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Existence and Uniqueness for a Third Order Non-Linear Partial Differential Equation

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Introduction

The purpose of this paper is to investigate the existence and uniqueness of a solution of the equation

$$(1) \qquad \left(\frac{\partial}{\partial t} - u_y \frac{\partial}{\partial x} + u_x \frac{\partial}{\partial y}\right)(\Delta u - \lambda^2 u) = 0$$

where u is a real valued function of the real variables x, y, and t; $\Delta = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$; and λ is a positive constant.

Equation (1) has arisen as an elementary mathematical model in meteorology [1]. In this model x and y are position variables in two dimensional Euclidean space, and t is the time. We may think of u as the effective depth of the atmosphere, of $(-u_y, u_x)$ as the velocity vector of the air particles, and of Δu as the vertical component of vorticity. We will thus speak of the solutions of the ordinary differential equations $\frac{dx}{dt} = -u_y(x,y,t)$ and $\frac{dy}{dt} = u_x(x,y,t)$ as parametric representations for the curves followed by air particles in the xy - plane. It is then clear from (1) that the Helmholtzian, $\Delta u - \lambda^2 u$, is constant along the air particle paths.

For convenience we will restrict ourselves to the consideration of existence and uniqueness of a solution of (1) in $\mathcal{D} = \{(x,y,t) \mid -\infty < x < \infty,\ y \geq 0,\ 0 \leq t \leq c\}$ where c is a positive constant.

Let $\Delta u - \lambda^2 u = h$ where u is a solution of (1) in $\mathcal{D}$. If h is smooth enough, it is well known that when y > 0 then

$$(2) \qquad \begin{aligned} u(x,y,t) = {} & \frac{1}{2\pi} \iint_{\eta \geq 0} g(x,y;\xi,\eta) h(\xi,\eta,t)\, d\xi\, d\eta \\ & - \frac{\lambda y}{\pi} \int_{-\infty}^{\infty} u(\xi,0,t)\, \frac{1}{\nu} K'(\lambda\nu)\, d\xi \end{aligned}$$

where $\nu = \sqrt{(\xi - x)^2 + y^2}$ and where we have used the appropriate Green's function $g(x,y;\xi,\eta)$ and Bessel function $K(x)$. That is, we let $\rho = \sqrt{(\xi - x)^2 + (\eta - y)^2}$, $\bar{\rho} = \sqrt{(\xi - x)^2 + (\eta + y)^2}$, and $g(x,y;\xi,\eta) = K(\lambda\bar{\rho}) - K(\lambda\rho)$ where $K(x)$ is the modified Bessel function of the second kind of order zero. Then $g_{xx} + g_{yy} - \lambda^2 g = g_{\xi\xi} + g_{\eta\eta} - \lambda^2 g = 0$, $g(x,0;\xi,\eta) = 0$, and g behaves like $\log \rho$ for (ξ,η) near (x,y).

We will use the above physical terminology in the following heuristic derivation of the appropriate initial and boundary conditions for equation (1).

The right side of (2) depends on h and $u(x,0,t)$. Since h is constant along the air particle paths, we see that h can be given everywhere in $\mathcal{D}$ in terms of its values at points where air particle paths enter $\mathcal{D}$ (i.e. at points where $u_x(x,0,t) > 0$ or $t = 0$). In particular u can be expressed in terms of h at points where the air particle paths enter $\mathcal{D}$ and in terms of $u(x,0,t)$. It therefore seems natural to prescribe the values of u on the xt - plane, to prescribe the values of $\Delta u - \lambda^2 u$ on the half plane $t = 0$ and $y \geq 0$, and to prescribe $\Delta u - \lambda^2 u$ at points on the xt - plane where $u_x > 0$. That this prescription of initial and boundary values constituted a well posed problem was suggested by E. Isaacson; earlier workers in meteorology learned this from numerical experiments.

For convenience we will assume that air particle paths leave $\mathcal{D}$ (i.e. $u_x < 0$) at points in a simply connected open set of the xt - plane, and air particle paths enter $\mathcal{D}$ (i.e. $u_x > 0$) at points of the xt - plane exterior to the above mentioned simply connected open set.

We will find it convenient to consider a solution which at infinity does not deviate "too much" from a uniform flow parallel to the y - axis. Such a uniform flow, $u^* = ax + b$ where a and b are constants and $a > 0$, satisfies (2) and hence any function u which satisfies (2) will also satisfy (3).

$$(3) \quad \begin{aligned} u(x,y,t) = {} & \frac{1}{2\pi} \iint_{\eta \geq 0} g(x,y;\xi,\eta)[h(\xi,\eta,t) + \lambda^2(a\xi + b)]\, d\xi d\eta \\ & - \frac{\lambda y}{\pi} \int_{-\infty}^{\infty} [u(\xi,0,t) - a\xi - b] \frac{1}{\nu} K'(\lambda\nu) d\xi + ax + b \end{aligned}$$

when $y > 0$. We will choose to work with (3) rather than (2) since we will be placing certain restrictions at infinity on $u - ax - b$ and $h + \lambda^2(ax + b)$.

Next we define what we mean by a weak solution of (1) in $\mathcal{D}$. In Part I we will show that a weak solution satisfying certain initial and boundary conditions exists with relatively weak restrictions placed on the prescribed initial and boundary conditions. In Part II we will show that as we gradually strengthen the restrictions placed on the initial and boundary conditions the solution is also gradually strengthened until we have existence of an ordinary solution of (1) satisfying the prescribed initial and boundary conditions. In Part III we prove a uniqueness theorem.

Let U be any real valued function with domain $\mathcal{D}$ such that U_x and U_y are continuous. We require that the solutions to the ordinary differential equations $\frac{dx}{dt} = -U_y(x,y,t)$ and $\frac{dy}{dt} = U_x(x,y,t)$ exist in the large in $\mathcal{D}$ and are unique. The curves in $\mathcal{D}$ described by the vector $[x(t),y(t),t]$ will be called the air

particle paths of U. Let H be any real valued function with domain $\mathcal{D}$ such that along each air particle path of U, H is constant (excepting possibly at points where the air particle path is tangent to the xt - plane). We will call H a pseudo-Helmholtzian of U. We also require that $\iint_{\eta\geq 0} g(x,y;\xi,\eta)[H(\xi,\eta,t)+\lambda^2(a\xi+b)]d\xi d\eta$ exists for (x,y,t) in $\mathcal{D}$. If (3) is valid for u replaced by U and h replaced by some such H, then we call U a weak solution of (1) in $\mathcal{D}$.

We note that for U to be a weak solution of (1) in $\mathcal{D}$, U_x and U_y are the only derivatives whose existence we are assuming. In the remainder of this paper we will use the notation u (and h) for genuine and weak solutions (for Helmholtzians and pseudo-Helmholtzians) and the reader should be forewarned.

Part I

Existence of a Weak Solution*

We will let ϕ and ψ_1 denote the prescribed values of u and h respectively on the plane $y = 0$, and ψ_2 will denote the prescribed values of h on the plane $t = 0$. We will prove the existence of a weak solution in Theorem 1 below for $0 \leq t \leq c_1$ (where $c_1 > 0$ is introduced in the statement of the theorem). Heuristically the proof is based on the following construction. For each n=1,2,3,... we define functions h_n, u_n, x_n, and y_n inductively in the strips $\frac{kc_1}{n} \leq t \leq \frac{(k+1)c_1}{n}$ for k=0,1,...,n-1. u_n may be thought of as an approximate weak solution, x_n and y_n describe the air particle paths of u_n, and h_n may be thought of as an approximate pseudo-Helmholtzian of u_n. We show that a subsequence of $\{x_n\}$ and $\{y_n\}$ converges to limit functions x and y respectively. We use the functions x and y to define functions h and u. We then show that the curves described by x and y are the air particle paths of u, that h is a pseudo-Helmholtzian of u, that u is a weak solution of (1), and that u and h satisfy the prescribed initial and boundary conditions.

Theorem 1. Let ϕ be a real valued function whose domain is $\{(x,t) \mid -\infty < x < \infty \text{ and } 0 \leq t \leq c \text{ where c is a positive constant}\}$. Let ϕ also satisfy (1_A), (1_B), and (1_C).

(1_A) ϕ, ϕ_x, and ϕ_{xx} are continuous. Also for some constants L and i such that $L > 0$ and $0 < i < 1$ we have $|\phi_{xx}(\bar{x},t)-\phi_{xx}(x,t)| \leq L|\bar{x}-x|^i$ for all $(\bar{x},t)$ and (x,t) in the domain of ϕ.

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* Certain symbols are used throughout a large part of this report, and a glossary of such symbols is contained at the end of this report.

(1_B) $\phi(x,t) - ax - b$, $\phi_x(x,t) - a$, and $\phi_{xx}(x,t)$ are bounded where a and b are real constants with $a > 0$.

(1_C) Let the boundary of the region of outgoing particles on the (x,t) plane be given by $x_1(t)$ and $x_2(t)$. That is, let x_1 and x_2 satisfy a uniform Lipschitz condition with $x_1(t) < x_2(t)$ for $0 \leq t \leq c$, call C_1 the curve consisting of the points $[x_1(t), 0, t]$ for $0 \leq t \leq c$, and call C_2 the curve consisting of the points $[x_2(t), 0, t]$ for $0 \leq t \leq c$. Let $\phi_x(x,t) = 0$ for $(x,0,t)$ on C_1 or C_2, let $\phi_x(x,t) < 0$ for $x_1(t) < x < x_2(t)$, and let $\phi_x(x,t) > 0$ for $x < x_1(t)$ or $x > x_2(t)$.

Let ψ_1 be a real valued function whose domain is $\{(x,t) \mid (x,t)$ is in the domain of ϕ and $\phi_x(x,t) \geq 0\}$. Let ψ_1 also satisfy (2_A).

(2_A) ψ_1 is continuous and $\psi_1(x,t) + \lambda^2(ax + b)$ is bounded where λ is a positive constant.

Let ψ_2 be a real valued function whose domain is $\{(x,y) \mid -\infty < x < \infty \text{ and } y \geq 0\}$. Let ψ_2 also satisfy (3_A) and (3_B).

(3_A) ψ_2 is continuous and $\psi_2(x,y) + \lambda^2(ax + b)$ is bounded.

(3_B) $\psi_2(x,0) = \psi_1(x,0)$ for $(x,0)$ in the domain of both ψ_1 and ψ_2.

Then for all small enough positive c_1 there exists a real valued function u with domain $\mathcal{D}_1 = \{(x,y,t) \mid -\infty < x < \infty,\ y \geq 0,\ 0 \leq t \leq c_1\}$ such that u satisfies (4_A), (4_B), and (4_C).

(4_A) $u(x,0,t) = \phi(x,t)$.

(4_B) There exists a pseudo-Helmholtzian h of u such that $h(x,0,t) = \psi_1(x,t)$ when (x,t) is in the domain of ψ_1, $h(x,y,0) = \psi_2(x,y)$, and (3) is valid for u and this h.

(4_C) $u(x,y,t) - ax - b$, $u_x(x,y,t) - a$, $u_y(x,y,t)$, and $h(x,y,t) + \lambda^2(ax + b)$ are all bounded.

We will start the proof of Theorem 1 by examining the second integral in (3).

Lemma (1.1). Let w be the function with domain $\mathcal{D}$ defined by

$$w(x,y,t) = \frac{\lambda y}{\pi}\int_{-\infty}^{\infty} [\phi(\xi,t) - a\xi - b]\,\frac{1}{\nu}\,K'(\lambda\nu)d\xi$$ when $y > 0$ where $\nu = \sqrt{(\xi-x)^2+y^2}$ and $K(x)$ is the modified Bessel function of the second kind of order zero, and by $w(x,0,t) = -\phi(x,t) + ax + b$ for $y = 0$. Then w and its first and second derivatives with respect to x and y are continuous in (x,y,t) and are bounded. Also $\triangle w - \lambda^2 w = 0$.

Proof of Lemma (1.1). Since $\phi(\xi,t) - a\xi - b$ is bounded and continuous, we could easily show that w is continuous for $y > 0$.

Now consider a fixed point $(x_o,0,t_o)$. For $y > 0$ we have

$$w(x,y,t) - w(x,0,t) = \frac{\lambda y}{\pi}\int_{-\infty}^{\infty} [\phi(\xi,t) - \phi(x,t) + a(x-\xi)]\,\frac{1}{\nu}\,K'(\lambda\nu)d\xi + [\phi(x,t) - ax - b]\Big[\frac{\lambda y}{\pi}\int_{-\infty}^{\infty}\frac{1}{\nu}\,K'(\lambda\nu)d\xi + 1\Big].$$

We observe that

$$\frac{\lambda y}{\pi}\int_{-\infty}^{\infty}\frac{1}{\nu}\,K'(\lambda\nu)d\xi = \frac{1}{\pi}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \lambda y \sec\theta\, K'(\lambda y\sec\theta)d\theta.$$

Since $\lambda y \sec\theta\, K'(\lambda y \sec\theta)$ is a measurable function of θ for each $y > 0$, since $|\lambda y \sec\theta\, K'(\lambda y\sec\theta)| \le M$, and since $\lim_{y\to 0+} \lambda y \sec\theta\, K'(\lambda y \sec\theta) = -1$ for almost all θ, then by the Lebesgue convergence theorem we have $\frac{\lambda y}{\pi}\int_{-\infty}^{\infty}\frac{1}{\nu}\,K'(\lambda\nu)d\xi \to -1$ as $y \to 0+$. Furthermore the convergence is uniform with respect to (x,t) since

$\frac{1}{\pi}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \lambda y \sec\theta\, K'(\lambda y \sec\theta)d\theta$ does not depend on (x,t). Now given

$\varepsilon > 0$ we can choose $\delta > 0$ so that $0 < y \leq \delta$ implies $|w(x,y,t) - w(x,0,t)| \leq \bar{M}\varepsilon + \frac{\lambda y}{\pi}\int_{-\infty}^{\infty} |\phi(\xi,t) - \phi(x,t) + a(x-\xi)| \frac{1}{\nu} |K'(\lambda\nu)| d\xi$ for all (x,t) where $\bar{M}$ is chosen so that $|\phi(x,t) - ax - b| \leq \bar{M}$. Now let R be any positive number such that $R > 2|x_0|$. Since $\phi(\xi,t) - a\xi$ is uniformly continuous in (ξ,t) for $|\xi| \leq R$ and $0 \leq t \leq c$, then we can choose $\delta^* > 0$ so that $|\phi(\xi,t) - \phi(x,t) + a(x-\xi)| \leq \varepsilon$ for $|\xi - x| \leq \delta^* \leq \frac{R}{2}$, $|x| \leq \frac{R}{2}$, and $0 \leq t \leq c$. Then for $|x| \leq \frac{R}{2}$, $0 < y \leq \delta$, and $0 \leq t \leq c$ we have $|w(x,y,t)-w(x,0,t)| \leq \bar{M}\varepsilon + \frac{\lambda y}{\pi}\int_{-\infty}^{\infty} \frac{\varepsilon}{\nu}\frac{M}{\lambda\nu} d\xi + \frac{\lambda y}{\pi}\left(\int_{-\infty}^{x-\delta^*} + \int_{x+\delta^*}^{\infty}\right) \frac{2\bar{M}}{\nu}\frac{M}{\lambda\nu} d\xi \leq \bar{M}\varepsilon + M\varepsilon + \frac{4y\bar{M}M}{\pi}\int_{x+\delta^*}^{\infty} \frac{d\xi}{(\xi-x)^2} \leq (\bar{M}+M)\varepsilon + \frac{4y\bar{M}M}{\pi\delta^*}$. Now choose $\bar{\delta} > 0$ so that $\bar{\delta} < \delta$ and $\frac{4\bar{\delta}\bar{M}M}{\pi\delta^*} < \varepsilon$. Then for $|x| \leq \frac{R}{2}$, $0 < y \leq \bar{\delta}$, and $0 \leq t \leq c$ we have $|w(x,y,t) - w(x,0,t)| \leq (\bar{M} + M + 1)\varepsilon$. Now for $|x| \leq \frac{R}{2}$, $0 < y \leq \bar{\delta}$, and $0 \leq t \leq c$ we have $|w(x,y,t)-w(x_0,0,t_0)| \leq |w(x,y,t)-w(x,0,t)| + |w(x,0,t)-w(x_0,0,t_0)| \leq (\bar{M}+M+1)\varepsilon + |\phi(x,t)-\phi(x_0,t_0) + a(x_0-x)| \leq (\bar{M}+M+2)\varepsilon$ for all (x,y,t) near enough to $(x_0,0,t_0)$. Thus w is continuous at $(x_0,0,t_0)$. This completes the proof that w is continuous.

To see that w is bounded we observe that[(1)] $|w(x,y,t)| \leq \frac{\lambda y}{\pi}\int_{-\infty}^{\infty} \bar{M}\frac{M}{\lambda\nu^2} d\xi = \bar{M}M$ for $y > 0$ and $|w(x,0,t)| \leq \bar{M}$.

We observe that the only hypothesis used, to show that w is continuous and bounded, was $\phi(x,t) - ax - b$ is continuous and bounded.

Next we will show that $w_x(x,y,t)$ exists and is continuous and bounded. Since $\phi(x,t) - ax - b$ is continuous and bounded, we could

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(1) See page 13 for the choice of M.

show that differentiation under the integral sign with respect to x is permitted for $y > 0$. For $y > 0$ we have $w_x(x,y,t) = \frac{\lambda y}{\pi}\int_{-\infty}^{\infty} [\phi(\xi,t) - a\xi - b] \frac{\partial}{\partial x} [\frac{1}{\nu} K'(\lambda\nu)]d\xi = - \frac{\lambda y}{\pi}\int_{-\infty}^{\infty} [\phi(\xi,t)-a\xi-b] \frac{\partial}{\partial \xi} [\frac{1}{\nu} K'(\lambda\nu)]d\xi = \frac{\lambda y}{\pi}\int_{-\infty}^{\infty} [\phi_x(\xi,t)-a] \frac{1}{\nu} K'(\lambda\nu)d\xi$. Since $w(x,0,t) = - \phi(x,t) + ax + b$, we have $w_x(x,0,t) = -\phi_x(x,t) + a$. We list this as

$$(1.1.1)\quad w_x(x,y,t) = \begin{cases} \frac{\lambda y}{\pi}\int_{-\infty}^{\infty} [\phi_x(\xi,t)-a] \frac{1}{\nu} K'(\lambda\nu)d\xi \text{ for } y > 0 \\ - \phi_x(x,t) + a \text{ for } y = 0. \end{cases}$$

Since $\phi_x(\xi,t) - a$ is continuous and bounded, the continuity and boundedness of w_x follows exactly as it did for w.

Next we will show that w_y exists and is continuous and bounded. Again we could show that differentiation under the integral sign with respect to y is permitted for $y > 0$ since $\phi(x,t) - ax - b$ is continuous and bounded. For $y > 0$ we have $w_y(x,y,t) = \frac{1}{\pi}\int_{-\infty}^{\infty} [\phi(\xi,t)-a\xi-b] \frac{\partial}{\partial y} [\frac{\lambda y}{\nu} K'(\lambda\nu)] d\xi =$ $\frac{1}{\pi}\int_{-\infty}^{\infty} [\phi(\xi,t)-a\xi-b][\frac{(\xi-x)^2}{\nu^3} \lambda K'(\lambda\nu) + \frac{y^2}{\nu^2} \lambda^2 K''(\lambda\nu)]d\xi$. Next we observe that $\frac{\partial^2}{\partial \xi^2} K(\lambda\nu) = \frac{y^2}{\nu^3} \lambda K'(\lambda\nu) + \frac{(\xi-x)^2}{\nu^2} \lambda^2 K''(\lambda\nu)$. Hence for $y > 0$, $w_y(x,y,t) = \frac{1}{\pi}\int_{-\infty}^{\infty} [\phi(\xi,t)-a\xi-b][\frac{\lambda}{\nu} K'(\lambda\nu) + \lambda^2 K''(\lambda\nu) - \frac{\partial^2}{\partial \xi^2} K(\lambda\nu)] d\xi = \frac{1}{\pi}\int_{-\infty}^{\infty} [\phi(\xi,t) - a\xi - b] [\lambda^2 K(\lambda\nu) - \frac{\partial^2}{\partial \xi^2} K(\lambda\nu)] d\xi =$

$= \frac{1}{\pi} \int_{-\infty}^{\infty} [\lambda^2\phi(\xi,t) - \lambda^2(a\xi+b) - \phi_{xx}(\xi,t)]K(\lambda\nu)d\xi$. We observe that this last integral exists for $y \geq 0$. Since $\lambda^2\phi(x,t)-\lambda^2(ax+b)-\phi_{xx}(x,t)$ is continuous and bounded, we could show that this last integral is continuous in (x,y,t) for $-\infty < x < \infty$, $y \geq 0$, and $0 \leq t \leq c$. Hence w_y exists for $y > 0$, and w_y coincides for $y > 0$ with a function which is continuous for $y \geq 0$. It follows that w_y exists and is continuous for $y \geq 0$, and we have

$$(1.1.2)\quad w_y(x,y,t) = \frac{1}{\pi} \int_{-\infty}^{\infty} [\lambda^2\phi(\xi,t)-\lambda^2(a\xi+b)-\phi_{xx}(\xi,t)]K(\lambda\nu)d\xi \ .$$

To see that w_y is bounded we choose $\overline{M}$ so that $|\lambda^2\phi(\xi,t)-\lambda^2(a\xi+b)-\phi_{xx}(\xi,t)| \leq \overline{M}$. Then $|w_y(x,y,t)| \leq \frac{\overline{M}}{\pi}\int_{-\infty}^{\infty} |K(\lambda\nu)|d\xi = \frac{\overline{M}}{\pi}\int_{-\infty}^{\infty} |K(\lambda\sqrt{z^2+y^2})|dz = \frac{2\overline{M}}{\pi}\int_{0}^{\infty} |K(\lambda\sqrt{z^2+y^2})|dz$. If $y \geq \frac{1}{4}$, then $|w_y(x,y,t)| \leq \frac{2\overline{M}}{\pi}\int_0^\infty M\, e^{-\lambda\sqrt{z^2+y^2}}\,dz \leq \frac{2\overline{M}M}{\pi}\int_0^\infty e^{-\lambda z}\,dz$. If $0 \leq y \leq \frac{1}{4}$, then $|w_y(x,y,t)| \leq \frac{2\overline{M}}{\pi}\int_0^{\sqrt{\frac{1}{4}-y^2}} (-M\log\lambda\sqrt{z^2+y^2}\,\lambda z + \frac{2\overline{M}}{\pi}\int_{\sqrt{\frac{1}{4}-y^2}}^{\infty} Me^{-\lambda\sqrt{z^2+y^2}}\,dz \leq -\frac{2\overline{M}M}{\pi}\int_0^{\frac{1}{2}}\log\lambda z dz + \frac{2\overline{M}M}{\pi}\int_0^\infty e^{-\lambda z}\,dz$. Hence w_y is bounded.

Using (1.1.1) and the fact that $\phi_{xx}(x,t)$ is continuous and bounded we can show that w_{xx} is continuous and bounded in the same way we showed w_x was continuous and bounded. Also we obtain

$$(1.1.3)\quad w_{xx}(x,y,t) = \begin{cases} \frac{\lambda y}{\pi}\int_{-\infty}^{\infty} \phi_{xx}(\xi,t)\,\frac{1}{\nu}\,K'(\lambda\nu)d\xi & \text{for } y > 0 \\ -\phi_{xx}(x,t) & \text{for } y = 0\ . \end{cases}$$

We could show that w_{yx} is continuous for $y > 0$ and

$$w_{yx}(x,y,t) = \frac{1}{\pi} \int_{-\infty}^{\infty} [\phi_x(\xi,t)-a][\lambda^2 K(\lambda\nu) - \frac{\partial^2}{\partial\xi^2} K(\lambda\nu)]d\xi$$ for $y > 0$ in the same way we obtained the similar result for w_y. Hence

$$w_{yx}(x,y,t) = \frac{\lambda^2}{\pi} \int_{-\infty}^{\infty} [\phi_x(\xi,t)-a]K(\lambda\nu)d\xi + \frac{1}{\pi} \int_{-\infty}^{\infty} \phi_{xx}(\xi,t) \frac{\xi-x}{\nu} \lambda K'(\lambda\nu)d\xi$$

for $y > 0$. For $y > 0$, $\frac{1}{\pi} \int_{-\infty}^{\infty} \phi_{xx}(\xi,t) \frac{\xi-x}{\nu} \lambda K'(\lambda\nu)d\xi =$

$$\frac{1}{\pi} \int_{-\infty}^{\infty} \phi_{xx}(x+z,t) \frac{\lambda z}{\sqrt{z^2+y^2}} K'(\lambda \sqrt{z^2+y^2})dz = \frac{1}{\pi} \int_{0}^{\infty} [\phi_{xx}(x+z,t)-\phi_{xx}(x-z,t)]$$

$\frac{\lambda z}{\sqrt{z^2+y^2}} K'(\lambda \sqrt{z^2+y^2})dz$. Thus we have

$$\begin{aligned} w_{yx}(x,y,t) &= \frac{\lambda^2}{\pi} \int_{-\infty}^{\infty} [\phi_x(\xi,t)-a]K(\lambda\nu)d\xi \\ &+ \frac{1}{\pi} \int_{0}^{\infty} [\phi_{xx}(x+z,t)-\phi_{xx}(x-z,t)] \frac{\lambda z}{\sqrt{z^2+y^2}} K'(\lambda \sqrt{z^2+y^2})dz . \end{aligned} \tag{1.1.4}$$

So far we have claimed that (1.1.4) is valid for $y > 0$. Now we notice that the integrals in (1.1.4) converge for $y \geq 0$, and we could show that they are continuous for $y \geq 0$. Hence $w_{yx}(x,y,t)$ coincides when $y > 0$ with a function which is continuous for $y \geq 0$. Hence w_{yx} exists for $y \geq 0$, and (1.1.4) is valid for $y \geq 0$.

We can show that the first integral in (1.1.4) is bounded in the same way we showed w_y is bounded. For the second integral in

(1.1.4) we have $\left|\frac{1}{\pi}\int_0^\infty [\phi_{xx}(x+z,t)-\phi_{xx}(x-z,t)]\frac{\lambda z}{\sqrt{z^2+y^2}} K'(\lambda\sqrt{z^2+y^2})dz\right|$

$$\leq \frac{1}{\pi}\int_0^\infty L\ 2^i\ z^i\ \lambda|K'(\lambda\sqrt{z^2+y^2})|dz$$

$$\leq \frac{2^iL}{\pi}\begin{cases}\int_0^{\sqrt{1-y^2}} \frac{Mz^i}{\sqrt{z^2+y^2}}dz + \int_{\sqrt{1-y^2}}^\infty Mz^i\ e^{-\lambda\sqrt{z^2+y^2}}dz \text{ for } 0 \leq y \leq 1 \\ \int_0^\infty Mz^i\ e^{-\lambda\sqrt{z^2+y^2}}dz \text{ for } y \geq 1\end{cases}$$

$$\leq \frac{2^iL}{\pi}\left(\int_0^1 \frac{M}{z^{1-i}}dz + \int_0^\infty Mz^i\ e^{-\lambda z}dz\right).$$ This completes the proof that w_{yx} is bounded and continuous.

Since w_x, w_y, and w_{yx} are continuous, then w_{xy} exists and is continuous and $w_{xy} = w_{yx}$.

We could show that (1.1.2) can be differentiated under the integral sign with respect to y for y > 0. Hence we obtain

$$w_{yy}(x,y,t) = \frac{\lambda y}{\pi}\int_{-\infty}^\infty [\lambda^2\phi(\xi,t)-\lambda^2(a\xi+b)-\phi_{xx}(\xi,t)]\ \frac{1}{\nu}\ K'(\lambda\nu)d\xi \text{ for}$$

$y > 0$.

The function defined by the last integral for $y > 0$ and by $-\lambda^2\phi(x,t)+\lambda^2(ax+b)+\phi_{xx}(x,t)$ for $y = 0$ is continuous and bounded for $y \geq 0$. The proof of this is the same as the proof that w is continuous since $\lambda^2\phi(\xi,t)-\lambda^2(a\xi+b)-\phi_{xx}(\xi,t)$ is continuous and bounded. Hence w_{yy} is continuous and bounded for $y \geq 0$ and

(1.1.5)
$$w_{yy}(x,y,t) = \begin{cases}\frac{\lambda y}{\pi}\int_{-\infty}^\infty [\lambda^2\phi(\xi,t)-\lambda^2(a\xi+b)-\phi_{xx}(\xi,t)]\frac{1}{\nu}K'(\lambda\nu)d\xi \text{ for } y > 0 \\ -\lambda^2\phi(x,t)+\lambda^2(ax+b)+\phi_{xx}(x,t) \text{ for } y = 0.\end{cases}$$

From (1.1.3) and (1.1.5) we easily obtain $\triangle w - \lambda^2 w = 0$. This completes the proof of Lemma (1.1).

Next we will choose several constants which we will be using. Using the properties of K, ψ_1, and ψ_2 we see that there is a real constant M such that $|K(\lambda x)| \leq M|\log x|$ for $0 < x \leq \frac{1}{2}$, $|K(\lambda x)| \leq M\, e^{-\lambda x}$ for $x \geq \frac{1}{2}$, $|\frac{d}{dx} K(\lambda x)| = \lambda |K'(\lambda x)| \leq \frac{M}{x}$ for $x > 0$, $|\frac{d}{dx} K(\lambda x)| = \lambda |K'(\lambda x)| \leq M\, e^{-\lambda x}$ for $x \geq 1$, $|\frac{d^2}{dx^2} K(\lambda x)| = \lambda^2 |K''(\lambda x)| \leq \frac{M}{x^2}$ for $x > 0$, $|\frac{d^2}{dx^2} K(\lambda x)| = \lambda^2 |K''(\lambda x)| \leq M\, e^{-\lambda x}$ for $x \geq 1$, $|\psi_1(x,t)+\lambda^2(ax+b)| \leq M$, and $|\psi_2(x,y)+\lambda^2(ax+b)| \leq M$.

Let W be an upper bound of the absolute values of the first and second derivatives of w with respect to x and y. Let $D_1 = 4\, M^2(1 + \frac{1}{\lambda^2}) + W + a$ and $D_2 = 5\; 2M^2 + \frac{16M^2}{\lambda^2} + 2W$. Let c_1 be any positive number such that $c_1 \leq c$, $a\lambda^2 D_1 c_1 \leq M$, and $2\exp(-2D_2 c_1) > 1$.

We are now ready to construct the functions h_n, u_n, x_n, and y_n.

For each positive integer n let $h_n(x_o,y_o,t_o) = \psi_2(x_o,y_o)$ for $-\infty < x_o < \infty$, $y_o \geq 0$, and $0 \leq t_o \leq \frac{c_1}{n}$.

<u>Lemma (1.2)</u>. h_n is a continuous function of (x_o,y_o,t_o) at almost all points on each plane t_o = constant. Also $|h_n(x_o,y_o,t_o) + \lambda^2(ax_o+b)| \leq 2M$.

The proof of Lemma (1.2) follows immediately from the definition of h_n. Clearly we could omit the word "almost", and we could replace 2M by M. We have stated the lemma as we did so that it remains valid when we get to larger values of t_o which will be shown as we extend the construction to later time intervals.

Let $v_n(x,y,t) = \frac{1}{2\pi} \iint_{\eta \geq 0} g(x,y;\xi,\eta)[h_n(\xi,\eta,t)+\lambda^2(a\xi+b)]\,d\xi\, d\eta$ for $-\infty < x < \infty$, $y \geq 0$, and $0 \leq t \leq \frac{c_1}{n}$, where $g(x,y;\xi,\eta) = K(\lambda\bar{\rho}) - K(\lambda\rho)$ and ρ and $\bar{\rho}$ are defined as $\rho = \sqrt{(\xi-x)^2+(\eta-y)^2}$, $\bar{\rho} = \sqrt{(\xi-x)^2+(\eta+y)^2}$.

Lemma (1.3). v_n, v_{nx}, and v_{ny} are continuous. $|v_{nx}| < 4M^2(1+\frac{1}{\lambda^2})$ and $|v_{ny}| < 4M^2(1+\frac{1}{\lambda^2})$. When $0 < s = \sqrt{(\bar{x}-x)^2+(\bar{y}-y)^2} \leq \frac{1}{4}$ we have

$$|v_{nx}(\bar{x},\bar{y},t)-v_{nx}(x,y,t)| < -(52M^2+\frac{16M^2}{\lambda^2})s \log s \text{ and}$$

$$|v_{ny}(\bar{x},\bar{y},t)-v_{ny}(x,y,t)| < -(52M^2+\frac{16M^2}{\lambda^2})s \log s.$$

These estimates are weaker than a Lipschitz condition and stronger than a Hölder condition and are used later to establish the uniqueness of air particle paths.

Proof of Lemma (1.3). We could show that v_x and v_y exist and are continuous since h is continuous almost everywhere on each horizontal plane, but we omit the proof.

For (x,y,t) in the domain of v_n we have

$$|v_{nx}(x,y,t)| = \left|\frac{1}{2\pi}\iint\limits_{\eta\geq 0} g_x(x,y;\xi,\eta)[h_n(\xi,\eta,t)+\lambda^2(a\xi+b)]d\xi d\eta\right|$$

$$\leq \frac{M}{\pi}\iint\limits_{\eta\geq 0}|g_x(x,y;\xi,\eta)|d\xi d\eta < \frac{M}{\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}|g_x(x,y;\xi,\eta)|d\xi d\eta$$

$$\leq \frac{M}{\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\left\{\left|\frac{\lambda(x-\xi)}{\bar{\rho}}K'(\lambda\bar{\rho})\right| + \left|\frac{\lambda(x-\xi)}{\rho}K'(\lambda\rho)\right|\right\} d\xi d\eta$$

$$= \frac{2M}{\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\left|\frac{\lambda(x-\xi)}{\rho}K'(\lambda\rho)\right|d\xi d\eta \leq \frac{2M}{\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}|\lambda K'(\lambda\rho)|d\xi d\eta$$

$$\leq \frac{2M}{\pi}\iint\limits_{\rho\leq 1}\frac{M}{\rho}d\xi d\eta + \frac{2M}{\pi}\iint\limits_{\rho\geq 1}M e^{-\lambda\rho}d\xi d\eta$$

$$\leq \frac{2M^2}{\pi}\int_0^{2\pi}\int_0^{1}d\rho d\theta + \frac{2M^2}{\pi}\int_0^{2\pi}\int_0^{\infty}\rho e^{-\lambda\rho}d\rho d\theta$$

$$= 4M^2 + 4M^2\left(-\frac{\rho}{\lambda}-\frac{1}{\lambda^2}\right)e^{-\lambda\rho}\Big|_0^{\infty} = 4M^2(1+\frac{1}{\lambda^2}).$$

Similarly $|v_{ny}(x,y,t)| < 4M^2(1+\frac{1}{\lambda^2})$.

Let (x,y,t) and $(\bar{x},\bar{y},t)$ be in the domain of v_n. Let $s = \sqrt{(\bar{x}-x)^2+(\bar{y}-y)^2}$ and $\rho_1 = \sqrt{(\xi-\bar{x})^2+(\eta-\bar{y})^2}$. For $0 < s \le \frac{1}{4}$ we have $|v_{nx}(\bar{x},\bar{y},t)-v_{nx}(x,y,t)|$

$$= \left|\frac{1}{2\pi}\iint_{\eta\ge 0}[g_x(\bar{x},\bar{y};\xi,\eta)-g_x(x,y;\xi,\eta)]\ [h_n(\xi,\eta,t)+\lambda^2(a\xi+b)]d\xi d\eta\right|$$

$$< \frac{M}{\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} |g_x(\bar{x},\bar{y};\xi,\eta)-g_x(x,y;\xi,\eta)|d\xi d\eta$$

$$\le \frac{2M}{\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \left|\frac{\bar{x}-\xi}{\rho_1}\lambda K'(\lambda\rho_1) - \frac{x-\xi}{\rho}\lambda K'(\lambda\rho)\right|d\xi d\eta$$

$$\le \frac{2M}{\pi}\iint_{\rho\le 2s} \lambda|K'(\lambda\rho_1)|d\xi d\eta + \frac{2M}{\pi}\iint_{\rho\le 2s} \lambda|K'(\lambda\rho)|d\xi d\eta$$

$$+ \frac{2M}{\pi}\iint_{\rho\ge 2s}\left\{\left|\frac{\bar{x}-\xi}{\rho_1}-\frac{x-\xi}{\rho_1}+\frac{x-\xi}{\rho_1}-\frac{x-\xi}{\rho}\right|\ |\lambda K'(\lambda\rho_1)|+\lambda\left|\frac{x-\xi}{\rho}[K'(\lambda\rho_1)-K'(\lambda\rho)]\right|\right\} d\xi d\eta$$

$$\le \frac{2M}{\pi}\iint_{\rho_1\le 3s} \frac{M}{\rho_1}\,d\xi d\eta + \frac{2M}{\pi}\iint_{\rho\le 2s}\frac{M}{\rho}\,d\xi d\eta$$

$$+ \frac{2M}{\pi}\iint_{2s\le\rho\le 1+s} \left[\left(\frac{|\bar{x}-x|}{\rho_1}+|x-\xi|\frac{|\rho-\rho_1|}{\rho_1\rho}\right)\frac{M}{\rho_1}+\lambda^2|\rho_1-\rho|\,|K''(\lambda\rho^*)|\right]d\xi d\eta$$

$$+ \frac{2M}{\pi}\iint_{\rho\ge 1+s} \left[\left(\frac{|\bar{x}-x|}{\rho_1}+|x-\xi|\frac{|\rho-\rho_1|}{\rho_1\rho}\right)M\,e^{-\lambda\rho 1}+\lambda^2|\rho_1-\rho|\,|K''(\lambda\rho^{\#})|\right]d\xi d\eta$$

(where ρ^* and $\rho^{\#}$ are between ρ and ρ_1)

$$\le 12M^2s + 8M^2s + \frac{2M}{\pi}\iint_{2s\le\rho\le 1+s}\left[\frac{2Ms}{\rho_1^2}+\frac{Ms}{(\rho^*)^2}\right]d\xi d\eta$$

$$+ \frac{2M}{\pi}\iint_{\rho\ge 1+s}\left(\frac{2Ms}{\rho_1}e^{-\lambda\rho_1} + Mse^{-\lambda\rho^{\#}}\right)d\xi d\eta$$

$$\le 20M^2s + \frac{2M^2s}{\pi}\iint_{2s\le\rho\le 1+s}\left(\frac{3}{\rho_1^2}+\frac{1}{\rho^2}\right)d\xi d\eta + \frac{2M^2s}{\pi}\iint_{\rho\ge 1+s}(3e^{-\lambda\rho_1}+e^{-\lambda\rho})d\xi d\eta$$

$$\leq 20M^2s + \frac{6M^2s}{\pi}\int_0^{2\pi}\int_s^{3/2} \frac{1}{\rho_1}\, d\rho_1 d\theta + \frac{2M^2s}{\pi}\int_0^{2\pi}\int_{2s}^{3/2} \frac{1}{\rho}\, d\rho d\theta$$

$$+ \frac{6M^2s}{\pi}\int_0^{2\pi}\int_0^{\infty} \rho_1\, e^{-\lambda\rho_1}\, d\rho_1 d\theta + \frac{2M^2s}{\pi}\int_0^{2\pi}\int_0^{\infty} \rho\, e^{-\lambda\rho}\, d\rho d\theta$$

$$\leq 20M^2s + 12M^2s \log \rho_1 \Big|_s^{3/2} + 4M^2s \log \rho \Big|_{2s}^{3/2}$$

$$+ 12M^2s(-\frac{\rho_1}{\lambda} - \frac{1}{\lambda^2})e^{-\lambda\rho_1}\Big|_0^{\infty} + 4M^2s(-\frac{\rho}{\lambda} - \frac{1}{\lambda^2})e^{-\lambda\rho}\Big|_0^{\infty}$$

$$\leq 20M^2s + 16M^2s \log 3/2 - 16M^2s \log s - 4M^2s \log 2 + \frac{16M^2s}{\lambda^2}$$

$$\leq (20 + \frac{16}{\lambda^2})M^2s + 16M^2s - 16M^2s \log s = (36 + \frac{16}{\lambda^2})M^2s - 16M^2s \log s$$

$$\leq -(52 + \frac{16}{\lambda^2})M^2s \log s.$$

Similarly $|v_{ny}(\bar{x},\bar{y},t)-v_{ny}(x,y,t)| < -(52 + \frac{16}{\lambda^2})M^2s \log s$ for $0 < s \leq \frac{1}{4}$.

Now let $u_n(x,y,t) = v_n(x,y,t)-w(x,y,t)+ax+b$ for $-\infty < x < \infty$, $y \geq 0$, and $0 \leq t \leq \frac{c_1}{n}$.

Lemma (1.4). u_n, u_{nx}, and u_{ny} are continuous. $|u_{nx}| < D_1$ and $|u_{ny}| < D_1$. When $0 < s = \sqrt{(\bar{x}-x)^2+(\bar{y}-y)^2} \leq \frac{1}{4}$ we have $|u_{nx}(\bar{x},\bar{y},t)-u_{nx}(x,y,t)| < -D_2 s \log s$ and $|u_{ny}(\bar{x},\bar{y},t)-u_{ny}(x,y,t)| < -D_2 s \log s$.

The proof of Lemma (1.4) is obvious using Lemmas (1.2) and (1.3).

To make it easier to discuss the behavior of the air particle paths of u_n at the boundary $y = 0$ we would like to extend the air

particle paths of u_n into the region where $y < 0$. To do this we introduce new functions F_{n1} and F_{n2}. Let $F_{n1}(x,y,t) = -u_{ny}(x,y,t)$ and $F_{n2}(x,y,t) = u_{nx}(x,y,t)$ for $-\infty < x < \infty$, $y \geq 0$, and $0 \leq t \leq \frac{c_1}{n}$. Let $F_{n1}(x,y,t) = -u_{ny}(x,-y,t)$ and $F_{n2}(x,y,t) = u_{nx}(x,-y,t)$ for $-\infty < x < \infty$, $y \leq 0$, and $0 \leq t \leq \frac{c_1}{n}$. That is, F_{n1} and F_{n2} are the even extensions of $-u_{ny}$ and u_{nx} respectively across the (x,t) plane.

Lemma (1.5). F_{n1} and F_{n2} are continuous. $|F_{ni}| < D_1$ for $i=1,2$. When $0 < s = \sqrt{(\bar{x}-x)^2+(\bar{y}-y)^2} \leq \frac{1}{4}$ we have $|F_{ni}(\bar{x},\bar{y},t)-F_{ni}(x,y,t)| < -D_2 s \log s$ for $i=1,2$.

The proof of Lemma (1.5) follows trivially from Lemma (1.4).

Lemma (1.6). Let (x_o,y_o,t_o) be any point in the domain of F_{n1} and F_{n2}. Then there exist unique functions $x_n(t)$ and $y_n(t)$ defined for $0 \leq t \leq \frac{c_1}{n}$ such that $x_n(t_o) = x_o$, $y_n(t_o) = y_o$, and $\frac{dx_n(t)}{dt} = F_{n1}[x_n(t), y_n(t), t]$ and $\frac{dy_n(t)}{dt} = F_{n2}[x_n(t), y_n(t), t]$ for $0 \leq t \leq \frac{c_1}{n}$. Since $x_n(t)$ and $y_n(t)$ also depend on (x_o,y_o,t_o), we also use the notation $x_n(x_o,y_o,t_o,t)$ for $x_n(t)$ and $y_n(x_o,y_o,t_o,t)$ for $y_n(t)$.

Proof of Lemma (1.6). The existence of x_n and y_n follows since $F_{ni}(i=1,2)$ is continuous and bounded [2]. The uniqueness of x_n and y_n follows since $|F_{ni}(\bar{x},\bar{y},t)-F_{ni}(x,y,t)| < -D_2 s \log s$ $(i=1,2)$ for $0 < s = \sqrt{(\bar{x}-x)^2+(\bar{y}-y)^2} \leq \frac{1}{4}$ [3].

Lemma (1.7). Let $(\bar{x}_o,\bar{y}_o,\bar{t}_o,t)$ and (x_o,y_o,t_o,t) be any points in the domain of x_n and y_n. Let $s = \sqrt{(\bar{x}_o-x_o)^2+(\bar{y}_o-y_o)^2+(\bar{t}_o-t_o)^2}$ and

let $S_n(t) =$

$$\sqrt{[x_n(\overline{x}_o,\overline{y}_o,\overline{t}_o,t)-x_n(x_o,y_o,t_o,t)]^2+[y_n(\overline{x}_o,\overline{y}_o,\overline{t}_o,t)-y_n(x_o,y_o,t_o,t)]^2}\ .$$

Then $S_n(t) \leq [2(D_1+1)s]^{\exp(-2D_2c_1)}$ when $s<s_o = \frac{1}{2(D_1+1)}(\frac{1}{4})^{\exp(2D_2c)}$.

<u>Proof of Lemma (1.7)</u>. Let $z(t) = [2(D_1+1)s]^{\exp[2D_2(t_o-t)]}$ for $t_o \leq t \leq \frac{c_1}{n}$ and $0 < s < s_o$. Then $z(t_o) = 2(D_1+1)s$,

$z'(t) = -2D_2z(t) \log z(t)$, and $z(t) \leq [2(D_1+1)s]^{\exp(-2D_2c_1)}$ since

$2(D_1+1)s < (\frac{1}{4})^{\exp(2D_2c)} < 1$ and $\exp[2D_2(t_o-t)] > \exp(-2D_2c_1)$.

We will show that $S_n(t) < z(t)$ thus establishing the lemma for $0 < s < s_o$ and $t_o \leq t \leq \frac{c_1}{n}$.

For $s < s_o$ we have $[2(D_1+1)s]^{\exp(-2D_2c_1)} <$ $[(\frac{1}{4})^{\exp(2D_2c)}]^{\exp(-2D_2c_1)} \leq \frac{1}{4}$. Hence $z(t) < \frac{1}{4}$ for $s < s_o$.

For $0 < s < s_o$ we have $|x_n(\overline{x}_o,\overline{y}_o,\overline{t}_o,t_o)-x_n(x_o,y_o,t_o,t_o)| =$ $|\overline{x}_o + \int_{\overline{t}_o}^{t_o} F_{n1}[x_n(\overline{x}_o,\overline{y}_o,\overline{t}_o,\xi),\ y_n(\overline{x}_o,\overline{y}_o,\overline{t}_o,\xi),\xi]d\xi - x_o| <$ $|\overline{x}_o-x_o| + D_1|\overline{t}_o-t_o|$(note $|F_{n1}| < D_1$) $\leq (D_1+1)s$. Similarly $|y_n(\overline{x}_o,\overline{y}_o,\overline{t}_o,t_o)-y_n(x_o,y_o,t_o,t_o)| < (D_1+1)s$, and hence $S_n(t_o) < 2(D_1+1)s = z(t_o)$ for $0 < s < s_o$.

Suppose $S_n(t^*) \geq z(t^*)$ for some s and t^* such that $0 < s < s_o$ and $t^* > t_o$. Since $S_n(t_o) < z(t_o)$, $S_n(t^*) \geq z(t^*)$, and S_n and z are continuous in t, then there is a t_1 such that $t_1 > t_o$, $S_n(t) < z(t)$ for $t_o \leq t < t_1$, and $S_n(t_1) = z(t_1)$. For $t_o \leq t \leq t_1$, we have $S_n(t) \leq z(t) < \frac{1}{4}$ and hence $|x_n(\overline{x}_o,\overline{y}_o,\overline{t}_o,t)-x_n(x_o,y_o,t_o,t)|$

$$= |x_n(\overline{x}_o,\overline{y}_o,\overline{t}_o,t_o)+\int_{t_o}^{t} F_{n1}[x_n(\overline{x}_o,\overline{y}_o,\overline{t}_o,\xi),y_n(\overline{x}_o,\overline{y}_o,\overline{t}_o,\xi),\xi]d\xi$$

$$- x_n(x_o,y_o,t_o,t_o) - \int_{t_o}^{t} F_{n1}[x_n(x_o,y_o,t_o,\xi), y_n(x_o,y_o,t_o,\xi),\xi]d\xi |$$

$< (D_1+1)s - D_2 \int_{t_o}^{t} S_n(\xi) \log S_n(\xi)d\xi$. Similarly

$$|y_n(\bar{x}_o,\bar{y}_o,\bar{t}_o,t)-y_n(x_o,y_o,t_o,t)| < (D_1+1)s-D_2\int_{t_o}^{t} S_n(\xi) \log S_n(\xi)d\xi$$

and hence $S_n(t) < 2(D_1+1)s - 2D_2 \int_{t_o}^{t} S_n(\xi) \log S_n(\xi)d\xi$ for $t_o \leq t \leq t_1$.

For $t_o \leq \xi < t_1$ we have $S_n(\xi) < z(\xi) < \frac{1}{4}$, $-S_n(\xi) \log S_n(\xi)$ $< -z(\xi) \log z(\xi)$, and $S_n(t_1) < 2(D_1+1)s-2D_2\int_{t_o}^{t_1} S_n(\xi) \log S_n(\xi)d\xi$

$$< 2(D_1+1)s-2D_2 \int_{t_o}^{t_1} z(\xi) \log z(\xi)d\xi = 2(D_1+1)s + \int_{t_o}^{t_1} z'(\xi)d\xi = z(t_1).$$

Since this contradicts $S_n(t_1) = z(t_1)$, we have $S_n(t) < z(t)$ $\leq [2(D_1+1)s]^{\exp(-2D_2c_1)}$ for $t \geq t_o$ and $0 < s < s_o$.

Similarly the lemma can be proved when $t \leq t_o$ and $0 < s < s_o$.

Lemma (1.8). x_n and y_n are uniformly continuous functions of (x_o,y_o,t_o,t) in their domain.

The proof of Lemma (1.8) follows easily from Lemma (1.7) and the fact that $|x_{nt}| < D_1$ and $|y_{nt}| < D_1$.

Let (x_o,y_o,t_o) be in the domain of u_n. We wish to define functions α_n, β_n, γ_n so that $[\alpha_n(x_o,y_o,t_o),\beta_n(x_o,y_o,t_o),$ $\gamma_n(x_o,y_o,t_o)]$ is the most recent point where the air particle path of u_n through (x_o,y_o,t_o) "enters" the domain of u_n (either β_n is zero or γ_n is zero depending on whether the particle path hits the (x,t) plane of the (x,y) plane).

For $-\infty < x_o < \infty$, $y_o > 0$, and $0 < t_o \leq \frac{c_1}{n}$ let γ_{no} be the largest number such that $\gamma_{no} \leq t_o$ and $y_n(x_o,y_o,t_o,\gamma_{no}) = 0$. If no such γ_{no} exists, let $\gamma_{no} = 0$.

When $-\infty < x_o < \infty$, $y_o = 0$, and $0 < t_o \leq \frac{c_1}{n}$ let $\gamma_{no} = t_o$ if $\phi_x(x_o,t_o) \geq 0$. If $\phi_x(x_o,t_o) < 0$, let γ_{no} be the largest number such that $\gamma_{no} < t_o$ and $y_n(x_o,y_o,t_o,\gamma_{no}) = 0$. If no such γ_{no} exists let $\gamma_{no} = 0$.

When $-\infty < x_o < \infty$, $y_o \geq 0$, and $t_o = 0$, let $\gamma_{no} = 0$.

We have now associated a number γ_{no} with each point (x_o,y_o,t_o) such that $-\infty < x_o < \infty$, $y_o \geq 0$, and $0 \leq t_o \leq \frac{c_1}{n}$. Let α_n, β_n, and γ_n be the functions defined by $\alpha_n(x_o,y_o,t_o) = x_n(x_o,y_o,t_o,\gamma_{no})$, $\beta_n(x_o,y_o,t_o) = y_n(x_o,y_o,t_o,\gamma_{no})$, and $\gamma_n(x_o,y_o,t_o) = \gamma_{no}$ for $-\infty < x_o < \infty$, $y_o \geq 0$, and $0 \leq t_o \leq \frac{c_1}{n}$. Then $(\alpha_n,\beta_n,\gamma_n)$ is a point where the curve, generated by $[x_n(x_o,y_o,t_o,t),y_n(x_o,y_o,t_o,t),t]$ enters the domain of u_n as t increases (except possibly when $\beta_n = 0$ and $\phi_x(\alpha_n,\gamma_n) = 0$).

From here on we let $\alpha_{no} = \alpha(x_o,y_o,t_o)$, $\overline{\alpha}_{no} = \alpha_n(\overline{x}_o,\overline{y}_o,\overline{t}_o)$, $\beta_{no} = \beta_n(x_o,y_o,t_o)$, $\overline{\beta}_{no} = \beta_n(\overline{x}_o,\overline{y}_o,\overline{t}_o)$, $\gamma_{no} = \gamma_n(x_o,y_o,t_o)$, and $\overline{\gamma}_{no} = \gamma_n(\overline{x}_o,\overline{y}_o,\overline{t}_o)$ where (x_o,y_o,t_o) and $(\overline{x}_o,\overline{y}_o,\overline{t}_o)$ are any points in the domain of u_n.

<u>Lemma (1.9)</u>. α_n,β_n, and γ_n are continuous at points (x_o,y_o,t_o) for which $\gamma_{no} = 0$ or $\gamma_{no} > 0$ with $(\alpha_{no},0,\gamma_{no})$ not on C_1 or C_2.

It is clear from the definitions of α_n, β_n, and γ_n that α_n and β_n are continuous at those points where γ_n is continuous. It is easy to show that the statement of Lemma (1.9) about γ_n is true using the uniform continuity x_n and y_n and the definition of γ_n.

We now wish to extend h_n in the t direction so that its domain is $\left\{(x_o,y_o,t_o) \mid -\infty < x_o < \infty,\ y_o \geq 0,\ 0 \leq t_o \leq \frac{2c_1}{n}\right\}$. Let (x_o,y_o,t_o) be a point in the new domain such that $\frac{c_1}{n} < t_o \leq \frac{2c_1}{n}$. Go straight down to the point $(x_o,y_o,t_o - \frac{c_1}{n})$ which is in the region where u_n and its air particle paths are defined. Follow the air particle path of u_n from $(x_o,y_o,t_o - \frac{c_1}{n})$ down to the nearest boundary point of the domain of u_n. We define h_n at (x_o,y_o,t_o) to be the value of ψ_1 or ψ_2 at this boundary point.

More precisely, when $-\infty < x_o < \infty$, $y_o \geq 0$, and $\frac{c_1}{n} < t_o \leq \frac{2c_1}{n}$, we extend the definition of h_n by letting $h_n(x_o,y_o,t_o) = \psi_1[\alpha_n(x_o,y_o,t_o - \frac{c_1}{n}),\gamma_n(x_o,y_o,t_o - \frac{c_1}{n})]$ if $\gamma_n(x_o,y_o,t_o - \frac{c_1}{n}) > 0$, and by letting $h_n(x_o,y_o,t_o) = \psi_2[\alpha_n(x_o,y_o,t_o-\frac{c_1}{n}),\beta_n(x_o,y_o,t_o-\frac{c_1}{n})]$ if $\gamma_n(x_o,y_o,t_o-\frac{c_1}{n}) = 0$.

We will now show that Lemma (1.2) remains valid for the extended h_n.

<u>Proof of Lemma (1.2)</u>. Since we have already observed that the lemma is true for planes $t = c^*$ where $0 \leq c^* \leq \frac{c_1}{n}$, we will prove the lemma for planes $t = c^*$ where $\frac{c_1}{n} \leq c^* \leq \frac{2c_1}{n}$. A similar argument can then be used to extend the proof to planes $t = c^*$ for larger c^* as the definition of h_n is extended further.

Consider a fixed plane $t = c^*$ where $\frac{c_1}{n} \leq c^* \leq \frac{2c_1}{n}$. Let $x_{ni}(\tau) = x_n[x_i(\tau),0,\tau,c^* - \frac{c_1}{n}]$ and $y_{ni}(\tau) = y_n[x_i(\tau),0,\tau,c^* - \frac{c_1}{n}]$ for i=1,2 (see glossary for $x_1(\tau)$ and $x_2(\tau)$). Let $R_i = \left\{(x_o,y_o,c^*) \mid x_o = x_{ni}(\tau) \text{ and } y_o = y_{ni}(\tau) \text{ for some } \tau \text{ such that } 0 \leq \tau \leq \frac{c_1}{n}\right\}$ for i=1,2. Then clearly the set of points on the plane $t = c^*$, at which h_n is discontinuous in (x_o,y_o,t_o), is a subset of $R_1 \cup R_2$. We will show that R_1 and R_2 have measure zero.

Choose ℓ so that $|x_1(\bar{\tau})-x_1(\tau)| \leq \ell|\bar{\tau}-\tau|$ for $0 \leq \bar{\tau}, \tau \leq c_1$. Then for $|\bar{\tau}-\tau| < \frac{s_o}{\ell+1}$ [1] and $0 \leq \tau, \bar{\tau} \leq \frac{c_1}{n}$ we have

$|x_{n1}(\bar{\tau})-x_{n1}(\tau)| = |x_n[x_1(\bar{\tau}),0,\bar{\tau},c^*-\frac{c_1}{n}]-x_n[x_1(\tau),0,\tau,c^*-\frac{c_1}{n}]|$

$\leq \left\{2(D_1+1)[|x_1(\bar{\tau})-x_1(\tau)|+|\bar{\tau}-\tau|]\right\}^{\exp(-2D_2c_1)}$ (see Lemma 1.7)

$\leq [2(D_1+1)(\ell+1)|\bar{\tau}-\tau|]^{\exp(-2D_2c_1)}$. Let

$H = [2(D_1+1)(\ell+1)]^{\exp(-2D_2c_1)}$. Then $|x_{n1}(\bar{\tau})-x_{n1}(\tau)|$

$\leq H|\bar{\tau}-\tau|^{\exp(-2D_2c_1)}$ for $0 \leq \bar{\tau}, \tau \leq \frac{c_1}{n}$ and $|\bar{\tau}-\tau| \leq \frac{s_o}{\ell+1}$.

Let k be a positive integer and choose k_o so that $k > k_o$ implies $\frac{c_1}{kn} < \frac{s_o}{\ell+1}$. Fix $k > k_o$. Let $\tau_i = \frac{ic_1}{kn}$ for $i=0,1,2,\ldots,k$. Let S_μ be the set of points (x_o,y_o,c^*) within and on the circle in the plane $t = c^*$ with center at $[x_{n1}(\tau_\mu),y_{n1}(\tau_\mu),c^*]$ and radius $2H(\frac{c_1}{kn})^{\exp(-2D_2c_1)}$ for $\mu=1,2,\ldots,k-1$. For $\tau_{\mu-1} \leq \tau \leq \tau_{\mu+1}$ we have

$|\tau-\tau_\mu| \leq \frac{c_1}{kn} < \frac{s_o}{\ell+1}$ and hence $\sqrt{[x_{n1}(\tau)-x_{n1}(\tau_\mu)]^2+[y_{n1}(\tau)-y_{n1}(\tau_\mu)]^2}$

$\leq |x_{n1}(\tau)-x_{n1}(\tau_\mu)| + |y_{n1}(\tau)-y_{n1}(\tau_\mu)| \leq 2H(\frac{c_1}{kn})^{\exp(-2D_2c_1)}$ so that $[x_{n1}(\tau),y_{n1}(\tau),c^*]$ is in S_μ for $\tau_{\mu-1} \leq \tau \leq \tau_{\mu+1}$. Clearly

$R_1 \subset S_1 \cup S_2 \cup \ldots \cup S_{k-1}$ and $m(S_\mu) = 4\pi H^2(\frac{c_1}{kn})^{2\exp(-2D_2c_1)}$ for $\mu=1,2,\ldots,k-1$ where $m(S_\mu)$ is the plane Lebesgue measure of S_μ.

Hence $\bar{m}(R_1) \leq \sum_{\mu=1}^{k-1} m(S_\mu) < 4\pi H^2(\frac{c_1}{n})^{2\exp(-2D_2c_1)} k^{1-2\exp(-2D_2c_1)}$ for

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(1) See glossary for s_o.

each positive integer $k > k_o$ where $\overline{m}(R_1)$ is the plane exterior measure of R_1. Since $1-2\exp(-2D_2c_1) < 0$ by the choice of c_1, then $k^{1-2\exp(-2D_2c_1)} \longrightarrow 0$ as $k \to \infty$. Therefore $\overline{m}(R_1) = 0$ and R_1 has measure zero. Similarly R_2 has measure zero. This completes the proof that h_n is a continuous in (x_o, y_o, t_o) at almost all points on each plane t_o = constant.

We have yet to show that $|h_n(\xi,\eta,t)+\lambda^2(a\xi+b)| \leq 2M$ for $\frac{c_1}{n} \leq t \leq \frac{2c_1}{n}$. If $\gamma_n(\xi,\eta,t-\frac{c_1}{n}) = 0$, then $|h_n(\xi,\eta,t)+\lambda^2(a\xi+b)|$

$$\leq |\psi_2[\alpha_n(\xi,\eta,t-\tfrac{c_1}{n}),\beta_n(\xi,\eta,t-\tfrac{c_1}{n})] + \lambda^2[a\alpha_n(\xi,\eta,t-\tfrac{c_1}{n})+b]|$$

$$+ a\lambda^2|\xi - \alpha_n(\xi,\eta,t-\tfrac{c_1}{n})|$$

$$\leq M + a\lambda^2|x_n(\xi,\eta,t-\tfrac{c_1}{n},t-\tfrac{c_1}{n}) - x_n[\xi,\eta,t-\tfrac{c_1}{n},\gamma_n(\xi,\eta,t-\tfrac{c_1}{n})]|$$

$\leq M + a\lambda^2D_1|t-\frac{c_1}{n} - \gamma_n(\xi,\eta,t-\frac{c_1}{n})| \leq M + a\lambda^2D_1c_1 \leq 2M$ where we have used the fact that $|\psi_2(x,y) + \lambda^2(ax+b)| \leq M$ and $a\lambda^2D_1c_1 \leq M$.

Similarly we obtain $|h_n(\xi,\eta,t)+\lambda^2(a\xi+b)| \leq 2M$ when $\gamma_n(\xi,\eta,t-\frac{c_1}{n}) > 0$.

From Lemma (1.2) we see that $\frac{1}{2\pi}\iint_{\eta\geq 0} g(x,y;\xi,\eta)[h_n(\xi,\eta,t) +\lambda^2(a\xi+b)]d\xi d\eta$ exists for $-\infty < x < \infty$, $y \geq 0$, and $0 \leq t \leq \frac{2c_1}{n}$. We extend the definition of v_n by letting

$$v_n(x,y,t) = \frac{1}{2\pi}\iint_{\eta\geq 0} g(x,y;\xi,\eta)[h_n(\xi,\eta,t) + \lambda^2(a\xi+b)]d\xi d\eta$$

for $-\infty < x < \infty$, $y \geq 0$, and $0 \leq t \leq \frac{2c_1}{n}$.

Lemma (1.3) remains valid for the extended v_n.

Next we extend the definition of u_n by letting $u_n(x,y,t) = v_n(x,y,t)-w(x,y,t)+ax+b$ for $-\infty < x < \infty$, $y \geq 0$, and $0 \leq t \leq \frac{2c_1}{n}$.

Lemma (1.4) remains valid for the extended u_n.

Next we extend the definition of F_{n1} and F_{n2} by replacing $\frac{c_1}{n}$ by $\frac{2c_1}{n}$ in the previous definition of F_{n1} and F_{n2}. Then Lemma (1.5) remains valid for the extended F_{n1} and F_{n2}.

Now replace $\frac{c_1}{n}$ by $\frac{2c_1}{n}$ in Lemma (1.6). The lemma remains valid and it extends the domain of x_n and y_n to $\left\{(x_o,y_o,t_o,t) \mid -\infty < x_o < \infty,\ -\infty < y_o < \infty,\ 0 \leq t_o \leq \frac{2c_1}{n},\ 0 \leq t \leq \frac{2c_1}{n}\right\}$. Lemmas (1.7) and (1.8) remain valid for the extended x_n and y_n.

Next we extend the definitions of α_n, β_n, and γ_n by replacing $\frac{c_1}{n}$ by $\frac{2c_1}{n}$ in their previous definition. Then Lemma (1.9) remains valid for the extended α_n, β_n, and γ_n.

We can thus extend the functions h_n, v_n, u_n, F_{n1}, F_{n2}, x_n, y_n, α_n, β_n, and γ_n stepwise in time until $0 \leq t_o \leq c_1$ and $0 \leq t \leq c_1$. That is, to define h_n at a point P in a new time strip we go back a distance $\frac{c_1}{n}$ in time to a point P_o. We define h_n at P to be $\psi_2(\alpha_n,\beta_n)$ at P_o if $\gamma_n = 0$ at P_o, and we define h_n at P to be $\psi_1(\alpha_n,\gamma_n)$ at P_o if $\beta_n = 0$ at P_o. We then define the remaining functions at P as previously. Lemmas (1.2) through (1.9) remain valid for these extended functions.

We will show that a subsequence of $\{u_n\}$ converges to a weak solution of (1) in $\mathcal{D}_1$ which has the properties mentioned in the theorem.

<u>Lemma (1.10)</u>. There is a subsequence, $\{n_k\}$, of the positive integers such that $\{x_{n_k}(x_o,y_o,t_o,t)\}$ and $\{y_{n_k}(x_o,y_o,t_o,t)\}$ converge for all (x_o,y_o,t_o,t) in the domain of x_{n_k} and y_{n_k} and such that the convergence is uniform in every bounded subset.

Proof of Lemma (1.10). Since $|x_n(x_o,y_o,t_o,t)-x_o|$

$= |\int_{t_o}^{t} F_{n1}[x_n(x_o,y_o,t_o,\xi),y_n(x_o,y_o,t_o,\xi),\xi]d\xi| < D_1|t-t_o| \leq D_1c_1,$

then the sequence $\{x_n(x_o,y_o,t_o,t)-x_o\}$ is bounded uniformly with respect to (x_o,y_o,t_o,t) and n.

For any $(\bar{x}_o,\bar{y}_o,\bar{t}_o,\bar{t})$ and (x_o,y_o,t_o,t) in the domain of x_n let $s=\sqrt{(\bar{x}_o-x_o)^2+(\bar{y}_o-y_o)^2+(\bar{t}_o-t_o)^2}$. For $s < s_o$ (see glossary) we have $|x_n(\bar{x}_o,\bar{y}_o,\bar{t}_o,\bar{t})-\bar{x}_o-x_n(x_o,y_o,t_o,t)+x_o|$

$\leq |\bar{x}_o-x_o|+|x_n(\bar{x}_o,\bar{y}_o,\bar{t}_o,\bar{t})-x_n(x_o,y_o,t_o,\bar{t})|+|x_n(x_o,y_o,t_o,\bar{t})$

$- x_n(x_o,y_o,t_o,t)| \leq s+[2(D_1+1)s]^{\exp(-2D_2c_1)} + D_1|\bar{t}-t|$. It follows that the sequence $\{x_n(x_o,y_o,t_o,t)-x_o\}$ is uniformly equicontinuous in (x_o,y_o,t_o,t).

Similarly the sequence $\{y_n(x_o,y_o,t_o,t)-y_o\}$ is uniformly bounded and uniformly equicontinuous.

It follows from well known arguments that a subsequence, $\{n_k\}$, of the positive integers exists having the properties listed in Lemma (1.10).

Let $x(x_o,y_o,t_o,t) = \lim_{k\to\infty} x_{n_k}(x_o,y_o,t_o,t)$ and $y(x_o,y_o,t_o,t)$ $= \lim_{k\to\infty} y_{n_k}(x_o,y_o,t_o,t)$ for $-\infty < x_o < \infty$, $-\infty < y_o < \infty$, $0 \leq t_o \leq c_1$, and $0 \leq t \leq c_1$.

Lemma (1.11). Let $(\bar{x}_o,\bar{y}_o,\bar{t}_o,t)$ and (x_o,y_o,t_o,t) be any points in the domain of x and y. Let

$S(t)=\sqrt{[x(\bar{x}_o,\bar{y}_o,\bar{t}_o,t)-x(x_o,y_o,t_o,t)]^2+[y(\bar{x}_o,\bar{y}_o,\bar{t}_o,t)-y(x_o,y_o,t_o,t)]^2}$

and let $s=\sqrt{(\bar{x}_o-x_o)^2+(\bar{y}_o-y_o)^2+(\bar{t}_o-t_o)^2}$. Then

$S(t) \leq [2(D_1+1)s]^{\exp(-2D_2c_1)}$ when $s < s_o = \frac{1}{2(D_1+1)} (\frac{1}{4})^{\exp(2D_2c)}$.

Also $|x(x_o,y_o,t_o,\bar{t})-x(x_o,y_o,t_o,t)| \leq D_1|\bar{t}-t|$ and $|y(x_o,y_o,t_o,\bar{t})-y(x_o,y_o,t_o,t)| \leq D_1|\bar{t}-t|$ for $0 \leq \bar{t}$, $t \leq c_1$.

Lemma (1.11) follows easily from Lemma (1.7) and the fact that $(F_{ni}) < D_i$ for $i=1,2$.

Lemma (1.12). $x(x_o,y_o,t_o,t)$ and $y(x_o,y_o,t_o,t)$ are uniformly continuous functions of (x_o,y_o,t_o,t) in their domain

Lemma (1.12) follows easily from Lemma (1.11).

For (x_o,y_o,t_o) in $\mathcal{D}_1$ with $y_o > 0$ and $t_o > 0$ let γ_o be the largest number such that $\gamma_o \leq t_o$ and $y(x_o,y_o,t_o,\gamma_o) = 0$. If no such γ_o exists, let $\gamma_o = 0$.

For $(x_o,0,t_o)$ in $\mathcal{D}_1$ with $t_o > 0$ let $\gamma_o = t_o$ if $\phi_x(x_o,t_o) \geq 0$. If $\phi_x(x_o,t_o) < 0$, let γ_o be the largest number such that $\gamma_o < t_o$ and $y(x_o,y_o,t_o,\gamma_o) = 0$. If no such γ_o exists, let $\gamma_o = 0$.

For $(x_o,y_o,0)$ in $\mathcal{D}_1$ let $\gamma_o = 0$.

We have associated a number γ_o with each (x_o,y_o,t_o) in $\mathcal{D}_1$. We define functions α, β, and γ with domain $\mathcal{D}_1$ by $\alpha(x_o,y_o,t_o) = x(x_o,y_o,t_o,\gamma_o)$, $\beta(x_o,y_o,t_o) = y(x_o,y_o,t_o,\gamma_o)$, and $\gamma(x_o,y_o,t_o) = \gamma_o$. Then (α,β,γ) is the most recent point before time $t = t_o$ where the curve $[x(x_o,y_o,t_o,t),y(x_o,y_o,t_o,t),t]$ enters $\mathcal{D}_1$ as t increases except possibly when $\beta = 0$ and $(\alpha,0,\gamma)$ is on C_1 or C_2.

In the following we will let $\alpha_o = \alpha(x_o,y_o,t_o)$, $\overline{\alpha}_o = \alpha(\overline{x}_o,\overline{y}_o,\overline{t}_o)$, $\beta_o = \beta(x_o,y_o,t_o)$, $\overline{\beta}_o = \beta(\overline{x}_o,\overline{y}_o,\overline{t}_o)$, $\gamma_o = \gamma(x_o,y_o,t_o)$, and $\overline{\gamma}_o = \gamma(\overline{x}_o,\overline{y}_o,\overline{t}_o)$.

Lemma (1.13). Let (x_o,y_o,t_o) be any point in $\mathcal{D}_1$ such that $\gamma_o = 0$ or $\gamma_o > 0$ with $(\alpha_o,0,\gamma_o)$ not on C_1 or C_2. Then $\lim_{k\to\infty} \alpha_{n_k o} = \alpha_o$, $\lim_{k\to\infty} \beta_{n_k o} = \beta_o$, and $\lim_{k\to\infty} \gamma_{n_k o} = \gamma_o$.

Lemma (1.13) follows from the fact that for each (x_o,y_o,t_o) we have $x_{n_k}(x_o,y_o,t_o,t) \longrightarrow x(x_o,y_o,t_o,t)$ and $y_{n_k}(x_o,y_o,t_o,t) \longrightarrow y(x_o,y_o,t_o,t)$ uniformly in t as $k \longrightarrow \infty$.

Lemma (1.14). α, β, and γ are continuous at points (x_o,y_o,t_o) for which $\gamma_o = 0$ or $\gamma_o > 0$ with $(\alpha_o,0,\gamma_o)$ not on C_1 or C_2.

Lemma (1.14) follows from the fact that $x(x_o,y_o,t_o,t)$ and $y(x_o,y_o,t_o,t)$ are uniformly continuous.

Let h be the function with domain $\mathcal{D}_1$ defined by $h(x_o,y_o,t_o) = \psi_1(\alpha_o,\gamma_o)$ when $\gamma_o > 0$ and $h(x_o,y_o,t_o) = \psi_2(\alpha_o,\beta_o)$ when $\gamma_o = 0$.

Lemma (1.15). Let (x_o,y_o,t_o) be any point in $\mathcal{D}_1$ such that $\gamma_o = 0$ or $\gamma_o > 0$ with $(\alpha_o,0,\gamma_o)$ not on C_1 or C_2. Then h is continuous at (x_o,y_o,t_o) and $\lim_{k\to\infty} h_{n_k}(x_o,y_o,t_o) = h(x_o,y_o,t_o)$.

Lemma (1.15) follows easily using Lemmas (1.13) and (1.14) and the definition of h.

Lemma (1.16). h is a continuous function of (x_o,y_o,t_o) almost everywhere on each plane t_o = constant. Also $|h(x_o,y_o,t_o) + \lambda^2(ax_o+b)| \leq 2M$.

The proof of Lemma (1.16) is similar to the proof of Lemma (1.2) for extended h_n.

From Lemma (1.16) we see that

$\frac{1}{2\pi}\iint\limits_{\eta\geq 0} g(x,y;\xi,\eta)[h(\xi,\eta,t)+\lambda^2(a\xi+b)]d\xi d\eta$ exists for each (x,y,t) in $\mathcal{D}_1$. We define v to be the function with domain $\mathcal{D}_1$ whose values are given by

$$v(x,y,t) = \frac{1}{2\pi}\iint\limits_{\eta\geq 0} g(x,y;\xi,\eta)[h(\xi,\eta,t)+\lambda^2(a\xi+b)]d\xi d\eta .$$

Lemma (1.17). v, v_x, and v_y are continuous. $|v_x| < 4M^2(1+\frac{1}{\lambda^2})$ and $|v_y| < 4M^2(1+\frac{1}{\lambda^2})$. For $0 < s = \sqrt{(\bar{x}-x)^2+(\bar{y}-y)^2} \leq \frac{1}{4}$,

$|v_x(\bar{x},\bar{y},t)-v_x(x,y,t)| < -(52M^2+\frac{16M^2}{\lambda^2})s \log s$ and

$|v_{ny}(\bar{x},\bar{y},t)-v_{ny}(x,y,t)| < -(52M^2+\frac{16M^2}{\lambda^2})s \log s$.

The proof of Lemma (1.17) is the same as that of Lemma (1.3).

Let $u(x,y,t) = v(x,y,t)-w(x,y,t)+ax+b$ for (x,y,t) in $\mathcal{D}_1$.

Lemma (1.18). u, u_x, and u_y are continuous. $|u_x| < D_1$ and $|u_y| < D_1$. When $0 < s = \sqrt{(\bar{x}-x)^2+(\bar{y}-y)^2} \leq \frac{1}{4}$, then $|u_x(\bar{x},\bar{y},t)-u_x(x,y,t)| < -D_2 s \log s$ and $|u_y(\bar{x},\bar{y},t)-u_y(x,y,t)| < -D_2 s \log s$.

Lemma (1.18) follows from the definition of u.

Let (x,y,t) be any point in $\mathcal{D}_1$. It is then clear that

$\lim\limits_{k\to\infty} g(x,y;\xi,\eta)[h_{n_k}(\xi,\eta,t) + \lambda^2(a\xi+b)]$

$= g(x,y;\xi,\eta)[h(\xi,\eta,t) + \lambda^2(a\xi+b)]$ for almost all (ξ,η) with $\eta \geq 0$,

$|g(x,y;\xi,\eta)[h_{n_k}(\xi,\eta,t) + \lambda^2(a\xi+b)]| \leq 2M|g(x,y;\xi,\eta)|$ for all (ξ,η)

with $\eta \geq 0$ and for all k, and $g(x,y;\xi,\eta)[h_{n_k}(\xi,\eta,t)+\lambda^2(a\xi+b)]$ is a measurable function of (ξ,η) for all k. Hence by the Lebesgue convergence theorem we have $\lim_{k\to\infty} u_{n_k}(x,y,t) = u(x,y,t)$. Similarly $\lim_{k\to\infty} u_{n_k x}(x,y,t) = u_x(x,y,t)$ and $\lim_{k\to\infty} u_{n_k y}(x,y,t) = u_y(x,y,t)$.

Let $F_1(x,y,t) = -u_y(x,y,t)$ and $F_2(x,y,t) = u_x(x,y,t)$ for (x,y,t) in $\mathcal{D}_1$ and let $F_1(x,y,t) = -u_y(x,-y,t)$ and $F_2(x,y,t) = u_x(x,-y,t)$ for $(x,-y,t)$ in $\mathcal{D}_1$. Then $\lim_{k\to\infty} F_{n_k 1}(x,y,t) = F_1(x,y,t)$ and $\lim_{k\to\infty} F_{n_k 2}(x,y,t) = F_2(x,y,t)$.

Lemma (1.19). Let (x_o,y_o,t_o) be in $\mathcal{D}_1$ and choose $\bar{t} \geq t_o$ so that $y(x_o,y_o,t_o,t) \geq 0$ for $t_o \leq t \leq \bar{t}$. Then the curve described by $[x(x_o,y_o,t_o,t),y(x_o,y_o,t_o,t),t]$ for $\gamma_o \leq t \leq \bar{t}$ is the unique air particle path of u through (x_o,y_o,t_o).

Proof of Lemma (1.19). For fixed (x_o,y_o,t_o) in $\mathcal{D}_1$ let $Z_k(\xi) = \sqrt{[x_{n_k}(x_o,y_o,t_o,\xi)-x(x_o,y_o,t_o,\xi)]^2+[y_{n_k}(x_o,y_o,t_o,\xi)-y(x_o,y_o,t_o,\xi)]^2}$ for $0 \leq \xi \leq c_1$. Then $Z_k(\xi) \to 0$ as $k \to \infty$ for each ξ. Given ξ choose k_o so that $k > k_o$ implies $Z_k(\xi) < \frac{1}{4}$. Then for $k > k_o$ we have

$$|F_{n_k 1}[x_{n_k}(x_o,y_o,t_o,\xi),y_{n_k}(x_o,y_o,t_o,\xi),\xi]-F_1[x(x_o,y_o,t_o,\xi),y(x_o,y_o,t_o,\xi),\xi]|$$

$$\leq |F_{n_k 1}[x_{n_k}(x_o,y_o,t_o,\xi),y_{n_k}(x_o,y_o,t_o,\xi),\xi] - F_{n_k 1}[x(x_o,y_o,t_o,\xi),y(x_o,y_o,t_o,\xi),\xi]|$$
$$+ |F_{n_k 1}[x(x_o,y_o,t_o,\xi),y(x_o,y_o,t_o,\xi),\xi] - F_1[x(x_o,y_o,t_o,\xi),y(x_o,y_o,t_o,\xi),\xi]|$$

$$\leq - D_2\, Z_k(\xi) \log Z_k(\xi)$$

each $\xi \geq 0$ and for all t [illegible] a measurable function [illegible] for all t. Hence [illegible] component [illegible] (t, η, z) [illegible]. Clearly [illegible]

Let [illegible] Γ_n [illegible] and [illegible] $x_n(t)$ [illegible] Then [illegible]

[illegible]

Lemma [illegible]. Let [illegible]. Then [illegible] for [illegible] $x \in \Gamma$. Then the [illegible] for [illegible]

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$$+ \left| F_{n_{k}1}[x(x_o,y_o,t_o,\xi),y(x_o,y_o,t_o,\xi),\xi] - F_1[x(x_o,y_o,t_o,\xi),y(x_o,y_o,t_o,\xi),\xi] \right| \longrightarrow 0 \quad \text{as} \quad k \to \infty.$$

Thus $\lim_{k\to\infty} F_{n_{k}1}[x_{n_k}(x_o,y_o,t_o,\xi),y_{n_k}(x_o,y_o,t_o,\xi),\xi]$ $= F_1[x(x_o,y_o,t_o,\xi),y(x_o,y_o,t_o,\xi),\xi]$ for each ξ, and $F_{n_{k}1}[x_{n_k}(x_o,y_o,t_o,\xi),y_{n_k}(x_o,y_o,t_o,\xi),\xi]$ is a measurable function of ξ and its absolute value is less than D_1 for each k and ξ. Therefore by the Lebesgue convergence theorem

$$x(x_o,y_o,t_o,t) = \lim_{k\to\infty} x_{n_k}(x_o,y_o,t_o,t)$$

$$= x_o + \lim_{k\to\infty}\int_{t_o}^{t} F_{n_{k}1}[x_{n_k}(x_o,y_o,t_o,\xi),y_{n_k}(x_o,y_o,t_o,\xi),\xi]d\xi$$

$$= x_o + \int_{t_o}^{t} F_1[x(x_o,y_o,t_o,\xi),y(x_o,y_o,t_o,\xi),\xi]d\xi .$$

Similarly we obtain

$$y(x_o,y_o,t_o,t) = y_o + \int_{t_o}^{t} F_2[x(x_o,y_o,t_o,\xi),y(x_o,y_o,t_o,\xi),\xi]d\xi .$$

Thus x_t and y_t exist and

$x_t(x_o,y_o,t_o,t) = F_1[x(x_o,y_o,t_o,t),y(x_o,y_o,t_o,t),t]$ and

$y_t(x_o,y_o,t_o,t) = F_2[x(x_o,y_o,t_o,t),y(x_o,y_o,t_o,t),t]$.

When (x_o,y_o,t_o) is in $\mathcal{D}_1$ and $\gamma_o \le t \le \overline{t}$, we have $x_t(x_o,y_o,t_o,t) = -u_y[x(x_o,y_o,t_o,t),y(x_o,y_o,t_o,t),t]$ and $y_t(x_o,y_o,t_o,t) = u_x[x(x_o,y_o,t_o,t),y(x_o,y_o,t_o,t),t]$. From the inequalities in Lemma (1.18) it is clear that $[x(x_o,y_o,t_o,t), y(x_o,y_o,t_o,t),t]$ represents the unique air particle path of u through (x_o,y_o,t_o).

<u>Proof of Theorem I</u>: On each air particle path of u, h by definition is constant except possibly at points where the air particle path meets C_1 or C_2. Hence h is a pseudo-Helmholtzian of u. Since u and h satisfy (3) by the definition of u, then u is a weak solution of (1). We now observe that $u(x,0,t) = \phi(x,t)$, $h(x,0,t) = \psi_1(x,t)$ when (x,t) is in the domain of ψ_1, $h(x,y,0) = \psi_2(x,y)$, $|u_x(x,y,t)-a| < D_1+a$, $|u_y(x,y,t)| < D_1$, and $|h(x,y,t)+\lambda^2(ax+b)| \leq 2M$.

To complete the proof of Theorem I we have yet to show that $u(x,y,t)-ax-b$ is bounded. We have

$$|u(x,y,t) - ax - b| = |v(x,y,t) - w(x,y,t)|$$

$$\leq \frac{1}{2\pi} \iint\limits_{\eta \geq 0} |g(x,y;\xi,\eta)| 2M d\xi d\eta + \overline{W} \text{ (where } \overline{W} \text{ is an upper bound of } |w|)$$

$$< \frac{M}{\pi} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} |g(x,y;\xi,\eta)| d\xi d\eta + \overline{W}$$

$$\leq \frac{2M}{\pi} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} |K(\lambda\rho)| d\xi d\eta + \overline{W} = \frac{2M}{\pi} \int_{0}^{2\pi} \int_{0}^{\infty} \rho |K(\lambda\rho)| d\rho d\theta + \overline{W}$$

which is a constant.

Thus for all small enough $c_1 > 0$ there is a weak solution with domain $\mathcal{D}_1$ satisfying the conditions of Theorem I.

Part II

Getting Stronger Solutions by Strengthening Hypotheses

For the rest of this report we let u be a weak solution constructed as in the proof of Theorem I, and we let M, W, D_1, D_2, and c_1 be fixed numbers choosen as in the proof of Theorem I. We also let v, F_1, F_2, $x(x_o,y_o,t_o,t)$, $y(x_o,y_o,t_o,t)$, α, β, γ, and h denote the same functions as in the proof of Theorem I.

Theorem II. Let ϕ, ψ_1, and ψ_2 satisfy the hypothesis of Theorem I with the exception that (2_A) and (3_A) are replaced by $(2_A')$ and $(3_A')$.

$(2_A')$ ψ_1 is uniformly Hölder continuous and $\psi_1(x,t)+\lambda^2(ax+b)$ is bounded.

$(3_A')$ ψ_2 is uniformly Hölder continuous and $\psi_2(x,y)+\lambda^2(ax+b)$ is bounded.

Let (x_o,y_o,t_o) be any point in $\mathcal{D}_1$ such that $\beta_o > 0$ or $\beta_o = 0$ with $(\alpha_o,0,\gamma_o)$ not on C_1 or C_2. Then the second derivatives of u with respect to x and y exist and are continuous at (x_o,y_o,t_o) and they satisfy $\Delta u - \lambda^2 u = h$. Thus h is the true Helmholtzian of u at (x_o,y_o,t_o).

Theorem II is proved with the aid of several lemmas which follow.

For (x_o,y_o,t_o) in $\mathcal{D}_1$ and $\delta > 0$ let

$$R_\delta = \left\{(x,y,t)\,|\,(x,y,t)\,\varepsilon\,\mathcal{D}_1,\ |x-x_o| < \delta,\ |y-y_o| < \delta, \text{ and } |t-t_o| < \delta\right\}.$$

Let $\alpha_o^* = \alpha(x_o^*,y_o^*,t_o^*)$, $\beta_o^* = \beta(x_o^*,y_o^*,t_o^*)$, and $\gamma_o^* = \gamma(x_o^*,y_o^*,t_o^*)$.

Lemma (2.1). Let (x_o, y_o, t_o) be any point in $\mathcal{O}_1$ such that $\beta_o > 0$, or $\beta_o = 0$ with $(\alpha_o, 0, \gamma_o)$ not on C_1 or C_2. Then there are constants $H > 0$ and $\delta > 0$ such that (x_o^*, y_o^*, t_o^*) and $(\bar{x}_o, \bar{y}_o, \bar{t}_o)$ in R_δ implies $|\alpha_o^* - \bar{\alpha}_o| \leq H\, s^{\exp(-2D_2 c_1)}$, $|\beta_o^* - \bar{\beta}_o| \leq H\, s^{\exp(-2D_2 c_1)}$, and $|\gamma_o^* - \bar{\gamma}_o| \leq H\, s^{\exp(-2D_2 c_1)}$ where $s = \sqrt{(x_o^* - \bar{x}_o)^2 + (y_o^* - \bar{y}_o)^2 + (t_o^* - \bar{t}_o)^2}$.

Proof of Lemma (2.1).

Case I ($\beta_o > 0$). Since β is continuous at (x_o, y_o, t_o), we can choose δ small enough so that $(\bar{x}_o, \bar{y}_o, \bar{t}_o)$ in R_δ implies $\bar{\beta}_o > 0$. Then $(\bar{x}_o, \bar{y}_o, \bar{t}_o)$ in R_δ implies $\bar{\gamma}_o = 0$, $\bar{\alpha}_o = x(\bar{x}_o, \bar{y}_o, \bar{t}_o, 0)$, and $\bar{\beta}_o = y(\bar{x}_o, \bar{y}_o, \bar{t}_o, 0)$. If we also choose δ small enough so that the diameter of R_δ is less than s_o, then Lemma (2.1) follows easily from Lemma (1.7).

Case II [$\beta_o = 0$, $\gamma_o > 0$, and $(\alpha_o, 0, \gamma_o)$ is not on C_1 or C_2]. Since y_t is continuous and $y_t(x_o, y_o, t_o, \gamma_o) = \phi_x(\alpha_o, \gamma_o) > 0$ (note $(\alpha_o, 0, \gamma_o)$ is not on C_1 or C_2), we can choose positive constants δ, ε_1, and ε_2 so that $y_t(\bar{x}_o, \bar{y}_o, \bar{t}_o, t) \geq \varepsilon_1 > 0$ for $(\bar{x}_o, \bar{y}_o, \bar{t}_o)$ in R_δ and $|t - \gamma_o| \leq \varepsilon_2$.

Since γ is continuous at (x_o, y_o, t_o), we can choose δ smaller if necessary so that $(\bar{x}_o, \bar{y}_o, \bar{t}_o)$ in R_δ implies $\bar{\gamma}_o > 0$ and $|\bar{\gamma}_o - \gamma_o| \leq \varepsilon_2$.

Now let $(\bar{x}_o, \bar{y}_o, \bar{t}_o)$ and (x_o^*, y_o^*, t_o^*) be in R_δ. Assume without loss of generality that $\gamma_o^* \leq \bar{\gamma}_o$. Then since $y(x_o^*, y_o^*, t_o^*, \gamma_o^*) = \beta_o^* = 0 (\gamma_o^* > 0)$, we have $y(x_o^*, y_o^*, t_o^*, \bar{\gamma}_o) = y(x_o^*, y_o^*, t_o^*, \bar{\gamma}_o) - y(x_o^*, y_o^*, t_o^*, \gamma_o^*) = (\bar{\gamma}_o - \gamma_o^*) y_t(x_o^*, y_o^*, \gamma_o^*, t^*)$ where t^* is between γ_o^* and $\bar{\gamma}_o$. Since

$|\gamma_o^*-\gamma_o| \leq \varepsilon_2$ and $|\bar{\gamma}_o-\gamma_o| \leq \varepsilon_2$, it follows that $|t^*-\gamma_o| \leq \varepsilon_2$ and hence $y(x_o^*,y_o^*,t_o^*,\bar{\gamma}_o) = (\bar{\gamma}_o-\gamma_o^*)y_t(x_o^*,y_o^*,\gamma_o^*,t^*) \geq \varepsilon_1(\bar{\gamma}_o-\gamma_o^*) = \varepsilon_1|\bar{\gamma}_o-\gamma_o^*|$. Using $y(\bar{x}_o,\bar{y}_o,\bar{t}_o,\bar{\gamma}_o) = \bar{\beta}_o = 0(\bar{\gamma}_o>0)$ we now have $|\bar{\gamma}_o-\gamma_o^*| \leq \frac{1}{\varepsilon_1} |y(x_o^*,y_o^*,t_o^*,\bar{\gamma}_o)-y(\bar{x}_o,\bar{y}_o,\bar{t}_o,\bar{\gamma}_o)|$. Now choose δ smaller if necessary so that the diameter of R_δ is less than s_o. Then from Lemma (1.7) we have $|\bar{\gamma}_o-\gamma_o^*| \leq \frac{1}{\varepsilon_1} [2(D_1+1)s]^{\exp(-2D_2c_1)}$. The results for α and β follow in an obvious manner.

Case III ($\beta_o = \gamma_o = 0$ and $(\alpha_o,0,\gamma_o)$ is not on C_1 or C_2). As in Case II we can choose positive constants δ, ε_1, and ε_2 so that $y_t(\bar{x}_o,\bar{y}_o,\bar{t}_o,t) \geq \varepsilon_1 > 0$ for $(\bar{x}_o,\bar{y}_o,\bar{t}_o)$ in R_δ and $|t-\gamma_o| \leq \varepsilon_2$, so that $|\bar{\gamma}_o-\gamma_o| \leq \varepsilon_2$ for $(\bar{x}_o,\bar{y}_o,\bar{t}_o)$ in R_δ, and so that the diameter of R_δ is less than s_o.

Now let $(\bar{x}_o,\bar{y}_o,\bar{t}_o)$ and (x_o^*,y_o^*,t_o^*) be in R_δ. If $\bar{\gamma}_o = \gamma_o^* = 0$, our conclusion follows as in Case I. If $\bar{\gamma}_o > 0$ and $\gamma_o^* > 0$, our conclusion follows as in Case II. If $\bar{\gamma}_o = 0$ and $\gamma_o^* > 0$, the continuity of γ in R_δ can be used to conclude that there is a $(\hat{x}_o,\hat{y}_o,\hat{t}_o)$ on the straight line segment from $(\bar{x}_o,\bar{y}_o,\bar{t}_o)$ to (x_o^*,y_o^*,t_o^*) such that $\gamma(\hat{x}_o,\hat{y}_o,\hat{t}_o) = 0$ but $\gamma(x,y,t) > 0$ (hence $\beta(x,y,t) = 0$) on the straight line segment between $(\hat{x}_o,\hat{y}_o,\hat{t}_o)$ and (x_o^*,y_o^*,t_o^*). Since β is continuous in R_δ, it follows that $\beta(\hat{x}_o,\hat{y}_o,\hat{t}_o) = 0$. The methods used in Case II can be used to show that $|\gamma(\hat{x}_o,\hat{y}_o,\hat{t}_o)-\gamma_o^*|$

$$\leq \frac{1}{\varepsilon_1} [2(D_1+1)]^{\exp(-2D_2c_1)} [(\hat{x}_o-x_o^*)^2+(\hat{y}_o-y_o^*)^2+(\hat{t}_o-t_o^*)^2]^{\frac{1}{2}\exp(-2D_2c_1)}$$

$< \frac{1}{\varepsilon_1} [2(D_1+1)s]^{\exp(-2D_2c_1)}$. But $|\bar{\gamma}_o-\gamma_o^*| = \gamma_o^* = |\gamma(\hat{x}_o,\hat{y}_o,\hat{t}_o)-\gamma_o^*|$ and our lemma follows.

Lemma (2.2). Let (x_o, y_o, t_o) be in $\mathcal{D}_1$ such that $\beta_o > 0$ or $\beta_o = 0$ with $(\alpha_o, 0, \gamma_o)$ not on C_1 or C_2. Then h is uniformly Hölder continuous in a neighborhood of (x_o, y_o, t_o).

Proof of Lemma (2.2). Using Lemma (2.1) and the fact that ψ_1 and ψ_2 are uniformly Hölder continuous we can show that there are constants $H > 0$, $\delta > 0$, and $0 < \varepsilon < 1$ such that (x_o^*, y_o^*, t_o^*) and $(\bar{x}_o, \bar{y}_o, \bar{t}_o)$ in R_δ implies $|h(x_o^*, y_o^*, t_o^*) - h(\bar{x}_o, \bar{y}_o, \bar{t}_o)| \leq Hs^\varepsilon$ where $s = \sqrt{(x_o^* - \bar{x}_o)^2 + (y_o^* - \bar{y}_o)^2 + (t_o^* - \bar{t}_o)^2}$. It is then clear that there are positive constants $\delta_1 < \delta$ and δ_2 such that $(\bar{x}_o, \bar{y}_o, \bar{t}_o)$ in R_{δ_1} and $s < \delta_2$ implies that (x_o^*, y_o^*, t_o^*) is in R_δ. Hence for $(\bar{x}_o, \bar{y}_o, \bar{t}_o)$ in R_{δ_1} we have $|h(x_o^*, y_o^*, t_o^*) - h(\bar{x}_o, \bar{y}_o, \bar{t}_o)| \leq Hs^\varepsilon$ for $s < \delta_2$. That is, h is uniformly Hölder continuous in R_{δ_1}.

Proof of Theorem II. Let (x_o, y_o, t_o) be in $\mathcal{D}_1$ such that $\beta_o > 0$ or $\beta_o = 0$ with $(\alpha_o, 0, \gamma_o)$ not on C_1 or C_2. For arbitrary (x, y, t_o) in $\mathcal{D}_1$ we have

$$\text{(II.1)} \quad v_x(x,y,t_o) = \frac{1}{2\pi} \iint_{\eta \geq 0} g_x(x,y;\xi,\eta)[h(\xi,\eta,t_o) + \lambda^2(a\xi + b)]d\xi d\eta$$

$$= \frac{1}{2\pi} \iint_{\eta \geq 0} g_x(x,y;\xi,\eta)[h(\xi,\eta,t_o) - h(x_o,y_o,t_o) + a\lambda^2(\xi - x_o)]d\xi d\eta$$

$$\text{(II.2)} \quad v_y(x,y,t_o) = \frac{1}{2\pi} \iint_{\eta \geq 0} g_y(x,y;\xi,\eta)[h(\xi,\eta,t_o) + \lambda^2(a\xi + b)]d\xi d\eta$$

$$= \frac{1}{2\pi} \iint_{\eta \geq 0} g_y(x,y;\xi,\eta)[h(\xi,\eta,t_o) - h(x_o,y_o,t_o) + a\lambda^2(\xi - x_o)]d\xi d\eta$$

$$- [h(x_o,y_o,t_o) + \lambda^2(ax_o + b)] \frac{1}{\pi} \int_{-\infty}^{\infty} K(\lambda\nu)d\xi \ .$$

To get the second representation of v_x we added and subtracted $[h(x_o,y_o,t_o)+\lambda^2(ax_o+b)]\ \frac{1}{2\pi}\iint\limits_{\eta\geq 0} g_x(x,y;\xi,\eta)d\xi d\eta$ to the first integral of (II.1), and then we observed that $\iint\limits_{\eta\geq 0} g_x(x,y;\xi,\eta)d\xi d\eta$ $= -\iint\limits_{\eta\geq 0} g_\xi(x,y;\xi,\eta)d\xi d\eta = 0$. We obtained the second representation of v_y in a similar manner by observing that

$$\frac{1}{2\pi}\iint\limits_{\eta\geq 0} g_y(x,y;\xi,\eta)d\xi d\eta = \frac{1}{2\pi}\iint\limits_{\eta\geq 0} \frac{\partial}{\partial\eta}[K(\lambda\bar{\rho})+K(\lambda\rho)]d\xi d\eta = -\frac{1}{\pi}\int_{-\infty}^{\infty} K(\lambda\nu)d\xi \ .$$

Since h is Hölder continuous at (x_o,y_o,t_o), we could show that differentiation with respect to x and y at $(x,y,t_o) = (x_o,y_o,t_o)$ is permitted under the integral sign in the second representations of (II.1) and (II.2). The resulting expressions are also valid for (x,y,t) in some neighborhood of (x_o,y_o,t_o) since h is also Hölder continuous in some neighborhood of (x_o,y_o,t_o). Hence for (x,y,t) in some neighborhood of (x_o,y_o,t_o) we have

$$\text{(II.3)}\quad v_{xx}(x,y,t) = \frac{1}{2\pi}\iint\limits_{\eta\geq 0} g_{xx}(x,y;\xi,\eta)[h(\xi,\eta,t)-h(x,y,t)+a\lambda^2(\xi-x)]d\xi d\eta \ ,$$

$$\text{(II.4)}\quad v_{xy}(x,y,t) = \frac{1}{2\pi}\iint\limits_{\eta\geq 0} g_{xy}(x,y;\xi,\eta)[h(\xi,\eta,t)-h(x,y,t)+a\lambda^2(\xi-x)]d\xi d\eta \ ,$$

$$\text{(II.5)}\quad v_{yy}(x,y,t) = \frac{1}{2\pi}\iint\limits_{\eta\geq 0} g_{yy}(x,y;\xi,\eta)[h(\xi,\eta,t)-h(x,y,t)+a\lambda^2(\xi-s)]d\xi d\eta$$

$$+ [h(x,y,t)+\lambda^2(ax+b)] \begin{cases} \dfrac{-\lambda y}{\pi}\displaystyle\int_{-\infty}^{\infty} \frac{1}{\nu}K'(\lambda\nu)d\xi & \text{if } y > 0 \\ 1 & \text{if } y = 0 \end{cases} .$$

Since the Hölder continuity of h is uniform in some neighborhood of (x_o, y_o, t_o), we can show with the aid of (II.3), (II.4), and (II.5) that v_{xx}, v_{xy}, and v_{yy} are continuous at (x_o, y_o, t_o).

Next we wish to show that $\triangle v - \lambda^2 v = h(x,y,t)+\lambda^2(ax+b)$ at points (x,y,t) where (II.3), (II.4), and (II.5) are valid. To aid us here we introduce $\overline{w}(x,y,t) = \begin{cases} -\frac{\lambda y}{\pi} \int_{-\infty}^{\infty} \frac{1}{\nu} K'(\lambda\nu)d\xi \text{ if } y > 0 \\ 1 \text{ if } y = 0 \end{cases}$ and $\overline{v}(x,y,t) = \frac{1}{2\pi} \iint_{\eta \geq 0} g(x,y;\xi,\eta)d\xi d\eta$. We will first show that $\overline{w}(x,y,t) = e^{-\lambda y}$ and $\overline{v}(x,y,t) = \frac{1}{\lambda^2}(e^{-\lambda y}-1)$.

Replacing $\phi(\xi,t)-(a\xi+b)$ by 1 in Lemma (1.1), we see $\overline{w}$ has continuous bounded first and second derivatives with respect to x and y for $y \geq 0$ and $\triangle\overline{w} - \lambda^2\overline{w} = 0$. Since $\overline{w}(x,y,t)$ $= \frac{\lambda y}{\pi} \int_{-\infty}^{\infty} \frac{1}{\sqrt{z^2+y^2}} K'(\lambda\sqrt{z^2+y^2})d\xi$, we have $\overline{w}_x = 0$ and hence $\overline{w}_{yy} - \lambda^2\overline{w} = 0$. Therefore $\overline{w} = c_1 e^{-\lambda y} + c_2 e^{\lambda y}$. Since $\overline{w}$ is bounded, then $c_2 = 0$. Since $\overline{w}(x,0,t) = 1$, $c_1 = 1$ and $\overline{w} = e^{-\lambda y}$.

Replacing $h(\xi,\eta,t)+\lambda^2(a\xi+b)$ by 1 in (II.1) and (II.5) we see that $\overline{v}_x = 0$ and $\overline{v}_{yy}(x,y,t) = \overline{w}(x,y,t) = e^{-\lambda y}$. Hence $\overline{v} = \frac{1}{\lambda^2} e^{-\lambda y} + \overline{v}_1 y + \overline{v}_2$. Since $\overline{v}$ is bounded, then $\overline{v}_1 = 0$. Since $\overline{v}(x,0,t) = 0$, then $\overline{v}_2 = \frac{-1}{\lambda^2}$ and $\overline{v} = \frac{1}{\lambda^2}(e^{-\lambda y}-1)$.

We now have $\triangle v - \lambda^2 v$

$$= \frac{1}{2\pi} \iint_{\eta \geq 0} \triangle g(x,y;\xi,\eta)[h(\xi,\eta,t)-h(x,y,t)+a\lambda^2(\xi-x)]\, d\xi d\eta$$
$$+ \overline{w}(x,y,t)[h(x,y,t)+\lambda^2(ax+b)]$$
$$- \frac{\lambda^2}{2\pi} \iint_{\eta \geq 0} g(x,y;\xi,\eta)[h(\xi,\eta,t)+\lambda^2(a\xi+b)]d\xi d\eta.$$ Using the fact that

$\Delta g(x,y;\xi,\eta) = \lambda^2 g(x,y;\xi,\eta)$ we have $\Delta v - \lambda^2 v$
$= [h(x,y,t)+\lambda^2(ax+b)][\overline{w}(x,y,t)-\lambda^2\overline{v}(x,y,t)] = h(x,y,t)+\lambda^2(ax+b)$.

Since $u = v-w+ax+b$, then u has continuous second derivatives with respect to x and y at (x_o,y_o,t_o), and $\Delta u(x_o,y_o,t_o) - \lambda^2 u(x_o,y_o,t_o) = h(x_o,y_o,t_o)+\lambda^2(ax_o+b) - 0 - \lambda^2(ax_o+b) = h(x_o,y_o,t_o)$. This completes the proof of Theorem II.

In our next theorem we assume that the prescribed values of the Helmholtzian are constant in a strip along the curves C_1 and C_2. We can then show that u_{xx}, u_{xy}, and u_{yy} exist and are continuous for all small enough t.

Theorem III. Let ϕ, ψ_1 and ψ_2 satisfy the hypothesis of Theorem II. Let ϕ_x satisfy a uniform Lipschitz condition with respect to t. For the functions $x_1(t)$ and $x_2(t)$ of (1_C) in Theorem I let $d_1 = \min_{0\leq t\leq c} x_1(t)$, $d_2 = \max_{0\leq t\leq c} x_1(t)$, $d_3 = \min_{0\leq t\leq c} x_2(t)$, $d_4 = \max_{0\leq t\leq c} x_2(t)$, and assume $d_2 < d_3$. Let (3_C) and (2_B) also be satisfied.

(3_C) For some positive number τ and real numbers p_1 and p_2 we have $d_2+\tau<d_3-\tau$, $\psi_2(x,y) = p_1$ for $d_1-\tau \leq x \leq d_2+\tau$ and $0 \leq y \leq \tau$, and $\psi_2(x,y) = p_2$ for $d_3-\tau \leq x \leq d_4+\tau$ and $0 \leq y \leq \tau$.

(2_B) There is a positive number σ such that $\sigma < \tau$ and such that $(x_o,0,t_o)$ on C_i implies $\psi_1(x,t) = p_i$ when (x,t) is in the domain of ψ_1 and both $|x-x_o| \leq \sigma$ and $|t-t_o| \leq \sigma$ $(i=1,2)$.

Then there is a c_2 such that $0 < c_2 \leq c_1$, u has continuous second derivatives with respect to x and y in $\mathcal{D}_2 = \{(x,y,t) \mid -\infty < x < \infty,\ y \geq 0,\ 0 \leq t \leq c_2\}$, and $\Delta u-\lambda^2 u = h$ in $\mathcal{D}_2$ so that h is the true Helmholtzian of u in $\mathcal{D}_2$.

Theorem III is proved using the lemmas which follow.

Choose M_1 so that $|\phi_x(x,\bar{t})-\phi_x(x,t)| \leq M_1|\bar{t}-t|$ for all $(x,\bar{t})$ and (x,t) in the domain of ϕ.

Let $\omega = \text{glb}\ \phi_x(x,t)$ where the greatest lower bound is taken over all (x,t) such that $\phi_x(x,t) \geq 0$, $|x-x_o| \geq \sigma$ or $|t-t_o| \geq \sigma$ for each $(x_o,0,t_o)$ on C_1 or C_2, and $d_1-(2D_1c+s_o+\tau) \leq x \leq d_4+2D_1c+s_o+\tau$. Then $\omega > 0$.

Choose c_2 so that $0 < c_2 \leq c_1$, $D_1c_2 \leq \frac{1}{4}$, $D_1c_2 < \tau$, and $c_2(WD_1+M_1-D_1D_2 \log D_1c_2) \leq \frac{\omega}{2}$.

Lemma (3.1). h is uniformly Hölder continuous in some neighborhood of each point in $\mathcal{D}_2$.

Proof of Lemma (3.1). Let (x_o,y_o,t_o) be any point in $\mathcal{D}_2$. If $\beta_o > 0$ or $\beta_o = 0$ with $(\alpha_o,0,\gamma_o)$ not on C_1 or C_2, then h is uniformly Hölder continuous in some neighborhood of (x_o,y_o,t_o) by Lemma (2.2). The only remaining case is the one in which $\beta_o = 0$ and $(\alpha_o,0,\gamma_o)$ is on C_1 or C_2.

Case I [$\beta_o = 0$, $\gamma_o > 0$, and $(\alpha_o,0,\gamma_o)$ is on C_1]. Suppose there is a point $(\bar{x}_o,\bar{y}_o,\bar{t}_o)$ in $\mathcal{D}_2$ such that $s = \sqrt{(\bar{x}_o-x_o)^2+(\bar{y}_o-y_o)^2+(\bar{t}_o-t_o)^2} < s_o$, $\bar{\gamma}_o > 0$, and either $|\bar{\alpha}_o-x| > \sigma$ or $|\bar{\gamma}_o-t| > \sigma$ for each $(x,0,t)$ on C_1. Then $\bar{\beta}_o = 0$ and $|y(\bar{x}_o,\bar{y}_o,\bar{t}_o,z)| = |y(\bar{x}_o,\bar{y}_o,\bar{t}_o,z)-y(\bar{x}_o,\bar{y}_o,\bar{t}_o,\bar{\gamma}_o)| \leq D_1|z-\bar{\gamma}_o|$ $\leq D_1c_2 \leq \frac{1}{4}$ for $0 \leq z \leq c_2$. Thus for $0 \leq z \leq c_2$ we have

$$|y_t(\bar{x}_o,\bar{y}_o,\bar{t}_o,z)-y_t(\bar{x}_o,\bar{y}_o,\bar{t}_o,\bar{\gamma}_o)|$$
$$= |F_2[x(\bar{x}_o,\bar{y}_o,\bar{t}_o,z),\ y(\bar{x}_o,\bar{y}_o,\bar{t}_o,z),z] - F_2(\bar{\alpha}_o,0,\bar{\gamma}_o)|$$
$$\leq |F_2[x(\bar{x}_o,\bar{y}_o,\bar{t}_o,z),\ y(\bar{x}_o,\bar{y}_o,\bar{t}_o,z),z] - F_2[x(\bar{x}_o,\bar{y}_o,\bar{t}_o,z),0,z]|$$
$$+\ |\phi_x[x(\bar{x}_o,\bar{y}_o,\bar{t}_o,z),z]-\phi_x(\bar{\alpha}_o,\bar{\gamma}_o)|$$

$\leq -D_2|y(\bar{x}_o,\bar{y}_o,\bar{t}_o,z)| \log|y(\bar{x}_o,\bar{y}_o,\bar{t}_o,z)|+W|x(\bar{x}_o,\bar{y}_o,\bar{t}_o,z)-\bar{a}_o|+M_1|z-\bar{\gamma}_o|$

(since $|y(\bar{x}_o,\bar{y}_o,\bar{t}_o,z)| \leq \frac{1}{4}$ and $|\phi_{xx}| = |w_{xx}|_{y=0} \leq W$)

$\leq -D_2D_1c_2 \log D_1c_2+W|x(\bar{x}_o,\bar{y}_o,\bar{t}_o,z)-x(\bar{x}_o,\bar{y}_o,\bar{t}_o,\bar{\gamma}_o)| + M_1|z-\bar{\gamma}_o|$

(since $|y(\bar{x}_o,\bar{y}_o,\bar{t}_o,z)| \leq D_1c_2$)

$\leq -D_2D_1c_2 \log D_1c_2+WD_1|z-\bar{\gamma}_o| + M_1|z-\bar{\gamma}_o|$

$\leq c_2(WD_1+M_1-D_2D_1 \log D_1c_2) \leq \frac{\omega}{2}$.

Also $|a_o-\bar{a}_o| = |(a_o-x_o)+(x_o-\bar{x}_o)+(\bar{x}_o-\bar{a}_o)| \leq D_1c_2+s_o+D_1c_2$ so that $\bar{a}_o \geq a_o-(2D_1c_2+s_o) \geq d_1-(2D_1c+s_o+\tau)$. Similarly $\bar{a}_o \leq d_4+2D_1c+s_o+\tau$ and hence from the definition of ω we have $\phi_x(\bar{a}_o,\bar{\gamma}_o) \geq \omega$.

Since $|y_t(\bar{x}_o,\bar{y}_o,\bar{t}_o,z)-y_t(\bar{x}_o,\bar{y}_o,\bar{t}_o,\bar{\gamma}_o)| \leq \frac{\omega}{2}$ for $0 \leq z \leq c_2$ and since $\phi_x(\bar{a}_o,\bar{\gamma}_o) \geq \omega$, then $y_t(\bar{x}_o,\bar{y}_o,\bar{t}_o,z) \geq y_t(\bar{x}_o,\bar{y}_o,\bar{t}_o,\bar{\gamma}_o)-\frac{\omega}{2}$ $= \phi_x(\bar{a}_o,\bar{\gamma}_o)-\frac{\omega}{2} \geq \omega-\frac{\omega}{2} = \frac{\omega}{2}$ for $0 \leq z \leq c_2$.

Since $(a_o,0,\gamma_o)$ is on C_1 we have either $|\bar{\gamma}_o-\gamma_o| > \sigma$ or $|\bar{a}_o-a_o| > \sigma$ by the choice of $(\bar{x}_o,\bar{y}_o,\bar{t}_o)$. Thus either

$$\sigma < |\bar{\gamma}_o-\gamma_o| = \frac{2}{\omega}\left|\int_{\gamma_o}^{\bar{\gamma}_o} \frac{\omega}{2}\,d\xi\right| \leq \frac{2}{\omega}\left|\int_{\gamma_o}^{\bar{\gamma}_o} y_t(\bar{x}_o,\bar{y}_o,\bar{t}_o,z)dz\right|$$

$= \frac{2}{\omega}|y(\bar{x}_o,\bar{y}_o,\bar{t}_o,\bar{\gamma}_o)-y(\bar{x}_o,\bar{y}_o,\bar{t}_o,\gamma_o)| = \frac{2}{\omega}|y(\bar{x}_o,\bar{y}_o,\bar{t}_o,\gamma_o)|$

$= \frac{2}{\omega}|y(\bar{x}_o,\bar{y}_o,\bar{t}_o,\gamma_o)-y(x_o,y_o,t_o,\gamma_o)|$

$\leq \frac{2}{\omega}[2(D_1+1)s]^{\exp(-2D_2c_1)}$, or $\sigma < |\bar{a}_o-a_o|$

$= |x(\bar{x}_o,\bar{y}_o,\bar{t}_o,\bar{\gamma}_o)-x(x_o,y_o,t_o,\gamma_o)|$

$\leq |x(\bar{x}_o,\bar{y}_o,\bar{t}_o,\bar{\gamma}_o)-x(x_o,y_o,t_o,\bar{\gamma}_o)| + |x(x_o,y_o,t_o,\bar{\gamma}_o)-x(x_o,y_o,t_o,\gamma_o)|$

$\leq [2(D_1+1)s]^{\exp(-2D_2c_1)}+D_1|\bar{\gamma}_o-\gamma_o| \leq (1+\frac{2D_1}{\omega})[2(D_1+1)s]^{\exp(-2D_2c_1)}$.

From this we see that all small enough neighborhoods of (x_o,y_o,t_o) do not contain any such points $(\bar{x}_o,\bar{y}_o,\bar{t}_o)$. Therefore for all small enough neighborhoods of (x_o,y_o,t_o) we have (x_o^*,y_o^*,t_o^*) in the neighborhood and $\gamma_o^* > 0$ implies $h(x_o^*,y_o^*,t_o^*) = \psi_1(a_o^*,\gamma_o^*) = p_1$.

Now suppose there is a point $(\bar{x}_0,\bar{y}_0,\bar{t}_0)$ in $\mathcal{D}_2$ such that $\bar{\gamma}_0 = 0$ and $s < s_0$. Then $0 \leq \bar{\beta}_0 = \bar{\beta}_0-\beta_0$
$\leq |y(\bar{x}_0,\bar{y}_0,\bar{t}_0,\bar{\gamma}_0)-y(\bar{x}_0,\bar{y}_0,\bar{t}_0,\gamma_0)|+|y(\bar{x}_0,\bar{y}_0,\bar{t}_0,\gamma_0)-y(x_0,y_0,t_0,\gamma_0)|$
$\leq D_1|\bar{\gamma}_0-\gamma_0|+[2(D_1+1)s]^{\exp(-2D_2c_1)}$. Since $D_1|\bar{\gamma}_0-\gamma_0| \leq D_1c_2 < \tau$, then $0 \leq \bar{\beta}_0 \leq \tau$ for all such $(\bar{x}_0,\bar{y}_0,\bar{t}_0)$ near enough to (x_0,y_0,t_0). Also $\bar{\alpha}_0 = x(\bar{x}_0,\bar{y}_0,\bar{t}_0,\bar{\gamma}_0)-x(\bar{x}_0,\bar{y}_0,\bar{t}_0,\gamma_0)+x(\bar{x}_0,\bar{y}_0,\bar{t}_0,\gamma_0)$
$- x(x_0,y_0,t_0,\gamma_0)+\alpha_0 \leq D_1|\bar{\gamma}_0-\gamma_0|+[2(D_1+1)s]^{\exp(-2D_2c_1)} + \alpha_0$. Again $D_1|\bar{\gamma}_0-\gamma_0| < \tau$. Also $\alpha_0 \leq d_2$ since $(\alpha_0,0,\gamma_0)$ is on C_1. Hence $\bar{\alpha}_0 \leq d_2+\tau$ and similarly $\bar{\alpha}_0 \geq d_1-\tau$ for all such $(\bar{x}_0,\bar{y}_0,\bar{t}_0)$ near enough to (x_0,y_0,t_0). It follows that for all such $(\bar{x}_0,\bar{y}_0,\bar{t}_0)$ near enough to (x_0,y_0,t_0) we have $d_1-\tau \leq \bar{\alpha}_0 \leq d_2+\tau$, $0 \leq \bar{\beta}_0 \leq \tau$, and hence $h(\bar{x}_0,\bar{y}_0,\bar{t}_0) = \psi_2(\bar{\alpha}_0,\bar{\beta}_0) = p_1$.

Therefore in Case I $h(\bar{x}_0,\bar{y}_0,\bar{t}_0) = p_1$ for all $(\bar{x}_0,\bar{y}_0,\bar{t}_0)$ near enough to (x_0,y_0,t_0). Hence h is uniformly Hölder continuous in a neighborhood of (x_0,y_0,t_0).

Case II [$\beta_0 = 0$, $\gamma_0 = 0$, and $(\alpha_0,0,\gamma_0)$ is on C_1]. Since α, β, and γ are continuous at (x_0,y_0,t_0), then for $(\bar{x}_0,\bar{y}_0,\bar{t}_0)$ in $\mathcal{D}_2$ near enough to (x_0,y_0,t_0) we have $h(\bar{x}_0,\bar{y}_0,\bar{t}_0) = p_1$, and h is uniformly Hölder continuous in some neighborhood of (x_0,y_0,t_0).

Similarly h is uniformly Hölder continuous in a neighborhood of (x_0,y_0,t_0) when $\beta_0 = 0$ and $(\alpha_0,0,\gamma_0)$ is on C_2. This completes the proof of Lemma (3.1).

Theorem III follows from Lemma (3.1) as Theorem II followed from Lemma (2.2).

Next we want u_{xx}, u_{xy}, and u_{yy} to be bounded at infinity. This is accomplished in Theorem IV.

Theorem IV. Let ϕ, ψ_1, and ψ_2 satisfy the hypothesis of Theorem III. Let ϕ also satisfy (1_D).

(1_D) glb $\phi_x(x,t) > 0$ where the greatest lower bound is taken over the set of all (x,t) such that $\phi_x(x,t) \geq 0$ and either $|x-x_o| \geq \sigma$ or $|t-t_o| \geq \sigma$ for each $(x_o,0,t_o)$ on C_1 or C_2.

Then there is a c_3 such that $0 < c_3 \leq c_2$ and u has bounded second derivatives with respect to x and y in $\mathcal{D}_3 = \{(x,y,t) | -\infty < x < \infty,\ y \geq 0,\ 0 \leq t \leq c_3\}$.

Let $\bar{\omega} = \text{glb}\ \phi_x(x,t)$ where the greatest lower bound is taken over the set specified in 1_D of Theorem IV. Then $0 < \bar{\omega} \leq \omega$.

Let c_3 satisfy $0 < c_3 \leq c_2$ and $c_3(WD_1+M_1-D_1D_2 \log D_1c_3) \leq \frac{\bar{\omega}}{2}$.

Again we prove several lemmas to aid us with the proof of the theorem.

Lemma (4.1). The Hölder continuity of h in (x,y) is uniform in $\bar{\mathcal{D}}_3 = \{(x,y,t) | (x,y,t) \in \mathcal{D}_3$ with $x \leq d_1-2\sigma-D_1c_3-1$ or $x \geq d_4+2\sigma+D_1c_3+1$ or $y \geq 2D_1c_3+1\}$ with respect to (x,y) and t.

Proof of Lemma (4.1). Let (x_o,y_o,t_o) be any point in $\bar{\mathcal{D}}_3$ and suppose $\beta_o = 0$. We will show that then $y_t(x_o,y_o,t_o,t) \geq \frac{\bar{\omega}}{2}$ for $0 \leq t \leq c_3$. We have $y_o = y(x_o,y_o,t_o,t_o)-y(x_o,y_o,t_o,\gamma_o) \leq D_1|t_o-\gamma_o| \leq D_1c_3$. Therefore since (x_o,y_o,t_o) is in $\bar{\mathcal{D}}_3$ we have $x_o \leq d_1-2\sigma-D_1c_3-1$ or $x_o \geq d_4+2\sigma+D_1c_3+1$. Hence $a_o = x(x_o,y_o,t_o,\gamma_o)-x(x_o,y_o,t_o,t_o)+x_o \leq D_1c_3 + x_o \leq d_1-2\sigma-1$ or $a_o \geq d_4+2\sigma+1$, and thus $\phi_x(a_o,\gamma_o) \geq \bar{\omega}$. Then for $0 \leq t \leq c_3$ we have $|y_t(x_o,y_o,t_o,t)-y_t(x_o,y_o,t_o,\gamma_o)|$

$$\leq |F_2[x(x_o,y_o,t_o,t),\ y(x_o,y_o,t_o,t),t]- F_2[x(x_o,y_o,t_o,t),0,t]|$$
$$+ |\phi_x[x(x_o,y_o,t_o,t),t]-\phi_x(a_o,\gamma_o)|$$

$\leq -D_2|y(x_o,y_o,t_o,t)|\log|y(x_o,y_o,t_o,t)|+W|x(x_o,y_o,t_o,t)-x(x_o,y_o,t_o,\gamma_o)|$
$+ M_1|t-\gamma_o|$

$\leq -D_2D_1c_3 \log D_1c_3+WD_1c_3+M_1c_3 \leq \frac{\overline{\omega}}{2}$ and hence

$y_t(x_o,y_o,t_o,t) \geq y_t(x_o,y_o,t_o,\gamma_o)- \frac{\overline{\omega}}{2} = \phi_x(\alpha_o,\gamma_o)- \frac{\overline{\omega}}{2} \geq \frac{\overline{\omega}}{2}$ for $0 \leq t \leq c_3$.

Now let (x_o,y_o,t_o) and $(\overline{x}_o,\overline{y}_o,\overline{t}_o)$ be in $\overline{\mathcal{D}}_3$ with $s = \sqrt{(\overline{x}_o-x_o)^2+(\overline{y}_o-y_o)^2+(\overline{t}_o-t_o)^2} < s_o$. Consider the case where $\beta_o = 0$. When $\overline{\gamma}_o > \gamma_o$, then $\overline{\beta}_o = 0$ and $0 < \overline{\gamma}_o-\gamma_o = \frac{2}{\overline{\omega}}\int_{\gamma_o}^{\overline{\gamma}_o} \frac{\overline{\omega}}{2}\, d\xi$

$\leq \frac{2}{\overline{\omega}}\int_{\gamma_o}^{\overline{\gamma}_o} y_t(x_o,y_o,t_o,\xi)d\xi = \frac{2}{\overline{\omega}}\, y(x_o,y_o,t_o,\overline{\gamma}_o)$

$= \frac{2}{\overline{\omega}}\, [y(x_o,y_o,t_o,\overline{\gamma}_o)-y(\overline{x}_o,\overline{y}_o,\overline{t}_o,\overline{\gamma}_o)] \leq \frac{2}{\overline{\omega}}\, [2(D_1+1)s]^{\exp(-2D_2c_1)}$.

When $\overline{\gamma}_o \leq \gamma_o$, then $0 \leq \gamma_o-\overline{\gamma}_o = \frac{2}{\overline{\omega}}\int_{\overline{\gamma}_o}^{\gamma_o} \frac{\overline{\omega}}{2}\, d\xi$

$\leq \frac{2}{\overline{\omega}}\int_{\overline{\gamma}_o}^{\gamma_o} y_t(x_o,y_o,t_o,\xi)d\xi = -\frac{2}{\overline{\omega}}\, y(x_o,y_o,t_o,\overline{\gamma}_o)$

$\leq \frac{2}{\overline{\omega}}\, [y(\overline{x}_o,\overline{y}_o,\overline{t}_o,\overline{\gamma}_o)-y(x_o,y_o,t_o,\overline{\gamma}_o)] \leq \frac{2}{\overline{\omega}}\, [2(D_1+1)s]^{\exp(-2D_2c_1)}$.

Hence $|\overline{\gamma}_o-\gamma_o| \leq \frac{2}{\overline{\omega}}\, [2(D_1+1)s]^{\exp(-2D_2c_1)}$ when $\beta_o = 0$ and $s < s_o$.

Now consider the case where $\beta_o > 0$. If $\overline{\beta}_o = 0$ for some (x_o,y_o,t_o) in $\overline{\mathcal{D}}_3$ with $s < s_o$, we obtain $|\overline{\gamma}_o-\gamma_o| \leq \frac{2}{\overline{\omega}}\, [2(D_1+1)s]^{\exp(-2D_2c_1)}$ as in the previous case. If $\overline{\beta}_o > 0$, we have $\overline{\gamma}_o = \gamma_o = 0$.

Thus $|\overline{\gamma}_o-\gamma_o| \leq \frac{2}{\overline{\omega}}\, [2(D_1+1)s]^{\exp(-2D_2c_1)}$ for $s < s_o$. Since $\alpha_o = x(x_o,y_o,t_o,\gamma_o)$ and $\beta_o = y(x_o,y_o,t_o\gamma_o)$, a similar result follows for α and β. Since ψ_1 and ψ_2 are uniformly Hölder continuous, we can easily obtain the conclusion for Lemma (4.1).

Proof of Theorem IV. Using the integral representations given in (II.3), (II.4), and (II.5) of the proof of Theorem II; the fact that $h(\xi,\eta,t)+\lambda^2(a\xi+b)$ is bounded; and the result of Lemma (4.1), we could show that v_{xx}, v_{xy}, and v_{yy} are bounded in $\bar{\mathcal{D}}_3$. It follows that u_{xx}, u_{xy}, and u_{yy} are bounded in $\bar{\mathcal{D}}_3$. Since u_{xx}, u_{xy}, and u_{yy} are continuous in $\mathcal{D}_3$ by Theorem III, then u_{xx}, u_{xy}, and u_{yy} are bounded in the closure of $\mathcal{D}_3 - \bar{\mathcal{D}}_3$. Hence u_{xx}, u_{xy}, and u_{yy} are bounded in $\mathcal{D}_3$.

We now come to our final existence theorem.

Theorem V. Let ϕ, ψ_1, and ψ_2 satisfy the hypotheses of Theorem IV. Let ϕ, ψ_1, and ψ_2 also satisfy the following assumptions some of which are repetitions.

($1_A'$) ϕ, ϕ_x, and ϕ_{xx} are continuous and have continuous bounded first derivatives with respect to x and t. Also $|\phi_{xxx}(\bar{x},t)-\phi_{xxx}(x,t)| \leq L|\bar{x}-x|^i$ and $|\phi_{xxt}(\bar{x},t)-\phi_{xxt}(x,t)| \leq L|\bar{x}-x|^i$ for all $(\bar{x},t)$ and (x,t) in the domain of ϕ.

($2_A''$) ψ_1, ψ_{1x}, and ψ_{1t} are continuous. ψ_{1x} and ψ_{1t} are bounded and uniformly Hölder continuous.

($3_A''$) ψ_2, ψ_{2x}, and ψ_{2y} are continuous. ψ_{2x} and ψ_{2y} are bounded and uniformly Hölder continuous.

($3_B'$) $\psi_1(x,0) = \psi_2(x,0)$ and

$$\psi_{1t}(x,0) = \frac{1}{2\pi}\,\psi_{2x}(x,0) \iint\limits_{\eta\geq 0} g_y(x,0;\xi,\eta)\psi_2(\xi,\eta)d\xi d\eta$$

$$-\frac{1}{\pi}\,\psi_{2x}(x,0)\int_{-\infty}^{\infty}[\lambda^2\phi(\xi,0)-\phi_{xx}(\xi,0)]K(\lambda|\xi-x|)d\xi-\phi_x(x,0)\psi_{2y}(x,0)$$

for $(x,0)$ in the domain of both ψ_1 and ψ_2.

Then u satisfies ($4_A'$), ($4_B'$), ($4_C'$), and ($4_D'$) in $\mathcal{D}_3$.

($4_A'$) u and its first and second partial derivatives with respect to x and y are continuous, and they all have continuous first partial derivatives with respect to x, y, and t.

($4_B'$) $(\frac{\partial}{\partial t} - u_y \frac{\partial}{\partial x} + u_x \frac{\partial}{\partial y})(\triangle u - \lambda^2 u) = 0.$

($4_C'$) $u(x,0,t) = \phi(x,t), \triangle u(x,0,t) - \lambda^2 u(x,0,t) = \psi_1(x,t)$ when (x,t) is in the domain of ψ_1, and $\triangle u(x,y,0) - \lambda^2 u(x,y,0) = \psi_2(x,y)$.

($4_D'$) u(x,y,t)-ax-b and its first and second partial derivatives with respect to x and y are bounded.

Again we break up the proof of the theorem into several lemmas.

Let $\mathcal{F}_1$ and $\mathcal{F}_2$ be defined by $\mathcal{F}_1(x,y,t) = -u_y(x,y,t)$ and $\mathcal{F}_2(x,y,t) = u_x(x,y,t)$ for (x,y,t) in $\mathcal{D}_3$, and $\mathcal{F}_1(x,y,t) = u_y(x,-y,t) - 2u_y(x,0,t)$ and $\mathcal{F}_2(x,y,t) = 2\phi_x(x,t) - u_x(x,-y,t)$ when (x,-y,t) is in $\mathcal{D}_3$. Then $\mathcal{F}_1$ and $\mathcal{F}_2$ are continuous and have continuous first derivatives with respect to x and y. For (x_o,y_o,t_o) in the domain of $\mathcal{F}_1$ and $\mathcal{F}_2$ let X(t) and Y(t) be functions such that $X(t_o) = x_o$, $Y(t_o) = y_o$, and $\frac{dX(t)}{dt} = \mathcal{F}_1[X(t),Y(t),t]$ and $\frac{dY(t)}{dt} = \mathcal{F}_2[X(t),Y(t),t]$ for $0 \leq t \leq c_3$. X(t) and Y(t) exist for $0 \leq t \leq c_3$ since $\mathcal{F}_1$ and $\mathcal{F}_2$ are continuous and bounded. X and Y are unique since $\mathcal{F}_1$ and $\mathcal{F}_2$ have continuous bounded first derivatives with respect to x and y. Since X(t) and Y(t) also depend on (x_o,y_o,t_o), we use the notation $X(x_o,y_o,t_o,t)$ for X(t) and $Y(x_o,y_o,t_o,t)$ for Y(t). We also observe that $X(x_o,y_o,t_o,t)$ and $Y(x_o,y_o,t_o,t)$ have continuous bounded first derivatives since $\mathcal{F}_1$ and $\mathcal{F}_2$ have continuous bounded first derivatives with respect to x and y.

Let (x_o,y_o,t_o) be in $\mathcal{D}_3$ and let t vary in an interval containing t_o such that $[X(t),Y(t),t]$ is in $\mathcal{D}_3$. For such t we have $\mathcal{F}_i[X(t),Y(t),t] = F_i[X(t),Y(t),t]$ for $i=1,2$. Hence $X(x_o,y_o,t_o,t) = x(x_o,y_o,t_o,t)$ and $Y(x_o,y_o,t_o,t) = y(x_o,y_o,t_o,t)$ for such (x_o,y_o,t_o,t). Therefore $\alpha_o = X(x_o,y_o,t_o,\gamma_o)$ and $\beta_o = Y(x_o,y_o,t_o,\gamma_o)$ for (x_o,y_o,t_o) in $\mathcal{D}_3$.

<u>Lemma (5.1)</u>. α, β, and γ have continuous first derivatives at (x_o,y_o,t_o) in $\mathcal{D}_3$ if $\beta_o > 0$ or $\gamma_o > 0$ with $(\alpha_o,0,\gamma_o)$ not on C_1 or C_2.

<u>Proof of Lemma (5.1)</u>. We will show that γ has continuous derivatives at the points mentioned. Since $\alpha_o = X(x_o,y_o,t_o,\gamma_o)$ and $\beta_o = Y(x_o,y_o,t_o,\gamma_o)$, the conclusion regarding α and β follows from the fact that X and Y have continuous first derivatives.

Case I ($\beta_o > 0$). Since β is continuous at (x_o,y_o,t_o), we can choose a neighborhood R_δ of (x_o,y_o,t_o) such that $(\bar{x}_o,\bar{y}_o,\bar{t}_o)$ in R_δ implies $\bar{\beta}_o > 0$. In such a neighborhood we have $\bar{\gamma}_o = 0$ so that γ has continuous first derivatives at (x_o,y_o,t_o).

Case II ($\gamma_o > 0$ with $(\alpha_o,0,\gamma_o)$ not on C_1 or C_2). Since α and γ are continuous at (x_o,y_o,t_o), we can choose a neighborhood R_δ of (x_o,y_o,t_o) such that $(\bar{x}_o,\bar{y}_o,\bar{t}_o)$ in R_δ implies $\bar{\gamma}_o > 0$ and $(\bar{\alpha}_o,0,\bar{\gamma}_o)$ is not on C_1 or C_2. Hence $\bar{\beta}_o = Y(\bar{x}_o,\bar{y}_o,\bar{t}_o,\bar{\gamma}_o) = 0$ for $(\bar{x}_o,\bar{y}_o,\bar{t}_o)$ in R_δ. Since $Y_t(\bar{x}_o,\bar{y}_o,\bar{t}_o,\bar{\gamma}_o) = \phi_x(\bar{\alpha}_o,\bar{\gamma}_o) > 0$ for $(\bar{x}_o,\bar{y}_o,\bar{t}_o)$ in R_δ, we conclude from the implicit function theorem that γ has continuous first derivatives at (x_o,y_o,t_o).

<u>Lemma (5.2)</u>. h has continuous first derivatives in $\mathcal{D}_3$.

<u>Proof of Lemma (5.2)</u>. The proof follows from Lemma (5.1) if $\beta_o>0$ or $\gamma_o>0$ with $(\alpha_o,0,\gamma_o)$ not on C_1 or C_2 since then $h(x_o,y_o,t_o) = \psi_1(\alpha_o,\beta_o)$ or $\psi_2(\alpha_o,\gamma_o)$.

If $\beta_o = 0$ and $(\alpha_o, 0, \gamma_o)$ is on C_1 or C_2, then h is a constant in some neighborhood of (x_o, y_o, t_o).

The remaining case is where $\beta_o = \gamma_o = 0$ and $(\alpha_o, 0, 0)$ is not on C_1 or C_2. We note that α, β, and γ are continuous at (x_o, y_o, t_o) and $\phi_x(\alpha_o, 0) \neq 0$. Suppose there is a sequence $\{(x_n, y_o, t_o)\}$ of points in $\mathcal{D}_3$ such that $\gamma(x_n, y_o, t_o) = 0$, $x_n - x_o \neq 0$, and $x_n \to x_o$ as $n \to \infty$. Then

$$\frac{\alpha(x_n, y_o, t_o) - \alpha_o}{x_n - x_o} = \frac{X(x_n, y_o, t_o, 0) - X(x_o, y_o, t_o, 0)}{x_n - x_o} \to X_{x_o}(x_o, y_o, t_o, 0) \text{ as}$$

$n \to \infty$, $\dfrac{\beta(x_n, y_o, t_o) - \beta_o}{x_n - x_o} \to Y_{x_o}(x_o, y_o, t_o, 0)$ as $n \to \infty$, and hence

$$\frac{h(x_n, y_o, t_o) - h(x_o, y_o, t_o)}{x_n - x_o} = \frac{\psi_2[\alpha(x_n, y_o, t_o), \beta(x_n, y_o, t_o)] - \psi_2(\alpha_o, 0)}{x_n - x_o}$$

$$\to X_{x_o}(x_o, y_o, t_o, 0)\psi_{2x}(\alpha_o, 0) + Y_{x_o}(x_o, y_o, t_o, 0)\psi_{2y}(\alpha_o, 0) \text{ as } n \to \infty.$$

Suppose there is a sequence $\{(x_n, y_o, t_o)\}$ of points in $\mathcal{D}_3$ such that $\gamma(x_n, y_o, t_o) > 0$, $x_n - x_o \neq 0$, and $x_n \to x_o$ as $n \to \infty$. Then

$$\frac{Y[x_o, y_o, t_o, \gamma(x_n, y_o, t_o)]}{x_n - x_o} = \frac{Y[x_o, y_o, t_o, \gamma(x_n, y_o, t_o)] - Y(x_o, y_o, t_o, \gamma_o)}{x_n - x_o}$$

$= \dfrac{\gamma(x_n, y_o, t_o) - \gamma_o}{x_n - x_o} Y_t(x_o, y_o, t_o, \bar{\gamma}_n)$ where $\bar{\gamma}_n$ is between $\gamma(x_n, y_o, t_o)$ and γ_o.

Also $Y[x_n, y_o, t_o, \gamma(x_n, y_o, t_o)] = \beta(x_n, y_o, t_o) = 0$ and

$$\frac{Y[x_o, y_o, t_o, \gamma(x_n, y_o, t_o)]}{x_n - x_o} = \frac{Y[x_o, y_o, t_o, \gamma(x_n, y_o, t_o)] - Y[x_n, y_o, t_o, \gamma(x_n, y_o, t_o)]}{x_n - x_o}$$

$= - Y_{x_o}[\bar{x}_n, y_o, t_o, \gamma(x_n, y_o, t_o)]$ where $\bar{x}_n$ is between x_n and x_o.

Therefore $\dfrac{\gamma(x_n, y_o, t_o) - \gamma_o}{x_n - x_o} Y_t(x_o, y_o, t_o, \bar{\gamma}_n)$

$= - Y_{x_o}[\bar{x}_n, y_o, t_o, \gamma(x_n, y_o, t_o)]$. As $n \to \infty$ $Y_t(x_o, y_o, t_o, \bar{\gamma}_n)$

$\to Y_t(x_o, y_o, t_o, \gamma_o) \neq 0$. Therefore for n large enough we have

$Y_t(x_o,y_o,t_o,\bar{\gamma}_n) \neq 0$ and $\dfrac{\gamma(x_n,y_o,t_o)-\gamma_o}{x_n-x_o}$

$$= -\frac{Y_{x_o}[\bar{x}_n,y_o,t_o,\gamma(x_n,y_o,t_o)]}{Y_t(x_o,y_o,t_o,\bar{\gamma}_n)} \longrightarrow -\frac{Y_{x_o}(x_o,y_o,t_o,0)}{\phi_x(\alpha_o,0)} \text{ as } n \to \infty .$$

Also $\dfrac{\alpha(x_n,y_o,t_o)-\alpha_o}{x_n-x_o} = \dfrac{X[x_n,y_o,t_o,\gamma(x_n,y_o,t_o)]-X(x_o,y_o,t_o,\gamma_o)}{x_n-x_o}$

$$= X_{x_o}[\bar{x}_n,y_o,t_o,\gamma(x_n,y_o,t_o)] + \frac{\gamma(x_n,y_o,t_o)-\gamma_o}{x_n-x_o} X_t(x_o,y_o,t_o,\bar{\gamma}_n)$$

where $\bar{x}_n$ is between x_o and x_n and $\bar{\gamma}_n$ is between γ_o and $\gamma(x_n,y_o,t_o)$.

Hence $\dfrac{\alpha(x_n,y_o,t_o)-\alpha_o}{x_n-x_o} \to X_{x_o}(x_o,y_o,t_o,0) + \dfrac{Y_{x_o}(x_o,y_o,t_o,0)}{\phi_x(\alpha_o,0)} u_y(\alpha_o,0,0)$

as $n \to \infty$.

Let $\bar{\psi}_1(x,t) = \psi_1(x,t)$ when (x,t) is in the domain of ψ_1 and, when (x,t) is not in the domain of ψ_1, define $\bar{\psi}_1$ so that $\bar{\psi}_1$ is continuous and has continuous derivatives everywhere. Then

$$\frac{h(x_n,y_o,t_o)-h(x_o,y_o,t_o)}{x_n-x_o} = \frac{\bar{\psi}_1[\alpha(x_n,y_o,t_o),\gamma(x_n,y_o,t_o)]-\bar{\psi}_1(\alpha_o,\gamma_o)}{x_n-x_o}$$

$$= \frac{\alpha(x_n,y_o,t_o)-\alpha_o}{x_n-x_o} \bar{\psi}_{1x}[\bar{\alpha}_n,\gamma(x_n,y_o,t_o)] + \frac{\gamma(x_n,y_o,t_o)-\gamma_o}{x_n-x_o} \bar{\psi}_{1t}(\alpha_o,\bar{\gamma}_n)$$

(where $\bar{\alpha}_n$ is between $\alpha(x_n,y_o,t_o)$ and α_o and $\bar{\gamma}_n$ is between $\gamma(x_n,y_o,t_o)$ and γ_o)

$$\longrightarrow \left\{X_{x_o}(x_o,y_o,t_o,0) + \frac{Y_{x_o}(x_o,y_o,t_o,0)}{\phi_x(\alpha_o,0)} u_y(\alpha_o,0,0)\right\} \psi_{1x}(\alpha_o,0)$$

$$- \frac{Y_{x_o}(x_o,y_o,t_o,0)}{\phi_x(\alpha_o,0)} \psi_{1t}(\alpha_o,0) \text{ as } n \to \infty .$$

From $(3_B')$ of the theorem we obtain

$\psi_{1t}(\alpha_o,0) = \psi_{1x}(\alpha_o,0)u_y(\alpha_o,0,0)-\phi_x(\alpha_o,0)\psi_{2y}(\alpha_o,0)$. Hence

$$\frac{h(x_n,y_o,t_o)-h(x_o,y_o,t_o)}{x_n-x_o} \longrightarrow X_{x_o}(x_o,y_o,t_o,0)\psi_{2x}(\alpha_o,0)$$

$$+ Y_{x_o}(x_o,y_o,t_o,0)\psi_{2y}(\alpha_o,0) \text{ as } n \to \infty .$$

We may now conclude that $h_{x_o}(x_o,y_o,t_o)$ exists and

$$h_{x_o}(x_o,y_o,t_o) = X_{x_o}(x_o,y_o,t_o,0)\psi_{2x}(\alpha_o,0)+Y_{x_o}(x_o,y_o,t_o,0)\psi_{2y}(\alpha_o,0)$$

(we use h_{x_o}, h_{y_o}, and h_{t_o} to denote the derivatives of h since h was defined as a function of (x_o,y_o,t_o)).

The continuity of h_{x_o} in $\mathcal{D}_3$ follows easily using $(3_B')$ of the theorem.

Similarly we can show that h_{y_o} and h_{t_o} exist and are continuous

Lemma (5.3). The first partial derivatives of h are bounded in $\mathcal{D}_3$.

Proof of Lemma (5.3). By examining the expressions for the first derivatives of h we can easily show that the first derivatives are bounded in any set such that if (x_o,y_o,t_o) is the set and $\beta_o = 0$, then $\phi_x(\alpha_o,\gamma_o) \geq \bar{\omega}$. Since the set of points (x_o,y_o,t_o) for which $\beta_o = 0$ and $\phi_x(\alpha_o,\gamma_o) \leq \bar{\omega}$ is a bounded set, and since the first derivatives of h are continuous everywhere, it follows that the first derivatives of h are bounded.

Lemma (5.4). $(4_A')$ is valid in $\mathcal{D}_3$.

Proof of Lemma (5.4). We have already shown that u, u_x, u_y, u_{xx}, u_{xy}, and u_{yy} are continuous in $\mathcal{D}_3$. We have yet to show that u_t, u_{tx}, and u_{ty} exist and are continuous in $\mathcal{D}_3$ and that u_{xx}, u_{xy}, and u_{yy} have continuous first derivatives with respect to x, y, and t in $\mathcal{D}_3$.

We could show that w and its first and second derivatives with respect to x and y have continuous bounded first derivatives with respect to x, y, and t using the same methods used to prove Lemma (1.1).

In a straight forward manner we can show that v_t, v_{tx}, and v_{ty} exist and are continuous since h_t is continuous and bounded. Hence we may conclude that u_t, u_{tx}, and u_{ty} exist and are continuous.

Since h has bounded first derivatives, h is Hölder continuous in (x,y) where the Hölder continuity is uniform with respect to (x,y) and t. Hence, using (II.3), (II.4), and (II.5) of the proof of Theorem II, we can show that v_{xx}, v_{xy}, and v_{yy} are Hölder continuous in (x,y) where the Hölder continuity is uniform with respect to both (x,y) and t. This can be shown with arguments similar to those used in proving Lemma (1.3) for all the integrals excepting the last. We can show that the last integral has continuous bounded first derivatives with respect to x and y so the result follows for the last integral also.

Since w_{xx}, w_{xy}, and w_{yy} have bounded first derivatives with respect to x and y in $\mathcal{A}_3$, then w_{xx}, w_{xy}, and w_{yy} are Hölder continuous in (x,y) where the Hölder continuity is uniform with respect to both (x,y) and t.

Since $u = v-w+ax+b$, it follows in $\mathcal{A}_3$ that u_{xx}, u_{xy}, and u_{yy} are Hölder continuous in (x,y) and that the Hölder continuity is uniform with respect to both (x,y) and t.

Next we will show that the first derivatives of $X(x_o,y_o,t_o,t)$ and $Y(x_o,y_o,t_o,t)$ with respect to x_o, y_o, and t_o are Hölder continuous in (x_o,y_o) and that the Hölder continuity is uniform with respect to (x_o,y_o), t_o, and t. Let $(\bar{x}_o,\bar{y}_o,t_o)$ and (x_o,y_o,t_o) be any points in $\mathcal{D}_3$ with $s = \sqrt{(\bar{x}_o-x_o)^2+(\bar{y}_o-y_o)^2}$. Let $z_1(t) = |X_{x_o}(\bar{x}_o,\bar{y}_o,t_o,t)-X_{x_o}(x_o,y_o,t_o,t)|$ and $z_2(t) = |Y_{x_o}(\bar{x}_o,\bar{y}_o,t_o,t)-Y_{x_o}(x_o,y_o,t_o,t)|$. Then $X_{x_o}(x_o,y_o,t_o,t)$

$$= 1 + \int_{t_o}^{t} X_{x_o}(x_o,y_o,t_o,\xi)\mathcal{F}_{1x}[X(x_o,y_o,t_o,\xi),Y(x_o,y_o,t_o,\xi),\xi]d\xi$$

$$+ \int_{t_o}^{t} Y_{x_o}(x_o,y_o,t_o,\xi)\mathcal{F}_{1y}[X(x_o,y_o,t_o,\xi),Y(x_o,y_o,t_o,\xi),\xi]d\xi.$$ Then there are constants $\bar{M}$ and ε $(0 < \varepsilon < 1)$ such that for s small enough we have $z_1(t)$

$$= \left|\int_{t_o}^{t}[X_{x_o}(\bar{x}_o,\bar{y}_o,t_o,\xi)-X_{x_o}(x_o,y_o,t_o,\xi)]\mathcal{F}_{1x}[X(\bar{x}_o,\bar{y}_o,t_o,\xi),Y(\bar{x}_o,\bar{y}_o,t_o,\xi),\xi]d\xi\right.$$

$$+ \int_{t_o}^{t} X_{x_o}(x_o,y_o,t_o,\xi)\Big\{\mathcal{F}_{1x}[X(\bar{x}_o,\bar{y}_o,t_o,\xi),Y(\bar{x}_o,\bar{y}_o,t_o,\xi),\xi]$$

$$\left.-\mathcal{F}_{1x}[X(x_o,y_o,t_o,\xi),Y(x_o,y_o,t_o,\xi),\xi]\Big\}\, d\xi + \text{etc.}\right|$$

$$\leq \left|\int_{t_o}^{t} [\bar{M}z_1(\xi)+\bar{M}s^{\varepsilon}+\bar{M}z_2(\xi)+\bar{M}s^{\varepsilon}]d\xi\right| \leq \bar{M}\left|\int_{t_o}^{t} [z_1(\xi)+z_2(\xi)]d\xi\right|+2\bar{M}c_3s^{\varepsilon} .$$

Similarly $z_2(\xi) \leq \bar{M}\left|\int_{t_o}^{t} [z_1(\xi)+z_2(\xi)]d\xi\right|+2\bar{M}c_3s^{\varepsilon}$. Let

$R(t) = \left|\int_{t_o}^{t} [z_1(\xi)+z_2(\xi)d\xi\right|$. For $t \geq t_o$ we have

$$R'(t) \leq 4\bar{M}c_3s^{\varepsilon} + 2\bar{M}R(t),\ \frac{d}{dt} R(t)\, e^{-2\bar{M}(t-t_o)} \leq 4\bar{M}c_3s^{\varepsilon}\, e^{-2\bar{M}(t-t_o)} ,$$

$R(t)\, e^{-2\overline{M}(t-t_o)} \le -2c_3 s^{\varepsilon}(e^{-2\overline{M}(t-t_o)}-1)$, $R(t) \le 2c_3 s^{\varepsilon}(e^{2\overline{M}(t-t_o)}-1)$

$\le 2c_3 s^{\varepsilon}(e^{2\overline{M}c}-1)$. Thus $z_1(t) \le \overline{M}R(t)+2\overline{M}c_3 s^{\varepsilon} \le 2\overline{M}c_3 s^{\varepsilon} e^{2\overline{M}c}$. We obtain the same result when $t \le t_o$. In a similar way we can show that the other first derivatives of X and Y are Hölder continuous in (x_o,y_o) uniformly with respect to (x_o,y_o), t_o, and t.

Now we could show that in some neighborhood of a point $(\overline{x}_o,\overline{y}_o,\overline{t}_o)$ the first derivatives of $\alpha(x_o,y_o,t_o)$, $\beta(x_o,y_o,t_o)$, and $\gamma(x_o,y_o,t_o)$ with respect to x_o, y_o, and t_o are Hölder continuous in (x_o,y_o) where the Hölder continuity is uniform with respect to (x_o,y_o) and t_o provided that $\beta(\overline{x}_o,\overline{y}_o,\overline{t}_o) > 0$ or $\beta(\overline{x}_o,\overline{y}_o,\overline{t}_o) = 0$ with $[\alpha(\overline{x}_o,\overline{y}_o,\overline{t}_o),0,\gamma(\overline{x}_o,\overline{y}_o,\overline{t}_o)]$ not on C_1 or C_2.

Next we could show that in some neighborhood of each point in $\mathcal{D}_3$ the first derivatives of $h(\xi,\eta,t)$ are Hölder continuous in (ξ,η) where the Hölder continuity is uniform with respect to (ξ,η) and t.

For an arbitrary point (x_o,y_o,t) we have

$$(5.4.1)\qquad v_{xx}(x,y,t) = \frac{1}{2\pi}\iint_{\eta\ge 0} g_x(x,y;\xi,\eta)[h_\xi(\xi,\eta,t)+a\lambda^2]d\xi d\eta$$

$$= \frac{1}{2\pi}\iint_{\eta\ge 0} g_x(x,y;\xi,\eta)[h_\xi(\xi,\eta,t)-h_\xi(x_o,y_o,t)]d\xi d\eta,$$

$$(5.4.2)\qquad v_{xy}(x,y,t) = \frac{1}{2\pi}\iint_{\eta\ge 0} g_y(x,y;\xi,\eta)[h_\xi(\xi,\eta,t)+a\lambda^2]d\xi d\eta$$

$$= \frac{1}{2\pi}\iint_{\eta\ge 0} g_y(x,y;\xi,\eta)[h_\xi(\xi,\eta,t)-h_\xi(x_o,y_o,t)]d\xi d\eta$$

$$- h_\xi(x_o,y_o,t)\,\frac{1}{\pi}\int_{-\infty}^{\infty} K(\lambda\nu)d\xi \text{ , and}$$

$$(5.4.6) \quad v_{yy}(x,y,t) = -\frac{1}{2\pi}\iint\limits_{\eta\geq 0} g_\eta(x,y;\xi,\eta)h_\eta(\xi,\eta,t)d\xi d\eta$$

$$= -\frac{1}{2\pi}\iint\limits_{\eta\geq 0} g_\eta(x,y;\xi,\eta)[h_\eta(\xi,\eta,t)-h_\eta(x_o,y_o,t)]d\xi d\eta$$

$$-\frac{\lambda y}{\pi}\int_{-\infty}^{\infty}\frac{1}{\nu}K'(\lambda\nu)[h(\xi,0,t)+\lambda^2(a\xi+b)]d\xi d\eta \; .$$

Since h_ξ and h_η are Hölder continuous in (x,y) uniformly with respect to (x,y) and t for (x,y,t) in some neighborhood of (x_o,y_o,t), we can differentiate under the integral signs with respect to x and y at (x_o,y_o,t), and we can show that the resulting derivatives are continuous at (x_o,y_o,t).

We could show that we can differentiate under the integral sign with respect to t in (II.3), (II.4), and (II.5) (contained in the proof of Theorem II), and from the resulting expressions we could show that v_{txx}, v_{txy}, and v_{tyy} are continuous.

Finally it follows that u_{xx}, u_{xy}, and u_{yy} have continuous first derivatives with respect to x, y, and t. We remark that it would also be possible to show that the first derivatives of u_{xx}, u_{xy}, and u_{yy} with respect to x, y, and t are Hölder continuous in (x,y).

<u>Lemma (5.5)</u>. $(4_B')$ is valid in $\mathcal{S}_3$.

Lemma (5.5) is obvious since h is constant along the air particle paths of u.

We have previously shown that $(4_C')$ and $(4_D')$ are valid, and hence this completes the proof of Theorem V.

Part III

Uniqueness

Uniqueness Theorem. Let ϕ, ψ_1 and ψ_2 satisfy the hypothesis of Theorem V. Let $\bar{u}$ be any real valued function with domain $\mathcal{D}_3$ such that $(4_A')$, $(4_B')$, $(4_C')$, and $(4_D')$ are valid with u replaced by $\bar{u}$. Then $\bar{u} = u$ in $\mathcal{D}_3$.

Proof. Let $\bar{h} = \triangle\bar{u} - \lambda^2\bar{u}$. From $(4_D')$ we see that $\bar{u}(x,y,t)-ax-b$, $\bar{u}_x(x,y,t)-a$, $\bar{u}_y(x,y,t)$, and $\bar{h}(x,y,t)+\lambda^2(ax+b)$ are bounded in $\mathcal{D}_3$, and hence we can show that (3) is valid with u and h replaced by $\bar{u}$ and $\bar{h}$ respectively. This result follows from

$$\iint[\triangle\bar{u} - \lambda^2\bar{u} + \lambda^2(a\xi+b)]\ g(x,y;\xi,\eta)d\xi d\eta$$

$$=\iint[\bar{h}(\xi,\eta,t)+\lambda^2(a\xi+b)]g(x,y;\xi,\eta)d\xi d\eta = \int[g\frac{d(\bar{u}-a\xi-b)}{dn} - (\bar{u}-a\xi-b)\frac{dg}{dn}]ds$$

where the double integration is over the region defined by $\eta \geq 0$, $\xi^2 + \eta^2 \leq R^2$, and $\rho \leq \varepsilon$, and the single integral is taken along the boundary of the above region in the positive sense. Letting $R \to \infty$ and then $\varepsilon \to 0$ we obtain (3) with u and h replaced by $\bar{u}$ and $\bar{h}$ respectively.

Obtain functions $\bar{\mathcal{F}}_1$, $\bar{\mathcal{F}}_2$, $\bar{X}$, $\bar{Y}$, $\bar{\alpha}$, $\bar{\beta}$, and $\bar{\gamma}$ from $\bar{u}$ as $\mathcal{F}_1$, $\mathcal{F}_2$, X, Y, α, β, and γ respectively were obtained from u. Methods similar to those used previously can be used to show that $\bar{\mathcal{F}}_1$, $\bar{\mathcal{F}}_2$, $\bar{X}$, and $\bar{Y}$ have bounded first derivatives with respect to all their variables. Choose D to be an upper bound in $\mathcal{D}_3$ of the absolute values of $\mathcal{F}_1$, $\mathcal{F}_2$, $\bar{\mathcal{F}}_1$, $\bar{\mathcal{F}}_2$ and the first partial derivatives of $\mathcal{F}_1$, $\mathcal{F}_2$, $\bar{\mathcal{F}}_1$, $\bar{\mathcal{F}}_2$, X, $\bar{X}$, Y, $\bar{Y}$, and h.

Assume $u(x,y,t) \not\equiv \bar{u}(x,y,t)$ in $\mathcal{D}_3$. Let $c^* = \sup \bar{t}$ where the sup is taken over all $\bar{t} \geq 0$ such that $\bar{u}(x,y,t) = u(x,y,t)$ for (x,y,t) in $\mathcal{D}_3$ and $0 \leq t \leq \bar{t}$. Possibly $c^* = 0$. If $c^* = 0$, then $\bar{u}(x,y,c^*) = u(x,y,c^*)$ follows from (3). If $c^* > 0$, then $\bar{u}(x,y,c^*) = u(x,y,c^*)$ follows from the continuity of $\bar{u}$ and u and the fact that $\bar{u}(x,y,t) = u(x,y,t)$ for $0 \leq t < c^*$.

Assume $c^* < c_3$. Then we will arrive at a contradiction by showing that there is an $\varepsilon > 0$ such that $\bar{u}(x,y,t) \equiv u(x,y,t)$ for $c^* \leq t \leq c^*+\varepsilon$. It follows then that $c^* = c_3$, and Theorem VI is proved.

We have shown that h is identically p_i in some neighborhood of each point on C_i $(i=1,2)$. Hence we can choose $\delta_1 > 0$ so that $h(x,y,c^*) = p_i$ for $|x-x_i(c^*)| \leq \delta_1$ $(i=1,2)$ and $0 \leq y \leq \delta_1$, and also $h(x,0,t) = p_i$ when $|x-x_i(c^*)| \leq \delta_1$ $(i=1,2)$ and $c^* \leq t \leq c^*+\delta_1$. Then we choose $\delta_2 > 0$ so that $\delta_2 < \delta_1$ and $|x_i(t)-x_i(c^*)| \leq \frac{\delta_1}{3}$ $(i=1,2)$ for $c^* \leq t \leq c^*+\delta_2$. Then $h(x,0,t) = p_i$ when (x,t) is in the domain of ψ_1 if $|x-x_i(c^*)| \leq \delta_1$ $(i=1,2)$ and $c^* \leq t \leq c^*+\delta_2$.

Since $\bar{u}(x,y,t) = u(x,y,t)$ for $0 \leq t \leq c^*$, then $\bar{h}(x,y,c^*) = h(x,y,c^*)$. Also $\bar{h}(x,0,t) = \psi_1(x,t) = h(x,0,t)$ when (x,t) is in the domain of ψ_1. Therefore $\bar{h}(x,y,c^*) = p_i$ for $|x-x_i(c^*)| \leq \delta_1$ $(i=1,2)$ and $0 \leq y \leq \delta_1$, and $\bar{h}(x,0,t) = p_i$ when (x,t) is in the domain of ψ_1 if $|x-x_i(c^*)| \leq \delta_1$ $(i=1,2)$ and $c^* \leq t \leq c^*+\delta_2$.

Let $w^* = \text{glb}\ \phi_x(x,t)$ where the greatest lower bound is taken over all (x,t) such that $\phi_x(x,t) \geq 0$, $c^* \leq t \leq c^*+\delta_2$, and either $x \leq x_1(c^*) - \frac{2\delta_1}{3}$ or $x \geq x_2(c^*) + \frac{2\delta_1}{3}$. Then $w^* > 0$.

Choose $\varepsilon > 0$ so that $\varepsilon < \delta_2$, $3D\varepsilon(2D+1) \leq \frac{\omega^*}{2}$, $2D\varepsilon \leq \frac{\delta_1}{3}$, and $12MD\varepsilon(1+\frac{1}{\lambda^2})(1+\frac{2D+1}{\omega^*})e^{2Dc} \leq \frac{1}{3}$.

Let $N(\bar{u}-u) = \|\bar{u}_x-u_x\| + \|\bar{u}_y-u_y\|$ with $\|\bar{u}_x-u_x\| = \sup |\bar{u}_x(x,y,t)-u_x(x,y,t)|$ and $\|\bar{u}_y-u_y\| = \sup |\bar{u}_y(x,y,t)-u_y(x,y,t)|$ where the sup is taken over all (x,y,t) in $\mathcal{D}_3$ such that $c^* \leq t \leq c^*+\varepsilon$.

We now insert several lemmas.

Lemma (uT.1). $|\bar{X}(x_o,y_o,t_o,t)-X(x_o,y_o,t_o,t)| \leq 3\varepsilon e^{2Dc} N(\bar{u}-u)$ and $|\bar{Y}(x_o,y_o,t_o,t)-Y(x_o,y_o,t_o,t)| \leq 3\varepsilon e^{2Dc} N(\bar{u}-u)$ for $c^* \leq t_o \leq c^*+\varepsilon$ and $c^* \leq t \leq c^*+\varepsilon$.

Proof of Lemma (uT.1). For any fixed (x_o,y_o,t_o) with $c^* \leq t_o \leq c^*+\varepsilon$ let $z_1(t) = |\bar{X}(x_o,y_o,t_o,t)-X(x_o,y_o,t_o,t)|$ for $c^* \leq t \leq c^*+\varepsilon$ and $z_2(t) = |\bar{Y}(x_o,y_o,t_o,t)-Y(x_o,y_o,t_o,t)|$ for $c^* \leq t \leq c^*+\varepsilon$.

Then
$$z_1(t) = \Big|x_o+\int_{t_o}^{t} \bar{\mathcal{F}}_1[\bar{X}(x_o,y_o,t_o,\xi), \bar{Y}(x_o,y_o,t_o,\xi),\xi]d\xi - x_o - \int_{t_o}^{t} \mathcal{F}_1[X(x_o,y_o,t_o,\xi), Y(x_o,y_o,t_o,\xi),\xi]d\xi\Big|$$
$$\leq \Big|\int_{t_o}^{t}\Big\{\bar{\mathcal{F}}_1[\bar{X}(x_o,y_o,t_o,\xi),\bar{Y}(x_o,y_o,t_o,\xi),\xi]-\bar{\mathcal{F}}_1[X(x_o,y_o,t_o,\xi),Y(x_o,y_o,t_o,\xi),\xi]\Big\}d\xi\Big|$$
$$+ \Big|\int_{t_o}^{t}\Big\{\bar{\mathcal{F}}_1[X(x_o,y_o,t_o,\xi),Y(x_o,y_o,t_o,\xi),\xi] - \mathcal{F}_1[X(x_o,y_o,t_o,\xi), Y(x_o,y_o,t_o,\xi),\xi]\Big\} d\xi\Big|$$
$$\leq D\Big|\int_{t_o}^{t} [z_1(\xi)+z_2(\xi)]d\xi\Big| + 3\|\bar{u}_y-u_y\|\ |t-t_o|$$
$$\leq 3\varepsilon\ \|\bar{u}_y-u_y\| + D\Big|\int_{t_o}^{t} [z_1(\xi)+z_2(\xi)]d\xi\Big|.$$ Similarly we obtain

$z_2(t) \leq 3\varepsilon \|\bar{u}_x - u_x\| + D\left|\int_{t_0}^{t} [z_1(\xi)+z_2(\xi)]d\xi\right|$ so that

$z_1(t)+z_2(t) \leq 3\varepsilon N(\bar{u}-u)+2D\left|\int_{t_0}^{t} [z_1(\xi)+z_2(\xi)]d\xi\right|.$

Let $R(t) = \left|\int_{t_0}^{t} [z_1(\xi)+z_2(\xi)]d\xi\right|$ for $c^* \leq t \leq c^*+\varepsilon$. For $t \geq t_0$ we have $R'(t) = z_1(t)+z_2(t) \leq 3\varepsilon N(\bar{u}-u)+2DR(t)$,

$R'(t)-2DR(t) \leq 3\varepsilon N(\bar{u}-u)$, $\frac{d}{dt}[R(t)e^{-2D(t-t_0)}] \leq 3\varepsilon N(\bar{u}-u)e^{-2D(t-t_0)}$,

$R(t)e^{-2D(t-t_0)}-R(t_0) \leq -\frac{3\varepsilon}{2D} N(\bar{u}-u)[e^{-2D(t-t_0)}-1]$, and

$R(t) \leq \frac{3\varepsilon}{2D} N(\bar{u}-u)[e^{2D(t-t_0)}-1] \leq \frac{3\varepsilon}{2D} N(\bar{u}-u)(e^{2Dc}-1)$. Similarly we obtain the same result when $t \leq t_0$. Therefore $z_i(t) \leq z(t)+z_2(t) \leq 3\varepsilon N(\bar{u}-u)+2DR(t) \leq 3\varepsilon e^{2Dc} N(\bar{u}-u)$ for $i=1,2$.

<u>Lemma (Tu.2)</u>. $|\bar{h}(x_0,y_0,t_0)-h(x_0,y_0,t_0)| \leq 6D\varepsilon(1+\frac{2D+1}{\omega^*})e^{2Dc} N(\bar{u}-u)$ for (x_0,y_0,t_0) in $\mathcal{D}_3$ with $c^* \leq t_0 \leq c^*+\varepsilon$.

Let (x_0,y_0,t_0) be any point in $\mathcal{D}_3$ with $c^* \leq t_0 \leq c^*+\varepsilon$. If $y_0 > 0$ and $t_0 > c^*$ let t_b (t sub-boundary) be the largest number such that $c^* \leq t_b \leq t_0$ and $Y(x_0,y_0,t_0,t_b) = 0$. If no such t_b exists, let $t_b = c^*$.

If $y_0 = 0$, $t_0 > c^*$, and $\phi_x(x_0,t_0) \geq 0$, let $t_b = t_0$. If $\phi_x(x_0,t_0) < 0$, let t_b be the largest number such that $c^* \leq t_b < t_0$ and $Y(x_0,y_0,t_0,t_b) = 0$. If no such t_b exists let $t_b = c^*$.

If $t_0 = c^*$, let $t_b = c^*$.

Let $x_b = X(x_0,y_0,t_0,t_b)$ and $y_b = Y(x_0,y_0,t_0,t_b)$. Then (x_b,y_b,t_b) is a point where the air particle path of u enters the slab $c^* \leq t \leq c^*+\varepsilon$.

In a similar manner we obtain numbers $\bar{x}_b$, $\bar{y}_b$, and $\bar{t}_b$ using $\bar{X}$ and $\bar{Y}$.

Consider the case where $t_b > c^*$ and $\phi_x(x_b,t_b) \geq \omega^*$. Then for $c^* \leq t \leq c^*+\varepsilon$ we have $y_b = 0$ and $|Y_t(x_o,y_o,t_o,t)-Y_t(x_o,y_o,t_o,t_b)|$

$$\leq |\mathcal{F}_2[X(x_o,y_o,t_o,t),\ Y(x_o,y_o,t_o,t),t]$$
$$-\mathcal{F}_2[X(x_o,y_o,t_o,t),\ Y(x_o,y_o,t_o,t_b),t]|$$
$$+\ |\phi_x[X(x_o,y_o,t_o,t),t]-\phi_x(x_b,t_b)|$$
$$\leq 3D[\,|Y(x_o,y_o,t_o,t)-Y(x_o,y_o,t_o,t_b)|+|X(x_o,y_o,t_o,t)-x_b|+|t-t_b|\,]$$

$\leq 3D[2D+1]|t-t_b| \leq 3D\varepsilon\,(2D+1) \leq \frac{\omega^*}{2}$, and $Y_t(x_o,y_o,t_o,t)$ $\geq Y_t(x_o,y_o,t_o,t_b)-\frac{\omega^*}{2} \geq \frac{\omega^*}{2}$. Therefore if $\bar{t}_b \leq t_b$ we have

$$0 \leq t_b-\bar{t}_b = \frac{2}{\omega^*}\int_{\bar{t}_b}^{t_b} \frac{\omega^*}{2}\,d\xi \leq \frac{2}{\omega^*}\int_{\bar{t}_b}^{t_b} Y_t(x_o,y_o,t_o,\xi)d\xi$$

$$= -\frac{2}{\omega^*}Y(x_o,y_o,t_o,\bar{t}_b) \leq \frac{2}{\omega^*}\,|\bar{Y}(x_o,y_o,t_o,\bar{t}_b)-Y(x_o,y_o,t_o,\bar{t}_b)|$$

$\leq \frac{6\varepsilon}{\omega^*}e^{2Dc}N(\bar{u}-u)$. When $\bar{t}_b > t_b$ we have $\bar{y}_b = 0$ and

$$0 < \bar{t}_b-t_b = \frac{2}{\omega^*}\int_{t_b}^{\bar{t}_b} \frac{\omega^*}{2}\,d\xi \leq \frac{2}{\omega^*}\int_{t_b}^{\bar{t}_b} Y_t(x_o,y_o,t_o,\xi)d\xi$$

$$= \frac{2}{\omega^*}Y(x_o,y_o,t_o,\bar{t}_b) = \frac{2}{\omega^*}\,[Y(x_o,y_o,t_o,\bar{t}_b)-\bar{Y}(x_o,y_o,t_o,\bar{t}_b)]$$

$\leq \frac{6\varepsilon}{\omega^*}e^{2Dc}N(\bar{u}-u)$. Hence $|\bar{t}_b-t_b| \leq \frac{6\varepsilon}{\omega^*}e^{2Dc}N(\bar{u}-u)$. When $t_b > c^*$ and $\phi_x(x_b,t_b) \geq \omega^*$ we now have $|\bar{h}(x_o,y_o,t_o)-h(x_o,y_o,t_o)|$ $= |\bar{h}(\bar{x}_b,\bar{y}_b,\bar{t}_b)-h(x_b,y_b,t_b)| = |h(\bar{x}_b,\bar{y}_b,\bar{t}_b)-h(x_b,y_b,t_b)|$ (since $\bar{h}(x,y,c^*) = h(x,y,c^*)$ and $\bar{h}(x,0,t) = h(x,0,t)$ for (x,t) in the domain of ψ_1)

$$\leq D(|\bar{x}_b-x_b| + |\bar{y}_b-y_b| + |\bar{t}_b-t_b|) \leq D[\bar{X}(x_o,y_o,t_o,\bar{t}_b)-X(x_o,y_o,t_o,\bar{t}_b)|$$
$$+\ |X(x_o,y_o,t_o,\bar{t}_b)-X(x_o,y_o,t_o,t_b)|]$$
$$+\ D[\,|\bar{Y}(x_o,y_o,t_o,\bar{t}_b)-Y(x_o,y_o,t_o,\bar{t}_b)|$$
$$+\ |Y(x_o,y_o,t_o,\bar{t}_b)-Y(x_o,y_o,t_o,t_b)|] + D|\bar{t}_b-t_b|$$

$\leq 6D\varepsilon e^{2Dc} N(\bar{u}-u) + 2D^2|\bar{t}_b-t_b| + D|\bar{t}_b-t_b|$

$\leq 6D\varepsilon e^{2Dc} N(\bar{u}-u) + D(2D+1) \frac{6\varepsilon}{\omega^*} e^{2Dc} N(\bar{u}-u) = 6D\varepsilon(1+\frac{2D+1}{\omega^*})e^{2Dc} N(\bar{u}-u).$

Similarly we obtain the same result when $\bar{t}_b > c^*$ and $\phi_x(\bar{x}_b,\bar{t}_b) \geq \omega^*$.

When $t_b > c^*$, $\bar{t}_b > c^*$, $\phi_x(x_b,t_b) < \omega^*$, and $\phi_x(\bar{x}_b,\bar{t}_b) < \omega^*$, then $\bar{h}(x_o,y_o,t_o) = p_1$ or p_2 and $h(x_o,y_o,t_o) = p_1$ or p_2. Suppose $\bar{h}(x_o,y_o,t_o) = p_1$. Then $x_b = X(x_o,y_o,t_o,t_b)-X(x_o,y_o,t_o,t_o)$ $+ \bar{X}(x_o,y_o,t_o,t_o)-\bar{X}(x_o,y_o,t_o,\bar{t}_b)+\bar{x}_b \leq D|t_b-t_o| + D|\bar{t}_b-t_o| + \bar{x}_b$ $\leq 2D\varepsilon + x_1(c^*) + \frac{2\delta_1}{3} \leq x_1(c^*) + \delta_1$. Thus we must have $h(x_o,y_o,t_o)=p_1$. Similarly if $\bar{h}(x_o,y_o,t_o) = p_2$, then $h(x_o,y_o,t_o) = p_2$. Hence $|\bar{h}(x_o,y_o,t_o)-h(x_o,y_o,t_o)| = 0$ in this case.

Next we consider the case in which $t_b = c^*$, $\bar{t}_b > c^*$, and $\phi_x(\bar{x}_b,\bar{t}_b) < \omega^*$. Then $\bar{h}(x_o,y_o,t_o) = p_1$ or p_2. Assume $\bar{h}(x_o,y_o,t_o) = p_1$. Then $x_b \leq x_1(c^*)+\delta_1$ as in the previous case. Also $x_b = X(x_o,y_o,t_o,t_b)-X(x_o,y_o,t_o,t_o)+\bar{X}(x_o,y_o,t_o,t_o)$ $-\bar{X}(x_o,y_o,t_o,\bar{t}_b)+\bar{x}_b \geq -D|t_b-t_o|-D|\bar{t}_b-t_o|+\bar{x}_b \geq -2D\varepsilon+\bar{x}_b$ $\geq x_1(c^*) - \frac{2\delta_1}{3} - 2D\varepsilon \geq x_1(c^*)-\delta_1$, and $Y_b = Y(x_o,y_o,t_o,t_b)$ $-Y(x_o,y_o,t_o,t_o)+\bar{Y}(x_o,y_o,t_o,t_o)-\bar{Y}(x_o,y_o,t_o,\bar{t}_b)+\bar{y}_b \leq 2D\varepsilon+\bar{y}_b$ $= 2D\varepsilon$ (since $\bar{Y}_b = 0$). Hence $h(x_o,y_o,t_o)$ $h(x_b,y_b,t_b) = p_1$, and $|\bar{h}(x_o,y_o,t_o)-h(x_o,y_o,t_o)| = 0$. We get the same result when $\bar{h}(x_o,y_o,t_o) = p_2$.

Similarly when $t_b > c^*$, $\bar{t}_b = c^*$, and $\phi_x(x_b,t_b) < \omega^*$, then $|\bar{h}(x_o,y_o,t_o)-h(x_o,y_o,t_o)| = 0$.

The only remaining case is the one where $t_b = \bar{t}_b = c^*$. In this case $|\bar{h}(x_o,y_o,t_o)-h(x_o,y_o,t_o)| = |\bar{h}(\bar{x}_b,\bar{y}_b,c^*)-h(x_b,y_b,c^*)|$ $= |h(\bar{x}_b,\bar{y}_b,c^*)-h(x_b,y_b,c^*)| \leq D(|\bar{x}_b-x_b|+|\bar{y}_b-y_b|)$ $= D[|\bar{X}(x_o,y_o,t_o,c^*)-X(x_o,y_o,t_o,c^*)|+|\bar{Y}(x_o,y_o,t_o,c^*)$ $-Y(x_o,y_o,t_o,c^*)|] \leq 6D\varepsilon e^{2Dc} N(\bar{u}-u)$ from Lemma (uT.1).

This completes the proof of Lemma (uT.2).

We now continue the proof of our uniqueness theorem. Using (3) with $c^* \leq t \leq c^*+\varepsilon$ we obtain $|\bar{u}_x(x,y,t)-u_x(x,y,t)|$

$$= \left|\frac{1}{2\pi} \iint_{\eta \geq 0} g_x(x,y;\xi,\eta)\ [\bar{h}(\xi,\eta,t)-h(\xi,\eta,t)]d\xi d\eta\right|$$

$$\leq \frac{3D\varepsilon}{\pi} \left(1+\frac{2D+1}{\omega^*}\right) e^{2Dc}\ N(\bar{u}-u) \iint_{\eta \geq 0} |g_x(x,y;\xi,\eta)|d\xi d\eta$$

$\leq 12MD\,\varepsilon \left(1+\frac{2D+1}{\omega^*}\right)\left(1+\frac{1}{\lambda^2}\right) e^{2Dc}\ N(\bar{u}-u) \leq \frac{1}{3}\ N(\bar{u}-u)$ where we have used $\frac{M}{\pi} \iint_{\eta \geq 0} |g_x(x,y;\xi,\eta)|d\xi d\eta \leq 4M^2\left(1+\frac{1}{\lambda^2}\right)$ from the proof of Lemma (1.3). Therefore $\|\bar{u}_x-u_x\| \leq \frac{1}{3}\ N(\bar{u}-u)$.

Similarly $\|\bar{u}_y-u_y\| \leq \frac{1}{3}\ N(\bar{u}-u)$, and hence $N(\bar{u}-u) \leq \frac{2}{3}\ N(\bar{u}-u)$. It follows that $N(\bar{u}-u) = 0$, $\bar{u}_x(x,y,t) = u_x(x,y,t)$, and $\bar{u}_y(x,y,t) = u_y(x,y,t)$ for $c^* \leq t \leq c^*+\varepsilon$. Hence $\bar{u}(x,y,t) = u(x,y,t)+z(t)$ for $c^* \leq t \leq c^*+\varepsilon$ and for some function $z(t)$. Since $\bar{u}(x,0,t) = \phi(x,t) = u(x,0,t)$, then $z(t) = 0$ and $\bar{u}(x,y,t) = u(x,y,t)$ for $c^* \leq t \leq c^*+\varepsilon$. But this contradicts the choice of c^*. Hence $c^* = c_3$ and $\bar{u} = u$ in $\mathcal{D}_3$.

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