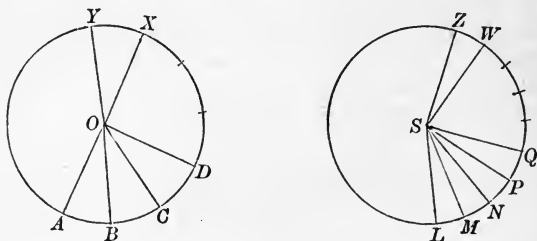
From *O* draw any number of radii *OC, OD, ..., OX, OY* making the angles *BOC, COD, ..., XOY* each equal to $\angle AOB$.

Also from *S* draw any number of radii *SN, SP, SQ, ..., SW, SZ* making the angles *MSN, NSP, PSQ, ..., WSZ* each equal to $\angle LSM$.

Then, since $\angle AOB = \angle BOC = \angle COD = ...,$

$$arc\ AB = arc\ BC = arc\ CD = ...$$

Hence the arc *ABCY* is the same multiple of the arc *AB* that the $\angle AOY$ is of the $\angle AOB$.

Similarly the arc *LMNZ* is the same multiple of the arc *LM* that the $\angle LSZ$ is of the $\angle LSM$.

Moreover, since the two circles have equal radii,

$$\text{the arc } ABY \gtreqless \text{arc } LMZ$$

according as $\qquad$ the $\angle AOY \gtreqless \angle LSZ.$

Hence of four magnitudes, namely the $\angle AOB$, the $\angle LSM$, the arc AB and the arc LM, we have taken *any* equimultiples of the first and third, and also *any* equimultiples of the second and fourth; and we have shewn that the multiple of the first is always greater than, equal to, or less than the multiple of the second according as the multiple of the third is greater than, equal to, or less than the multiple of the fourth.

Hence by definition

$$\text{arc } AB : \text{arc } LM = \angle AOB : \angle LSM.$$

Since an angle at the circumference of a circle is half the angle at the centre on the same arc, the ratio of the angles at the circumferences which stand on the arcs AB, LM respectively is equal to $\angle AOB : \angle LSM$, and therefore also equal to arc AB : arc LM.

Again, since in the same circle, or in equal circles, sectors with equal angles are equal*, it follows that sector AOY is the same multiple of sector AOB that $\angle AOY$ is of $\angle AOB$, and that sector LSZ is the same multiple of sector LSM that $\angle LSZ$ is of $\angle LSM$.

Moreover $\qquad$ sector $AOY \gtreqless$ sector LSZ

according as $\qquad$ $\angle AOY \gtreqless \angle LSZ.$

Hence by definition

$$\text{sector } AOB : \text{sector } LSM = \angle AOB : \angle LSM.$$

N.B. In the above proof it will be seen that angles not merely greater than two right angles, but greater than any number of right angles, must be considered; for the proof is invalid unless we can take **any** multiples whatever (millions, for example) of the angles AOB and LSM.

It is sometimes asserted that Euclid did not recognise angles greater than two right angles, but this is not true. He, however, omitted to give an extended definition of an angle, when it was required by him in III. 20 and again in VI. 33.

* This is obvious by superposition.

ADDITIONAL PROPOSITIONS.

I.   *If an angle of a triangle be bisected by a straight line which cuts the opposite side, the sum of the rectangle contained by the two segments of that side and the square on the bisecting line is equal to the rectangle contained by the other two sides of the triangle.*

Let the ∠ *ABC* of the △ *ABC* be bisected by the line *BD*, which cuts *AC* in *D*.   Then *it is required to prove that the sum of rect. AD, DC and sq. on BD is equal to the rect. AB, BC.*
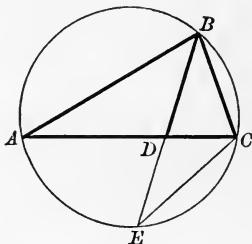
Describe the circle *ABCE* about the △ *ABC*, and produce *BD* to cut the circumference at the point *E*.   Join *EC*.

Then, by hyp.,   ∠ *ABD* = ∠ *EBC*

and   ∠ *BAD* = ∠ *BEC*,

for they are in the same segment.

Hence △ˢ *ABD, EBC* are equiangular and are ∴ similar, so that

$$AB : BD = EB : BC;$$ [VI. 4.

$$\therefore \text{ rect. } AB, BC = \text{rect. } BD, BE$$ [VI. 16.

$$= \text{rect. } BD, DE \text{ and sq. on } BD$$ [II. 3.

$$= \text{rect. } AD, DC \text{ and sq. on } BD.$$

Similarly, if the bisector of the exterior angle at *B* cut the base at *F* and the circle again in *G*, it will be easily seen that the triangles *ABF*, *GBC* are similar, and therefore

$$AB : BF = GB : BC;$$

$$\therefore AB \cdot BC = BF \cdot GB = BF \cdot GF - BF^2$$

$$= AF \cdot CF - BF^2.$$

II. *If a perpendicular be drawn from a vertex of a triangle to the opposite side, the rectangle contained by the other sides of the triangle is equal to the rectangle contained by the perpendicular and the diameter of the circle described about the triangle.*

Let BD be the perp. drawn from the vertex B to the side AC, and let BE be the diameter of the circle described about ABC. Then *it is required to prove that*

$$\text{rect. } AB, BC = \text{rect. } BD, BE.$$
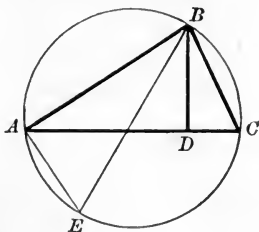
Describe the circle ABC, and let BE be the diameter through B. Join AE.

Then in the $\triangle^s \, BAE, \, BDC$

$$\angle AEB = \angle DCB \text{ in the same segment;}$$

and, since BE is a diameter,

$$\angle BAE = \text{rt. } \angle = \angle BDC.$$

Hence the $\triangle^s \, BAE, \, BDC$ are equiangular, and are $\therefore$ similar, so that

$$BA : BE = DB : BC; \qquad \text{[VI. 4.}$$

$$\therefore \text{rect. } BA, BC = \text{rect. } DB, BE. \qquad \text{[VI. 16.}$$

Since

$$2\triangle \, ABC = \text{rect. } CA, BD;$$

$$\therefore 2\triangle : AB \, . \, BC = CA \, . \, BD : AB \, . \, BC$$

$$= CA \, . \, BD : BE \, . \, BD$$

$$= CA : BE;$$

$$\therefore 4R \, . \, \triangle = BC \, . \, CA \, . \, AB,$$

where R is the radius of the circle.

III. PTOLEMY'S THEOREM.

The rectangle contained by the diagonals of a quadrilateral inscribed in a circle is equal to the sum of the rectangles contained by the two pairs of opposite sides.

Let $ABCD$ be any quadrilateral inscribed in a circle; then, if AC, BD be drawn, *it is required to prove that rect. AC, BD is equal to the sum of rect. AB, CD and rect. AD, BC.*
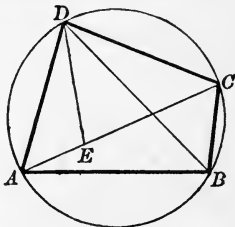
Make the $\angle ADE$ equal to the $\angle BDC$, E being on AC.

Then, in the $\triangle^s$ ADE, BDC, $\angle ADE = \angle BDC$, and $\angle DAE = \angle DBC$ in the same segment; $\therefore$ the $\triangle^s$ ADE, BDC are equiangular, and are $\therefore$ similar, so that

$$AD : AE = BD : BC;$$

$$\therefore \text{ rect. } AD, BC = \text{rect. } BD, AE. \qquad \text{[VI. 16.}$$

Again, to each of the equal $\angle^s$ ADE, BDC add $\angle EDB$, then $\angle ADB = \angle CDE$; also $\angle ABD = \angle ECD$ in the same segment; $\therefore \triangle^s$ ADB and EDC are equiangular, and are $\therefore$ similar, so that

$$AB : BD = EC : CD,$$

and $\therefore$ $\qquad\qquad$ rect. AB, $CD =$ rect. BD, EC.

Hence sum of rect. AD, BC and rect. AB, CD

$\qquad =$ sum of rect. BD, AE and rect. BD, EC

$\qquad =$ rect. BD, AC.

The following theorem, which includes the converse of Ptolemy's Theorem, is of great importance.

Theorem. *If A, B, C, D be any four points on a plane, and the lines AB, BC, &c. be joined, then any two of rect. AB, CD, rect. AC, BD and rect. AD, BC are together greater than the third, unless the four points lie on a circle.*

Make $\angle^s$ ADE, DAE equal respectively to the $\angle^s$ BDC, DBC.

Then the $\triangle^s$ ADE, BDC are equiangular, and are $\therefore$ similar, so that

$$AD : AE = BD : BC;$$
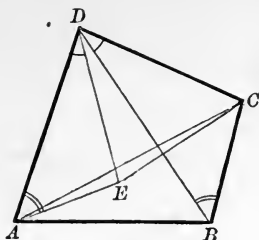
$$\therefore \text{ rect. } AD, BC = \text{rect. } BD, AE.$$

Also $\qquad AD : DE = BD : DC$;

or $\qquad AD : DB = ED : DC$.

But $\qquad \angle ADB = \angle EDC$, since $\angle ADE = \angle BDC$.

Thus in the $\triangle^s$ ADB, EDC the sides about a pair of equal angles are $::^1$, and $\therefore$ the $\triangle^s$ are similar, so that

$$AB : BD = EC : CD;$$

$$\therefore \text{rect. } AB, CD = \text{rect. } BD, EC.$$

Hence the sum of rect. AD, BC and rect. AB, CD

$\qquad$ = sum of rect. BD, AE and rect. BD, EC

$\qquad$ = rect. contained by BD and the sum of AE and EC.

Hence the sum of rect. AD, BC and rect. AB, CD is always greater than the rect. AC, BD unless the sum of AE and EC is equal to AC, that is unless the point E is on the line AC, in which case A, B, C, D are cyclic, for $\angle DAC$ would then be equal to $\angle DBC$. [See also page 374.]

Ex. 1. *If ABC be an equilateral triangle and D any point on its circumcircle, then will one of the three distances DA, DB, DC be equal to the sum of the other two.*

For, if DA cut BC, then by Ptolemy's Theorem,

$$DA \cdot BC = DB \cdot CA + DC \cdot AB.$$

Ex. 2. *If $ABCDE$ be a regular pentagon, and O any point on the arc of its circumcircle cut off by EA; then will $OA + OC + OE = OB + OD$.*

Apply Ptolemy's Theorem to the quadrilaterals $OABC$, $OBCD$, $OCDE$, $ODEA$, $OEAB$ in order; then we have, since all the sides and all the diagonals are equal,

$$d \cdot OB = a \cdot OA + a \cdot OC, \quad a \cdot OB + a \cdot OD = d \cdot OC,$$
$$d \cdot OD = a \cdot OC + a \cdot OE, \quad a \cdot OD = d \cdot OE + a \cdot OA$$

and $\qquad a \cdot OB = a \cdot OE + d \cdot OA,$

where a is the length of a side and d of a diagonal.

Hence $\qquad (2a + d)(OB + OD) = (2a + d)(OA + OC + OE)$,

and therefore $\qquad OB + OD = OA + OC + OE$.

The above method of proof can be extended to the case of any polygon with an odd number of sides. Thus we have the following theorem :—

If $ABCD \ldots\ldots XY$ be any regular polygon with an odd number of sides, and O be any point on its circumcircle between Y and A, the sum of OA, OC, $OE, \ldots\ldots OY$ is equal to the sum of OB, $OD, \ldots\ldots OX$.

IV. HARMONIC RANGES AND PENCILS.

1. Def. *Points which lie on a straight line are said to be* **Collinear,** *and straight lines which pass through a point are said to be* **Concurrent.** *A set of points lying on a straight line is called a* **Range,** *and a set of straight lines passing through a point is called a* **Pencil.**

Four collinear points A, B, C, D are said to form a **Harmonic Range** when

$$AB : BC = AD : CD,$$

that is when AC is divided internally and externally in the same ratio at the points B and D.

[Since $\qquad\qquad AB : BC = AD : CD$;

$\therefore$ alternately, $\qquad AB : AD = BC : CD,$

so that BD is also divided internally and externally in the same ratio at the points A and C.]

If A, B, C, D form a harmonic range, the pair A and C, and also the pair B and D, are called **harmonic conjugates.**

2. *If A, B, C, D form a harmonic range, and U, V be the middle points of AC, BD; then will $UC^2 = UB \cdot UD$ and $VB^2 = VA \cdot VC$.*

If A, B, C, D is a harmonic range

$$AB : BC = AD : CD.$$
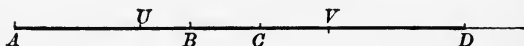
Hence $\qquad AB + BC : AB - BC = AD + CD : AD - CD.$

Hence $\qquad\qquad 2UC : 2UB = 2UD : 2UC$;

$$\therefore\ UC^2 = UB \cdot UD.$$

And, since $\qquad AB : BC = AD : CD,$

$\therefore$ alternately, $\qquad AB : AD = BC : CD$;

$\qquad \therefore\ AB + AD : AD - AB = BC + CD : CD - BC.$

$$\therefore\ 2AV : 2BV = 2BV : 2CV;$$

$$\therefore\ BV^2 = AV \cdot CV.$$

Conversely, if U is the middle point of AC, and $UC^2 = UB \cdot UD$; then A, B, C, D must be a harmonic range.

For, since $\quad UC^2 = UB \cdot UD$, $\quad 2UB : 2UC = 2UC : 2UD$;

$\qquad$ i.e. $AB - BC : AB + BC = AD - CD : AD + CD$;

$$\therefore\ 2AB : 2BC = 2AD : 2CD,$$

$$\therefore\ AB : BC = AD : CD.$$

3. *If A, B, C, D is a harmonic range and O be any point, and if the line through C parallel to OA cut OB, OD in X, Y respectively; then will XC = CY.*
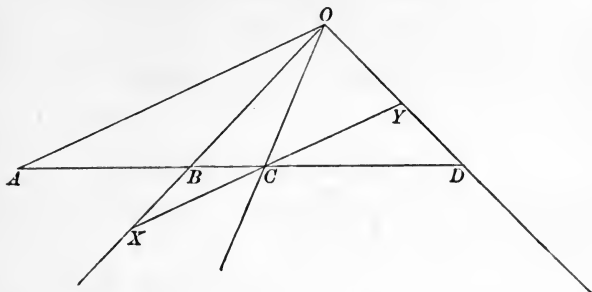
The $\triangle^s$ *ABO, CBX* are similar; $\therefore$ *AB* : *BC* = *AO* : *XC*.
The $\triangle^s$ *ADO, CDY* are similar; $\therefore$ *AD* : *CD* = *AO* : *CY*.
But *AB* : *BC* = *AD* : *CD*, since *A, B, C, D* is harmonic;

$$\therefore AO : XC = AO : CY;$$
$$\therefore XC = CY.$$

4. *If A, B, C, D be a harmonic range and O be any point, and if the lines OA, OB, OC, OD be cut by any other straight line in the points A', B', C', D' respectively; then will A', B', C', D' be also a harmonic range.*
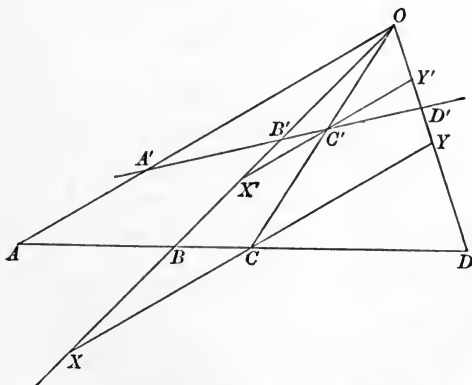
For, if *XCY* be drawn parallel to *OA* to cut *OB, OD* in *X, Y* respectively; then, by (3), *XC* = *CY*. And, if *X'C'Y'* be drawn parallel to *OA* to cut *OB, OD* in *X', Y'* respectively; then, since *XCY* is parallel to *X'C'Y'*, *X'C'* : *C'Y'* = *XC* : *CY*, so that *X'C'* = *C'Y'*.

Also $\qquad A'B' : B'C' = A'O : X'C'$,

and $\qquad A'D' : C'D' = A'O : C'Y'$;

$$\therefore \ A'B' : B'C' = A'D' : C'D',$$

so that A', B', C', D' form a harmonic range.

Def. *A pencil of four lines which are cut by any transversal in four points forming a harmonic range, is called a* **Harmonic Pencil.**

V. THE CIRCLE OF APOLLONIUS.

The locus of a point whose distances from two fixed points are in a constant ratio is a circle.

Let A, B be the given points.
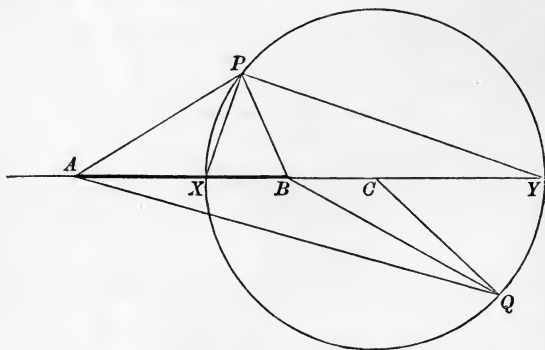
The points X, Y which divide the straight line AB internally and externally in the given ratio are clearly points on the locus. Let P be any other point on the locus. Then, by supposition,

$$AP : PB = AX : XB = AY : BY = \text{given ratio}.$$

Hence PX, PY are the internal and external bisectors of the angle APB, and are therefore at rt. $\angle$ˢ.

Hence P must be on the circle whose diameter is XY.

Conversely. Since $AX : XB = AY : BY$, if C be the middle point of XY,

$$CB . CA = CX^2 \qquad\qquad [\text{IV. 2}$$
$$= CQ^2,$$

where Q is any point on the circle XPY.

Hence CQ touches the circumcircle of the $\triangle ABQ$, and therefore

$$\angle CQB = \angle CAQ.$$

The triangles BCQ, QCA are therefore similar, and

$$AQ : BQ = CQ : CB.$$

Similarly, since $\quad\quad CA \cdot CB = CP^2,$

$$AP : BP = CP : CB = CQ : CB;$$

$$\therefore \ AQ : BQ = AP : BP.$$

Hence every point of the circle XPY is on the locus.

The following form of the theorem is important:

If A, B are two points on a straight line through the centre C of a circle such that rect. $CA \cdot CB$ is equal to the square on the radius of the circle, then will the ratio of the distances of any point on the circle from A and B be constant.

For, let Q be any point on the circle;

then since $\quad\quad CA \cdot CB = CQ^2,$

$$CA : CQ = CQ : CB.$$

Hence the $\triangle^s ACQ$, QCB are similar, and therefore

$$AQ : BQ = AC : CQ = \text{const.}$$

[See page 357.]

Ex. 1. Construct a triangle having given the base, the vertical angle, and the ratio of the other two sides.

Ex. 2. Construct a triangle having given the base, the ratio of the other two sides, and the length of the median corresponding to the base.

Ex. 3. Construct a triangle having given the base, the ratio of the other two sides, and the length of the bisector of the vertical angle cut off by the base.

Ex. 4. Construct a triangle having given the base, the ratio of the other two sides, and the area.

Ex. 5. ABC is a triangle, find a point O such that $AO : BO$ and $BO : CO$ may be equal to given ratios.

Ex. 6. A, B, C, D are four collinear points, find the locus of a point at which AB and CD subtend equal angles.

Let P be a point such that $\angle APB = \angle CPD$. Make $\angle APO = \angle PDC$, and let PO cut $ABCD$ in O. Then, since $\angle OPA = \angle PDC$, OP touches the circumcircle of APD. And

$$\angle BCP = \angle CPD + \angle CDP = \angle APB + \angle OPA = \angle OPB;$$

$\therefore$ OP also touches the $\odot BPC$. Hence $OA \cdot OD = OP^2 = OB \cdot OC$, and there is only **one** point on $ABCD$ such that $OA \cdot OD = OB \cdot OC$. Hence the locus of P is a circle whose centre is O.

VI. CEVA'S THEOREM.

If through any point O the lines AOD, BOE, COF be drawn so as to cut the sides BC, CA, AB of the triangle ABC, produced if necessary, in the points D, E, F respectively ; then the ratio compounded of the ratio of the segments of the sides taken in order will be equal to unity.
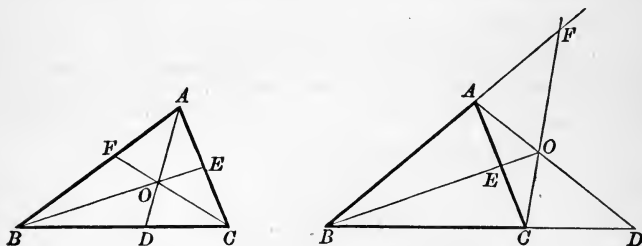
For $\triangle ADB : \triangle CDA = BD : DC,$

and $\triangle ODB : \triangle CDO = BD : DC;$

$\therefore \triangle AOB : \triangle COA = BD : DC.$

Similarly $\triangle COA : \triangle BOC = AF : FB,$

and $\triangle BOC : \triangle AOB = CE : EA.$

Hence the ratio compounded of $BD : DC$, $AF : FB$, and $CE : EA$ is equal to the ratio compounded of

$$\triangle AOB : \triangle COA, \quad \triangle COA : \triangle BOC \text{ and } \triangle BOC : \triangle AOB,$$

that is equal to $\triangle AOB : \triangle AOB$, which is equal to unity.

If the sides are divided into segments which are *commensurable*, the ratios of the segments can be expressed as the ratios of whole numbers, i.e. as vulgar fractions, and the ratio compounded of their ratios will then be the continued product of these vulgar fractions. Thus, Ceva's Theorem can, in this case, be enunciated in the form

$$\frac{BD}{DC} \cdot \frac{CE}{EA} \cdot \frac{AF}{FB} = 1.$$

The Converse of Ceva's Theorem is very important.

Conversely. *If points D, E, F be taken on the sides BC, CA, AB respectively of the triangle ABC so that*

$$\frac{BD}{DC} \cdot \frac{CE}{EA} \cdot \frac{AF}{FB} = 1,$$

then will AD, BE, CF meet in a point.

For, if AD, BE meet in O and CO cut AB in F', by Ceva's Theorem

$$\frac{BD}{DC} \cdot \frac{CE}{EA} \cdot \frac{AF'}{F'B} = 1.$$

Hence $AF : FB = AF' : F'B$, and therefore F and F' coincide.

It should be carefully noticed that **two**, if any, of the three points D, E, F must be on the productions of the sides on which they lie, so that of the three pairs BD, DC ; CE, EA ; AF, FB the segments are both in the same direction in all *three* cases, or in only *one* case ; and therefore, *if the segments of the same line be considered to be of* **opposite sign** *when they are drawn in* **opposite directions**, then the converse of Ceva's Theorem asserts that AD, BE, CF, will meet in a point, if

$$\frac{BD}{DC} \cdot \frac{CE}{EA} \cdot \frac{AF}{FB} = +1.$$

Ex. 1. If D, E, F are the middle points of the sides BC, CA, AB of the triangle ABC ; shew that AD, BE, CF will meet in a point.

Ex. 2. Shew that the bisectors of the three angles of a triangle meet in a point.

Ex. 3. Shew that the bisectors of two of the exterior angles of a triangle and of the remaining interior angle meet in a point.

Ex. 4. The in-circle of the triangle ABC touches the sides BC, CA, AB in the points D, E, F respectively ; shew that AD, BE, CF meet in a point.

Ex. 5. An escribed circle of the triangle ABC touches the sides BC, CA, AB, or these sides produced, in the points D, E, F respectively; shew that AD, BE, CF meet in a point.

Ex. 6. The three escribed circles of the triangle ABC touch the sides BC, CA, AB externally in the points D, E, F respectively; shew that AD, BE, CF are concurrent.

Ex. 7. Lines drawn through a point meet the sides BC, CA, AB of the triangle ABC in X, Y, Z and on these sides the points X', Y', Z' are taken such that $BX = X'C$, $CY = Y'A$ and $AZ = Z'B$. Shew that AX', BY', CZ' are concurrent.

Ex. 8. The three lines AO, BO, CO meet BC, CA, AB respectively in D, E, F and the circle DEF cuts BC, CA, AB again in D', E', F' respectively; shew that AD', BE', CF' will meet in a point.

VII. MENELAUS' THEOREM.

If a straight line cut the sides of a triangle, produced if necessary, the ratio compounded of the ratio of the segments of the sides, taken in order, is equal to unity.
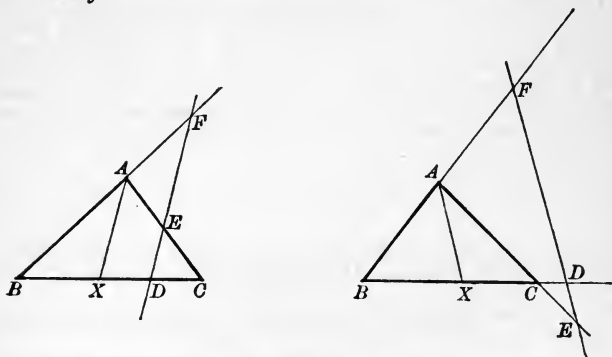
Let a st. line cut the sides BC, CA, AB of the $\triangle ABC$ in D, E, F respectively. Then we have to prove that the ratio compounded of $BD : DC$, CE to EA and AF to FB is equal to unity.

Through A draw $AX \parallel$ to DEF and cutting BC in X.

Then, since AX is $\parallel$ to DEF,

$$CE : EA = CD : DX$$

and also $AF : BF = XD : BD.$

Hence the ratio compounded of $CE : EA$ and $AF : BF$ is equal to the ratio compounded of CD to DX and $DX : BD$, that is equal to the ratio $CD : BD$.

Hence the ratio compounded of $CE : EA$, $AF : BF$ and BD to CD is equal to the ratio compounded of $CD : BD$ and $BD : CD$, i.e. to the ratio $CD : CD$, which is unity.

If the sides are divided into segments which are *commensurable*, the ratios of the segments can be expressed as the ratios of whole numbers, i.e. as vulgar fractions, and the ratio compounded of their ratios will then be the continued product of these vulgar fractions.   Thus Menelaus' Theorem can, in this case, be enunciated in the form

$$\frac{BD}{DC} \cdot \frac{CE}{EA} \cdot \frac{AF}{BF} = 1.$$

The converse of the theorem can be proved in the same manner as the converse of Ceva's Theorem.

It should be carefully noticed that *one* or *three* of the points D, E, F must be on the productions of the sides on which they lie, so that of the three pairs BD, DC; CE, EA; AF, FB the segments are both in the same direction in *two* or in *none*; and therefore, *if the segments of the same line be considered to be of* **opposite sign** *when they are in* **opposite directions**, then the converse of the Theorem of Menelaus asserts that D, E, F lie on a straight line, if

$$\frac{BD}{DC} \cdot \frac{CE}{EA} \cdot \frac{AF}{FB} = -1.$$

From the theorems of Ceva and Menelaus it follows that, if AD, BE, CF are concurrent and if EF cuts BC in G, then will G, B, D, C form a harmonic range.
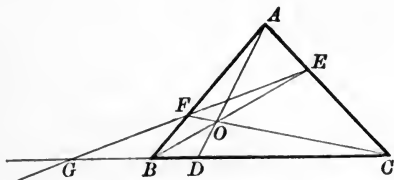
For, from Ceva's Theorem

$$\frac{BD}{DC} \cdot \frac{CE}{EA} \cdot \frac{AF}{FB} = 1,$$

and from Menelaus' Theorem

$$\frac{GB}{GC} \cdot \frac{CE}{EA} \cdot \frac{AF}{FB} = 1.$$

Hence

$$GB : GC = BD : DC.$$

Ex. 1. D, E, F are points on the sides BC, CA, AB respectively such that AD, BE, CF meet in a point, and the lines EF, FD, DE cut BC, CA, AB in X, Y, Z respectively; shew that X, Y, Z are collinear.

Ex. 2. The in-circle of the △ ABC touches BC, CA, AB respectively in D, E, F, and EF, FD, DE cut BC, CA, AB in X, Y, Z respectively; shew that X, Y, Z are collinear.

Ex. 3. The tangents at A, B, C to the circumcircle of the △ ABC meet the sides BC, CA, AB in the points D, E, F respectively; shew that D, E, F are collinear.

$$[BD : CD = BA^2 : CA^2.]$$

VIII. RADICAL AXIS.   CO-AXAL CIRCLES.

1.   Let the tangents OP, OQ to the two circles whose centres are A and B respectively, be equal in length.
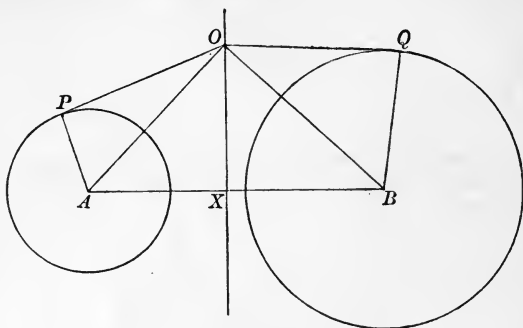
Join AP, AO, BQ, BO, and draw $OX \perp$ to AB.

Then, since $\angle$ s OPA, AXO, OQB are rt. $\angle$ s,

$$AX^2 + XO^2 = AO^2 = AP^2 + PO^2,$$

and $$XB^2 + XO^2 = BO^2 = BQ^2 + OQ^2.$$

Hence, if $$OP = OQ,$$

$$AX^2 - XB^2 = AP^2 - BQ^2 = \text{const.}$$

Hence X is a *fixed* point, and therefore O is on the fixed straight line $\perp$ to AB and which divides AB in X so that $AX^2 - XB^2$ is equal to the difference of the squares on the radii.

Conversely, if X be the point on AB such that

$$AX^2 - XB^2 = AP^2 - BQ^2,$$

and O be any point on the line through X perpendicular to AB, the tangents from O to the two circles (or, if O be within the circles, the rectangle contained by the segments of chords through O) will be equal.

Thus the locus of a point from which the tangents drawn to two given circles are equal in length is a straight line perpendicular to the line joining their centres.   This line is called the **Radical Axis** of the circles.

If the circles cut one another, the radical axis must pass through the points of intersection, for the tangents to the two circles from a point of intersection are equal, being both of zero length.   That the radical axis of intersecting circles is their common chord can, however, be at once seen from Euclid III. 36.

When two circles touch one another, their radical axis is the tangent at their point of contact.

It is easily seen that the intersection of the radical axis and the line of centres is within both circles or without both circles according as the circles do or do not cut one another.

A *point* may be regarded as a circle of zero radius, and the radical axis of the circle whose centre is A and radius AP and the *point-circle B* is the line perpendicular to AB and which cuts AB in X so that

$$AX^2 - XB^2 = AP^2.$$

2. *The radical axes of three circles taken in pairs meet in a point.*

For, if A, B, C be the centres of the three circles, and if the radical axis of the circles A and B meets the radical axis of the circles A and C in the point O; then, since O is on the radical axis of A and B, the tangents from O to the circles A and B are equal; and, since O is on the radical axis of the circles A and C, the tangents from O to the circles A and C are equal. Hence the tangents from O to the circles B and C must be equal, whence it follows that O is on the radical axis of the circles B and C.

If the centres A, B, C are on a straight line, the three radical axes are parallel.

Def. *The point of intersection of the three radical axes of three circles taken in pairs is called their* **Radical Centre.**

Since two straight lines can only meet in *one* point, unless they altogether coincide, it follows that three given circles can only have *one* radical centre, unless the given circles are co-axal.

Ex. *Draw the radical axis of two given circles.*

If the given circles intersect, the radical axis is their common chord.

If the given circles do not intersect, draw any other circle which intersects them both, and let PQ, RS be the common chords. Then, O, the point of intersection of PQ and RS, is on the radical axis of the given circles. Hence the required straight line is the line through O perpendicular to the join of the centres of the given circles.

Def. *Circles which are such that any pair have the same radical axis are called* **Co-axal Circles.**

Since the radical axis of two circles is $\perp^r$ to the line joining their centres, it follows that *the centres of a system of co-axal circles are collinear.*

3. It is obvious that all circles through two given points are co-axal; and that all circles co-axal with two intersecting circles must pass through their points of intersection.

Thus, if two circles of a co-axal system intersect one another, they will all intersect one another in the same two points; and, if two circles of a co-axal system do not intersect one another, no two circles of the system can intersect one another.

In a system of non-intersecting co-axal circles there are two *point-circles*, that is circles of zero radius. These are the points L, L' on the line of centres such that $LX^2 = L'X^2 = XB^2 - BQ^2$, where X is the intersection of the radical axis and the line of centres, and B is the centre and BQ the radius of any one of the circles.

Def. *In a system of co-axal circles, the two circles of zero radius are called the* **Limiting Points**.

In a system of intersecting co-axal circles there can be no limiting points, for a circle of zero radius cannot pass through the two points common to all the circles of the system.

When the co-axal circles touch one another at a point, the two limiting points coincide with the point of contact.

If O be any point on the radical axis of a system of co-axal circles, the tangents from O to all the circles of the system will be of equal length, and therefore the circle described with O as centre and the length of these tangents as radius will cut all the circles of the co-axal system at right angles.

Thus corresponding to a given system of co-axal circles there is a system of circles each of which cuts all the circles of the given system orthogonally. Moreover these orthogonal circles are co-axal with one another, their radical axis being the line of centres of the original co-axal system; for, if any number of circles cut a given circle orthogonally, the tangents to them from the centre of the given circle are all equal to its radius, so that the centre of the given circle is on the radical axis of every pair of the orthogonal circles.

Since $BL \cdot BL' = BX^2 - XL^2 = BQ^2$, it follows that *any circle through the limiting points of a co-axal system cuts every circle of the system orthogonally*.
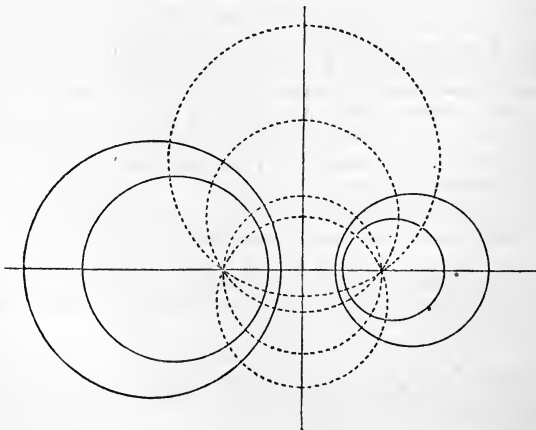
The figure shows two systems of co-axal circles, every circle of one system cutting all the circles of the other system orthogonally. In one system all the co-axal circles have two common points which are the limiting points of the orthogonal system.

A co-axal system is determined (1) when two of the circles are given, (2) when one circle and the common radical axis are given, (3) when the two common points are given if the circles intersect each other, or (4) when the limiting points are given if the circles do not intersect each other.

We have seen that co-axal circles are cut orthogonally by an infinite number of other circles which are co-axal with one another, and

Conversely. *Three, or more, circles which are cut orthogonally by more than one circle must be co-axal.*

For, the centre of any circle which cuts two given circles orthogonally must be on their radical axis, and therefore the centre of a circle which cuts three given circles orthogonally must be at their radical centre; and if there is more than *one* such orthogonal circle, there must be more than *one* radical centre of the given circles, which must therefore be co-axal.

4. *One circle of a given co-axal system will pass through any given point.*

For, take any point O on the given radical axis, and draw OK the tangent to one of the circles; then join O to P the given point, and on OP take the point Q such that $OP \cdot OQ = OK^2$. Now take any other point on the radical axis, and find a point R in a similar manner. Then the circle PQR is the circle required.

[If the co-axal circles have two common points, the circle through these points and the given point is the circle required.]

5. *Two circles of a co-axal system will touch a given straight line.*

Let the radical axis cut the given straight line in the point O, and let OQ be the tangent from O to one of the co-axal circles. Take P, P' on the given line such that $PO = OP' = OQ$.

Then the two circles required are the circles through P and P', which can be constructed as in the previous case.

Ex. 1.   Draw a circle to cut three given circles orthogonally.

[The circle whose centre is the radical centre of the three given circles, and whose radius is the length of the tangent from the radical centre to any one of the circles, is the circle required.]

Ex. 2.   Draw a circle to cut two given circles orthogonally and to pass through a given point.

[If A, B be the centres of the given circles and P the given point, the tangents from A and B to the required circle are equal to the radii of the circles A and B respectively. Hence, if Q be taken on AP, such that $AP \cdot AQ =$ sq. on rad. of circle A, and if R be taken on BP such that $BP \cdot BR =$ sq. on radius of circle B, the circle PQR will be the circle required.]

Ex. 3.   Draw a circle to cut a given circle orthogonally and to pass through two given points.

6.   *The difference between the squares on the tangents drawn from any given point to two given circles is equal to twice the rectangle contained by the distance between the centres of the circles and the perpendicular distance of the given point from their radical axis.*
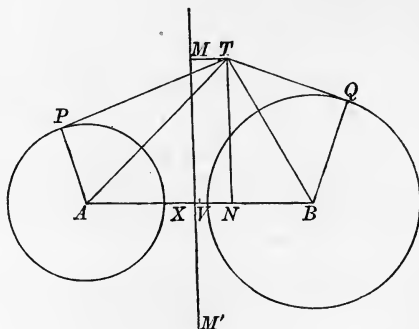
Let A, B be the centres of the given circles and MXM' their radical axis.

Draw the tangents TP, TQ from any point T. Draw TN, $TM \perp^r$ to AB and MXM' respectively, and join AP, AT, BQ, BT.

Then $TP^2 = AT^2 - AP^2 = AN^2 + NT^2 - AP^2,$

and $TQ^2 = BT^2 - BQ^2 = BN^2 + NT^2 - BQ^2;$

           $\therefore\ TP^2 - TQ^2 = AN^2 - BN^2 + BQ^2 - AP^2.$

Let V be the middle point of AB ;

then $\qquad AN^2 - NB^2 = (AN + NB)(AN - NB) = 2AB . VN.$

Also

$$BQ^2 - AP^2 = BX^2 - XA^2 = (BX + XA)(BX - XA) = 2BA . VX.$$

Hence $\qquad TP^2 - TQ^2 = 2AB . VN + 2AB . XV$

$$= 2AB . XN = 2AB . MT.$$

Cor. If T be on the co-axal circle whose centre is C, then will

$$TP^2 = 2AC . MT,$$

and $\qquad TQ^2 = 2BC . MT.$

Hence $\qquad TP^2 : TQ^2 = AC : BC.$

Thus we have the following very important theorem (which includes the Theorem of Apollonius as a particular case):

The tangents drawn to two circles from any point on a co-axal circle are in a constant ratio.

Conversely. *The locus of a point from which the tangents drawn to two given circles are in a given ratio, is a co-axal circle.*

For, let A, B be the centres of the given circles, and let P be any point on the locus. Draw through P a circle co-axal to the given circles and let C be its centre. Then the squares of the tangents drawn from P to the given circles are in the ratio $AC : BC$, whence it follows that C is a *fixed* point, and therefore any point P on the locus is on a fixed co-axal circle.

IX. POLES AND POLARS.

1. Through the fixed point O draw any straight line so as to cut a given circle, whose centre is C, in the points P, Q, and let the tangents at P, Q intersect in T.

Join CT and OC, and draw $TN \perp^r$ to OC. Let TC, TN cut OPQ in the points V, K respectively. Join CP, CQ and NP, NQ.
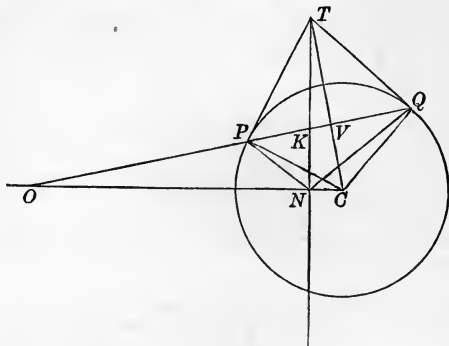
[Now CT bisects PQ and is $\perp^r$ to PQ, for PTQ and PCQ are isosceles triangles.

The $\triangle^s$ CVQ, CQT are therefore similar, and $CV : CQ = CQ : CT$, so that $CV . CT = CQ^2$.]

Then, since $\angle TVO = \angle TNO = $ rt. $\angle$, the points T, V, N, O are cyclic.

Hence $\qquad CN . CO = CV . CT = CQ^2.$

And, since $CN . CO = $ sq. on radius of circle, N is a fixed point if O is fixed, so that T is always on a fixed line $\perp^r$ to OC.

Thus we have the following theorem:

If a chord of a circle passes through a given point, the tangents at its extremities meet on a fixed straight line, which is called the **polar** *of the given point.*

Conversely. *If tangents be drawn to a circle from any point on a given straight line, their chord of contact will pass through a fixed point, which is called the* **pole** *of the given straight line.*

For, if T be any point on the given line TN, and if the corresponding chord of contact PQ cut the line through the centre of the circle $\perp^r$ to the given line in the point O; then, as above, $CN . CO = $ sq. on rad. Hence, as N is fixed, O is a fixed point.

A figure should be drawn for the case in which the pole is *within* the circle, in which case the polar will not cut the circle.

[For, since $CN \cdot CO = $ sq. on rad., if CO be less than the radius, CN must be greater.]

When the pole is *on* the circumference of the circle, the polar is the tangent at that point.

2. Since TPC, TQC, TNC are right angles, the points T, P, N, C, Q lie on a circle; and, since $TP = TQ$ and T, P, N, Q are cyclic, TN and the $\perp^{r}$ line ON will be the bisectors of the angles between PN and QN.

Hence $\qquad OP : OQ = PN : NQ = PK : KQ,$

so that we have the following theorem :

A chord of a circle is divided harmonically by any point on it and the polar of that point.

3. *If the polar of P passes through Q, the polar of Q will pass through P.*
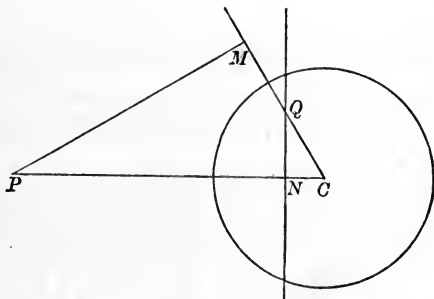
Let Q be any point on the polar of P with respect to the circle whose centre is C.

Then CP will cut the polar of P at rt. $\angle^{s}$ in N, and $CN \cdot CP = $ sq. on radius.

Join CQ, and draw $PM \perp^{r}$ to CQ.

Then, since PMQ, PNQ are rt. $\angle^{s}$, P, N, Q, M are cyclic.

$$\therefore CQ \cdot CM = CN \cdot CP = \text{sq. on radius.}$$

Since PM is $\perp^{r}$ to CQ and meets CQ in M so that $CQ \cdot CM = $ sq. on rad., PM must be the polar of Q.

4. Salmon's Theorem. *The perpendiculars from each of two given points on the polar of the other with respect to a circle are in the ratio of the distances of the points from the centre of the circle.*

Let P, Q be the given points, and C the centre of the circle.
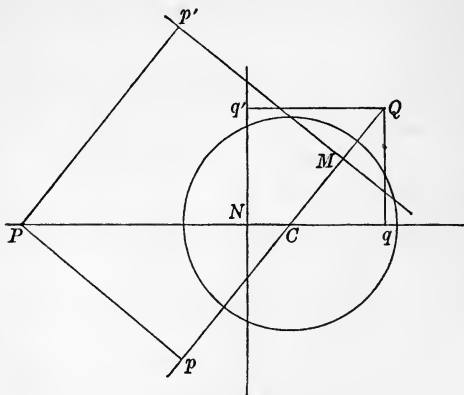
From P draw Pp, Pp' $\perp^r$ to QC and to the polar of Q respectively.

Also from Q draw Qq, Qq' $\perp^r$ to PC and to the polar of P respectively.

Then, since $\angle PpQ = $ rt. $\angle = \angle PqQ$, P, p, q, Q are cyclic, and therefore

$$PC \cdot Cq = QC \cdot Cp.$$

But $PC \cdot NC = $ sq. on radius $= QC \cdot MC$;

$$\therefore PC (NC + Cq) = QC (MC + Cp), \text{ i.e. } PC \cdot Nq = QC \cdot Mp;$$

$$\therefore PC \cdot Qq' = QC \cdot Pp',$$

$$\text{or } Pp' : Qq' = CP : CQ.$$

5. *If ABCD is a cyclic quadrilateral, the rectangles contained by the perpendiculars drawn from any point on the circle to AB and CD, AC and BD, and AD and BC, are all equal.*

Let the perpendiculars OP, OQ, OR, OS, OT, OU be drawn from any point O on the circle $ABCD$ to AB, BC, CD, DA, AC and BD respectively.

Join OA, OB, OC, OD. Join PQ and SR.

Since O, P, B, Q are cyclic, and S, O, R, D are cyclic,

$$\angle POQ = \text{supplement of } \angle PBQ$$

$$= \dots\dots\dots\dots \angle SDR = \angle SOR.$$
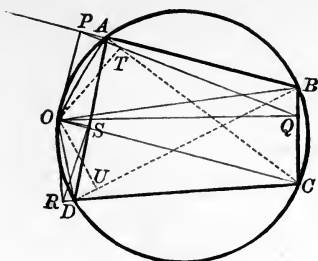
Also $\qquad \angle OQP = \angle OBP = \angle ODS = \angle ORS$;

$\therefore \triangle^s\ POQ$ and SOR are similar, and

$$PO\ :\ QO = SO\ :\ RO;$$

$$\therefore\ PO\ .\ RO = QO\ .\ SO.$$

It can be proved in a similar manner that $\triangle^s\ POU$ and TOR are similar, so that

$$PO\ :\ UO = TO\ :\ RO,$$

and therefore $\qquad PO\ .\ RO = UO\ .\ TO.$

If C moves up to and ultimately coincides with B, and D moves up to and ultimately coincides with A, so that BC and DA become the tangents at B and A respectively and DC coincides with AB; then we have as a particular case [which can easily be proved independently] the theorem :

6. *If from any point on a circle perpendiculars be drawn on any two tangents and on their chord of contact, the rectangle contained by the perpendiculars on the tangents is equal to the square on the perpendicular on the chord of contact.*

The theorem can be extended to the case of any inscribed polygon with an even number of sides.

For let $p_1, p_2, p_3, p_4, p_5, p_6$ be the perpendiculars from any point of a circle on the sides AB, BC, CD, DE, EF, FA respectively of the inscribed hexagon $ABCDEF$, and let x be the perpendicular from O on AD. Then, from the quad. $ABCD$, $p_1 \cdot p_3 = p_2 \cdot x$; and, from the quad. $DEFA$, $x \cdot p_5 = p_4 \cdot p_6$; $\therefore\ p_1 \cdot p_3 \cdot p_5 = p_2 \cdot p_4 \cdot p_6$.

7. *If ABCD is any quadrilateral described about a circle, the rectangle contained by the perpendiculars from A and C on any other tangent to the circle is in a constant ratio to the rectangle contained by the perpendiculars from B and D on the same tangent.*

Let P, Q, R, S be the points of contact of the sides AB, BC, CD, DA, with the circle whose centre is X.

Let O be any other point on the circle, and let Aa, Bb, Cc, Dd be the perpendiculars from A, B, C, D on the tangent at O, and let Oa, $O\beta$, $O\gamma$, $O\delta$ be the perpendiculars from O on SP, PQ, QR, RS respectively.

Then, by Salmon's Theorem,
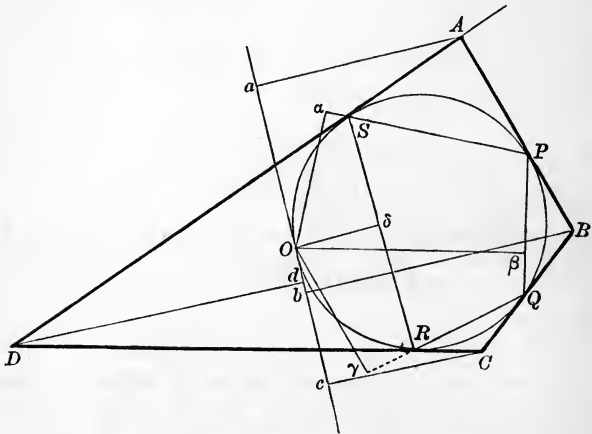
$$Aa : XA = Oa : XO,$$

$$Cc : XC = O\gamma : XO;$$

$$\therefore Aa . Cc : XA . XC = Oa . O\gamma : XO^2.$$

Similarly $Bb . Dd : XB . XD = O\beta . O\delta : XO^2.$

But, from (a), $Oa . O\gamma = O\beta . O\delta;$

$$\therefore Aa . Cc : XA . XC = Bb . Dd : XB . XD;$$

$$\therefore Aa . Cc : Bb . Dd = XA . XC : XB . XD.$$

The theorem can be extended to the case of any circumscribed polygon with an even number of sides.

X. CENTRES OF SIMILITUDE.

1. In Euclid VI. 8 the triangles ABC, DBA and DAC are proved to be similar. It will, however, be seen that when the perimeter of the triangle DBA (figure, page 314) is described in the order of the letters D, B, A, the rotation is in the same direction as that of the hands of a watch, and that this is also the case for the triangle DAC; but in the triangle ABC the rotation is in the opposite direction.

Def. *Two similar polygons $ABCD$ ... and $A'B'C'D'$ ... are said to be* **directly similar** *or* **reversely similar** *according as the directions of rotation are the same or opposite when the two perimeters are described in the order of the corresponding angular points A, B, C, D, ... and A', B', C', D', ... respectively.*

Thus, for example, all the triangles in the figure to Euclid VI. 26 are *directly* similar.

Def. *The* **Centre of Similitude** *of two similar polygons ABC...,* $A'B'C'$ ... *is the point O such that*

$$OA : OA' = OB : OB' = OC : OC' = ... = AB : A'B'.$$

The centre of similitude of two similar figures can always be found as a point of intersection of two circles of Apollonius; we shall, however, only consider the following simple and important case.

2. *When two similar triangles have their corresponding sides parallel, the lines joining their corresponding angular points will meet in a point which is the centre of similitude of the two triangles.*

Let ABC, $A'B'C'$ be the two similar triangles having $AB \parallel$ to $A'B'$, $BC \parallel$ to $B'C'$ and $CA \parallel$ to $C'A'$.
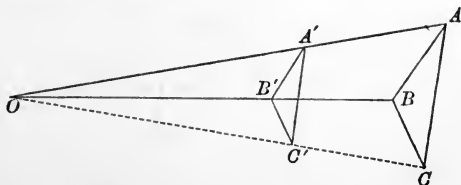
Draw AA', BB' and let them be produced if necessary to meet in O. Then we have to prove that $OC'C$ is a straight line.

Since $A'B'$ is $\parallel$ to AB, the $\triangle^s$ $OA'B'$, OAB are similar;

$$\therefore \quad OB' : OB = A'B' : AB.$$

And, since $\triangle^s$ $A'B'C'$, ABC are similar,

$$A'B' : AB = B'C' : BC ;$$

$$\therefore \ OB' : OB = B'C' : BC.$$

Also $\angle C'B'O = \angle CBO.$

Hence $\triangle^s$ $OC'B'$, OCB are similar and $\therefore \angle B'OC' = \angle BOC$, so that $OC'C$ is a straight line.

The point O is the centre of similitude, for it is easily seen that

$$OA' : OA = OB' : OB = OC' : OC = A'B' : AB.$$

It should be noticed that if the three sides of one triangle are parallel respectively to the three sides of another, the triangles must be *directly* similar.

Conversely. *If any point O be joined to the three vertices of any triangle ABC, and the lines OA, OB, OC be divided in the points A', B', C' so that $OA' : OA = OB' : OB = OC' : OC$, then will the triangles $A'B'C'$, ABC be similar and O will be their centre of similitude.*

Ex. Shew that, if two pairs of corresponding sides of two similar polygons are parallel, all the pairs of corresponding sides will be parallel, and the lines joining corresponding vertices will all meet in a point.

3. *If a straight line be drawn from a fixed point O to any point P on a given circle, and if OP be divided in Q so that $OQ : OP$ is constant, the locus of Q will be a circle.*
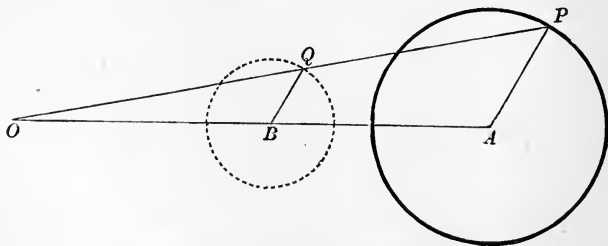
For, let A be the centre of the circle on which P lies, and let OA be joined and divided in B so that $OB : OA$ may be equal to the given ratio $OQ : OP$. Join BQ.

Then the triangles OBQ, OAP are similar (for the angle at O is common and $OQ : OB = OP : OA$).

Hence $BQ : OB = AP : OA$, so that BQ is constant, and therefore Q is on a fixed circle whose centre is B.

Ex. Shew that, if one side of a triangle and the opposite angle be given, the loci of the centroid, orthocentre and nine-point centre are circles.

For, if BC be the given side, and the opposite angle BAC be also given, the point A will lie on the arc of a circle through B and C. If A' be the middle point of BC, the centroid G is on $A'A$ and is such that $A'G = \frac{1}{3}A'A$; hence, as the locus of A is a circle, the locus of G is a circle of one-third the radius. Now S the circumcentre of ABC is fixed, and we know that $SGNO$ is a straight line and $SN : SG$ and $SO : SG$ are constant ratios; hence, as the locus of G is a circle, the loci of N and O are also circles.

4. Let A, B be the centres of two circles whose radii are a, b respectively; and let C, D be the points on AB such that

$$DA : DB = AC : CB = a : b.$$

Draw any line through D cutting the circle A in the points P, P'. Join AP, AP' and draw BQ, $BQ' \parallel$ to AP, AP' to meet DPP' in Q, Q' respectively.
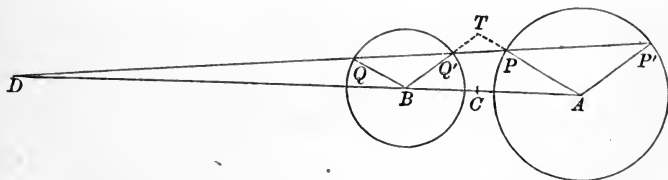
Then the $\triangle^s\ DBQ$, DAP are equiangular and therefore similar;

$$\therefore\ BQ : AP = DB : DA = b : a.$$

Hence Q, and similarly Q', are on the circle B.

Hence also $\qquad DQ : DP = DQ' : DP' = b : a.$

The proof is precisely the same when the line is drawn through C the internal centre of similitude.

Def.   *The two points which divide the line joining the centres of two given circles internally and externally in the ratio of the radii are called the* **centres of similitude** *of the circles.*

It is easily seen that the centres of similitude of two circles are the points where common tangents to the circles (if there are any common tangents) cut the line joining their centres; and that if two circles touch one another the point of contact is a centre of similitude.

It should be noticed that the tangents at the corresponding points Q, P, and also Q', P', are parallel, since the radii to which they are respectively perpendicular are parallel; and that the tangents at the non-corresponding points Q, P', and also Q', P, meet on the radical axis of the given circles [for the tangents at P, P' make equal angles with $DQQ'PP'$ and the tangent at Q is parallel to that at P, so that the tangents at Q and P' make equal angles with $DQQ'PP'$, and if T be the point of intersection $TQ = TP'$].

It will also be easily proved that if another line be drawn to cut the circles in q, q' and p, p' respectively; then corresponding chords Qq and Pp, and also $Q'q'$ and $P'p'$, will be parallel, and non-corresponding chords Qq and $P'p'$, or $Q'q'$ and Pp, will meet on the radical axis.

5. By means of the Theorem of Menelaus it is easy to prove the following theorem:

The six centres of similitude of three given circles when taken in pairs lie by threes on four straight lines.

Def.   *A straight line on which three of the centres of similitude of three given circles lie, is called an* **axis of similitude.**

6. *The centres of similitude of the circum-circle and the nine-point circle of any triangle are the centroid and the orthocentre.*

For, if S be the circum-centre, N the nine-point centre, O the orthocentre and G the centroid; then we know [see page 280] that the four points S, G, N, O are on a straight line in the order named.   We know also that $SO = 2 NO$, and $SG = \frac{1}{3} SO = \frac{2}{3} SN = 2 GN$.   Hence the points G and O divide SN in the ratio of the radii of the circles, for the circum-radius is twice the nine-point radius.

7. Since rect. $DQ \cdot DQ'$ is constant for all chords through D, and $DQ : DP = DQ' : DP' = b : a$, it follows that

$$\text{rect. } \mathbf{DQ} \cdot \mathbf{DP'} = \text{rect. } \mathbf{DQ'} \cdot \mathbf{DP} = \text{const.,}$$

where Q, P' and Q', P are points on each circle the radii to which are not parallel, and which are therefore called non-corresponding points.
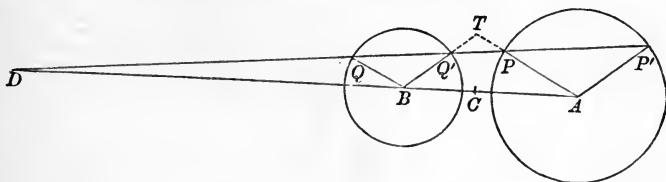
Again, if BQ', AP be produced to meet in T,

$$\angle TQ'P = \angle AP'P, \text{ since } BQ' \parallel AP'$$

$$= \angle TPQ'.$$

Hence $TP = TQ'$, and therefore a circle whose centre is T and radius TP will touch the given circles at P, Q'.

So also a circle will touch the given circles at Q and P'.

Hence the following theorem :

If a line be drawn through a centre of similitude of two given circles so as to cut the circles in two pairs of points, two circles can be drawn to touch the given circles in non-corresponding points.

Conversely. *If a circle touch two given circles the line joining its points of contact with the circles will pass through one or other of the centres of similitude of the given circles.*

It will be easily seen that when the given circles both touch the variable circle externally, or both internally, the line joining the points of contact will pass through the external centre of similitude ; and that when one of the given circles touches the variable circle internally and the other touches it externally, the line joining the points of contact will pass through the internal centre of similitude.

Ex. 1. The radical axis of any two circles, each of which touches two given circles both internally or both externally, passes through the external centre of similitude of the given circles.

Ex. 2. The radical axis of any two circles, each of which touches two given circles one internally and the other externally, passes through the internal centre of similitude of the two given circles.

Ex. 3.   Any circle which touches two given circles is cut orthogonally by one or other of two fixed circles.

8.   *The circle whose diameter is the line joining the centres of similitude of two circles is sometimes called their* **circle of similitude.**
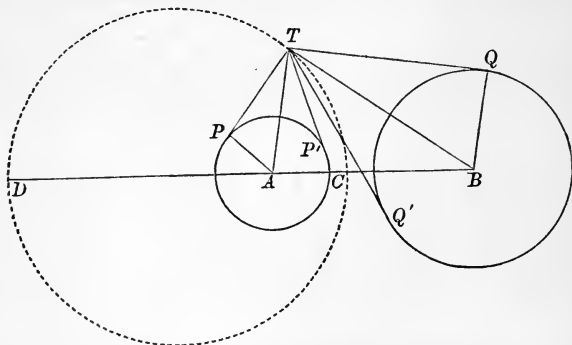
Let C, D be the centres of similitude of the circles whose centres are A, B and radii a, b respectively.

Let T be any point on the circle whose diameter is CD, and join TA, TB.   Draw TP, TP' touching the circle A and TQ, TQ' touching the circle B.   Join AP, BQ.

Then, since $DA : DB = AC : CB = a : b$,

by the Theorem of Apollonius

$$TA : TB = a : b = PA : QB.$$

Also the angles APT, BQT are *right angles*;

$$\therefore \triangle^s \ ATP, \ BTQ \ \text{are similar.}$$

Thus $\angle PTA = \angle QTB$, and $\therefore \angle PTP' = \angle QTQ'$;

also $TP : TQ = PA : QB.$

Thus *the tangents drawn to two given circles, from any point on the circle whose diameter is the line joining their centres of similitude, are in the ratio of the radii of the given circles.*

Since the tangents drawn from T to the given circles are in a fixed ratio, it follows from VIII. (3) that

The circle of similitude is co-axal with the given circles.

XI. TO DRAW A CIRCLE TO TOUCH THREE GIVEN CIRCLES.

Let A, B, C be the centres of the given circles. Then, if O be the centre of a circle which touches all the given circles externally, it is easily seen that a circle whose centre is O and radius OC will touch the two circles whose centres are A, B respectively, and whose radii are the differences of the radii of the circles A and C, and B and C respectively. Thus the problem is reduced to that of drawing a circle through the *point C* so as to touch two given circles.
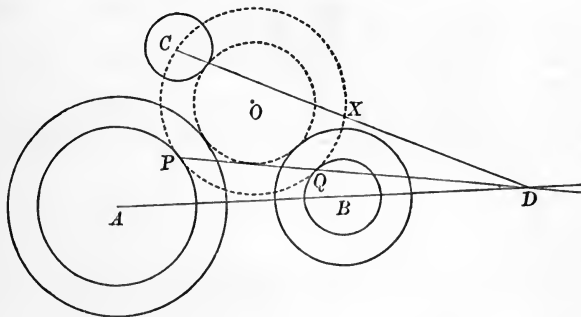
Now, if the required circle touches the two circles in P, Q respectively, the line PQ will pass through D the external centre of similitude; moreover the rectangle $DQ \cdot DP$ is known [p. 367]. If then, the point X be taken on the line DC such that $DX \cdot DC$ is equal to the known rectangle $DQ \cdot DP$, the point X will be on the required circle. Thus the problem is further reduced to that of drawing a circle through two known points C, X so as to touch a fixed circle, and this problem has already been solved [page 235].

There will in general be *eight* different circles which touch the three given circles, for each of the given circles may touch internally or externally. The above construction will have to be slightly modified to apply to the other cases.

Ex. 1. Draw a circle to pass through a given point and touch a given circle and a given straight line.

[The centres of similitude of a circle and a straight line (that is of a circle of infinite radius) are the two extremities of the diameter of the circle which is perpendicular to the straight line.]

Ex. 2. Draw a circle to touch two given circles and a given straight line.

XII. INVERSION.

Def. If P, Q be two points on the radius OA of a circle whose centre is O, such that $OP . OQ = OA^2$, the points P, Q are said to be **inverse points** with respect to the circle whose centre is O and radius OA.

The point O is called the **pole of inversion** and OA is called the **radius of inversion**.

If the point P trace out a curve, the curve which is the locus of Q is called the **inverse** of the locus of P with respect to the pole O.

1. *The inverse of a straight line, with respect to any pole without it, is a circle which passes through the pole of inversion.*

Let O be the pole of inversion, and from O draw $OP \perp^r$ to the given st. line.

Let Q be the inverse of P with respect to the circle whose radius is OA, and let S be the inverse of *any* other point R on the given straight line.
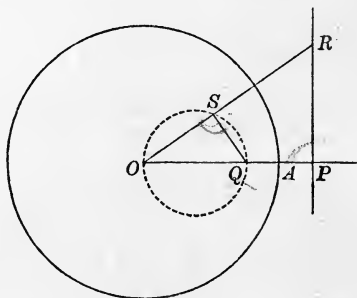
Then, by definition, $OQ . OP = OA^2 = OS . OR$.

Hence Q, P, R, S are cyclic;

$$\therefore \ \angle OSQ = \angle OPR = \text{rt. } \angle .$$

Hence S is on the circle whose diameter is OQ.

Thus the inverse of the straight line PR is a circle through O the pole of inversion.

Conversely, *the inverse of the circle OSQ with reference to a pole O on its circumference is a straight line perpendicular to the diameter through O.*

It should be noticed that a straight line through the pole inverts into itself.

2.   *The inverse of a circle is another circle, and the pole of inversion is a centre of similitude of a circle and its inverse.*

Let P be any point on a circle whose centre is A, and let O be the pole of inversion.

Let Q be the inverse of P, so that $OQ \cdot OP = k^2$, where k is the radius of the circle of inversion.

[The circle of inversion is rarely drawn.]
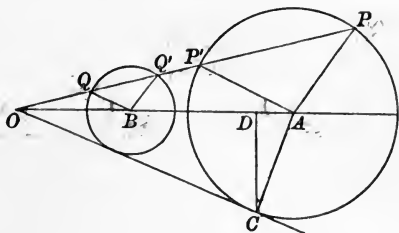
Now, if OP cut the given circle again in P', since $OP \cdot OQ = k^2$ and $OP \cdot OP'$ is also constant, it follows that $OQ : OP'$ is const.

Hence, if QB be drawn $\parallel$ to $P'A$ to cut OA in B,

$$OB : OA = BQ : AP' = OQ : OP' = \text{const.}$$

Hence B is a fixed point and BQ is of constant length.

Thus as P moves round the circle, the inverse point Q describes another circle; moreover the origin O is the external centre of similitude of the given circle and its inverse, since $OB : OA = BQ : AP'$.

If OC be the tangent from O to the given circle, then

$$QB : AP = OQ : OP'$$

$$= OQ \cdot OP : OP' \cdot OP$$

$$= k^2 : OC^2.$$

Cor. *If the circle of inversion cuts the given circle orthogonally* (in which case the tangent from O to the given circle is equal to the radius of the circle of inversion) *the given circle inverts into itself.*

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3. *Two inverse curves cut the radius vector through corresponding points at equal angles.*
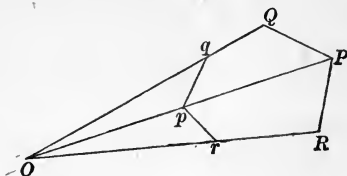
Let p, q, r be the inverse points of P, Q, R with respect to the point O.

Then since $Op \cdot OP = Oq \cdot OQ$, the points p, P, Q, q are cyclic.

Hence $\angle OPQ = \angle Oqp$.

Now, if Q move up to P and ultimately coincides with it, the line QP will ultimately be the *tangent* to the curve on which P and Q lie. So also qp in its ultimate position will be the tangent at p to the inverse of the curve PQ.

But, when $\angle POQ$ is made smaller and smaller and ultimately vanishes, $\angle qpP = \angle Oqp$; $\therefore$ ultimately the tangent at P to the curve PQ and the tangent at p to the inverse curve pq make the same angles with OpP.

If two curves meet at the point P, and PQ, PR are the tangents at P and pq, pr the tangents at the corresponding point of their inverses, it follows from the above that the angles between the two tangents at P are equal to the angles between the tangents to the inverse curves at p.

Hence *two curves cut one another at a common point at the same angle as their inverses cut one another at the inverse point.*

Two special cases are important, namely:

If two curves *touch* one another their inverses with respect to any point will *touch* one another.

Also, if two curves cut one another at right angles their inverses with respect to any point will cut one another at right angles.

4. *Coaxal circles invert into coaxal circles.*

[When the coaxal system has two real common points, it is at once obvious that their inverses have two real common points, and are therefore coaxal. The following proof applies to both types of coaxal circles.]

We know that coaxal circles are cut orthogonally by an infinite number of circles.

Hence, if we invert with respect to any point, the inverse circles will [by 3] be cut orthogonally by an infinite number of circles.

But we know that when three, or more, circles are cut orthogonally by more than *one* circle they must be coaxal.

5. *Coaxal circles with real limiting points invert into concentric circles if one of the limiting points be the pole of inversion.*

For we know that *any* circle through the limiting points L, L' of a coaxal system will cut all the circles of the system orthogonally.

Hence, if we invert with respect to L, any *straight line* through the inverse of L' will cut all the inverse circles orthogonally, whence it follows that the inverse circles are concentric, the inverse of L' being the common centre.

Conversely, *a system of concentric circles invert into coaxal circles of which the pole of inversion is a limiting point.*

Ex. 1.   Shew that three circles can be inverted into themselves if the radical centre be taken as the pole of inversion.

Ex. 2.   Shew how to invert two given circles into equal circles.

[If r_1, r_2 be the radii of the given circles, κ the constant of inversion, and τ_1, τ_2 the lengths of the tangents from the pole to the two circles; then $\rho_1 : r_1 = k^2 : \tau_1^2$ and $\rho_2 : r_2 = k^2 : \tau_2^2$. Hence, in order that ρ_1 may be equal to ρ_2,

$$\tau_1^2 : \tau_2^2 = r_1 : r_2.$$

Hence the pole must be anywhere on a certain coaxal circle.]

Ex. 3.   Shew how to invert three given non-coaxal circles into equal circles.

Ex. 4. If OA, OC be two equal rods, and AB, BC, CD, DA four other equal rods, the six rods being hinged together at the points O, A, B, C, D; then, if O be kept fixed, B and D will describe inverse curves.
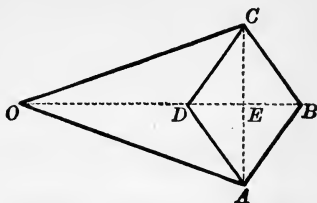
For, since $OC = OA$, $DC = DA$ and $BC = BA$, the points O, D, B are on the straight line which bisects AC perpendicularly. Also CA bisects BD.

Hence
$$OD \cdot OB = OE^2 - DE^2$$
$$= OC^2 - DC^2 = \text{const.}$$

Hence if any curve is described by B, the inverse curve will be described by D; and, in particular, if one of the points describes part of a circle through O the other will describe part of a straight line.

The above arrangement of jointed rods is called a **Peaucellier Cell.**

Ex. 5. Prove that, if A, B, C be any three points, and A', B', C' their inverses with respect to the point O, then will

$$\frac{BC}{B'C' \cdot OA'} = \frac{CA}{C'A' \cdot OB'} = \frac{AB}{A'B' \cdot OC'}.$$

Deduce Ptolemy's Theorem and its converse. [See page 342.]

For, since $OA' \cdot OA = OB' \cdot OB = OC' \cdot OC = k^2$, the points A', A, B, B' are cyclic and the $\triangle$s OAB, $OB'A'$ are similar. Hence

$$A'B' : AB = OB' : OA = OA' \cdot OB' : OA \cdot OA'.$$

Hence
$$A'B' \cdot OC' : AB = OA' \cdot OB' \cdot OC' : k^2.$$

Whence
$$\frac{A'B' \cdot OC'}{AB} = \frac{B'C' \cdot OA'}{BC} = \frac{C'A' \cdot OB'}{CA}.$$

Now, if A, B, C are on a straight line, the points O, A', B', C' are cyclic; and since one of the three segments BC, CA, AB is equal to the sum of the other two, it follows that one of the three rectangles $B'C' \cdot OA'$, $C'A' \cdot OB'$, $A'B' \cdot OC'$ is equal to the sum of the other two.

If, however, the points A, B, C are not on a straight line, it is obvious that the sum of any two of the three lines BC, CA, AB is greater than the third. Hence, if O, A', B', C' be four points not on a circle, the sum of any two of the rectangles $B'C' \cdot OA'$, $C'A' \cdot OB'$, $A'B' \cdot OC'$ is greater than the third rectangle.

XIII. MAXIMA AND MINIMA.

1. The following theorems have already been proved.

(i) *If the base and the sum of the two other sides of a triangle are given, the area is greatest when the triangle is isosceles.* [page 98.

(ii) *If the base and the area of a triangle are given, the sum of the two other sides is least when the triangle is isosceles.* - [page 97.

(iii) *If the sum of two straight lines is given, the rectangle contained by them is greatest when they are equal.* [II. 5, Cor.

(iv) *If the area of a rectangle is given, the perimeter is least when the rectangle is a square.* [II. 14, Cor.

(v) *If two sides of a triangle are given in length, the area is greatest when the given sides are at right angles.*

[For the altitude of the triangle PBA is less than PB unless PBA is a right angle.]
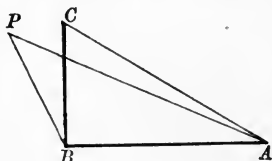
Now let AB be a fixed chord of a circle and let C, D be the two points at which the tangents are parallel to AB.

Then, if a point P travel round the circle, the area of the triangle APB will change continuously, and it will be easily seen that on one side of AB the triangle APB will be greatest when P is at C, and on the other side of AB the triangle will be greatest when P is at D.
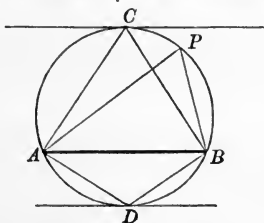
Def. *When a geometrical line, angle, or area is drawn so as to satisfy certain conditions under which it can change continuously in magnitude, it is said to have a* **maximum** *value when it has increased up to a certain limit after which it diminishes; and it is said to have a* **minimum** *value when it has diminished to a certain limit after which it increases.*

Thus the area of the triangle APB, of which the base AB is fixed and P is any point on a circle through A and B, has the *two* maxima values ACB and ADB, the tangents at C and D being parallel to AB; and there are also *two* minima values when P is on the circle but indefinitely near to A or to B, in which cases the triangle APB has zero area.

In the above example it will be seen that for many positions of P on the circle the triangle APB will be greater than the triangle ADB (the arc ADB being supposed to be smaller than the arc ACB), so that a *maximum* value of a magnitude is by no means necessarily the *greatest* value; if, however, there is only *one* maximum value, this must be greater than any other value of the magnitude in question.

There are two important points to notice :

(1) that maxima and minima values occur *alternately*,

(2) that a magnitude is either a maximum or a minimum at *a position of symmetry*.

Ex. 1. Through a fixed point O, within a circle whose centre is C, a chord POQ is drawn. Find when the triangle PCQ is a maximum or a minimum.

Since the two sides CP, CQ of the triangle PCQ are given, the area will be the greatest when the angle PCQ is a right angle. Hence, if O is at such a distance from the centre that one chord, and therefore two chords, can be drawn through it which will subtend a right angle at the centre, this chord will give the greatest triangle.

Hence as the chord is turned about O, the area of the triangle is a *minimum* when the chord is the diameter through O, it is a *maximum* when the chord subtends a right angle at the centre of the circle, a *minimum* when it is in the position of symmetry perpendicular to the diameter, a *maximum* again when it again subtends a right angle at the centre, and a *minimum* when it again lies along the diameter.

If, however, the point is so near to the centre that the shortest chord through it subtends at the centre an angle equal or greater than a right angle [it will be easily seen that in this case $2CO^2 \not> \text{rad}^2$], the triangle will be a *minimum* when the chord lies along the diameter and a *maximum* when the chord is perpendicular to the diameter.

Ex. 2. A, B are two fixed points on the same diameter of a circle, and PQ is any chord through A; find when the triangle PBQ is a maximum or a minimum.

Ex. 3. POQ is any chord of a circle through a fixed point O; find the position of the chord when the sum of the squares of PO and QO is minimum.

Ex. 4. P is any point on the arc ACB of a circle. Shew that the sum of the chords AP and BP is greatest when they are equal.

Ex. 5. Find the points on a given straight line at which another given finite line subtends a maximum angle. Where are the points at which the second given line subtends a minimum angle?

2. *If all the sides of a polygon except one be of given lengths, the area of the polygon will be greatest when its vertices all lie on a circle whose diameter is the variable side of the polygon*.*

Let AB, BC, CD be each of given length. Then the area of $ABCD$ will be a maximum when B and C are on the circle whose diameter is AD.
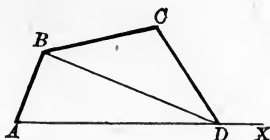
Join BD. Then, if ABD is not a right angle, turn the area BCD about B until BD is perpendicular to AB. Then the area BCD will be unchanged but the area ABD will be increased.

Hence when the area $ABCD$ is greatest, the angle ABD must be a right angle. Similarly the angle ACD must be a right angle, and therefore B and C must be on a circle whose diameter is AD.

[The figure has been drawn for the case of a quadrilateral, but the proof will apply to all cases.]

3. *If a figure be bounded by a curved line of given length and a straight line of indefinite length, the included area will be a maximum when the figure is a semi-circle.*

Let $AXCYB$ be a curved line of given length, and let C be any point on it such that ACB is not a right angle.
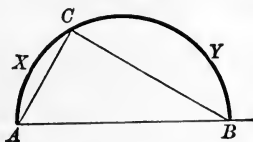
Then, keeping the figures AXC, CYB unchanged, turn CYB about C until ACB becomes a right angle, and join AB. Then the figure so formed will be greater than before, for the areas AXC, CYB are unchanged, but the triangle ACB is increased.

Hence the enclosed area can always be increased unless the bounding straight line subtends a right angle at every point on the curve, that is unless the curve is the arc of the semi-circle whose diameter is AB.

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* The interesting methods of sections 2—6 appear to have been first given by Thomas Simpson in 1747.

Cor.   *A circle encloses a maximum area for a given perimeter.*

Let the perimeter of the curve *ABCD* be given, and let the points
A, C be such that arc *ABC*=arc *CDA.*
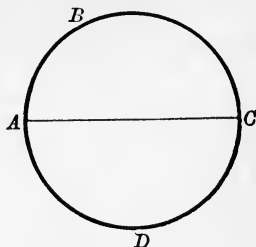
Join *AC.*

Then, if the whole is greatest, the areas *ABCA* and *ADCA* must be
equal; for, if they were unequal the smaller could clearly be made equal
in all respects to the greater, and the whole area would thereby be
increased.

Thus the area *ABCA* bounded by the curved line *ABC* of given length
and the indefinite straight line *AC* must be a maximum, and must
therefore be a semi-circle.

4.   *If any number of straight lines be given, each of which is less than
the sum of all the others, there is a certain circle in which the given lines
can be the sides of an inscribed polygon.*

Take any circle and place in it in succession chords equal respectively
to the given straight lines, and let the sum of the angles subtended at
the centre of the circle by these chords be less than four right angles;
then, if the radius of the circle be continually diminished, each chord
will subtend a greater and greater angle at the centre, and therefore, by
sufficiently diminishing the radius of the circle, the sum of all the angles
at the centre can be made greater than four right angles.   It is therefore
clear that there must be one circle, and only one, the radius of which is
such that the sum of the angles subtended by chords equal respectively
to the given straight lines is equal to four right angles, and this circle is
the circle required.

The actual construction of the circle in which a polygon can be
inscribed whose sides are equal respectively to given straight lines cannot
in general be effected by ruler and compasses.   The construction can,
however, be made in the case of a quadrilateral.

5. *To find the circle in which a quadrilateral can be inscribed whose sides are equal respectively to four given straight lines each of which is less than the sum of the other three.*

Let AB, X, Y, Z be the four given straight lines, and suppose the construction effected, so that $BC = X$, $CD = Y$ and $DA = Z$, A, B, C, D being cyclic.
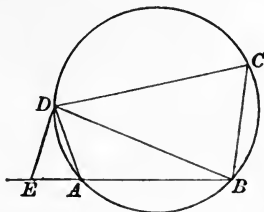
Then, if BA be produced, $\angle DAE = \angle DCB$.

If therefore $\angle DEA$ be made equal to $\angle CBD$, then $\triangle^s DAE$, DCB will be similar, so that

$$EA : AD = BC : CD,$$

i.e. $EA : Z = X : Y$, which gives the length of AE.

Also $ED : DA = DB : DC$;

$$\therefore ED : DB = DA : DC = Z : Y.$$

Hence D is on the circle of Apollonius given by $ED : DB = Z : Y$, E and B being fixed points.

But D is also at a fixed distance Z from A.

Thus D is found, and knowing D the rest of the construction is obvious.

Ex. Construct a quadrilateral having given the lengths of the four sides and the sum of a pair of opposite angles.

The above construction will apply to this case; if instead of producing BA, we make the $\angle BAE$ equal to the known sum of the angles A and C.

6. *The area of a quadrilateral whose sides are of given lengths is greatest when the quadrilateral is cyclic.*

Construct the circle in which the quadrilateral $ABCD$ can be inscribed whose sides are equal respectively to the four given straight lines.
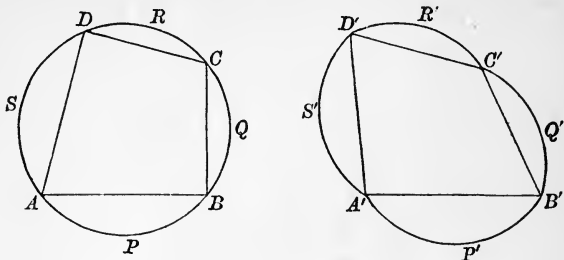
Now, let $A'B'C'D'$ be any other quadrilateral formed by lines of the given lengths. Then on $A'B'$, $B'C'$, $C'D'$, $D'A'$ describe segments similar and therefore equal in all respects to the segments APB, BQC, CRD and DSA respectively.

Then the perimeter $A'P'B'Q'C'R'D'S'$ is equal to the perimeter $APBQCRDS$.

Hence the area $A'P'B'Q'C'R'D'S'$ is [by 3] less than the circle $APBQCRDSA$.

But the sum of the segments $A'P'B'$, &c. is equal to the sum of the segments APB, etc.

Hence the area of the quadrilateral $A'B'C'D'$ is less than the area of the quadrilateral $ABCD$.

7. The theorem that the area of a cyclic quadrilateral is greater than that of any other quadrilateral with the same sides, is of very great importance. The following is an independent proof.

Let $ABCD$ be a cyclic quadrilateral, and let $AB'C'D$ be any other quadrilateral with equal sides, the side AD being common to the two.

Let the lines bisecting the angles BAB', CDC' meet in O. Join OB, OB', OC, OC'.

Then, $AB = AB'$ and AO bisects the $\angle BAB'$; whence it follows that $BO = B'O$ and AO bisects $\angle BOB'$.

So also $CO = C'O$ and DO bisects $\angle COC'$.

Then, since $BO = B'O$, $CO = C'O$ and $BC = B'C'$,

$$\triangle BOC = \triangle B'OC' \quad \text{and} \quad \angle BOC = \angle B'OC'.$$

Now the sum of quad. $ABCD$, $\triangle OBC$ and fig. $ABOB'$ = the sum of quad. $AB'C'D$, $\triangle OB'C'$ and fig. $DCOC'$.

But $\triangle OBC = \triangle OB'C'$.

Hence quad. $ABCD$ is greater than quad. $AB'C'D$,

if   figure $DCOC'$ is greater than fig. $ABOB'$.

Take β, γ on AO, DO respectively so that $O\beta = OB$ and $O\gamma = OC$.

Since

$$\angle BOC = \angle B'OC', \quad \angle BOB' = \angle COC';$$

and OA, OD are the bisectors of BOB' and COC' respectively;

$\therefore \angle AOB = \angle DOC$ and $\angle \beta O\gamma = \angle BOC$.

Hence $\triangle \beta O\gamma$ is equal in all respects to $\triangle BOC$;

$\therefore \angle \beta\gamma O = \angle BCO$

$\qquad < BCX < BAD$, since $ABCD$ is a circle,

$\qquad < \beta AD$.

Make then $\angle O\gamma Z = \angle \beta AD$; then Z will fall between β and A.

And A, Z, γ, D will be cyclic;

$$\therefore O\gamma \cdot OD = OZ \cdot OA > O\beta \cdot OA,$$

$$\text{i.e. } OC \cdot OD > OB \cdot OA.$$

Hence, as the angles BOA and COD are equal, $\triangle COD > \triangle BOA$.

But $2\triangle COD = \text{fig. } DCOC'$ and $2\triangle BOA = \text{fig. } ABOB'$;

$$\therefore \text{ fig. } DCOC' > \text{fig. } ABOB',$$

and $\therefore$ quad. $ABCD > $ quad. $A'B'C'D'$.
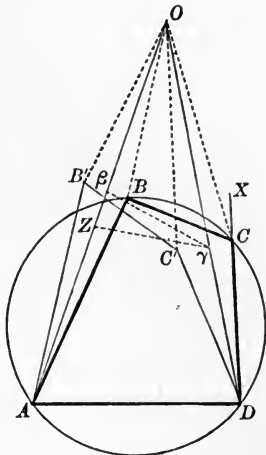
[The student should draw a figure in which the bisectors of the angles BAB', CDC' meet on the other side of AB, in which case a very slight modification of the proof is required.]

Or thus: It is easily proved that, if $ABCD$ be a quadrilateral, and UV be the projection of BD on AC; then

$$AB^2 + CD^2 - BC^2 - DA^2 = 2AC \cdot UV.$$

Hence　　$AB^2 + CD^2 - BC^2 - DA^2 : 2AC \cdot BD = UV : BD.$

But, by the converse of Ptolemy's Theorem, if the sides of a quad. be given, the rectangle contained by the diagonals is greatest when the quad. is cyclic.

Hence the above equal ratios are least when the quad. is cyclic.

But $UV : BD$ is least when the angle between AC and BD is greatest.

Hence *the rectangle contained by the diagonals and also the angle between them are both greatest when a quadrilateral of given sides is cyclic.*

From this it easily follows that the area of a quadrilateral whose sides are of given length is a maximum when the quad. is cyclic.

8.　*The area of a polygon with a given number of sides and a given perimeter is greatest when the polygon is regular.*

For, if any two of the sides, AB and BC suppose, are unequal; then the triangle ABC will be increased by taking instead of AB and BC two other sides each equal to half the sum of AB and BC, all the other sides of the polygon remaining unchanged.

Hence, when the area of the polygon is a maximum, every pair of consecutive sides, and therefore all the *sides*, must be equal.

Again, if the sides AB, BC, CD,... of the polygon $ABCD...XYZ$ be all equal, the area of $ABCD$, and therefore the whole area, can be increased unless A, B, C, D are cyclic; and when AB, BC, CD are equal chords of a circle it is easily seen that $\angle ABC = \angle BCD$.

Thus when the area of the polygon is a maximum, any two consecutive angles and therefore all the *angles* must be equal.

Cor.　*The area of a regular polygon of given perimeter increases as the number of the sides increases.*

For let $ABCD...$ be a regular polygon of n sides. Take any point F on CD, and make the isosceles triangle FXE so that $FX + XE = FD + DE$; then $\triangle FXE$ is greater than $\triangle FDE$, and therefore the polygon $ABCFXE...$ of $n+1$ sides is greater than $ABCD...$. But the polygon $ABCFXE...$ is less than the regular polygon of $n+1$ sides and the same perimeter.
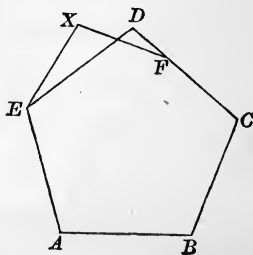
Hence the regular polygon of n sides is less than the regular polygon of $n+1$ sides with the same perimeter.

9. *The area and the perimeter of a polygon of a given number of sides which circumscribes a given circle, are least when the polygon is regular.*

Let AX, AY be two tangents to a circle whose centre is O. Let D be the point such that, if the tangent at D cuts AX, AY in B, C respectively, $BD = DC$ and $\therefore OB = OC$.

Let the tangent at any other point Q cut AX, AY in P, R respectively. Join OB, OC, OP, OR and let OP cut BC in K.
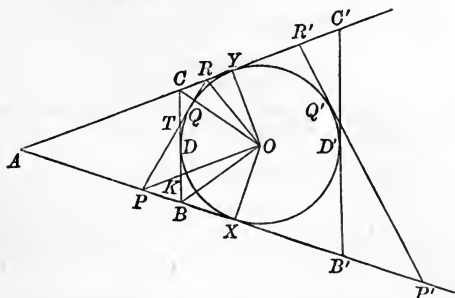
Then, since the tangents from any point to a circle are equal and subtend equal angles at the centre, it follows that

$$\angle BOC = \tfrac{1}{2} \angle XOY = \angle POR.$$

Hence $\qquad \angle BOP = \angle COR.$

But $\qquad OB = OC$ and $\angle OBK = \angle OCB = \angle OCR$;

$$\therefore \triangle BOK = \triangle COR.$$

Hence $\qquad \triangle BOP > \triangle COR$;

$$\therefore \triangle BOP + \triangle BOX + \triangle ROY > \triangle COR + \triangle BOX + \triangle ROY;$$

i.e. $\qquad \triangle POX + \triangle ROY > \triangle COY + \triangle BOX.$

But $\qquad 2 \triangle POX + 2 \triangle ROY = \text{figure } OXPRY,$

since $PX = PQ$ and $RQ = RY$.

Also $\qquad 2 \triangle COY + 2 \triangle BOX = \text{figure } OXBCY.$

Hence $\qquad \text{figure } OXPRY > \text{figure } OXBCY.$

Again, it is easily seen that

$$\text{figure } OXPRY = \text{rect. } PR . OY,$$

and $\qquad \text{figure } OXBCY = \text{rect. } BC . OY;$

$$\therefore PR > BC.$$

Thus the area enclosed by two fixed tangents to a circle and a variable tangent is least, and the length of the intercepted portion of the variable tangent is also least when the variable tangent is bisected at its point of contact.

From this the general theorem readily follows. For, if $ABCD...$ be a polygon circumscribing a circle, the sides AB, BC, CD,... touching the circle at P, Q, R,....

Then, when the area of the polygon is least each side must be bisected at its point of contact.

Hence $$AP = PB, \quad BQ = QC, \quad CR = RD,...$$

But $$BP = BQ, \quad CQ = CR,...$$

Hence $$AB = BC = CD = ...$$

Also, since the tangents from A, B, C,... to the circle are all equal, the angles at A, B, C,... must be all equal.

The polygon $ABCD...$ must therefore be regular.

Again, if any tangent be drawn to the circle between P and Q so as to cut AB, BC respectively in X and Y, it is clear that the area and also the perimeter of the circumscribing polygon $AXYCD...$ will be less than the area and perimeter of the polygon $ABCD...$.

But the area (or perimeter) of the regular polygon which circumscribes the circle and has the same number of sides as the polygon $AXYBCD...$ is less than the area (or perimeter) of $AXYBCD...$.

Hence *the area (or perimeter) of a regular polygon which circumscribes a given circle decreases as the number of its sides increases.*

Ex. Through a fixed point O a line is drawn cutting the fixed straight lines XAX', YAY' in the points P, Q respectively. Shew that the area PAQ is a minimum when PQ is bisected in O, and find when the length of PQ is a minimum.

10.   Fermat's Problem. *To find the point within a triangle the sum of whose distances from the angular points of the triangle is a minimum.*

Let ABC be the triangle.   On the side of BC opposite to A describe the equilateral triangle BDC.
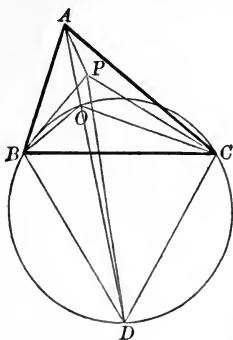
Let AD meet the circum-circle of BCD in the point O.   Then O is the required point.

Join OB, OC.

Take any other point P within the triangle ABC, and join PA, PB, PC, PD.

Then, by Ptolemy's Theorem

$$DO \cdot BC = OB \cdot DC + OC \cdot BD;$$
$$\therefore \ DO = OB + OC, \text{ since } BC = CD = BD.$$

Hence $OA + OB + OC = AD.$

And, by the converse of Ptolemy's Theorem

$$BP \cdot DC + CP \cdot DB \text{ is not less than } DP \cdot BC;$$

and $BC = CD = DB;$

$$\therefore \ BP + CP \text{ is not less than } DP.$$

Hence $AP + BP + CP$ is not less than DP and $AP;$

$$\therefore \ AP + BP + CP > AD$$
$$> OA + OB + OC.$$

[The above elegant solution of Fermat's problem is due to R. Chartres, *Nature*, Feb. 2, 1888.]

It is easily seen that each of the sides of ABC subtends at O an angle equal to one-third of four right angles.

[It should be noticed that the above reasoning fails if any one of the angles of the given triangle is greater than four-thirds of a right angle.]

XIV. THE CIRCUM-CIRCLE, THE IN-CIRCLE AND THE NINE-POINT CIRCLE.

1. *If S, I be the centres and R, r the radii of the circum-circle and the in-circle respectively of a triangle, then will $SI^2 = R^2 - 2Rr$.*

Let ABC be the triangle, and let AI cut the circum-circle again in P.
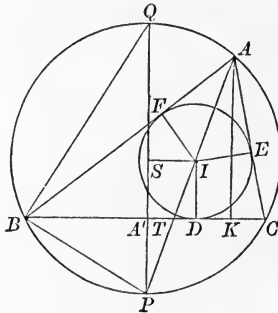
Then $SI^2 = R^2 - AI \cdot IP.$ [Euclid III. 35.

Now $\angle PIB = \angle IAB + \angle IBA = \tfrac{1}{2}A + \tfrac{1}{2}B,$

and $\angle PBI = \angle PBC + \angle CBI = \tfrac{1}{2}A + \tfrac{1}{2}B\,;$

$$\therefore PI = PB.$$

Let PSQ be the diameter through P, then the rt.-angled $\triangle^s$ PBQ, IFA are equiangular, for $\angle BQP = \angle FAI$.

Hence $IF : AI = BP : PQ = IP : PQ\,;$

$$\therefore PQ \cdot IF = AI \cdot IP.$$

Hence $SI^2 = R^2 - 2Rr.$

Cor. *The diameter of the inscribed circle cannot be greater than the radius of the circum-circle.*

[For, since $SI^2 = R^2 - 2Rr$, R^2 cannot be less than $2Rr$, i.e. R cannot be less than $2r$. When $R = 2r$, SI is zero, so that the two centres coincide, and the triangle must in this case be equilateral.]

2. *If in the triangle ABC, the bisector of the angle BAC cut BC in T, and if AK is the perpendicular from A to BC, A' the middle point of BC and D the point of contact of the in-circle; then will $A'D^2 = A'T \cdot A'K$.*

Since P is the middle point of the arc BPC, PA' is perp. to BC.

We know that $PB = PI$.

But PB touches the $\odot ATB$, since

$$\angle PBT = \angle PAC = \angle TAB;$$
$$\therefore PT \cdot PA = PB^2 = PI^2,$$
$$\therefore PT : PI = PI : PA.$$

Hence, as PA', ID, AK are parallel,

$$A'T : A'D = A'D : A'K;$$
$$\therefore A'D^2 = A'T \cdot A'K.$$

3. *If one triangle can be inscribed in one given circle and circumscribed to another, an infinite number of triangles can be so drawn.*

For, if one triangle can be inscribed in the circle whose centre is S and radius R, and circumscribed about the circle whose centre is I and radius r; then we know that

$$SI^2 = R^2 - 2Rr.$$

Hence, if A be any point on the circle S, and AI cut the circle again in P,

$$AI \cdot IP = R^2 - SI^2 = 2Rr.$$
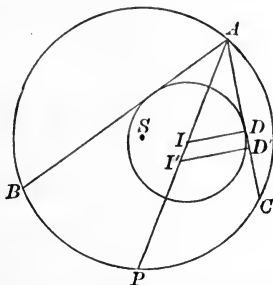
Now let the tangents from A to the circle I cut the circle S in B, C.

Then, if I' be the centre of the circle inscribed in the triangle ABC, I' must be on the line AIP which bisects the angle BAC; and if r' be the radius of the circle inscribed in ABC,

$$AI' \cdot I'P = 2Rr'.$$

Hence $\qquad AI' : r' = 2R : I'P,$

and $\qquad AI : r = 2R : IP.$

But, from similar triangles

$$AI' : r' = AI : r;$$

$\therefore\ IP = I'P$, so that I and I' must coincide.

Thus, if from *any* point A on the circle S tangents be drawn to the circle I which cut S again in the points B, C; then the line BC will also touch the circle I. $\qquad$ [See also page 400.]

Ex. Shew that, if a triangle be inscribed in one given circle and two of its sides touch another given circle, the locus of the in-centre of the triangle is a circle.

For, let ABC be inscribed in the circle whose centre is S and radius R, and let the sides AB, AC touch the circle whose centre is I and radius r, and let I' be the in-centre of ABC and r' its radius.

Let AII' cut the circle S in P.

Then, since I is a fixed point

$$AI \cdot IP = \text{const.}$$

and $\qquad AI' \cdot I'P = 2R \cdot r'.$

Also $\qquad AI : AI' = r : r',$

whence $IP : I'P$ is const., and $\therefore\ IP : II'$ is const.

Hence the locus of I' is similar to the locus of P, and is therefore a circle.

Def. *A* **Porism** *is a problem such that no solution is possible unless a certain relation between the geometrical magnitudes concerned holds good, and when the problem admits of one solution there are an* **infinite number** *of solutions.*

The Porism that, if a single triangle can be inscribed in one given circle and circumscribed to another then an infinite number of such triangles can be so drawn, is due to W. Chapple [*Lady's Diary*, 1746]. Euler's investigation was given in 1765.

4. The following properties of a triangle are often useful.

It will be easily seen that if S be the circumcentre and O the orthocentre of the triangle ABC, then $\angle SAB = \angle OAC$.

$\qquad$ [See figure, page 279.]

[For $\angle C'SA = \frac{1}{2} \angle ASB = \angle ACB$, and rt. $\angle SC'A = $ rt. $\angle ADC$.]

Hence $\angle SAO = \angle BAO - \angle SAB = \angle BAO - \angle CAD$

$$= (90° - B) - (90° - C) = C - B.$$

Also, if I be the in-centre, AI will bisect $\angle BAC$, and $\therefore$ also $\angle SAO$. Hence $\angle SAI = \angle IAO = \frac{1}{2}(C - B)$.

Again, since $A'X$ is parallel to SA,

$$\angle A'XD = \angle SAD = C - B,$$

so that *the chord of the nine-point circle cut off from BC subtends an angle C − B at any point on that circle.*

5. Feuerbach's Theorem. *The nine-point circle of a triangle touches the inscribed and the three escribed circles of the triangle.*

In the triangle ABC let A', B', C' be the middle points of the sides and D, E, F the feet of the perpendiculars; also let K, L, M be the points of contact of the in-circle and a, β, γ the middle points of IA, IB, IC.

Let P be the middle point of the arc $A'D$ of the nine-point circle, and let PK cut the nine-point circle of ABC in O.

[To avoid great complication certain lines are not drawn in the figure. Their absence will not, however, lead to any difficulty in following the reasoning.]
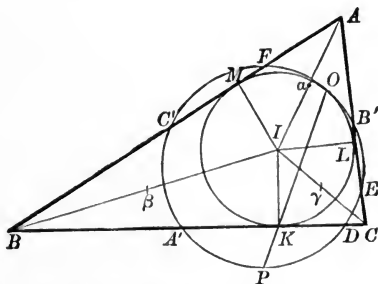
Then $$\angle A'OP = \tfrac{1}{2} A'OD = \tfrac{1}{2}(C - B).$$

Also $$A'\gamma K = \gamma KC - \gamma A'K = \tfrac{1}{2}C - \tfrac{1}{2}B.$$

$$\therefore O \text{ is on the circle } A'\gamma K,$$

i.e. O is on the nine-point circle of the $\triangle BIC$.

Hence $$\angle A'O\beta = \angle A'\gamma\beta = \tfrac{1}{2}B,$$

and $$\angle A'OC' = \angle A'B'C' = B;$$

$$\therefore \angle \beta OC' = \tfrac{1}{2}B = \beta aC';$$

$$\therefore O \text{ is on the circle } \beta aC',$$

i.e. O is on the nine-point circle of $\triangle AIB$.

Hence $$\angle MO\beta = \angle Ma\beta = \angle aMA = \angle IAM = \angle ILM,$$
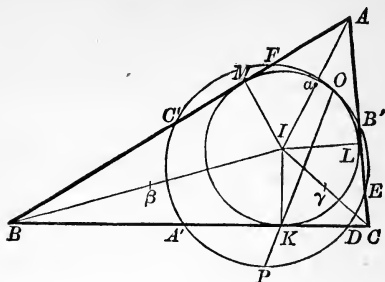
And, since O is on the $\odot \beta \gamma K$,

$$\angle \beta OK = \angle \beta \gamma K = \angle \gamma KC = \angle ICK = \angle ILK.$$

Hence $\angle MOK = \angle MLK;$

$\therefore$ O is *on the inscribed circle* KLM.

And, since the tangents at P and K to the nine-point circle and the in-circle are parallel, and tangents to a circle make equal angles with their common chord, it follows that the tangents at their common point O are coincident, so that the two circles touch at O.

From the above proof it appears that *the nine-point circles of the triangles* BIC, CIA, AIB *pass through the point of contact of the in-circle and the nine-point circle of the triangle* ABC.

[The above proof is substantially the same as that given in the *Géométrie Élémentaire* of Rouché and Comberousse.

The following theorem can be proved in a similar manner.

If A_1, A_2, A_3, A_4 *be any four points on a plane and* P_1, Q_1, R_1 *be the feet of the perpendiculars from* A_1 *on the sides of the triangle* $A_2 A_3 A_4$, *and similarly for* A_2, A_3 *and* A_4; *then will the four nine-point circles of* $A_1 A_2 A_3$ &c. *and the four circum-circles of* $P_1 Q_1 R_1$ &c. *all meet in a point.*]

6.   It can be proved in a similar manner that the nine-point circle of a triangle touches each of the escribed circles.   Moreover, since the nine-point circle of the triangle ABC is also the nine-point circle of each of the triangles BCO, CAO, ABO, where O is the orthocentre of ABC, it follows that

If O *be the orthocentre of the triangle* ABC, *the nine-point circle of the triangle* ABC *touches the sixteen circles which touch the sides of the four triangles* ABC, BCO, CAO, ABO.

7. The following interesting extension of Feuerbach's Theorem was given by T. T. Wilkinson in the *Lady's and Gentleman's Diary* for 1857.

It can be proved without difficulty that *the six radical axes of the inscribed and escribed circles of any triangle ABC intersect perpendicularly in pairs at the middle points of the sides of ABC.*

From this it follows that, if K_1, K_2, K_3, K_4 be the radical centres of the four circles taken in threes, the middle points of the sides of ABC are the feet of the perpendiculars of the triangles $K_1 K_2 K_3$, &c.; and therefore *the nine-point circle of ABC is also the nine-point circle of each of the triangles $K_1 K_2 K_3$, &c.*

Hence, by Feuerbach's Theorem, *the nine-point circle of ABC touches the sixteen circles which touch the sides of the triangles $K_1 K_2 K_3$, &c.*

We can now take the radical centres of the inscribed and escribed circles of $K_1 K_2 K_3$, and prove in the same manner that the nine-point circle of ABC touches sixteen other circles, and so on indefinitely.

8. **Def.** *If each of the sides of a triangle is the polar with respect to a circle of the opposite angular point, the triangle is said to be* **self-polar** *with respect to the circle, and the circle is called the* **polar circle** *of the triangle.*

Since the line joining the centre of a circle to any point is perpendicular to the polar of the point with respect to the circle, it follows that the centre of the polar circle of a given triangle is the orthocentre of the triangle; and, since a point and its polar with respect to a circle are on the same side of the centre of the circle, it follows that an acute-angled triangle cannot be self-polar with respect to any circle.

In the case of a right-angled triangle the polar circle reduces to a point-circle at the right angle, and in this case it is easily seen that the circum-circle and the nine-point circle touch at the right angle. This is a particular case of the following theorem:

9. *The circum-circle, the nine-point circle, and the polar-circle of a triangle have a common radical axis.*

Let S, N, O be the centres of the circum-circle, the nine-point circle and the polar-circle respectively of the triangle ABC.

Then, since O is the orthocentre, we know that $ON = NS$ [p. 279], and that $OD = DL$ [p. 277] where $OADL$ is perpendicular to BC.

Now let P be a point common to the circum-circle and the polar-circle.

Then, since A is the pole of BC with respect to the polar-circle,

$$OD . OA = \text{square of radius of polar-circle} = OP^2.$$
$$\therefore \ 2OP^2 = 2OD . OA = OL . OA = OS^2 - SP^2.$$
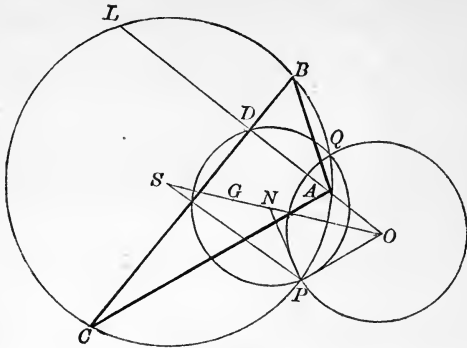
And, since $SN = NO$,

$$4NP^2 + 4SN^2 = 2SP^2 + 2PO^2;$$

$$\therefore\ 4NP^2 + SO^2 = 2SP^2 + OS^2 - SP^2;$$

$$\therefore\ 4NP^2 = SP^2,\ \text{and}\ \therefore\ NP = \tfrac{1}{2}SP.$$

Thus NP is equal to the radius of the nine-point circle, and therefore either point of intersection of the circum-circle and the polar-circle is also on the nine-point circle, which proves the theorem.

Or thus: Let the tangents at A, B, C to the circum-circle cut the opposite sides of the $\triangle$ in L, M, N respectively, and let U, V, W be the middle points of AL, BM, CN respectively.

Let AL cut the polar circle in the points P, P'. Then, since A is the pole of BC with respect to the polar circle, the range $APLP'$ is harmonic.

Hence $$UA^2 = UP \cdot UP'.$$

But, if B', C' are the middle points of CA, AB respectively (and $\therefore$ points on the nine-point circle) $B'C'U$ is a straight line; and

$$UA = \tfrac{1}{2}LA, \quad UB' = \tfrac{1}{2}LC, \quad UC' = \tfrac{1}{2}LB.$$

But $$LA^2 = LB \cdot LC;$$

$$\therefore\ UA^2 = UB' \cdot UC'.$$

Hence the tangents from U to the three circles are all equal; and similarly the tangents from V and from W to the three circles are all equal.

Hence UVW is the radical axis of the three circles.

The circle whose diameter is GO is the circle of similitude of the circum-circle and the nine-point circle, and is therefore co-axal with them [p. 368].

XV. PROPERTIES OF A QUADRILATERAL.

The figure formed by four straight lines indefinitely produced is called a **four-line** or a **complete quadrilateral**.

The four lines have six intersections, namely the points A, B, C, D, E, F in the figure.

There are *three* diagonals (that is lines which join the intersection of two of the given lines to the intersection of the other two), namely the lines AC, BD, EF.

The triangle PQR whose sides lie along the diagonals is called the **diagonal triangle** of the quadrilateral.

1. *Each of the three diagonals of a quadrilateral is divided harmonically by the other two diagonals.*

Consider the triangle AEF; then, since AQ, FB, ED meet in a point,

$$\frac{FQ}{QE} \cdot \frac{EB}{BA} \cdot \frac{AD}{DF} = 1. \quad \text{[Ceva's Theorem.}$$

And, since BDR is a straight line,

$$\frac{FR}{ER} \cdot \frac{EB}{AB} \cdot \frac{AD}{FD} = 1. \quad \text{[Menelaus.}$$
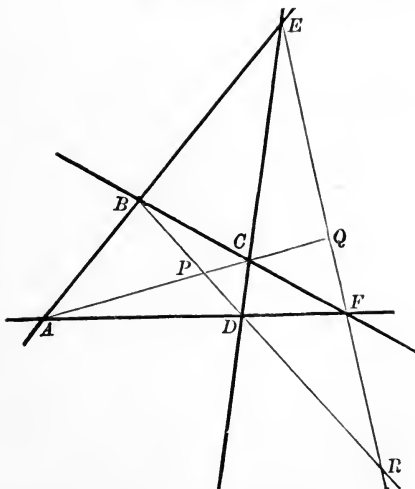
Hence $\quad\quad\quad FQ : QE = FR : ER.$

Thus E, Q, F, R is a harmonic range, and similarly A, P, C, Q and B, P, D, R are harmonic ranges.

Cor. *If ABCD is a cyclic quadrilateral, the triangle PEF is such that each of its angular points is the pole of the opposite side with respect to the circle.*

For, since $APCQ$ is harmonic, and AC is a chord of the circle, the polar of P must pass through Q [p. 359], and it must similarly pass through R.

Hence EF is the polar of P with respect to the circle $ABCD$.

Also, if PE, PF be drawn, the pencil PE, PQ, PF, PR would be cut by ECD and by EBA in points forming harmonic ranges, since E, Q, F, R is a harmonic range; whence it follows that PF is the polar of E and EP the polar of F.

2. *If the sides AB, BC, CD, DA of the quadrilateral ABCD touch a circle in the points α, β, γ, δ respectively; then will αγ and βδ pass through the intersection of the diagonals AC, BD.*

Let $\alpha\beta$, $\gamma\delta$ meet in Q, $\alpha\gamma$, $\beta\delta$ in P, and $\alpha\delta$, $\beta\gamma$ in R.

Then we know that P, Q, R are the poles of QR, RP, PQ respectively with respect to the circle.
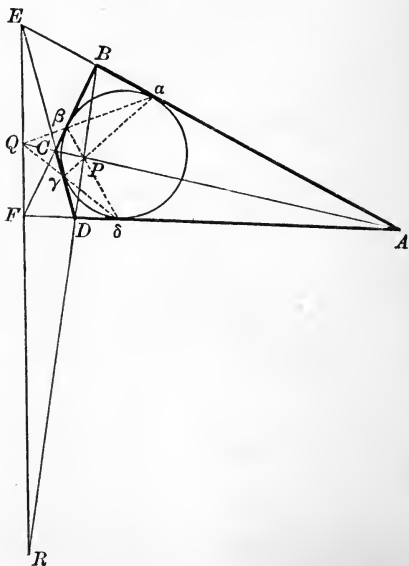
But B is the pole of $\alpha\beta$, and D is the pole of $\gamma\delta$;

∴ BD is the polar of Q, the point of intersection of $\alpha\beta$ and $\gamma\delta$.

But PR is the polar of Q.

Hence

$BPDR$ is a straight line.

Similarly

$APCQ$ is a straight line,

and

$EQFR$ is a straight line.

3. *The middle points of the three diagonals of a quadrilateral lie on a straight line.*

Let a, β, γ be the middle points of the sides BC, CE, EB of the triangle BCE, and U, V, W the middle points of the diagonals AC, BD, EF.
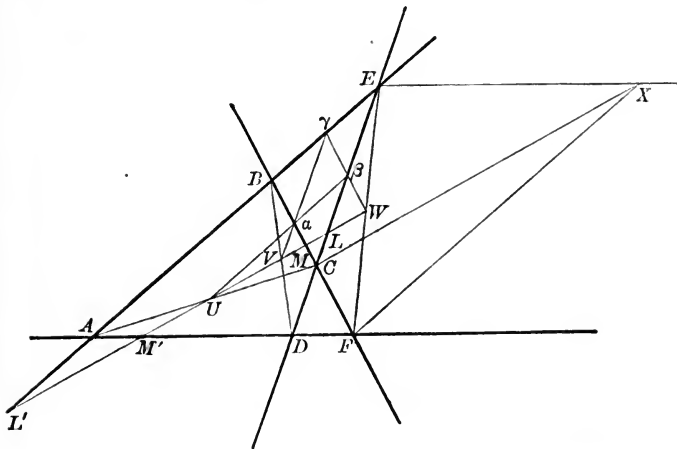
Then, since $E\gamma = \gamma B$, $E\beta = \beta C$ and $EW = WF$, $\gamma\beta W$ is a straight line, and

$$\beta W : \gamma W = CF : BF.$$

Similarly $\gamma a V$ and $a\beta U$ are straight lines, and

$$\gamma V : aV = ED : CD,$$

and

$$aU : \beta U = BA : EA.$$

Hence

$$\frac{\beta W}{\gamma W} \cdot \frac{\gamma V}{aV} \cdot \frac{aU}{\beta U} = \frac{CF}{BF} \cdot \frac{ED}{CD} \cdot \frac{BA}{EA}.$$

But, by the Theorem of Menelaus, since ADF is a straight line

$$\frac{CF}{BF} \cdot \frac{ED}{CD} \cdot \frac{BA}{EA} = 1.$$

Hence

$$\frac{\beta W}{\gamma W} \cdot \frac{\gamma V}{aV} \cdot \frac{aU}{\beta U} = 1,$$

whence it follows that UVW is a straight line.

If the parallelogram $EAFX$ be completed, W will be the middle point of the diagonal AX, and therefore UVW will be parallel to CX, a result which is sometimes useful.

Ex. 1. If U, V, W be the middle points of the diagonals of a quadrilateral, W being on the external diagonal EF, and if the line UVW cut the opposite sides BC, AD in M, M' and the opposite sides CD, AB in L, L'; then will $WM.WM' = WL.WL' = WU.WV$. Shew also that each of these rectangles is equal to WF^2, if $ABCD$ is cyclic.

Since γa is $\parallel$ to EL, and $\beta a \parallel$ to EB,

$$WL : WV = W\beta : W\gamma = WU : WL';$$

$$\therefore \ WL.WL' = WU.WV.$$

Similarly $$WM.WM' = WU.WV.$$

Now, if $ABCD$ is cyclic, it is easily seen that $FCEX$ is also cyclic.

Hence $\angle ELW = \angle ECX = \angle EFX = \angle AEW$;

$$\therefore \ \angle EL'W = \angle LEW.$$

Hence WE touches the circle ELL';

$$\therefore \ WL.WL' = WE^2.$$

Ex. 2. If $ABCD$ be a cyclic quadrilateral, the exterior diagonal will subtend supplementary angles at the middle points of the diagonals AC and BD.

4. (1) *The orthocentres of the four triangles formed by four straight lines are on a straight line,*

(2) *the three circles whose diameters are the diagonals of the quadrilateral are co-axal,*

(3) *the middle points of the diagonals are on a straight line,*

and (4) *the circles on the diagonals cut the polar circles of the four triangles and also the circum-circle of the diagonal triangle orthogonally.*

Let O_1 be the orthocentre of the triangle EBC.

Then, since $\angle EeC = \text{rt.} \angle = \angle EcC$, the points E, e, c, C are cyclic, and $O_1e.O_1E = O_1c.O_1C$.

Hence O_1 is on the radical axis of the circles whose diameters are EF and AC respectively.

Similarly, if O_2, O_3, O_4 are the orthocentres of the triangles ABF, FCD, DAE respectively, the points O_2, O_3, O_4 will also be on the radical axis of the circles whose diameters are EF and AC.

Hence *the four orthocentres are on a straight line.*

Also the line of orthocentres is the radical axis of the circles EF and AC, and it can be proved in a similar manner that the orthocentres are on the radical axis of the circles EF and BD.

Hence *the circles whose diameters are the diagonals of any quadrilateral have a common radical axis.*
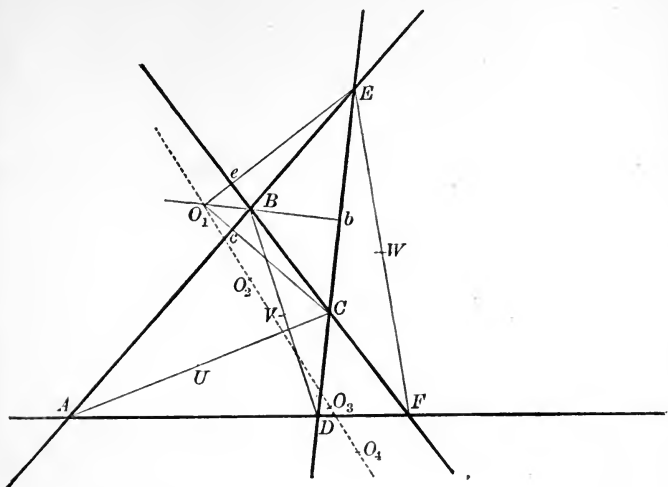
But the centres of all co-axal circles lie on a straight line.

Hence *the middle points of the three diagonals of any quadrilateral lie on a straight line.*

Again, the square on the radius of the polar circle of the triangle EBC is equal to $O_1B \cdot O_1b = O_1C \cdot O_1c = O_1E \cdot O_1e$, so that the radius of the polar circle of EBC is equal to the tangent from its centre to any one of the circles whose diameters are the diagonals of the quadrilateral.

Hence *the circles whose diameters are EF, AC, BD all cut orthogonally the polar circle of the triangle EBC, and therefore similarly the polar circles of the triangles EAD, CDF and ABF.*

And, since $APCQ$ is harmonic [figure, p. 393], $UC^2 = UP \cdot UQ$, whence it follows that the circle PQR is cut orthogonally by the circle whose diameter is AC, and similarly by the circles on the other diagonals.

Also, since the four polar circles and the circum-circle of the diagonal triangle are cut orthogonally by more than one circle, *these five circles must be co-axal.*

[The above elegant proof was first given by M. Mention in *Nouvelles Annales,* t. xii.]

5. Another proof that the middle points of the diagonals lie on a straight line depends upon the following theorem :

If AB and CD are two given straight lines, the locus of a point P which moves so that the sum of the areas of the triangles PAB, PCD is constant, is a straight line.

Let O be the point of intersection of the given straight lines AB and CD.

Then, if OX, OY be measured along OA, OC so that $OX = AB$ and $OY = CD$, it is obvious that $\triangle OPX = \triangle APB$ and $\triangle OPY = \triangle PCD$.
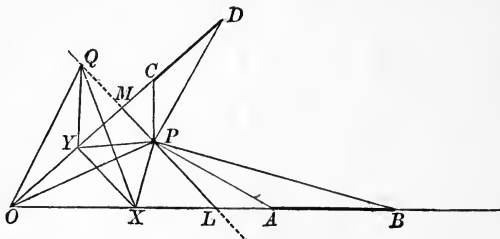
Hence P must move so that the area $OXPY$ is constant.

Hence $\triangle XPY$ must be constant, and therefore *the locus of P is a straight line parallel to XY.*

It is important to notice that, if the line on which P moves cut the given lines in L, M, and Q be any point on LM produced, it will easily be seen that the *difference* of the triangles OQX, OYQ is equal to $OXPY$.

Hence we may say that the locus of P which moves so that the sum of $\triangle^s$ APB and CPD is constant is a straight line, provided that either of the triangles APB, CPD is considered to change from *positive* to *negative* when P crosses AB or CD, or when the direction of rotation in APB, or CPD, changes.

Let U, V, W be the middle points of the diagonals AC, BD, EF of a complete quadrilateral.

Then $\triangle AUB = \frac{1}{2} \triangle ACB$ and $\triangle DUC = \frac{1}{2} \triangle DAC$.

Hence $\triangle AUB + \triangle DUC =$ half figure $ABCD$.

Similarly $\triangle AVB + \triangle DVC =$ half figure $ABCD$.

Again, $\triangle AWB = \frac{1}{2} \triangle ABF$ and $\triangle DWC = \frac{1}{2} \triangle DFC$;

          $\therefore \triangle AWB - \triangle DWC =$ half figure $ABCD$;

but W and U are on opposite sides of CD.

Hence W is on the line UV.

If a circle whose centre is O can be inscribed in $ABCD$, it is easily seen that $\triangle AOB + \triangle COD =$ half figure $ABCD$, and $\therefore$ O is on UVW.

Thus, *if a quadrilateral be circumscribed to a circle, the centre of the circle is on the line through the middle points of the diagonals.*

[NEWTON'S THEOREM.

XVI. PONCELET'S THEOREM.

1. *If the chords AB, A'B' of one circle touch a second circle in the points P, P', then will the chords AA', BB' touch a third circle co-axal with the two former circles, the points of contact lying along PP'.*

Let AA', BB' cut PP' in the points Q', Q respectively.
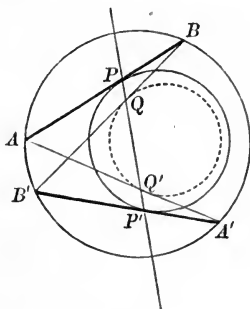
Then the tangents AB, $A'B'$ make equal angles with PP', and $\angle ABQ = \angle AA'B'$.

Hence $\angle BQQ' = \angle A'Q'Q$, and therefore a circle can be drawn to touch AA', BB' at Q', Q respectively.

Also the $\triangle^s$ PBQ, $P'A'Q'$ are similar, and the $\triangle^s$ APQ', $B'P'Q$ are similar;

$$\therefore\ BP : BQ = A'P' : A'Q',$$

and $$AP : AQ' = B'P' : B'Q.$$

Moreover, if AX be drawn parallel to BB' so as to cut PP' in X, $\angle AXP = \angle BQP = \angle AQ'Q$, so that $AX = AQ'$; and, since AX is $\parallel$ to BQ,

$$BP : BQ = AP : AX = AP : AQ'.$$

The ratios $BP : BQ$, $AP : AQ'$, $B'P' : B'Q$ and $A'P' : A'Q'$ are therefore all equal.

The circle through A, B, A', B' is therefore co-axal with the other two circles. [Page 357.]

It can be shewn in the same manner that another co-axal circle will touch the lines AB' and BA'. [Hart, *Q. J.* vol. II.

2. **Poncelet's Theorem.** *If a triangle be inscribed in one given circle and two of its sides touch another given circle, the third side will always touch a fixed co-axal circle.*

Let ABC, $A'B'C'$ be any two triangles inscribed in one given circle, and let the sides BA, BC, $B'A'$, $B'C'$ touch a second given circle in the points p, q, p', q' respectively.
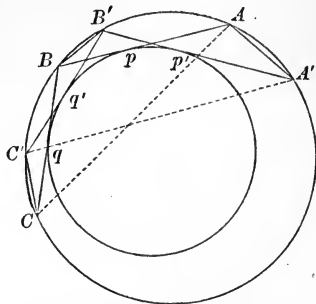
Then, by the preceding theorem, AA' and BB' will both touch a co-axal circle, the points of contact being on pp'.

So also BB' and CC' will both touch a co-axal circle, the points of contact being on qq'.

Now B is the pole of pq, and B' the pole of $p'q'$ with respect to the inner circle. Hence BB' is the polar of the point of intersection of pq and $p'q'$. But pp' and qq' meet on the polar of the point of intersection of pq and $p'q'$ [p. 394], and therefore on BB'.

Hence the co-axal circle which touches both AA' and BB', and the co-axal circle which touches both BB' and CC' touch BB' in the same point, and must therefore be the same co-axal circle. [Page 355.]

Thus AA' and CC' both touch the same co-axal circle.

Hence, by the preceding theorem, AC and $A'C'$ touch another co-axal circle.

Thus AC in all its positions *touches a fixed co-axal circle.*

It follows that, if AC in any one of its positions touches the circle to which BA, BC are tangents, it will touch the same circle in all positions; so that, *if* **one** *triangle be inscribed in one given circle and circumscribed to another, an* **infinite number** *of such triangles can be drawn.*

The theorems can easily be extended to the following :

If any polygon be inscribed in a given circle, and all its sides but one touch another given circle, the remaining side will touch a fixed co-axal circle.

If any **one** *polygon can be inscribed in one given circle and circumscribed to another, there will be an* **infinite number** *of such polygons.*

3. In the same way can be proved the following theorem :

If a triangle be inscribed in one given circle of a co-axal system, and if two of its sides touch two other given circles of the system, then will the third side touch one or other of two fixed circles of the co-axal system.

4. Let the circles whose centres are S, I and radii R, r respectively be such that triangles can be inscribed in the first whose sides touch the second, and let ABC be one such triangle which touches the circle I in the points P, Q, R.
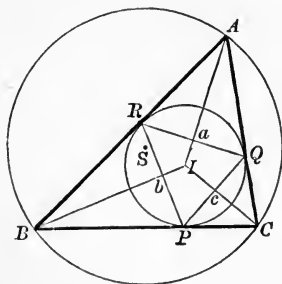
Then IA will pass through a, the middle point of QR, and $Ia . IA = r^2$.

Hence the circum-circle of abc, that is *the nine-point circle of PQR, is the inverse of the circle S with respect to the circle I, and is therefore a fixed circle.*

This fixed nine-point circle of PQR always touches the in-circle of PQR. Hence *the in-circle of PQR touches a fixed circle.*

Again, the centroid and the orthocentre of the triangle PQR are the centres of similitude of the circle PQR and its nine-point circle; hence, as these circles are both fixed, *the centroid and the orthocentre of the triangle PQR are fixed points.*

Also, if the tangents at A, B, C form another triangle XYZ, it will easily be seen that the circum-circle of XYZ is the inverse of the nine-point circle of ABC with respect to S.

But the nine-point circle of ABC touches the circle PQR.

Hence the circum-circle of XYZ will touch the inverse of the circle PQR with respect to S; so that *the circum-circle of XYZ for all possible positions of ABC will touch a fixed circle.*

5.   *If chords of a circle subtend a right angle at a given point O ; then* (i) *the locus of the middle points of the chords is a circle,* (ii) *the locus of the poles of the chords is a circle, and* (iii) *these three circles belong to a co-axal system of which O is a limiting point.*

Let POp, $P'Op'$ be any two perpendicular chords of the given circle through the given point O.

Let C be the centre of the circle, and M the middle point of OC.
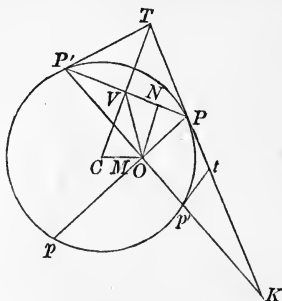
Let the tangents at P, P' meet in T; then CT will be perpendicular to PP' and will bisect it in V.

Since POP' is a right angle, and V is the middle point of PP', $PV = VP' = VO$.

Now $2CM^2 + 2MV^2 = CV^2 + OV^2 = CV^2 + VP^2 =$ square on radius.

Hence MV is of constant length.

Thus *the locus of the middle point of PP' is a circle whose centre is the middle point of CO.*

[If ON be the $\perp^r$ from O on PP', it is easily seen that $MN = MV$, so that the locus of N coincides with that of V.]

Again, since $CV \cdot CT = $ (radius)^2, the locus of T is the inverse of the locus of V with respect to the given circle. Hence *the locus of T is also a circle.*

Produce $P'Op'$ to meet the tangent at P in K, and let the tangent at p' cut TPK in t.

Then, since $P'T$ and $p't$ make equal angles with $P'p'K$,

$$TK : tK = TP' : tp'$$
$$= TP : Pt.$$

Hence, by the theorem of Apollonius, the ratio of the distances from T and t of all points on the circle whose diameter is PK, is equal to $TP : Pt$; and O is on this circle since POK is a rt. angle.

Hence $$TO : tO = TP : tP,$$

or $$TO : TP = tO : tP,$$

whence it follows that the circle on which T, t lie is co-axal with the given circle and a point-circle at O.

Again, since $VO^2 = PV \cdot VP'$, it follows that the locus of V is co-axal with the given circle and a point-circle at O.

Thus *the circles on which V and T lie belong to the co-axal system determined by the given circle and a point-circle at O.*

6. Let $PQRS$ be a quadrilateral inscribed in one given circle and whose sides touch another given circle in the points K, L, M, N.
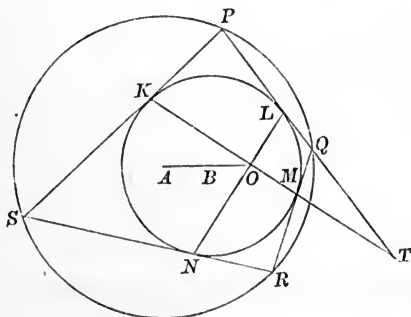
It is easily proved that KM and LN intersect at right angles, at O suppose.

Also it can be proved as above that $PO : PL = QO : QL$, and therefore also that $PO : PL = QO : QL = RO : RM = SO : SN$.

Hence O is a point-circle belonging to the co-axal system defined by the given circles.

Hence *the diagonals of the quadrilateral formed by joining the points of contact of any one of the quadrilaterals which are inscribed in one of the given circles and circumscribed to the other meet in a* **fixed point** *on the line joining the centres of the given circles.*

If A, B are the centres of the two given circles, and we draw two perpendicular chords through O equally inclined to the straight line ABO, meeting the circle which is circumscribed by the quadrilaterals in K, M and L, N, two of the angular points of the quadrilaterals will be at the extremities of the diameter through A, B and O.
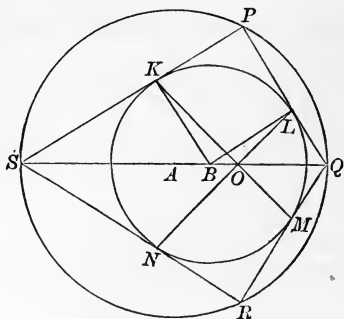
Hence in that case $KBLP$ will be a square, and

$$\frac{KB^2}{SB^2} + \frac{BL^2}{BQ^2} = \frac{PQ^2}{SQ^2} + \frac{SP^2}{SQ^2} = 1.$$

Hence

$$\frac{r^2}{(R+d)^2} + \frac{r^2}{(R-d)^2} = 1,$$

where R, r are the radii of the two circles and d the distance between their centres.

N.B. It should be noticed that in the above investigation it has been understood that $PQRS$ is a *convex* quadrilateral.

If $PQRS$ is a crossed quadrilateral, the above relation between R, r and d does not hold good. It can, however, be easily proved that, in this case, the centre of the circle $PQRS$ is on the circumference of the other circle. Conversely, *if the centre of one circle be on the circumference of a second, an infinite number of quadrilaterals, in which one pair of opposite sides intersect, can be inscribed in the first circle each of whose sides will touch the second.*

XVII. MISCELLANEOUS PROBLEMS.

1. *To inscribe a square in a given triangle.*

It is required to inscribe a square in the triangle ABC so that two of the angular points of the square may be on one of the sides of the triangle (on BC suppose) and the remaining angular points on CA, AB respectively.
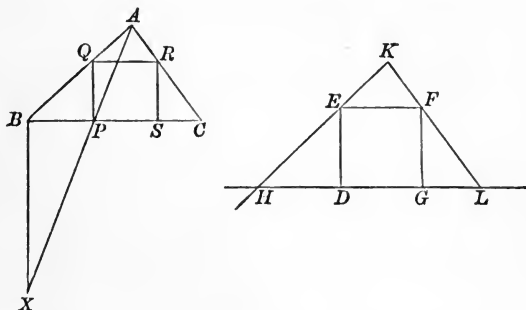
Let $PQRS$ be the square required.

Then, since $QR : BC = AQ : AB$, and $QR = QP$,

$$QP : BC = AQ : AB.$$

Now, if AP be produced to cut the line through B perpendicular to BC in X,

$$QP : BX = AQ : AB.$$

Hence BX must be equal to BC, so that the following is the construction required :

Draw BX perpendicular and equal to BC. Join AX cutting BC in P. Then draw $PQ \parallel$ to BX cutting BA in Q, $QR \parallel BC$ cutting AC in R, and RS perpendicular to BC. Then it can be easily proved that $PQRS$ is a square, and it is described in the required manner.

Or thus : Instead of the problem proposed consider the following :

About a given square describe a triangle similar to a given triangle.

The necessary construction in this case is obvious ; for if *DEFG* is the given square and *ABC* the given triangle, we have only to draw through *E* and *F* lines *KEH*, *KFL* making with *DG* produced angles equal to *ABC*, *ACB* respectively.

To complete the original problem we have now only to **alter the scale.**

Divide *AB* in *Q* so that $AQ : QB = KE : EH$. Draw *QR* parallel to *BC* and complete the figure.

Then it is easily seen that

$$QR : EF = AQ : KE = BQ : HE = QP : ED.$$

Hence $QR = QP$, for $EF = ED$.

Thus the figure *PQRS*, which is a rectangle by construction, is a square.

2. In many cases a problem can be solved more readily by first solving a cognate problem and then altering the scale of the diagram, as in the above case.

For example : *To inscribe a square in a given regular pentagon.*

It is easily seen that one side of the square must be parallel to a side of the pentagon.

Now take *any* square *ABCD*, having the side *DA* parallel to the side *TP* of the given pentagon *PQRST*.

Draw through *A*, *B*, *C*, *D* lines parallel to *PQ*, *QR*, *RS*, *ST* respectively.

Then the pentagon circumscribing *ABCD* can easily be completed, and a square can then be inscribed in the given pentagon by altering the scale.

Again : *To find three points P, Q, R, one on each of the circumferences of three given concentric circles, so that the triangle PQR may be equilateral (or of given species).*

The solution of the cognate problem

To find a point the ratios of whose distances from the angular points of a given triangle may be equal to given ratios (namely to the ratios of the radii of the given circles) by means of circles of Apollonius suggests itself at once, and an alteration of the scale gives the solution of the original problem.

The following is another problem which is readily solved in this manner :

To describe a quadrilateral of given species (that is, similar to a given quadrilateral) *so that its angular points may lie on the sides of another given quadrilateral.*

In order to construct a quadrilateral $PQRS$ similar to a given quadrilateral and whose sides PQ, QR, RS, SP pass respectively through the given points A, B, C, D, it is at once obvious that P, Q, R, S lie on arcs of known circles through D and A, A and B, B and C, and C and D respectively, since the angles at P, Q, R, S are given angles. Moreover, it is easily seen that we may start with *any* point P of the arc on which P lies and complete a figure $PQRS$ which will be equiangular to the given quadrilateral [cf. Book III., page 241].

In order, however, that the quadrilateral $PQRS$ may be *similar* to the given quadrilateral, it is also necessary, and sufficient, that the diagonal PR should make a given angle with PQ, and therefore also with QR.

Hence PR must cut the circles DPA, BRC, in points X, Y suppose, such that the angles XPA and YRB are each equal to known angles ; the points X and Y can therefore be found.

Find the points X, Y ; then XY will cut the circles DPA, BRC in two of the vertices of the required quadrilateral, and the other two vertices can be at once found.

To complete the construction for the problem originally proposed, we have only to divide the sides of the given quadrilateral corresponding to PQ, QR, &c. in the ratios $PB : BQ$, $QC : CR$, &c., and join the points of section. The proof that the quadrilateral formed by joining these points of section is similar to $ABCD$ is obvious.

N.B. The above construction becomes porismatic when three (and therefore all four) of the circles DPA, AQB, BRC, CSD meet in a point.

For let O be on the three circles DPA, AQB and BRC, and let PR cut the circle DPA in X.

Then, since $XPAO$ and $OAQB$ are circles,

$$\angle OXR = \angle PAO = \angle OBQ ;$$

$\therefore$ X is on the circle $OBRC$.

Hence, when the circles on which P, Q, R, S lie have a common point, the diagonal PR always passes through the other point of intersection of the circles DPA and BRC, and therefore PR makes a constant angle with PA, whence it follows that *all quadrilaterals through A, B, C, D which are equiangular to the given quadrilateral are similar to one another.*

Hence, in this case, if *one* quadrilateral can be drawn through A, B, C, D similar to the given quadrilateral, there will be an *infinite number* of such quadrilaterals.

For example, if A, B, C, D be four points such that AC is perpendicular to BD, all rectangles whose sides in order pass through A, B, C, D respectively will be similar to one another, and therefore an infinite number of squares or no square at all can be drawn to circumscribe $ABCD$ when the diagonals AC and BD are at right angles.

3. Castillon's Problem. *In a given circle to inscribe a triangle whose sides pass respectively through three given points.*

Let P, Q, R be the three given points, and suppose that ABC is the triangle required.
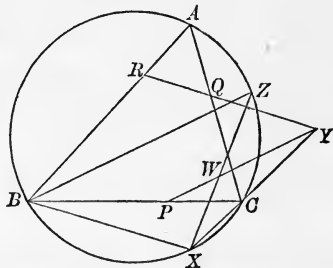
Draw the chord BX parallel to RQ. Join XC and produce XC and RQ to meet in Y.

Then $\angle ARQ = \angle ABX = \angle ACY$.

Hence A, R, C, Y are cyclic,

and $\therefore RQ \cdot QY = AQ \cdot QC = \text{constant}$,

since Q is a fixed point.

Hence Y is a *fixed* point. Hence in the triangle BXC the sides BC, XC pass through fixed points P, Y respectively, and the third side is parallel to RQ.

Now draw the chord BZ parallel to PY. Join XZ cutting PY in W.

Then $\angle PWX = \angle BZX = \angle BCX$.

Hence P, W, C, X are cyclic, and therefore,

$$YW \cdot YP = YC \cdot YX = \text{constant},$$

since Y is a fixed point.

Hence W is a *fixed* point.

But, since BX and BZ are ∥ to the fixed lines RQ, PY respectively, the angle XBZ is constant, whence it follows that XZ is of fixed length, and therefore XZ touches a fixed concentric circle, and it also passes through the fixed point W. The line XZ can therefore be drawn, and when XZ is found the triangle ABC is determined.

MISCELLANEOUS EXERCISES.

1. Shew that the sum of the lengths of the perpendiculars drawn on the sides of an equilateral triangle from any point within it is constant.

2. Shew that the sum of the lengths of the perpendiculars drawn on the sides of any regular polygon from any point within it is constant.

3. E, F, G, H are the middle points of the sides AB, BC, CD, DA of the parallelogram $ABCD$. The lines AG, CH meet in X, and the lines AF, CE meet in Y. Shew that X is a point of trisection of each of the lines GA, HC, DB and that $AXCY$ is a parallelogram whose area is one-third that of $ABCD$.

4. Find a point O such that if it be joined to the extremities of three given straight lines AB, CD and EF, the three triangles AOB, COD and EOF will all be equal.

5. Two circles intersect in the points A, B and through A a line PAQ is drawn to cut the circles in P, Q respectively; shew that the ratio $BP : BQ$ is constant.

6. Through a given point on the circumference of a circle draw two chords inclined at a given angle and in a given ratio.

7. Find a straight line such that the perpendiculars on it from three given points may be in given ratios.

8. On the circle circumscribing the triangle ABC, two points D, E are taken such that $AD^2 = AE^2 = AB \cdot AC$. Shew that, if DE cut BC in K, AK will bisect the angle BAC.

9. The internal and external bisectors of the angle BAC meet BC the opposite side of the triangle in the points D, E respectively; shew that the circles ABD and ABE cut one another at right angles.

10. Shew that, if two circles cut one another orthogonally, any line through the centre of either which cuts both circles is divided harmonically.

11. Construct a triangle having given the base, the ratio of the other sides, and the sum of the squares on these sides.

12. One vertex of a triangle of given species is fixed, and another moves on a given straight line; find the locus of the third vertex.

13. The bisector of the angle BAC of the triangle ABC cuts the circum-circle in G, and GL, GM are drawn perpendicular to AB, AC respectively; shew that $2AL = 2AM = AB + AC$.

14. In a given triangle inscribe another with its sides parallel respectively to three given straight lines.

15. A quadrilateral is divided into four triangles by its diagonals; shew that the centroids of these four triangles are at the angular points of a parallelogram.

16. Having given the inscribed and the circumscribed circles of a triangle, shew that the centres of the escribed circles lie on a fixed circle.

17. Shew that, if I is the in-centre of the triangle ABC, and O_1 the centre of the escribed circle corresponding to the side BC, then will

$$AI \cdot AO_1 = AB \cdot AC.$$

18. Shew that, if three circles touch each other, the three lines joining the centre of each to the point of contact of the other two will meet in a point.

19. In a given triangle inscribe a parallelogram similar to a given parallelogram.

20. Shew that, if ABC be any triangle and PQR any triangle circumscribing ABC, and XYZ the triangle circumscribing PQR with its sides parallel to the corresponding sides of ABC, then will

$$\triangle XYZ : \triangle PQR = \triangle PQR : \triangle ABC.$$

21. Shew that if the nine-point circle of a triangle be given and also one of the angular points, the loci of the orthocentre, the centroid and the circum-centre are circles.

22. The tangents at A, B, C to the circum-circle of the triangle ABC meet the sides BC, CA, AB in P, Q, R respectively; shew that P, Q, R lie on a straight line.

23. Shew that, if the lines AO, BO, CO meet the sides of the triangle ABC in X, Y, Z respectively; then will

$$\frac{AO}{AX} + \frac{BO}{BY} + \frac{CO}{CZ} = 2.$$

24. AA', BB', CC' are diameters of a circle, and they meet the sides BC, CA, AB of the triangle ABC respectively in X, Y, Z; shew that

$$\frac{A'X}{XA} + \frac{B'Y}{YB} + \frac{C'Z}{ZC} = 1.$$

25. Construct a triangle having given one angle, the ratio of the sides containing that angle, and the length of the diameter of the circum-circle.

26. If I, O_1, O_2, O_3 are the centres of the four circles which touch the sides of a triangle whose circum-centre is S, and if G, G_1, G_2, G_3 be the centroids of the triangles $O_1O_2O_3$, O_2O_3I, O_3IO_1, IO_1O_2 respectively, then will ISG, O_1SG_1, O_2SG_2 and O_3SG_3 be straight lines.

27. Having given the circum-circle of a triangle and one of its sides, find the loci of the centroid, the orthocentre and the nine-point centre.

28. Find the locus of the middle points of the sides of all triangles which have a given orthocentre and are inscribed in a given circle.

29. Shew that, if two circles cut one another orthogonally, the chord of contact of the tangents drawn to one circle from any point P on the other will pass through the other extremity of the diameter through P.

30. Shew that the polar of a fixed point with respect to a system of co-axal circles passes through a fixed point.

31. Shew that, if two circles cut orthogonally, the rectangle contained by their common chord and the maximum chord of the two circles which can be drawn through a common point is equal to the rectangle contained by the diameters of the circles.

32. Prove that, if a circle pass through two fixed points O, O' and cut a given circle in the points Q, Q', then will the ratio $OQ \cdot OQ'$ to $O'Q \cdot O'Q'$ be constant.

33. The lines joining each of the centres of three given circles to the centres of similitude of the other two pass by threes through four points.

34. Two circles touch at the point A and any points P, P' are taken on the two circles such that PAP' is a right angle. Shew that PP' passes through a fixed point.

35. Draw a line cutting the sides BC, CA, AB of the triangle ABC, or these sides produced, in the points P, Q, R respectively so that PQ and QR may be equal to given straight lines.

36. Two points B, C are taken on two given straight lines AX, AY respectively such that $AB + AC$ is a given length; shew that the locus of the centre of the circle ABC is a straight line.

37. From any point on the radical axis of two given circles a tangent is drawn to each of the circles; shew that the line joining the points of contact will pass through one or other of the centres of similitude of the two given circles.

38. Shew that, if a circle cut two given circles orthogonally, the line joining either point in which it cuts one of the given circles to either of the points in which it cuts the other will pass through one of the centres of similitude of the given circles.

39. A line AB of constant length slides with its ends on two fixed lines CAX, CBY. Prove that the locus of the orthocentre of the triangle CAB is a circle.

40. Shew that the three radical axes of the in-circle and the three escribed circles of any triangle are the three lines through the middle points of the sides perpendicular to the internal bisectors of the opposite angles of the triangle.

41. Find the side of a square having given the lengths of the three lines joining a point to three consecutive angular points of the square.

42. Shew that, if G is the centroid of the triangle ABC, the algebraic sum of the perpendiculars from A, B, C on any line through G is zero.

43. Shew that, if the lines AO, BO, CO be produced to cut the sides BC, CA, AB of the triangle ABC in the points A', B', C' respectively, and the circle $A'B'C'$ cuts the sides of ABC again in A'', B'', C'' respectively, then will AA'', BB'', CC'' meet in a point.

44. From any point O the perpendiculars OX, OY, OZ are drawn to the sides BC, CA, AB of the triangle ABC, and the circle XYZ cuts the sides again in X', Y', Z'. Shew that the perpendiculars from X', Y', Z' to the sides on which they lie will also meet in a point.

45. Points A', B', C' are taken on the sides BC, CA, AB respectively of the triangle ABC such that AA', BB', CC' meet in a point. Shew that the lines joining A, B, C respectively to the middle points of $B'C'$, $C'A'$, $A'B'$ will also meet in a point.

46. Shew that, if lines from the vertices of a triangle to the opposite sides pass through the circum-centre, the circles on these lines as diameters will touch the nine-point circle.

47. Describe a circle to pass through two given points and to subtend a given angle at a third given point.

48. Shew that, if four points A, B, C, D be taken on a circle of radius R (1) the centroids of the triangles BCD, CDA, DAB, ABC all lie on a circle of radius $\frac{R}{3}$, (2) the orthocentres lie on a circle of radius R, and (3) the nine-point centres are on a circle of radius $\frac{R}{2}$.

49. In a given triangle inscribe a rectangle one of whose sides is parallel to the base, and which is such that the difference of two adjacent sides is equal to a given straight line.

50. If from each of the vertices of a quadrilateral inscribed in a circle perpendiculars are drawn on the two opposite sides, and the feet of these perpendiculars are joined, prove that the four lines so found are equal.

51. Draw a circle of given radius so as to cut two given circles orthogonally.

52. Describe a circle passing through a given point and cutting two given straight lines so that the intercepted chords may subtend given angles at the centre.

53. Shew that the distance between the polars of the centres of similitude of two circles with respect to one of the circles is equal to the distance between the polars with respect to the other circle.

54. From a point without a given circle draw a straight line to cut the circle so that the rectangle contained by the whole line and the part within the circle may be equal to the square on the part without the circle.

55. O is any point within the triangle ABC. Find points D, E on AB, AC respectively such that $DO : EO$ may be equal to a given ratio and that DOE may be equal to a given angle.

56. Shew that, if O be the orthocentre of the triangle ABC, and if P, Q, R be the circum-centres of the triangles BOC, COA, AOB, then will the triangle PQR be equal in all respects to ABC, and the lines AP, BQ, CR will all pass through the nine-point centre of ABC.

57. In a given triangle ABC inscribe a rhombus having one of its angular points at a given point on BC, and the other angular points on BC, CA, AB respectively.

58. In a given triangle ABC inscribe another triangle $A'B'C'$ similar to a given triangle PQR and having one of its sides parallel to a given straight line.

59. Find points A, B on the two given straight lines OX, OY respectively, such that AB may subtend a given angle at the given point P, and that the distances AP, BP may be in a given ratio.

60. The tangents to a circle at the points A, B intersect at right angles at D, and the tangent at any point P of the circle cuts AB in Q. Shew that DA and DB are the bisectors of the angle PDQ.

61. The opposite sides of a cyclic quadrilateral meet in P, Q and about the four triangles thus formed circles are described; prove that the tangents to these circles at P and Q form a quadrilateral equal in all respects to the original.

62. Two circles intersect in A, B and any straight line through A cuts them again in P and Q respectively. Shew that the locus of the point which divides PQ in a given ratio is a circle.

63. If each side of a parallelogram, produced if necessary, pass through one of four fixed points which lie on a given straight line, then will each diagonal also pass through a fixed point on the given straight line.

64. A point P being given in the side BC of the triangle ABC, draw a straight line parallel to BC and cutting AB, AC in B', C' so that the angle $B'PC'$ may be a right angle.

65. Two circles touch one another at C, and any straight line is drawn cutting one circle in P, Q and the other circle in R, S. Prove that the ratio $PR \cdot PS : PC^2$ is constant.

66. Through one of the points of intersection of two circles a line is drawn, and the points in which it meets the circles are joined to the other point of intersection; prove that the orthocentre of the triangle so formed lies on a fixed circle.

67. On a given circle are taken two fixed points A, B and a variable point P. On BP or its production a point Q is taken such that $BQ : AP$ is a given ratio. Prove that the locus of Q consists of two circular arcs.

68. In a given triangle ABC inscribe a triangle $A'B'C'$ so that the lines joining A, B, C to the middle points of $B'C'$, $C'A'$, $A'B'$ respectively may pass through a given fixed point.

69. Find a pair of points on a given circle concyclic with each of two given pairs of points.

70. Describe an equilateral triangle with one of its angular points at a given point, a second on a given straight line, and a third on a given circle.

71. Construct a triangle whose angular points shall be one on each of three given circles, and whose sides shall pass through three of their centres of similitude.

72. A', B', C' are the middle points of the sides BC, CA, AB of the triangle ABC, and $A'P$, $B'Q$, $C'R$ are tangents from A', B', C' to the in-circle whose centre is I; shew that AP, BQ, CR are parallel to $A'I$, $B'I$, $C'I$ respectively, and that they meet in a point.

73. ABC, $A'B'C'$ are two directly similar triangles but are not similarly situated. Find a point O such that

$$AO : A'O = AB : A'B = BO : B'O = CO : C'O.$$

74. Describe a triangle equal in all respects to a given triangle, and with its angular points respectively on three given straight lines which meet in a point.

75. Describe when possible a triangle equal in all respects to a given triangle, and with its angular points respectively on the sides of a given triangle.

76. Through a given point P draw a line cutting two given circles in the points A, B so that the rectangle $PA . PB$ may be equal to a given square.

77. Construct a triangle having given the lengths of two of its sides and the length of the line bisecting the angle between them and terminated by the third side.

78. Construct a triangle having given the lengths of two of its sides and also of the line from their intersection to a point which divides the base in a given ratio.

79. Find three points A, B, C, one on each of three given circles whose centres are P, Q, R respectively, such that BC, CA, AB are parallel respectively to QR, RP, PQ.

80. Describe two equal circles to touch one another and also to touch two given straight lines at given points respectively.

81. Describe two circles whose radii are in a given ratio to touch one another and also to touch two given straight lines at given points.

82. Draw tangents, one to each of two given circles, inclined at a given angle and such that the sum of their lengths is equal to a given straight line.

83. Shew that, if AOL, BOM, CON be drawn through any point O to cut BC, CA, AB in L, M, N respectively, and if A', B', C', P, Q, R be the middle points of BC, CA, AB, AL, BM, CN respectively; then will $A'P$, $B'Q$, $C'R$ meet in a point.

84. Describe a circle such that the tangents to it from three given points may be of given lengths.

85. Draw a straight line parallel to a given straight line so that the part of it intercepted by two other given straight lines may subtend a given angle at a given point.

86. AD is perpendicular to the side BC of the triangle ABC, and the lines BO, CO are drawn to any point O on AB and are produced to meet AC, AB in E, F respectively. Shew that AD bisects the angle EDF.

87. Shew that, if the median AD of the triangle ABC be a mean proportional to the sides AB, AC it will be equal to AK, where K is on BC produced and AK makes equal angles with AB and AC.

88. Having given the base and the vertical angle of a triangle, shew that the loci of the in-circle and of the centre of the circle through the centres of the three escribed circles are circles.

89. Inscribe a square in a given quadrilateral.

90. From the middle points of the sides of a cyclic quadrilateral perpendiculars are drawn to the opposite side; shew that these four perpendiculars meet in a point.

91. The bisectors of the angle BAC of the triangle ABC meet BC in D, G, also E, H and F, K are similar points on CA, AB respectively. Shew (1) that GHK is a straight line, (2) that the circles whose diameters are DG, EH, FK respectively have a common radical axis.

92. Construct a triangle having given the circum-centre, the in-centre and the orthocentre.

93. T is any point on the radical axis of two given circles O, O', and a common tangent touches the circles in P, P' respectively. OP, OP' cut the circles O, O' respectively again in the points Q, Q'; shew that the circle TQQ' will touch the given circles.

94. Shew that the lines joining the angular points of a triangle to the points of contact of the corresponding escribed circles with the opposite sides meet in a point on a line through the in-centre and the centroid of the triangle.

95. On the sides BC, CA, AB of the triangle ABC and external to the triangle the similar isosceles triangles $BA'C$, $CB'A$, $AC'B$ are described; shew that AA', BB', CC' meet in a point.

96. On the sides BC, CA, AB of the triangle ABC and external to the triangle the equilateral triangles $BA'C$, $CB'A$, $AC'B$ are described; shew that, if X, Y, Z are the centroids of the equilateral triangles, the triangle XYZ is equilateral.

97. The straight line AB is divided into any two parts at the point C, and on AB, BC, CA the equilateral triangles $AC'B$, $BA'C$, $CB'A$ are described, the triangle $AC'B$ being on the side opposite to that on which the other triangles lie. Shew that AA', BB', CC' are equal and meet in a point, and that, if X, Y, Z are the centroids of the three equilateral triangles, the triangle XYZ is equilateral.

98. Construct a triangle having given the vertices of the three equilateral triangles drawn on its sides external to the triangle.

99. A line is drawn cutting two given circles so that the intercepted chords are equal; shew that the tangents at the extremities of one of the intercepted chords meet the tangents at the extremities of the other in four points on the circle of similitude of the given circles.

100. If the vertices of one triangle are at the middle points of the sides of another, the in-centre of the first triangle is the radical centre of the circles escribed to the second.

101. Having given an angle of a triangle in position, and the sum of the containing sides of the triangle; shew that the locus of the circum-centre of the triangle is a straight line.

102. Prove that the radical centre of the three escribed circles of a triangle is the centre of a circle inscribed in the triangle formed by joining the middle points of the sides of the given triangle.

103. Shew that, if A, B, C be any three points on a straight line, and O be any other point, then will

$$AO^2 . BC + BO^2 . CA + CO^2 . AB + BC . CA . AB = 0.$$

104. Shew that, if A, B, C be any three points on a straight line, and AP, BQ, CR the three tangents from them to any circle, then will

$$AP^2 . BC + BQ^2 . CA + CR^2 . AB + BC . CA . AB = 0.$$

105. On the sides BC, CA, AB of the triangle ABC points P, Q, R are respectively taken. Shew that the circum-centres of the triangles AQR, BRP, CPQ are the angular points of a triangle similar to ABC.

106. If A', B', C' are the middle points of the sides BC, CA, AB of the triangle ABC, and if O be any point within the triangle, and D, E, F be the middle points of AO, BO, CO respectively; then will the lines $A'D$, $B'E$, $C'F$ meet in a point.

107. If from any point O within a triangle ABC, the three perpendiculars OP, OQ, OR are drawn to the sides BC, CA, AB respectively, the three perpendiculars from A, B, C on OR, RP, PQ respectively will also meet in a point.

108. The bisectors of the angles A, B, C of the triangle ABC meet BC, CA, AB respectively in the points D, D'; E, E' and F, F'. Shew that the circles whose diameters are DD', EE', FF' respectively have two common points P, P'. Shew also that, if PX, PY, PZ be the perpendiculars from P on BC, CA, AB respectively, the triangle XYZ will be equilateral.

109. Three circles with their centres on the sides BC, CA, AB respectively of the triangle ABC are described so as to cut the circumcircle of ABC orthogonally at A, B, C respectively; shew that the three circles are co-axal.

110. $ABCD$ is a quadrilateral and any line parallel to CD is drawn which cuts BC, DA, BD, CA, AB, or these lines produced, in the points P, P', Q, Q', O respectively. Shew that $OP \cdot OP' = OQ \cdot OQ'$.

111. AB is a fixed chord of a circle and PQ is any other chord whose middle point is on AB. Shew that the locus of the point of intersection of the tangents at P, Q is a circle.

112. Shew that, if either of two given circles be inverted into the other, their circle of similitude will invert into their radical axis.

113. Shew that, if any two intersecting straight lines be inverted with respect to a point P, and if PM, PN be drawn perpendicular to the bisectors of the angles between the given lines, and PM, PN be produced to Q, R respectively so that $PM = MQ$ and $PN = NR$, then will the points Q and R invert into the centres of similitude of the inverses of the given straight lines.

114. Draw a circle through a given point so as to cut two given circles at given angles.

115. Construct a rectangle of given area whose sides pass respectively through four given points.

116. The sides DA, CB of the cyclic quadrilateral $ABCD$ are produced to meet in E, and the sides AB, DC to meet in F. Shew that the circle whose diameter is EF will cut the circle $ABCD$ orthogonally.

117. If A, B, C, D be any four points in a plane, and if P, Q, R, S be the centroids of the triangles BCD, CDA, DAB, ABC respectively, shew that the quadrilateral $PQRS$ is similar to $ABCD$.

118. Through a given point P draw a straight line cutting two given straight lines AB, AC in the points D, E respectively so that the sum of AD and AE may be of given length.

119. Through a given point P draw a straight line cutting two given straight lines AB, AC in the points D, E respectively so that the difference of AD and AE may be of given length.

120. Draw a straight line in a given direction so as to intersect two given straight lines AB, AC in D, E respectively so that the triangle ADE may be of given area.

121. Draw a straight line through a given point so as to cut off a triangle of given area from two given straight lines.

122. A quadrilateral is divided into four triangles by its diagonals; shew that the quadrilaterals having for angular points (i) the ortho-centres and (ii) the circum-centres of the four triangles are similar parallelograms; and that, if their areas be respectively Δ_1 and Δ_2 and Δ be the area of the original quadrilateral, then $2\Delta + \Delta_1 = 4\Delta_2$.

123. If E, F are two conjugate points with respect to a given circle, the circles whose centres are E, F and which cut the given circle orthogonally will cut each other orthogonally.

124. Shew that, if A be a fixed point on a given circle and AB, AC two chords through A such that the sum of their squares is constant, then will the middle point of BC lie on a fixed straight line.

125. From any point T pairs of tangents TP, TP' and TQ, TQ' are drawn to two concentric circles. Shew that QP, $Q'P$ make equal angles with the tangent TP.

126. Find two points Q, Q' one on each of two given circles such that QQ' may be parallel to a given straight line and that QQ' may be (1) of maximum and (2) of minimum length.

127. AD, BE, CF are the perpendiculars from A, B, C on the sides BC, CA, AB respectively of the triangle ABC, and points D', E', F' are taken on these sides so that $BD' = DC$, $CE' = EA$, $AF' = FB$. Shew that, if O be the circum-centre of ABC, OA, OB, OC will bisect $E'F'$, $F'D'$, $D'E'$ respectively.

128. Shew that, if a circle be drawn so as to touch one given circle and to cut another given circle at right angles, it will touch another fixed circle whose centre is on the line joining the centres of the two given circles.

129. The centre of the circle through the centres of three given circles, the radical centre of the circles, and the centre of the circle which bisects their circumferences, are in a straight line.

130. AK, BL, CM are the bisectors of the angles of the triangle ABC, and the in-circle and the three escribed circles touch BC in the points D, D_1, D_2, D_3 respectively, shew that AD_1, D_2L, D_3M pass through the other extremity of the diameter DD' of the in-circle.

131. Shew that, if the in-circle of the triangle ABC touch the sides BC, CA, AB in D, E, F respectively, and if O_1, O_2, O_3 are the centres of the corresponding escribed circles, then will O_1D, O_2E and O_3F meet in a point.

132. Draw a line parallel to a given straight line and cutting two given circles so that the intercepted chords may be in a given ratio.

133. Three circles A, B, C are touched externally by a circle whose centre is P, and internally by a circle whose centre is Q. Shew that PQ passes through the point of concurrence of the radical axes of A, B and C taken in pairs.

134. Draw through a given point on one of two given circles a straight line such that the chords of the circles intercepted on it may be (1) equal and (2) in a given ratio.

135. Through a fixed point O any two chords POP', QOQ' of a given circle are drawn; shew that the locus of the second point of intersection of the circles POQ, $P'OQ'$ is a circle.

136. $ABCD$ is a cyclic quadrilateral and the sides AB, CD meet in E, the sides AD, BC in F and the diagonals AC, BD in G. Shew that, if GK is drawn perpendicular to EF, K will be on the circles ADE and CDF.

137. On the sides AB, AC of the triangle ABC find two points P, Q respectively such that $BP = PQ = QC$.

138. On the sides AB, AC of the triangle ABC find two points P, Q respectively such that $BP : PQ$ and $PQ : QC$ are given ratios.

139. Shew that, if a circle cut two of the diagonals of a quadrilateral harmonically, the circle must be one of a co-axal system whose radical axis is the line through the middle points of the diagonals, and that it will cut the other diagonal harmonically.

140. Shew that, if AA', BB', CC' be the perpendiculars from the angular points of the triangle ABC on the opposite sides, and if α, β, γ be the orthocentres of the triangles $AB'C'$, $BC'A'$, $CA'B'$; then will the triangle $\alpha\beta\gamma$ be equal in all respects to $A'B'C'$ and $A'\alpha$, $B'\beta$, $C'\gamma$ will meet in a point and bisect each other.

141. Two circles touch each other internally at O, and a line cuts them in A, D and B, C respectively. The tangent at A intersects the tangents at B and C in G and E respectively, and the tangent at D intersects the tangents at B and C in F and H respectively. Prove that OA, OD bisect the angles GOE, FOH respectively, and that E, F, G, H are on a circle which touches the given circles at O.

142. The locus of the point of contact of two circles which touch one another and also two given circles is a circle co-axal with the given circles.

143. Draw a straight line to cut four given straight lines so that the three intercepted portions may be proportional to three given straight lines.

144. Draw through a given point P a line cutting the sides BC, CA, AB of the triangle ABC, or these sides produced, in the points D, E, F respectively such that $DE : EF$ may be equal to a given ratio.

145. From the angular points of a triangle ABC tangents AP, BQ, CR are drawn to a given circle. Shew that, if one of the rectangles $AP . BC$, $BQ . CA$, $CR . AB$ be equal to the sum of the other two, the circle ABC will touch the given circle.

146. Two given circles are such that a triangle ABC can be inscribed in one and circumscribed about the other. The tangents at A, B, C form another triangle PQR. Shew that the circum-circle of PQR, for all possible positions of ABC, will touch a fixed circle.

147. Construct a triangle having given the base, the ratio of the other sides, and the angle between the medians through the extremities of the base.

148. Prove that, if a straight line $PQRS$ be drawn intersecting the sides of a square in order in P, Q, R, S so that $PQ . RS = PS . QR$, then will $PQRS$ touch the circle inscribed in the square.

149. Shew that, if a variable circle pass through a given point and cut a given circle at a given angle, it will touch another fixed circle.

150. Draw a circle of a given co-axal system so as to cut a given circle orthogonally.

151. Shew that, if the polars of any point P with respect to two given circles meet in P', then will PP' be bisected by the radical axis of the given circles.

152. Shew that, if P be a given point within the angle formed by two given straight lines AX, AY, and if the tangent at P to the circle which passes through P and touches AX, AY cut the lines in B, C respectively; then will the perimeter of ABC be less than that of any other triangle cut off from the given lines by a line through P.

153. Shew that all circles whose centres lie on a given straight line and which cut a given circle orthogonally are co-axal.

154. From any point P on the bisector of the angle A of the triangle ABC the perpendiculars PA', PB', PC' are drawn to the sides BC, CA, AB of the triangle; shew that PA' intersects $B'C'$ on the median through A.

155. Shew that the lines joining the middle points of the sides of a triangle to the middle points of the corresponding perpendiculars meet in a point.

156. If AD, BE, CF be any three lines through a common point which meet the sides BC, CA, AB of the triangle ABC in D, E, F respectively, and if A', B', C' be the middle points of BC, CA, AB respectively, then the lines joining A', B', C' to the middle points of AD, BE, CF will meet in a point.

157. Shew that, if the parallelogram $PQRS$ be inscribed in the triangle ABC so that PQ is on BC and the sides QR, SP are parallel to AK, then will the centre of the parallelogram be on a line parallel to AK and midway between AK and the middle point of BC.

158. Shew that all circles which cut two given circles at equal angles are cut orthogonally by the same circle.

159. Shew that, if a circle touch two given circles, its radius is in a constant ratio to the distance of its centre from the radical axis of the given circles.

160. Any circle is drawn through one of the limiting points of a co-axal system so as to touch one of the circles of the system; shew that it cuts any other circle of the system at a constant angle.

161. Any circle which touches two given non-intersecting circles will cut a fixed circle co-axal to the given circles at a constant angle.

162. All circles which cut two given non-intersecting circles at given angles will touch two fixed circles co-axal with the given circles.

163. A straight line cuts the sides BC, CA, AB of the triangle ABC, produced if necessary, in the points D, E, F respectively. Shew that if $FD : DE$ be equal to a given ratio, the circles ABC, FBD, DEC, FAE will meet in a point.

164. Draw through a given point a straight line such that the rectangle contained by the perpendiculars upon it from two other given points may be equal to a given square.

165. A given line EF is such that its square is equal to the sum of the squares of the tangents drawn from E and F to a given circle which is not cut by EF or EF produced. Prove that an infinite number of quadrilaterals can be inscribed in the circle of which EF is the exterior diagonal.

166. Shew that the locus of a point whose polars with respect to three given circles meet in a point, is the circle which cuts the three given circles orthogonally.

167. The centres of similitude of two circles are joined to any point on one of the circles and the joining lines intersect the other circle in the points P, P' and Q, Q' respectively. Shew that one of the two pairs of chords PQ, $P'Q'$ and PQ', $P'Q$ will meet on the radical axis of the circle, and the other pair will cut the line of centres in fixed points.

168. Shew that, if a triangle ABC is inscribed in a circle of a co-axal system, and if the sides BC, CA, AB touch circles of the system in the points X, Y, Z respectively, then will AX, BY, CZ meet in a point.

169. Shew that, if A', B', C' be the points of contact of the in-circle of the triangle ABC, $\triangle ABC \gtrless 4 \triangle A'B'C'$.

170. Two straight lines cut a pair of opposite sides of a cyclic quadrilateral in four points which lie on a circle; shew that these lines will cut the other pair of opposite sides, and also the diagonals of the quadrilateral, in four cyclic points, and that all these circles are co-axal.

171. Two straight lines are drawn to cut two given circles in the points P, Q, R, S and P', Q', R', S', so that P, Q, P', Q' are on one circle and R, S, R', S' on the other. Shew that PQ', $P'Q$ cut RS', $R'S$ in four points which lie on a circle co-axal with the given circles.

172. If any st. line be drawn to cut two given circles in four points and the tangents to the circles at the points of section be drawn, the four points in which a tangent to one of the circles intersects a tangent to the other all lie on a circle co-axal with the given circles.

173. If a st. line cut two given circles so that the intercepted chords are in a given ratio and the tangents to the circles at the points of intersection be drawn, a tangent to one of the circles will meet a tangent to the other circle on a fixed co-axal circle.

174. Shew that the chords of contact of the four common tangents of two given non-intersecting circles which are not perpendicular to the line of centres will pass through the limiting points of the circles.

175. A chord AB of the outer of two circles touches the inner in C, and cuts their radical axis in D; shew that $AD : BD = AC^2 : CB^2$.

176. Through one of the limiting points of a co-axal system any straight line is drawn cutting a fixed circle of the system in two points; shew that the rectangle contained by the distances of these points from the radical axis is constant.

177. Shew that, if the in-circle and the circum-circle of a triangle be given, the loci of the nine-point centre, the orthocentre and the centroid of the triangle are all circles.

178. The in-circle of the triangle ABC touches BC in D, and P is the pole with respect to the in-circle of the line which bisects AB and AC; shew that DP is equal to the radius of the escribed circle which touches BC externally.

179. A is one of the points of intersection of two given circles, and AP, AQ are chords of the two circles which make a constant angle with one another; shew that if the parallelogram $PAQT$ be completed the locus of T is a circle, and that the locus of a point which divides PQ in a constant ratio is also a circle.

180. O_1, O_2, O_3 are the centres of the escribed circles of a triangle, and A', B', C' are the middle points of its corresponding sides; shew that O_1A', O_2B', O_3C' meet in a point.

181. Describe a circle to cut two given circles orthogonally and to touch a third given circle.

182. Describe a circle to touch two given circles and to cut a given circle orthogonally.

183. Shew that if the transverse common tangents to two given circles be perpendicular to the direct common tangents, the eight points of contact of the four tangents will lie on two straight lines.

184. Points P, Q are taken one on each of two given non-intersecting circles and PQ subtends a right angle at a limiting point of the given circles; shew that the tangents at P and Q intersect on a fixed circle co-axal with the given circles.

185. The line joining the centres of the circum-circle and the in-circle of a triangle will pass through the orthocentre and the centroid of the triangle formed by joining the points of contact of the in-circle.

186. The in-circle of the triangle ABC touches the sides in D, E, F. Shew that the algebraic sum of the perpendiculars from D, E, F on the line joining the circum-centre and the in-centre is zero.

187. Shew that, if the vertices of a variable triangle of constant species move on fixed straight lines, every point invariably connected with it moves also on a fixed straight line. And, if the sides of a variable triangle of given species pass through fixed points, every line invariably connected with the triangle will also pass through a fixed point.

188. In a given circle inscribe a quadrilateral whose three diagonals are of given lengths.

189. Shew that an infinite number of triangles can be described such that each has the same circum-circle, nine-point circle and polar circle as a given triangle.

190. From two points P, P' which are inverse points with respect to a circle, perpendiculars PX, PY, PZ, and $P'X'$, $P'Y'$, $P'Z'$ are drawn on the sides of any triangle inscribed in the circle; shew that the triangles XYZ, $X'Y'Z'$ will be similar.

191. AOD, BOE, COF are the perpendiculars of the triangle ABC and A', B', C' are the middle points of the sides, also X, Y, Z are the middle points of EF, FD, DE respectively; prove that the pedal lines of D, E, F with respect to the triangle $A'B'C'$ and the pedal lines of A', B', C' with respect to the triangle DEF all pass through the in-centre of the triangle XYZ.

192. Shew that an infinite number of triangles can be inscribed in a circle so that each of its sides will pass through one of the vertices of a triangle which is self-polar with respect to the circle.

193. X and Y are two circles and O is one of their limiting points; a variable tangent to Y is drawn cutting X in the points P, Q. Prove that the circle OPQ touches a fixed circle concentric with X.

194. Two given circles O, O' intersect in the points A, B, and from any point P on the circle O the lines PA, PB are drawn cutting the circle O' again in the points Q, R respectively; shew that (1) the loci of the circum-centre, the orthocentre, and the centroid of the triangle PQR are circles, and (2) that the circle PQR touches a fixed circle.

195. Prove that with four given straight lines, three distinct cyclic quadrilaterals can be constructed, that their areas are equal, that the six diagonals which intersect within the circle are equal in pairs; and that, if l, m, n be the lengths of these different diagonals, S the area of the quadrilateral and R the radius of the circle, $4RS = lmn$.

196. Shew that, if A', B', C' are the middle points of the sides of the triangle ABC, and D, E, F the feet of the perpendiculars, and if EC', FB' intersect in α and similarly for β and γ, also if $B'C'$, EF intersect in α' and similarly for β' and γ'; then will (1) the points α, β, γ lie on SN, (2) the lines $A\alpha'$, $B\beta'$, $C\gamma'$ will all be perpendicular to SN, (3) the points A, α, β', γ' will lie on a straight line, and (4) $\alpha'\beta'\gamma'$ is self-polar with respect to the nine-point circle.

197. Prove that the polar circles of all the triangles formed by five straight lines are cut orthogonally by the same circle.

198. AD, BE, CF are the perpendiculars of the triangle ABC. The projections of E and F on BC are X, X' respectively, the projections of F and D on CA are Y, Y' respectively, and the projections of D and E on AB are Z, Z' respectively. Shew that a circle will pass through the six points X, X', Y, Y', Z, Z'.

199. D, E, F are the feet of the perpendiculars of the acute-angled triangle ABC. From any point P on BC, PQ is drawn parallel to DE to meet CA in Q, then QR parallel to EF to meet AB in R, then RS parallel to FD to meet BC in S, ST parallel to DE to meet CA in T and TU parallel to EF to meet AB in U. Shew that UP is parallel to FD, and that the perimeter of $PQRSTU$ is double the perimeter of the triangle DEF.

200. $ABCD$ is a cyclic quad. and from the intersection of the diagonals perpendiculars OP, OQ, OR, OS are drawn to the sides AB, BC, CD, DA respectively. Shew that the sides of $PQRS$ are equally inclined to the sides of $ABCD$ on which they meet. Shew also that, if points P', Q', R', S' are taken on AB, BC, CD, DA respectively, such that $P'Q'$ is parallel to PQ, $Q'R'$ to QR and $R'S'$ to RS, then $S'P'$ will be parallel to SP, and the rectangle contained by the perimeter of $P'Q'R'S'$ and the radius of the circle $ABCD$ will be equal to the rectangle $AC.BD$.

BOOK XI.

DEFINITIONS.

1. *A finite portion of space regarded as separated from the rest, is called a* **solid.**

2. *The boundary of a portion of space, that is of a solid, is called a* **surface.**

3. *A* **plane surface,** *or a* **plane,** *is a surface such that the straight line joining any two points on the surface will lie entirely on the surface.*

4. *Two straight lines which are in the same plane, and which do not meet however far they are produced, are said to be* **parallel.**

5. **Parallel planes** *are planes which do not meet however far they are produced.*

6. **A straight line** *is said to be* **parallel to a plane** *when it does not meet the plane however far they are both produced.*

7. **A straight line** *is said to be at* **right angles to a plane** *when it is at right angles to every straight line which lies in the plane and meets it.*

Postulate I. *A plane can be drawn through any straight line.*

Postulate II. *A plane can be turned round any indefinite straight line upon it until it passes through any given point.*

PROPOSITION I.

One part of a straight line cannot lie in a plane and another part without it.

This follows immediately from the definition by which a straight line given in the plane lies in it throughout its whole extent. It has been taken for granted from the beginning. For example, in Euclid I. 2, it is assumed that *BD* when produced must intersect the circle *CEF*, which implies that it cannot leave the plane of the circle.

Again, in Book I, Prop. 4, *AB* is placed on *DE* so that the lines coincide to begin with, and it is necessarily assumed that they cannot afterwards separate.

PROPOSITION II.

Two straight lines which intersect are in the same plane, and three straight lines which intersect two and two are in the same plane.

Let *PQ, RS* be two straight lines which cut one another in *A*, and let *XY* be a third straight line which cuts *PQ, RS* in *B, C* respectively.   Then *it is required to prove that PQ, RS are in the same plane, and that PQ, RS, XY are in the same plane.*
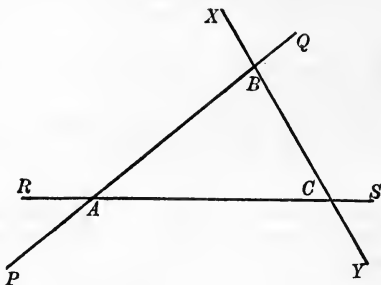
Let any plane pass through the straight line *PQ*, and let this plane be turned about *PQ* produced indefinitely until it passes through the point *C*.

Then, since the points *A, C* are on this plane, the straight line *AC* is wholly on the plane.

Hence the two intersecting lines *PQ* and *RS* lie on a plane.

Also, since *B* and *C* are in the plane through *PQ* and *RS*, the straight line *BC* is wholly in that plane.

Hence the three straight lines *PQ, RS, XY*, which intersect two and two, lie on the same plane.

It follows from the above proof that one, and only one, plane can be drawn:

(1) to pass through any three points which are not in the same straight line;

(2) to pass through a given straight line and a given point not on the line;

(3) to pass through two intersecting straight lines;

(4) to pass through two parallel lines.

Lines or points which are in the same plane are said to be **co-planar.**

Since any three points lie on a plane, a triangle is necessarily a plane figure. Four points are, however, not necessarily co-planar, and therefore a quadrilateral is not necessarily a plane figure.

A quadrilateral which is not in a plane is called a **skew quadrilateral.**

PROPOSITION III.

If two planes cut one another their common section is a straight line.

Let A, B be any two points common to two planes.

Then, by definition, the straight line AB throughout its whole extent will lie on both planes.

Moreover no point which is not on the straight line AB can be on both planes, for only *one* plane can be drawn through a straight line and a point not on that line.

PROPOSITION IV.

A straight line which is perpendicular to each of two intersecting straight lines at their point of intersection, is perpendicular to the plane in which they lie.

Let AB and AC be two intersecting straight lines and let PA be perpendicular to each of them. Then *it is required to prove that PA is perpendicular to the plane BAC.*
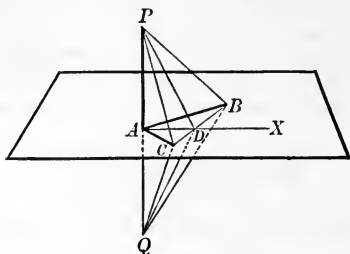
Through A draw any other line AX in the plane BAC.

Join BC and let it cut AX in D.

Produce PA below the plane to Q so that $AQ = PA$.

Join PC, PD, PB, QC, QD, QB.

Then, in the $\triangle^s$ PAC, QAC

$PA = QA$, AC is common, and rt. $\angle PAC =$ rt. $\angle QAC$;

$$\therefore PC = QC.$$

Similarly $PB = QB$.

Again in $\triangle^s$ PBC and QBC,

$$PB = QB, \ BC = BC \text{ and } PC = QC;$$
$$\therefore \angle PBC = \angle QBC.$$

Hence, in $\triangle^s$ PBD and QBD,

$PB = QB$, $BD = BD$ and included $\angle PBD =$ included $\angle QBD$;

$$\therefore PD = QD.$$

And, since $PA = QA$, $AD = AD$ and $PD = QD$;

$\therefore \angle PAD = \angle QAD$, and $\therefore$ both are rt. $\angle^s$.

$\therefore PAQ$ is $\perp^r$ to AD.

Thus, if PA is $\perp^r$ to AB and to AC, it is $\perp^r$ to any other straight line through A which is in the plane BAC, and is therefore $\perp^r$ to the plane BAC.

PROPOSITION V.

If three straight lines meet in a point and if a fourth straight line through that point is perpendicular to each of them, the three straight lines must lie in a plane.

Let the straight line *AP* be perpendicular to each of the three straight lines *AB*, *AC*, *AD*. Then *it is required to prove that AB, AC, AD lie in a plane.*
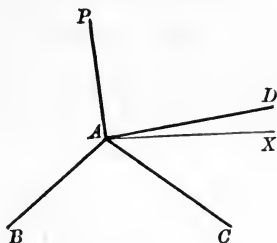
For, if possible, let *AD* be not in the plane through *AB* and *AC*.

Let a plane be drawn through *AP* and *AD* and let it cut the plane *BAC* in the line *AX*.

Then, since *AP* is ⊥ʳ to both *AB* and *AC*, it is ⊥ʳ to the line *AX* which lies in the plane *BAC*.

Hence the angles *PAD* and *PAX* are both right angles, which is impossible, since they are both in the plane *PAD*.

Therefore *AD* must itself lie in the plane *BAC*.

The proposition can be enunciated as follows:

'*The locus of the lines drawn perpendicular to a given straight line through a given point on it, is a plane.*'

The following constructions are important and their results of frequent use. The solution of 5 gives the centre of a sphere through four given points.

Ex. 1. Through a given point on a given straight line draw a plane perpendicular to the given straight line.

Ex. 2. Through any point draw a plane perpendicular to a given straight line.

Ex. 3. The locus of a point which is equally distant from two given points is the plane which passes through the middle point of the straight line joining the two given points and its perpendicular to that straight line.

Ex. 4. Find the locus of the points which are equidistant from three given points. When are there no such points?

Ex. 5. Find the point which is equidistant from four given points. When is this impossible?

PROPOSITION VI.

Two straight lines which are perpendicular to the same plane must be parallel.

Let the straight lines AB and CD be perpendicular to the same plane. Then *it is required to prove that AB and CD are parallel.*
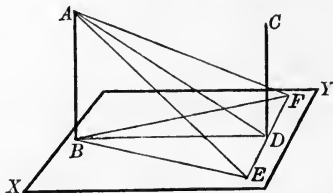
Let AB, CD meet the plane XY, to which they are both perpendicular, in the points B, D respectively.

Join BD. Then AB, CD are both $\perp^r$ to the plane XY, and BD is in the plane XY;

$$\therefore \ \angle^s \ ABD \text{ and } CDB \text{ are rt. } \angle^s.$$

And, since $\angle^s ABD$ and CDB are together equal to two rt. $\angle^s$, it follows that AB and CD must be parallel *provided that they are in the same plane.*

Through D draw EDF in the plane XY and $\perp^r$ to BD, making $DE = DF$.

Join BE, BF, AE, AD, AF.

Then the sides BD, DE and the included $\angle BDE$ of the $\triangle BDE$ are equal respectively to the sides BD, DF and the included $\angle BDF$ of the $\triangle BDF$.

Hence $BE = BF$.

Since AB is $\perp^r$ to the plane XY, $\angle ABE = \text{rt. } \angle = \angle ABF$.

Thus AB, BE and the included $\angle\ ABE$ of the $\triangle\ ABE$ are equal respectively to AB, BF and the included $\angle\ ABF$ of the $\triangle\ ABF$.

Hence $AE = AF$.

Then, because AD, DE, AE are equal respectively to AD, DF, AF;

$\therefore\ \angle\ ADE = \angle\ ADF$, so that ED is at rt. $\angle^s$ to AD.

But $\angle\ EDB =$ rt. $\angle$ (const.), and $\angle\ EDC =$ rt. $\angle$, since CD is $\perp^r$ to plane BDE.

Thus ED is $\perp^r$ to BD, AD and CD;

$\therefore\ CD$ lies in the plane through AD and BD. [XI. 5.

But AB is in the plane through AD and BD; [XI. 2.

$\therefore\ AB$ and CD are both in the plane ADB.

And the $\angle^s\ ABD$ and CDB are rt. $\angle^s$;

$\therefore\ AB$ is parallel to CD.

[The student will find it helpful to construct solid figures with cardboard, thread and sticks or wires to illustrate this and some other of the more difficult propositions.]

PROPOSITION VII.

If two straight lines be parallel, the straight line drawn from any point in the one to any point in the other, is in the plane of the parallels.

This follows from the definition of a plane.

PROPOSITION VIII.

If a straight line is at right angles to a plane, any parallel straight line will be at right angles to the same plane.

Let AB, CD be any two parallel lines, and let AB be perpendicular to the plane XY. Then *it is required to prove that CD is perpendicular to the plane XY.*
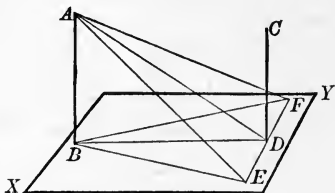
Let the parallel lines meet the plane XY in the points B, D respectively. Join BD.

In the plane XY draw $EDF \perp^r$ to BD, and make $ED = DF$.

Join BE, BF, AE, AD, AF.

Then, since BD, DE and the included $\angle BDE$ are equal respectively to BD, DF and the included $\angle BDF$,

$$\therefore BE = BF.$$

And, since AB is $\perp^r$ to the plane in which BE and BF lie, $\angle ABE = $ rt. $\angle = \angle ABF$.

And, since AB, BE and the included $\angle ABE$ are equal respectively to AB, BF and the included $\angle ABF$,

$$\therefore AE = AF.$$

Then, since AD, DE, EA are equal respectively to AD, DF, FA,

$$\angle ADE = \angle ADF, \text{ and } \therefore \text{ each is a rt. } \angle.$$

$\therefore ED$ is $\perp^r$ to AD, and it was drawn $\perp^r$ to BD.

$\therefore ED$ is $\perp^r$ to the plane ADB.

But since AB and CD are parallel, CD must be in the plane ABD;

$$\therefore CD \text{ is } \perp^r \text{ to } DE.$$

But, since AB and CD are ‖, and AB is $\perp^r$ to BD, CD must also be $\perp^r$ to BD.

Hence CD is $\perp^r$ to the plane through DB and DE, that is to the plane to which AB is perpendicular.

PROPOSITION IX.

Straight lines which are parallel to the same straight line are parallel to one another.
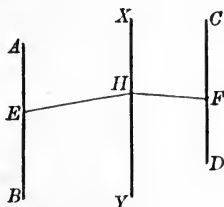
Let each of the lines AB, CD be parallel to the line XY. Then *it is required to prove that AB and CD are parallel to one another.*

[The case when the three lines are all in one plane has already been proved.] [I. 30.

Take any point H in XY, and in the plane of the ‖s XY, CD draw $HF \perp^r$ to XY cutting CD in F.

Also in the plane of the ‖s XY, AB draw $HE \perp^r$ to XY cutting AB in E.

Then, since XH is $\perp^r$ to HF and to HE, XY must be $\perp^r$ to the plane EHF.

But AB and CD are each of them ‖ to XY;

$\therefore AB$ and CD are also $\perp^r$ to the plane EHF; [XI. 8.

$\therefore AB$ is ‖ to CD. [XI. 6.

Ex. 1. Shew that the middle points of the sides of a skew quadrilateral are at the angular points of a parallelogram.

PROPOSITION X.

The angles between any two intersecting straight lines are equal to the angles between any other two intersecting straight lines which are parallel to them respectively.
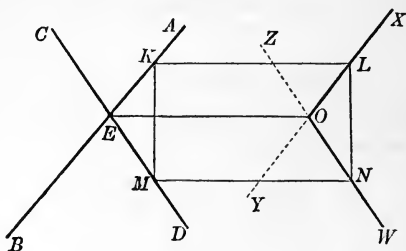
Let the straight lines AB, CD which intersect at E be ∥ respectively to the straight lines XY, ZW which intersect at O. Then *it is required to prove that the angles between AB and CD are equal respectively to the angles between XY and ZW.*

Join EO, and in the plane of the ∥ˢ AB, XY, draw any line KL ∥ to EO so as to cut AB, XY in the points K, L respectively.

Also in the plane of the parallels CD, ZW draw any line MN ∥ to EO so as to cut CD, ZW in the points M, N respectively.

Then, $EOLK$ is a ∥ᵐ by construction,

∴ KL is equal and parallel to EO.

Similarly MN is equal and parallel to EO.

Hence KL is equal and parallel to MN;   [XI. 9.

∴ $KLNM$ is a ∥ᵐ and $KM = LN$.

Then, since $KE = LO$, $EM = ON$ and $KM = LN$;

∴ ∠ KEM = ∠ LON,

and ∴ also ∠ KEC = ∠ LOZ.

Cor. *If through any point P on the line of intersection of two given planes, two straight lines be drawn, one on each plane, perpendicular to the line of intersection, the angles between these lines will be constant for all positions of P.*

Def. *The* **angles between two planes** *are the angles between the straight lines drawn in the planes through any point of their line of intersection and perpendicular to that line. Such an angle is called a* **dihedral angle.**

Def. *Two* **planes are at right angles** *when the two lines, drawn in the planes through the same point on their line of intersection and perpendicular to that line, are at right angles to one another.*

If P is any point on the line of intersection AB of two perpendicular planes, and PQ, PR are drawn in the planes perpendicular to APB, then PQ and PR are at right angles, by definition; also PQ and PR are at right angles to APB.

Hence PQ, PR are at right angles to the planes ABR, ABQ respectively.

Thus we have the alternative definition.

Def. *Two planes are at right angles when a line drawn in one of the planes perpendicular to their line of intersection is perpendicular to the other plane.*

Def. *The angle between two straight lines which are not parallel, and which do not intersect, is the angle between two straight lines drawn through any point parallel respectively to those straight lines.*

It should be noticed that the last definition could not have been given until it had been proved that the angle between two straight lines drawn through *any* point, parallel respectively to two given straight lines, is a constant angle.

PROPOSITION XI.

To draw a straight line perpendicular to a plane from a given point without it.

Let P be the given point without the given plane MN. Then *it is required to draw a line through P perpendicular to the plane MN.*
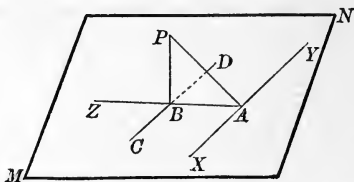
Draw any line XY in the given plane MN, and draw $PA \perp^r$ to XY.

Through A draw AZ in the plane $MN \perp^r$ to XY.

Then, if PA were $\perp^r$ to AZ, it would be $\perp^r$ to two lines in the plane MN, and would $\therefore$ be the line required.

But, if PA be not $\perp^r$ to AZ, draw $PB \perp^r$ to AZ.

Then PB will be the line required.

In the plane MN draw $CBD \parallel$ to XY.

Since by construction XY is $\perp^r$ to AP and to AZ, XY is $\perp^r$ to the plane PAZ.

Hence CBD, which is $\parallel$ to XY, is also $\perp^r$ to the plane PAZ;

$$\therefore PB \text{ is } \perp^r \text{ to } CB.$$

But PB is also $\perp^r$ to AB;

$$\therefore PB \text{ is } \perp^r \text{ to the plane } ABC.$$

Def. *The **projection** of a point on a plane is the foot of the perpendicular drawn from the point to the plane.*

If a straight line PA meet a plane MN in the point A, and B is the projection of P on the plane, it is easily seen

that the projection on the plane of any other point on the line
PA will lie on the line *AB*. Thus *the projection of a straight
line on a plane is a straight line.*

Def. *The* **angle between a straight line and a
plane** *is the angle between the straight line and its projection
on the plane.*

Ex. 1. Shew that the shortest straight line drawn from a given point
to a given plane is the perpendicular from the point to the plane. Shew
also that all the lines through a given point which make the same angle
with a given plane are equal in length; and that the line which makes a
greater angle with the perpendicular is greater than that which makes
a less angle.

Ex. 2. A line *AB* meets a plane at *B*, and *BC* is projection of *AB* on
the plane. Shew that the angle *ABC* is less than the angle between *AB*
and any other line on the plane drawn through the point *B*.

Ex. 3. Shew that the angles between two planes are equal to the
angles between the perpendiculars drawn to the planes from any point.

Ex. 4. Any number of planes have a common line of intersection.
Shew that the feet of the perpendiculars on the planes from any point lie
on a circle.

Ex. 5. Shew that parallel straight lines make equal angles with the
same plane.

Ex. 6. Shew that the angle between two given planes is equal to the
angle between any two planes parallel respectively to the given planes.

PROPOSITION XII.

To draw a straight line perpendicular to a given plane from a given point on the plane.

Let P be the given point on the given plane MN. Then *it is required to draw a line through P perpendicular to the plane MN.*
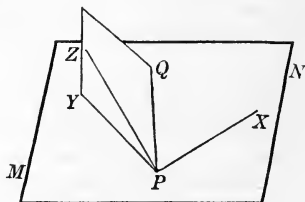
Through P draw in the given plane any line PX and a $\perp^r$ line PY.

In any other plane through PX, draw the line $PZ \perp^r$ to PX.

In the plane through PY and PZ draw $PQ \perp^r$ to PY.

Then PQ will be $\perp^r$ to the given plane MN.

For, since PX is $\perp^r$ to PY and to PZ,

PX is $\perp^r$ to the plane YPZ, and $\therefore$ to PQ.

Hence PQ is $\perp^r$ to PX and to PY, and $\therefore$ to the plane XPY, which is the given plane MN.

Ex. 1. Through a given point draw a plane perpendicular to a given plane.

Shew that an infinite number of such planes can be drawn. Draw a plane through two given points perpendicular to a given plane.

Ex. 2. Through two given points on a given plane draw a plane making a given angle with the given plane.

Ex. 3. Draw the planes which bisect the angles between two given planes.

Ex. 4. Shew that the locus of a point whose perpendicular distances from two given planes are equal is one or other of the two planes which bisect the angles between the given planes.

Ex. 5. Find the locus of points which are equidistant from three given planes. [Shew that the complete locus is four straight lines.]

Ex. 6. Find all the points which are equidistant from four given planes. How many such points are there?

PROPOSITION XIII.

From the same point there cannot be drawn two straight lines perpendicular to a given plane.

Let A be the given point and MN the given plane. Then *it is required to prove that the two lines AB, AC cannot both be perpendicular to MN.*
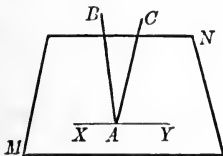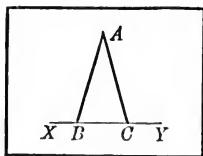
Fig. 1.   Fig. 2.

Draw a plane through AB, AC and let it cut the plane MN in the straight line XY.

Then, in fig. 1, where A is on the plane MN, if AB and AC were both $\perp^r$ to MN, the $\angle^s$ BAY, CAY which lie in a plane would both be rt. $\angle^s$, and this is impossible.

And, in fig. 2, where A is not on the plane, if AB and AC were both $\perp^r$ to MN, the $\angle^s$ ABC, ACB in the $\triangle$ ABC would both be rt. $\angle^s$, which is impossible.

PROPOSITION XIV.

Planes to which the same straight line is perpendicular are parallel.

Let AB be perpendicular to each of the planes XY, ZW. Then *it is required to prove that the planes XY, ZW are parallel.*
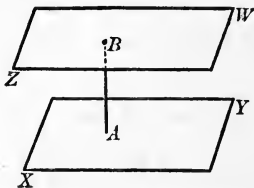
For if the planes were not parallel they would meet. And, if any common point, C suppose, were joined to A and B, AB would be $\perp^r$ to AC and to BC since it is $\perp^r$ to both planes, so that two angles in the $\triangle$ ABC would be right angles, and this is impossible.

Hence the planes XY, ZW cannot have a common point, that is, they must be parallel.

PROPOSITION XV.

If a pair of intersecting straight lines be parallel respectively to another pair of intersecting straight lines, the plane through the first pair is parallel to the plane through the second pair.

Let the straight lines AB, BC be ‖ respectively to the straight lines DE, EF. Then *it is required to prove that the plane ABC is parallel to the plane DEF.*
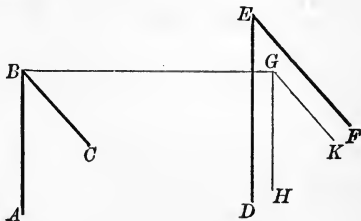
Through B draw a line $BG \perp^r$ to the plane DEF.

In the plane DEF draw GH, GK parallel to ED, EF respectively.

Then, since AB and HG are both $\parallel$ to DE, they are $\parallel$ to one another.

Hence sum of $\angle^s$ GBA, $BGH = 2$ rt. $\angle^s$.

But $\angle$ BGH is a rt. $\angle$, since BG is $\perp^r$ to plane HGK.

Hence $\angle$ GBA is a rt. $\angle$.

So also $\angle$ GBC is a rt. $\angle$.

Hence GB is $\perp^r$ to the plane ABC.

But GB is also $\perp^r$ to the plane DEF; $\therefore$ the planes ABC, DEF, which have a common perpendicular, must be parallel.

PROPOSITION XVI.

Two parallel planes are cut by a third plane in parallel straight lines.

Let the parallel planes AB, CD be cut by any other plane in the lines EF, GH. Then *it is required to prove that EF and GH are parallel.*
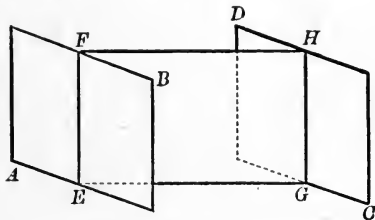
For every point on EF is on the plane AB, and every point on GH is on the plane CD. Hence, if EF, GH had a common point when produced, the planes AB and CD would have a common point, and this is impossible, since the planes AB and CD are $\parallel$.

Hence EF and GH cannot meet when produced, and by supposition they are in the same plane.

$\therefore EF$ is $\parallel$ to GH.

Ex. 1. Two planes which are parallel to a third plane are parallel to each other.

Ex. 2. Through a given point draw a plane parallel to a given plane.

Ex. 3. Shew that, if a straight line is perpendicular to a plane, it is perpendicular to any parallel plane.

Ex. 4. Shew that, if a straight line is parallel to a plane, it is parallel to any parallel plane.

Ex. 5. Shew that, if a straight line is parallel to any straight line on a given plane, it is parallel to the plane.

Ex. 6. Shew that, if a given straight line be parallel to a given plane, any plane through the line will cut the given plane in another straight line parallel to the given line.

Ex. 7. Shew that, if a straight line be parallel to a plane, a parallel line drawn through any point on the plane will lie entirely on the plane.

Ex. 8. Shew that, if a straight line is parallel to each of two planes, it is parallel to their line of intersection.

Ex. 9. Shew that the lines of intersection of three planes are either parallel or concurrent.

Ex. 10. Shew that, if AB, CD are parallel straight lines, any plane through AB meets any plane through CD in a line parallel to AB and CD.

PROPOSITION XVII.

If two straight lines are cut by parallel planes, they are cut in the same ratio.

Let the straight lines ABC, DEF be cut by the parallel planes PQ, RS, UV in the points A, B, C and D, E, F respectively. Then *it is required to prove that*

$$AB : BC = DE : EF.$$
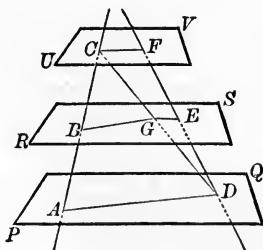
Join *CD*, and let *CD* cut the plane *RS* in the point *G*. Join *AD*, *BG*, *GE*, *CF*.

Then the ∥ planes *PQ*, *RS* are cut by the plane *ACD* in the lines *AD*, *BG* ;

$$\therefore AD \text{ is } \| \text{ to } BG.$$

So also *CF* is ∥ to *GE*.

Since *BG* is ∥ to *AD*, *AB* : *BC* = *DG* : *GC*.

Also, since *GE* is ∥ to *CF*, *DG* : *GC* = *DE* : *EF*.

But ratios which are equal to the same ratio are equal to one another ;

$$\therefore AB : BC = DE : EF.$$

Ex. Shew that parallel planes intercept equal lengths from parallel lines.

PROPOSITION XVIII.

Every plane which passes through a line at right angles to a given plane is at right angles to that plane.

Let *AB* be ⊥ʳ to the plane *MN*, and let *ABC* be any plane through *AB* which cuts the plane *MN* in the line *BC*. Then *it is required to prove that the plane ABC is at right angles to the plane MN.*
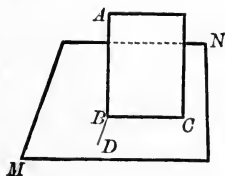
In the plane *MN* draw the line *BD* ⊥ʳ to *BC*.

Then, since *AB* is ⊥ʳ to the plane *MN*, *AB* is ⊥ʳ to *BD* which lies in that plane.

Hence the lines *AB*, *BD*, which are drawn in the planes *ABC*, *MN* respectively at right angles to *BC* their line of intersection, are ⊥ʳ to one another.

Hence, by definition, the planes *ABC* and *MN* are at right angles to one another.

PROPOSITION XIX.

If each of two planes be at right angles to a third plane, their line of intersection will be at right angles to the third plane.

Let AB be the line of intersection of two planes each of which is $\perp^r$ to the plane MN. Then *it is required to prove that* AB *is* $\perp^r$ *to the plane* MN.
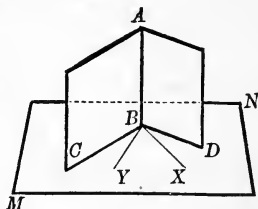
Let BC, BD be the lines of intersection of MN with the two planes through AB.

In the plane CBD draw BX, $BY \perp^r$ to BC, BD respectively.

Then, since the planes ABC, CBD are at right angles, and BX is drawn in the plane $CBD \perp^r$ to their line of intersection, BX is $\perp^r$ to the plane ABC.

Hence AB is $\perp^r$ to BX.

So also AB is $\perp^r$ to BY.

Hence AB is $\perp^r$ to the plane in which BX, BY lie, that is to the plane MN.

Definitions. *A* **polyhedron** *is a solid bounded on all sides by planes.*

The planes which bound a polyhedron are called its **faces.**

The straight lines in which the faces of a polyhedron intersect one another are called its **edges.**

The points where three or more edges of a polyhedron meet are called its **vertices.**
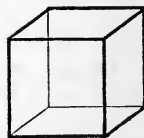
A **solid angle** *is bounded by three or more planes which meet in a point.*

A **convex solid** *is a solid no plane face of which would if produced cut the solid.*

PROPOSITION XX.

If a solid angle be contained by three plane angles, the sum of any two of them is greater than the third.
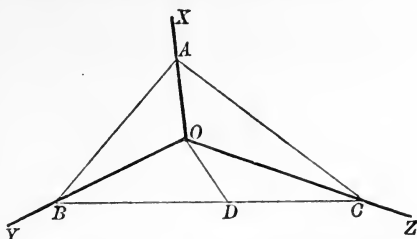
Let the solid angle at O be bounded by the three planes YOZ, ZOX, XOY which intersect in pairs in the lines OX, OY, OZ. Then *it is required to prove that the sum of any two of the three angles YOZ, ZOX, XOY is greater than the third.*

Let the $\angle YOZ$ be greater than either of the other angles, then it is only necessary to prove that $\angle YOZ$ is less than the sum of the $\angle^s ZOX$ and XOY.

In the plane YOZ make $\angle YOD = \angle YOX$.

Through D draw a line in the plane YOZ cutting OY, OZ in the points B, C respectively. Take a point A on OX such that $OA = OD$.

Then OA, OB and included $\angle AOB$ are equal to OD, OB and the included $\angle DOB$;

$$\therefore BD = BA.$$

But BC is less than the sum of BA and AC;

$$\therefore DC \text{ is less than } AC.$$

Then, $\because OD = OA$, OC is common, and $DC < AC$;

$$\therefore \angle DOC < \angle AOC.$$

But $\angle BOD = \angle BOA$;

$$\therefore \text{ whole } \angle BOC < \text{ sum of } \angle^s BOA \text{ and } AOC.$$

Cor. *If a solid angle is bounded by any number of plane angles any one of these is less than the sum of all the others.*

PROPOSITION XXI.

Every solid angle, of a convex solid, is contained by plane angles which are together less than four right angles.

Let any plane cut the planes bounding the solid angle at O in the lines AB, BC, ..., FA.
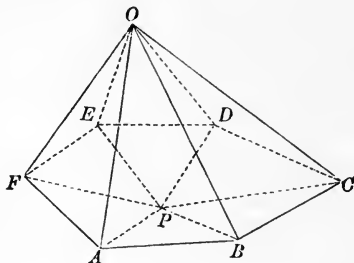
Take P any point in the plane $ABCDEF$, and join PA, PB, ..., PF.

Then the sum of all the angles of the $\triangle^s$ AOB, BOC, ..., FOA is equal to the sum of all the angles of the $\triangle^s$ APB, BPC, ..., FPA, for the $\triangle^s$ are equal in number.

But, by the previous proposition, if $ABCDEF$ is a *convex* polygon, $\angle$ FAB is less than the sum of $\angle^s$ FAO and BAO, and similarly at B, C, D, E, F.

Hence the sum of the angles of the $\triangle^s$ AOB, BOC, ..., FOA excluding the angles at O is greater than the sum of the angles of the $\triangle^s$ APB, BPC, ..., FPA excluding the angles at P.

Hence the sum of all the angles at O is less than the sum of all the angles at P, that is less than four right angles.

ADDITIONAL PROPOSITIONS.

I. POLYHEDRA.

1. Polyhedra are named according to the number of their plane faces. For example, a polyhedron bounded by four plane faces is called a **tetrahedron**, one bounded by six planes a **hexahedron**, by eight an **octahedron**, by twelve a **dodecahedron**, and by twenty an **icosahedron**.

2. *A solid bounded by three pairs of parallel planes is called a* **parallelepiped**.
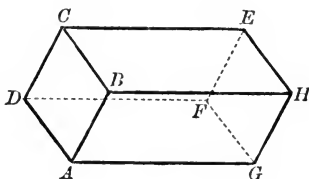
Since two parallel planes are cut in parallel lines by any other plane, it is easily seen that the six faces of a parallelepiped are all *parallelograms*.

Conversely, if the six faces of a hexahedron are all parallelograms, it follows from Euclid XI. 15 that the planes of opposite faces are parallel.

A parallelepiped in which each of the three planes which meet at a vertex is perpendicular to the other two, is called a **rectangular parallelepiped**.

A rectangular parallelepiped in which the three edges which meet in a point are equal, is called a **cube**.

It will be easily seen that the polyhedron bounded by six rectangles must be a *rectangular parallelepiped*, and that the polyhedron bounded by six squares must be a *cube*.

The lines joining opposite vertices of a parallelepiped are called its **diagonals**.

Thus AE, BF, CG, DH are the diagonals in the above figure.

The following properties of a parallelepiped will be easily proved :

(i) The diagonals of any parallelepiped meet in a point and bisect each other.

(ii) The section of a parallelepiped by any plane which cuts two pairs of opposite faces, is a parallelogram.

(iii) The square on a diagonal of a rectangular parallelepiped is equal to the sum of the squares on the three edges which meet at a vertex.

(iv) If the diagonals of a parallelepiped are all equal, the parallelepiped must be rectangular.

3.   *A **pyramid** is a polyhedron all whose faces but one meet in a point, which is called the **vertex** of the pyramid, the face opposite to the vertex being called its **base**.*
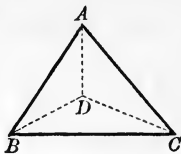

The length of the perpendicular drawn from the vertex of a pyramid on the base is called the *altitude* of the pyramid.

A pyramid on a triangular base is called a *triangular pyramid,* and a pyramid whose base is a square (like the pyramids of Egypt) is called a *square* pyramid ; and so on.

A triangular pyramid is generally called a **tetrahedron**.

It is easily seen that any plane section of a pyramid parallel to its base is similar to the base.

4.   *A **prism** is a polyhedron all but two of whose faces are parallel to the same straight line ; the faces which are parallel to the straight lines*
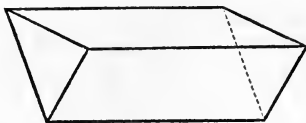
*are called the **sides**, and the other two faces are called the **ends** of the prism.*

If two planes are both parallel to a given straight line, their line of intersection must also be parallel to the given straight line.   It therefore follows that, if the two ends of a prism are parallel to one another, the sides are all parallelograms.

It should be noted that a parallelepiped is a prism with parallel ends.

It is easily seen that all parallel plane sections of a prism, which do not meet either of the ends, are similar and equal polygons.

II. TWO NON-INTERSECTING STRAIGHT LINES.

To draw a straight line perpendicular to each of two given non-intersecting straight lines.

Let AB and CD be the given st. lines.
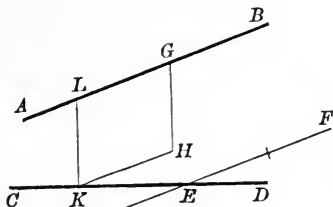
Through any point E on CD draw the st. line $EF \parallel$ to AB.

From any pt. G in AB draw $GH \perp^r$ to the plane CDF, meeting the plane in H.

Through H draw HK in the plane $CDF \parallel$ to FE or AB to cut CD in K; then, since AB and HK are parallel, $AGHK$ is a plane. Complete the $\parallel^m$ $GHKL$.

Then, since KL and GH are $\parallel$, and GH is $\perp^r$ to the plane CDF, KL must also be $\perp^r$ to the plane CDF. Hence LK is $\perp^r$ to CD and to KH, and $\therefore$ also to AB which is $\parallel$ to KH.

Thus *the st. line KL meets AB and CD and is perpendicular to both.*

The straight line KL is the shortest straight line which joins two points one on each of the two given lines AB and CD. For, if G be any other point on AB, the $\perp^r$ from G on the plane through CD and a line parallel to AB will, as we have seen, be equal to LK, and therefore all points on CD are at a distance from G greater than LK.

Thus *the* **shortest distance** *between two non-intersecting straight lines is perpendicular to both lines.*

For example, if KL be the line of intersection of two sides of a room, K being on the ceiling and L on the floor; then KL is the shortest distance between any two straight lines drawn on the ceiling and floor respectively through K and L.

Ex. Having given three straight lines no two of which lie in a plane, draw a straight line parallel to one of the lines and intersecting the other two.

[Let AB, CD, EF be the given lines. Draw $EG \parallel$ to AB and let CD cut the plane FEG in X. In the plane FEG draw $XY \parallel$ to EF, and let the plane CXY cut AB in Z. Then a line through $Z \parallel$ to YX will intersect CDX and will be the line required.]

III. PROPERTIES OF A SPHERE.

Def. *A* **sphere** *is the surface generated by the complete revolution of a semi-circle about its diameter.*

It will easily be seen that the distance of any point on the surface of a sphere from the centre of the semi-circle is equal to its radius, so that we have the alternative definition:—

A **sphere** *is the locus of a point which moves in space so that its distance from a certain fixed point, called the* **centre**, *is always equal to a given length, which is called the* **radius** *of the sphere.*

Any straight line through the centre of a sphere whose extremities are on the surface is called a **diameter** *of the sphere.*

It is obvious that all diameters of a sphere are equal.

It follows from the definition of a sphere that any plane section through its centre is a circle whose radius is the radius of the sphere.

Def. *The section of a sphere by any plane passing through its centre is called a* **great circle** *of the sphere.*

Any two planes through the centre of a sphere must intersect along a straight line through the centre, that is along a diameter of the sphere, and this diameter is also a diameter of each of the sections; hence *any two great circles of a sphere must bisect each other.*
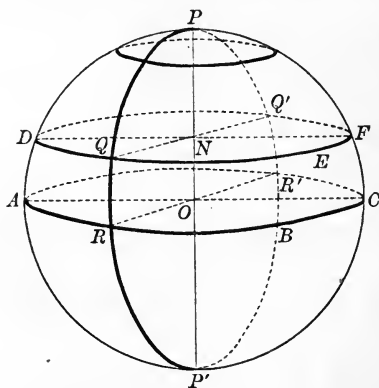
1. Let ABC be any great circle of a sphere, and let DEF be any plane section parallel to ABC. Draw the diameter POP' $\perp^r$ to the plane ABC, and let it meet the plane DEF at rt. $\angle^s$ in the pt. N.

Let Q be any point on the section of the sphere by the plane DEF. Join NQ and QO.

Then, since $\angle ONQ$ is a rt. $\angle$,
$$NQ^2 = OQ^2 - ON^2.$$

Hence NQ is constant for all positions of Q on the section.

Thus *any plane section of a sphere is a circle whose centre is the projection of the centre of the sphere on the plane of the section.*

It follows from the relation
$$NQ^2 = OQ^2 - ON^2$$
that the radius of any plane section of a sphere becomes smaller and smaller as the perpendicular distance of the plane from the centre of the sphere is increased, and that when this perpendicular distance is equal to the radius of the sphere the section becomes a circle of *zero* radius.

Thus *the plane drawn through a point P on a sphere perpendicular to the diameter of the sphere through P will* **touch** *the sphere at P.*

Def. *The extremities of the diameter of a sphere which is perpendicular to any circular section of the sphere are called the* **poles** *of that circle.*

Thus P, P' are the *poles* of the circles ABC or DEF.

Since the arcs of great circles are proportional to the angles they subtend at the centre of the sphere, it follows from Euclid XI. 20 and 21 that

(i)   The sum of any two sides of a spherical triangle, whose sides are arcs of great circles, is greater than the third side.

(ii)   The sum of all the sides of a convex spherical polygon, whose sides are arcs of great circles, is less than a great circle of the sphere.

The student will have no difficulty in proving the following properties of a sphere :

Ex. 1.   Any line drawn through a point P on a sphere perpendicular to the radius OP will touch the sphere.

Ex. 2.   If two spheres are concentric, any tangent plane to the inner will cut the outer in a circle of constant radius.

Ex. 3.   The planes of all small circles of a sphere which are of equal radius are equally distant from the centre of the sphere and touch a concentric sphere.

Ex. 4.   If two circles on a sphere bisect each other, they must both be great circles.

Ex. 5.   All points on a plane section of a sphere are equally distant from the poles of the section.

Ex. 6.   If O be the centre of a sphere, Q any external point, and PQ a tangent line to the sphere at the point P which passes through Q, then will P lie on the plane perpendicular to OQ which meets it in the point N such that $ON \cdot OQ =$ Sq. on radius of the sphere.

Conversely, if Q be any point external to a sphere whose centre is O and the point N be taken on OQ such that $ON \cdot OQ =$ Sq. on radius of the sphere; then, if P be any point on the section of the sphere by the plane through N perpendicular to ONQ, the line QP will touch the sphere.

Ex. 7. If the distance between the centres of two spheres is less than the sum and greater than the difference of their radii, the two spheres will intersect in a circle.

Ex. 8. If the distance between the centres of two spheres is equal to the sum of their radii, the two spheres will touch one another externally; and if the distance between the centres is equal to the difference of the radii, the two spheres will touch internally.

Ex. 9. The locus of the centres of all plane sections of a sphere which pass through a given point is a sphere.

Ex. 10. The locus of the centres of all plane sections of a sphere which pass through a given straight line is a circle.

Ex. 11. The locus of a point whose distances from two given points are in a given ratio, is a sphere.

Ex. 12. If through a given point O any straight line is drawn which cuts a given sphere in the points P, Q, the rectangle OP, OQ is constant.

The following problems should also be noted :

Ex. 1. To draw a tangent plane to a sphere from a given external point.

Let O be the centre of the sphere, P the external point, and V the middle point of OP. Then the sphere whose centre is V and radius OV or VP will cut the given sphere in a circle, and the tangent plane to the sphere at any point of this circle will pass through P.

Ex. 2. To draw a tangent plane to a sphere through a given straight line which does not cut the sphere.

Let O be the centre of the sphere and KL the given straight line. Through O draw a plane $\perp^r$ to KL and cutting the sphere in a great circle and KL in M. Then if P, Q be the tangent lines from M to the great circle, the planes at PKL, QKL will be the planes required.

Ex. 3. Draw a plane so as to touch three given spheres.

It is easily seen that a common tangent plane to two spheres will cut the line joining their centres in one or other of the two points which divide that line in the ratio of the radii of the spheres. These points are called the **centres of similitude** of the spheres.

It is also easily seen that a plane which passes through a centre of similitude of two spheres and touches one of the spheres will also touch the other sphere.

Hence, if K be a centre of similitude of the spheres A and B, and L a centre of similitude of the spheres A and C, the planes through the line KL which touch the sphere A will touch the three spheres A, B and C.

Ex. 4. Through a given point draw a plane to touch two given spheres.

IV. THE TETRAHEDRON.

1. *The four lines from the vertices of a tetrahedron to the centroids of the opposite faces meet in a point.*
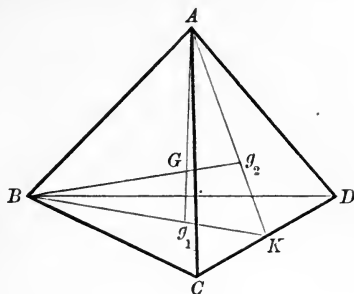
Let g_1, g_2 be the centroids of the faces BCD, CDA respectively. Then, Bg_1 and Ag_2 will both pass through K, the middle point of CD.

Hence Ag_1, Bg_2 are both in the plane ABK, and will therefore meet, in G suppose.

Then, since $Bg_1 = 2g_1K$ and $Ag_2 = 2g_2K$,

$$\triangle AGB = 2 \triangle KGB = 3 \triangle g_1GB;$$
$$\therefore \quad AG = 3Gg_1.$$

Thus Ag_1 is met by Bg_2 in a point G such that $4g_1G = g_1A$, and it can be proved in a similar manner that Cg_3 and Dg_4 also pass through the point G.

Def. *The point of intersection of the four lines from the vertices of a tetrahedron to the centroids of the opposite faces is called the* **centroid** *of the tetrahedron.*

2. *The four lines through the circum-centres of the faces of a tetrahedron and perpendicular to the faces will meet in a point.*

3. *The six planes through the middle points of the edges of a tetrahedron and perpendicular to the edges will meet in a point.*

4. *Through each edge of a tetrahedron a plane is drawn bisecting the angle between the two planes which intersect along that edge; shew that these six planes have a common point which is the centre of the sphere which touches the faces of the tetrahedron.*

5. *Eight spheres will touch four given planes which do not meet in a point and no three of which intersect in a straight line.*

6. *One sphere will pass through any four points which do not lie on a plane.*

7. *If two pairs of opposite edges of a tetrahedron are at right angles, the four lines through the vertices perpendicular respectively to the opposite faces will meet in a point.*

Let Aa, Bb, Cc, Dd be the perpendiculars from A, B, C, D on the opposite faces of the tetrahedron $ABCD$, and let AB be perpendicular to CD, and AC perpendicular to BD.
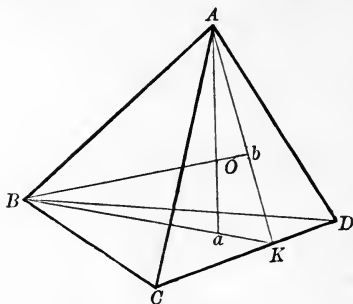
Join Ba and produce it to cut CD in K.

Then, since Aa is $\perp^r$ to plane BCD,

$$CD \text{ is } \perp^r \text{ to } Aa, \text{ and it is also } \perp^r \text{ to } AB;$$

$\therefore$ CD is $\perp^r$ to the plane ABa, and $\therefore$ to AK and BK.

Hence CD is $\perp^r$ to AK, AB and Bb;

$\therefore$ Bb must be in the plane ABK, so that Aa and Bb lie in a plane and must therefore intersect.

Again, if Ca cut BD in L, BD can be proved in a similar manner to be $\perp^r$ to CL, and it is also $\perp^r$ to Cc, Aa and AC, whence it follows that Aa, Cc, AC and CL lie in a plane, so that Aa and Cc intersect.

And, since CD is $\perp^r$ to BK and BD $\perp^r$ to CL, the point a must be the orthocentre of ABC, and BC is $\perp^r$ to aD. But BC is also $\perp^r$ to Aa; $\therefore$ BC must be $\perp^r$ to the plane AaD and therefore $\perp^r$ to AD. [Thus if two pairs of opposite edges be $\perp^r$, the third pair will also be at rt. $\angle^s$.]

Then, as before, we can shew that Aa meets Dd.

Thus Aa meets Bb, Cc and Dd, and so also Bb meets Aa, Cc and Dd.

But, if each of Aa, Bb and Cc meets the other two, they must all meet in a point, for they cannot all lie on a plane.

Hence *the four 'perpendiculars' of a tetrahedron meet in a point provided two pairs of opposite edges are at right angles.*

It should be noticed that when two pairs of opposite edges of a tetrahedron are at rt. $\angle^s$ (and therefore, as we have seen, the third pair also at rt. $\angle^s$), *the sum of the squares of one pair of opposite edges is equal to the sum of the squares of either of the other pairs.*
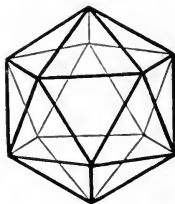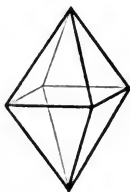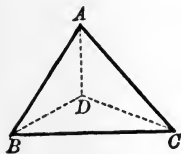
V.   THE REGULAR SOLIDS.

A **regular solid** *is a solid bounded by plane faces which are all equal regular polygons.*

We know (1) that three planes at least must meet at any solid angle, and (2) that the sum of the plane angles at the solid angle must be less than four right angles.

Since each of the angles of an equilateral triangle is two-thirds of a right angle, it follows that three, or four, or five (but not more than five) equilateral triangles can meet at a point and form a solid angle.

Thus there can only be three (and it can be proved that there are really three) regular solids whose faces are equilateral triangles.

The regular solids whose faces are equilateral triangles are the **tetrahedron**, the **octahedron** and the **icosahedron**.

Only *three* squares can meet at a point to form a solid angle, for the sum of all the plane angles at a solid angle must be less than four right angles.

The regular solid whose faces are squares is the **cube**.
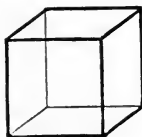
Three, but not more than three, regular pentagons can meet at a point to form a solid angle, for an angle of a regular pentagon is six-fifths of a right angle.

There is a regular solid whose faces are regular pentagons, namely the **dodecahedron.**
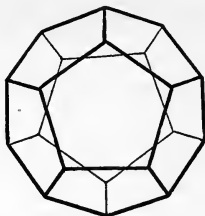
Three of the angles of a regular hexagon are together equal to four right angles, and three of the angles of a regular polygon of more than six sides are greater than four rt. angles.

Hence no regular solid can be formed whose faces are regular polygons of more than five sides.

Hence *there can only be five regular solids.*

MISCELLANEOUS EXERCISES.

1. Shew that every plane section of a parallelepiped which cuts two pairs of opposite faces is a parallelogram.

2. Shew that the middle points of the four diagonals of a parallelepiped are coincident.

3. The corners of a triangle *ABC* are joined to a point *O* outside its plane, and the joining lines *OA*, *OB*, *OC* are cut by a plane parallel to that of the triangle in *D*, *E*, *F* respectively; prove that the triangle *DEF* is similar to the triangle *ABC*.

If the plane *DEF* is not parallel to *ABC*, prove that the intersections of *BC*, *EF*, of *CA*, *FD* and of *AB*, *DE* lie in a straight line.

4. A number of unlighted candles stand upon a table, and another lighted candle, shorter than any of the former, stands on the same table; prove that the shadows formed on the ceiling by the unlighted candles, if produced, will all meet in a point.

5. Shew that, if a given straight line is parallel to a given plane, the shortest distance between the given line and any line on the given plane, which is not parallel to it, is constant.

6. Find the locus of points which are equally distant from two intersecting straight lines.

7. *OA*, *OB*, *OC* are three straight lines which meet in a point. Find a line *OP* such that $\angle AOP = \angle BOP = \angle COP$.

8. Shew that, if the opposite edges of a tetrahedron are equal in pairs, each of the solid angles is bounded by three plane angles whose sum is equal to two right angles.

9. Three lines *OA*, *OB*, *OC* meet in a point. Shew that, if the angles *AOB*, *AOC* are equal, the planes *AOB*, *AOC* make equal angles with the plane *BOC*.

10. *A*, *B* are two points on the same side of a given plane. Find the point *P* on the plane such that the sum of the straight lines *PA*, *PB* is a minimum.

11. *AB* is the line of intersection of two planes, and *P*, *Q* are two given points one on each of the planes. Find the point *X* on *AB* such that the sum of *PX* and *QX* is a minimum.

12. *OA*, *OB*, *OC* are three lines on a plane and *OP* is such that $\angle POA = \angle POB = \angle POC$. Shew that *OP* is perpendicular to the plane *OABC*.

13. If a point within a spherical triangle whose sides are arcs of great circles be joined to the three angles by arcs of great circles, prove that the sum of the lengths of these three arcs is intermediate between the perimeter and the semiperimeter of the triangle.

14. *AB, CD* are two parallel lines, and four lines through *A, B, C, D* parallel to one another are cut by any plane in *a, b, c, d* respectively; shew that *ab* is parallel to *cd*.

15. *P, Q* are any two points on two given non-coplanar lines; shew that the middle point of *PQ* is on a fixed plane.

16. *P, Q* are any two points on two given non-coplanar lines; shew that the point which divides *PQ* in a given ratio lies on a fixed plane.

17. Shew that the square on a diagonal of a cube is three times the square on one of its edges.

18. Shew that the locus of a point, the sum of the squares of whose distances from two given points is constant, is a sphere.

19. Shew that the locus of a point, the difference of the squares of whose distances from two given points is constant, is a plane.

20. Through a given point draw a straight line to meet two given non-intersecting straight lines.

21. Find the shortest path on the walls of a room from a given point on one wall to a given point on the adjacent wall.

22. Shew that the three lines joining the middle points of opposite edges of a tetrahedron meet in a point and bisect each other.

23. Of the three rectangles contained by pairs of opposite edges of a tetrahedron, the sum of any two is greater than the third.

24. The middle points of the edges of a regular tetrahedron are the vertices of a regular octahedron.

25. Shew how to cut any four given straight lines *OA, OB, OC, OD* meeting in the point *O* by a plane, so that the four points of intersection may be the angular points of a parallelogram.

26. The middle points of the six edges of a cube which do not meet a particular diagonal of the cube all lie on a plane to which the diagonal is perpendicular.

27. Shew how to cut a cube by a plane so that the section may be a regular hexagon.

28. Shew that, if the pairs of opposite edges of a tetrahedron are at right angles, the shortest distances between opposite edges meet at the point of intersection of the perpendiculars.

29. A tetrahedron is cut by a plane parallel to a pair of opposite edges; shew that the section is a parallelogram, and find the position of the plane when the area of the parallelogram is a maximum.

30. Shew that if a plane section of a tetrahedron be a parallelogram it must be parallel to a pair of opposite edges.

31. Shew that if a plane section of a regular tetrahedron be a parallelogram its perimeter is constant.

32. Shew that if each edge of a tetrahedron is equal to the opposite edge, the straight line which joins the middle points of two opposite edges will be at right angles to those edges.

33. If points A, B, C be taken on three conterminous edges of a cube, the triangle ABC will be acute angled.

34. If A, B, C, D are four points not on a plane, the sum of the angles ABC, BCD, CDA, DAB will be less than four right angles.

35. If in a tetrahedron each edge is equal to the opposite edge, the faces will all be acute-angled triangles.

36. $ABCD$ is one face of a cube, and a plane through D cuts the edges through A, B, C which are perpendicular to the plane $ABCD$ in the points A', B', C' respectively; shew that $A'B'C'D$ is a parallelogram which may be a rhombus but not a square.

37. In a regular tetrahedron the line joining the middle points of a pair of opposite edges is the shortest distance between them.

38. If AD is a diagonal of a cube, and PQ an edge which does not meet AD, the shortest distance between AD and PQ is the line joining their middle points.

39. Shew that an edge of a regular octahedron is perpendicular to an edge which meets it but is not on the same face of the octahedron.

40. If a sphere touch the six edges of a tetrahedron, the sum of one pair of opposite edges is equal to the sum of either of the other pairs of opposite edges.

41. If a sphere touch the six edges of a tetrahedron, the three lines joining the points of contact of opposite edges will meet in a point.

42. Shew that the line joining the middle points of a pair of opposite edges of a tetrahedron passes through the centroid of the tetrahedron.

43. Shew that the six planes, each passing through one edge of a tetrahedron and bisecting the opposite edge, meet in a point.

44. Through the middle points of every edge of a tetrahedron a plane is drawn perpendicular to the opposite edge; shew that the six planes so drawn will meet in a point such that the centroid of the tetrahedron is midway between it and the centre of the circumscribing sphere.

45. D, E, F are any pts. on the sides BC, CA, AB respectively of the triangle ABC, and O is any point not on the plane of the triangle; shew that the sum of the angles BOC, COA, AOB is greater than the sum of the angles EOF, FOD, DOE.

46. If X is any pt. on the base BCD of the tetrahedron $ABCD$, the sum of the angles BAX, CAX, DAX is less than the sum but greater than half the sum of the angles BAC, CAD, DAB.

47. If O be a point within the tetrahedron $ABCD$, the sum of the three angles BOC, COD, DOB is greater than the sum of BAC, CAD, DAB.

Shew also that the sum of the areas of the $\triangle^s$ BOC, COD, DOB is less than the sum of the areas of the $\triangle^s$ BAC, CAD, DAB.

[Cf. Euclid I. 21.]

48. If from any point outside a sphere lines be drawn to all the points of a small circle of the sphere, prove that these lines meet the sphere again in a circle.

49. Prove Euler's Theorem, namely that if E be the number of edges, S the number of solid angles and F the number of faces of a polyhedron; then will $F + S = E + 2$.

50. The sum of all the plane angles of a convex polyhedron is double the sum of the angles of a plane polygon having the same number of vertices as the polyhedron.

ELEMENTARY ALGEBRA

BY

CHARLES SMITH, M.A.

MASTER OF SIDNEY SUSSEX COLLEGE, CAMBRIDGE.

Second Edition. Ninth Impression. Revised and Enlarged.

For the Second Edition the whole book has been thoroughly revised, and the early Chapters remodelled and simplified, and Chapters on Logarithms and Scales of Notation have been added. The number of examples has been very greatly increased.

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The *Saturday Review* says:—" One could hardly desire a better beginning on the subject of which it treats than Mr Charles Smith's *Elementary Algebra*. It is instinct with the merits which distinguished his previous ventures, and has the same *lucidus ordo*...A very carefully selected collection of exercises adds considerably to the work. The author has certainly acquired—unless it growed—the knack of writing text-books which are not only easily understood by the junior student, but which also commend themselves to the admiration of more matured ones."

The *Educational Times* says:—" Mr Charles Smith, Tutor of Sidney Sussex College, Cambridge, whose *Elementary Treatise on Conic Sections* is so well known to most students of Mathematics, has done us very good service in publishing an Elementary Algebra. There is a logical clearness about his expositions and the order of his chapters for which both schoolboys and schoolmasters should be, and will be, very grateful. His treatment of the Theory of Indices, for instance, though really a very simple matter, is admirable for the way in which it sets forth the difficulties of the subject, and then solves them."

The *Oxford Magazine* says:—" Mr Charles Smith, whose books are well known to all who are concerned in teaching of Analytical Geometry, plane or solid, has followed up the work he has begun by bringing out a no less excellent text-book on *Elementary Algebra* with a promise of a larger work dealing with more advanced branches of the subject. The present work is characteristic in its clear explanations and copious supply of exercises ; it ought to, and will no doubt be well received."

The *School Guardian* says:—" Mr Charles Smith's *Elementary Algebra*, we are glad to see, has reached a new edition. It is an excellent book, and specially adapted to the use of beginners and private students. In the second edition the work has been thoroughly revised, while the early chapters have been remodelled and simplified ; the number of examples has been considerably increased ; and chapters have been added on Logarithms and Scales of Notation."