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An Isospectral Family of Random Processes

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Abstract

We construct a family of random step functions $\{x_n(t)\}$ whose members all have the same power spectrum and such that as $n \to \infty$, $x_n(t)$ converges to $x_\infty(t)$, the Gaussian process with the same spectrum. We illustrate the procedure for calculating the general multivariate distribution of the processes $\{x_n(t)\}$ by calculating the univariate, bivariate and trivariate distributions. We show how a suitably constructed univariate entropy can serve as an index of the extent to which $x_n(t)$ has approached the Gaussian limit $x_\infty(t)$.

Table of Contents

I. Introduction

Consider a zero-mean stationary random process $x(t)$ with correlation function $C(\tau)$, i.e. a process $x(t)$ for which

$$E\,x(t) = 0, \quad E\,x(t)x(t+\tau) = C(\tau) \quad ,$$

where E is the expectation operator. A considerable part of noise theory is devoted to study of the case where $x(t)$ is Gaussian. In this case, the law of $x(t)$, i.e. the probability that $x(t_n) < c_n$, $1 \leq n \leq N$, $1 \leq N < \infty$, where the c_n are arbitrary real numbers, is the familiar N-variate Gaussian distribution whose moment matrix can be expressed simply in terms of $C(\tau)$[1]. Thus, in the Gaussian case, $C(\tau)$ uniquely specifies $x(t)$, which is, of course, the particular beauty of this case. More generally, $C(\tau)$ gives only a more or less incomplete characterization of $x(t)$.

The object of the present paper is to describe and study an infinite family $\{x_n(t)\}$, $1 \leq n < \infty$, of non-Gaussian random processes, which converge to a Gaussian process $x_\infty(t)$ as $n \to \infty$, and whose laws are calculable, at least in principle. The family $\{x_n(t)\}$ will be constructed in such a way that all its members have the same correlation function $C(\tau)$, or equivalently, the same power spectrum

$$\Phi(\omega) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \exp(-i\omega\tau)C(\tau)d\tau \quad .$$

Briefly, we shall say that $\{x_n(t)\}$ is an isospectral family. Since as $n \to \infty$, the general appearance of the sample functions of $x_n(t)$ changes and approaches that of the Gaussian process $x_\infty(t)$ with the same power spectrum (or correlation function), the general inadequacy of $\Phi(\omega)$ or $C(\tau)$ as a means of characterizing random processes and the general need for higher-order statistics are put into rather strong focus.

The method we use to construct $x_n(t)$ from an underlying shot noise (Poisson process) is a familiar one. The novelty of our treatment consists in observing that when the shots are rectangular the members of $\{x_n(t)\}$ can be described in theoretically complete detail.

II. Construction of $\{x_n(t)\}$

Let the sequence t_j ($j = \ldots,-1,0,1,\ldots$) be the occurrence times of the shots in a stationary shot noise (Poisson process) with an average rate of ρ shots per second. More specifically, we have the following five properties (among others):

1. The occurrence of m shots in the interval I and the occurrence of n shots in the interval I' ($m,n=0,1,2,\ldots$) are independent events if I and I' do not overlap.

2. The probability of one shot in an infinitessimal interval of length Δt is $\rho\Delta t + o(\Delta t)$, whereas the probability of more than one shot is $o(\Delta t)$.

3. The probability of m shots in any interval of length T is given by the Poisson distribution

$$p(m;\lambda) = \frac{\lambda^m}{m!} e^{-\lambda} \quad , \; m = 0,1,2,\ldots ,$$

where $\lambda = \rho T$, the parameter of the Poisson distribution, is the common value of the mean and variance of a random variable which takes the values $m = 0,1,2,\ldots,$ with probabilities $p(m;\lambda)$.

4. Given that m shots have occurred in an interval of length T, their occurrence times $t_1,\ldots,t_m$ (without regard to order of appearance) are independent, identically distributed random variables with the uniform probability density $1/T$.

5. The probability density of the interval between successive shots is $\rho \exp(-\rho t)$.

For discussion and derivation of these properties we refer the reader elsewhere[2],[3],[4].

Now let $h(t;\alpha)$ be a step function of unit height and width α, i.e.

$$(1) \qquad \begin{aligned} h(t;\alpha) &= 1\ , \quad 0 \le t \le \alpha\ , \\ h(t;\alpha) &= 0\ , \quad -\infty < t < 0,\ \alpha < t < \infty. \end{aligned}$$

To construct the isospectral family $\{x_n(t)\}$ we write

$$(2) \qquad x_n(t) = (1/\sqrt{n}) \sum_{j=-\infty}^{\infty} h(t - t_j^{(n)};\alpha) - \sqrt{n}\,\alpha, \quad 1 \le n < \infty\ ,$$

where the $t_j^{(n)}$ are random times belonging to a shot noise with an average rate of n shots per second. It is clear that each $x_n(t)$ is a stationary random step function. To calculate expectations of lagged products of $x_n(t)$ we follow the usual procedure: First we use property 4 (above) to calculate conditional expectations for a long finite interval of length T containing exactly N shots and then we average over N using property 3. Again we refer to the literature for details[5]. In particular we find

$$E\ x_n(t) = 0, \quad E\ x_n(t)x_n(t+\tau) = C(\tau), \quad 1 \le n < \infty\ ,$$

where $C(\tau)$ is given by

$$(3) \qquad \begin{aligned} C(\tau) &= \int_{-\infty}^{\infty} h(t;\alpha)h(t+\tau;\alpha)dt = \alpha - |\tau|,\ 0 \quad |\tau| \le \alpha\ , \\ C(\tau) &= 0\ , \quad |\tau| > \alpha\ . \end{aligned}$$

The corresponding common power spectrum of the isospectral family $\{x_n(t)\}$ is

$$\Phi(\omega) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \exp(-i\omega\tau)C(\tau) = (2/\pi\omega^2)\ \sin^2(\omega\alpha/2)\ .$$

The constant α is still at our disposal. In order to make the members of $\{x_n(t)\}$ corresponding to small values of n drastically non-Gaussian, we can choose $\alpha \ll 1$, so that there is negligible overlap of the shots making up $x_n(t)$ if $n \ll 1/\alpha$. Then for $n \sim 1/\alpha$ overlap of the shots begins to be appreciable and as n increases further the process $x_n(t)$ approaches the Gaussian process $x_\infty(t)$ with the same power spectrum*.

III. Typical Sample Functions of $\{x_n(t)\}$

To construct typical sample functions of $x_n(t)$ we first use random number tables [6] to find sample values of a random variable ξ which is uniformly distributed in the unit interval (0,1). We then observe that the new random variable

$$\eta = \frac{1}{\rho} \log \frac{1}{1-\xi} \tag{4}$$

has the probability density $\rho \exp(-\rho t)$ of the interval between successive shots in a Poisson process with average rate ρ. To see this, note that

$$\mathrm{Prob}\left[F^{-1}(\xi) < t\right] = \mathrm{Prob}\left[\xi < F(t)\right] = F(t) \quad ,$$

so that $F^{-1}(\xi)$ has the distribution function $F(t)$. If $F(t)$ is to be the distribution function corresponding to the probability density which is $\rho \exp(-\rho t)$ for $t \geq 0$ and zero otherwise, then $F(t) = 1 - \exp(-\rho t)$, whence (4) follows at once. Moreover, if ξ is uniformly distributed in (0,1) so is $1-\xi$. Thus, sample values of the random variable

$$\eta_n = \frac{1}{n} \log \frac{1}{\xi} \tag{5}$$

generate time markers locating the shots in a Poisson process with an average rate of ρ shots per second. Specifically, if $\eta_{n1}, \ldots, \eta_{nm}$ are m sample

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* Specifically, the type of convergence we have in mind is convergence in distribution, i.e. the law of $x_n(t)$ converges to that of $x_\infty(t)$.

values of (5), we get m random times $t_1^{(n)},\ldots,t_m^{(n)}$ by writing

$$t_1^{(n)} = \eta_{n1},\ t_2^{(n)} = \eta_{n1}+\eta_{n2},\ldots,\ t_m^{(n)} = \eta_{n1}+\eta_{n2} + \ldots \eta_{nm}.$$

Finally, with these values of the random times we use (2) to construct sample functions of $x_n(t)$.

Figs. 1, 2 and 3 show the results of this procedure for the cases of low, medium and high density shot noise, respectively. In each case α was chosen to be 0.25. Fig. 1 shows a 4 second sample of the process

$$x_1(t) = \sum_{j=-\infty}^{\infty} h(t - t_j^{(1)};0.25) - 0.25 \quad ,$$

Fig. 2 shows a 4 second sample of the process

$$x_4(t) = \frac{1}{2}\sum_{j=-\infty}^{\infty} h(t-t_j^{(4)};0.25) - 0.50 \quad ,$$

and Fig. 3 shows a 1 second sample (drawn on a different scale) of the process

$$x_{16}(t) = \frac{1}{4}\sum_{j=-\infty}^{\infty} h(t - t_j^{(16)};0,25) - 1.00 \quad .$$

The circle in Fig. 2 shows a level where the sojourn of $x_4(t)$ was so brief that it could not be indicated on the scale of Fig. 2. A similar remark applies to the three circles appearing in Fig. 3. In each case the sample functions were given the initial value of $-\sqrt{n}\,\alpha$, i.e. -0.25 for $x_1(t)$, -0.50 for $x_4(t)$ and -1.00 for $x_{16}(t)$. When starting the sample functions of $x_4(t)$ and $x_{16}(t)$, atypical portions lasting about $\alpha = 0.25$ seconds occur, due to the fact that there are no shots before $t = 0$*. (Theoretically, each $x_n(t)$ should begin in the infinitely remote past). To eliminate this transient behavior, we suppressed the first second of the record of $x_4(t)$ and the first 0.4 second of the record of $x_{16}(t)$.

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* This effect is unimportant for $x_1(t)$, since in this case the overlap of shots is slight.

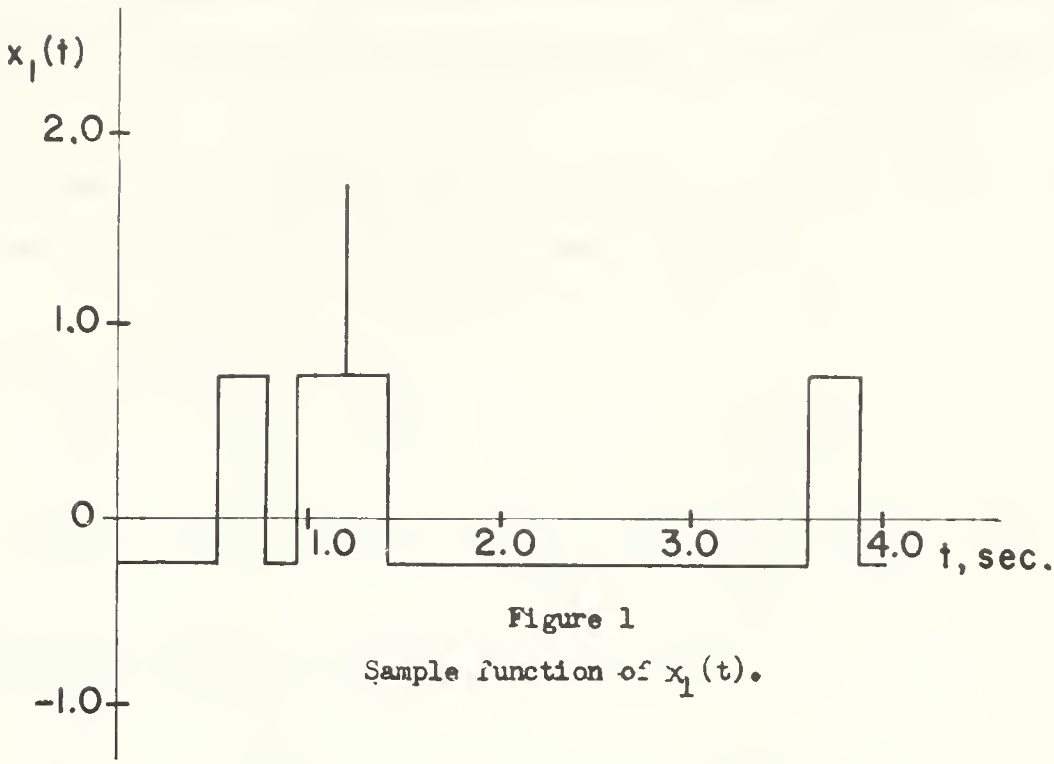

Figure 1

Sample function of $x_1(t)$.

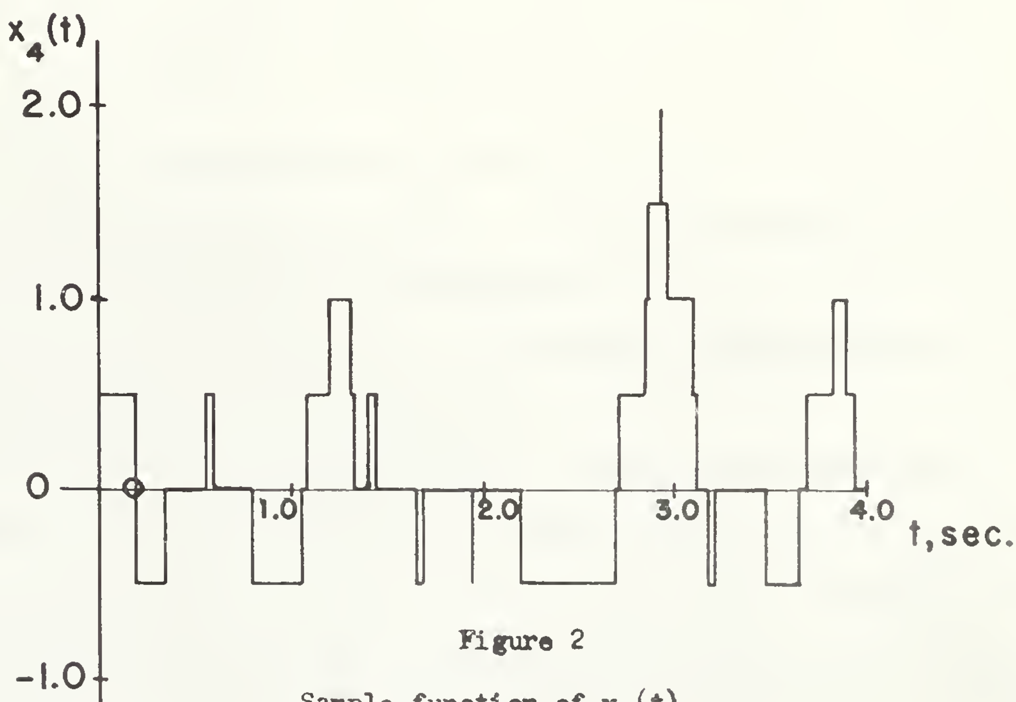

Figure 2

Sample function of $x_4(t)$.

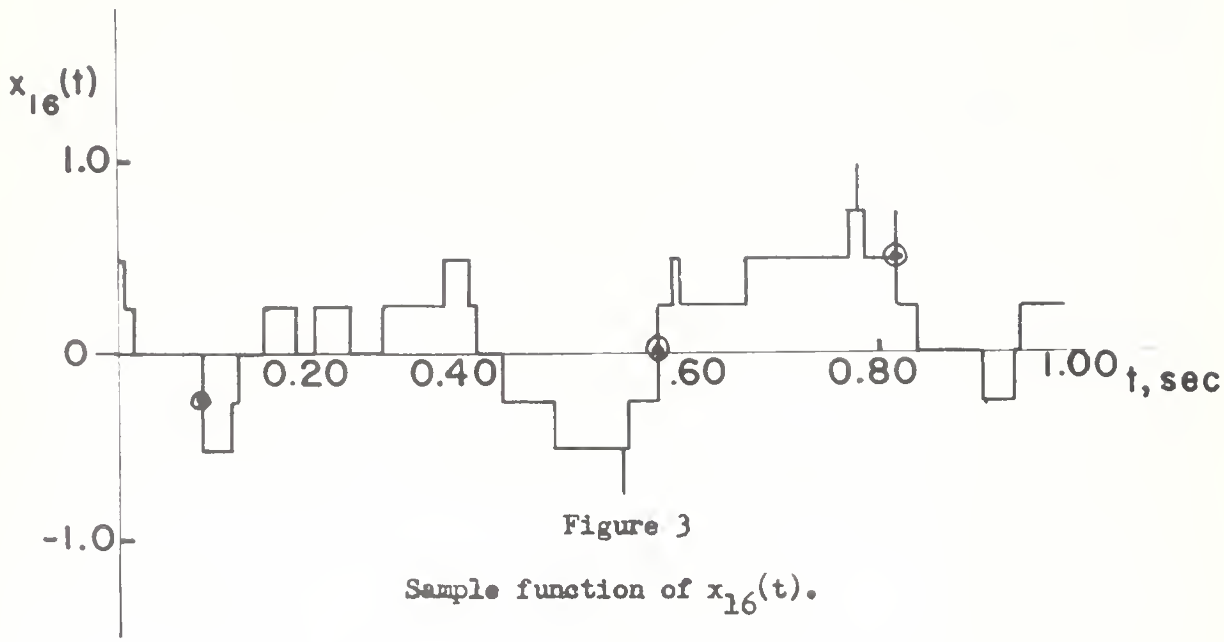

Figure 3

Sample function of $x_{16}(t)$.

IV. Probability Distributions of $\{x_n(t)\}$

The family $\{x_n(t)\}$ has the desirable feature that we can calculate the law of each $x_n(t)$, i.e. all the multivariate probability distributions of $x_n(t)$, although the amount of work required to calculate high-order probability distributions is enormous. We shall illustrate the general procedure by deriving formulas for the univariate, bivariate and trivariate probability distributions of $x_n(t)$.

1. Univariate distribution

By (1) and (2) the range of possible values of the random variable $x_n(t)$, t fixed*, is the lattice

$$y_{m,n} = m/\sqrt{n} - \sqrt{n}\,\alpha \ , \quad m = 0,1,2,\ldots$$

The probability that $x_n = y_{m,n}$ is just the probability that in the α seconds preceding t precisely m shots occur. Consequently, denoting the probability that $x_n = y_{m,n}$ by $P_n(m)$, we have

$$P_n(m) = p(m;n\alpha) = \frac{(n\alpha)^m}{m!} e^{-n\alpha} \tag{6}$$

where $p(m;n\alpha)$ is the Poisson distribution with parameter $n\alpha$. Elementary calculations verify that

$$\sum_{m=0}^{\infty} y_{m,n} P_n(m) = 0 \ , \quad \sum_{m=0}^{\infty} y_{m,n}^2 P_n(m) = \alpha \quad , \ 1 \leq n < \infty \quad ,$$

as required. Note that $P_n(m)$ can be regarded as the sum of n independent, identically distributed Poisson random variables with parameter α. Thus, the convergence of $x_n(t)$ to a zero-mean Gaussian random variable with variance α is a particularly simple case of the central limit theorem.** Note also that $x_n(t)$ can take arbitrarily large positive values, but no

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* The value of t is irrelevant since $x_n(t)$ is stationary, so that we can drop the argument t.

** The fact that the Poisson distribution $p(m;\lambda)$ approximates the Gaussian distribution for large values of λ is noted by Feller, op. cit., p. 176.

negative values less than $-\sqrt{n}\,a$; this asymmetry of the sample functions disappears as $n \to \infty$.

2. Bivariate distribution

The probability that $x_n(t) = y_{\ell,n}$ while $x_n(t+\tau) = y_{m,n}$ will be denoted by $P_n(\ell,m;\tau)$ and is independent of t, since $x_n(t)$ is stationary. The event that precisely m shots occur in the interval (u,v) will be denoted by $E_m(u,v)$. Then clearly $P_n(\ell,m;\tau)$ is the joint probability of the events $E\,(-a,0)$ and $E_m(\tau - a,\tau)$. If $|\tau| \geq a$, these two events are independent and we have

$$P_n(\ell,m;\tau) = P_n(\ell)P_n(m) = \frac{(na)^\ell}{\ell!}\,\frac{(na)^m}{m!}\,e^{-2na} \quad .$$

On the other hand, if $\tau = 0$, we obviously have

$$P_n(\ell,m;0) = P_n(\ell)\delta_{\ell m} \quad ,$$

where $\delta_{\ell m}$ is the Kronecker delta. The interesting case is $0 < |\tau| < a$, for then $E_\ell(-a,0)$ and $E_m(\tau - a,\tau)$ are not independent events, since the intervals $(-a,0)$ and $(\tau - a,\tau)$ overlap. In this case we have

$$P_n(\ell,m;\tau) = \sum \text{Prob}\left\{E_{\ell-s}(-a,\tau-a),E_s(\tau-a,0),E_{m-s}(0,\tau)\right\}$$

if $0 < \tau < a$, and

$$P_n(\ell,m;\tau) = \sum \text{Prob}\left\{E_{m-s}(\tau-a,-a),E_s(-a,\tau),E_{\ell-s}(\tau,0)\right\}$$

if $-a < \tau < 0$; the summations are over all values of the non-negative integer s compatible with the given values of ℓ and m. In either case, we find

$$P_n(\ell,m;\tau) = \sum_{s=0}^{\min(\ell,m)} \frac{(n|\tau|)^{\ell-s}(n(a-|\tau|))^s(n|\tau|)^{m-s}}{(\ell-s)!\;s!\;(m-s)!}$$

(7)

$$\cdot \exp(-n|\tau|-na) \quad , \quad 0 < |\tau| < a \;.$$

Of course, $P_n(\ell,m;\tau)$ must satisfy the normalization condition

$$\sum_{\ell,m=0}^{\infty} P_n(\ell,m;\tau) = 1 \tag{8}$$

for all n and τ. For $\tau = 0$ or $|\tau| \geq a$, it is trivial to verify (8) directly, but for $0 < |\tau| < a$ it appears much harder to give an independent proof of (8). $P_n(\ell, m;\tau)$ must also satisfy

$$\sum_{\ell=0}^{\infty} P_n(\ell,m;\tau) = P_n(m) .$$

For small values of m this relation can be verified directly, but the algebra becomes quite complicated for large m. A more complicated identity stems from the fact that $C(\tau)$ as given by (3) must be the correlation function of every $x_n(t)$, whence

$$\sum_{\ell,m=0}^{\infty} y_{\ell,n} y_{m,n} P_n(\ell,m;\tau) = C(\tau) \ , \ 1 \leq n < \infty \ . \tag{9}$$

For $\tau = 0$ or $|\tau| \geq a$ it is trivial to verify (9) directly, but for $0 < |\tau| < a$, (9) becomes highly non-transparent. A typical identity derivable from (9) is

$$\sum_{\ell,m=0}^{\infty} \frac{\ell m}{2^{\ell+m}} \sum_{s=0}^{\min(\ell,m)} \frac{2^s}{(\ell-s)!\, s!\, (m-s)!} = \frac{3}{2}\exp(3/2).$$

(Set $n = a = 1$, $\tau = 1/2$.)

3. Trivariate distribution

It is clear by now how the construction proceeds in general. Therefore, we calculate only the probability $P_n(k,\ell,m;\tau,\tau')$ of the joint event that $x_n(t) = y_{k,n}$, $x_n(t+\tau) = y_{\ell,n}$ and $x_n(t+\tau+\tau') = y_{m,n}$, assuming that $0 < \tau+\tau' < a$. In this case we have

$$\begin{aligned} P_n(k,\ell,m;\tau,\tau') &= \mathrm{Prob}\left\{E_k(-a,0), E_\ell(\tau-a,\tau), E_m(\tau+\tau'-a,\tau+\tau')\right\} \\ &= \sum \mathrm{Prob}\left\{E_{k-r-s}(-a,\tau-a), E_r(\tau-a,\tau+\tau'-a), E_s(\tau+\tau'-a,0),\right. \\ &\qquad\qquad \left. E_{\ell-r-s}(0,\tau), E_{m-\ell+r}(\tau,\tau+\tau')\right\} \ , \end{aligned}$$

where we have gone over to non-overlapping events, and the summation is over all values of the non-negative integers r and s compatible with the given values of k, ℓ and m. Specifically, we find

$$P_n(k,\ell,m;\tau,\tau') =$$

$$\sum_{r=-\min(m-\ell,0)}^{\min(k-s,\ell-s)} \sum_{s=0}^{\min(k,\ell,m)} \frac{(n\tau)^{k-r-s}}{(k-r-s)!} \frac{(n\tau')^r}{r!} \frac{(n(a-\tau-\tau'))^s}{s!}$$

$$\cdot \frac{(n\tau)^{\ell-r-s}}{(\ell-r-s)!} \frac{(n\tau')^{m-\ell+r}}{(m-\ell+r)!} \exp(-n\tau-n\tau'-na) \quad .$$

One can use the fact that $P_n(k,\ell,m;\tau,\tau')$ is a probability distribution to generate a series of very complicated identities, e.g.

$$\sum_{k,\ell,m=0}^{\infty} P_n(k,\ell,m;\tau,\tau') = 1,$$

$$\sum_{m=0}^{\infty} P_n(k,\ell,m;\tau,\tau') = P_n(k,\ell;\tau) \quad ,$$

etc.

The examples given make it clear how to calculate $P_n(k,\ell,m;\tau,\tau')$ for other ranges of the parameters τ,τ' and how to calculate the general N-variate distribution $P_n(k_1,\ldots,k_N;\tau_1,\ldots,\tau_{N-1})$. Of course, for large N, calculation of the N-variate distribution is very formidable and would require the services of a high speed electronic computer.

V. Entropy of $\{x_n(t)\}$

The fact that as $n \to \infty$, $x_n(t)$ converges in distribution to the Gaussian process $x_\infty(t)$ with the same power spectrum follows from the work of Fortet [7]. An index of the closeness of $x_n(t)$ to $x_\infty(t)$ is furnished by the entropy of $x_n(t)$; for simplicity we consider only the univariate entropy of $x_n(t)$.

Since the entropy of the Gaussian random variable x_∞ with mean zero and variance α as defined in the usual way* is

$$H_\infty(\alpha) = -\int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi\alpha}} e^{-x^2/2\alpha} \log\left(\frac{1}{\sqrt{2\pi\alpha}} e^{-x^2/2\alpha}\right) dx = \log\sqrt{2\pi e\alpha} \quad ,$$

we must insist that the univariate entropy of x_n converge to $H_\infty(\alpha)$ as $n \to \infty$.** As we have seen, the univariate distribution of x_n is given by

$$P_n(m) \equiv \text{Prob}\left\{x_n = y_{m,n} \equiv m/\sqrt{n} - \sqrt{n}\,\alpha\right\} = p(m;n\alpha) \equiv \frac{(n\alpha)^m}{m!} e^{-n\alpha} \; .$$

Thus, at first it might seem that the appropriate entropy to consider is the entropy

$$\text{(10)} \qquad H_n(\alpha) = -\sum_{m=0}^{\infty} p(m;n\alpha)\log p(m;n\alpha)$$

of the discrete random variable x_n. However, $H_n(\alpha)$ as defined by (10) is independent of α, since α is involved only in the combination $n\alpha$, and therefore $H_n(\alpha)$ cannot converge to $H_\infty(\alpha) = \log\sqrt{2\pi e\alpha}$. In fact, $H_n(\alpha)$ is actually logarithmically divergent, as the following simple qualitative argument shows: As already noted, as $n \to \infty$ the random variable x_n converges in distribution to the Gaussian random variable x_∞ with mean zero and variance α. It follows that for large n most of the distribution of x_n is concentrated in the interval $(-\sqrt{\alpha}, +\sqrt{\alpha})$, i.e. within one standard deviation of zero, and that the values of $p(m;n\alpha)$ are approximately equal in this interval. Since the possible values of x_n are a distance $1/\sqrt{n}$ apart,

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* C.E. Shannon, "The mathematical theory of communication", reprinted in the book of the same title by C.E. Shannon and W. Weaver, Univ. of Illinois Press, Urbana, p. 54; 1949. Shannon's discussion of the difference between the entropy of a discrete random variable and that of a continuous random variable is highly relevant to the present analysis.

** Since $x_n(t)$ and $x_\infty(t)$ are stationary we need not retain the argument t in discussing univariate quantities.

there are approximately $2\sqrt{n\alpha}$ values of $p(m;n\alpha)$, all approximately equal to $1/2\sqrt{n\alpha}$. The resulting estimate of the entropy $H_n(\alpha)$ of the discrete distribution $p(m;n\alpha)$ is

$$-2\sqrt{n\alpha}\ \frac{1}{2\sqrt{n\alpha}}\ \log\frac{1}{2\sqrt{n\alpha}} = \log 2\sqrt{n\alpha}\quad,$$

which diverges logarithmically as $n \to \infty$.

The appearance of this difficulty is not surprising since the entropy of a continuous random variable is defined as $-\int_{-\infty}^{\infty} p(x)\log p(x)dx$, where $p(x)$ is the probability density of the random variable, whereas the discrete random variable x_n used to define $H_n(\alpha)$ has no probability density, in spite of the fact that x_n converges in distribution to the continuous random variable x_∞. What has to be done is to replace the random variable x_n by a related continuous random variable x_n^* which converges to x_∞ in <u>density</u> as well as in distribution(*). To achieve this we define x_n^* as the continuous random variable with the probability density

$$p_n(x) = \sum_{m=0}^{\infty} \sqrt{n}\ p(m;n\alpha)\ h(x - y_{m,n};1/\sqrt{n}) \quad, \tag{11}$$

where h is the step function defined by (1) and as usual $y_{m,n} = m/\sqrt{n} - \sqrt{n\alpha}$. In other words, we replace the discrete random variable x_n by the continuous random variable x_n^* where the "mass" formerly concentrated at the point $y_{m,n}$ is now uniformly distributed over the interval $(y_{m,n}, y_{m,n+1})$; of course, this requires adjustment of the step height, which accounts for the factor $\sqrt{n}$ in (11).

(*) As usual, we say that the random variable x_n converges in distribution as $n \to \infty$ to the random variable x_∞ if F_n, the distribution function of x_n converges as $n \to \infty$ to F_∞, the distribution function of x_∞, at every continuity point of the latter. If x_n and x_∞ have probability densities, we say that x_n converges in density to x_∞ if p_n, the density of x_n converges to p_∞, the density of x_∞, at every continuity point of the latter. Of course, when the density $p(x)$ exists, the distribution function $F(x)$ is given by

$$F(x) = \int_{-\infty}^{x} p(z)dz.$$

Clearly, it follows from the central limit theorem that as $n \to \infty$, x_n^* converges in density to the Gaussian random variable x_∞. The entropy of x_n^* is defined in the usual way as

$$H_n^*(\alpha) = -\int_{-\infty}^{\infty} p_n(x) \log p_n(x)\, dx \;.$$

Substituting for $p_n(x)$ from (11) we find that

$$H_n^*(\alpha) = -\sum_{m=0}^{\infty} p(m;n\alpha) \log p(m;n\alpha) - \log \sqrt{n} \;,$$

so that the logarithmic divergence of $H_n(\alpha)$ is cancelled out and the dependence on α is restored. Since the entropy of a continuous random variable is a continuous functional of the probability density, we must have

$$H_n^*(\alpha) \to H_\infty(\alpha) = \log \sqrt{2\pi e \alpha}$$

as $n \to \infty$.

In Table 1 we show the results of calculating $H_n^*(\alpha)$ for the value $\alpha = 0.25$ used in constructing Figs. 1,2 and 3. (All logarithms are to the base e.) The table shows quite clearly that as n increases the entropy of x_n^* rapidly approaches the entropy of the limiting Gaussian random variable x_∞.(*)

TABLE 1(**)

n	$H_n^*(0.25)$
1	0.617
2	.581
4	.612
8	.664
16	.700
32	.714
64	.721
400 (***)	.722
∞	.726

(*) Note, however, the initial decrease of $H_n^*(0.25)$.

(**) The numerical work was done by Miss P.A. Smith, using E.C. Molina's tables, " Poisson's Exponential Binomial Limit " , D.Van Nostrand Co., Inc.; 1942.

(***) The value 400 corresponds to the largest value of p(m;nα) available in Molina's tables.

The univariate entropies of the processes $x_1(t)$, $x_4(t)$ and $x_{16}(t)$ represented in Figs. 1,2 and 3 are all close to the limiting value of 0.726, despite the fact that these cases correspond to low, medium and high density shot noise, respectively. However, the univariate entropy still seems to be a useful supplement to the power spectrum as an index of the structure of non-Gaussian random processes. (It will be recalled that all information about the relative phases of the harnomic components of a process are suppressed in its power spectrum, which limits the utility of the power spectrum as a means of characterizing non-Gaussian processes.)

VI. Conclusion

Every noise theorist is familiar with the way in which dense shot noise approximates a Gaussian process. We have given concrete form to this model by introducing an isospectral family $\{x_n(t)\}$ of non-Gaussian processes corresponding to progressively denser shot noise. The merit of our family $\{x_n(t)\}$ is that we know how to calculate in detail the statistical properties of its members. We have shown how to do this in cases where the amount of work required is not prohibitive.

ACKNOWLEDGEMENT

The author wishes to thank Prof. Peter Elias for his very helpful remarks concerning the entropy of $\{x_n(t)\}$.

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