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FORCED VIBRATION OF BEAMS AND FLUTTER OF BEAM-FOIL SYSTEMS

by

E. H. Kennard
and
Ralph C. Leibowitz

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NOTATION

A	Cross-sectional area of stressed material
$a_{11}, a_{12}, a_{21}, a_{22}$	Influence or response coefficients (for certain cases these quantities will have a prime or a bar above them)
B	Lift constant of hydrofoil (equals $\pi \rho b \ell_f$)
b	Semichord length of hydrofoil
$b_{11}, b_{12}, b_{21}, b_{22}$	Inertia coefficients (for certain cases these quantities will have a prime or a bar above them)
C	Lift constant of hydrofoil (equals $B\left(\frac{3}{2}b + e\right)$); $\cosh q_2\ell$; a constant
c	Internal damping constant; $\cos q_1\ell$
c_1, c_2	Still-water damping constants (structural damping plus viscous damping at $S = 0$)
c_a	External damping constant
$\tilde{c}$	External damping constant
dx	Thickness of beam slice (differential length)
E	Young's modulus of elasticity; lift constant of hydrofoil $\left(\text{equals } B\left(\frac{b}{2} + e\right)^2\right)$
EI	Flexural rigidity of beam
e	A constant; base of natural logarithms (2.718.....); distance from midchord line to axis of rotation positive toward approaching stream
F	Force on beam
F_L	Oscillatory lift force on hydrofoil
G	Modulus of elasticity in shear; external moment or couple on beam positive in same direction of positive θ or $\frac{\partial v}{\partial x}$

G_e Equals G for shear modulus

h Distance from axis of rotation to effective center of mass of hydrofoil, positive when center of mass is displaced from axis toward the approaching stream, as in Figure 3

h_o Distance from axis of rotation to center of mass of rigid body, positive toward approaching stream, as in Figure 5

I Area moment of inertia of beam section about axis perpendicular to the plane of bending; effective moment of inertia of foil about axis through its center of mass and parallel to axis of rotation (used for foil not attached to beam)

I_o Effective moment of inertia of foil as just defined when foil is attached to a beam; similar moment of a rigid body

I_θ Equals $I_o + h^2 m$ for foil attached to a beam

i $\sqrt{-1}$

K Numerical factor dependent upon geometry of beam cross section; $K \leqq 1$

KAG Shear rigidity

k, k_1, k_2 Elastic constants of structure connecting foil to beam; k is also the reduced frequency $\frac{\omega b}{S}$

L A constant in the lift formula assumed equal to $\frac{1}{2} b - e$; distance from effective center of mass of foil to lift line

ℓ Length of beam

ℓ_f Length of hydrofoil

M Total moment acting on cross section of beam material lying toward $x = 0$, taken positive in same direction as G or θ

M_b Moment due to bending of beam

M_s Moment due to shear warping of beam

M_θ Total moment on hydrofoil about axis of rotation

m, m_o Effective mass of foil and of a rigid body, respectively (including virtual mass due to surrounding fluid)

P Shear force on beam taken positive with F and **v**

q, q_1, q_2 Equals $\left(\frac{\mu\omega^2}{EI}\right)^{1/4}$; $q\sqrt{\sqrt{1+\xi^2}+\xi}$; $q\sqrt{\sqrt{1+\xi^2}-\xi}$ respectively

S Uniform speed of stream of fluid; $\sinh q_2\ell$

S_c Critical flutter speed

s $\sin q_1\ell$

t Time

v Displacement of beam, positive in same direction as F; similar displacement of foil at axis of rotation

v_o Equals v at $x = 0$, or at line of attachment of foil

x, y Rectangular coordinates with x axis always parallel to the beam axis and y along a perpendicular principal axis

γ When vibrations are due to bending flexibility only, ($KAG = \infty$) γ is the equivalent rotation of a cross section of the beam about an axis perpendicular to the xy-plane due to bending and rigid-body motion; positive in direction of positive $\frac{dy}{dx}$

For a vibrating beam with finite bending and shear flexibility, γ is no longer the slope of the beam but represents an equivalent rotation of the cross section about an axis perpendicular to the xy-plane. For the meaning of equivalent rotation, see Appendix A and Equation [A27] of Reference 11

ϵ Bending strain; a number allowing for effect of variation in E or shape of cross section

ζ Specific gravity of fluid

η Damping constant; equals $\frac{1}{\lambda^2} \frac{\rho_b A}{\rho'_b A'}$ (See Section 7)

θ Equals $\frac{\partial v}{\partial x}$; angle of rotation of hydrofoil about axis of rotation lying in plane of foil and parallel to its length, positive in same direction as G, and measured from zero when both foil and beam are in their neutral positions

θ_o Equals $\frac{\partial v}{\partial x}$ at $x = 0$; local rotation of beam; angle of rotation of a rigid body; positive like θ

λ $\pm iq_1$, or $\pm q_2$; ratio of length of any beam to standard beam

μ Effective mass per unit length of beam (including an appropriate allowance for virtual mass of surrounding water)

ξ Equals $\frac{\mu\omega^2}{q^2} \sigma = \sigma EIq^2$

ρ, ρ_o Density of fluid bathing the foil and density of pure water, respectively

σ Bending stress; equals $\frac{1}{2KAG}$

τ A number allowing for the effect of a change of either beam density or cross-sectional area of beam

$\bar{\Omega}$ Equals $\mu(\omega^2 - i\omega\tilde{c})$

ω Circular frequency of vibration

ABSTRACT

Equations are developed for two-dimensional harmonic transverse forcing at any point of an undamped uniform beam having bending and shearing flexibilities and for harmonic end forcing of a uniform beam having also external and internal damping. Variations of beam influence and inertia coefficients with frequency are discussed for an undamped beam without shear warping. Equations are also developed for the determination of the flutter speed of a rigid hydrofoil flexibly attached to such a beam. The flutter process is illustrated by a detailed discussion of two simple cases of flutter.

1. INTRODUCTION

Sea trials of USS FOREST SHERMAN (DD 931) indicated that during a steady horizontal maneuver severe vibrations were transmitted by the rudders to the hull.[1] To anticipate the occurrence of severe vibrations and possible control-surface flutter within the range of ship operating speeds, the authors undertook a series of studies of the flutter phenomenon.[2] A theory was advanced for treating the vibration characteristics of a control surface (e.g., rudder or diving plane) hull system subject to hydrodynamic forces on the control surfaces.[3*] This theory has

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1 References are listed on page 92.

* This theory has been applied to USS ALBACORE (AGSS 569) and to the motor gunboat PGM. The results for these two cases will be reported separately.

been shown to have some degree of verification.[4] The method of determination of the hydroelastic parameters for these control surfaces is given in References 5, 6, and 7. Methods for determining certain damping terms from observations, originally omitted from the flutter equations in Reference 2, can now be included and are given in Reference 8. Related studies of the static and dynamic loads on the rudder of a ship during a steady horizontal maneuver, which is also of interest in the treatment of flutter, were undertaken in References 9 and 10.

The Bureau of Ships, recognizing the parallel need for the exploration of the possibility of the occurrence of flutter in hydrofoil craft, requested the David Taylor Model Basin to undertake a similar hydroelastic study for these craft. To achieve this objective, in this report equations are derived for predicting the critical flutter speed of a rigid foil flexibly attached to a uniform mass-elastic (i.e., nonrigid) free-free beam immersed in a fluid moving with uniform velocity. The analysis includes the two-dimensional quasi-steady expression for hydrodynamic force and moment on the foil. The relative influences upon the vibrations, critical flutter speeds, and frequencies of the values of the various parameters of the beam-foil system are discussed in detail. Certain special cases are also considered. A survey of the work performed in this report is given in the Summary.

2. HARMONIC END FORCING OF UNDAMPED UNIFORM BEAM WITH SHEAR WARPING

Consider a uniform beam that is maintained in vibration by a transverse force F and a couple G, acting in a principal plane containing the principal axes of all beam cross sections; see p. 188 of Reference 11. It will be assumed that F and G are both harmonic functions of the time; nonharmonic forcing is much more complicated.

In developing the theory, F and G will be assumed to act at one end of the beam, taken as $x = 0$; generalization for other positions will be given later. Furthermore, F and G will be assumed to vary at the same frequency and in phase, so that

$$F = A \sin \omega t; \qquad G = B \sin \omega t \qquad [1a,b]$$

in terms of constants A and B. The resulting formulas can then be applied to a case in which the frequencies of F and G differ by first putting $B = 0$, then $A = 0$, and adding the two motions thus obtained. If, on the other hand, F and G have the same frequency but differ in phase ($F = A \sin \omega t$, $G = B \sin(\omega t + \phi)$), it is readily seen in the same way that the general formulas are all valid as they stand (hence, for instance, in Equations [12a,b] v_o and θ_o are intermediate in phase between F and G).

Let $v(x, t)$ denote the displacement of the beam, positive in the same direction as F, and let G be positive in the same direction as positive $\partial v/\partial x$. F and G are assumed to be applied in such a way that significant *distortion* of the end of the beam, near $x = 0$, is avoided except for the normal distortions of cross sections accompanying bending and shear. Let the other end of the beam, at $x = \ell$, be entirely free.

The *general* harmonic case of a transverse force and couple acting on one end of the beam can be resolved into *two* sets of forces and couples like the set just described, with the two sets acting in *perpendicular principal planes*.

Although *nonharmonic* forcing is much more complicated, solutions to such forcing can be found using Fourier Series or Fourier Integral methods.

Basic equations for the harmonic forcing are developed, including the shear-warping effect; however, rotary inertia is ignored because of its small effect in practice (see p. 187 of Reference 11).

Let M denote the moment acting across any cross section on material lying toward $x = 0$, taken positive in the same direction as G or θ, where θ denotes $\partial v/\partial x$ for convenience, P the corresponding shear force taken positive with F and v, and μ the mass of the beam per unit length. Then the appropriate differential equations are as follows (see Equations [2.12] to [2.15] in Reference 11, where $V = -P$ because V was taken positive toward negative v, and the I_{mz} term is to be omitted); the reader can also find the basis for these equations in many texts on vibration theory:

$$\mu \frac{\partial^2 v}{\partial t^2} = \frac{\partial P}{\partial x} \quad ; \quad \frac{\partial M}{\partial x} = -P \qquad [2a,b]$$

$$\theta = \frac{\partial v}{\partial x} = \gamma + 2\sigma P; \quad \frac{\partial \gamma}{\partial x} = \frac{M}{EI} \qquad [2c,d]$$

Here $\frac{1}{2\sigma}$ is written for the constant usually denoted by KAG, the shear rigidity, and this parameter as well as μ and EI are assumed ***uniform*** along the beam.

In harmonic vibration, Equation [2a] becomes:

$$\mu\omega^2 v = -\frac{\partial P}{\partial x}$$

An equation containing M alone is easily deduced, thus:

$$P = -\frac{\partial M}{\partial x}\ ; \quad \therefore\ \mu\omega^2 v = \frac{\partial^2 M}{\partial x^2}$$

$$\gamma = \frac{\partial v}{\partial x} - 2\sigma P;\quad \therefore\ \frac{\partial \gamma}{\partial x} = \frac{\partial^2 v}{\partial x^2} + 2\sigma\frac{\partial^2 M}{\partial x^2}$$

and, substituting this last expression for $\partial\gamma/\partial x$ in Equation [2d] and then substituting for v yields:

$$\frac{M}{EI} = \frac{1}{\mu\omega^2}\frac{\partial^4 M}{\partial x^4} + 2\sigma\frac{\partial^2 M}{\partial x^2}$$

or

$$\frac{\partial^4 M}{\partial x^4} + 2\sigma\mu\omega^2\frac{\partial^2 M}{\partial x^2} - \frac{\mu\omega^2}{EI} M = 0 \qquad [3a]$$

For convenience, introduce q and ξ, defined thus:*

$$q^4 = \frac{\mu\omega^2}{EI}\ ; \quad \xi = \frac{\mu\omega^2}{q^2}\sigma = \sigma EIq^2 \qquad [4a,b]$$

Then, substituting $M \propto e^{\lambda x}\sin\omega t$ and also $2\sigma\mu\omega^2 = 2q^2\xi$ into Equation [3a], and dividing through by M, yields the following equation as the necessary and sufficient condition for λ:

$$\lambda^4 + 2q^2\xi\lambda^2 - q^4 = 0$$

Thus, by solving $\lambda^2 = -q^2\xi \pm \sqrt{q^4\xi^2 + q^4}$ and taking first the minus and then the plus sign before the radical and again extracting the square root, it is found that either $\lambda = \pm iq_1$, or $\lambda = \pm q_2$, where

$$q_1 = q\sqrt{\sqrt{1+\xi^2} + \xi}\ ; \quad q_2 = q\sqrt{\sqrt{1+\xi^2} - \xi} \qquad [5a,b]$$

*Compare with equations in Appendix [D2] of Reference 11.

All square roots are meant to be *positive*. The values $\lambda = \pm iq_1$ lead to $\sin q_1x$ and $\cos q_1x$ as alternative real factors, whereas $\lambda = \pm q_2$ leads to $\sinh q_2x$ and $\cosh q_2x$.

The following useful auxiliary formulas are easily obtained:

$$q_1^2 + q_2^2 = 2q^2\sqrt{1+\xi^2}\ ; \quad q_1^2 - q_2^2 = 2q^2\xi; \quad q_1q_2 = q^2 \qquad [6a,b,c]$$

$$\frac{q_1^3}{q_2} + \frac{q_2^3}{q_1} = \frac{q_1^4 + q_2^4}{q_1q_2} \quad ; \quad q_1^4 + q_2^4 = 2q^4(1+2\xi^2)\ . \qquad [6d,e]$$

Thus, in harmonic motion, M can be expressed in terms of the four independent solutions of Equation [3a] as follows:

$$M = e_1 \sin q_1x + e_2 \cos q_1x + e_3 \sinh q_2x + e_4 \cosh q_2x \quad [7]$$

in which each coefficient e_n is the product of $\sin \omega t$ and an arbitrary constant. Corresponding series for P, v, and $\partial v/\partial x$ or θ are easily obtained by substitution for M in equations previously written, differentiating the series for v to obtain θ.

At $x = \ell$, $P = M = 0$; whereas at $x = 0$, $P = -F$ and $M = -G$, since F and G represent actions on material lying toward positive x from the terminal cross section and P and M represent actions toward negative x. Thus, the boundary conditions for the present problem are* (since $P = -\partial M/\partial x$):

$$\text{at } x = 0: \qquad \frac{\partial M}{\partial x} = F \quad ; \quad M = -G$$

$$\text{at } x = \ell: \qquad \frac{\partial M}{\partial x} = 0 \quad ; \quad M = 0$$

Substituting the series for M and then setting first $x = 0$ and then $x = \ell$ gives the four equations:

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* Compare with Equations [D8] of Reference 11.

$$q_1 e_1 + q_2 e_3 = F \quad ; \quad e_2 + e_4 = -G \qquad [8a,b]$$

$$q_1 e_1 \cos q_1 \ell - q_1 e_2 \sin q_1 \ell + q_2 e_3 \cosh q_2 \ell + q_2 e_4 \sinh q_2 \ell = 0 \qquad [8c]$$

$$e_1 \sin q_1 \ell + e_2 \cos q_1 \ell + e_3 \sinh q_2 \ell + e_4 \cosh q_2 \ell = 0 \qquad [8d]$$

These equations can now be solved for e. It appears to be more useful, however, to relate F and G to values of v and $\partial v/\partial x$ at $x = 0$, denoted, respectively, by v_o and θ_o. Since $\mu\omega^2 v = EIq^4 v = \partial^2 M/\partial x^2$,

$$EIq^4 v_o = \left(\frac{\partial^2 M}{\partial x^2}\right)_{x=0} \quad ; \quad EIq^4 \theta_o = \left(\frac{\partial^3 M}{\partial x^3}\right)_{x=0}$$

or, substituting the series for M;

$$EIq^4 v_o = -q_1^2 e_2 + q_2^2 e_4 \qquad [9a]$$

$$EIq^4 \theta_o = -q_1^3 e_1 + q_2^3 e_3 \qquad [9b]$$

Solving Equations [8a,b] and [9a,b] for e gives:

$$q_1(q_1^2 + q_2^2)e_1 = q_2^2 F - EIq^4 \theta_o$$

$$(q_1^2 + q_2^2)e_2 = -q_2^2 G - EIq^4 v_o$$

$$q_2(q_1^2 + q_2^2)e_3 = q_1^2 F + EIq^4 \theta_o$$

$$(q_1^2 + q_2^2)e_4 = -q_1^2 G + EIq^4 v_o$$

These values of e may then be substituted into Equations [8c,d] and multiplied through for convenience by $q_1^2 + q_2^2$. Hereafter, it will also be convenient to shorten the notation by writing:

$$s = \sin q_1 \ell \quad ; \quad c = \cos q_1 \ell \quad ; \quad S = \sinh q_2 \ell \quad ; \quad C = \cosh q_2 \ell$$

Then Equations [8c,d], taken in reverse order, may be written:

$$(C-c)EIq^4 v_o + \left(\frac{S}{q_2} - \frac{s}{q_1}\right)EIq^4 \theta_o = -\left(\frac{q_1^2}{q_2}S + \frac{q_2^2}{q_1}s\right)F + (q_1^2 C + q_2^2 c)G \qquad [10a]$$

$$(q_2 S + q_1 s)EIq^4 v_o + (C-c)EIq^4 \theta_o = -(q_1^2 C + q_2^2 c)F + q_1 q_2(q_1 S - q_2 s)G \qquad [10b]$$

Finally, these equations may be solved, for convenience, either for v_o and θ_o in terms of F and G, or for F and G in terms of v_o and θ_o. For this solution, use may be made of Equations [6a-e]. Furthermore, besides verifying that $c^2 + s^2 = 1$ and $C^2 - S^2 = 1$, it can be easily verified that:

$$(C \pm c)^2 - (S^2 - s^2) = 2(1 \pm cC) \qquad [11a]$$

$$s^2 C^2 - c^2 S^2 - s^2 S^2 = 1 - c^2 C^2 \qquad [11b]$$

The resulting alternative sets of formulas are:*

$$v_o = a_{11}F + a_{12}G; \qquad \theta_o = a_{21}F + a_{22}G \qquad [12a,b]$$

$$a_{11} = -\frac{1}{D_a}\sqrt{1+\xi^2}\left(\frac{1}{q_1}sC - \frac{1}{q_2}cS\right)\frac{1}{EIq^2}$$

$$a_{12} = \frac{1}{D_a}[sS + \xi(1 - cC)]\frac{1}{EIq^2}$$

$$a_{21} = \frac{1}{D_a}[(1 + 2\xi^2)sS - \xi(1-cC)]\frac{1}{EIq^2}$$

$$a_{22} = -\frac{1}{D_a}\sqrt{1 + \xi^2}(q_1 sC + q_2 cS)\frac{1}{EIq^2}$$

$$D_a = 1 - cC - \xi sS$$

* Compare with Equations [A67a,b] of Reference 11 with $Y = \Gamma = 0$.

$$F = b_{11}v_o + b_{12}\theta_o; \quad G = b_{21}v_o + b_{22}\theta_o \qquad [13a,b]$$

$$b_{11} = -\frac{1}{D_b}\sqrt{1+\xi^2}\,(q_1 sC + q_2 cS)\,EIq^2$$

$$b_{12} = -\frac{1}{D_b}[sS + \xi(1-cC)]\,EIq^2$$

$$b_{21} = -\frac{1}{D_b}[(1+2\xi^2)sS - \xi(1-cC)]\,EIq^2$$

$$b_{22} = -\frac{1}{Db}\sqrt{1+\xi^2}\,\left(\frac{1}{q_1}sC - \frac{1}{q_2}cS\right)\,EIq^2$$

$$D_b = 1 + cC + \xi sS + 2\xi^2$$

Here both q and ξ increase as ω increases. ξ is a dimensionless quantity and may serve as a relative measure of the effect of shear warping on the vibrations of the beam. *The increase of ξ with increasing ω means that the effect of shear warping increases with increasing frequency, as is well known in other connections.* *

Using a common terminology, the coefficients a_{11}, a_{12}, a_{21}, and a_{22} may be called *influence coefficients* for forcing of the beam at one end. Similarly b_{11}, b_{12}, b_{21}, and b_{22} might be called *inertia coefficients.* ** Because of the factors $\sin q_1\ell$ and $\cos q_1\ell$ in the formulas, the values of both sets of coefficients, as ω increases from 0, range in roughly cyclic fashion over all values between $-\infty$ and $+\infty$.

These coefficients may be of use in treating the vibrations of a system composed of a uniform beam attached at one end to another structure.

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* See pp. 9 and 180 of Reference 11.

** The a_{ij}'s refer to the *influence* of forces in producing vibrational displacements; the b_{ij}'s to the effects of *beam mass* in making F and G necessary.

3. HARMONIC END FORCING OF AN UNDAMPED UNIFORM BEAM WITHOUT SHEAR WARPING

Because of the two independent parameters q and ξ in the formulas, the discussion of special cases is complicated. Accordingly, the variation of the coefficients with ω will be discussed in greater detail only in the case of $\xi = 0$; that is, when shear warping is neglected.

When $\xi = 0$, the formulas become, with two useful additions:

$$v_o = a_{11}F + a_{12}G \ ; \qquad \theta_o = a_{12}F + a_{22}G \qquad [14a,b]$$

$$a_{11} = -\frac{sC-cS}{1-cC}\,\frac{1}{EIq^3} \ ; \quad a_{12} = a_{21} = \frac{sS}{1-cC}\,\frac{1}{EIq^2}$$

$$a_{22} = -\frac{sC+cS}{1-cC}\,\frac{1}{EIq} \ ; \quad a_{11}a_{22} - a_{12}^2 = \frac{1+cC}{1-cC}\,\frac{1}{(EI)^2q^4}$$

$$F = b_{11}v_o + b_{12}\theta_o \ ; \qquad G = b_{12}v_o + b_{22}\theta_o \qquad [15a,b]$$

$$b_{11} = -\frac{sC+cS}{1+cC}\,EIq^3 \ ; \quad b_{12} = b_{21} = -\frac{sS}{1+cC}\,EIq^2$$

$$b_{22} = -\frac{sC-cS}{1+cC}\,EIq \ ; \quad b_{11}b_{22}-b_{12}^2 = \frac{1-cC}{1+cC}\,(EI)^2q^4$$

The added formulas are obtained with the use of Formula [11b]. It will be noted that when $\xi = 0$, $a_{21} = a_{12}$ and $b_{21} = b_{12}$.

As $q\ell$ increases, sinh $q\ell$ and cosh $q\ell$ soon become large and nearly equal; for example, sinh 3 = 10.018, cosh 3 = 10.068. Hence, at least from $q\ell = 2\pi$ upward, the following simplified formulas may be sufficiently accurate; they are obtained by dividing numerator and denominator by C and then replacing S/C by unity, provided $\xi = 0$:

$$a_{11} \doteq \frac{s-c}{c-(1/C)} \frac{1}{EIq^3} \; ; \quad a_{12} = a_{21} = -\frac{s}{c-(1/C)} \frac{1}{EIq^2}$$

$$a_{22} \doteq -\frac{s+c}{c-(1/C)} \frac{1}{EIq} \; ; \quad a_{11}a_{22} - a_{12}^2 = -\frac{c+(1/C)}{c-(1/C)} \frac{1}{(EI)^2q^4}$$

If also $|c| >> 1/C$, the still simpler approximations become available:

$$a_{11} \doteq (\tan q\ell - 1) \frac{1}{EIq^3} \; ; \quad a_{12} = a_{21} \doteq -\tan q\ell \frac{1}{EIq^2}$$

$$a_{22} \doteq (\tan q\ell + 1) \frac{1}{EIq} \; ; \quad a_{11}a_{22} - a_{12}^2 = -\frac{1}{(EI)^2q^4}$$

Several interesting cases are now discussed in the order of increasing $q\ell$, assuming that $\xi = 0$.

(a) *Uniform rigid beam, $q\ell$ << 1:* Smallness of q may arise from *smallness of either* ℓ or ω, the latter occurring in q. Then use may be made of the series (note that when $\xi = 0$, $q_1 = q_2 = q$):

$$s = \sin q\ell = q\ell - \frac{1}{6}(q\ell)^3 \ldots ; \; c = \cos q\ell = 1 - \frac{1}{2}(q\ell)^2 + \frac{1}{24}(q\ell)^4 \ldots$$

$$S = \sinh q\ell = q\ell + \frac{1}{6}(q\ell)^3 \ldots ; C = \cosh q\ell = 1 + \frac{1}{2}(q\ell)^2 + \frac{1}{24}(q\ell)^4 \ldots$$

Keeping only the lowest power of $q\ell$:

$$1 + cC \doteq 2; \qquad 1 - cC \doteq \frac{1}{6}(q\ell)^4; \qquad sS \doteq (q\ell)^2$$

$$sC + cS \doteq 2q\ell; \qquad sC - cS \doteq \frac{2}{3}(q\ell)^3$$

In this case, it may be enlightening to return also to $\mu\omega^2$ by substituting $q^4 = \frac{\mu\omega^2}{EI}$. Then, provided $q\ell$ << 1:

$$a_{11} \doteq -\frac{4}{EIq^4\ell} = -\frac{4}{\mu\ell\omega^2}\ ; \qquad a_{12} = a_{21} \doteq \frac{6}{EIq^4\ell^2} = \frac{6}{\mu\ell^2\omega^2}\ ;$$

$$a_{22} \doteq -\frac{12}{EIq^4\ell^3} = -\frac{12}{\mu\ell^3\omega^2}$$

$$b_{11} \doteq -\ \mu\ell\omega^2\ ; \quad b_{12} = b_{21} \doteq -\frac{1}{2}\mu\ell^2\omega^2\ ;\ b_{22} = -\frac{1}{3}\mu\ell^3\omega^2$$

In b_{11}, $\mu\ell$ is the total mass of the beam; in b_{22}, $\frac{1}{3}\mu\ell^3$ is its moment of inertia as a rigid body rotating about one end; in b_{12}, $\frac{1}{2}\cdot\mu\ell^2$ equals mass times distance from either end of the beam to the center of mass. The validity of Equations [15a,b] for F and G in this approximation is easily verified from elementary mechanics. Thus, when $q\ell << 1$, the beam behaves, as it should, approximately like a uniform rigid rod of length ℓ and mass $\mu\ell$.

As $q\ell \to 0$, a_{11}, a_{12}, a_{22}, and $a_{11}a_{22} - a_{12}^2$ all become infinite, whereas b_{11}, b_{12}, b_{22}, and $b_{11}b_{22} - b_{12}^2$ all go to zero.

In the following discussion of other cases, the notation is often shortened by denoting the first fraction in each formula preceding the EI fraction by the symbol for the coefficient with a bar over it. For example, $\bar{a}_{11} = -\dfrac{sC - cS}{1 - cC}$. Also, $\bar{A} = \dfrac{1 + cC}{1 - cC}$; or if $q\ell >> 1$,

$\bar{a}_{11} = \dfrac{s - c}{c - (1/C)}$ and $\bar{A} = -\dfrac{c + (1/C)}{c - (1/C)}$; also if $|c| >> (1/C)$,

$\bar{a}_{11} \doteq \tan q\ell - 1$ and $\bar{A} \doteq -1$.

(b) *First "built-in" frequency:* When $q\ell = 0.597\pi = 1.875$, calculation shows that $cC = \cos q\ell \cosh q\ell = -1$. This is the familiar frequency equation for a uniform beam vibrating with one end built in and the other end free.* Calculation from the formulas following Equations [14a,b] gives:

$$\bar{a}_{11} = -2.069; \quad \bar{a}_{12} = 1.520; \quad \bar{a}_{22} = -1.116; \quad \bar{A} = 0$$

The corresponding values of b_{11}, b_{12}, and b_{22} are all infinite. It does not follow, however, from Equations [15a,b] that v_o and θ_o are necessarily zero. This would be the case if only one of these variables were present; for Equations [14a,b] furnish finite values of v_o and θ_o to match any finite values of F and G. The ratio θ_o/v_o, however, is fixed and cannot be varied by changing F and G. For, since $\bar{A} = 0$, $a_{11}a_{22} - a_{12}^2 = 0$ and, hence:

$$\frac{a_{12}}{a_{11}} = \frac{a_{22}}{a_{12}} = -0.735\ q$$

Thus, the ratio of the right-hand members of Equations [14a,b] is -0.735 q for all finite values of F and G; it follows that $\theta_o/v_o = -0.735\ q$ also.

If, however, the ratio F/G = + 0.735 q, then $F/G = -a_{12}/a_{11} = -a_{22}/a_{12}$, so that $a_{11}F + a_{12}G = a_{12}F + a_{22}G = 0$ and Equations [14a,b] give $v_o = \theta_o = 0$. The beam is then vibrating as if it were built-in at x = 0. If it actually is built-in, F/G or 0.735 q is the ratio of the reactions -F and -G on the supporting structure.

Further illumination results from considering what happens when cC is merely very close to -1. Then b_{11}, b_{12}, and b_{22} are finite but large so that, in general, large forces are required to produce arbitrarily assigned values of v_o and θ_o. The same conclusion can be derived from

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* The equation cC = -1 is assumed to be exactly true, the value of $q\ell$ being within 0.001 of that stated. Similarly, the same is assumed in other cases.

Equations [14a,b] by noting that the result of solving [14a] for F and substituting for F in [14b], or solving for G and substituting for G, is:

$$\theta_o = \frac{a_{12}}{a_{11}} v_o + (a_{11}a_{22}-a_{12}^2)\frac{G}{a_{11}} = \frac{a_{22}}{a_{12}} v_o - (a_{11}a_{22}-a_{12}^2)\frac{F}{a_{12}}$$

Since $a_{11}a_{22} - a_{12}^2$ is almost zero when cC is close to -1, moderate-sized F and G must leave θ_o/v_o close to a_{12}/a_{11} and also to a_{22}/a_{12}. To produce considerably different values of θ_o/v_o requires large-sized F and G.

(c) *First sliding-end frequency:* When $q\ell = 0.753\pi = 2.365$, calculation shows that $sC = - cS$ or (multiplied by cC) $\tan q\ell = - \tanh q\ell$ and

$$\bar{a}_{11} = -1.558 \; ; \quad \bar{a}_{12} = 0.767 \; ; \quad \bar{a}_{22} = 0$$

Also, $b_{11} = 0$. If $F = 0$, then Equation [14b] gives $\theta_o = 0$. The beam is then vibrating as if free to slide transversely (with $F = 0$) at $x = 0$; however, it is constrained against rotation. Also, in this case, $v_o = a_{12}G$, and $-G$ or $-v_o/a_{12}$ represents the reaction upon the constraining structure. Different vibratory motions occur if F is not zero.

(d) As $q\ell$ passes through π or 3.142, a_{12} passes through zero and changes sign (because $\sin q\ell$ does), and b_{12} does likewise. Thus, at $q\ell = \pi$, $v_o = a_{11}F$ and $\theta_o = a_{22}G$. Calculation gives:

$$\bar{a}_{11} = -0.917 \; ; \quad \bar{a}_{22} = 0.917 \; ; \quad \bar{A} = -0.842$$

(e) *First pin-end frequency:* At $q\ell = 1.250\pi = 3.927$, $\tan q\ell = \tanh q\ell$ and

$$a_{11} = b_{22} = 0 \; ; \quad \bar{a}_{12} = -0.946 \; ; \quad \bar{a}_{22} = 1.894 \; ; \quad \bar{A} = -0.894$$

The interesting special case at this value of $q\ell$ is a pin-ended beam vibrating with $G = 0$, $v_o = 0$, and $F = \theta_o/a_{12}$. If $G \neq 0$, then $v_o = -0.946\ G/(EIq^2)$.

(f) *Second built-in frequency:* As $q\ell$ increases toward 1.5π, c is negative but decreases numerically until $cC = -1$. Because of the huge size of S and C, c must be very small; $cC = -1$ at $q\ell = 1.495\pi = 4.695$, with $c = -0.017$, $S = C = 54.7$ to three figures, and

$$\bar{a}_{11} = 26.9\ ;\quad \bar{a}_{12} = -27.3\ ;\quad \bar{a}_{22} = 27.8\ ;\quad \bar{A} = 0$$

whereas b_{11}, b_{12}, and b_{22} are all infinite. As in Case (b), it follows from $a_{11}a_{22} - a_{12}^2 = 0$ that, in general, θ_o/v_o is fixed at the value a_{12}/a_{11}, or here at $-1.017\ q$. If, however, either $F = G = 0$ or $F/G = -a_{12}/a_{11} = +1.017\ q$, then $v_o = \theta_o = 0$ as for a built-in beam.

(g) *At $q\ell = 1.5\pi = 4.712$:* $cC = 0$ and $\bar{a}_{11} = -\bar{a}_{12} = \bar{a}_{22} = 55.6$ (to 3 figures), and $\bar{A} = 1$.

(h) *First free-free frequency:* At (very nearly) $q\ell = 1.505\pi = 4.730$, $cC = 1$, as for free-free vibration. Now, finally, it is the turn of a and A to become infinite. Also, $b_{11}b_{22} - b_{12}^2 = 0$, hence $b_{11}/b_{12} = b_{12}/b_{22}$, so that the right-hand members of Equations [15a,b], if not zero, are in the ratio b_{11}/b_{12}, and, if $G \neq 0$:

$$\frac{F}{G} = \frac{b_{11}}{b_{12}} = (\cot q\ell + \coth q\ell)\ q = 0.982\ q$$

a finite quantity. Thus, when a beam is forced so that $F = 0.982qG$, the beam amplitude remains finite! An attempt to force with F/G in a different ratio, or with either F or G alone, will result in infinite amplitudes of vibration. With F = G = 0, however, the beam may execute a free vibration with

$$\frac{\theta_o}{v_o} = - \frac{b_{11}}{b_{12}} = - 0.982\ q$$

(i) *At $q\ell = 1.55\pi = 4.870$:* $\bar{a}_{11} = - 8.11$, $\bar{a}_{12} = 7.00$, $\bar{a}_{22} = - 5.89$, and $\bar{A} = - 1.22$. As $q\ell$ increases further, the a's continue to decrease numerically.

(j) *Second sliding-end frequency:* Just beyond $q\ell = 1.75\pi$ or 5.50, $\tan q\ell = - \tanh q\ell$ and $\bar{a}_{11} = - 2.02$; $\bar{a}_{12} = 1.01$, $\bar{a}_{22} = 0$, and $\bar{A} = - 1.02$. The general situation is as in Case (b).

(k) *At $q\ell = 2\pi = 6.283$:* $\bar{a}_{11} = - 1.004$, $\bar{a}_{12} = 0$, $\bar{a}_{22} = 1.004$, and $\bar{A} = - 1.008$.

During each further 2π range of $q\ell$ ($2\pi < q\ell \leqq 4$, $4\pi < q\ell \leqq 6\pi$, etc.), a cycle of zero and infinite values of $\bar{a}_{ij}$ occurs similar to that in the range $0 < q\ell \leqq 2\pi$ except that during the first quadrant of each 2π range, two additional points occur that have no analog in the first 2π range. Successive points of interest are now spaced almost exactly $\pi/4$ apart, and the peaks in the curve that contains the infinities become extremely narrow.

The two additional points in the fifth quadrant and two other points illustrating the concentrated occurrence of large a's are as follows:

(l) *Second pin-end frequency:* $\tan q\ell = \tanh q\ell$, $q\ell = 2.25\pi = 7.07$ (within 0.01). Here $s = c = 0.707$, $C = 588$, $\bar{a}_{11} = 0$, $\bar{a}_{12} = -1.00$, $\bar{a}_{22} = 2.00$, and $\bar{A} = -1.00$ (within 0.01).

(m) *At $q\ell = 2.490\pi = 7.823$:* $\bar{a}_{11} = 31.6$, $\bar{a}_{12} = -32.7$, $a_{22} = 33.7$, and $\bar{A} = -1.05$.

(n) *Second free-free frequency:* $cC = 1$; $q\ell = 2.500\pi = 7.851$ (within 0.001). Here a_{11}, a_{12}, and a_{22} are all infinite. In analogy with Case (g), finite amplitudes may occur only if neither F nor G vanishes and $F/G = b_{11}/b_{12} = 1.001\,q$, or if $F = G = 0$.

(o) *Third built-in frequency:* $cC = -1$; hence $q\ell = 2.50025\pi = 7.855$. Here $C = 1290$ and $\bar{a}_{11} = \bar{a}_{22} = -645$, $\bar{a}_{12} = 645$, and $\bar{A} = 0$ (exactly).

To find $q\ell$ and $\bar{a}_{12}$, let $cC = -1$; but $C \approx S$, hence $c = -\dfrac{1}{1290} = -0.000775$. Also $c = -\theta$, the slight angle beyond 2.5π or 7.85398, therefore $\theta = 0.000775 = 0.000247\pi$ and $q\ell = 7.85398 + 0.000775 = 7.855$ or 2.50025π. $\bar{a}_{12} = -\frac{1}{2}\tan q\ell - \frac{1}{2}\left(\frac{1.00000\ldots}{c}\right) = -\frac{1}{2}(-1290) = 645.$

Alternatively, $\bar{a}_{12} = \dfrac{sS}{1-cC} = \frac{1}{2}sS = \frac{1}{2}S = \frac{1}{2}\sinh 7.855$ because $cC = -1$, $s \approx 1$. Hence, $\bar{a}_{12} = \frac{1}{2}\left(e^{7.855} - e^{-7.855}\right) \approx \frac{1}{2}e^{7.855} = 1290$ and $\bar{a}_{12} = 645$.

(p) *At $q\ell = 2.51\pi = 7.885$:* $C = 1322$, $\bar{a}_{11} = -32.1$, $\bar{a}_{12} = 31.2$, $\bar{a}_{22} = -30.2$, and $\bar{A} = -0.95$.

Above $q\ell = 3$, at least, curves representing $\bar{a}_{11}$, $\bar{a}_{12}$, and $\bar{a}_{22}$ lie close to those defined by $\bar{a}_{11} = \tan q\ell - 1$, $\bar{a}_{12} = -\tan q\ell$, and

$\bar{a}_{22}$ = tan $q\ell$ + 1. The effect of the small omitted term 1/C is chiefly to shift the position of the infinite values very slightly along the $q\ell$ axis, by $\Delta q\ell = \pm 1/C$, since, when $q\ell \ll 1$, $\cos q\ell - (1/C) \doteq \cos [q\ell \pm (1/C)]$.

At the *free-free frequencies defined by cC = 1,* resonance may be said to occur, since the beam can vibrate at these frequencies even if F = G = 0. In the one-dimensional cases, an attempt to force at a resonant frequency necessarily results in an infinite amplitude. In a two-dimensional (or two degrees of freedom, (v,θ)) case like the case under discussion, however, and presumably in any multidimensional case, finite amplitudes will occur, provided the applied reactions are in certain ratios to each other.

Conversely, at the *built-in frequencies defined by cC = -1,* there is "antiresonance." In a one-dimensional case of this kind, the forced amplitude cannot be budged from zero at the forcing point; however, in similar multidimensional cases, the amplitudes of displacement or rotation at the forcing point merely stand necessary in certain fixed ratios to each other, these ratios being independent of the ratios of the applied forces or moments.

The detailed discussion of individual cases is greatly simplified when shear warping is neglected because then, in the formulas, only the single parameter q varies with frequency. If shear warping is included, the parameter ξ also varies and the variation of the coefficients becomes more complicated. It may reasonably be surmised, however, that in this case, also, the a's and b's will vary widely with frequency. In roughly cyclic fashion, the infinite a's will occur at the natural free-free frequencies and the infinite b's at the frequencies for built-in vibration.*

The presence of damping should replace the infinities by finite peaks in the curves.

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* Practically speaking, the effect of shear is small below the first natural frequency, but must be considered beyond this frequency.

4. HARMONIC FORCING AT AN INTERMEDIATE POINT

When the external harmonic force F and moment G act at an intermediate point instead of at the end of the uniform beam, the problem is more complicated but can be handled by a double application of the equations for forcing at one end.

First the fundamental beam equations are generalized so as to include *distributed* forces and moments of respective magnitudes F_1 and G_1 per unit length, acting on the beam in the same principal plane; see Figure 1. On a slice dx thick there is then a net force $dP + F_1 dx$ and a net moment $dM + G_1 dx$ so that, after dividing through by dx, the equations of motion become in place of Equations [2a,b]:

$$\mu \frac{\partial^2 v}{\partial t^2} = \frac{\partial P}{\partial x} + F_1 \; ; \qquad \frac{\partial M}{\partial x} = - P - G_1 \qquad [16a,b]$$

Rotary inertia is omitted here. Equations [2c,d] and the elastic analysis leading to them require no change.

Now let the external forces and moments act only on a very thin slice of the beam. See Figure 2, where the thickness of the slice is greatly exaggerated for clarity and at its faces, the common device of drawing the beam as separated is adopted. Let the values of the shear force P and moment M in the beam be P' and M' at the left side of the slice (toward negative x) and P'', M'' at the right side. Then the associated reactions on the slice are -P' and -M' at the left side but P'' and M'' at the right.

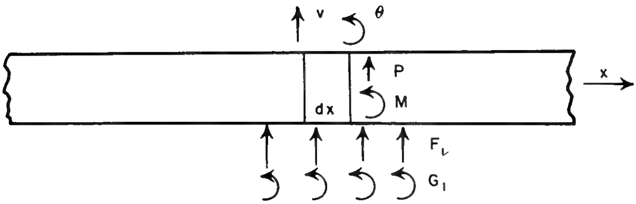

Figure 1 - Forces and Moments on a Beam

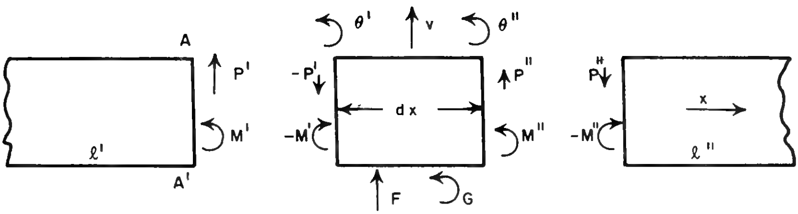

Figure 2 - Forces and Moments on a Beam Slice

Integration of Equation [16a] through the slice gives:

$$\int \mu \frac{\partial^2 v}{\partial t^2}\, dx = \int \frac{\partial P}{\partial x}\, dx + \int F_1 dx$$

The first integral is negligible because of the thinness of the slice, whereas:

$$\int \left(\frac{\partial P}{\partial x}\right) dx = P'' - P' \quad ; \quad \int F_1 dx = F$$

Hence, the total external force is*

$$F = P' - P'' \qquad [17a]$$

Similar treatment of Equation [16b] gives, since $\int P dx$ is negligible:

$$G = M' - M'' \qquad [17b]$$

Also, differentiation of Equation [2c], elimination of γ by means of Equation [2d], and integration with respect to x give:

$$\int \frac{\partial \theta}{\partial x}\, dx = \int \frac{M}{EI}\, dx + 2\sigma \int \frac{\partial P}{\partial x}\, dx$$

Here the M integral is negligible, whereas $\int (\partial\theta/\partial x) dx = \theta'' - \theta'$, where θ' and θ'' denote the slope of the beam or $\partial v/\partial x$ at the left and right sides of the slice, respectively. Hence, using Equation [17a]:

$$\theta'' - \theta' = 2\sigma(P'' - P') = -2\sigma F \qquad [18]$$

The equations for a beam forced at one end can now be applied to the two sections into which the beam has been divided. Let the lengths of the sections be ℓ' and ℓ'', so that $\ell' + \ell'' = \ell$, the total length. Let all quantities referring to the left or ℓ' section be distinguished by one prime and those referring to the ℓ'' section, which extends toward positive x, by two primes.

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* Compare with Equation [18] of Reference 12.

The quantities v_o, θ_o, F, and G in the forcing equations now have the respective values for the ℓ'' section:

$$v''_o = v \; ; \quad \theta''_o = \theta'' \; ; \quad F'' = - P'' \; ; \quad G'' = - M''$$

Here F'' and G'' are the negatives of P'' and M'', which act on the slice; and v is the transverse displacement of the beam at the slice.

The ℓ' section, however, is forced on the end (AA') *facing positive* x. A beam whose forced end *faces negative* x, as in the previously developed theory, can be brought into the ℓ' position by a rotation through 180 degrees about an axis parallel to F. This rotation, however, reverses the spatial directions of θ_o and G. Hence, for the ℓ' section:

$$v'_o = v \; ; \quad \theta'_o = - \theta' \; ; \quad F' = P' \; ; \quad G' = - M'$$

(Note that G becomes -G by the rotation, but -G' = M'.)

Substitution into Equations [17a,b] then gives:*

$$F = F' + F'' \; ; \quad G = - G' + G'' \qquad [19a,b]$$

These equations relate the *actual* external force and moment F and G to the quantities which were denoted by F and G in the equations for forcing at one end as applied to the two sections.

Now, likewise, let the b's for the ℓ' and ℓ'' sections be distinguished by one or two primes. Then Equations [13a,b] of the general end-forcing theory become, for the two sections:

$$F' = b'_{11} \, v + b'_{12} \, \theta'_o \; ; \quad F'' = b''_{11} \, v + b''_{12} \, \theta''_o$$

$$G' = b'_{21} \, v + b'_{22} \, \theta'_o \; ; \quad G'' = b''_{21} \, v + b''_{22} \, \theta''_o$$

* Compare with Equation [19] of Reference 12 where G = 0. Note the use of two coordinate systems (Figures 7 and 8) in that reference.

In the formulas for b, ℓ is to be replaced by ℓ' in calculating b'_{11}, b'_{12}, b'_{21}, and b'_{22} but by ℓ'' in calculating b''_{11}, b''_{12}, b''_{21}, and b''_{22}. Thus, the quantities s, c, S, and C have different values for the two sections, but q and ξ are the same. Substituting $\theta'_o = -\theta'$ and $\theta''_o = \theta''$, and then substituting in Equations [19a,b] gives:

$$F = (b'_{11} + b''_{11})\,v - b'_{12}\,\theta' + b''_{12}\,\theta''$$

$$G = -(b'_{21} - b''_{21})\,v + b'_{22}\,\theta' + b''_{22}\,\theta''$$

The difference $\theta'' - \theta'$ represents a *jump* in slope due to *shear warping* (a jump being naturally assumed positive from negative toward positive x). Thus, the beam has *no definite slope* where F and G are applied (if shear warping is not neglected). If at this point it is desired to treat interaction with another structure, so that a unique value of θ is needed, perhaps the mean of θ' and θ'' may be used, or

$$\theta = \frac{1}{2}(\theta' + \theta'')$$

Then, replacing $\theta'' - \theta'$ by $-2\sigma F$ as in Equation [18] yields:

$$\theta' = \theta - \frac{1}{2}(\theta'' - \theta') = \theta + \sigma F$$

$$\theta'' = \theta + \frac{1}{2}(\theta'' - \theta') = \theta - \sigma F$$

The last equations for F and G then become:

$$[1 + \sigma(b'_{12} + b''_{12})]\,F = (b'_{11} + b''_{11})\,v - (b'_{12} - b''_{12})\theta \qquad [20a]$$

$$-\sigma(b'_{22} - b''_{22})\,F + G = -(b'_{21} - b''_{21})v + (b'_{22} + b''_{22})\theta \qquad [20b]$$

The final formulas derived by solving these equations, first for F and G, then, alternatively, for v and θ, are:

$$F = b_{11}\,v + b_{12}\,\theta \;; \qquad G = b_{21}\,v + b_{22}\,\theta \qquad [21a,b]$$

$$D_b b_{11} = b'_{11} + b''_{11} \;; \qquad D_b b_{12} = -\,b'_{12} + b''_{12}$$

$$b_{21} = -b'_{21} + b''_{21} + \sigma b_{11}(b'_{22} - b''_{22})$$

$$b_{22} = b'_{22} + b''_{22} + \sigma b_{12}(b'_{22} - b''_{22})$$

$$D_b = 1 + \sigma(b'_{12} + b''_{12})$$

(The absence of primes on b_{11} and b_{12} after σ is correct.)

Or,

$$v = a_{11}F + a_{12}G \;; \qquad \theta = a_{21}F + a_{22}G \qquad [22a,b]$$

$$D_a a_{11} = b'_{22} + b''_{22} + 2\sigma(b'_{12}b''_{22} + b'_{22}b''_{12})$$

$$D_a a_{12} = b'_{12} - b''_{12}$$

$$D_a a_{21} = b'_{21} - b''_{21} + \sigma[(b'_{12} + b''_{12})(b''_{21} - b''_{21}) - (b'_{11} + b''_{11})(b'_{22} - b''_{22})]$$

$$D_a a_{22} = b'_{11} + b''_{11}$$

$$D_a = (b'_{11} + b''_{11})(b'_{22} + b''_{22}) - (b'_{12} - b''_{12})(b'_{21} - b''_{21})$$

The coefficients b_{11}, b_{12}, b_{21}, and b_{22} are inertia coefficients and a_{11}, a_{12}, a_{21}, and a_{22} are response coefficients for the beam of length ℓ under forcing at a distance ℓ' from either end. If $\sigma = 0$, so that $\xi = 0$ also, then $b_{21} = b_{12}$ and $a_{21} = a_{12}$.

As a check, if $\ell' \rightarrow 0$, Equations [21a,b] reduce to the end-forcing equations for the ℓ'' section or $F = b''_{11} v + b''_{12} \theta''$ and $G = b''_{21} \theta'' + b''_{22} \theta''$, as they must do. This reduction is not obvious, for θ and b_{11}, b_{12}, b_{21}, and b_{22} do not become, respectively, equal to θ'', b''_{11}, b''_{12}, b''_{21}, and b''_{22}. Some algebraic juggling is necessary to establish the reduction.

5. HARMONIC END FORCING OF A UNIFORM BEAM HAVING EXTERNAL AND INTERNAL DAMPING

End forcing of an undamped uniform beam was considered previously in this report. The same problem will now be attacked with the beam subjected to both external and internal damping.

Let the external damping be due to a uniform force per unit length of magnitude $-\tilde{c}\mu\partial v/\partial t$, with μ denoting the mass per unit length of the beam, v its displacement parallel to a principal plane, and t the time. Here for convenience $\tilde{c}\mu$ is written instead of the usual c.

As in Reference 13, pp. 6 and 7, the internal damping is assumed to be due to a resistance to variation of the bending stress such that the instantaneous value of the stress is $\sigma = E(\varepsilon + \eta\partial\varepsilon/\partial t)$, where E denotes Young's modulus, η a damping constant, and ε the bending strain. (In Reference 13, $E\eta$ was denoted by ν.) Thus, the moment M_b due to bending is:*

$$M_b = EI\left(\frac{\partial^2 v}{\partial x^2} + \eta\,\frac{\partial^3 v}{\partial t\partial x^2}\right)$$

---

* From elementary beam theory, if c is the distance from the neutral surface to the fiber whose strain is ξ and ρ is the radius of curvature of the elastic curve at the section for which the bending moment is M, then

$$M = \frac{\sigma I}{c} = \frac{I}{c}E\left(\varepsilon + \eta\,\frac{\partial\varepsilon}{\partial t}\right) = EI\,\frac{\partial^2 v}{\partial x^2} + b\,\frac{\partial^3 v}{\partial t\partial x^2}$$

because

$$\frac{\varepsilon}{c} = \frac{1}{\rho} = \frac{\partial^2 v}{\partial x^2}\,; \qquad \frac{1}{c}\frac{\partial\varepsilon}{\partial t} = \frac{\partial^3 v}{\partial t\partial x^2}\,; \qquad b = E\eta I$$

I being the areal "moment of inertia" of the cross section and x denoting the distance along the beam.

In an actual beam there may well be resistance to variation of the shear strains also. Inclusion of such an effect as an independent parameter, however, greatly complicates the theory. Accordingly, the resistance of the shearing stresses to time variation is assumed to be such that the moment M_s due to variation with time of shear warping along the beam retains its usual value as stated on p. 175 of Reference 11, namely

$$M_s = - EI \frac{\partial}{\partial x} \frac{P}{KAG}$$

in terms of the shearing force P, the area of cross section A, a dimensionless constant K depending on the shape of the cross section, and the modulus of rigidity G where $G = E/[2(1 + \nu)]$.

The total moment (or "bending moment") M then equals $M_b + M_s$. As usual, $P = - \partial M/\partial x$, but the equation of motion now reads

$$\mu \frac{\partial^2 v}{\partial t^2} = \frac{\partial P}{\partial x} - \tilde{c}\mu \frac{\partial v}{\partial t}$$

Thus the basic equations for the damped uniform beam can be written as:

$$P = - \frac{\partial M}{\partial x} \qquad [23]$$

$$M = EI \left(\frac{\partial^2 v}{\partial x^2} + \eta \frac{\partial^3 v}{\partial t\, \partial x^2} \right) + \frac{EI}{KAG} \frac{\partial^2 M}{\partial x^2} \qquad [24]$$

$$\mu \frac{\partial^2 v}{\partial t^2} + \tilde{c}\mu \frac{\partial v}{\partial t} + \frac{\partial^2 M}{\partial x^2} = 0 \qquad [25]$$

It is assumed that an external force F and an external moment G act on the beam at one end, where x = 0, in the same principal plane with v.

F and v are taken positive in the same direction, and G is positive in the direction of $\partial v/\partial x$ viewed as a rotation θ. Then, the boundary conditions for a beam of length ℓ are:

$$\text{at } x = 0: \quad M = -G; \; \frac{\partial M}{\partial x} = -P = F \qquad [26a,b]$$

$$\text{at } x = \ell: \quad M = 0; \; \frac{\partial M}{\partial x} = -P = 0 \qquad [26c,d]$$

The final results of the investigation will be relations between v_o and θ_o, the values of v and of θ or $\partial v/\partial x$ at $x = 0$, and F and G.

(a) Harmonic motion. This problem is solved only for the case in which all time-dependent variables vary harmonically with time at circular frequency ω. In developing the solution, complex quantities are employed because of their algebraic compactness. Complex quantities are distinguished from real ones by adding a bar over the symbol. All time-dependent complex variables are assumed to vary with time in proportion to $e^{i\omega t}$ $(i = \sqrt{-1})$.

The complex analogs of Equations [24], [25], [26a,b,c,d] thus become:

$$\overline{M} = EI\left(\frac{\partial^2 \overline{v}}{\partial x^2} + \frac{\partial^3 \overline{v}}{\partial t \partial x^2}\right) + \frac{EI}{KAG}\frac{\partial^2 \overline{M}}{\partial x^2} \qquad [27]$$

$$\mu\frac{\partial^2 \overline{v}}{\partial t^2} + \tilde{c}\mu\frac{\partial \overline{v}}{\partial t} + \frac{\partial^2 \overline{M}}{\partial x^2} = 0 \qquad [28]$$

$$\text{at } x = 0: \quad \overline{M} = -\overline{G}; \; \frac{\partial \overline{M}}{\partial x} = \overline{F} \qquad [29a,b]$$

$$\text{at } x = \ell: \quad \overline{M} = 0; \; \frac{\partial \overline{M}}{\partial x} = 0 \qquad [29c,d]$$

Here $\overline{F}$ and $\overline{G}$ can be expressed in terms of real amplitudes A and B as:

$$\overline{F} = Ae^{i\omega t}; \quad \overline{G} = Be^{i\omega t} \qquad [29e,f]$$

Furthermore, Equations [27] and [28] can also be written more explicitly because of the assumed time variation as $e^{i\omega t}$, thus:

$$\overline{M} = EI\ (1 + i\omega\eta)\ \frac{\partial^2 \overline{v}}{\partial x^2} + \frac{EI}{KAG}\ \frac{\partial^2 \overline{M}}{\partial x^2} \qquad [27\alpha]$$

$$(-\ \mu\omega^2 + i\omega\tilde{c}\mu)\ \overline{v} + \frac{\partial^2 \overline{M}}{\partial x^2} = 0 \qquad [28\alpha]$$

An equation for $\overline{M}$ alone can also be obtained by differentiating Equation [28α] relative to x and substituting from the resulting equation for $\partial^2\overline{v}/\partial x^2$ in Equation [27α]. Slightly rearranged, the result is:

$$\frac{EI}{\mu\omega^2}\ \frac{1 + i\omega\eta}{1 - i(\tilde{c}/\omega)}\ \frac{\partial^4 \overline{M}}{\partial x^4} + \frac{EI}{KAG}\ \frac{\partial^2 \overline{M}}{\partial x^2} - \overline{M} = 0$$

Now it is convenient to introduce the notation:

$$q^4 = \frac{\mu\omega^2}{EI}; \quad \xi = \frac{EI}{2KAG}\ q^2$$

$$\frac{1 - i(\tilde{c}/\omega)}{1 + i\omega\eta} = e - ig = \frac{1 - \tilde{c}\eta}{1 + \omega^2\eta^2} - i\ \frac{(\tilde{c}/\omega) + \omega\eta}{1 + \omega^2\eta^2}$$

The symbols e and g represent real quantities whose values can be read from the last member of the equation, and always $g > 0$. The equation obtained for $\overline{M}$, multiplied through by $q^4\ (e - ig)$, then reads:

$$\frac{\partial^4 \overline{M}}{\partial x^4} + 2\xi q^2\ (e - ig)\ \frac{\partial^2 \overline{M}}{\partial x^2} - q^4\ (e - ig)\ \overline{M} = 0 \qquad [30]$$

(Here, the symbols q and ξ represent the same real quantities as in the section on the undamped beam; and, if $e = 1$ and $g = 0$, Equation [30] becomes the same as Equation [3a] in that section.)

Basic solutions for Equation [30] must now be sought. Assume that $\overline{M}$ varies as $e^{i\omega t + \overline{\lambda}x}$. Then Equation [30] divided through by $\overline{M}$ gives for $\overline{\lambda}$:

$$\overline{\lambda}^4 + 2\xi q^2 (e - ig) \overline{\lambda}^2 - q^4 (e - ig) = 0$$

Solving as a quadratic in $\overline{\lambda}^2$ gives:

$$\overline{\lambda}^2 = - \xi q^2 (e - ig) \pm [\xi^2 q^4 (e - ig)^2 + q^4 (e - ig)]^{1/2}$$

Since the square root indicated here has two possible values, each the negative of the other, plus or minus alternatives are indicated twice over, whereas $\overline{\lambda}^2$ has really only two alternative values. It is convenient, therefore, to select a particular value of the square root, which will be denoted by a subscript + and will be defined presently. This square root is chosen so that when $e = 1$ and $g = 0$ it becomes the quantity $q^2 \sqrt{1 + \xi^2}$ occurring in the theory of the undamped beam. In this latter theory, it was found convenient to write $\lambda^2 = q_2^2$ when the positive sign in the plus or minus alternative was chosen but $\lambda^2 = - q_1^2$ when the negative sign was chosen. Analoguously, the alternative values of $\overline{\lambda}^2$ obtained from the last equation are taken to be $\overline{\lambda}^2 = - \overline{q}_1^2$ or $\overline{q}_2^2$ where:

$$\overline{q}_1^2 = q^2 \left\{ \xi(e - ig) + [e - ig + \xi^2 (e - ig)^2]_+^{1/2} \right\} \qquad [31a]$$

$$\overline{q}_2^2 = q^2 \left\{ -\xi(e - ig) + [e - ig + \xi^2 (e - ig)^2]_+^{1/2} \right\} \qquad [31b]$$

An explicit expression for the square root occurring in Equations [31a,b] can be found from the general formula:

$$(a + ib)^{1/2} = \pm \left(u + \frac{ib}{2u}\right) \; ; \quad u = \sqrt{\frac{1}{2}\left(a + \sqrt{a^2 + b^2}\right)} \qquad [32a,b]$$

Here a and b are any two real numbers (except b = 0 if $a \leqq 0$) and $\sqrt{}$ denotes, as usual, the positive square root of a positive real number. If $a \leqq 0$ and b = 0, $(a + ib)^{1/2} = \pm\, i \sqrt{|a|}$. It will be convenient to denote by $(a + ib)_+^{1/2}$ the value given by [32a] with use of the plus sign. [Note that $(a + ib)_+^{1/2}\,(a + ib)_+^{1/2} = a + ib$.] $(a + ib)_+^{1/2}$ is a continuous function of a and b except for a discontinuous jump as b varies past zero with a < 0, the jump as b rises from negative to positive being readily found to be from $-i\sqrt{|a|}$ to $i\sqrt{|a|}$. [To verify [32a], solve [32b] for b^2 and substitute in [32a] squared.]

Putting a = e and b = − g into Equations [32a,b] gives:

$$(e - ig)_+^{1/2} = u - \frac{ig}{2u}, \quad u = \sqrt{\frac{1}{2}\left(e + \sqrt{e^2 + g^2}\right)} \qquad [33a,b]$$

Since for a damped beam g > 0, no difficulty can occur if e < 0 (which is quite unlikely).

Also, from Equations [32a,b] with $a = e + \xi^2 (e^2 - g^2)$, $b = -\, g\,(1 + 2\xi^2 e)$, the square root indicated in Equations [31a,b] has the value

$$[e - ig + \xi^2 (e - ig)^2]_+^{1/2} = w - \frac{ig}{2w}(1 + 2\xi^2 e) \qquad [34a]$$

where

$$w = \sqrt{\frac{1}{2}\left(a + \sqrt{a^2 + b^2}\right)} \qquad [34b]$$

Here, to lessen confusion, w is written in place of u but the lengthy values of a and b have not been inserted into Equation [34b].

For moderate damping, $e > 0$; in any case $b < 0$ and the formula holds if $e > -1/(2\xi^2)$. Very likely, numerically greater values of negative e need not be admitted. If they are, the continuous values of the square root can be extended by choosing the negative sign in Formula [32a] and, hence, using the right-hand member of [34a] with signs reversed.

$\bar{q}_1$ and $\bar{q}_2$ may now be found as $(\bar{q}_1^2)_+^{1/2}$ and $(\bar{q}_2^2)_+^{1/2}$. For clarity, replace the symbols a, b, and u in Equations [32a,b] by α, β, and r, respectively. Substitute from Equation [34a] for the square root in Equations [31a,b] and then, in the brace in each of these equations, collect the real terms as α and the imaginary terms as $i\beta$. Then Equations [32a,b] give:

$$\bar{q}_1 = q\left(r + \frac{i\beta}{2r}\right); \quad \alpha = \xi e + w; \quad \beta = -g\left[\xi + \frac{1}{2w}(1 + 2\xi^2 e)\right] \quad [35a,b,c]$$

$$\bar{q}_2 = q\left(r + \frac{i\beta}{2r}\right); \quad \alpha = -\xi e + w; \quad \beta = -g\left[-\xi + \frac{1}{2w}(1 + 2\xi^2 e)\right] \quad [35d,e,f]$$

where in either case

$$r = \sqrt{\frac{1}{2}\left(\alpha + \sqrt{\alpha^2 + \beta^2}\right)}$$

and w is given by Equation [34b]. Here in both $\bar{q}_1$ and $\bar{q}_2$, $\alpha > 0$ provided $e > \xi^2 g^2$. For then in [34b], $a = e + \xi^2(e^2 - g^2) > \xi^2 e^2 \geqq 0$ so that $w^2 > a > \xi^2 e^2$ and $w > |\xi e|$. Otherwise it may be necessary again to extend the range of continuous values by reversing signs.

Without finding $\bar{q}_1$ or $\bar{q}_2$, however, the following useful relations can be inferred directly from the expressions given for $\bar{q}_1^2$ and $\bar{q}_2^2$ in Equations [31a,b]:

$$\bar{q}_1^2 + \bar{q}_2^2 = 2q^2 \left[e - ig + \xi^2 (e - ig)^2\right]_+^{1/2} \qquad [36a]$$

$$\bar{q}_1^2 - \bar{q}_2^2 = 2\xi q^2 (e - ig) \qquad [36b]$$

$$\bar{q}_1^2 \bar{q}_2^2 = q^4 (e - ig), \text{ hence } \bar{q}_1 \bar{q}_2 = q^2 (e - ig)_+^{1/2} \qquad [36c]$$

That the square root of $(e - ig)$ occurring in $\bar{q}_1\bar{q}_2$ really is the one denoted by $(e - ig)_+^{1/2}$ can easily be verified by means of the argument from continuity, which holds also for the individual values of $\bar{q}_1$ and $\bar{q}_2$. Also:

$$\bar{q}_1^4 + \bar{q}_2^4 = 2q^4 \left[e - ig + 2\xi^2 (e - ig)^2\right] \qquad [36d]$$

$$\frac{\bar{q}_1^3}{\bar{q}_2} + \frac{\bar{q}_2^3}{\bar{q}_1} = \frac{\bar{q}_1^4 + \bar{q}_2^4}{\bar{q}_1 \bar{q}_2} = 2q^2 \left[1 + 2\xi^2 (e - ig)\right] (e - ig)_+^{1/2} \qquad [36e]$$

In the further development of the analysis some relations among the exponential-trigonometric-hyperbolic functions are useful. A systematic list of some of the relations for sin iz, cos iz, sinh iz, cosh iz, $e^{\pm z}$, $e^{\pm iz}$, $\sin (x \pm y)$, $\cos (x \pm y)$, $\sinh (x \pm y)$, $\cosh (x \pm y)$, $\sin (x \pm iy)$, $\cos (x \pm iy)$, $\sinh (x \pm iy)$, and $\cosh (x \pm iy)$ can be found in an elementary calculus textbook. Here $z = x + iy$, x and y being any real numbers or 0.

The four possible values of $\bar{\lambda}$ will then be, since $\bar{\lambda}^2 = -\bar{q}_1^2$ or $\bar{q}_2^2$, $\bar{\lambda} = \pm i\bar{q}_1$ or $\pm \bar{q}_2$. The corresponding values of $\bar{M}$ are $e^{i\omega t + i\bar{q}_1 x}$, $e^{i\omega t - i\bar{q}_1 x}$, $e^{i\omega t + \bar{q}_2 x}$, $e^{i\omega t - \bar{q}_2 x}$.

Further work will be simplified, however, as in the theory of the undamped beam, if combinations of these four functions are used such that two of them vanish when x = 0. The latter two combinations of the exponential factors are:

$$\frac{1}{2i}\left(e^{i\bar{q}_1 x} - e^{-i\bar{q}_1 x}\right) = \frac{1}{i}\sinh i\bar{q}_1 x = \sin \bar{q}_1 x;$$

$$\left(e^{\bar{q}_2 x} - e^{-\bar{q}_2 x}\right)/2 = \sinh \bar{q}_2 x \qquad [37]$$

The general solution of Equation [30] may then be written thus:

$$\bar{M} = e^{i\omega t}\left(\bar{d}_1 \sin \bar{q}_1\, x + \bar{d}_2 \cos \bar{q}_1\, x + \bar{d}_3 \sinh \bar{q}_2\, x + \bar{d}_4 \cosh \bar{q}_2 x\right) \qquad [38]$$

Here $\bar{d}_1 \ldots \bar{d}_4$ are four adjustable constants, probably complex.

Substitution for $\bar{M}$ from Equation [38] and from Equations [29e,f] for $\bar{F}$ and $\bar{G}$ in the boundary equations, Equations [29a,b,c,d], then gives the following four equations, after canceling out $e^{i\omega t}$:

$$\bar{d}_1\bar{q}_1 + \bar{d}_3\bar{q}_2 = A; \quad \bar{d}_2 + \bar{d}_4 = -B \qquad [39a,b]$$

$$\bar{d}_1 \sin \bar{q}_1\ell + \bar{d}_2 \cos \bar{q}_1\ell + \bar{d}_3 \sinh \bar{q}_2\ell + \bar{d}_4 \cosh \bar{q}_2\ell = 0 \qquad [39c]$$

$$\bar{d}_1\bar{q}_1 \cos \bar{q}_1\ell - \bar{d}_2\bar{q}_1 \sin \bar{q}_1\ell + \bar{d}_3\bar{q}_2 \cosh \bar{q}_2\ell + \bar{d}_4\bar{q}_2 \sinh \bar{q}_2\ell = 0 \qquad [39d]$$

The treatment may now proceed formally just as for the undamped beam. Equations for the complex displacement $\bar{v}$ and the complex slope $\bar{\theta}$ $(= \partial\bar{v}/\partial x)$ are as follows (derived from Equation [28α]):

$$\mu(\omega^2 - i\omega\tilde{c})\,\bar{v} = \frac{\partial^2\bar{M}}{\partial x^2}; \quad \mu(\omega^2 - i\omega\tilde{c})\,\bar{\theta} = \frac{\partial^3\bar{M}}{\partial x^3}$$

Substitute here for $\bar{M}$ from Equation [38] and then let $x \to 0$. Write $\bar{v}_o$ and

$\bar{\theta}_o$ for the values of $\bar{v}$ and $\bar{\theta}$ at x = 0, and also $\bar{v}_o = \bar{v}_{oa} e^{i\omega t}$ and $\bar{\theta}_o = \bar{\theta}_{oa} e^{i\omega t}$ in terms of complex amplitudes $\bar{v}_{oa}$ and $\bar{\theta}_{oa}$. Then, after canceling out $e^{i\omega t}$, it is found that:

$$\mu(\omega^2 - i\omega\tilde{c})\ \bar{v}_{oa} = -\ \bar{q}_1^2 \bar{d}_2 + \bar{q}_2^2 \bar{d}_4$$

$$\mu(\omega^2 - i\omega\tilde{c})\ \bar{\theta}_{oa} = -\ \bar{q}_1^3 \bar{d}_1 + \bar{q}_2^3 \bar{d}_3$$

(Note that $\partial^2 \sin \bar{q}_1/\partial x^2 = -\ \bar{q}_1^2 \sin \bar{q}_1 x$, = 0 when x = 0, etc.) These two equations and Equations [39a,b] are easily solved for the $\bar{d}$'s with the following results:

$$\bar{q}_1\ (\bar{q}_1^2 + \bar{q}_2^2)\ \bar{d}_1 = \bar{q}_2^2\ A - \mu\ (\omega^2 - i\omega\tilde{c})\ \bar{\theta}_{oa}$$

$$(\bar{q}_1^2 + \bar{q}_2^2)\ \bar{d}_2 = -\ \bar{q}_2^2\ B - \mu\ (\omega^2 - i\omega\tilde{c})\ \bar{v}_{oa}$$

$$\bar{q}_2\ (\bar{q}_1^2 + \bar{q}_2^2)\ \bar{d}_3 = \bar{q}_1^2\ A + \mu\ (\omega^2 - i\omega\tilde{c})\ \bar{\theta}_{oa}$$

$$(\bar{q}_1^2 + \bar{q}_2^2)\ \bar{d}_4 = -\ \bar{q}_1^2\ B + \mu\ (\omega^2 - i\omega\tilde{c})\ \bar{v}_{oa}$$

The $\bar{d}$'s can now be eliminated entirely by multiplying Equations [39c,d] by $(\bar{q}_1^2 + \bar{q}_2^2)$ and then substituting in these equations the values of $(\bar{q}_1^2 + \bar{q}_2^2)\bar{d}_{1,2,3,4}$ given by the last four equations. Also, introduce the following notation:

$$\bar{\Omega} = \mu\ (\omega^2 - i\omega\tilde{c}) \qquad \bar{S} = \sinh \bar{q}_2\ell$$

$$\bar{s} = \sin \bar{q}_1\ell \qquad \bar{C} = \cosh \bar{q}_2\ell$$

$$\bar{c} = \cos \bar{q}_1\ell$$

Then Equations [39c,d] divided by $\bar{\Omega}$ become:

$$(\bar{C} - \bar{c})\ \bar{v}_{oa} + \left(\frac{\bar{S}}{\bar{q}_2} - \frac{\bar{s}}{\bar{q}_1}\right)\bar{\theta}_{oa}$$

$$= -\left(\frac{\bar{q}_1^2}{\bar{q}_2}\bar{S} + \frac{\bar{q}_2^2}{\bar{q}_1}\bar{s}\right)\frac{A}{\bar{\Omega}} + (\bar{q}_1^2\bar{C} + \bar{q}_2^2\bar{c})\frac{B}{\bar{\Omega}} \qquad [40a]$$

$$(\bar{q}_2\bar{S} + \bar{q}_1\bar{s})\ \bar{v}_{oa} + (\bar{C} - \bar{c})\ \bar{\theta}_{oa}$$

$$= -\ (\bar{q}_1^2\bar{C} + \bar{q}_2^2\bar{c})\frac{A}{\bar{\Omega}} + \bar{q}_1\bar{q}_2(\bar{q}_1\bar{S} - \bar{q}_2\bar{s})\frac{B}{\bar{\Omega}} \qquad [40b]$$

These equations are easily solved formally for $\bar{v}_{oa}$ and $\bar{\theta}_{oa}$ in terms of A and B. The determinant of the coefficients of $\bar{v}_{oa}$ and $\bar{\theta}_{oa}$ is:

$$\bar{D} = (\bar{C} - \bar{c})^2 - \bar{S}^2 + \bar{s}^2 - \bar{s}\,\bar{S}\left(\frac{\bar{q}_1}{\bar{q}_2} - \frac{\bar{q}_2}{\bar{q}_1}\right)$$

or, using Formulas (36b,c]:

$$\bar{D} = 2\ [1 - \bar{c}\bar{C} - \xi\bar{s}\bar{S}\ (e - ig)_{+}^{1/2}] \qquad [41]$$

The results of this solution may be written in the form:

$$\bar{v}_{oa} = \frac{1}{\bar{\Omega}\bar{D}}\left\{ Q_p\left(-\frac{\bar{s}\bar{C}}{\bar{q}_1} + \frac{\bar{c}\bar{S}}{\bar{q}_2}\right)A\right.$$

$$\left. + [Q_m\ (1 - \bar{c}\bar{C}) + 2Q_{12}\bar{s}\bar{S}]B\right\} \qquad [42a]$$

$$\bar{\theta}_{oa} = \frac{1}{\bar{\Omega}\bar{D}}\left\{[-Q_m\ (1 - \bar{c}\bar{C}) + Q_4\bar{s}\bar{S}]A\right.$$

$$\left. -\ Q_p\ (\bar{q}_1\bar{s}\bar{C} + \bar{q}_2\bar{c}\bar{S})\ B\right\} \qquad [42b]$$

where, using Equations [36a - e]:

$$Q_p = \bar{q}_1^2 + \bar{q}_2^2 = 2q^2 \, [e - ig + \xi^2 \, (e - ig)^2]_+^{1/2}$$

$$Q_m = \bar{q}_1^2 - \bar{q}_2^2 = 2\xi q^2 \, (e - ig)$$

$$Q_{12} = \bar{q}_1 \bar{q}_2 = q^2 \, (e - ig)_+^{1/2}$$

$$Q_4 = \frac{\bar{q}_1^4 + \bar{q}_2^4}{\bar{q}_1 \bar{q}_2} = 2q^2 \, (e - ig)_+^{1/2} \, [1 + 2\xi^2 \, (e - ig)]$$

Thus, the analytic solution of the complex problem is formally completed.

(b) Real results. For practical use, real relations must now be inferred from the complex relations.

The original equations requiring real solution were Equations [24], [25], and [26a,b,c,d]. As final results, relations between v_o and θ_o and the applied force F and moment G appear to be the most useful.

For convenience, the complex Equations [42a,b] may be summarized thus:

$$\bar{v}_{oa} = (a'_{11} + ia''_{11}) \, A + (a'_{12} + ia''_{12}) \, B$$

$$\bar{\theta}_{oa} = (a'_{21} + ia''_{21}) \, A + (a'_{22} + ia''_{22}) \, B$$

the primed coefficients a'_{11}, a''_{11}, etc., being all real. Then

$$\bar{v}_o = \bar{v}_{oa} e^{i\omega t} = (a'_{11} + ia''_{11}) \, Ae^{i\omega t} + (a'_{12} + ia''_{12}) \, Be^{i\omega t}$$

$$\bar{\theta}_o = \bar{\theta}_{oa} e^{i\omega t} = (a'_{21} + ia''_{21}) \, Ae^{i\omega t} + (a'_{22} + ia''_{22}) \, Be^{i\omega t}$$

Here $\bar{v}_o$ and $\bar{\theta}_o$ are the values at $x = 0$ of $\bar{v}$ and $\bar{\theta}$ (or $\partial\bar{v}/\partial x$).

For $\overline{v}$ and $\overline{\theta}$ and also for $\overline{M}$, expressions have been found that satisfy Equations [27α] and [28α] and, hence, also Equations [27] and [28]. But in Equations [27] and [28], all coefficients are real. Consequently, the real parts of $\overline{v}$, $\overline{\theta}$, and $\overline{M}$ by themselves must satisfy these equations and must also be harmonic solutions of Equations [24] and [25].

The real parts of the end displacements $\overline{v}_o$ and $\overline{\theta}_o$ must, therefore, represent v_o and θ_o as functions of the time in a possible forced vibration. Keeping only the real parts in both members of the last two formulas for $\overline{v}_o$ and $\overline{\theta}_o$ gives, since $e^{i\omega t} = \cos \omega t + i \sin \omega t$:

$$v_o = a'_{11} A \cos \omega t - a''_{11} A \sin \omega t$$

$$+ a'_{12} B \cos \omega t - a''_{12} B \sin \omega t \qquad [43a]$$

$$\theta_o = a'_{21} A \cos \omega t - a''_{21} A \sin \omega t$$

$$+ a'_{22} B \cos \omega t - a''_{22} B \sin \omega t \qquad [43b]$$

Selecting real parts in Equations [29e,f] gives as the corresponding real external force and moment:

$$F = A \cos \omega t \; ; \quad G = B \cos \omega t$$

Thus the displacement v_o and slope θ_o produced by force actions proportional to $\cos \omega t$ alone contain terms proportional to both $\cos \omega t$ and $\sin \omega t$ when damping is present. Equations [43a,b] might be accepted as the final expressions for end displacement and slope produced by harmonic end forcing.

If, however, interactions of the beam with attached structures are to be considered, it may be preferable to avoid explicit mention of $\sin \omega t$ and $\cos \omega t$.

Note that:

$$A \sin \omega t = -\frac{1}{\omega}\frac{dF}{dt} \; ; \quad B \sin \omega t = -\frac{1}{\omega}\frac{dG}{dt}$$

Thus Equations [43a,b] can also be written in the form:

$$v_o = a'_{11}F + \frac{a''_{11}}{\omega}\frac{dF}{dt} + a'_{12}G + \frac{a''_{12}}{\omega}\frac{dG}{dt} \tag{44a}$$

$$\theta_o = a'_{21}F + \frac{a''_{21}}{\omega}\frac{dF}{dt} + a'_{22}G + \frac{a''_{22}}{\omega}\frac{dG}{dt} \tag{44b}$$

The eight coefficients a'_{11}, a''_{11}/ω, etc., may be regarded as an extended set of influence coefficients for the damped beam under harmonic end forcing, corresponding to the four coefficients a_{11}, a_{12}, a_{21}, and a_{22} for the undamped beam. As an alternative, inertia coefficients corresponding to b_{11}, b_{12}, b_{21}, and b_{22} for the undamped beam could be calculated.

The following extension of the results is perhaps obvious. F and G have been assumed to vibrate *in phase*, but *this restriction is easily removed*. Let B = 0. Then the motion of the beam is excited only by F, and all formulas will obviously hold if ωt is replaced by $(\omega t + \alpha)$ in which α is an arbitrary phase angle. Similarly, if A = 0, only G is active and ωt may be replaced by $(\omega t + \beta)$, in which β is another arbitrary phase angle. Since the equations of motion are linear, these two motions and the exciting force actions can be superposed. Hence, Equations [43a,b] will remain valid if the external force and moment are assumed to be

$$F = A \cos (\omega t + \alpha) \; ; \quad G = B \cos (\omega t + \beta)$$

and provided that in Equations [43a,b] ωt is changed to $\omega t + \alpha$ in all A terms but to $\omega t + \beta$ in all B terms, α and β being independently arbitrary. Equations [44a,b] require no change.

(c) Calculation of a'_{11}, a''_{11}, etc. The coefficients a'_{11}, etc., in Equations [43a,b] and [44a,b] can be calculated by evaluating the coefficients of A and B in Equations [42a,b] and separating real and imaginary parts. In preparation for this calculation, each complex number may be replaced by the sum of a real and an imaginary part. Write

$$\overline{q}_1 = q'_1 + iq''_1 \;;\; \overline{q}_2 = q'_2 + iq''_2$$

in which q'_1, etc., represent qr or $q\beta/(2r)$ in Formulas [35a,d]; then

$$\frac{1}{\overline{q}_1} = \frac{1}{q'_1 + iq''_1} = \frac{q'_1 - iq''_1}{q'^2_1 + q''^2_2} \;;\; \frac{1}{\overline{q}_2} = \frac{q'_2 - iq''_2}{q'^2_1 + q''^2_1}$$

and from Formulas [37m-p] applied to the definitions of $\overline{s}$, $\overline{c}$, $\overline{S}$, $\overline{C}$:

$$\overline{s} = \sin q'_1\ell \cosh q''_1\ell + i \cos q'_1\ell \sinh q''_1\ell$$

$$\overline{c} = \cos q'_1\ell \cosh q''_1\ell - i \sin q'_1\ell \sinh q''_1\ell$$

$$\overline{S} = \sinh q'_2\ell \cos q''_2\ell + i \cosh q'_2\ell \sin q''_2\ell$$

$$\overline{C} = \cosh q'_2\ell \cos q''_2\ell + i \sinh q'_2\ell \sin q''_2\ell$$

Similarly, Q_p, Q_m, and Q_4 can be separated into parts by using Equations [33a] and [34a].

Denominators can be handled most conveniently by transferring i to the numerator. Write $\overline{D} = D' + iD''$, determining D' and D'' from Equation [41]. Then:

$$\frac{1}{D} = \frac{1}{D' + iD''} \; \frac{D' - iD''}{D' - iD''} = \frac{D' - iD''}{D'^2 + D''^2}$$

$$\frac{1}{\Omega} = \frac{1}{\mu\,(\omega^2 - i\omega\tilde{c})} = \frac{\omega^2 + i\omega\tilde{c}}{\mu\omega^2\,(\omega^2 + \tilde{c}^2)}$$

The final calculation of a'_{11}, a''_{11}, etc., from the coefficients of A and B in Equations [42a,b] then requires only a massive process of multiplication and collection of real and imaginary parts. Here, it does not seem worthwhile to elaborate the final expressions.

6. FLUTTER OF A RIGID FOIL ATTACHED TO A UNIFORM BEAM

6.1 DERIVATION OF EQUATIONS OF MOTION

A rigid foil attached to a uniform beam of length ℓ will be considered in this section. The foil is assumed to be equivalent to a straight flat strip of rectangular shape, with length much greater than width, attached to the beam so that its length is perpendicular to a principal plane of the beam, with this plane being midway between the ends of the beam. In the undeflected position of foil and beam, the plane of the foil is assumed to pass through the principal axis of the beam. Elastic deflections of foil and beam, where they are connected, will be allowed but the affected part of the beam is assumed to be much smaller than the whole beam. The relatively small part of the foil that lies within the beam may be absent, being replaced perhaps by a through-shaft similar to the mounting of a pair of diving planes.

There is a certain axis of rotation lying in the plane of the foil and parallel to its length about which relative rotation of foil and beam evokes only elastic moments G acting on the beam and -G on the foil. Let θ denote a small angle of rotation of the foil about the axis, measured from zero when both foil and beam are in their neutral positions; in addition, the local part of the beam itself, due to vibration and aside from distortions of the attaching structure, may be rotated similarly through a small angle θ_o. The positive directions are assumed the same for θ, θ_o, and G. The foil may also have a small translational displacement v perpendicular to its plane and the beam at the axis for θ_o a small displacement v_o, associated with a force F on the beam and -F on the foil; let v, v_o, and F be positive in the same direction. In terms of elastic constants k_1 and k_2 of the connecting structure:

$$F = k_1 (v - v_o) \;;\quad G = k_2 (\theta - \theta_o)$$

Figure 3 represents a section in the principal plane of the beam and shows positive values except where the associated symbol is preceded by a minus sign (as in -F). A heavy line represents a section through the foil plane; the foil itself, of width 2b, being perhaps reduced to a connecting structure here. The axis A in the foil plane and the axis A_o through the principal axis of the beam coincide when foil and beam are undeflected.

Let beam and foil be immersed in a stream of fluid approaching at uniform speed S, as shown in Figure 3. Expressions for the resulting lift forces on the foil are adapted from Theodorsen's formulas for a uniform foil of infinite length vibrating harmonically. An approximation closer than the common steady-motion approximation is used (see p. 841 of Reference 4 (Appendix H)), the additional complexity being only moderate. In the case of ship rudders or control foils, Theodorsen's parameter 1/k or S/bω, in which b is the half-chord length and ω the circular frequency, is less than 1 (or at least < 2); then *Theodorsen's functions** *"F" and "G"* may be replaced without great error by their values at S = 0 or 1/2 and 0, respectively (see pp. 840-841 of Reference 4). The effects of certain other terms may be included in the effective mass m and the effective moment of inertia I_o (taken about the effective center of mass);

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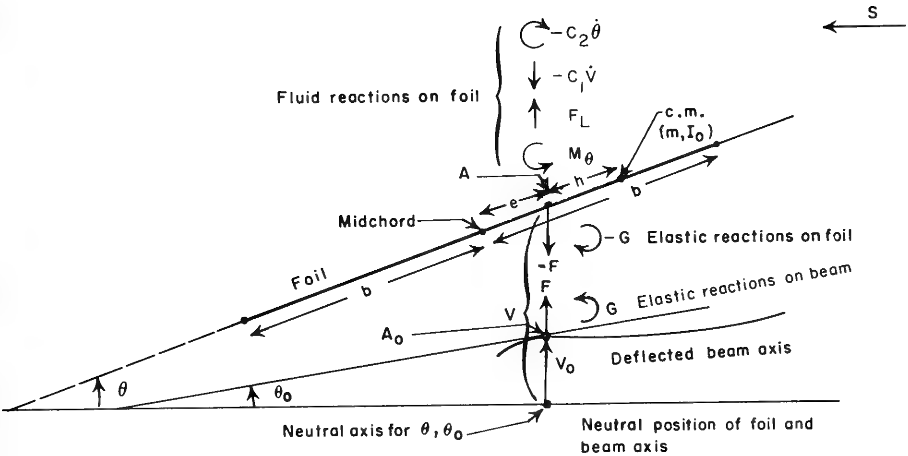
* Not to be confused with external force and moment respectively on beam.



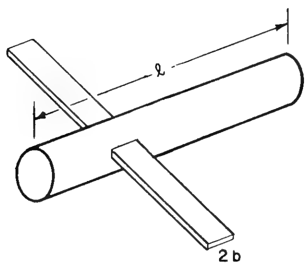


Figure 3 - Forces on and Motions of a Beam-Foil System Immersed in a Moving Fluid

m and I_o are further defined later.

There remain then a total resultant lift force F_L and a total moment M_θ about the θ axis whose magnitudes can be written:**

$$F_L = BS^2\theta - BS\dot{v} + CS\dot{\theta} \qquad [45a]$$

$$M_\theta = LBS^2\theta - LBS\dot{v} - E_L S\dot{\theta} \qquad [45b]$$

Here B, C, L, and E_L are constants (defined below), L having the dimension of length. F_L and M_θ are positive in the respective directions of v and θ; $\dot{v} = dv/dt$ and $\dot{\theta} = d\theta/dt$. Since M_θ denotes the total moment about the θ axis, F_L may be supposed (for concreteness of thought) to act through this axis, as drawn in Figure 3 (i.e., Theodorsen gives the total force F and total moment M about a certain axis. Hence, we cause no error if we *arbitrarily* assume F *to act through the axis.*)

According to Theodorsen's calculations;

$$B = \pi\rho b\ell_f \; ; \quad C = B\left(\frac{3}{2}b + e\right) \; ; \quad L = \frac{1}{2}b - e \; ; \quad E_L = B\left(\frac{1}{2}b + e\right)^2$$

in terms of the density ρ of the surrounding fluid, the half-chord length b and length ℓ_f of the foil, and the distance e that the axis of rotation or the θ axis lies ahead of the midchord line (here ahead means toward the approaching stream). The first two terms of M_θ may be regarded as arising from a force equal to the first two terms of F_L acting at the forward quarter-chord point. With these values inserted, Equations [45a,b] might be called the *low-speed approximation* to Theodorsen's expressions. For a derivation of the approximation, see Reference 4. In this reference it is shown that if e = b/2, Equations [45a,b] agree with the "Modified

** These values of F_L and M_θ agree with those of P_a and M_α given, respectively, on pp. 841 and 842 of Reference 4.

Theodorsen Analysis" described on p. 37 of Reference 14 by McGoldrick and Jewell.

For greater generality, allowance is made for possible additional damping due to other causes by adding a force $- c_1 \dot{v}$ acting on the foil along the same line as F_L and positive in the same direction, and also a moment $- c_2 \dot{\theta}$ about the θ axis. Without too great a complexity, the more general expressions $- c_{11} \dot{v} - c_{12} \dot{\theta}$ and $- c_{21} \dot{v} - c_{22} \dot{\theta}$ could be used* but this was not thought worthwhile for the present purpose.

Equations of motion may now be written for the foil. Let the effective center of the foil be at a distance h ahead of the θ-axis, h being positive toward the approaching stream. The total effective mass of the foil (including virtual mass) will be denoted by m, and its total effective moment of inertia about an axis drawn through its center of mass and parallel to θ-axis by I_o. The displacement of its center of mass is $v + h\theta$ and the total upward force on the foil is $- F - c_1 \dot{v} + F_L$, $- F$ being due to the attachment structure that exerts the force F on the beam. The total moment about a line drawn through the center of mass parallel to the θ axis positive in the same direction as θ is, similarly:

$$- G - c_2 \dot{\theta} + M_\theta - h\ (- F - c_1 \dot{v} + F_L)$$

Hence

$$m\ (\ddot{v} + h\ddot{\theta}) = - F - c_1 \dot{v} + F_L$$

$$I_o \ddot{\theta} = - G - c_2 \dot{\theta} + M_\theta - h\ (- F - c_1 \dot{v} + F_L)$$

A simpler solution, however, is to add h times the first equation to the second equation, thus obtaining for the second equation:

* See p. 78 of Reference 3, for example. c_{11}, c_{22} and c_{12}, c_{21} are direct and cross damping constants, respectively.

$hm\ddot{v} + (I_o + h^2m)\,\ddot{\theta} = -\,G - c_2\dot{\theta} + M_\theta$. For brevity, write

$$I_\theta = I_o + h^2m$$

Then, inserting the expressions written previously for F_L and M_θ:

$$m\ddot{v} + hm\ddot{\theta} + F + (c_1 + BS)\,\dot{v} - CS\dot{\theta} - BS^2\theta = 0 \qquad [46a]$$

and

$$hm\ddot{v} + I_\theta\ddot{\theta} + G + LBS\dot{v} + (c_2 + E_LS)\,\dot{\theta} - LBS^2\theta = 0 \qquad [46b]$$

With these equations may be associated the F and G equations written previously, and also the equations furnished by beam-forcing theory for the response of the beam:

$$F = k_1\,(v - v_o)\ ;\quad G = k_2\,(\theta - \theta_o) \qquad [47a,b]$$

$$v_o = a_{11}F + a_{12}G\ ;\quad \theta_o = a_{21}F + a_{22}G \qquad [48a,b]$$

Here a_{11}, a_{12}, a_{21}, and a_{22} are response coefficients to be calculated from the formulas for end forcing if the foil is attached at the end of the beam, otherwise from the formulas for forcing at an intermediate point.

Thus are obtained six fundamental equations in the six variable functions of the time: v, θ, v_o, θ_o, F, and G. At the critical flutter speed, all of these variables will vary harmonically at the same frequency but probably not in the same phase. It is only for such variation that the lift formulas are reliable.

Equations involving only v and θ as time variables are easily obtained by eliminating the other four variables. To shorten the notation write:

$$s_1 = 1 + a_{11}k_1 \;;\quad s_2 = 1 + a_{22}k_2$$

Then substitution for v_o and θ_o from Equations [48a,b] into [47a,b] gives:

$$s_1F + a_{12}k_1G = k_1v,\; a_{21}k_2F + s_2G = k_2\theta \qquad [49a,b]$$

These equations may be solved for F and G in the form:

$$DF = s_2k_1v - a_{12}k_1k_2\theta \qquad [50a]$$

$$DG = -\, a_{21}k_1k_2v + s_1k_2\theta \qquad [50b]$$

$$D \;= s_1s_2 - a_{12}a_{21}k_1k_2 \qquad [50c]$$

Direct substitution for F and G from Equations [50a,b] into [46a,b] then gives as equations of motion, provided $D \neq 0$:

$$m\ddot{v} + hm\ddot{\theta} + (c_1 + BS)\,\dot{v} - CS\dot{\theta}$$

$$+ D^{-1}\,(s_2k_1v - a_{12}k_1k_2\theta) - BS^2\theta = 0 \qquad [51a]$$

$$hm\ddot{v} + I_\theta\ddot{\theta} + LBS\dot{v} + (c_2 + E_LS)\,\dot{\theta}$$

$$+ D^{-1}\,(-a_{21}k_1k_2v + s_1k_2\theta) - LBS^2\theta = 0 \qquad [51b]$$

Similar substitution for F and G in Equations [48a,b] gives, provided $D \neq 0$:

$$v_o = (1 - D^{-1}s_2)\;v + D^{-1}a_{12}k_2\theta \;;\quad \theta_o = D^{-1}a_{21}k_1v + (1 - D^{-1}s_1)\theta \qquad [52a,b]$$

The special case $D = 0$ requires other methods as will be explained later.

6.2 CRITICAL FLUTTER SPEED

For typical flutter to be possible, there must exist one or more values of the speed S at which a steady vibration of the foil and beam is possible in spite of existing damping actions. The lowest such speed is the critical flutter speed, which is the item of principal practical interest.

To seek a critical flutter speed for the foil-beam system by the usual method, assume that in Equations [51a,b] v and θ are complex functions of the time proportional to $e^{i\omega t}$ ($i = \sqrt{-1}$). Then $\dot{v} = i\omega v$, $\ddot{v} = -\omega^2 v$, etc., and, after canceling $e^{i\omega t}$, the remaining equations are linear in v and θ. Nonzero values of v or θ are possible only if the determinant of the coefficients of v and θ in these equations vanishes; that is, if

$$[-\omega^2 m + D^{-1}s_2k_1 + i\omega(c_1 + BS)]\ [-\omega^2 I_\theta + D^{-1}s_1k_2 - LBS^2 + i\omega(c_2 + E_LS)]$$

$$-\ (-\omega^2 hm - D^{-1}a_{12}k_1k_2 - BS^2 - i\omega CS)\quad (-\omega^2 hm - D^{-1}a_{21}k_1k_2 + i\omega LBS) = 0$$

The real and imaginary parts of this equation must hold separately. The imaginary equation may be divided by $i\omega$ if $\omega \neq 0$, and the D^{-2} term that occurs in it may be simplified by means of Equation [50c], thus:

$$D^{-2}\ (s_1s_2k_1k_2 - a_{12}a_{21}k_1^2k_2^2) = D^{-1}k_1k_2$$

As a result the following two simultaneous equations in ω^2 and S are obtained:

$$\omega^4 mI_o - \omega^2 \Big\{ D^{-1} \; [ms_1k_2 + I_\theta s_2k_1 + hm\,(a_{12} + a_{21})\,k_1k_2]$$

$$+ (c_1 + BS)(c_2 + E_LS) + [(h - L)\,m + LC]\,BS^2 \Big\}$$

$$+ D^{-1}k_1k_2 - D^{-1}k_1\,(a_{21}k_2 + s_2L)\,BS^2 = 0 \qquad [53a]$$

$$-\omega^2\,[c_1I_\theta + c_2m + (I_\theta - hLm)\,BS + hmCS + mE_LS]$$

$$+ D^{-1}\,(c_1s_1k_2 + c_2s_2k_1 + D^{-1}k_2\,(s_1 + La_{12}k_1)\,BS$$

$$- D^{-1}a_{21}k_1k_2CS + D^{-1}s_2k_1E_LS - c_1\,LBS^2 = 0 \qquad [53b]$$

In calculations, however, it may be more convenient when D is small to multiply the equations through by D, thus:

$$-\omega^2 \Big\{ ms_1k_2 + I_\theta s_2k_1 + hm\,(a_{12} + a_{21})\,k_1k_2$$

$$+ D\,(c_1 + BS)\,(c_2 + E_LS) + D\,[(h - L)\,m + LC]BS^2 \Big\}$$

$$+ D\omega^4 mI_o + k_1k_2 - k_1\,(a_{21}k_2 + s_2L)\,BS^2 = 0 \qquad [53c]$$

$$c_1s_1k_2 + c_2s_2k_1 + k_2\,(s_1 + La_{12}k_1)\,BS - a_{21}k_1k_2CS$$

$$+ s_2k_1E_LS - D\omega^2\,[c_1I_\theta + c_2m + (I_\theta - hLm)\,BS$$

$$+ hMCS + mE_LS] - Dc_1LBS^2 = 0 \qquad [53d]$$

It will be shown presently that Equations [53c,d] are valid even when $D = 0$.

If a harmonic solution of the equations of motion, Equations [46a,b], exists when the stream speed is S, its circular frequency being ω, then ω^2 and S must solve Equations [53c,d], or, if $D \neq 0$, Equations [53a,b]. Conversely, any solution of these equations in which $\omega^2 > 0$ leads to a harmonic motion.

The problem of finding such solutions is complicated by the fact that the parameters a_{11}, a_{12}, a_{21}, a_{22}, and D all vary with ω. In numerical computation, the only feasible procedure seems to be to assume successive values of ω^2, to calculate the paramaters for each value and then to attempt to find a value of S that satisfies both equations. Perhaps the work may be facilitated by rewriting the equations as quadratics in S; perhaps for Equations [53a,b], with all coefficients reversed in sign for convenience, the following will result:

$$b_1 S^2 + b_2 S + b_3 = 0, \quad e_1 S^2 + e_2 S + e_3 = 0 \qquad [54a,b]$$

where

$$b_1 = B\,[D^{-1}(a_{21}k_1k_2 + Ls_2k_1) + \omega^2 (hm - Lm + LC + E_L)]$$

$$b_2 = \omega^2 (c_1E_L + c_2B)$$

$$b_3 = -D^{-1}k_1k_2 + \omega^2 \Big\{D^{-1}\,[ms_1k_2 + I_\theta s_2k_1 + hm\,(a_{12} + a_{21})\,k_1k_2] + c_1c_2\Big\} - mI_o\omega^4$$

$$e_1 = c_1LB$$

$$e_2 = -BD^{-1}k_2\,(s_1 + La_{12}k_1) + CD^{-1}a_{21}k_1k_2 - E_LD^{-1}s_2k_1 + \omega^2\,[(I_\theta - hLm)\,B + hmC + mE_L]$$

$$e_3 = -\, c_1 D^{-1} s_1 k_2 - c_2 D^{-1} s_2 k_1 + \omega^2\,(c_1 I_\theta + c_2 m)$$

Or where, to match Equations [53c,d] with signs reversed:

$$b_1 = B\,[a_{21} k_1 k_2 + L s_2 k_1 + D\omega^2\,(hm - Lm + LC + E_L)]$$

$$b_2 = \omega^2 D\,(c_1 E_L + c_2 B)$$

$$b_3 = -\, k_1 k_2 + \omega^2\,[m s_1 k_2 + I_\theta s_2 k_1$$

$$+ hm\,(a_{12} + a_{21})\,k_1 k_2 + D c_1 c_2] - DmI_o \omega^4$$

$$e_1 = Dc_1 LB$$

$$e_2 = -\,(s_1 + La_{12} k_1)\,k_2 B + a_{21} k_1 k_2 C - s_2 k_1 E_L$$

$$+ \omega^2 D \left[(I_\theta - hLm)\,B + hmC + mE_L \right]$$

$$e_3 = -\, c_1 s_1 k_2 - c_2 s_2 k_1 + \omega^2 D\,(c_1 I_\theta + c_2 m)$$

S must be a common root of both of the quadratic Equations [54a,b]. Two such equations can have a common root only for special values of the coefficients. If a common root exists, a simple formula is easily found by eliminating S^2, thus, from Equation [54a,b]:

$$e_1\,(b_1 S^2 + b_2 S + b_3) - b_1\,(e_1 S^2 + e_2 S + e_3)$$

$$\equiv (e_1 b_2 - b_1 e_2)\,S + e_1 b_3 - b_1 e_3 = 0 \qquad [55]$$

A general procedure for seeking a positive common root of Equations [54a,b] may now be given as follows:

If, at the ω chosen in calculating the parameters, $e_1b_2 - b_1e_2 \neq 0$, determine S from Equation [55] using the second form of the left member; and, if $S > 0$, test this value by substituting it into Equation [54a] if $b_1 \neq 0$ or into Equation [54b] if $e_1 \neq 0$. If the chosen one of Equations [54a,b] is satisfied, simple reasoning from Equation [55], using the first form of the left member, shows that the other one of Equations [54a,b] is also satisfied, so that S is a common root. If this procedure cannot be used, one of the following procedures is available.

If $e_1b_2 - b_1e_2 = 0$, then $e_1b_3 - b_1e_3 = 0$ also, to satisfy Equation [55]; otherwise there is no common root. If $b_1 \neq 0$ and $e_1 \neq 0$, it follows that $e_2 = (e_1/b_1)\, b_2$ and $e_3 = (e_1/b_1)\, b_3$, and, of course, $e_1 = (e_1/b_1)\, b_1$. Thus the two Equations [54a,b] are proportional and have the same roots, which may be found by solving either equation provided it contains S. If either $b_1 = 0$ or $e_1 = 0$ but not both, one of Equations [54a,b] reduces to $0 = 0$ and the other must be solved for S.

Finally, if $b_1 = e_1 = 0$, Equations [54a,b] are, at most, linear in S; and S determined from one of these equations may be tested in the other equation.

In the very rare case that $b_1 = b_2 = b_3 = e_1 = e_2 = e_3 = 0$, steady vibration of the system may occur at the assumed ω with any value of S.

If one of these procedures does not yield a common root for Equations [54a,b], then there is no common root and steady vibration cannot occur at the chosen ω. Furthermore, for our present purpose any negative S is to be rejected. Usually no acceptable S will be found; by repeated trials, using interpolation wherever useful, the rare values of ω^2 at which an acceptable S occurs must be ascertained. For a complete solution, the entire range of positive values of ω must be explored (see the case of small ql discussed later).

6.3 THE SINGULAR CASE: $s_1s_2 - a_{12}a_{21}k_1k_2 = 0$, HENCE $D = 0$

If a chosen value of ω^2 makes $D = 0$, the deduction given for Equations [53c,d] with corresponding values of $b_1 \ldots e_3$ is not valid because at some stage it is necessary to divide by D. Equations [45a,b] through [50a,b] are all valid; but now Equations [50a,b] yield only the ratio v/θ or θ/v, the two equations being obviously equivalent if s_1, s_2, a_{12}, and a_{21} are all nonzero. The same relations follow from Equations [51a,b] and [52a,b] if these equations are first multiplied through by D and D is then made zero. The relations that hold when $D = 0$ are, from Equations [50a,b]:

$$s_2 v = a_{12}k_2\theta \;;\quad a_{21}k_1 v = s_1\theta \qquad [56a,b]$$

The difficulty may be overcome by choosing a different pair of variables from among v, θ, F, and G. When $s_1 \neq 0$, a convenient choice is the pair v and G. From Equations [49a] and [56b]:

$$s_1 F = k_1 v - a_{12}k_1 G \;;\quad s_1\theta = a_{21}k_1 v \qquad [57a,b]$$

Multiplication of Equations [46a,b] by s_1 and then elimination of θ and F by means of [57a,b] gives as equations of motion for the system in terms of v and G as variables:

$$m(s_1 + a_{21}k_1h)\ \ddot{v} + [s_1\ (c_1 + BS) - a_{21}k_1CS]\ \dot{v}$$

$$+ k_1\ (1 - a_{21}BS^2)\ v - a_{12}k_1G = 0$$

$$(s_1hm + a_{21}k_1I_\theta)\ \ddot{v} + [s_1LBS + a_{21}k_1\ (c_2 + E_LS)]\ \dot{v}$$

$$- a_{21}k_1LBS^2v + s_1G = 0$$

and if v and G are proportional to $e^{i\omega t}$:

$$\{- \omega^2m\ (s_1 + a_{21}k_1h) + k_1 - a_{21}k_1BS^2$$

$$+ i\omega\ [s_1\ (c_1 + BS) - a_{21}k_1CS]\}\ v - a_{12}k_1G = 0$$

$$\{- \omega^2\ (s_1hm + a_{21}k_1I_\theta) - a_{21}k_1LBS^2$$

$$+ i\omega\ [s_1LBS + a_{21}k_1\ (c_2 + E_LS)]\}\ v + s_1G = 0$$

Equating to zero the determinant of the coefficients of v and G and dividing the imaginary part by $i\omega$ yields:

$$s_1\ [-\omega^2m\ (s_1 + a_{21}k_1h)\ + k_1 - a_{21}k_1BS^2]$$

$$- a_{12}k_1\ [\omega^2\ (s_1hm + a_{21}k_1I_\theta) + a_{21}k_1LBS^2] = 0$$

$$s_1\ [s_1\ (c_1 + BS) - a_{21}k_1CS]$$

$$+ a_{12}k_1\ [s_1LBS + a_{21}k_1\ (c_2 + E_LS)] = 0$$

Now multiply these two equations by k_2, substitute into them $a_{12}a_{21}k_1k_2 = s_1s_2$ from $D = 0$, and then divide the equations thus obtained by s_1 on the assumption that $s_1 \neq 0$. Equations [53c,d] with D set equal to zero are the result.

This proof fails, however, if $s_1 = 0$. Then, as a third alternative, θ and F may be used as variables. For the present purpose, it is assumed that $s_1 = 0$ and, hence, also that $a_{12}a_{21} = 0$ (since $D = 0$ and $k_1k_2 > 0$). Then Equations [49b] and [56a] give the formulas:

$$s_2G = k_2\theta - a_{21}k_2F \ ; \quad s_2v = a_{12}k_2\theta$$

Substitution from these formulas into Equations [46a,b], multiplied by s_2 gives, if $\theta \propto F \propto e^{i\omega t}$:

$$[-\omega^2 m\,(a_{12}k_2 + s_2h) - s_2BS^2$$

$$+ i\omega a_{12}k_2\,(c_1 + BS) - i\omega s_2CS]\ \theta + s_2F = 0$$

$$[-\omega^2\,(s_2I_\theta + a_{12}k_2hm) + k_2 - s_2LBS^2$$

$$+ i\omega s_2\,(c_2 + E_LS) + i\omega a_{12}k_2LBS]\ \theta - a_{21}k_2F = 0$$

and the usual procedure of setting the determinant equal to zero gives (since $a_{12}a_{21} = 0$ here), all signs being reversed:

$$-\omega^2\,[s_2^2I_\theta + s_2hm\,(a_{12} + a_{21})\,k_2]$$

$$+ s_2k_2 - s_2\,(a_{21}k_2 + s_2L)\,BS^2 = 0$$

$$s_2a_{12}k_2LBS + s_2^2\,(c_2 + E_LS) - s_2a_{21}k_2CS = 0$$

If now $s_2 \neq 0$ and these equations are multiplied by k_1/s_2, they become what is left of Equations [53c,d] in case $D = 0$ and $s_1 = 0$.

If $s_1 = s_2 = 0$, however, neither of these two proofs holds. Then again $a_{12}a_{21} = 0$, and three further alternatives may occur.

If $a_{12} = 0$ but $a_{21} \neq 0$, Equations [49a,b] give $v = 0$, $F = \theta/a_{21}$. Then [46a] requires $hm\omega^2 = (1/a_{21}) - BS^2$ and also, to remove $\dot{\theta}$, $CS = 0$. If $C \neq 0$, $S = 0$; if $C = 0$, any S may occur together with the proper ω. Equation [46b] merely gives the value of G.

If $a_{21} = 0$ but $a_{12} \neq 0$, then Equations [49a,b] give $\theta = 0$, $G = v/a_{12}$. From [46b], $hm\omega^2 = 1/a_{12}$, $LBS = 0$. Equation [46a] gives F.

If $a_{12} = a_{21} = 0$, then $v = \theta = 0$ and, hence, by Equations [46a,b] $F = G = 0$; and by Equations [47a,b] $v_o = \theta_o = 0$. Thus, if $s_1 = s_2 = a_{12} = a_{21} = 0$, there can be no harmonic vibration at the ω for which these parameters were calculated.

Examination now shows that all of these conclusions as to ω and S follow also from Equations [53c,d] when $D = 0$ and the parameters have the specified values.

The general validity of Equations [53c,d] is thus established.

6.4 DISCUSSION OF DAMPING EFFECTS

Components of force or moment on the foil that are proportional to velocities have a damping effect, positive or negative. Included here are the force $- c_1\dot{v}$ and the components $- BS\dot{v}$ and $CS\dot{\theta}$ in F_L, assumed to act through the θ axis, whose velocity is $\dot{v}$. Then there are also the moment $- c_2\dot{\theta}$ and the terms $- LBS\dot{v}$ and $- E_LS\dot{\theta}$ in M_θ, acting at the velocity $\dot{\theta}$. These forces and moments do work on the beam-foil system at the total rate (see Section 8.3 of Reference 3 and Appendix H of Reference 4):

$$[- (c_1 + BS)\,\dot{v} + CS\dot{\theta}]\,\dot{v} + (-c_2\dot{\theta} - LBS\dot{v} - E_LS\dot{\theta})\,\dot{\theta}$$

or

$$- (c_1 + BS)\, \dot{v}^2 + (C - LB)\, S\dot{v}\dot{\theta} - (c_2 + E_L S)\, \dot{\theta}^2$$

This expression may be positive or negative. If, however, the Theodorsen relations are used, namely:

$$B = \pi\rho b \ell_f$$

$$C = B\left(\frac{3}{2} b + e\right) \;;\; L = \frac{1}{2} b - e \;;\; E_L = B\left(\frac{1}{2} b + e\right)^2$$

then $C - LB = B\,(b + 2e)$, and the rate of work is:

$$-c_1\dot{v}^2 - c_2\dot{\theta}^2 - BS\left[\dot{v}^2 - (b + 2e)\,\dot{v}\dot{\theta} + \left(\frac{1}{2} b + e\right)^2 \dot{\theta}^2\right]$$

$$= - c_1\dot{v}^2 - c_2\dot{\theta}^2 - BS\left[\dot{v} - \left(\frac{1}{2} b + e\right)\dot{\theta}\right]^2$$

With these values, therefore, if $c_1 \geqq 0$, $c_2 \geqq 0$, and $BS > 0$, there is no gain but, in general, a loss of energy from the beam-foil system during harmonic motion. Somehow, such a loss must be compensated for through work done by the BS^2 terms in F_L and M_θ.

If $s = 0$, the c_1 and c_2 terms alone will cause a loss of energy unless $c_1 = c_2 = 0$; and when $S = 0$, the equations of motion are correct for any type of motion (whether harmonic or not).* Consequently any existing motion must die out unless maintained by additional forces. The same conclusion will probably hold at first as S increases from 0. Eventually, however, the rate of loss may decrease and become zero at a certain speed

* The Theodorsen terms are derived only for *harmonic* motion. If, however, $S = 0$, the Theodorsen terms disappear and, in general, the equations of motion hold. General equations are needed to deal accurately with damped motion.

S_c', and it may safely be assumed that the motion will then be harmonic. As S is increased above S_c, it is almost certain that the damping will be negative, so that any vibration tends to build up. At one or more higher speeds, harmonic motion may again become possible; however, this is of little practical interest. Thus S_c, the lowest value of S at which steady harmonic motion can occur, constitutes the critical flutter speed that must not be exceeded in practical operation.

6.5 THREE SPECIAL CASES

6.5.1 Damping Due to Lift Only ($c_1 = c_2 = 0$)

If $c_1 = c_2 = 0$, so that damping arises only from terms in F_L and M_θ, then S may be canceled out of Equations [53b] and [53d] entirely because every term is then linear in S. Also, only the S^2 term occurs now in Equation [53c] or [53a] but not in Equations [53b] and [53d]. These features open the way to the following simpler mode of solution: having assumed a value of ω and calculated a_{11}, a_{12}, a_{21}, and a_{22}, calculate ω^2 from either [53b] or [53d]; if this value of ω^2 agrees with ω as assumed, then harmonic motion is possible at this ω, provided S has a value that satisfies either Equations [53a] or [53c]. This value is easily found.

If the two values of ω do not agree or if S^2 turns out to be zero or negative, then the assumed ω must be rejected.

6.5.2 Foil Rigidly Mounted on the Beam

The equations may be adapted to the assumption of rigid mounting by letting $k_1 \to \infty$ and $k_2 \to \infty$. Since $s_1/k_1 = (1/k_1) + a_{11}$ and $s_2/k_2 = (1/k_2) + a_{22}$ and $D/(k_1k_2) = (s_1s_2 - a_{12}a_{21}k_1k_2)/(k_1k_2)$, the following reduction formulas hold as $k_1 k_2 \to \infty$

$$\frac{s_1}{k_1} \to a_{11} \; ; \quad \frac{s_2}{k_2} \to a_{22}$$

$$\frac{D}{k_1 k_2} \to D_o = a_{11}a_{22} - a_{12}a_{21} \; ; \quad D^{-1}k_1 k_2 \to D_o^{-1}$$

The limiting forms of most equations become evident after suitable manipulations to make these formulas applicable, Equations [53c,d] being divided through by $k_1 k_2$.

Equations [46a,b] and [48a,b] require no change. Equations [47a,b] divided by k_1 or k_2 give $v_o = v$ and $\theta_o = \theta$. In Equations [49a,b], [50a,b], [51a,b], [53a,b,c,d], and [54a,b], the final effect is simply to replace D by D_o, s_1 by a_{11}, and s_2 by a_{22}, and to erase k_1 and k_2 everywhere, the term $k_1 k_2$ in [53c] being replaced by 1.

The equations do not seem, however, to be simpler in this case or modified in a mathematically interesting way.

6.5.3 Low-Frequency Approximation

In the theory of a uniform beam under forcing, as given in Section 2, formulas were found for a_{11}, a_{12}, a_{21}, and a_{22} in terms of E, I, the length ℓ of the beam, and two dimensionless parameters $q\ell$ and ξ defined thus:

$$q\ell = \left(\frac{\mu}{EI}\right)^{1/4} \omega^{1/2} \ell; \qquad \xi = \frac{EI}{2KAG_e} \frac{1}{\ell^2} (q\ell)^2 \qquad [58a,b]$$

Here μ is the effective mass of the beam per unit length,*

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* μ includes the effect of virtual mass when the beam is immersed in a fluid.

E is Young's modulus,

G_e is the shear constant usually denoted by G,

A is the area of the cross section,

I is its area moment of inertia about its principal axis, and

K is the usual shear-warping constant.

If ω is small enough, $q\ell$ and ξ will both be much less than unity. Specifically, $q\ell << 1$ and $\xi << 1$ if

$$\omega << \frac{1}{\ell^2}\left(\frac{EI}{\mu}\right)^{1/2} \quad ; \quad \omega << 2KAG_e/\sqrt{\mu EI} \qquad [59a,b]$$

The second limit on ω is obtained upon substituting for $q\ell$ in the expression given for ξ and setting the result << 1.

When ξ is so small, all shear effects in the beam may be ignored and the approximations derived in Section 3 for $q\ell << 1$ may be used, namely:

$$a_{11} \doteqdot -\frac{4}{\mu\ell\omega^2} \; ; \quad a_{12} \doteqdot a_{21} \doteqdot \frac{6}{\mu\ell^2\omega^2} \; ; \quad a_{22} \doteqdot -\frac{12}{\mu\ell^3\omega^2} \qquad [60a,b,c]$$

Let ω be small enough so that

$$\omega^2 << \frac{4k_1}{\mu\ell} \; ; \quad \omega^2 << \frac{12k_2}{\mu\ell^3} \qquad [61a,b]$$

Then, using the approximations for a_{11} and a_{22}, $|a_{11}k_1| >> 1$ and $|a_{22}k_2| >> 1$. Hence:

$$s_1 \doteqdot a_{11}k_1 \; ; \quad s_2 \doteqdot a_{22}k_2 \; ; \quad D \doteqdot \frac{12k_1k_2}{\mu^2\ell^4\omega^4}$$

After inserting the approximations into the formula:

$$D = s_1s_2 - a_{12}a_{21}k_1k_2$$

Upon substituting these approximations and also writing $\ddot{v}$ for $-\omega^2 v$ and $\ddot{\theta}$ for $-\omega^2\theta$, Equations [50a,b] and [46a,b] divided by $k_1 k_2$ take the approximate forms:

$$F = \mu\ell\ddot{v} + \frac{1}{2}\mu\ell^2\ddot{\theta} \; ; \quad G = \frac{1}{2}\mu\ell^2\ddot{v} + \frac{1}{3}\mu\ell^3\ddot{\theta} \qquad [62a,b]$$

$$m'\ddot{v} + h'm'\ddot{\theta} - BS^2\theta + (c_1 + BS)\dot{v} - CS\dot{\theta} = 0 \qquad [62c]$$

$$h'm'\ddot{v} + I'_\theta\ddot{\theta} - LBS^2\theta + LBS\dot{v} + (c_2 + E_L S)\dot{\theta} = 0 \qquad [62d]$$

where

$$m' = m + \mu\ell \; ; \quad h'm' = hm + \frac{1}{2}\mu\ell^2 \; ; \quad I'_\theta = I_\theta + \frac{1}{3}\mu\ell^3 \qquad [63a,b,c]$$

Also, Equations [52a,b] become:

$$v_o = \left(1 + \frac{\mu\ell\omega^2}{k_1}\right)v + \frac{\mu\ell^2\omega^2}{2k_1}\theta$$

$$\theta_o = \frac{\mu\ell^2\omega^2}{2k_2} v + \left(1 + \frac{\mu\ell^3\omega^2}{3k_2}\right)\theta$$

Treated in the usual way (i.e., equating determinant to zero), Equations [62c,d] give as approximate conditions for harmonic vibration:

$$\omega^2\,[m'I'_\theta - (h'm')^2] - c_1c_2 - (c_1E_L + c_2B)S$$

$$- [h'm' + E_L + L\,(C - m')]\,BS^2 = 0 \qquad [64a]$$

$$\omega^2\,[(c_2 + BS)\,I'_\theta + (c_2 + E_LS)\,m' + h'm'CS - Lh'm'BS]$$

$$+ c_1LBS^2 = 0 \qquad [64b]$$

These equations are simpler than Equations [53a,b,c,d], especially because ω affects the coefficients here only through the factor ω^2. The final search for a useable S, however, is about as complicated.

A useful minimum for ω^2 can be obtained from Equation [64a] provided the coefficients of S and S^2 are not negative, namely:

$$\omega^2 \geqq c_1 c_2 \left[m'I'\theta - (h'm')^2\right]^{-1}$$

Otherwise, the determination of a minimum ω is more complicated.

In Section 3 it was remarked that the approximate values of a_{11}, a_{12}, a_{22}, and a_{21} are the same as the values for a slender uniform rigid rod of length ℓ and mass $\mu\ell$ forced at one end. For the rod, $a_{12} = a_{21}$. Thus the equations derived here for $q\ell << 1$ are applicable also to a foil attached at one end of such a rod.

It is interesting also that the equations still hold if $\ell = 0$, which corresponds to removal of the beam altogether. Thus the foil may exhibit flutter all by itself. Usually, any low-frequency harmonic vibration of the foil that can occur with the beam attached can also occur without the beam, and at the same ω and S, provided the mass of the foil is increased from m to m' and its moment of inertia from I_θ to I'_θ, and provided h is changed to a value h' such that $h'm' = hm + \frac{1}{2}\mu\ell^2$.

6.6 SIMILITUDE

An interesting question is the relative influence of the various parameters of the foil-beam system upon the cricitcal flutter speed. Relevant *foil parameters* are m, I_o, h, and L; the lift constants B, C, and E_L (or the foil constants $\rho\ell_f$, b, and e); and c_1 and c_2. Also, there are the attachment elasticities k_1 and k_2. The influence of the *beam* is represented by the values of a_{11}, a_{12}, a_{21}, and a_{22}. Apparently, in general, the relative influence of these parameters can be ascertained only by making extensive numerical calculations. A major source of difficulty lies in the possible variation of the a_{ij}'s over all real values.

A simple form of similitude, however, is readily discovered. In Section 2 formulas are given for forced vibration of a uniform beam when F and G vibrate at the same circular frequency ω but perhaps in different phases (for example, $F = A \sin \omega t$, $G = B \sin (\omega t + \phi)$). The mode of vibration of the beam was shown there to depend upon two *dimensionless parameters* $q\ell$ and ξ whose definitions, as given in Equations [58a,b], may conveniently be rewritten as follows:

$$\xi = \left(\frac{EI}{2KAG_e} \frac{1}{\ell^2} \right) (q\ell)^2 \quad ; \quad \omega = \sqrt{\frac{EI}{\mu}} \cdot \frac{1}{\ell^2} (q\ell)^2 \qquad [65a,b]$$

Here ℓ denotes the beam length and $q\ell$ may have any positive value. The formulas obtained for the a_{ij}'s can be written thus:

$$a_{11} = \bar{a}_{11} \frac{\ell^3}{EI} \; ; \quad a_{12} = \bar{a}_{12} \frac{\ell^2}{EI} \; ; \quad a_{21} = \bar{a}_{21} \frac{\ell^2}{EI} \; ; \quad a_{22} = \bar{a}_{22} \frac{\ell}{EI} \qquad [66a,b,c,d]$$

where $\bar{a}_{11}$, $\bar{a}_{12}$, $\bar{a}_{21}$, and $\bar{a}_{22}$ are complicated functions of $q\ell$ and ξ but are otherwise independent of the beam parameters.*

To obtain simple similitude, variation of $q\ell$ and ξ must be avoided. A group of foil-beam systems that can vibrate at the same $q\ell$ and ξ will be compared, and the comparison will include only vibrations of these systems

* To infer Formulas [66a,b,c,d] from those given below Equations [12a,b] in Section 2, q_1 and q_2 being defined there by Equations [5a,b], note that $q_1\ell$ and $q_2\ell$ are functions of $q\ell$ and ξ and, consequently, the same is true of the functions s, c, S, and C. Elsewhere, replace q_1 and q_2 by $(q_1\ell)/\ell$ or $(q_2\ell)/\ell$, and replace q^2 by multiplying EI by $(q\ell)^2/\ell^2$.

at a chosen $q\ell$ and ξ.

To facilitate thinking, choose one system out of the chosen group as a reference system and indicate the quantities that refer to it by adding a prime. For any other system of the group write $\ell = \lambda\ell'$, $EI = \varepsilon\lambda^4(EI)'$. Thus within the group:

$$\ell \propto \lambda \; ; \quad EI \propto \varepsilon\lambda^4$$

with λ and ε denoting arbitrary numbers that equal unity for the reference system. The factor λ^4 is introduced here because, in a simple change of scale with all dimensions increased in ratio λ, $I \propto \lambda^4$. The factor thus represents the effect on EI of any change in the shape of the beam and perhaps in E. Then, from Equations [66a,b], among the beams of the chosen group:

$$a_{11} \propto \frac{1}{\varepsilon\lambda} \; ; \quad a_{12} \propto a_{21} \propto \frac{1}{\varepsilon\lambda^2} \; ; \quad a_{22} \propto \frac{1}{\varepsilon\lambda^3} \qquad [67a,b,c]$$

To make ξ the same for all of the vibrations considered, the factor $KAG_e\ell^2/(EI)$ in Equation [65] must be the same for all beams. Hence, it must be that $KAG_e \propto \varepsilon\lambda^2$. Also, $\mu = \rho_b A$ in terms of the density ρ_b of the beam material. Thus

$$\frac{\mu}{\mu'} = \frac{\rho_b A}{\rho'_b A'} = \frac{\rho_b K'G'_e}{\rho'_b KG_e} \; \frac{KAG_e}{K'A'G'_e}$$

or

$$\frac{\mu}{\mu'} = \tau\varepsilon\lambda^2 \; ; \quad \tau = \frac{\rho_b}{\rho'_b} \; \frac{K'G'_e}{KG_e}$$

For the reference beam, $\tau = 1$. Thus among the vibrating beams it must be that

$$KAG_e \propto \varepsilon\lambda^2 \; ; \quad \mu \propto \tau\varepsilon\lambda^2 \; ; \quad \omega \propto \frac{1}{\lambda\sqrt{\tau}} \qquad [68a,b,c]$$

The variation of ω is inferred from Equation [65b]. For any two beams made of the same material and with cross sections of the same shape, so that KG_e and τ are the same, $A \propto \varepsilon\lambda^2$ and $I \propto \varepsilon\lambda^4$.

The validity of the fundamental Equations [46a,b], [47a,b], and [48a,b] for all beams must now be secured. Tentatively, the rule that all terms in a given equation must vary in the same ratio from one vibrating system to another will be followed.

Equations [48a,b] read, for the reference system (denoted here by primes) and for any other chosen system, in view of Equations [67a,b,c]:

$$v_o' = a_{11}'F' + a_{12}'G' \; ; \qquad \theta_o' = a_{21}'F' + a_{22}'G'$$

$$v_o = \frac{a_{11}'}{\varepsilon\lambda} F + \frac{a_{12}'}{\varepsilon\lambda^2} G \; ; \qquad \theta_o = \frac{a_{21}'}{\varepsilon\lambda^2} F + \frac{a_{22}'}{\varepsilon\lambda^3} G$$

Here it can be assumed that the time factor in F is the same for all beams (perhaps "sin ωt"). Furthermore, since the amplitude of any vibration may be arbitrarily varied, it does no harm to assume that the amplitudes are such that $(a_{11}'/\varepsilon\lambda)\,F$ is the same for all beams and hence equal to $a_{11}'F'$. Then $F \propto \varepsilon\lambda F'$. Also, according to the general rule being followed, the second term $(a_{12}'/\varepsilon\lambda^2)\,G$ must then equal $a_{12}'G'$, so that $G = \varepsilon\lambda^2 G'$. It follows that the phase of G is the same for all beams (perhaps $G \propto \varepsilon\lambda^2 \sin(\omega t + \phi)$, ϕ constant).

Thus $v = v_o'$ for the vibrations considered, but substitution for F and G gives $\theta_o = \theta_o'/\lambda$. Summarizing, among the chosen vibrating beams:

$$F \propto \varepsilon\lambda \; ; \quad G \propto \varepsilon\lambda^2 \; ; \quad v_o \propto 1 \; ; \quad \theta_o \propto 1/\lambda \qquad [68d,e,f,g]$$

It should be noted that v_o is in the same phase on all beams, and so is θ_o, but these phases may differ from those of F and G.

Consider next the beam-foil relations expressed by Equations [47a,b], which may be written:

$$v = v_o + \frac{F}{k_1} \; ; \quad \theta = \theta_o + \frac{G}{k_2}$$

To make F/k_1 and G/k_2 vary the same as v_o and θ_o, respectively, assume $k_1 \propto F$, $k_2 \propto \lambda G$. Then

$$k_1 \propto \varepsilon\lambda \; ; \quad k_2 \propto \varepsilon\lambda^3 \; ; \quad v \propto v_o \propto 1 \; ; \quad \theta \propto \theta_o \propto 1/\lambda \quad [68h,i,j,k]$$

Turning now to the foil, assume that the values F', G', v', and θ' for the reference system satisfy Equations [46a,b] for a certain set of values of the foil parameters. Make these equations valid for all other systems of the group in such a way that all terms of each equation vary in the same ratio; that is, so that they are $\propto F \propto \varepsilon\lambda$ in [46a] and $\propto G \propto \varepsilon\lambda^2$ in [46b]. The necessary variation of the foil parameters and of S is found to be:

$$m \propto \tau\varepsilon\lambda^3 ; \quad h \propto \lambda \; ; \quad I_\theta \propto I_o \propto \tau\varepsilon\lambda^5$$

$$c_1 \propto \varepsilon\lambda^2 \sqrt{\tau} \; ; \quad c_2 \propto \varepsilon\lambda^4 \sqrt{\tau} \; ; \quad S \propto 1/\sqrt{\tau}$$

$$B \propto \tau\varepsilon\lambda^2 \; ; \quad C \propto \tau\varepsilon\lambda^3 \; ; \quad E_L \propto \tau\varepsilon\lambda^4 \; ; \quad L \propto \lambda$$

Or, if B, C, and E_L have Theodorsen's values as stated following Equations [45a,b], then

$$b \propto \lambda \; ; \quad e \propto \lambda \; ; \quad \ell_f \propto \tau\varepsilon\lambda/\zeta$$

where ζ is the specific gravity of the ambient fluid.

Examples of the calculations leading to these expressions are:

(a) Since $\ddot{v} = -\omega^2 v, m\ddot{v} \propto m\omega^2 v \propto (m/\lambda^2\tau) \cdot 1$; this is $\propto \varepsilon\lambda$ if $m \propto \tau\varepsilon\lambda^3$. I_θ is obtained similarly.

(b) To make $(c_1 + BS)\,\dot{v} \propto \varepsilon\lambda$, where $\dot{v} \propto \omega v \propto 1/(\lambda\sqrt{\tau})$, it is necessary that $(c_1 + BS) \propto \varepsilon\lambda^2\sqrt{\tau}$. This is most simply secured by making $c_1 \propto BS \propto \varepsilon\lambda^2\sqrt{\tau}$.

(c) To make $BS^2\theta \propto \varepsilon\lambda$, let $BS^2 \propto \varepsilon\lambda^2$. Dividing this by $BS \propto \varepsilon\lambda^2\sqrt{\tau}$ gives $S \propto 1/\sqrt{\tau}$, and dividing $S \propto 1\sqrt{\tau}$ into $BS \propto \varepsilon\lambda^2\sqrt{\tau}$ gives $B \propto \tau\varepsilon\lambda^2$.

(d) If Theodorsen's values for the lift are used,

$$\frac{3}{2}b + e = C/B \propto \lambda \; ; \quad \frac{1}{2}b + e = \sqrt{E_L/B} \propto \lambda$$

whence

$$b = C/B - \sqrt{E_L/B} \propto \lambda \; ; \quad e = (C/B) - (3/2)\,b \propto \lambda$$

Also, $\rho\ell_f = B/(\pi b) \propto \tau\varepsilon\lambda$. Write $\rho = \zeta\rho_o$, ζ being the specific gravity of the fluid bathing the foil and ρ_o the density of pure water. Then, ρ_o being the same for all foils, $\ell_f \propto \tau\varepsilon\lambda/\zeta$.

As a special case, if the systems actually differ only in scale, all linear dimensions varying as λ, then $\varepsilon = \tau = 1$ and the similitude just described requires that $m \propto \lambda^3$, $I_\theta \propto \lambda^5$, $c_1 \propto \lambda^2$, $c_2 \propto \lambda^4$, $L \propto \lambda$, $B \propto \lambda^2$, $C \propto \lambda^3$, $E_L \propto \lambda^4$ (or $b \propto e \propto \ell_f \propto \lambda$), with S and ζ remaining constant for all systems. Also, $\omega \propto 1/\lambda$.

The general similitude described here might conceivably serve as the basis for a model test, although the limited variability of S is likely to be inconvenient.*

In practical cases the influence of shear warping tends to be small. It may be sufficiently accurate to omit this effect by letting $K \to \infty$ (hence $\xi = 0$). For this case write the relation $\mu = (\rho_b A/\rho_b' A')\ \mu'$ in the form:

$$\mu = \eta \lambda^2 \mu' \quad ; \qquad \eta = \frac{1}{\lambda^2} \ \frac{\rho_b A}{\rho_b' A'}$$

Thus $\tau\varepsilon$ is replaced by η or τ by η/ε in all formulas, and

$$\mu \propto \eta \lambda^2 \ ; \qquad \omega \propto \frac{1}{\lambda} \sqrt{\frac{\varepsilon}{\eta}} \ ; \qquad S \propto \sqrt{\frac{\varepsilon}{\eta}}$$

Since $\varepsilon \propto EI/\lambda^4$, ε can easily be varied without affecting A or η; for example, by changing all linear dimensions of the cross sections parallel to the principal plane of the beam in a ratio r and the perpendicular dimensions in the ratio 1/r, thus A is left unchanged. Variation of S in any desired ratio can be effected in the same manner.

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* With $q\ell$ invariant, $\tau = \dfrac{\rho_b}{\rho_b'} \ \dfrac{K' G_e'}{K G_e}$. Since ρ_b, K, and G_e cannot vary much, then the variation of S is limited.

7. TWO SIMPLE CASES OF FLUTTER

The mathematical theory of flutter is so complicated in realistic cases that it is not easy to form intuitive ideas of the flutter process. For this reason a certain interest may attach to simple cases that are easily understood. It apprears that typical flutter can occur only if at least two degrees of freedom are coupled together; and commonly, but not always, there is a difference of phase between the two motions.

Two relatively simple examples are described in detail. Although the theory given here might conceivably be roughly applicable to some actual problem, the principal aim in devising these cases has been to make them simple.

In both cases a rigid foil is assumed to be immersed in a uniform stream, and for the lift F_L on the foil the simple steady-motion approximation is used:*

$$F_L = - BS\dot{v} + BS^2\theta$$

In steady motion, θ is conveniently assumed to represent merely a small inclination of the foil to the approaching stream and $\dot{v}$, a slow perpendicular velocity of translation. This formula is known to provide a fair approximation also when θ varies slowly, as is assumed here. Then θ must represent rotation about an axis parallel to the foil and perpendicular to the stream, while v may be regarded as denoting a displacement of a line on the foil located at the axis of rotation, v being

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* In Appendix H of Reference 4, it is shown that the effect of the $\dot{\theta}$ term is relatively small if $\frac{b\omega}{S}$ is small. Hence, the $\dot{\theta}$ term is dropped here.

perpendicular to both the stream and the foil. The assumed axis of rotation may be displaced into any parallel position in the plane of the foil without affecting the motion provided an appropriate change is made in v.

The foil is assumed to be attached to some structure but it is assumed to be so long that disturbance of the lift by the presence of the structure may reasonably be neglected.

7.1 FOIL ATTACHED ELASTICALLY WITH DAMPING TO AN IMMOBILE BASE

Let the foil be attached so that it can undergo displacements v and θ as described and, to simplify the formulas, locate the θ axis so that the lift acts through it. When the foil is displaced, let an elastic force $-k_1 v$ and a frictional force $-c_1 \dot{v}$ act on it along the same line as F_L and positive in the same direction. Also let an elastic moment $-k_2\theta$ and a frictional moment $-c_2\dot{\theta}$ act on it about the θ axis and positive in the same direction as θ.

Let the foil be so positioned that the lift acts at a distance L from the center of mass of the foil,* L being positive when the lift is displaced from the center of mass toward the approaching stream. Denote by m the foil mass and by I its moment of inertia** about an axis drawn through its center of mass and parallel to the θ axis. See Figure 4 where directions are shown for positive v, $\dot{v}$, θ, $\dot{\theta}$, F_L and L.

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* With $\dot{\theta}$ terms present, no single location for action of F_L was considered. In the present case such a location is specified. L here is not the same as in Equation [45b].

** Note that here the moment of inertia of the foil is denoted by I instead of I_o, to allow use of I_o for a rigid body in Section 7.2.

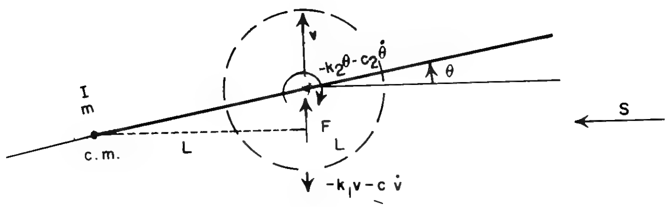

Figure 4 – Forces on and Motions of a Foil Attached Elastically to an Immobile Base

Since the displacement of the center of mass of the foil is $v - L\theta$, then the equations of motion for the foil will be:

$$m\,(\ddot{v} - L\ddot{\theta}) = -\,k_1 v - c_1\dot{v} + F_L$$

$$I\ddot{\theta} = -\,k_2\theta - c_2\dot{\theta} + L\,(\,-\,k_1 v - c_1\dot{v} + F_L)$$

The second equation may be simplified by subtracting from it L times the first equation. Then, inserting also the value of F_L, the equations of motion take the form:

$$m\ddot{v} - Lm\ddot{\theta} + k_1 v + (c_1 + BS)\,\dot{v} - BS^2\theta = 0 \qquad [69a]$$

$$(I + L^2m)\,\ddot{\theta} - Lm\ddot{v} + k_2\theta + c_2\dot{\theta} = 0 \qquad [69b]$$

The v and θ motions are thus coupled together inertially through the Lm terms.

To search for harmonic motion, make the usual mathematical assumption that $v \propto \theta \propto e^{i\omega t}$. Canceling $e^{i\omega t}$ results in the equations:

$$[-\omega^2 m + k_1 + i\omega\,(c_1 + BS)]\,v + (\omega^2 Lm - BS^2)\,\theta = 0 \qquad [70a]$$

$$\omega^2 Lmv + [-\omega^2\,(I + L^2m) + k_2 + i\omega c_2]\,\theta = 0 \qquad [70b]$$

Equating to zero the determinant of these two equations gives:

$$[-\omega^2 m + k_1 + i\omega\,(c_1 + BS)]\,[-\omega^2\,(I + L^2m) + k_2 + i\omega c_2]$$
$$-\omega^2 Lm\,(\omega^2 Lm - BS^2) = 0$$

Multiplying out, then equating to zero separately the real and imaginary parts and dividing the latter by $i\omega$, on the assumption that $\omega \neq 0$, yields:

$$\omega^4 mI - \omega^2 [mk_2 + (I + L^2 m) k_1$$

$$+ c_2 (c_1 + BS) - LmBS^2] + k_1 k_2 = 0 \qquad [71a]$$

$$-\omega^2 [(c_1 + BS)(I + L^2 m) + c_2 m] + (c_1 + BS) k_2 + c_2 k_1 = 0 \qquad [71b]$$

The second equation can also be written

$$\omega^2 = \frac{(c_1 + BS) k_2 + c_2 k_1}{(c_1 + BS)(I + L^2 m) + c_2 m} \qquad [71b']$$

In both equations all symbols denote positive quantities except that it may be interesting to try to make c_1, c_2, k_1, k_2, or S zero. Hence, Equation [71b] cannot require a negative value of ω^2.

Equations [71a,b] fix ω^2 and S but, in general, an algebraic solution is not practical. Substituting ω^2 from [71b'] into [71a] and clearing of fractions gives a quartic equation in S, or a cubic if either k_1 or k_2 is zero (or both), so that the term $k_1 k_2$ disappears and Equation [71a] can be divided through by ω^2. The most feasible method of numerical solution would probably be to assume a value of S, calculate ω^2 by [71b'], and then calculate the value of the left member of [71a]. Successive approximation and interpolation would then be used in order to find the values of S for which Equation [71a] holds.

The smallest positive value of S thus found will undoubtedly be a critical flutter speed, provided $L \neq 0$ so as to couple the v and θ motions.

If S = 0 but Equation [71b'] furnishes a positive value for ω^2, then v and θ will vibrate in a certain ratio, decreasing to zero with time if either $c_1 > 0$ or $c_2 > 0$. If, then S is made slightly positive, the term $BS\dot{v}$ will predominate over $- BS^2\theta$ in Equation [69a] and will itself damp out the motion even if $c_1 = c_2 = 0$. As S increases further, however, the damping effect may decrease because of the increase in the value of the $- BS^2\theta$ term until, finally, at a certain speed S_c harmonic vibration may again become possible. Then S_c is the critical flutter speed. As S increases above S_c, the vibration will probably become unstable.

In special cases further conclusions can be drawn, and it is these conclusions that may give a certain interest to the case of a foil mounted on a rigid base:

(a) *Assume* $c_2 = 0$ but $k_2 > 0$, $L \neq 0$, and either $S > 0$ or $c_1 > 0$ or both. Then the factor $c_1 + BS$ can be canceled out of Equation [71b']; it is found when substituting for ω^2 that k_1k_2 disappears from [71a]; finally Equations [71a,b'] can be written:

$$\omega^2 = \frac{k_2}{I + L^2m} ; \qquad BS^2 = \frac{Lmk_2}{I + L^2m} = Lm\omega^2 \qquad [72a,b]$$

Inspection of Equations [69a,b] shows why this case is so simple. If BS^2 has the value specified in Equation [72b], the two θ terms in Equations [69a] cancel each other. Hence, if $k_1 > 0$, v must either be zero or execute a damped oscillation ending in zero. Or, if $k_1 = 0$, v must ultimately reduce to a constant value since the only other possible solution of [69a] is then the nonharmonic solution $v \propto e^{-\mu t}$, where

$\mu = (c_1 + BS)/m$.* In either case, ultimately $\ddot{v} = 0$ and the term $- Lm\ddot{v}$ in Equation [69b] disappears; and the v and θ equations are independent of each other. According to Equation [69b], θ vibrates steadily with ω, as given by Equation [72a].

(b) *If $k_2 = 0$* and either $c_2 = 0$ or $k_1 = 0$ or both, and if also at least one of c_1 and S is positive, then $\omega^2 = 0$ according to Equation [71b'], and no harmonic vibration with $\omega \neq 0$ is possible.

When $k_2 = 0$ and $c_2 = 0$, solutions of Equations [69a,b] are easily found in the form of finite series in powers of t. The simplest of these may be mentioned; namely, $\theta = \alpha$ and, if $k_1 \neq 0$, $v = BS^2\alpha/k_1$; or, if $k_1 = 0$ but $c_1 + BS > 0$, $v = \dfrac{BS^2\alpha t}{c_1 + BS} + \beta$, where α and β are arbitrary constants and t denotes the time. If $k_1 \neq 0$, $F_L = - BS\dot{v} + BS^2\theta = BS^2\alpha = k_1 v$. If $k_1 = 0$, $(c_1 + BS)\dot{v} = BS^2\alpha$, so that $BS\dot{v} = BS^2\alpha - c_1\dot{v}$ and $F_L = c_1\dot{v}$. Thus, in either case, a steady lift on the foil serves to balance another force.

(c) *If $L = 0$*, Equation [71a] does not contain BS^2 and it is more illuminating to return either to Equations [70a,b] or to the more specific equations of motion, Equations [69a,b], which now read:

$$m\ddot{v} + k_1 v + (c_1 + BS)\,\dot{v} = BS^2\theta$$

$$I\ddot{\theta} + k_2\theta + c_2\dot{\theta} = 0$$

Now there are just two possibilities for steady harmonic vibration:

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* Here, this definition of μ, is restricted to the case at hand and should not be confused with the definition of μ generally used throughout this report.

(a) $c_2 > 0;\ k_1 > 0,\ c_1 = 0,\ S = 0:\ \omega^2 = \frac{k_1}{m};\ \theta = 0$

(b) $c_2 = 0,\ k_2 > 0:\ \omega^2 = \frac{k_2}{I}$

Case (b) is not a typical flutter case since S may have any value, but it may serve to illustrate in a simple case a typical feature of flutter vibrations in that the term $BS^2\theta$ forces a vibration of v in shifted phase, the energy abstracted by damping being furnished by the stream. This feature may be worth exploring in detail.

By properly choosing the zero of time, θ can be expressed thus:

$$\theta = C \sin \omega t \ ; \ \omega^2 = k_2/I$$

where C denotes an arbitrary constant. The accompanying forced vibration of v can be written, in terms of amplitude factors α and β that remain to be found, as follows:

$$v = \alpha C \sin \omega t + \beta C \cos \omega t$$

hence

$$\dot{v} = - \omega\beta C \sin \omega t + \omega\alpha C \cos \omega t$$

$$\ddot{v} = - \omega^2\alpha C \sin \omega t - \omega^2\beta C \cos \omega t$$

When these expressions are substituted into the $m\ddot{v}$ equation, the sin ωt and cos ωt terms must balance separately. Hence, after canceling out C sin ωt or C cos ωt:

$$(k_1 - m\omega^2)\ \alpha - (c_1 + BS)\ \omega\beta = BS^2 \quad \text{(for } \sin \omega t\text{)}$$

$$(c_1 + BS)\ \omega\alpha + (k_1 - m\omega^2)\ \beta = 0 \quad \text{(for } \cos \omega t\text{)}$$

Solving for α and β gives:

$$\alpha = D^{-1} (k_1 - m\omega^2) BS^2 \; ; \quad \beta = - D^{-1}\omega(c_1 + BS) BS^2$$

where

$$D = (k_1 - m\omega^2)^2 + \omega^2 (c_1 + BS)^2$$

The phase angle of v relative to θ is thus $\tan^{-1} (\beta/\alpha)$ or $- \tan^{-1} \omega(c_1 + BS)/(k_1 - m\omega^2)$. *This phase shift enables* $BS^2\theta$ *to supply the energy lost from the foil through the* $\dot{v}$ *term.* It may be worthwhile to verify this compensation.

The rate of energy loss is $(c_1 + BS) \dot{v}^2$; the rate of energy supply is $BS^2 \theta\dot{v}$. Now $\dot{v}^2 = \omega^2 C^2 (\beta^2 \sin^2 \omega t + \alpha^2 \cos^2 \omega t - 2\alpha\beta \sin \omega t \cos \omega t)$. The time average, over a cycle, of $\sin^2\omega t$ or $\cos^2\omega t$ is 1/2, whereas $\sin \omega t \cos \omega t$ averages zero. Hence, $\dot{v}^2$ averages $(1/2)\, \omega^2 C^2 (\alpha^2 + \beta^2)$, but $\alpha^2 + \beta^2 = D^{-1} (BS^2)^2$. The average rate of energy loss is, therefore:

$$\frac{1}{2} (c_1 + BS)\, \omega^2 C^2 D^{-1} (BS^2)^2$$

On the other hand,

$$\theta\dot{v} = - \omega\beta C^2 \sin^2 \omega t + \omega\alpha C^2 \sin \omega t \cos \omega t$$

Thus the rate of energy gain, or $BS^2\theta\dot{v}$, averages:

$$- \frac{1}{2} \omega\beta C^2 BS^2 = \frac{1}{2} \omega^2 C^2 D^{-1} (c_1 + BS)(BS^2)^2$$

This equals the average rate of energy loss.

In conclusion, to have a typical flutter situation with a positive critical flutter speed, it is clearly necessary in the arrangement under discussion that the center of mass of the foil does not lie on the axis of rotation; hence, that $L \neq 0$ and that, consequently, the v and θ motions are mechanically coupled together. Furthermore, some elastic attachment to the base is necessary, hence, either $k_2 > 0$ or $k_1 > 0$; if $k_2 = 0$, it is also necessary that the rotational motion be damped ($c_2 > 0$) to keep v from wandering off.

7.2 FOIL ELASTICALLY MOUNTED WITH DAMPING ON A FREE RIGID BODY

Let the foil be attached to a free rigid body so that it will turn about an axis fixed in the body, drawn parallel to the foil axis and perpendicular to the stream. Let the center of mass of the foil lie on the axis, and denote the distances from the axis to the center of mass of the body* and to the center of lift on the foil by h_o and L, respectively. These three points are assumed to lie on a line that is parallel to the stream in the undisplaced position of the system, and h_o and L are positive toward the approaching stream. Positive directions are shown in Figure 5.

Let the body have a principal axis of inertia at its center of mass parallel to the axis of rotation, and denote the masses of foil and body* by m and m_o and their moments of inertia about their center of mass by I and I_o, respectively. Assume that motion of the axis of rotation in the direction of the stream is somehow prevented.

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* h_o and m_o refer now to the rigid body, not the foil; hence the change of notation from h and m used in previous sections for the foil.

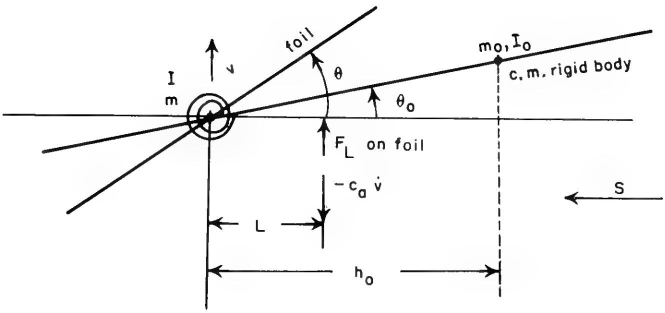

Figure 5 - Forces on and Motions of a Foil Elastically Mounted with Damping on a Free Rigid Body

In Section 7.2 only (see notations) m and m_o denote the masses of the foil and body and I and I_o their moments about their centers of mass, respectively.

The motions of foil and body will then be two dimensional. Let v denote displacement of the axis of rotation in a direction perpendicular to the foil and to the stream and let θ and θ_o denote the small rotations of foil and body, respectively, about the axis. θ and θ_o are zero in the undisplaced position of the system and positive from a direction opposite to that of the stream toward positive v; see Figure 2.

Assume that a spring and also an *internal damping mechanism* are present, exerting turning moments $- k (\theta - \theta_o) - c (\dot{\theta} - \dot{\theta}_o)$ on the foil and $k (\theta - \theta_o) + c (\dot{\theta} - \dot{\theta}_o)$ on the body. Assume also an *external damping force* $- c_a \dot{v}$ acting on the foil along the same line as the lift F_L.

In general, there is a force F, positive toward positive v, acting on the body at the axis and a corresponding reaction -F acting on the foil. The displacement of the center of mass of the body is $v + h_o\theta_o$.

The primary equations of motion for foil and body will then be:

$$m\ddot{v} = - F - c_a\dot{v} + F_L \;; \quad I\ddot{\theta} = - k (\theta - \theta_o) - c (\dot{\theta} - \dot{\theta}_o) + L (- c_a\dot{v} + F_L)$$

$$m_o (\ddot{v} + h_o\ddot{\theta}_o) = F \;; \quad I_o\ddot{\theta}_o = k (\theta - \theta_o) + c (\dot{\theta} - \dot{\theta}_o) - h_oF$$

Eliminating F and inserting $F_L = - BS\dot{v} + BS^2\theta$ gives:

$$(m + m_o) \ddot{v} + h_om_o\ddot{\theta}_o + (c_a + BS) \dot{v} - BS^2\theta = 0 \qquad [73a]$$

$$I\ddot{\theta} + c (\dot{\theta} - \dot{\theta}_o) + k (\theta - \theta_o) + L [(c_a + BS) \dot{v} - BS^2\theta] = 0 \qquad [73b]$$

$$(I_o + h_o^2m_o) \ddot{\theta}_o - c (\dot{\theta} - \dot{\theta}_o) - k (\theta - \theta_o) + h_om_o\ddot{v} = 0 \qquad [73c]$$

A more simple equation may be obtained by subtracting L times Equation [73a] from the sum of Equations [73b] and [73c], namely:

$$[h_o m_o - L(m + m_o)]\,\ddot{v} + I\ddot{\theta} + (I_o + h_o^2 m_o - h_o L m_o)\,\ddot{\theta}_o = 0 \qquad [74]$$

This last equation can be integrated at once as follows:

$$[h_o m_o - L(m + m_o)]\,v + I\theta + (I_o + h_o^2 m_o - h_o L m_o)\,\theta_o = \alpha + \beta t$$

where α and β are arbitrary constants and t denotes the time. A corresponding special solution of [73a,b,c] exists if $c_a = 0$, namely:

$$\theta = \theta_o = \gamma \;; \qquad v = \delta + \gamma S t$$

where γ and δ are constants which are easily expressible in terms of α and β. This solution represents a steady lateral motion in which the foil does not disturb the stream.* By means of the integrated equation, it is possible to eliminate θ_o and thereby reduce the equations of motion from three to two. It seems simpler, however, to work with all three equations.

In looking for harmonic motion, it is apparently less complicated to use as equations of motion Equations [73a] and [73c], with signs reversed, and Equation [74]. Assuming that in these equations $\mathbf{v}$, $\boldsymbol{\theta}$, and θ_o are proportional to $e^{i\omega t}$, then canceling $e^{i\omega t}$, and also dividing Equation [74] by $-\omega^2$, gives the following result, provided $\omega^2 \neq 0$:

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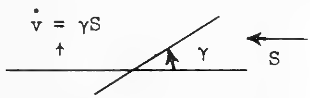
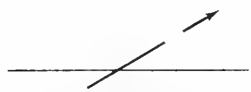
* The foil does not disturb the stream because $v = \delta$ represents a steady displacement with $\theta = 0$, whereas $\theta = \gamma$ and $v = \gamma S t$ represent a steady motion in which the foil slips chordwise through the water thus:



as seen by observer



as seen by water

$$[-\omega^2 (m + m_o) + i\omega (c_a + BS)]\, v - BS^2\theta - \omega^2 h_o m_o \theta_o = 0$$

$$\omega^2 h_o m_o v + (k + i\omega c)\, \theta + [\omega^2 (I_o + h_o^2 m_o) - k - i\omega c]\, \theta_o = 0$$

$$[h_o m_o - L (m + m_o)]\, v + I\theta + (I_o + h_o^2 m_o - h_o L m_o)\, \theta_o = 0$$

The determinant of the coefficients of v, θ, and θ_o in these equations has the value:

$$[-\omega^2 (m + m_o) + i\omega (c_a + BS)] (k + i\omega c)(I_o + h_o^2 m_o - h_o L m_o)$$

$$- BS^2 [\omega^2 (I_o + h_o^2 m_o) - k - i\omega c][h_o m_o - L (m + m_o)]$$

$$-\omega^4 h_o^2 m_o^2 I + \omega^2 h_o m_o (k + i\omega c)[h_o m_o - L (m + m_o)]$$

$$+ BS^2 \omega^2 h_o m_o (I_o + h_o^2 m_o - h_o L m_o) - I [\omega^2 (I_o + h_o^2 m_o) - k - i\omega c]$$

$$[-\omega^2 (m + m_o) + i\omega (c_a + BS)]$$

Multiplying out and equating to zero separately the real part and the imaginary part divided by $i\omega$ yields:

$$\omega^4 I [I_o (m + m_o) + h_o^2 m m_o] - \omega^2 \{ k[(I + I_o)(m + m_o) + h_o^2 m m_o]$$

$$c (c_a + BS)(I + I_o + h_o^2 m_o - h_o L m_o) \} + BS^2 \{ k[h_o m_o - L (m + m_o)]$$

$$+ \omega^2 L [I_o (m + m_o) + h_o^2 m m_o] \} = 0 \qquad [75a]$$

$$-\omega^2 \{ (c_a + BS)\, I\, (I_o + h_o^2 m_o) + c[(I + I_o)(m + m_o) + h_o^2 m m_o] \}$$

$$+ k(c_a + BS)(I + I_o + h_o^2 m_o - h_o L m_o) + cBS^2 [h_o m_o - L(m + m_o)] = 0 \quad [75b]$$

or, provided $c > 0$ and $c_a + BS > 0$:

$$\omega^2 = \frac{k\,(c_a + BS)(I + I_o + h_o^2 m_o - h_o L m_o) + c\,BS^2\,[h_o m_o - L\,(m + m_o)]}{(c_a + BS)\,I\,(I_o + h_o^2 m_o) + c\,[(I + I_o)(m + m_o) + h_o^2 m m_o]} \qquad [75b']$$

If $cc_a = 0$* and $S = 0$, Equation [75a] gives, as the natural frequency of vibration of the undamped system without the stream:

$$\omega^2 = \omega_o^2 = \frac{k}{I}\left(1 + \frac{I\,(m + m_o)}{I_o\,(m + m_o) + h_o^2 m m_o}\right)$$

It should be noted that the mathematical complexity of Equations [75a,b] is no greater than that of Equations [71a,b] in the case of the foil attached elastically to an immobile base, although here three variables (v, θ, θ_o) and three corresponding equations are involved as against two in the previous case. The reason for the simplicity here lies in the occurrence of the factor ω^2 in all three terms of Equation [74]. It is not clear in what general class of systems this sort of reduction in complexity will occur.

The smallest positive value of S for which both [75a] and [75b] are true will undoubtedly be a critical speed for the inception of flutter. At a lower speed any motion will be damped out due to the $BS\dot{v}$ term and, perhaps, other causes if $S > 0$; or, if $S = 0$, provided at least one of the constants c_a and c is positive,

The case: $c = 0$. If $c = 0$ but $c_a + BS > 0$, Equation [75b'] gives:

$$\omega^2 = \frac{k}{I}\left(\frac{I + I_o + h_o^2 m_o - L h_o m_o}{I_o + h_o^2 m_o}\right) \qquad [76]$$

* $cc_a = 0$ means $c = 0$ or $c_a = 0$ or both.

The necessary value of S^2 may be found most easily by rearranging Equation [75a], giving the value of the determinant as follows:

$$- \omega^2 (m + m_o) \{ k(I_o + h_o^2 m_o - Lh_o m_o) - I[\omega^2 (I_o + h_o^2 m_o) - k] \}$$

$$+ \omega^2 h_o m_o \{- \omega^2 I h_o m_o + BS^2 [I_o + h_o^2 m_o - Lh_o m_o] \}$$

$$+ [h_o m_o - L(m + m_o)] \{\omega^2 k\, h_o m_o - BS^2 [\omega^2 (I_o + h_o^2 m_o) - k]\} = 0$$

(Here the leading factors are the coefficients of v in the v, θ, θ_o equations, the coefficients of v having been chosen because they contain the only imaginary term when c = 0.)

Now the value of ω^2 given by Equation [76] causes the first brace (i.e., the coefficient of $-\omega^2(m + m_o)$) to vanish. The second brace can be made to vanish by assigning the proper value to BS^2. Then it is easily seen that the third brace also vanishes. For, the ratio of the first term in the third brace to the first term in the second brace, and the similar ratio of the second terms, are, respectively;

$$- \frac{k}{I} \text{ and } - \frac{\omega^2 (I_o + h_o^2 m_o) - k}{I_o + h_o^2 m_o - Lh_o m_o}$$

But the vanishing of the first brace makes these fractions equal. Hence, the third brace is proportional to the second, so that if the second is made zero, the third brace also vanishes, and, consequently, the whole determinant is zero.

The value of BS^2 obtained in this way, when c = 0 but $c_a + BS > 0$, is:

$$BS^2 = \omega^2 \frac{h_o m_o I}{I_o + h_o^2 m_o - Lh_o m_o} \qquad [77]$$

Incidentally, it may be remarked that if the values of ω^2 and of BS^2 given by [76] and [77] are substituted into the v, θ, and θ_o equations with $c = 0$, it is found that

$$v = 0 \; ; \quad \theta = - \theta_o \; [I_o + h_o^2 m_o - Lh_o m_o]/I$$

If Equations [76] and [77] yield positive values for ω^2 and S^2, then S is the critical flutter speed for the system. Even if $c_a = 0$, any motion at a smaller speed will be damped out by the $- BS\dot{v}$ term in the lift.

A critical flutter speed will not exist for all values of h_o and L. In Equations [76] and [77], only h_o and L can be negative. Hence to make ω^2 and S^2, as given by these equations, positive, it is necessary that:

$$I + I_o + h_o^2 m_o - Lh_o m_o > 0 \; ; \quad \frac{I_o + h_o^2 m_o}{h_o m_o} - L > 0$$

(The sign of a fraction is not changed if the fraction is inverted or if numerator and denominator are multiplied or divided by the same number.)

If $h_o > 0$, the second inequality requires that:

$$L < h_o + \frac{I_o}{h_o m_o}$$

and then the first inequality is satisfied also. Here, L may be positive or negative.

If $h_o < 0$, substitute $h_o = - |h_o|$. Then

$$- |h_o| - \frac{I + I_o}{m_o|h_o|} < L < - |h_o| - \frac{I_o}{m_o|h_o|}$$

with the left-hand inequality coming from the first of the basic inequalities and the right-hand inequality from the second. (Note that $h_o^2/|h_o| = |h_o|$.)

Thus. h_o and L may both be positive, as drawn in Figure 4; if $h_o < 0$, L must be negative also.

8. SUMMARY

A theory has been advanced for determining the vibrations, including flutter, of a hydrofoil craft. In general, the craft is treated as a rigid foil flexibly attached to a uniform mass-elastic free-free beam immersed in a fluid moving with uniform velocity. The hydrodynamic force and moment on the foil are represented by two-dimensional quasi-steady expressions. The sequence of steps undertaken in devising this theory is summarized as follows:

1. In Introduction the background and objective of this report is discussed.

2. In Section 2 influence and inertia coefficients are derived for a uniform beam that is maintained in harmonic vibration with two degrees of freedom by a transverse force F and a moment or couple G acting at *one end* of the beam and in the same principal plane of the beam. Shear warping of the beam is included but rotary inertia and damping are not. These coefficients are useful in treating the vibrations of a system composed of a uniform beam attached at one end to another structure (e.g., a foil).

3. In Section 3, for convenience of analysis, the formulation for these coefficients is simplified by neglecting shear warping. Various interesting cases are discussed, in the order of increasing frequency, for the vibrations of a forced, undamped, uniform beam without shear warping and rotary inertia.

4. In Section 4, the analysis is extended to provide formulas for these coefficients in the case where the external force F and the moment

G act at an *intermediate* point instead of at an end of the uniform beam; shear warping is included but rotary inertia and damping are ignored.

5. In Section 5, the analysis is further extended to include the effects of both external (Rayleigh) and internal damping, the latter being associated with bending strains; shear warping is included but rotary inertia is ignored. It is shown that the equations derived are valid even if the external force F and the moment G are not vibrating in phase. Formulations are developed from which the final calculation of the coefficients can be made. The procedure for making these calculations is clearly indicated.

6. In Section 6, flutter of a rigid foil flexibly attached to a uniform beam is considered, the entire system being immersed in a stream of fluid approaching at uniform speed. Expressions for the resulting lift forces and moment on the foil are adapted from Theodorsen's equations for a uniform foil of infinite length vibrating harmonically. An approximation closer than the common steady-motion approximation is used. For greater generality, allowance is made for still-water damping (structural plus fluid) as well as for hydrodynamic damping. Equations of motion for the foil are related (or coupled) to the equations previously derived for the response of the beam subject to harmonic loads; i.e., the coefficients previously derived for the beam and Hooke's law for the flexible connecting structure serve to couple the motions of the foil to the harmonic loads imposed upon it by the motions of the beam. A method is described for finding the critical flutter speed and frequency for this system, at which a steady vibration of the foil and beam is possible in

spite of existing damping actions. The effects of various damping terms are discussed and three special cases for which the determination of the critical flutter speed and frequency are of interest are treated. These cases occur for (1) damping which arises only from terms in the lift force and associated moment, damping due to friction in the surrounding fluid being excluded; (2) a foil rigidly mounted on a beam; and (3) a low-frequency approximation, in which shear effects in the beam may be ignored. For the latter case, it is shown that any low-frequency harmonic vibration of the foil that can occur with the beam attached can also occur without the beam and at the same frequency and stream speed provided certain simple changes specified in the text are made in the foil. In subsection 6.6, a simple form of similitude is described. The parameters of the foil-beam system are assumed to be changed in certain ratios with resulting changes in the frequency of vibration and in the critical flutter speed. The necessary restrictions on the ratios are specified in detail. The relations derived here may be of interest in designing a model to represent a much larger system.

7. In Section 7, two simple cases of flutter are treated in order to reduce the mathematical complexity to a point where intuitive ideas of the flutter process are relatively easy to arrive at. The cases discussed are for (1) a foil attached elastically to an immobile base; and (2) a foil elastically mounted with damping on a free rigid body.

For these cases various concepts, relatively simple equations, and criteria for flutter are presented; and methods are described for determining the critical flutter speed and frequency from these equations.

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