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On Diurnal Variation of Barometric Pressure.

(Contribution II. from the Geophysical Seminary in the Physical Institute, College of Science).

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1. While the amplitudes and phases of the semidiurnal wave of barometric pressure show a very regular distribution over the entire surface of the earth, those of the diurnal component depend remarkably on secondary local conditions, as was fully illustrated by the classical investigations of A. Angot[1] and J. Hann.[2]

According to the recent aerological investigations,[3] the diurnal component of the daily variation of temperature is conspicuous chiefly in the lowest kilometer of the atmosphere, and it seems quite natural that the corresponding diurnal component of the barometric pressure is influenced by the variety of the nature of the underlying earth's surface within comparatively narrow extent. Imagine for a moment the picture of the isobaric surface at about 2 km. over a land with irregular patches of water, desert *etc.*, while the earth's surface is rapidly heated up by the solar radiation. The isobaric surface will be scattered over with numerous hills and dales, according to the nature of the substratum, and the resulting horizontal gradient of pressure cannot subsist without the flow of air tending to annul the gradient. When the heating

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1.) A. Angot, Annales du Bureau Central Météorologique de France, 1887.

2.) J. Hann, Denkschriften d. kais. Akad. d. Wiss., math.-naturwiss. Kl., **55**, pp. 49-121.

3.) Reger, Arbeiten d. kön. preuss. aeronautischen Observatoriums bei Lindenberg, **8**, p. 229.

is sufficiently rapid, the state can be maintained, since the flow is partly hindered by the friction of the underlying layer and also the deviating influence of the Coriolis's force. The irregularity of small scale due to the influence of the immediate neighbourhood will be gradually smoothed down as we proceed higher and higher, and the influence of the remoter substratum will gradually come into play.

The above idea is not at all essentially new, being entertained by the authorities such as Hann and Angot among others.[1] It seems indeed otherwise impossible to account for the irregular nature of the geographical distribution of the diurnal components as actually observed. As far as we are aware, there were however as yet no serious attempt made to consider these effects of local conditions a little more closely and to deduce anything in way of finding some rules or laws from among the apparently intractable chaos of materials. The present communication is the results of some trials dared in this direction. Though far from laying any serious claim on the rigorousness of the method employed, nor on the exhaustiveness of the materials utilized, the essential features of some of the results given below may be of some interests for meteorologists.

2. The beginning of the present investigation dates back to several years ago when the attention of the one of the author was drawn by the simple chart constructed by Buchan[2], showing roughly the geographical distribution of the daily amplitude of barometric pressure over the world. The dependency of the amplitude on the distribution of land and water was so conspicuous that it seemed quite feasible to infer some quantitative relation between the amplitude and the proportion of land and water over a certain definite area. To carry out the comparison, the following procedure was taken. The amplitudes of the pressure variation at different points on the circles of latitude 20° and 40° respectively were estimated from the chart by interpolation and plotted in a diagram

1.) *e. g.* Börnstein, Wiener Berichte, **113**, 2a, 1904, p. 721; Met. Z.S., **7**, p. 837.

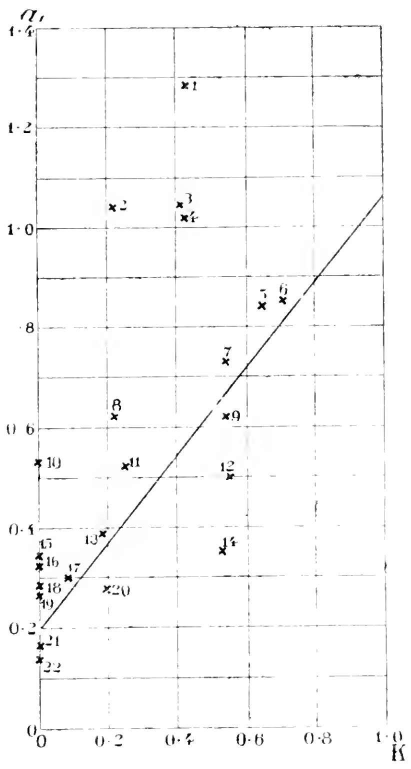
2.) Buchan, Challenger Report, Phys. and Chem. **2**, Report on Atmospheric Circulation, p. 21, Fig. 2.

with the longitudes as abscissa. On the other hand, the percentage of land area included within an area with the points in question

Diurnal Component and Continentality.

Fig. 1.

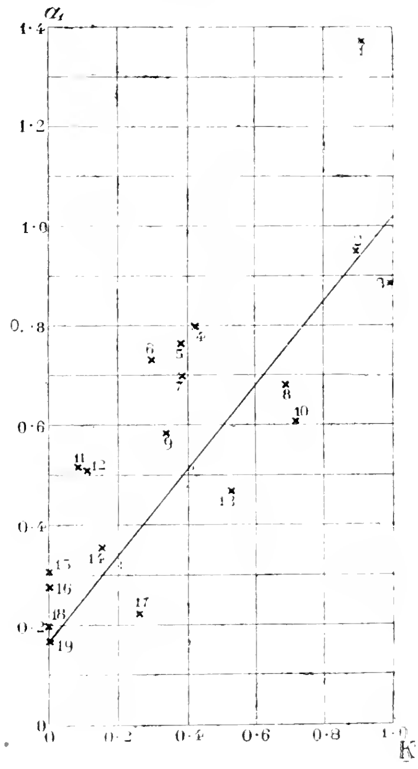
Latitude: 0°–10°



1. Quixeramobim.
2. Payta.
3. Boma.
4. S. Paul de Loanda.
5. Kibwezi.
6. Angola.
7. Gabun.
8. Batavia.
9. Quito.
10. Nauru.
11. Singapore.
12. Zanzibar.
13. Trevendrum.
14. Christiansborg.
15. Jaluit.
16. Indian and Pacific Ocean (8.7°N.)
17. Bay of Bengal.
18. Ascension.
19. Pac. Ocean (6.4°S.)
20. Costa Rica.
21. Atlantic Ocean (0°–5°N.)
22. Atlantic Ocean (5°–10°N.)

Fig. 2.

Latitude: 10°–20°

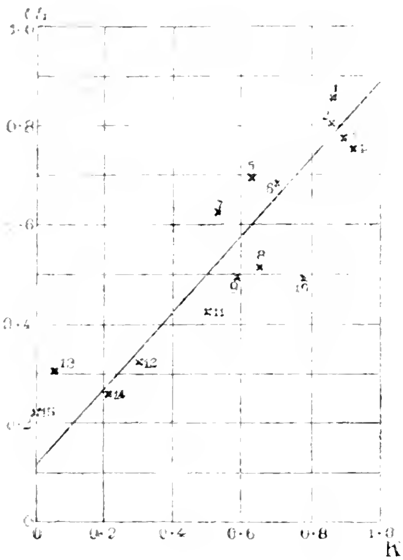


1. Boroma.
2. Kartum.
3. Cuyaba.
4. Joal.
5. Mexico.
6. Acapulco.
7. Port Darwin.
8. Caetete.
9. Madras.
10. Puno.
11. Manila.
12. Port au Prince.
13. Bombay.
14. Kingston.
15. Pac. Ocean (16.3°S.)
16. Samoa.
17. Dodabetta.
18. Tahiti.
19. St. Helena.

Diurnal Component and Continentality *(Continued)*.

Fig. 3.

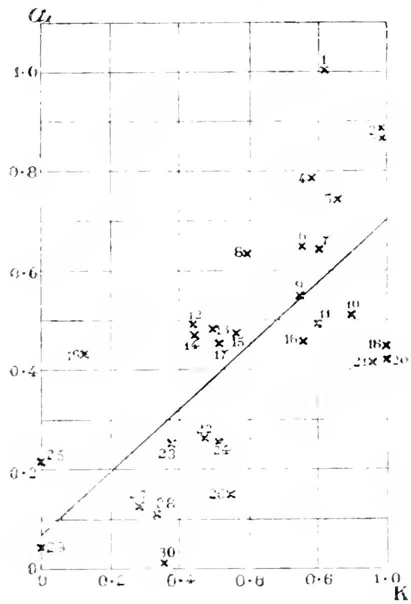
Latitude: 20°–30°



1. Goalpara.
2. Patna.
3. Allahabad.
4. Asuncion.
5. Kimberley.
6. Calcutta.
7. San Paolo.
8. Galveston.
9. New Orleans.
10. Hazaribagh.
11. Hongkong.
12. Taihoku.
13. Mauritius.
14. Habana.
15. Mangewa I.

Fig. 4.

Latitude: 30°–40°



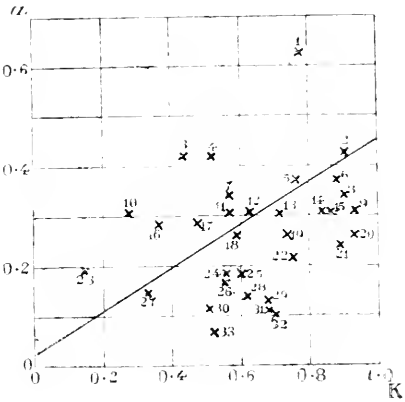
1. Cordoba.
2. Yarkand.
3. Leh.
4. Rocky. Mt.
5. Peking.
6. Rosario.
7. Memphis.
8. Yuma.
9. Kairo.
10. St. Louis.
11. Cincinnati.
12. San Francisco.
13. San Diego.
14. Savannah.
15. Washington.
16. Hankow.
17. Philadelphia.
18. Dodge City.
19. Tôkyô.
20. Denver.
21. Santa Fé.
22. Zikawei.
23. Melbourn.
24. San Fernando.
25. Pacific Ocean (33.3°S.)
26. Santiago.
27. Cape Town.
28. Lisbon.
29. Azores.
30. Cape Northumberland.

at the centres and bounded by two circles of latitude and two meridians, each 20° or 40° apart. Plotting the values of the percentage on the same diagram in a proper scale, a remarkable parallelism was observed, especially in the case of the land percentage in 40°

Diurnal Component and Continentality *(Continued).*

Fig. 5.

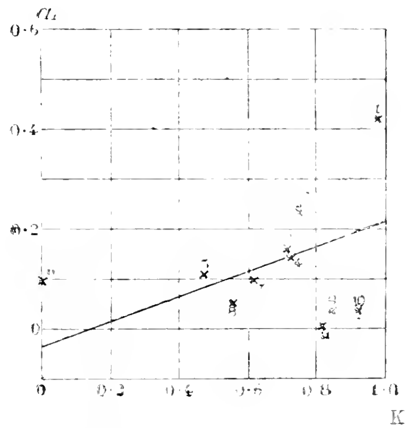
Latitude: 40°–50°



1. Tiflis.
2. St. Paul.
3. Madrid.
4. New York.
5. Toronto.
6. Duluth.
7. Cleveland.
8. Chicago.
9. Salt Lake City.
10. Sapporo.
11. Portland.
12. Milano.
13. Bucharest.
14. Nukuss.
15. Alpena.
16. Halifax.
17. Boston.
18. Genève.
19. Kremsmünster.
20. Bismarck.
21. Winnipeg.
22. Wien.
23. Hobarton.
24. Albany.
25. Esquimault.
26. Paris.
27. Coimbra.
28. Lesina.
29. Triest.
30. St. Martin de Hinx.
31. München.
32. Pola.
33. Napoli.

Fig. 6.

Latitude: 50°–60°



1. Irkutsk.
2. Prag.
3. Leipzig.
4. Magdeburg.
5. Greenwich.
6. South Georgia I.
7. Upsala.
8. Bruxelles.
9. Petrograd.
10. Moscou.
11. Dorpat.

square. Moreover it was clearly shown that the influence of the land distribution was remarkably less for the latitude 40° than for 20.° The result was communicated in a meeting of "Meteorologisches Colloquium" in the Meteorological Institute of Berlin. The publication was however refrained, since the Buchan's chart was far from being up to date and moreover the amplitudes refers to the total amount of variation, but not to the diurnal component. Recently the subject was resumed for a more detailed investigation, choosing as the materials those compiled by Angot and Hann in their classical papers above cited and also those given in the current numbers of the "Meteorologische Zeitschrift", from which the diurnal amplitudes a_1 were taken and their dependency on the distribution of land and water in a definite area enclosing the stations was to be examined. A circle with the radius of 10° was drawn, with each station in question as the centre, on a suitable globe, and the percentage of land area was determined by means of planimeter. The value of the percentage was called the "continentality" of the station for simplicity's sake and denoted by K. It may be remarked that the area is roughly of the same order of magnitude as Australia. Different stations were then classified into groups according to the 10° zones of latitude to which they belong. For each group, a diagram was constructed in which the values of a_1 were plotted with the corresponding continentality as abscissa (Figs. 1–6). The points thus obtained, representing different stations, though rather capriciously scattered over the diagram, showed still an undeniable tendency to be arranged within a certain belt inclined to the axis of the continentality. Moreover, the inclination of the belt to the axis of abscissa seems to become less as the latitude increases. The result shows at least that among the numerous factors determining a_1, the continentality as defined above may play not quite an unimportant rôle. To proceed a little further, a straight line was drawn in each diagram, roughly representing the median line of the belt supposed. The ensemble of such straight lines for different latitudes taken together was adjusted among themselves, so as to show a regular transition according to latitude. From lines reduced (the full lines in Figs. 1–6) a diagram was constructed in which the ordinate represents a_1 and the abscissa the latitude, and the system of curves was drawn representing the dependency of a_1 on the latitude for different values of the continentality. It was found that the curves may be roughly represented by an empirical formula of the form

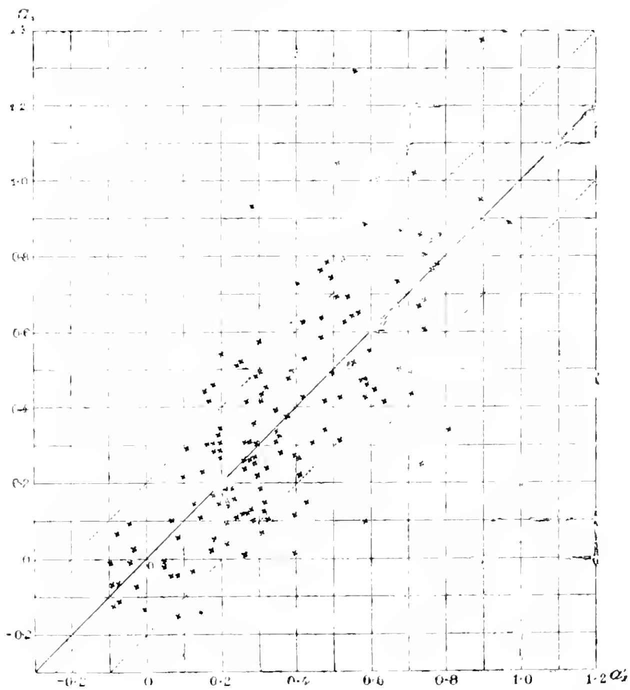
$$a' = (a + bK)\cos^2\varphi - c,$$

$$a \doteqdot 0.35, \quad b \doteqdot 0.86, \quad c \doteqdot 0.15 \text{ in mm.}.$$

where φ denotes the latitude, K the continentality. a_1' becomes negative for φ near $90°$, which means that for a higher latitude, the phase of the variation is inverted. The calculated value a'_1 was then compared with the actual value a_1. The differences $a_1 - a_1' = \Delta$ are generally considerable as might have been expected from the outset. Nevertheless, it may be scarcely hoped that any empirical formula in which the term depending on the continentality is put out of account, will show a less discrepancy than here met with. In Fig. 7 the actual values of a_1 is plotted against the calculated

Fig. 7.

a_1 : observed amplitude of the diurnal component.
a_1' : calculated value.



value a_1'. It will be seen that the great majority of the points lie within the belt with a breadth of 0.2 mm. on both sides of the line $a_1' = a_1$, while the value of a_1 varies from 0.1 to 1.0.

Next, on plotting the values of Δ for different stations upon a chart, it was found that for European stations negative Δ's remarkably predominate, in other words the variation is of the oceanic type :

TABLE I.

Stations.	Δ.	Stations.	Δ.
Åbo	−.161	Genève	−.093
Petrograd	−.213	Milano	−.085
Upsala	−.093	Triest	−.288
Ekatelinburg	−.246	Pola	−.320
Moscow	−.301	Bucharest	−.132
Magdeburg	.182	St. Martin de Hinx	−.233
Utrecht	−.262	Lesina	−.263
Oxford	−.086	Napoli	−.323
Greenwich	−.144	Coimbra	−.148
Leipzig	−.186	Lisbon	−.203
Bruxelles	−.235	Bosskop	+.009
Prag	−.138	Klagenfurt	+.175
Paris	−.133	Bozen	+.547
Wien	−.183	Tiflis	+.113
München	.253	Madrid	+.074

These negative discrepancies are associated with the small values of the daily temperature variation, as will be shown in subsequent paragraphs. The abnormal values for Bozen and Klagenfurt are evidently due to the altitude of the stations.

In Northern America, negative Δ prevails in the inland, while positive values are frequent in the coastal stations :

TABLE II.

	Positive Δ.		Negative Δ.	
Coast.	Eastport	.096	New Orleans	−.004
	New York	.115	Galveston	−.027
	Philadelphia	.051	Sitka	−.142
	Washington	.033		
	Savannah	.095		
	Port au Prince	.268		
	Kingston	.071		
	Portland	.038		
	San Francisco	.114		
	San Diego	.189		
	Yuma	.171		

	Positive Δ.		Negative Δ.	
Inland.	St. Paul	.008	Winnipeg	−.078
	Cincinnati	.108	Chicage	−.134
	Memphis	.095	St. Louis	−.025
			Bismarck	−.126
			Saltlake City	−.205
			Denver	−.151
			Dodge City	−.155
			Santa Fé	−.218

The same tendency is suspected also in South America, though the data availed of were too scanty. For Asian continental stations, the data are still more wanting; Nertinsk and Tomsk give negative, but Irkutsk a positive Δ. While Japanese stations and also Peking give positive Δ, the Chinese stations Hankow and Shanghai as well as Hongkong and Tonkin show negative values.

Again, it is interesting to observe that the positive anomaly prevails in the Pacific Ocean, while the negative values seem to be frequent in the Atlantic, though for the latter the data are scanty:

TABLE III.

Pacific Stations		Atlantic Stations	
Tahiti	+.221	Ascension	+.088
Mangarewa	+.057	St. Helena	−.019
Jaluit	+.143	Ocean 5°—10°N.	−.056
Nauru	+.340	,, 0° — 5°N.	.035
Manila	+.258		
Samoa	+.276		
Batavia	+.221		
Ocean 6.4°S.	+.067		
,, 16.3°	+.117		
,, 33.3°	+.074		

From the above examples, we may realize that there remains still some important local factors determining the amount of the diurnal component, beside the "continuality" as artificially defined above. The above discrepancies are quite systematic in character and cannot be generally accounted for by the local topography within a small extent. The last comparison of the two oceans, suggests a considerable influence of the continent lying far beyond the limit of the 10° circle here adopted for the determination of K.

3. The very unsatisfactory results of the above attempt to account for the variety of a_1 by the simple consideration of the "continentality", led us to consider the matter more closely under the light of some elementary theoretical considerations.[1]

Take first of all, for simplicity's sake, only the two-dimensional problem, in which the earth's surface considered as plane is taken for the xy-plane, and the isobaric as well as the isothermal lines are all straight and parallel to the y-axis. Suppose now the isobaric surface originally plane at the height z, be swelled up by $\zeta(x, z)$ on account of the heating of the underlying strata. The horizontal pressure gradient at the level z will be $g\rho \frac{\partial \zeta}{\partial x}$. If we may assume the steady state soon established, this will be equilibriated by the friction, *i.e.*

$$g\rho \frac{\partial \zeta}{\partial x} = -\mu \frac{\partial^2 u}{\partial z^2}, \tag{1}$$

where u is the horizontal velocity toward x and μ the coefficient of friction.[2] Now, ζ may be roughly given by

$$\zeta = \int_0^z \frac{\theta}{\Theta} dz, \tag{2}$$

where Θ is the mean absolute temperature at z and assumed independent of x, and θ is the variable part of the temperature which is to be considered as a function of both x and z, and also of time t. The above holds only on the assumption that the air columns are kept by vertical partition walls from expanding horizontally: but

1.) Prof. Okada kindly drew our attention to a paper by Hill, Met. Zs, **5**, 1888, p. 340, on the annual variation of barometric pressure in India, in which a similar idea as followed in this section can be traced.

2.) The influence of the rotation of earth is here provisionally put out of account. The effect will be referred to later.

when the heating is effected sufficiently rapid, we may take (2) as a rough approximation. Next, let us assume that θ is of the form

$$\theta(x, z) = \theta_o(x) f(z),$$

where $\theta_o(x)$ is the variation of temperature at sea-level and is a function of x and t only, while $f(z)$ is the function of z only. The latter assumption amounts to saying that the variation of temperature at z level is proportional to the variation at sea-level. Then we have

$$\frac{\partial^2 u}{\partial z^2} = -\frac{g\rho}{\mu}\frac{\partial \zeta}{\partial x} = -\frac{g\rho}{\mu}\frac{d\theta_o}{dx}\int_o^z \frac{f(z)}{\Theta(z)}dz.$$

To simplify the matter, we idealize the case still further and assume that the elevation ζ is nearly constant for a small finite altitude near the surface. Then, if we are considering only the initial stage of the motion where the pressure gradient near the earth's surface is still insignificant, we may put $u = o$, $\frac{\partial u}{\partial z} = o$ at $z = o$. Hence we have

$$u = -\frac{g}{\mu}\frac{d\theta_o}{dx}\int_o^z dz\int_o^z \rho dz\int_o^z \frac{f(z)}{\Theta(z)}dz. \tag{3}$$

In this integral, $\rho(z)$ may for the present purpose be considered to be the mean value independent of the variable part of the temperature. It may therefore be regarded for the rough approximation, as the function of z only, say $F(z)$,

$$F(z) = \int_o^z dz\int_o^z \rho dz\int_o^z \frac{f(z)}{\Theta(z)}dz. \tag{4}$$

Then the total flow of mass through the infinite vertical plane parallel to yz, is given by

$$U = \int_o^\infty u\rho dz = -\frac{g}{\mu}\frac{d\theta_o}{dx}\int_o^\infty \rho F(z)dz. \tag{5}$$

The latter integral is a constant and may be denoted by I. The rate of change of the barometic pressure p at sea-level will be approximately given by

$$\frac{dp}{dt} = -g\frac{\partial U}{dx} = \frac{g^2 I}{\mu}\frac{d^2\theta_o}{dx^2}. \qquad (6)$$

The above holds of course only for the highly idealized case considered; but, if the total amount of U be small and the variation of ζ sufficiently rapid, the essential feature of the variation of pressure may roughly be represented by (6), provided that the stationary state expressed by (1) is established instantaneously.

In the above calculation, the effect of the Coriolis's force was entirely put out of account. The component flow in y-direction due to this influence will become considerable compared with u, especially for high latitudes and in high levels. However, as long as the assumption of parallel isobars and the uniform deviating force is adhered, this component will bring no essential modification to the form of the expression (6), except the value of the constant factor. In applying the above theory to the actual problems, however, the nature of the abstractions made in the assumption must always be borne in mind.

Again, in the above, we have tacitly neglected the influence of the topography in affecting the thickness of the air layer subjected to the daily variation of temperature. In actual cases, ζ will be generally diminished with the elevation of the earth's surface, if we provisionally assume that the temperature reduced to the sea-level is everywhere uniform and the variation takes place as in the case there were no elevation of land. In this case we must replace $\int_0^z$ in the above equations by $\int_h^z$. This makes $\frac{dp}{dt}$ proportional to

$$\frac{d^2\theta_o}{dx^2}\cdot h + \frac{d\theta_o}{dx}\frac{dh}{dx},$$

where $h(x)$ represent the elevation of the surface as a function of x.

Remembering the nature of the assumptions made at the outset, let us simply take the form

$$\frac{dp}{dt} = C\frac{d^2\theta_o}{dx^2}, \qquad C = \text{const}, \qquad (6')$$

as at least qualitatively legitimate, neither entering upon the form of $\theta(z)$ and thence the value of I, nor upon the influence of $h(x)$ which we assume small, and carry through the application a little further on some appropriate actual problems, to see how far the formula may be utilized for the explanation of the phenomena in question.

4. Suppose a temperature wave proceeding toward the positive direction of x, West say, over the earth's surface idealized as in the preceding paragraph, such that

$$\theta_o = a \cos \frac{2\pi}{T} \left(t - \frac{x}{v} + \varphi\right), \tag{7}$$

where $T=24$ hours and $v=2\pi/24$ per hour. Taking the origin of time at noon at the origin of x, $-\varphi$ gives the retardation of the temperature maximum after noon. Let us assume φ constant and only a as a function of x. Then

$$\frac{1}{C}\,\frac{dp}{dt} = \frac{d^2\theta}{dx^2} = \left(\frac{d^2a}{dx^2} - a\right)\cos\frac{2\pi}{T}\left(t - \frac{x}{v} + \varphi\right) + 2\,\frac{da}{dx}\sin\frac{2\pi}{T}\left(t - \frac{x}{v} + \varphi\right).$$

Integrating, we obtain

$$\frac{2\pi}{CT}\left(p - p_o\right) = \left(\frac{d^2a}{dx^2} - a\right)\sin\frac{2\pi}{T}\left(t - \frac{x}{v} + \varphi\right) - 2\frac{da}{dx}\cos\frac{2\pi}{T}\left(t - \frac{v}{x} + \varphi\right), \tag{8}$$

where p_o is the mean value of p. Putting the expression of $p - p_o$ in the form as given by Angot, and transferring the origin of time to midnight, we have

$$p - p_o = a_1 \cos(m + \psi_1) \tag{9}$$

where

$$\left.\begin{aligned} a_1 &= C\sqrt{\left(\frac{d^2a}{dx^2}\cdot\frac{T}{2\pi} - \frac{a}{v}\right)^2 + \frac{4}{v^2}\left(\frac{da}{dx}\right)^2}, \\ \psi_1 &= \tan^{-1}\frac{\dfrac{d^2a}{dx^2}\cdot\dfrac{T}{2\pi} - \dfrac{a}{v}}{\dfrac{2}{v}\quad\dfrac{da}{dx}} + \frac{2\pi\varphi}{T} + \pi. \end{aligned}\right\} \tag{10}$$

If a be constant in a special case, and φ be -2^h [1], we obtain $\psi_1 = \frac{\pi}{3} + \pi$ or $240°$, *i.e.* the maximum must occur at 8^h a.m.[2]

1) Taking into account the retardation of the temperature maximum in higher levels, -3^h may have been preferable, which gives $\psi_1 = 225°$, i.e. the maximum occurs at 9^h a.m.

2) The result is not very far from truth, except for some elevated stations and for high latitude. The inversiŏn of the phase in the latter case will be considered later.

Next, if
$$\alpha = a + bx, \tag{11}$$

then
$$a_1 = \frac{C}{v}\sqrt{(a + bx)^2 + 4b^2};$$

if b/a be small, we have for large value of x

$$a_1 \doteqdot \frac{C}{v}(a + bx). \tag{11'}$$

5. The above formula (7)—(10) may most plausibly be applied to the case when long parallel strips of land and water bounded by meridian lines are arranged alternately. For the simplest case when the breadth of the strips are all equal, we may assume provisionally

$$\alpha = a + b\cos\frac{2\pi x}{l}, \tag{12}$$

where a and b are constant, l is the breadth of the strip and x denotes the longitude counted positive toward West. Then

$$\frac{d\alpha}{dx} = -\frac{2\pi}{l}b\sin\frac{2\pi x}{l}, \qquad \frac{d^2\alpha}{dx^2} = -\left(\frac{2\pi}{l}\right)^2 b\cos\frac{2\pi x}{l}.$$

Hence putting $vT = \lambda = 2\pi$, we obtain

$$a_1 \propto \sqrt{\left\{\frac{a^2}{\lambda^2} + \frac{b^2}{2l^2}\left(\frac{l^2}{\lambda^2} + 6 + \frac{\lambda^2}{l^2}\right)\right.}$$
$$\left. + 2ab\left(\frac{1}{\lambda^2}+\frac{1}{l^2}\right)\cos\frac{2\pi x}{l} + \frac{b^2}{2l}\left(\frac{l}{\lambda}-\frac{\lambda}{l}\right)^2\cos\frac{4\pi x}{l}\right\}. \tag{13}$$

Now suppose for very rough approximation that the part of Western Europe, the Atlantic Ocean and Northern America can be compared with the ideal arrangement of land and water as above considered, if we take the meridians 0°, 60°W. and 120°W. as the ideal coast lines, each strip of land or water having a breadth of 60° in longitude. The comparison may in some measure be justified, if we apply the above formula only to the narrow belt along the European coast. Take the origin of x at 30 E°., corresponding to the assumption that the amplitude of the temperature variation is minimum at the centre of Atlantic Ocean. In (12), we assume $a=5$°C. and $l=\frac{2\pi}{3}$.

Then

$$a_1^2 \propto \frac{1}{2}\left(\frac{3l^2}{\lambda^2}+6+\frac{\lambda^2}{l^2}\right)+2\left(\frac{l^2}{\lambda^2}+1\right)\cos\frac{2\pi x}{l} + \frac{1}{2}\left(\frac{l}{\lambda}-\frac{\lambda}{l}\right)^2\cos\frac{4\pi x}{l}. \qquad (14)$$

Carrying out the numerial calculation for each ten degree of longitude, we obtained the second line of the following Table, in which the observed value of a_1, as the mean values for different stations included within successive strips with the breadth of 10°, are given in the third line. In the fourth line, the number of stations taken for evaluating the third line is given.

TABLE IV.

Longitude.	20° W.		10° W.		0° W.		10° E.		20° E.		30° E.
Calc.	2.74		2.18		2.03		2.65		3.36		3.67
Mean a_1.		.227		.159		.245		.317		.203	
No. of Stations.		4		15		21		16		3	

Some parallelism is to be observed between the second and the third lines, if we put range 20°–30°E. out of account, in which the number of stations is scanty.

6. For the next trial, we chose all stations falling within the zone of latitude 20°–60°N. and the mean value of the observed a_1 were calculated for successive areas with the breadth of 20° in longitude. To compare this with the result to be expected from the present theory, it will be most natural to take for a the mean value of the amplitude of daily temperature variation for each area mentioned and estimate the coefficients $\frac{da}{dx}$ and $\frac{d^2a}{dx^2}$ from these

data. Since the necessary materials were, however, not at our disposal, we were compelled to resort to a provisory assumption : *That the mean a for a given zone of latitude as the function of longitude is proportional to the mean annual amplitude of the temperature for the same zone.*

The latter assumption may appear at first sight utterly unjustifiable, since the daily amplitude generally decreases with latitude, whereas the annual amplitude increases with the distance from the equator. If we, however, confine our attention to a given narrow belt of latitude and consider the dependency of the amplitude on the longitude, it will be rather plausible to assume for the purpose of rough approximation, as is aimed at throughout the present investigation, a parallelism between the two amplitudes, since both depend on the nature of the earth's surface in a similar way, in spite of the difference in the periods of the periodic heating and cooling.[1)]

---

1) when we consider the average daily fluctuation of temperature, we may consider the sun at the equator. Then the solar radiation per unit area of the surface will be proportional to $\cos\varphi$, where φ is the latitude. The temperature amplitude of the earth's surface depends not only on the intensity of the radiation, but largely on the nature of the surface. Assume it to be proportional to $F(\lambda)$. Then the daily amplitude will be roughly of the form $F\cos\varphi$.

On the other hand, the annual amplitude is determined by the difference of insolation in different seasons which will be proportional to

$$\cos(\varphi-\delta_0) - \cos(\varphi+\delta_0) = 2\sin\varphi\sin\delta_0,$$

where $\delta_0=23.5°$. Hence, the annual amplitude may be put

$$F'(\lambda)\sin\varphi\sin\delta_0.$$

The ratio of the amplitudes, for constant φ, becomes $F(\lambda)/F'(\lambda)$ as the function of λ. Now, consider F and F' expanded by Fourier's series of the form

$$\sum_{m=0}^{\infty} A_m \sin m\lambda + \sum_{0}^{\infty} B_m \cos m\lambda.$$

It is evident that A'_m and B'_m for F' converge more rapidly than the corresponding coefficients for F, since the daily variation of temperature, being greater for the lower layer of atmosphere, will be influenced conspicuously by the minor irregularities of the surface, while the annual variation extending to a higher strata, will depend on the general condition of the wide area surrounding the station in question. Taking however the mean value of the daily amplitude over a wide area, the terms depending on the large values of m will tend to cancel each other, and only the first few terms will remain as the leading terms. Our assumption actually amounts to assuming the proportionality of A_m, B_m with A'_m, B'_m for the small values of m, which will be in any case not very far from the truth.

The data used were the small chart giving the "lines of equal annual temperature variation" in Berghaus's Physikalisches Atlas, III. Abt., Meteorologie, No. I., from which the mean annual amplitude for the zone of latitude 20°—60° N. for every ten degrees of longitude was determined, from the mean values for the three latitudes 20°, 40° and 60°N. at the corresponding longitudes. The value a' thus obtained was plotted in a diagram with the longitude x as abscissa and the resulting curve was smoothed down by taking for each the mean values of a''s at $x-20°$, $x-10°$, x, $x+10°$ and $x+20°$. From the smoothed curve thus obtained, the value of $\frac{da'}{dx}$ was determined for every ten degrees of longitude by drawing the tangent at the corresponding point of the curve. The resulting curve for $\frac{da'}{dx}$ was again smoothed down in a similar manner and then $\frac{d^2a'}{dx^2}$ was determined[1]. The values of the differential co-efficients were substituted for $\frac{da}{dx}$ and $\frac{d^2a}{dx^2}$ in (10) and a_1'[2] and ψ_λ calculated for each 20° of longitude. The result is shown in the following table and also plotted in Fig. 8., in which the calculated value proportional to a_1' is given in an arbitrary scale. The agreement of the calculated and observed curves are rather remarkable, if we remember the nature of the assumptions made in the derivation of the formula (10).

---

1) It may be remarked that the values of the differential coefficients thus obtained are actually given by

$$\frac{d\alpha'}{dx}=\frac{1}{2d_1}\left(\alpha_{x+d_1}-\alpha_{x-d}\right) ;$$

$$\frac{d^2\alpha'}{dx^2}=\frac{1}{4d_1d_2}\left(\alpha_{x+d_1+d_2}-\alpha_{x-d_1+d_2}-\alpha_{x+d_1-d_2}+\alpha_{x-d_1-d_2}\right)$$

as far as we consider the part of the curve to be smoothed may be regarded as linear within the range $2d_1$ or $2d_2$. In our case $2d_1$ and $2d_2$ are equal to 50°.

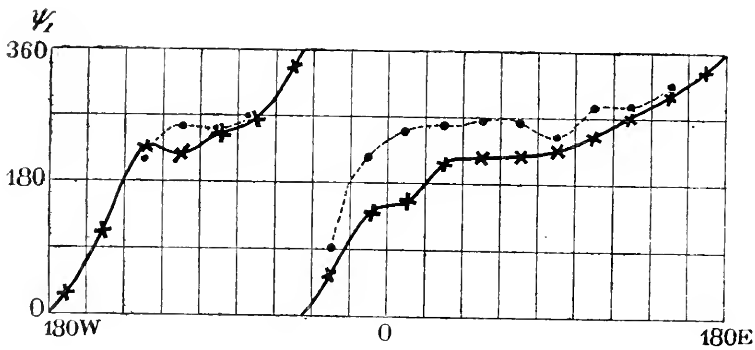
2) Of course we may obtain only the value proportional to a_1.

Fig. 8.

Observed and calculated values of the amplitude a_1 and the phase ψ_1 of the diurnal component, as depending on the longitude.

·····•·····•·····•····· observed.
—×—×—×— calculated.

ψ_1



a_1

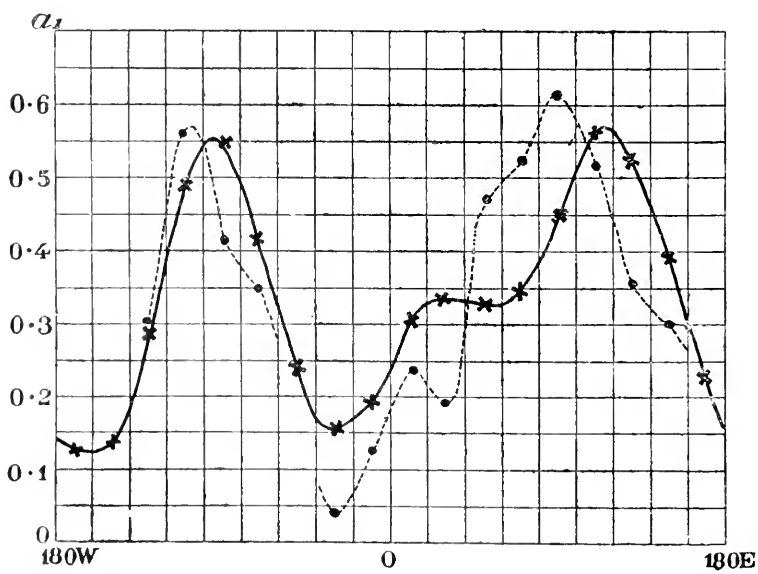


TABLE V.

Observed and calculated values of a_1 and ψ_1 as the function of the longitude.

Longitude.	Stations.	Mean a_1.	Mean ψ_1.	Calc. Value ppl. a_1.	Calc. ψ_1.*
180°–160°E.		—	—	230	335°
160°–140°E.	Sapporo.	0.301	313°	394	301°
140°–120°E.	Tôkyô Zikawei.	0.352	285°	528	272°
120°–100°E.	Irkutsk, Nertschinsk, Peking, Hongkong, Tonkin.	0.519	284°	562	247°
100° – 80°E.	Barnaul, Goalpara, Patna, Allahabad, Hazaribagh, Calcutta.	0.614	245°	452	229°
80° – 60°E.	Ekaterinburg, Yarkand, Leh, Simla.	0.524	265°	347	219°
60° – 40°E.	Nukuss, Tiflis.	0.465	265°	329	215°
40° – 20°E.	Petrograd, Moscow, Tarnopol, Bucharest, Athen, Cairo.	0.199	261°	334	210°
20° – 0°E.	Christiania, Upsala, Berlin, Magdeburg, Utrecht, Leipzig, Bruxelles, Prag, Paris, Wien, München, Kremsmünster, Budapest, Klagenfurt, Bozen, Genève, Aosta, Milano, Triest, Pavia, Pola, Lesina, Pelagosa, Napoli, Etna.	0.239	251°	307	155°
0°–20°W.	Makerstown, Dublin, Oxford, Greenwich, Jersey Island, St. Martin de Hinx, Madrid, Coimbra, Lisbon, San Fernando.	0.121	220°	197	142°
20°–40°W.	Ponta Delgada.	0.046	92°	157	59°
40°–60°W.				241	334°
60°–80°W.	Eastport, Halifax, Toronto, Albany, Boston, New York, Philadelphia, Washington.	0.348	270°	407	267°
80°–100°W.	Winnipeg, Duluth, Alpena, St. Paul, Chicago, Cleveland, Cincinnati, St. Louis, Memphis, Savannah, New Orleans, Galveston, Havana.	0.418	250°	548	249°
100°–120°W.	Assiniboine, Bismarck, Salt Lake City, Denver, Pikes Peak, Colorado Spring, Rocky Mt., Dodge City, Santa Fé, Yuma, San Diego.	0.563	257°	493	218°
120°–140°W.	Sitka, Esquimault, Portland, San Francisco.	0.300	213°	284	229°
140°–160°W.		—	—	138	111°
160°–180°W.		—	—	127	23°

* It must be remarked that a difficulty is often met with in taking the mean value of ψ_1, since the actual values for different stations in the same group vary widely and it becomes in such cases uncertain whether ψ_1 as such or $2\pi-\psi_1$ is to be taken in calculating the mean value. Some of the values here tabulated are therefore more or less affected by an arbitrary choice in the case of ambiguity. The influence is not serious for groups with large number of stations.

7. By the rather unexpected success of the above trial, we were tempted to extend the theory for the case of three-dimensional problems and put

$$\frac{dp}{dt} = C\left(\frac{d^2\theta}{dx^2} + \frac{d^2\theta}{dy^2}\right), \tag{15}$$

where x denotes the longitude and y the latitude. Putting

$$\theta = a\cos\frac{2\pi}{T}\left(t-\frac{x}{v}+\varphi\right), \tag{16}$$

where a is considered as the function of y only, we obtain

$$\frac{2\pi}{CT}\left(p-p_0\right)=\left(\frac{d^2a}{dy^2}-a\right)\cos\frac{2\pi}{T}\left(t-\frac{x}{v}+\varphi-\frac{\pi}{4}\right) \tag{17}$$

Hence

$$\left.\begin{aligned} a_1 &= \frac{CT}{2\pi}\left(\frac{d^2a}{dy^2}-a\right), \\ \varphi_1 &= \frac{2\pi\varphi}{T}+\frac{\pi}{2}, \qquad \text{if} \quad \frac{d^2a}{dy^2} > a, \\ &= \frac{4\pi}{3} \quad , \qquad ,, \qquad ,, \; < \; ,, . \end{aligned}\right\} \tag{18}$$

If we may represent the average diurnal amplitude of temperature as the function of latitude only and of the form

$$a = \cos^2(y-\delta), \tag{19}$$

where δ is the sun's declination, we obtain

$$\frac{d^2a}{dy^2}-a = -\frac{5\cos 2(y-\delta)+1}{2}.$$

Thus a_1 vanishes for

$$5\cos 2(y-\delta)+1 = 0 \qquad \text{or} \qquad y-\delta = 50^\circ\, 45'.$$

In fact a_1 becomes smaller at higher latitudes and shows a tendency to change its sign.

8. Returning to the case of the two dimentional problem,

consider the case where the land and water are represented by semi-infinite planes, the coast line being represented by the y-axis. In this case, it will not be far from truth, if we assume

$$\alpha = \alpha_1 \pm \alpha_2 \tanh a(x \mp x_0) \tag{20}$$

where the upper or the lower sign is to be taken according as the water lies on the east or west side of the y-axis. We have then

$$\frac{d\alpha}{dx} = \pm a \, \alpha_2 \operatorname{sech}^2 a\,(x \mp x_0),$$

$$\frac{d^2\alpha}{dx^2} = \mp 2\, a^2 \alpha_2 \tanh a\,(x \mp x_0) \operatorname{sech}^2 a\,(x \mp x_0).$$

Hence,

$$\left.\begin{aligned} a_1^2 \propto\ & \alpha_1^2 \pm 2\alpha_1 \alpha_2 \tanh a\,(x \mp x_0)\,\{1+2\,a^2 \operatorname{sech}^2 a(x \mp x_0)\} \\ & + \quad \alpha_2^2 \tanh^2 a(x \mp x_0)\,\{1+2\,a^2 \operatorname{sech}^2 a(x \mp x_0)\}^2 \\ & + 4a^2\alpha_2^2 \operatorname{sech}^4 a(x \mp x_0), \\ \psi_1 =\ & \tan^{-1} \frac{-\alpha_1 \mp \alpha_2 \tanh a(x \mp x_0)\,\{1+2a^2 \operatorname{sech}^2 a\,(x \mp x_0)\}}{\pm 2a\alpha_2 \operatorname{sech}^2 a\,(x \mp x_0)} \\ & + \frac{2\pi\varphi}{T} + \pi. \end{aligned}\right\} \tag{21}$$

In order to get some idea of the magnitudes of the constants α_1, α_2, a and x_o, the data for some South African stations given by J. R. Sutton were utilized, according to which we obtain

for	$x = 0$,	$\alpha \doteqdot 4°$C.,
,,	$x = 117$ km.,	$\alpha \doteqdot 7°$C.,
,,	$x = 420$ km.,	$\alpha \doteqdot 8°$C.,

Let us assume also $\alpha \doteqdot 8°$ for $x=\infty$ and $\alpha \doteqdot 1°$ for $x=-\infty$; then we may put approximately

$\alpha_1 = 4°.5$, $\alpha_2 = 3.°5$,

$a = 0.00855$ 1/km. or $54300 \cos\varphi$ radian^{-1},

$x_0 = 16.9$ km. or $\dfrac{26.5 \times 10^{-7}}{\cos\varphi}$ radian of longitude.

1) J. R. Sutton, Trans. South African Phil. Soc., 11, 1902; Met. Zs., 1904, p. 40.

Since a has a very large value for not very high latitude, the term

$$a_2^2 \tanh^2 a(x-x_0)[2a^2 \operatorname{sech}^2 a(x-x_0)]^2$$

is predominant in the expression of a_1^2; hence the graph of a_1, with x as abscissa takes an M-shape with the minimum lying in the sea at a distance 16.9 km. from the coast. Carrying out the calculation for some values of x, we obtain Table VI.

TABLE VI.

$a(x \pm x_0)$.	a_1.
0.0	12.0×10^4
0.5	74.8×10^3
1.0	65.8 ,,
1.5	42.6 ,,
2.0	14.0 ,,
2.5	5.4 ,,
3.0	2.1 ,,
3.5	0.22 ,,

Here, the unit increase of $a(x \mp x_0)$ corresponds to 119 km. in the increment of x.

In applying the above to the actual case, a difficulty is met with in assigning a suitable value of x for each station, since in the actual case the coast line is by no means straight even approximately, or even if portions of the coast line could be regarded as nearly straight, the cases are rare in which we may directly apply the above theory with confidence. However, let us consider the geometrical distribution of land and water expressed by means of infinite series, for example in terms of spherical harmonics and retain only the first few terms, neglecting the remaining terms pertaining to the minor irregularities. The resulting coast lines will be devoid of all irregular zigzags. For such an ideal distribution, the applicability of our theory becomes more justifiable, since the portions of the coast lines with the length of several hundred km. may be regarded as nearly straight, or at least as portions of a plane curve of second degree with no remarkable curvature. In such a case, it may be in some measure plausible to consider the coast as straight. In applying the above formula, we will take for x, *i.e.* the distance of the station from the coast, the value proportional to the continentality of the station, instead of taking the distance from the nearest coast, since the latter distance is very uncertain for the most cases. For, it may be easily shown that when the coast is actually straight, the continentality as defined in the previous paragraph, is given by

$$K = \frac{1}{\pi}\left(\frac{\pi}{2} + \alpha + \frac{\sin 2\alpha}{2}\right), \qquad \alpha = \frac{x}{R}; \qquad x < R, \tag{22}$$

where x is the nearest distance of the station from the coast considered positive when the station is on the land side of the coast. The above value of K is of course not proportional to x, but nearly so for $x<R$, when only very rough approximation is concerned. For $x>R$, we have always $K=1$. As far as the qualitative verification of our theory for a stations not very far from the coast is concerned, we must expect that the observed values of a_1 considered as the function of K will have a minimum near the coast, or for $K=0.5$, and two maxima on both sides. To test the point, the data for 33 North American stations were chosen[1]. The continentality for each station was evaluated and compared with the corresponding amplitude a_1. Grouping these according to the magnitudes of K, and taking the mean value for each group, we obtained the following result :

TABLE VII.

Range of K.	0–0.2	0.2–0.4	0.4–0.6	0.6–0.8	0.8–1.0
Mean K.	0.132	0.288	0.526	0.682	0.907
Mean a_1.	0.436	0.270	0.356	0.500	0.400
Number of Stations	2	2	12	5	13

In Fig. 9. the values of mean a_1 are plotted, with the mean K as abscissa[2]. The result is qualitatively in accordance with the theory, in so far as there is a minimum of a_1 near $K=0.3$ or 0.4 and two apparent maxima on both side of it.

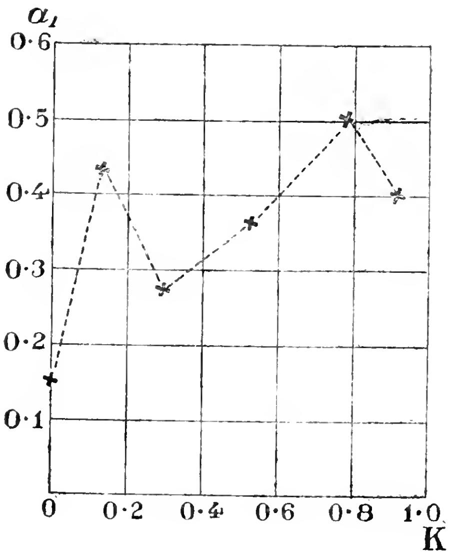
If the assumption (20) were actually the case, the values of $\dfrac{da}{dx}$ and $\dfrac{d^2a}{dx^2}$ would be practically zero for greater value of x and the amplitude a_1 in the interior of a large continent would be the same as in the ocean. In the actual case, this latter is never the

1) Hann, *loc. cit.;* also Met. Zs., 1899, p. 421.

2) The curve (a_1, K) may be considered as transformed from the curve (a_1, x) by suitably varying the scale of the abscissa. The value of a_1 for $K=0$ is taken from the mean value for Atlantic Ocean 0°–10°N.

case, a_1 increasing *generally* towards the interior of the continent as was evidently shown in the first part of the present communication. The reason is that the assumption (20) roughly represent the course of a only near the coast, while for the interior of the continent, there must be added another term on the right side of (20), which is to represent the gradual increase of a towards inside, for an example such linear term as in (11). For the present, we are still at a loss, on account of the want of necessary data, how to determine the best expression of a applicable for the majority of cases. In actual cases, a will depend not only on the distance of the station from sea, but on many other conditions, for examples, cloudiness, prevailing winds *etc.*, *etc.*

Fig. 9.



a_1 : mean diurnal amplitude.
K : continentality.

9. It must be remarked that the value defined by us as the continentality is a quantity of very arbitrary character.

We could have better chosen the diameter of the circular area smaller or greater than 10°. Probably there may be found a more adequate magnitude of the radius for which the relation between the continentality K and the amplitude a_1 may turn out more conspicuous.

Remembering that the irregularity in the daily variation of temperature must gradually disappear with the altitude, and that the higher the layer, the more conspicuous relatively will become the influence of the remoter area of the earth's surface, it will be more appropriate to consider a determined by a quantity, A say, instead of K, which is defined by

$$A = \underset{R\ large}{Lt} \frac{1}{\pi R^2} \int_0^R L_r \, \phi(r) \, dr, \tag{23}$$

where L_r is the area of land in a circular belt with the radius r and the breadth dr. $\phi(r)$ is to be considered as a function of r which continuously tends to zero for the large value of r. Let ϕ be practically zero for $r > R_0$, then

$$A = \frac{1}{\pi R_0^2} \int_0^{R_0} L_r \, \phi(r) \, dr = \frac{\phi(\theta R_0)}{\pi R_0^2} \int_0^{R_0} L_r \, dr \tag{24}$$

where θ is a fraction between 0 and 1. But the factor of $\phi(\theta R_0)$ is what we have called the continentality in the preceding paragraphs, if R_0 could be put$=10°$. The value of θ varies of course from station to station. For an equal value of the integral, θ will be greater or less according as the land is concentrated near the centre, or near the margin of the area in question. The case in which the land is convex toward the sea will have a greater A than the case when it is concave, K being the same for both cases. It has been found in the first part of the paper that a_1 for Western European stations are generally small compared with that for the stations in the other continents, with nearly the same K. This may of course be explained chiefly by the effects of the prevailing West wind from the Atlantic, but probably also partly by the fact that for most of these stations the water surface is found intruding more or less near the central part of the circular area.

10. Thus far, we have treated exclusively the annual means of the diurnal amplitude and took no notice of the rather remarkable variability during the course of a year. Angot applied the method of harmonic analysis for representing the annual variation of a_1 as well as of ψ_1, the results of which are tabulated in his memoirs cited. On actually plotting the monthly values of a_1 and ψ_1 in diagrams, however, we came to the conviction that for a large number of stations, the annual variation is so irregular that the real physical significance of only the first terms of Fourier's expansion seems quite doubtful, being small compared with the terms of higher orders left out. It seemed to us interesting to review

the actual annual course of the daily variation under the light of the present theory.

Referring to § 5, we have, when $\alpha = a+b\cos\frac{2\pi x}{l}$,

$$a_1^2 \propto a^2+\frac{b^2}{2}(p^4+6p^2+1)+2ab(p^2+1)\cos\frac{2\pi x}{l}+\frac{b^2}{2}(p^2-1)^2\cos\frac{4\pi x}{l},$$

putting $\lambda/l=p$ in (13). If p is large compared with 1 and we may neglect 1 against p^2, we have approximately

$$|a_1| \propto \left| a+bp^2\cos\frac{2\pi x}{l} \right|.$$

The assumption that p is large, corresponds to the case when we are considering the local irregularities of the daily amplitude of temperature within a narrow extent of land and water, whose linear dimension is negligibly small in comparison with the earth's circumference. In such a case, the quantity b, representing the deviation of the amplitude from the mean value, will necessarily be small in comparison with a.

If $bp^2<a$, a_1 will have one maximum at $x=0$ and a minimum at $x=l/2$. But if $bp^2>a$, $|a_1|$ may have two maxima at 0 and π and two zero between them. Taking the equation for the general value of p, compare two special points $x=0$ and $x=l/2$, for which the amplitudes are a_1 and a_1' respectively. Then

$$\left|\frac{a_1}{a_1'}\right| = \left|\frac{1+\frac{b}{a}(p^2+1)}{1-\frac{b}{a}(p^2+1)}\right|.$$

Now consider b as a periodic function of *season*, and put

$$b=b_0+b_1\cos n(t+\varphi),$$

then we have

$$\left|\frac{a_1}{a_1'}\right| = \left|\frac{1+\frac{b_0}{a}(p^2+1)+\frac{b_1}{a}(p^2+1)\cos n(t+\varphi)}{1-\frac{b_0}{a}(p^2+1)-\frac{b_1}{a}(p^2+1)\cos n(t+\varphi)}\right|.$$

a) If both $b_0(p^2+1)/a$ and $b_1(p^2+1)/a$ be small compared with unity,

$$1-\frac{b_0}{a}(p^2+1)>\frac{b_1}{a}(p^2+1),$$

and a_1 and a_1' will each attain one maximum and minimum during the course of a year. Moreover, the maximum of a_1 will correspond to the minimum of a_1.

b) If $1-\frac{b_0}{a}(p^2+1)<\frac{b_1}{a}(p^2+1)$, but $1+\frac{b_0}{a}(p^2+1)>\frac{b_1}{a}(p^2+1)$, a_1 will have one maximum and a_1' two maxima.

c) If $1-\frac{b_1}{a}(p^2+1)<\frac{b_1}{a}(p^2+1)$, and $1+\frac{b_0}{a}(p^2+1)<\frac{b_1}{a}(p^2+1)$, a_1 as well as a_1' will have two maxima.

The position of zero will vary with the value of a, b, p.

Again consider the case when a varies only in the direction of meridian. Putting in § 7, (18)

$$a=c+d\cos\frac{2\pi}{L}y\,, \tag{25}$$

we have

$$a_1 \propto c+d\left\{1-\left(\frac{2\pi}{L}\right)^2\right\}\cos\frac{2\pi}{L}y\,. \tag{26}$$

When we consider d as a periodic function of seasons, it will be evident that the annual course will be inverted at the two stations apart $L/2$ in the direction of y.

In order that we may apply the above to actual examples, we must of course have a detailed knowledge about the geographical and seasonal distribution of the daily variation of temperature. Since we are not yet in possession of the sufficient data, we will merely mention here some facts with regard to the actual examples to which our theory may find application, and suggest a method of investigation convenient for the purpose.

We will choose the data for British Isles, because the seasonal variation of a_1 is very irregular, apparently owing to the rather complicated distribution of land and water in this region. The seasonal variation of a_1 was taken from Angot's paper. Our first procedure was to obtain the mean seasonal variation (Fig. 10) for the ten stations, and then to obtain for each station the deviation curve (Figs. 11–18). On comparing these deviation curves among themselves, several interesting facts may be noticed:

Annual Variation of the Diurnal Components a_1

(Fig. 11 to 18 show the deviation from the mean value given in Fig. 10).

Fig. 10.

Mean of 10 stations.

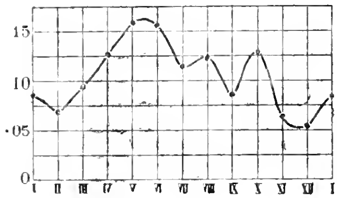


Fig. 11.

Aberdeen.

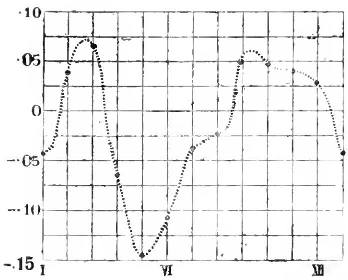


Fig. 12.

Oxford.

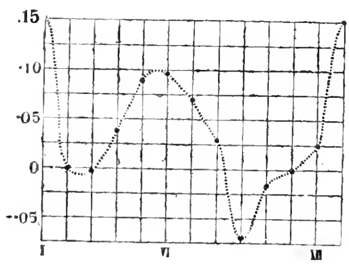


Fig. 13.

Armagh (upper curve) and Glasgow (lower).

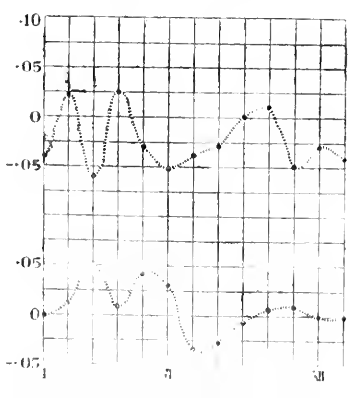


Fig. 14.

Liverpool (upper) and Stonyhurst (lower).

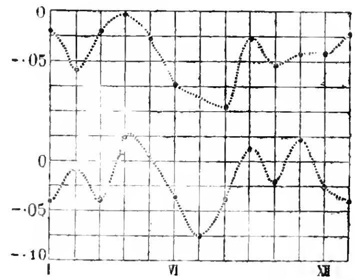


Fig. 15.

Greenwich.

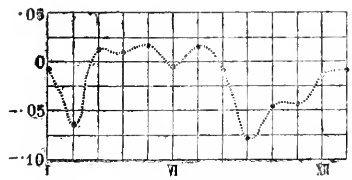


Fig. 16.

Kew.

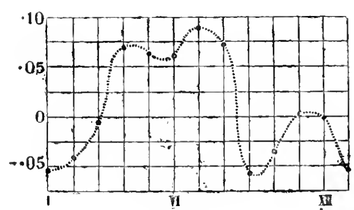


Fig. 17.

Falmouth.

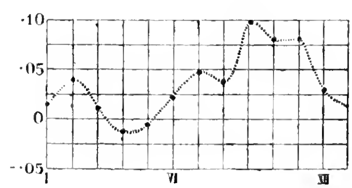
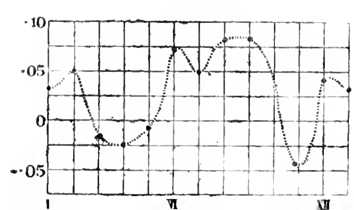


Fig. 18.

Valencia.



a) How much the daily variation may be affected by the local influence, may be seen by the comparison of the two curves for Kew and Greenwich.

b) Deviation curves for Aberdeen and Oxford have two annual maxima and show very nearly opposite course. Falmouth and Valencia resemble Aberdeen in some measure, while the mean of Kew and Greenwich is rather of the Oxford type. This may probably be accounted for, referring to the ideal case corresponding to the equation (26) above.

c) Comparing the deviation curve for Armagh and Glasgow, we observe that they are reversed to each other between February and July, while they are parallel in the interval July to September. Moreover, taking the mean of Armagh and Glasgow, it is found to be nearly parallel to Stonyhurst curve in the interval January to September and also parallel to Liverpool curve between March and September.

d) Shifting the deviation curve for the mean of Kew and Greenwich two months later, the curve becomes similar to Valencia curve.

e) The mean of Aberdeen and Oxford deviation curves is inverted to the mean curve of the ten stations.

It will be very interesting to study the actual origin of these complicated anomalies under the light of an elementary theory as propounded in the previous paragraphs. The results may in any case be instructive for elucidating the actual mechanisms of the daily exchange of the atmosphere over a region with various geographical conditions. We hope we will be able to resume the subject when the necessary data for the temperature variations are at our disposal.

In conclusion, we wish to express our best thanks to Prof. T. Okada of the Central Meteorological Observatory for many valuable informations.

SUMMARY.

1. The geographical distribution of the diurnal component a_1 of daily barometric change is compared with the distribution of land and water, a quantity called "continentality" being introduced which is the percentage of land in a definite area surrounding each station.

2. A linear relation between the amplitude and the continentality was assumed and a systematic discrepancy was discovered.

3. An elementary theory based on a number of simplifying assumptions is proposed and applied to actual examples.

4. The variation of the amplitudes and phases as functions of longitude could be explained in its essential features.

5. The inversion of phase near the pole is explained.

6. The minimum of amplitude near coast is pointed out and explained.

7. The influence of geographical conditions on the seasonal variation of amplitude is discussed and different possibilities pointed out.

8. A method of investigating the complicated variety of the seasonal variation under the light of the above theory is suggested, referring to some examples.

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Mesotomisation of Diammine=dinitro=oxalo=cobalt Complex and Determination of the Configurations of this Complex and of Diammine=tetranitro=cobalt Complex.

by

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and

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In a recent communication [Yuji Shibata, *Recherches sur les spectres d'absorption des ammines-complexes métalliques I. Spectres d'absorption des solutions aqueuses des ammines-complexes cobaltiques ;* Journ. Coll. Scien. Imp. Univ. Tokio, vol. XXXVII, Art. 2, 1915.] one of the authors discussed the relation between the constructions of various cobalt ammine complexes and the absorption spectra of their aqueous solutions.

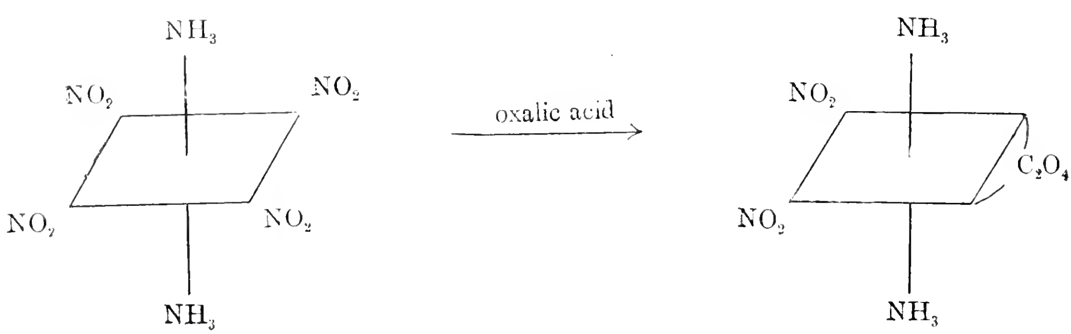
Among those studied, some nitrammine cobalt complexes were also included, and it was found that their absorption spectra are strongly influenced by the difference of the space positions of nitro-groups in the complex ions. Thus, if at least two nitro-groups occupy the furthest corners of a regular octahedron, which is considered to represent the space model of such metal-ammine complexes, three distinct bands appears in their absorption spectra, the absorption maxima existing respectively at about 2000, 3000 and 4000 of frequencies. Another group of nitrammine cobalt complexes, which have their nitro-groups only in the adjacent positions, shows merely two absorption bands and lacks the one in the most refrangible region of the spectrum.

From these diversities in the optical behavier of these complexes, the author was able to determine the spacial arrangements of the nitro-groups in several nitrammine cobalt complexes. Thus, the result of the investigation of the absorption spectrum of diammine-dinitro-oxalo-cobalt complex $[Co(NH_3)_2(NO_2)_2C_2O_4]'$ $Me^{\cdot}$ easily led him to ascertain that its two nitro-groups are in the adjacent positions (cis-position), this complex showing only two bands at about 2000 and 3000 of frequencies.

Now, if we could determine the space positions of the two ammonia molecules in this complex, the configuration of the salt would become quite clear, the oxalic acid-residue necessarily occupying the cis-position [A. Werner, *Ann.*, **386**, 10, 1912]. Further, the configuration of diammine-dinitro-oxalo-cobalt complex having once been determined, that of its mother substance, diammine-tetranitro-cobalt complex $[Co(NH_3)_2(NO_2)_4]'$ $Me^{\cdot}$, will be easily ascertained, the one being readily obtainable from the other by the action of oxalic acid [Jörgensen, *Zeitsch. anorg. Chem.*, **11**, 440, 1896]. Thus:—

$$[Co(NH_3)_2(NO_2)_4]Me + C_2O_4H_2 = [Co(NH_3)_2(NO_2)_2C_2O_4]Me + 2NO_2H$$

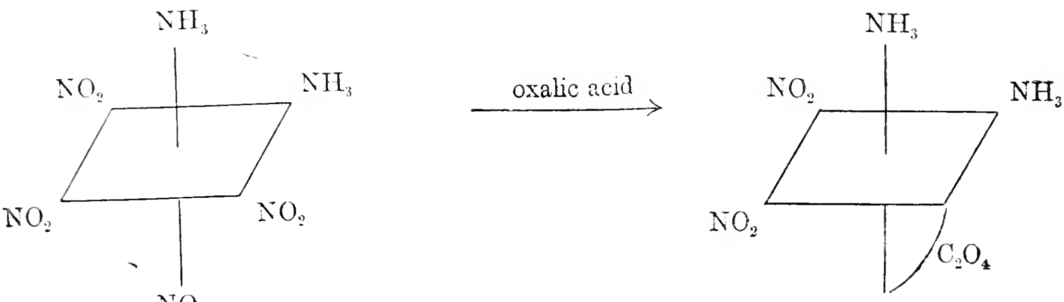
Now, owing to the fact that there are two possibilities in the spacial arrangements of the ammonia molecules in the complex ion, we may express the above chemical reaction in either of the following ways:—



(Ammonia molecules in the trans-position.)

(I)

or



(Ammonia molecules in the cis-position.)

(II)

The choice between these two formulae for diammine-dinitro-oxalo-cobalt complex offers little difficulty, when we take their constructions into account. As may be seen, formula (I) is constructed quite symmetrically, which allows no possibility of the existence of enanthiomorphs, while with formula (II), the construction of which is obviously asymmetric, the resolution of the complex into optically active isomers has to be expected.

Accordingly, mesotomisation of diammine-dinitro-oxalo-cobalt complex has been tried, with the result that both of the optical isomers have actually been obtained. Details of our procedure will be described in the experimental part, but its essential points may here be outlined. The cobalt complex, which was initially prepared as an ammonium salt, was first transformed into the corresponding barium salt. To a concentrated solution of this salt, sulphate of an alkaloid, such as brucine, strychinine or cinchonine, was added, the precipitated barium sulphate was filtered off, and the alkaloid salt of the complex, which readily crystallised out from the filtrate, was then fractionated in the usual manner.

The most easily and difficultly soluble portions of the alkaloid complexes, which were thus obtained by repeated fractionations, were then converted into the potassium salts by the action of potassium iodide. The two potassium diammine-dinitro-oxalo-cobaltiates $[Co(NH_3)_2(NO_2)_2C_2O_4]K$ thus obtained from the two extreme fractions were both proved to be optically active, as had been anticipated, and the one was found to be the optical antipode of the other, having the following specific rotatory power:

$$[\alpha]_D = \pm 115^\circ.$$

The possibility of the mesotomisation of this complex shows that it is asymmetrically constructed and that, therefore, its two ammonia molecules must be in the adjacent positions, corresponding to the space formula (I) (see page 2). Consequently the position of the two ammonia molecules in diammine-tetranitro-cobalt complex, the mother substance, must also be adjacent.

The mesotomisation of diammine-dinitro-oxalo-cobalt complex not only affords us a means of determining its configuration in the manner above described, but it also offers us some interesting facts concerning the stereochemistry of metal complexes. In so far as this asymmetrically constructed complex ion is found to be an anion, this is virtually the first example of a mesotomised complex anion containing a cobalt atom[1], the large number of optically active cobalt complexes hitherto obtained by A. Werner and his pupils being all of them cations. Moreover the type of the optically active anion $[Co(NH_3)_2(NO_2)_2C_2O_4]'$ is entirely new in respect to the molecular asymmetry of such metal complexes.

When A. Werner found optical activity in some cobalt-ammine complexes, he drew attention to the fact that the activity is due sometimes to the asymmetric cobalt atom, sometimes to an asymmetric structure of the complex ion as a whole. Those in which optical activity is due to the asymmetric structure of the complex ion as a whole were further classified by him into the two types $[MeA_2B^I_2]$ (or $[MeA_2C^{II}]$) and $[MeA_3]$[2], calling them respectively the type of molecular asymmetry I and the type of molecular asymmetry II. For example, *cis*-dinitro-diethylene-diamine-cobalt complex $[Co\,\underline{en}_2(NO_2)_{2(2)}^{(1)}]^{\cdot}$, and carbonato-diethylene-

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1) A. Werner mesotomised formerly a chromium complex containing a complex anion $[Cr(C_2O_4)_3]'''$. [*Ber.*, **45**, 3061, 1912].

2) In these general formulae, A means a molecule of organic amine bases which corresponds to two molecules of ammonia, or a bivalent acid radical, while B and C represent respectively mono-and bivalent acid radicals.

diamine-cobalt complex $[Co\underline{en}_2CO_3]^{\cdot}$ belong to the type of molecular asymmetry I, while triethylene-diamine-cobalt complex $[Co\underline{en}_3]^{\cdot\cdot\cdot}$ and trioxalo-chromium complex $[Cr(C_2O_4)_3]'''$ belong to that of molecular asymmetry II.

Now, it will be noticed that the optical activity of diammine-dinitro-oxalo-cobalt complex is due not to the asymmetry of the cobalt atom but to a molecular asymmetry, the central cobalt atom in this complex ion which may be expressed by the general formula $[CoE_2F_2D^{II}]'$ being by no means asymmetric, as is the case with $[Co\underline{en}_2{\cdot}NO_2{\cdot}Cl]^{\cdot}$, $[Co\underline{en}_2{\cdot}NH_3{\cdot}Br]^{\cdot\cdot}$ etc [compare A. Werner, *Ber.*, **44**, 1887 and 2445, 1911]. This new type of molecular asymmetry to which attention has just been called is, therefore, to be introduced into the stereochemistry of inorganic complex compounds and, in extension of Werner's classification, it may be called the type of molecular asymmetry III.

Experimental.

1) Preparation of Barium Diammine-dinitro-oxalo-cobaltiate.

To start with, ammonium diammine-tetranitro-cabaltiate $[Co(NH_3)_2(NO_2)_4]NH_4$ was first of all prepared according to the method described by Jörgensen [*Zeitsch. anorg. Chem.*, **17**, 476, 1898], this substance having the advantage of being easily converted into the barium salt and the barium salt being required in later operations.

The large yellowish-brown coloured crystals of the ammonium salt thus obtained were carefully purified by repeated recrystallisation, and subsequently converted into ammonium diammine-dinitro-oxalo-cobaltiate $[Co(NH_3)_2(NO_2)_2C_2O_4]{\cdot}NH_4$ by the action of oxalic acid upon a concentrated aqueous solution of the original salt [Jörgensen, *Zeitsch. anorg. Chem.*, **11**, 440, 1896].

The oxalo-complex thus obtained was then dissolved in water, and finely pulverised barium chloride was added in portions. When one molecular proportion of barium chloride was added to two molecular proportions of the complex, barium diammine-dinitro-oxalo-cobaltiate began to crystallise out from the solution in the form of fine brownish red needles containing 3 molecules of water [Jörgensen, *ibid.*, 445]. This important salt was also repeatedly recrystallised, until the estimation of cobalt in the complex gave a quite satisfactory value.

2) Mesotomisation of Diammine-dinitro-oxalo-cobalt Complex as Brucine Salt.

To a saturated aqueous solution of barium diammine-dinitro-oxalo-cobaltiate, which was kept at about 40°C, finely pulverised brucine sulphate $(C_{23}H_{26}N_2O_4)_2SO_4H_2$, $7H_2O$ was added in small portions under constant agitation. The barium sulphate formed was filtered off after the whole of the calculated amount of brucine sulphate was added.

As the filtrate became cooler, brown needles aggregating in a radial form gradually began to appear here and there on the walls of the vessel. After allowing the solution to stand over night at the ordinary temperature, these crystals were gathered on a filter as the first fraction. The mother liquor was then kept in a vacuum over sulphuric acid, and the crystals, which separated out from it, were gathered from time to time. In this way four fractions in all were obtained.

An estimation of cobalt and water with one of these fractions gave the following results:

obs	calc. for $[Co(NH_3)_2(NO_2)_2C_2O_4]\cdot H(C_{23}H_{26}N_2O_4)$, H_2O
Co=8·52%	Co=8·59%
H_2O=2·72 "	H_2O=2·63 "

The four fractions gave the following values for their specific rotations in an aqueous solution (0.3%, 10 cm):

I	$[\alpha]_D^{29°} = -21{\cdot}3°$
II	$[\alpha]_D^{28.5°} = +21{\cdot}3°$
III	$[\alpha]_D^{28°} = +27°$
IV	$[\alpha]_D^{27°} = +29.3°$

The first fraction which consisted of the least soluble crystals was found to be fairly unstable when dissolved in water, for on standing the insoluble free alkaloid gradually separated out from the solution. However, by working rapidly but cautiously we were able to effect its further fractional crystallisation and purification, and ultimately a crop of crystals which had as high a specific rotation as

$$[\alpha]_D^{27.5^\circ} = -70{\cdot}7^\circ$$

was obtained, a value which has not been increased by further recrystallisation. These crystals we have designated as fraction Ia.

Fractions II, III and IV, having been ascertained to show comparatively small differences in their rotatory power, were now together dissolved in the mother liquor from fraction Ia, and this solution was again kept in a vacuum in order to repeat the fractionation. Two more fractions were thus obtained, fractions V and VI, which had the following specific rotations:

V	$[\alpha]_D^{27^\circ} = 0^\circ$
VI	$[\alpha]_D^{27^\circ} = +45^\circ$

Fraction VI was once more dissolved in a small quantity of water and refractionated by partial crystallisation in a vacuum (fractions VII and VIII). Fraction VIII, which was most easily soluble in water, had a specific rotation of $[\alpha]_D^{28^\circ} = +68{\cdot}3$. As this value remained unchanged even after further recrystallisations, it was concluded that the fractions Ia and VIII were essentially the *l*-brucine salt of *l*-and *d*-diammine-dinitro-oxalo-cobalt complex respectively.

Elimination of the brucine molecule from these fractions was then proceeded with. For this purpose, the example set by A. Werner, who mesotomised trioxalo-chromium complex $[Cr(C_2O_4)_3]'''\ Me^{\bullet}_3$ (*loc. cit.*) as strychinine salt and subsequently decomposed it by means of potassium iodide in order to eliminate strychinine as its insoluble iodide, was followed.

The operation was carried out in the following manner: the brucine complex was first dissolved in a small quantity of water, and to this solution a calculated amount of solid potassium iodide was then added. On vigorous agitation the hardly soluble brucine hydroiodide completely separated out from the solution, and the precipitates thus formed were immediately filtered off. In order to avoid autoracemisation, which might occur in the dissolved active complex, absolute alcohol was added to the filtrate in small portions, until a slight but permanent turbidity was established. The walls of the vessel were then strongly rubbed with platinum spatula to accelarate crystallisation, when fine brownish red needles were readily formed. These crystals were collected and well sucked on a filter, washed first with a small quantity of water and then with absolute alcohol, until they had no more a bitter, but slightly sweet taste.

An estimation of cobalt, potassium and water in the product thus obtained from fraction Ia gave the following numbers, proving that the desired potassium salt of the complex was obtained:

obs.	calc. for $[Co(NH_3)_2(NO_2)_2C_2O_4]K\cdot 1\frac{1}{2}H_2O$
Co=17·88%	Co=17·38%
K=10·96 "	K=11·53 "
H_2O= 6·45 "	H_2O= 7·96 "

Its specific rotation was found to be:

$$[\alpha]_D^{27°} = -115° \quad (0·1\%, 10 \text{ cm})$$

The salt prepared from fraction VIII in the same manner gave the following analytical and optical data:

obs.	theor.
Co=17·95%	Co=17·38%
K=11·59 "	K=11·53 "
H_2O= 7·17 "	H_2O= 7.96 "

Sp. rot.: $[\alpha]_D^{2?°} = +115°$ (0.1%, 10 cm)

It is evident, therefore, that as already pointed out, the least soluble fraction consisted of *l*-brucine-*l*-diammine-dinitro-oxalo-cobalt complex, while the most easily soluble fraction consisted of *l*-brucine-*d*-diammine-dinitro-oxalo-cobalt complex.

3) Mesotomisation of Diammine-dinitro-oxalo-cobalt Complex as Strychinine Salt.

Mesotomisation of diammine-dinitro-oxalo-cobalt complex with strychinine sulphate $(C_{21}H_{22}N_2O_2)SO_4H_2 \cdot 6H_2O$ was next tried.

The mode of proceedure was quite the same as before; but in view of the fact that the solubility of strychinine sulphate is considerably less than that of brucine sulphate a slight modification was introduced in taking a less concentrated solution of the barium salt of the complex and in keeping its temperature at about 50°C, instead of at 40°. The filtrate from the precipitated barium sulphate soon yielded a quantity of pale brown needles, showing that the strychinine salt of the complex is likewise much less soluble than the corresponding brucine salt.

An analysis of these crystals gave the following numbers:

obs.	calc. for $[Co(NH_3)_2(NO_2)_2C_2O_4]\cdot H\cdot(C_{21}H_{22}N_2O_2)\cdot H_2O$
Co=9·60%	Co=9·41%
H_2O=3·62 "	H_2O=2·88 "

In the course of the fractionations of the strychinine salt of the complex, it was observed that the least soluble fraction, which contained *l*-strychinine-*l*-diammine-dinitro-oxalo-cobalt complex, was, in the state of a solution, even more unstable than the corresponding brucine salt; consequently, an accurate measurement of its rotatory power in a dilute solution could not be made. The strychinine salt after two recrystallisations was, therefore, converted into the potassium salt in the manner already described, and the specific rotation of the potassium diammine-dinitro-oxalo-cobaltiate thus obtained was determined with the following result:

$$[\alpha]_D^{27°} = -104° \quad (0\cdot1\%, 10\text{ cm})$$

Due perhaps to the small solubility of the strychinine salt the *d*-variety of the complex could not be separated even by repeated fractionations.

4) Mesotomisation of Diammine-dinitro-oxalo-cobalt Complex as Cinchonine Salt.

As the alkaloids used in the two preceeding cases were themselves laevorotatory, the optically active cobalt complex obtained as the first fraction by using a salt of these alkaloids was also, in each case, found to be laevorotatory. With the view, therefore, of obtaining a *d*-variety of the optically active complex as the first fraction, mesotomisation of diammine-dinitro-oxalo-cobalt complex was next tried with cinchonine sulphate $(C_{19}H_{22}N_2O)_2SO_4H_2$, which is itself dextrorotatory.

The method of proceedure was quite the same as in the previous cases. The cinchonine salt of the complex forms fine pale brown needles which are anhydrous. The results of the analysis were

obs.	calc. for $[Co(NH_3)_2\cdot(NO_2)_2\cdot C_2O_4]\cdot H\cdot(C_{19}H_{22}N_2O)$
Co=10.64%	Co=10.40%
H_2O= 0 "	H_2O= 0 "

At first, the cinchonine salt was separated in three fractions; the first fraction consisted of the crystals obtained directly from the solution, while the second and third fractions consisted of those obtained by partial crystallisation in a vacuum.

The three fractions had the following specific rotations:

I	$[\alpha]_D^{27°}=+88.5°$	(0.2 %, 10 cm)
II	$[\alpha]_D^{29°}=+88°$	(0.25%, ")
III	$[\alpha]_D^{29°}=-28.8°$	(" , ")

The first two of these were then mixed together and recrystallised from water. The least soluble portion of the crystals thus obtained gave the value

$$[\alpha]_D^{27°}=+149°$$

for its specific rotation, which remained unaltered even after further recrystallisations. This fraction was then transformed into the potassium salt. It had the following specific rotation:

$$[\alpha]_D^{27°}=+111° \quad (0.1\%,\ 10\ cm)$$

On account of the scantiness of the material, we were unable to obtain from the more soluble portions, the corresponding *l*-variety of a sufficiently strong rotatory power.

5) Optically Active Ammonium Diammine-dinitro-oxalo-cobaltiate.

Optically active ammonium salts of this cobalt complex

$[Co(NH_3)_2(NO_2)_2C_2O_4]NH_4$ were prepared from *l*-brucine-*l*-diammine-dinitro-oxalo-cobalt complex and *d*-cinchonine-*d*-diammine-dinitro-oxalo-cobalt complex, in quite the same manner as in the case of the potassium salts, only using ammonium iodide instead of potassium iodide.

The *l*-and *d*-ammonium salts of the complex thus obtained had the following specific rotations:

$$[\alpha]_D^{27°} = -107° \quad (0{\cdot}1\%,\ 10\ \text{cm})$$
$$\text{,,} \quad = +116° \quad (\text{,,} \quad , \quad \text{,,} \quad)$$

These values are practically the same as those of the potassium salts. This is probably due to the fact that the difference between the molecular weights of the potassium- (=302) and ammonium- (=291) salts is not large enough to exert any distinct influence upon their rotatory powers, which were always measured in so diluted a solution as 0·1%, because of a fairly intense colour of the solution.

Sammary.

1. Diammine-dinitro-oxalo-cobalt complex $[Co(NH_3)_2(NO_2)_2 C_2O_4]' Me^{\bullet}$ has been resolved into the optically active isomers by fractionation of the salts of brucine, strychinine and cinchonine.

2. In the fractionation of the alkaloid salts of the complex, either the *l*-alkaloid-*l*-cobalt complex or *d*-alkaloid-*d*-cobalt complex was always found to separate out as the least soluble fraction, while either the *l*-alkaloid-*d*-cobalt complex or *d*-alkaloid-*l*-cobalt complex always constituted the most easily soluble fraction.

3. The specific rotations of potassium and ammonium diammine-dinitro-oxalo-cobaltiates, which have been obtained by replacing the alkaloid molecules with potassium-and ammonium ions respectively, were measured, and found to have the values of *ca* ±115°, using sodium light.

4. As a results of the mesotomisation of diammine-dinitro-oxalo-cobalt complex, the configurations of this complex and also of its mother substance, diammine-tetranitro-cobalt complex, have been made clear, the possibility of mesotomisation indicating that

the complex ion $[Co(NH_2)_2(NO_2)_2C_2O_4]'$ is constructed asymmetrically, which means that the two ammonia molecules in this complex ion occupy the adjacent spacial position. As to the two nitro-groups, their position was previously determined, by one of the authors (Shibata), by a spectroscopic study.

5. The molecular asymmetry of the above mentioned complex ion, which is the first example of a mesotomised complex anion containing a cobalt atom, belongs to a new type and has been introduced into the stereochemistry of the metal complexes, calling it the type of molecular asymmetry III, in extension of Werner's classification.

The senior author, Yuji Shibata, has profound sorrow in recording here the death of his collaborator Mr. Toshio Maruki,. *Rigakushi*, which occurred during the progress of the investigation recorded in this paper. At the same time, he has much pleasure in expressing his hearty thanks to Messrs. K. Matsuno, *Rigakushi*, and S. Mitsukuri for the great assistance they have given him in completing this work.

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Researches on the Distribution of the Mean Motions of the Asteroids.

By

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With 1 plate.

In accordance with the views of some astronomers the fall of meteors, the zodiacal light and the *gegenschein* suggest the possibility of some kind of resistance to the planetary and satellite motions. It may be exceedingly small for the major planets. But for small bodies like asteroids or satellites[1], it may not be entirely negligible if the interval of time be sufficiently long, say hundreds or thousands of years.

Last year while at Yale University I considered the theoretical effect of a resisting medium, supposedly motionless, on the libration of asteroids, and tried to explain the gaps of the asteroid distribution on that hypothesis. But I did not succeed.

Recently I have worked on the supposition of another kind of resistance suggested by Prof. E. W. Brown.[2] According to this, resisting materials having the size of ordinary meteors are supposed to move around the central body in circular orbits. The result of my study seems satisfactory to explain the gaps in the first approximation. I shall present the course of my investigation in the following chapters.

The numerical computations throughout this investigation were duplicated by Mr. S. Terada, to whom the writer desires to express his sincere thanks.

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1) For comets see § 8.

2) Sir G. Darwin seems to have had similar idea. See A. N. 184. p. 263.

Chapter I.

Effect on the Elliptic Motion, of Resisting Materials supposed to move around the Central Body in Circular Orbits.

1. I shall assume that *the resisting particles and the asteroid move in one and the same plane.* The components of the velocities of two bodies at a common point, referred to the center of the sun and a system of fixed axes, are

	Asteroid	Particle
$\frac{dr}{dt}$	$\frac{nae}{\sqrt{1-e^2}}\sin w$	0
$r\frac{dw}{dt}$	$\frac{na^2\sqrt{1-e^2}}{r}$	$n_0 r$

where r, w, n, a and e are respectively the common radius vector, the true anomaly, the mean motion, the semi-major axis and the eccentricity of the asteroid and n_0, the circular mean motion of the particle. We have

$$n^2a^3 = n_0^2 r^3$$

or

$$n_0 r = na\sqrt{\frac{a}{r}}$$

Hence the components of the relative velocity of the asteroid are

$$(1)\qquad \begin{cases} V_s = \dfrac{nae}{\sqrt{1-e^2}}\sin w \\ V_t = na\left(\dfrac{a\sqrt{1-e^2}}{r} - \sqrt{\dfrac{a}{r}}\right) = na\dfrac{1+e\cos w - \sqrt{1+e\cos w}}{\sqrt{1-e^2}} \end{cases}$$

Assuming that *the resistance is proportional to the hth power of the relative velocity,* and denoting by S and T, the components of the resistance in the direction of the radius vector and of the perpendicular to it, we may write

$$(2)\qquad S = -c\rho V^{h-1}V_s \qquad T = -c\rho V^{h-1}V_t$$

where c is a constant depending only on the size of the asteroid, ρ, the density of the particles and V, the resultant relative velocity of the asteroid. The constants c and ρ are essentially positive.

2. Let ρ be a *continuous function* of r, so that it may be developed in the convergent series,

$$\rho=\rho_0+\frac{d\rho}{da}(r-a)+\frac{d^2\rho}{da^2}\frac{(r-a)^2}{2}+\dots\dots\dots$$

Put

$$k_1=\frac{a}{\rho_0}\frac{d\rho}{da},\qquad k_2=\frac{1}{2}\frac{a^2}{\rho_0}\frac{d^2\rho}{da^2},\ \dots\dots\dots\dots$$

then

$$\rho=\rho_0\left\{1+k_1\frac{r-a}{a}+k_2\left(\frac{r-a}{a}\right)^2+\dots\dots\dots\dots\right\}$$

Or developing $\frac{r-a}{a}$ in powers of e,

$$\rho=\rho_0(1-k_1e\cos w-k_1e^2\sin^2 w+k_2e^2\cos^2 w+\dots\dots\dots) \tag{3}$$

3. The equations for the variations of the elements are[1)]

$$\frac{da}{dt}=\frac{2}{n\sqrt{1-e^2}}\left[Se\sin w+T(1+e\cos w)\right]$$

$$\frac{de}{dt}=\frac{\sqrt{1-e^2}}{na}\left[S\sin w+T\left(\cos w+\frac{\cos w+e}{1+e\cos w}\right)\right]$$

$$e\frac{d\varpi}{dt}=\frac{\sqrt{1-e^2}}{na}\left[-S\cos w+T\left(1+\frac{1}{1+e\cos w}\right)\sin w\right]$$

$$\frac{d\varepsilon}{dt}=-\frac{2(1-e^2)}{na(1+e\cos w)}S+\frac{e^2}{1+\sqrt{1-e^2}}\frac{d\varpi}{dt}$$

where ϖ and ε are the mean longitude of perihelion and the mean longitude at the epoch. Since V contains the 1st power of e as factor, S and T will contain e^h as factor. Hence, if we confine ourselves to the order of e^{h+1} in the development of the differential coefficients, we may neglect e^2 terms in the coefficients of S and T. Thus,

1) Tisserand, Mécanique Céleste, I p. 433 and IV p. 218.

(4)
$$\left\{\begin{aligned} \frac{da}{dt} &= \frac{2}{n}[Se \sin w + T(1+e\cos w)] \\ \frac{de}{dt} &= \frac{1}{na}[S \sin w + T(2\cos w + e\sin^2 w)] \\ e\frac{d\varpi}{dt} &= \frac{1}{na}[-S\cos w + T(2-e\cos w)\sin w] \\ \frac{d\varepsilon}{dt} &= -\frac{2}{na}S(1-e\cos w) + \frac{e^2}{2}\frac{d\varpi}{dt} \end{aligned}\right.$$

The equations (1) become, keeping two orders of e.

$$V_s = nae \sin w \qquad V_t = \frac{1}{2}nae\cos w\left(1+\frac{e}{4}\cos w\right)$$

whence

$$V = nae\sqrt{1-\frac{3}{4}\cos^2 w + \frac{e}{8}\cos^3 w}$$

or neglecting e^3

(5)
$$V = nae\left(1+\frac{e}{16}\,\frac{\cos^3 w}{1-\frac{3}{4}\cos^2 w}\right)\sqrt{1-\frac{3}{4}\cos^2 w}$$

The equation (3) becomes simply

(6)
$$\rho = \rho_0(1-k_1 e\cos w)$$

4. We have to change the independent variable from t to w in the equations (4). The relation between the differentials is

$$dt = \frac{(1-e^2)^{\frac{3}{2}}}{n(1+e\cos w)^2}dw$$

or neglecting e^2

(7)
$$dt = \frac{1}{n}(1-2e\cos w)\,dw$$

5. Combining the equations (2), (4), (5), (6) and (7), and putting

$W=$

$$-c\rho_0 n^{k-2}a^{k-1}e^k(1-2e\cos w)(1-k_1 e\cos w)\left(1-\frac{3}{4}\cos^2 w\right)^{\frac{k-1}{2}}\left(1+\frac{1}{16}\,\frac{e\cos^3 w}{1-\frac{3}{4}\cos^2 w}\right)^{k-1}$$

I obtain

$$\frac{da}{dw}=Wa\left[2e\sin^2w+\cos w\left(1+\frac{e}{4}\cos w\right)\left(1+e\cos w\right)\right]$$

$$\frac{de}{dw}=W\left[\sin^2w+\cos w\left(1+\frac{e}{4}\cos w\right)\left(\cos w+\frac{e}{2}\sin^2w\right)\right]$$

$$e\frac{d\varpi}{dw}=W\left[-\sin w\cos w+\sin w\cos w\left(1+\frac{e}{4}\cos w\right)\left(1-\frac{e}{2}\cos w\right)\right]$$

$$\frac{d\varepsilon}{dw}=-2W\sin w(1-e\cos w)+\frac{e^2}{2}\frac{d\varpi}{dw}$$

Or neglecting e^{h+2} and simplifying

$$W=-c\rho_0 n^{k-2}a^{k-1}e^h\left(1-\frac{3}{4}\cos^2w\right)^{\frac{h-1}{2}}\left\{1-\left(2+k_1-\frac{h-1}{16}\,\frac{\cos^2w}{1-\frac{3}{4}\cos^2w}\right)e\cos w\right\}$$

$$(8)\quad\left\{\begin{aligned}\frac{da}{dw}&=Wa(\cos w+2e-\frac{3}{4}e\cos^2w)\\ \frac{de}{dw}&=W(1+\frac{e}{2}\cos w-\frac{e}{4}\cos^3w)\\ e\frac{d\varpi}{dw}&=-W\frac{e}{4}\sin w\cos^2w\\ \frac{d\varepsilon}{dw}&=-2W(\sin w-e\sin w\cos w)+\frac{e^2}{2}\frac{d\varpi}{dw}\end{aligned}\right.$$

6. To find the secular variations of the elements in a unit interval of time we have to evaluate

$$\left[\frac{da}{dt}\right]=\frac{n}{2\pi}\int_0^{2\pi}\frac{da}{dw}\,dw \qquad \text{etc.} \qquad \text{etc.}$$

Since the effect of the resistance is supposed to be very small, we may neglect $c^2\rho_0^2$ within a single revolution, so that the quantities n, a and e may be integrated as constants.

Now if we put

$$X=(1-q\cos^2w)^{\frac{s}{2}}\sin^iw\cos^jw$$

where $q<1$ and s, i, j are any positive integers, it may be easily seen that

$$\int_0^{2\pi} X dw = \{1+(-1)^i\}\{1+(-1)^j\}\int_0^{\frac{\pi}{2}} X dw$$

Hence the integral vanishes when either i or j is odd, and becomes

$$4\int_0^{\frac{\pi}{2}} X dw$$

when both are even.

7. Applying this result to the equations (8), I obtain after some simplifications

$$(9)\quad \begin{cases} \left[\dfrac{da}{dt}\right] = -c\rho_0 n^{h-1} a^h e^{h+1}\left\{2I_1 - \left(\dfrac{11}{4}+k_1\right)I_2 + \dfrac{h-1}{16}I_3\right\} \\[2ex] \left[\dfrac{de}{dt}\right] = -c\rho_0 n^{h-1} a^{h-1} e^h I_1 \\[2ex] \left[\dfrac{d\varpi}{dt}\right] = 0 \qquad\qquad \left[\dfrac{d\varepsilon}{dt}\right] = 0 \end{cases}$$

where

$$I_1 = \frac{2}{\pi}\int_0^{\frac{\pi}{2}}\left(1-\frac{3}{4}\cos^2 w\right)^{\frac{h-1}{2}} dw$$

$$I_2 = \frac{2}{\pi}\int_0^{\frac{\pi}{2}}\left(1-\frac{3}{4}\cos^2 w\right)^{\frac{h-1}{2}} \cos^2 w\, dw$$

$$I_3 = \frac{2}{\pi}\int_0^{\frac{\pi}{2}}\left(1-\frac{3}{4}\cos^2 w\right)^{\frac{h-3}{2}} \cos^4 w\, dw$$

Taking the first integral and expanding

$$I_1 = \frac{2}{\pi}\int_0^{\frac{\pi}{2}}\left\{1 - \frac{h-1}{2}\left(\frac{3}{4}\right)\cos^2 w + \frac{(h-1)(h-3)}{2\cdot 4}\left(\frac{3}{4}\right)^2 \cos^4 w - \dots\dots\right\} dw$$

Now
$$\frac{2}{\pi}\int_0^{\frac{\pi}{2}} \cos^i w\, dw = \frac{1\cdot 3 \dots\dots (i-1)}{2\cdot 4\dots\dots i}$$

when i is even. Therefore

$$I_1 = 1 - \frac{h-1}{2}\,\frac{1}{2}\left(\frac{3}{4}\right) + \frac{(h-1)(h-3)}{2\cdot 4}\,\frac{1\cdot 3}{2\cdot 4}\left(\frac{3}{4}\right)^2$$
$$- \frac{(h-1)(h-3)(h-5)}{2\cdot 4\cdot 6}\,\frac{1\cdot 3\cdot 5}{2\cdot 4\cdot 6}\left(\frac{3}{4}\right)^3 + \dots\dots$$

Similarly

$$I_2=\frac{1}{2}-\frac{h-1}{2}\,\frac{1\cdot3}{2\cdot4}\left(\frac{3}{4}\right)+\frac{(h-1)(h-3)}{2\cdot4}\,\frac{1\cdot3\cdot5}{2\cdot4\cdot6}\left(\frac{3}{4}\right)^2-\ldots\ldots$$

$$I_3=\frac{1\cdot3}{2\cdot4}-\frac{h-3}{2}\,\frac{1\cdot3\cdot5}{2\cdot4\cdot6}\left(\frac{3}{4}\right)+\frac{(h-3)(h-5)}{2\cdot4}\,\frac{1\cdot3\cdot5\cdot7}{2\cdot4\cdot6\cdot8}\left(\frac{3}{4}\right)^2-\ldots\ldots$$

These series stop at a finite number of terms when h is odd, and continue infinitely when h is even. If we put

(10) $$\alpha_1=2I_1-\frac{11}{4}I_2+\frac{h-1}{16}I_3 \qquad \alpha_2=I_2 \qquad \beta_1=I_1$$

we get finally

(11) $$\begin{cases} \left[\dfrac{da}{dt}\right]=-(\alpha_1-\alpha_2k_1)c\rho_0n^{h-1}a^he^{h+1} & \left[\dfrac{d\varpi}{dt}\right]=0 \\ \left[\dfrac{de}{dt}\right]=-\qquad\beta_1c\rho_0n^{h-1}a^{h-1}e^h & \left[\dfrac{d\varepsilon}{dt}\right]=0 \end{cases}$$

The numerical values of α_1, α_2, and β_1, for h=1, 2, 3 are computed as follows :—

h	α_1	α_2	β_1
1	0·62	0·50	1·00
2	0·69	0·32	0·77
3	0.70	0.22	0·62

8. If we put $h=2$ in (11), as it seems most natural, $\left[\frac{de}{dt}\right]$ is proportional to e^2, and $\left[\frac{da}{dt}\right]$ to e^3. Hence the effect must be very small for the bodies whose orbits have small eccentricities. Or in other words if the effect be appreciable in the motion of asteroids which have small eccentricities in general, it must be remarkably great for the motion of the comets. This appears to be almost fatal to our assumption, supposing the resistance still to exist, because the effect of the resistance on the cometary motion, whatever may be its law, is known to be very small if it exists at all.

But there is an answer to this objection. The comet, as far as we know, is not a single body rigidly bound like a planet. It seems to be a loose aggregation of small bodies[1], perhaps composed of the same kind of material as meteors, with rare gaseous envelope. Most of the particles which are supposed to effect the resistance will pass freely through this meteoric swarm. Rarely it may occur that some resisting particle strikes a cometary particle. Then the latter will be projected outside of the swarm and take on an individual motion. Gradual degeneration will follow this action if repeated frequently but there is no effect on the motion of the main body. Yet one more thing is conceivable, viz. an indirect effect coming in through the gaseous envelope. This, however, would be very small owing to the tenuity of the latter.

9. The assumption that the particles move in circular orbits in a definite plane is nothing but an imaginary convention to make the problem simpler. Practically this may be said to be that at a point in or near the plane, the resultant composite velocity of the particles passing through that point is equal to the velocity of the circular motion.

As for the density of the particles it is natural to assume a certain amount of decrease as the distance from the sun increases, that is, to assume a negative value for k_1. This is not all, for there is some reason to believe that the particles are not numerous near the path of the major planets. They cannot move in orbits of small eccentricity in the neighborhood of the planets. If they did, they would be disturbed a great deal by the action of the latter, or they might even be swept up, except those moving about the triangular equilibrium points.

Chapter II.

Motion of the Asteroids whose Mean Motions are nearly Commensurable to the Mean Motion of Jupiter.

1) Young compares it with "pin-heads several hundred feet apart."

10. In this chapter I shall develop a simple theory of planetary librations according to the method of Prof. Brown.[1]

Assuming that *the asteroid moves in the plane of the orbit of Jupiter and that Jupiter moves in a circular orbit*, let n, a, e, ε and ϖ be the elements of the asteroid as before, n', a', and ε', the circular elements of Jupiter and R the perturbative function, as usual. Take the mass of the sun as unity and the unit of length, so as to make

(1) $$n^2a^3=1,$$

neglecting the mass of the asteroid, and also

$$n'^2a'^3=1+m'$$

m' being the mass of Jupiter which is about 1/1047.

11. Let n_0 be the mean motion commensurable to n' and let

(2) $$n=n_0(1+x) \qquad \text{or} \qquad a=a_0(1+x)^{-\frac{2}{3}}$$

The quantity x is supposed to be small, of the same order as e^2 at most. Let also

(3) $$\frac{n'}{n_0}=\frac{s'}{s}$$

where s and s' are positive integers prime to each other.

12. As a consequence of the assumptions mentioned above, any argument of long period-terms, or critical argument as usually called, takes the form

$$is'l-isl'+jw$$

where i and j are any integers positive or negative, and l and l' are mean longitudes. By the properties of R we have

$$is'-is+j=0 \qquad \text{or} \qquad j=i(s-s')$$

Hence the critical argument is

1) Month. Notices of the R. A. S., lxxii, p. 609.

(4) $$i\{s'l-sl'+(s-s')\varpi\}\equiv i\theta$$

The corresponding term will be factored by $e^{i(s-s')}$, so that for the principal term we have to put $i=1$. Now

$$l=\int ndt+\varepsilon=n_0t+n_0\int xdt+\varepsilon$$

by (2) and $$l'=n't+\varepsilon'$$

whence

(5) $$\theta=s'n_0\int xdt+s'\varepsilon-s\varepsilon'+(s-s')\varpi$$

13. According to the theory of perturbations we have

(6) $$\begin{cases}\dfrac{da}{dt}=(a,\varepsilon)\dfrac{\partial R}{\partial\varepsilon} & \dfrac{d\varepsilon}{dt}=(\varepsilon,a)\dfrac{\partial R}{\partial a}+(\varepsilon,e)\dfrac{\partial R}{\partial e}\\[2ex] \dfrac{de}{dt}=(e,\varepsilon)\dfrac{\partial R}{\partial\varepsilon}+(e,\varpi)\dfrac{\partial R}{\partial\varpi} & \dfrac{d\varpi}{dt}=\qquad(\varpi,e)\dfrac{\partial R}{\partial e}\end{cases}$$

in which

(7) $$\begin{cases}(a,\varepsilon)=-(\varepsilon,a)=2na^2\\[1ex] (e,\varepsilon)=-(\varepsilon,e)=-\dfrac{na\sqrt{1-e^2}}{e}(1-\sqrt{1-e^2})\\[2ex] (e,\varpi)=-(\varpi,e)=-\dfrac{na\sqrt{1-e^2}}{e}\end{cases}$$

Neglecting all short period-terms in R, let

$$R=R_0+R_c$$

and

$$R_0=\frac{m'}{a'}\varphi\left(\frac{a}{a'},e^2\right)$$

$$R_c=\frac{m'}{a'}\psi_1\left(\frac{a}{a'},e^2\right)e^{s-s'}\cos\theta+\frac{m'}{a'}\psi_2\left(\frac{a}{a'},e^2\right)e^{2(s-s')}\cos 2\theta+\ldots\ldots$$

The functions φ, ψ_1, ψ_2,...are developable in terms of powers and products of $\frac{a}{a'}$ and e^2. The second and succeeding terms of R_c are not important, being of higher orders with respect to e. Since ε and ϖ are contained in R through θ, we have

$$\frac{\partial R}{\partial \varepsilon}=s'\frac{\partial R}{\partial \theta} \qquad \frac{\partial R}{\partial \varpi}=(s-s')\frac{\partial R}{\partial \theta}$$

The elements ε and ϖ may be eliminated by these relations, viz.

$$(8)\quad \begin{cases} \dfrac{da}{dt}=s'(a,\varepsilon)\dfrac{\partial R}{\partial \theta} \qquad \dfrac{de}{dt}=[s'(e,\varepsilon)+(s-s')(e,\varpi)]\dfrac{\partial R}{\partial \theta} \\ \dfrac{d\theta}{dt}=s'n_0x+s'\dfrac{d\varepsilon}{dt}+(s-s')\dfrac{d\varpi}{dt} \\ \qquad =s'n_0x+s'(\varpi,a)\dfrac{\partial R}{\partial a}+[s'(\varepsilon,e)+(s-s')(\varpi,e)]\dfrac{\partial R}{\partial e} \end{cases}$$

14. The first two equations give

$$\frac{de}{dt}=\frac{s'(e,\varepsilon)+(s-s')(e,\varpi)}{s'(a,\varepsilon)}\frac{da}{dt}$$

or

$$2e\frac{de}{dt}=-\frac{\sqrt{1-e^2}}{s'a}(s-s'\sqrt{1-e^2})\frac{da}{dt}=\frac{2}{3}\frac{n_0}{n}\sqrt{1-e^2}\;\frac{s-s'\sqrt{1-e^2}}{s'}\frac{dx}{dt}$$

whence

$$\frac{2s'}{\sqrt{1-e^2}(s-s'\sqrt{1-e^2})}e\frac{de}{dt}=\frac{2}{3}\frac{n_0}{n}\frac{dx}{dt}$$

Expanding the coefficient of $e\dfrac{de}{dt}$ in terms of e^2, and n, in terms of x, and integrating we get

$$(9)\qquad \frac{s'}{s-s'}e^2=\frac{2}{3}x+\text{Const.}$$

in which x^2 and e^4, and higher powers are neglected.

15. From the three equations of (8), we get

$$\frac{dR}{dt}=\frac{\partial R}{\partial a}\frac{da}{dt}+\frac{\partial R}{\partial e}\frac{de}{dt}+\frac{\partial R}{\partial \theta}\frac{d\theta}{dt}=s'n_0x\frac{\partial R}{\partial \theta}=\frac{n_0}{(a,\varepsilon)}x\frac{da}{dt}$$

But

$$\frac{da}{dt}=-\frac{2}{3}a\frac{n_0}{n}\frac{dx}{dt}$$

whence

$$\frac{dR}{dt} = -\frac{1}{3a}\frac{n_0^2}{n^2}x\frac{dx}{dt}$$

Developing n and a in terms of powers of x and integrating

$$R = -\frac{x^2}{6a_0} + \text{Const.}$$

in which x^3 and higher powers are neglected. Or

$$x^2 + 6a_0R_0 + 6a_0R_c = \text{Const.}$$

R_0 is a function of a and e^2. and e^2 is a function of x containing an arbitrary constant. Hence developing

$$R_0 = (R_0)_0 + \left(\frac{\partial R_0}{\partial a}\frac{da}{dx} + \frac{\partial R_0}{\partial e^2}\frac{de^2}{dx}\right)_0 x + \ldots\ldots$$

where the suffix 0 outside of the parentheses means the value for $x=0$. The terms with x^2 and higher powers may be neglected, since they are multiplied by m'. Now

$$\left(\frac{da}{dx}\right)_0 = -\frac{2}{3}a_0 \qquad \left(\frac{de^2}{dx}\right)_0 = \frac{2}{3}\frac{s-s'}{s'}$$

whence

$$R_0 = (R_0)_0 - \frac{2}{3}a_0\left(\frac{\partial R_0}{\partial a}\right)_0 x + \frac{2}{3}\frac{s-s'}{s'}\left(\frac{\partial R_0}{\partial e^2}\right)_0 x$$

Substituting this expression in the integral we get

$$x^2 - 4a_0^2\left(\frac{\partial R_0}{\partial a}\right)_0 x + 4\frac{s-s'}{s'}a_0\left(\frac{\partial R_0}{\partial e^2}\right)_0 x + 6a_0R_c = \text{Const.} - 6a_0(R_0)_0$$

or putting

(10) $$x_0 = 2a_0^2\left(\frac{\partial R_0}{\partial a}\right)_0 - 2\frac{s-s'}{s'}a_0\left(\frac{\partial R_0}{\partial e^2}\right)_0$$

the integral becomes

(11) $$(x-x_0)^2 + 6a_0R_c = C$$

where C is an arbitrary constant.

16. By the last equation of (8)

$$\frac{d\theta}{dt}=s'n_0x-2s'na^2\frac{\partial R}{\partial a}+\frac{na\sqrt{1-e^2}}{e}(s-s'\sqrt{1-e^2})\frac{\partial R}{\partial e}$$

Neglecting $m'x$, $m'e$ and higher powers we may write

$$\frac{d\theta}{dt}=s'n_0x-2s'n_0a_0^2\left(\frac{\partial R_0}{\partial a}\right)_0+2(s-s')n_0a_0\left(\frac{\partial R_0}{\partial e^2}\right)_0+(s-s')\frac{n_0a_0}{e}\frac{\partial R_c}{\partial e}$$

or introducing x_0 by (10)

(12) $$\frac{d\theta}{dt}=s'n_0(x-x_0)+(s-s')\frac{n_0a_0}{e}\frac{\partial R_c}{\partial e}$$

The equation (9) may be written also

(13) $$e^2=E+\frac{2}{3}\frac{s-s'}{s'}(x-x_0)$$

where E is an arbitrary constant.

17. The constant x_0 is a quantity depending on $\frac{a_0}{a'}$ and E, multiplied by m'. Now E is supposed to be small, of the same order as e^2. Hence, neglecting $m'E$ in x_0, the latter becomes a constant depending only on $\frac{a_0}{a'}$.

Since x is always diminished by x_0, if we change the origin by that amount, we may write x for $x-x_0$ and x_0 disappears in the equations (11), (12) and (13). Thus

(14) $$\begin{cases} x^2=C-6a_0R_c \\ e^2=E+\frac{2}{3}\frac{s-s'}{s'}x \\ \frac{1}{n_0}\frac{d\theta}{dt}=s'x+(s-s')\frac{a_0}{e}\frac{\partial R_c}{\partial e} \end{cases}$$

18. The expression of x_0 in terms of the powers of $\frac{a_0}{a'}$ may be obtained easily by the usual process. The result is

$$\frac{x_0}{m'}=\alpha_0^2\frac{db^{(0)}}{d\alpha_0}-\frac{\nu\alpha_0^2}{2}\left(\frac{db^{(0)}}{d\alpha_0}+\frac{\alpha_0}{2}\frac{d^2b^{(0)}}{d\alpha_0^2}\right)$$

$$=(4-3\nu)\left(\frac{1}{2}\right)^2\alpha_0^3+2(4-5\nu)\left(\frac{1\cdot3}{2\cdot4}\right)^2\alpha_0^5+3(4-7\nu)\left(\frac{1\cdot3\cdot5}{2\cdot4\cdot6}\right)\alpha_0^7+\ldots\ldots$$

where $\alpha_0=\frac{a_0}{a'}$ and $\nu=\frac{s-s'}{s'}$

The numerical values are

s/s'	$x_0\times10^2$
2/1	-0.005
3/2	$+0.027$
3/1	-0.012

which are too small to be considered in our problem.

The equations (14) represent the motion in the librating region with remarkable simplicity. This will be shown geometrically.

19. *General Case of the First Order*, $s-s'=1$. The principal term of R_c, becomes in this case

$$a_0R_c=-p_1^{(s)}\,e\cos\theta$$

in which $$p_1^{(s)}=\frac{m'\alpha_0}{2}\left(2sb^{(s)}+\alpha_0\frac{db^{(s)}}{d\alpha_0}\right)$$

The three equations of (14) become

(15) $$x^2=C+6p_1^{(s)}e\cos\theta$$

(16) $$e^2=E+\frac{2}{3s'}x$$

(17) $$\frac{1}{n_0}\frac{d\theta}{dt}=s'x-\frac{p_1^{(s)}\cos\theta}{e}$$

Suppose the quantities x and e are two rectangular coordinates. The equation (16), then, represents a *parabola* whose axis is the axis of x. Let this parabola be designated by A. The equation (15), also, if we put a definite value for θ, represents a parabola whose axis is the axis of e. For the limiting values of $\cos\theta$, ± 1, we have two *limiting parabolas* whose equations are

$$x^2 = C \pm 6p_1 e \quad (18)$$ [1]

Let these limiting parabolas be designated by B_0 and B_1, respectively. B_0 and B_1 meet at the same points $(\pm\sqrt{C}, 0)$ with the axis of x. Two straight lines passing through these points and parallel to the axis of e represent the equation (15) for $\theta = \frac{\pi}{2}$ and $\frac{3\pi}{2}$.

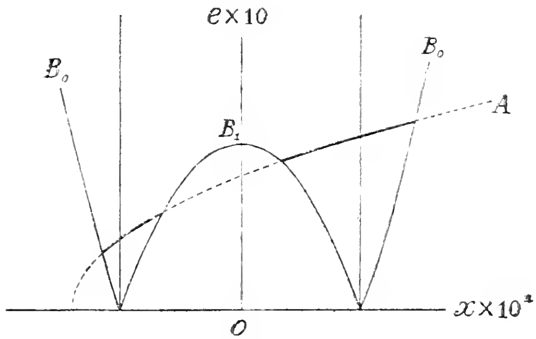

Fig. 1

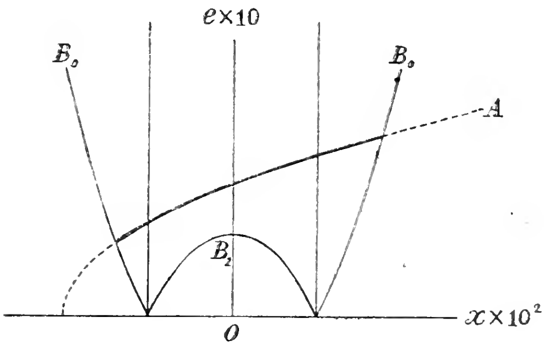

Fig. 2

1) Writing p_1 for $p_1^{(s)}$ and $p_{s-s'}$ for $p_{s-s'}^{(s)}$ generally, for the sake of brevity.

The parts, or the part as the case may be, of A inside of B_0 and outside of B_1 are the real paths of the point (x, e). In Fig. 1 two parts, one on the negative side of x and the other on the positive side, are separate. The argument θ may take any value in each part.

But when E becomes larger or C becomes smaller, these two parts approach each other until A touches B_1, in which case they are connected. After this A does not intersect B_1, as in Fig. 2, and the argument θ never takes the value π or $-\pi$. It increases from 0 to the maximum value θ_0 and decreases, passing through 0 again, until it reaches the minimum value $-\theta_0$. Then it increases again and so forth. This is a kind of *libration.*[1)]

20. We can distinguish *six types of the motion*, namely:—

1	*Revolution*	on the negative side (of x),
2	,,	extending on both sides,
3	,,	on the positive side,
1	*Libration*	on the negative side,
2	,,	extending on both sides,
3	,,	on the positive side.

21. In order to determine the limits of C and E for each type, we need to consider some singular cases.

1. Condition of the contact of A and B_0 or B_1. Differentiating the equations (16) and (18) with respect to x and equating $\frac{de}{dx}$ we get the condition

$$ex = \pm \frac{p_1}{s'} \tag{19}$$

The same equation results from (17) putting $\cos\theta = \pm 1$ and $\frac{d\theta}{dt} = 0$ in it. To find the relation between C and E, eliminating x and e from the equations (16), (18) and (19), we get

1) Libration is defined as a case of motion in which the value of the argument is limited. This definition is slightly different from that of Prof. Brown (Month. Not. lxxii p. 618). The term *revolution* is used after M. Callandreau (Annal. de l'Obs. de Paris, Mémoires Tome XXII).

(20) $$12s'^2p_1^2E^3-\frac{s'^2}{3}E^2C^2-6p_1^2EC+\frac{9}{s'^2}p_1^4+\frac{4C^3}{27}=0$$

2. Condition of the contact of the second order of A and B_0 or B_1. The equation (20) gives three real roots for E when C is great. Let them be denoted by E_1, E_2 and E_3 in which E_1 is the smallest and E_3, the greatest, algebraically. E_1 is always real. E_2 and E_3 are positive when real and may be equal. The condition of the equal roots is

(21) $$C=3\left(\frac{3}{s'}p_1^2\right)^{\frac{2}{3}}\equiv C_1 \qquad E=\frac{1}{s'}\left(\frac{3}{s'}p_1^2\right)^{\frac{1}{3}}$$

which is the condition of the contact of the second order, geometrically. When C is smaller than C_1 two roots E_2 and E_3 become imaginary.

3. Case in which A and B_0 or B_1 intersect on the axis of x. Putting $e=0$ in the equations (16) and (18), and equating x, we get

(22) $$E=\pm\frac{2}{3s'}\sqrt{C}\equiv\pm F_1$$

4. Case in which A and B_0 or B_1 intersect on the axis of e.

(23) $$E=\left(\frac{C}{6p_1}\right)^2\equiv F_0$$

5. Singular case in which A and B_0 or B_1 intersect on the axis of x and touch each other, simultaneously. Putting the condition (22) in the equation (20) we get easily

(24) $$C=\frac{9}{4}\left(\frac{6}{s'}p_1^2\right)^{\frac{2}{3}}\equiv C_2 \qquad E=\frac{1}{s'}\left(\frac{6}{s'}p_1^2\right)^{\frac{1}{3}}$$

22. The following table is constructed to show how different types of the motion are related to the magnitudes of the arbitrary constants C and E.

Type	Limiting Curves	Limits of x	Limits of C: $-\infty$ 0	0 C_1	C_1 C_2	C_2 $+\infty$
			Limits of E			
Revol. 1	B_0 B_1	$-$ $-$	impossible	impossible	impossible	$+F_1$ E_3
,, 2	B_1 B_0	$-$ $+$	,,	F_0 $+F_1$	F_0 $+F_1$	F_0 E_3
,, 3	,,	$+$ $+$	,,	$-F_1$ F_0	$-F_1$ F_0	$-F_1$ F_0
Libr. 1	B_1 B_1	$-$ $-$	,,	impossible	E_2 E_3	E_2 $+F_1$
,, 2	B_0 B_0	$-$ $+$	F_0 $+\infty$	$+F_1$ $+\infty$	$+F_1$ $+\infty$	E_1 $+\infty$
,, 3	,,	$+$ $+$	E_1 F_0	E_1 $-F_1$	E_1 $-F_1$	E_1 $-F_1$

This will be very easily understood by the geometrical representation of the motion.

23. The sign of $\frac{d\theta}{dt}$ for different types will be discussed now. The point at which $\frac{d\theta}{dt}$ changes its sign is determined by the equation

$$s'ex-p_1\cos\theta=0$$

combined with the equations (15) and (16). Eliminating e and $\cos\theta$, we get easily

$$x^2+2s'Ex+\frac{C}{3}=0$$

or

$$x=-s'E\pm\sqrt{s'^2E^2-\frac{C}{3}} \tag{25}$$

The corresponding values of e are given by

$$e^2=\frac{E}{3}\pm\frac{2}{3s'}\sqrt{s'^2E^2-\frac{C}{3}} \tag{26}$$

Since e must be positive, two points, real or imaginary, will be determined by these equations. Now evidently only an odd number of zero-points in the librations and an even number in the revolutions, are possible. Accordingly, since there cannot be more than two, *one and only one zero-point of* $\frac{d\theta}{dt}$ *will exist in the case of the librations.*

The conditions necessary in order that two zero-points may be real, are

$$E>0, \qquad \frac{C}{3s'^2}<E^2<\frac{4}{9s'^2}C$$

by (25) and (26). Hence C must be positive and therefore both values of x must be negative. Accordingly there exists no zero-point in the revolution of Type 3. For the revolution of Type 1, we have the condition

$$E>+F_1\left(=\frac{2}{3s'}\sqrt{C}\right)$$

which contradicts with one of the above conditions. Hence no zero-point exists in the revolution of Type 1, and therefore the *only type of the motion, in which two zero-points of* $\frac{d\theta}{dt}$ *may exist, is the revolution of Type 2.*

Evidently the sign of $\frac{d\theta}{dt}$ is *negative in the revolution of Type 1* and *positive in the revolution of Type 3.*

24. *General Case of the Second Order*, $s-s'=2$. We have in this case

$$a_0R_c=p_2e^2\cos\theta$$

where $$p_2=p_2^{(s)}=\frac{m'a_0}{8}\left[(4s^2-5s)b^{(s)}+(4s-2)a_0\frac{db^{(s)}}{da_0}+a_0^2\frac{d^2b^{(s)}}{da_0^2}\right]$$

Putting $$e^2=y$$

we have by (14)

$$
(27)\quad \begin{cases} x^2 = C - 6p_2 y \cos\theta \\[2ex] y = E + \dfrac{4}{3s'}x \\[2ex] \dfrac{1}{n_0}\dfrac{d\theta}{dt} = s'x + 4p_2 \cos\theta \end{cases}
$$

The curve A becomes *a straight line* in this case. The condition of the contact of A and B_0 or B_1 is

$$x = \mp\frac{4p_2}{s'}$$

and the relation between C and E becomes simply

$$C \mp 6p_2 E + \frac{16}{s'^2}p_2^2 = 0$$

There is *no contact of the second order*. The condition that A and B_0 or B_1 may touch each other on the axis of x is

$$C = \frac{16}{s'^2}p_2^2 \equiv C_2 \qquad\qquad E = \pm\frac{16}{3s'^2}p_2$$

25. The limits of C and E for different types of the motion are as follows:—

Type		Limiting Curves		Limits of x		Limits of C: $-\infty$	0	0	C_2	C_2	$+\infty$
						Limits of E					
Revol.	1	B_1	B_0	$-$	$-$	impossible		impossible		$+F_1$	E_3
,,	2	B_0	B_1	$-$	$+$	,,		F_0	$+F_1$	F_0	E_3
,,	3	,,		$+$	$+$	,,		$-F_1$	F_0	$-F_1$	F_0
Libr.	1	——		——		,,		impossible		impossible	
,,	2	B_1	B_1	$-$	$+$	F_0	$+\infty$	$+F_1$	$+\infty$	F_3	$+\infty$
,,	3	,,		$+$	$+$	E_1	F_0	E_1	$-F_1$	impossible	

The limits are simpler than in the case of the first order. *The libration of Type 1 is impossible throughout* and the librations are restricted to that about π.

26. The point at which $\frac{d\theta}{dt}$ changes its sign is given by the equations

$$x = -\frac{3s'}{4}E \pm \sqrt{\left(\frac{3s'}{4}E\right)^2 - C}$$

$$e^2 = \pm\sqrt{E^2 - \left(\frac{4}{3s'}\right)^2 C}$$

Only one point is possible on the positive side of e. Hence, there is *no zero-point in the revolutions* and one and only one point, in the librations.

27. *General Case of the Third or Higher Orders, $s-s' > 2$.* We have in this case

$$a_0 R_c = (-1)^{s-s'} p_{s-s'} e^{s-s'} \cos\theta$$

where $p_{s-s'}$ is a function of a_0 with a positive value. Putting

$$y = e^{s-s'}$$

we have by (14)

$$(28)\quad \begin{cases} x^2 = C - (-1)^{s-s'} 6 p_{s-s'} y \cos\theta \\ y = \left(E + \frac{2}{3}\frac{s-s'}{s'}x\right)^{\frac{s-s'}{2}} \\ \frac{1}{n_0}\frac{d\theta}{dt} = s'x + (-1)^{s-s'}(s-s')^2 p_{s-s'} y^{\frac{s-s'-2}{s-s'}} \cos\theta \end{cases}$$

The curve A becomes a parabola of various degrees with the axis parallel to the axis of y, and the vertex on the axis of x. *The constant C_2 becomes* 0. The table of the limits is much simplified by this change. The libration of Type 1 is always impossible and the librations are restricted to those about 0 when $s-s'$ is odd and those about π when $s-s'$ is even. The libration of Type 3 becomes impossible when $C>0$, in the cases of the fourth and higher orders.

Chapter III.

Theoretical Effect of the Resistance on the Motion of the Asteroids in the Librating Region.

28. Putting $h=2$, the equations (11) of Chapter I become

$$(1)\qquad \begin{cases} \frac{1}{a}\left[\frac{da}{dt}\right]=-\alpha e^3 & \left[\frac{de}{dt}\right]=-\beta e^2 \\ \left[\frac{d\varepsilon}{dt}\right]=0 & \left[\frac{d\varpi}{dt}\right]=0 \end{cases}$$

where

$$\alpha=(\alpha_1-\alpha_2k_1)c\rho_0na \qquad \beta=\beta_1c\rho_0na$$

Now, by the equations (1) and (2) of Chapter II, we have

$$\frac{1}{a}\left[\frac{da}{dt}\right]=-\frac{2}{3}\frac{1}{n}\left[\frac{dn}{dt}\right]$$

and

$$\left[\frac{dx}{dt}\right]=\frac{1}{n_0}\left[\frac{dn}{dt}\right]$$

Since n and n_0 are supposed to be very nearly equal

$$\frac{1}{a}\left[\frac{da}{dt}\right]=-\frac{2}{3}\left[\frac{dx}{dt}\right]$$

Hence

$$(2)\qquad \left[\frac{dx}{dt}\right]=\frac{3\alpha}{2}e^3$$

We have, also, by (5) of Chapter II

$$\frac{d\theta}{dt}=s'n_0x+s'\frac{d\varepsilon}{dt}-s\frac{d\varepsilon'}{dt}+(s-s')\frac{d\varpi}{dt}$$

and, since x is not affected directly by the resistance, we have by (1)

(3) $$\left[\frac{d\theta}{dt}\right]=0$$

The differential coefficients of x, e and θ are composed of two parts, viz; the differential coefficients due to the perturbation and those due to the resistance which are denoted by $\left[\frac{dx}{dt}\right]$, etc. Hence, if we denote the former by $\left(\frac{dx}{dt}\right)$, etc.

$$\frac{dx}{dt}=\left(\frac{dx}{dt}\right)+\left[\frac{dx}{dt}\right] \qquad \text{etc.}$$

Or, substituting the expressions of $\left[\frac{dx}{dt}\right]$, etc. we get

(4) $$\begin{cases} \frac{dx}{dt}=\left(\frac{dx}{dt}\right)+\frac{3\alpha}{2}e^3 \\ \frac{de}{dt}=\left(\frac{de}{dt}\right)-\beta e^2 \\ \frac{d\theta}{dt}=\left(\frac{d\theta}{dt}\right) \end{cases}$$

Since x is supposed to be very small, we can write

$$\alpha=(a_1-a_2k_1)c\rho_0n_0a_0 \qquad \beta=\beta_1c\rho_0n_0a_0$$

As it will be seen in Chapter I, β is essentially *positive* and α may be positive or negative according as k_1 is less or greater than $\frac{a_1}{a_2}$ which is positive. We shall consider the case in which k_1 is negative and consequently α is *positive.*

29. *General Case of the First Order.* Differentiating the equations (15) and (16) of Chapter II completely with respect to t, we get

$$2x\frac{dx}{dt}=\frac{dC}{dt}+6p_1\cos\theta\frac{de}{dt}-6p_1e\sin\theta\frac{d\theta}{dt}$$

$$2e\frac{de}{dt}=\frac{dE}{dt}+\frac{2}{3s'}\frac{dx}{dt}$$

Differentiating the same equations with respect to t, supposing the effect of the resistance is null,

$$2x\left(\frac{dx}{dt}\right)=6p_1\cos\theta\left(\frac{de}{dt}\right)-6p_1e\sin\theta\left(\frac{d\theta}{dt}\right)$$

$$2e\left(\frac{de}{dt}\right)=\frac{2}{3s'}\left(\frac{dx}{dt}\right)$$

Substituting the expressions of $\frac{dx}{dt}$, etc. by (4) and taking the differences between the two sets of equations, we obtain

$$(5)\qquad\left\{\begin{aligned}\frac{dC}{dt}&=3\alpha e^3x+6\beta p_1e^2\cos\theta\\ \frac{dE}{dt}&=-\left(\frac{\alpha}{s'}+2\beta\right)e^3\end{aligned}\right.$$

which are the equations for the variations of the arbitrary constants. These equations show that E *always decreases* while C increases or decreases depending on x and $\cos\theta$. Now the signs of x and $\cos\theta$ are both negative in the libration of Type 1 and both positive in the libration of Type 3. Hence *C always decreases in the libration of Type 1 and increases in that of Type 3.*

30. There are three more cases in which the sign of $\frac{dC}{dt}$ may be determined easily, namely; the libration of Type 2 when C is negative, the revolution of Type 1, and the revolution of Type 3 when E is negative. In the first of these cases the motion being a kind of libration about 0, θ may increase from 0 to an angle θ_0 and decrease to $-\theta_0$. The limit θ_0 is not greater than $\frac{\pi}{2}$ so long as C is negative and consequently the second term of $\frac{dC}{dt}$ is always positive. Now, we have by the equation (17) of Chapt. II

$$n_0\int e^3x\,dt=\int\frac{e^4x\,d\theta}{s'ex-p_1\cos\theta}$$

and

$$n_0\int_0^T e^3 x dt = \left(\int_0^{\theta_0} + \int_{\theta_0}^0 + \int_0^{-\theta_0} + \int_{-\theta_0}^0\right) \frac{e^4 x d\theta}{s'ex - p_1 \cos\theta}$$

where T is the period of the libration. Writing x' and e' for x and e in the part in which $\frac{d\theta}{dt}$ is negative, we have

$$n_0\int_0^T e^3 x dt = 2\int_0^{\theta_0}\left(\frac{e^4 x}{s'ex - p_1 \cos\theta} - \frac{e'^4 x'}{s'e'x' - p_1 \cos\theta}\right)d\theta \equiv 2\int_0^{\theta_0} X d\theta$$

X is evidently positive when both x and x' are positive. Also,

$$X = \frac{s'(e^3 - e'^3)ee'xx' - p_1 \cos\theta(e^4 x - e'^4 x')}{(s'ex - p_1 \cos\theta)(s'e'x' - p_1\cos\theta)}$$

The denominator is negative, $e^3 - e'^3$ is positive, $\cos\theta$ and $e^4x - e'^4x'$ are also positive and therefore X is positive when x' is negative. Hence the integral is always positive and accordingly $\int_0^T \frac{dC}{dt}dt$ is positive. Thus C *increases algebraically when negative.*

31. The sign of the first term of $\frac{dC}{dt}$ is exclusively negative in the revolution of Type 1 and positive in that of Type 3. For the second term we have

$$n_0\int_0^T e^2 \cos\theta dt = \pm\int_0^{2\pi} \frac{e^3\cos\theta}{s'ex - p_1\cos\theta} d\theta$$

in which T is the period of the revolution and the double sign corresponds to the positive and negative values of $\frac{d\theta}{dt}$, i. e., to Type 3 and Type 1 respectively. We may write

$$\int_0^{2\pi} \frac{e^3 \cos\theta d\theta}{s'ex - p_1 \cos\theta} = 2\int_0^{\frac{\pi}{2}}\left(\frac{e^3}{s'ex - p_1\cos\theta} - \frac{e'^3}{s'e'x' + p_1 \cos\theta}\right)\cos\theta d\theta$$

$$\equiv 2\int_0^{\frac{\pi}{2}} Y \cos\theta d\theta$$

in which x and e are written x' and e' where $\cos\theta$ is negative, and

$$Y=\frac{s'ee'(e^2x'-e'^2x)+p_1\cos\theta(e^3+e'^3)}{(s'ex-p_1\cos\theta)(s'e'x'+p_1\cos\theta)}$$

$$=\frac{s'ee'(x'-x)E+p_1\cos\theta(e^3+e'^3)}{(s'ex-p_1\cos\theta)(s'e'x'+p_1\cos\theta)}$$

The denominator is positive in both cases. The quantity $x'-x$ is positive in Type 1 and negative in Type 3. E is positive in Type 1 and may be positive or negative in Type 3. Hence, for the *revolution of Type 1* taking the negative sign for the double sign, the integral becomes negative and consequently C *decreases.* For the *revolution of Type 3, if E is negative or less than a certain positive value,* Y is positive and, taking the positive sign for the double sign, the integral becomes positive and therefore C *increases.*

32. Since e cannot be zero permanently in the general case of the first order, E decreases without limit so that it will become negative after a certain epoch. Now, when E is negative, the type of the motion is limited to two kinds, namely; the revolution and the libration, of Type 3. Hence we see that *the revolutions and the librations, of Type 1 and Type 2, are not permanent forms of the motion ; but will change ultimately either to the revolution or the libration, of Type 3.* Also, since C increases algebraically when negative, *it will become positive if it was initially negative,* and *it cannot become negative if it was initially positive ;* but will decrease to some limiting value which is positive and then will increase without limit.

33. It becomes necessary before proceeding further to find the limiting values of x and e in the libration or the revolution of Type 2, in which $\frac{d\theta}{dt}=0$ for $\theta=\pi$, supposing the effect of the resistance is null. Let x_1 and e_1 be the values of x and e at the zero-point of $\frac{d\theta}{dt}$, then

$$x_1^2=C-6p_1e_1 \qquad e_1^2=E+\frac{2}{3s'}x_1$$

and $$s'e_1x_1+p_1=0$$

Evidently, the limiting values of x and e will be attained when $\theta=0$ or π. Denoting these values by $\bar{x}$ and $\bar{e}$, we have

$$\bar{x}^2=C\pm 6p_1\bar{e} \qquad \bar{e}^2=E+\frac{2}{3s'}\bar{x}$$

in which the upper sign corresdonds to $\theta=0$ and the lower to $\theta=\pi$. Eliminating C, E and $\bar{e}$, we have

$$(\bar{x}^2-x_1^2)^2-12p_1e_1(\bar{x}^2-x_1^2)-\frac{24}{s'}p_1^2(\bar{x}-x_1)=0$$

which becomes by the use of the relation $s'e_1x_1+p_1=0$

$$(\bar{x}-x_1)^2\{(\bar{x}+x_1)^2-12p_1e_1\}=0$$

Hence, $\qquad \bar{x}=x_1 \qquad$ or $\qquad -x_1(\pm)\sqrt{12p_1e_1}$

that is, $\qquad \bar{x}=-\dfrac{p_1}{s'e_1} \qquad$ or $\qquad \dfrac{p_1}{s'e_1}(\pm)\sqrt{12p_1e_1}$

the double sign enclosed in the parentheses meaning that it is independent of the value of θ. Similarly, we obtain

$$\bar{e}=\mp e_1 \qquad \text{or} \qquad \pm e_1(\pm)\sqrt{\frac{4p_1}{3s'^2e_1}}$$

The limiting values of x and e will thus be tabulated as follows:—

(6)

θ	x	e
0, π	$+\frac{p_1}{s'e_1}-\sqrt{12p_1e_1}$	$\pm\left(e_1-\sqrt{\frac{4p_1}{3s'^2e_1}}\right)$
π	$-\frac{p_1}{s'e_1}$	e_1
0	$+\frac{p_1}{s'e_1}+\sqrt{12p_1e_1}$	$+e_1+\sqrt{\frac{4p_1}{3s'^2e_1}}$

The following conditions become necessary:—

$$0<e_1-\sqrt{\frac{4p_1}{3s'^2e_1}} \qquad \text{for libration}$$

$$0<\sqrt{\frac{4p_1}{3s'^2e_1}}-e_1<e_1 \qquad \text{for revolution}$$

that is,

$$(7)\qquad \begin{cases} 4p_1<3s'^2e_1^3 & \text{for libration} \\ p_1<3s'^2e_1^3<4p_1 & \text{for revolution} \end{cases}$$

It may be remarked that when $3s'^2e_1^3=p_1$, then $C=C_1$, and when $3s'^2e_1^3=4p_1$, than $C=C_2$, according to the equations (21) and (24) of Chapter II.

34. Let $\bar{x}$ and $\bar{e}$ be the values of x and e corresponding to a definite value $\bar{\theta}$ of θ and put

$$H=\bar{e}\,\bar{x}$$

Supposing $\bar{x}$ and $\bar{e}$ are functions of C and E, and differentiating H,

$$\frac{dH}{dt}=\left(\bar{e}\frac{\partial\bar{x}}{\partial C}+\bar{x}\frac{\partial\bar{e}}{\partial C}\right)\frac{dC}{dt}+\left(\bar{e}\frac{\partial\bar{x}}{\partial E}+\bar{x}\frac{\partial\bar{e}}{\partial E}\right)\frac{dE}{dt}$$

Now, $\qquad \bar{x}^2=C+6p_1\bar{e}\cos\bar{\theta} \qquad \bar{e}^2=E+\frac{2}{3s'}\bar{x}$

and therefore,

$$2\bar{x}\frac{\partial\bar{x}}{\partial C}=1+6p_1\cos\bar{\theta}\frac{\partial\bar{e}}{\partial C} \qquad 2\bar{e}\frac{\partial\bar{e}}{\partial C}=\frac{2}{3s'}\frac{\partial\bar{x}}{\partial C}$$

$$2\bar{x}\frac{\partial\bar{x}}{\partial E}=6p_1\cos\bar{\theta}\frac{\partial\bar{e}}{\partial E} \qquad 2\bar{e}\frac{\partial\bar{e}}{\partial E}=1+\frac{2}{3s'}\frac{\partial\bar{x}}{\partial E}$$

whence

$$2(s'\bar{e}\bar{x}-p_1\cos\bar{\theta})\left(\bar{e}\frac{\partial\bar{x}}{\partial C}+\bar{x}\frac{\partial\bar{e}}{\partial C}\right)=s'\bar{e}^2+\frac{\bar{x}}{3}$$

$$2(s'\bar{e}\bar{x}-p_1\cos\bar{\theta})\left(\bar{e}\frac{\partial\bar{x}}{\partial E}+\bar{x}\frac{\partial\bar{e}}{\partial E}\right)=s'(\bar{x}^2+3p_1\bar{e}\cos\bar{\theta})$$

Substituting these expressions and the expressions of $\frac{dC}{dt}$ and $\frac{dE}{dt}$ of (5) in the equation of $\frac{dH}{dt}$,

$$2(s'\bar{e}\bar{x}-p_1\cos\bar{\theta})\frac{dH}{dt}=(3s'\bar{e}^2+\bar{x})(ae^3x+\varepsilon\beta p_1e^2\cos\theta)-(\bar{x}^2+3p_1\bar{e}\cos\bar{\theta})(a+2s'\beta)e^3$$

Or, reducing by the known relations we obtain finally,

$$(8)\quad 2\frac{dH}{dt}=M\bar{e}e+3(a+2s'\beta)\bar{e}e(e^2-\bar{e}^2)+\frac{N}{2}[Me(e^2-\bar{e}^2)+\frac{3}{2}(2a+s'\beta)e(e^2-\bar{e}^2)^2]$$

where

$$M=3a\bar{e}^2-2\beta\bar{x}\qquad\qquad N=\frac{s'(3s'\bar{e}^2+\bar{x})}{s'\bar{e}\bar{x}-p_1\cos\bar{\theta}}$$

35. Putting $\bar{\theta}=\pi$ and changing x, e and H to x_1, e_1 and H_1 respectively, apply the above result to the case of the revolutions of Type 1 and Type 2 very near to the libration of Type 2. In this case, (a) $s'e_1x_1+p_1$ is very small, negative in Type 1 and positive in Type 2, (b) $e^2-e_1^2$, is always negative in Type 1 and positive in Type 2, (c) M is positive by (a), (d) the quantity

$$3s'e_1^2+x_1=\frac{3s'^2e_1^3-p_1}{s'e_1}$$

is positive by (7) and therefore (e) N is negative in Type 1 and positive in Type 2. Since N is very great, we may write

$$\frac{dH_1}{dt}=\frac{N}{4}\left[Me(e^2-e_1^2)+\frac{3}{2}(2a+s'\beta)e(e^2-e_1^2)^2\right]$$

In the revolution of Type 2, M, N and $e^2-e_1^2$, are all positive and therefore $\frac{dH_1}{dt}$ is always positive. In the revolution of Type 1, N and $e^2-e_1^2$, are negative. Writing

$$(9)\qquad \frac{dH_1}{dt}=\frac{3N}{8}e(e^2-e_1^2)\left[(2a+s'\beta)e^2-s'\beta e_1^2-\frac{4}{3}\beta x_1\right]$$

and denoting the smallest value of e by e_0, $\frac{dH_1}{dt}$ is always positive if

$$(2a+s'\beta)e_0^2>s'\beta e_1^2+\frac{4}{3}\beta x_1$$

But
$$x_1=-\frac{p_1}{s'e_1}$$

and
$$e_0=e_1-\sqrt{\frac{4p_1}{3s'^2e_1}}$$

by (6) approximately. Hence in the revolution of Type 1, $\frac{dH_1}{dt}$ is always positive if the condition

$$e_1^3>\frac{4p_1}{3s'^2}\left(1+\frac{s'\beta}{a}\right)^2 \qquad \text{or} \qquad \frac{e_1-e_0}{e_0}<\frac{a}{s'\beta}$$

be fulfilled.

36. Since $s'e_1x_1+p_1$ is zero when the path A touches the curve B_1, and negative or positive according as the revolution belongs to Type 1 or Type 2, the quantity H_1 is equal to $-\frac{p_1}{s'}$ when the path is in contact with B_1 and is less or greater algebraically than $-\frac{p_1}{s'}$ according as the revolution belongs to Type 1 or Type 2. Now it has been proved that H_1 increases in the revolution of Type 2, and in that of Type 1 if e_1 is greater than a certain positive value. Hence the path A recedes from the position of the contact in the revolution of Type 2, and approaches to that position in the revolution of Type 1 if e_1 is greater than a certain positive value.

37. To see how the librations are changed by the resistance, let us find the variation of the minimum or the maximum value u of $\cos\theta$. Let $\bar{x}$ and $\bar{e}$ be the values of x and e corresponding to $\cos\theta=u$. We have then

$$s'\bar{e}\bar{x}-p_1u=0$$

and $x^2-\bar{x}^2=6p_1(e\cos\theta-\bar{e}u)$ $\qquad e^2-\bar{e}^2=\frac{2}{3s'}(x-\bar{x})$

Differentiating these equations and eliminating $\frac{dx}{dt}$ and $\frac{d\bar{e}}{dt}$, we get

$$-p_1\bar{e}\frac{du}{dt}=\frac{x-\bar{x}}{3}\left[\frac{dx}{dt}\right]+(s'e\bar{x}-p_1\cos\theta)\left[\frac{de}{dt}\right]$$

Or substituting the expressions of $\left[\frac{dx}{dt}\right]$ and $\left[\frac{de}{dt}\right]$ by (4) and eliminating x and $\cos\theta$ by the known relations, we get

$$(10)\qquad -4p_1\bar{e}\frac{du}{dt}=s'Me(e^2-\bar{e}^2)+\frac{3}{2}s'(2\alpha+s'\beta)e(e^2-\bar{e}^2)^2$$

where $\qquad M=3\alpha\bar{e}^2-2\beta\bar{x}$

as before.

38. The second term of the right hand member of (10) is positive for all values of e. The sign of the first term depends on the sign of M and $e^2-\bar{e}^2$. Now, if we confine ourselves to the libration of Type 2 in which u is very nearly equal to -1, M is positive since x is negative, and the integral

$$\int_0^T e(e^2-\bar{e}^2)dt$$

may be proved to be positive supposing the quantities C and E are constants within a single revolution. To prove this I shall proceed as follows:—

We have, using the symbol of § 28,

$$\frac{1}{n_0}\left(\frac{de}{dt}\right)=-p_1\sin\theta$$

and therefore

$$n_0\int(e^2-\bar{e}^2)dt=-\int\frac{e^2-\bar{e}^2}{p_1\sin\theta}de$$

Now, since $\bar{\theta}$ is very nearly equal to π,

$$x^2-\bar{x}^2=6p_1e\cos\theta+6p_1\bar{e} \qquad e^2-\bar{e}^2=\frac{2}{3s'}(x-\bar{x})$$

and

$$s'\bar{e}\bar{x}=-p_1$$

Accordingly

$$-\frac{e^2-\bar{e}^2}{p_1\sin\theta}=\mp\frac{4e}{s'\sqrt{12p_1\bar{e}-(x+\bar{x})^2}}$$

in which the negative sign will be taken if $e^2-\bar{e}^2$ and $\sin\theta$ have the same sign, and the positive sign if otherwise. Hence

$$n_0\int_0^T(e^2-\bar{e}^2)dt=\frac{4}{s'}\left(\int_{\bar{e}}^{e_0}-\int_{e_0}^{\bar{e}}+\int_{\bar{e}}^{e'_0}-\int_{e'_0}^{\bar{e}}\right)\frac{ede}{\sqrt{12p_1\bar{e}-(x+\bar{x})^2}}$$

where e_0 and e'_0 are the limiting values of e determined by the equations

$$e_0=\bar{e}+\sqrt{\frac{4p_1}{3s'^2\bar{e}}} \qquad e'_0=\bar{e}-\sqrt{\frac{4p_1}{3s'^2\bar{e}}}$$

Putting

$$f(e)=12p\bar{e}-(x+\bar{x})^2$$

and

$$e=\bar{e}\pm\varepsilon \qquad \varepsilon>0$$

we have

$$n_0\int_0^T(e^2-\bar{e}^2)dt=\frac{8}{s'}\int_0^{\sqrt{\frac{4p_1}{3s'^2\bar{e}}}}\left(\frac{\bar{e}+\varepsilon}{\sqrt{f(\bar{e}+\varepsilon)}}-\frac{\bar{e}-\varepsilon}{\sqrt{f(\bar{e}-\varepsilon)}}\right)d\varepsilon$$

$$=\frac{8}{s'}\int_0^{\sqrt{\frac{4p_1}{3s'^2\bar{e}}}}\frac{(\bar{e}+\varepsilon)^2f(\bar{e}-\varepsilon)-(\bar{e}-\varepsilon)^2f(\bar{e}+\varepsilon)}{\sqrt{f(\bar{e}+\varepsilon)f(\bar{e}-\varepsilon)}\{(\bar{e}+\varepsilon)\sqrt{f(\bar{e}-\varepsilon)}+(\bar{e}-\varepsilon)\sqrt{f(\bar{e}+\varepsilon)}\}}d\varepsilon$$

Now

$$f(e)=\frac{6p_1}{\bar{e}}(e^2+\bar{e}^2)-\frac{9s'^2}{4}(e^2-\bar{e}^2)-\frac{4p_1^2}{s'^2\bar{e}^2}$$

and

$$f(\bar{e}\pm\varepsilon)=\frac{6p_1}{\bar{e}}(2\bar{e}^2\pm 2\bar{e}\varepsilon+\varepsilon^2)-\frac{9s'^2}{4}\varepsilon^2(4\bar{e}^2\pm 4\bar{e}\varepsilon+\varepsilon^2)-\frac{4p_1{}^2}{s'^2\bar{e}^2}$$

whence

$$(\bar{e}+\varepsilon)^2 f(\bar{e}-\varepsilon)-(\bar{e}-\varepsilon)^2 f(\bar{e}+\varepsilon)$$

$$=\frac{s'^2\varepsilon}{\bar{e}}\left(6\bar{e}^3-3\bar{e}\varepsilon^2-\frac{4p_1}{s'^2}\right)\left(\frac{4p_1}{s'^2}-3\bar{e}\varepsilon^2\right)$$

$$=\frac{s'^2\varepsilon}{\bar{e}}\left(6\bar{e}^3-\frac{8p_1}{s'^2}+\frac{4p_1}{s'^2}-3\bar{e}\varepsilon^3\right)\left(\frac{4p_1}{s'^2}-3\bar{e}\varepsilon^2\right)$$

The quantity $\frac{4p_1}{s'^2}-3\bar{e}\varepsilon^2$ is positive within the limits and $3\bar{e}^3-\frac{4p_1}{s'^2}$ is also positive by (7). Hence the integral

$$\int_0^T (e^2-\bar{e}^2)dt$$

is positive. Now

$$\int_0^T e(e^2-\bar{e}^2)dt=\bar{e}\int_0^T (e^2-\bar{e}^2)dt+\int_0^T (e-\bar{e})(e^2-\bar{e}^2)dt$$

Hence

$$\int_0^T e(e^2-\bar{e}^2)dt>0$$

and therefore

$$\int_0^T \frac{du}{dt}dt<0$$

This result shows that the quantity u, when it is very nearly equal to -1, approaches to -1. Hence we may conclude that *the libration of Type 2 in which the maximum value of θ is very nearly equal to π changes to the revolution of Type 1 or Type 2.*

Now it has been proved that the revolution of Type 1, when e_1 is greater than a certain positive value, approaches the position of contact, and that the revolution of Type 2 recedes from that position. Hence it may be seen that the *revolution of Type 1 changes to the libration of Type 2, when e_1 is greater than a certain positive value, and immediately later to the revolution of Type 2.*

39. The equation (8) becomes, when $\bar{e}$ is very small and negligible,

$$\frac{dH}{dt} = -\frac{s'\bar{x}}{4p_1 \cos\bar{\theta}} \left[-2\beta\bar{x} + \frac{3}{2}(2\alpha + s'\beta)e^2 \right] e^3$$

Accordingly, if $\bar{x}$ be negative, the sign of $\frac{dH}{dt}$ is determined by the sign of $\cos\bar{\theta}$. Applying this result to the limiting cases of the revolution and the libration, of Type 1, we see at once that the quantity H at the point very near to the axis of x, increases in the revolution and decreases in the libration. Now, the quantity H is very small and negative in both cases. Accordingly it approaches 0 in the revolution and recedes from 0 in the libration. Hence we conclude that the *revolution of Type 1 in its limiting case changes to the libration of Type 1.*

The same result may be obtained in the limiting cases of the revolution and the libration, of Type 2, when C is positive and less than C_2. Thus we see that the *libration of Type 2 changes to the revolution of Type 2, in this case.*

In the limiting cases of the revolution and the libration, of Type 3, the sign of the quantity in the brackets is not definite. It becomes positive when β is very small compared with α. It this case H increases in the revolution and decreases in the libration, and, since H is very small and positive, it approaches 0 in the libration and recedes in the revolution. Hence, *if β be very small compared with α, the libration of Type 3 changes to the revolution of Type 3.*

40. Let us next consider the extreme case of the libration of Type 1 in which $\frac{d\theta}{dt}$ is very small for $\theta = \pi$. We have by (9)

$$\frac{dH_1}{dt} = \frac{3}{8} Ne(e^2 - e_1^2) \left[(2\alpha + s'\beta)e^2 - s'\beta e_1^2 - \frac{4}{3}\beta x_1 \right]$$

Now $s'e_1x_1 + p_1$ is negative, $3s'e_1^2 + x_1$ is positive by (7), and therefore N is negative. The quantity $e^2 - e_1^2$ is negative for all values of e. The lower limit of e is $\sqrt{\frac{4p_1}{3s'^2e_1}} - e_1$ by (6), and

$$(2\alpha+s'\beta)\left(\sqrt{\frac{4p_1}{3s'^2e_1}}-e_1\right)^2-s'\beta e_1^2-\frac{4}{3}\beta x_1$$
$$=2\alpha\left(\sqrt{\frac{4p_1}{3s'^2e_1}}-e_1\right)^2+2s'\beta\sqrt{\frac{4p_1}{3s'^2e_1}}\left(\sqrt{\frac{4p_1}{3s'^2e_1}}-e_1\right)>0$$

Hence $\frac{dH_1}{dt}>0$, and therefore the *libration of Type 1 changes to the revolution of Type 2.*

41. When the motion changes from the revolution or the libration, of Type 1, to the revolution of Type 2, the limits of x and e increase discontinuously. Referring back to the table (6), it may be seen that the lower limits change from the second row to the third row, and the upper limits from the third row to the fourth row. Hence the lower limits of x and e increase by the amounts

$$\sqrt{12p_1e_1}-\frac{2p_1}{s'e_1} \qquad \text{and} \qquad e_1\mp\left(e_1-\sqrt{\frac{4p_1}{3s'^2e_1}}\right)$$

respectively, and the upper limits by

$$\sqrt{12p_1e_1}+\frac{2p_1}{s'e_1} \qquad \text{and} \qquad \sqrt{\frac{4p_1}{3s'^2e_1}}$$

respectively.

The upper limits become minimum when $3s'^2e_1^3=p_1$, and the minimum values of x and e are

$$3\left(\frac{3p_1^2}{s'}\right)^{\frac{1}{3}} \qquad \text{and} \qquad 3\left(\frac{p_1}{3s'^2}\right)^{\frac{1}{3}} \qquad \text{respectively.}$$

The corresponding lower limits are

$$-\left(\frac{3p_1^2}{s'}\right)^{\frac{1}{3}} \qquad \text{and} \qquad \left(\frac{p_1}{3s'^2}\right)^{\frac{1}{3}}$$

Numerical values of these limits for different cases of the first order are as follows:—

s/s'	$p_1\times10^3$	$-\left(\frac{3p_1^2}{s'}\right)^{\frac{1}{3}}\times10^2$	$3\left(\frac{3p_1^2}{s'}\right)^{\frac{1}{3}}\times10^2$	$\left(\frac{p_1}{3s'^2}\right)^{\frac{1}{3}}\times10$	$3\left(\frac{p_1}{3s'^2}\right)^{\frac{1}{3}}\times10$
2/1	0.716	−1.15	3.46	0.62	1.86
3/2	1.474	−1.48	4.45	0.50	1.49
4/3	2.237	−1.71	5.13	0.45	1.34

The variations of the constants, C and E, become rapid by this increase of the eccentricity.

42. *The revolution of Type 2 will change to the revolution of Type 3* sooner or later, since E decreases and C cannot become negative. It has been proved that C increases when E is less than a certain positive value. Hence C begins to increase without limit just as E decreases.

43. The equation (10) may be integrated by expanding in series when $\bar{\theta}$ is small. Let us first take the integral

$$\int_0^T e(e^2-\bar{e}^2)dt$$

where T is the period of the libration as before. Putting

$$\xi=x-\bar{x} \qquad \eta=e-\bar{e}$$

we have $$e^2-\bar{e}^2=2\bar{e}\eta+\eta^2=\frac{2}{3s'}\xi$$

and $$x^2-\bar{x}^2=2\bar{x}\xi+\xi^2=6p_1(e\cos\theta-\bar{e}\cos\bar{\theta})$$

Eliminating ξ and making use of the relation

$$s'\bar{e}\bar{x}=p_1\cos\bar{\theta}$$

we get

$$s'\bar{x}\eta^2+\frac{3s'^2}{4}(2\bar{e}\eta+\eta^2)^2=4p_1e\left(\sin^2\frac{\bar{\theta}}{2}-\sin^2\frac{\theta}{2}\right)$$

whence, neglecting η^3, $\eta\theta^2$, $\eta\bar{\theta}^2$ and higher powers, we obtain

$$\eta^2 = \frac{p_1\bar{e}(\bar{\theta}^2-\theta^2)}{3s'^2\bar{e}^2+s'\bar{x}}$$

Now

$$n_0\int e(e^2-\bar{e}^2)dt=\int\frac{e^2(e^2-\bar{e}^2)}{s'ex-p_1\cos\theta}d\theta$$

and

$$s'ex-p_1\cos\theta = s'(3s'\bar{e}^2+\bar{x})\eta+\frac{s'}{2\bar{e}}(6s'\bar{e}^2-\bar{x})\eta^2+\ldots\ldots$$

Also

$$e^2(e^2-\bar{e}^2)=2\bar{e}^3\eta+5\bar{e}^2\eta^2+\ldots\ldots$$

Hence

$$\frac{e^2(e^2-\bar{e}^2)}{s'ex-p_1\cos\theta}=\frac{2\bar{e}^3+5\bar{e}^2\eta+\ldots\ldots}{s'(3s'\bar{e}^2+\bar{x})+\frac{s'}{2\bar{e}}(6s'\bar{e}^2-\bar{x})\eta+\ldots\ldots}$$

$$=\frac{2\bar{e}^3}{s'(3s'\bar{e}^2+\bar{x})}\left(1+\frac{3}{2}\,\frac{3s'\bar{e}^2+2\bar{x}}{3s'\bar{e}^2+\bar{x}}\,\frac{\eta}{\bar{e}}\right)$$

neglecting η^2. Accordingly we can write

$$n_0\int e(e^2-\bar{e}^2)dt=P\int d\theta\pm Q\int\sqrt{\bar{\theta}^2-\theta^2}\,d\theta$$

Since $\bar{x}$ is positive in this case the coefficients P and Q are finite and positive. The double sign will be taken the same as that of η. Now

$$\int_0^T\frac{d\theta}{dt}dt=0$$

and the sign of η is the same as that of $\frac{d\theta}{dt}$ throughout. Hence

$$n_0\int_0^T e(e^2-\bar{e}^2)dt=2Q\int_{-\bar{\theta}}^{\bar{\theta}}\sqrt{\bar{\theta}^2-\theta^2}\,d\theta=Q\pi\bar{\theta}^2$$

Again

$$n_0\int e(e^2-\bar{e}^2)^2dt=\int\frac{e^2(e^2-\bar{e}^2)d\theta}{s'ex-p_1\cos\theta}=\frac{4\bar{e}^4}{s'(3s'\bar{e}^2+\bar{x})}\int\eta d\theta$$

neglecting η^2. Hence

$$n_0\int e(e^2-\bar{e}^2)^2dt=\pm Q'\int\sqrt{\bar{\theta}^2-\theta^2}d\theta$$

in which Q' is a quantity finite and positive, and the double sign will be taken the same as that of η. Consequently

$$n_0\int_0^T e(e^2-\bar{e}^2)^2dt=2Q'\int_{-\bar{\theta}}^{\bar{\theta}}\sqrt{\bar{\theta}^2-\theta^2}d\theta=Q'\pi\bar{\theta}^2$$

and

$$n_0\int_0^T\frac{du}{dt}dt=-S\pi\bar{\theta}^2$$

where S is a finite quantity, positive when M is positive. Also

$$n_0\int dt=\int\frac{ed\theta}{s'ex-p_1\cos\theta}=\frac{\bar{e}}{s'(3s'\bar{e}^2+\bar{x})}\int\frac{d\theta}{\eta}=\pm P'\int\frac{d\theta}{\sqrt{\bar{\theta}^2-\theta^2}}$$

where P' is a quantity finite and positive, and

$$n_0T=n_0\int_0^T dt=2P'\int_{-\bar{\theta}}^{\bar{\theta}}\frac{d\theta}{\sqrt{\bar{\theta}^2-\theta^2}}=2\pi P'$$

Hence

$$\frac{1}{T}\int_0^T\frac{du}{dt}dt=-\frac{S}{2P'}\bar{\theta}^2=-S'\bar{\theta}^2$$

and

$$\frac{1}{T}\int_0^T\frac{d\bar{\theta}}{dt}dt=S'\bar{\theta}$$

Consequently the variation of $\bar{\theta}$ is slow when $\bar{\theta}$ is small. The same proposition may be proved for the libration of Type 1 when $\bar{\theta}$ is very nearly equal to π. Hence in general *the amplitude of the libration varies very slowly when the amplitude is very small.*

44. Now when the amplitude of the libration is very small x and e are connected approximately by the relation $s'ex=p_1$ in the libration of Type 3. Hence the point (x, e) moves in a rectangular hyperbola, and therefore when e is great the point moves down nearly parallel to the axis of e until e becomes small and consequently the variation becomes very slow.

In the case of the libration of Type 1 the point (x, e) will move in another rectangular hyperbola represented by the equation $s'ex=-p_1$. It will move nearly parallel to the axis of x with a small eccentricity until reaches at the point

$$x=-\left(\frac{3p_1^2}{s'}\right)^{\frac{1}{3}} \qquad e=\left(\frac{p_1}{3s'^2}\right)^{\frac{1}{3}}$$

and the motion abruptly changes to the revolution of Type 2.

45. The conclusions for the general case of the first order may be stated as follows:—

1. The revolution of Type 1 will change to the revolution of Type 2 either directly or indirectly and finally to that of Type 3. The limits of the mean motion and eccentricity increase discontinuously when the motion changes to the revolution of Type 2, whence the variations of the constants become rapid, and the asteroids of this class will not stay long near the critical point.

2. The libration of Type 2, when C is not negative and relatively great, changes to the revolution of Type 2 either directly or indirectly and finally to that of Type 3.

3. The libration of Type 2, when C is negative and relatively great, changes to the libration of Type 3 which may change to the revolution of Type 3 if the constant β be

sufficiently small compared with α. The amplitude of the libration varies very slowly when the amplitude is small, and the asteroids of this class will stay long near the critical point with small eccentricities.

46. *General Case of the Second Order.* Putting $\theta=\pi+\theta'$ in the equations (27) of Chapter II. we have

$$(11)\qquad \begin{cases} x^2=C+6p_2e^2\cos\theta' \qquad e^2=E+\dfrac{4}{3s'}x \\ \dfrac{1}{n_0}\dfrac{d\theta'}{dt}=s'x-4p_2\cos\theta' \end{cases}$$

The equations for the variations of the constants become

$$\frac{dC}{dt}=3\alpha e^3x+12\beta p_2e^3\cos\theta' \qquad \frac{dE}{dt}=-2\left(\frac{\alpha}{s'}+\beta\right)e^3$$

The quantity *E always decreases* as in the case of the first order. *C increases in the libration of Type 3*, since x and $\cos\theta'$ are always positive. We can prove also that *C increases if negative*, that *it decreases in the revolution of Type 1* and that *it increases in the revolution of Type 3 if E be less than a certain positive value.*

47. The limits of the mean motion and eccentricity, when the path A is in contact with the limiting curve B_0, may be obtained as follows:—

θ'	x	e^2
π	$-\dfrac{4p_2}{s'}$	$E-\dfrac{16}{3s'^2}p_2$
0	$\dfrac{4p_2}{s'}\pm\sqrt{12p_2E}$	$E+\dfrac{16}{3s'^2}p_2\pm\sqrt{\dfrac{64}{3s'^2}p_2E}$

For the limiting case in which $e_0=e_1=0$ we have

θ'	x	e^2
$\pi,\ 0$	$-\frac{4p_2}{s'}$	0
0	$\frac{12p_2}{s'}$	$\frac{64p_2}{3s'^2}$

Numerical values of these limits for different cases of the second order become as follows: —

s/s'	$p_2\times 10^2$	$-\frac{4p_2}{s'}\times 10^2$	$\frac{12}{s'}p_2\times 10^2$	$\left(\frac{64}{3s'^2}p_2\right)^{\frac{1}{2}}\times 10$
3/1	0·0275	−0·110	0·330	0·766
5/3	0·222	−0·296	0·888	0·725

48. The equation corresponding to (8) becomes

$$\frac{d\bar{x}}{dt}=\frac{3}{2}\left\{\alpha\bar{e}^2+(\alpha+s'\beta)(e^2-\bar{e}^2)\right\}e$$
$$+\frac{9s'^2}{8(s'\bar{x}-4p_2\cos\bar{\theta}')}\left\{\alpha\bar{e}^2+\left(\alpha+\frac{s'\beta}{2}\right)(e^2-\bar{e}^2)\right\}e(e^2-\bar{e}^2)$$

Putting $\bar{\theta}'=\pi$ and applying this equation to the case of the revolutions very near to the contact, it may be seen that $\bar{x}$ increases in the revolution of Type 2 and that it also increases in the revolution of Type 1 if

$$\bar{e}^2>\frac{s'\beta}{\alpha}\left(1+\frac{1}{2}\,\frac{s'\beta}{\alpha}\right)\frac{32}{3s'^2}p_2$$

Since the libration of Type 1 is impossible in the case of the second order, the revolution of Type 1, if it does not change to the revolution of Type 2, will remain unchanged.

49. The equation corresponding to (10) becomes in this case

$$\frac{8}{3}p_2\bar{e}^2\frac{du}{dt}=-s'\alpha\bar{e}^2e(e^2-\bar{e}^2)-s'\left(\alpha+\frac{s'\beta}{2}\right)e(e^2-\bar{e}^2)^2$$

where $u=\cos\bar{\theta}'$ is the minimum value of $\cos\theta'$ and $\bar{e}$ is the corresponding value of e. Now

$$n_0\int_0^T(e^2-\bar{e}^2)dt=\frac{4n_0}{3s'}\int_0^T(x-\bar{x})dt$$

$$=\frac{8}{3s'}\int_0^{\bar{\theta}'}\left(\frac{x-\bar{x}}{s'x-4p_2\cos\theta'}-\frac{x'-\bar{x}}{s'x'-4p_2\cos\theta'}\right)d\theta'$$

$$\equiv\frac{8}{3s'}\int_0^{\bar{\theta}'}Zd\theta'$$

x' corresponding to the portion in which $\frac{d\theta'}{dt}$ is negative, and

$$Z=\frac{(s'\bar{x}-4p_2\cos\theta')(x-x')}{(s'x-4p_2\cos\theta')(s'x'-4p_2\cos\theta')}=\frac{-4p_2(\cos\theta'-\cos\bar{\theta}')(x-x')}{(s'x-4p_2\cos\theta')(s'x'-4p_2\cos\theta')}$$

The denominator is negative. The differences $x-x'$ and $\cos\theta'-\cos\bar{\theta}'$ are positive and therefore Z is always positive. Hence

$$\int_0^T(e^2-\bar{e}^2)dt>0$$

Now

$$\int_0^T e(e^2-\bar{e}^2)dt=\int_0^T(e-\bar{e})(e^2-\bar{e}^2)dt+\bar{e}\int_0^T(e^2-\bar{e}^2)dt$$

and therefore

$$\int_0^T e(e^2-\bar{e}^2)dt>0$$

Hence

$$\int_0^T\frac{du}{dt}dt<0$$

always. Accordingly *the librations ultimately change to revolutions.* The libration of Type 2 changes to the revolution of Type 2 or to that of Type 1, and the libration of Type 3 to the revolution of Type 3.

50. We can prove also, as in the case of the first order, that

$$\frac{1}{T}\int_0^T \frac{d\bar{\theta}'}{dt}dt = S'\bar{\theta}'$$

where S' is a quantity finite and *positive*. *The amplitude of the libration therefore increases very slowly when the amplitude is very small.*

51. The conclusions for the general case of the second order are as follows:—

1. The revolution of Type 1 may change to the revolution of Type 2 and finally to that of Type 3. The limits of the mean motion and eccentricity increase discontinuously when the motion changes to the revolution of Type 2, whence the variations of the constants become rapid and the asteroids of this class will not stay long near the critical point.
2. The libration of Type 2, when C is not negative and relatively great, changes either to the revolution of Type 1 or to the revolution of Type 2 which changes finally to that of Type 3.
3. The libration of Type 2, when C is negative and relatively great, changes to the libration of Type 3 and finally to the revolution of Type 3. The amplitude of the libration increases slowly when the amplitude is small, and the asteroids of this class will stay long near the critical point with small eccentricities.

52. *General Case of the Third or Higher Orders.* Restoring e and putting

$$s - s' = i \qquad \text{and} \qquad \theta = (i-1)\pi + \theta'$$

in the equations (28) of Chapter II, we get

$$(12) \qquad \begin{cases} x^2 = C + 6p_i e^i \cos\theta' \qquad e^2 = E + \dfrac{2i}{3s'}x \\ \dfrac{1}{n_0}\dfrac{d\theta'}{dt} = s'x - i^2 p_i e^{i-2}\cos\theta' \end{cases}$$

The equations for the variations of the constants become

$$\frac{dC}{dt}=3\alpha e^3x+6{}_i\beta ip_ie^{i+1}\cos\theta' \qquad \frac{dE}{dt}=-\left(\frac{i\alpha}{s'}+2{}_i\beta\right)e^3$$

E always decreases, as in the previous cases. We can prove also that *C increases if negative,* that *it decreases in the revolution of Type 1* and that *it increases in the revolution of Type 3 if E be less than a certain positive value.*

53. We have to consider the *orders* of the small quantities in this case. The eccentricity e may be supposed to be a quantity of the order of 10^{-1}, and the constant p_i, being a quantity multiplied by $m'=1/1047$, may therefore be supposed to be the third order of e. Hence, assuming that the orders of the three terms in the first equation of (12) are the same, x becomes a quantity of the order $\frac{i+3}{2}$. Consequently if $i>3$, the order of x will be higher than that of e^2 by a unit order at least, so that e may be supposed to be a constant in the first approximation. In the second member of the third equation of (12), the order of p_ie^{i-2}, being $i+1$, is higher than that of x by a unit order at least, if $i>3$. Hence we may write

$$\frac{1}{n_0}\frac{d\theta'}{dt}=s'x-g\cos\theta'$$

where g is a constant.

54. The equation corresponding to (8) becomes in this case

$$\frac{d}{dt}(\bar{x}\bar{e}^{-i+2})=\left[\frac{3\alpha}{2}\bar{e}^2+\frac{3\alpha}{2}(e^2-\bar{e}^2)+{}_i\beta(i-2)\bar{x}+\frac{3s'{}_i\beta}{i}(e^2-\bar{e}^2)\right]\bar{e}^{-i+2}e$$

$$+\frac{N}{2}\left[\frac{3\alpha}{2}\bar{e}^2+\frac{3\alpha}{2}(e^2-\bar{e}^2)+{}_i\beta(i-2)\bar{x}+\frac{3s'{}_i\beta}{4}(e^2-\bar{e}^2)\right]e(e^2-\bar{e}^2)$$

in which

$$N=\frac{s'}{i}\,\frac{3s'\bar{e}^2-i(i-2)\bar{x}}{s'\bar{x}-i^2p_i\bar{e}^{i-2}\cos\theta'}\bar{e}^{-i}$$

Or neglecting the quantities of higher orders we get

$$\frac{d}{dt}(\bar{x}\bar{e}^{-i+2})=\frac{3a}{2}\bar{e}^{-i+4}e+\frac{3a}{4}N\bar{e}^2e(e^2-\bar{e}^2)$$

where

$$N=\frac{s'}{i}\ \frac{3s'\bar{e}^{-i+2}}{s'\bar{x}-i^2p_i\bar{e}^{i-2}\cos\bar{\theta}'}$$

If we put $\bar{\theta}'=\pi$ in this equation, $N(e^2-\bar{e}^2)$ is always positive, and hence

$$\frac{d}{dt}(\bar{x}\bar{e}^{-i+2})>0$$

Accordingly the revolution of Type 1 in its limiting case approaches the position of contact, and the revolution of Type 2 recedes from that position.

55. The equation corresponding to (10) takes the form

$$2ip_i\bar{e}^i\frac{du}{dt}=-s'\left[\frac{3a}{2}\bar{e}^2+\frac{3a}{2}(e^2-\bar{e}^2)+{}_i\beta(i-2)\bar{x}+\frac{3s'{}_i\beta}{4}(e^2-\bar{e}^2)\right]e(e^2-\bar{e}^2)$$

$$=-\frac{3s'}{2}a\bar{e}^2e'e^2-\bar{e}^2)$$

neglecting the quantities of higher orders. Now

$$n_0\int e(e^2-\bar{e}^2)dt=\int\frac{e(e^2-\bar{e}^2)}{s'x-g\cos\theta'}d\theta'$$

Accordingly we can prove that

$$\int_0^T e(e^2-\bar{e}^2)dt>0$$

as in the case of the second order. Hence

$$\int_0^T \frac{du}{dt}dt<0$$

and therefore, *the librations ultimatety change to revolutions.*

We can prove also that *the amplitude of the libration increases very slowly when it is very small,* as in the previous case.

Thus we obtain the same conclusions as for the general case of the second order.

Chapter IV.

Peculiarities in the Distribution of the Mean Motions of the Asteroids and their Possible Explanations.

56. We shall first examine the nature of the *gaps.* The ratios of the mean motions up to the seventh order and lying within the denser portion of the asteroids are as follows:—

Order	n_0/n'	Order	n_0/n'	Order	n_0/n'
1	2/1	5	9/4	7	13/6
2	3/1		8/3		12/5
3	5/2		7/2		11/4
4	7/3	6	11/5		10/3

The simplest way of determining the width of the gaps is to find the difference of the two mean motions nearest to n_0 in both directions. But this will be very rough, especially when one or both of the mean motions is not reliable, as in the case of (132) of the class 3/1. The following method will answer for this defect. Denoting by

$$n_{-15}, n_{-14}, \ldots\ldots n_{-2}, n_{-1}; n_1, n_2, \ldots\ldots n_{14}, n_{15}$$

the mean motions arranged in ascending order and interposing n_0 between n_{-1} and n_1, the quantities

$$Q_1 = \frac{1}{6}(n_1 + n_2 + \ldots\ldots + n_9 - n_{13} - n_{14} - n_{15}) - n_0$$

and

$$Q_{-1}=n_0-\frac{1}{6}(n_{-1}+n_{-2}+\ldots\ldots+n_{-9}-n_{-13}-n_{-14}-n_{-15})$$

may represent the width in the positive and negative directions of x respectively. The sum of these two quantities becomes zero when the distribution is uniform, and becomes negative when there is some condensation near n_0. The number of the mean motions in each direction may be taken more or less. But if this number be too small the result will be inaccurate, while if too many it may be affected by other irregularities of the distribution.

57. The width and position of the gaps thus determined, according to the *Berliner Jahrbuch* for 1917, are as follows:—

n_0/n'	Order	n_0	Q_{-1}	Q_1	Width	Displacement of Center
		″	″	″	″	
2/1	1	598·26	18·76	9·58	28·34	−4·59
13/6	7	648·12	0·67	0·06	0·73	−0·30
11/5	6	658·09	3·43	0·81	4·24	−1·31
9/4	5	673·04	1·56	0·96	2·52	−0·30
7/3	4	697·97	2·66	5·73	8·39	+1·54
12/5	7	717·91	0·52	1·93	2·45	+0·70
5/2	3	747·82	6·17	4·33	10·50	−0·92
8/3	5	797·68	2·76	1·26	4·02	−0·75
11/4	7	822·61	0·17	−0·99	−0·82	−0·58
3/1	2	897·39	12·12	10·92	23·04	−0·60
10/3	7	997·10	0·80	−3·22	−2·42	−2·01
7/2	5	1046·96	1·69	5·24	6·93	+1·78

The existence of the gaps up to the fifth order seems established beyond doubt. It may also be observed that the width becomes narrower as the order advances and also as the denominator or the numerator of the ratio increases.

58. In the portion with smaller mean motions than 500″, though the number of the asteroids is very small we see at once that the character is *reversed* (PLATE). No asteroid can be found

except near the commensurable points in which the gaps are found in the other portion. The positions of the condensations and the number of the asteroids are as follows:—

	Order	n_0	No. of Asteroids
1/1	0	299″·13	4
4/3	1	398·84	1
3/2	1	448·70	6

If the distribution of the mean motions can be compared with the spectrum, the portion with greater mean motions than 600″ will correspond to an absorption spectrum, and that with smaller mean motions than 500″ to an emission spectrum. The portion intermediate between these portions may possibly belong to the former class, although the character is not distinct owing to the scarcity of the asteroids.

59. That the eccentricity of the asteroids near the gaps is smaller than its mean value was remarked by Prof. Brown.[1] To verify this I have taken ten asteroids on each side of the gap and computed the mean angle of eccentricity as follows:—

n_0/n'	Order	φ			
		Outer	Inner	Mean	Diff.
2/1	1	7°·49	6°·09	6°·79	+1°·40
11/5	6	9·08	8·57	8·82	+0·51
9/4	5	7·16	6·24	6·70	+0·92
7/3	4	8·94	6·10	7·52	+2·84
5/2	3	8·41	6·54	7·48	+1·87
8/3	5	8·15	7·57	7·86	+0·58
3/1	2	7·89	7·87	7·88	+0·02
7/2	5	6·65	9·47	8·06	−2·82
Mean		7·97	7·31	7·64	+0·66

1) Science, Jan. 20. 1911.

The mean value of the angle of eccentricity of all the asteroids, 791 in number, is $8^\circ.50$. Accordingly the mean value in the outer portion is less than the total mean by $0^\circ.53$ and that in the inner portion, by $1^\circ.19$. Thus it seems quite certain that *the eccentricity in the portion near the gaps, especially on the inner side, is smaller than that in the other portions.*

60. In order to determine the types of the motion near the commensurable points I have computed the quantities C, E, F_0, F_1 and E_3 for each asteroid by the formulæ—

$$x=\frac{n}{n_0}-1 \qquad \theta=s'(l-l')+(s-s')(\varpi-l')$$

$$C=x^2+(-1)^{s-s'}6p^{(s)}_{s-s'}e^{s-s'}\cos\theta \qquad E=e^2-\frac{2}{3}\,\frac{s-s'}{s'}x$$

$$F_0=\left(\frac{C}{6p^{(s)}_{s-s'}}\right)^{\frac{2}{s-s'}} \qquad F_1=\frac{2}{3}\,\frac{s-s'}{s'}\sqrt{C}$$

$$E_3=F_0+\frac{(s-s')^3}{3s'^2}\,p^{(s)}_{s-s'}\left(\frac{C}{6p^{(s)}_{s-s'}}\right)^{\frac{s-s'-2}{s-s'}} \qquad \text{(approximately)}$$

which may be easily deduced from the equations in Chapter II. The constants $p^{(s)}_{s-s'}$ were computed by Leverrier's formulæ, and the elements of the asteroids, n, e, ϖ and l, according to the data in the *Berliner Jahrbuch* for 1917.

61. For the class 2/1 taking 41 asteroids with $n-n_0$ within the limits $-35''$ and $+25''$, the results are as follows:—

$$2/1 \qquad n_0=598''.26 \quad p_1^{(2)}\times10^3=0.716$$

No.	Epoch	ϖ	l	l'	n	$x\times10^2$	$e\times10$	θ	$C\times10^4$	$E\times10^2$	$F_0\times10^2$	$F_1\times10^2$	Type
		°	°	°	"			°					
790	1914 VII 10·5	285·6	274·0	319·0	564·31	−5·68	1·48	281·6	+30·99	+ 5·97	52·03	3·71	R_1
76	1911 VII 6·0	87·5	309·6	227·5	564·54	−5·64	1·73	302·1	+27·86	+ 6·76	42·06	3·52	R_1

No.	Epoch			ϖ	l	l'	n	$x\times10^2$	$e\times10$	θ	$C\times10^4$	$E\times10^2$	$F_0\times10^2$	$F_1\times10^2$	Type
713	1911	IV	28·5	349°·4	209°·6	221°·9	565″·33	−5·50	1·59	115°·2	+33·17	+ 6·20	59·61	3·84	R_1
733	1912	IX	19·5	152·6	8·5	264·3	566·13	−5·37	0·59	352·5	+26·33	+ 3·93	37·56	3·42	R_1
225	1903	XI	5·0	298·5	27·2	354·8	567·59	−5·13	2·64	336·1	+15·95	+10·40	13·79	2·67	R_1
528	1913	IX	13·0	52·2	9·6	294·0	567·84	−5·09	0·21	193·8	+26·78	+ 3·44	38·86	3·45	L_1
566	1905	VI	1·5	17·0	260·3	42·6	570·18	−4·69	1·36	192·1	+27·70	+ 4·99	41·58	3·51	R_1
692	1910	V	30·5	111·8	194·2	194·1	570·82	−4·59	1·65	277·8	+20·10	+ 5·78	21·89	2·99	R_1
168	1899	V	29·0	23·8	242·2	220·2	571·69	−4·44	0·76	185·6	+22·97	+ 3·54	28·59	3·20	R_1
466	1915	V	26·0	198·1	246·5	345·6	575·95	−3·73	0·74	113·4	+15·17	+ 3·04	12·47	2·60	R_1
643	1907	IX	12·5	90·2	9·5	111·8	577·58	−3·46	0·77	236·1	+13·82	+ 2·90	10·35	2·48	R_1
525	1904	III	18·5	47·4	116·7	6·0	581·34	−2·83	3·71	152·1	+22·09	+15·66	26·44	3·13	R_1
401	1913	III	17·0	239·3	164·5	279·0	584·39	−2·32	0·49	205·8	+ 7·28	+ 1·79	2·87	1·80	L_1
745	1913	III	7·5	129·2	152·6	278·2	606·78	+1·42	0·90	85·4	+ 1·71	− 0·13		0·87	R_3
781	1914	I	25·5	267·6	136·1	305·2	608·78	+1·76	0·86	153·3	+ 6·40	− 0·43		1·69	R_3
175	1914	I	11·0	330·5	90·4	303·9	609·57	+1·89	1·87	173·1	+11·55	+ 2·24	7·23		R_3
530	1911	IX	3·5	323·0	323·7	232·5	610·21	+2·00	1·77	181·7	+11·60	+ 1·80	7·29		R_3
777	1914	I	28·5	167·1	121·8	305·5	611·31	+2·18	1·46	37·9	− 0·20	+ 0·68	0·00		L_2
756	1908	IV	26·5	194·8	217·6	130·7	612·32	+2·35	1·20	151·0	+10·03	− 0·12		2·11	R_3
758	1912	VI	9·5	53·2	256·2	255·8	612·61	+2·40	1·12	160·8	+10·31	− 0·35		2·14	R_3
122	1911	V	7·0	189·9	214·4	222·5	614·37	+2·69	0·56	319·3	+ 5·42	− 1·48		1·55	R_3
581	1905	XII	24·5	63·5	92·1	59·7	615·96	+2·96	0·44	36·2	+ 7·24	− 1·78		1·79	R_3
300	1895	VII	10·0	325·4	302·1	102·3	617·27	+3·18	0·43	62·9	+ 9·27	− 1·93		2·03	R_3
318	1912	IV	11·0	78·4	186·5	250·7	617·67	+3·24	0·59	123·5	+11·91	− 1·81		2·30	R_3
108	1911	IX	24·0	164·9	324·5	234·1	617·91	+3·28	1·05	21·2	+ 6·55	− 1.08		1·70	R_3
667	1908	VIII	24·5	98·4	334·7	140·6	618·03	+3·30	1·71	151·9	+17·37	+ 0·73	16·35		R_3
325	1913	XII	2·0	60·4	69·9	300·6	618·24	+3·34	1·65	249·1	+13·68	+ 0·50	10·14		R_3
580	1906	II	12·5	54·9	86·8	63·8	618·61	+3·40	1·33	14·1	+ 6·01	− 0.50		1·63	R_3
755	1915	VIII	14·0	217·5	316·2	352·2	619·88	+3·61	1·27	189·3	+18·42	− 0·79		2·86	R_3
595	1906	V	18·5	289·5	221·1	71·7	620·18	+3·66	0·75	7·2	+10·20	− 1·88		2·13	R_3
645	1907	IX	29·5	89·9	14·6	113·2	620·25	+3·68	1·55	238·1	+17·06	− 0·05		2·76	R_3
491	1903	I	0·0	41·1	21·8	329·1	620·55	+3·73	0·65	124·7	+15·51	− 2·06		2·62	R_3
381	1906	III	14·0	268·4	174·9	66·3	620·62	+3·74	1·26	310·7	+10·46	− 0·90		2·16	R_3
286	1905	VI	7·0	32·8	244·8	43·0	620·63	+3·75	0·13	191·6	+14·61	− 2·48		2·55	R_3
702	1910	VIII	4·5	345·3	316·0	199·6	621·86	+3·95	0·15	262·1	+15·70	− 2·61		2·64	R_3
696	1910	II	1·5	37·9	92·6	184·4	621·91	+3·95	2·41	121·7	+21·04	+ 3·18	23·99		R_3
618	1906	X	25·5	346·6	19·7	85·0	622·09	+3·98	0·60	196·3	+18·32	− 2·29		2·85	R_3
435	1906	II	2·0	15·4	106·1	62·9	622·10	+3·99	0·83	355·7	+12·37	− 1·97		2·34	R_3
184	1910	XII	18·0	191·0	75·6	210·8	622·48	+4·05	0·60	205.0	+18·74	− 2·34		2·89	R_3
92	1904	II	13·0	323·4	105·9	3·1	622·68	+4·08	0·94	63·1	+14·84	− 1·85		2·57	R_3
316	1912	V	1·0	75·4	229·0	252·4	623·00	+4·14	1·20	159·6	+22·33	− 1·09		3·15	R_3

Two asteroids (401) and (528) seem to make the libration of Type 1, but as the difference of E and F_1 is very small the motion may be changeable to the revolution of Type 1 depending on the amount of the smaller inequalities. The asteroid (777) has a negative value of C and the motion is the libration of Type 2. All the remaining 38 asteroids make the revolution of Type 1 or Type 3.

62. For the class 3/2 six asteroids only may be taken. The results of computation are as follows:—

$$3/2 \qquad n_0 = 448''\!\cdot\!70 \qquad p_1^{(3)} \times 10^3 = 1\cdot47$$

No.	Epoch			ϖ	l	l'	n	$x\times10^2$	$e\times10$	θ	$C\times10^4$	$E\times10^2$	$F_0\times10^2$	$F_1\times10^2$	Type
				°	°	°	″			°					
153	1911	III	28·0	282·6	207·9	219·2	449·46	+0·17	1·62	40·8	−10·79	+2·57	1·49		L_2
748	1913	III	8·5	103·0	160·9	278·3	451·35	+0·59	1·36	309·9	− 7·34	+1·65	0·69		L_2
361	1914	XI	27·0	93·4	61·6	330·6	453·60	+1·09	2·08	304·8	− 9·26	+3·96	1·10		L_2
190	1910	XI	8·0	103·7	71·0	207·5	453·69	+1·11	1·67	343·2	−12·87	+2·42	2·13		L_2
499	1911	I	30·5	92·5	112·4	214·5	457·15	+1·88	2.14	33·8	−12·15	+3·95	1·90		L_2
334	1913	IV	26·0	8·5	225·4	282·3	459·51	+2·41	0·15	332·4	+ 4·63	−0·78		0·72	L_3

Five asteroids out of six have negative values of C and make the libration of Type 2. The asteroid (334) makes the libration of Type 3 with a positive value of C.

63. Only one asteroid may be taken near the point 4/3 with the following result:—

$$4/3 \qquad n_0 = 398''\!\cdot\!84 \qquad p_1^{(4)} \times 10^3 = 2\cdot24$$

No.	Epoch			ϖ	l	l'	n	$x\times10^2$	$e\times10$	θ	$C\times10^4$	$E\times10^2$	$F_0\times10^2$	$F_1\times10^2$	Type
				°	°	°	″			°					
279	1913	VI	17·5	296·1	294·7	286·7	397·60	−0·31	0·64	33·4	−7·08	+0·48	0·28		L_2

The type of the motion is the same as those of the five asteroids of the class 3/2, i.e. the libration of Type 2 with a negative value of C.

64. For the class 3/1, taking 25 asteroids with $n-n_0$ within the limits $-25''$ and $+25''$, the following results are obtained:—

$$3/1 \qquad n_0=897''\!\cdot\!39 \qquad p_2^{(3)}\times 10^2=0\cdot0275$$

No.	Epoch	ϖ	l	l'	n	$x\times10^2$	$e\times10$	θ	$C\times10^4$	$E\times10^2$	$F_0\times10^2$	$F_1\times10^2$	Type
765	1913 X 3·5	36°·7	21°·3	295°·7	874″·04	−2·60	2·81	287°·6	+7·16	+11·36	43·4		R_1
714	1911 V 25·5	102·7	214·2	224·1	874·17	−2·59	0·45	107·3	+6·70	+ 3·65	40·6		R_1
472	1908 III 23·0	62·2	177·7	127·8	875·74	−2·41	0·98	278·7	+5·83	+ 4.17	35·3		R_1
787	1914 IV 22·5	309·5	230·8	312·4	876·73	−2·30	1·24	272·6	+5·30	+ 4·60	32·1		R_1
355	1905 I 2·5	86·9	99·3	30·1	877·28	−2·24	1.08	182·8	+4·82	+ 4·15	29·2		R_1
695	1909 XI 7·5	353·4	40·6	177·2	877·30	−2·24	1·56	215·8	+4·69	+ 5·42	28·4		R_1
660	1908 I 12·5	264·0	126·0	121·9	877·99	−2·16	1·03	288·3	+4·72	+ 3·94	28·6		R_1
421	1912 VIII 29·0	34·6	349·8	262·4	878·56	−2·10	2·92	351·8	+5·79	+11·32	35·1		R_1
292	1902 IV 4·0	331·4	206·7	306·6	881·55	−1·77	0·29	309·7	+3·14	+ 2·44	19·0		R_1
46	1910 XI 28·0	354·5	62.6	209·2	884·45	−1.44	1·67	144·0	+1·71	+ 4·71	10·4		R_1
518	1903 X 20.5	322·5	10·2	353·5	885·77	−1·29	2·20	314·7	+2·24	+ 6·56	13·6		R_1
619	1906 X 22·5	2·4	37·7	84·8	886·62	−1·20	0·75	148·1	+1·35	+ 2·16	8·2		R_1
132	1895 XI 30·5	152·4	123·2	114·2	903·69	+0·70	3·31	85·4	+0·64	+10·02	3·9*		L_2
495	1902 XI 21·5	26·5	47·4	325·9	910·12	+1·42	1·47	202·7	+1·68	+ 0·27	10·2		R_3
329	1901 VIII 27·0	217·0	337·1	288·3	912·13	+1·64	0·28	266·2	+2·69	− 2·11		2·2	R_3
335	1906 II 2·0	288·3	134·3	62·9	912·66	+1·70	1·80	163·2	+2·38	+ 0·97	14·4		R_3
17	1911 VII 26·0	263·0	290·0	229·1	913·55	+1·80	1·33	128·7	+3·06	− 0·63		2·3	R_3
248	1905 VIII 6·0	247·8	319·5	48·0	913·94	+1·84	0·64	311·1	+3·43	− 2·05		2·5	R_3
556	1905 I 16·5	101·0	116·6	31·2	915·85	+2·06	1·01	225·0	+4·13	− 1·73		2·7	R_3
752	1913 V 10·5	105·8	212·5	283·6	917·80	+2·27	0·74	293·3	+5·18	− 2·48		3·0	R_3
623	1907 II 5·5	71·7	123·0	93·6	918·32	+2·33	1·15	345·6	+5·63	− 1·79		3·2	R_3
650	1907 X 4·5	31·7	34·8	113·6	918·48	+2·35	1·87	117·4	+5·25	+ 0·36	31·8		R_3
732	1912 IV 24·5	236·9	212·8	252·0	919·07	+2·42	0·46	290·6	+5·87	− 3·02		3·2	R_3
178	1910 III 13·0	261·3	178·1	187·6	919·41	+2·45	0·45	137·9	+5·98	− 3·07		3·3	R_3
198	1910 VII 31·0	356·4	310·6	199·2	920·05	+2·53	2·28	65·8	+6·75	+ 1·82	40·9		R_3

* $E_3\times10^2=4\cdot0$

All the asteroids except (132) make the revolution of Type 1 or Type 3. The asteroid (132) seems to make the libration of Type 2. But this asteroid has not been observed since its discovery in 1873 and is known as one of the lost planets.[1] So the existence

1) Mr. D. Alter corrected the mean motion of this asteroid to 883″.47 on the assumption of its identity with the object observed at the Lowell Observatory in 1913. See the Lick Obs. Bull. Nos. 275 and 285.

of the librating asteroids near the point 3/1 is doubtful.

65. Only two asteroids will be taken near the point 5/3 with the following results:—

$$5/3 \qquad n_0 = 498''{\cdot}55 \quad p_2^{(5)} \times 10^2 = 0{\cdot}2222$$

No.	Epoch			ϖ	l	l'	n	$x\times10^2$	$e\times10$	θ	$C\times10^4$	$E\times10^2$	$F_0\times10^2$	$F_1\times10^2$	Type
522	1913	IV	6·0	1°·3	228°·3	280°·6	513″·62	+3·02	0·78	4°·5	+ 9·93	−0·74		1·4	R_3
721	1911	X	18·5	29·0	19·2	236·2	526·85	+5·68	1·18	14·6	+34·05	−1·13		2·6	R_3

Both make the revolution of Type 3.

66. In the cases of the third or higher orders, the effect of the neglected terms being relatively great, it becomes very hard to determine the types of the motion accurately. As a rough approximation however the same method was applied for the classes 5/2 and 7/4, arriving at the following results:—

All the asteroids near the point 5/2 ($n_0=747''{\cdot}82$) make the revolution of Type 1 or Type 3, except (464) which seems to make the libration of Type 2. But this asteroid has not been observed since its discovery in 1901 and therefore full weight cannot be assigned to its result. For the class 7/4 ($n_0=523''{\cdot}48$) two asteroids (522) and (721) only will be taken. The former seems to make the revolution of Type 1 and the latter, the libration of Type 2 with a positive value of C.

67. It is a well known fact that all of the four asteroids of the class 1/1 make the libration about the triangular equilibrium points. Hence, including this case, the results will be summarized as follows:—

1. *All the astreoids with smaller mean motions than 500″ make the libration and form a series of the groups near the commensurable points 1/1, 3/2 and 4/3.*

2. *The asteroids with greater mean motions than 580″ do not make the libration* (except a few cases which are mostly doubtful) *and form a series of the gaps at the commensurable points 2/1, 3/1, 5/2 etc.*

68. The first of these remarkable peculiarities may be accounted for by a consideration of gravitation only. Taking the cases of the first order, we have

$$\theta = s(l-l') - (l-\varpi)$$

If we put $$l'=l$$

in this equation, then

$$l=\varpi-\theta$$

which shows that the conjunction of the mean longitudes occurs at the point $l=\varpi-\theta$. Now in the libration of Type 2 with a negative value of C and in that of Type 3, the argument θ oscillates about 0 and the amplitude of the oscillation is less than $\frac{\pi}{2}$. Hence in these cases the conjunction takes place near the point $l=\varpi$, that is, near the perihelion of the asteroid. Contrarily if the type of the motion be the libration of Type 1 or the revolution of any type, the conjunction may take place near the aphelion of the asteroid.

69. The linear distance of the asteroid from Jupiter when the conjunction occurs at the aphelion of the asteroid is

$$a'-a(1+e)$$

This expression becomes zero when

$$\frac{a'}{a}=1+e=\left(\frac{n}{n'}\right)^{\frac{2}{3}}$$

The values of the eccentricity satisfying this condition are

n	350″	400″	450″	500′
e	0·11	0·21	0·31	0·41

This shows how the asteroid with a moderate eccentricity approaches Jupiter when the conjunction occurs near the aphelion of the asteroid. The eccentricity of the asteroid is generally smaller than 0·20, but this will not remain always small if the orbit of the asteroid be sufficiently close to that of Jupiter. So the motion of the asteroids with smaller mean motions, something like 450″ or less, if the conjunctions occur near the aphelia, will be disturbed a great deal by the attraction of Jupiter.

In the cases of the asteroids which make the libration about zero with the amplitudes less than $\frac{\pi}{2}$, since the conjunctions always take place near the perihelia, they will not suffer large disturbances and the motions will be stable. Contrarily the asteroids librating about π, or making the revolution will suffer large disturbances, since the conjunctions may take place near the aphelia. So their motions will be unstable.

Theoretically speaking the libration of Type 3 is possible for any positive value of x. But if the value of x be great the range of E for the libration becomes very small. Consequently the libration may be changeable to the revolution by a slight variation of E due to the smaller inequalities. The libration of Type 3 is thus practically impossible when x is not small. The fact that the asteroids do not exist at the intermediate positions is thus explicable.

70. The asteroids of the class 1/1, in spite of the proximity of their orbits to that of Jupiter, never approach very near to the latter and consequently their motions are stable. This fact is in perfect agreement with the above explanation.

An instance analogous to the librating asteroids of the 3/2 and 4/3 classes may be found in the Saturnian system. Hyperion, the seventh satellite of Saturn, has a period of revolution very nearly commensurable to that of Titan, the sixth and the largest satellite of Saturn. It has been shown in theory and by observations that the argument $4l-3l'-\varpi$[1] of Hyperion makes a libration about π, so that the conjunctions always take place near the aposaturnium of Hyperion. Consequently these satellites, in spite of the

1) Here l, ϖ and l' denote the mean longitude, the mean longitude of perisaturnium, of Hyperion, and the mean longitude of Titan, respectively.

proximity of their orbits, never approach very near to each other.

71. To explain the second character of the distribution of the mean motions, I shall introduce the hypothesis of resisting materials. Assuming the existence of the resisting materials moving around the sun in circular orbits, it was proved in Chapter III that, in the general cases of the second and higher orders, the librations ultimately change to revolutions, while the revolutions do not change to librations. In the general case of the first order it was proved that the revolutions do not change to librations except that the revolution of Type 1 may temporarily change to the libration of Type 1, that the librations ultimately change to the revolution or the libration, of Type 3, and also that, if the constant α be sufficiently great compared witn β, the libration of Type 3 changes to the revolution of the same type. Now in the class 2/1, the unique case of the first order with greater mean motions, we may naturally suppose, as it was noticed in § 9, that the density of the resisting materials rapidly decreases as the distance from the sun increases, so that α becomes great in comparison with β. Hence in general, in the cases of greater mean motions, the librations ultimately change to revolutions, while the revolutions do not change to librations, except the revolution of Type 1 in the general case of the first order, which may temporarily change to the libration of Type 1. In the cases of smaller mean motions, we have sufficient reason to believe that the density of the resisting materials is very rare, as was noticed in § 9, so that the effect on the librating motions is insignificant.

The fact that the eccentricities of the asteroids near the gaps, especially on the inner side, are generally smaller than those in the other portions (§ 59), may be explained in the following manner:—On the negative (outer) side of x, the eccentricity cannot be great so long as the type of the motion is confined to the revolutions. Even when the eccentricity is moderate, the motion will sometimes change to the libration on account of the smaller inequalities and the asteroid will remove to the positive side of x. Hence in order that the revolution of Type 1 may be stable, the eccentricity must always be very small. On the positive (inner) side of x, the asteroids which made the libration previously with negative value of C remain near the critical point with small eccentricities, as was shown in Chapter III. The asteroids which were removed from the negative side do not stay long near the critical point, and therefore will not affect the mean value of the eccentricity, although they may have unusually great values.

In concluding the present paper I wish to express my gratitude to Prof. Brown of Yale University, and to Prof. Eichelberger of the Naval Observatory at Washington, both of whom read the first part of this paper and gave valuable and encouraging suggestions in connection with my investigation.

Astronomical Observatory, Tokyo, Nov. 6, 1917.

List of Notations Used.

Numbers refer to the articles in which the notations are introduced. The parentheses enclosing numbers indicate that the notations are used temporarily.

Notation	Article
A, B_0, B_1	19
C	15
C_1	21
C_2	21, 24
E	16
E_1, E_2	21
E_3, F_0, F_1	21, 60
H	34
H_1	35
I_1, I_2, I_3	(7)
M	34
N	34, 54
P, P', Q, Q'	(43)
Q_1, Q_{-1}	56
R	10
R_0, R_c	13
S	1, (43)
S'	(43), (50)
T	1,30, 31)
V, V_s, V_t	1
W	5
X	(6), (30)
Y	(31)
Z	(49)
a	1
a'	10
a_0	11
$b^{(\ )}$	18, 19
c	1
e	1
e'	(30), (31)
e_0	35
e_1	33
$\bar{e}$	(33),34,37
f	(38)
g	53
h	1
i	(6),(12),52
j	(6), (12)
k_1, k_2	2
l, l'	12
m', n'	10
n_0	1, 11
n_{-15}, …, n_{-1}, n_1, …, n_{15}	56
$p_{s-s'}$, $p^{(s)}_{s-s'}$	19, 24, 27
q	(6)
r	1
s	(6), 11
s'	11
t	1
u	37
w	1
x	11
x'	(30), (31)
x_0	15
x_1	33
$\bar{x}$	(33),34,37
y	(24), (27)
α	28
α_0	18
α_1, α_2	7
β	28
β_1	7
ε	3, (38)
ε'	10
η	(43)
θ	12
θ'	46, 52
θ_0	19
$\bar{\theta}$	34
$\bar{\theta}'$	48
ν	(18)
ξ	(43)
ϖ	3
ρ	1
ρ_0	2
φ, ψ_1, ψ_2	(13)

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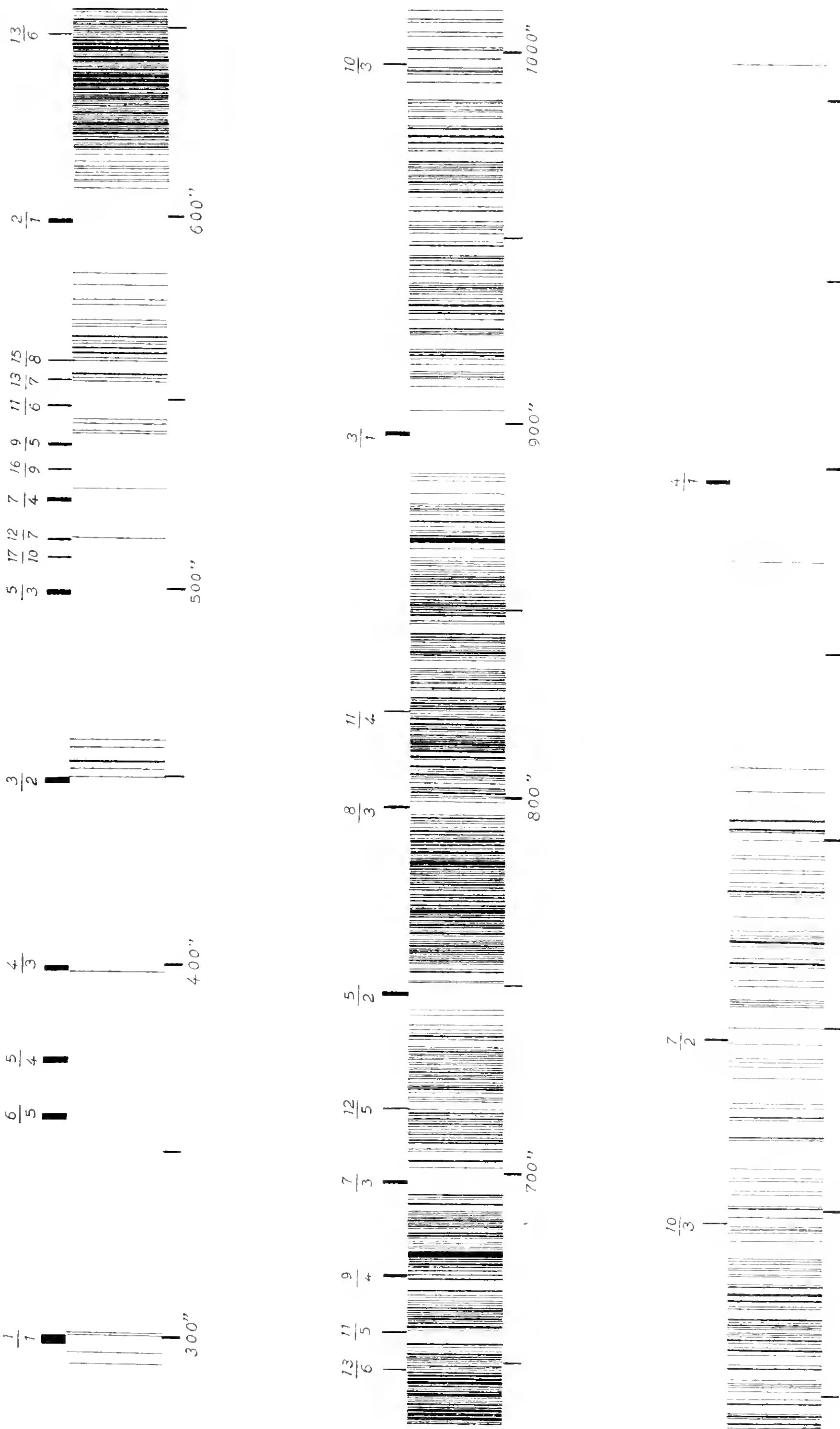

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Asymptotic Formulae for oscillating Dirichlet's Integrals and Coefficients of Power Series*

By

Motoji KUNIYEDA, *Rigakushi.*

Introduction.

1. G. H. Hardy, in his interesting papers† with the title "Oscillating Dirichlet's Integrals", has discussed almost completely the oscillating nature of an integral of the form

$$\int_0^{\xi} f(x)\frac{\sin \lambda x}{x} dx \qquad (\xi > 0),$$

when λ tends to infinity. Hereby $f(x)$ is of the form

$$\rho(x)\, e^{i\sigma(x)},$$

where $\rho(x)$ and $\sigma(x)$ are logarithmico-exponential functions (or L-functions) and $\sigma(x)$ tends to infinity as $x\to 0$.

As he remarks, the problem is equivalent to that of investigating the convergence or divergence of the Fourier's series defined by a function which has a single oscillating discontinuity of the type specified by

$$\rho(x) \begin{matrix}\cos\\ \sin\end{matrix}\, \sigma(x).$$

It is also closely related to that of determining asymptotic formulae for the coefficients a_n of a power series

$$\sum_{n=0}^{\infty} a_n z^n,$$

* This paper was worked out at the suggestion of Mr. Hardy, to whom I wish to express my sincere thanks for valuable advices.

† *Quarterly Journal*, Vol. XLIV, *pp. 1—40 and 242—263*. We shall refer to these papers as "*O. D. I. 1.*" and "*O. D. I. 2.*" respectively.

convergent for $|z|<1$, and representing a function $f(z)$ which has, on the circle of convergence, one singular point only, at $z=1$. In the investigation of this problem, we are often led to consider integrals of the types

$$\int_0^{\xi} f(x)\frac{\sin\lambda x}{x}dx, \qquad \int_0^{\xi} f(x)\frac{\cos\lambda x}{x}dx.$$

2. In Part I of this paper, I consider the integrals

$$(1) \qquad S(\lambda)=\int_0^{\xi}\rho(x)e^{i\sigma(x)}\frac{\sin\lambda x}{x}dx$$

$$(2) \qquad C(\lambda)=\int_0^{\xi}\rho(x)e^{i\sigma(x)}\frac{\cos\lambda x}{x}dx \qquad (\xi>0),$$

where, ρ and σ denote L-functions and $\sigma\succ 1$* as $x\to 0$, ξ being a positive number chosen sufficiently small so as to ensure that ρ and σ are monotonic and continuous in the interval $0<x\leqq\xi$. These integrals (1) and (2) will be called "the *sine-integral*" and "the *cosine-integral*" respectively.

Hardy, following Du Bois-Reymond, distinguishes the following three cases:

$$(A) \qquad \sigma(x)\prec l(1/x),$$

$$(B) \qquad \sigma(x)\asymp l(1/x),$$

$$(C) \qquad \sigma(x)\succ l(1/x).$$

The results arrived at concerning the sine-integral are designated as theorems A, B, C' in his papers.

As will be seen from these results, Hardy has principally considered the cases in which the sine-integral oscillates as $\lambda\to\infty$; but he did not went into a minute discussion of the cases in which the integral tends to zero. It will be interesting to find asymptotic formulae for $S(\lambda)$ in the latter cases; and it appears quite natural that the formulae obtained by him are also available to a certain extent in such cases. I have succeeded in extending the range of validity of his formulae considerably—roughly speaking, to all cases in which the order of $S(\lambda)$ is greater than $\frac{1}{\lambda}$.

* Throughout this paper, I will entirely adopt the symbols and notations defined in Section II of "*O. D. I. I.*"

As for the cosine-integral $C(\lambda)$, it will be seen that, in the two cases (A) and (B), it always tends to zero as $\lambda \to \infty$, when convergent, and, in Case (C), its behaviour is very similar to that of the sine-integral. I have found for it asymptotic formulae whose range of validity is the same as that of the formulae for $S(\lambda)$.

That the formulae should cease to hold when the order of the integrals sinks as low as $\frac{1}{\lambda}$ is to be expected. For, it is easy to see that the parts of the integrals away from $x=0$ are in general of the order $\frac{1}{\lambda}$, so that in such cases the behaviour of $S(\lambda)$ or $C(\lambda)$ is no longer dominated by the parts near $x=0$.

3. The principal results arrived at are as follows: Writing

$$\rho(x) = x^{-a}\theta(x), \qquad x^{\delta} < \theta < (1/x)^{\delta},$$

in Case (A),

$$S(\lambda) \sim -\Gamma(-a)\sin(\tfrac{1}{2}a\pi)\,\rho(1/\lambda)e^{i\sigma(1/\lambda)} \qquad (-1<a<0),$$

$$S(\lambda) \sim \tfrac{1}{2}\pi\rho(1/\lambda)e^{i\sigma(1/\lambda)} \qquad (a=0,\ \rho<1).$$

Combining these results with Theorem A, we obtain the theorem:

If $1<\sigma<l(1/x)$, $\rho<\sigma'$ *and*

$$\rho = x^{-a}\theta(x), \qquad x^{\delta} < \theta < (1/x)^{\delta}, \qquad a \leqq 1,$$

then we have, as $\lambda \to \infty$,

$$(3)\begin{cases} S(\lambda) = O(\lambda^{-1+\delta}) & (a \leqq -1), \\ S(\lambda) \sim -\Gamma(-a)\sin(\tfrac{1}{2}a\pi)\,\rho(1/\lambda)e^{i\sigma(1/\lambda)} & (-1<a<1), \\ S(\lambda) \sim \lambda\, T(1/\lambda) & (a=1), \end{cases}$$

where

$$T(x) = \int_0^x \rho(t)e^{i\sigma(t)}\,dt.$$

In the particular case $a=0$, *the factor* $-\Gamma(-a)\sin(\tfrac{1}{2}a\pi)$ *is to be replaced by its limiting value* $\tfrac{1}{2}\pi$.

The corresponding formalae for $C(\lambda)$ are as follows:—

$$
(4)\begin{cases}
C(\lambda) = O(1/\lambda) & (a<-1 \text{ or } a = -1,\ x\theta' \leqq 1,\ \rho\sigma' \leqq 1), \\
C(\lambda) \sim -(\tfrac{1}{2}\pi/\lambda)\{(1/\lambda)\theta'(1/\lambda)+i\rho(1/\lambda)\sigma'(1/\lambda)\}e^{i\sigma(1/\lambda)} & \\
 & (a = -1,\ x\theta' \text{ or } \rho\sigma' > 1), \\
C(\lambda) \sim \Gamma(-a)\cos(\tfrac{1}{2}a\pi)\,\rho(1/\lambda)e^{i\sigma(1/\lambda)} & (-1<a<0), \\
C(\lambda) \sim T(1/\lambda) & (a=0),
\end{cases}
$$

where

$$T(x) = \int_0^x \rho(t)e^{i\sigma(t)}\frac{dt}{t}.$$

In Case (B),

$$S(\lambda) \sim -\Gamma(-a-bi)\sin\{\tfrac{1}{2}(a+bi)\pi\}\ \rho(1/\lambda)e^{i\sigma(1/\lambda)}$$
$$(-1<a<0 \text{ or } a=0,\ \rho < 1).$$

Combining this result with Theorem B, we obtain the theorem:

If $\sigma \sim bl(1/x)$ $(b\neq 0)$, $\rho<\dfrac{1}{x}$, *and*

$$\rho = x^{-a}\theta(x), \qquad x^{\delta}<\theta<(1/x)^{\delta}, \qquad a\leqq 1,$$

then, as $\lambda\to\infty$, *we have*

$$
(5)\begin{cases}
S(\lambda) = O(\lambda^{-1+\delta}) & (a\leqq -1), \\
S(\lambda) \sim -\Gamma(-a-bi)\sin\{\tfrac{1}{2}(a+bi)\pi\}\ \rho(1/\lambda)e^{i\sigma(1/\lambda)} & (-1<a\leqq 1).
\end{cases}
$$

The corresponding formulae for $C(\lambda)$ are

$$
(6)\begin{cases}
C(\lambda) = O(1/\lambda) & (a<-1 \text{ or } a = -1,\ \theta\leqq 1), \\
C(\lambda) \sim \Gamma(-a-bi)\cos\{\tfrac{1}{2}(a+bi)\pi\}\ \rho(1/\lambda)e^{i\sigma(1/\lambda)} & \\
 & (-1<a\leqq 0 \text{ or } a=-1,\ \theta>1).
\end{cases}
$$

In Case (C), Hardy gave formulae which were shewn to be valid when $x\sqrt{\sigma''}\leqq \rho < x\sigma'$. I have succeeded in proving that they are valid for

$$x\sqrt{\sigma''/\sigma'} < \rho < x\sigma',$$

thus extending the range of validity of the formulae considerably. It will be seen that this lower limit of ρ (namely $\rho\asymp x\sqrt{\sigma''/\sigma'}$) corresponds to our natural limit $\dfrac{1}{\lambda}$ of the order of the integrals $S(\lambda)$ and $C(\lambda)$.

Combining these results with Theorem C, we obtain the theorem:

The integrals $S(\lambda)$ *and* $C(\lambda)$ *are convergent when* $l(1/x) \prec \sigma \prec (1/x)^{\Delta}$ *and* $\rho \prec x\sigma'$. *The behaviour of these integrals, as* $\lambda \to \infty$, *is determined asymptotically as follows :*

If $x^{\Delta} \prec \rho \preceq x\sqrt{\sigma''/\sigma'}$,

$$S(\lambda) = O(1/\lambda), \qquad C(\lambda) = O(1/\lambda);$$

if $x\sqrt{\sigma''/\sigma'} \prec \rho \prec x\sigma'$,

$$(7) \qquad S(\lambda) \sim \frac{\rho(\theta)}{\theta\sqrt{\{2\sigma''(\theta)\}}} e^{(\beta-\frac{1}{4}\pi)i}\sqrt{\pi},$$

$$(8) \qquad C(\lambda) \sim \frac{\rho(\theta)}{\theta\sqrt{\{2\sigma''(\theta)\}}} e^{(\beta+\frac{1}{4}\pi)i}\sqrt{\pi},$$

where

$$\beta = \lambda\theta + \sigma(\theta)$$

and θ *is determined as a function of* λ *by the equation*

$$\sigma'(\theta) + \lambda = 0.$$

In the course of the proof of this theorem, we are led to the comparison of the order of magnitude of the functions

$$\varphi(a) = \frac{\rho(a)}{a\{\lambda - \sigma'(a)\}}, \qquad \psi(\theta) = \frac{\rho(\theta)}{\theta\sqrt{\{2\sigma''(\theta)\}}}$$

as $\lambda \to \infty$, when $x \prec \rho \prec x\sqrt{\sigma''}$ or $\rho = Ax\{1-\bar{\rho}(x)\}$, where $A>0, \bar{\rho}>0$ and $\bar{\rho}<1$, a and θ being functions of λ determined respectively by the equations

$$\frac{d}{da}\left[\frac{\rho(a)}{a\{\lambda-\sigma'(a)\}}\right] = 0, \qquad \sigma'(\theta)+\lambda = 0.$$

It will be proved that

$$\varphi(a) \prec \psi(\theta)$$

as $\lambda \to \infty$. The proof of this relation plays an important rôle in the discussion of Case (*C*).

The integral $S(\lambda)$ is still convergent when $x\sigma' \preceq \rho \prec \sigma'$. Hardy did not went into the discussion of this case, his method ceasing to be applicable in this case. I have succeeded in proving that formula (7) holds also in this case generally, the proof being left incomplete only in a few special cases.

4. In Part II, I will give applications of the results, obtained in Part I, to the determination of the behaviour of the coefficients a_n of a power series

$$\sum_{n=0}^{\infty} a_n z^n,$$

as $n \to \infty$, whose radius of convergence is unity and which represents a function $f(z)$ having, on the circle of convergence, one singular point only, at $z=1$.

As is well known, this problem was first systematically treated by Darboux*. Particularly he considered the case in which $f(z)$ has a singularity of the type

$$f(z) = \frac{\varphi(z)}{(1-z)^p},$$

$\varphi(z)$ being a function regular for $z=1$ and p denoting any real constant other than zero or a negative integer. His results were extended by Hamy† who considered the case in which $f(z)$ has a singularity of the type

$$f(z) = \frac{1}{(1-z)^p}\left(\log \frac{1}{1-z}\right)^q,$$

where q is a positive integer. These two authors did not attack the case in which $f(z)$ has an essential singularity for $z=1$. This was first done by Fejèr,‡ who considered the case where

$$f(z) = \frac{1}{(1-z)^p} e^{\frac{1}{z-1}},$$

and shewed that

$$a_n \sim \frac{1}{\sqrt{(e\pi)}} n^{-(\frac{3}{4}-\frac{1}{2}p)} \sin\{2\sqrt{n}+(\tfrac{3}{4}-\tfrac{1}{2}p)\pi\},$$

p being any real constant.

I have considered still more general cases in which the function $f(z)$ has a singularity of the following types:

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* *Journal de Math.* Série 3, *t.* 4 (1878) *pp.* 5—57, 377—417.

† " Série 6, *t.* 4 (1908) *pp.* 203—283.

‡ *Comptes Rendus*, 30 Nov., 1908; and "*Asymptotikus értékek megatározásáról*" (1909) Budapest.

$$\text{(i)} \quad f(z) = \frac{1}{(1-z)^p} e^{A/(1-z)^q},$$

$$\text{(ii)} \quad f(z) = \frac{1}{(1-z)^p} e^{A/(1-z)^q} \left(\log \frac{1}{1-z}\right)^r,$$

$$\text{(iii)} \quad f(z) = \frac{1}{(1-z)^p} e^{A/(1-z)^q} \left(l_1 \frac{1}{1-z}\right)^{r_1} \left(l_2 \frac{1}{1-z}\right)^{r_2} \cdots\cdots \left(l_h \frac{1}{1-z}\right)^{r_h},$$

A being a certain constant of the form $A = ae^{\alpha i}$.

My results in Case (i) are as follows:—

Let $a > 0$ *and* p *be any real constant, then the behaviour of* a_n *as* $n \to \infty$, *is determined asymptotically as follows:*

If $q = 1$, $\alpha = \pi$,

$$(9) \quad a_n \sim \frac{1}{\sqrt{\pi}} a^{-\frac{1}{2}p+\frac{1}{4}} e^{-\frac{1}{2}a} n^{\frac{1}{2}p-\frac{3}{4}} \sin \{2a^{\frac{1}{2}} n^{\frac{1}{2}} - (\tfrac{1}{2}p - \tfrac{3}{4})\pi\}^* ;$$

if $0 < q < 1$, $\alpha = (1+q)\frac{\pi}{2}$,

$$(10) \quad a_n \sim \frac{1}{\sqrt{\{2(1+q)\pi\}}} (qa)^{-\frac{p-\frac{1}{2}}{1+q}} n^{\frac{p-1-\frac{1}{2}q}{1+q}} \; exp\left[\{kn^{\frac{q}{1+q}} - (\tfrac{1}{2}p - \tfrac{1}{4})\pi\} i\right],$$

where
$$k = (1+q) q^{-\frac{q}{1+q}} a^{\frac{1}{1+q}} ;$$

if $0 < q < 1$, $\alpha = (3-q)\frac{\pi}{2}$,

$$(11) \quad a_n \sim \frac{1}{\sqrt{\{2(1+q)\pi\}}} (qa)^{-\frac{p-\frac{1}{2}}{1+q}} n^{\frac{p-1-\frac{1}{2}q}{1+q}} \; exp\left[-\{kn^{\frac{q}{1+q}} - (\tfrac{1}{2}p - \tfrac{1}{4})\pi\} i\right],$$

k being the same as that in the above formula.

Similar results were obtained in Case (ii) and Case (iii).

Now it may be remarked that, owing to the restricted applicability of the method, asymptotic formulae were obtained only in the following three cases:

$(1°) \quad q = 1, \qquad \alpha = \pi,$

$(2°) \quad 0 < q < 1, \quad \alpha = (1+q)\frac{\pi}{2},$

$(3°) \quad 0 < q < 1, \quad \alpha = (3-q)\frac{\pi}{2}.$

---

* Observe that this becomes Fejér's formula, if we put $a = 1$.

While I was working out this paper at Cambridge, Hardy, being struck with this fact, tried to attack the problem with a quite different method and obtained the following result*:

If $$f(z) = \frac{1}{(1-z)^p} e^{a(1-z)^q}, \qquad a>0,\ 0<q<1,$$

then

$$(12) \quad a_n \sim \frac{1}{\sqrt{\{2(1+q)\pi\}}} (qa)^{-\frac{p-\frac{1}{2}}{1+q}} n^{\frac{\frac{1}{2}q+p}{1+q}-1} exp\left\{ \frac{q+1}{q} (qa)^{\frac{1}{q+1}} n^{\frac{q}{q+1}} \right\}.$$

It is very probable that formula (12) also holds for complex values of a. If, in this formula, we replace a by

$$A = ae^{\alpha i},$$

where

$$\alpha = (1+q)\frac{\pi}{2} \quad \text{or} \quad (3-q)\frac{\pi}{2},$$

then we get (10) and (11). This shows that (12) holds also when a takes these special complex values.

PART I.†

Oscillating Dirichlet's Integrals

1. *Division of the Problem into three Cases.*

5. We have to consider the integrals

$$(1) \qquad S(\lambda) = \int_0^{\varepsilon} \rho(x)\, e^{i\sigma(x)} \frac{\sin \lambda x}{x} dx,$$

$$(2) \qquad C(\lambda) = \int_0^{\varepsilon} \rho(x)\, e^{i\sigma(x)} \frac{\cos \lambda x}{x} dx,$$

where $\rho(x)$ and $\sigma(x)$ are L-functions and $\sigma \succ 1$ as $x \to 0$. As was already mentioned, these integrals will be called "the sine-integral" and "the cosine-integral" respectively. It will be supposed that ε is a positive number so small that the range of integration does not

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* *Messenger of Math.* Vol. XLVI (1916) *pp.* 70–73.

† A preliminary notice of this Part appeared in the *Quarterly Journal*, Vol. XLVIII (1918) *pp.* 113–135.

include any point at which the subject of integration possesses any irrelevant discontinuity or other singularity. We distinguish as in Hardy's papers, the following three cases:

$$\text{(A)} \qquad \sigma \prec l(1/x),$$

$$\text{(B)} \qquad \sigma \asymp l(1/x),$$

$$\text{(C)} \qquad \sigma \succ l(1/x).$$

II. *Lemma for Case (A).*

6. The proofs of the theorems A and B of "O. D. I. 1." are principally carried out by means of H-lemma* 29. By examining the proof of this lemma, we can easily extend the range of validity of the formula given there.

In fact, the integrals

$$\int_0^\infty u^{-a\pm\delta-bi}\left(\frac{\sin u}{u}\right)^2 du,$$

there considered, are absolutely convergent also in the case $-1<a\leqq 0$. Hence the argument of "O. D. I. 1." for the case $0<a<1$ of this lemma holds also in the case $-1<a\leqq 0$. Thus we easily obtain the following modification of this lemma.

Lemma 1. *Let*

$$(13) \qquad J(\lambda)=\int_0^\tau x^{-a-bi}\,\phi(x)\left(\frac{\sin\frac{1}{2}\lambda x}{\frac{1}{2}\lambda x}\right)^2 \tfrac{1}{2}\lambda\,dx,$$

where $a\leqq 1$ *and*

$$\phi(x)=\theta(x)e^{i\psi(x)},$$

$$x^\delta < \theta <(1/x)^\delta, \qquad \psi(x)<l(1/x),$$

---

* The work of Mr. Hardy is chiefly included in the proofs of a great number of lemmas. Naturally, in my paper, these lemmas will be used freely, being referred as "H-lemma 1", "H-lemma 2", ..., in order to distinguish them from new lemmas which will be established here.

as $x \to 0$. *Then, if* $a \leqq -1$,

$$J(\lambda) = O(\lambda^{-1+\delta});$$

if $-1 < a < 1$,

$$(14) \qquad J(\lambda) \sim \Gamma(-1-a-bi) \sin\{\tfrac{1}{2}(a+bi)\pi\} \lambda^{a+bi} \Phi(1/\lambda),$$

except when $a=b=0$, *in which case the right-hand side of (14) is to be replaced by* $\frac{1}{2}\pi\,\Phi(1/\lambda)$; *if* $a=1$, $b \neq 0$, *this result (14) still holds, provided that the integral*

$$T(x) = \int_0^x t^{-1-bi}\,\Phi(t)\,dt$$

is convergent; and if $a=1$, $b=0$, *and* $T(x)$ *is still convergent,*

$$J(\lambda) \sim \tfrac{1}{2}\lambda\, T(1/\lambda).$$

III. *Discussion of Case (A)*: $\sigma(x) \prec l(1/x)$.

7. At first we shall consider the sine-integral

$$S(\lambda) = \int_0^\xi \rho(x) e^{i\sigma(x)} \frac{\sin \lambda x}{x}\,dx.$$

As is given in the paper "O. D. I. 1.", the necessary and sufficient condition for the convergence of this integral is

$$\rho(x) \prec \sigma'(x)$$

as $x \to 0$. As in the same paper, by performing integration by parts, we have

$$S(\lambda) = \frac{\rho(\xi)}{\xi} e^{i\sigma(\xi)} \frac{1-\cos \lambda\xi}{\lambda} + J(\lambda)$$

$$(15) \qquad = O(1/\lambda) + J(\lambda),$$

where

$$\begin{cases} J(\lambda) = \displaystyle\int_0^\xi (R_1 - iR_2) e^{i\sigma} \left(\frac{\sin \frac{1}{2}\lambda x}{\frac{1}{2}\lambda x}\right)^2 \tfrac{1}{2}\lambda\,dx, \\ R_1 = \rho - x\rho', \\ R_2 = x\rho\sigma'; \end{cases}$$

and hence

(16) $$J(\lambda) = J^{(1)}(\lambda) - iJ^{(2)}(\lambda),$$

where

(17) $$\begin{cases} J^{(1)}(\lambda) = \int_0^{\xi} R_1 e^{i\sigma}\left(\frac{\sin\frac{1}{2}\lambda x}{\frac{1}{2}\lambda x}\right)^2 \frac{1}{2}\lambda\, dx, \\ J^{(2)}(\lambda) = \int_0^{\xi} R_2 e^{i\sigma}\left(\frac{\sin\frac{1}{2}\lambda x}{\frac{1}{2}\lambda x}\right)^2 \frac{1}{2}\lambda\, dx. \end{cases}$$

Now we can write

$$\rho = x^{-a}\theta(x),$$

where $a \leqq 1$ and $x^{\delta} \prec \theta \prec (1/x)^{\delta}$ as $x \to 0$. Then, if $a \neq -1$,

$$\begin{cases} R_1 = \rho - x\rho' \sim (1+a)\, x^{-a}\theta(x) \quad \text{[by H-lemma 4]}, \\ R_2 = x\rho\sigma' = x^{-a}\bar{\theta}(x), \end{cases}$$

where $\bar{\theta}$ is a function of the same type as θ and $\bar{\theta} \prec \theta$ as $x \to 0$*; if $a = -1$,

$$R_1 = -x^2\theta' = -x\theta_1, \qquad R_2 = x\bar{\theta},$$

where $\theta_1 = x\theta'$ is a function of the same type as θ and $\theta_1 \prec \theta$. Hence we have:

(i) Let $a \leqq -1$. Then, applying Lemma 1 to the integrals (17), we obtain

$$J^{(1)}(\lambda) = O(\lambda^{-1+\delta}), \qquad J^{(2)}(\lambda) = O(\lambda^{-1+\delta});$$

hence, by (15) and (16),

$$S(\lambda) = O(\lambda^{-1+\delta}).$$

(ii) Let $-1 < a < 0$. Then, by Lemma 1, we obtain

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* Since $1 \prec \sigma \prec l(1/x)$, we easily see that σ is a function of the same type as Θ. It can also easily be proved that the function $x\Theta'$ is a function of the same type as Θ and a product of two functions of this type is also of the same type. Hence it follows that $\bar{\Theta}(x) = x\sigma'\Theta$ is a function of the type Θ; and since $x\sigma' \prec 1$, we have $\bar{\Theta} \prec \Theta$.

$$J^{(1)}(\lambda) \sim -\Gamma(-a)\sin(\tfrac{1}{2}a\pi)\,\lambda^a\Theta(1/\lambda)\,e^{i\sigma(1/\lambda)},$$
$$J^{(2)}(\lambda) \sim -\Gamma(-1-a)\sin(\tfrac{1}{2}a\pi)\,\lambda^a\,\bar{\Theta}(1/\lambda)\,e^{i\sigma(1/\lambda)};$$

and since $\bar{\Theta} \prec \Theta$, we have

$$J(\lambda) \sim J^{(1)}(\lambda),$$

and hence

$$S(\lambda) \sim -\Gamma(-a)\sin(\tfrac{1}{2}a\pi)\,\rho(1/\lambda)\,e^{i\sigma(1/\lambda)},$$

for, in this case, $\rho(1/\lambda) \succ \dfrac{1}{\lambda}$ as $\lambda \to \infty$.

(iii) Let $a = 0$ and $\Theta \prec 1$. Then

$$R_1 \sim \rho = \Theta, \qquad R_2 \prec \rho = \Theta.$$

Applying Lemma 1, we have also $J^{(2)} \prec J^{(1)}$ and

$$S(\lambda) \sim \tfrac{1}{2}\pi\rho(1/\lambda)\,e^{i\sigma(1/\lambda)}.$$

Combining these results with Theorem A, we obtain

Theorem I. *The integral*

$$S(\lambda) = \int_0^\xi \rho(x)\,e^{i\sigma(x)}\,\frac{\sin\lambda x}{x}\,dx,$$

where $1 \prec \sigma \prec l(1/x)$ and $\rho \prec \sigma'$, *is convergent. If* $\rho = x^{-a}\Theta(x)$, *where* $x^\delta \prec \Theta \prec (1/x)^\delta$, *so that* $a \leqq 1$, *the behaviour of* $S(\lambda)$, *as* $\lambda \to \infty$, *is determined asymptotically by the following formulae:*

$$(3)\quad \begin{cases} S(\lambda) = O(\lambda^{-1+\delta}) & (a \leqq -1), \\ S(\lambda) \sim -\Gamma(-a)\sin(\tfrac{1}{2}a\pi)\,\rho(1/\lambda)\,e^{i\sigma(1/\lambda)} & (-1 < a < 1), \\ S(\lambda) \sim \lambda\,T(1/\lambda) & (a = 1), \end{cases}$$

where

$$T(x) = \int_0^x \rho(t)\,e^{i\sigma(t)}\,dt.$$

In the particular case $a = 0$, *the factor* $-\Gamma(-a)\sin(\tfrac{1}{2}a\pi)$ *is to be replaced by its limiting value* $\tfrac{1}{2}\pi$.

8. Now we pass to the cosine-integral

$$C(\lambda)=\int_0^{\xi}\rho(x)e^{i\sigma(x)}\frac{\cos\lambda x}{x}\,dx.$$

The integral

$$\int_0^{\xi}\rho(x)e^{i\sigma(x)}\frac{dx}{x}$$

is convergent if, and only if,

$$(18)\qquad\qquad \rho\prec x\sigma'$$

as $x\to 0$. This involves $\rho<1$, as $\sigma'<\dfrac{1}{x}$. As the factor $\cos\lambda x$ of the subject of integration of $C(\lambda)$ is ultimately monotonic, this condition (18) is the necessary and sufficient condition for the convergence of the cosine-integral $C(\lambda)$.

Performing integration by parts, we obtain

$$C(\lambda)=\int_0^{\xi}\frac{\rho}{x}e^{i\sigma}\frac{d}{dx}\left(\frac{\sin\lambda x}{\lambda}\right)dx$$

$$(19)\qquad =\frac{\rho(\xi)}{\xi}e^{i\sigma(\xi)}\frac{\sin\lambda\xi}{\lambda}+J(\lambda)=O(1/\lambda)+J(\lambda),$$

where

$$(20)\qquad \begin{cases} J(\lambda)=\dfrac{1}{\lambda}\{J^{(1)}(\lambda)-i\,J^{(2)}(\lambda)\}, \\ J^{(1)}(\lambda)=\displaystyle\int_0^{\xi}R_1 e^{i\sigma}\frac{\sin\lambda x}{x}\,dx, \\ J^{(2)}(\lambda)=\displaystyle\int_0^{\xi}R_2 e^{i\sigma}\frac{\sin\lambda x}{x}\,dx, \\ R_1=\dfrac{\rho}{x}-\rho', \qquad R_2=\rho\sigma'. \end{cases}$$

The integrals $J^{(1)}$ and $J^{(2)}$ are of the type of $S(\lambda)$.

As before, we can write

$$\rho=x^{-\alpha}\,\theta(x),$$

where $a \leqq 0$ and $x^\delta < \theta < (1/x)^\delta$ as $x \to 0$.*

Then, if $a \leqq 0$ and $a \neq -1$,

$$R_1 = \frac{\rho}{x} - \rho' \sim (a+1)\, x^{-(a+1)}\theta = (a+1)\frac{\rho}{x};$$

if $a = -1$,

$$R_1 = -x\,\theta' = -\theta_1,$$

where θ_1 is a function of the same type as θ and $\theta_1 < \theta$. Hence applying Theorem I, we obtain

$$\begin{cases} J^{(1)}(\lambda) = o(1) & (a < -1), \\ J^{(1)}(\lambda) \sim -(\tfrac{1}{2}\pi/\lambda)\theta'(1/\lambda)\, e^{i\sigma(1/\lambda)} & (a = -1), \\ J^{(1)}(\lambda) \sim -(a+1)\,\Gamma(-a-1)\sin\{\tfrac{1}{2}(a+1)\pi\}\lambda\,\rho(1/\lambda)e^{i\sigma(1/\lambda)} & \\ & (-1 < a < 0), \\ J^{(1)}(\lambda) \sim \lambda\, T(1/\lambda) & (a = 0), \end{cases}$$

where

$$T(x) = \int_0^x \rho(t) e^{i\sigma(t)} \frac{dt}{t}.$$

Observing that $R_2 = \rho\sigma' = x^{-(a+1)}\bar{\theta}(x)$, where $\bar{\theta}$ is a function of the same type as θ and $\bar{\theta} < \theta$, similarly we obtain

$$\begin{cases} J^{(2)}(\lambda) = o(1) & (a < -1), \\ J^{(2)}(\lambda) \sim \tfrac{1}{2}\pi\rho(1/\lambda)\,\sigma'(1/\lambda)\, e^{i\sigma(1/\lambda)} & (a = -1), \\ J^{(2)}(\lambda) \sim -\Gamma(-a-1)\sin\{\tfrac{1}{2}(a+1)\pi\}\rho(1/\lambda)\sigma'(1/\lambda)e^{i\sigma(1/\lambda)} & (-1 < a < 0), \\ J^{(2)}(\lambda) \sim \lambda\bar{T}(1/\lambda) & (a = 0), \end{cases}$$

where

$$\bar{T}(x) = \int_0^x \bar{\theta}(t)\, e^{i\sigma(t)} \frac{dt}{t}.$$

Hence we obtain the following results:

* Observe that, when $a = 0$, $x^\delta \prec \Theta \prec x\sigma' \prec 1$.

(i) Let $a<-1$. Then, by (20),

$$J(\lambda)=\frac{1}{\lambda}\{o(1)-i\,o(1)\} = o(1/\lambda),$$

and hence, by (19),

$$C(\lambda) = O(1/\lambda).$$

(ii) Let $a = -1$. Then

$$J(\lambda) \sim -(\tfrac{1}{2}\pi/\lambda)\{(1/\lambda)\theta'(1/\lambda)+i\rho(1/\lambda)\,\sigma'(1/\lambda)\}e^{i\sigma(1/\lambda)}\,.$$

Hence, if $x\,\theta' \leq 1$ and $\rho\sigma' \leq 1$,

$$J(\lambda) \leq \frac{1}{\lambda},$$

and $$C(\lambda) = O(1/\lambda);$$

if $x\theta'$ or $\rho\sigma'>1$,

$$C(\lambda) \sim J(\lambda) \sim -(\tfrac{1}{2}\pi/\lambda)\{(1/\lambda)\theta'(1/\lambda)+i\rho(1/\lambda)\sigma'(1/\lambda)\}e^{i\sigma(1/\lambda)}.$$

(iii) Let $-1<a<0$. Since $\bar{\theta}<\theta$, we have

$$J(\lambda) \sim \frac{1}{\lambda}J^{(1)}(\lambda),$$

and hence

$$C(\lambda) \sim \Gamma(-a)\cos(\tfrac{1}{2}a\pi)\,\rho(1/\lambda)\,e^{i\sigma(1/\lambda)}\,.$$

(iv) Let $a=0$. Since $\bar{\theta}<\theta$, by H-lemma 10, we have

$$\bar{T}(1/\lambda) \prec T(1/\lambda),$$

and $$J^{(2)}(\lambda) \prec J^{(1)}(\lambda).$$

Hence $$C(\lambda) \sim T(1/\lambda).$$

We can now state

Theorem II. *The integral*

$$C(\lambda) =\int_0^{\xi} \rho(x)e^{i\sigma(x)}\,\frac{\cos\lambda x}{x}\,dx,$$

where $1 \prec \sigma \prec l(1/\lambda)$ *and* $\rho \prec x\sigma'$, *is convergent. If* $\rho = x^{-a}\Theta(x)$, *where* $x^{\delta} \prec \Theta \prec (1/\lambda)^{\delta}$, *so that* $a \leqq 0$, *the behaviour of* $C(\lambda)$, *as* $\lambda \to \infty$, *is determined asymptotically by the following formulae:**

$$(4)\quad \begin{cases} C(\lambda) = O(1/\lambda) & (a < -1 \text{ or } a = 1,\ x\Theta' \underline{\prec} 1,\ \rho\sigma' \underline{\prec} 1), \\ C(\lambda) \sim -(\tfrac{1}{2}\pi/\lambda)\{(1/\lambda)\Theta'(1/\lambda) + i\rho(1/\lambda)\sigma'(1/\lambda)\}e^{i\sigma(1/\lambda)} & \\ & (a = -1,\ x\Theta' \text{ or } \rho\sigma' \succ 1), \\ C(\lambda) \sim \Gamma(-a)\cos(\tfrac{1}{2}a\pi)\,\rho(1/\lambda)e^{i\sigma(1/\lambda)} & (-1 < a < 0), \\ C(\lambda) \sim T(1/\lambda) & (a = 0), \end{cases}$$

where

$$T(x) = \int_0^x \rho(t)e^{i\sigma(t)}\frac{dt}{t}.$$

IV. *Discussion of Case* (*B*) : $\sigma(x) \asymp l(1/\lambda)$.

9. In this case, we can write

$$(21)\qquad \sigma(x) = b\,l(1/x) + \bar{\sigma}(x),$$

where $b \neq 0$ and $\bar{\sigma} \prec l(1/x)$. Then

$$\rho e^{i\sigma} = x^{-a-bi}\,\Theta(x)e^{i\bar{\sigma}(x)}.$$

Hence, as Hardy remarks in the paper "*O. D. I. 1.*", the treatment of the sine-integral $S(\lambda)$ in Case (B) may be done by precisely the same method as in Case (A), by applying Lemma 1. Thus we can easily see:

If $a \leqq -1$,

$$S(\lambda) = O(\lambda^{-1+\delta});$$

if $- < a < 0$, or if $a = 0$ and $\rho \prec 1$,

$$S(\lambda) \sim -\Gamma(-a-bi)\sin\{\tfrac{1}{2}(a+bi)\pi\}\rho(1/\lambda)e^{i\bar{\sigma}(1/\lambda)}.$$

Combining these results with Theorem B, we obtain

* Observe that here always $C(\lambda) = o(1)$ as $\lambda \to \infty$.

Theorem III. *The integral*

$$S(\lambda) = \int_0^{\xi} \rho(x) e^{i\sigma(x)} \frac{\sin \lambda x}{x}\, dx,$$

where $\sigma \sim b\,l(1/x)$ $(b \neq 0)$ *and* $\rho \prec 1/x$, *is convergent. If* $\rho = x^{-a}\,\theta(x)$, *where* $x^{\delta} \prec \theta \prec (1/x)^{\delta}$, *so that* $a \leqq 1$, *the behaviour of* $S(\lambda)$, *as* $\lambda \to \infty$, *is determined asymptotically by the following formulae:*

$$(5)\quad \begin{cases} S(\lambda) = O(\lambda^{-1-\delta}) & (a \leqq -1), \\ S(\lambda) \sim -\Gamma(-a-bi) \sin\{\tfrac{1}{2}(a+bi)\pi\}\, \rho(1/\lambda) e^{i\sigma(1/\lambda)} & (-1 < a \leqq 1). \end{cases}$$

10. We now consider the cosine-integral

$$C(\lambda) = \int_0^{\xi} \rho(x) e^{i\sigma(x)} \frac{\cos \lambda x}{x}\, dx.$$

Since the function σ has the form (21), the condition for the convergence of this integral $C(\lambda)$ is

$$\rho \prec 1$$

as $x \to 0$.

As in Case (A), we have

$$C(\lambda) = O(1/\lambda) + J(\lambda),$$

where

$$\begin{cases} J(\lambda) = \dfrac{1}{\lambda} \displaystyle\int_0^{\xi} (R_1 - i\,R_2) e^{i\sigma} \frac{\sin \lambda x}{x}\, dx, \\ R_1 = \dfrac{\rho}{x} - \rho', \quad R_2 = \rho\sigma'. \end{cases}$$

If we write, as before,

$$\rho = x^{-a}\,\theta(x),$$

where $\qquad a \leqq 0, \qquad x^{\delta} \prec \theta \prec (1/x)^{\delta}$,*

we have

$$R_1 - i\,R_2 \sim (a+1+bi)\, x^{-(a+1)}\, \theta(x).$$

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* Observe that, when $a = 0$, $x^{\delta} \prec \Theta \prec 1$.

Hence, applying Theorem III, we obtain:
If $a<-1$, or if $a=-1$ and $\theta\leq 1$,

$$J(\lambda)=O\,(1/\lambda);$$

if $a=-1$ and $\theta>1$, or if $-1<a\leqq 0$,

$$J(\lambda)\sim -(a+1+bi)\,\Gamma(-a-1-bi)\sin\{\tfrac{1}{2}(a+1+bi)\pi\}\,\rho(1/\lambda)e^{i\sigma(1/\lambda)}.$$

Therefore we have the theorem.

Theorem IV. *The integral*

$$C(\lambda)=\int_0^{\xi}\rho(x)\,e^{i\sigma(x)}\,\frac{\cos\lambda x}{x}\,dx,$$

where $\sigma\sim bl(1/x)$ $(b\neq 0)$ *and* $\rho<1$, *is convergent. If* $\rho=x^{-a}\theta(x)$, *where* $x^{\delta}<\theta<(1/x)^{\delta}$, *so that* $a\leqq 0$, *the behaviour of* $C(\lambda)$, *as* $\lambda\to\infty$, *is determined asymptotically by the following formulae:**

$$(6)\begin{cases} C(\lambda)=O(1/\lambda) & (a<-1 \text{ or } a=-1,\ \theta\leq 1), \\ C(\lambda)\sim\Gamma(-a-bi)\cos\{\tfrac{1}{2}(a+bi)\pi\}\,\rho(1/\lambda)e^{i\sigma(1/\lambda)} & \\ & (-1<a\leqq 0 \text{ or } a=-1,\ \theta>1). \end{cases}$$

V. *Examples† of the Cases (A) and (B).*

11. As examples of the two cases (A) and (B), we shall give some discussion about the behaviour of the integral

$$J(\lambda)=\int_0^1 e^{i\lambda x}\,x^{r-1}\left(\log\frac{1}{x}\right)^{s-1}dx$$

as $\lambda\to\infty$, where

$$R(r)>0,\quad R(s)>0.$$

At first, we consider the case in which

* Observe that, here also always $C(\lambda)=o(1)$ as $\lambda\to\infty$.

† In the followings, I have given examples and verifications, quite similar to those given in Hardy's papers, for the purpose of parallelism.

$$(22)\qquad \begin{cases} r = -a, & -1<a<0, \\ s = a+\text{m}i, & 0<a<1, \qquad \text{m}\neq 0. \end{cases}$$

Then

$$J(\lambda) = \int_0^1 e^{i\lambda x}\, x^{-a-1}\left(\log\frac{1}{x}\right)^{\alpha-1} e^{\text{m}i\log\log(1/x)}dx$$

$$= I(\lambda)+i\,\bar{I}(\lambda),$$

where

$$I(\lambda) = \int_0^1 x^{-a}\left(\log\frac{1}{x}\right)^{\alpha-1} e^{\text{m}i\log\log(1/x)}\frac{\cos\lambda x}{x}\,dx,$$

$$\bar{I}(\lambda) = \int_0^1 x^{-a}\left(\log\frac{1}{x}\right)^{\alpha-1} e^{\text{m}i\log\log(1/x)}\frac{\sin\lambda x}{x}\,dx.$$

Now

$$I(\lambda) = \left(\int_0^{\frac{1}{2}}+\int_{\frac{1}{2}}^1\right)x^{-a}\left(\log\frac{1}{x}\right)^{\alpha-1} e^{\text{m}i\log\log(1/x)}\frac{\cos\lambda x}{x}\,dx$$

$$= I_1(\lambda)+I_2(\lambda)$$

say. Evidently the integral $I_1(\lambda)$ is convergent, if $a<0$. The integral $I_2(\lambda)$ may be written in the form

$$I_2(\lambda) = \int_0^{\frac{1}{2}}(1-x)^{-a-1}\left(\log\frac{1}{1-x}\right)^{\alpha-1} e^{\text{m}i\log\log\{1/(1-x)\}}\cos\lambda(1-x)dx,$$

and, when x is small, we have

$$\log\frac{1}{1-x} = x\{1+O(x)\}.$$

Hence the integral $I_2(\lambda)$ is convergent if $a>0$.

The integral $I_1(\lambda)$ may be divided into the two parts

$$I_1(\lambda) = \left(\int_0^{\xi}+\int_{\xi}^{\frac{1}{2}}\right)x^{-a}\left(\log\frac{1}{x}\right)^{\alpha-1} e^{\text{m}i\log\log(1/x)}\frac{\cos\lambda x}{x}\,dx$$

$$= I_1'(\lambda)+I_1''(\lambda)$$

say, ξ being a sufficiently small positive number. Then

$$I_1'(\lambda) = \int_0^{\xi}\rho(x)e^{i\theta(x)}\frac{\cos\lambda x}{x}dx,$$

where $\rho = x^{-a}\left(\log\frac{1}{x}\right)^{\kappa-1}, \quad \sigma = m \log\log(1/x).$

Applying Theorem II, we obtain

$$I_1'(\lambda) \sim \Gamma(-a)\cos(\tfrac{1}{2}a\pi)\,\lambda^a(\log\lambda)^{\kappa-1}e^{mi\log\log\lambda} \qquad (-1<a<0).$$

If we put $f(x) = x^{-a-1}\left(\log\frac{1}{x}\right)^{\kappa-1} e^{mi\log\log(1/x)},$

then, by performing integration by parts, we have

$$I_1''(\lambda) = O(1/\lambda) - \frac{1}{\lambda}\int_\xi^{\frac{1}{2}} f'(x)\sin\lambda x\,dx.$$

Evidently $f'(x)$ has no singularity and is absolutely integrable in the interval $(\xi, \frac{1}{2})$. Hence by a well known theorem*

$$\int_\xi^{\frac{1}{2}} f'(x)\sin\lambda x\,dx = o(1)$$

as $\lambda \to \infty$. Hence we have

$$I_1''(\lambda) = O(1/\lambda).$$

Since $-1<a<0$, we have $I_1' > I_1''$ as $\lambda \to \infty$.
Thus we obtain

$$\begin{aligned} I_1(\lambda) &= \Gamma(-a)\cos(\tfrac{1}{2}a\pi)\lambda^a(\log\lambda)^{\kappa-1}e^{mi\log\log\lambda}(1+\varepsilon_\lambda) \\ &= \Gamma(r)\cos(\tfrac{1}{2}r\pi)\,\lambda^{-r}(\log\lambda)^{s-1}(1+\varepsilon_\lambda), \end{aligned}$$

where $\lim_{\lambda\to\infty} \varepsilon_\lambda = 0.$

The integral $I_2(\lambda)$ may also be divided into the two parts

$$\begin{aligned} I_2(\lambda) &= \left(\int_0^\xi + \int_\xi^{\frac{1}{2}}\right)(1-x)^{-a-1}\left(\log\frac{1}{1-x}\right)^{\kappa-1} e^{mi\log\log\{1/(1-x)\}}\cos\lambda(1-x)\,dx \\ &= I_2'(\lambda) + I_2''(\lambda) \end{aligned}$$

say. As in the case of $I_1''(\lambda)$, we easily see that

$$I_2''(\lambda) = O(1/\lambda).$$

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* Hobson, Theory of Functions of a Real Variable, p. 672.

In considering the integral $I_2'(\lambda)$, introduce the relations

$$\log\frac{1}{1-x} = x\{1+O(x)\},$$
$$(1-x)^{-a-1} = 1+O(x),$$
$$e^{\text{mi}\log\{1+O(x)\}} = 1+O(x).$$

Then we have

$$I_2'(\lambda) = \int_0^{\varepsilon} x^a e^{-\text{mi}\log(1/x)}\{1+O(x)\}\frac{\cos\lambda(1-x)}{x}\,dx$$
$$= \cos\lambda\int_0^{\varepsilon} x^a e^{-\text{mi}\log(1/x)}\{1+O(x)\}\frac{\cos\lambda x}{x}\,dx$$
$$+\sin\lambda\int_0^{\varepsilon} x^a e^{-\text{mi}\log(1/x)}\{1+O(x)\}\frac{\sin\lambda x}{x}\,dx$$
$$= j(\lambda)+\bar{j}(\lambda)$$

say. Then, by Theorems III and IV, we have, for $0<a<1$,

$$\int_0^{\varepsilon} x^a e^{-\text{mi}\log(1/x)}\frac{\cos\lambda x}{x}\,dx \sim \Gamma(a+\text{mi})\cos\{\tfrac{1}{2}(a+\text{mi})\pi\}\,\lambda^{-a}e^{-\text{mi}\log\lambda},$$
$$\int_0^{\varepsilon} x^a e^{-\text{mi}\log(1/x)}\frac{\sin\lambda x}{x}\,dx \sim \Gamma(a+\text{mi})\sin\{\tfrac{1}{2}(a+\text{mi})\pi\}\,\lambda^{-a}e^{-\text{mi}\log\lambda}.$$

Hence, we can easily see that

$$j(\lambda) \sim \cos\lambda\,\Gamma(a+\text{mi})\cos\{\tfrac{1}{2}(a+\text{mi})\pi\}\,\lambda^{-a}e^{-\text{mi}\log\lambda},$$
$$\bar{j}(\lambda) \sim \sin\lambda\,\Gamma(a+\text{mi})\sin\{\tfrac{1}{2}(a+\text{mi})\pi\}\,\lambda^{-a}e^{-\text{mi}\log\lambda},$$

and

$$I_2'(\lambda) \sim \Gamma(a+\text{mi})\cos\{\lambda-\tfrac{1}{2}(a+\text{mi})\pi\}\,\lambda^{-a}e^{-\text{mi}\log\lambda}.$$

Since $0<a<1$, evidently we have $I_2' > I_2''$ as $\lambda\to\infty$. Thus we obtain

$$I_2(\lambda) = \Gamma(a+\text{mi})\cos\{\lambda-\tfrac{1}{2}(a+\text{mi})\pi\}\,\lambda^{-a}e^{-\text{mi}\log\lambda}(1+\varepsilon_\lambda')$$
$$= \Gamma(s)\cos(\lambda-\tfrac{1}{2}s\pi)\,\lambda^{-s}(1+\varepsilon'_\lambda),$$

where
$$\lim_{\lambda\to\infty}\varepsilon_\lambda' = 0.$$

Hence we have

$$I(\lambda) = \Gamma(r)\cos\left(\tfrac{1}{2}r\pi\right)\lambda^{-r}(\log\lambda)^{s-1}(1+\varepsilon_\lambda)$$
$$+\Gamma(s)\cos\left(\lambda-\tfrac{1}{2}s\pi\right)\lambda^{-s}(1+\varepsilon_\lambda').$$

Similarly we can prove that

$$\bar{I}(\lambda) = \Gamma(r)\sin\left(\tfrac{1}{2}r\pi\right)\lambda^{-r}(\log\lambda)^{s-1}(1+\varepsilon_\lambda'')$$
$$+\Gamma(s)\sin\left(\lambda-\tfrac{1}{2}s\pi\right)\lambda^{-s}(1+\varepsilon_\lambda''').$$

Thus we obtain

$$J(\lambda) = \int_0^1 e^{i\lambda x}\, x^{r-1}\left(\log\frac{1}{x}\right)^{s-1} dx$$
$$= \Gamma(r)\, e^{\frac{1}{2}r\pi i}\lambda^{-r}(\log\lambda)^{s-1}(1+\varepsilon)$$
$$+\Gamma(s)\,\lambda^{-s}\, e^{(\lambda-\frac{1}{2}s\pi)i}(1+\varepsilon'),$$

where

$$\lim_{\lambda\to\infty}\varepsilon = 0,\quad \lim_{\lambda\to\infty}\varepsilon' = 0,$$

r and s having the values of (22).

12. This result may be verified as follows.

Hardy proved* that, if $R(r)>0$ and $R(s)>0$, then, for pure imaginary values of t, we have

$$f_{r,s}(t) = \Gamma(s)\sum_{\nu=0}^{\infty}\frac{t^\nu}{(\nu+r)^s\,\nu!} = \int_0^1 e^{tx}x^{r-1}\left(\log\frac{1}{x}\right)^{s-1}dx$$
$$= \Gamma(r)(-t)^{-r}\{\log(-t)\}^{s-1}(1+\varepsilon_t)$$
$$+\Gamma(s)\,t^{-s}e^t(1+\varepsilon'_t),$$

where

$$(-t)^{-r} = \exp\{-r\log(-t)\} = \exp[-r\{\log|t| - \tfrac{1}{2}\varepsilon\pi i\}],$$
$$t^{-s} = \exp\{-s\log t\} = \exp[-s\{\log|t| + \tfrac{1}{2}\varepsilon\pi i\}],$$

Proc. London Math. Soc. Ser. 2, Vol. 2, pp. 401 et seq.

and $\varepsilon=+1$ or $\varepsilon=-1$ according as $\frac{t}{i}>0$ or $\frac{t}{i}<0$.

Herein put $\quad t=\lambda i \qquad (\lambda>0),$

then $$(-t)^{-r}=e^{\frac{1}{2}r\pi i}\lambda^{-r}, \qquad t^{-s}=e^{-\frac{1}{2}s\pi i}\lambda^{-s}.$$

Therefore

$$(23)\quad \int_0^1 e^{i\lambda x}x^{r-1}\left(\log\frac{1}{x}\right)^{s-1}dx = \Gamma(r)e^{\frac{1}{2}r\pi i}\lambda^{-r}(\log\lambda)^{s-1}(1+\varepsilon)$$
$$+\Gamma(s)\lambda^{-s}e^{(\lambda-\frac{1}{2}s\pi)i}(1+\varepsilon'),$$

where $$\lim_{\lambda\to\infty}\varepsilon=0, \qquad \lim_{\lambda\to\infty}\varepsilon'=0.$$

This formula quite agrees with our result obtained for the case in which

$$r=-a, \quad -1<a<0\,; \quad s=\alpha+mi, \quad 0<\alpha<1, \quad m\neq 0.$$

Thus our result is verified.

13. Next consider the case in which

$$\begin{cases} r=-a-bi, & -1<a<0, \quad b\neq 0, \\ s=\alpha, & 0<\alpha<1. \end{cases}$$

In this case

$$J(\lambda)=\int_0^1 e^{i\lambda x}x^{-a-1}\left(\log\frac{1}{x}\right)^{\alpha-1}e^{bi\log(1/x)}dx$$
$$=\int_0^{\frac{1}{2}}+\int_{\frac{1}{2}}^1=J_1(\lambda)+J_2(\lambda)$$

say. Then

$$J_1(\lambda)=\int_0^{\frac{1}{2}}x^{-a}\left(\log\frac{1}{x}\right)^{\alpha-1}e^{bi\log(1/x)}\frac{\cos\lambda x}{x}dx$$
$$+i\int_0^{\frac{1}{2}}x^{-a}\left(\log\frac{1}{x}\right)^{\alpha-1}e^{bi\log(1/x)}\frac{\sin\lambda x}{x}dx.$$

Applying Theorems III and IV, we can easily see that

$$J_1(\lambda) = \Gamma(-a-bi)\, e^{-\frac{1}{2}(a+bi)\pi i}\, \lambda^{a+bi}\, (\log \lambda)^{\alpha-1}\, (1+\varepsilon)$$
$$= \Gamma(r) e^{\frac{1}{2}r\pi i} \lambda^{-r} (\log \lambda)^{s-1} (1+\varepsilon),$$

where $\lim\limits_{\lambda\to\infty} \varepsilon = 0.$

The integral $J_2(\lambda)$ may be written in the form

$$J_2(\lambda) = \int_0^{\frac{1}{2}} e^{i\lambda(1-x)}\, (1-x)^{-a-1} \left(\log \frac{1}{1-x}\right)^{\alpha-1} e^{bi \log\{1/(1-x)\}}\, dx.$$

If we make use of the equations

$$\left\{\begin{aligned} &\log \frac{1}{1-x} = x\{1+O(x)\}, \quad (1-x)^{-a-1} = 1+O(x), \\ &e^{bi \log\{1/(1-x)\}} = 1+O(x), \\ &\int_0^{\infty} x^{\alpha-1} \cos \lambda x\, dx = \Gamma(\alpha)\lambda^{-\alpha} \cos \tfrac{1}{2}\alpha\pi \\ &\int_0^{\infty} x^{\alpha-1} \sin \lambda x\, dx = \Gamma(\alpha)\lambda^{-\alpha} \sin \tfrac{1}{2}\alpha\pi \end{aligned}\right. \qquad (0<\alpha<1),$$

we can easily prove that

$$J_2(\lambda) = \Gamma(\alpha)\, \lambda^{-\alpha}\, e^{(\lambda-\frac{1}{2}\alpha\pi)i}(1+\varepsilon')$$
$$= \Gamma(s)\, \lambda^{-s} e^{(\lambda-\frac{1}{2}s\pi)i}(1+\varepsilon'),$$

where $\lim\limits_{\lambda\to\infty} \varepsilon' = 0.$

Thus we obtain

$$J(\lambda) = J_1(\lambda)+J_2(\lambda) = \Gamma(r) e^{\frac{1}{2}r\pi i} \lambda^{-r} (\log \lambda)^{s-1}(1+\varepsilon)$$
$$+ \Gamma(s)\, \lambda^{-s}\, e^{(\lambda-\frac{1}{2}s\pi)i}(1+\varepsilon'),$$

where $\lim\limits_{\lambda\to\infty} \varepsilon = 0, \qquad \lim\limits_{\lambda\to\infty} \varepsilon' = 0.$

Here again we have obtained the result which quite agrees with the formula (23) for $f_{r,s}(\lambda i)$.

14. Finally consider the case in which

$$(24)\qquad \begin{cases} r=-a-bi, & -1<a<0, \quad b\neq 0, \\ s=\alpha+mi, & 0<\alpha<1, \quad m\neq 0. \end{cases}$$

In this case we have

$$J(\lambda)=\int_0^1 e^{i\lambda x}x^{-r-1}\left(\log\frac{1}{x}\right)^{\alpha-1}e^{\{b\log(1/x)+m\log\log(1/x)\}i}dx$$

$$=\int_0^{\frac{1}{2}}+\int_{\frac{1}{2}}^1=J_1(\lambda)+J_2(\lambda)$$

say. Then, proceeding as before, we easily see that the discussion of the integral $J_1(\lambda)$ may be carried out by means of the integrals

$$\int_0^{\xi}x^{-a}\left(\log\frac{1}{x}\right)^{\alpha-1}e^{\{b\log(1/x)+m\log\log(1/x)\}i}\frac{\sin\lambda x}{x}dx,$$

$$\int_0^{\xi}x^{-a}\left(\log\frac{1}{x}\right)^{\alpha-1}e^{\{b\log(1/x)+m\log\log(1/x)\}i}\frac{\cos\lambda x}{x}dx,$$

and that of $J_2(\lambda)$ by means of the integrals

$$\int_0^{\xi}x^{\alpha}e^{-mi\log(1/x)}\frac{\sin\lambda x}{x}dx,$$

$$\int_0^{\xi}x^{\alpha}e^{-mi\log(1/x)}\frac{\cos\lambda x}{x}dx.$$

Thus by another application of Theorems III and IV, we obtain

$$J(\lambda)=\Gamma(r)e^{\frac{1}{2}r\pi i}\lambda^{-r}(\log\lambda)^{s-1}(1+\varepsilon)$$
$$+\Gamma(s)\lambda^{-s}e^{(\lambda-\frac{1}{2}s\pi)i}(1+\varepsilon'),$$

where $\qquad \lim_{\lambda\to\infty}\varepsilon=0, \qquad \lim_{\lambda\to\infty}\varepsilon'=0,$

r and s having the values of (24).

This result is nothing but the formula (23) for the case (24).

IV. *Lemmas for Case (C).*

15. Among the lemmas given in the paper "*O. D. I. 2.*", the most important ones are H-lemmas 32 and 33. They give important properties concerning the variations of the functions

$$\frac{\rho(x)}{x\{\lambda-\sigma'(x)\}}, \qquad \frac{\rho(x)}{x\{\lambda+\sigma'(x)\}}$$

for sufficiently large values of λ, provided that

$$\sigma \succ l(1/x),$$

and

$$x \prec \rho \prec x\sigma'.^{*}$$

I will give two more lemmas of a similar nature, concerning the variations of these two functions, for the case in which

$$\sigma \succ l(1/x),$$

and

$$x^{\Delta} \prec \rho \preceq x.$$

16. Lemma 2. *Let* $\qquad \sigma \succ l(1/x).$

(i) *If* $\rho \prec x$, *or if* $\qquad \rho = Ax\{1+\bar{\rho}(x)\}$,

where A *is a positive constant and*

$$\bar{\rho} \succeq 0, \quad \bar{\rho} \prec 1,$$

then the function

$$\varphi = \frac{\rho}{x(\lambda-\sigma')}$$

is a steadily increasing function of x *throughout the interval* $0<x<\xi$.

(ii) *If* $\qquad \rho = Ax\{1-\bar{\rho}(x)\}$,

where $A>0$, $\bar{\rho}>0$ *and* $\bar{\rho}\prec 1$, *then the function* φ *has, for sufficiently large fixed values of* λ, *one and only one stationary value in the range* $0<x<\xi$, *which is a maximum and tends to zero as* $\lambda \to \infty$.

Proof. In the case (i), $\frac{\rho}{x}$ is evidently a steadily increasing function of x (or a constant when $\rho=0$) in the interval $(0, \xi)$ and so also is the function

* In the following investigation of Case (C), we shall assume that

$$\rho > 0, \qquad \sigma \succeq 0.$$

We can easily see that, by this assumption, no loss of generality will be introduced.

$$\frac{1}{\lambda-\sigma'(x)},$$

since $\sigma'<0$, $\sigma'>1$. Hence the first part of the lemma follows immediately.

Now consider the second case (ii) of the lemma, in which

$$\rho = Ax\{1-\rho(x)\},$$

where $$A>0, \qquad \rho>0, \qquad \bar{\rho}<1.$$

Then we have

$$\varphi = \frac{A(1-\bar{\rho})}{\lambda-\sigma'},$$

and $\frac{d\varphi}{dx}=0$ gives

$$\lambda = \frac{\sigma''(1-\rho)}{\rho'}+\sigma'. \tag{25}$$

Let us write $$\bar{\rho} = \sigma'\gamma,$$

so that $$\gamma<0, \qquad \sigma'\gamma<1.$$

Then $$\frac{\sigma''(1-\rho)}{\bar{\rho}'}+\sigma' = \frac{\sigma''+\sigma'^2\gamma'}{\rho'};$$

and from the relation $\sigma'\gamma\prec 1$, we obtain $\gamma\prec\frac{1}{\sigma'}$ and, by differentiation,

$$\sigma'^2\gamma'<\sigma''.$$

Hence (25) becomes

$$\lambda = \frac{\sigma''+\sigma'^2\gamma'}{\rho'} \sim \frac{\sigma''}{\rho'}. \tag{26}$$

Here we have

$$\sigma''\succ\frac{1}{x^2},$$

since $\sigma\succ l(1/x)$; and, since $\rho\prec 1$, we have $x\rho\prec x$ and, by differentiation,

$$x\rho'+\rho<1.$$

But $\rho>0, \quad \rho<1, \quad \rho'>0,$

and hence $x\rho'<1.$

Therefore $$\frac{\sigma''}{\rho'}>\frac{1}{x^2\rho'}>\frac{1}{x}>1,$$

whence it follows that, for sufficiently large fixed values of λ, the equation (25) has one, and only one, root. Thus the function φ has one stationary value; and as φ is positive and $\varphi<1$, this value is plainly a maximum.

If the root of the equation (25) is $x=a$, then the value of $\varphi(a)$ is given by

$$(27) \qquad \varphi(a)=\frac{A\rho'(a)}{\sigma''(a)}=\frac{\rho(a)-a\rho'(a)}{a^2\sigma''(a)}.$$

For $$\varphi(a)=\frac{\rho(a)}{a\{\lambda-\sigma'(a)\}}=\frac{A\{1-\rho(a)\}}{\lambda-\sigma'(a)},$$

and, by (25), $$\lambda-\sigma'(a)=\frac{\sigma''(a)\{1-\rho(a)\}}{\rho'(a)}.$$

Hence $$\varphi(a)=\frac{A\rho'(a)}{\sigma''(a)};$$

and $$A\rho'(x)=-\frac{d}{dx}\left(\frac{\rho}{x}\right)=\frac{\rho-x\rho'}{x^2},$$

which proves the equation (27).

Since $\frac{\sigma''}{\rho'}>1$, we see that $\varphi(a)$ tends to zero as $\lambda\to\infty$

The proof of the lemma is thus completed.

17. Lemma 3. *Let* $\sigma>l(1/x)$.

(i) *If* $\rho<x$ *or if* $\rho=Ax\{1+\bar{\rho}(x)\}$,

where A *is a positive constant and*

$$\rho>0, \qquad \rho<1,$$

then the function

$$\varphi=\frac{\rho}{x(\lambda+\sigma')}$$

has, for sufficiently large fixed values of λ, *one and only one stationary value in the range* $0<x<\xi$, *which is a minimum and tends to zero as* $\lambda\to\infty$. *The value* α *of* x *corresponding to this minimum is greater than the value* θ *of* x *for which the function* φ *becomes infinite,* θ *being given by the equation* $-\sigma'(\theta)=\lambda$; *so that the function* φ *is continuous, except for this value* $x=\theta$, *and is a steadily decreasing function of* x *in the interval* $0<x<\alpha$ *and a steadily increasing function in the interval* $\alpha<x<\xi$.

(ii) *If*
$$\rho = Ax\{1-\rho(x)\},$$
where $A>0$, $\rho\geq 0$ *and* $\overline{\rho}<1$, *then the function* φ *has no stationary value in the interval* $0<x<\xi$. *It becomes infinite for one value* θ *of* x, *given by* $-\sigma'(\theta)=\lambda$, *and otherwise it is continuous and is a steadily decreasing function of* x *throughout the interval* $0<x<\xi$.

Proof. We observe that
$$\sigma'<0,$$
and hence
$$\varphi>0,$$
if $x>\theta$, θ being the root of the equation $\lambda+\sigma'(\theta)=0$.

(i) At first, consider the case in which
$$\rho < x,$$
and write
$$\rho = x\gamma,$$
so that
$$\gamma > 0, \quad \gamma < 1.$$
Then
$$\varphi = \frac{\gamma}{\lambda+\sigma'},$$
and $\dfrac{d\varphi}{dx}=0$ gives

$$\lambda = \frac{\sigma''\gamma}{\gamma'}-\sigma'. \qquad (28)$$

Now
$$\sigma'<0, \quad \sigma''>0, \quad \gamma>0, \quad \gamma'>0, \quad \frac{\sigma''\gamma}{\gamma'}>0,$$
so that we have

$$\frac{\sigma''\gamma}{\gamma'} - \sigma' > -\sigma' > 1,$$

whence it follows that, for sufficiently large fixed values of λ, the equation (28) has one, and only one, root α. Thus the function φ has one stationary value.

By (28), we have

$$\lambda + \sigma'(\alpha) = \frac{\sigma''(\alpha)\,\gamma(\alpha)}{\gamma'(\alpha)} > 0,$$

which proves that $$\alpha > 0.$$

Also we have

$$\varphi(\alpha) = \frac{\gamma'(\alpha)}{\sigma''(\alpha)} = -\frac{\rho(\alpha) - \alpha\,\rho'(\alpha)}{\alpha'^2\,\sigma''(\alpha)}. \tag{29}$$

Since $\gamma < 1$, we have $$x\gamma' < 1,$$

and, from the property of σ, $$\sigma'' > \frac{1}{x^2},$$

whence it follows that $$\frac{\gamma'(x)}{\sigma''(x)} < x^2\gamma'(x) < x < 1,$$

so that $\varphi(\alpha)$ tends to zero as $\lambda \to \infty$.

Next consider the case in which

$$\rho = Ax\,\{1 + \bar{\rho}(x)\},$$

where $$A > 0, \quad \bar{\rho} > 0, \quad \bar{\rho} < 1.$$

Then $$\varphi = \frac{A(1+\bar{\rho})}{\lambda + \sigma'},$$

and $\dfrac{d\varphi}{dx} = 0$ gives

$$\lambda = \frac{\sigma''(1+\bar{\rho})}{\bar{\rho}'} - \sigma'. \tag{30}$$

Now $$\sigma' < 0, \quad \sigma'' > 0, \quad \bar{\rho} > 0, \quad \bar{\rho}' > 0,$$

so that we have

$$\frac{\sigma''(1+\rho)}{\rho'} - \sigma' > -\sigma' > 1,$$

whence it follows that, for sufficiently large fixed values of λ, the equation (30) has one, and only one, root α. Thus the function φ has one stationary value.

By (30), we have

$$\lambda + \sigma'(\alpha) = \frac{\sigma''(\alpha)\{1+\rho(\alpha)\}}{\rho'(\alpha)} > 0,$$

which proves that $\alpha > \theta$.

Also we have

(31) $$\varphi(\alpha) = \frac{A\rho'(\alpha)}{\sigma''(\alpha)} = -\frac{\rho(\alpha)-\alpha\rho'(\alpha)}{\alpha^2\sigma''(\alpha)}.$$

Easily we see that

$$\frac{\rho'}{\sigma''} < 1,$$

so that $\varphi(\alpha)$ tends to zero as $\lambda \to \infty$.

(ii) Let $\rho = Ax\{1-\rho(x)\}$,

where $A > 0, \quad \rho > 0, \quad \rho < 1.$

Then $$\varphi = \frac{A(1-\rho)}{\lambda+\sigma'}.$$

and $\dfrac{d\varphi}{dx} = 0$ gives

$$\lambda = -\frac{\sigma''(1-\rho)+\sigma'\rho'}{\rho'}.$$

If we write $\rho = \sigma'\gamma,$

so that $\gamma < 0, \quad \sigma'\gamma < 1,$

then we have

$$\lambda = -\frac{\sigma'' + \sigma'^2\gamma'}{\rho'}. \tag{32}$$

From the relation $\sigma'\gamma < 1$, we obtain

$$\sigma'' > \sigma'^2\gamma'.$$

Hence
$$\frac{\sigma'' + \sigma'^2\gamma'}{\rho'} \sim \frac{\sigma''}{\rho'} > 0,$$

since $\sigma'' > 0$ and $\rho' > 0$. Therefore the right-hand side of (32) is ultimately negative and hence there is no stationary value of φ.

If $\rho = 0$, then
$$\varphi = \frac{A}{\lambda + \sigma'},$$

and evidently there is no stationary value of φ.

We easily see that, as $x \to 0$, φ tends to zero by negative values, and that, as $x \to \theta$ from below, $\varphi \to -\infty$ and, as $x \to \theta$ from above, $\varphi \to +\infty$. Thus in the case (i), the function φ is a steadily decreasing function of x in the interval $0 < x < a$, except for the value $x = \theta$, and it is a steadily increasing function of x in the interval $a < x < \xi$, the stationary value for $x = a$ being plainly a minimum. In the case (ii), φ is a steadily decreasing function of x throughout the interval $0 < x < \xi$, except for the value $x = \theta$.

Evidently φ is continuous throughout the interval $0 < x < \xi$, except for $x = \theta$.

The proof of the lemma is thus completed.

18. I will give other lemmas of a different type.

Lemma 4. *Let $f(y)$ and $f_1(y)$ be L-functions such that*

$$y^\delta > f(y) > (1/y)^\delta, \quad y^\delta > f_1(y) > (1/y)^\delta,$$

$$f_1(y) \sim f(y) > 0$$

as $y \to \infty$. If $y = \theta$ and $y = \theta_1$ are respectively the roots of the equations

$$yf(y) = \lambda, \quad yf_1(y) = c\lambda,$$

for large values of λ, c being a positive constant independent of λ, then

$$\theta_1 \sim c\,\theta$$

as $\lambda \to \infty$.

Proof. Evidently $yf(y)$ and $yf_1(y)$ are ultimately monotonic and tend to infinity as $y \to \infty$. Hence each of the equations

$$yf(y) = \lambda, \quad yf_1(y) = c\,\lambda$$

has, for sufficiently large values of λ, one and only one root which tends to infinity as $\lambda \to \infty$.

By hypothesis, we have

$$\theta f(\theta) = \lambda, \quad \theta_1 f_1(\theta_1) = c\,\lambda\,;$$

and, since $f_1(y) \sim f(y)$, we have

$$f_1(\theta_1) = f(\theta_1)(1+\varepsilon),$$

where $\varepsilon \to 0$ as $\theta_1 \to \infty$ or $\lambda \to \infty$. Hence we obtain

$$c\,\theta f(\theta) = \theta_1 f(\theta_1)(1+\varepsilon).$$

Let η be a function of λ such that

$$\theta_1 = \theta\,\eta.$$

Then we have

$$(33) \qquad c f(\theta) = \eta f(\theta_1)(1+\varepsilon),$$

where $\varepsilon \to 0$ as $\lambda \to \infty$.

We have to prove that

$$\eta \sim c$$

as $\lambda \to \infty$.

Evidently η is positive and continuous for all sufficiently large values of λ, and it might tend to infinity or zero, or might oscillate finitely or infinitely as $\lambda \to \infty$.

If we suppose that $\eta > 1$ or η oscillates in an infinite range of values, then, corresponding to any prescribed positive number P, however great, there will exist a sequence (E) of values of λ tending to infinity, namely,

$$(E) : \lambda_1, \lambda_2, \ldots, \lambda_n, \ldots \quad (\lim_{n\to\infty} \lambda_n = \infty),$$

such that, for every λ_n, we have

$$\eta > P,$$

all values of λ in (E) being greater than a certain positive number λ_0 which can be determined corresponding to each given P.

Since $\theta = \dfrac{\theta_1}{\eta} > 1$ as λ tends to infinity, taking the values of the sequence (E), we can always choose a number a such that

$$1 < a < \eta < \theta_1/a.$$

Easily we can see that H-lemma 24 is available in our case. Hence we have

$$f(\theta) = f(\theta_1/\eta) < f(\theta_1) \qquad (f > 1),$$
$$< Kf(\theta_1)/f(\eta) \quad (f < 1).$$

Therefore, by (33), we have

$$\eta f(\theta_1)(1+\varepsilon) < cf(\theta_1) \qquad (f > 1),$$
$$< cKf(\theta_1)/f(\eta) \quad (f < 1),$$

or

$$(34) \qquad \begin{cases} \eta(1+\varepsilon) < c & (f > 1), \\ \eta f(\eta)(1+\varepsilon) < cK & (f < 1). \end{cases}$$

But $\qquad \eta > P, \quad \eta f(\eta) > \eta^{1-\delta} > P^{1-\delta},$

and the value of P may be chosen as large as we please. Hence neither of the inequalities of (34) can be true.

Thus η cannot take values which become indefinitely great as $\lambda \to \infty$.

Next, if we suppose that $\eta < 1$, or η oscillates in such a manner that it takes indefinitely small values as $\lambda \to \infty$, then, corresponding to any prescribed positive number p, however small, there will exist a sequence (E) of values of λ tending to infinity such that, for every value of λ of this sequence, we have

$$\eta < p.$$

In this case, if we write η_1 for $\frac{1}{\eta}$, then

$$\eta_1 > 1/p = P$$

for every value of λ in the sequence (E). Hence we can proceed quite similarly as in the above case, observing that

$$\frac{\theta}{\eta_1} = \theta_1 > 1$$

as $\lambda \to \infty$. Thus we see that η cannot take values which become indefinitely small as $\lambda \to \infty$.

Therefore there must exist two certain positive constants p and P such that

$$p < \eta < P.$$

But in this case we have

$$f(\theta\eta) \sim f(\theta)$$

as $\theta \to \infty$ (or $\lambda \to \infty$), since $y^\delta > f(y) > (1/y)^\delta$ as $y \to \infty$.*

Hence, by (33), we obtain

$$cf(\theta) \sim \eta f(\theta),$$

whence it follows that

$$\eta \sim c$$

as $\lambda \to \infty$. Thus the lemma is proved.

Let $f(y)$, $f_1(y)$ and c be the same as in our Lemma 4, then we have the following corollary, n denoting any positive constant.

Corollary. *If $y = \theta$, $y = \theta_1$ are respectively the roots of*

$$y^n f(y) = \lambda, \quad y^n f_1(y) = c\lambda$$

for large values of λ, then

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* This can be easily shown by means of H-lemma 18.

$$\theta_1 \sim c^{\frac{1}{n}} \theta$$

as $\lambda \to \infty$.

The truth of this corollary can be inferred immediately by writing our equations in the forms

$$y\{f(y)\}^{\frac{1}{n}} = \lambda^{\frac{1}{n}}, \quad y\{f_1(y)\}^{\frac{1}{n}} = c^{\frac{1}{n}} \lambda^{\frac{1}{n}}.$$

19. Lemma 5. *Let $f(x)$ and $f_1(x)$ be L-functions such that*

$$f > 0, \quad f_1 > 0, \quad f_1 \succ f \succ 1$$

as $x \to 0$. If $x = \theta$, $x = \theta_1$ are respectively the roots of the equations

$$f(x) = \lambda, \quad f_1(x) = \lambda$$

for large values of λ, then

$$\theta > \theta_1$$

for every sufficiently large value of λ.

If we notice that f and f_1 are ultimately monotonic and $f < f_1$ for every sufficiently small value of x, then our lemma follows immediately.

VII *Discussion of Case (C)* : $\sigma(x) \succ l(1/x)$.

20. We now pass to the discussion of the behaviour of the integrals*

$$C(\lambda) = \int_0^{\xi} \rho(x) e^{i\sigma(x)} \frac{\cos \lambda x}{x} dx,$$

$$S(\lambda) = \int_0^{\xi} \rho(x) e^{i\sigma(x)} \frac{\sin \lambda x}{x} dx,$$

as $\lambda \to \infty$, when $l(1/x) \prec \sigma \prec (1/x)^{\Delta}$. It will in this case be convenient to separate the real and imaginary parts of the integrals. Thus we have to consider

* Although the sine-integral $S(\lambda)$ has already been treated by Hardy, we shall discuss it again, reproducing briefly his analysis, because, for the purpose of this paper, it is necessary to modify his argument and to extend it to the case $x^{\Delta} \prec \rho \prec x$, while the same argument applies to the discussion of the cosine-integral $C(\lambda)$.

$$\begin{cases} I_1(\lambda) = \int_0^{\xi} \rho(x) \cos \sigma(x) \dfrac{\cos \lambda x}{x}\, dx, & I_2(\lambda) = \int_0^{\xi} \rho(x) \sin \sigma(x) \dfrac{\cos \lambda x}{x}\, dx, \\ I_3(\lambda) = \int_0^{\xi} \rho(x) \cos \sigma(x) \dfrac{\sin \lambda x}{x}\, dx, & I_4(\lambda) = \int_0^{\xi} \rho(x) \sin \sigma(x) \dfrac{\sin \lambda x}{x}\, dx. \end{cases}$$

All these integrals are convergent if

$$(35) \qquad x^{\Delta} \prec \rho(x) \prec x\sigma'(x).$$

Hence we shall suppose that this is satisfied. Then, if we put

$$\begin{cases} J_1(\lambda) = \int_0^{\xi} \dfrac{\rho(x)}{x} \cos \{\lambda x + \sigma(x)\}\, dx, & J_2(\lambda) = \int_0^{\xi} \dfrac{\rho(x)}{x} \cos \{\lambda x - \sigma(x)\}\, dx, \\ J_3(\lambda) = \int_0^{\xi} \dfrac{\rho(x)}{x} \sin \{\lambda x + \sigma(x)\}\, dx, & J_4(\lambda) = \int_0^{\xi} \dfrac{\rho(x)}{x} \sin \{\lambda x - \sigma(x)\}\, dx, \end{cases}$$

we have

$$\begin{cases} I_1(\lambda) = \tfrac{1}{2}\{J_1(\lambda) + J_2(\lambda)\}, & I_2(\lambda) = \tfrac{1}{2}\{J_3(\lambda) - J_4(\lambda)\}, \\ I_3(\lambda) = \tfrac{1}{2}\{J_3(\lambda) + J_4(\lambda)\}, & I_4(\lambda) = \tfrac{1}{2}\{-J_1(\lambda) + J_2(\lambda)\}, \end{cases}$$

and

$$(36) \qquad \begin{cases} C(\lambda) = \tfrac{1}{2}[J_1(\lambda) + J_2(\lambda) + i\{J_3(\lambda) - J_4(\lambda)\}], \\ S(\lambda) = \tfrac{1}{2}[J_3(\lambda) + J_4(\lambda) - i\{J_1(\lambda) - J_2(\lambda)\}]. \end{cases}$$

21. *Integrals J_2 and J_4.* At first we shall consider the integral

$$J_2(\lambda) = \int_0^{\xi} \frac{\rho(x)}{x} \cos y\, dx,$$

where $y = \lambda x - \sigma(x)$. As x increases from 0 to ξ, y increases from $-\infty$ to $\eta = \lambda\xi - \sigma(\xi)$, and η tends to infinity with λ. Also

$$J_2(\lambda) = \int_{-\infty}^{\eta} \frac{\rho(x)}{x\{\lambda - \sigma'(x)\}} \cos y\, dy.$$

Let
$$\varsigma = \frac{\rho(x)}{x\{\lambda - \sigma'(x)\}},$$

then, by II-lemma 32 and Lemma 2, we have to separate the following cases.

(i) Let $\sigma < x$ or $\rho = Ax(1+\rho)$, where $A>0$, $\bar{\rho} \geqq 0$, $\bar{\rho} < 1$. Then φ is a steadily increasing function of x throughout the interval $0 < x < \xi$. Hence, by the Second Mean Value Theorem,

$$J_2(\lambda) = \frac{\rho(\xi)}{\xi\{\lambda - \sigma'(\xi)\}} \int_{\eta_1}^{\eta} \cos y \, dy \quad (-\infty < \eta_1 < \eta).$$

Therefore $$J_2(\lambda) = O(1/\lambda).$$

(ii) Let $x < \rho < x\sigma'$ or $\rho = Ax(1-\rho)$, where $A>0$, $\rho > 0$, $\bar{\rho} < 1$. Then φ has one stationary value in the interval $0 < x < \xi$, which is a maximum given by $x = \alpha$, α being the root of the equation $\dfrac{d\varphi}{dx} = 0$.

We now write

$$J_2(\lambda) = \left(\int_{-\infty}^{\beta} + \int_{\beta}^{\eta} \right) \frac{\rho(x)}{x\{\lambda - \sigma'(x)\}} \cos y \, dy = J_2' + J_2''$$

say, where $\beta = \lambda\alpha - \sigma(\alpha)$. Then, by another application of the Second Mean Value Theorem,

$$J_2' = \frac{\rho(\alpha)}{\alpha\{\lambda - \sigma'(\alpha)\}} \int_{\beta_1}^{\beta} \cos y \, dy \qquad (-\infty < \beta_1 < \beta).$$

Therefore $$J_2' = \frac{\rho(\alpha)}{\alpha\{\lambda - \sigma'(\alpha)\}} . O(1) = O\{\varphi(\alpha)\}.$$

Similarly we obtain

$$J_2'' = \frac{\rho(\alpha)}{\alpha\{\lambda - \sigma'(\alpha)\}} . O(1) = O\{\varphi(\alpha)\}.$$

Hence we have

$$J_2(\lambda) = \frac{\rho(\alpha)}{\alpha\{\lambda - \sigma'(\alpha)\}} . O(1) = O\{\varphi(\alpha)\}.$$

The same argument applies to the integral $J_4(\lambda)$. Hence, if $\rho < x$ or $\rho = Ax(1+\rho)$, where $A>0$, $\bar{\rho} > 0$, $\rho < 1$,

$$J_4(\lambda) = O(1/\lambda);$$

if $x < \rho < x\sigma'$ or $\rho = Ax(1-\bar{\rho})$, where $A > 0$, $\rho > 0$, $\bar{\rho} < 1$,

$$J_4(\lambda) = \frac{\rho(\alpha)}{\alpha\{\lambda - \sigma'(\alpha)\}} \cdot O(1) = O\{\varphi(\alpha)\}.$$

22. *Integrals J_1 and J_3.* The same methods apply to both of the integrals J_1, J_3, so that we shall consider the integral

$$(37) \qquad J_1(\lambda) = \int_0^{\varepsilon} \frac{\rho(x)}{x} \cos y \, dx,$$

where $y = \lambda x + \sigma(x)$. This function y has one stationary value, which is a minimum given by

$$\lambda + \sigma'(x) = 0,$$

or, say,

$$x = \theta(\lambda) = \theta,$$

being a positive function of λ which tends steadily to zero as $\lambda \to \infty$.

As in the paper "*O. D. I. 2.*", in the following discussion, we shall use an auxiliary function ε of λ such that

$$(38) \qquad \varepsilon > 0, \qquad \varepsilon < \theta, \qquad \rho(\theta) < \varepsilon\sigma'(\theta), \qquad \varepsilon^2\sigma''(\theta) > 1.$$

Since $x\sigma'' \sim A\sigma'$ in our case (C), the last condition of (38) is equivalent to

$$\varepsilon^2 > \theta/\sigma'(\theta).$$

Let $\sigma'(x) = \dfrac{\mu(x)}{x}$, so that $\mu(x) > 1$, and let $\rho(x) = x\sigma'(x)\nu(x)$, so that $\nu(x) < 1$. Then the above conditions (38) for ε are equivalent to

$$(38') \qquad \varepsilon > 0, \qquad \varepsilon < \theta, \qquad \varepsilon > \theta\nu(\theta), \qquad \varepsilon > \frac{\theta}{\sqrt{\mu(\theta)}},$$

and evidently such a choice of the function ε is always possible.

It is convenient to divide the discussion into the following two cases (i) and (ii).

(i) The case in which $x < \rho < x\sigma'$, or $\rho = Ax(1-\rho)$, where $A>0$, $\bar{\rho}\geq 0$, $\rho<1$.

In this case, by H-lemma 33 and Lemma 3, the function

$$\varphi = \frac{\rho(x)}{x\{\lambda+\sigma'(x)\}} \tag{39}$$

is a steadily decreasing function of x, except for the value $x=\theta$, throughout the interval $0<x<\xi$; and we divide the integral (37) into the three parts

$$J_1(\lambda) = \int_0^{\theta-\varepsilon} + \int_{\theta-\varepsilon}^{\theta+\varepsilon} + \int_{\theta+\varepsilon}^{\xi} = J_1^{(1)} + J_1^{(2)} + J_1^{(3)}. \tag{40}$$

(ii) The case in which $x^1 < \rho < x$, or $\rho = Ax(1+\bar{\rho})$, where $A>0$, $\rho>0$, $\rho<1$.

In this case, by Lemma 3, the function φ has one stationary value, which is a minimum given by $x=\alpha$, α being the root of the equation $\frac{d\varphi}{dx}=0$. Hereby α is greater than θ, so that the function φ is a steadily decreasing function of x in the interval $0<x<\alpha$, except only for the value $x=\theta$, and it is a steadily increasing function of x in the interval $\alpha<x<\xi$. As it will be proved presently, we have

$$\theta+\varepsilon<\alpha.$$

Hence we divide the integral J_1 into the four parts

$$J_1(\lambda) = \int_0^{\theta-\varepsilon} + \int_{\theta-\varepsilon}^{\theta+\varepsilon} + \int_{\theta+\varepsilon}^{\alpha} + \int_{\alpha}^{\xi} = J_1^{(1)} + J_1^{(2)} + J_1^{(3)} + J_1^{(4)}. \tag{41}$$

23. *Integrals* $J_1^{(1)}$ and $J_1^{(3)}$. As x increases from 0 to $\theta-\varepsilon$, the function $y=\lambda x+\sigma(x)$ decreases from ∞ to $\lambda(\theta-\varepsilon)+\sigma(\theta-\varepsilon)$, which is large and positive when θ is small and ε smaller. Also

$$J_1^{(1)} = \int_{\lambda(\theta-\varepsilon)+\sigma(\theta-\varepsilon)}^{\infty} \left[-\frac{\rho(x)}{x\{\lambda+\sigma'(x)\}}\right]\cos y\,dy.$$

The factor $-\frac{\rho(x)}{x\{\lambda+\sigma'(x)\}}$ which multiplies $\cos y$ is positive and monotonic, as we have already seen. Hence, by the Second Mean Value Theorem, we obtain

$$|J_1^{(1)}| < K\left[-\frac{\rho(\theta-\varepsilon)}{(\theta-\varepsilon)\{\lambda+\sigma'(\theta-\varepsilon)\}}\right]$$

$$< K\left[\frac{\rho(\theta-\varepsilon)}{(\theta-\varepsilon)\,.\,\varepsilon\sigma''(\theta-\varepsilon_1)}\right],$$

where $$0 < \varepsilon_1 < \varepsilon.$$

Now ρ, σ and all their derivatives satisfy the condition

$$x^{\Delta} \prec f \prec (1/x)^{\Delta};$$

and so each of them satisfies the relation

$$f(\theta \pm \varphi) \sim f(\theta),$$

if $\varphi \prec \theta$, in virtue of H-lemma 11.
Hence we have

$$J_1^{(1)} = \frac{\rho(\theta)}{\varepsilon\,\theta\,\sigma''(\theta)}\,.\,O(1).$$

Similarly, in both of the cases (i) and (ii), we obtain

$$J_1^{(3)} = \frac{\rho(\theta)}{\varepsilon\,\theta\,\sigma''(\theta)}\,.\,O(1).$$

Now $$\frac{\rho(\theta)}{\varepsilon\,\theta\,\sigma''(\theta)} = \frac{\rho(\theta)}{\theta\sqrt{\{2\sigma''(\theta)\}}}\cdot\frac{\sqrt{2}}{\varepsilon\sqrt{\sigma''(\theta)}} \prec \frac{\rho(\theta)}{\theta\sqrt{\{2\sigma''(\theta)\}}},$$

since, by (38), $$\varepsilon^2\sigma''(\theta) \succ 1.$$

Therefore, in both of the cases (i) and (ii), we have

$$\begin{cases} J_1^{(1)} \prec \dfrac{\rho(\theta)}{\theta\sqrt{\{2\sigma''(\theta)\}}}, \\[2ex] J_1^{(3)} \prec \dfrac{\rho(\theta)}{\theta\sqrt{\{2\sigma''(\theta)\}}}. \end{cases}$$

24. In the case (ii), we have assumed the relation

$$\theta + \varepsilon < a,$$

which may be proved as follows.

When $\rho \prec x$, $x = \alpha$ is the positive root of the equation

$$f(x) = \frac{\sigma''(x)\,\gamma(x)}{\gamma'(x)} - \sigma'(x) - \lambda = 0,$$

where
$$\rho = x\,\gamma(x).$$

This equation has, for sufficiently large fixed values of λ, one and only one positive root α. Hence the function $f(x)$ changes its sign when x passes through the value α.

Now, since $\quad \sigma''(\theta) > 0, \quad \gamma(\theta) > 0, \quad \gamma'(\theta) > 0, \quad \sigma'(\theta) + \lambda = 0,$
we have

$$f(\theta) = \frac{\sigma''(\theta)\,\gamma(\theta)}{\gamma'(\theta)} > 0.$$

And
$$\begin{aligned} f(\theta+\varepsilon) &= \frac{\sigma''(\theta+\varepsilon)\,\gamma(\theta+\varepsilon)}{\gamma'(\theta+\varepsilon)} - \sigma'(\theta+\varepsilon) - \lambda \\ &= \frac{\sigma''(\theta+\varepsilon)\,\gamma(\theta+\varepsilon)}{\gamma'(\theta+\varepsilon)} - \varepsilon\,\sigma''(\theta+\varepsilon_1) \qquad (0 < \varepsilon_1 < \varepsilon) \\ &\sim \frac{\sigma''(\theta)\,\gamma(\theta)}{\gamma'(\theta)} - \varepsilon\,\sigma''(\theta) \\ &= \frac{\sigma''(\theta)\{\gamma(\theta) - \varepsilon\,\gamma'(\theta)\}}{\gamma'(\theta)}. \end{aligned}$$

Since $\gamma \prec 1$, we have

$$\theta\,\gamma'(\theta) \prec \gamma(\theta),$$

and, since $\varepsilon \prec \theta$, $\qquad \varepsilon\,\gamma'(\theta) \prec \gamma(\theta).$

Hence ultimately $\qquad \gamma(\theta) - \varepsilon\,\gamma'(\theta) > 0,$

and therefore $\qquad f(\theta+\varepsilon) > 0.$

Thus $f(\theta)$ and $f(\theta+\varepsilon)$ have ultimately the same sign.
Therefore it follows that $\qquad \theta < \theta + \varepsilon < \alpha.$

Next, when $\rho = Ax(1-\rho)$, $x = \alpha$ is the positive root of the equation

$$f(x) = \{1 + \rho(x)\}\frac{\sigma''(x)}{\rho'(x)} - \sigma'(x) - \lambda = 0.$$

Now
$$f(\theta) = \{1+\bar{\rho}(\theta)\}\frac{\sigma''(\theta)}{\rho'(\theta)} > 0,$$

since
$$\sigma''(\theta) > 0, \quad \rho'(\theta) > 0, \quad \bar{\rho}(\theta) < 1, \quad \sigma'(\theta) + \lambda = 0.$$

And
$$f(\theta + \varepsilon) = \{1 + \bar{\rho}(\theta + \varepsilon)\}\frac{\sigma''(\theta + \varepsilon)}{\rho'(\theta+\varepsilon)} - \sigma'(\theta + \varepsilon) - \lambda$$

$$= \{1 + \bar{\rho}(\theta+\varepsilon)\}\frac{\sigma''(\theta+\varepsilon)}{\rho'(\theta+\varepsilon)} - \varepsilon\,\sigma''(\theta+\varepsilon_1) \quad (0<\varepsilon_1<\varepsilon)$$

$$\sim \frac{\sigma''(\theta)}{\rho'(\theta)} + \sigma''(\theta)\,.\,\frac{\bar{\rho}(\theta)-\varepsilon\bar{\rho}'(\theta)}{\rho'(\theta)} > 0,$$

since
$$\sigma''(\theta) > 0, \quad \bar{\rho}'(\theta) > 0, \quad \varepsilon\,\rho'(\theta) < \bar{\rho}(\theta).$$

Therefore it follows that $\theta < \theta + \varepsilon < a.$

25. *Integral* $J_1^{(2)}$. We now consider the integral

$$J_1^{(2)} = \int_{\theta-\varepsilon}^{\theta+\varepsilon} \frac{\rho(x)}{x} \cos y\, dx = J_1' + J_1'',$$

where
$$\begin{cases} J_1' = \displaystyle\int_{\beta}^{\lambda(\theta+\varepsilon)+\sigma(\theta+\varepsilon)} \left\{ \frac{\rho(x)}{x} \cdot \frac{\cos y}{\lambda+\sigma'(x)} \right\} dy, \\ J_1'' = \displaystyle\int_{\beta}^{\lambda(\theta-\varepsilon)+\sigma(\theta-\varepsilon)} \left\{ -\frac{\rho(x)}{x} \cdot \frac{\cos y}{\lambda+\sigma'(x)} \right\} dy, \end{cases}$$

$$\beta = \lambda\theta + \sigma(\theta).$$

Now let us consider the following difference of integrals

$$(42) \qquad j' = \int_{\beta}^{\lambda(\theta+\varepsilon)+\sigma(\theta+\varepsilon)} \frac{\rho(x)}{x\{\lambda+\sigma'(x)\}} \cos y\, dy$$
$$- \int_{\beta}^{\lambda(\theta+\varepsilon)+\sigma(\theta+\varepsilon)} \frac{\rho(\theta)}{\theta\sqrt{\{2\sigma''(\theta)\,(y-\beta)\}}} \cos y\, dy$$

which may be written in the form

$$(43) \qquad j' = \int_{x=\theta}^{x=\theta+\varepsilon} \frac{r(x)-r(\theta)}{\lambda+\sigma'(x)} \cos y\, dy + \frac{\rho(\theta)}{\theta} \int_{x=\theta}^{x=\theta+\varepsilon} \chi(y) \cos y\, dy,$$

where
$$r(x) = \frac{\rho(x)}{x}, \quad \chi(y) = \frac{1}{\lambda+\sigma'(x)} - \frac{1}{\sqrt{\{2\sigma''(\theta)\,(y-\beta)\}}}.$$

By the analysis of §§ 33—35 of "*O. D. I. 2.*", we see that

$$\int_{x=\theta+\varepsilon_1}^{x=\theta+\varepsilon_2} \chi(y)\cos y\, dy < \frac{1}{\sqrt{\sigma''(\theta)}},$$

$$\int_{x=\theta+\varepsilon_1}^{x=\theta+\varepsilon_2} \frac{\cos y\, dy}{\lambda+\sigma'(x)} < \frac{K}{\sqrt{\sigma''(\theta)}},$$

where $$0 \leqq \varepsilon_1 \leqq \varepsilon_2 \leqq \varepsilon.$$

Hence we have

$$(44) \qquad \frac{\rho(\theta)}{\theta}\int_{x=\theta}^{x=\theta+\varepsilon} \chi(y)\cos y\, dy < \frac{\rho(\theta)}{\theta\sqrt{\{2\sigma''(\theta)\}}}.$$

If we put $$k' = \int_{x=\theta}^{x=\theta+\varepsilon} \frac{r(x)-r(\theta)}{\lambda+\sigma'(x)}\cos y\, dy,$$

then as in "*O. D. I. 2.*", we obtain

$$k' = \{r(\theta+\varepsilon)-r(\theta)\}\int_{x=\theta+\varepsilon_1}^{x=\theta+\varepsilon} \frac{\cos y}{\lambda+\sigma'(x)}\, dy \qquad (0<\varepsilon_1<\varepsilon),$$

and hence $$|k'| < \frac{K\varepsilon r'(\theta)}{\sqrt{\sigma''(\theta)}};$$

and, if $x < \rho < x\sigma'$,

$$\varepsilon r'(\theta) < r(\theta).$$

If $x^4 < \rho \leq x$, we have $r < 1$ and we easily see that $xr' \leq r$, whence it also follows that

$$\varepsilon r'(\theta) < r(\theta).$$

Therefore, in both of the cases (i) and (ii), we have

$$|k'| < \frac{r(\theta)}{\sqrt{\sigma''(\theta)}},$$

or $$k' < \frac{\rho(\theta)}{\theta\sqrt{\{2\sigma''(\theta)\}}}.$$

Introducing (44) and this result into (43), we obtain

$$j' < \frac{\rho(\theta)}{\theta\sqrt{\{2\sigma''(\theta)\}}}.$$

Hence, by (42), we have

$$J_1' = \frac{\rho(\theta)}{\theta\sqrt{\{2\sigma''(\theta)\}}}\left\{\int_{\beta}^{\lambda(\theta+\varepsilon)+\sigma(\theta+\varepsilon)} \frac{\cos y}{\sqrt{(y-\beta)}}\,dy + o(1)\right\}.$$

Similarly we obtain

$$J_1'' = \frac{\rho(\theta)}{\theta\sqrt{\{2\sigma''(\theta)\}}}\left\{\int_{\beta}^{\lambda(\theta-\varepsilon)+\sigma(\theta-\varepsilon)} \frac{\cos y}{\sqrt{(y-\beta)}}\,dy + o(1)\right\}.$$

Now $$\lambda(\theta+\varepsilon)+\sigma(\theta+\varepsilon) = \beta + \tfrac{1}{2}\varepsilon^2\sigma''(\theta+\varepsilon_1),$$

where $0<\varepsilon_1<\varepsilon$, so that $$\sigma''(\theta+\varepsilon_1) \sim \sigma''(\theta);$$

and, by (38), $$\varepsilon^2\sigma''(\theta) \succ 1.$$
Hence we have

$$\int_{\beta}^{\lambda(\theta+\varepsilon)+\sigma(\theta+\varepsilon)} \frac{\cos y}{\sqrt{(y-\beta)}}\,dy = \int_{\beta}^{\infty} \frac{\cos y}{\sqrt{(y-\beta)}}\,dy + o(1).$$

But $$\int_{\beta}^{\infty} \frac{\cos y}{\sqrt{(y-\beta)}}\,dy = \cos(\beta+\tfrac{1}{4}\pi)\sqrt{\pi}.$$

Therefore we obtain

$$J_1' = \frac{\rho(\theta)}{\theta\sqrt{\{2\sigma''(\theta)\}}}\{\cos(\beta+\tfrac{1}{4}\pi)\sqrt{\pi}+o(1)\}.$$

Similarly

$$J_1'' = \frac{\rho(\theta)}{\theta\sqrt{\{2\sigma''(\theta)\}}}\{\cos(\beta+\tfrac{1}{4}\pi)\sqrt{\pi}+o(1)\}.$$

Hence, in both of the cases (i) and (ii), we obtain

$$J_1^{(2)} = \frac{2\,\rho(\theta)}{\theta\sqrt{\{2\sigma''(\theta)\}}}\{\cos(\beta+\tfrac{1}{4}\pi)\sqrt{\pi}+o(1)\}.$$

26. *Integral* $J_1^{(4)}$. Finally we consider the integral

$$J_1^{(4)} = \int_a^{\xi} \frac{\rho(x)}{x}\cos y\,dx = \int_{x=a}^{x=\xi} \frac{\rho(x)}{x\{\lambda+\sigma'(x)\}}\cos y\,dy.$$

In this integral, $\dfrac{\rho(x)}{x\{\lambda+\sigma'(x)\}}$ is a steadily increasing function of x in the interval $a<x<\xi$. Hence, by the Second Mean Value Theorem, we obtain

$$J_1^{(4)} = \frac{\rho(\xi)}{\xi\{\lambda+\sigma'(\xi)\}} \cdot O(1) = O(1/\lambda).$$

27. Introducing the results of §§ 23, 25 and 26 into (40) and (41), we obtain:

In the case (i)

$$J_1(\lambda) = \frac{2\,\rho(\theta)}{\theta\sqrt{\{2\sigma''(\theta)\}}}\,\{\cos(\beta+\tfrac{1}{4}\pi)\sqrt{\pi}+o(1)\}\,;$$

in the case (ii)

$$J_1(\lambda) = \frac{2\,\rho(\theta)}{\theta\sqrt{\{2\sigma''(\theta)\}}}\,\{\cos(\beta+\tfrac{1}{4}\pi)\sqrt{\pi}+o(1)\}+O(1/\lambda).$$

Similarly we obtain:

In the case (i)

$$J_3(\lambda) = \frac{2\,\rho(\theta)}{\theta\sqrt{\{2\sigma''(\theta)\}}}\,\{\sin(\beta+\tfrac{1}{4}\pi)\sqrt{\pi}+o(1)\}\,;$$

in the case (ii)

$$J_3(\lambda) = \frac{2\,\rho(\theta)}{\theta\sqrt{\{2\sigma''(\theta)\}}}\,\{\sin(\beta+\tfrac{1}{4}\pi)\sqrt{\pi}+o(1)\}+O(1/\lambda).$$

Now we can state

Theorem V. *If $x^A < \rho < x$ or $\rho = Ax\{1+\rho(x)\}$, where $A>0$, $\rho>0$ and $\rho<1$, then*

$$J_1(\lambda)=\frac{2\,\rho(\theta)}{\theta\sqrt{\{2\sigma''(\theta)\}}}\,\{\cos(\beta+\tfrac{1}{4}\pi)\sqrt{\pi}+o(1)\}+O(1/\lambda),\qquad J_2(\lambda)=O(1/\lambda),$$

$$J_3(\lambda)=\frac{2\,\rho(\theta)}{\theta\sqrt{\{2\sigma''(\theta)\}}}\,\{\sin(\beta+\tfrac{1}{4}\pi)\sqrt{\pi}+o(1)\}+O(1/\lambda),\qquad J_4(\lambda)=O(1/\lambda)\,;$$

if $x < \rho < x\sigma'$ or $\rho = Ax\{1-\rho(x)\}$, where $A>0$, $\rho>0$ and $\rho<1$, then

$$J_1(\lambda)=\frac{2\,\rho(\theta)}{\theta\sqrt{\{2\sigma''(\theta)\}}}\,\{\cos(\beta+\tfrac{1}{4}\pi)\sqrt{\pi}+o(1)\},\qquad J_2(\lambda)=\frac{\rho(\alpha)}{\alpha\{\lambda-\sigma'(\alpha)\}}\cdot O(1),$$

$$J_3(\lambda)=\frac{2\,\rho(\theta)}{\theta\sqrt{\{2\sigma''(\theta)\}}}\,\{\sin(\beta+\tfrac{1}{4}\pi)\sqrt{\pi}+o(1)\},\qquad J_4(\lambda)=\frac{\rho(\alpha)}{\alpha\{\lambda-\sigma'(\alpha)\}}\cdot O(1);$$

finally, if $\rho = Ax$, *where* $A>0$, *then*

$$J_1(\lambda) = \frac{2\rho(\theta)}{\theta\sqrt{\{2\sigma''(\theta)\}}}\{\cos(\beta+\tfrac{1}{4}\pi)\sqrt{\pi}+o(1)\}, \quad J_2(\lambda) = O(1/\lambda),$$

$$J_3(\lambda) = \frac{2\rho(\theta)}{\theta\sqrt{\{2\sigma''(\theta)\}}}\{\sin(\beta+\tfrac{1}{4}\pi)\sqrt{\pi}+o(1)\}, \quad J_4(\lambda) = O(1/\lambda);$$

and, in these formulae, θ *and* α *are functions of* λ *determined respectively by the equations*

$$\sigma'(\theta)+\lambda = 0, \qquad \frac{d}{d\alpha}\left[\frac{\rho(\alpha)}{\alpha\{\lambda-\sigma'(\alpha)\}}\right] = 0,$$

and $$\beta = \lambda\theta+\sigma(\theta).$$

Corollary 1. *If* $x^{\Delta} < \rho < x$ *or* $\rho = Ax\{1+\bar{\rho}(x)\}$, *where* $A>0$, $\bar{\rho} \geqq 0$ *and* $\bar{\rho} < 1$, *then*

$$\begin{cases} C(\lambda) = \dfrac{\rho(\theta)}{\theta\sqrt{\{2\sigma''(\theta)\}}}\{e^{(\beta+\frac{1}{4}\pi)i}\sqrt{\pi}+o(1)\}+O(1/\lambda), \\ S(\lambda) = \dfrac{\rho(\theta)}{\theta\sqrt{\{2\sigma''(\theta)\}}}\{e^{(\beta-\frac{1}{4}\pi)i}\sqrt{\pi}+o(1)\}+O(1/\lambda); \end{cases}$$

if $x < \rho < x\sigma'$ *or* $\rho = Ax\{1-\bar{\rho}(x)\}$, *where* $A>0$, $\bar{\rho}>0$ *and* $\bar{\rho}<1$, *then*

$$\begin{cases} C(\lambda) = \dfrac{\rho(\theta)}{\theta\sqrt{\{2\sigma''(\theta)\}}}\{e^{(\beta+\frac{1}{4}\pi)i}\sqrt{\pi}+o(1)\}+\dfrac{\rho(\alpha)}{\alpha\{\lambda-\sigma'(\alpha)\}}\cdot O(1), \\ S(\lambda) = \dfrac{\rho(\theta)}{\theta\sqrt{\{2\sigma''(\theta)\}}}\{e^{(\beta-\frac{1}{4}\pi)i}\sqrt{\pi}+o(1)\}+\dfrac{\rho(\alpha)}{\alpha\{\lambda-\sigma'(\alpha)\}}\cdot O(1); \end{cases}$$

θ, α, β *being the same as in the theorem.*

By H-lemma 32 and Lemma 2, we know that

$$\frac{\rho(\alpha)}{\alpha\{\lambda-\sigma'(\alpha)\}} \prec 1$$

as $\lambda\to\infty$, and evidently $\dfrac{1}{\lambda}\to 0$ as $\lambda\to\infty$. But $\dfrac{\rho(\theta)}{\theta\sqrt{\{2\sigma''(\theta)\}}}$ never tends to zero as $\lambda\to\infty$, if $\rho \succeq x\sqrt{\sigma''}$. Hence we have

Corollary 2. *If* $x\sqrt{\sigma''} \preceq \rho < x\sigma'$, *then*

$$(45)\qquad \begin{cases} C(\lambda) \sim \dfrac{\rho(\theta)}{\theta\sqrt{\{2\sigma''(\theta)\}}}\, e^{(\beta+\frac{1}{4}\pi)i}\sqrt{\pi}, \\[2ex] S(\lambda) \sim \dfrac{\rho(\theta)}{\theta\sqrt{\{2\sigma''(\theta)\}}}\, e^{(\beta-\frac{1}{4}\pi)i}\sqrt{\pi}, \end{cases}$$

θ, β having the meanings defined in the theorem.

This formula for $S(\lambda)$ is nothing but the one obtained in "*O. D. I. 2*". In the followings we are going to prove that the formulae (45) hold also when $\rho < x\sqrt{\sigma''}$ so long as $C(\lambda) \succ 1/\lambda$ and $S(\lambda) \succ 1/\lambda$ as $\lambda \to \infty$.

28. For the purpose of this paper, it is necessary to compare the order of magnitude of

$$\frac{1}{\lambda}, \qquad \frac{\rho(\alpha)}{\alpha\{\lambda-\sigma'(\alpha)\}}, \qquad \frac{\rho(\theta)}{\theta\sqrt{\{2\sigma''(\theta)\}}}$$

as $\lambda \to \infty$, when $\rho < x\sqrt{\sigma''}$.

At first, we consider the first and the last of these functions. Now, since $\lambda + \sigma'(\theta) = 0$, we have

$$\frac{\rho(\theta)}{\theta\sqrt{\{2\sigma''(\theta)\}}} \Big/ \frac{1}{\lambda} = -\frac{\rho(\theta)\,\sigma'(\theta)}{\theta\sqrt{\{2\sigma''(\theta)\}}}.$$

Hence, if $\rho \preceq x\sqrt{\sigma''}/\sigma'$, then

$$\frac{\rho(\theta)}{\theta\sqrt{\{2\sigma''(\theta)\}}} \prec \frac{1}{\lambda};$$

if $\rho \succ x\sqrt{\sigma''}/\sigma'$, then

$$\frac{\rho(\theta)}{\theta\sqrt{\{2\sigma''(\theta)\}}} \succ \frac{1}{\lambda}.$$

We know that, if $\sigma \succ l(1/x)$, then

$$\sigma' \succ \sqrt{\sigma''}. \qquad \text{[H-lemma 31]}$$

Hence, if $\rho \succ x$, then

$$\frac{\rho(\theta)}{\theta\sqrt{\{2\sigma''(\theta)\}}} \succ \frac{1}{\lambda}.$$

Therefore we obtain :

If $\rho \preceq x\sqrt{\sigma''}/\sigma'$, *then*

$$C(\lambda) = O(1/\lambda), \qquad S(\lambda) = O(1/\lambda);$$

if $x\sqrt{\sigma''}/\sigma' \prec \rho \prec x$ *or if* $\rho = Ax\{1+\bar{\rho}(x)\}$, *where* $A>0$, $\bar{\rho}\geqq 0$ *and* $\bar{\rho} \prec 1$, *then the formulae (45) hold, i.e.,*

$$\begin{cases} C(\lambda) \sim \dfrac{\rho(\theta)}{\theta\sqrt{\{2\sigma''(\theta)\}}} e^{(\beta+\frac{1}{4}\pi)i}\sqrt{\pi}, \\ S(\lambda) \sim \dfrac{\rho(\theta)}{\theta\sqrt{\{2\sigma''(\theta)\}}} e^{(\beta-\frac{1}{4}\pi)i}\sqrt{\pi}, \end{cases} \qquad (\lambda\to\infty).$$

29. It now remains to compare the order of magnitude of

$$\varphi(\alpha) = \frac{\rho(\alpha)}{\alpha\{\lambda-\sigma'(\alpha)\}}, \qquad \psi(\theta) = \frac{\rho(\theta)}{\theta\sqrt{\{2\sigma''(\theta)\}}}$$

as $\lambda\to\infty$, when $x \prec \rho \prec x\sqrt{\sigma''}$ or $\rho = Ax\{1-\bar{\rho}(x)\}$, where $A>0$, $\bar{\rho}>0$ and $\bar{\rho}\prec 1$.

When $x \prec \rho \prec x\sqrt{\sigma''}$, we write, as before,

(46) $$\rho(x) = x^{-a}\,\theta(x),$$

where $a\geqq -1$ and $x^{\delta} \prec \theta \prec (1/x)^{\delta}$ as $x\to 0$. We observe that $\theta \succ 1$, if $a = -1$.

Under the supposition that $l(1/x) \prec \sigma \prec (1/x)^{\Delta}$, we can write

(47) $$\sigma'(x) = -\,x^{-1-b}\,\theta_1(x),$$

where $b\geqq 0$ and $x^{\delta} \prec \theta_1 \prec (1/x)^{\delta}$ as $x\to 0$. We observe that $\theta_1 \succ 1$, if $b = 0$, since $\sigma' \succ 1/x$.

From the condition $\rho \prec x\sqrt{\sigma''}$, we obtain

$$a < \tfrac{1}{2}b,$$

or

$$a = \tfrac{1}{2}b, \quad \theta \prec \theta_1^{\frac{1}{2}}.$$

We have to separate the discussion into several cases as follows.

30. (i) Let $a > -1$.

At first we consider the case in which

$$b>0 \qquad \text{or} \qquad b=0, \quad -1<a<0.$$

Then we have
$$a < b,$$
since $a \leqq \frac{1}{2}b$.

Let us write
$$\frac{\rho}{x} = \sigma' \gamma(x),$$
so that
$$\gamma = - x^{b-a}\, \Theta/\Theta_1.^*$$
Then
$$\gamma < 0, \quad \gamma < 1$$
since $a < b$. Now
$$\varphi(x) = \frac{\rho}{x(\lambda - \sigma')} = \frac{\sigma' \gamma}{\lambda - \sigma'},$$
and $\dfrac{d\varphi}{dx} = 0$ gives

$$\lambda = \frac{\sigma'^2 \gamma'}{\sigma' \gamma' + \sigma'' \gamma}. \tag{48}$$

From (47) and the above expression for γ, we obtain
$$x\gamma' \sim (b-a)\gamma, \qquad x\sigma'' \sim -(1+b)\sigma'.$$
and
$$\frac{\sigma'^2 \gamma'}{\sigma' \gamma' + \sigma'' \gamma} \sim -\frac{b-a}{a+1}\sigma'.$$
Hence (48) becomes
$$\lambda \sim -\frac{b-a}{a+1}\sigma',$$
or
$$-\sigma'\{1 + e(x)\} = \frac{a+1}{b-a}\lambda,$$
where $e < 1$ as $x \to 0$. Now $x = \alpha$ is the root of this equation, while $x = \theta$ is that of the equation
$$-\sigma' = \lambda.$$

Therefore, by the corollary to Lemma 4, we obtain
$$\alpha \sim c\theta, \qquad c = \left(\frac{b-a}{a+1}\right)^{\frac{1}{1+b}}$$
as $\lambda \to \infty$.

Now
$$\varphi(\alpha) = \frac{\sigma'(\alpha)\gamma(\alpha)}{\lambda - \sigma'(\alpha)} \sim -\frac{a+1}{1+b}\gamma(\alpha).$$

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* Observe that $\Theta > 0$ and $\Theta_1 > 0$, which follow from the assumption $\rho > 0$ and $\sigma > 0$.

and
$$\gamma(\alpha) = - \alpha^{b-a} \frac{\Theta(\alpha)}{\Theta_1(\alpha)}$$

$$\sim - c^{b-a}\, \theta^{b-a} \frac{\Theta(\theta)}{\Theta_1(\theta)} = c^{b-a}\gamma(\theta),$$

since
$$\begin{cases} \alpha^{b-a} \sim (c\theta)^{b-a} \\ \Theta(\alpha) \sim \Theta(c\theta) \sim \Theta(\theta), \\ \Theta_1(\alpha) \sim \Theta_1(c\theta) \sim \Theta_1(\theta). \end{cases}$$

Hence we have
$$\varphi(\alpha) \sim -\frac{a+1}{b+1} c^{b-a} \gamma(\theta) = K \frac{\rho(\theta)}{\theta\, \sigma'(\theta)}.$$

Therefore
$$\frac{\varphi(\alpha)}{\psi(\theta)} \sim K \frac{\sqrt{\{2\sigma''(\theta)\}}}{\sigma'(\theta)} \prec 1$$

since $\sqrt{\sigma''} \prec \sigma'$. Thus we obtain
$$\varphi(\alpha) \prec \psi(\theta)$$

as $\lambda \to \infty$.

Next consider the case in which
$$b = 0, \quad a = 0.$$

Then we have
$$\begin{cases} \rho = \Theta(x), & \sigma' = -x^{-1}\Theta_1(x), \\ \Theta \prec \sqrt{\Theta_1}, & \Theta_1 > 1. \end{cases}$$

In this case
$$\varphi = \frac{\Theta}{x(\lambda - \sigma')} = \frac{\sigma' \gamma}{\lambda - \sigma'},$$

$$\gamma = -\frac{\Theta}{\Theta_1}, \quad x\gamma' \prec \gamma, \quad \sigma'' \sim -x^{-1}\sigma'.$$

Hence
$$\frac{\sigma'^2 \gamma'}{\sigma'\gamma' + \sigma''\gamma} \sim \sigma' \frac{x\gamma'}{\gamma} \prec \sigma',$$

and the equation (48) takes the form

$$-\sigma'(x)\,t(x) = \lambda,$$

where
$$t \sim \frac{x\gamma'}{\gamma} < 1$$

as $x \to 0$.

As $x = \alpha$ is the root of this equation, we have

$$\frac{\lambda}{\sigma'(\alpha)} = -\,t(\alpha) < 1,$$

whence

$$\varphi(\alpha) = \frac{\gamma(\alpha)}{\lambda/\sigma'(\alpha) - 1} \sim -\,\gamma(\alpha) = \Theta(\alpha)/\Theta_1(\alpha)\,;$$

and

$$\alpha < \theta$$

by Lemma 5.

Now
$$\psi(\theta) = \frac{\rho(\theta)}{\theta\sqrt{\{2\sigma''(\theta)\}}} \sim \frac{\Theta(\theta)}{\sqrt{\{2\,\Theta_1(\theta)\}}}\,,$$

and hence we have

$$\frac{\varphi(\alpha)}{\psi(\theta)} \sim \sqrt{2}\cdot\frac{\Theta(\alpha)}{\Theta_1(\alpha)}\cdot\frac{\sqrt{\Theta_1(\theta)}}{\Theta(\theta)}.$$

We may write

$$\frac{\Theta(\alpha)}{\Theta_1(\alpha)}\cdot\frac{\sqrt{\Theta_1(\theta)}}{\Theta(\theta)} = \frac{\Theta(\alpha)}{\Theta(\theta)}\cdot\frac{\Theta_1(\theta)}{\Theta_1(\alpha)}\cdot\frac{1}{\sqrt{\Theta_1(\theta)}}$$

$$= \frac{\Theta(\alpha)}{\sqrt{\Theta_1(\alpha)}}\cdot\left\{\frac{\Theta_1(\theta)}{\Theta_1(\alpha)}\right\}^{\frac{1}{2}}\cdot\frac{1}{\Theta(\theta)}\,;$$

and we have

$$\frac{\Theta(\alpha)}{\sqrt{\Theta_1(\alpha)}} < 1, \qquad \frac{1}{\sqrt{\Theta_1(\theta)}} < 1, \qquad \frac{\Theta_1(\theta)}{\Theta_1(\alpha)} < 1.$$

For $\theta < \sqrt{\Theta_1}$, $\Theta_1 > 1$ and $\alpha < \theta$; and

if $\theta < 1$,
$$\frac{\Theta(\alpha)}{\Theta(\theta)} < 1\,;$$

if $\theta \sim A$, $$\frac{\Theta(\alpha)}{\Theta(\theta)} \sim 1;$$

if $\theta > 1$, $$\frac{1}{\Theta(\theta)} < 1.$$

Hence it follows that in all cases we have

$$\frac{\varphi(\alpha)}{\chi(\theta)} < 1.$$

Thus we obtain $$\varphi(\alpha) < \psi(\theta)$$

as $\lambda \to \infty$.

31. (ii) Let $a = -1$ and $\theta > 1$.

In this case we have

$$\rho = x\Theta, \qquad \varphi = \frac{\Theta}{\lambda - \sigma'};$$

and $\dfrac{d\varphi}{dx} = 0$ gives

$$\lambda = -\frac{\sigma''\Theta}{\Theta'} + \sigma'.$$

Now $$-\frac{\sigma''\Theta}{\Theta'} \sim (1+b)\,\sigma'.\,\frac{\Theta}{x\Theta'} > \sigma'$$

since $\theta > x\Theta'$. Hence the equation for $x = \alpha$ takes the form

$$-\sigma'(x)\,t\,(x) = \lambda,$$

where $t \sim -(1+b)\dfrac{\Theta}{x\Theta'} > 1$. Hence we have

$$\frac{-\sigma'(\alpha)}{\lambda} = \frac{1}{t(\alpha)} < 1,$$

whence

$$\varphi(\alpha) = \frac{\Theta(\alpha)}{\lambda - \sigma'(\alpha)} \sim \frac{\Theta(\alpha)}{\lambda};$$

and $$\alpha > \theta$$

by Lemma 5.

Now $$\psi(\theta) = \frac{\rho(\theta)}{\theta\sqrt{\{2\sigma''(\theta)\}}} = \frac{\Theta(\theta)}{\sqrt{\{2\sigma''(\theta)\}}},$$

and hence

$$\frac{\varphi(\alpha)}{\psi(\theta)} \sim \frac{\Theta(\alpha)}{\Theta(\theta)} \cdot \frac{\sqrt{\{2\sigma''(\theta)\}}}{\lambda}$$

$$= -\frac{\Theta(\alpha)}{\Theta(\theta)} \cdot \frac{\sqrt{\{2\sigma''(\theta)\}}}{\sigma'(\theta)} < 1,$$

since $\sqrt{\sigma''} < \sigma'$ and $\frac{\Theta(\alpha)}{\Theta(\theta)} < 1$, as $\theta > 1$ and $\alpha > \theta$.

Thus we obtain $$\varphi(\alpha) < \psi(\theta)$$

as $\lambda \to \infty$.

32. (iii) Let $\rho = Ax\{1-\bar{\rho}(x)\}$, where $A > 0$, $\bar{\rho} > 0$ and $\bar{\rho} < 1$.

In this case

$$\varphi = \frac{A(1-\bar{\rho})}{\lambda-\sigma'},$$

and $\frac{d\varphi}{dx} = 0$ gives

$$\lambda = \frac{\sigma''(1-\bar{\rho})}{\bar{\rho}'} + \sigma'.$$

In § 16, we have seen that

$$\frac{\sigma''(1-\bar{\rho})}{\bar{\rho}'} + \sigma' \sim \frac{\sigma''}{\bar{\rho}'}, \qquad x\bar{\rho}' < 1.$$

Hence, observing that $x\sigma'' \sim -(1+b)\sigma'$, we have

$$\frac{\sigma''}{\bar{\rho}'} \sim -(1+b)\sigma' . \frac{1}{x\bar{\rho}'} > \sigma'.$$

Therefore the equation for $x = \alpha$ takes the form

$$-\sigma'(x)t(x) = \lambda,$$

where $t \sim \frac{1+b}{x\bar{\rho}'} > 1$; and hence

$$\frac{-\sigma'(\alpha)}{\lambda} = \frac{1}{t(\alpha)} < 1,$$

whence $$\varphi(\alpha) = \frac{A\{1-\bar{\rho}(\alpha)\}}{\lambda-\sigma'(\alpha)} \sim \frac{A}{\lambda}.$$

Now $$\psi(\theta) = \frac{\rho(\theta)}{\theta\sqrt{\{2\sigma''(\theta)\}}} \sim \frac{A}{\sqrt{\{2\sigma''(\theta)\}}},$$

and hence, observing that $\lambda = -\sigma'(\theta)$, we have

$$\frac{\varphi(\alpha)}{\psi(\theta)} \sim \frac{\sqrt{\{2\sigma''(\theta)\}}}{\lambda} = \frac{\sqrt{\{2\sigma''(\theta)\}}}{-\sigma'(\theta)} \prec 1$$

since $\sqrt{\sigma''} \prec \sigma'$. Thus we obtain

$$\varphi(\alpha) \prec \psi(\theta)$$

as $\lambda \rightarrow \infty$.

Thus we have completely proved the following proposition.

Let $x \prec \rho \prec x\sqrt{\sigma''}$ *or* $\rho = Ax\{1-\rho(x)\}$, *where* $A > 0$, $\bar{\rho} > 0$ *and* $\rho \prec 1$. *Then*

$$\frac{\rho(\alpha)}{\alpha\{\lambda-\sigma'(\alpha)\}} \prec \frac{\rho(\theta)}{\theta\sqrt{\{2\sigma''(\theta)\}}}$$

as $\lambda \rightarrow \infty$, α *and* θ *being functions of* λ *determined respectively by the equations*

$$\frac{d}{d\alpha}\left[\frac{\rho(\alpha)}{\alpha\{\lambda-\sigma'(\alpha)\}}\right] = 0, \quad \sigma'(\theta)+\lambda = 0.$$

33. In the above arguments, except in a few special cases, the whole thing depends on the fundamental Lemma 4. We can also prove the same proposition, with exception of a few special cases, by a more direct method without recourse to this lemma. The principal object of the method is to find such asymptotic expressions in terms of λ for the functions $\varphi(\alpha)$ and $\psi(\theta)$, which are of convenient forms for the purpose of comparing their order of magnitude as $\lambda \rightarrow \infty$. The analysis is not very difficult and I content myself with giving only the following results.

Du Bois-Reymond proved* that, *if* y *be the root of the equation*

* *Math. Annalen* Bd. VIII (1875), *pp.* 394 *et seq.* Du Bois-Reymond does not state clearly the conditions to which his functions are subjected.

$$y f(y) = \lambda,$$

where $f(y)$ is an L-function such that $y^\delta \succ f(y) \succ (1/y)^\delta$ as $y \to \infty$, then, for large values of λ, we have

$$y = \lambda \{f(\lambda)\}^{-1+\nu}$$

where ν is a certain function of λ, tending to zero as $\lambda \to \infty$.

Easily we can prove that

$$\lambda^\delta \succ \{f(\lambda)\}^{-1+\nu} \succ (1/\lambda)^\delta$$

as $\lambda \to \infty$. Hence we have

Lemma 6. *Let $f(y)$ be an L-function such that*

$$y^\delta \succ f(y) \succ (1/y)^\delta$$

as $y \to \infty$. If $y = \theta$ is the root of the equation

$$y f(y) = \lambda$$

for large values of λ, then θ can be expressed in the form

$$\theta = \lambda\, g(\lambda),$$

where g is a certain function of the same type as f, namely,

$$y^\delta \succ g(y) \succ (1/y)^\delta$$

as $y \to \infty$.

As before, write

$$\rho = x^{-a}\, \Theta(x), \qquad \sigma' = -x^{-(1+b)}\, \Theta_1(x).$$

Then, applying Lemma 6, we arrive at the following result.

Let $x \prec \rho \prec x\sqrt{\sigma''}$ or $\rho = Ax\{1-\bar{\rho}(x)\}$, where $A > 0$, $\bar{\rho} > 0$ and $\bar{\rho} \prec 1$. Then, if $b > 0$, we have, for large values of λ,

$$\varphi(a) = \frac{\rho(a)}{a\{\lambda - \sigma'(a)\}} = \lambda_1^{a-b}\, g(\lambda_1),$$

$$\psi(\theta) = \frac{\rho(\theta)}{\theta\sqrt{\{2\sigma''(\theta)\}}} = \lambda_1^{a-\frac{1}{2}b}\, h(\lambda_1),$$

where $\lambda_1 = \lambda^{\frac{1}{1+b}}$, *and* g, h *are certain functions satisfying the condition*

$$y^{\delta} > f(y) > (1/y)^{\delta}$$

as $y \to \infty$.

Hence we have

$$\frac{\varphi(a)}{\psi(\theta)} = \lambda_1^{-\frac{1}{2}b} \frac{g(\lambda_1)}{h(\lambda_1)} < 1$$

as $\lambda_1 \to \infty$, since $b > 0$. Thus we obtain

$$\varphi(a) < \psi(\theta)$$

as $\lambda \to \infty$.

34. We can now state

Theorem VI. *The integrals*

$$S(\lambda) = \int_0^{\xi} \rho(x)\, e^{i\sigma(x)} \frac{\sin \lambda x}{x}\, dx,$$

$$C(\lambda) = \int_0^{\xi} \rho(x)\, e^{i\sigma(x)} \frac{\cos \lambda x}{x}\, dx,$$

where $l(1/x) < \sigma < (1/x)^{\Delta}$ *and* $\rho < x\sigma'$, *are convergent. The behaviour o these integrals, as* $\lambda \to \infty$, *is determined asymptotically as follows.*

If $x^{\Delta} < \rho < x\sqrt{\sigma''}/\sigma'$, *then*

$$S(\lambda) = O(1/\lambda), \qquad C(\lambda) = O(1/\lambda);$$

if $x\sqrt{\sigma''}/\sigma' < \rho < x\sigma'$, *then*

$$S(\lambda) \sim \frac{\rho(\theta)}{\theta\sqrt{\{2\sigma''(\theta)\}}} e^{(\beta - \frac{1}{4}\pi)i} \sqrt{\pi}, \tag{7}$$

$$C(\lambda) \sim \frac{\rho(\theta)}{\theta\sqrt{\{2\sigma''(\theta)\}}} e^{(\beta + \frac{1}{4}\pi)i} \sqrt{\pi}, \tag{8}$$

where
$$\beta = \lambda\theta + \sigma(\theta),$$

and θ *is determined as a function of* λ *by the equation*

$$\sigma'(\theta) + \lambda = 0.$$

Corollary. *If*

$$\begin{cases} I_1(\lambda) = \int_0^\varepsilon \rho(x) \cos\sigma(x) \frac{\cos \lambda x}{x}\, dx, & I_2(\lambda) = \int_0^\varepsilon \rho(x) \sin\sigma(x) \frac{\cos \lambda x}{x}\, dx, \\ I_3(\lambda) = \int_0^\varepsilon \rho(x) \cos\sigma(x) \frac{\sin \lambda x}{x}\, dx, & I_4(\lambda) = \int_0^\varepsilon \rho(x) \sin\sigma(x) \frac{\sin \lambda x}{x}\, dx, \end{cases}$$

then

$$\begin{cases} I_1(\lambda) = O(1/\lambda), & I_2(\lambda) = O(1/\lambda), \\ I_3(\lambda) = O(1/\lambda), & I_4(\lambda) = O(1/\lambda), \end{cases}$$

or

$$(49) \quad \begin{cases} I_1(\lambda) \sim \dfrac{\rho(\theta)}{\theta\sqrt{\{2\sigma''(\theta)\}}} \cos(\beta + \tfrac{1}{4}\pi)\sqrt{\pi}, \\ I_2(\lambda) \sim \dfrac{\rho(\theta)}{\theta\sqrt{\{2\sigma''(\theta)\}}} \sin(\beta + \tfrac{1}{4}\pi)\sqrt{\pi}, \\ I_3(\lambda) \sim \dfrac{\rho(\theta)}{\theta\sqrt{\{2\sigma''(\theta)\}}} \sin(\beta + \tfrac{1}{4}\pi)\sqrt{\pi}, \\ I_4(\lambda) \sim -\dfrac{\rho(\theta)}{\theta\sqrt{\{2\sigma''(\theta)\}}} \cos(\beta + \tfrac{1}{4}\pi)\sqrt{\pi}, \end{cases}$$

under conditions the same as those of Theorem VI.

It may be remarked that all the integrals $S(\lambda)$, $C(\lambda)$, $I_1(\lambda)$, $I_2(\lambda)$, $I_3(\lambda)$ and $I_4(\lambda)$ tend to zero as $\lambda \to \infty$, when $\rho < x\sqrt{\sigma''}$.

35. In Case (C), the sine-integral $S(\lambda)$ is still convergent when

$$x\sigma' \leqq \rho < \sigma'$$

as $x \to 0$. Hardy has not proceeded to the discussion in this case, his method ceasing to be applicable, as he remarks.* I have succeeded in proving that the formula (7) of Theorem VI holds also in this case generally.

The following proof is not complete, having some inaccuracy in a few special cases. At first it will be shown that the proof may be carried out in its full generality if we make an assumption

* "*O. D. I. 2*" *p. 260.*

which appears to be quite probable ; and after that, a rigorous proof will be given, with exception of a few special cases.

Let

$$(50) \quad \begin{cases} S(\lambda) = \int_0^{\xi} \rho(x)\, e^{i\sigma(x)} \dfrac{\sin \lambda x}{x}\, dx, \\ \overline{S}(\lambda) = \int_0^{\xi} \overline{\rho}(x)\, e^{i\sigma(x)} \dfrac{\sin \lambda x}{x}\, dx, \end{cases}$$

where σ, ρ, $\overline{\rho}$ are L-functions such that

$$(51) \quad l(1/x) \prec \sigma \prec (1/x)^{\Delta}, \qquad x\sigma' \preceq \rho \prec \sigma', \qquad \overline{\rho} \prec \rho$$

as $x \to 0$.

Now we shall assume that, for all sufficiently large values of λ,

$$(52) \qquad \left| \frac{\overline{S}(\lambda)}{S(\lambda)} \right| < K,$$

K being a certain positive constant, independent of λ.

We observe that this relation (52) evidently holds when σ, ρ are the functions treated in Theorems I, III and VI; namely when they belong to each one of the cases

(i) $\qquad \sigma \prec l(1/x), \qquad \rho \prec \sigma', \qquad S(\lambda) \succ 1/\lambda\,;$

(ii) $\qquad \sigma \sim A\,l(1/x), \qquad \rho \prec \sigma', \qquad S(\lambda) \succ 1/\lambda\,;$

(iii) $\qquad \sigma \succ l(1/x), \qquad x\sqrt{\sigma''/\sigma'} \prec \rho \prec x\sigma'.$

In the followings it will be seen that the same relation holds also in our case (51), except in a few special cases. Hence the above assumption seems very likely to be admissible.

36. With the above assumption, we can prove the lemma.

Lemma 7. *If $S(\lambda)$, $\overline{S}(\lambda)$ are the integrals of (50), then*

$$\overline{S}(\lambda) \prec S(\lambda)$$

as $\lambda \to \infty$.

Proof. It is convenient to separate our integrals into the real and imaginary parts; the same methods apply to both parts. Thus we consider

$$I(\lambda) = \int_0^{\xi} \rho(x) \cos\sigma(x) \frac{\sin \lambda x}{x} dx,$$

$$\bar{I}(\lambda) = \int_0^{\xi} \bar{\rho}(x) \cos\sigma(x) \frac{\sin \lambda x}{x} dx.$$

These integrals are convergent, if $\rho < \sigma'$.

Put
$$\varepsilon(x) = \frac{\bar{\rho}(x)}{\rho(x)}.$$

Then $\varepsilon(x)$ is ultimately monotonic and tends to zero as $x \to 0$; and we assume that ξ is chosen sufficiently small to ensure that $\varepsilon(x)$ is monotonic in the interval $0 < x < \xi$.

We may write

$$\bar{I}(\lambda) = \int_0^{\xi} \varepsilon(x) \rho(x) \cos \sigma(x) \frac{\sin \lambda x}{x} dx$$

$$= \int_0^{\xi} \varepsilon(x) f(x) \sin \lambda x \, dx,$$

where
$$f(x) = \frac{\rho(x) \cos \sigma(x)}{x},$$

so that
$$I(\lambda) = \int_0^{\xi} f(x) \sin \lambda x \, dx.$$

Now, corresponding to any prescribed positive number δ, however small, there always exists a positive number ξ', independent of λ, such that

$$0 < \varepsilon(\xi') < \delta, \qquad (0 < \xi' < \xi).$$

We have

$$\bar{I}(\lambda) = \left(\int_0^{\xi'} + \int_{\xi'}^{\xi} \right) \varepsilon(x) f(x) \sin \lambda x \, dx$$

$$= J^{(1)}(\lambda) + J^{(2)}(\lambda)$$

say. Then, in the integral

$$J^{(2)}(\lambda) = \int_{\xi'}^{\xi} \varepsilon(x) f(x) \sin \lambda x \, dx,$$

the coefficient of $\sin \lambda x$ in the subject of integration is absolutely integrable in the range of integration $\xi' \leq x \leq \xi$. Hence, by a well-known theorem,* we have

$$J^{(2)}(\lambda) = o(1)$$

as $\lambda \to \infty$.

In the integral

$$J^{(1)}(\lambda) = \int_0^{\xi'} \epsilon(x) f(x) \sin \lambda x \, dx,$$

$\epsilon(x)$ is monotonic and, being an L-function, has a differential coefficient with a constant sign in the range of integration. Hence, by the Second Mean Value Theorem, we have

$$J^{(1)}(\lambda) = \epsilon(\xi') \int_{\xi_1}^{\xi'} f(x) \sin \lambda x \, dx \qquad (0 < \xi_1 < \xi')$$

$$= \epsilon(\xi') \left(\int_0^{\xi} - \int_0^{\xi_1} \right) f(x) \sin \lambda x \, dx$$

$$= \epsilon(\xi') \{ j(\lambda) - j'(\lambda) \}$$

say. Then

$$j(\lambda) = \int_0^{\xi} f(x) \sin \lambda x \, dx = I(\lambda) - \int_{\xi}^{\xi} f(x) \sin \lambda x \, dx,$$

and

$$\int_{\xi'}^{\xi} f(x) \sin \lambda x \, dx = o(1)$$

as $\lambda \to \infty$, by the same reason as in the case of $J^{(2)}(\lambda)$. Hence

$$j(\lambda) = I(\lambda) + o(1).$$

As to the integral

$$j'(\lambda) = \int_0^{\xi_1} f(x) \sin \lambda x \, dx \qquad (0 < \xi_1 < \xi' < \xi),$$

we observe that the upper limit ξ_1 of integration is a function of λ, and it may be inferred that

$$| j'(\lambda) | < K \, | I(\lambda) |$$

* Hobson, *Theory of Functions of a Real Variable*, p. 672.

for all sufficiently large values of λ, K being a constant independent of λ. For, if not, corresponding to any given K, there would exist some large values of λ for which

$$|j'(\lambda)| > K|I(\lambda)|,$$

and

$$|J^{(1)}(\lambda)| = |\varepsilon(\xi')\{j(\lambda)-j'(\lambda)\}| > \varepsilon(\xi')\Big|(K-1)|I(\lambda)| - |o(1)|\Big|.$$

Hence
$$|J^{(1)}(\lambda)| > \Big|K|I(\lambda)| - o(1)\Big|, *$$
and we obtain

$$|\bar{I}(\lambda)| > K|I(\lambda)|.$$

Therefore it follows that

$$|\bar{S}(\lambda)| > K|S(\lambda)|,$$

contrary to our assumption (52).

Thus we have

$$|j'(\lambda)| < K|I(\lambda)|$$

for all sufficiently large values of λ. Hence we have

$$|J^{(1)}(\lambda)| < \delta K|I(\lambda)| + o(1),$$

and

$$|\bar{I}(\lambda)| < \delta K|I(\lambda)| + o(1),$$

whence it follows that

$$|\bar{S}(\lambda)| < \delta K|S(\lambda)| + o(1).$$

As will be seen presently,† $S(\lambda)$ does not tend to zero as $\lambda \to \infty$. Therefore

$$\left|\frac{\bar{S}(\lambda)}{S(\lambda)}\right| < \delta K + o(1) < (K+1)\delta$$

by choosing λ sufficiently large. Now K is a constant independent of λ and δ may be chosen as small as we please. Hence

* Here K is written for $\varepsilon(\xi')\ (K-1)$ and, as $\varepsilon(\xi')$ is a constant independent of λ, K in this expression may take any large value as we please.

† See the next paragraph 37.

$(K+1)\delta$ may be made as small as we please. Hence it follows that

$$S(\lambda) < S(\lambda)$$

as $\lambda \to \infty$.

37. Now we consider the integral

$$S(\lambda) = \int_0^{\xi} \rho(x)\, e^{i\sigma(x)} \frac{\sin \lambda x}{x} dx.$$

Performing integration by parts, we have

$$S(\lambda) = \quad -i \left[\frac{\rho}{\sigma'} e^{i\sigma} \frac{\sin \lambda x}{x}\right]_0^{\xi} + i \int_0^{\xi} e^{i\sigma} \frac{d}{dx} \left\{ \frac{\rho}{\sigma'} \frac{\sin \lambda x}{x} \right\} dx.$$

Since $\rho < \sigma'$, we obtain

$$S(\lambda) = O(1) + C_1(\lambda) + i\, S_1(\lambda), \tag{53}$$

where

$$(54) \quad \begin{cases} C_1(\lambda) = i\lambda \displaystyle\int_0^{\xi} \frac{\rho}{\sigma'} e^{i\sigma} \frac{\cos \lambda x}{x} dx, \\ S_1(\lambda) = \displaystyle\int_0^{\xi} \rho_1\, e^{i\sigma} \frac{\sin \lambda x}{x} dx, \\ \rho_1 = x \dfrac{d}{dx} \left(\dfrac{\rho}{x\sigma'} \right) = -\dfrac{\rho}{x\sigma'} + \dfrac{\rho'}{\sigma'} - \dfrac{\rho\sigma''}{(\sigma')^2}. \end{cases}$$

Then, in the integral $C_1(\lambda)$, we have

$$x < \frac{\rho}{\sigma'} < x\sigma',$$

since σ, ρ satisfy the first two conditions of (51). Hence Theorem VI may be applied to this integral. Thus we have

$$C_1(\lambda) = i\lambda \frac{\rho(\theta)}{-\theta\sigma'(\theta)\sqrt{\{2\sigma''(\theta)\}}} \{e^{(\beta+\frac{1}{4}\pi)i} \sqrt{\pi} + o(1)\},$$

where $\qquad \sigma'(\theta) + \lambda = 0, \qquad \beta = \lambda\theta + \sigma(\theta).$

Hence we obtain

$$C_1(\lambda) = \frac{\rho(\theta)}{\theta\sqrt{\{2\sigma''(\theta)\}}} \{e^{(\beta-\frac{1}{4}\pi)i} \sqrt{\pi} + o(1)\}. \tag{55}$$

We observe that
$$C_1(\lambda) > 1$$
as $\lambda \to \infty$, since $x\sigma' \leq \rho$.

Now take the integral
$$S_1(\lambda) = \int_0^{\xi} \rho_1 e^{i\sigma} \frac{\sin \lambda x}{x} dx,$$
where
$$\rho_1 = -\frac{\rho}{x\sigma'} + \frac{\rho'}{\sigma'} - \frac{\rho\sigma''}{(\sigma')^2}.$$
If we write
$$(56) \qquad \begin{cases} \rho = x^{-a}\theta, & a \geqq 0, \quad x^{\delta} < \theta < (1/x)^{\delta}, \\ \sigma' = -x^{-(1+b)}\theta_1, & b \geqq 0, \quad x^{\delta} < \theta_1 < (1/x)^{\delta}, \end{cases}$$
we have, by the condition $x\sigma' \leq \rho < \sigma'$,
$$(57) \qquad b \leqq a \leqq 1 + b,$$
and, if $a = b$,
$$\theta_1 \leq \theta;$$
if $a = 1 + b$,
$$\theta_1 > \theta.$$
And we observe that, if $b = 0$,
$$\theta_1 > 1,$$
since $\sigma > l(1/x)$.

From (56), we have
$$x\sigma'' \sim -(1+b)\sigma',$$
and
$$\begin{cases} x\rho' \sim -a\rho & (a > 0), \\ x\rho' < \rho & (a = 0). \end{cases}$$
Hence
$$\rho_1 = -\frac{\rho}{x\sigma'} + \frac{x\rho'}{x\sigma'} - \frac{\rho}{x\sigma'} \cdot \frac{x\sigma''}{\sigma'}$$
$$\sim -(a-b)\frac{\rho}{x\sigma'} < \rho$$
since $x\sigma' > 1$.

Now $S(\lambda)$ cannot tend to zero as $\lambda \to \infty$. For, if $S(\lambda) \prec 1$, then, by the relation (52),* we have

$$S_1(\lambda) \prec 1,$$

and, by (53),
$$C_1(\lambda) + O(1) \prec 1,$$

contradictory to the above result $C_1(\lambda) \succ 1$. Thus $S(\lambda)$ does not tend to zero as $\lambda \to \infty$. Hence, by Lemma 7, we have

$$S_1(\lambda) \prec S(\lambda) \tag{58}$$

as $\lambda \to \infty$. Hence, from (53) and (55), we obtain

$$S(\lambda) \sim \frac{\rho(\theta)}{\theta\sqrt{\{2\sigma''(\theta)\}}}\, e^{(\beta - \frac{1}{4}\pi)i} \sqrt{\pi},$$

which is nothing but the formula (7) in our case. Thus we may state, by combining this result with theorem VI,

Theorem VII. *The integral*

$$S(\lambda) = \int_0^{\xi} \rho(x)\, e^{i\sigma(x)} \frac{\sin \lambda x}{x}\, dx,$$

where $l(1/x) \prec \sigma \prec (1/x)^{\Delta}$ *and* $\rho \prec \sigma'$, *is convergent. The behaviour of* $S(\lambda)$, *as* $\lambda \to \infty$, *is determined asymptotically as follows:*

If $x^{\Delta} \prec \rho \preceq x\sqrt{\sigma''/\sigma'}$, *then*

$$S(\lambda) = O(1/\lambda);$$

if $x\sqrt{\sigma''/\sigma'} \prec \rho \prec \sigma'$, *then*

$$S(\lambda) \sim \frac{\rho(\theta)}{\theta\sqrt{\{2\sigma''(\theta)\}}}\, e^{(\beta - \frac{1}{4}\pi)i} \sqrt{\pi}, \tag{7}$$

where
$$\beta = \lambda\,\theta + \sigma(\theta),$$

and θ *is determined as a function of* λ *by the equation*

$$\sigma'(\theta) + \lambda = 0.$$

---

* Here $S_1(\lambda)$ is replaced for $S(\lambda)$.

38. We now pass to another proof which is quite independent of the assumption (52).

We have to prove the relation (58), or

$$S_1(\lambda) \prec C_1(\lambda)$$

as $\lambda \to \infty$.

At first we consider the case in which

$$b > 0.$$

Now in the integral

$$S_1(\lambda) = \int_0^{\xi} \rho_1(x)\, e^{i\sigma(x)} \frac{\sin \lambda x}{x}\, dx,$$

we have

$$\begin{cases} \rho_1 \sim -(a-b)\dfrac{\rho}{x\sigma'} & (a > b), \\ \rho_1 \prec \dfrac{\rho}{x\sigma'} & (a = b). \end{cases}$$

By (56), $\quad x\sigma' = -x^{-b}\theta_1, \qquad \dfrac{\rho}{x\sigma'} = -x^{-(a-b)}\dfrac{\theta}{\theta_1}.$

Hence, if $a-b<b$, we have

$$\frac{x\sqrt{\sigma''}}{\sigma'} \prec \rho_1 \preceq \frac{\rho}{x\sigma'} \prec x\sigma',$$

and Theorem VI may be applied to the integral $S_1(\lambda)$. Thus we have

$$S_1(\lambda) \sim \frac{\rho_1(\theta)}{\theta\sqrt{\{2\sigma''(\theta)\}}}\, e^{(\beta-\frac{1}{4}\pi)i}\sqrt{\pi}\,;$$

and, as $\rho_1 \prec \rho$, we obtain

$$S_1(\lambda) \prec C_1(\lambda).$$

If $a-b>b$, then, by performing integration by parts, we obtain

$$S_1(\lambda) = O(1) + C_2(\lambda) + i\,S_2(\lambda),$$

where

$$\begin{cases} C_2(\lambda) = i\lambda \int_0^{\xi} \frac{\rho_1}{\sigma'} e^{i\sigma} \frac{\cos \lambda x}{x} dx \sim \frac{\rho_1(\theta)}{\theta\sqrt{\{2\sigma''(\theta)\}}} e^{(\beta-\frac{1}{4}\pi)i}\sqrt{\pi}, \\ S_2(\lambda) = \int_0^{\xi} \rho_2 e^{i\sigma} \frac{\sin \lambda x}{x} dx, \\ \rho_2 = -\frac{\rho_1}{x\sigma'} + \frac{\rho_1'}{\sigma'} - \frac{\rho_1\sigma''}{(\sigma')^2}. \end{cases}$$

Now

$$\begin{cases} \rho_2 \sim -(a-2b)\frac{\rho_1}{x\sigma'} & (a > 2b), \\ \rho_2 \prec \frac{\rho_1}{x\sigma'} & (a = 2b), \end{cases}$$

and

$$\frac{\rho_1}{x\sigma'} \sim (a-b)\, x^{-(a-2b)} \frac{\Theta}{\theta_1^2}.$$

If $a-2b < b$, then

$$\rho_2 \preceq \frac{\rho_1}{x\sigma'} \prec x\sigma',$$

and, by another application of Theorem VI, we obtain

$$S_1(\lambda) \prec C_1(\lambda),$$

since $S_2(\lambda) \prec C_2(\lambda) \prec C_1(\lambda)$ as $\lambda \to \infty$.

If $a-2b \geqq b$, then repeat a similar process.

Since $b > 0$ and $b \leqq a \leqq 1+b$ by (57), there exists a positive integer n such that

$$(n-1)\, b \leqq a < nb.$$

Hence, after repeating n times the above process, we are led to the equation

$$S_{n-1}(\lambda) = O(1) + C_n(\lambda) + i\, S_n(\lambda),$$

where

$$\begin{cases} C_n(\lambda) \sim \frac{\rho_{n-1}(\theta)}{\theta\sqrt{\{2\sigma''(\theta)\}}} e^{(\beta-\frac{1}{4}\pi)i}\sqrt{\pi}, \\ S_n(\lambda) = \int_0^{\xi} \rho_n e^{i\sigma} \frac{\sin \lambda x}{x} dx, \\ \rho_n \preceq \frac{\rho_{n-1}}{x\sigma'} \sim -(a-b)(a-2b)\cdots(a-\overline{n-1}b)\, x^{-(a-nb)} \frac{\Theta}{\theta_1^{\,n}}. \end{cases}$$

Since $a < nb$, Theorem VI may be applied to the integral $S_n(\lambda)$, and we obtain

$$S_n(\lambda) \prec C_n(\lambda) \sim S_{n-1}(\lambda).$$

Since $\rho \succ \rho_1 \succ \cdots \succ \rho_{n-1}$, we have

$$C_1(\lambda) \succ C_2(\lambda) \succ \cdots \succ C_n(\lambda),$$

and hence we obtain

$$S_1(\lambda) \prec C_1(\lambda).$$

Thus, in the case $b > 0$, always we have

$$S(\lambda) \sim C_1(\lambda) \sim \frac{\rho(\theta)}{\theta\sqrt{\{2\sigma''(\theta)\}}} e^{(\beta-\frac{1}{4}\pi)i}\sqrt{\pi}.$$

Thus the proof is completed for the case $b > 0$.

39. Next we consider the case in which

$$b = 0.$$

Thus
$$\sigma' = -x^{-1}\theta_1, \qquad \theta_1 \succ 1,$$
$$\rho = x^{-a}\theta.$$

We observe that, if $a = 1$, $\quad \theta \prec \theta_1$;

if $a = 0$, $\quad \theta \succeq \theta_1$.

In this case $\quad x\sigma' = -\theta_1$.

(i) If $a > 0$, we have

$$\rho_1 = -\frac{\rho}{x\sigma'} + \frac{\rho'}{\sigma'} - \frac{\rho\sigma''}{(\sigma')^2} \sim -a\, x^{-a}\frac{\theta}{\theta_1},$$

and, for any positive integer n,

$$\rho_n \sim (-a)^n\, x^{-a} \frac{\theta}{\theta_1{}^n} \succ -\theta_1 = x\sigma'.$$

Hence, if $a > 0$, then the method of the last paragraph fails.

(ii) Now consider the case in which

$$a = 0.$$

First, let $$\theta \sim A\,\theta_1.$$

Then $$\theta = A\,\theta_1\{1+\varepsilon(x)\},$$

where $\varepsilon(x)$ is an L-function such that $\varepsilon(x) \prec 1$ as $x \to 0$; and we have

$$S(\lambda) = A\int_0^{\xi} \theta_1\, e^{i\sigma}\frac{\sin \lambda x}{x}\,dx + A\int_0^{\xi} \varepsilon\,\theta_1\, e^{i\sigma}\frac{\sin \lambda x}{x}\,dx$$

$$= A\{I_1(\lambda)+I_2(\lambda)\}$$

say. Then, performing integration by parts, we have

$$I_1(\lambda) = O(1) - i\lambda\int_0^{\xi} e^{i\sigma}\cos \lambda x\,dx$$

$$= O(1) - i\lambda\frac{1}{\sqrt{\{2\sigma''(\theta)\}}}\{e^{(\beta+\frac{1}{4}\pi)i}\sqrt{\pi}+o(1)\} \quad \text{[by Theorem VI]},$$

$$\sim \frac{\theta_1(\theta)}{\theta\sqrt{\{2\sigma''(\theta)\}}}e^{(\beta-\frac{1}{4}\pi)i}\sqrt{\pi}.$$

In the integral $I_2(\lambda)$, we have

$$\varepsilon\,\theta_1 \prec \theta_1 = -x\sigma'.$$

Hence, by Theorem VI,

$$I_2(\lambda) \sim \frac{\varepsilon(\theta)\,\theta_1(\theta)}{\theta\sqrt{\{2\sigma''(\theta)\}}}e^{(\beta-\frac{1}{4}\pi)i}\sqrt{\pi},$$

or $$I_2(\lambda) = O(1/\lambda);$$

and we have

$$I_2(\lambda) \prec I_1(\lambda).$$

Therefore

$$S(\lambda) \sim \frac{\rho(\theta)}{\theta\sqrt{\{2\sigma''(\theta)\}}}e^{(\beta-\frac{1}{4}\pi)i}\sqrt{\pi},$$

since $\rho \sim A\,\theta_1$.

Thus, in the case when $a=0$ and $\theta \sim A\,\theta_1$, the truth of the formula (7) is proved.

Next let $$\theta \succ \theta_1.$$

Then in the integral

$$S_1(\lambda) = \int_0^{\varepsilon} e^{i\sigma} \frac{d}{dx}\left(\frac{\rho}{x\sigma'}\right) \sin \lambda x \, dx$$

we have
$$\frac{\rho}{x\sigma'} = -\frac{\theta}{\theta_1} > 1.$$

Let us write

$$\frac{\theta(x, 1)}{\theta_1} = -x\frac{d}{dx}\left(\frac{\theta}{\theta_1}\right),$$

so that
$$\theta(x, 1) = \theta\left\{\frac{x\theta_1'}{\theta_1} - \frac{x\theta'}{\theta}\right\} < \theta$$

since $x\theta' < \theta$ and $x\theta_1' < \theta_1$. We observe that, since $\dfrac{\theta}{\theta_1} > 1$, we have $\theta(x, 1) > 0$, and $\theta(x, 1)$ is a function of the same type as θ, namely

$$x^{\delta} < \theta(x, 1) < (1/x)^{\delta}.$$

Thus we have

$$S_1(\lambda) = \int_0^{\varepsilon} \rho_1 e^{i\sigma} \frac{\sin \lambda x}{x} dx,$$

where
$$\rho_1 = \frac{\theta(x, 1)}{\theta_1}.$$

If $\theta(x, 1) < \theta_1^2$, then $\rho_1 < x\sigma'$ and, as before, applying Theorem VI, we see that

$$S_1(\lambda) < C_1(\lambda) \sim \frac{\rho(\theta)}{\theta\sqrt{\{2\sigma''(\theta)\}}} e^{(\beta - \frac{1}{4}\pi)i}\sqrt{\pi}.$$

If $\theta(x, 1) \sim A\theta_1^2$, then, by proceeding as in the case $\theta \sim A\theta_1$, we easily arrive at the same result.

If $\theta(x, 1) > \theta_1^2$, then repeat a similar process.

Thus we have to consider successively the functions $\theta(x, 1)$, $\theta(x, 2)$, ..., $\theta(x, n)$ defined by the equations

$$
(59)\qquad
\begin{cases}
\dfrac{\theta(x,1)}{\theta_1} = -x\dfrac{d}{dx}\left\{\dfrac{\theta}{\theta_1}\right\}, \\[2ex]
\dfrac{\theta(x,2)}{\theta_1^2} = -x\dfrac{d}{dx}\left\{\dfrac{\theta(x,1)}{\theta_1^2}\right\}, \\[2ex]
\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\[2ex]
\dfrac{\theta(x,n)}{\theta_1^n} = -x\dfrac{d}{dx}\left\{\dfrac{\theta(x,n-1)}{\theta_1^n}\right\},
\end{cases}
$$

where

$$\theta(x, n-1) > \theta_1^n.$$

We easily see that

$$\theta > \theta(x,1) > \theta(x,2) > \cdots\cdots > \theta(x,n).$$

There are two different cases.

(a) For a certain integer n, we have

$$\theta(x,n) \leq \theta_1^{n+1}.$$

In this case, applying Theorem VI, we obtain

$$S_n(\lambda) < S_{n-1}(\lambda) < \cdots\cdots < S_1(\lambda) < C_1(\lambda).$$

Thus, in this case, the formula (7) holds also.

(b) For any integer n, however great, we have always

$$\theta(x,n) > \theta_1^{n+1}.$$

In this case the above method again fails.

We have thus proved that the formula (7) holds also when $x\sigma' \leq \rho < \sigma'$, with the exception of the following special cases.

(i) $\qquad b = 0, \qquad 0 < a \leq 1;$

(ii) $\qquad b = 0, \qquad a = 0, \qquad \theta(x,n) > \theta_1^{4},$

for any integer n, $\theta(x,n)$ being the function defined by the equations (59).

I have already got a certain proof for some of these special cases, but not yet completed it. Perhaps I may return to this problem on another occasion.

40. Here I will give another lemma which will be useful in Part II.

Lemma 8. *Let $\rho(x)$ and $\sigma(x)$ be the L-functions which are treated in Theorem VI, and let $\varpi(x)$ be a real function, not necessarily an L-function, but continuous and differentiable in the interval $(0, \xi)$ save for $x = 0$, satisfying the relation $\varpi(x) \sim \rho(x)$ in such a way that*

$$\varpi(x) = \rho(x)\{1 + \varepsilon(x)\},$$

where $\varepsilon(x)$ is ultimately monotonic and tends to zero as $x \to 0$. If there exists an L-function $\gamma(x)$ such that

$$\varepsilon(x) \sim \gamma(x),$$

then, under the conditions the same as those of Theorem VI, the same asymptotic formulae (7), (8) and (49) hold respectively for the integrals obtained by replacing $\varpi(x)$ for $\rho(x)$ in $S(\lambda)$, $C(\lambda)$, $I_1(\lambda)$, $I_2(\lambda)$, $I_3(\lambda)$ and $I_4(\lambda)$.

Proof. If $\varepsilon(x)$ be an L-function, then the lemma follows immediately from Theorem VI and its corollary.

If $\varepsilon(x)$ is not an L-function, still it behaves like an L-function under our hypothesis and hence the truth of the lemma may be conjectured from Theorem VI.

Take the integral

$$J(\lambda) = \int_0^\xi \varpi(x) \cos\sigma(x) \frac{\cos\lambda x}{x}\, dx$$

$$= \int_0^\xi \rho(x) \cos\sigma(x) \frac{\cos\lambda x}{x}\, dx + \int_0^\xi \varepsilon(x)\, \rho(x) \cos\sigma(x) \frac{\cos\lambda x}{x}\, dx.$$

Let $\bar{\gamma}(x)$ be an L-function such that

$$\gamma(x) < \bar{\gamma}(x) < 1^*$$

as $x \to 0$; and write

$$\bar{\rho}(x) = \rho(x)\,\bar{\gamma}(x), \qquad \bar{\varepsilon}(x) = \frac{\varepsilon(x)}{\bar{\gamma}(x)},$$

so that $$\bar{\rho} < \rho, \qquad \bar{\varepsilon} < 1,$$

since $\varepsilon \sim \gamma$. Then we have

* For instance, we may take $\bar{\gamma} = \{\gamma(x)\}^{\frac{1}{2}}$.

$$J(\lambda) = I_1(\lambda) + \int_0^{\xi} \varepsilon(x)\, \rho(x) \cos \sigma(x)\, \frac{\cos \lambda x}{x}\, dx,$$

where, by the corollary to Theorem VI, we have

$$I_1(\lambda) = O(1/\lambda) \qquad\qquad (\rho \prec x\sqrt{\sigma''}/\sigma'),$$

$$I_1(\lambda) \sim \frac{\rho(\theta)}{\theta\sqrt{\{2\sigma''(\theta)\}}} \cos(\beta + \tfrac{1}{4}\pi)\sqrt{\pi} \qquad (x\sqrt{\sigma''}/\sigma' \prec \rho \prec x\sigma'),$$

θ, β being the same as those in Theorem VI.

Since $\bar\rho \prec \rho$, the integral

$$\int_0^{\xi} \bar\rho(x) \cos \sigma(x)\, \frac{\cos \lambda x}{x}\, dx$$

is evidently convergent when $\rho \prec x\sigma'$; and since, by hypothesis, ϖ is differentiable, ε is also differentiable and $\dfrac{d\varepsilon}{dx}$ has a constant sign in the interval $(0, \xi)$, ξ being chosen sufficiently small. Hence by the Second Mean Value Theorem, we obtain

$$\begin{aligned}
\bar J(\lambda) &= \int_0^{\xi} \bar\varepsilon(x)\, \bar\rho(x) \cos \sigma(x)\, \frac{\cos \lambda x}{x}\, dx \\
&= \varepsilon(\xi) \int_{\xi_1}^{\xi} \bar\rho(x) \cos \sigma(x)\, \frac{\cos \lambda x}{x}\, dx \qquad (0 < \xi_1 < \xi) \\
&= \varepsilon(\xi) \left(\int_0^{\xi} - \int_0^{\xi_1} \right) \bar\rho(x) \cos \sigma(x)\, \frac{\cos \lambda x}{x}\, dx \\
&= \bar\varepsilon(\xi)\, \{ j(\lambda) - j'(\lambda) \}
\end{aligned}$$

say. Then, by the corollary to Theorem VI, we have

$$j(\lambda) = O(1/\lambda),$$

if $\bar\rho \prec x\sqrt{\sigma''}/\sigma'$; and, if $x\sqrt{\sigma''}/\sigma' \prec \bar\rho \prec x\sigma'$,

$$j(\lambda) \sim \frac{\bar\rho(\theta)}{\theta\sqrt{\{2\sigma''(\theta)\}}} \cos(\beta + \tfrac{1}{4}\pi)\sqrt{\pi},$$

θ, β being the same as those in the above formula for $I_1(\lambda)$.

Therefore, from the relation $\bar\rho \prec \rho$, it follows that

$$j(\lambda) \prec I_1(\lambda)$$

as $\lambda \to \infty$, when $x\sqrt{\sigma''/\sigma'} < \rho < x\sigma'$.

In the integral

$$j'(\lambda) = \int_0^{\xi_1} \bar{\rho}(x) \cos \sigma(x) \frac{\cos \lambda x}{x} dx \qquad (0 < \xi_1 < \xi),$$

we observe that ξ_1 varies with λ; still, if we examine the proof of Theorem VI. we can see without difficulty that

$$j'(\lambda) \leq K j(\lambda).$$

Hence

$$J(\lambda) = \epsilon(\xi)\{j(\lambda) - j'(\lambda)\} < I_1(\lambda),$$

if $x\sqrt{\sigma''/\sigma'} < \rho < x\sigma'$. Therefore we have:

If $\rho \leq x\sqrt{\sigma''/\sigma'}$, then $\qquad J(\lambda) = O\{1/\lambda\}$;

if $x\sqrt{\sigma''/\sigma'} < \rho < x\sigma'$, then $\qquad J(\lambda) \sim I_1(\lambda)$.

The same argument applies to the other integrals.

Thus the lemma is completely proved.

VIII *Examples of Case (C)*

41. Let us consider the case in which

$$\rho = x^{-a}, \qquad \sigma = \frac{m}{x},$$

where m is positive, so that

$$I_1(\lambda) = \int_0^{\xi} x^{-a} \cos\left(\frac{m}{x}\right) \frac{\cos \lambda x}{x} dx, \quad I_2(\lambda) = \int_0^{\xi} x^{-a} \sin\left(\frac{m}{x}\right) \frac{\cos \lambda x}{x} dx,$$

$$I_3(\lambda) = \int_0^{\xi} x^{-a} \cos\left(\frac{m}{x}\right) \frac{\sin \lambda x}{x} dx, \quad I_4(\lambda) = \int_0^{\xi} x^{-a} \sin\left(\frac{m}{x}\right) \frac{\sin \lambda x}{x} dx,$$

$$S(\lambda) = \int_0^{\xi} x^{-a} e^{(m/x)i} \frac{\sin \lambda x}{x} dx, \qquad C(\lambda) = \int_0^{\xi} x^{-a} e^{(m/x)i} \frac{\cos \lambda x}{x} dx.$$

Since $\sigma = \dfrac{m}{x} \succ l(1/x)$, Theorems VI and VII are applicable.

Now

$$\sigma' = -\frac{m}{x^2}, \qquad \sigma'' = \frac{2m}{x^3}.$$

The conditions $\rho \leq x\sqrt{\sigma''}/\sigma'$, $\rho < x\sigma'$, $\rho < \sigma'$ give respectively

$$a \leqq -\tfrac{3}{2}, \qquad a < 1, \qquad a < 2.$$

The equation $\lambda + \sigma'(\theta) = 0$ gives

$$\theta = \left(\frac{m}{\lambda}\right)^{\frac{1}{2}},$$

and we have

$$\beta = \lambda\theta + \sigma(\theta) = 2(m\lambda)^{\frac{1}{2}},$$

$$\frac{\rho(\theta)}{\theta\sqrt{\{2\sigma''(\theta)\}}} = \frac{\theta^{\frac{1}{2}-a}}{2\sqrt{m}} = \tfrac{1}{2}m^{-\frac{1}{2}a-\frac{1}{4}}\,\lambda^{\frac{1}{2}a-\frac{1}{4}}.$$

Hence we have:

If $-\frac{3}{2} < a < 1$, then

$$\begin{cases} I_1(\lambda) \sim \frac{1}{2}\pi^{\frac{1}{2}}m^{-\frac{1}{2}a-\frac{1}{4}}\lambda^{\frac{1}{2}a-\frac{1}{4}}\cos\left(2m^{\frac{1}{2}}\lambda^{\frac{1}{2}}+\frac{1}{4}\pi\right), \\ I_2(\lambda) \sim \frac{1}{2}\pi^{\frac{1}{2}}m^{-\frac{1}{2}a-\frac{1}{4}}\lambda^{\frac{1}{2}a-\frac{1}{4}}\sin\left(2m^{\frac{1}{2}}\lambda^{\frac{1}{2}}+\frac{1}{4}\pi\right), \\ C(\lambda) \sim \frac{1}{2}\pi^{\frac{1}{2}}m^{-\frac{1}{2}a-\frac{1}{4}}\lambda^{\frac{1}{2}a-\frac{1}{4}}\exp\left\{\left(2m^{\frac{1}{2}}\lambda^{\frac{1}{2}}+\frac{1}{4}\pi\right)i\right\}; \end{cases}$$

if $-\frac{3}{2} < a < 2$, then

$$\begin{cases} I_3(\lambda) \sim \frac{1}{2}\pi^{\frac{1}{2}}m^{-\frac{1}{2}a-\frac{1}{4}}\lambda^{\frac{1}{2}a-\frac{1}{4}}\cos\left(2m^{\frac{1}{2}}\lambda^{\frac{1}{2}}-\frac{1}{4}\pi\right), \\ I_4(\lambda) \sim \frac{1}{2}\pi^{\frac{1}{2}}m^{-\frac{1}{2}a-\frac{1}{4}}\lambda^{\frac{1}{2}a-\frac{1}{4}}\sin\left(2m^{\frac{1}{2}}\lambda^{\frac{1}{2}}-\frac{1}{4}\pi\right), \\ S(\lambda) \sim \frac{1}{2}\pi^{\frac{1}{2}}m^{-\frac{1}{2}a-\frac{1}{4}}\lambda^{\frac{1}{2}a-\frac{1}{4}}\exp\left\{\left(2m^{\frac{1}{2}}\lambda^{\frac{1}{2}}-\frac{1}{4}\pi\right)i\right\}. \end{cases}$$

We observe that all these integrals tend to zero as $\lambda \to \infty$ if $a < \frac{1}{2}$, and they oscillate if $a \geq \frac{1}{2}$.

42. These results may be verified as follows.

Hardy proved* that

$$\int_0^\infty \cos u \cos\left(\frac{\beta^2}{u}\right)\frac{du}{u^{1-\nu}}$$

$$= \frac{\pi\beta^\nu}{4\sin\frac{1}{2}\nu\pi}\left\{-J^\nu(2\beta)+J^{-\nu}(2\beta)-e^{-\frac{1}{2}\nu\pi i}J^\nu(2i\beta)+e^{\frac{1}{2}\nu\pi i}J^{-\nu}(2i\beta)\right\}.$$

* *Messenger of mathematics* Vol. XL, *pp.* 44 *et seq.*

From this we can easily deduce

$$\int_0^\infty \cos u \sin\left(\frac{\beta^2}{u}\right)\frac{du}{u^{1-\nu}}$$

$$= \frac{\pi\beta^\nu}{4\cos\frac{1}{2}\nu\pi}\{J^\nu(2\beta)+J^{-\nu}(2\beta)+e^{-\frac{1}{2}\nu\pi i}J^\nu(2i\beta)-e^{\frac{1}{2}\nu\pi i}J^{-\nu}(2i\beta)\}.$$

These formulae hold respectively for

$$-1<\nu<1,$$

and for
$$-1<\nu<2,$$

it being understood that, in certain special cases, the expression of the right-hand side must be replaced by its limits. For $\nu=\frac{1}{2}$ they assume the forms

$$\int_0^\infty \cos u \cos\left(\frac{\beta^2}{u}\right)\frac{du}{\sqrt{u}}=\tfrac{1}{2}\sqrt{(\tfrac{1}{2}\pi)}\,(-\sin 2\beta+\cos 2\beta+e^{-2\beta}),$$

$$\int_0^\infty \cos u \sin\left(\frac{\beta^2}{u}\right)\frac{du}{\sqrt{u}}=\tfrac{1}{2}\sqrt{(\tfrac{1}{2}\pi)}\,(\sin 2\beta+\cos 2\beta-e^{-2\beta});$$

and their values are expressible in terms of elementary functions also when

$$\nu=-\tfrac{1}{2},\ \tfrac{3}{2}$$

(the last value, of course, only in the second integral).

Now write λm for β^2, and put $u=\lambda x$ in the integrals. We obtain

$$\int_0^\infty \cos\lambda x\cos\left(\frac{m}{x}\right)\frac{dx}{x^{1-\nu}}=\frac{\pi}{4\sin\frac{1}{2}\nu\pi}\left(\frac{m}{\lambda}\right)^{\frac{1}{2}\nu}\Big[-J^\nu\{2\sqrt{(m\lambda)}\}$$

$$+J^{-\nu}\{2\sqrt{(m\lambda)}\}-e^{-\frac{1}{2}\nu\pi i}J^\nu\{2i\sqrt{(m\lambda)}\}+e^{\frac{1}{2}\nu\pi i}J^{-\nu}\{2i\sqrt{(m\lambda)}\}\Big],$$

$$\int_0^\infty \cos\lambda x\sin\left(\frac{m}{x}\right)\frac{dx}{x^{1-\nu}}=\frac{\pi}{4\cos\frac{1}{2}\nu\pi}\left(\frac{m}{\lambda}\right)^{\frac{1}{2}\nu}\Big[J^\nu\{2\sqrt{(m\lambda)}\}$$

$$+J^{-\nu}\{2\sqrt{(m\lambda)}\}+e^{-\frac{1}{2}\nu\pi i}J^\nu\{2i\sqrt{(m\lambda)}\}-e^{\frac{1}{2}\nu\pi i}J^{-\nu}\{2i\sqrt{(m\lambda)}\}\Big].$$

Now, when β is large,

$$J^{\nu}(2_i\beta) = \frac{1+\epsilon_\beta}{\sqrt{(\pi_i\beta)}} \cos\{2_i\beta - \tfrac{1}{4}(1+2\nu)\pi\},$$

where
$$|\epsilon_\beta| < \frac{K}{\beta}.$$

Hence, when β is large, $-J^{\nu}(2_i\beta)+J^{-\nu}(2_i\beta)$ and $J^{\nu}(2_i\beta)+J^{-\nu}(2_i\beta)$ behave respectively like

$$-\frac{2\sin\frac{1}{2}\nu\pi}{\sqrt{(\pi_i\beta)}}\sin(2_i\beta-\tfrac{1}{4}\pi),$$

and
$$\frac{2\cos\frac{1}{2}\nu\pi}{\sqrt{(\pi_i\beta)}}\cos(2_i\beta-\tfrac{1}{4}\pi).$$

On the other hand

$$e^{-\frac{1}{2}\nu\pi i}J^{\nu}(2i_i\beta)-e^{\frac{1}{2}\nu\pi i}J^{-\nu}(2i_i\beta) = \frac{i}{\sin\nu\pi}e^{\frac{1}{2}\nu\pi i}H_1^{\nu}(2i_i\beta)$$

tends *exponentially* to zero as $\beta\to\infty$ i.e., as $\lambda\to\infty$, so that this term is negligible in comparing with the remaining terms. Hence we obtain the results:—

The integral

$$A(\lambda) = \int_0^\infty x^{\nu}\cos\left(\frac{m}{x}\right)\frac{\cos\lambda x}{x}\,dx$$

is convergent if $-1<\nu<1$, and, as $\lambda\to\infty$, we have

$$A(\lambda) = -\tfrac{1}{2}\pi^{\frac{1}{2}}m^{\frac{1}{2}\nu-\frac{1}{4}}\lambda^{-\frac{1}{2}\nu-\frac{1}{4}}\{\sin(2m^{\frac{1}{2}}\lambda^{\frac{1}{2}}-\tfrac{1}{4}\pi)+o(1)\}.$$

The integral

$$B(\lambda) = \int_0^\infty x^{\nu}\sin\left(\frac{m}{x}\right)\frac{\cos\lambda x}{x}\,dx$$

is convergent if $-1<\nu<2$, and, as $\lambda\to\infty$, we have

$$B(\lambda) = \tfrac{1}{2}\pi^{\frac{1}{2}}m^{\frac{1}{2}\nu-\frac{1}{4}}\lambda^{-\frac{1}{2}\nu-\frac{1}{4}}\{\cos(2m^{\frac{1}{2}}\lambda^{\frac{1}{2}}-\tfrac{1}{4}\pi)+o(1)\}.$$

43. Now put $\nu = -a$ and write

$$A(\lambda) = \left(\int_0^\xi + \int_\xi^\infty\right)x^{-a}\cos\left(\frac{m}{x}\right)\frac{\cos\lambda x}{x}\,dx \qquad (-1<a<1),$$

$$= I_1(\lambda) + J_1(\lambda)$$

say. Then, by writing $f(x)=x^{-1-a}\cos\left(\frac{m}{x}\right)$, we have

$$J_1(\lambda)=-\frac{f(\xi)\sin\lambda\xi}{\lambda}-\frac{1}{\lambda}\int_\xi^\infty f'(x)\sin\lambda x\,dx$$

and
$$f'(x)=-(1+a)\,x^{-(a+2)}\cos\left(\frac{m}{x}\right)+mx^{-(3+a)}\sin\left(\frac{m}{x}\right),$$

whence

$$\left|\int_\xi^\infty f'(x)\sin\lambda x\,dx\right|\leqq a+1\int_\xi^\infty x^{-(2+a)}\,dx+m\int_\xi^\infty x^{-(3+a)}\,dx$$
$$<K,$$

since $a>-1$. Therefore we obtain

$$J_1(\lambda)=O(1/\lambda),$$

and evidently

$$J_1(\lambda)\prec\lambda^{\frac12 a-\frac14}\qquad(-1<a<1),$$

whence it follows that

$$A(\lambda)\sim I_1(\lambda)\qquad\left(\begin{matrix}\nu=-a,\\ -1<a<1\end{matrix}\right).$$

Similarly

$$B(\lambda)\sim I_2(\lambda)\qquad\left(\begin{matrix}\nu=-a\\ -1<a<1\end{matrix}\right).$$

Therefore we obtain

$$\begin{cases} I_1(\lambda)\sim\frac12\pi^{\frac12}m^{-\frac12 a-\frac14}\lambda^{\frac12 a-\frac14}\cos\left(2m^{\frac12}\lambda^{\frac12}+\frac14\pi\right) & (-1<a<1),\\ I_2(\lambda)\sim\frac12\pi^{\frac12}m^{-\frac12 a-\frac14}\lambda^{\frac12 a-\frac14}\sin\left(2m^{\frac12}\lambda^{\frac12}+\frac14\pi\right) & (-1<a<1),\\ C(\lambda)\sim\frac12\pi^{\frac12}m^{-\frac12 a-\frac14}\lambda^{\frac12 a-\frac14}\exp\left\{\left(2m^{\frac12}\lambda^{\frac12}+\frac14\pi\right)i\right\} & (-1<a<1),\end{cases}$$

which agree with the results obtained from our general Theorem VI, only the difference being that the lower limit of a is -1 instead of $-\frac32$, this limitation being introduced from the condition for convergence of the integral $A(\lambda)$.

As Hardy gives in "*O. D. I. 2.*", from the values of the integrals

$$\int_0^\infty\sin u\cos\left(\frac{\beta^2}{u}\right)\frac{du}{u^{1-\nu}},\qquad\int_0^\infty\sin u\sin\left(\frac{\beta^2}{u}\right)\frac{du}{u^{1-\nu}},$$

we can infer that

$$\begin{cases} I_3(\lambda) \sim \frac{1}{2}\pi^{\frac{1}{2}} m^{-\frac{1}{2}a-\frac{1}{4}} \lambda^{\frac{1}{2}a-\frac{1}{4}} \cos(2m^{\frac{1}{2}}\lambda^{\frac{1}{2}} - \frac{1}{4}\pi) & (-1 < a < 2), \\ I_1(\lambda) \sim \frac{1}{2}\pi^{\frac{1}{2}} m^{-\frac{1}{2}a-\frac{1}{4}} \lambda^{\frac{1}{2}a-\frac{1}{4}} \sin(2m^{\frac{1}{2}}\lambda^{\frac{1}{2}} - \frac{1}{4}\pi) & (-1 < a < 2), \\ S(\lambda) \sim \frac{1}{2}\pi^{\frac{1}{2}} m^{-\frac{1}{2}a-\frac{1}{4}} \lambda^{\frac{1}{2}a-\frac{1}{4}} \exp\{(2m^{\frac{1}{2}}\lambda^{\frac{1}{2}} - \frac{1}{4}\pi)i\} & (-1 < a < 2), \end{cases}$$

which also agree with the results obtained from Theorem VII, only the difference being that the lower limit of a is -1 instead of $-\frac{3}{2}$.

Thus our theorems VI and VII are verified.

PART II

Coefficients of Power Series

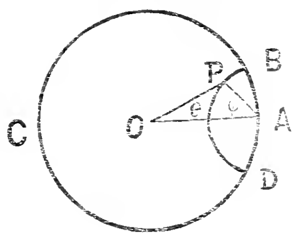
I. *Preliminaries.*

44. Consider a power series

$$\sum_{n=0}^{\infty} a_n z^n, \tag{60}$$

whose radius of convergence is unity, representing a function $f(z)$ which has, on the circle of convergence, one singular point only at $z = 1$, being regular at every other point on it.

Let $ABCD$ be the circle of convergence of the series (60), A being the point $z = 1$ and O the centre. Draw a circular arc BPD inside the circle of convergence, cutting it at the points B and D, with the centre at A and the radius $r_1 < 1$.



Let $I(r_1)$ denote the integral

$$I(r_1) = \frac{1}{2\pi i}\int_{\widehat{DPB}} \frac{f(z)}{z^{a+1}}\,dz, \tag{61}$$

where the path of integration is the arc DPB, starting at the point D, turning round the point A in the clock-wise direction along this arc and ending at the point B.

Lemma 9. *If*

(62) $$\lim_{r_1 \to 0} I(r_1) = 0,$$

then
$$a_n = \frac{1}{2\pi i} \int_{(C)} \frac{f(z)}{z^{n+1}}\, dz,$$

provided that this integral is convergent, the contour (C) of integration being the circle of convergence.

Proof. Since the function $f(z)$ is regular at every point on the circle of convergence except only at the point A, the integral

$$\int_{\widehat{BCD}} \frac{f(z)}{z^{n+1}}\, dz,$$

where the path of integration is the arc BCD, is convergent and so also is the integral $I(r_1)$ for every r_1 such that $0 < r_1 < 1$. Hence, by means of Cauchy's Theorem, we have

$$a_n = \frac{1}{2\pi i} \int_{BCDPB} \frac{f(z)}{z^{n+1}}\, dz$$

$$= \frac{1}{2\pi i} \int_{\widehat{BCD}} \frac{f(z)}{z^{n+1}}\, dz + I(r_1).$$

Now let r_1 tend to zero. Then the arc BCD tends to the whole circle of convergence and we have

$$a_n = \frac{1}{2\pi i} \int_{(C)} \frac{f(z)}{z^{n+1}}\, dz + \lim_{r_1 \to 0} I(r_1) = \frac{1}{2\pi i} \int_{(C)} \frac{f(z)}{z^{n+1}}\, dz$$

since, by hypothesis, $\lim_{r_1 \to 0} I(r_1) = 0$ and the last integral is convergent.

45. Lemma 10. *Let ξ be a small positive constant and*

(63) $$I(n) = \frac{1}{2\pi} \left(\int_0^{\xi} + \int_{2\pi-\xi}^{2\pi} \right) f(e^{\theta i})\, e^{-n\theta i}\, d\theta \qquad (0 < \xi < \pi).$$

Then, if the conditions of Lemma 9 are satisfied, the behaviour of the coefficient a_n of the power series (60), as $n \to \infty$, is asymptotically determined as follows:

If $I(n) \prec \frac{1}{n}$, *then* $a_n = O(1/n)$;

if $I(n) \succ \frac{1}{n}$, *then* $a_n \sim I(n)$.

Proof. By Lemma 9, we have

$$a_n = \frac{1}{2\pi i}\int_{(c)} \frac{f(z)}{z^{n+1}}\,dz = \frac{1}{2\pi}\int_0^{2\pi} f(e^{\theta i})\,e^{-n\theta i}\,d\theta$$

$$= \frac{1}{2\pi}\left(\int_0^{\xi} + \int_{2\pi-\xi}^{2\pi}\right) f(e^{\theta i})\,e^{-n\theta i}\,d\theta + \frac{1}{2\pi}\int_{\xi}^{2\pi-\xi} f(e^{\theta i})\,e^{-n\theta i}\,d\theta$$

$$(0 < \xi < \pi)$$

$$= I(n) + I'(n)$$

say. Then

$$I'(n) = \frac{1}{2\pi}\int_{\xi}^{2\pi-\xi} f(e^{\theta i})\,e^{-n\theta i}\,d\theta$$

$$= \frac{1}{2\pi}\int_{\xi}^{2\pi-\xi} (U+iV)\,(\cos n\theta - i\sin n\theta)\,d\theta,$$

where $$f(e^{\theta i}) = U+iV,$$

U and V denoting real functions of θ. Since $f(z)$ is regular on the circle of convergence, except at $z=1$, the functions

$$U, \quad V, \quad \frac{dU}{d\theta}, \quad \frac{dV}{d\theta}$$

have no singularities and are integrable in the interval $(\xi, 2\pi-\xi)$. Hence

$$\int_{\xi}^{2\pi-\xi} U\cos n\theta\,d\theta = \left|\frac{U\sin n\theta}{n}\right|_{\xi}^{2\pi-\xi} - \frac{1}{n}\int_{\xi}^{2\pi-\xi}\frac{dU}{d\theta}\sin n\theta\,d\theta$$

$$= O(1/n).$$

Similarly the integrals

$$\int_{\xi}^{2\pi-\xi} U\sin n\theta\,d\theta, \quad \int_{\xi}^{2\pi-\xi} V\,{\cos \atop \sin}\,n\theta\,d\theta$$

have values of the same type. Therefore we obtain

$$I'(n) = O(1/n).$$

Thus we have

$$a_n = I(n) + O(1/n).$$

Hence, if $I(n) \leq \frac{1}{n}$, then

$$a_n = O(1/n);$$

if $I(n) > \frac{1}{n}$, then

$$a_n \sim I(n).$$

46. Now we may write

$$I(n) = J(n) + \bar{J}(n), \tag{64}$$

where

$$(65)\begin{cases} J(n) = \dfrac{1}{2\pi}\displaystyle\int_0^{\xi} f(e^{\theta i})\, e^{-n\theta i}\, d\theta, \\ \bar{J}(n) = \dfrac{1}{2\pi}\displaystyle\int_{2\pi-\xi}^{2\pi} f(e^{\theta i})\, e^{-n\theta i}\, d\theta = \dfrac{1}{2\pi}\displaystyle\int_0^{\xi} f\{e^{(2\pi-\theta)i}\}\, e^{n\theta i}\, d\theta, \end{cases}$$

n being a positive integer.

If, in the neighbourhood of $\theta = 0$, both of the functions $f(e^{\theta i})$ and $f\{e^{(2\pi-\theta)i}\}$ take the form

$$\varphi(\theta)\, e^{i\psi(\theta)},$$

where $\varphi(x)$ and $\psi(x)$ are real functions of x such that

$$\varphi(x) \sim \frac{\rho(x)}{x}, \quad \psi(x) \sim \sigma(x)$$

as $x \to 0$, ρ and σ denoting certain L-functions, then the behaviour of $J(n)$ and $\bar{J}(n)$, as $n \to \infty$, may be determined by applying the results of Part I of this paper.

II Case in which the Singularity is of the Type $\frac{1}{(1-z)^p} e^{A/(1-z)^q}$.

47. Let us consider the case in which the function $f(z)$ has a singularity of the type

$$f(z) = \frac{1}{(1-z)^p} e^{A/(1-z)^q},$$

where
$$A = a\, e^{\alpha i},$$

and p, q, a, α are real constants such that

$$p >, =, <0, \quad a > 0, \quad q > 0, \quad 0 \leqq \alpha < 2\pi.$$

It is to be understood that, when p and q are not integers, $(1-z)^p$ and $(1-z)^q$ assume respectively the values

$$e^{p \log (1-z)}, \qquad e^{q \log (1-z)},$$

where $\log (1-z)$ assumes its principal value.

At first we consider the integral

$$I(r_1) = \frac{1}{2\pi i} \int_{\widehat{DPB}} \frac{f(z)}{z^{n+1}}\, dz$$

Let P be any point $z = r\, e^{\theta i}$ on the arc DPB and let ς denote the angle between the radius OA and the straight line AP, namely

$$\varsigma = \angle\, OAP.$$

Then
$$1-z = 1-r\cos\theta - ir\sin\theta = r_1 \cos\varsigma - ir_1 \sin\varsigma$$
$$= r_1\, e^{-\varsigma i},$$

$$\frac{1}{(1-z)^p} = \frac{1}{r_1^{\,p}}\, e^{p\varsigma i},$$

$$\frac{A}{(1-z)^q} = \frac{a}{r_1^{\,q}}\, e^{(\alpha+q\varsigma)i} = \frac{a}{r_1^{\,q}} \{\cos(\alpha+q\varsigma) + i\sin(\alpha+q\varsigma)\},$$

$$f(z) = \frac{1}{r_1^{\,p}} e^{a\cos(\alpha+q\varsigma)/r_1^{\,q}} \cdot e^{\{p\varsigma + a\sin(\alpha+q\varsigma)/r_1^{\,q}\}i},$$

$$dz = i\, r_1\, e^{-\varsigma i}\, d\varsigma.$$

Hence

(66)

$$I(r_1)=\frac{1}{2\pi r_1^{p-1}}\int_{-(\frac{\pi}{2}-\varepsilon)}^{\frac{\pi}{2}-\varepsilon} e^{ar_1^{-q}\cos(\alpha+q\varphi)}\cdot e^{\{(p-1)\varphi+ar_1^{-q}\sin(\alpha+q\varphi)\}i}\cdot\frac{d\varphi}{(1-r_1e^{-\varphi i})^{n+1}},$$

where ε denotes the difference of the angle OAB and a right-angle, namely

$$\frac{\pi}{2}-\varepsilon=\angle OAB=\angle OAD,$$

and we observe that $$\lim_{r_1\to 0}\varepsilon=0.$$

Now, if $\cos(\alpha+q\varphi)>0$ in any part of the range

$$-(\tfrac{\pi}{2}-\varepsilon)\leqq\varphi\leqq\tfrac{\pi}{2}-\varepsilon,$$

then $e^{ar_1^{-q}\cos(\alpha+q\varphi)}$ tends exponentially to infinity as $r_1\to 0$, so that $I(r_1)$ does not necessarily tend to zero. Hence we shall put aside this case and confine ourselves to the case in which

$$\cos(\alpha+q\varphi)\leqq 0,$$

or

(67) $$(4m+1)\frac{\pi}{2}\leqq\alpha+q\varphi\leqq(4m+3)\frac{\pi}{2}\qquad(-\tfrac{\pi}{2}+\varepsilon\leqq\varphi\leqq\tfrac{\pi}{2}-\varepsilon),$$

m being a positive integer or zero.

If we observe that, by hypothesis,

$$0\leqq\alpha<2\pi,$$

and that the condition (67) is to be satisfied by all values of φ in the interval $-\frac{\pi}{2}+\varepsilon\leqq\varphi\leqq\frac{\pi}{2}-\varepsilon$, where ε takes any value corresponding to r_1 which tends to zero, it can easily be inferred that

$$\begin{cases}0<q\leqq 1,\\(1+q)\dfrac{\pi}{2}\leqq\alpha\leqq(3-q)\dfrac{\pi}{2}.\end{cases}$$

Now we can prove the lemma.

Lemma 11. *If* $0<q\leqq 1$, $(1+q)\frac{\pi}{2}\leqq\alpha\leqq(3-q)\frac{\pi}{2}$ *and* $p<1+q$, *then*

$$\lim_{r_1\to 0}I(r_1)=0.$$

Proof. By hypothesis

$$(1+q)\frac{\pi}{2} \leqq \alpha \leqq (3-q)\frac{\pi}{2} \qquad (0 < q \leqq 1).$$

Hence, if we put $\alpha = \frac{\pi}{2} + q\alpha'$,

we obtain

$$\frac{\pi}{2} \leqq \alpha' \leqq \frac{\pi}{q} - \frac{\pi}{2},$$

$$\cos(\alpha + q\varphi) = \cos\{\tfrac{1}{2}\pi + q(\alpha' + \varphi)\} = -\sin q(\alpha' + \varphi),$$

and

$$q\varepsilon \leqq q(\alpha' + \varphi) \leqq \pi - q\varepsilon,$$

since $-\frac{\pi}{2} + \varepsilon \leqq \varphi \leqq \frac{\pi}{2} - \varepsilon$. Hence we have

$$\sin q(\alpha' + \varphi) > 0$$

as $\varepsilon > 0$.

Now, by (66),

$$|I(r_1)| < \frac{1}{2\pi\, r_1^{p-1}} \int_{-\frac{\pi}{2}+\varepsilon}^{\frac{\pi}{2}-\varepsilon} e^{a r_1^{-q} \sin q(\alpha' + \varphi)} \frac{d\varphi}{(1-r_1)^{n+1}}$$

$$< \frac{1}{2q\pi\, r_1^{p-1}(1-r_1)^{n+1}} \int_0^{\pi} e^{-a r_1^{-q} \sin t}\, dt,$$

and

$$\int_0^{\pi} e^{-a r_1^{-q} \sin t}\, dt = 2\int_0^{\frac{\pi}{2}} e^{-a r_1^{-t} \sin t}\, dt$$

$$= \frac{2r_1^q}{a} \int_0^{a/r_1^q} \frac{e^{-u}\, du}{\sqrt{\left(1 - \frac{r_1^{2q}}{a^2} u^2\right)}}.$$

The last integral is convergent and tends to

$$\int_0^{\infty} e^{-u}\, du = 1$$

as $r_1 \to 0$. Hence

$$\int_0^{\pi} e^{-a r_1^{-q} \sin t}\, dt < K\, r_1^q,$$

and we obtain

$$I(r_1) < \frac{K\,r_1^{\,q}}{2q\,\pi\,r_1^{\,p-1}(1-r_1)^{n+1}} = \frac{K\,r_1^{\,1+q-p}}{2q\pi\,(1-r_1)^{n+1}} \to 0$$

as $r_1 \to 0$, provided that $p < 1+q$.

Thus the lemma is proved.

48. We now consider the integral $I(n)$ under the supposition

(68) $$0 < q \leqq 1, \quad (1+q)\frac{\pi}{2} \leqq \alpha \leqq (3-q)\frac{\pi}{2}, \quad p < 1+q.$$

We have

$$1-e^{\theta i} = 2\sin\tfrac{1}{2}\theta\; e^{-\left(\frac{\pi}{2}-\frac{1}{2}\theta\right)i},$$

$$\frac{1}{(1-e^{\theta i})^p} = \frac{1}{(2\sin\frac{1}{2}\theta)^p}\, e^{\frac{1}{2}p(\pi-\theta)i},$$

$$\frac{A}{(1-e^{\theta i})^q} = \frac{a}{(2\sin\frac{1}{2}\theta)^q}\, e^{\left\{\alpha+\frac{1}{2}q(\pi-\theta)\right\}i}.$$

Hence we may write

(69) $$f(e^{\theta i}) = \varphi(\theta)\, e^{i\psi(\theta)},$$

where

(70) $$\begin{cases} \varphi(\theta) = \dfrac{1}{(2\sin\frac{1}{2}\theta)^p}\, e^{a\cos\left\{\alpha+\frac{1}{2}q(\pi-\theta)\right\}/(2\sin\frac{1}{2}\theta)^q}, \\[2ex] \psi(\theta) = \frac{1}{2}p(\pi-\theta) + \dfrac{a\sin\left\{\alpha+\frac{1}{2}q(\pi-\theta)\right\}}{(2\sin\frac{1}{2}\theta)^q}; \end{cases}$$

and

(71) $$f\left\{e^{(2\pi-\theta)i}\right\} = \bar{\varphi}(\theta)\, e^{i\bar{\psi}(\theta)},$$

where

(72) $$\begin{cases} \bar{\varphi}(\theta) = \dfrac{1}{(2\sin\frac{1}{2}\theta)^p}\, e^{a\cos\left\{\alpha-\frac{1}{2}q(\pi-\theta)\right\}/(2\sin\frac{1}{2}\theta)^q}, \\[2ex] \bar{\psi}(\theta) = -\frac{1}{2}p(\pi-\theta) + \dfrac{a\sin\left\{\alpha-\frac{1}{2}q(\pi-\theta)\right\}}{(2\sin\frac{1}{2}\theta)^q}; \end{cases}$$

so that we have, by (64) and (65),

(73) $$\begin{cases} J(n) = \dfrac{1}{2\pi}\displaystyle\int_0^{\varepsilon} f(e^{\theta i})\, e^{-n\theta i}\, d\theta = \dfrac{1}{2\pi}\displaystyle\int_0^{\varepsilon} \varphi(\theta)\, e^{\left\{\psi(\theta)-n\theta\right\}i}\, d\theta, \\[2ex] \bar{J}(n) = \dfrac{1}{2\pi}\displaystyle\int_0^{\varepsilon} f\left\{e^{(2\pi-\theta)i}\right\} e^{n\theta i}\, d\theta = \dfrac{1}{2\pi}\displaystyle\int_0^{\varepsilon} \bar{\varphi}(\theta)\, e^{\left\{\bar{\psi}(\theta)+n\theta\right\}i}\, d\theta, \end{cases}$$

(64) $$I(n) = J(n) + J(n).$$

49. *Integral* $J(n)$. First of all, we shall consider the integral $J(n)$.

Observe that, when θ is very small, we have

$$\frac{1}{2\sin\frac{1}{2}\theta} = \frac{1}{\theta}\{1+O(\theta^2)\},$$

$$\cos\{\alpha+\tfrac{1}{2}q(\pi-\theta)\} = \cos(\alpha+\tfrac{1}{2}q\pi)\{1+O(\theta^2)\}$$
$$+\sin(\alpha+\tfrac{1}{2}q\pi).\,\tfrac{1}{2}q\theta\{1+O(\theta^2)\},$$

$$\sin\{\alpha+\tfrac{1}{2}q(\pi-\theta)\} = \sin(\alpha+\tfrac{1}{2}q\pi)\{1+O(\theta^2)\}$$
$$-\cos(\alpha+\tfrac{1}{2}q\pi).\,\tfrac{1}{2}q\theta\{1+O(\theta^2)\}.$$

Hence the equations (70) may be written in the forms

(70′)
$$\begin{cases} \varphi(\theta) = \dfrac{1}{\theta^p}\{1+O(\theta^2)\}e^{a\theta^{-q}\{\cos(\alpha+\frac{1}{2}q\pi)+\frac{1}{2}q\theta\sin(\alpha+\frac{1}{2}q\pi)\}+O(\theta^{2-q})}, \\ \psi(\theta) = \tfrac{1}{2}p(\pi-\theta)+\dfrac{a}{\theta^q}\{\sin(\alpha+\tfrac{1}{2}q\pi)-\tfrac{1}{2}q\theta\cos(\alpha+\tfrac{1}{2}q\pi)\}+O(\theta^{2-q}). \end{cases}$$

Under the conditions (68), the discussion may be divided into the four cases

(i)	$q=1,$	$\alpha=\pi,$	$p<2,$
(ii)	$0<q<1,$	$\alpha=(1+q)\dfrac{\pi}{2},$	$p<1+q,$
(iii)	$0<q<1,$	$\alpha=(3-q)\dfrac{\pi}{2},$	$p<1+q,$
(iv)	$0<q<1,$	$(1+q)\dfrac{\pi}{2}<\alpha<(3-q)\dfrac{\pi}{2},$	$p<1+q.$

50. (i) The case in which $q=1,\ \alpha=\pi,\ p<2.$ In this case

$$\alpha+\tfrac{1}{2}q\pi = \tfrac{3}{2}\pi.$$

Hence, by (70),

$$\varphi(\theta) = \frac{1}{\theta^p}e^{-\frac{1}{2}a}\{1+O(\theta^2)\},\quad \psi(\theta) = -\frac{a}{\theta}+\tfrac{1}{2}p\pi+O(\theta),$$

and, if we write

$$\rho(x) = x^{-(p-1)}, \quad \sigma(x) = ax^{-1},$$

then

$$J(n) = \frac{1}{2\pi} e^{-\frac{1}{2}a+\frac{1}{2}p\pi i} \int_0^{\xi} \rho(x)\{1+\varepsilon_1(x)+i\varepsilon_2(x)\} e^{-\{nx+\sigma(x)\}i} \frac{dx}{x}$$

$$= \frac{1}{2\pi} e^{-\frac{1}{2}a+\frac{1}{2}p\pi i}\{J_1(n)-iJ_2(n)\},$$

where

$$J_1(n) = \int_0^{\xi} \frac{\rho(x)}{x}\{1+\varepsilon_1(x)+i\varepsilon_2(x)\} \cos\{nx+\sigma(x)\}\, dx,$$

$$J_2(n) = \int_0^{\xi} \frac{\rho(x)}{x}\{1+\varepsilon_1(x)+i\varepsilon_2(x)\} \sin\{nx+\sigma(x)\}\, dx,$$

$\varepsilon_1(x)$ and $\varepsilon_2(x)$ being real functions such that

$$\varepsilon_1(x) = O(x), \quad \varepsilon_2(x) = O(x).$$

We easily see* that there exist certain L-functions $\gamma_1(x)$ and $\gamma_2(x)$ such that

$$\varepsilon_1 \sim \gamma_1, \qquad \varepsilon_2 \sim \gamma_2$$

as $x \to 0$, and also that $\frac{d\varepsilon_1}{dx}$ and $\frac{d\varepsilon_2}{dx}$ have ultimately constant signs.

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* From (69) and (70), we see that

$$\varepsilon_1(x) = \left(\frac{x}{2\sin\frac{1}{2}x}\right)^p \cos\left(\frac{a}{x} - \frac{1}{2}a\cot\frac{x}{2} - \frac{1}{2}px\right) - 1$$

$$= \left(1+\frac{1}{24}x^2+\cdots\right)^p \cos\left\{\left(\frac{a}{12}-\frac{p}{2}\right)x+\cdots\right\} - 1,$$

$$\varepsilon_2(x) = \left(\frac{x}{2\sin\frac{1}{2}x}\right)^p \sin\left(\frac{a}{2} - \frac{1}{2}a\cot\frac{x}{2} - \frac{1}{2}px\right)$$

$$= \left(1+\frac{1}{24}x^2+\cdots\right)^p \sin\left\{\left(\frac{a}{12}-\frac{p}{2}\right)x+\cdots\right\}.$$

Hence ε_1 and ε_2 may be expressed as power series of x, which are uniformly convergent for sufficiently small values of x. Thus the first terms of these series may respectively be taken as γ_1 and γ_2, and it immediately follows that $\frac{d\varepsilon_1}{dx}$ and $\frac{d\varepsilon_2}{dx}$ have ultimately constant signs.

The integrals $J_1(n)$, $J_2(n)$ and $J(n)$ are all convergent, if

$$\rho < x\sigma'$$

as $x \to 0$. Introducing the above expressins of ρ and σ, this condition becomes

$$p < 2,$$

which is nothing but our hypothesis.

Thus, ε_1 and ε_2 having the above properties, Lemma 8 of Part I may be applied to our integrals, and we obtain

$$J_1(n) \sim \int_0^{\xi} x^{-p} \cos\left(nx + \frac{a}{x}\right) dx,$$

$$J_2(n) \sim \int_0^{\xi} x^{-p} \sin\left(nx + \frac{a}{x}\right) dx$$

as $n \to \infty$, and hence we have

$$J_1(n) \sim \pi^{\frac{1}{2}} a^{-\frac{1}{2}p+\frac{1}{4}} n^{\frac{1}{2}p-\frac{3}{4}} \cos(2a^{\frac{1}{2}} n^{\frac{1}{2}} + \tfrac{1}{4}\pi) \qquad (-\tfrac{1}{2} < p < 2),$$

$$J_2(n) \sim \pi^{\frac{1}{2}} a^{-\frac{1}{2}p+\frac{1}{4}} n^{\frac{1}{2}p-\frac{3}{4}} \sin(2a^{\frac{1}{2}} n^{\frac{1}{2}} + \tfrac{1}{4}\pi) \qquad (-\tfrac{1}{2} < p < 2).$$

Hence we have

$$(74) \quad \begin{cases} J(n) \sim \tfrac{1}{2}\pi^{-\frac{1}{2}} a^{-\frac{1}{2}p+\frac{1}{4}} c^{-\frac{1}{2}} n^{\frac{1}{2}p-\frac{3}{4}} \exp\{-(2a^{\frac{1}{2}} n^{\frac{1}{2}} - \tfrac{1}{2}p\pi + \tfrac{1}{4}\pi)i\}, \\ \qquad\qquad (-\tfrac{1}{2} < p < 2). \end{cases}$$

51. (ii) The case in which $0 < q < 1$, $\alpha = (1+q)\frac{\pi}{2}$, $p < 1+q$. In this case

$$\alpha + \tfrac{1}{2}q\pi = \tfrac{1}{2}\pi + q\pi,$$

and this lies between $\frac{1}{2}\pi$ and $\frac{3}{2}\pi$, since $0 < q < 1$. By (70), we have:

If $q \neq \frac{1}{2}$,

$$\begin{cases} \varphi(\theta) = \dfrac{1}{\theta^p} e^{-a\theta^{-q} \sin q\pi} \{1 + O(\theta^{1-q})\}, \\ \psi(\theta) = \dfrac{a}{\theta^q} \cos q\pi + \tfrac{1}{2}p\pi + O(\theta^{1-q}); \end{cases}$$

if $q = \frac{1}{2}$.

$$\begin{cases} \varphi(\theta) = \dfrac{1}{\theta^p} e^{-a\theta^{-q}}\{1+O(\theta^{2-q})\}, \\ \psi(\theta) = \frac{1}{2}p\pi + O(\theta^{\frac{1}{2}}). \end{cases}$$

Hence we have

$$J(n) = \frac{1}{2\pi} e^{\frac{1}{2}p\pi i} \int_0^{\xi} \varpi(x) e^{-\{nx+\sigma(x)\}i} dx,$$

where

$$\begin{cases} \sigma(x) = ax^{-q}\cos q\pi, \\ \varpi(x) = x^{-p} e^{-ax^{-q}\sin q\pi}\{1+\varepsilon_1(x)+i\varepsilon_2(x)\}, \end{cases}$$

ε_1 and ε_2 being real, continuous and differentiable functions of x in the interval $(0, \xi)$, such that

$$\lim_{x\to 0} \varepsilon_1(x) = 0, \qquad \lim_{x\to 0} \varepsilon_2(x) = 0.$$

Since $\sin q\pi > 0$, $\varpi(x)$ tends exponentially to zero as $x \to 0$. Therefore the integral $J(n)$ is absolutely convergent. Now

$$(75) \quad \int_0^{\xi} \varpi(x) e^{-\{nx+\sigma(x)\}i} dx = \int_0^{\xi} \varpi(x)\cos\sigma(x)\cos nx\, dx$$

$$- \int_0 \varpi(x)\sin\sigma(x)\sin nx\, dx$$

$$-i \int_0 \varpi(x)\cos\sigma(x)\sin nx\, dx$$

$$-i \int_0^{\xi} \varpi(x)\sin\sigma(x)\cos nx\, dx.$$

First we consider the integral

$$\int_0^{\xi} \varpi(x)\cos\sigma(x)\cos nx\, dx$$

$$= \frac{\varpi(\xi)\cos\sigma(\xi)\sin n\xi}{n} - \frac{1}{n}\int_0^{\xi} \frac{d}{dx}\{\varpi(x)\cos\sigma(x)\}\sin nx\, dx,$$

and $\quad \left|\int_0^{\xi} \frac{d}{dx}\{\varpi(x)\cos\sigma(x)\}\sin nx\, dx\right| \leqq \int_0^{\xi}\{|\varpi'(x)| + |\varpi(x)\sigma'(x)|\}dx.$

But $|\varpi'(x)| + |\varpi(x)\sigma'(x)|$ tends exponentially to zero as $x\to 0$. Hence the last integral is convergent. Therefore we have

$$\int_0^\xi \varpi(x)\cos\sigma(x)\cos nx\,dx = O(1/n).$$

Similarly the other three integrals on the right-hand side of (75) assume values of the same type.
Thus we obtain

(76) $$J(n) = O(1/n).$$

We observe that, in this case, p may take any value, positive or negative.

52. (iii) The case in which $0<q<1$, $\alpha=(3-q)\frac{\pi}{2}$, $p<1+q$. In this case

$$\alpha+\tfrac{1}{2}q\pi = \tfrac{3}{2}\pi.$$

Hence, by (70′),

$$\varphi(\theta) = \frac{1}{\theta^p}\{1+O(\theta^{1-q})\}, \qquad \psi(\theta) = -\frac{a}{\theta^q}+\tfrac{1}{2}p\pi+O(\theta),$$

and, if we write

$$\rho(x) = x^{-(p-1)}, \qquad \sigma(x) = ax^{-q},$$

then

$$J(n) = \frac{1}{2\pi}e^{\frac{1}{2}p\pi i}\int_0^\xi \frac{\rho(x)}{x}\{1+\epsilon_1(x)+i\epsilon_2(x)\}e^{-\{nx+\sigma(x)\}i}\,dx$$

$$= \frac{1}{2\pi}e^{\frac{1}{2}p\pi i}\{J_1(n)-iJ_2(n)\},$$

where

$$J_1(n) = \int_0^\xi \frac{\rho(x)}{x}\{1+\epsilon_1(x)+i\epsilon_2(x)\}\cos\{nx+\sigma(x)\}\,dx,$$

$$J_2(n) = \int_0^\xi \frac{\rho(x)}{x}\{1+\epsilon_1(x)+i\epsilon_2(x)\}\sin\{nx+\sigma(x)\}\,dx,$$

$\epsilon_1(x)$ and $\epsilon_2(x)$ being real functions such that

$$\epsilon_1(x) = O(x^{1-q}), \quad \epsilon_2(x) = O(x^{1-q}).$$

We easily see* that there exist certain L-functions $\gamma_1(x)$ and $\gamma_2(x)$ such that

$$\varepsilon_1 \sim \gamma_1, \quad \varepsilon_2 \sim \gamma_2$$

as $x \to 0$, and also that $\dfrac{d\varepsilon_1}{dx}$ and $\dfrac{d\varepsilon_2}{dx}$ have ultimately constant signs.

By hypothesis $$p < 1+q,$$

hence the condition $$\rho < \pi\sigma'$$

is satisfied, so that the integrals $J_1(n)$, $J_2(n)$ and $J(n)$ are convergent.

Thus, applying Lemma 8 of Part I, we obtain

$$J_1(n) \sim \int_0^{\xi} x^{-p} \cos\{nx + ax^{-q}\}\, dx,$$

$$J_2(n) \sim \int_0^{\xi} x^{-p} \sin\{nx + ax^{-q}\}\, dx$$

as $n \to \infty$. Hence, we have

$$\begin{cases} J_1(n) \sim \left(\dfrac{2\pi}{1+q}\right)^{\frac{1}{2}} (qa)^{-\frac{p-\frac{1}{2}}{1+q}} n^{\frac{p-1-\frac{1}{2}q}{1+q}} \cos(kn^{\frac{q}{1+q}} + \frac{1}{4}\pi), \\ J_2(n) \sim \left(\dfrac{2\pi}{1+q}\right)^{\frac{1}{2}} (qa)^{-\frac{p-\frac{1}{2}}{1+q}} n^{\frac{p-1-\frac{1}{2}q}{1+q}} \sin(kn^{\frac{q}{1+q}} + \frac{1}{4}\pi), \end{cases}$$

and hence

$$(77)\quad \begin{cases} J(n) \sim \left\{\dfrac{1}{2(1+q)\pi}\right\}^{\frac{1}{2}} (qa)^{-\frac{p-\frac{1}{2}}{1+q}} n^{\frac{p-1-\frac{1}{2}q}{1+q}} \exp\{-(kn^{\frac{q}{1+q}} - \frac{1}{2}p\pi + \frac{1}{4}\pi)i\}, \\ k = (1+q)\, q^{-\frac{q}{1+q}} a^{\frac{1}{1+q}}, \qquad (-\frac{1}{2}q < p < 1+q). \end{cases}$$

---

* The exact forms of ε_1 and ε_2 are

$$\varepsilon_1(x) = \left(\frac{x}{2\sin\frac{1}{2}x}\right)^p e^{-a\sin\frac{1}{2}qx\,(2\sin\frac{1}{2}x)^q} \cos\left\{\frac{a}{x^q} - \frac{a\cos\frac{1}{2}qx}{(2\sin\frac{1}{2}x)^q} - \frac{1}{2}px\right\} - 1,$$

$$\varepsilon_2(x) = \left(\frac{x}{2\sin\frac{1}{2}x}\right)^p e^{-a\sin\frac{1}{2}qx\,(2\sin\frac{1}{2}x)^q} \sin\left\{\frac{a}{x^q} - \frac{a\cos\frac{1}{2}qx}{(2\sin\frac{1}{2}x)^q} - \frac{1}{2}px\right\}.$$

These functions are continuous and differentiable when x is sufficiently small, including the value $x=0$. And we can easily obtain the said result.

53. (iv) The case in which $0 < q < 1$, $(1+q)\frac{\pi}{2} < \alpha < (3-q)\frac{\pi}{2}$, $p < 1+q$. In this case

$$\tfrac{1}{2}\pi + q\pi < \alpha + \tfrac{1}{2}q\pi < \tfrac{3}{2}\pi.$$

Hence
$$\cos(\alpha + \tfrac{1}{2}q\pi) < 0,$$

and therefore, by (70′), we see that $\varphi(\theta)$ tends exponentially to zero as $\theta \to 0$, so that, by proceeding as in the case (ii), we easily obtain

$$(78) \qquad J(n) = O(1/n).$$

Thus we have established the following results.

(i) *If* $q = 1$, $\alpha = \pi$, $p < 2$, *then*

$$\begin{cases} J(n) \sim \tfrac{1}{2}\pi^{-\frac{1}{2}}a^{-\frac{1}{2}p+\frac{1}{4}}e^{-\frac{1}{2}a}n^{\frac{1}{2}p-\frac{3}{4}}\exp\{-(2a^{\frac{1}{2}}n^{\frac{1}{2}} - \tfrac{1}{2}p\pi + \tfrac{1}{4}\pi)i\}, \\ \qquad\qquad (-\tfrac{1}{2} < p < 2); \end{cases}$$

(ii) *if* $0 < q < 1$, $\alpha = (1+q)\frac{\pi}{2}$, $p < 1+q$, *then*

$$J(n) = O(1/n);$$

(iii) *if* $0 < q < 1$, $\alpha = (3-q)\frac{\pi}{2}$, $p < 1+q$, *then*

$$\begin{cases} J(n) \sim \left\{\dfrac{1}{2(1+q)\pi}\right\}^{\frac{1}{2}}(qa)^{-\frac{p-\frac{1}{2}}{1+q}}n^{\frac{p-1-\frac{1}{2}q}{1+q}}\exp\{-(kn^{\frac{q}{1+q}} - \tfrac{1}{2}p\pi + \tfrac{1}{4}\pi)i\}, \\ \qquad k = (1+q)q^{-\frac{q}{1+q}}a^{\frac{1}{1+q}}, \qquad (-\tfrac{1}{2}q < p < 1+q); \end{cases}$$

(iv) *if* $0 < q < 1$, $(1+q)\frac{\pi}{2} < \alpha < (3-q)\frac{\pi}{2}$, $p < 1+q$, *then*

$$J(n) = O(1/n).$$

54. *Integral* $J(n)$. The discussion is quite similar as in the case of $J(n)$.

We have

$$\bar{\varphi}(\theta) = \frac{1}{\theta^p}\{1 + O(\theta^2)\}e^{a\theta^{-q}\{\cos(\alpha - \frac{1}{2}q\pi) - \frac{1}{2}q\theta\sin(\alpha - \frac{1}{2}q\pi)\} + O(\theta^{2-q})},$$

$$\psi(\theta)=-\tfrac{1}{2}p(\pi-\theta)+\frac{a}{\theta^q}\{\sin(a-\tfrac{1}{2}q\pi)+\tfrac{1}{2}q\theta\cos(a-\tfrac{1}{2}q\pi)\}+O(\theta^{2-q});$$

and, without difficulty, we obtain the following results.

(i) *If* $q=1$, $a=\pi$, $p<2$, *then*

$$\begin{cases} J(n)\sim\tfrac{1}{2}\pi^{-\frac{1}{2}}a^{-\frac{1}{2}p+\frac{1}{4}}e^{-\frac{1}{2}a}n^{\frac{1}{2}p-\frac{3}{4}}\exp\{2a^{\frac{1}{2}}n^{\frac{1}{2}}-\tfrac{1}{2}p\pi+\tfrac{1}{4}\pi\ i\}. \\ \qquad\qquad (-\tfrac{1}{2}<p<2); \end{cases}$$

(ii) *if* $0<q<1$, $a=(1+q)\frac{\pi}{2}$, $p<1+q$, *then*

$$\begin{cases} J(n)\sim\left\{\dfrac{1}{2(1+q)\pi}\right\}^{\frac{1}{2}}(qa)^{-\frac{p-\frac{1}{2}}{1+q}}n^{\frac{p-1-\frac{1}{2}q}{1+q}}\exp\{(k\,n^{\frac{1}{1+q}}-\tfrac{1}{2}p\pi+\tfrac{1}{4}\pi)i\}, \\ \qquad k=(1+q)\,q^{-\frac{q}{1+q}}\,a^{\frac{1}{1+q}} \qquad (-\tfrac{1}{2}q<p<1+q); \end{cases}$$

(iii) *if* $0<q<1$, $a=(3-q)\frac{\pi}{2}$, $p<1+q$, *then*

$$J(n)=O(1/n);$$

(iv) *if* $0<q<1$, $(1+q)\frac{\pi}{2}<a<(3-q)\frac{\pi}{2}$, $p<1+q$, *then*

$$J(n)=O(1/n).$$

55. Hence, by (64) and Lemma 10, we obtain:

If $q=1$, $a=\pi$, *and* $-\frac{1}{2}<p<2$, *then*

(79) $a_n\sim\pi^{-\frac{1}{2}}a^{-\frac{1}{2}p+\frac{1}{4}}e^{-\frac{1}{2}a}n^{-\frac{1}{2}p+\frac{3}{4}}\sin\{2a^{\frac{1}{2}}n^{\frac{1}{2}}-(\frac{1}{2}p-\frac{3}{4})\pi\}$;

if $0<q<1$, $a=(1+q)\frac{\pi}{2}$, *and* $-\frac{1}{2}q<p<1+q$, *then*

$$(80)\quad\begin{cases} a_n\sim\left\{\dfrac{1}{2(1+q)\pi}\right\}^{\frac{1}{2}}(qa)^{-\frac{p-\frac{1}{2}}{1+q}}n^{\frac{p-1-\frac{1}{2}q}{1+q}}\exp\left[\{k\,n^{\frac{1}{1+q}}-(\tfrac{1}{2}p-\tfrac{1}{4})\pi\}i\right], \\ \qquad k=(1+q)\,q^{-\frac{q}{1+q}}\,a^{\frac{1}{1+q}}; \end{cases}$$

if $0<q<1$, $a=(3-q)\frac{\pi}{2}$, *and* $-\frac{1}{2}q<p<1+q$, *then*

$$(81)\quad\begin{cases} a_n\sim\left\{\dfrac{1}{2(1+q)\pi}\right\}^{\frac{1}{2}}(qa)^{-\frac{p-\frac{1}{2}}{1+q}}n^{\frac{p-1-\frac{1}{2}q}{1+q}}\exp\left[-\{k\,n^{\frac{1}{1+q}}-(\tfrac{1}{2}p-\tfrac{1}{4}\pi\}i\right], \\ \qquad k=(1+q)\,q^{-\frac{q}{1+q}}\,a^{\frac{1}{1+q}}; \end{cases}$$

$0 < q < 1,\ (1+q)\frac{\pi}{2} < \alpha < (3-q)\frac{\pi}{2}$, *and* $p < 1+q$, *then*

(82) $$a_n = O(1/n).$$

56. We have thus obtained asymptotic formulae for a_n, as $n \to \infty$, in the three cases*

(i)	$q = 1,$	$\alpha = \pi,$
(ii)	$0 < q < 1,$	$\alpha = (1+q)\frac{\pi}{2},$
(iii)	$0 < q < 1,$	$\alpha = (3-q)\frac{\pi}{2},$

always with the condition

$$-\tfrac{1}{2}q < p < 1+q,$$

which is introduced from the conditions that the integral $I(n)$ is convergent and $I(n) \succ 1/n$ as $n \to \infty$. But, by proceeding as follows, it will be seen that this restriction about the value of p may be removed.

Now, in a certain region near the point $z = 1$ and interior to the circle of convergence, we may put

$$\frac{1}{(1-z)^p} e^{A/(1-z)^q} = \sum_{n=0}^{\infty} a_n z^n,$$

and we may differentiate this equation with respect to z, since our series $\sum a_n z^n$ is uniformly convergent in the said region. Thus we obtain

$$\frac{p}{(1-z)^{p+1}} e^{A/(1-z)^q} + \frac{qA}{(1-z)^{p+q+1}} e^{A/(1-z)^q} = \sum_{n=0}^{\infty} (n+1) a_{n+1} z^n.$$

Observing that a_n is a function of n and p, we write

$$a_n = a(n, p).$$

Then we have

$$p\sum_{n=0}^{\infty} a(n, p+1) z^n + qA\sum_{n=0}^{\infty} a(n, p+q+1) z^n = \sum_{n=0}^{\infty} (n+1) a(n+1, p) z^n,$$

* Our method fails to determine the asymptotic formulae in other cases.

whence

$$(83) \qquad p\,a(n, p+1)+qAa(n, p+q+1) = (n+1)\,a(n+1, p)$$
$$(n = 0, 1, 2, \cdots\cdots).$$

Hence

$$(84) \qquad a(n, p+q+1) = \frac{n+1}{qA}\,a(n+1, p)-\frac{p}{qA}\,a(n, p+1).$$

For instance, take the case

$$0 < q < 1, \qquad \alpha = (1+q)\frac{\pi}{2}.$$

Then, by (80), we have

$$\begin{cases} a(n, p) \sim C(q\alpha)^{-(p-\frac{1}{2})/(1+q)} n^{(p-1-\frac{1}{2}q)/(1+q)} \exp\{(kn^{q/(1+q)} -\frac{1}{2}p\pi+\frac{1}{4}\pi)i\}, \\ (-\frac{1}{2}q < p < 1+q), \end{cases}$$

where $$C = \left\{\frac{1}{2(1+q)\pi}\right\}^{\frac{1}{2}}, \quad k = (1+q)\,q^{-\frac{q}{1+q}}\,\alpha^{\frac{1}{1+q}}$$

are independent of n and p.

Let us suppose that

$$p < q,$$

so that $$1+p < 1+q,$$

and we have

$$a(n, p+1) \sim C(q\alpha)^{-(p+\frac{1}{2})/(1+q)} n^{(p-\frac{1}{2}q)/(1+q)} \exp[\{kn^{q/(1+q)} -\tfrac{1}{2}(p+1)\pi+\tfrac{1}{4}\pi\}i],$$

whence

$$a(n, p+1) = O\left\{n^{(p-\frac{1}{2}q)/(1+q)}\right\}.$$

We have also

$$(n+1)\,a(n+1, p) = O\left\{(n+1)^{(p+\frac{1}{2}q)/(1+q)}\right\} = O\left\{n^{(p+\frac{1}{2}q)/(1+q)}\right\}.$$

Hence we have

$$(n+1)\,a(n+1, p) \succ a(n, p+1).$$

Thus we obtain, by (84),

$$a(n, p+q+1) \sim \frac{1}{qA}\,(n+1)\,a(n+1, p).$$

Now

$$qA = qa\, e^{(1+q)\frac{\pi}{2} i},$$

$$a(n+1, p) \sim C(qa)^{-(p-\frac{1}{2})/(1+q)}(n+1)^{(p-1-\frac{1}{2}q)/(1+q)} \exp\left[\{k(n+1)^{q/(1+q)} - \tfrac{1}{2}p\pi + \tfrac{1}{4}\pi\} i\right],$$

and

$$e^{ik(n+1)^{q/(1+q)}} = \exp\left[ik\left\{n^{\frac{q}{1+q}} + O(n^{-\frac{1}{1+q}})\right\}\right]$$

$$= e^{ikn^{q/(1+q)}}\left\{1 + O(n^{-\frac{1}{1+q}})\right\}$$

$$\sim e^{ikn^{q/(1+q)}}.$$

Hence, writing $p_1 = p+q+1$, we obtain

$$a(n, p_1) \sim C(qa)^{-(p_1-\frac{1}{2})/(1+q)} n^{(p_1-1-\frac{1}{2}q)/(1+q)} \exp\left[\{kn^{q/(1+q)} - \tfrac{1}{2}p_1\pi + \tfrac{1}{4}\pi\} i\right],$$

where $$p_1 < 1+2q.$$

Thus, in the formula (80), the upper limit of p is increased by q. By repeating this process m times, the upper limit of p in (80) may be increased by mq. Thus we see that, in the formula (80), p may take any positive value, whatever.

Next, by (83), we have

$$a(n+1, p) = \frac{p}{n+1}\, a(n, p+1) + \frac{qA}{n+1}\, a(n, p+q+1).$$

By repeated applications of this formula, the lower limit of p in the formula (80), may be decreased as much as we please. Thus we see that, in the formula (80), p may take any negative value, whatever.

Therefore the formula (80) holds for all real values of p.

The same argument applies to the other two formulae. Hence we can state

Theorem VIII. *Let $\sum_{n=0}^{\infty} a_n z^n$ be a power series, whose radius of convergence is unity, representing a function $f(z)$ which has on the circle of convergence, one singular point only at $z=1$, being regular at every other point on it. If the singularity is of the type*

$$f(z) = \frac{1}{(1-z)^p}\, e^{A/(1-z)^q}, \quad A = a\, e^{\alpha i},$$

where p is any real constant, $a>0$ and q, α are certain constants, then the behaviour of the coefficient a_n, as $n\to\infty$, is determined asymptotically as follows:

(i) *If $q=1$, $\alpha=\pi$, or the singularity is of the type*

$$f(z)=\frac{1}{(1-z)^p}e^{a/(z-1)} \qquad \begin{pmatrix} p>,=,<0\,; \\ a>0 \end{pmatrix},$$

then

(9) $$a_n\sim\frac{1}{\sqrt{\pi}}a^{-\frac{1}{2}p+\frac{1}{4}}e^{-\frac{1}{2}a}n^{\frac{1}{2}p-\frac{3}{4}}\sin\{2a^{\frac{1}{2}}n^{\frac{1}{2}}-(\tfrac{1}{2}p-\tfrac{3}{4})\pi\}$$

(ii) *If $0<q<1$, $\alpha=(1+q)\frac{\pi}{2}$, or the singularity is of the type*

$$f(z)=\frac{1}{(1-z)^p}e^{-a(\sin\frac{1}{2}q\pi-i\cos\frac{1}{2}q\pi)/(1-z)^q} \qquad \begin{pmatrix} p>,=,<0\,; \\ a>0,\ 0<q<1 \end{pmatrix}.$$

then

(10) $$a_n\sim\left\{\frac{1}{2(1+q)\pi}\right\}^{\frac{1}{2}}(qa)^{-\frac{p-\frac{1}{2}}{1+q}}n^{\frac{p-1-\frac{1}{2}q}{1+q}}\exp\left[\{k\,n^{\frac{q}{1+q}}-(\tfrac{1}{2}p-\tfrac{1}{4})\pi\}i\right],$$

where $$k=(1+q)\,q^{-\frac{q}{1+q}}a^{\frac{1}{1+q}}.$$

(iii) *If $0<q<1$, $\alpha=(3-q)\frac{\pi}{2}$, or the singularity is of the type*

$$f(z)=\frac{1}{(1-z)^p}e^{-a(\sin\frac{1}{2}q\pi+i\cos\frac{1}{2}q\pi)/(1-z)^q} \qquad \begin{pmatrix} p>,=,<0\,; \\ a>0,\ 0<q<1 \end{pmatrix},$$

then

(11) $$a_n\sim\left\{\frac{1}{2(1+q)\pi}\right\}^{\frac{1}{2}}(qa)^{-\frac{p-\frac{1}{2}}{1+q}}n^{\frac{p-1-\frac{1}{2}q}{1+q}}\exp\left[-\{k\,n^{\frac{q}{1+q}}-(\tfrac{1}{2}p-\tfrac{1}{4})\pi\}i\right],$$

k being the same as in the case (ii).

III *Case in which the Singularity is of the Type*

$$\frac{1}{(1-z)^p}e^{A/(1-z)^q}\left(\log\frac{1}{1-z}\right)^r.$$

57. We now pass to the case in which $f(z)$ has a singularity of the type

$$f(z)=\frac{1}{(1-z)^p}e^{A/(1-z)^q}\left(\log\frac{1}{1-z}\right)^r,$$

where $$A = a\, e^{\alpha i},\ a > 0,\ 0 \leqq \alpha < 2\pi,\ q > 0,$$

and p, r denote arbitrary real constants.

First of all, in considering the integral $I(r_1)$, we observe that

$$\log \frac{1}{1-z} = \log\left(\frac{1}{r_1} e^{\varphi i}\right) = \log \frac{1}{r_1} + \varphi i.$$

Hence, without difficulty, we can prove that Lemma 11 holds also in this case; namely, if $0 < q \leqq 1$, $(1+q)\frac{\pi}{2} \leqq \alpha \leqq (3-q)\frac{\pi}{2}$ and $p < 1+q$, then

$$\lim_{r_1 \to 0} I(r_1) = 0$$

Next, when θ is small, we have

$$\begin{aligned}\log \frac{1}{1-e^{\theta i}} &= \log \frac{1}{2 \sin \frac{1}{2}\theta} + (\tfrac{1}{2}\pi - \tfrac{1}{2}\theta) i \\ &= \log \frac{1}{\theta} + \tfrac{1}{2}\pi i + O(\theta) \\ &= \log \frac{1}{\theta}\left\{1 + O\left(1 / \log \frac{1}{\theta}\right)\right\},\end{aligned}$$

whence

$$\left(\log \frac{1}{1-e^{\theta i}}\right)^r = \left(\log \frac{1}{\theta}\right)^r \left\{1 + O\left(1 / \log \frac{1}{\theta}\right)\right\}.$$

Similarly

$$\left(\log \frac{1}{1-e^{(2\pi-\theta)i}}\right)^r = \left(\log \frac{1}{\theta}\right)^r \left\{1 + O\left(1 / \log \frac{1}{\theta}\right)\right\}.$$

Hence the dicussion may be carried out quite similarly as in the preceding case, the presence of the logarithmic factor producing no great change in the analysis, and we content ourselves with giving only the following results.

Theorem IX. *Let* $\sum_{n=0}^{\infty} a_n z^n$ *be a power series, whose radius of convergence is unity, representing a function* $f(z)$ *which has, on the circle of convergence, one singular point only at* $z = 1$, *being regular at every other point on it. If the singularity is of the type*

$$f(z) = \frac{1}{(1-z)^p} e^{A/(1-z)^q} \left(\log \frac{1}{1-z}\right)^r, \quad A = a\, e^{\alpha i},$$

where $a > 0$, q *and* α *denote certain constants, and* p, r *arbitrary real constants, then the behaviour of the coefficient* a_n, *as* $n \to \infty$, *is determined asymptotically as follows.*

(i) *If* $q = 1$, $\alpha = \pi$, *or the singularity is of the type*

$$f(z) = \frac{1}{(1-z)^p} e^{a/(z-1)} \left(\log \frac{1}{1-z}\right)^r,$$

then

$$a_n \sim 2^{-r} \pi^{-\frac{1}{2}} a^{-\frac{1}{2}p+\frac{1}{4}} e^{-\frac{1}{2}a} n^{\frac{1}{2}p-\frac{3}{4}} (\log n)^r \sin \{2a^{\frac{1}{2}} n^{\frac{1}{2}} - (\tfrac{1}{2}p - \tfrac{3}{4})\pi\}.$$

(ii) *If* $0 < q < 1$, $\alpha = (1+q)\frac{\pi}{2}$, *or the singularity is of the type*

$$f(z) = \frac{1}{(1-z)^p} e^{-a(\sin \frac{1}{2}q\pi - i \cos \frac{1}{2}q\pi)/(1-z)^q} \left(\log \frac{1}{1-z}\right)^r,$$

then

$$a_n \sim \frac{1}{\sqrt{(2\pi)}} (1+q)^{-(r+\frac{1}{2})} (qa)^{-\frac{p-\frac{1}{2}}{1+q}} n^{\frac{p-1-\frac{1}{2}q}{1+q}} (\log n)^r \exp\left[\{k\, n^{\frac{q}{1+q}} - (\tfrac{1}{2}p - \tfrac{1}{4})\pi\} i\right],$$

where
$$k = (1+q)\, q^{-\frac{q}{1+q}} a^{\frac{1}{1+q}}.$$

(iii) *If* $0 < q < 1$, $\alpha = (3-q)\frac{\pi}{2}$, *or the singularity is of the type*

$$f(z) = \frac{1}{(1-z)^p} e^{-a(\sin \frac{1}{2}q\pi + i \cos \frac{1}{2}q\pi)/(1-z)^q} \left(\log \frac{1}{1-z}\right)^r,$$

then

$$a_n \sim \frac{1}{\sqrt{(2\pi)}} (1+q)^{-(r+\frac{1}{2})} (qa)^{-\frac{p-\frac{1}{2}}{1+q}} n^{\frac{p-1-\frac{1}{2}q}{1+q}} (\log n)^r \exp\left[-\{k\, n^{\frac{q}{1+q}} - (\tfrac{1}{2}p - \tfrac{1}{4})\pi\} i\right],$$

k *being the same as in the case (ii).*

IV *Case in which the Singularity is of the Type*

$$\frac{1}{(1-z)^p} e^{A/(1-z)^q} \left(l_1 \frac{1}{1-z}\right)^{r_1} \left(l_2 \frac{1}{1-z}\right)^{r_2} \cdots \left(l_h \frac{1}{1-z}\right)^{r_h}.$$

58. More generally we obtain the following theorem, the argument being quite similar as in the preceding case.

Theorem X. *Let* $\sum_{n=0}^{\infty} a_n z^n$ *be a power series, whose radius of convergence is unity, representing a function* $f(z)$ *which has, on the* *circle of convergence, one singular point only at $z = 1$, being regular at every other point on it. If the singularity is of the type*

$$f(z) = \frac{1}{(1-z)^p} e^{A/(1-z)^q} g\left(\frac{1}{1-z}\right),$$

where $g(x) = \{l_1(x)\}^{r_1} \{l_2(x)\}^{r_2} \cdots \{l_h(x)\}^{r_h}$, $A = a\, e^{\alpha i}$, $a > 0$, q *and* α *denote certain constants, and* p *and all* r's *arbitrary real constants, then the behaviour of the coefficient* a_n, *as* $n \to \infty$, *is determined asymptotically as follows.*

(i) *If* $q = 1$, $\alpha = \pi$, *or the singularity is of the type*

$$f(z) = \frac{1}{(1-z)^p} e^{a/(z-1)} g\left(\frac{1}{1-z}\right),$$

then

$$a_n \sim 2^{-r_1} \pi^{-\frac{1}{2}} a^{-\frac{1}{2}p+\frac{1}{4}} e^{-\frac{1}{2}a} n^{\frac{1}{2}p-\frac{3}{4}} g(n) \sin\{2a^{\frac{1}{2}} n^{\frac{1}{2}} - (\tfrac{1}{2}p - \tfrac{3}{4})\pi\}.$$

(ii) *If* $0 < q < 1$, $\alpha = (1+q)\dfrac{\pi}{2}$, *or the singularity is of the type*

$$f(z) = \frac{1}{(1-z)^p} e^{-a(\sin\frac{1}{2}q\pi - i\cos\frac{1}{2}q\pi)/(1-z)^q} g\left(\frac{1}{1-z}\right),$$

then

$$a_n \sim \frac{1}{\sqrt{(2\pi)}} (1+q)^{-(r_1+\frac{1}{2})} (qa)^{-\frac{p-\frac{1}{2}}{1+q}} n^{\frac{p-1-\frac{1}{2}q}{1+q}} g(n) \exp\left[\{k\, n^{\frac{q}{1+q}} - (\tfrac{1}{2}p - \tfrac{1}{4})\pi\} i\right],$$

where $$k = (1+q)\, q^{-\frac{q}{1+q}} a^{\frac{1}{1+q}}.$$

(iii) *If* $0 < q < 1$, $\alpha = (3-q)\dfrac{\pi}{2}$, *or the singularity is of the type*

$$f(z) = \frac{1}{(1-z)^p} e^{-a(\sin\frac{1}{2}q\pi + i\cos\frac{1}{2}q\pi)/(1-z)^q} g\left(\frac{1}{1-z}\right),$$

then

$$a_n \sim \frac{1}{\sqrt{(2\pi)}} (1+q)^{-(r_1+\frac{1}{2})} (qa)^{-\frac{p-\frac{1}{2}}{1+q}} n^{\frac{p-1-\frac{1}{2}q}{1+q}} g(n) \exp\left[-\{k\, n^{\frac{q}{1+q}} - (\tfrac{1}{2}p - \tfrac{1}{4})\pi\} i\right],$$

k being the same as in the case (ii).

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1. As to the general distribution of precipitation in Japan, there is an early investigation of Prof. Kiyoo Nakamura[1] He pointed out a marked difference of the annual course of the precipitation on the Pacific and Japan Sea sides. According to his results, Japan Sea side has abundunt precipitation in autumn and winter compared with spring and summer; the maximum falls in December and the minimum in May; but the seasonal fluctuation is generally small on this side. On the contrary, the Pacific side is characterized by abundunt precipitation in summer and autumn and also by a large fluctuation. He divided the Pacific side into five districts and described the peculiarities of each district in some details. Besides, he alluded to the remarkable effect of topography in some examples.

Recently, a Decennial Report of Precipitation,[2] 1901–1910, was published by the Central Meteorological Observatory, in which monthly records of observations in 1570 stations are given. In a note appended to the Report, Prof. Fujiwhara gave a brief account of the general distribution of precipitation in entire Japan, and confirmed in the main the results obtained by Prof. Nakamura. He also discussed the dependency of precipitation on the latitude

1) K. Nakamura, Dai-Nippon Hûdohen 大日本風土編 (Climatology of Japan), 1897, Chapter VI.
2) Uryô-Zyûnenhô 雨量十年報. 1914.

and pointed out a peculiar fact that the decrease with the latitude is comparatively small in Southern Japan, but remarkable in the Northern, the dependency being quite different from those obtained by Murray and Supan.[1] He compared the relation with that of the temperature and precipitation and suggested an intimate physical connection between the latter elements. Moreover, he gave brief but suggestive discussions on the influences of the shielding mountains, altitude of the station, the slope of land, the distance from the sea *etc*.

In a previous communication,[2] we have shown a remarkable influence of topographical condition on the distribution of rain accompanying cyclone. The present note which gives a resumé of some statistical investigations on the relation of the geographical distribution of the yearly and mean monthly amounts of precipitation with the prevailing barometric gradients, may be regarded as a supplement to the previous note.

It may be remarked that the subject in question is not without interests also from the seismological point of view, since as already shown by Prof. Omori,[3] there exists a correlation between the yearly seismic frequency of some localities and the annual amount of precipitation in some other districts. It seems, however, still an open question whether the precipitation is the direct agent acting as a secondary cause of earthquakes, or it is rather the barometric gradient which affects the seismic origin and the precipitation at the same time. We will add later a few remark on this latter point, though unfortunately we were not yet able to trace the relation in any conclusive manner, on account of the want of data.

2. The data used for the yearly mean barometric pressure and precipitation were taken from Kisyôyôran of the Central Meteorological Observatory, the epoch ranging from 1900 to 1917, while for the monthly means, the materials were taken from the

1) Hann. Lehrbuch, 2. Aufl. p. 295.
2) Terada, Yokota and Otuki, Journ. Coll. Sci. 37, Art. 4, (1916).
3) F. Omori, Bull. I.E.I.C. Vol. II. No. 2.

Climatological Table of Japan[1] recently issued by the Central Meteorological Observatory.

Since it was our main purpose to study the effect of the discontinuity of wind velocity on land and sea, only those stations were chosen which are situated not far from the open coast facing either to the Pacific or to Japan Sea. The stations of which the yearly courses of rainfall were to be studied were grouped into the following six regions:

I. Akita and Niigata, representing the Northern Japan Sea coast.

II. Miyako, Isinomaki and Tyôsi, the Northern Pacific coast.

III. Husiki, Kanazawa and Hukui, the central Japan Sea coast.

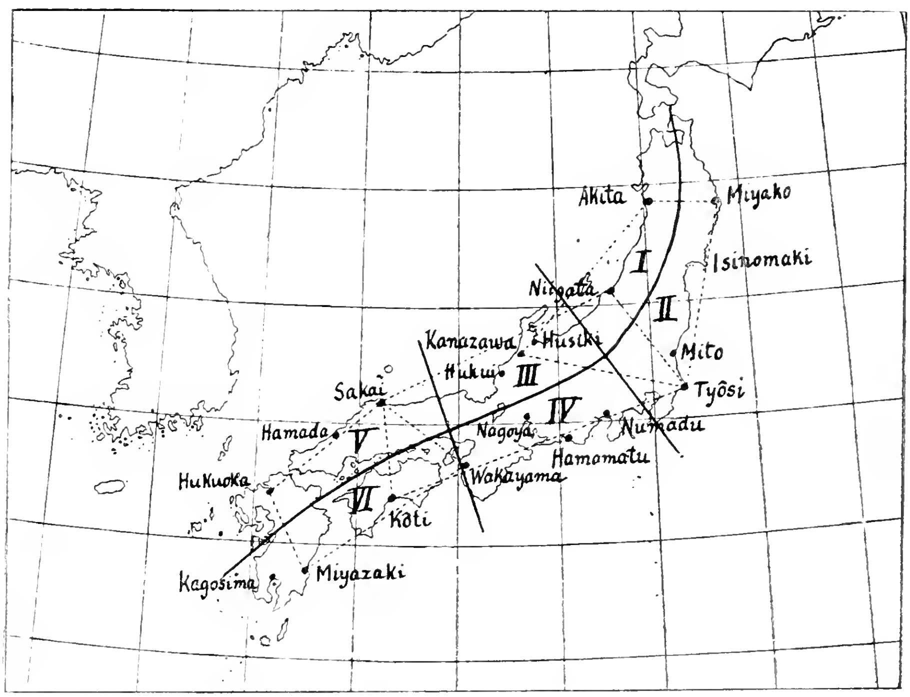

Fig. 1.

1) Kisyô-Zassan (氣象雜纂) Vol. I. No. 4, (1918).

IV. Numadu, Hamamatu and Nagoya, the central Pacific coast.

V. Sakai, Hamada and Hukuoka, the Southern Japan Sea coast.

VI. Kôti, Miyazaki and Kagosima, the Southern Pacific coast.

The distribution of the stations are shown in Fig. 1.

The choice of the stations may seem somewhat arbitrary, but we were led to it by different subsidiary considerations.

In some stations, the observations were interrupted once or twice during the interval of eighteen years taken. In order to fill up the gap, the following procedure was taken. When the amount of precipitation is wanting for a certain year at a given station, the ratio of the mean amount for the remaining seventeen years for the other stations in the same group to that of the station in question was taken and multiplied to the mean value of the year concerned of the other stations and the result was assumed as the reduced value to be replaced for the wanting data.

Though the annual amounts of precipitation recorded in Kisyô-yôran are given in mm. and its fractions so that the numbers are made up of five figures, it was considered convenient and rather reasonable for the present purpose to cut the figures to only three, stopping at the place of cm.

The data thus reduced are given in Table I.

TABLE I.*

Group	Year	1900	1901	1902	1903	1904	1905	1906	1907	1908	1909	1910	1911	1912	1913	1914	1915	1916	1917	Mean
I	Akita	186	165	185	214	196	196	154	155	154	148	184	220	175	178	198	177	167	234	183
	Niigata	195	186	169	188	(204)	213	171	174	161	169	204	223	181	189	191	172	192	232	190
	Mean	191	176	177	201	200	205	163	165	158	159	194	222	178	184	195	175	180	233	186
	%	103	95	95	108	108	110	88	89	85	86	104	119	96	99	105	94	97	125	M.a. = 8.3
	100 Anom. / Mean Anom.	34	-57	-57	91	91	114	-136	-125	170	-159	45	216	-45	11	57	-68	-43	284	
II	Miyako	97	160	133	179	146	144	124	124	109	140	161	192	146	114	98	151	134	133	138
	Isinomaki	91	123	101	124	128	(118)	104	92	112	118	137	133	120	111	103	122	130	112	116
	Mito	130	124	162	163	128	151	120	153	142	154	174	168	170	166	146	163	773	133	151
	Mean	106	136	132	155	134	138	116	123	121	137	157	164	145	130	116	145	146	126	135
	%	79	101	98	115	99	102	83	91	90	102	116	121	107	96	86	107	108	93	M.a. = 8.9
	100 Anom. / Mean Anom.	-236	11	-22	169	-11	22	-157	-101	-112	22	180	236	79	-45	-157	79	90	-79	
III	Husiki	218	211	201	256	222	267	218	(197)	200	241	252	(253)	210	215	237	182	235	301	229
	Kanazawa	254	256	253	299	268	287	263	223	254	274	258	294	242	268	252	212	272	348	269
	Hukui	230	221	239	245	265	230	226	218	250	250	269	271	238	237	203	222	264	339	245
	Mean	234	229	231	267	252	261	236	213	235	255	260	273	230	240	231	205	257	329	247
	%	95	93	94	108	102	106	96	86	95	103	105	111	93	97	94	83	104	133	M.a. = 8.1
	100 Anom. / Mean Anom.	-62	-86	-74	99	25	74	-49	-173	-62	37	62	136	-86	-37	-74	210	49	407	
IV	Numadu	177	168	215	279	176	235	192	233	222	203	306	250	185	157	200	331	219	188	213
	Hamamatu	193	203	201	250	176	210	174	248	212	213	264	259	169	141	189	194	214	207	207
	Nagoya	156	144	188	213	209	212	155	192	175	168	176	168	136	130	134	179	183	172	172
	Mean	175	172	201	247	187	219	174	224	203	195	249	226	163	143	174	201	205	189	197
	%	89	87	102	125	95	111	88	114	103	99	126	115	83	73	88	102	104	96	M.a. = 11.3
	100 Anom. / Mean Anom.	-97	-115	18	221	-44	97	-112	124	27	9	230	133	-150	-239	-106	18	35	-35	
V	Sakai	172	175	209	193	215	223	192	197	178	183	198	209	187	178	190	203	226	191	196
	Hamada	144	147	157	159	155	178	174	149	137	164	165	182	155	131	161	153	198	178	160
	Hukuoka	180	171	174	146	152	220	161	150	172	180	172	(187)	159	126	182	190	168	168	170
	Mean	165	164	180	166	174	207	176	165	162	176	178	193	167	145	178	182	197	179	175
	%	94	94	103	95	99	118	101	94	93	101	102	110	95	83	102	104	113	102	M.a. = 6.1
	100 Anom. / Mean Anom.	-98	-98	49	-82	-16	295	16	-98	-115	16	33	164	-82	-279	33	66	213	33	
VI	Kôti	300	260	335	313	180	324	243	252	284	241	230	323	250	234	263	281	338	220	271
	Miyazaki	261	261	276	261	150	345	270	(238)	252	274	201	254	266	206	271	265	266	262	254
	Kagosima	192	274	279	217	148	355	283	220	228	232	205	249	284	160	239	275	182	191	234
	Mean	251	265	297	264	159	341	265	237	255	249	212	275	267	200	258	274	262	234	253
	%	99	105	117	104	63	135	105	94	101	98	84	109	106	79	102	108	104	89	M.a. = 10.6
	100 Anom. / Mean Anom.	-9	47	160	38	-349	330	47	-57	9	19	-151	85	57	-198	19	75	38	-104	

* For each station, the amount of precipitation is given in cm. The line % gives the amount in percentage of the mean value averaged over 18 years. The next line with the entry, 100 Anom./Mean Anom., is the deviation of the percentage value given in the preceding line from 100 expressed in percentage of the mean anomaly or deviation given in the last column. ( ) means the reduced value for the year **devoid of the datum.**

3. Referring to the above Table, it will be remarked that the mean annual precipitation is most abundunt in the region VI and least in II. In the middle Japan, the Japan Sea side III has decidedly larger precipitation than the Pacific side IV. On the contrary, in the SW Japan, the Pacific side VI has 1.44 times more precipitation than the Japan Sea side. In the NE part of Japan, it is again the Japan Sea side which has more precipitation, though the contrast is comparatively more pronounced than in the case of III and IV. The mean values of I-II, III-IV and V-VI are respectively 161, 222 and 214, while the difference, Japan Sea side minus Pacific side are respectively 51, 50 and −78.

According to our previous theory, this result might have been explained at least qualitatively, if we could assume an area of inland high pressure near the junction of the central and SW parts of Japan. That this is really the case may be seen from the annual isobar map. In the map, we see a general fall of pressure from the Asiatic continent toward the Pacific. The general slope is, however, disturbed by a remarkable tong or promontory protruding from Corea toward the central part of Japan along the axial line of the land. Such a distribution may conveniently be interpreted as due to the superposition of an elongated area of high pressure having its ridge near the western part of Honsiu, upon the nearly uniform slope from the continent to the ocean. Hence, according to our rule, the Japan Sea side has more and the Pacific less precipitation on the NE side of high area, compared with the ideal case where no such high existed. The reverse may be said with regard to the SW side of the high. The very marked contrast between III-IV and V-VI, shows that the above effect of the "tong" is rather prominent.

4. In discussing the matter more minutely, it is desirable to take the seasonal distribution of precipitation instead of the annual. In Table II. the mean monthly amounts of precipitation are given for the six regions concerned:

Table II.*

Group	Month	I	II	III	IV	V	VI	VII	VIII	IX	X	XI	XII	M = Mean S = sum
	Akita	122	99	106	111	113	146	198	187	191	162	184	171	S=1789
	Niigata	193	127	108	109	92	132	166	130	183	154	184	235	S=1813
	Mean	158	113	107	110	103	139	182	159	187	158	184	203	M=150
	%	105	75	71	73	69	93	121	106	125	105	123	135	M.a.=19.9
	100 Anom. / Mean Anom.	25	-126	-146	-136	-156	-35	106	30	126	25	116	176	
II	Miyako	71	66	88	99	124	135	143	179	206	161	69	63	S=1404
	Isinomaki	47	49	76	95	121	124	148	122	156	119	56	44	S=1157
	Mito	61	59	119	141	161	163	150	165	211	159	67	55	S=1509
	Mean	60	58	94	112	135	141	147	155	191	146	64	54	M=113
	%	53	51	83	99	119	125	130	137	169	129	57	48	M.a.=34.8
	100 Anom. / Mean Anom.	-135	-141	-49	-3	55	72	86	106	198	83	-124	-149	
III	Husiki	239	164	137	134	114	151	193	160	204	151	200	323	S=2172
	Kanazawa	269	186	164	170	147	181	202	180	226	194	259	357	S=2534
	Hukui	249	218	163	159	151	192	190	168	188	166	213	335	S=2391
	Mean	252	189	155	154	137	175	195	169	206	170	224	338	M=197
	%	128	96	79	78	70	89	99	86	105	86	114	172	M.a.=19.7
	100 Anom. / Mean Anom.	142	-20	-107	-112	-152	-56	-5	-71	25	-71	71	366	
IV	Numadu	83	77	150	200	187	220	234	229	280	172	104	75	S=2012
	Hamamatu	63	68	147	203	212	240	214	205	271	157	100	69	S=1949
	Nagoya	60	64	129	170	166	225	192	181	243	148	81	52	S=1712
	Mean	69	70	142	191	188	228	213	205	265	159	95	65	M=158
	%	44	44	90	121	119	144	135	130	168	101	60	41	M.a.=36.6
	100 Anom. / Mean Anom.	-153	-153	-27	57	52	120	96	82	186	3	-109	161	
V	Sakai	203	143	144	136	113	166	163	131	221	162	157	195	S=1931
	Hamada	115	98	112	138	123	193	164	128	177	121	100	114	S=1592
	Hukuoka	68	81	112	136	125	249	245	140	189	101	72	76	S=1583
	Mean	129	74	123	136	120	203	191	133	196	128	110	128	M=142
	%	91	52	87	96	85	143	135	93	138	90	77	90	M.a.=21.3
	100 Anom. / Mean Anom.	-42	226	-61	-19	-70	202	164	-33	178	-47	-108	-47	
VI	Koti	69	98	186	300	292	355	328	279	419	225	125	79	S=2754
	Miyazaki	81	101	187	245	271	383	270	255	349	226	125	74	S=2566
	Kagosima	92	92	158	232	229	415	291	175	240	139	91	84	2240
	Mean	81	97	177	259	264	384	296	236	336	197	114	79	M=210
	%	39	46	84	123	126	183	141	112	160	94	54	38	M.a.=40.8
	100 Anom. / Mean Anom.	-150	-132	-39	56	64	203	100	29	147	-15	-113	-152	

* Precipitation is given in mm. The line % gives the monthly mean of different stations expressed in % of the mean for all month. The next line gives the deviation of the % values from 100 expressed in % of the mean value of diviation (m.a.)

Table III.

Mean & Diff.	I	II	III	IV	V	VI	VII	VIII	IX	X	XI	XII
½(I+II)	109	86	101	111	119	140	165	157	189	152	124	129
½(III+IV)	161	130	149	173	163	202	204	187	236	195	160	202
½(V+VI)	105	86	150	198	192	294	244	185	266	163	112	104
I—II	98	55	13	**−2**	**−32**	**−2**	35	4	**−4**	12	120	149
III—IV	183	119	13	**−37**	**−51**	**−53**	**−18**	**−36**	**−59**	11	129	273
V—VI	48	**−23**	**−54**	**−123**	**−144**	**−181**	**−105**	**−103**	**−140**	**−69**	**−4**	49

In Table III., we give the mean values and the difference of the Japan Sea and Pacific sides of the NE, Central and SW parts of Japan respectively. Referring to the latter, we may remark several interesting facts. Firstly, the mean values of the both sides, are greatest in the Middle part III-IV, during the colder months Oct.-Feb., while it is so in the SW part during the warmer months, Mar.-July and Sept. While during the colder months, the NE and SW regions have nearly same amounts of precipitation, contrasted with the large amount in the Middle Japan, the amount of precipitation during the warmer season continously falls off from the SW towards the NE districts. Secondly, the difference, Japan Sea side minus Pacific side show quite parallel course, but the mean value successively falls off from the Northern part to the Southern. Thirdly, among different possible combinations of cases,

I>II, III>IV, V>VI occurs in January and December,
I>II, III>IV, V<VI occurs in February, March, October and November,
I<II, III<IV, V<VI occurs in April, May, June, and September,
I>II, III<IV, V<VI occurs in July and August.

But the remaining two cases:

I<II, III<IV, V>VI, and I<II, III>IV, V>VI

do not occur actually.

A full discussion of these characteristic distributions will be given in the later part of this paper. Here, it may suffice to draw the attention of the readers to the remarkable mode of distribution in which the influence of topography plays an important rôle.

5. Turning our attention to the amount of the yearly fluctuation, the most prominent feature is the difference of the mean anomalies on the both sides of the land, the Japan Sea side showing generally less fluctuation than the Pacific. On the latter side the amount is least in II and greatest in VI, while on the Japan Sea side it is the least in V.

In Table I, we have given the yearly amount of precipitation also in percentages of the mean value and besides, the percentage fluctuation of the yearly percentage values are given in the next line. We reproduce here in Table IV. the mean anomalies of the percentage values for the six regions. On the Pacific side the

TABLE IV.

	NE	Middle	SW	Mean
Japan Sea Side	I 8.8	III 8.1	V 6.1	7.7
Pacific Side	II 8.9	IV 11.3	VI 10.6	10.3
Mean	8.85	9.7	8.35	9.0
Difference	0.1	3.2	4.5	2.6

anomaly is the least on the northern part of Japan whereas in the Japan Sea side it is least in the Southern. While the mean values of the anomalies of the both sides are not very different for the different parts, the contrast between the both sides is most prominent in the southern part.

6. The mean of the amount of precipitation for the Japan Sea side I, III, V is 203 and for the Pacific side II, IV, VI 195. The difference is not remarkable. On the other hand, the mean anomaly for the Japan Sea side is 7.7 while is it 10.3 for the Pacific. Thus it appears that the Pacific side is generally more "sensitive" to the main causes of the fluctuation, among which

the barometric gradiet must be one of the most important. If such be the case, due consideration must be paid to this point, if we are to attempt to compare the amount of precipitation on both sides of the land and thereby deduce some topographical relation with respect to the barometric gradient. If this precaution is not made, the difference between the both sides will be largely determined by the side on which the amount is decidedly larger. Hence the deviation of the yearly percentage value from the mean, *i. e.* 100, was divided by the mean anomaly. These quotients (Table I.) were adopted for the final data to be used in the comparison with the barometric gradient.

7. The barometric gradients of the different parts of Japan, to be used for the comparison with the precipitation may be obtained in different ways. Referring to the mean annual isobar chart, the isobaric surface over the land is far from being nearly plane, chiefly due to the protruding area of high pressure lying along the axial line of the land. On account of the latter, the actual gradients on both sides in different parts of Japan may differ considerably from each other and may even have nearly opposite directions. In the present investigation, however, we will at first put the axial high out of consideration, which is in all probability very shallow phenomena, brought out by the reduction to sea level, and take the gradient obtained from the coastal stations. The general procedure taken is as follows. For the Northern Japan I II, for example, we choose four coastal stations, say Akita, Kanazawa on the Japan Sea side and Miyako, Tyôsi on the Pacific side, forming the angular points of a quadrilateral, as nearly rectangular as possible, including the region in question. The difference of the pressure on both sides divided by the mean distance may be taken as the measure of the gradient in the direction combining the centre of the opposite sides.[1] If the quadrilateral be nearly rectangular, we obtain thus the two rectangular components of the gradient. The stations chosen for the purpose are as follows (see Fig. 1) :

1) In the case when the isobaric surface is nearly plane, it will be plausible to take a triangle for the determination of gradient. But in such a case as is here concerned, the procedure mentioned above seems more advisable as giving a kind of mean gradient.

Akita, Kanazawa, Miyako and Tyôsi for Northern Japan I-II.

Niigata, Sakai, Tyôsi and Kôti for Middle Japan III-IV.

Sakai, Hukuoka, Wakayama and Miyazaki for Southern Japan V-VI.

The difference of the means, Japan Sea pair minus the Pacific, reduced to the gradient corresponding to 111 km. distance was taken as the x-component. For the y-component, the difference was taken between the mean value on the opposite sides of the rectangle running transverse to the axis of the land, the positive sign corresponding to the case when the SW side is higher than the NE side. The mean distance between the opposite side was roughly estimated on a map, scale 1 : 16,000,000.

Thus the component gradients x_1, y_1, x_2 y_2, x_3 y_3, for the three regions I-II, III-IV, V-VI, respectively were obtained according to the following schema:

$$x_1=\left(\frac{\text{Akita}+\text{Kanazawa}}{2}-\frac{\text{Miyako}+\text{Tyôsi}}{2}\right)\times 0.386$$

$$y_1=\left(\frac{\text{Kanazawa}+\text{Tyôsi}}{2}-\frac{\text{Akita}+\text{Miyako}}{2}\right)\times 0.248$$

$$x_2=\left(\frac{\text{Niigata}+\text{Sakai}}{2}-\frac{\text{Tyôsi}+\text{Kôti}}{2}\right)\times 0.435$$

$$y_2=\left(\frac{\text{Sakai}+\text{Kôti}}{2}-\frac{\text{Niigata}+\text{Tyôsi}}{2}\right)\times 0.165$$

$$x_3=\left(\frac{\text{Sakai}+\text{Hukuoka}}{2}-\frac{\text{Wakayama}+\text{Miyazaki}}{2}\right)\times 0.463$$

$$y_3=\left(\frac{\text{Hukuoka}+\text{Miyazaki}}{2}-\frac{\text{Sakai}+\text{Wakayama}}{2}\right)\times 0.267$$

It must be remarked that in the case of x_1 y_1, the two components are not even nearly rectangular, so that this point must be remembered when a quantitative relation is concerned. But for the most purpose in the present investigation, the general qualitative relations are not seriously modified by treating the components as rectangular. The values of the components thus obtained are contained in Table V.

Table V.

Region	Year / Component	1900	1901	1902	1903	1904	1905	1906	1907	1908	1909	1910	1911	1912	1913	1914	1915	1916	1917	Mean
I, II.	x_1	.15	.15	.15	.31	.17	.10	.19	.12	.17	.15	.21	.29	.31	.29	.25	.14	.17	-.10	.18
	y_1	.10	.05	.05	.07	.06	.11	.10	.07	.06	.05	.01	.09	.07	.09	.11	.09	.01	-.09	.06
III, IV.	x_2	.00	.22	.17	.17	.07	.11	.22	.15	.17	.20	.28	.28	.33	.30	.15	.22	.22	.13	.19
	y_2	.10	.12	.08	.10	.07	.02	.10	.04	.10	.07	.11	.06	.12	.13	.12	.07	.07	.20	.11
V, VI.	x_3	.16	.19	.09	.12	.16	.09	.16	.19	.19	.14	.16	.14	.19	.23	.07	.14	.14	.09	.15
	y_3	-.01	.00	-.03	-.01	.04	-.05	-.01	.00	.03	.00	.04	.03	.05	.08	.04	0.3	.00	.16	.02

Table VI.

Diff. / Year	1900	1901	1902	1903	1904	1905	1906	1907	1908	1909	1910	1911	1912	1913	1914	1915	1916	1917	Mean
I—II	270	-68	-35	78	102	92	21	-24	-58	-181	-135	-20	-124	34	214	-147	-124	-363	+5.4
III—IV	35	29	92	122	69	-23	63	297	89	46	168	3	64	202	32	228	14	442	1.1
V—VI	-89	145	111	-120	333	-35	-31	-41	124	35	184	79	139	-81	14	-9	175	137	+1.8

8. In comparing the yearly barometric gradient with the difference of yearly precipitation on both sides of the land, several procedures were tried among which the following seems to be most convenient for demonstrating the topographical influence.

To take, for example, the case of the Northern Japan I-II, a vector diagram (Fig. 2) of the yearly barometric gradient was plotted on a sheet of coordinate paper, the end point of the vector for each year being marked and numbered according to the number of the year. On the other hand the *difference* of the values:

$$\frac{\text{Yearly anomaly of the percentage value of precipitation}}{\text{Mean anomaly} \quad ,, \quad ,, \quad ,, \quad ,,}$$

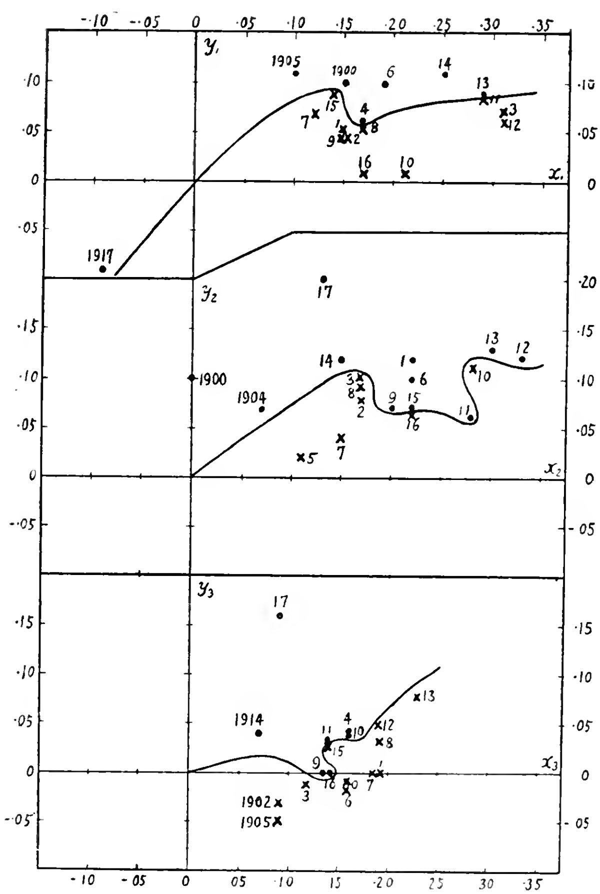

Fig. 2.

of the region I and II, was taken (Table VI) for successive years. Then the points on the vector diagram were classified according as the above difference of the both sides is positive or negative. Carrying out the procedure, it was found that the points belonging to the two classes may be separated by a boundary line dividing the area of the diagram into two halves. Though the line of demarcation is more or less sinuous, it may be compared for very rough approximation with a straight line making an angle of about 20° or 30° with the axis of x.

This latter result is exactly what may be expected from our previous elementary theory, except that the angle expected from the calculation was 13° instead of 20 or 30°. The fact that the actual angle is decidedly larger than the calculated, points to the conclusion that the difference of the angles between the wind and the barometric gradient on land and sea must be assumed much larger than in the previous paper. This discrepancy may probably be explained if we consider that the angle between the gradient and the wind increases generally with the height and also that in calculating the vertical displacement of air caused by the discontinuity of the horizontal flow on the boundary of land and sea, it is reasonable to take the data of calculation from a certain mean or "effective" height from the sea level. If the years of observation be multiplied several times, we may obtain sufficient number of points on the vector diagram for drawing a more definite line of demarcation and thence deduce something about the effective height above mentioned. Moreover, it seems quite possible that the sinuosity of the boundary curve may have a real significance, since the "effective height" may be a function of the velocity of wind as well as the direction of wind determined by the irregularity of topography. These latter points of interests must be postponed to a future when a sufficient materials would have accumulated.

9. In passing, it may be observed that vector diagram above described is very convenient for investigating the sensibility of different stations with respect to the barometric gradient, or in other worlds the mode and degree in which the topographic conditions affect the precipitation of the stations. Taking the

diagram for a given region, we classify different years according as the precipitation of a certain station situated in the said region, is above or below the normal value, and mark the corresponding points with proper signs. Then if the station is especially sensitive to the gradient, the points belonging to the two classes will be separated by a definite line of demarcation on the area of the diagram. Carrying out the procedure for the different stations chosen above, it was found that some Pacific stations such as Numadu, Hamamatu, Kôti *etc.* show striking dependency on the topographical rain, whereas most of the Japan Sea stations show rather irregular aspects. It was suspected that the latter fact could be explained by the influence of the high pressure zone along the axial line of the land. Choosing a number of stations near the axial lines, we have calculated the gradient between these axial stations and the coastal stations on both sides of the land respectively. It was found that the gradient between the axial part and the Pacific coast, say s, is generally parallel or at least have the same sign with the x-component above obtained, whereas the pressure difference between the axial and the Japan Sea regions, say n, has frequently different signs with x. On plotting n-y diagram instead of x-y diagram, however, the distribution of the years belonging to the above two classes for the case of the Japan Sea stations is still very irregular. We may therefore infer that on this side of Japan some other terms predominates over the effect above considered, as far as the annual amount of precipitation is concerned. We will resume the question in a later section in connection with the seasonal distribution.

10. A similar comparison can be made with respect to the

TABLE VII.

Region	Month / Compt.	I	II	III	IV	V	VI	VII	VIII	XI	X	XI	XII	Year
I, II.	x_1	.66	.66	.50	.17	.06	-.17	-.14	-.27	-.15	.17	.42	.61	.23
	y_1	.27	.27	.14	.06	.04	-.04	-.06	-.10	-.10	-.06	.14	.33	.07
III, IV.	x_2	.43	.54	.52	.20	.02	-.15	-.15	-.15	.11	.41	.39	.30	.17
	y_2	.36	.34	.15	-.02	-.01	-.06	-.04	-.11	-.11	.01	.18	.40	.10
V, VI.	x_3	.21	.32	.25	.09	.00	-.14	-.21	-.12	.19	.37	.23	.21	.09
	y_3	.20	.16	.01	-.03	-.03	-.03	-.04	-.09	-.13	-.05	.08	.31	.04

TABLE VIII.

Month / Diff.	I	II	III	IV	V	VI	VII	VIII	IX	X	XI	XII
I—II	160	15	−97	−133	−211	−107	20	−76	−72	−58	240	325
III—IV	295	133	−80	−167	−204	−176	−101	−153	−161	−74	180	527
V—VI	108	−94	−22	−75	−134	−1	64	−62	31	−32	5	105

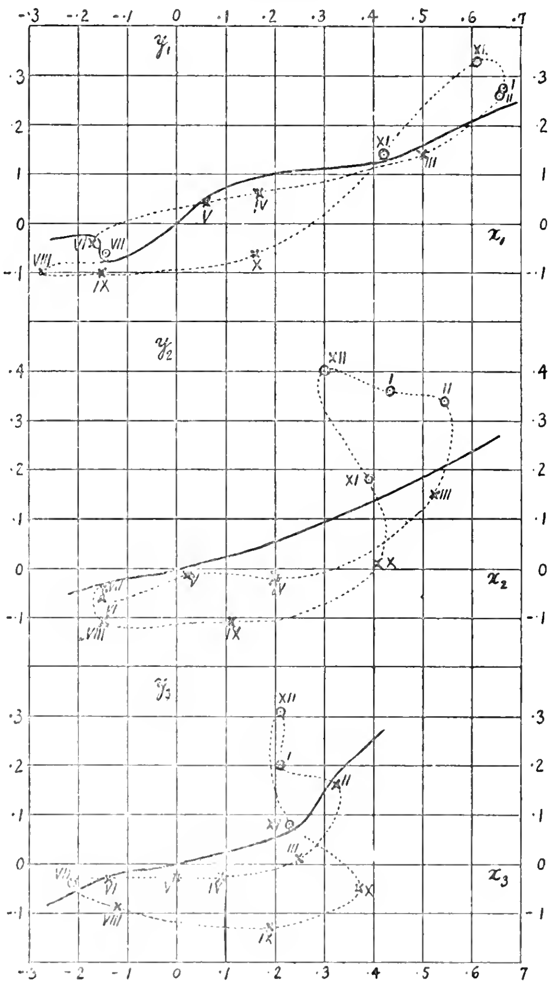

Fig. 3.

monthly value of precipitation and the barometric gradient. Table VII gives the mean monthly gradient to be compared with the precipitation given above in Table II. In Table VIII, the differences of the percentage anomaly, obtained in similar manner as in Table VI are given. The vector diagrams plotted in exactly same manner are given in Fig. 3. Here also the months with more than usual excess of precipitation on the Japan Sea side, are distributed on the one side of a line of demarcation, similar as in the case of the yearly amount.

From the results thus far obtained, it seems certainly worth while to apply the vector diagram method to the relation of the precipitation and barometric gradient of each month of each year, instead of taking the monthly or the yearly means, since as is well known the fluctuation of the monthly means of the both elements are considerable. At present, we must refrain from the task on account of the want of time, with a hope that in some days we will be able to resume the investigation.

11. In section 3, we have already alluded to the peculiar distribution of the precipitation of the different regions in different seasons. We will now make an attempt to interprete or analyse the actual distribution in terms of the elementary theory proposed.

Let us put at first for a trial:

$$\begin{aligned} r_1 &= R_1 + c_1, & r_2 &= R_1 - c_1'. \\ r_3 &= R_2 + c_2, & r_4 &= R_2 - c_2', \\ r_5 &= R_3 + c_3, & r_6 &= R_3 - c_3', \end{aligned} \qquad \text{(A)}$$

where c_1, c_2, c_3 *etc* represent the effects of the topography which will be considered as some functions of the barometric gradients, varying from month to month. R_1, R_2, R_3 represent the principal terms which will prevail in absence of the topographical influence. It seems plausible to assume that the c's are proportional to c''s, if not equal, so that we will put

$$c_1' = ac_1, \quad c_2' = ac_2, \quad c_3' = ac_3.$$

The factor a must vary with season, but it must be generally positive if our theory be valid. Further, on the basis of the approximate equality of the annual amounts of $r_3 + r_5$ and $r_4 + r_6$, we will assume provisionally that $R_2=R_3=R'$ Putting R for R_1, the above expressions (A) assume the following forms:

$$r_1=R+c_1, \qquad r_2=R-ac_1,$$
$$r_3=R'+c_2, \qquad r_4=R'-ac_2, \qquad \text{(B)}$$
$$r_5=R'+c_3, \qquad r_6=R'-ac_3,$$

from which we may determine the values of R. R', a, c_1, c_2 and c_3:

$$a=\frac{r_3-r_5}{r_6-r_4},$$
$$c_1=\frac{r_1-r_2}{1+a}, \qquad R=r_1-c_1,$$
$$c_2=\frac{r_3-r_4}{1+a}, \qquad R'=r_3-c_2. \qquad \text{(C)}$$
$$c_3=\frac{r_5-r_6}{1+a}$$

The results of calculation are given in Table IX. and Fig. 4.

TABLE IX.

Month	a	c_1	c_2	c_3	R	R'
I	0.098	89.2	166.7	43.7	69	85
II	0.235	44.5	96.3	−18.6	68	93
III	1.094	6.2	6.2	−25.8	101	149
IV	3.780	−0.4	−7.7	−25.7	110	162
V	4.471	−6.0	−9.3	−26.3	109	146
VI	−5.570	0.4	11.6	39.6	139	163
VII	20.75	1.6	−0.8	−4.8	180	196
VIII	0.861	2.2	−19.3	−55.4	157	188
IX	7.100	−0.5	7.3	−17.3	188	213
X	0.905	6.3	5.8	−36.2	152	176
XI	0.167	102.8	110.5	−3.4	81	113
XII	0.067	139.6	255.9	45.9	63	82

It may be remarked that a comes out generally positive, except in June. It is generally less than unity in the colder season and greater than unity in warmer months. Each of c_1, c_2, c_3,

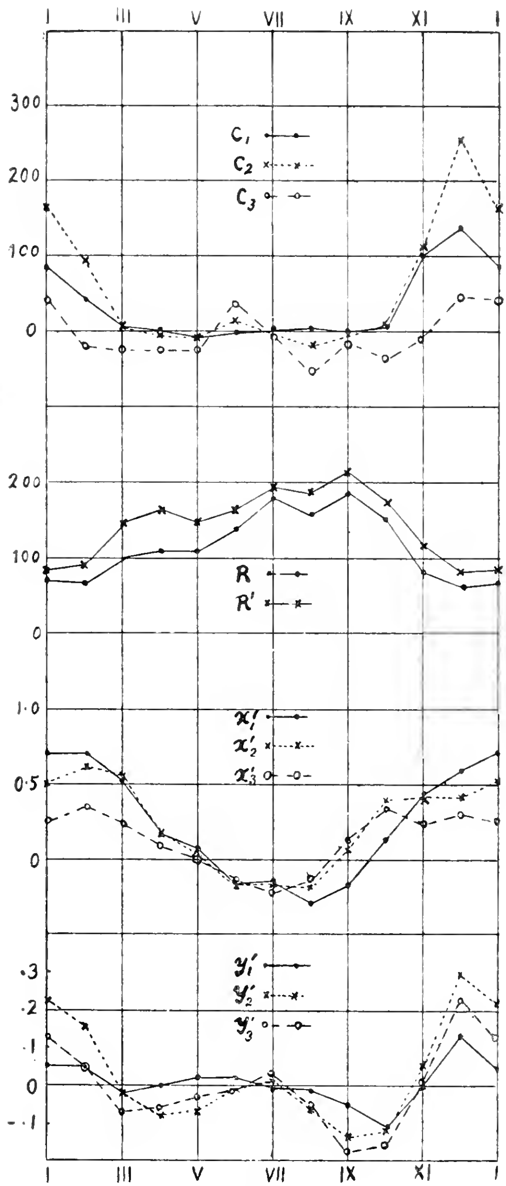

Fig. 4.

which run nearly parallel to each other, shows a maximum in winter and a secondary maximum in summer. We may also remark a parallelism of R and R'.

The fact that a for June becomes negative is no serious objection to our theory, but rather due to the inadequacy of the simple assumption (B). This might have been easily avoided if for an instance we distinguish R_2 and R_3 instead of putting them equal to each other. Assuming $a=12.6$ which is the mean values of a's for May and July, and putting

$$r_5=R'+R''+c_3,$$
$$r_6=R'+R''-ac_3$$

we obtain $R''=39$, $c_1=-.15$, $c_2=-3.90$ and $c_3=-13.30$. Thus the positive values of c's disappears. Substituting these values of c's, however, the general feature of the c-curves is not altered.

According to our previous theory, the values of c's must chiefly depend on the component of the barometric gradient parallel to the y'-axis making an angle of about 20° with the y-axis. Calculating the x'- and y'-components by

$$x' = x \cos 18^\circ + y \sin 18^\circ$$
$$y' = y \cos 18^\circ - x \sin 18^\circ$$

it was found that the general course of the y'-component (Table X and Fig. 4) is quite similar to that of e.

TABLE X.

Region	Compt. \ Month	I	II	III	IV	V	VI	VII	VIII	IX	X	XI	XII	Year
I, II.	x_1'	.71	.71	.52	.18	.07	-.17	-.15	- 29	-.17	.14	.44	.59	.24
	y_1'	.05	.05	-.02	.00	.02	.02	-.01	-.01	-.05	-.11	.00	.13	.01
III, IV.	x_2'	.52	.62	.54	.18	.02	-.16	-.16	-.18	.07	.39	.43	.41	.19
	y_2'	.22	.16	-.02	-.08	-.07	-.01	.01	-.06	-.14	-.12	.05	.29	.02
V, VI.	x_3'	.26	.35	.24	.08	-.01	-.14	-.22	-.14	.14	.34	.24	.30	.10
	y_3'	.13	.05	-.07	-.06	-.03	.01	.03	-.05	-.18	-.16	.01	.23	-.01
Mean	x'	.50	.56	.43	.15	.03	-.16	-.18	-.24	.01	.29	.37	.43	.18
	y'	.13	.09	-.04	-.05	-.03	.01	.01	-.04	-.12	-.13	.02	.22	.01

As to the terms R and R' which represent the parts of the precipitation independent of the local topography, we may remark that the general annual variation is such as to be easily explained by yearly course of temperature and humidity, though the high values in September and October may probably be due to the cyclones prevalent in this season.

From the above results, it may be seen that on the Pacific side, the seasonal fluctuations of the general term R and R′ are generally enhanced by the topographical effect in warmer as well as in the colder seasons, whereas on the Japan Sea side the seasonal variation is partially compensated by the topographical influence. This is the formal explanation based on the present theory why the seasonal variation is small on the latter side.

In a previous paragraph we have mentioned that on the Japan Sea side, the yearly fluctuation of the precipitation shows no regular relation with the gradient components x and y, and inferred that on this side some other agents must prodominate over the above gradient. It is interesting to observe that here in the seasonal fluctuation, the effect of topography appears conspicuous, the effects of the other agents being apparently eliminated by taking average of different years.

The formula (B) fails in the case of the yearly variations, the values of c's obtained showing no regular relation with the components of the barometric gradient. Besides, their values are generally a small fractions of the general terms R and R'. The simple assumption is therefore inadequate in this case. One point of interest is that the values of a thus obtained are generally positive and their mean value is very nearly equal to unity, being, 1.002, which points to the inference that on an average $r_3-r_5=r_6-r_4$. The latter fact could only be explained by the presence of the high pressure area on the axial line of the land, as already mentioned in § 3.

From the mere mathematical points of view, the above method of analysis is nothing more than the substitution of the six given amounts of precipitation by the new six quantities R, R', a and c's. Neither is the substitution the unique one possible. The chief physical interests, however, consist in the fact that the apparently irregular distributions of the precipitation may thus be explained at least in its main feature by the combination of the topographical effects on the basis of the simple elementary theory. These results may turn out useful for the practical purpose, as soon as the long period forcast of the barometric conditions becomes a matter of practice, the possibility of which is nowaday anything more than the dream of the modern meteorologists.

12. As already cited at the beginning of this note, Prof. Omori shew that there exists a remarkable parallelism between the yearly amount of precipitation at Niigata, a Japan Sea station, and that of the earthquake at Tôkyô. In the present invesigation, the earthquakes originating in the submarine zone extending from Kinkwazan to Idu were chosen from the Kisyô-yôran. The yearly number of occurrence is given in the first line of the Table XI and plotted in Fig. 5. Comparing the graph with that of precipitation in different stations or regions, it was found that the correlation is rather remarkable in the case of the mean fluctuation of precipita-

TABLE XI.

Year	1900	1901	1902	1903	1904	1905	1906	1907	1908	1909	1910	1911	1912	1913	1914	1915	1916	1917
Earthquake No.	—	6	24	26	40	48	42	28	23	42	37	44	31	24	29	64	26	22
Bar. Diff.	-.20	-.53	-.50	-.43	-.20	-.38	-.23	-.28	-.40	-.43	-.18	-.25	-.18	-.35	-.53	-.33	-.23	-.38
Precip. (II+IV+VI)/3.	-1.14	-.19	.52	1.43	1.35	1.50	-.74	-.11	-.25	-.02	.86	1.51	-.05	-1.61	-.81	.57	.54	-.73

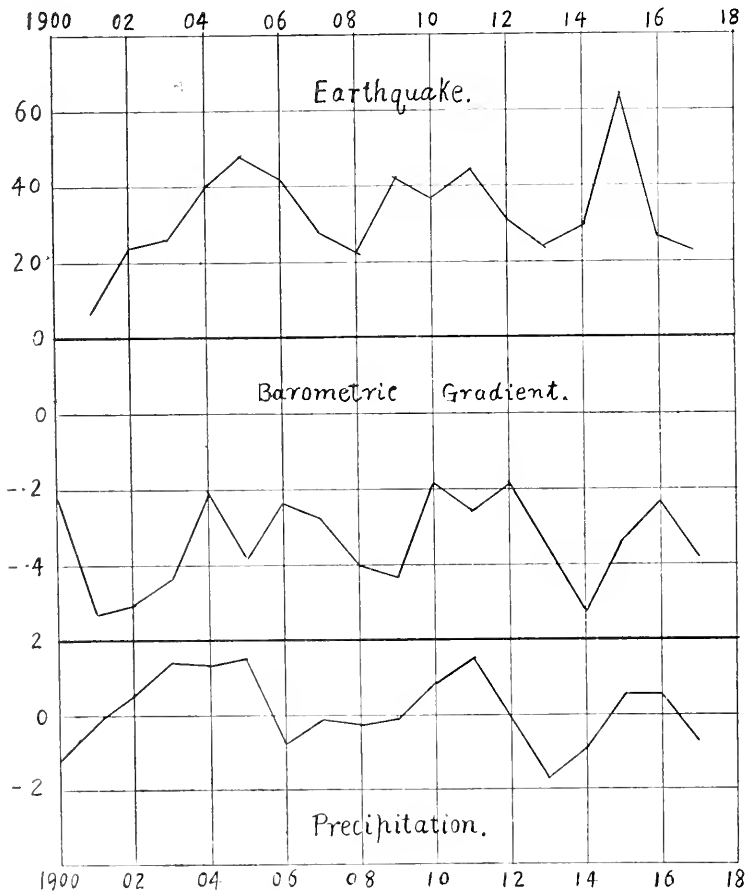

Fig. 5.

tion of the entire Pacific regions, *i.e.* (II+IV+VI)/3,[1] which is given in the third line of the Table. The Japan Sea side, (I+III+V)/3 shows nearly similar yearly variation as the Pacific, except in a few years.

Since the intimate relation of the earthquake frequency and the barometric gradient has been already fully established by the investigations[2] of one of the authors and also of K. Hasegawa and Saemontarô Nakamura, it was suspected that it may be also the barometric gradient that directly determines the rainfall on the one hand and the earthquake on the other hand. Hence the earthquake frequency was compared with the different components of the gradient prevailing in different parts of the land. After a series of trials, it was found that the difference of the barometric pressure, the Japan Sea stations minus the inland stations, shows a parallel course with the earthquake frequency. Taking for the inland stations Matumoto, Takayama, Hikone and Osaka, and for the Japan Sea coastal stations Niigata, and Sakai, the difference is given in the second line of the Table. XI.

Comparing this with the earthquake frequency (Fig. 5), it will be seen that the earthquake curve shifted about one year earlier, shows a rather remarkable parallelism with the gradient curve. Whether the very curious correlation is merly accidental or not, cannot be ascertained at present from such a scanty material. Though the apparent relation may appear rather absurd, the possibility of such a coincidence cannot be excluded by a superfical consideration, if we consider for an instance that the precipitation of the last year may affect the barometric pressure of the year concerned.

A further investigation in this line is now in progress and we hope will be able to clear up the apparent mystery in a near future.

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1) Here the values of (anomaly)/(mean anomaly) was averaged for the three regions.

2) T. Terada, Proc. Tôkyô Math.-Phys. Soc., vol. IV (1908) p. 454; K. Hasegawa, *ibid.* Vol. VII (1913), p. 181; S. Nakamura. *Ibid.* Vol. VIII (1915), p. 69.

Summary.

1. The remarkable influence of topography on the distribution of precipitation is demonstrated with respect to the yearly as well as the monthly amounts.

2. A vector diagram method of investigating the different relations of the precipitation with the barometric gradient is illustrated.

3. The neccesity of taking the percentage values of the precipitation is emphasized.

4. The rainfalls in different districts are analysed in terms of the topographical effects. It is shown that the effect is largely determined by the component of the barometric gradient taken in a direction a little inclined to the axial line of the land, as was expected from the elementary theory proposed.

5. A peculiar relation between the earthquake frequency and the barometric gradient is pointed out.

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Recherches sur les spectres d'absorption des ammine-complexes métalliques.

III.

Spectres d'absorption des sels complexes de nickel, de chrome et de cuivre.

Par

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Avec 18 figures.

Nous avons récemment publié quelques notes sur les spectres d'absorption des solutions aqueuses des ammine-complexes cobaltiques (Journ. Coll. Scien. Imp. Univ. Tokio, Vol. **37**, Art. 2 et 8.). On peut résumer les résultats de ces travaux dans les quelques règles suivantes:

I) L'absorption de rayons d'un sel complexe est exclusivement effectuée par son ion complexe, ou pour parler exactement, c'est le point de la connection de l'atome métallique central avec les radicaux acidiques, ou les molécules de l'eau ou de l'ammoniaque etc. dans l'ion complexe, qui est le vrai siège de l'absorption.

II) Les ions complexes semblablement construits absorbent semblablement les uns et les autres.

III) Si les atomes métalloides, qui s'enchâinent directement avec un atome métallique central dans un ion complexe, sont égaux, la constitution des groupes des métalloides n'a pas d'importance pour l'absorption.

IV) Dans ces deux derniers cas, ni la différence des signes de l'ion, ni la différence des valences, n'influence jamais l'absorption.

V) L'ion simple, qui n'absorbe pas lui-même et s'accouple avec un ion complexe, n'influence guère l'absorption.

VI) Le remplacement d'un groupe atomique dans un ion complexe par un autre exerce une influence bien caractéristique d'après les substituants sur l'absorption.

VII) Les radicaux fortement acidiques dans un ion complexe sont généralement remplacés par les molécules d'eau, quand on dissout le sel dans ce même liquide.

Le but du travail présent est donc de rechercher si les conclusions cités ci-dessus peuvent être appliquées aussi aux sels complexes des autres métaux, comme le chrome, le nickel ou le cuivre, Or, nous avons étudié spectroscopiquement les solutions des 48 sels complexes de chrome, de nickel et de cuivre, qui sont donnés ci-dessous:

Sels complexes chromiques.	Sels complexes nickeleux.	Sels complexes cuivriques.
$[Cr(NH_3)_6]Cl_3$	$[Ni(NH_3)_6]Cl_2$	$[Cu(Py)_6](NO_3)_2$
$[Cr\ en_3]Cl_3$	$[Ni(NH_3)_6]SO_4$	$[Cu(Py)_4](NO_3)_2$
$[Cr\ en_3](SCN)_3$	$[Ni(N_2H_4)_3]Cl_2$	$[Cu(Py)_3](NO_3)_2$
$[Cr(NH_3)_5Cl]Cl_2$	$[Ni\ en_3]Cl_2\cdot 2H_2O$	$[Cu(Py)_2](NO_3)_2$
$[Cr(NH_3)_5H_2O]Cl_3$	$(NH_4)_2Ni(SO_4)_2\cdot 6NH_3\cdot H_2O$	$Cu(NH_3)_2(C_2H_3O_2)_2$
$[Cr(NH_3)_5(SCN)](SCN)_2$	$2NiSO_4\cdot 5NH_3\cdot 7H_2O$	$Cu(NH_3)_3I(C_2H_3O_2)$
$[Cr(NH_3)_4Cl\cdot H_2O]Cl_2$	$[Ni(NH_3)_4]SO_4\cdot 2H_2O$	$Cu(NH_3)_3Cl(C_2H_3O_2)\cdot H_2O$
$[Cr(NH_3)_4Cl\cdot H_2O]SO_4$	$[Ni(N_2H_4)_2]Cl_2$	$NH_4Cu_2(C_2H_3O_2)_5\cdot H_2O$
$[Cr\ en_2Cl_2]Cl$	$[Ni\ en_2]Cl_2\cdot H_2O$	$Cu(C_2H_3O_2)_2\cdot H_2O$
$[Cr(NH_3)_3Cl_2\cdot H_2O]NO_3$	$[Ni(NH_3)_3]Cl_2\cdot 3H_2O$	$Cu(NH_3)_5\cdot C_2O_4$
$Cr(NH_3)_3(SCN)_3$	$NiSO_4\cdot 7H_2O$	$Cu(NH_3)_4\cdot C_2O_4\cdot 2H_2O$
$[Cr(NH_3)_2Br_2(H_2O)_2]Br$	$NiCl_2\cdot 6H_2O$	$Cu(NH_3)_2\cdot C_2O_4$
$[Cr(NH_3)_2(SCN)_4]NH_4$		$CuNH_3\cdot C_2O_4$
$Cr_2(SO_4)_3\cdot 18H_2O$(Sel violet)		$Cu(NH_3)_5SO_4$
$Cr_2(SO_4)_3\cdot 6H_2O$(Sel vert)		$Cu(NH_3)_4SO_4$
$[Cr(C_2O_4)_3]K_3$		$CuSO_4\cdot 5H_2O$
$[Cr(SCN)_6]K_3$		$CuCl_2\cdot 2H_2O$
$CrO_4\cdot 3NH_3$		
$CrO_4\cdot 3KCN$		

Parmi ces complexes, ceux de chrome sont les plus stables et montrent une ressemblance remarquable avec les complexes cobaltiques. En conséquence, toutes les conclusions citées plus haut sont, sans exception, appliquables aux sels chromiques, tandis que les complexes de nickel diffèrent très sensiblement des deux précédents. L'atome de nickel central des ions complexes nickeleux est toujours bivalent, et le nombre de coordination n'est pas nécessairement six, comme dans le cas du cobalt et du chrome. Surtout, leurs solutions aqueuses ne sont pas du tout stables, car elles montrent la tendance de l'hydrolyse. Toutefois, on se trouve, dans ce cas même, en présence d'une bonne concordance avec tout ce qu'on a observé dans les expériences spectroscopique sur les complexes cobaltiques.

Quant aux sels complexes cuivriques, ils manifestent une instabilité plus avancée que ceux de nickel, et de même leurs absorptions sont distinctement influencées par les anions, qui s'accouplent avec les cations complexes. Nous discuterons sur la propriété optique spéciale aux complexes cuivriques, plus bas, dans la partie expérimentale.

Les solutions aqueuses des complexes chromiques, qui sont bien stables et satisfont complètement la loi de Beer sur l'absorption de rayons, donnent deux ou trois bandes d'absorption très nettes dans l'échelle spectrale visible et ultraviolette. Dans le cas du cobalt où l'on a aussi observé deux ou trois bandes bien distinctes, on a remarqué que l'une de ces bandes qui se place toujours à 2000 de fréquence n'est pas influencée sensiblement par une substitution quelconque dans les ions complexes (Y. Shibata: Journ. Coll. Scien. Imp. Univ. Tokio, Vol. **37**, Art. 2, P. 5). Il n'en est cependant pas de même dans le cas du chrome; les bandes d'absorption des complexes chromiques sont bien sensibles et mobiles d'après les substitutions dans l'ion complexe. Comparé au cobalt, les sels de chrome possèdent généralement beaucoup moins d'abilité d'absorption et quelques complexes chromiques ne montrent leurs absorptions que dans les solutions assez concentrées, par exemple, décinormale (en employant un tube de 10 cm de longueur), tandis que les complexes cobaltiques donnent leurs absorptions dans les solutions très étendues, quelquefois en dix-millième normale.

Les solutions des complexes de nickel absorbent aussi faiblement, et leurs bandes ne paraîssent qu'à la concentration de décinormale, tandis que les sels cuivriques, soit les complexes, soit les simples, ne donnent aucune bande d'absorption; ils absorbent pourtant continuellement dans les deux parties du rouge et du violet.

Partie Spéciale.

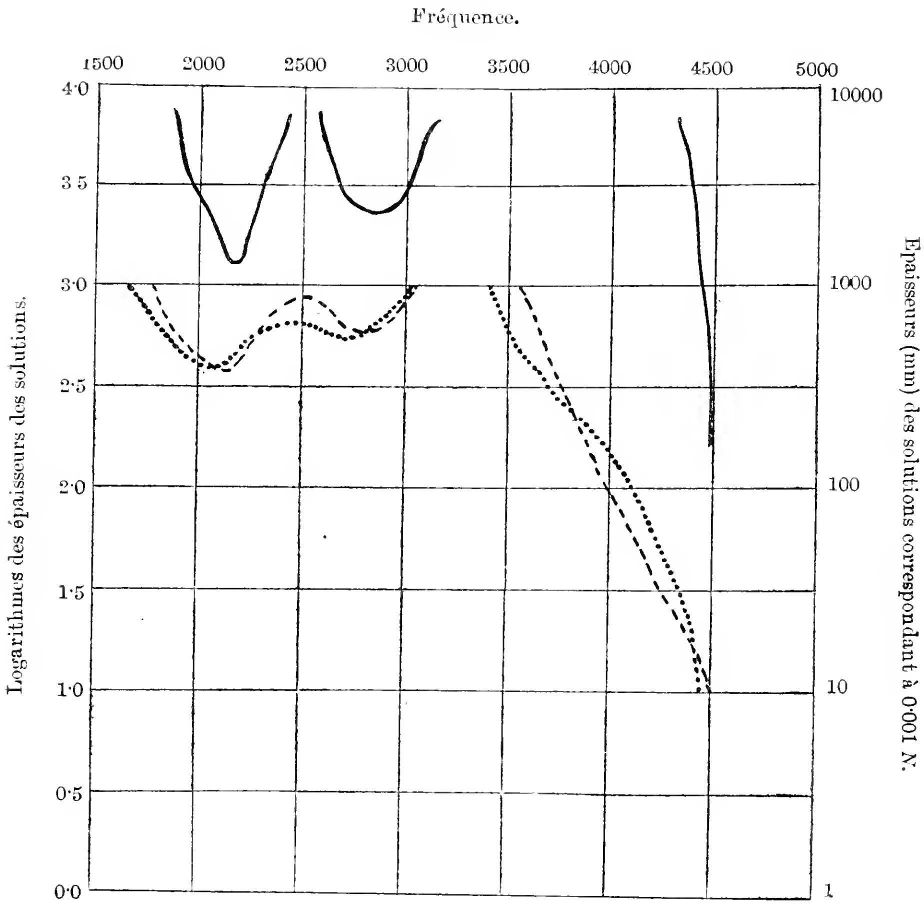
I. Absorption de rayons des sels complexes chromiques.

1) Complexes héxammoniés chromiques.

Les trois sels de cette catégorie $[Cr(NH_3)_6]Cl_3 \cdot H_2O$ [Jörgensen: J. prakt. Chem., 1884, [2], **30**, 12], $[Cr\ en_3]Cl_3 \cdot 3\frac{1}{3}H_2O$ [Pfeiffer: Z. anorg. Chem., 1900, **24**, 286] et $[Cr\ en_3](SCN)_3 \cdot H_2O$ [Pfeiffer: Ibid., 294] sont des corps bien cristallisés avec les couleurs jaunes ou jaunes rougeâtres. Ils sont facilement solubles dans l'eau et donnent des solutions bien stables.

Comme on le voit dans la figure I, les abilités d'absorption de rayons du complexe héxammonié proprement dit et des complexes, dont une partie des molécules d'ammoniaque est rempli par les molécules d'éthylènediamine, sont considérablement différentes; le premier absorbe beaucoup moins fortement que les derniers, c'est-à-dire que, bien que leurs deux bandes d'absorption se placent respectivement aux positions des mêmes logueurs d'onde (2150 et 2870 de fréquence), les concentrations, dans lesquelles on peut avantageusement observer les bandes, ne sont pas de même. Les deux bandes du corps $[Cr(NH_3)_6]Cl_3$ ne paraîssent bien nettement que dans la solution de décinormale, tandis qu'on les observe, dans le cas des complexes contenant d'éthylènediamine, déjà dans la concentration de centinormale.

Fig. I.



$[Cr(NH_3)_6]Cl_3$ ———————— $\left(\frac{N}{10}\right)$

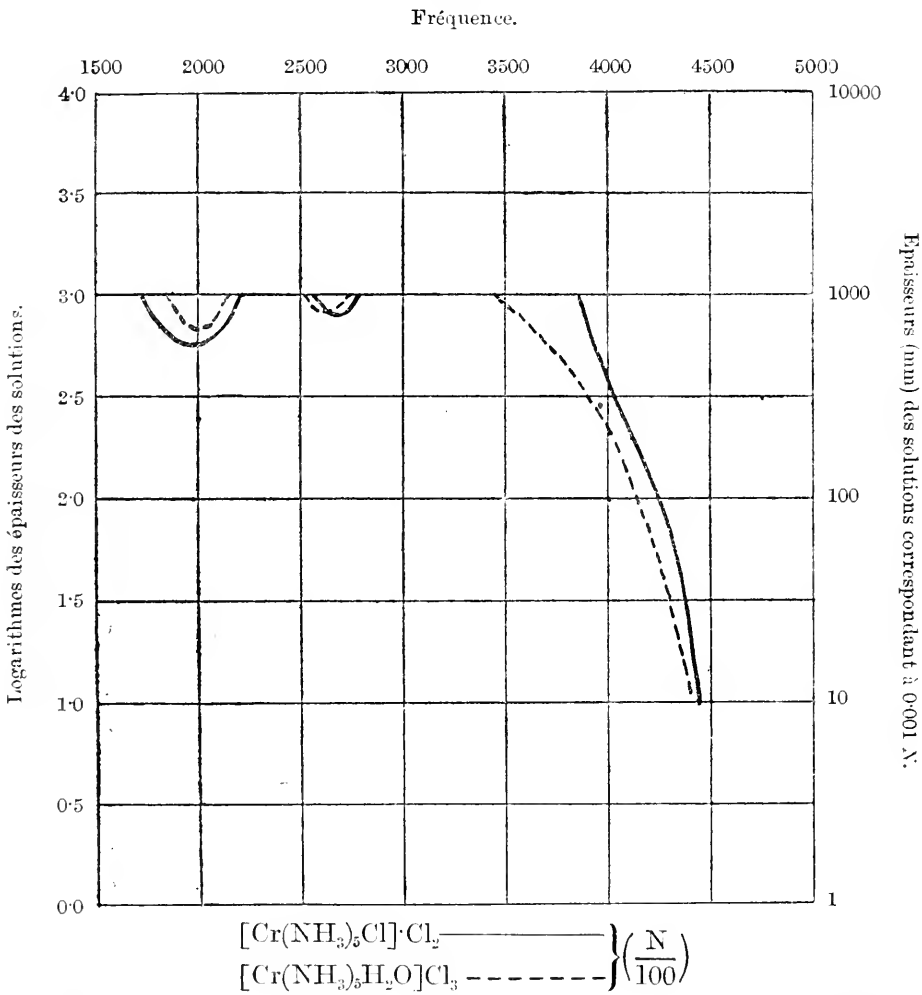
$[Cr\ en_3]Cl_3$ - - - - - - - - - - $\Big\}\left(\frac{N}{100}\right)$

$[Cr\ en_3]\ (SCN)_3$ ....

Nous nous rappelons que, dans le cas des complexes cobaltiques, la différence des intensités d'absorption entre le complexe héxammonié proprement dit et celui qui contient de l'éthylènediamine à la place de l'ammoniaque n'existait pas. Ainsi, on remarquera deux faits importants; premièrement les complexes chromiques sont influencés optiquement bien delicatement par les substituants dans leurs ions complexes et deuxièmement, conformément aux complexes cobaltiques, l'absorption de ceux de chrome est pratiquement indifférente aux ions simples qui s'accouplent avec les ions complexes.

2) Complexes péntammoniés chromiques.

Fig. II.



Le chlorure du péntammine-monochloro-chrome, $[Cr(NH_3)_5Cl]Cl_2$ [Christensen: J. prakt. Chem., 1881, [2], **23**, 54], est un corps cristallin coloré rouge, tandis que, le chlorure du péntammine-monoaquo-chrome, $[Cr(NH_3)_5H_2O]\ Cl_3$ [Christensen: Ibid., 28], est composé de cristaux rouges jaunâtres. Ces deux corps sont solubles dans l'eau et donnent des solutions ayant la même couleur que les corps solides.

Comme la figure II l'indique, les deux sels absorbent pareillement l'un et l'autre; leurs premières bandes se trouvent à 2000 de fréquence, tandis que les deuxièmes se placent à 2650.

Par les courbes, on remarquera que ce qui était vrai pour les complexes cobaltiques, l'est aussi pour ceux de chrome, c'est-à-dire que les chloro- et aquo-complexes absorbent semblablement dans les solutions aqueuses.

Dans l'eau, les deux corps sont, sans doute, dans l'état d'équilibre suivant:

$$[Cr(NH_3)_5Cl]\,Cl_2 + H_2O \rightleftarrows [Cr(NH_3)_5H_2O]\,Cl_3$$

Dans une solution assez étendue, comme centinormale, il faut considérer que c'est l'aquo-complexe qui prédomine. Or, les courbes qui sont données dans la figure II représentent très probablement celles des aquo-complexes. En comprant ces courbes d'absorption avec celles des complexes héxammoniés, on peut facilement apercevoir que le remplacement d'une molécule d'ammoniaque par une molécule d'eau exerce une influence bathochromique, c'est-à-dire que les deux bandes sont deplacées bien sensiblement vers le rouge. Il nous semble que ce phenomème est bien général, car nous avons toujours remarqué que l'importation des molécules d'eau dans un ion complexe cobaltique exercait aussi une influence dans le même sens.

3) Complexes tétrammoniés chromiques.

Le chlorure et le sulfate du tétrammine-monoaquo-monochloro-chrome, qui sont des cristaux colorés en pourpre, ont été préparés par des voies tout à fait différentes l'une de l'autre. Le chlorure a été synthetisé d'après Pfeiffer (Ber. deutch. chem. Gesell., 1905, **38**, 3594) qui a donné le procédé préparatif suivant:

$$\underset{\text{Chlorure vert}}{[CrCl_2(H_2O)_4]Cl} + 2C_5H_5N \longrightarrow [Cr(Py)_2(H_2O)_2(OH)_2]Cl + 2ClH$$

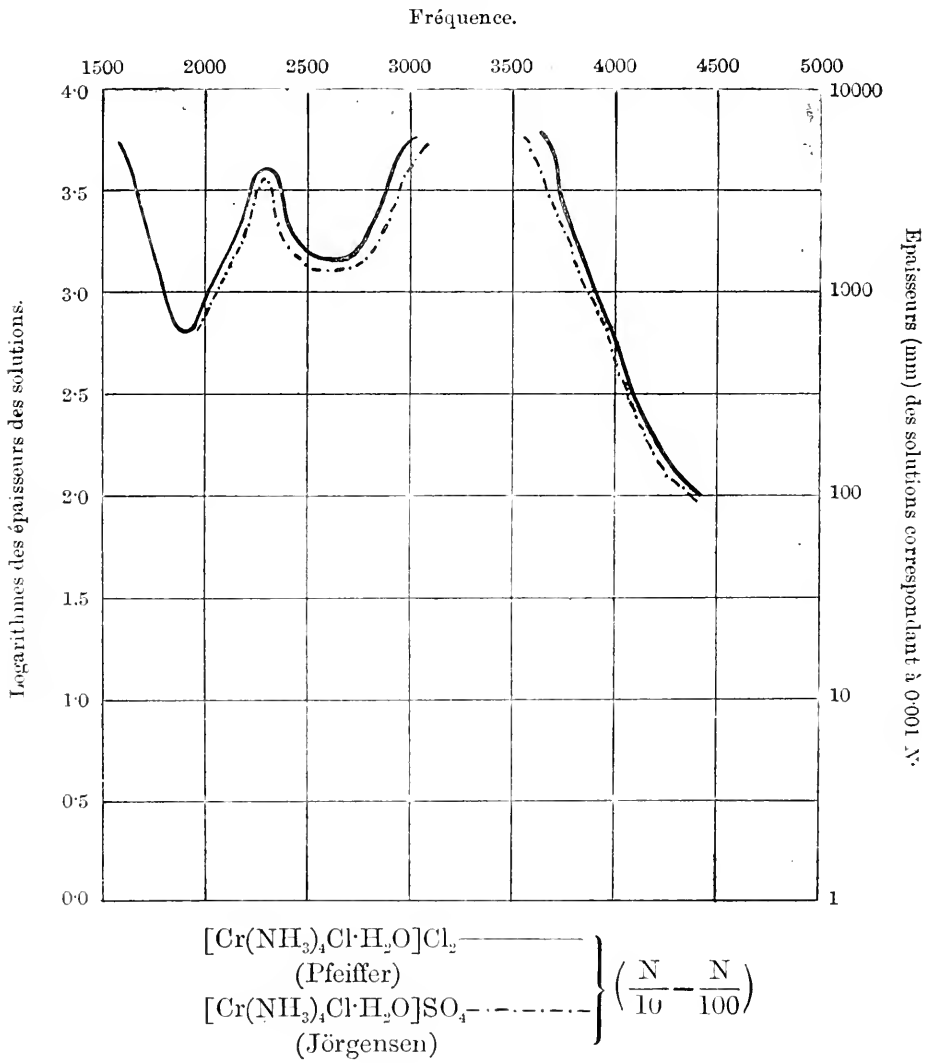
$$[Cr(Py)_2(H_2O)_2(OH)_2]Cl \xrightarrow{NH_3 \text{ et } HCl} [Cr(NH_3)_4Cl\,H_2O]Cl$$

Cependant Jörgensen (Journ. prakt. Chem., 1899, [2,] **20**, 110) a préparé son sulfate en opérant la réduction d'un mélange de bichromate d'ammoniaque, d'acide chlorohydrique et de chlorure d'ammonium par l'alcool.

Quoi que l'écart des manières préparatives de ces deux corps soit assez grand, leurs absorptions, qui sont traduites graphiquement dans la figure III ne diffèrent presque pas l'une de l'autre. Voici l'équilibre qui s'établit:

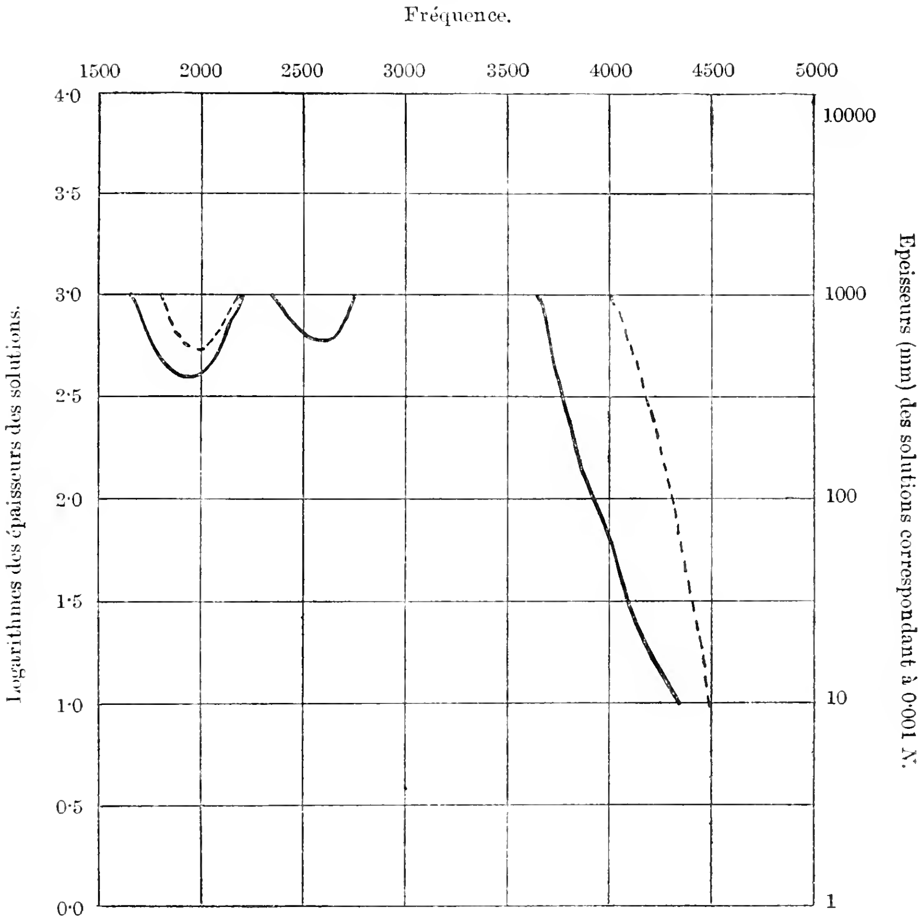
$$[Cr(NH_3)_4Cl{\cdot}H_2O]Cl_2 \xrightarrow{H_2O} [Cr(NH_3)_4(H_2O)_2]Cl_3$$

Fig. III.



Conformément au cas précédent, il faut que les courbes représentent l'absorption du diaquo-complexe chromique. Les courbes indiquent que les bandes sont déplacées davantage vers le rouge que dans le cas du monaquo-complexe.

Fig. IV.



$[Cr\ en_2Cl_2]Cl$,

—— la solution fraiche (violet).

- - - - - - - la solution laissée (rouge).

Le chlorure du diéthylènediamine-dichloro-chrome, $[Cr\ en_2 Cl_2^{(2)}]Cl\cdot H_2O$ [Pfeiffer u. Lando: Ber., 1904, **37**, 4278], dont les deux atomes de chlore dans l'ion complexe se placent à la position de cis, est un corps cristallin violet. Il donne une solution bien labile et, quand on la laisse, le changement de la couleur a lieu de telle sorte que sa coloration violette initiale disparâit peu à peu, et en même temps la coloration rouge prend naissance. Cette transformation de tonalité continue jusqu'à ce qu'après cinq ou six heures, elle soit devenue tout à fait rouge.

La courbe, tracée par la ligne noire dans la figure IV, représente l'absorption de la solution nouvelle (violette), tandis que celle de la ligne brisée représente la solution qui est laissée pendant une nuit (rouge). La raison, pour laquelle cette dernière courbe montre une bonne concordance avec celle du diaquo-complexe (fig. III.), c'est qu'il y a eu aussi la substitution de deux molécules d'eau dans l'ion complexe diaquo-chromique:

$$[Cr\ en_2Cl_{2}{}^{(1)}_{(2)}]Cl \xrightarrow{2H_2O} [Cr\ en_2(H_2O)_{2}{}^{(1)}_{(2)}]Cl_3$$

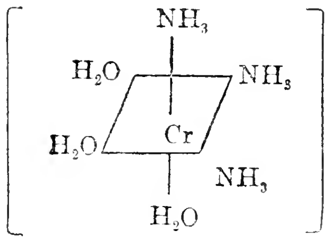
4) Complexes tri- et diammoniés chromiques.

Le nitrate du triammine-dichloro-monoaquo-chrome, $[Cr(NH_3)_3Cl_2H_2O]NO_3$ [Werner: Ber., 1906, **39**, 2663], est un corps cristallin gris verdâtre, presque insoluble dans l'eau. Mais si on le laisse assez longtemps en contact avec de l'eau, ou si on le chauffe un peu, il se dissout et donne une solution violette qui est vraisemblablement formée par la réaction suivante:

$$[Cr(NH_3)_3Cl_2H_2O]NO_3 \xrightarrow{H_2O} [Cr(NH_3)_3Cl(H_2O)_2]^{Cl}_{NO_3} \xrightarrow{H_2O} [Cr(NH_3)_3(H_2O)_3]^{Cl_2}_{NO_3}$$

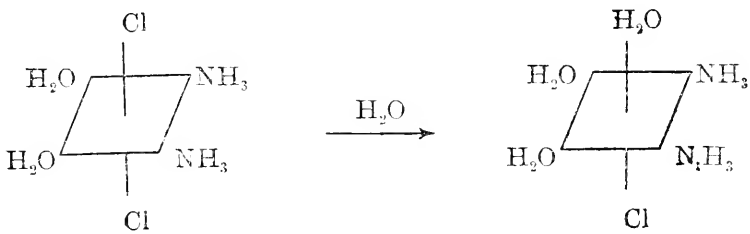
On sait déjà que le triammine-triaquo-complexe cobaltique, dont trois molécules d'eau dans l'ion complexe se placent dans la position consécutive, donne une solution violette, tandis que son isomère, dont deux molécules d'eau parmi trois se trouvent dans la position diagonale (ou la position de trans) change la couleur de sa solution spontanément du violet verdâtre jusqu'au violet clair dans l'éspace de quelques instants (Comparer K. Matsuno: Journ. Tokio Chem. Soc., 1917, Vol. **38**, 664).

De même, il est bien probable que la solution violette du triammine-triaquo-complexe chromique prend la configuration, qui contient les trois molécules d'eau dans la position consécutive:



Le bromure du diammine-dibromo-diaquo-chrome, $[Cr(NH_3)_2Br_2(H_2O)_2]Br$ [Werner u. Dubsky: Ber., 1907, **40**, 4090], est un corps vert brillant. On peut légitimement penser, à cause de cette couleur, que les deux atomes de brome dans l'ion complexe de ce corps se trouvent à la position de trans. Il est facilement soluble dans l'eau et donne une solution verte, qui change pourtant peu à peu et devient enfin tout-à-fait rouge après quelques heures.

Une pareille combinaison en cobalt, $[Co(NH_3)_2(H_2O)_2Cl_2]X$, est aussi un corps vert, mais sa solution verte nouvelle, après quelques instants, prend la couleur bleue verdâtre; puis le changement s'arrête et ne marche plus (Matsuno: ibid.). Cette transformation intermédiare de couleur est, sans doute, provoquée par le remplacement d'un seul atome de chlore:

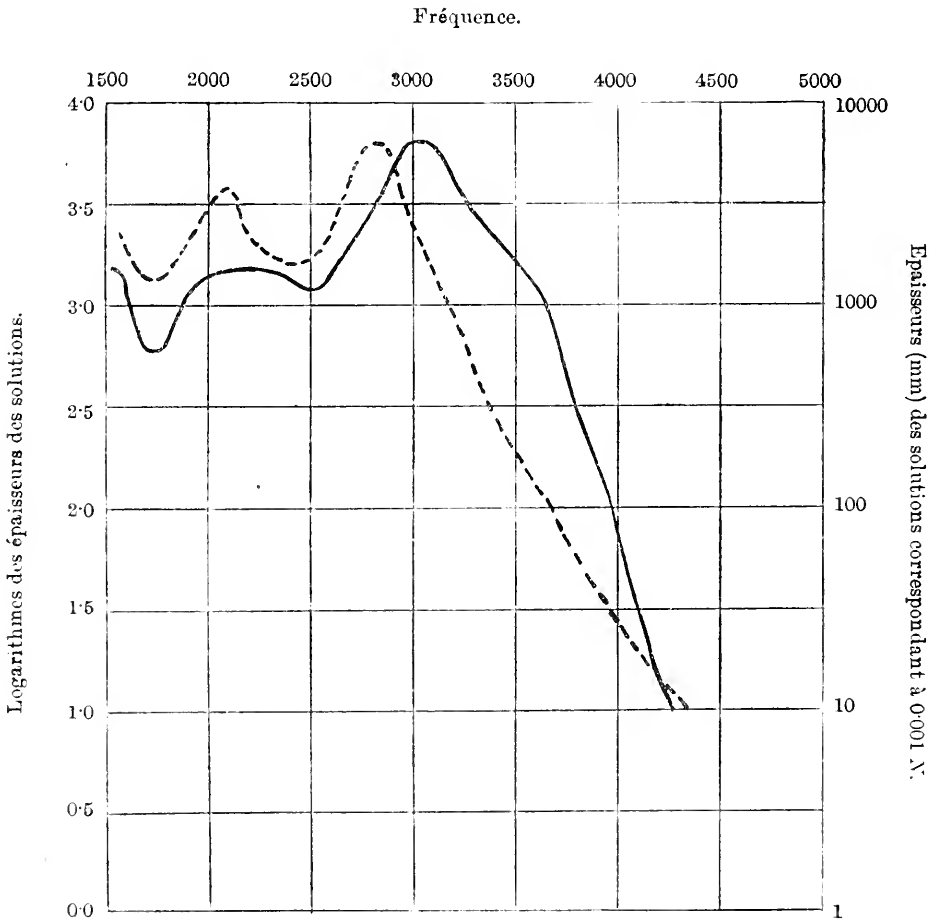


Il faut, de là, conclure que, dans le cas de chrome, où l'on obtient facilement une solution rouge, le remplacement des atomes d'halogène dans son ion complexe par les molécules d'eau est complète, et donne tétraquo-complexe:

$$[Cr(NH_3)_2(H_2O)_2Br_2]Br \xrightarrow{2H_2O} [Cr(NH_3)_2(H_2O)_4]Br_3$$

Les courbes dans la figure V représentent les absorptions des solutions aqueuses des deux complexes chromiques triammonié et diammonié, dont les atomes d'halogène sont très probablement tout remplacés par les molécules d'eau. De ce point de vue, on peut considérer que les courbes données ci-dessous sont di- et triaquo-complexes chromiques.

Fig. V.



$[Cr(NH_3)_3Cl_2H_2O]NO_3$ ——
$[Cr(NH_3)_2Br_2(H_2O)_2]Br$ - - - $\left(\frac{N}{10} - \frac{N}{100}\right)$

Or, on trouve ici encore une fois l'influence bathochromique des molécules d'eau dans un ion complexe, parce que le tétraquo-complexe possède sa deuxième band à 2490 de fréquence, tandis que le triaquo-complexe l'a à 2550, et encore, si l'on compare leurs absorptions d'extrémité, on remarquera facilement que l'influence s'y manifeste davantage en ce même sens.

5) Complexes ammoniés chromiques qui contiennent les radicaux de sulfocyanate.

Nous avons étudié spectroscopiquement les cinq sels, donnés ci-dessous:

a) $[Cr(NH_3)_5(SCN)](SCN)_2$ [Werner u. v. Halban: Ber., 1906, **39**, 2670]
b) $[Cr\ en_2(SCN)_{2(6)}^{(1)}]$ SCN [Pfeiffer: Ber., 1901, **34**, 4307.]
c) $[Cr(NH_3)_3(SCN)_3]$ [Werner u. v. Halban: Ber., 1906, **39**, 2668]
d) $[Cr(NH_3)_2(SCN)_4]NH_4{\cdot}H_2O$ [S. Christensen: J. prakt. Chem., 1892, [2], **45**, 216]
e) $[Cr(SCN)_6]K_3{\cdot}3H_2O$ [Joseph Roesler: Ann. d. Chem., 1867, **141**, 185]

Le corps (a) est formé de cristaux rouges violets; (b) orange carmin fade; (c) rouge fade; (d) rose foncé et (e) pourpre rougeâtre. Ils sont facilement solubles dans l'eau, sauf (c) qui est pourtant soluble dans l'acétone. Les solutions ont la même couleur rouge.

Il a déjà été observé par l'un de nous (Y. Shibata: Journ. Coll. Scien Imp. Univ. Tokio, Vol. **37**, Art. 2, P. 23) que le sulfocyano-ion dissocié dans l'eau n'absorbe pas du tout, tandis que, s'il se trouve dans un ion complexe, il donne toujours une bande bien caractéristique à 3400 de fréquence. On en trouvera encore quelques unes dans les fig. VI et fig. VII. De même, on remarquera l'influence extraordinaire du radical de sulfocyanate dans la direction bathochromique, si l'on examine les courbes d'un complexe héxammonié (fig. I.) et du sulfocyanate de péntammine-monosulfocyanato-chrome (fig. VI.).

La première bande à 2150 de fréquence de complexes héxammoniés chromiques est alors déplacée jusqu'à 1750 vers le rouge, par la substitution d'un seul radical du sulfocyanate au lieu d'une molécule d'ammoniaque dans l'ion complexe, tandis que la deuxième est poussée de 2870 à 2400 pour la même raison. Quant à la troisième bande du péntammine-monosulfocyanato-complexe chromique, elle est, comme il a déjà été indiqué plus haut, pour le radical du sulfocyanate dans l'ion complexe; c'est ce qu'on a aussi observé pour la même sel complexe cobaltique. Faisons attention seulement, concernant cette dernière bande, que le radical de sulfocyanate dans l'ion complexe chromique montre une propriété hyperchromique bien remarquable: la bande en question du sulfocyanate complexe chromique paraît déjà dans l'épaisseur de 5 mm de la solution centinormale, tandis que celle du complexe cobaltique ne commence à paraître que dans l'épaisseur de 40 mm de la solution, dans la même concentration.

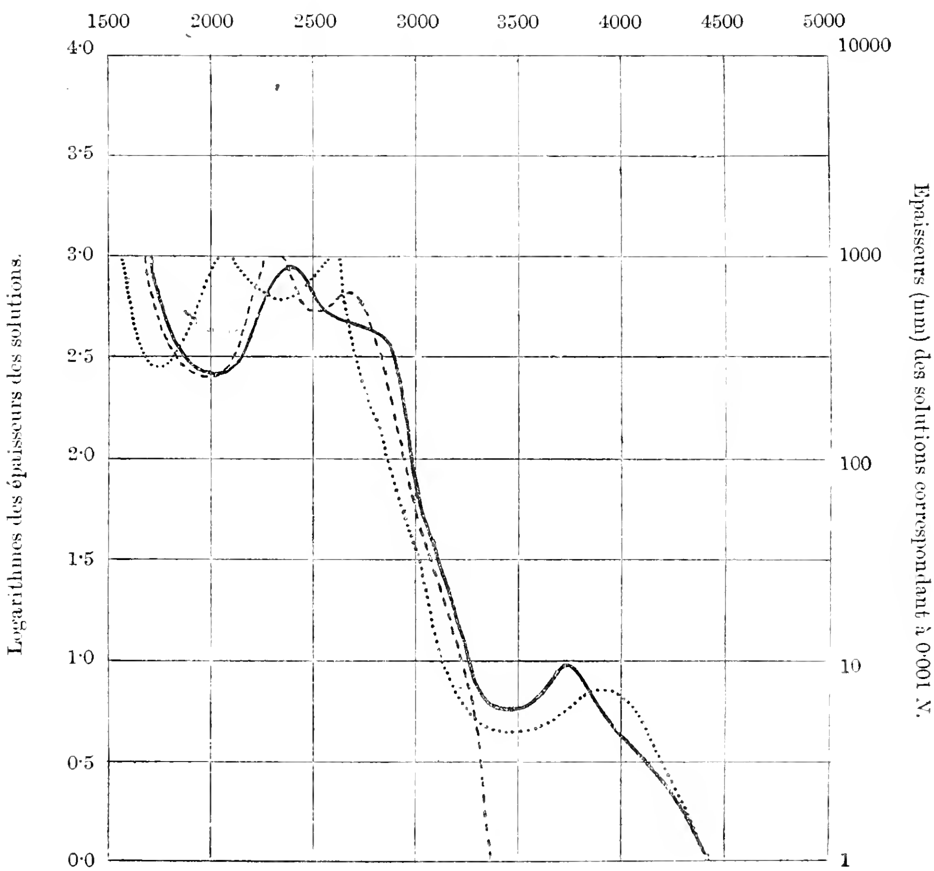
La courbe d'absorption du diéthylènediamine-disulfocyanato-complexe, dont les deux radicaux de sulfocyanate sont à la position de trans, est tracée, dans la figure VI, avec la ligne pointillée. On y remarquera un fait bien curieux; c'est que l'influence du deuxième radical de sulfocyanate est plutôt hypsochromique, c'est-à-dire que sa première bande est retrogradée jusqu'à 2100 de fréquence (vers violet), quoi que l'absorption du disulfocyanate-complexe (trans) soit distinctement bathochromique comparée à celle du héxammine-complexe.

De même, sa deuxième bande est très insignifiante. Il faut donc considérer que cette anomalie apparente est provoquée probablement par l'influence stéreochimique de deux radicaux de sulfocyanate, qui se placent, dans un ion complexe, à la position de trans l'un et l'autre. Ce qui est ainsi observé dans le cas du complexe chromique, c'est ce que nous avons déjà remarqué en certains complexes cobaltiques, bien que le phenomène dans les deux cas ne soit pas tout à fait identique. Nous nous rappellerons que les deux isomères de tétrammine-dinitro-complexes cobaltiques, crocéo (trans)- et flavo (cis) complexes, n'absorbent pas identiquement l'un et l'autre; le crocéo-complexe donne trois bandes dans toute la region de l'échelle spectrale, tandis que le flavo-complexe n'en possède que deux. Cette différence de la manière d'absorption doit être aussi due à l'influence stéreochimique. A notre grand regret, nous n'avons pu étudier, faute des matière necéssaire, le cis-disulfocyano-complexe chromique, qui nous mettra en évidence évidemment des choses bien intéressantes.

Le triammine-trisulfocyano-chrome a été étudié en une solution acétonique, à cause de son insolubilité dans l'eau. Comme

Fig. VI.

Fréquence.

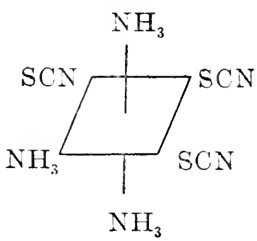


$Cr(NH_3)_3(SCN)_3$ (Solution dans l'acétone) - - - - - - - - - - -
$[Cr(NH_3)_5(SCN)](SCN)_2$ ·········
$(Cr\,en_2(SCN)_2^{(1)}_{(6)}]SCN$ ————————

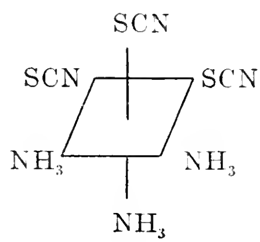
$\left(\frac{N}{100} - \frac{N}{1000}\right)$

on le voit dans la figure VI, ses deux bandes, qui se trouvent respectivement à 1900 et à 2500 de fréquence, sont encore sensiblement hypsochromiques relativement à celles du monosulfocyano-complexe, bien qu'elles soient logiquement bathochromiques, comparées aux complexes héxammoniés chromiques. Sa troisième bande, qui est due au radical de sulfocyanate dans l'ion complexe, ne paraît pas, ou elle est plutôt couverte par une absorption continuelle d'extrémité, causée par le solvant (acétone).

La similarité des positions des premières bandes du trans-disulfocyano-complexe et du trisulfocyano-triammine-complexe nous fait penser que la configuration I, dans laquelle deux radicaux de SCN parmi trois se trouvent dans la position de trans, est préferable pour le trisulfocyano-complexe, à la configuration II, dont les trois radicaux de SCN sont arrangés cosécutivement:



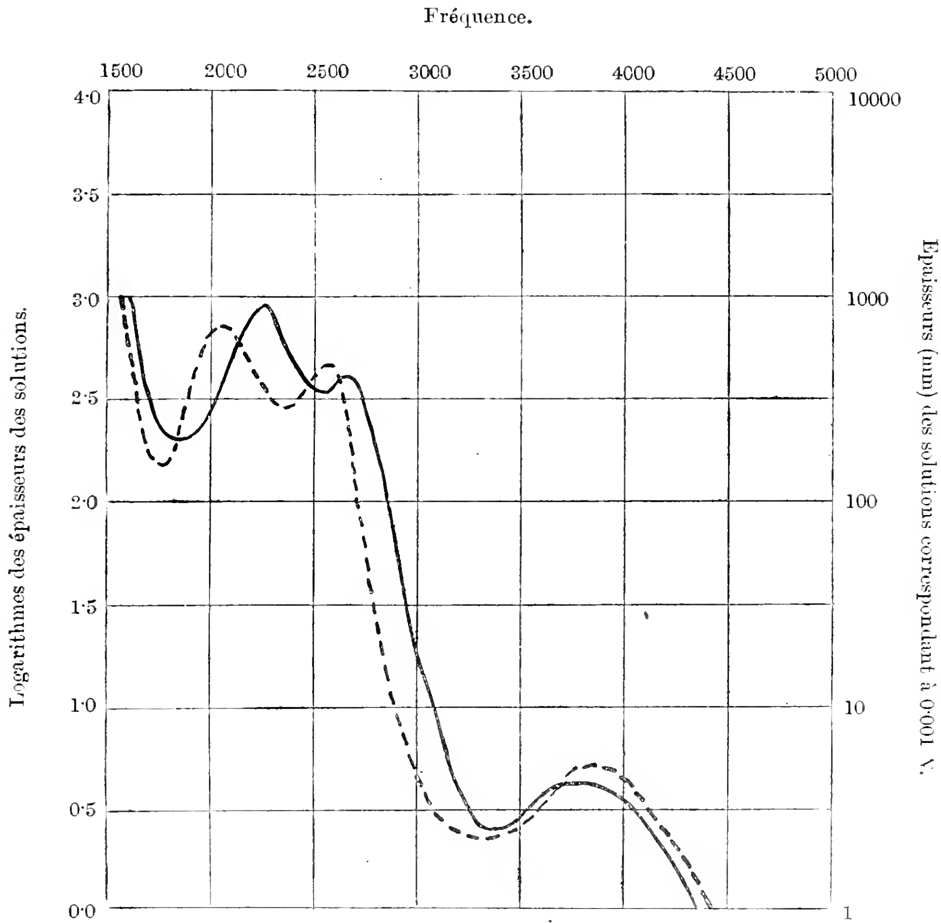
I.



II.

Les absorptions du diammine-térasulfocyano-complexe et de l'héxasulfocyano-complexe sont données dans la figure VII. Ces deux sels donnent aussi trois bandes très nettes. Leurs premières et deuxièmes bandes mettent en évidence bien clairement la nature bathochromique des radicaux de sulfocyanate dans l'ion complexe. Pourtant, par l'examen plus profond de ces deux courbes, nous pouvons facilement remarquer que le dégré de l'influence bathochromique est sensiblement moins important dans le tétrasulfocyano-complexe que dans le cas de l'héxacyano-complexe; en effet, la première et la deuxième bandes du tétrasulfocyano-complexe qui se placent respectivement à 1850 et à 2500 de fréquence n'atteignent pas même aux positions, où se trouvent les deux bandes du monosulfocyano-complexe (1750 et 2400). C'est peut-être la raison, pour laquelle le tétrasulfocyano-complexe absorbe moins bathochromiquement que le monosulfocyano-complexe, qu'au moins deux radicaux de SCN dans le premier corps doivent être necéssairement placés à la position de trans, qui, comme on a remarqué plus haut, exerce l'influence moins bathochromique.

Fig. VII.



$[Cr(NH_3)_2(SCN)_4]NH_4 \cdot H_2O$ ——— $\left.\right\}\left(\frac{N}{100} - \frac{N}{1000}\right)$
$[Cr(SCN)_6]K_3$ - - - - - - - - - -

Quant aux troisième bandes des deux complexes tétra- et héxasulfocyaniques, il suffira d'indiquer qu'elles montrent une bonne concordance avec celles d'autres sels de cette catégorie comme les bandes caractéristiques au radical de SCN.

6) Sulfates et oxalate chromiques.

Depuis long-temps il est bien connu que presque tout les sels chromiques existent en deux séries de composés: l'une représente des corps colorés violets tandis que l'autre contient ceux qui ont la couleur verte.

Leur constitution a été long-temps le sujet des recherches de beaucoup de chimistes; entre autres M. A. Werner et M. Recoura doivent etre spécialement nommés, car nous devons beaucoup à ces savants à propos du devéloppement de la chimie des sels chromiques dans cette direction.

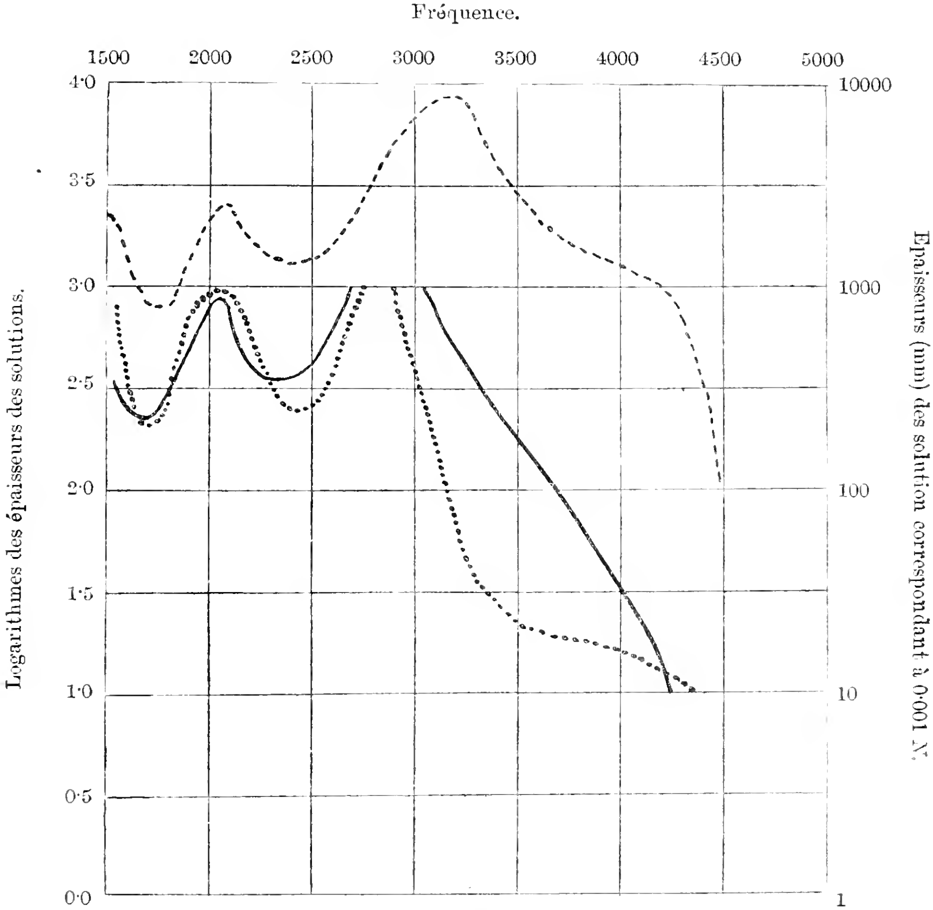
Donnons quelques exemples des sels appartenant aux deux séries:

Sels verts.	Sels violets.
$\left[Cr^{Cl}_{(H_2O)_5}\right]SO_4 \cdot H_2O$	$[Cr(H_2O)_6]Cl_3$
$\left[Cr^{Cl_2}_{(H_2O)_4}\right]Cl \cdot 2H_2O$	$[Cr(H_2O)_6](NO_3)_3 \cdot 3H_2O$
$\left[Cr^{SO_4}_{(H_2O)_5}\right]Cl \cdot H_2O$	$[Cr(H_2O)_6]^{Cl}_{SO_4}$
$\left[Cr_2{}^{(SO_4)_3}_{(H_2O)_6}\right]$ ou $\left[Cr_2{}^{(SO_4)_3}_{(H_2O)_8}\right] \cdot 2H_2O$	$[Cr_2(H_2O)_{12}](SO_4)_3 \cdot 5$ ou $6H_2O$

Nous avons remis, dans cette occasion, l'étude spectroscopique des membres de ces deux séries et, comme objets de nos recherches, nous avons choisi deux sulfates chromiques violet et vert.

L'examen comparatif des courbes d'absorption des deux corps, qui sont données dans la figure VIII, nous a permis de mettre en évidence des choses bien intéressantes. Approximativement, ils absorbent très semblablement l'un et l'autre (exactement, les deux bandes des sels verts et violet se trouvent respectivement à 1720 et à 2250 pour le premier et à 1750 et à 2400 pour le dernier), bien qu'ils fassent les solutions avec des couleurs très différentes. Toutefois, leurs abilités d'absorption ne sont pas du tout les mêmes, c'est-à-dire que le sel vert commence à absorber sélectivement à 28 mm de l'épaisseur de la solution pour sa prèmière bande et à 33 mm pour la deuxième dans la concentration centinormale, tandis que le sel violet commence à absorber sélectivement respectivement à 71 mm et à 130 mm de l'épaisseur. Cette absorption fortement hypochromique du sel violet est

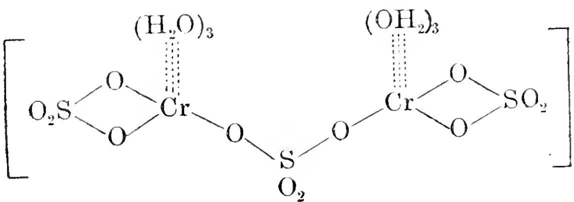
Fig. VIII.



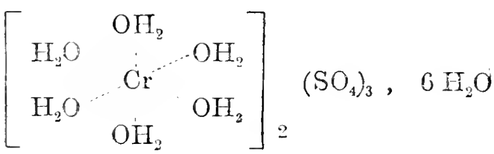
$Cr_2(SO_4)_3 \cdot 6H_2O$ (vert) ——— $\Big\}\left(\frac{N}{100}\right)$
$[Cr(C_2O_4)_3]\ K_3$ ··········
$Cr_2(SO_4)_3 \cdot 18H_2O$ (violet) - - - - - $\left(\frac{N}{10}\right)$

provoquée, il nous semble, par les molécules d'eau, qui occupent tous les nombres de coordination dans l'ion complexe, comme on le voit par sa formule dans la table donnée plus haut. Quant aux propriétés à la fois bathochromique et hypochromique des molécules d'eau dans un ion complexe, nous les avons déjà observé très souvent dans plusieurs aquo-complexes cobaltiques et chromiques. Il est pourtant bien naturel, que dans le sel vert, l'influence hypochromique ne soit pas très devéloppée, parce que les molécules d'eau, dans ce cas, n'occupe qu'une partie des nombres de coordination de son ion complexe. Leurs constitutions expliqueront davantage la rélation absorptive de ces deux sulfates.

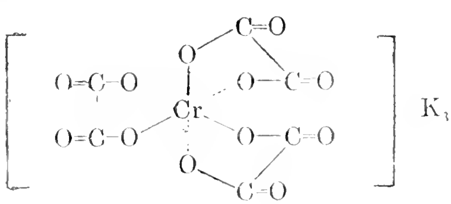
Sulfate vert.



Sulfate violet.



L'absorption de l'oxalate chromique, $[Cr(C_2O_4)_6]K_3$, qui est indiquée dans la figure VIII avec une ligne pointillée, montre aussi une similarité étonnante avec celles des sulfates, spécialement celle du sulfate vert. Comme la constitution de l'oxalate l'indique, les radicaux d'oxalate s'enchâinent directement à l'atome chromique central avec les atomes d'oxygène (Comparer Y. Shibata: Ibid. P. 13):

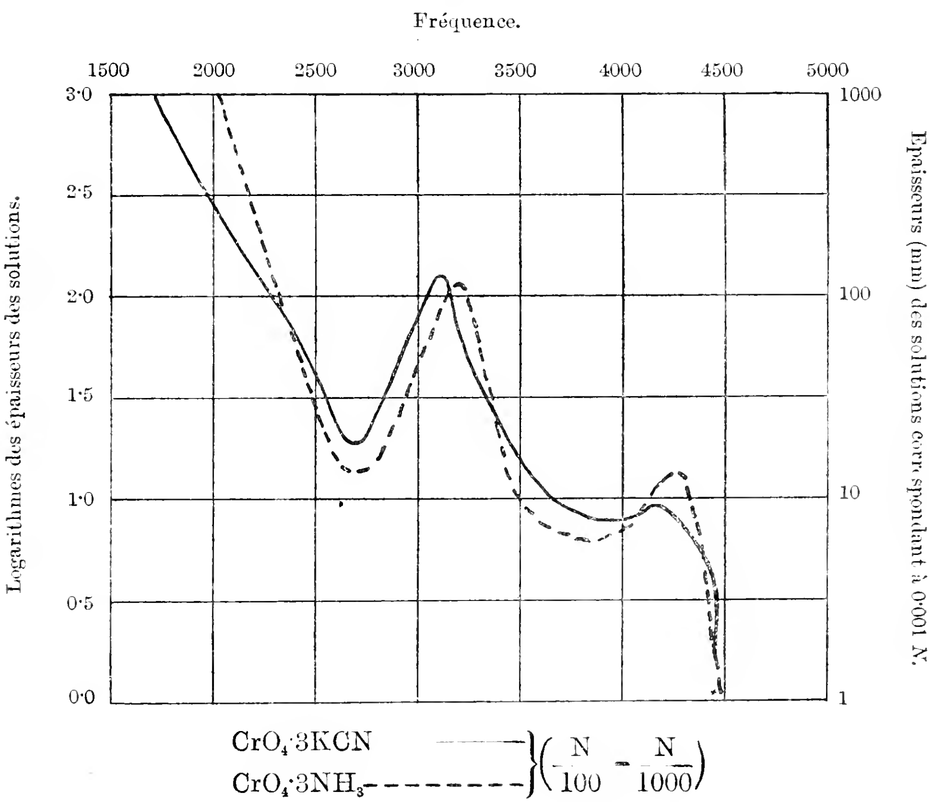


En revenant aux sulfates encore une fois, nous remarquerons que les radicaux du sulfate et les molécules d'eau dans le sel vert se sont liés à l'atome chromique central avec leurs atomes d'oxygène, tandis que l'atome chromique du sulfate violet, à son tour, s'enchâine directement aussi avec les atomes d'oxygène des molécules d'eau. C'est donc une raison pour que ces trois corps absorbent bien pareillement les uns et les autres.

7) Complexes de peroxydes chromiques.

On connaît une série de complexes chromiques bien curieux: $CrO_4 \cdot 3C_5H_5N$, $CrO_4 \cdot 3NH_3$ et $CrO_4 \cdot 3KCN$[K. A. Hofmann u. H. Hiendlmaier: Ber., 1905, **38,** 3059]. Parmi eux, celui qui contient trois molécules de pyridine, étant insoluble dans l'eau, n'a pas été étudié, tandis que les deux autres sont bien solubles, donnant des solution jaunes.

Fig. IX.



Comme on le voit dans la figure IX, les absorptions de ces deux complexes sont presque les mêmes l'une et l'autre, mais elles sont tout-à-fait différentes des autres complexes chromiques, que nous venons de décrire. Ils montrent deux bandes bien distinctes, comme les autres complexes, toutefois on n'aperçoit point de similarité entre les courbes d'absorption en question et celles des autres complexes. Les absorptions de ces deux complexes de peroxydes chromiques donnent leurs bandes à 2700 et à 3900 de fréquence, et dans l'allure tout le long de la courbe, on trouve une ressemblance parfaite entre celle-ci et celle des solutions aqueuses de chromates metalliques, qui a été observée par M. A. Hantzsch (Zeit. physik. Chem., 1910, **72,** 363).

Ce dernier fait nous indique que les chromates de potassium et d'ammonium sont faits dans les solutions des deux corps, d'apres les formules suivantes:

$$CrO_4{\cdot}3NH_3+3H_2O=CrO_4+3NH_4OH$$
$$CrO_4+H_2O=CrO_4H_2+O$$
$$CrO_4H_2+2NH_4OH=CrO_4(NH_4)_2+2H_2O$$

et

$$CrO_4{\cdot}3KCN+4H_2O=CrO_4H_2+3KOH+3CNH+O$$
$$CrO_4H_2+2KOH=CrO_4K_2+2H_2O$$

II. Absorption de rayons des sels de nickel.

1) Complexes héxammoniés nickeleux.

Les quatre sels complexes donnés ci-dessous ont été étudiés:

$[Ni(NH_3)_6]Cl_2$ [Sörensen: Z. anorg. Chem., 1894, 5, 363], un corps bleu violacé fade.

$[Ni(NH_3)_6]SO_4$ (Ibid.), un corps bleu violacé.

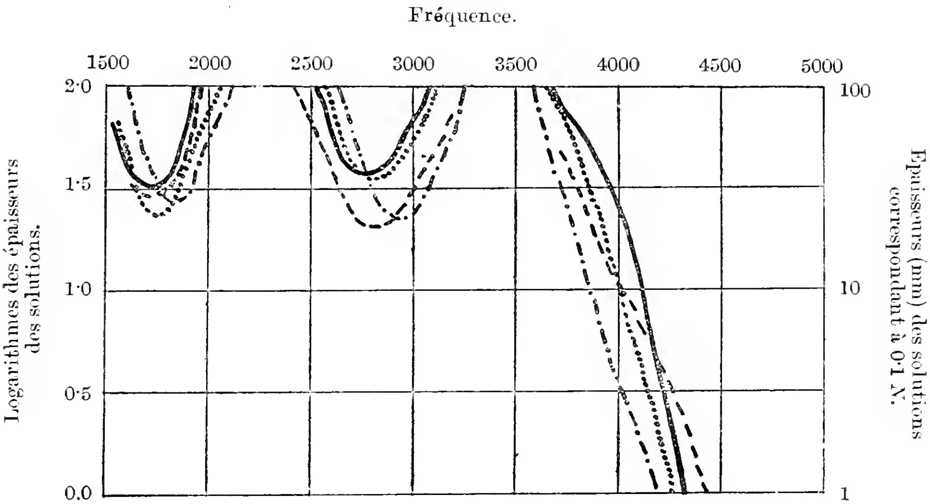
$[Ni(N_2H_4)_3]Cl_2$ [H. Franzen u. O. v. Mayer: Z. anorg. Chem., 1908, **60,** 262], un corps violet bleuâtre.

$[Ni\ en_3]Cl_2$ (Werner et Spruck: Zeit. anorg. Chem., 1899, **21,** 212), un corps violet bleuâtre.

Ils sont solubles dans l'eau, sauf celui qui contient trois molécules d'hydrazine. Ce dernier corps même, est bien soluble dans l'eau ammoniacale.

Leurs absorptions sont étroitement analogues les unes et les autres, comme elles sont montrées dans la figure X; les trois premiers corps donnent leurs deux bandes à 1700 et à 2750 de fréquence, tandis que seulement le triéthylènediamine-complexe, étant un peu hypsochromique, en donne respectivement à 1800 et à 2900 de fréquence. Or, on se trouve encore en présence d'exemples des propositions II, III et V, données en tête de cette note, qui annoncent que des complexes construits d'une façon analogue les uns et les autres absorbent semblablement et ne sont pas influencés par des ions simples qui s'accouplent avec les premiers.

Fig. X.



$[Ni(N_2H_4)_3]Cl_2$ ··············
$[Ni(NH_3)_6]Cl$ -
$[Ni(NH_3)_6]SO_4$ - - - - - -
$[Ni\ en_3]Cl_2 2H_2O$ —·—·—·—

(Solutions ammoniacales, $\frac{N}{10}$)

2) Ammine-complexes insaturés.

Dans les complexes nickeleux, il y en a quelques uns, dont les nombres de coordination ne sont pas saturés. Comme exemples des sels de cette catégorie, nous avons étudié les quatre sels suivants:

$[Ni(NH_3)_4]SO_4 \cdot 2H_2O$ [O. L. Erdmann: J. prakt. Chem., 1836, **7**, 264], bleu fade.

$[Ni\ en_2]Cl_2 \cdot H_2O$ [Grossmann u. Schück: Zeit. anorg. Chem., 1906, **50**, 9], violet rougeâtre.

$[Ni(N_2H_4)_2]Cl_2$ [H. Franzen u. O. v. Mayer: Zeit. anorg. Chem., 1908, **60**, 262], gris violacé.

$[Ni(NH_3)_3]Cl_2 \cdot 3H_2O$ [Andreé: Compt. rend., 1888, **106**, 937], bleu violacé.

Ils sont très bien solubles dans l'eau, mais ces solutions ne sont pas du tout stables; elles se décomposent assez rapidement et donnent des précipités d'hydroxyde nickeleux. Dans ce cas, quelques gouttes d'ammoniaque, ajoutées à la solution, suffiront pour arrêter cette décomposition hydrolytique.

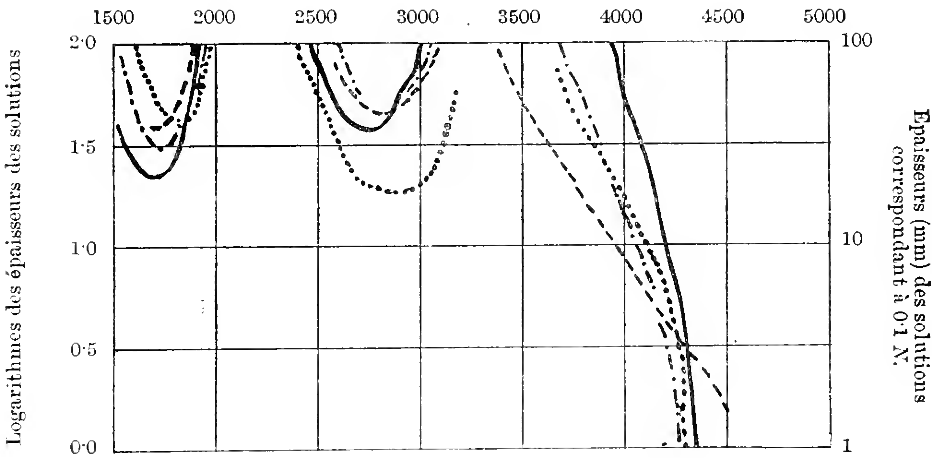
Comme on le voit dans la figure XI, les absorptions de ces quatre corps ne diffèrent guère les unes et les autres. De même, ces absorptions montrent une analogie bien notable avec celles des complexes précédents. La ressemblance de l'aspect des courbes des complexes appartenants à ces deux groupes nous indique que ceux qui sont insuffisamment ammoniés prennent quelques molécules d'ammoniaque dans la solution ammoniacale et deviennent des complexes héxammoniés en saturant des nombres de coordination.

3) Complexes nickeleux compliqués.

$2NiSO_4 \cdot 5NH_3 \cdot 7H_2O$[Andrée: Compt. rend., 1888, **106**, 937] est composés de cristaux bleus fades, qui se décomposent dans une solution aqueuse et donnent des précipités d'hydroxyde nickeleux. Ces précipités sont cependant solubles par l'addition de quelques quantités d'ammoniaque. L'autre corps $[NiSO_4(NH_4)_2SO_4 \cdot 6NH_3 \cdot 3H_2O$[Andrée: Ibid. 938] est composés de cristaux de la même couleur que le complexe précédent; il est toutefois bien soluble dans l'eau avec une couleur bleue et ne montre aucune tendance de décomposition.

Dans la figure XII, on voit les courbes d'absorption de ces deux complexes qui montrent encore une fois une analogie étroite non seulement entre elles, mais aussi avec celles des autres complexes nickeleux, dont les absorptions ont été discutées par nous jusqu'ici, c'est-à-dire que leurs bandes se placent à 1750 et à 2750 de fréquence.

Fig. XI.



$[Ni(NH_3)_3]Cl_2 \cdot 3H_2O$ ———

$[Ni\ en_2]Cl_2 \cdot H_2O$ - - - - -

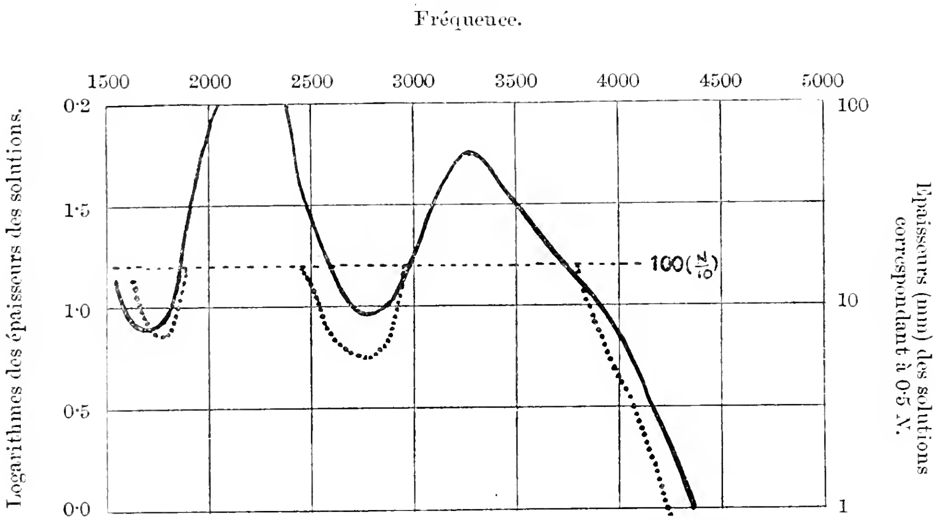
$[Ni(NH_3)_4]SO_4 \cdot 2H_2O$ ·····

$[Ni(N_2H_4)_2]Cl_2$ —·—·—·—

(Solutions ammoniacales, $\frac{N}{10}$)

Quant à la constitution du corps $2NiSO_4 \cdot 5NH_3 \cdot 7H_2O$, il faut tout d'abord déterminer, s'il prend la forme, dans la solution, d'un sel double entre le sulfate nickeleux avec 7 molécules de l'eau de cristallisation et le sulfate de péntammine-complexe nickeleux, $[Ni(NH_3)_5]SO_4$, ou s'il est devenu deux molécules d'un complexe $[Ni(NH_3)_6]SO_4$, en prenant encore 7 molécules d'ammoniaque dans une solution ammoniacale. S'il est vrai que ce complexe prend la formule d'un sel double contenant une molécule de sulfate nickeleux comme un composant, il devra donner l'absorption caractéristique à ce sel simple, dont l'absorption va être décrite tout de suite (la figure XIII.). Mais un coup d'œil jeté sur la figure suffit pour remarquer que ce n'est pas le

Fig. XII.



$2NiSO_4 \cdot 5NH_3 \cdot 7H_2O$——(Solution ammoniacales, $\frac{N}{2}$)

$(NH_4)_2SO_4 \cdot NiSO_4 \cdot 6NH_3 \cdot 3H_2O$······(Solution dans l'eau pure, $\frac{N}{10}$)

cas; l'aspect de sa courbes nous indique bien clairement qu'il est plutôt de la forme du complexe héxammonié.

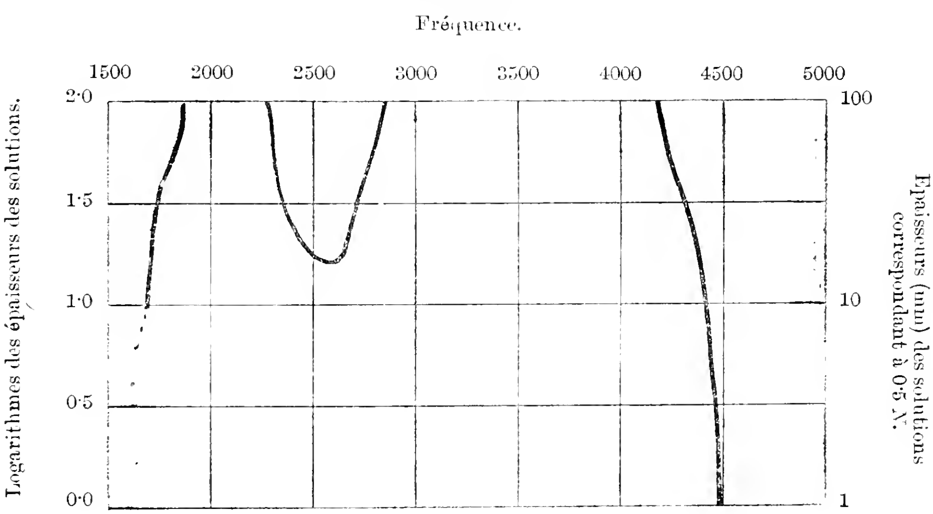
Le sel avec une formule brute $(NH_4)_2SO_4 \cdot NiSO_4 \cdot 6NH_3 \cdot 3H_2O$ est un corps bien stable dans l'eau, et sa courbe d'absorption ne diffère pas beaucoup de celle du complexe précédent. Dans la solution, le sel se dissocie évidemment dans les ions $NH_4^{\cdot}$, SO_4'' et peut-être $[Ni(NH_3)_6]^{\cdot\cdot}$, ce dernier donnant l'absorption caractéristique aux complexes ammoniés nickeleux. Le sel est bien stable dans l'eau contrairement à autres complexes, parce qu'il est protégé contre la décomposition hydrolytique par l'excès des ions SO_4'' et $NH_4^{\cdot}$, c'est-à-dire que les réactions hydrolytiques réversibles formulées ci-dessous sont poussées, d'après la loi d'action de masse, de droite à gauche respectivement par SO_4'' et $NH_4^{\cdot}$:

$$[Ni(NH_3)_n]SO_4 + 2H_2O \rightleftarrows [Ni(NH_3)_n](HO)_2 + SO_4'' + 2H^{\cdot}$$
$$[Ni(NH_3)_n](HO)_2 + nH_2O \rightleftarrows Ni(HO)_2 + nNH_4 \cdot HO,$$
$$n < 6.$$

4) Sulfate et chlorure de nickel.

Ils se cristallisent avec 7 molecules d'eau en sulfate et avec 6 molécules d'eau en chlorure, leur couleur étant vert foncé. La figure XIII montre les courbes d'absorption de ces deux sels. Bien que leurs anions soient tout-à-fait différents, les deux sels absorbent d'une manière analogue. Leur absorption se caracterise par sa très faible abilité. Son unique bande à 2550 de fréquence ne paraît que dans la solution d'une concentration demi-normale. On observe encore des absorptions continuelles du côté du rouge et de l'ultraviolet: il nous semble que la première doit être un fragment d'une bande dans l'infrarouge, dont il est impossible de faire l'observation complète avec une plaque panchromatique même. Si cette considération est correcte, elles devront correspondre aux deux bandes des complexes ammoniés nickeleux, qui se placent à 1750 et à 2750 de fréquence. Dans ce cas, elles sont alors déplacées considérablement vers le rouge, évidemment par l'influence bathochromique des molécules d'eau qui sont liées avec l'atome nickeleux central en donnant un ion complexe $Ni(H_2O)_n$.

Fig. XIII.



$NiCl_2 \cdot 6H_2O$

$NiSO_4 \cdot 7H_2O$

III. Absorption de rayons des sels cuivriques.

1) Complexes pyridinés cuivriques.

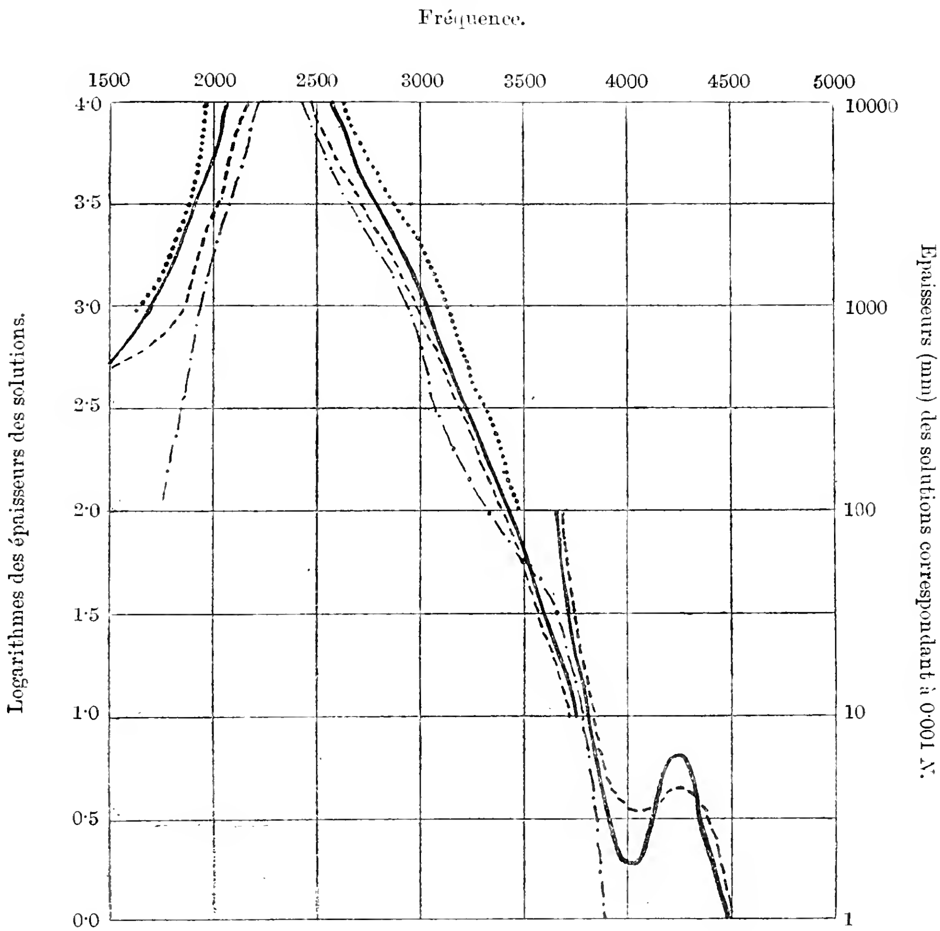
Les quatre sels complexes pyridinés donnés ci-dessous ont été étudiés:

$[Cu(C_5H_5N)_6](NO_3)_2 \cdot 3H_2O$ [Pfeiffer u. Pimmer: Z. anorg. Chem., 1906, 48, 107].
$[Cu(C_5H_5N)_4](NO_3)_2$ [Pfeiffer u. Pimmer: Ibid. 101].
$[Cu(C_5H_5N)_3](NO_3)_2$ [Pfeiffer u. Pimmer: Ibid. 103].
$[Cu(C_5H_5N)_4]SO_4$ [S. M. Jörgensen: J. prakt. Chem., 1886, [2], 33, 502].

Tous ces corps sont formés de cristaux bleus foncés et solubles dans l'eau sans montrer aucune tendance de décomposition. La figure XIV représente les courbes d'absorption d'un groupe de ces substances. C'est leur caractère bien notable qu'il n'y a aucune bande essentielle à ces corps, mais elles absorbent continuellement du côté du rouge et du violet. L'examen des courbes nous indique, que le complexe héxapyridiné est, en outre, le plus stable dans l'eau et la solution satisfait parfaitement la loi de Beer sur l'absorption. De même, ce corps montre l'abilité d'absorption la plus forte; cela est bien naturel, car c'est un complexe, dont les nombres de coordination sont complètement saturés par 6 molécules de pyridine.

Le nitrate du complexe tétrapyridiné cuivrique est aussi assez stable dans l'eau, lorsque la solution n'est pas encore très étendue, mais la courbe perd sa continuité déjà dans les concentrations entre centinormale et millinormale. Il est bien intéressant qu'on observe une bande, dans la solution millinormale, à 4000 de fréquence, ce qui est caractéristique de la solution aqueuse de pyridine (Hartley: Journ. Chem. Soc., 1885, **47**, 685). La naissance d'une bande caratéristique à pyridine n'indique pas autre chose que ce qui a paru dans la solution assez étendue, comme millinormale, par la décomposition partielle de l'ion complexe tétrapyriné $[Cu(Py)_4]$.

Fig. XIV.



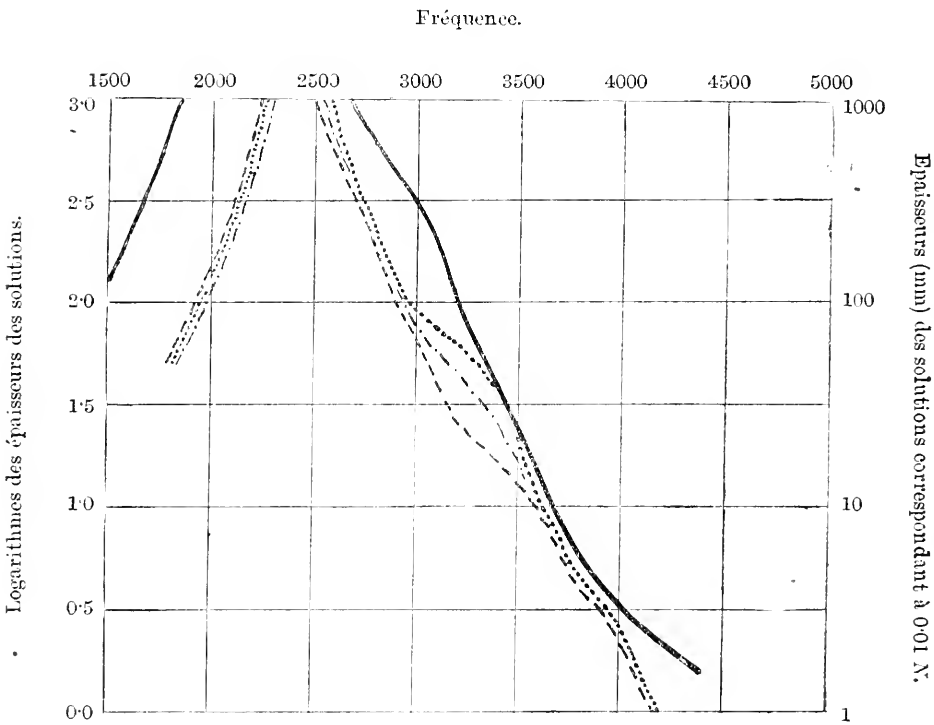
$[Cu(C_5H_5N)_3](NO_3)_2$ ——
$[Cu(C_5H_5N)_4](NO_3)_2$ - - - - - -
$[Cu(C_5H_5N)_4]SO_4$ ··············
$[Cu(C_5H_5N)_6](NO_3)_2$ -·-·-·-·-·-

$\left(\frac{N}{10} - \frac{N}{1000}\right)$

Quant à la stabilité du complexes tripyriné, il nous semble qu'elle est peut-être moins avancée, parce que la bande de pyridine dans la solution de la concentration de millinormale, est beaucoup plus nette que celle du complexe précédent. Le sulfate du complexe tétrapyridiné n'est plus inaltérable dans la solution de la concentration de centinormale; si on la laisse, elle donne des précipités d'hydroxyde cuivrique, qui nous ont empêché d'étudier les solutions plus diluées.

2) Acétates des complexes ammoniés cuivriques.

Fig. XV.



$\frac{1}{2}[NH_4 \cdot Cu_2(C_2H_3O_2)_5]H_2O$ —— (Solution dans l'eau pure, $\frac{N}{10}$ — $\frac{N}{100}$)

$Cu(NH_3)_2(C_2H_3O)_2$ - - - - - - -
$Cu(NH_3)_3(C_2H_3O_2)Cl \cdot H_2O$ ......
$Cu(NH_3)_3(C_2H_3O_2)I$ —·—·—·—
} (Solutions ammoniacales, $\frac{N}{10}$ — $\frac{N}{100}$)

Comme exemples des complexes de ce genre, nous avons étudié les quatre sels suivants:

$Cu(NH_3)_2(C_2H_3O_2)_2$ [D. W. Horn : Am. Chem. J., 1908, 39, 206].

$Cu(NH_3)_3Cl(C_2H_3O_2)\cdot H_2O$ [Th. W. Richards and Show : Am. Chem. J., 1893, 15, 645].

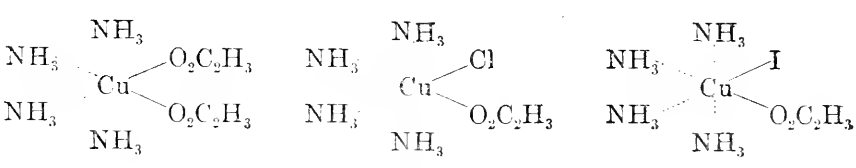
$Cu(NH_3)_3I(C_2H_3O_2)$ [Th. W. Richards and Oenslager : Am. Chem. J., 1895, 17, 298].

$NH_4Cu_2(C_2H_3O_2)_5\cdot H_2O$ [Th. W. Richards and Oenslager : Ibid., 304].

Les trois premiers complexes ont presque la même couleur de bleu violacé brillant, tandis que le dernier est coloré en vert bleuâtre.

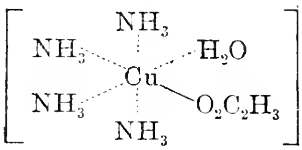
Les trois complexes propres sont hydrolysés dans les solutions aqueuses, sinon un peu d'ammoniaque y est ajouté, tandis que le sel double vert bleuâtre est bien stable dans l'eau. Par l'examen comparatif des courbes, on remarque que les trois acétates des complexes ammoniés absorbent bien fortement, tandis que le sel double se montre sensiblement inférieur en cette abilité.

Comme nous l'avons déjà remarqué dans l'introduction, les absorption des complexes ammoniés cuivriques sont curieusement influencées par des anions qui s'accouplent avec les cations complexes, et ce phenomème est le plus remarquable dans le cas des acétates. Prenant en considération ce fait, nous voulons donner les constitutions suivantes aux complexes en question:



(I) (II) (III)

Quant aux (II) et (III), ils subiront très probablement les substitutions de l'atome de chlrore ou d'iode respectivement par une molécule d'eau, lorsqu'ils seront dissolus dans l'eau:

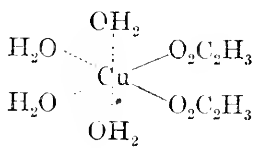


Cl ou I

Le sel double vert bleuâtre se dissocie, sans doute, dans l'eau, d'après les formules suivants:

$NH_4Cu_2(C_2H_3O_2)_5 \rightleftarrows NH_4(C_2H_3O_2)+2Cu(C_2H_3O_2)_2 \rightleftarrows NH_4^{\cdot}+ 2Cu^{\cdot\cdot}+5(C_2H_3O_2)'$, mais les ions de cuivre et d'acétate reconstruiront un nouvau sel complexe:

$Cu^{\cdot\cdot}+2(C_2H_3O_2)'+4H_2O \rightleftarrows Cu(H_2O)_4(C_2H_3O_2)_2$, dont la constitution sera



parce que ce sel double absorbe non seulement dans la region du violet, mais aussi à côté du rouge; cette dernière absorption ne paraît jamais dans l'acétate cuivrique simple, dont l'absorption sera traitée plus tard (la figure XVIII.).

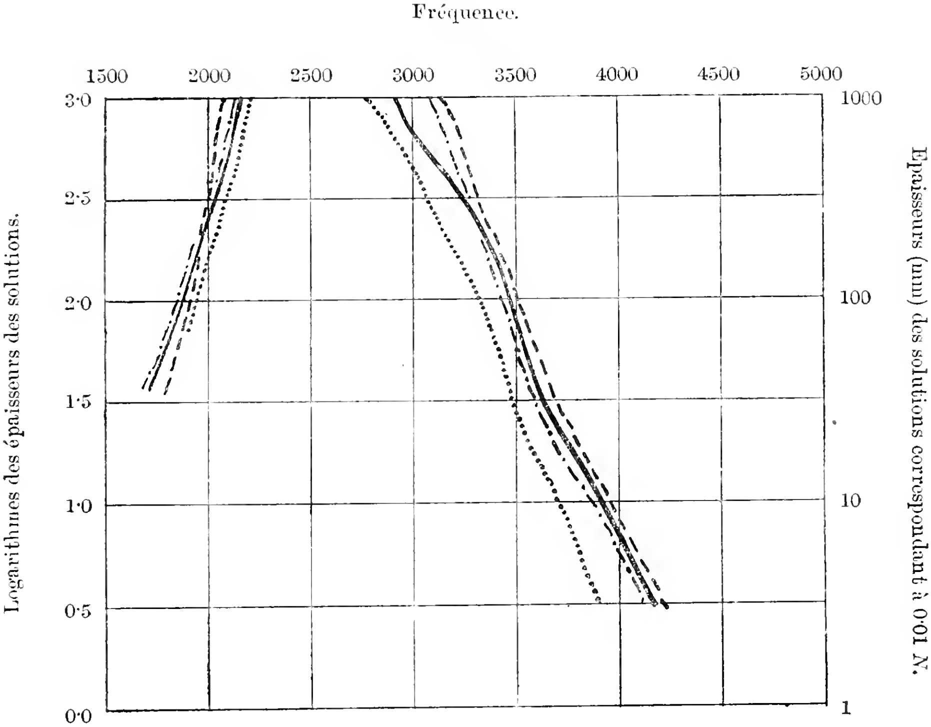
3) Oxalates des complexes ammoniés cuivriques.

Les quatre sels étudiés sont

$Cu(NH_3)C_2O_4$ [D. W. Horn u. M. A. Graham: Am. Chem. J., 1908, 39, 508]
$Cu(NH_3)_2C_2O_4$ [〃 〃]
$Cu(NH_3)_4C_2O_4 \cdot 2H_2O$ [〃 〃]
$Cu(NH_3)_5C_2O_4$ [D. W. Horn: Am. Chem. J., 1906, 35, 279]

Il sont colorés en bleu très vif, et soluble dans l'eau très difficilement. Si l'on emploie de l'eau ammoniacale comme solvant, les oxalates s'y dissolvent bien facilement.

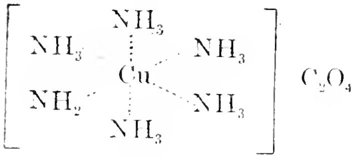
Fig. XVI.



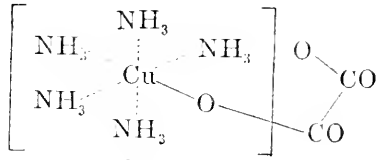
$Cu(NH_3)C_2O_4$ ———
$Cu(NH_3)_2C_2O_4$ - - - - - - - - -
$Cu(NH_3)_4C_2O_4 \cdot 2H_2O$ ....
$Cu(NH_3)_5C_2O_4$ —·—·—·—·—

(Solutions ammoniacales, $\frac{N}{10} - \frac{N}{100}$)

Comme la figure XVI l'indique, ces quatre corps absorbent bien semblablement les uns et les autres, et leur abilité est sensiblement inférieure vis-à-vis de tous les complexes, dont les absorptions ont été discutées jusqu'ici. Le fait que les quatre complexes montrent des absorptions bien analogues les uns et les autres, nous indique qu'ils prennent une constitution similaire. On ne sait pourtant s'ils sont de la forme (I) en recevant les molécules d'ammoniaque des solutions, ou s'ils cessent d'être un complexe péntammonié et prennent la forme (II):



(I)



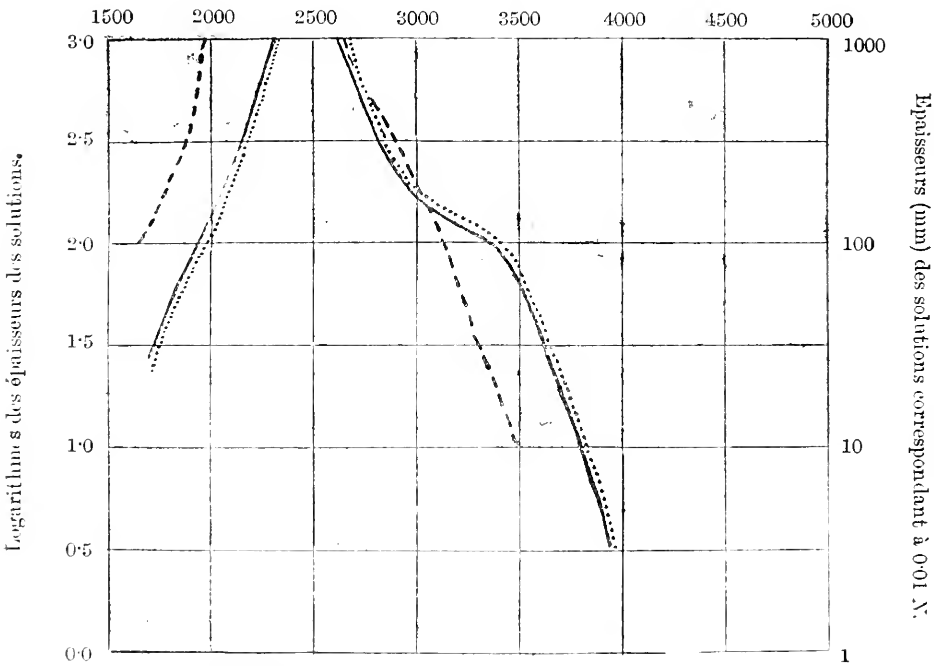
(II)

mais par la raison que des complexes cuivriques sont toujours optiquement influencés par les anions, nous voulons accepter la formule (II).

4) Sulfates des complexes ammoniés cuivriques.

Fig. XVII.

Fréquence.



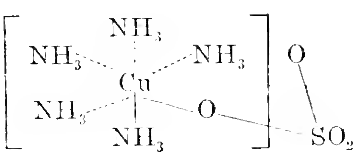
$Cu(NH_3)_4SO_4$ ——— } (Solutions ammoniacales, $\frac{N}{10}$ — $\frac{N}{100}$)

$Cu(NH_3)_5SO_4$ ······

$Cu(C_5H_5N)_4SO_4$ - - - - - - (Solution dans l'eau pure, $\frac{N}{10}$)

Les deux sels $Cu(NH_3)_4SO_4$ [Bouzat: Compt. rend., 1902, **134,** 1218] et $Cu(NH_3)_5SO_4$ [D. W. Horn: Am. Chem. J., 1908, **39,** 194] sont formés de cristaux bleus très foncés et bien solubles dans l'eau; leurs solutions deviennent cependant instables par dilution et donnent enfin des précipités d'hydroxyde cuivrique, qui sont encore dissous, quand on y ajoute quelques quantités d'ammoniaque.

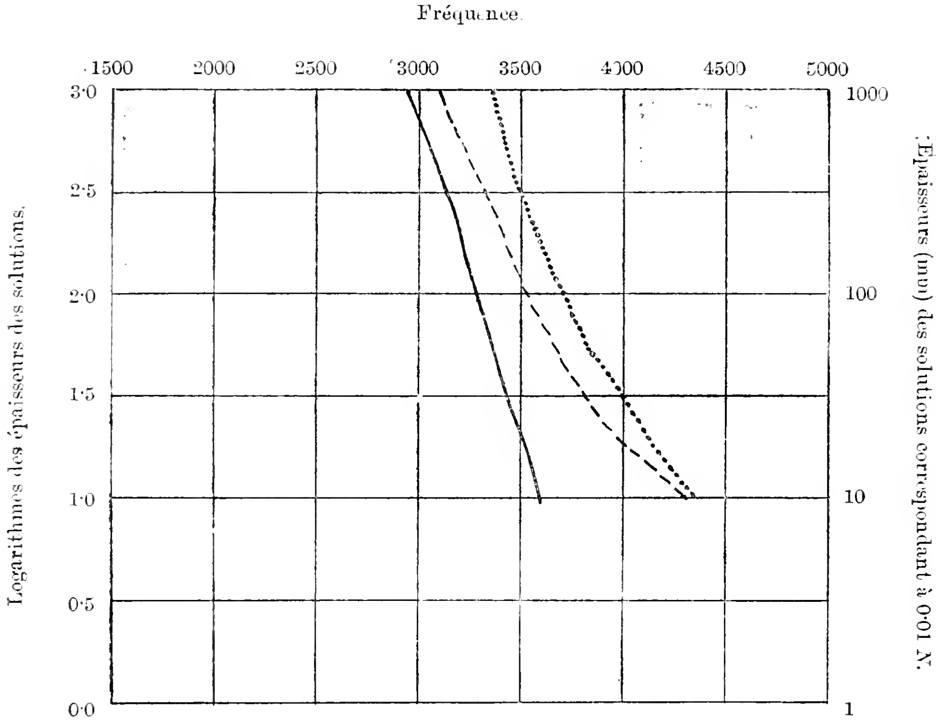
Les deux corps absorbent tout pareillement l'un et l'autre, comme on le remarque dans la figure XVII; l'absorption continuelle à côté du violet montre une allure très particulière, qui est peut-être caractéristique aux sulfates des complexes ammoniés. Ce qui n'a pas lieu dans le cas du sulfate du complexe pyridiné, dont la courbe est donnée aussi dans la figure XVII pour que l'on puisse faire la comparaison, c'est parce qu'il a une autre constitution que celles des sulfates des complexes ammoniés. Pour la même raison, que nous avons déjà annoncé dans le cas des acétates et des oxalates, nous donnons la constitution suivante au sulfate du complexe ammonié:



5) Sels cuivriques simples—Acétate, sulfate et chlorure.

Pour faire une étude comparative, nous avons choisi trois sels simples $Cu(C_2H_3O_2)_2 \cdot H_2O$, $CuCl_2 \cdot 2H_2O$ et $CuSO_4 \cdot 5H_2O$, dont les absorptions sont données dans la figure XVIII.

Fig. XVIII.



$Cu(C_2H_3O_2)_2 \cdot H_2O$ ——— } (Solutions aqueuses, $\frac{N}{10}$)
$CuCl_2 \cdot 2H_2O$ – – – – – – – – –
$CuSO_4 \cdot 5H_2O$ ……………

Chacun de ces trois sels, contrairement aux autres sels complexes cuivriques, ne montre, dans la solution décinormale, qu'une absorption continuelle dans la région du violet. S'il est vrai, comme on le croit généralement, que des sels dans les solutions aqueuses se dissocient eléctrolytiquement dans un dégré presque parfait, les trois sels étudiés ici doivent donner la même absorption l'un et l'autre, parce que, comme nous l'avons déjà très souvent indiqué, les anions, qui n'absorbent pas eux-mêmes, n'exercent pas d'influence quelconque sur les absorptions de ces sels. Dans le cas actuel pourtant les trois sels absorbent bien différemment, c'est-à-dire que, l'acétate, qui n'a qu'une moléeule d'eau, absorbe le plus fortement, le chlorure avec deux molécules d'eau montre une abilité d'absorption intermédiaire, tandis que le sulfate, qui possède autant de molécules d'eau que cinq, absorbe le moins fortement.

Comme on le connaît bien, la solution aqueuse de sulfate cuivrique réagit neutralement, tandis que les deux autres montrent une réaction acidique dans la solution aqueuse. On croit généralement que le chlorure et l'acétate de cuivre se décomposent hydrolytiquement dans l'eau, d'après les formules suivantes:

$$CuCl_2 \cdot 2H_2O \rightleftarrows [CuCl_2(OH)_2]'' + 2H^{\cdot}$$
$$CuCl_2 \cdot 2H_2O \rightleftarrows [Cu(H_2O)_2]^{\cdot\cdot} + 2Cl'$$
$$Cu(C_2H_3O_2)_2 \cdot H_2O \rightleftarrows Cu(C_2H_3O_2)_2(HO) + H^{\cdot}$$
$$'' + nH_2O \rightleftarrows [Cu(H_2O)_n] + (C_2H_3O_2)'$$

Ce sont certainement les ions complexes contenant les radicaux d'acétate ou de chlorure qui absorbent le plus fortement, tandis que le cation complexe qui ne contient que l'atome cuivrique et quelques molécules d'eau absorbe bien inférieurement.

RÉSUMÉ.

1) Les complexes chromiques sont toujours bien stables dans les solutions aqueuses et donnent généralement deux ou trois bandes d'absorption très nettes. Tout ce qui est vrai pour les complexes cobaltiques l'est aussi pour ceux de chrome.

2) Les complexes nickeleux sont généralement instables dans l'eau, par conséquent, il nous fallait les étudier dans les solutions ammoniacales. Ils montrent toujours deux bandes d'absorption et leurs abilités sont beaucoup moins fortes que celles des complexes précédents. Néanmoins les absorptions des complexes nickeleux obèissent à la même loi que le chrome et le cobalt.

3) Les complexes cuivriques sont les plus labiles dans les solutions; de même, les anions exercent quelque influence sur l'absorption dans ce cas. Ils ne donnent aucune bande dans l'échelle spectrale entière, mais ils absorbent continuellement à côté du rouge et du violet.

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Magnetic Separation of the Lines of Iron, Nickel and Zinc in Different Fields.

By

Kogoro YAMADA, *Rigakushi.*

With 20 *Plates.*

I. The Object of the Experiment.

Zeeman,[1] taking the "molecular current" of Ampere into consideration, inferred that the outer components of triplets of iron lines may differ in intensity. Experimentally, however, he confirmed that there is no evidence of a directing influence of the magnetic field on the orbits of the light-ions as Preston[2] believed. Becquerel and Deslandres,[3] Reese,[4] Hartmann,[5] van Bilderbeek-van Meurs,[6] Arthur King[7] and Graftdijk[8] studied the magnetic separations of iron lines and examined their characters with care; but all these authors assumed that the separations of iron lines are proportional to the strength of the magnetic fields. It is a question whether we may assume this proportionality in spite of the disagreement of $\frac{e}{m}$ of these lines with the value obtained by Lorentz's elementary theory. Professor Nagaoka, from a theoretical point of view, suggested that in many-lined spectra such as

1) Zeeman, Astrophys. Journ., **9** (1899), p. 47.

2) Preston, Phil. Mag., **45** (1898), p. 333.

3) Becquerel et Deslandres, C. R., **126** (1898), p. 997.

4) Reese, Astrophys. Journ., **12** (1900), p. 120.

5) Hartmann, Dissertation, Halle (1907).

6) Van Bilderbeek-van Meurs, Arch. Néerl., (2) **15** (1911), p. 353.

7) Arthur King, Astrophys. Jour., **31** (1910), p. 433; **34** (1911), p. 225; Carnegie Institute Papers of the Mount Wilson Solar Observatory, **2** (1912), Part I.

8) Graftdijk, Arch. Néerl., (3) **2** (1912) p. 192.

iron and tungsten, there may be numerous lines showing abnormal Zeeman effect. It was with the object of testing this point that the present investigation was undertaken.

On the other hand, the results obtained by various experimenters differ so greatly from one another that I attributed these disagreements to the different values of $\frac{\Delta\lambda}{H}$ (here $\Delta\lambda$ denotes the separation of two outer components of a triplet and H the magnetic field applied) of one and the same line, and the discrepancy of many results is chiefly due to the fact that magnetic separations were studied by applying different magnetic fields. In order to decide this question, magnetic fields of different strengths must be applied to test the magnetic separations. Kent[1] in studying this problem found the drooping of the magnetic separations in fields stronger than 30,000 gauss. In reviewing his result, I found that the separation of zinc line λ: 4680·138 is not linearly proportional to the magnetic fields applied, while it was confirmed that this line is separated lineary proportional to the fields by Cotton and Weiss,[2] Stettenheimer[3] and others. And I concluded that his determination of magnetic fields by the ballistic method bofore and after photographing the spectra may be different. Stettenheimer, comparing her result upon the separation of λ: 4680·138 of zinc with that of Kent, remarked that the value of Kent was smaller by 13·2 %. Hartmann also remarked that the value given by Kent was smaller by 8% as regards the iron lines. Van Bilderbeek-van Meurs[4] and Arthur King[5] are also of the same opinion. For the solution of this problem, we must have recourse to new experiments, and *determine as accurately as possible the field at which the magnetic separation takes place.*

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1) Kent., Astrophys. Journ., **13** (1901), p. 289.
2) Weiss et Cooton, Journ. d. Phys., (4) **6** (1907), p. 427.
3) Stettenheimer, Ann. d. Phys., (4) **24** (1907), p. 384.
4) Van Bilderbeek-van Meurs, loc. cit., p. 391.
5) Arthur King, Carnegie Institution Papers etc., p. 7.

II. The Method of Experiment.

1. **Light Source.** A spark discharge between nickel steel and zinc wires by an induction coil was used as the source of light throughout the experiment. The primary current of the coil was supplied at 100 volts and 50 cycles from the secondary of the pole transformer of the laboratory, run at 3500 volts of the city main. The current strength was usually from 3 to 4 amperes,

The spark discharge, as the source of light, was placed in the middle between the two poles of the electromagnet and the spark gap made as small or large as the case required. In Fig. 1. the spark can be displaced by means of screw A. With a view of changing the spark gap, C can be moved up and down by means of D, and a small adjustment made it possible to keep the ends at a constant distance from each other, when the ends of the electrodes move away during the sparking. B and C are insulated by pieces of ebonite e and e′.

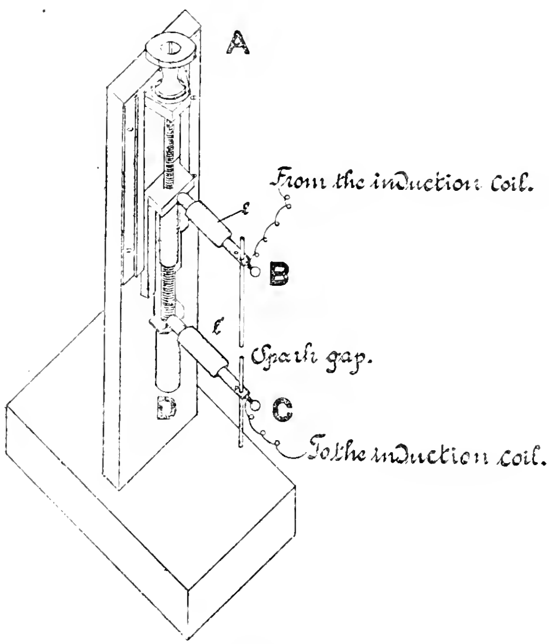

Fig. 1.

As the terminals for the spark must be put in high magnetic fields, it was necessary to use pieces which were made non-magnetic if possible. A non-magnetic nickle-steel alloy containing about 25% nickel[1] is most recommendable for this purpose as a portion of the photograph of

1) Nagaoka and Honda, Journ. Coll. Sc., Imperial University, Tokyo, **19** (1903–1904) Art. 11.

iron and nickel spectra can be taken on the same plate. Moreover, I soldered a piece of the above mentioned nickel-steel, about 10 mm in length and 2 mm in diameter, to a brass rod. When strong fields were required, the proximity of the magnet poles necessitated the filing of the terminals into flat ends as shown in Fig. 2. Even when the magnetic field was raised to 35000 gauss, these pieces were not found to be attracted by the magnet poles. In this investigation, as the zinc spectrum for determining the field was photographed simultaneously, *one of the terminals was always a zinc rod of the same diameter.*[1)]

Fig.2.

Similary as mentioned in many published papers, condensers were connected parallel in the spark circuit, which also contained selfinduction coils. In my experiment, three Leyden jars, each of which was about 0·00404 micro-farad, were connected in parallel and the whole was inserted parallel to the spark circuit. Afterwards, on the advice of Professor Nagaoka, I introduced another spark gap into the spark circuit in addition to the selfinduction coils. This is recommendable, especially in the case of long exposures. Very small selfinduction was sufficient in my experiment, the sound of the spark retained its shrillness, especially after the auxiliary spark gap was introduced in the circuit.

The spark gap of the light source varied from 1 to 3 mm. as the pole-pieces of the electromagnet were changed. From a long spark gap, I could get a brilliant light and a reduction in time of exposure was rendered possible.

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1) When I read this paper at the ordinary meeting of the Physico-mathematil Society of Japan on the 8th of March, Professor Nagaoka put the following question to me:

"The non-magnetic nickel-steel can positively have no effect upon the strength of the magnetic field in which the spark is discharged. But the iron dust grains scattered from the end of the spark electrodes are attracted by the magnet poles. How have you disposed of the influence of this dust upon the magnetic fields?"

I answered: "In order to get rid of this influence, I broke the magnetizing current and completely removed the dust several times during the photographing. Moreover, in my experiment, as the zinc line 4680·138 was simultaneously photographed to deterenine the field, the change of the field due to these powders, if any, also affected the separation of this zinc line at the same time; in this manner the real value of the magnetic field in which the centre of vibration of iron and nickel spectra is found, was obtained. To determine the field accurately is one of the chief purposes of my experiment."

2. **Electromagnet.** The magnetic fiele, in which the light source was placed, was excited by an electromagnet of Du-Bois half-ring type. Various fields were obtained by changing either the pole-gaps or the magnetizing current or both. I used four kinds of iron tips: 4 mm., 5·5 mm., 6 mm. and 6·5 mm. in end diameters according to the kind of spark desired as well as the width of the magnetic gap used. The length of the spark was always smaller than the diameter of the pole tips, and it was always placed in uniform fields. The gaps between the poles were 1·5 mm., 2·0 mm., 2·5 mm., 3·1 mm., 3·7 mm. and 4·0 mm. The magnetizing current varied from 2 to 13 amperes. The vertical angle of the conical pole-pieces was about 120° to attain the maximum field. The highest magnetic field thus attained was 34120 gauss. The electromagnet can be excited for as long as six hours at 10 amperes, but it must be cooled down after three continuous hours when a current of 13 amperes is used. The current was obtained from the secondary batteries in the laboratory. It was easily controlled when allowance was made for the increased resistance as the coil became warmer, and there was no difficulty in keeping the current constant as shown by the ammeter. The magnet was arranged broadside on, that is, the lines of force between the poles were in a direction perpendicular to the line joining the slit and the grating.

3. **Spectrograph.** The spectroscope used in this investigation was a Rowland concave grating whose focal length is $10\frac{1}{2}$ feet, the breadth and the height of the ruled surface are $3\frac{1}{2}$ and $1\frac{1}{2}$ inches respectively. The number of ruled lines per inch is 14,438. The resolving power $\frac{\lambda}{\delta\lambda}$ and the linear distance in mm. on the photographic plate, corresponding to a change of wave-length of one Ångstrom unit, was calculated by the formulae inserted in Baly's "Spectroscopy" (p. 171, 1905). Let λ be the wave-length in Å. U., and g the linear distance in mm. from the spectral line of wave-length λ to the slit, then we have the following data for the grating:

Order of Spectrum	$\frac{dg}{d\lambda}$	$\frac{d\lambda}{dg}$	$\frac{\lambda}{d\lambda}$
1	0·182	5·50	50,000
2	0·364	2·75	101,000
3	0·546	1·83	151,000
4	0·728	1·37	202,000
5	0·910	1·10	252,000

The mounting of the grating was of the usual Rowland type, in which the grating and the photographic camera ran on rectangular iron rails. The source of light in most cases was focussed on the slit by a lens. A quartz lens was used for the photographing of the ultra-violet region. The focal length of the lens was about 12 cm. and its diameter 5 cm. For the visible region, it was found profitable to use a short focus cylindrical glass lens together with a spherical one, in order to condense the light. To analyse the nature of the vibration in the magnetic field, a Wollaston or Rochon quartz prism was introduced between the lens and the slit. The optical arrangement is shown in Fig. 3.

The spectra produced by the Rowland grating almost always give rise to "ghosts," to which I have paid special attention. The distance Δ of the ghost from the main line divided by its wave-length λ, *i. e.* $\frac{\Delta}{\lambda}$ is constant for different orders of spectra[1] with respect to the instrument. Hence this constant is serviceable to decide the wave-length when its order is known. In our graring $\frac{\Delta}{\lambda} = 2{\cdot}568 \times 10^3$.

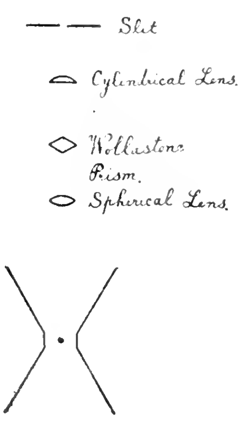


Fig. 3.

In this experiment, I succeeded in photographing as far as the fifth order spectra, by using a glass and a quartz lens separately to take the same region of the spectra. To distinguish between iron and nickel lines, the

1) Rowland, Physical Papers, p. 536 (1902) Baltimore.

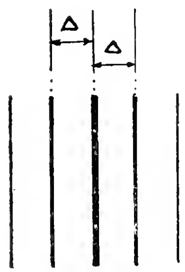


Fig. 4.

iron (completely free from nickel) spark was photographed and the spectra compared to distinguish the line due to these metals.

4. **Arrangement.** The room in which the experiment was undertaken was on a concrete basement. The iron rails on which the rod supporting grating and the photographic camera were placed were fixed on stone piers. This firm setting enabled an exposure of many hours duration to be undertaken without any risk from external shocks. The temperature did not vary so much as to disturb the sharpness of the lines. The arrangement is shown in Fig. 5.

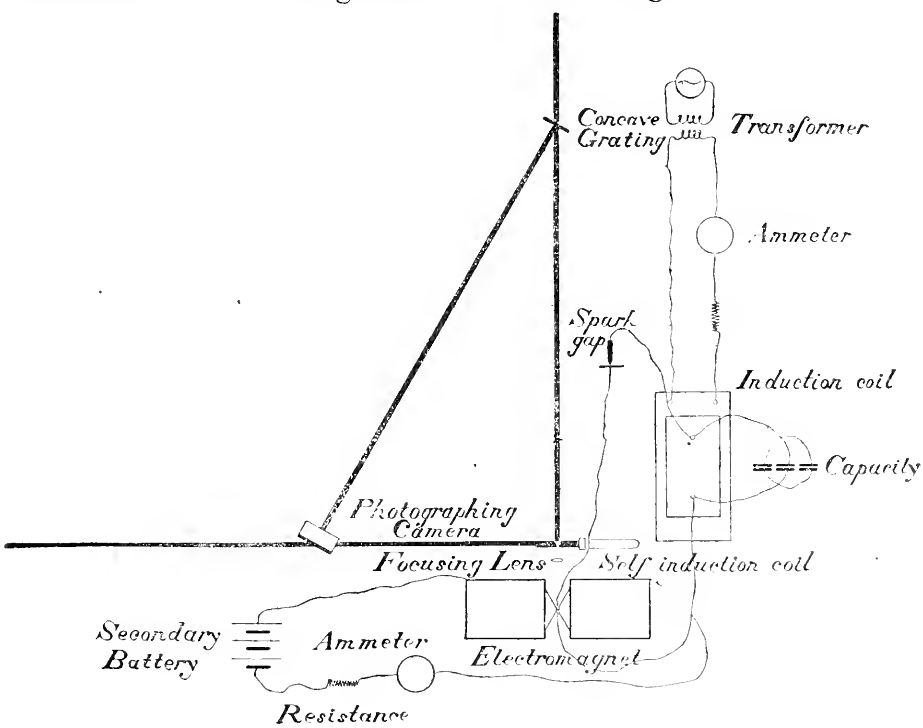


Fig. 5.

5. **Photography.** Many kinds of photographic dry plates were tested to find which would be the best for my purpose. Those tested were (1) Lion 20, (2) Lion 225, (3) Lion, Orthochrome Plates, non-filter, (4) ditto, backed, (5) Paget Prize Plate,

(6) Paget process Plate, (7) Chromo Isolar, (8) Wratten and Wainwright panchromatic plates, (9) Seed 27, (10) Agfa-Trocken-Platten, (11) Schleussnerplatten, (12) Cramer, (13) X Ray plate and (14) Ilford Process Plate. Of these fourteen kinds, I found the "Ilford Process Plate" the most recommendable for such accurate purposes. The peculiarity of this plate is its fine grain, though it has the drawback of requiring long exposure.

The developer was in almost all cases "Agfa Rodinal" and afterwards "Azol," with the addition of a few drops of potassium bromide. A hydroquinone developer was also used, but it had no advantage over those above mentioned.

6. **Identifications of Lines.** The identification of lines was not so easily worked out when three orders, the 3rd, 4th and 5th overlapped on a photographic plate. At first I photographed iron spectra together with λ : 4358·58, 4078·05 and 4046·78 of mercury lines. By the aid of this plate, I found to what region these spectra belong and by the help of Buisson and Fabry's charts inserted in "Recueil de Constantes Physiques (1913)", I could easily identify the lines and from the tables of line spectra[1] and those of iron lines[2], the values of weve-length in international unit are easily known. When two or three orders overlap on a plate, a mere calculation with the help of the tables enables us to detect the value of wave-length.

7. **Determination of the Magnetic Field.** The magnetic separation of zinc line λ: 4680·138 has been accurately examined and many experimenters have used it to determine the magnetic field. One of the chief purposes of this experiment was the simultaneous photography of zinc and iron lines. The groups "g," "h," "i" and "j" were taken in this manner. But the simultaneous photography thus obtained limited the number of lines, and I was obliged to photograph the zinc lines before and after taking the iron lines. The group "e" and "f" are those just mentioned. The ranges and orders photographed on these plates are given in Plates I. and II.

1) Kayser, Handbuch der Spectroscopie, VI (1912) pp. 935-1033.

2) Kayser, ditto, pp. 896-926.

8. **Measurement and Reduction.** To measure the magnetic separation of photographed lines, I used a dividing machine No. 0030[1] made by the Société Genevoise. Taking the tracelet off, I put a supporter of the photographic plate on the platform. Under this supporter, a plane mirror, inclined 45° to the platform, was fixed and by reflecting the light from an electric lamp, the plate was illuminated. The platform, and consequently the photographic plate were moved by the rotation of the handle, and the spectral line under examination was placed under a single straight wire of a fixed microscope whose magnification was 20. But when the line was very intense, it was put between two parallel wires. As the magnification was moderately large, the former method gave better results. The drum of the dividing machine was divided into 0·005 mm. and by the aid of the vernier it could be read to 0·001 mm.

A device was made so as to move the plate in two directions at right angle to each other. Hence the line was measured along its length as many times as possible. In the case of the fourth order spectra, as the line is long, measurement can be made on twenty different parts. As this work is very important for finding accurate values the measurements were made by myself. The result obtained by the measurements of one and the same line on different days agreed well with each other.

III. The Character of the Separation.

After the publication of Voigt's[2] asymmetry theory with regard to the magnetic separation, Zeeman[3] remarked that the asymmetrical intensity of the outer components of a triplet was also due to secondary circumstances, *i. e.* to the reflecting grating and the focussing quartz lens. As, in my case, these two secondary circumstances are related to each other I could make out the character distinctly, but it may be of interest to report that on one

1) Société Genevoise, Price List (1912), Fascicle 1, General Instruments of Measurements, p. 13.

2) W. Voigt, Ann. d. Phys. (4) **1** (1900), 376.

3) Zeeman, Versl. Amsterd. Acad., 26 oct. (1907).

and the same plate there are various features of intensity of separated lines. As to the triplet, I classify the following three cases: (*a*) the three components have equal intensities, (*b*) the inner component is stronger than the outer and (*c*) the outer components are stronger than the inner.

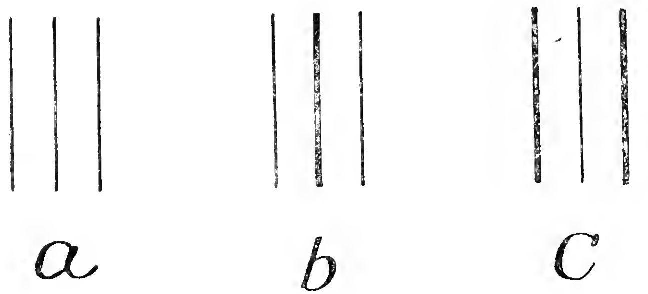


Fig. 6.

IV. The Magnetic Separation of Iron Lines.

The wave-lengths are represented in international units and the intensities of lines when they are not affected are taken from the 6th volume of Kayser's Spectroscopie.

(1) λ:4415·13.

Intensity 10. King took this for a septuplet. According to my measurements, it has many peculiarities; a remarkable character is the decrease of intensity in high fields, the cause of which may be the broadening of components, but further separations can neither be observed in the parallel nor in the normal component. The broadening of the central component is larger than the outer two.[1] The table contains the distance between two outer components. The fields were determined by the separation of λ: 4680·138 of zinc, photographed on the same plate. Plate I, Fig. 1; Plate III, Fig. 1 and Plate VII, Fig. 1.

1) Mr. Y. Takahashi privately told me that each component of this line was found to be further separated in his research with an echelon spectroscope of greater resolving power than the concave grating used by me.

Table I.

H	$\Delta\lambda$	$\frac{\Delta\lambda}{H}\times 10^6$	$\frac{\Delta\lambda}{H\lambda^2}\times 10^{13}$	Deviation from 10·88.	
				$\frac{\Delta\lambda}{H\lambda^2}\times 10^{13}$	$\frac{\text{Deviation}}{10·88}$
9325	0·204±0·0026	21·88	11·22	+0·34	+3·1
10850	0·235±0·0037	21·64	11·10	+0·22	+2·0
12330	0·259±0·0037	21·00	10·77	−0·11	−1·0
13900	0·292±0·0028	21·00	10·77	−0·11	−1·0
14140	0·298±0·0018	21·08	10·82	−0·06	−0·5
16600	0·353±0·0029	21·26	10·91	+0·03	+0·3
17800	0·374±0·0035	21·01	10·78	−0·10	−0·9
19000	0·399±0·0021	21·00	10·77	−0·11	−1·0
20100	0·416±0·0063	20·68	10·62	−0·26	−2·4
21540	0·459±0·0042	21·35	10·95	+0·07	+0·6
23680	0·528±0·0073	22·30	11·44	+0·56	−5·1
23860	0·506±0·0015	21·20	10·88	0·00	0·0
26100	0·536±0·0022	20·52	10·53	−0·35	−3·1
26800	0·536±0·0079	21·00	10·77	−0·11	−1·0
28500	0·637±0·0020	22·36	11·46	+0·58	+5·3
29920	0·666±0·0053	22·28	11·43	+0·55	+5·1
30780	0·704±0·0090	22·88	11·74	+0·86	+7·9
31460	0·720±0·0058	22·90	11·75	+0·87	+8·0
32300	0·752±0·0075	23·30	11·95	+1·07	+9·8
33800	0·767±0·0041	22·70	11·65	+0·77	+7·1
34120	0·792±0·0005	23·20	11·90	+1·02	+9·4

From the table we see that $\Delta\lambda$ increases as the fields become higher. The feature is seen from Fig. 1, Plate VII. The mean value of the first fourteen $\frac{\Delta\lambda}{H\lambda^2}\times 10^{13}$ is 10·88 and the deviations from this value are given in Table I.

For comparison, I insert the results of former observers:

Observer	Character of Separation	H	$\Delta\lambda$	$\frac{\Delta\lambda}{H}\times10^6$	$\frac{\Delta\lambda}{H\lambda^2}\times10^{13}$
King	Septuplet	16000	0·338	21·12	10·84
Kent	Triplet	28000	0·540	20·4	10·48
Van Bilderbeek-van Meurs	,,	32040	0·66	20·6	10·6
Graftdijk	,,	32040	0·684	21·38	10·95
Hartmann	,,			20·4	10·48

(2) λ: 4404·75.

Intensity 15; sharp triplet; character *b*. Plate I, Fig. 1; Plate III, Fig. 1. The determination of the fields is the same as in the case of λ: 4415·13.

TABLE II.

H	$\Delta\lambda$	$\frac{\Delta\lambda}{H}\times10^6$	$\frac{\Delta\lambda}{H\lambda^2}\times10^{13}$	Deviation from the Mean	
				$\frac{\Delta\lambda}{H\lambda^2}\times10^{13}$	$\frac{\text{Deviation}}{\text{Mean Value}}$
9325	0·190±0·0023	20·38	10·50	−0·14	−1·3
10800	0·220±0·0018	20·38	10·50	−0·14	−1·3
10850	0·227±0·0005	20·92	10·78	+0·14	+1·3
12330	0·252±0·0030	20·44	10·54	−0·10	−0·9
13900	0·289±0·0035	20·80	10·72	+0·08	+0·8
14140	0·289±0·0004	20·46	10·55	−0·09	−0·8
16600	0·346±0·0015	20·84	10·75	+0·11	+1·0
17800	0·370±0·0023	20·78	10·70	+0·06	+0·6
19000	0·396±0·0018	20·86	10·76	+0·12	+1·1
20100	0·416±0·0017	20·68	10·67	+0·03	+0·3
21540	0·450±0·0007	20·90	10·77	+0·13	+1·2
23680	0·497±0·0019	20·94	10·79	+0·15	+1·4
23760	0·493±0·0025	20·76	10·72	+0·08	+0·8
23860	0·496±0·0009	20·76	10·72	+0·08	+0·8
25300	0·523±0·0000	20·68	10·67	+0·03	+0·3
26100	0·538±0·0015	20·64	10·64	0·00	0·0
26200	0·545±0·0013	20·80	10·73	+0·09	+0·8
26800	0·549±0·0036	20·48	10·57	−0·07	−0·7
27400	0·571±0·0022	20·84	10·74	+0·10	+0·9
28320	0·589±0·0014	20·80	10·73	+0·09	+0·8
28500	0·594±0·0023	20·84	10·74	+0·10	+0·9

H	$\Delta\lambda$	$\frac{\Delta\lambda}{H}\times 10^6$	$\frac{\Delta\lambda}{H\lambda^2}\times 10^{13}$	Deviation from the Mean	
				$\frac{\Delta\lambda}{H\lambda^2}\times 10^{13}$	$\frac{\text{Deviation}}{\text{Mean Value}}$
29120	0·600±0·0013	20·62	10·64	0·00	0·0
29160	0·598±0·0036	20·52	10·58	−0·06	−0·6
29920	0·615±0·0021	20·58	10·61	−0·03	−0·3
30000	0·619±0·0015	20·64	10·64	0·00	0·0
30200	0·624±0·0030	20·76	10·70	+0·06	+0·6
30340	0·620±0·0013	20·44	10·54	−0·10	−0·9
30780	0·633±0·0009	20·60	10·62	−0·02	−0·2
31100	0·637±0·0024	20·48	10·56	−0·08	−0·8
31300	0·646±0·0024	20·66	10·65	+0·01	+0·1
31460	0·647±0·0009	20·58	10·61	−0·03	−0·3
31650	0·644±0·0023	20·37	10·50	−0·14	−1·3
32300	0·660±0·0014	20·44	10·54	−0·10	−0·9
32300	0·661±0·0017	20·48	10·57	−0·07	−0·7
32400	0·663±0·0016	20·48	10·57	−0·07	−0·7
32880	0·675±0·0022	20·59	10·61	−0·03	−0·3
32900	0·675±0·0015	20·52	10·58	−0·06	−0·6
33100	0·681±0·0012	20·58	10·60	−0·04	−0·4
33340	0·681±0·0038	20·44	10·54	−0·10	−0·9
33350	0·697±0·0022	20·90	10·78	+0·14	+0·3
33500	0·688±0·0030	20·56	10·60	−0·04	−0·4
33800	0·696±0·0021	20·60	10·63	−0·01	+0·1
34120	0·701±0·0025	20·58	10·61	−0·03	−0·3
Mean		20·62	10·64		

$\Delta\lambda$ is linearly proportional to H (Plate VII, Fig. 2). The results of former investigators are:

Observer	Character of Ceparation	H	$\Delta\lambda$	$\frac{\Delta\lambda}{H}\times 10^6$	$\frac{\Delta\lambda}{H\lambda^2}\times 10^{13}$
King	Triplet	16000	0·334	20·88	10·74
Kent	,,	28000	0·512	18·30	9·42
Van Bilderbeek-van Meurs	,,	32040	0·680	21·22	10·93
Graftdijk	,,	32040	0·678	21·18	10·91
Hartmann	,,	—	—	20·20	10·26

Within 1%, the results obtained by King and by me are in good agreement with each other.

(3) λ: 4383·55.

Intensity 20; sharp triplet; character *b*. Plate I, Fig. 1; Plate III, Fig. 1.

The determination of the fields is the same as in the case of λ: 4415·13.

Table III.

H	Δλ	$\frac{\Delta\lambda}{H}\times10^{6}$	$\frac{\Delta\lambda}{H\lambda^{2}}\times10^{13}$	Deviation from the Mean: $\frac{\Delta\lambda}{H\lambda^{2}}\times10^{13}$	Deviation from the Mean: $\frac{\text{Deviation}}{\text{Mean Value}}$
9325	0·193±0·0014	20·70	10·76	−0·03	−0·3
10800	0·220±0·0012	20·38	10·60	−0·19	−1·8
10850	0·224±0·0009	20·64	10·73	−0·06	−0·6
12330	0·256±0·0006	20·74	10·78	−0·01	−0·1
13900	0·287±0·0004	20·64	10·73	−0·06	−0·6
14140	0·296±0·0007	20·92	10·88	+0·09	+0·8
16600	0·344±0·0014	20·72	10·78	−0·01	−0·1
17800	0·371±0·0011	20·84	10·85	+0·06	+0·6
19000	0·393±0·0006	20·68	10·75	−0·04	−0·4
20100	0·417±0·0009	20·72	10·78	−0·01	−0·1
21540	0·449±0·0035	20·88	10·86	+0·07	+0·7
23680	0·487±0·0010	20·56	10·70	−0·09	−0·8
23760	0.498±0·0016	20·98	10·91	+0·12	+1·1
23860	0.494±0·0018	20·70	10·76	−0·03	−0·3
25300	0.523±0·0016	20·68	10·75	−0·04	−0·4
26100	0·544±0·0014	20·86	10·85	+0·06	+0·6
26200	0·547±0·0013	20·88	10·86	+0·07	+0·7
26800	0·556±0·0016	20·74	10·78	−0·01	−0·1
27400	0·561±0·0018	20·50	10·66	−0·13	−1·2
28320	0·589±0·0015	20·80	10·82	+0·03	+0·3
28500	0·588±0·0018	20·64	10·73	−0·06	−0·6
29120	0·604±0·0009	20·74	10·78	−0·01	−0·1
29160	0·599±0·0018	20·58	10·65	−0·14	−1·3
29920	0·622±0·0005	20·80	10·82	+0·03	+0·3
30000	0·621±0·0010	20·70	10·76	−0·03	−0·3
30200	0·638±0·0023	21·12	10·98	+0·19	+1·8
30340	0·628±0·0013	20·72	10·78	−0·01	−0·1
30780	0·637±0·0003	20·70	10·76	−0·03	−0·3
31100	0·642±0·0017	20·66	10·74	−0·05	−0·5
31300	0·653±0·0011	20·88	10·86	+0·07	+0·7
31460	0·651±0·0012	20·72	10·78	−0·01	−0·1
31650	0·653±0·0023	21·64	10·73	−0·06	−0·6
32300	0·667±0·0011	20·65	10·74	−0·05	−0·5
32300	0·673±0·0018	20·84	10·85	+0·06	+0·6
32400	0·671±0·0006	20·72	10·78	−0·01	−0·1
32880	0·678±0·0010	20·64	10·73	−0·06	−0·6
32900	0·687±0·0013	20·88	10·86	+0·07	+0·7
33100	0·680±0·0014	20·54	10·68	−0·11	−1·0
33340	0·695±0·0023	20·88	10·86	+0·07	+0·7
33350	0·694±0·0009	20·84	10·85	+0·06	+0·6
33500	0·702±0·0013	20·96	10·90	+0·11	+1·0
33800	0·701±0·0020	20·74	10·78	−0·01	−0·1
34120	0·712±0·0006	20·88	10·86	+0·07	+0·7
Mean		20·74	10·79		

From the table we see that $\Delta\lambda$ increases proportionally to the fields applied. Plate VIII, Fig. 1.

The results of former investigations are:

Observer	Character of Separation	H	$\Delta\lambda$	$\frac{\Delta\lambda}{H}\times 10^6$	$\frac{\Delta\lambda}{H\lambda^2}\times 10^{13}$
King	Triplet	16000	0·332	20·74	10·78
Kent	,,	28000	0·514	18·36	9·55
Van Bilderbeek-van Meurs	,,	32040	0·684	21·32	11·10
Graftdijk	,,	32040	0·674	21·04	10·95
Hartmann	,,	—	—	19·2	9·98

The value obtained by me is in full accord with King's result.

(4) λ:4325·78.[1]

Intensity 15; seems to be a triplet, but in high fields resembles more or less :4415·13. Character *b*, Plate I, Fig. 1; Plate III, Fig. 1. The determination of the fields is the same as in the case of λ:4415·13.

TABLE IV.

H	$\Delta\lambda$	$\frac{\Delta\lambda}{H}\times 10^6$	$\frac{\Delta\lambda}{H\lambda^2}\times 10^{13}$	Deviation from the Mean: $\frac{\Delta\lambda}{H\lambda^2}\times 10^{13}$	Deviation from the Mean: $\frac{\text{Deviation}}{\text{Mean Value}}$
9325	0·156±0·0008	16·76	8·95	+0·20	+2·3
10800	0·171±0·0025	15·83	8·47	−0·28	−3·2
10850	0·174±0·0011	16·03	8·57	−0·18	−2·1
12330	0·199±0·0013	16·14	8·63	−0·12	−1·4
12900	0·210±0·0011	15·76	8·42	−0·33	−3·8
14140	0·231±0·0026	16·33	8·73	−0·02	−0·2
16600	0·266±0·0013	16·03	8·57	−0·18	−2·1
17800	0·280±0·0011	15·73	8·41	−0·34	−3·9
19000	0·308±0·0009	16·22	8·67	−0·08	−0·9
20100	0·328±0·0013	16·33	8·73	−0·02	−0·2
21540	0·348±0·0015	16·17	8·64	−0·11	−1·3
23760	0·387±0·0024	16·29	8·70	−0·05	−0·6
23860	0·386±0·0015	16·17	8·64	−0.11	−1·3
25300	0·402±0·0016	15·89	8·49	−0.26	−3·0
26100	0·421±0·0024	16·13	8·62	−0.13	−1·5
26200	0·432±0·0014	16·50	8·82	+0·07	+0·8
26800	0·429±0·0001	16·03	8·57	−0·18	−2·1
27400	0·447±0·0033	16·30	8·71	−0·04	−0·5

1) In Mr. Y. Takahashi's experiment, this line also show complex separation.

H	$\Delta\lambda$	$\frac{\Delta\lambda}{H}\times10^6$	$\frac{\Delta\lambda}{H\lambda^2}\times10^{13}$	Deviation from the Mean	
				$\frac{\Delta\lambda}{H\lambda^2}\times10^{13}$	$\frac{\text{Deviation}}{\text{Mean Value}}$
28320	0·474±0·0016	16·75	8·95	+0·20	+2·3
28500	0·459±0·0026	16·21	8·61	−0·14	−1·6
29120	0·484±0·0007	16·63	8·89	+0·14	+1·6
29160	0·485±0·0027	16·58	8·86	+0·11	+1·3
30000	0·495±0·0023	16·50	8·82	+0·07	+0·8
30200	0·500±0·0038	16·57	8·85	+0·10	+1·1
30340	0·493±0·0012	16·27	8·70	−0·05	−0·6
30780	0·506±0·0007	16·45	8·79	+0·04	+0·5
31100	0·508±0·0021	16·34	8·73	−0·02	−0·2
31300	0·519±0·0023	16·59	8·86	+0·11	+1·3
31460	0·523±0·0013	16·63	8·88	+0·13	+1·5
31650	0·534±0·0025	16·87	9·01	+0·26	+3·0
32300	0·541±0·0012	16·76	8·95	+0·20	+2·3
32300	0·531±0·0007	16·45	8·79	+0·04	+0·5
32400	0·528±0·0015	16·32	8·72	−0·03	−0·3
32880	0·538±0·0032	16·36	8·74	−0·01	−0·1
32900	0·548±0·0047	16·65	8·90	+0·15	+1·7
33100	0·554±0·0022	16·74	8·95	+0·20	+2·3
33340	0·559±0.0047	16·78	8·97	+0·22	+2·5
33350	0·555±0.0027	16·65	8.90	+0·15	+1·7
33500	0·565±0·0039	16·87	9·01	+0·26	+3·0
33800	0·567±0.0014	16·79	8·97	+0·22	+2·5
34120	0·583±0·0030	17·10	9.14	+0·39	+4·5
Mean			8·75		

From the table we see that $\Delta\lambda$ slightly increases as the fields become higher, as in the case of λ:4415·13. Plate VIII, Fig. 2.

The results of the former investigators are:

Observer	Character of Separation	H	$\Delta\lambda$	$\frac{\Delta\lambda}{H}\times10^6$	$\frac{\Delta\lambda}{H}\times10^{13}$
King	Triplet	16000	0·245	15.31	8·18
Kent	,,	28000	0·390	13·93	7·44
Van Bilderbeek-van Meurs	,,	32040	0·524	16·37	8·74
Graftdijk	,,	32040	0·524	16·37	8·74
Hartmann	,,	—	—	16·0	8·62

(5) λ:4307·92.

Intensity 15; sharp triplet; character *b*. Plate I, Fig. 1; Plate III, Fig. 1. This line resembles λ:4404·75. The determination of the fields is the same as in the case of λ:4415·13.

Table V.

H	$\Delta\lambda$	$\frac{\Delta\lambda}{H}\times 10^6$	$\frac{\Delta\lambda}{H\lambda^2}\times 10^{13}$	Deviation from the Mean	
				$\frac{\Delta\lambda}{H\lambda^2}\times 10^{13}$	$\frac{\text{Deviation}}{\text{Mean Value}}$
9325	0·188±0·0010	20·15	10·86	+0·28	+2·6
10800	0·215±0·0014	19·90	10·73	+0·15	+1·4
10850	0·214±0·0007	19·73	10·63	+0·05	+0·5
12330	0·244±0·0004	19·78	10·66	+0·08	+0·8
13900	0·273±0·0011	19·65	10·59	+0·01	+0·1
14140	0·277±0·0013	19·59	10·55	−0·03	−0·3
16600	0·330±0·0017	19·87	10·71	+0·13	+1·2
17800	0·346±0·0010	19·44	10·48	−0·10	−0·9
19000	0·376±0·0011	19·79	10·66	+0·08	+0·8
20100	0·397±0·0015	19·76	10·65	+0·07	+0·7
21540	0·427±0·0007	19·84	10·69	+0·11	+1·0
23760	0·468±0·0014	19·71	10·63	+0·05	+0·5
23860	0·468±0·0009	19·62	10·57	−0·01	−0·1
25300	0·494±0·0005	19·54	10·53	−0·05	−0·5
26100	0·510±0·0005	19·54	10·53	−0·05	−0·5
26200	0·513±0·0011	19·59	10·55	−0·03	−0·3
26800	0·527±0·0023	19·67	10·60	+0·02	+0·2
27400	0·530±0·0016	19·35	10·43	−0·15	−1·4
28320	0·550±0·0027	19·43	10·47	−0·11	−1·0
28500	0·561±0·0026	19·70	10·62	+0·04	+0·4
29160	0·563±0·0014	19·32	10·42	−0·16	−1·5
30000	0·579±0·0007	19·30	10·41	−0·17	−1·6
30200	0·607±0·0026	20·10	10·84	+0·26	+2·5
30340	0·591±0·0029	19·47	10·50	−0·08	−0·8
30780	0·603±0·0021	19·60	10·56	−0·02	−0·2
31100	0·601±0·0036	19·33	10·42	−0·16	−1·5
31300	0·608±0·0037	19·43	10·47	−0·11	−1·0
31460	0·604±0·0012	19·20	10·35	−0·23	−2·2
31650	0·628±0·0012	19·84	10·69	+0·11	+1·0
32300	0·627±0·0025	19·42	10·46	−0·12	−1·1
32300	0·623±0·0016	19·30	10·41	−0·17	−1·6
32400	0·629±0·0026	19·40	10·45	−0·13	−1·2
32880	0·642±0·0026	19·54	10·53	−0·05	−0·5
32900	0·643±0·0023	19·54	10·53	−0·05	−0·5
33100	0·643±0·0031	19·43	10·47	−0·11	−1·0
33340	0·654±0·0025	19·63	10·58	0·00	0·0
33350	0·672±0·0012	20·14	10·84	+0·26	+2·5
33500	0·661±0·0026	19·75	10·64	+0·06	+0·6
33800	0·673±0·0021	19·91	10·73	+0·15	+1·4
34120	0·673±0·0042	19·74	10·64	+0·06	+0·6
Mean		19·63	10·58		

From the table we see that $\Delta\lambda$ increases proportionally to the fields applied, as in the case of 4404·75. Plate IX, Fig. 1.

The results of the former investigations are:

Observer	Character of Separation	H	$\Delta\lambda$	$\frac{\Delta\lambda}{H}\times 10^6$	$\frac{\Delta\lambda}{H}\times 10^{13}$
King	Triplet	16000	0·320	20·00	10·77
Kent	,,	28000	0·480	17·14	9·24
Van Bilderbeek-van Meurs	,,	32040	0·643	20·08	10·81
Graftdijk	,,	32040	0·643	20·08	10·81
Hartmann	,,	—	—	18·8	10·12

The nearest value to my result is that found by King, whose value is in agreement with mine with a discrepancy of 2%.

(6) λ: 3886·29.

Intensity 5; sharp triplet; character a. Plate I, Fig. 4; Plate IV, Fig. 4. The fields were determined before and after photographing this line by the aid of λ: 4680·138 of zinc. But the fields marked with an asterisk were determined by λ: 3856·38, taking $\frac{\Delta}{H}$ = 20·94, which we shall discuss hereafter. (Table IX and its discussion).

TABLE VI.

H	$\Delta\lambda$	$\frac{\Delta\lambda}{H\lambda^2}\times 10^6$	$\frac{\Delta\lambda}{H\lambda^2}\times 10^{13}$	Deviation from the Mean	
				$\frac{\Delta\lambda}{H\lambda^2}\times 10^{13}$	$\frac{\text{Deviation}}{\text{Mean Value}}$
8220	0·174±0·0009	21·16	14·01	+0·09	+0·6
9615	0·200±0·0005	20·80	13·77	−0·15	−1·1
10880	0·228±0·0005	20·94	13·87	−0·05	−0·4
13430	0·282±0·0010	20·98	13·90	−0·02	−0·1
14500	0·300±0·0017	20·70	13·70	−0·22	−1·6
16370	0·344±0·0009	21·00	13·91	−0·01	−0·1
17700	0·373±0·0010	21·08	13·95	+0·03	+0·2
19300	0·406±0·0005	21·04	13·93	+0·01	+0·1
20000	0·421±0·0008	21·05	13·94	+0·02	+0·1
21740	0·462±0·0014	21·26	14·09	+0·17	+1·2
*25020	0·530±0·0005	21·04	13·94	+0·02	+0·1
25250	0·532±0·0009	21·08	13·95	+0·03	+0·2
*25540	0·538±0·0008	21·08	13·95	+0·03	+0·2
*26100	0·547±0·0006	20·94	13·87	−0·05	−0·4
26520	0·556±0·0008	20·98	13·90	−0·02	−0·2
27240	0·569±0·0007	20·90	13·84	−0·08	−0·6
28680	0·611±0·0021	21·30	14·10	+1·8	+1·3
Mean		21·02	13·92		

From the table we see that $\Delta\lambda$ is proportional to H. Plate IX, Fig. 2.

The results of the previous investigations are:

Investigator	Character of Separation	H	$\Delta\lambda$	$\frac{\Delta\lambda}{H}\times 10^6$	$\frac{\Delta\lambda}{H\lambda^2}\times 10^{13}$
King	Triplet	16000	0·348	21·75	14·40
Kent	,,	28000	0·523	18·68	12·37
Reese	,,	28300	0·618	21·84	14·46
Van Bilderbeek-van Meurs	,,	32040	0·698	21·80	14·43

My result is smaller than those of King, Reese and van Bilderbeek-van Meurs, while the latter three are in good agreement with one another.

(7) λ : 3878·78.

Intensity 5; sharp triplet; character α. Plate II, Fig. 1; Plate IV, Fig. 4. The determination of the fields is the same as in the case of λ : 3886·29.

TABLE VII.

H	$\Delta\lambda$	$\frac{\Delta\lambda}{H}\times 10^6$	$\frac{\Delta\lambda}{H\lambda^2}\times 10^{13}$	Deviation from the Mean: $\frac{\Delta\lambda}{H\lambda^2}\times 10^{13}$	Deviation from the Mean: $\frac{\text{Deviation}}{\text{Mean Value}}$
8220	0·174±0·0020	21·18	14·07	+0·09	+0·6
9615	0·199±0·0006	20·70	13·76	−0·22	−1·6
10880	0·231±0·0006	20·22	14·12	+0·14	+1·0
13430	0·281±0·0009	20·90	13·90	−0·08	−0·6
16370	0·342±0·0007	20·80	13·83	−0·15	−1·1
17700	0·372±0·0009	21·00	13·95	−0·03	−0·2
19300	0·401±0·0014	20·74	13·78	−0·20	−1·4
20000	0·421±0·0020	21·05	13·98	0·00	0·0
21740	0·462±0·0018	21·26	14·14	+0·16	+1·1
*25020	0·538±0·0012	21·52	14·31	+0·33	+2·4
25250	0·531±0·0012	21·05	13·98	0·00	0·0
*25540	0·540±0·0012	21·14	14·05	+0·08	+0·6
*26100	0·545±0·0010	20·88	13·89	−0·09	−0·6
26520	0·556±0·0011	20·96	13·94	−0·04	−0·3
27240	0·568±0·0022	20·86	13·86	−0·12	−0·9
28680	0·612±0·0013	21·34	14·19	+0·21	+1·5
Mean		21·05	13·98		

$\Delta\lambda$ is nearly proportional to H. Plate V, Fig. 4. The results of former investigations are:

Investigators	Character of Separation	H	$\Delta\lambda$	$\frac{\Delta\lambda}{H}\times10^6$	$\frac{\Delta\lambda}{H\lambda^2}\times10^{13}$
King	Triplet	16000	0·346	21·64	14·37
Van Bilderbeek-van Meurs	,,	32040	0·698	21·80	14·49

The discrepancy between my value and that of King is somewhat large, amounting to 3%.

(8) λ:3859·90.

Intensity 6; sharp triplet; character a. Plate II, Fig. 1; Plate IV, Fig. 4. The determination of the fields is the same as in the case of λ:3886·29.

TABLE VIII.

H	$\Delta\lambda$	$\frac{\Delta\lambda}{H}\times10^6$	$\frac{\Delta\lambda}{H\lambda^2}\times10^{13}$	Deviation from the Mean	
				$\frac{\Delta\lambda}{H\lambda^2}\times10^{13}$	Deviation / Mean Value
8220	0·172±0·0005	20·90	14·04	+0·06	+0·4
9615	0·199±0·0009	20·70	13·89	−0·10	−0·7
10880	0·229±0·0006	21·04	14·13	+0·14	+1·0
13430	0·281±0·0009	20·90	14·04	+0·06	+0·4
14500	0·302±0·0009	20·84	13·98	−0·01	−0·1
16370	0·341±0·0007	20·84	13·98	−0·01	−0·1
17700	0·370±0·0012	20·90	14·04	+0·06	+0·4
19300	0·392±0·0007	20·30	13·63	−0·36	−2·6
20000	0·417±0·0008	20·84	13·98	−0·01	−0·1
21740	0·468±0·0010	21·06	14·14	+0·15	+1·1
25020	0·522±0·0006	20·86	14·02	+0·03	+0·2
25250	0·526±0·0003	20·84	13·98	−0·01	−0·1
25540	0·532±0·0005	20·82	13·98	−0·01	−0·1
26100	0·543±0·0006	20·80	13·97	−0·02	−0·1
26520	0·550±0·0009	20·74	13·93	−0·06	−0·4
27240	0·568±0·0005	20·86	14·02	+0·03	+0·2
28680	0·598±0·0013	20·88	14·03	+0·04	+0·3
Mean		20·83	13·99		

$\Delta\lambda$ is nearly proportional to H. Plate V. Fig. 5. The results of former investigations are:

Investigator	Character of Separation	H	$\Delta\lambda$	$\frac{\Delta\lambda}{H}\times 10^6$	$\frac{\Delta\lambda}{H\lambda^2}\times 10^{13}$
King	Triple	16000	0·341	21·30	14·30
Van Bilderbeek-van Meurs	,,	32040	0·681	21·26	14·26
Reese	,,	28300	0·622	21·98	14·76

(9) λ:3856·38.

Intensity 5; sharp triplet; character α. Plate II, Fig. 1; Plate IV, Fig. 4. The determination of the fields is the same as in the case of λ:3886·29.

TABLE IX.

H	$\Delta\lambda$	$\frac{\Delta\lambda}{H}\times 10^6$	$\frac{\Delta\lambda}{H\lambda^2}\times 10^{13}$	Deviation from the Mean	
				$\frac{\Delta\lambda}{H\lambda^2}\times 10^{13}$	$\frac{\text{Deviation}}{\text{Mean Value}}$
6700	0·144±0·0010	21·50	14·46	+0·36	+2·6
8220	0·173±0·0007	21·08	14·16	+0·06	+0·4
10880	0·227±0·0007	20·88	14·04	−0·06	−0·4
13430	0·280±0·0013	20·84	14·03	−0·07	−0·5
16370	0·340±0·0009	20·78	13·98	−0·12	−0·9
17700	0·370±0·0019	20·90	14·06	−0·04	−0·3
19300	0·401±0·0009	20·75	13·95	−0·15	−1·1
20000	0·422±0·0049	21·10	14·20	+0·10	+0·7
21740	0·458±0·0009	21·10	14·20	+0·10	+0·7
23130	0·482±0·0023	20·84	14·03	−0·07	−0·5
25250	0·526±0·0011	20·84	14·03	−0·07	−0·5
26520	0·551±0·0010	20·78	13·98	−0·12	−0·9
27240	0·571±0·0011	20·96	14·10	0·00	0·0
28680	0·607±0·0020	21·16	14·23	+0·13	+0·9
Mean		20·96	14·10		

$\Delta\lambda$ is nearly proportional to H. Plate XI, Fig. 1.

The mean value 20·94 of twelve $\frac{\Delta\lambda}{H}$ 10^6 from 8220 to 27240 was used to determine the fields *25020, *25540 and *26100. In the diagrams of the Plates, the points corresponding to these fields are denoted with ×.

The results of former investigators are:

Investigator	Character of Separation	H	$\Delta\lambda$	$\frac{\Delta\lambda}{H}\times 10^6$	$\frac{\Delta\lambda}{H\lambda^2}\times 10^{13}$
King	Triple	16000	0·341	21·30	14·32
Kent	,,	28000	0·501	17·89	12·03
Van Bilderbeek-van Meurs	,,	32040	0·689	21·50	14·46

The difference of my result from that of King is 1·6%

(10) λ:3827·83.

Intensity 8; triplet, resembles λ:4325·78; character *b*. Plate II, Fig. 1: Plate IV, Fig. 4. The determination of the fields is the same as in the case of λ:3886·29.

TABLE X.

H	$\Delta\lambda$	$\frac{\Delta\lambda}{H}\times 10^6$	$\frac{\Delta\lambda}{H\lambda^2}\times 10^{13}$	Deviation from 9·56	
				$\frac{\Delta\lambda}{H\lambda^2}\times 10^{13}$	$\frac{\text{Deviation}}{9{\cdot}56}$
8220	0·122±0·0014	14·84	10·13	+0·57	+6·0
9615	0·141±0 0010	14·68	10·02	+0·46	+5·0
10880	0·152±0·0005	13·98	9·54	−0·02	−0·2
13430	0·182±0·0007	13·55	9·25	−0·31	−3·2
16370	0·226±0·0011	13·80	9·42	−0·14	−1·5
17700	0·251±0·0012	14·18	9·68	+0·12	+1·3
19300	0·280±0·0007	14·50	9·90	+0·34	+3·6
20000	0·283±0·0020	14·15	9·66	+0·10	+1·2
21740	0·317±0·0005	14 59	9·97	+0·41	+4·3
23130	0·320±0·0009	13·84	9·45	−0·11	−1·1
*25020	0·341±0·0006	13·63	9·31	−0·25	−2·6
25250	0·348±0·0017	13·78	9·41	−0·15	−1·6
*25540	0·356±0·0010	13·94	9·51	−0·05	−0·5
*26100	0·363±0·0013	13·91	9·50	−0·06	−0·6
26520	0·367±0·0006	13·84	9·45	−0·11	−1·1
27240	0·389±0·0019	14·27	9·75	+0·19	+2·0
28680	0·404±0·0017	14·08	9·62	+0·06	+0·6

From the table we see that $\Delta\lambda$ is not proportional to the field applied. The mean of fifteen values of $\frac{\Delta\lambda}{H\lambda^2}\times 10^{13}$ from H=10880 to H=28680 is 9·56 and the deviations from this mean are calculated. The feature is seen from Fig. 2 of Plate XI.

The results of former investigations are:

Invertigator	Character of Separation	H	$\Delta\lambda$	$\frac{\Delta\lambda}{H}\times 10^6$	$\frac{\Delta\lambda}{H\lambda^2}\times 10^{13}$
King	Triple	16000	0·225	14·06	9.59
Kent	,,	28000	0·346	12·36	8.43
Van Bilderbeek-van Meurs	,,	32040	0·449	14·03	9.57

(11) λ: 3825·90.

Intensity 8; character *b;* King took this for a septuplet (?), but he could not exactly find the separation of parallel components and only measured the normal component. This resembles λ: 4415·13. I am of opinion that this may be further separated in a stronger field or by a spectroscope of higher resolving power. Plate II, Fig. 1; Plate IV, Fig. 4. The results of my observation are:

Table XI.

H	$\Delta\lambda$	$\frac{\Delta\lambda}{H\lambda^2}\times 10^6$	$\frac{\Delta\lambda}{H\lambda^2}\times 10^{13}$	Deviation from the Mean	
				$\frac{\Delta\lambda}{H\lambda^2}\times 10^{13}$	$\frac{\text{Deviation}}{\text{Mean Value}}$
8220	0·131±0·0008	15·94	10·89	+0·33	+3·1
10880	0·172±0·0010	15·80	10·79	+0·23	+2·2
13430	0·206±0·0009	15·33	10·47	−0·09	−0·9
14500	0·225±0·0008	15·52	10·60	+0·04	+0·4
16370	0·256±0·0007	15·64	10·68	+0·12	+1·1
17700	0·279±0·0012	15·76	10·77	+0·21	+2·0
19300	0·307±0·0008	15·90	10·86	+0·30	+2·8
20000	0·308±0·0020	15·40	10·52	−0·04	−0·4
23130	0·346±0·0016	14·96	10·23	−0·33	−3·1
*25020	0·380±0·0008	15·18	10·37	−0·29	−2·7
25250	0·375±0·0016	14·86	10·15	−0·41	−3·9
*25540	0·401±0·0008	15·69	10·72	+0·16	+1·5
*26100	0·403±0·0008	15·44	10·54	−0·02	−0·2
26520	0·396±0·0012	14·93	10·20	−0·36	−3·4
27240	0·425±0·0015	15·61	10·66	+0·10	+0·9
28680	0·439±0·0029	15·31	10·46	−0·10	−0·9
Mean			10·56		

$\Delta\lambda$ can not be said to vary linearly proportional to H, as the deviations from the mean seem to indicate. Plate XII, Fig. 1.

The results of the former investigations are:

Investigator	Character of Separation	H	$\Delta\lambda$	$\frac{\Delta\lambda}{H}\times 10^6$	$\frac{\Delta\lambda}{H\lambda^2}\times 10^{13}$
King	Septuple?	16000	0·274w₂	17·13	11·70
Kent	Triple	28000	0·386	13·78	9·42
Reese	,,	28300	0·452	15·97	10·91
Van Bilderbeek-van Meurs	,,	32040	0·495	15·46	10·54

(12) λ:3820·44.

Intensity 10; character *b*; Plate II, Fig. 1; Plate IV, Fig. 4.

King took this for a triplet, but on my plate this line resembles λ:3825·90. Although further separation of the inner and outer components could not be detected, this can not be a simple triplet. But here the separations $\Delta\lambda$ of two outer components are reported. The determination of the fields is the same as in the case of λ:3886·29.

TABLE XII.

H	$\Delta\lambda$	$\frac{\Delta\lambda}{H}\times 10^6$	$\frac{\Delta\lambda}{H\lambda^2}\times 10^{13}$	Deviation from the Mean	
				$\frac{\Delta\lambda}{H\lambda^2}\times 10^{13}$	$\frac{\text{Deviation}}{\text{Mean Value}}$
8220	0·147±0·0007	17·88	12·26	+0·77	+7·0
10880	0·184±0·0007	16·91	11·60	+0·11	+1·0
13430	0·222±0·0009	16·53	11·33	−0·16	−1·4
14500	0·237±0·0006	16·34	11·21	−0·28	−2·4
16370	0·278±0·0011	16·99	11·65	+0·16	+1·4
17700	0·303±0·0013	17·12	11·74	+0·25	+2·2
19300	0·329±0·0007	17·04	11·68	+0·19	+1·7
20000	0·338±0·0008	16·90	11·59	+0·10	+0·9
21740	0·370±0·0010	17·02	11·67	+0·18	+1·6
23130	0·376±0.0016	16·27	11·15	−0·34	−3·0
*25020	0·414±0·0007	16·53	11·34	−0·15	−1·3
25250	0·403±0·0008	16·10	11·03	−0·36	−3·1
*25540	0·435±0·0013	17·04	11·68	+0·19	+1·7
*26100	0·440±0·0009	16·87	11·57	+0·08	+0·7
26520	0·428±0·0016	16·14	11·07	−0·42	−3·7
27240	0·453±0·0011	16·64	11·42	−0·07	−0·6
28680	0·472±0·0011	16·46	11·28	−0·21	−1·8
Mean			11·49		

The increment of $\Delta\lambda$ is irregular as indicated by the deviations from the mean of $\frac{\Delta\lambda}{H\lambda^2}\times 10^{13}$. No definite conclusion can be drawn from it. Plate XII, Fig. 2.

The results of the former investigations are:

Investigator	Character of Separation	H	$\Delta\lambda$	$\frac{\Delta\lambda}{H} \times 10^6$	$\frac{\Delta\lambda}{H\lambda^2} \times 10^{13}$
King	Triple	16000	0·282	17·63	12·07
Kent	„	28000	0·425	15·18	10·40
Reese	„	28300	0·472	16·67	11·43
Van Bilderbeek-van Meurs	„	32040	0·536	16·73	11·47

(13) λ: 3815·84.

Intensity 10; sharp triplet; character *b*. Plate II, Fig. 1; Plate IV, Fig. 4. King is also of opinion that this is a triplet. The determination of the fields is the same as in the case of λ: 3886·29.

TABLE XIII.

H	$\Delta\lambda$	$\frac{\Delta\lambda}{H} \times 10^6$	$\frac{\Delta\lambda}{H\lambda^2} \times 10^{13}$	Deviation from the Mean	
				$\frac{\Delta\lambda}{H\lambda^2} \times 10^{13}$	$\frac{\text{Deviation}}{\text{Mean Value}}$
8220	0·132±0·0007	16·06	11·03	+0·32	+3·0
10880	0·168±0·0007	15·45	10·61	−0·10	−0·9
13430	0·204±0·0011	15·18	10·43	−0·28	−2·6
14500	0·227±0·0019	15·65	10·75	+0·04	+0·4
16370	0·254±0·0009	15·51	10·65	−0·06	−0·6
17700	0·274±0·0013	15·48	10·63	−0·08	−0·7
19300	0·305±0·0009	15·80	10·85	+0·14	+1·3
20000	0·316±0·0010	15·80	10·85	+0·14	+1·3
21740	0·341±0·0010	15·69	10·78	+0·07	+0·7
23130	0·361±0·0010	15·61	10·73	+0·02	+0·2
*25020	0·387±0·0007	15·46	10·62	−0·09	−0·8
25250	0·387±0·0008	15·33	10·53	−0·18	−1·7
*25540	0·399±0·0010	15·63	10·74	+0·03	+0·3
*26100	0·407±0·0007	15·59	10·71	0·00	0·0
26520	0·409±0·0007	15·42	10·60	−0·11	−1·0
27240	0·429±0·0016	15·73	10·81	+0·10	−0·9
28680	0·451±0·0013	15·72	10·80	+0·09	+0·8
Mean		15·59	10·71		

$\frac{\Delta\lambda}{H\lambda^2}$ is constant within 1 or 2%. Plate XII, Fig. 3. The results of the former investigations are:

Investigator	Character of Separation	H	$\Delta\lambda$	$\frac{\Delta\lambda}{H}\times 10^6$	$\frac{\Delta\lambda}{H\lambda^2}\times 10^{13}$
King	Triple	16000	0·264	16·50	11·33
Kent	,,	28000	0·382	13·64	9·37
Reese	,,	28300	0·478	16·90	11·62
Van Bilderbeek-van Meurs.	,,	32040	0·505	15·77	10·83

The nearest value to mine is that of van Bilderbeek-van Meurs.

(14) λ: 3763·80.

Intensity 6; very sharp triplet; character *a*. Plate II, Fig. 1; Plate IV, Fig. 5.

King also took this for a triplet. The determination of the fields is the same as in the case of λ: 3886·29.

Table XIV.

H	$\Delta\lambda$	$\frac{\Delta\lambda}{H}\times 10^6$	$\frac{\Delta\lambda}{H\lambda^2}\times 10^{13}$	Deviation from the Mean: $\frac{\Delta\lambda}{H\lambda^2}\times 10^{13}$	Deviation from the Mean: $\frac{\text{Deviation}}{\text{Mean Value}}$
8220	0·112±0·0011	13·63	9·61	+0·20	+2·1
9615	0·134±0·0008	13·94	9·82	+0·41	+4·3
10880	0·145±0·0007	13·33	9·40	−0·01	−0·1
13430	0·177±0·0007	13·18	9·30	−0·11	−1·2
16370	0·215±0·0008	13·13	9·26	−0·15	−1·6
17700	0·231±0·0011	13·04	9·20	−0·21	−2·1
19300	0·263±0·0008	13·62	9·61	+0·20	+2·1
21740	0·292±0·0008	13·43	9·47	+0·06	+0·6
23130	0·301±0·0020	13·02	9·18	−0·23	−2·4
25020	0·329±0·0005	13·14	9·27	−0·14	−1·5
25250	0·335±0·0008	13·27	9·35	−0·06	−0·6
26100	0·344±0·0007	13·18	9·30	−0·11	−1·2
26520	0·352±0·0014	13·28	9·36	−0·05	−0·5
27240	0·367±0·0026	13·47	9·50	+0·09	+1·0
28680	0·386±0·0010	13·45	9·49	+0·08	+0·9
Mean		13·34	9·41		

$\Delta\lambda$ may be said to increase proportionally to H, though the differences from the mean are somewhat great. Plate XIII, Fig. 1.

The results of the previous investigations are:

Investigator	Character of Separation	H	$\Delta\lambda$	$\frac{\Delta\lambda}{H}\times 10^6$	$\frac{\Delta\lambda}{H\lambda^2}\times 10^{13}$
King	Triple	16000	0·218	13·62	9·61
Kent	,,	28000	0·328	11·72	8·27
Van Bilderbeek-van Meurs	,,	32040	0·431	13·45	9·48

The results of both King and van Bilderbeek-van Meurs may be said to be in agreement with mine.

(15) λ: 3758·23.

Intensity 8*r*; sharp triplet; character *a*. Plate II, Fig. 1. The determination of the fields is the same as in the case of λ: 3886·29.

Table XV.

H	$\Delta\lambda$	$\frac{\Delta\lambda}{H}\times 10^6$	$\frac{\Delta\lambda}{H\lambda^2}\times 10^{13}$	Deviation from the Mean	
				$\frac{\Delta\lambda}{H\lambda^2}\times 10^{13}$	$\frac{\text{Deviation}}{\text{Mean Value}}$
8220	0·140±0·0006	17·03	12·06	+0·32	+2·7
9615	0·162±0·0012	16·86	11·93	+0·19	+1·6
10880	0·179±0·0006	16.45	11·65	−0 09	− 0.8
13430	0·225±0·0009	16·74	11·86	+0·12	+1·0
14500	0·239±0·0008	16·48	11·67	−0·07	−0·6
16370	0·267±0·0008	16·32	11·55	−0·19	−1·6
17700	0.291±0·0012	16·44	11·64	−0·10	−0·9
19300	0·321±0·0010	16·63	11·77	+0·03	+0·3
20000	0·334±0·0016	16·70	11·83	+0·09	+0·8
21740	0·366±0·0009	16·84	11·93	+0·19	+1·6
23130	0·379±0·0009	16·39	11·62	−0·12	−1·0
*25020	0·412±0·0006	16·45	11·65	−0·09	−0·8
25250	0·417±0·0006	16·52	11·70	−0·04	−0·3
*25540	0·422±0·0007	16·53	11·71	−0·03	−0·3
*26100	0·428±0·0006	16·40	11·62	−0·12	−1·0
26520	0·437±0·0009	16·48	11·68	−0·16	−1·4
27240	0·452±0·0010	16·60	11·75	+0·01	+0·1
28680	0·476±0·0005	16·57	11·74	0·00	0·0
Mean		16·57	11·74		

$\Delta\lambda$ varies linearly proportional to H. Plate XIII, Fig. 2.

The results of the former investigations are:

Investigator	Separation	H	$\Delta\lambda$	$\frac{\Delta\lambda}{H}\times10^6$	$\frac{\Delta\lambda}{H\lambda^2}\times10^{13}$
King	Triplet	16000	0·269	16·82	11·91
Kent	,,	28000	0·403	14·42	10·19
Reese	,,	28300	0·478	16·89	11·94
Van Bilderbeek-van Meurs	,,	32040	0·541	16·89	11·94

(16) λ: 3749·47.

Intensity 10; sharp triplet; character a. Plate II, Fig. 1; Plate IV, Fig. 5. The determination of the fields is the same as in the case of λ: 3886·29.

Table XVI.

H	$\Delta\lambda$	$\frac{\Delta\lambda}{H}\times10^6$	$\frac{\Delta\lambda}{H\lambda^2}\times10^{13}$	Deviation from the Mean	
				$\frac{\Delta\lambda}{H\lambda^2}\times10^{13}$	$\frac{\text{Deviation}}{\text{Mean Value}}$
8220	0·150±0·0015	18·25	12·98	+0·38	+3·0
10880	0·192±0·0006	17·65	12·56	−0·04	−0·3
13430	0·239±0·0013	17·78	12·65	+0·05	+0·4
14500	0·255±0·0004	17·58	12·51	−0·09	−0·7
16370	0·288±0·0008	17·60	12·52	−0·08	−0·6
17700	0·311±0·0014	17·57	12·50	−0·10	−0·8
19300	0·342±0.0005	17·72	12·61	+0·01	+0·1
20000	0·353±0 0015	17·65	12·56	−0·04	−0·3
21740	0·388±0·0006	17·86	12·70	+0·10	+0·8
23130	0·408±0·0012	17·65	12·56	−0·04	−0·3
*25020	0·443±0·0005	17·71	12·60	0·00	0·0
25250	0·440±0·0005	17·43	12·40	−0·20	−0·6
*25540	0·455±0·0005	17·83	12·68	+0·08	+0·6
*26100	0·458±0·0009	17·56	12·50	−0·10	−0·8
26520	0·468±0·0007	17·65	12·56	−0·04	−0·3
27240	0·484±0·0009	17·76	12·63	+0·03	+0·2
28680	0·512±0·0007	17·85	12·69	+0·09	+0·7
Mean		17·71	12·60		

$\Delta\lambda$ is linearly proportional to H. Plate XIII, Fig. 3.

The results of the former investigations are:

Investigator	Separation	H	$\Delta\lambda$	$\frac{\Delta\lambda}{H}\times 10^6$	$\frac{\Delta\lambda}{H\lambda^2}\times 10^{13}$
King	Triplet	16000	0·289	18·07	12·84
Kent	,,	28000	0·432	15·42	10·97
Reese	,,	28300	0·530	18·74	13·32
Van Bilderbeek-van Meurs	,,	32040	0·580	18·11	12·86

(17) λ: 3737·13.

Intensity 6; the separation seems to be a triplet, but each component is diffuse. King took this for a septuplet. Character *a*. Plate II, Fig. 1; Plate IV, Fig. 5.

The separations of the outer two components, being considered as a triplet, are as follows:

TABLE XVII.

H	$\Delta\lambda$	$\frac{\Delta\lambda}{H}\times 10^6$	$\frac{\Delta\lambda}{H\lambda^2}\times 10^{13}$	Deviation from the Mean	
				$\frac{\Delta\lambda}{H\lambda^2}\times 10^{13}$	$\frac{\text{Deviation}}{\text{Mean Value}}$
8220	0·126±0·0011	15·33	10·97	+0·17	+1·0
10880	0·163±0·0007	14·97	10·72	−0·08	−0·7
13430	0·204±0·0011	15·18	10·87	+0·07	+0·6
14500	0·221±0·0012	15·26	10·92	+0·12	+1·1
17700	0·274±0·0007	15·48	11·08	+0·28	+2·6
19300	0·291±0·0009	15·08	10·79	−0·01	−0·1
21740	0·331±0·0010	15·23	10·91	+0·11	+1·0
23130	0·340±0·0014	14·71	10·54	−0·26	−2·4
*23580	0·355±0·0016	15·06	10·78	−0·02	−0·2
*25020	0·367±0·0020	14·68	10·52	−0·28	−2·6
25250	0·363±0·0009	14·38	10·29	−0·51	−4·7
*25540	0·391±0·0008	15·31	10·96	+0·16	+1·5
*26100	0·406±0·0007	15·56	11·14	+0·34	+3·1
26520	0·392±0.0017	14·78	10·58	−0·22	−2·0
27240	0·415±0·0009	15·24	10·92	+0·12	+1·1
28680	0·435±0·0014	15·16	10·85	+0·05	+0·5
Mean			10·80		

We can not say that $\Delta\lambda$ varies linearly proportional to H, as in all other cases of diffuse triplets. The deviation from the mean can also be seen from Fig. 1 of Plate XIV.

The results of other investigators are:

Investigator	Separation	H	$\Delta\lambda$	$\frac{\Delta\lambda}{H}\times10^6$	$\frac{\Delta\lambda}{H\lambda^2}\times10^{13}$
King	Septuple?	16000	0·254 w_1	15·88	11·36
Kent	Triple	28000	0·371	13·24	9·49
Reese	,,	28300	0·418	14·76	10·57
Van Bilderbeek-van Meurs	,,	32040	0·467	14·58	10·42

(18) λ: 3734·86.

Intensity 10; sharp triplet; character *a*. Plate II, Fig. 1; Plate IV, Fig. 5. The determination of the fields is the same as in the case of λ: 3886·29.

TABLE XVIII.

H	$\Delta\lambda$	$\frac{\Delta\lambda}{H}\times10^6$	$\frac{\Delta\lambda}{H\lambda^2}\times10^{13}$	Deviation from the Mean	
				$\frac{\Delta\lambda}{H\lambda^2}\times10^{13}$	$\frac{\text{Deviation}}{\text{Mean Value}}$
8220	0·156±0·0009	18·97	13·60	+0·48	+3·7
10880	0·198±0·0007	18·21	13·05	−0·07	−0·5
13430	0·248±0·0012	18·46	13·24	+0·12	+0·9
14500	0·263±0·0009	18·14	13·01	−0·11	−0·8
16370	0·294±0·0013	18·24	13·08	−0·04	−0·3
17700	0·325±0·0014	18·37	13·16	+0·04	+0·3
19300	0·354±0·0007	18·34	13·15	+0·03	+0·2
20000	0·371±0·0009	18·55	13·30	+0·18	+1·4
21740	0·401±0·0010	18·45	13·23	+0·11	−0·8
23130	0·421±0·0008	18·21	13·06	−0·06	−0·5
*23580	0·430±0·0006	18·25	13·08	−0·04	−0·3
*25020	0·455±0·0006	18·18	13·04	−0·08	−0·6
25250	0·453±0·0006	17·97	12·88	−0·24	−1·8
*25540	0·468±0·0007	18·35	13·16	+0·04	+0·3
*26100	0·472±0·0006	18·10	12·97	−0·15	−1·1
26520	0·480±0·0005	18·13	12·99	−0·13	−1·0
27240	0·498±0·0009	18·30	13·12	0·00	0·0
28680	0·524±0·0009	18·27	13·10	−0·02	−0·2
Mean		18·30	13·12		

$\Delta\lambda$ may be said to be nearly proportional to H. The greatest deviation is 2%, except at the lowest field that was measured. Plate XIV, Fig. 2.

The results of the former investigations are:

Investigator	Character of Separation	H	$\Delta\lambda$	$\frac{\Delta\lambda}{H}\times 10^6$	$\frac{\Delta\lambda}{H\lambda^2}\times 10^{13}$
King	Triplet	16000	0·310	19·37	13·89
Kent	,,	28000	0·453	16·19	11·60
Reese	,,	28300	0·538	19·02	13·63
Van Bilderbeek-van Meurs	,,	32040	0·594	18·55	13·30

(19) λ: 3719·93.

Intensity 10; this line does not seem to be a triplet, but rather a sextet, although separations of all components can not be detected; character *c*. Plate II, Fig. 1; Plate IV, Fig. 5. The determination of the fields is the same as in the case of λ: 3886·29. The separations $\Delta\lambda$ of the outer two components are:

TABLE XIX.

H	$\Delta\lambda$	$\frac{\Delta\lambda}{H}\times 10^6$	$\frac{\Delta\lambda}{H\lambda^2}\times 10^{13}$	Deviation from the Mean	
				$\frac{\Delta\lambda}{H\lambda^2}\times 10^{13}$	$\frac{\text{Deviation}}{\text{Mean Value}}$
8220	0·133±0·0008	16·18	11·69	+0·19	+1·7
10880	0·176±0·0006	16·17	11·68	+0·18	+1·6
13430	0·216±0·0008	16·08	11·62	+0·12	+1·0
14500	0·225±0·0005	15·52	11·21	−0·29	−2·5
16370	0·266±0·0013	16·26	11·74	+0·24	+2·1
19300	0·311±0·0008	16·12	11·64	+0·14	+1·2
20000	0 313±0·0013	15·65	11·30	−0·20	−1·7
21740	0·353±0·0011	16·24	11·74	+0·24	+2·1
23130	0·359±0·0014	15 53	11·22	−0·28	−2·4
*23580	0·382±0·0010	16·20	11·70	+0·20	+1·9
*25020	0·405±0·0010	16·18	11·69	+0·19	+1·7
25250	0·386±0·0009	15·29	11·04	−0·46	−4·0
*25540	0·417±0·0010	16·33	11·80	+0·30	+2·6
*26100	0·423±0·0014	16·21	11·71	+0·21	+1·8
26520	0·406±0·0011	15·33	11·06	−0·44	−3·8
27240	0·429±0·0008	15·76	11·38	−0·12	−1·0
28680	0·449±0·0012	15·67	11·32	−0·18	−1·6
Mean			11·50		

We can not say that $\Delta\lambda$ is proportional to H. The deviations from the mean are also seen from Fig. 1 of Plate XV.

The results of the previous investigators are:

Investigator	Character of Separation	H	$\Delta\lambda$	$\frac{\Delta\lambda}{H}\times10^6$	$\frac{\Delta\lambda}{H\lambda^2}\times10^{13}$
King	Triplet ?	16000	0·268	16·75	12·10
Kent	„	28000	0·393	14·03	10·14
Reese	„	28300	0·448	15·83	11·43
Van Bilderbeek-van Meurs	„	32040	0·506	15·81	11·42

(20) λ: 3618·77.

Intensity 6; diffuse triplet; character *b*. This line is photographed on the same plate together with the zinc line λ: 4680·138 as field determination. 4680·138 is of the 3rd order spectrum and 3618·77 of the 4th. A line of other order also appeared close by this zinc line in its red side as seen from the photographs (Plate I, fig. 2 and Plate I, fig. 3). Also Plate III, Fig. 4.

In higher fields, as the separations of the zinc line are large, the determination of the field is not affected by the appearance of this line. In lower fields, however, the overlapping of this line with $+\delta\lambda$ component of 4680·138 was the cause of an error in the measurement of the field, when determined by measuring the distance of the outer two components of the zinc line; in such cases, twice the distance between λ_0 and $-\delta\lambda$ was taken instead of the separation of the outer two components, as this zinc line has been confirmed to be perfectly symmetrical. $\Delta\lambda$ denotes the separation of two outer components as before.

TABLE XX.

H	Δλ	$\frac{\Delta\lambda}{H} \times 10^6$	$\frac{\Delta\lambda}{H\lambda^2} \times 10^{13}$	Deviation from the Mean	
				$\frac{\Delta\lambda}{H\lambda^2} \times 10^{13}$	$\frac{\text{Deviation}}{\text{Mean Value}}$
19090	0·197±0·0023	10·32	7·88	+0·30	+4·0
22380	0·221±0·0014	9·87	7·54	−0·04	−0·5
23100	0·228±0·0015	9·87	7·54	−0·04	−0·5
*23580	0·235±0·0009	9·98	7·62	+0·04	+0·5
26440	0·251±0·0009	9·49	7·25	−0·33	−4·4
27750	0·272±0·0015	9 80	7·47	−0·11	−1·5
27920	0·275±0·0013	9·86	7·53	−0·05	−0·7
28700	0·285±0·0025	9·93	7·58	0·00	0·0
29200	0·288±0·0017	9·86	7·53	−0·05	−0·7
29400	0·299±0·0025	10·17	7·76	+0·18	+2·4
30080	0·301±0 0011	10·00	7·63	+0·05	+0·7
32100	0·315±0·0012	9·81	7·48	−0·10	−1·3
32240	0·323±0·0020	10·02	7·64	+0·06	+0·8
33100	0·333±0·0022	10·06	7·68	+0·10	+1·3
Mean			7·58		

From the last column of the table we see that the deviations from the mean are less than 2%, except for three points. Hence we can say that $\Delta\lambda$ is nearly proportional to the fields applied. Plate XV, Fig. 2. The results of former investigators are:

Investigator	Character of Separation	H	Δλ	$\frac{\Delta\lambda}{H} \times 10^6$	$\frac{\Delta\lambda}{H\lambda^2} \times 10^{13}$
Kent	Triplet	28000	0·245	8·75	6·68
Van Bilderbeek-van Meurs	,,	32040	0·311 .	9·70	7·41

(21) λ:3581·20.

Intensity 10; triplet, and each component is intense in comparison with that of other lines on the same plate; character *b*. The determination of the fields is the same as in the case of λ:3618·77. Plate I, Fig. 3; Plate III, Fig. 4.

Table XXI.

H	$\Delta\lambda$	$\frac{\Delta\lambda}{H}\times10^6$	$\frac{\Delta\lambda}{H\lambda^2}\times10^{13}$	Deviation from the Mean	
				$\frac{\Delta\lambda}{H\lambda^2}\times10^{13}$	$\frac{\text{Deviation}}{\text{Mean Value}}$
14370	0·207±0·0010	14·41	11·25	+0·28	+2·6
18080	0·257±0·0018	14·21	11·09	+0·12	+1·1
19090	0·270±0·0011	14·15	11·04	+0·07	+0·6
22380	0·318±0·0016	14·21	11·09	+0·12	+1·1
23100	0·329±0·0006	14·24	11·11	+0·14	+1·3
*23580	0·352±0·0009	14·94	11·65	+0·68	+6·2
26440	0·370±0·0009	13·98	10·90	−0·07	−0·6
27750	0·384±0·0006	13·83	10·79	−0·18	−1·6
27920	0·392±0·0007	14·05	10·96	−0·01	−0 1
28700	0·395±0·0011	13·77	10·74	−0·23	−2·1
29200	0·404±0·0008	13·83	10·79	−0·18	−1·6
29400	0·404±0·0012	13·76	10·73	−0·24	−2·2
30080	0·418±0·0010	13·91	10·84	−0·13	−1·2
31100	0·441±0·0009	14·18	11·06	+0·09	+0·8
31500	0·446±0·0013	14·16	11·05	+0·08	+0·7
32100	0·449±0·0009	13·98	10·90	−0·07	−0·6
32240	0·442±0·0011	13·70	10·68	−0·29	−2·6
33100	0·458±0·0009	13·84	10·80	−0·17	−1·5
Mean			10·97		

From the table we see that $\Delta\lambda$ varies nearly proportionally to the fields applied, though somewhat great discrepancies are found in three of these measurements. Plate XV, Fig. 3.

The results of the previous observers are:

Observer	Character of Separation	H	$\Delta\lambda$	$\frac{\Delta\lambda}{H}\times10^6$	$\frac{\Delta\lambda}{H\lambda^2}\times10^{13}$
Kent	Triplet	28000	0·359	12·83	10·00
Van Bilderbeek-van Meurs	,,	32040	0·458	14·30	11·14

(22) λ:3570·12.

Intensity 10; diffuse triplet; character *b*. In non-magnetic field, this line and λ:3581·20 have equal intensity 10; but in magnetic fields, the components of the former are more intense than those of the latter. The determination of the fields is the same as in the case of λ:3618·77. Plate I, Fig. 3; Plate III, Fig. 4.

TABLE XXII.

H	$\Delta\lambda$	$\frac{\Delta\lambda}{H}\times 10^{6}$	$\frac{\Delta\lambda}{H\lambda^2}\times 10^{13}$	Deviation from the Mean	
				$\frac{\Delta\lambda}{H\lambda^2}\times 10^{13}$	$\frac{\text{Deviation}}{\text{Mean Value}}$
14370	0·195±0·0023	13·58	10·65	+0·66	+6·6
18080	0·243±0·0012	13·43	10·54	+0·55	+5·5
19090	0·250±0·0015	13·10	10·28	+0·29	+2·9
22380	0·288±0·0014	12·87	10·10	+0·11	+1·1
23100	0·303±0·0008	13·12	10·30	+0·31	+3·1
*23580	0·310±0·0006	13·16	10·33	+0·34	+3·4
26440	0·329±0·0019	12·43	9·77	−0·22	−2·2
27750	0·347±0·0018	12·50	9·82	−0·17	−1·7
27920	0·352±0·0013	12·61	9·90	−0 09	−0·9
28700	0·359±0·0029	12·51	9·82	−0·17	−1·7
29200	0·363±0·0013	12·42	9·76	−0·23	−2·3
29400	0·368±0·0015	12·51	9·82	−0·17	−1·7
30080	0·375±0·0009	12·46	9·78	−0·21	−2·1
32100	0·395±0·0021	12·31	9·66	−0·33	−3·3
32240	0·399±0·0016	12·38	9·71	−0·28	−2·8
33100	0·408±0·0019	12·33	9·68	−0·31	−3·1
Mean			9·99		

$\frac{\Delta\lambda}{H}$ is larger in lower fields than in the higher, as is easily seen in the last column of the table. The deviations from the mean are also seen in Fig. 1 of Plate XVI. The results of the former investigators are:

Investigator	Character of Separation	H	$\Delta\lambda$	$\frac{\Delta\lambda}{H}\times 10^{6}$	$\frac{\Delta\lambda}{H\lambda^2}\times 10^{13}$
Kent	Triplet	28000	0·326	11·64	9·13
Van Bilderbeek-van Meurs	„	32010	0·407	12·71	9·96

(23) λ:3075·725.

Intensity 3; sharp triplet; character *b*. The third order spectrum of this line was photographed together with the second order of λ:4680·138 of zinc as field determination. Plate I, Fig. 4; Plate IV, Fig. 3.

TABLE XXIII.

H	$\Delta\lambda$	$\frac{\Delta\lambda}{H}\times10^6$	$\frac{\Delta\lambda}{H\lambda^2}\times10^{13}$
25150	0·330	13·12	13·87
25920	0·343	13·23	13·98
26420	0·346	13·08	13·82
28050	0·372	13·25	14·00
Mean		13·17	13·92

The data are too few to determine the relation between $\Delta\lambda$ and H. But in these regions $\frac{\Delta\lambda}{H}$ may be said to be constant. I could not find any published data of the separation of this line; perhaps it is the first time that its separation has been examined.

(24) λ:3059·08.

Intensity 3; sharp triplet; character *b*. Plate I, Fig. 4; Plate IV, Fig. 3. The determination of the fields is the same as in the case of λ:3075·725.

TABLE XXIV.

H	$\Delta\lambda$	$\frac{\Delta\lambda}{H}\times10^6$	$\frac{\Delta\lambda}{H\lambda^2}\times10^{13}$
25150	0·330	13·12	14·01
25920	0·340	13·12	14·01
26420	0·344	13·02	13·91
28050	0·365	13·02	13·91
Mean		13 07	13·96

In these regions $\frac{\Delta\lambda}{H}$ is constant. Van Bilderbeek-van Meurs observed this line and reported that

H	$\Delta\lambda$	$\frac{\Delta\lambda}{H}\times10^6$	$\frac{\Delta\lambda}{H\lambda^2}\times10^{13}$
32040	0·437	13·64	14·56

(25) λ:3047·60.

Intensity 3; triplet; character *b*. The determination of the field is the same as in the case of λ:3075·725. Plate I, Fig. 4: Plate IV, Fig. 3.

Table XXV.

H	$\Delta\lambda$	$\frac{\Delta\lambda}{H}\times 10^6$	$\frac{\Delta\lambda}{H\lambda^2}\times 10^{13}$
25150	0·321	12·75	13·73
25920	0·332	12·81	13·79
26420	0·332	12·55	13·51
28050	0·368	13·10	14·10

Evidently $\frac{\Delta\lambda}{H}$ is not constant within these narrow ranges. But the data are too few to decide the feature. For comparison, I give only:

Observer	H	$\Delta\lambda$	$\frac{\Delta\lambda}{H}\times 10^6$	$\frac{\Delta\lambda}{H\lambda^2}\times 10^{13}$
Van Bilderbeek-van Meurs	32040	0·432	13·49	14·50

(26) λ:2756·31.

Intensity 1; diffuse triplet; character *b*. Plate I, Fig. 2; Plate III, Fig. 2. The fifth order spectrum of this line is photographed on the plate together with the third order of λ:4680·138 of zinc.

Table XXVI.

H	$\Delta\lambda$	$\frac{\Delta\lambda}{H}\times 10^6$	$\frac{\Delta\lambda}{H\lambda^2}\times 10^{13}$
26210	0·220±0·0011	8·42	11·07
27940	0·237±0·0017	8·48	11·16
29600	0·256±0·0010	8·67	11·41
30000	0·254±0·0019	8·47	11·14
30100	0·251±0·0013	8·34	10·96

Further study is necessary to decide the character, but in these regions the variation may be considered as regular and approximately 11·15 adopted as the mean value of $\frac{\Delta\lambda}{H\lambda^2}\times 10^{13}$. It is to be regretted that I could not find any published data as to the separation of this line.

(27) λ:2755·73.

Intensity 15; a triplet, whose two outer components are diffuse and wider than the central. Character *b*. Plate I, Fig. 2; Plate III, Fig. 2. The determination of the fields is the same as in the case of λ:2756·31.

TABLE XXVII.

H	$\Delta\lambda$	$\frac{\Delta\lambda}{H}\times10^6$	$\frac{\Delta\lambda}{H\lambda^2}\times10^{13}$	Deviation from the Mean: $\frac{\Delta\lambda}{H\lambda^2}\times10^{13}$	Deviation from the Mean: $\frac{\text{Deviation}}{\text{Mean Value}}$
25800	0·226±0·0005	8·73	11·50	−0·06	−0·5
26210	0·227±0·0007	8·66	11·38	−0·18	−1·6
27940	0·247±0·0007	9·32	12·26	+0·70	+6·1
29600	0·258±0·0010	8·73	11·50	−0·06	−0·5
30000	0·260±0·0019	8·66	11·38	−0·18	−1·6
30100	0·261±0·0007	8·67	11·40	−0·16	−1·4
30350	0·263±0·0009	8·67	11·40	−0·16	−1·4
31320	0·280±0·0007	8 87	11·68	+0·12	+1·0
Mean			11·56		

$\frac{\Delta\lambda}{H}$ is not straight as in all other cases of diffuse triplets. The following I found reported:

Investigator	Character of Separation	H	$\Delta\lambda$	$\frac{\Delta\lambda}{H}\times10^6$	$\frac{\Delta\lambda}{H\lambda^2}\times10^{13}$
Van Bilderbeek-van Meurs	Triplet	32040	0·272	8·49	11·17

(28) λ:2746·98.

Intensity 8; diffuse triplet; the peculiarity of this line is that the violet side component is weaker in intensity than the red side one, as required by Voigt's[1] asymmetry theory. Plate I, Fig. 2; Plate III, Fig. 3. The determination of the fields is the same as in the case of λ:2756·31.

1) W. Voigt, Ann. d. Phys., **1** (1900), p. 376.

Table XXVIII.

H	$\Delta\lambda$	$\frac{\Delta\lambda}{H}\times10^6$	$\frac{\Delta\lambda}{H\lambda^2}\times10^{13}$	Deviation from the Mean	
				$\frac{\Delta\lambda}{H\lambda^2}\times10^{13}$	$\frac{\text{Deviation}}{\text{Mean Value}}$
25800	0·238±0·0023	9·21	12·20	−0·63	−4·9
26210	0·258±0·0009	9·82	13·01	+0·18	+1·4
27940	0·276±0·0009	9·88	13·10	+0·27	+2·1
29600	0·289±0·0009	9·75	12·92	+0·09	+0·7
30000	0·293±0·0010	9·76	12·93	+0·10	+0·8
30100	0·292±0·0011	9·72	12·87	+0·04	+0·3
30350	0·296±0·0014	9·76	12·93	+0·10	+0·7
31320	0·299±0·0015	9·55	12·65	−0·18	−1·4
Mean			12·83		

From the table we see that the variation of $\Delta\lambda$ is not proportional to H. Plate XVI, Fig. 3. According to van Bilderbeek-van Meurs:

Character of Separation	H	$\Delta\lambda$	$\frac{\Delta\lambda}{H}\times10^6$	$\frac{\Delta\lambda}{H\lambda^2}\times10^{13}$
Triplet	32040	0·305	9·52	12·60

She makes no remark as to the asymmetry of the separation.

(29) λ:2746·48.

Intensity 10; diffuse triplet, whose outer two components are broader and fainter than the central one. Character *b*. Plate I, Fig. 2; Plate III, Fig. 3. The determination of the fields is the same as in the case of λ:2756·31.

TABLE XXIX.

H	$\Delta\lambda$	$\frac{\Delta\lambda}{H}\times10^6$	$\frac{\Delta\lambda}{H\lambda^2}\times10^{13}$	Deviation from the Mean	
				$\frac{\Delta\lambda}{H\lambda^2}\times10^{13}$	Deviation / Mean Value
25800	0·185±0·0012	7·16	9·50	−0·13	−1·4
26210	0·187±0·0009	7·14	9·48	−0·15	−1·6
27940	0·206±0·0012	7·38	9·79	+0·16	+1·7
29600	0·216±0·0007	7·30	9·67	+0·04	+0·4
30000	0·221±0·0012	7·37	9·77	+0·14	+1·5
30100	0·220±0·0009	7·31	9·69	+0·06	0·6
30350	0·218±0·0010	7·17	9·52	0·09	−0·9
31320	0·227±0·0015	7·23	9·60	−0·03	−0·3
Mean			9·63		

Within these regions, $\frac{\Delta\lambda}{H}$ may be said to be constant. The result of van Bilderbeek-van Meurs is

Character of Separation	H	$\Delta\lambda$	$\frac{\Delta\lambda}{H}\times10^6$	$\frac{\Delta\lambda}{H\lambda^2}\times10^{13}$
Triplet	32040	0·221	6·90	9·14

This line 2746·48 and the line 2746·98 are so close that there may be some mutual action between them. The weakening of the violet component of 2746·98 may be caused by such action. The difference Δ of the central components of these two lines is worth mentioning (Table XXX). This is nearly constant and I can say that there is no such shift in the central components, as in D_1 and D_2 of sodium spectra.[1)]

TABLE XXX.

H	Δ
25800	0·512
26210	0·509
27940	0·509
29600	0·509
30000	0·506
30100	0·508
30350	0·503
31320	0·506

(30) λ: 2739·550.*

Intensity 15; triplet; character *a*. Plate I, Fig. 2; Plate III, Fig. 3. The determination of the fields is the same as in the case of 2756·31.

1) Paschen.

TABLE XXXI.

H	$\Delta\lambda$	$\frac{\Delta\lambda}{H}\times 10^6$	$\frac{\Delta\lambda}{H\lambda^2}\times 10^{13}$	Deviation from the Mean	
				$\frac{\Delta\lambda}{H\lambda^2}\times 10^{13}$	$\frac{\text{Deviation}}{\text{Mean Value}}$
25800	0.257 ± 0.0008	9.95	13.27	-0.16	-1.2
26210	0.264 ± 0.0005	10.07	13.41	-0.02	-0.2
27940	0.285 ± 0.0008	10.21	13.59	$+0.16$	$+1.2$
29600	0.295 ± 0.0006	9.97	13.28	-0.15	-1.1
30000	0.302 ± 0.0007	10.07	13.40	-0.03	-0.2
30100	0.305 ± 0.0004	10.13	13.50	$+0.07$	$+0.5$
30350	0.308 ± 0.0009	10.15	13.52	$+0.09$	$+0.7$
31320	0.316 ± 0.0006	10.10	13.45	$+0.02$	$+0.2$
Mean		10.08	13.43		

Within the range of the fields $\frac{\Delta\lambda}{H}$ is nearly constant. The results of van Bilderbeek-van Meurs are

Character of Separation	H	$\Delta\lambda$	$\frac{\Delta\lambda}{H}\times 10^6$	$\frac{\Delta\lambda}{H\lambda^2}\times 10^{13}$
Triplet	32040	0.325	10.13	13.50

(31) Separations in a single field.

Although the magnetic separations of iron lines in a single field have been investigated by many phyicists, it may not be out of place to report here the results obtained in the course of my study of the above mentioned problem. (Plate II, Figs. 2, 3, 4 and 5; Plate V, Figs. 1, 2, 3, 4 and 5; Plate VI, Figs. 1, 2, 3 and 4).

Table XXXII.

Wave-length (I. U.)	Intensity	Character of Separation	H	$\Delta\lambda$	$\frac{\Delta\lambda}{H}\times10^6$	$\frac{\Delta\lambda}{H\lambda^2}\times10^{13}$	$\frac{\Delta\lambda}{H\lambda^2}\times10^{13}$ (King)	Remark
4143·88	5		24560	⊥0·623 //	25·38	14·80	⊥14·30 w_1 // w_2	King took this for a septuplet?
4071·75	8	Triple	24560	⊥0·264 //0·000	10·77 0·00	6·50 0·00	⊥ 6·40 // 0·00	
4063·61	(6)	Triple	24560	⊥0·423 //0·000	17·21 0·00	10·43 0·00	⊥10·18 // 0·00	
4045·82	15	Triple	24560	⊥0·477 //0·000	19·47 0·00	11·90 0·00	⊥11·37 // 0·00	
3969·26	5		25020	⊥0·586 //	23·43	14·87	⊥14·04 w_1 // w_2	Septuple? (King)
3956·67	3	Triple	25020	⊥0·445 //0·000	17·80 0·00	11·37 0·00	⊥11·54 // 0·00	Red side component is weaker in intensity. $\frac{-\delta\lambda}{H\lambda^2}=5{\cdot}96,\ \frac{+\delta\lambda}{H\lambda^2}=5{\cdot}42.$
3930·30	4	Triple	25020	⊥0·551 //0·000	22·04 0·00	14·26 0·00	⊥14·25 // 0·00	
3927·94	4	Triple	25020	⊥0·550 //0·000	21·98 0·00	14·25 0·00	⊥14·26 // 0·00	
3922·92	4	Triple	25020	⊥0·550 //0·000	21·96 0·00	14·27 0·00	⊥14·26 // 0·00	
3920·26	4	Triple	25020	⊥0·545 //0·000	21·80 0·00	14·17 0·00	⊥14·17 // 0·00	
3906·481	(4)	Triple	25020	⊥0·543 //0·000	21·74 0·00	14·25 0·00	⊥10·20 // 4·00	

Wave-length (I. U.)	Intensity	Character of Separation	H	$\Delta\lambda$	$\frac{\Delta\lambda}{H} \times 10^6$	$\frac{\Delta\lambda}{H\lambda^2} \times 10^{13}$	$\frac{\Delta\lambda}{H\lambda^2} \times 10^{13}$ (King)	Remark
3902·95	5		25020	⊥ …… // 0·249	…… 9·93	…… 6·52	⊥ 11·40 w_3 // 6·24 w_1	10 comps? (King)
3899·70	4	Triple	25020	⊥ 0·538 // 0·000	21·50 0·00	14·20 0·00	⊥ 14·33 // 0·00	
3895·65	(4)	Triple	25020	⊥ 0·543 // 0·000	21·70 0·00	14·28 0·00	⊥ 14·29 // 0·00	
3887·05	(4)		25020	⊥ 0·547 // 0·185	21·88 7·38	14·48 4·89	⊥ 14·40 w_2 // 4·83	Septuple (King)
3865·527	4	Quintuple	25020	⊥ 0·532 ⊥ 0·000 // 0·532	21·24 0·00 21·24	14·20 0·00 14·20	⊥ 14·35 ⊥ 0·00 // 14·22	Becquerel and Deslandres took this for an inverse triplet.
3841·06	5	Triple	25020	⊥ 0·275 // 0·000	10·98 0·00	7·44 0·00	⊥ 6·95 // 0·00	
3834·23	6		25020	⊥ 0·308 // ……	12·31 ……	8·37 ……	⊥ 10·55 w_1 // w_2	Septuple (King)
3824·44	5	Triple	25020	⊥ 0·525 // 0·000	21·00 0·00	14·37 0·00	⊥ 14·75 // 0·00	
3821·18	3	Triple	25020	⊥ 0·325 // 0·000	13·00 0·00	8·90 0·00	⊥ 9·33 // 0·00	
3806·70	3	Triple	25020	⊥ 0·360 // 0·000	14·40 0·00	9·93 0·00	⊥ 9·74 // 0·00	
3805·346	3	Triple	25020	⊥ 0·312 // 0·000	12·45 0·00	8·59 0·00	⊥ 8·81 // 0·00	

Wave-length (I. U.)	Intensity	Character of Separation	H	$\Delta\lambda$	$\frac{\Delta\lambda}{H} \times 10^6$	$\frac{\Delta\lambda}{H\lambda^2} \times 10^{13}$	$\frac{\Delta\lambda}{H\lambda^2} \times 10^{13}$ (King)	Remark
3799·55	5	Triple	25020	⊥ 0·521 // 0·000	20·80 0·00	14·40 0·00	⊥ 14·12 // 0·00	
3798·50	4	Triple	25020	⊥ 0·516 // 0·000	20·62 0·00	14·28 0·00	⊥ 14·12 // 0·00	
3797·51	3	Triple	25020	⊥ 0·408 // 0·000	16·33 0·00	11·33 0·00	⊥ 11·30 // 0·00	
3787·88	4	Octuple	25020	⊥ {0·333 0·166 0·000 0·172 0·348} {0·166 0·000 0·172}	{13·30 6·63 0·00 6·88 13·90} {6·63 0·00 6·88}	{9·28 4·63 0·00 4·80 9·63} {4·63 0·00 4·80}	⊥ {9·53 4·83 0·00 4·71 9·32} // {4·75 0·00 4·83}	
3767·19	5	Unaffected	25020					
3765·54	3	Triple	25020	⊥ 0·355 // 0·000	14.19 0·00	10·01 0·00	⊥ 10·05 // 0·00	
3748·25	4	Octuple	25020	⊥ {Pair III 0·305 Pair II 0·158 Pair I 0·000} // 0·158	{12·95 6·68 0·00} 6·68	{9·22 4·76 0·00} 4·76	⊥ {14·06 10·05 4·49} // w.	9 Comps? (King) Octuple (Myself)
3745·91	4	Unaffected	25020				Unaffected	

Wave-length (I. U.)	Intensity	Character of Separation	H	$\Delta\lambda$	$\frac{\Delta\lambda}{H}\times10^6$	$\frac{\Delta\lambda}{H\lambda^2}\times10^{13}$	$\frac{\Delta\lambda}{H\lambda^2}\times10^{13}$ (King)	Remark
3745·55	5		25020	⊥ 0·277 //	11·08	7 89	⊥ 10·15 w_1 // ...... w_2	Septuple (King)
3743·37	6	Octuple	23580	⊥ {0·301, 0·151, 0·000, 0·151, 0·310} // {0·151, 0·000, 0·151}	{12·76, 6·40, 0·00, 6·40, 13·13} {6·10, 0·00, 6·40}	{9·10, 4·57, 0·00, 4·57, 9·38} {4·57, 0·00, 4·57}	⊥ {9·50, 4·99, 0·00, 4·64, 9·28} // {4·73, 0·00, 4·77}	
3732·39	(4)	Triple	23580	⊥ 0·487 // 0·000	19·43 0·00	13·96 0·00	⊥ 17·90 // 0·00	Character *C* (Fig. 6)
3727·63	5		23580	⊥ 0·492 //	20·34	14·99	⊥ 14·30 w_1 // w_2	Septuple ? (King)
3722·57	4		23580	⊥ 0·620 ⊥ 0·460 ⊥ 0·306 ⊥ 0·136 // 0·306	26·30 19·50 12·97 5·75 12·97	18·99 14·07 9·36 4·15 9·36	⊥ 18·70 ⊥ 14·03 ⊥ 9·51 ⊥ 4·64 // 8·80 // ?	12 Comps. (King)
3709·24	4	Triple	23580	⊥ 0·466 // 0·000	19·75 0·00	16·10 6·58	⊥ 14·16 // 0·00	
3705·56	4	Quadruple	25020	⊥ 0·554 // 0·226	22·11 9·03	14·36 0·00	⊥ 13·38 w_2 // 6·68	Violet side comp. of the pair II is weaker in intensity. 8 or 10 comps. (King)
3687·64	1	Triple	23580	⊥ 0·456 // 0·000	19·34 0·00	14·23 0·00	⊥ 14·30 // 0·00	

Wave-length (U. I.)	Intensity	Character of Separation	H	$\Delta\lambda$	$\frac{\Delta\lambda}{H} \times 10^6$	$\frac{\Delta\lambda}{H\lambda^2} \times 10^{13}$	$\frac{\Delta\lambda}{\Delta\lambda^2} \times 10^{13}$ (Bilderbeek)	Remark
3606·682	4	Triple	23580	⊥ 0·310 // 0·000	13·17 0·00	+ 10·12 + 0·00	⊥ 10·36 // 0·00	
3586·97	3	Double	23580	⊥ 0·370	15·70	12·21		
3572·00	2	Triple	23580	⊥ 0·292 // 0·000	12·40 0·00	9·73 0·00		
3558·52	4	Double	23580	⊥ 0·102	4·33	3·42	⊥ 5·56	According to Kent, ⊥ & // comps. have the same position. But I found that this is a doublet.
*3526·38	1	Double	23580	⊥ 0·314	13·34	10·73		
*3521·26	3	Double	23580	⊥ 0·273	11.58	9·34		
3497·83	3	Triple	23580	⊥ 0·549 // 0·000	23·20 0·00	19·03 0·00	⊥ 19·67 // 0·00	
3430·58	4	Sextuple?	23580	⊥ 0·432 // 0·118	18·33 5·01	15·05 4·11	⊥ 14·84 // doublet	Sextuple ?
*3480·66	1	Double	23580	⊥ 0·167	7·08	5·81		
3476·69	3	Triple	23580	⊥ 0·663 // 0·000	28·12 0·00	23·28 0·00	⊥ 25·27 // 0·00	
3475·44	3	Quadruple	23580	⊥ Δ 0·155 // Δ_1 0·165 ⊥ $-\Delta_2$ 0·142	6·56 6·99 6·04	5·43 5·78 5·00	⊥ 15·93 // 6·17	[Δ_2] [$-\Delta_2$] [Δ_1] The most remote red side comp. is strong in intensity.

Wave-length (I. U.)	Intensity	Character of Separation	H	$\Delta\lambda$	$\frac{\Delta\lambda}{H} \times 10^6$	$\frac{\Delta\lambda}{H\lambda^2} \times 10^{13}$	$\frac{\Delta\lambda}{H\lambda^2} \times 10^{13}$ (Bilderbeek)	Remark
3465·87	3	Octuple	23580	⊥ 0·647 ⊥ 0·399 // 0·181 // 0·090	27·44 16·92 7·67 3·84	22·86 14·08 6·39 3·19	⊥ 18·9 // 9·94	The separation of this line is surely complex; van Bilderbeek-van Meurs took it for a quartet.
*3424·29	2	Unaffected	23580					
2767·52	5r	Triple	26210	⊥ 0·199 // 0·000	7·62 0·00	9·94 0·00	⊥ 17·02 // 0·00	
2747·55	(4)	Triple	26210	⊥ 0·251 // 0·000	9·58 0·00	13·58 0·00		
*2744·52	1	Unaffected	23580					
*2743·54	(4)	Unaffected	23580					
2743·17	8	Unaffected	23580				Probably unaffected	
*2714·419	5	Triple	23580	⊥ 0·234 // 0·000	9·90 0·00	13·43 0·00		
*2697·01	5	Quadruple?	23580	⊥ 0·203 //	8·61	11·84		
*2684·78	4	Unaffected	23580					
*2673·22	(4)	Triple	23580	⊥ 0·217 // 0·000	9·20 0·00	12·88 0·00		

Wave-length (I. U.)	Intensity	Character of Separation	H	$\Delta\lambda$	$\frac{\Delta\lambda}{H} \times 10^6$	$\frac{\Delta\lambda}{H\lambda^2} \times 10^3$	$\frac{\Delta\lambda}{H\lambda^2} \times 10^3$ (Bilderbeck)	Remark
2631·34	3	Triple	23580	⊥ 0·266 / 0·000	11·27 0·000	16·29 0·00	⊥ 13·11 / 0·00	
2631·03	4	Triple	23580	⊥ 0·255 / 0·006	10·82 0·00	15·64 0·00	⊥ 13·29 / 0·00	
2628·296	8	Double	23580	⊥ 0·150	6·35	9·18	⊥ 10·02 / 0·00	Triplet (Bilderbeck).
2625·66	4	Triple	24700	⊥ 0·255 / 0·000	10·32 0·00	14·95 0·00	⊥ 14·30 / 0·00	
2621·66	4	Triple	23580	⊥ 0·502 / 0·000	21·33 0·00	31·12		Surely large separation.
*2613·82	8	Sextuple?	23580	{Outer Pair 0·398 {Inner Pair 0·139	16·82 5·90	24·60 8·64		Each comp. of the inner pair will be separated in two.
2607·08	10	Triple		⊥ 0·507 / 0·000	20·52 0·00	30·16 0·00	⊥ 14·53 / 0·00	
*2601·84	(4)			⊥ 0.277	11·76	17·82		

Of these lines, the wave-lengths marked with an asterisk were measured for the first time. Most of these values are in good agreement with King's result.

(32) Conclusion. (i) Among the iron lines investigated in different magnetic fields, 4404·75, 4383·55, 4307·92, 3886·29, 3878·78, 3859·90, 3856·38, 3815·84 3758·23 and 3749·47 are all sharp triplets and their separations $\Delta\lambda$ are linearly proportional to the fields applied. Arthur King, using larger grating, also showed that they are all triplets. (ii) 4415·13 and 3825·90 are diffuse triplets. Arthur King took them for septuplets (?). In these lines $\Delta\lambda$ is proportional to H when the fields are comparatively low, and when the fields become higher, $\frac{\Delta\lambda}{H}$ become larger than that in lower fields. 4325·78 and 3820·44 have similar features to these two, although Arthur King thinks they are triplets. 3737·13 is also a line which King took for a septuplet and which I observed as a diffuse triplet, as for example 4415·13 etc. The data are too few in higher fields to compare it with 4415·13 etc., but $\Delta\lambda$ varies so irregularly that it is difficult to consider $\frac{\Delta\lambda}{H}$ as constant. (iii) 3734·86 is taken for a septuplet by King, but in my photography it is a sharp triplet and the mean value of $\frac{\Delta\lambda}{H\lambda^2}\times 10^{13}=13{\cdot}12$ within 2% deviation from the mean. (iv) 3827·83 and 3763·80 are triplets and King is of the same opinion: but $\frac{\Delta\lambda}{H}$ is not constant and these two lines have similar features. (v) King reported 3719·93 to be a triplet (?), but in my photographs this line seems to be a sextet (?); $\Delta\lambda$ can not be said to vary linearly proportional to H. (vi) The lines from No. (20) to (30) are not reported by King. According to my investigation, $\Delta\lambda$ does not vary linearly proportional to the fields in the case of diffuse triplets and their features are similar to those which are discussed in (ii). (vii) 2746·98 is a line whose violet side component is fainter in intensity than the red side one. (viii) 3047·60 and 2756·31 are probably lines whose separations were studied for the first time.

V. The Magnetic Separations of Nickel Lines.

The wave-lengths are given in international unit, and the intensities of lines when they are not affected are taken from Kayser's Spectroscopie.

(1) λ: 3619·39.

Intensity 15; sharp triplet; character *b*. The fourth order of this line was photographed on the same plate with the third order of a zinc line λ: 4680·138 for field determination. Plate I, Fig. 3; Plate III, Fig. 4. The results obtained are:

TABLE XXXIII.

H	$\Delta\lambda$	$\frac{\Delta\lambda}{H}\times10^6$	$\frac{\Delta\lambda}{H\lambda^2}\times10^{13}$	Deviation from the Mean	
				$\frac{\Delta\lambda}{H\lambda^2}\times10^{13}$	Deviation / Mean Value
18080	0·244±0·0019	13·50	10·30	+0·18	+1·8
19090	0·257±0·0011	13·45	10·27	+0·15	+1·5
22380	0·301±0·0011	13·45	10·27	+0·15	+1·5
23100	0·306±0·0013	13·24	10·11	−0·01	−0·1
26440	0·347±0·0011	13·13	10·02	−0·10	−1·0
27750	0·363±0·0022	13·08	9·98	−0·14	−1·4
27920	0·370±0·0008	13·26	10·13	+0·01	+0·1
28700	0·378±0·0017	13·17	10·05	−0·07	−0·7
29200	0·386±0·0009	13·22	10·09	−0·03	−0·3
29400	0·385±0·0026	13·10	10·00	−0·12	−1·2
30080	0·398±0·0013	13·23	10·10	−0·02	−0·2
32100	0·426±0·0010	13·26	10·13	+0·01	+0·1
32240	0·423±0·0017	13·14	10·03	−0·09	−0·9
33100	0·439±0·0014	13·28	10·14	+0·02	+0·2
Mean			10·12		

$\frac{\Delta\lambda}{H}$ is constant, as is easily seen from the table (Plate XVII, Fig. 1).

The results of the former investigators are:

Investigator	Character of Separation	H	$\Delta\lambda$	$\frac{\Delta\lambda}{H}\times10^6$	$\frac{\Delta\lambda}{H\lambda^2}\times10^{13}$
Graftdijk	Triplet	26230	0·393	14·98	11·44
Reese	,,	28300	0·363	12·83	9·79
Kent	,,	32800	0·390	11·89	9·08

(2) λ: 3566·37.

Intensity 10; sharp triplet; character *b*. Plate I, Fig. 3; Plate III, Fig. 4. The determination of the fields is the same as in the case of λ:3619·39.

TABLE XXXIV.

H	$\Delta\lambda$	$\frac{\Delta\lambda}{H}\times10^6$	$\frac{\Delta\lambda}{H\lambda^2}\times10^{13}$	Deviation from the Mean	
				$\frac{\Delta\lambda}{H\lambda^2}\times10^{13}$	Deviation / Mean Value
18080	0·251±0·0012	13·98	10·91	+0·99	+10·0
19090	0·255±0·0011	13·36	10·50	+0·58	+ 5·9
22380	0·282±0·0016	12·60	9·91	−0·01	− 0·1
23100	0·290±0·0032	12·56	9·88	−0·04	− 0·4
26440	0·331±0·0022	12·51	9·84	−0·08	− 0·9
27750	0·339±0·0014	12·22	9·60	−0·32	− 3·2
27920	0·350±0·0009	12·54	9·87	−0·05	− 0·5
28700	0·347±0·0030	12·10	9·52	−0·40	− 4·0
29200	0·364±0·0012	12·34	9·70	−0·22	− 2·2
29400	0·362±0·0019	12·31	9·68	−0·24	− 2·4
30080	0·374±0·0029	12·43	9·78	−0·14	− 1·4
32100	0·402±0·0022	12·53	9·86	−0·06	− 0·6
32240	0·408±0·0026	12·67	9·96	+0·04	+ 0·4
33100	0·413±0·0024	12·48	9·81	−0·11	− 1·1
Mean			9·92		

$\frac{\Delta\lambda}{H}$ may be supposed to be constant when H is greater than 22380; but when it is smaller than this, $\frac{\Delta\lambda}{H}$ is larger than that in stronger fields. Plate XVII, Fig. 2. The results of the former investigations are:

Investigator	Character of Separation	H	$\Delta\lambda$	$\frac{\Delta\lambda}{H}\times10^6$	$\frac{\Delta\lambda}{H\lambda^2}\times10^{13}$
Graftdijk	Triplet	26230	0·356	13·57	10·67
Reese	,,	28300	0·338	11·95	9·40
Kent	,,	32800	0·318	9·54	7·50

(3) λ:3524·53.

Intensity 15; somewhat diffuse triplet; character *b*. Plate I, Fig. 2 and Plate I, Fig. 3; Plate III, Fig. 2; Plate IV, Fig. 1.

The determination of the fields is the same as in the case of λ:3619·39.

Table XXXV.

H	$\Delta\lambda$	$\frac{\Delta\lambda}{H}\times10^6$	$\frac{\Delta\lambda}{H\lambda^2}\times10^{13}$	Deviation from the Mean	
				$\frac{\Delta\lambda}{H\lambda^2}\times10^{13}$	$\frac{\text{Deviation}}{\text{Mean Value}}$
18080	0·262±0·0009	14·49	11·65	+0·62	+5·6
19090	0·269±0·0010	14·09	11·32	+0·29	+2·6
20340	0·285±0·0009	14·02	11·27	+0·24	+2.2
20600	0·289±0·0011	14·03	11·28	+0·25	+2·3
22380	0·311±0·0011	13·88	11·17	+0·14	+1·3
23100	0·321±0·0006	13·90	11·17	+0 14	+1·3
*23580	0·328±0·0006	13·90	11·17	+0·14	+1·3
25800	0·346±0·0010	13·40	10·77	–0·26	–2·4
26210	0·355±0·0008	13·55	10·90	–0·13	–1·2
26440	0·363±0·0011	13·73	11·05	+0·02	+0·2
27750	0·377±0·0014	13·59	10·94	–0·09	–0·8
27920	0·384±0·0010	13·76	10·06	+0·03	+0·3
27940	0·383±0·0008	13·71	11·03	0·00	0·0
28700	0·386±0·0008	13·45	10·82	–0·21	–1·9
29200	0·393±0·0009	13·45	10·82	–0·21	–1·9
29400	0·398±0·0010	13·47	10·83	–0·20	–1·8
29600	0·398±0·0010	13·44	10·82	–0·21	–1·9
30000	0·409±0·0013	13·63	10·96	–0·07	–0·6
30080	0·411±0·0006	13·67	10·99	–0·04	–0·4
30100	0·410±0·0010	13·63	10·96	–0·07	–0·6
30350	0·413±0·0016	13·63	10·96	–0·07	–0·6
31100	0·424±0·0013	13·65	10·98	–0·05	–0·5
31320	0·430±0·0008	13·73	11·01	–0·02	–0·2
31720	0·433±0·0012	13·66	10·99	–0·04	–0·4
32100	0·441±0·0011	13·75	11·05	+0·02	+0·2
32240	0·437±0·0016	13·56	10·91	–0·12	–1·1
33100	0·449±0·0015	13·58	10·93	–0·10	–0·9
Mean			11·03		

In weaker fields the variation of $\Delta\lambda$ is more or less irregular, but in strong fields it is regular and $\Delta\lambda$ is proportional to H. Plate XVII, Fig. 3. The results of former investigations at single fields are:

Investigator	Character of Separation	H	$\Delta\lambda$	$\frac{\Delta\lambda}{H}\times10^6$	$\frac{\Delta\lambda}{H\lambda^2}\times10^{13}$
Graftdijk	Triplet	26230	0·409	15·60	12·55
Reese	,,	28300	0·391	13·82	11·12
Kent	,,	32800	0·410	12·50	10·05

(4) λ: 3515·06.

Intensity 10; very sharp triplet; character *b*. Plate I, Fig. 2

and Plate I. Fig. 3. The determination of the fields is the same as in the case of λ: 3524·53.

TABLE XXXVI.

H	$\Delta\lambda$	$\frac{\Delta\lambda}{H}\times 10^6$	$\frac{\Delta\lambda}{H\lambda^2}\times 10^{13}$	Deviation from the Mean: $\frac{\Delta\lambda}{H\lambda^2}\times 10^{13}$	Deviation from the Mean: $\frac{\text{Deviation}}{\text{Mean Value}}$
18080	0·226±0·0002	12·50	10·12	+0·45	+4·7
19090	0·234±0·0017	12·26	9·91	+0·24	+2·5
20340	0·245±0·0011	12·05	9·75	+0·08	+0·8
20600	0·245±0·0008	11·89	9·62	−0·05	−0·5
22380	0·268±0·0005	11·98	9·70	+0·03	+0·3
23100	0·279±0·0007	12·08	9·78	+0·12	+1·2
23580	0·276±0·0004	11·71	9·48	−0·19	−2·0
25800	0·305±0·0006	11·82	9·57	−0·10	−1·0
26210	0·311±0·0008	11·86	9·60	−0·07	−0·7
27750	0·327±0·0012	11·78	9·54	−0·13	−1·3
27920	0·337±0·0010	12·08	9·77	+0·10	+1·0
27940	0·333±0·0009	11·92	9·65	−0·02	−0·2
28700	0·337±0·0015	11·74	9·50	−0·17	−1·8
29200	0·348±0·0010	11·92	9·65	−0·02	−0·2
29400	0·353±0·0020	12·01	9·72	+0·05	+0·5
29600	0·356±0·0008	12·02	9·73	+0·06	+0·6
30000	0·359±0·0014	11·97	9·69	+0·02	+0·2
30080	0·358±0·0010	11·90	9·63	−0·04	−0·4
30100	0·357±0·0009	11·86	9·60	−0·07	−0·7
31320	0·373±0·0021	11·91	9·64	−0·03	−0·3
31720	0·380±0·0012	11·99	9·70	+0·03	+0·3
32100	0·379±0·0015	11·80	9·55	−0·12	−1·2
32240	0·385±0·0021	11·93	9·66	−0·01	−0·1
33100	0·390±0·0023	11·78	9·54	−0·13	−1·3
Mean			9·67		

$\frac{\Delta\lambda}{H}$ is nearly constant as in all cases of sharp triplets. Plate XVIII. Fig. 1. The results of already published investigations are:

Investigator	Character of Separation	H	$\Delta\lambda$	$\frac{\Delta\lambda}{H}\times 10^6$	$\frac{\Delta\lambda}{H\lambda^2}\times 10^{13}$
Graftdijk	Triplet	26230	0·346	13·19	10·68
Reese	,,	28300	0·324	11·45	9·26
Kent	,,	32800	0·339	10·33	8·36

(5) λ: 3492·96.

Intensity 10; diffuse triplet; character *b*. Plate I, Fig. 2 and Plate I, Fig. 3; Plate III, Fig. 2; Plate IV, Fig. 1. The determination of the fields is the same as in the case of λ: 3524·53.

TABLE XXXVII.

H	$\Delta\lambda$	$\frac{\Delta\lambda}{H}\times10^6$	$\frac{\Delta\lambda}{H\lambda^2}\times10^{13}$	Deviation from the Mean	
				$\frac{\Delta\lambda}{H\lambda^2}\times10^{13}$	$\frac{\text{Deviation}}{\text{Mean Value}}$
18080	0·218±0·0021	12·05	9·88	+0·29	+3·0
20340	0·233±0·0010	11·47	9·40	−0·19	−2·0
20600	0·230±0·0015	11·16	9·15	−0·44	−4·6
22380	0·263±0·0017	11·75	9·63	+0·04	+0·4
23100	0·276±0·0014	11·95	9·80	+0·21	+2·2
*23580	0·270±0·0016	11·45	9·39	−0·20	−2·1
25800	0·297±0·0007	11·51	9·43	−0·16	−1·7
26210	0·301±0·0015	11·48	9·41	−0·18	−1·9
26440	0·312±0·0023	11·79	9·67	+0·08	+0·8
27750	0·328±0·0049	11·82	9·69	+0·10	+1·0
27920	0·334±0·0018	11·97	9·82	+0·23	+2·4
27940	0·328±0·0015	11·74	9·63	+0·04	+0·4
28700	0·339±0.0026	11·81	9·68	+0·09	+0·9
29200	0·345±0·0014	11·82	9·69	+0·10	+1·0
29400	0·344±0·0019	11·70	9·59	0·00	0·0
29600	0·339±0·0011	11·45	9·39	−0·20	−2·1
30000	0·358±0·0016	11·93	9·79	+0·20	+2·1
30100	0·347±0·0012	11·53	9·45	−0·14	−1·5
31320	0·369±0·0012	11·78	9·66	+0·07	+0·7
31720	0·372±0·0026	11·73	9·62	+0·03	+0·3
32100	0·380±0·0023	11·84	9·71	+0·12	+1·3
32240	0·380±0·0028	11·78	9·66	+0·07	+0·7
33100	0·382±0·0021	11·54	9·46	−0·13	−1·4
Mean			9·59		

$\Delta\lambda$ can not be said to be proportional to H. Plate XVIII, Fig. 2. The results of former investigations are:

Investigator	Character of Separation.	H	$\Delta\lambda$	$\frac{\Delta\lambda}{H}\times10^6$	$\frac{\Delta\lambda}{H\lambda^2}\times10^{13}$
Reese	Triplet	28300	0·301	10·64	8·72
Kent	"	32800	0·311	9·47	7·77

(6) λ: 3461·65.

Intensity 20; very sharp triplet; character *b*. Plate I, Fig. 2 and Plate I, Fig. 3; Plate III, Fig. 2; Plate IV, Fig. 1. The determination of the fields is the same as in the case of λ: 3619·39.

TABLE XXXVIII.

H	$\Delta\lambda$	$\frac{\Delta\lambda}{H}\times10^6$	$\frac{\Delta\lambda}{H\lambda^2}\times10^{13}$	Deviation from the Mean	
				$\frac{\Delta\lambda}{H\lambda^2}\times10^{13}$	Deviation / Mean Value
18080	0·257±0·0017	14·21	11·85	+0·50	+4·4
19090	0·262±0·0014	13·72	11·45	+0·10	+0·9
20340	0·277±0·0009	13·62	11·36	+0·01	+0·1
20600	0·281±0·0008	13·64	11·38	+0·03	+0·3
22380	0·304±0·0009	13·58	11·33	−0·02	−0·2
23100	0·318±0·0015	13·77	11·48	+0·13	+1·1
25800	0·349±0·0009	13·53	11·28	−0·07	−0·6
26210	0·355±0·0007	13·54	11·29	−0·06	−0·5
26440	0·351±0·0011	13·28	11·08	−0·27	−2·4
27750	0·371±0·0019	13·37	11·15	−0·20	−1·8
27920	0·381±0·0008	13·65	11·38	+0·03	+0·3
27940	0·382±0·0007	13·68	11·42	+0·07	+0·6
28700	0·385±0·0016	13·42	11·19	−0·16	−1·4
29200	0·397±0·0014	13·60	11·34	−0·01	−0·1
29400	0·392±0·0014	13·34	11·12	−0·23	−2·0
29600	0·404±0·0008	13·67	11·40	+0·05	+0·4
30000	0·406±0·0015	13·53	11·28	−0·07	−0·6
30080	0·410±0·0011	13·63	11·37	+0·02	+0·2
30100	0·411±0·0011	13·65	11·38	+0·03	+0·3
30350	0·417±0·0019	13·57	11·32	−0·03	−0·3
31320	0·424±0·0012	13·53	11·28	−0·07	−0·6
32100	0·440±0·0016	13·71	11·44	+0·09	+0·8
32240	0·441±0·0017	13·68	11·42	+0·07	+0·6
33100	0·453±0·0019	13·70	11·44	+0·09	+0·8
Mean			11·35		

$\Delta\lambda$ is linearly proportional to the fields applied as in all other cases of sharp triplets. Plate XVIII, Fig. 3. The results of former investigators are:

Investigator	Character of Separation	H	$\Delta\lambda$	$\frac{\Delta\lambda}{H}\times10^6$	$\frac{\Delta\lambda}{H\lambda^2}\times10^{13}$
Graftdijk	Triplet	26230	0·386	14·71	12·28
Reese	,,	28300	0·377	13·32	11·11
Kent	,,	32800	0·397	12·10	10·10

(7) λ: 3458·45.

Intensity 10; diffuse triplet; character *b*. Plate I, Fig. 2; Plate III, Fig. 2. The determination of the fields is the same as in the case of λ: 3619·39.

TABLE XXXIX.

H	$\Delta\lambda$	$\frac{\Delta\lambda}{H}\times10^6$	$\frac{\Delta\lambda}{H\lambda^2}\times10^{13}$	Deviation from the Mean	
				$\frac{\Delta\lambda}{H\lambda^2}\times10^{13}$	Deviation / Mean Value
20340	0·194±0·0013	9·54	7·98	−0·74	−8·5
22380	0·235±0·0028	10·48	8·77	+0·05	+0·6
25800	0·271±0·0013	10·48	8·77	+0·05	+0·6
26210	0·274±0·0009	10·45	8·74	+0·02	+0·2
27940	0·295±0·0013	10·65	8·91	+0·19	+2·2
28700	0·289±0·0020	10·16	8·43	−0·29	−3·3
29200	0·311±0·0015	10·63	8·89	+0·17	+1·9
29600	0·313±0·0014	10·58	8·86	+0·14	+1·6
30000	0·318±0·0013	10·60	8·86	+0·14	+1·6
30080	0·322±0·0027	10·45	8·74	+0·02	+0·2
30100	0·318±0·0014	10·58	8·86	+0·14	+1·6
33100	0·350±0·0025	10·56	8·84	+0·12	+1·4
Mean			8·72		

Deviations from the mean are too large to take $\frac{\Delta\lambda}{H}$ for a constant; but similar features are seen in the cases of diffuse triplets (Plate XIX, Fig. 1). The results of former investigators are:

Investigator	Character of Separaction	H	$\Delta\lambda$	$\frac{\Delta\lambda}{H}\times10^6$	$\frac{\Delta\lambda}{H\lambda^2}\times10^{13}$
Graftdijk	Triplet	26230	0·30	11·44	9·6
Reese	,,	28300	0·292	10·32	8·63

(8) λ: 3446·27.

Intensity 9; diffuse triplet, especially the central component is broad; character *b*. Plate I, Fig. 2; Plate III, Fig. 2. The determination of the fields is the same as in the case of λ: 3524·53.

Table XI.

H	$\Delta\lambda$	$\frac{\Delta\lambda}{H}\times10^6$	$\frac{\Delta\lambda}{H\lambda^2}\times10^{13}$	Deviation from the Mean	
				$\frac{\Delta\lambda}{H\lambda^2}\times10^{13}$	$\frac{\text{Deviation}}{\text{Mean Value}}$
20340	0·247±0·0012	12·15	10·24	−0·09	−0·9
20600	0·247±0·0011	11·97	10·09	−0·24	−2·3
*24700	0·309±0·0015	12·52	10·55	+0·22	+2·1
25800	0·322±0·0013	12·48	10·52	+0·19	+1·8
26210	0·321±0·0010	12·25	10·33	0·00	0·0
27940	0·347±0·0006	12·41	10·45	+0·12	+1·3
29600	0·358±0·0007	12·11	10·20	−0·13	−1·3
30000	0·367±0·0012	12·22	10·30	−0·03	−0·3
30100	0·369±0·0012	12·23	10·31	−0·02	−0·2
Mean			10·33		

$\Delta\lambda$ is nearly proportional to H. Plate XIX, Fig. 3. The results of former investigations are:

Investigator	Character of Separation	H	$\Delta\lambda$	$\frac{\Delta\lambda}{H}\times10^6$	$\frac{\Delta\lambda}{H\lambda^2}\times10^{13}$
Graftdijk	Triplet	26230	0·350	13·34	11·24
Reese	,,	28300	0·350	12·37	10·43
Kent	,,	32800	0·350	10·67	8·99

(9) λ: 3414·82.

Intensity 10; sharp triplet; character *b*. Plate I, Fig. 2; Plate III, Fig. 3. The determination of the fields is the same as in the case of λ: 3524·53.

Table XLI.

H	$\Delta\lambda$	$\frac{\Delta\lambda}{H}\times10^6$	$\frac{\Delta\lambda}{H\lambda^2}\times10^{13}$	Deviation from the Mean	
				$\frac{\Delta\lambda}{H\lambda^2}\times10^{13}$	$\frac{\text{Deviation}}{\text{Mean Value}}$
20340	0·272±0·0010	13·38	11·48	+0·10	+0·9
20600	0·273±0·0007	13·26	11·38	0·00	0·0
*24700	0·331±0·0014	13·40	11·50	+0·12	+1·1
25800	0·339±0·0005	13·13	11·27	−0·11	−1·0
26210	0·344±0·0004	13·18	11·30	−0·08	−0·7
27940	0·372±0·0005	13·32	11·43	+0·05	+0·4
29600	0·391±0·0005	13·21	11·34	−0·04	−0·4
30000	0·397±0·0010	13·23	11·36	−0·02	−0·2
30100	0·399±0·0004	13·26	11·38	0·00	0·0
30350	0·402±0·0007	13·23	11·36	−0·02	−0·2
31320	0·414±0·0006	13·22	11·35	−0·03	−0·3
Mean			11·38		

The variation of $\Delta\lambda$ is proportional to the fields applied as in all other cases of sharp triplets. Plate XIX, Fig. 2. The results of former investigators are:

Investigator	Character of Separation	H	$\Delta\lambda$	$\frac{\Delta\lambda}{H}\times10^6$	$\frac{\Delta\lambda}{H\lambda^2}\times10^{13}$
Graftdijk	Triplet	26230	0·39	14·88	12·76
Reese	,,	28300	0·374	13·22	11·34
Kent	,,	32800	0·406	12·37	10·61

(10) λ: 3392·97.

Intensity 7; triplet, but each component is somewhat wide; character *b*. Plate I, Fig. 2; Plate III, Fig. 3. The determination of the fields is the same as in the case of λ: 3619·39.

TABLE XLII.

H	$\Delta\lambda$	$\frac{\Delta\lambda}{H}\times10^6$	$\frac{\Delta\lambda}{H\lambda^2}\times10^{13}$	Deviation from the Mean	
				$\frac{\Delta\lambda}{H\lambda^2}\times10^{13}$	$\frac{\text{Deviation}}{\text{Mean Value}}$
20340	0·295±0·0020	14·53	12·62	+0·46	+3·8
20600	0·294±0·0009	14·27	12·40	+0·24	+2·0
25800	0·358±0·0012	13·88	12·05	−0·11	−0·9
26210	0·365±0·0008	13·92	12·10	−0·06	−0·5
27940	0·387±0·0009	13·85	12·03	−0·13	−1·1
29600	0·410±0·0012	13·85	12·03	−0·13	−1·1
30000	0·418±0·0016	13·93	12·11	−0·05	−0·4
30100	0·418±0·0008	13·88	12·05	−0·11	−0·9
30350	0·426±0·0023	14·03	12·19	+0·03	+0·2
31320	0·432±0·0011	13·80	11·99	−0·17	−1·4
Mean			12·16		

In weak fields, $\frac{\Delta\lambda}{H}$ is larger than that in the stronger. Results of former investigators are:

Investigator	Character of Separation	H	$\Delta\lambda$	$\frac{\Delta\lambda}{H}\times10^6$	$\frac{\Delta\lambda}{H\lambda^2}\times10^{13}$
Graftdijk	Triplet	26230	0·399	15·21	13·21
Kent	,,	32800	0·399	12·16	10·56

(11) λ: 3380·58.

Intensity 6; very sharp triplet, this is the sharpest line among the nickel lines which I have observed; character *b*. Plate I, Fig. 2; Plate III, Fig. 3. The determination of the fields is the same as in the case of λ: 3619·39.

TABLE XLIII.

H	$\Delta\lambda$	$\frac{\Delta\lambda}{H}\times10^6$	$\frac{\Delta\lambda}{H\lambda^2}\times10^{13}$	Deviation from the Mean	
				$\frac{\Delta\lambda}{H\lambda^2}\times10^{13}$	$\frac{\text{Deviation}}{\text{Mean Value}}$
20600	0·223±0·0009	10·82	9·47	−0·18	−1·9
25800	0·286±0·0010	11·08	9·69	+0·04	+0·4
26210	0·288±0·0008	10·99	9·61	−0·04	−0·4
27940	0·311±0·0013	11·13	9·73	+0·08	+0·8
29600	0·328±0·0012	11·08	9·69	+0·04	+0·4
30100	0·334±0·0009	11·09	9·70	+0·05	+0·5
31320	0·345±0·0015	11·02	9·63	−0·02	−0·2
Mean		11·02	9·65		

$\Delta\lambda$ is linearly proportional to the field applied, as in all other cases of sharp triplets. Plate XX, Fig. 1. Graftdijk is the only one who has studied this line; she obtained:

Character of Separation	H	$\Delta\lambda$	$\frac{\Delta\lambda}{H}\times10^6$	$\frac{\Delta\lambda}{H\lambda^2}\times10^{13}$
Triplet	26230	0·315	12·00	10·49

(12) λ: 3050·82.

Intensity 6; diffuse triplet: character *b*. Plate I, Fig. 4. The third order spectrum of this line is photographed on the same plate together with the second order of λ: 4680·138 of zinc.

TABLE XLIV.

H	$\Delta\lambda$	$\frac{\Delta\lambda}{H}\times10^6$	$\frac{\Delta\lambda}{H\lambda^2}\times10^{13}$
25150	0·233	9·28	9·97
25920	0·242	9·36	10·05
26420	0·244	9·24	9·93
28050	0·260	9·27	9·95
Mean		9·29	9·98

The data are too few for a discussion of the character. According to Graftdijk.

Character of Separation	H	Δλ	$\frac{\Delta\lambda}{H}\times10^6$	$\frac{\Delta\lambda}{H\lambda^2}\times10^{13}$
Triplet	26230	0·272	10·37	11·14

(13) Separations at single fields.

TABLE XLV.

Wave-length (I. U.)	Intensity	Character of Separation	H	Δλ	$\frac{\Delta\lambda}{H}\times10^6$	$\frac{\Delta\lambda}{H\lambda^2}\times10^{13}$	$\frac{\Delta\lambda}{H\lambda^2}\times10^{13}$ (Graftdijk)
3858·29	8	Triple	25020	⊥0·401 // 0·000	16·03 0·00	10·77 0·00	11·29 0·00
3807·14	8	Triple	25020	⊥0·209 // 0·000	11·96 0·00	8·26 0·00	16·07 0·00
3510·33	10	Triple	23580	⊥0·143 // 0·000	6·08 0·00	4·61 0·00	6·5 0·0
3483·78	4	Triple	23580	⊥0·217 // 0·000	9·18 0·00	7·57 0·00	7·60 0·00
3472·53	5	Triple	24700	⊥0·222 // 0·000	8·97 0·00	7·46 0·00	14·13 0·00
3452·88	5	Triple	24700	⊥0·357 // 0·000	14·46 0·00	12·14 0·00	12·50 0·00
3437·29	5	Triple	24700	⊥0·331 // 0·000	13·40 0·00	11·35 0·00	12·68 0·00
3433·58	6	Quadruple ?	24700	⊥0·235 // ...	9·52	8·07	12·49 4·98
3423·71	5	Unaffected	24700				+6·44 +0·00
3391·04	4	Triple	24700	⊥0·354 // 0·000	14·31 0·00	12·69 0·00	12·60 0·00
3380·58	6	Triple	24700	⊥0·296 // 0·000	11·57 0·00	10·12 0·00	10.51 0·00
3369·57	4	Triple	24700	⊥0·310 // 0·000	12·55 0·00	11·04 0·00	11·91 0·00

(14) (i) Among all nickel lines which I have studied in different magnetic fields, $\lambda\lambda$: 3619·39, 3515·06, 3461·65, 3414·82 and 3380·58 are sharp triplets and their separations are proportional to the fields applied, as in the case of all sharp triplets of iron lines. (ii) But the separations of the sharp triplet 3566·37 are linearly proportional to the fields applied at stronger fields, while $\frac{\Delta\lambda}{H}$ is larger in lower fields than in the higher. (iii) $\lambda\lambda$: 3524·53, 3492·96, 3458·45 and 3392·97 are diffuse triplets and their separations are not proportional to the fields, as we have already seen in the iron diffuse triplets. (iv) But 3446·27 is the only diffuse triplet whose separations are nearly proportional to the fields.

VI. The magnetic Separation of a *Zinc* Line.

λ: 3345·13.

λ: 3345·13 of zinc is a line of the first subordinate series of zinc, and its magnetic separation has already been studied by Miller[1] who obtained the result

$$\frac{\Delta\lambda}{H}\times 10^6=11{\cdot}27, \qquad \frac{\Delta\lambda}{H\lambda^2}\times 10^{13}=10{\cdot}11.$$

In the course of my investigation of iron and nickel lines in different magnetic fields, this line is also photographed on the plates, and I think it may be of interest to report the results of measurement. The fourth order of this line was photographed and the fields were measured with the aid of λ: 4680·138 (3rd order) on the same plate. Plate I, Fig. 2.

TABLE XLVI.

H	$\Delta\lambda$	$\frac{\Delta\lambda}{H}\times 10^6$	$\frac{\Delta\lambda}{H\lambda^2}\times 10^{13}$
20340	0·251±0·0016	12·34	11·04
20600	0·255±0·0011	12·37	11·06
25800	0·323±0·0004	12·52	11·19
26210	0·329±0·0019	12·55	11·22
27940	0·351±0·0023	12·56	11·23
29600	0·372±0·0015	12·57	11·23
30100	0·385±0·0012	12·79	11·43
30356	0·384±0·0022	12·65	11·31
31320	0·394±0·0021	12·58	11·24
Mean		12·54	11·22

1) Miller, Ann. d. Phys. (4), **24** (1907), pp. 105—136.

The discrepancy of the value obtained by Miller from that here given is about 10%, but the simultaneous and accurate determination of magnetic fields leaves no room for doubt as to the value here given. (Plate XX, Fig. 2).

Summary.

The object of the experiment was to investigate whether the separations of iron and nickel lines are proportional to the fields applied. For measuring the strength of the fields, the zinc line λ:4680·138 was adopted, as the separation of the line has been accurately determined by many observers. In groups "g", "h", "i" and "j", *this line was simultaneously photographed on the same plate with iron and nickel lines.* The simultaneous photography, however, limited the number of lines to be studied, consequently I was obliged to photograph this line before and after the photography of iron lines (groups "e" and "f").

Concave grating was used for obtaining spectra of the lines from the second to the fifth order.

The sharp triplets show separations of their outer two components proportional to the fields applied; but in the case of most of the diffuse triplets, these separations are not proportional to the fields; in some diffuse lines, $\frac{\Delta\lambda}{H}$ is constant in weak fields but becomes larger in the stronger; in other diffuse lines, the deviations from the mean are too large to assume $\frac{\Delta\lambda}{H}$ as constant. Arthur King observed these lines with larger grating and took these diffuse triplets for more complicated separated lines.

In the course of investigation, the range of spectra photographed was taken as wide as possible at single fields. The results thus obtained are compared with the published data.

λ:3345·13 of zinc is also measured in different fields. The separation of this line is linearly proportional to the fields.

In conclusion I express my sincere thanks to Professor Nagaoka, at whose suggestion I undertook this investigation and by whose kind guidance I have been able to carry out the experiment.

Published February 28th, 1921.

Fig. 1. (Group g) H = 34120

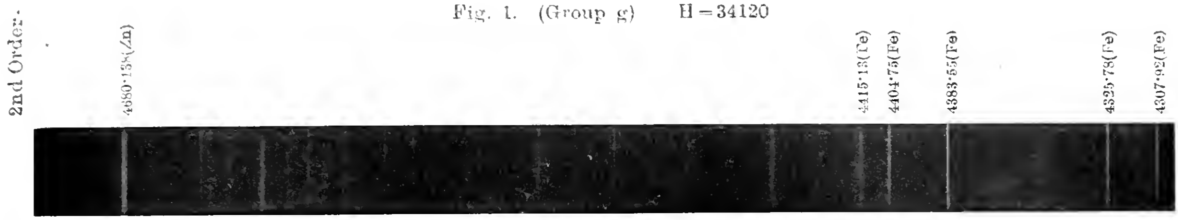


Fig. 2. (Group h) H = 30100

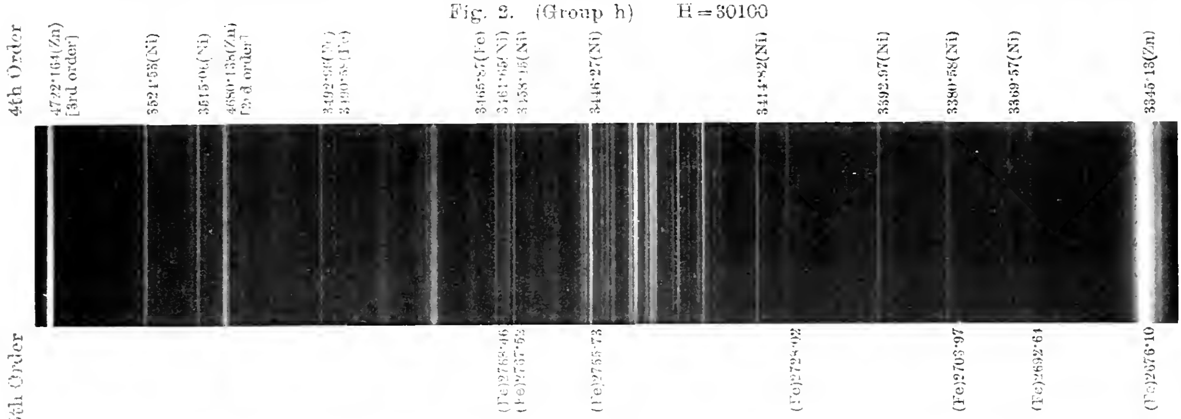


Fig. 3. (Group i) H = 23100

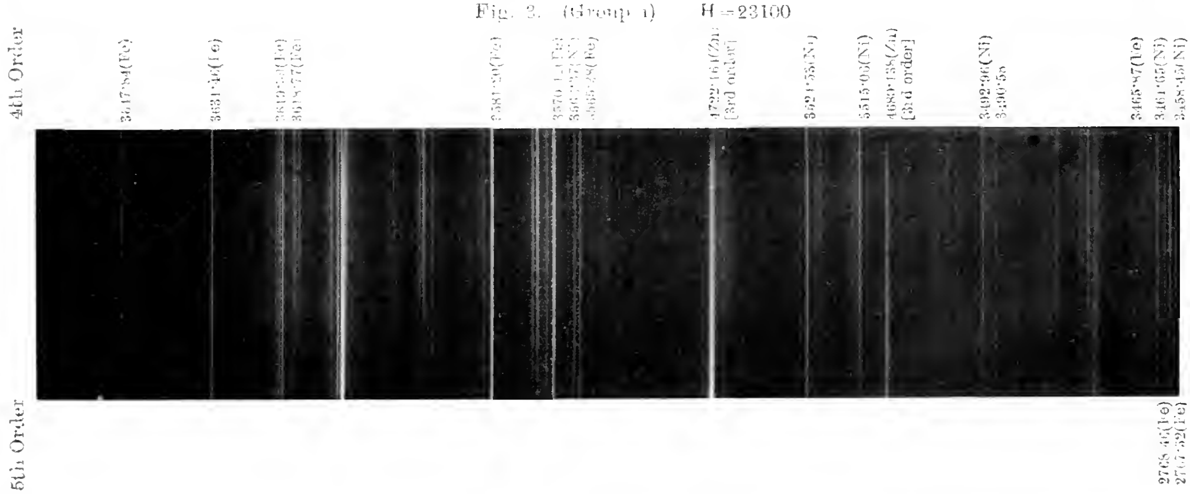
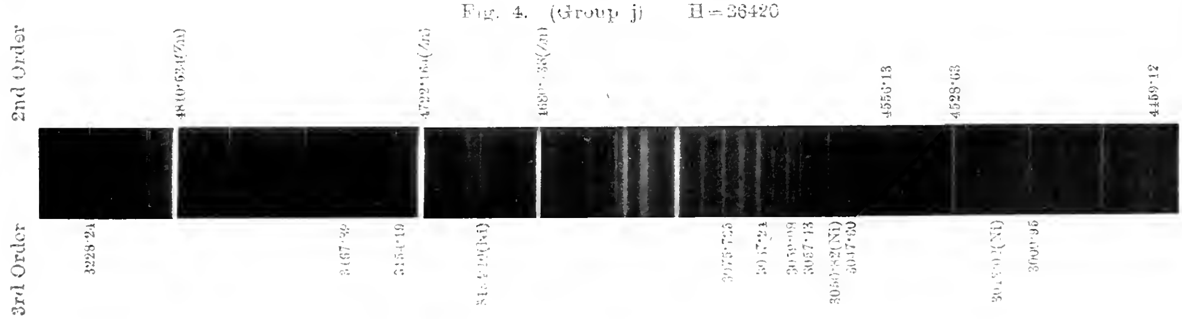


Fig. 4. (Group j) H = 36420



K. Yamada. Magnetic Separations of the Lines of Iron, Nickel and Zinc in Different Fields.

Fig. 1. (Group f) H=19300

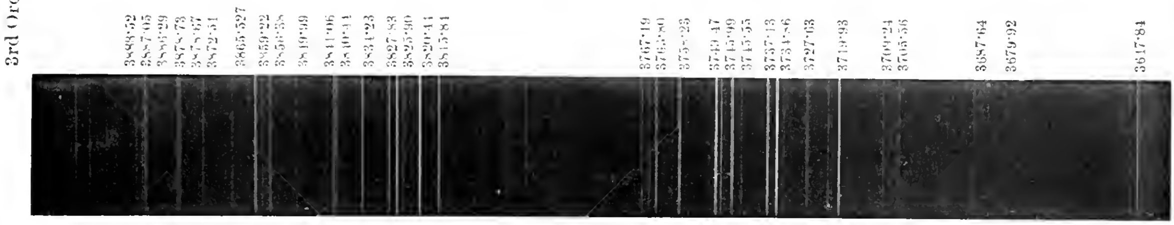


Fig. 2. (m 2) H=24560

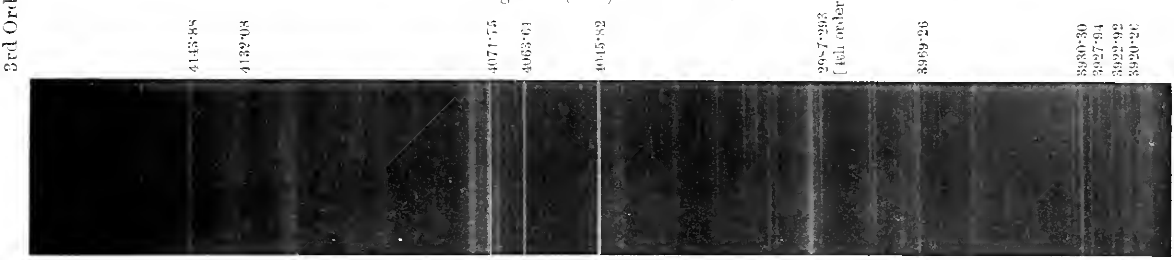


Fig. 3. (m 6) H=25540

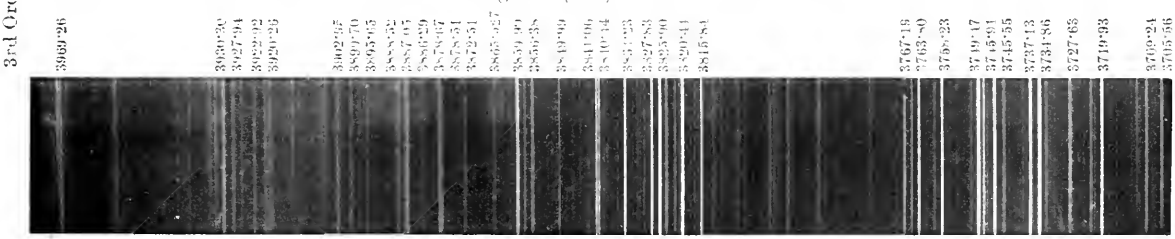


Fig. 4. (m 7) H=23580

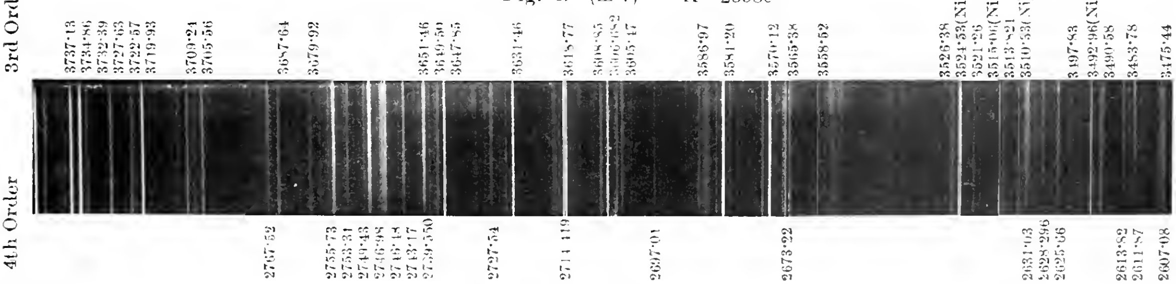
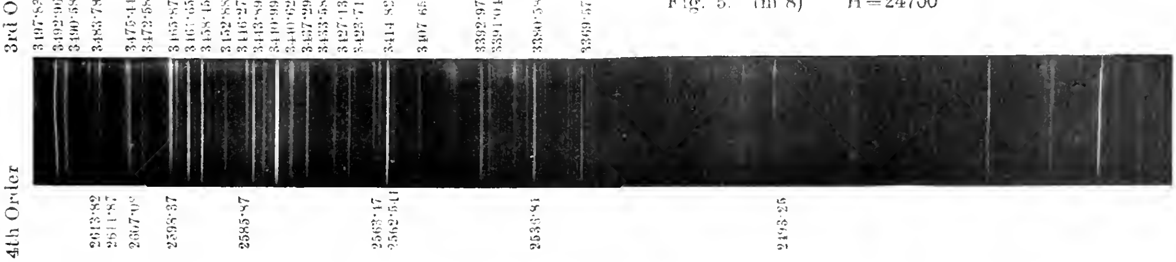


Fig. 5. (m 8) H=24700



K. Yamada. Magnetic Separations of the Lines of Iron, Nickel and Zinc in Differrent Fields.

Fig. 1 (Enlargement of Plate I, Fig. 1)

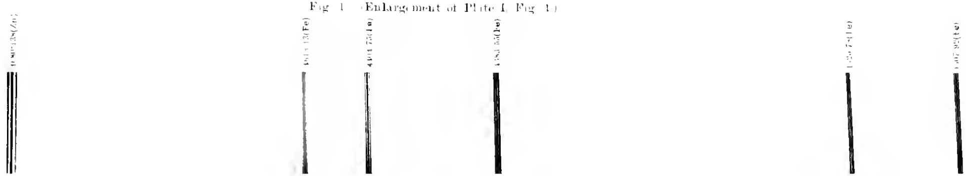


Fig. 2 (Enlargement of Plate I, Fig. 2)

Fig. 3 (Enlargement of Plate I, Fig. 2)

Fig. 4 (Enlargement of Plate I, Fig. ...)

S. Yamada: Magnetic Separations of the Lines of Iron, Nickel and Zinc in Different Fields.

Fig. 1 (Enlargement of Plate I, Fig. 3)

Fig. 2 (Enlargement of Plate I, Fig. 4)

Fig. 3 (Enlargement of Plate I, Fig. 4)

Fig. 4 (Enlargement of Plate II, Fig. 1)

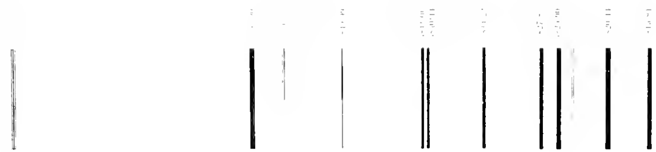
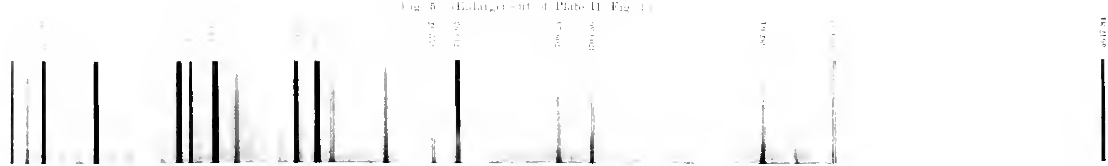


Fig. 5 (Enlargement of Plate II, Fig. 1)



K. Yamada: Magnetic Separations of the Lines of Iron, Nickel and Zinc in Different Fields.

Fig. 1. (Enlargement of Plate II, Fig. 2)

Fig. 2. (Enlargement of Plate II, Fig. 2)

Fig. 3. (Enlargement of Plate II, Fig. 3)

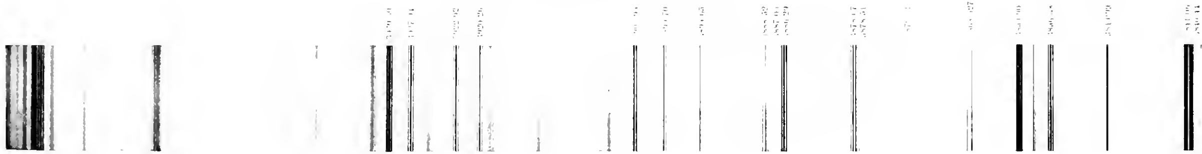


Fig. 4. (Enlargement of Plate II, Fig. 3)

Fig. 5. (Enlargement of Plate II, Fig. 4)

Fig. 1. (Enlargement of Plate II, Fig. 4)

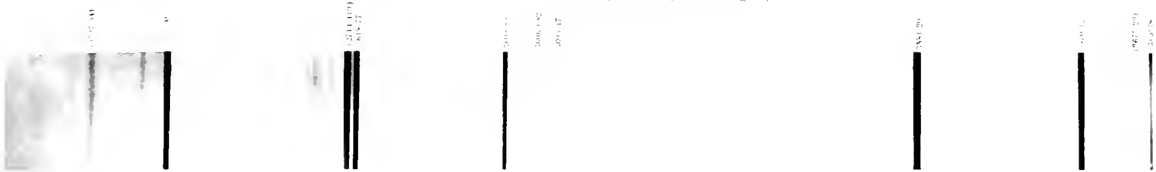


Fig. 2. (Enlargement of Plate II, Fig. 4)

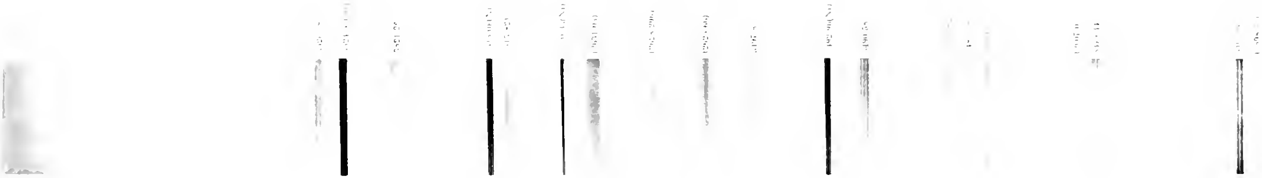


Fig. 3. (Enlargement of Plate II, Fig. 5)

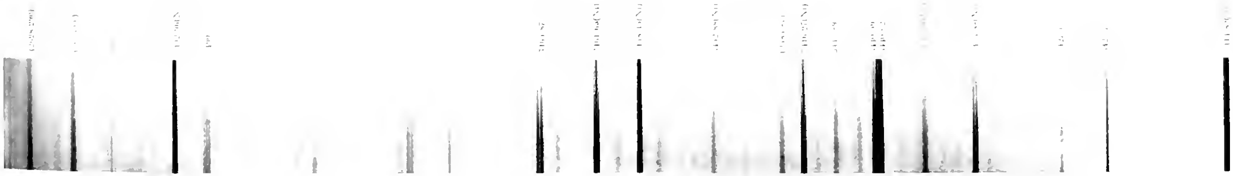
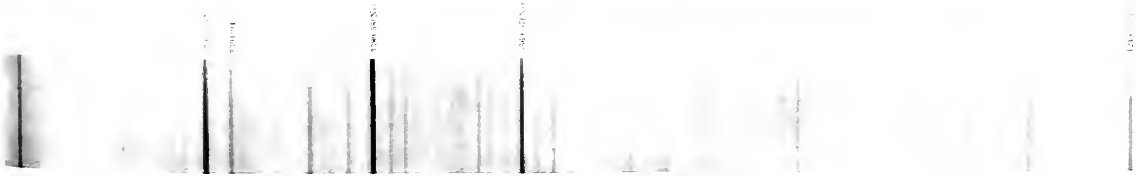


Fig. 4. (Enlargement of Plate II, Fig. 5)



K. Yamada: Magnetic Separations of the Lines of Iron, Nickel and Zinc in Different Fields.

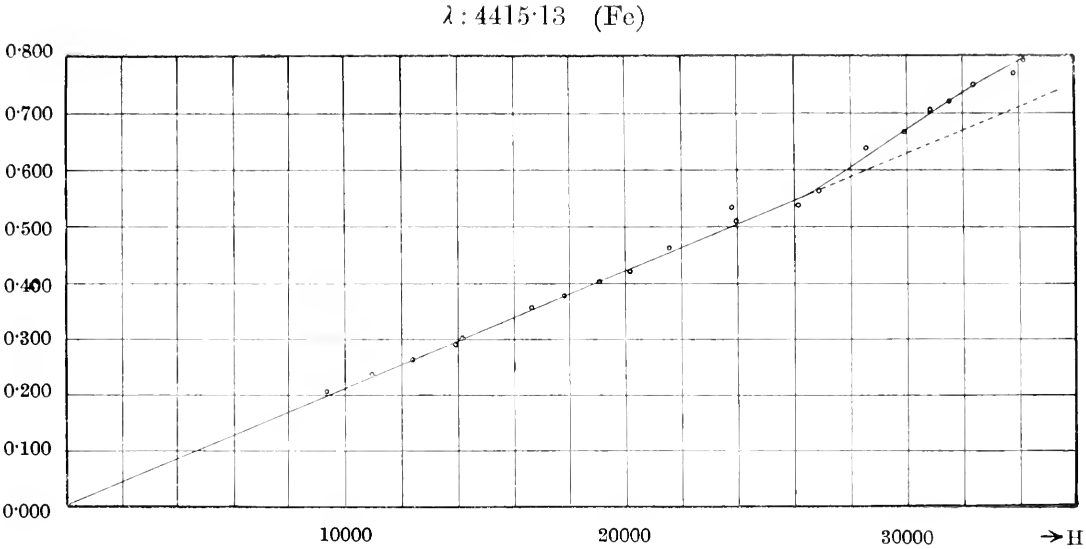


Fig. 1.

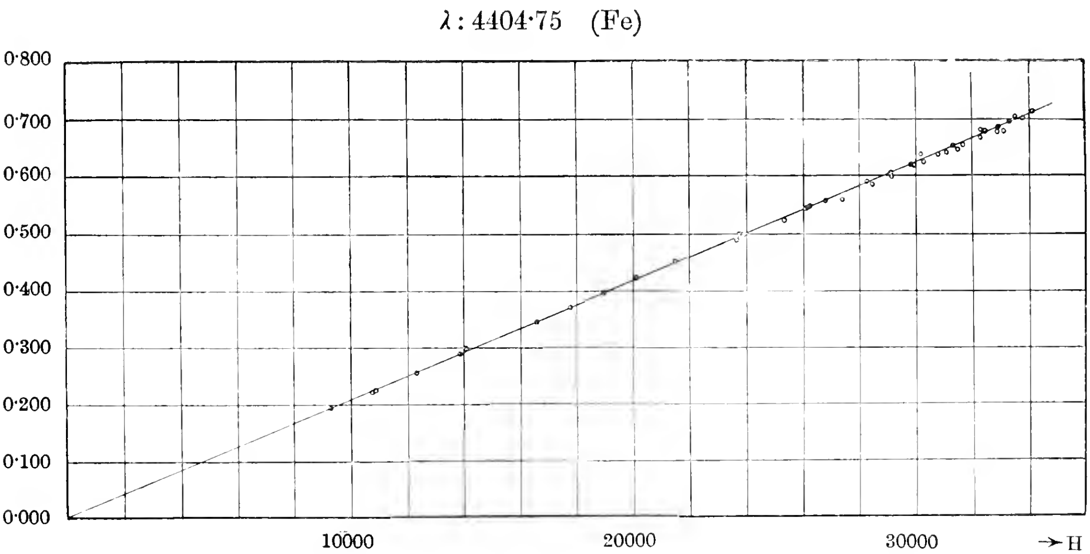


Fig. 2.

K. Yamada : Magnetic Separations of the Lines of Iron, Nickel and Zinc in Different Fields.

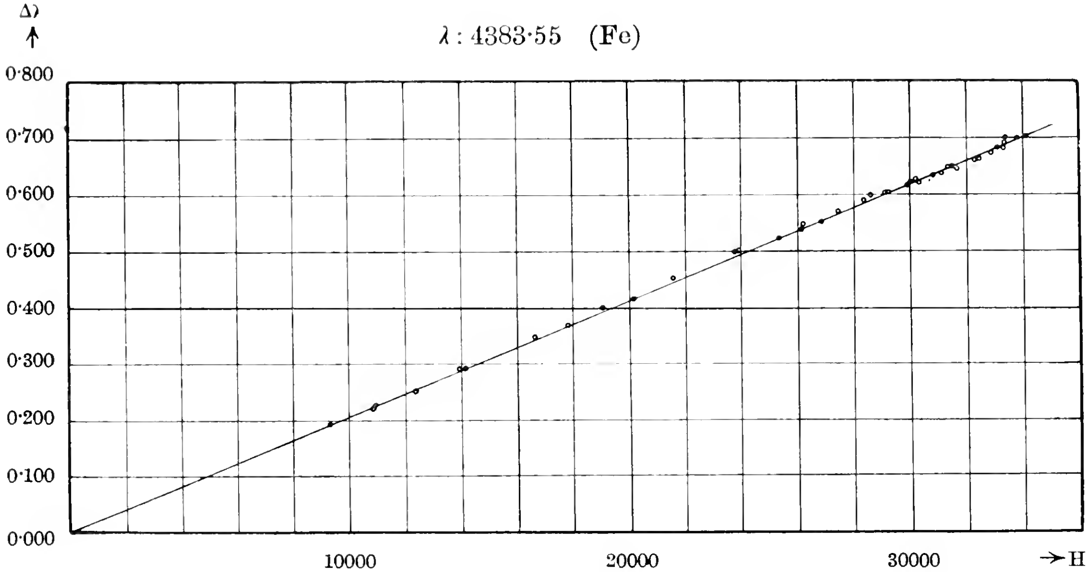


Fig. 1.

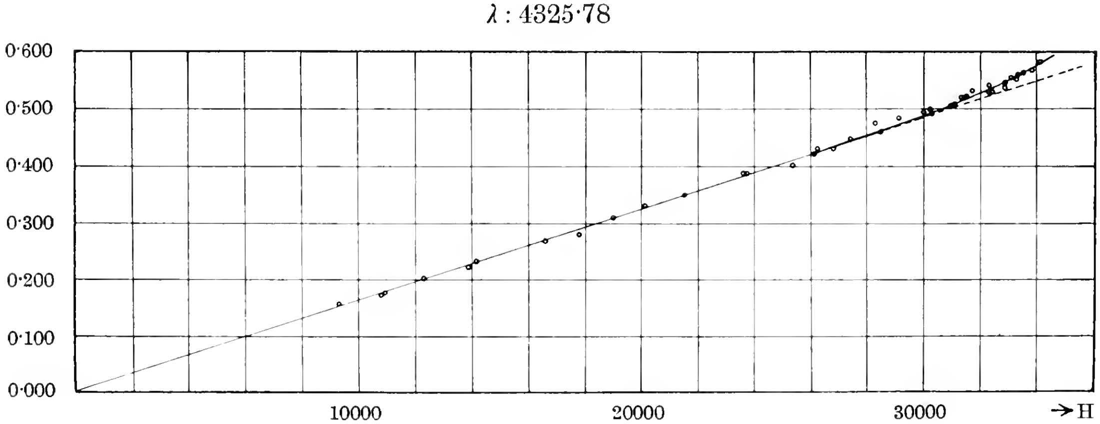


Fig. 2.

K. Yamada : Magnetic Separations of the Lines of Iron, Nickel and Zinc in Different Fields.

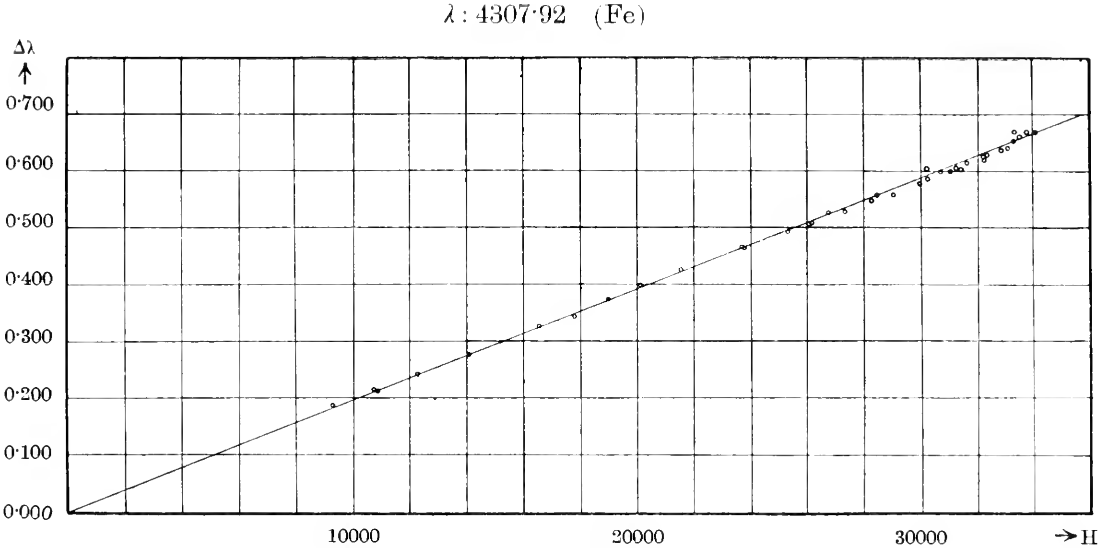

λ : 4307·92 (Fe)
Δλ
0·700
0·600
0·500
0·400
0·300
0·200
0·100
0·000
10000
20000
30000
→H

Fig. 1.

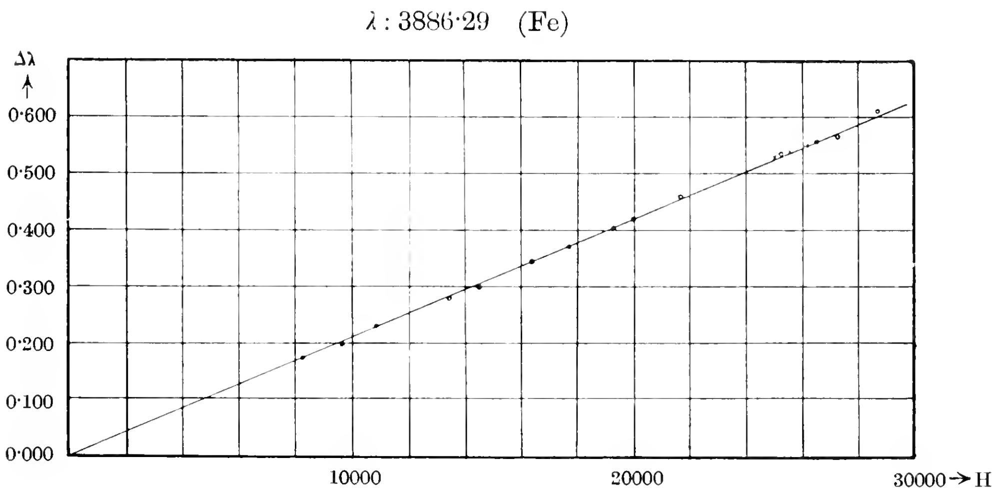

λ : 3886·29 (Fe)
Δλ
0·600
0·500
0·400
0·300
0·200
0·100
0·000
10000
20000
30000→H

Fig. 2.

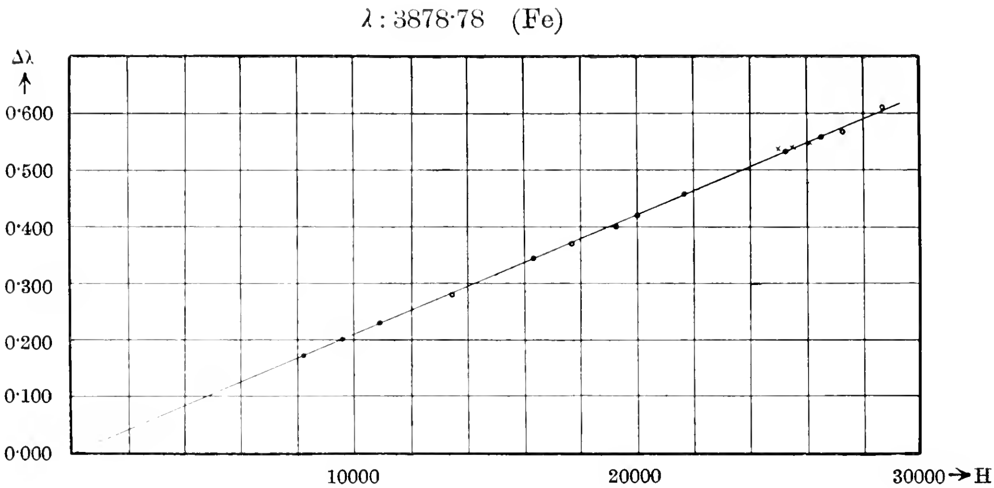

λ : 3878·78 (Fe)
Δλ
0·600
0·500
0·400
0·300
0·200
0·100
0·000
10000
20000
30000→H

Fig. 1.

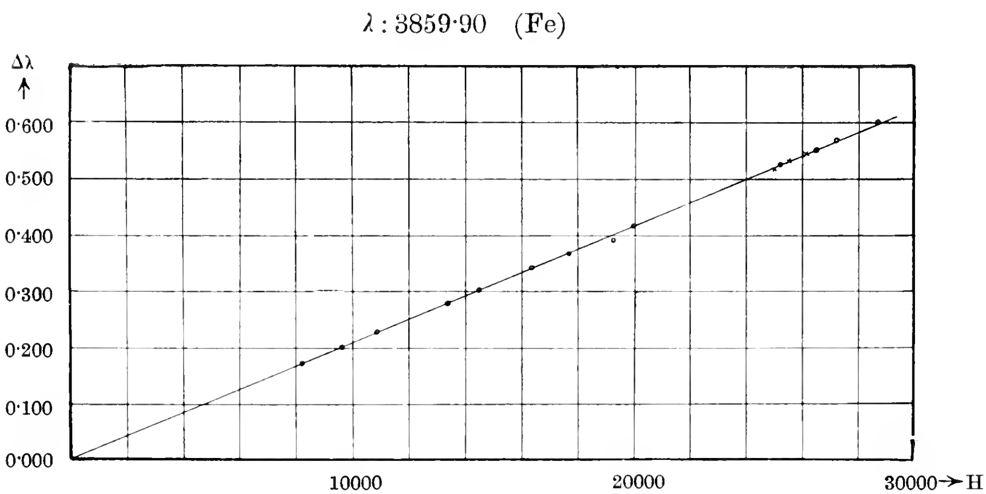

λ : 3859·90 (Fe)
Δλ
0·600
0·500
0·400
0·300
0·200
0·100
0·000
10000
20000
30000→H

Fig. 2.

λ : 3856·38 (Fe)

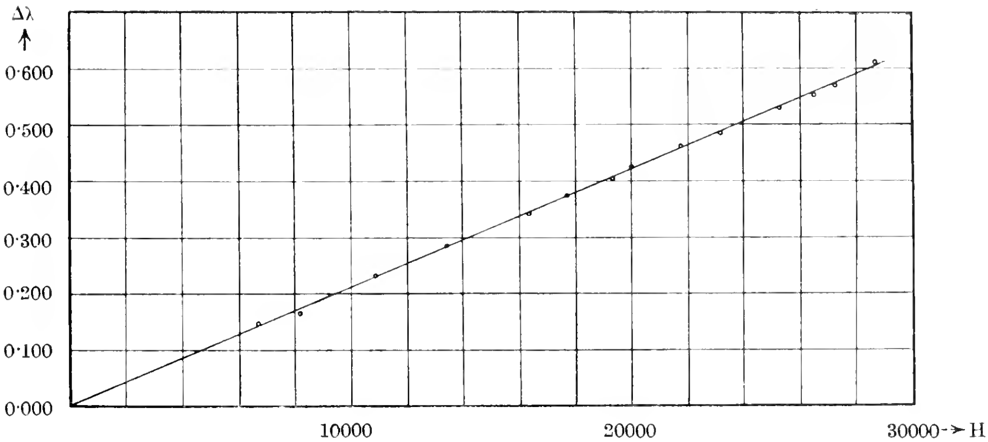


Fig. 1.

λ : 3827·83 (Fe)

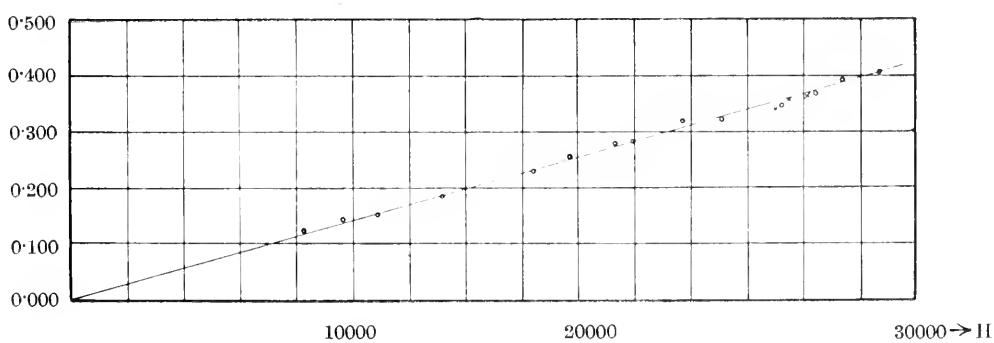


Fig. 2.

λ : 3825·90 (Fe)

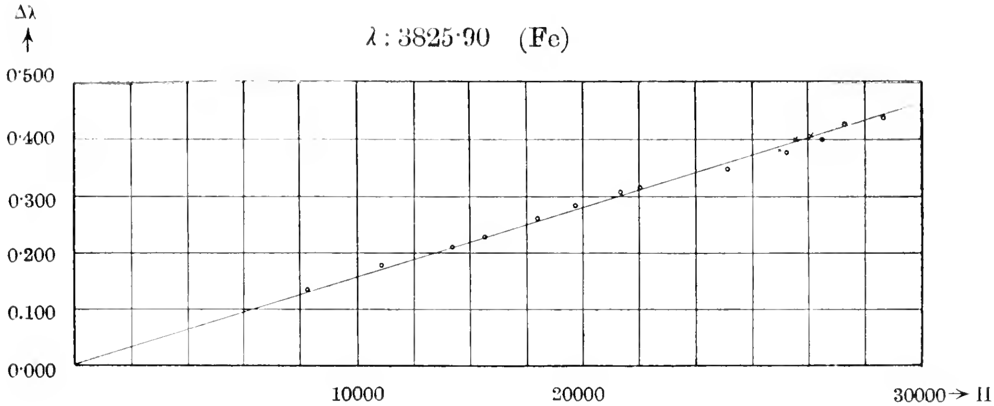


Fig. 1.

λ : 3820·44 (Fe)

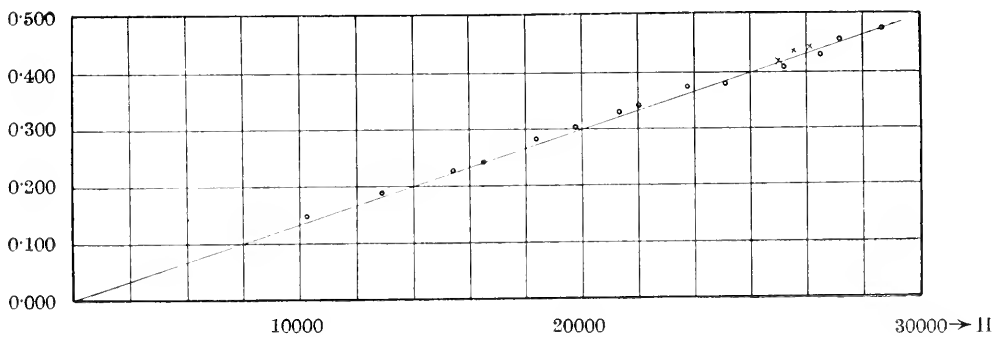


Fig. 2.

λ : 3815·84 (Fe)

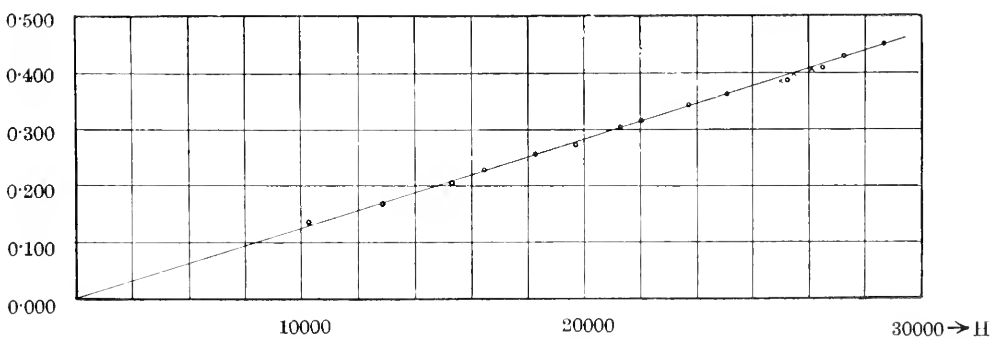


Fig. 3.

K. Yamada : Magnetic Separations of the Lines of Iron, Nickel and Zinc in Different Fields.

λ : 3763·80 (Fe)

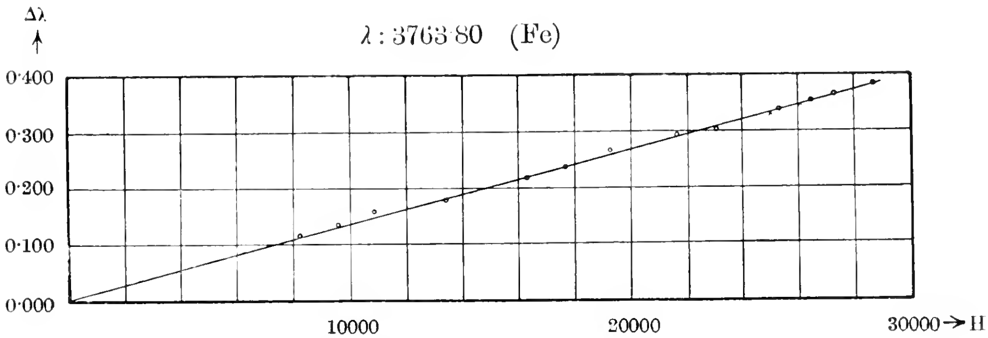


Fig. 1.

λ : 3758·23 (Fe)

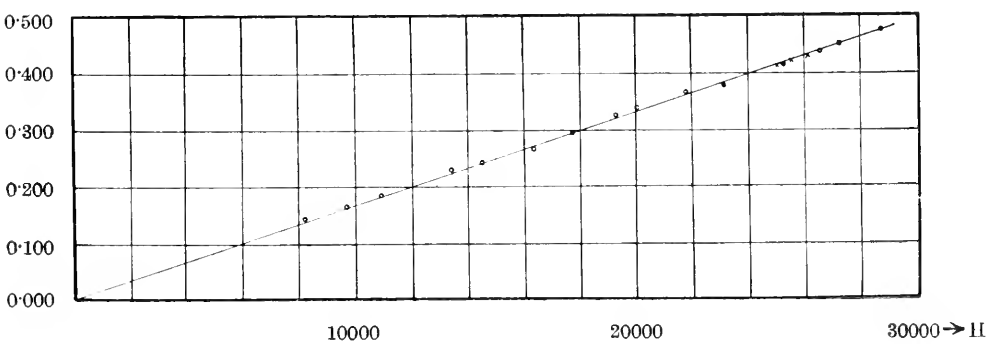


Fig. 2.

λ : 3749·47 (Fe)

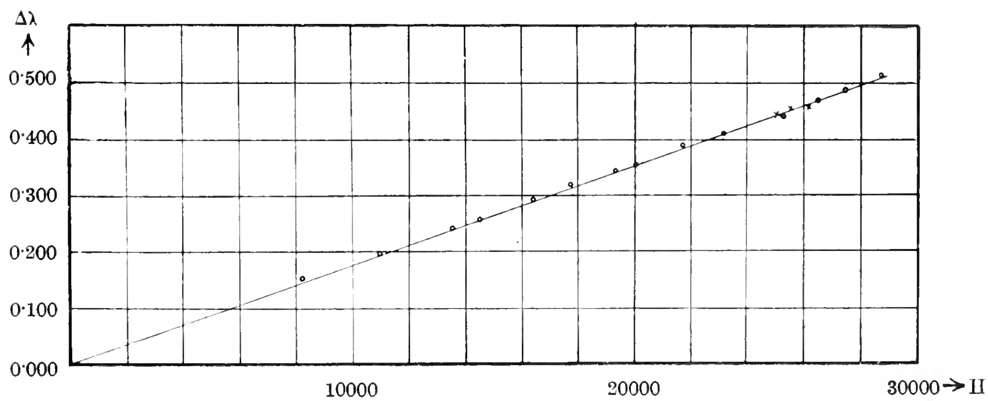


Fig. 3.

K. Yamada : Magnetic Separations of the Lines of Iron, Nickel and Zinc in Different Fields.

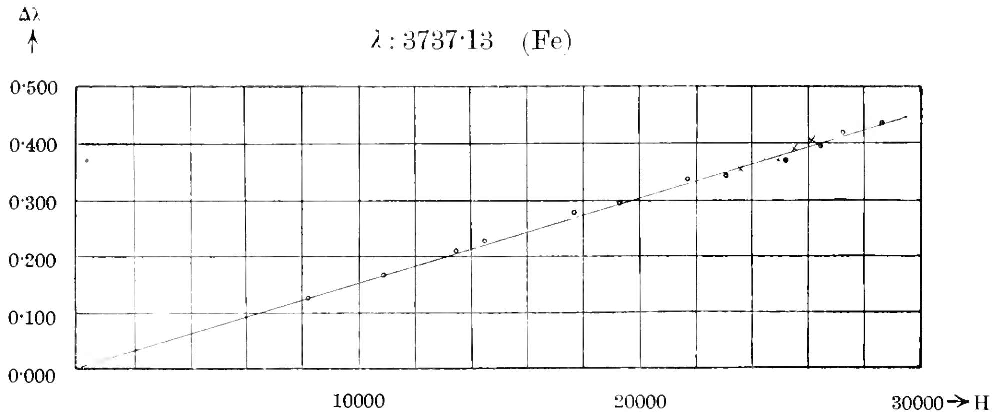

Δλ
λ : 3737·13 (Fe)
0·500
0·400
0·300
0·200
0·100
0·000
10000
20000
30000 → H

Fig. 1.

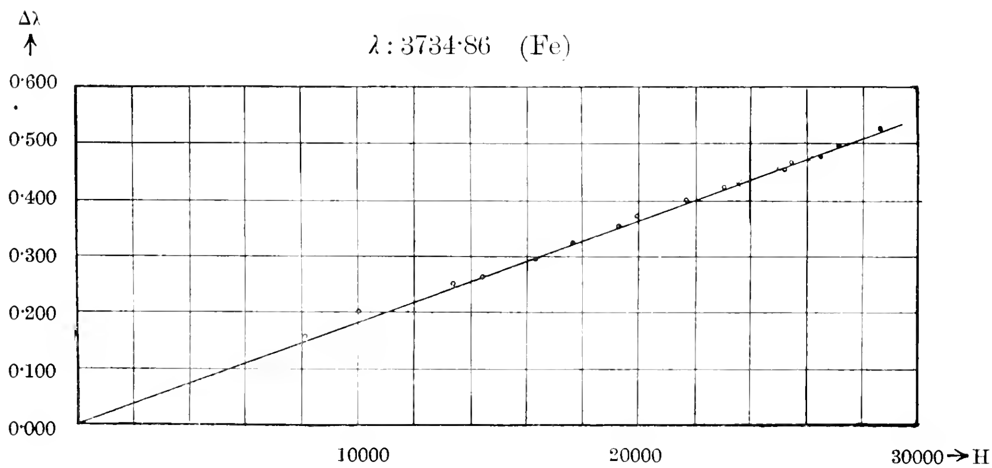

Δλ
λ : 3734·86 (Fe)
0·600
0·500
0·400
0·300
0·200
0·100
0·000
10000
20000
30000 → H

Fig. 2.

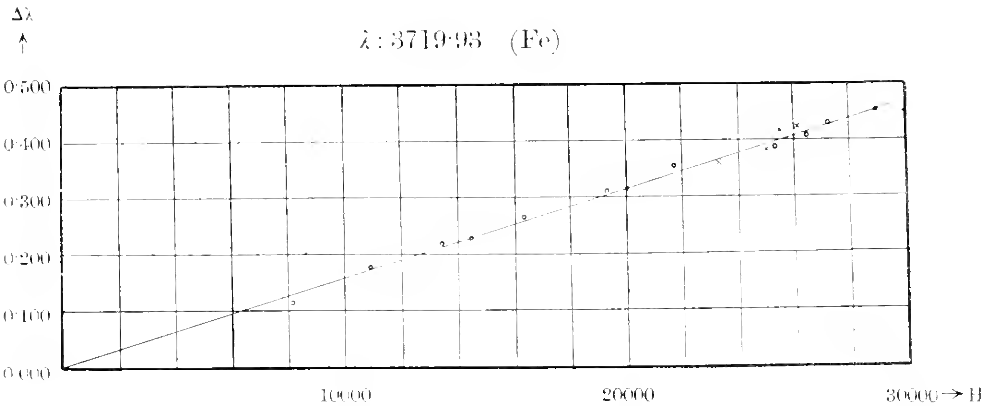


Fig. 1.

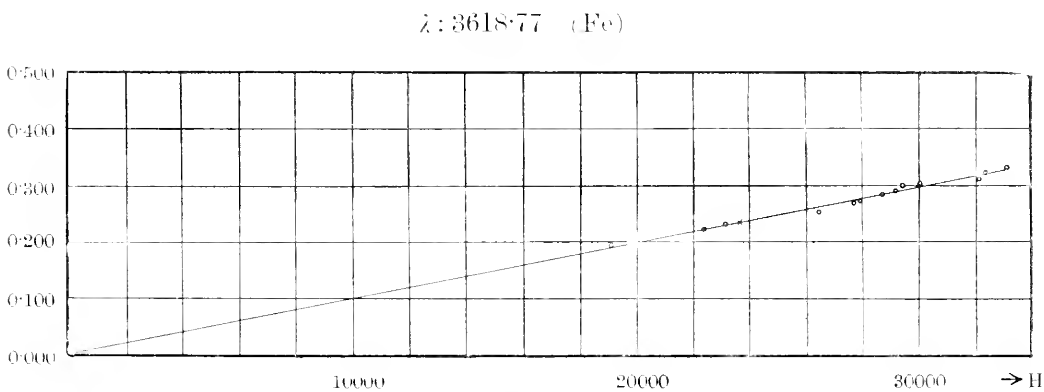


Fig. 2.

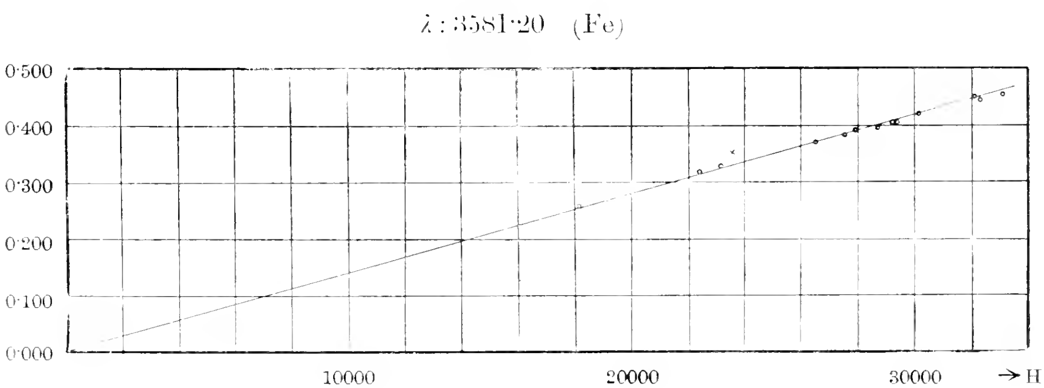


Fig. 3.

K. Yamada : Magnetic Separations of the Lines of Iron, Nickel and Zinc in Different Fields.

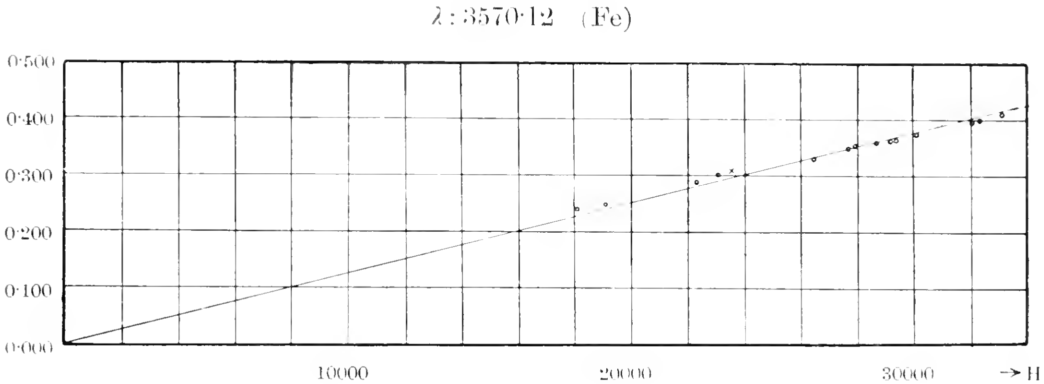


Fig. 1.

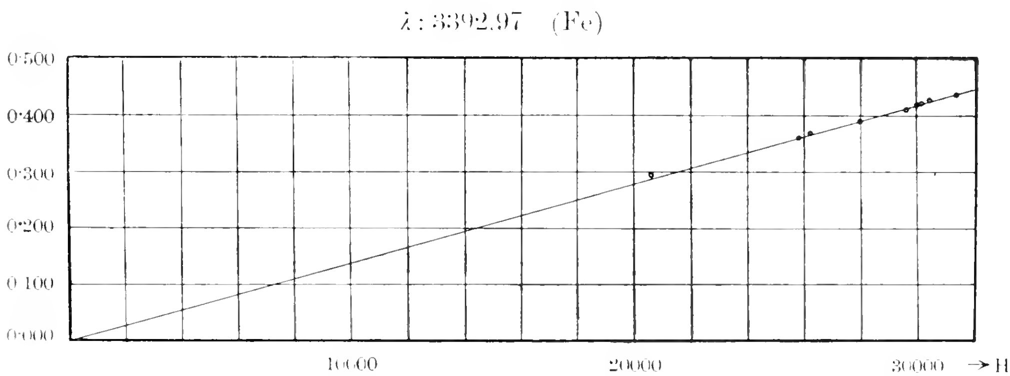


Fig. 2.

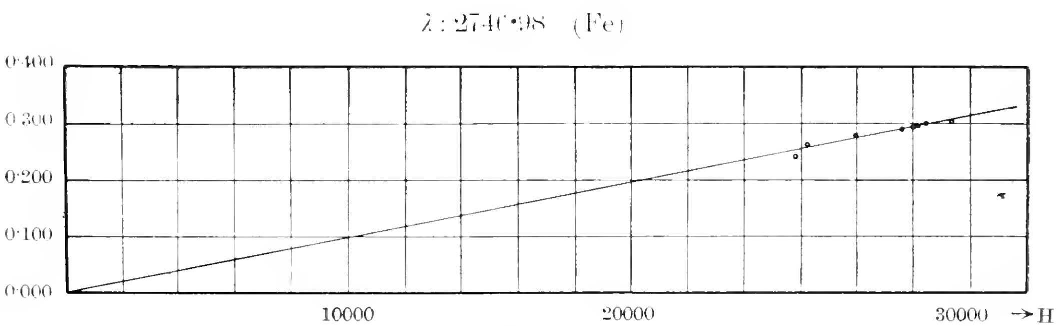


Fig. 3.

K. Yamada : Magnetic Separations of the Lines of Iron, Nickel and Zinc in Different Fields.

λ : 3619·39 (Ni)

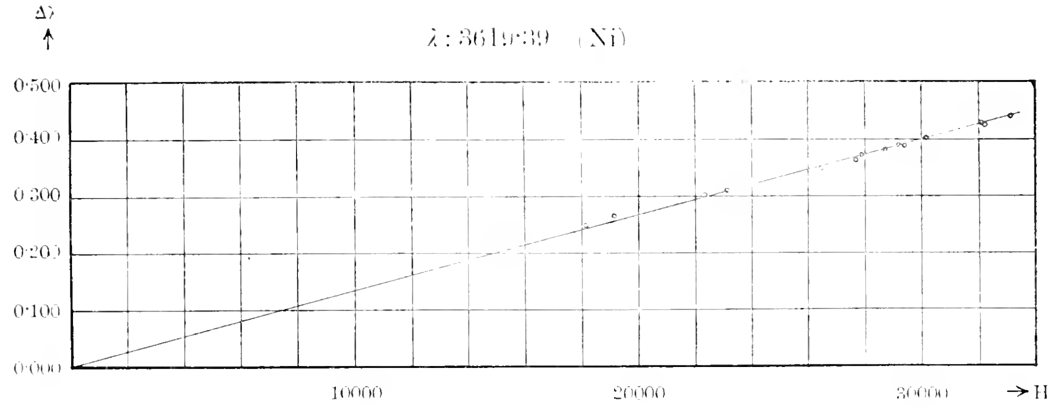


Fig. 1.

λ : 3566·37 (Ni)

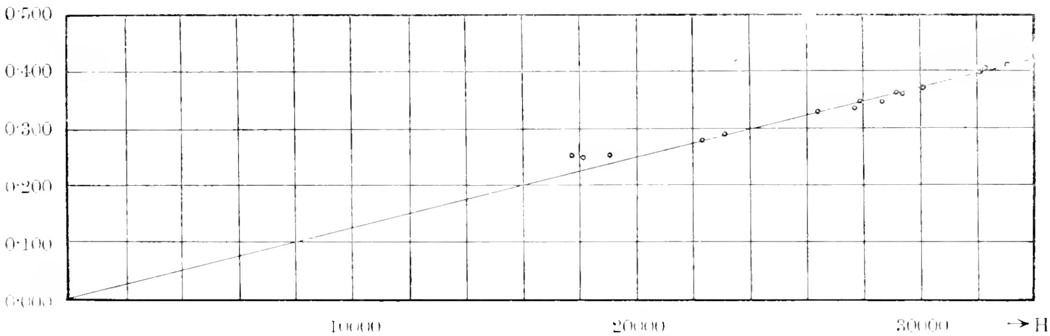


Fig. 2.

λ : 3524·53 (Ni)

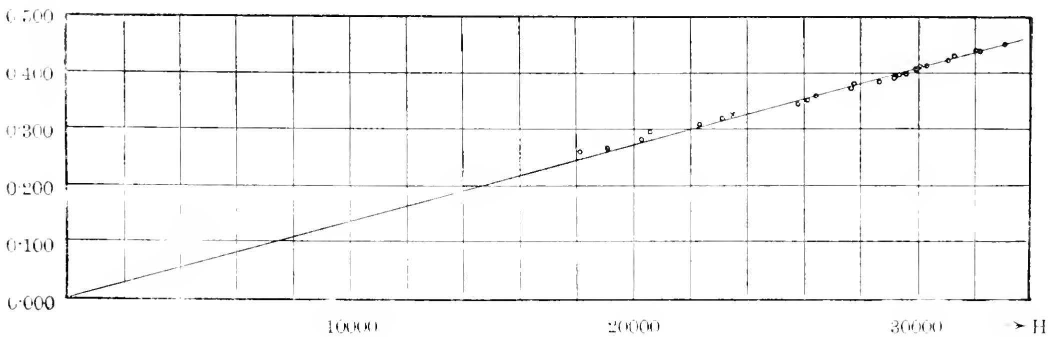


Fig. 3.

K. Yamada : Magnetic Separation of the Lines of Iron, Nickel and Zinc in Different Fields.

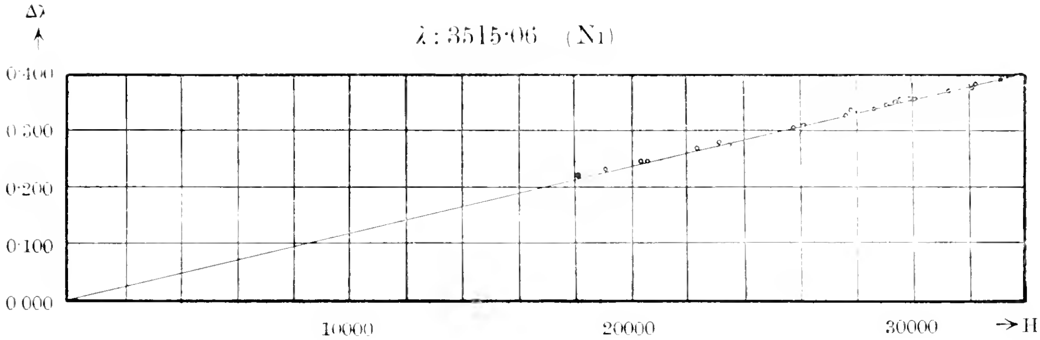


Fig. 1.

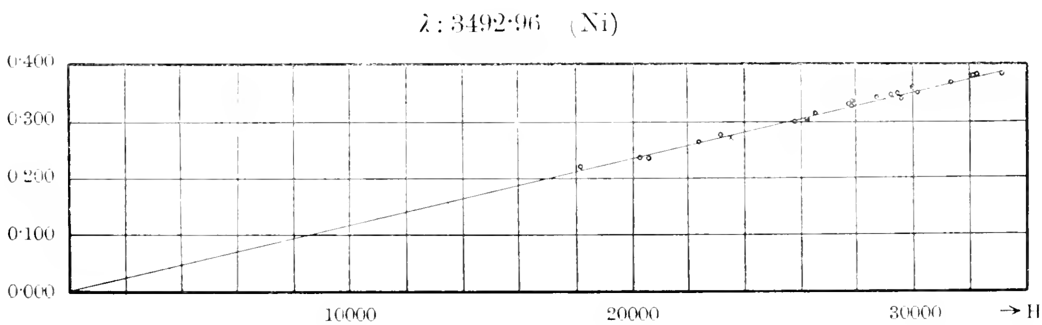


Fig. 2.

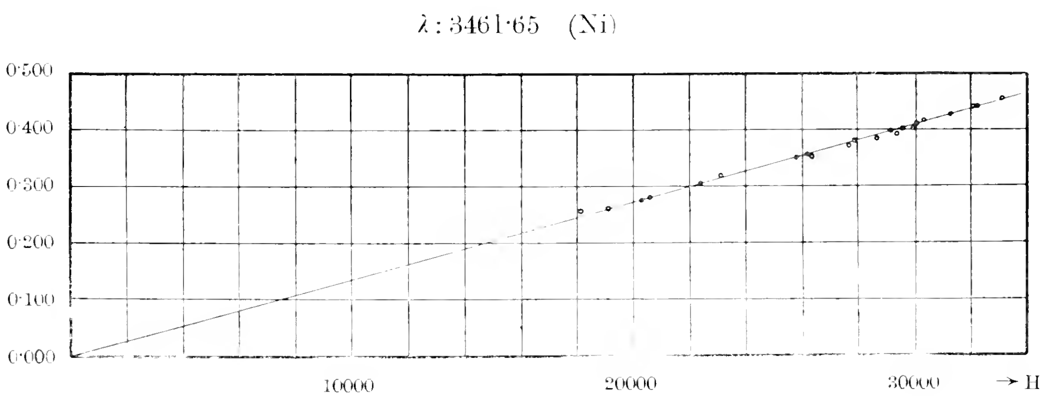


Fig. 3.

K. Yamada : Magnetic Separations of the Lines of Iron, Nickel and Zinc in Different Fields.

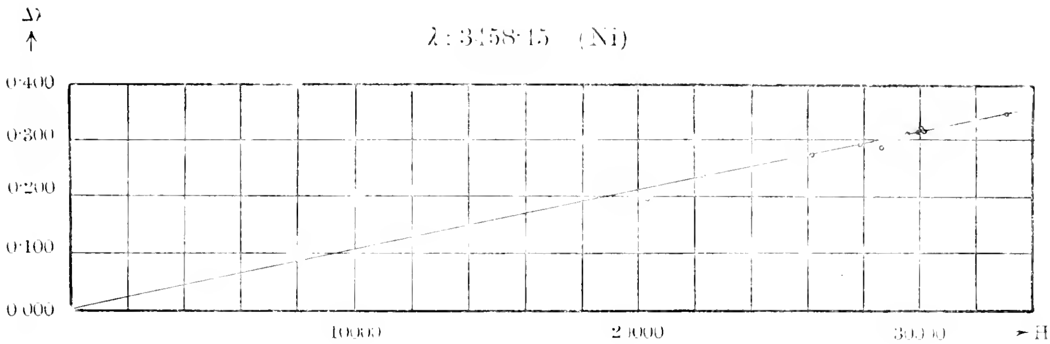


Fig. 1.

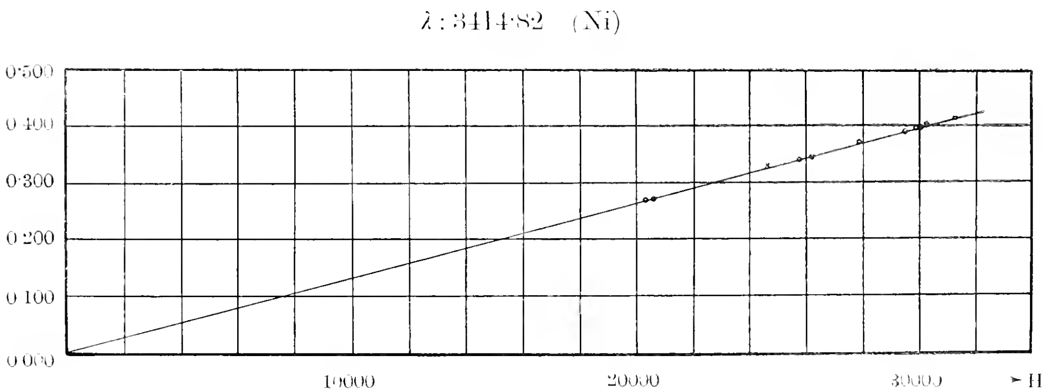


Fig. 2.

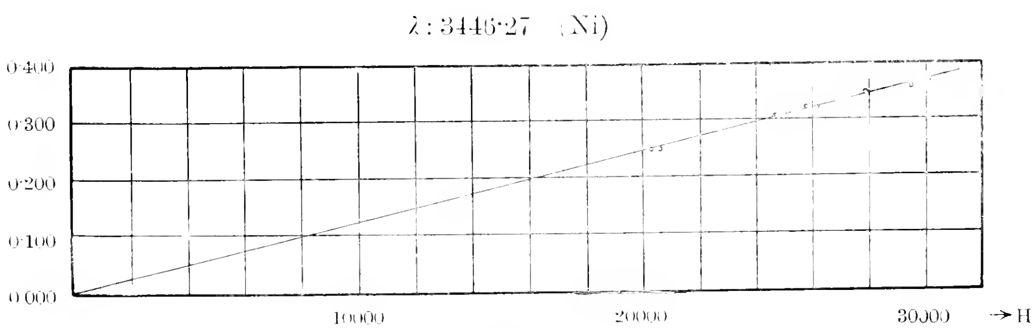


Fig. 3.

K. Yamada : Magnetic Separations of the Lines of Iron, Nickel and Zinc in Different Fields.

λ : 3380·58 (Ni)

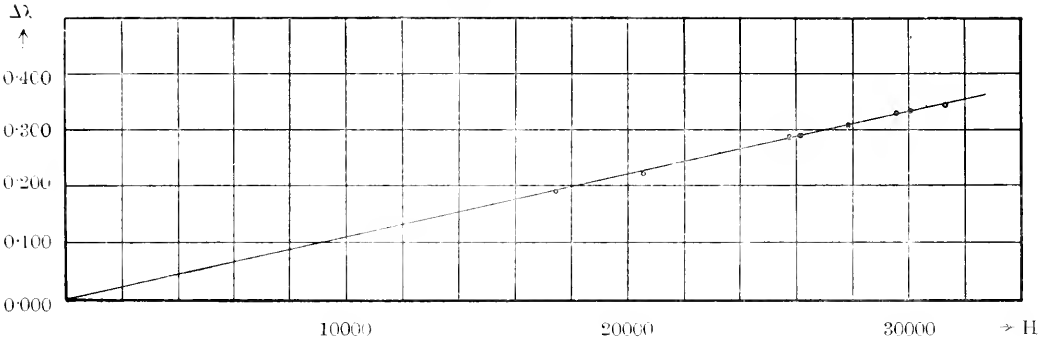


Fig. 1.

λ : 3345·13 (Zn)

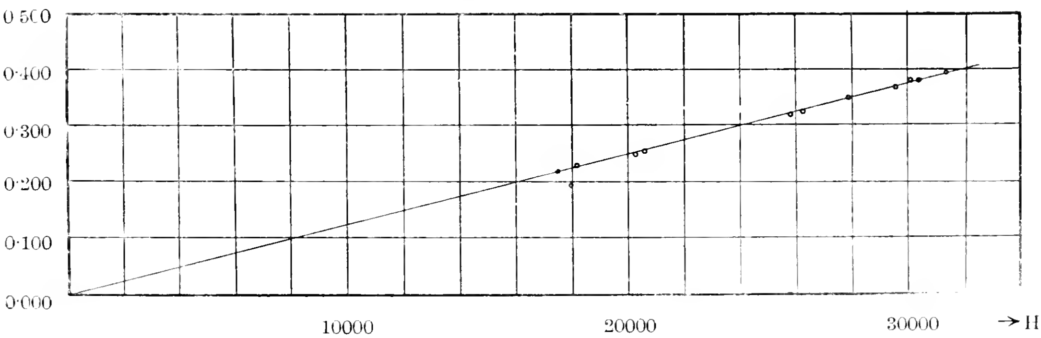


Fig. 2.

K. Yamada : Magnetic Separations of the Lines of Iron, Nickel and Zinc in Different Fields.

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Magnetic Separations of Iron Lines in Different Fields.

By

Yutaka TAKAHASHI, *Rigakushi.*

With 13 Plates.

Introduction.

As a recent development of the atomic theory, various hypotheses have been proposed to illustrate the atomic structure from the point of view as revealed by radiation. In spite of the divergence of the various theories, they all agree in attributing the origin of the spectrum lines to the moving electrons in the atom. Prof. Nagaoka has suggested that, if such an atom is placed in a magnetic field, the mutual influence between the electrons may give rise to the separation of some spectrum line, which is not proportional to the magnetic field.

In some series lines with complex structure many investigators[1] have observed anomalous Zeeman separations, and it is known as a general rule that each component, showing its own Zeeman separation in weak fields, either disappears or unites with others when the field is increased, and as a whole tends to the normal triplet in a sufficiently strong field: consequently the separation is not generally proportional to the field. But to

1) Paschen u. Back, Ann. d. Physik, **39** (1912) p. 897 ; **40** (1913) p. 960.
Fortrat, C. R., **156** (1913) p. 1607 ; **157** (1913) p. 636.
Nagaoka and Takamine, Proc. Tokyo Math.-Phys. Soc., [2], 7 (1913) p. 188 ; (1914) p. 331 ; Phil. Mag., **27** (1914) p. 333 ; **29** (1915) p. 241.
Wood and Kimura. Astrophys. Jour. **46** (1917) p. 197.

make exact measurements of such lines seems to be very difficult. As for the single lines, Reese[1] and Kent[2] found that the separations of λλ 4680, 4722 and 4810 of zinc are not proportional to the field, but Cotton and Weiss[3] showed that proportionality holds good for these lines. The deviations from the proportionality given by Reese and Kent for other lines are much smaller than those for the above lines. Several investigators showed that magnetic separations of single lines are in general nearly proportional to the field. So far, however, as the present writer is aware, those lines for which the separations are accurately measured over a wide range of magnetic fields are very few, so that it requires more extended experiments covering a wide range of magnetic fields as well as numerous spectrum lines to ascertain whether a linear relation between the separation and the magnetic field exists or not, which seems to be very interesting inasmuch as it may throw some light on the structure of the atom. The following investigation was undertaken with this object in view. For this purpose, elements rich in lines are convenient, as the chance of coupling among the electrons may be large in such an atom, for it is not probable that these lines are emitted from separate atoms each by itself; indeed they give many Zeeman triplets having divergent values of separations,[4] which might be the result of coupling.

Iron, manganese, calcium and titanium were examined, but the experiments with the latter three elements being yet incomplete, only the results obtained with the first are given in the present report.

Besides the fact that iron is rich in lines which give divergent values of magnetic separations, its ferromagnetic property—though we do not know the property at the temperature of the spark—may show some special character in its magnetic spectrum,

1) Reese, Astrophys. Jour., **12** (1900) p. 120.

2) Kent, Astrophys. Jour., **13** (1901) p. 289.

3) Cotton et Weiss, Jour. de Phys., [4], **6** (1907) p. 429.

4) Zeeman, Magnetooptics, (1913) p. 157.

and a comparison with other para-or diamagnetic elements, when sufficient data are obtained, will be interesting.

Source of Light.

The source of light used was the spark discharge between 25 percent nickel-steel terminals excited by an induction coil, the primary circuit being fed with 6 amperes of alternating current of 110 volts, 50 cycles, obtained from the city main. 4 Leyden jars were connected parallel to the spark gap directly between the poles of the induction coil to obtain a condensed discharge. An adjustable inductance and auxiliary spark gap with small capacity were inserted in the spark circuit in series; the increase of the inductance, giving the spark the character of an arc, made the spectrum lines sharper and put out the air lines, with, however, a considerable sacrifice in the intensity of the iron lines also; the auxiliary spark gap served to control the spark under examination in a favourable state for various magnetic fields. It is desirable to give the spark sonorous character with steady greenish appearance. When the magnetic field was not strong the spark continued in a favourable state for a long time, but in a strong field the terminals soon became dark red and the spark formed an arc of violet tint, making the iron lines weaker and increasing the luminosity of the continuous spectrum in the back ground, —this was especially remarkable in the green region—, so that it was necessary to polish the tips of the terminals every 5 or 10 minutes. The terminals, on being polished, recovered their ferromagnetic property, and it was very troublesome to place them in proper position between the poles of the strong electromagnet, which was, however, overcome by heating the polished terminals with a Bunsen burner.

Small cylinders about 1 mm. thick attached to brass rods were used as terminals for most of the experiments, and wedge shaped ones were used with the strongest fields to prevent the current from passing through the poles of the magnet.

During the whole course of the experiment, it was desirable to regulate the length of the spark gap, which was adjusted by clamping the terminals to the poles provided with an arrangement like a spark micrometer, by which the distance between the sparking terminals was changed so as to obtain a spark best suited for the exferiment.

Electromagnet.

The electromagnet used was so constructed that the cores with the coils can be displaced along their common axis and rotated as a whole about the vertical axis, which enabled us to bring the middle of the magnetic field on the line of collimation of the optical system, keeping this line and the magnetic lines of force at right angle with each other.

A current of 1·5 to 23 amperes from the secondary battery was used according to the magnetic field desired, and its constancy was carefully observed by means of a small adjustable resistance and an ammeter by Siemens & Halske.

Conical pole pieces ending in circular sections were used during the whole course of the experiments, the diameter of the faces and the air gap between them were 2 cm. and 1 cm. respectively for weak fields, and 0·3 cm. and 0·12 cm. for the strongest ones, the diameter and the gap being changed between these limits.

For the purpose of examining the character of the magnetic separation, it was desirable to extend the range of the magnetic field as wide as possible, but it was difficult to apply a field stronger than 37230 gauss with the electromagnet in our laboratory for the spark gap giving a source of light of considerable intensity, the diameter of the faces of the pole pieces and the air gap between them being limited as mentioned above, for further approach of the pole faces caused short-circuit of the sparking current through them, and further diminution of the diameter might have disturbed the uniformity of the magnetic field in the spark. An electromagnet with water-cooling arrangement and ferro-cobalt pole pieces would have been more suitable for experiments in stronger fields.

Echelon Grating.

The echelon grating by Hilger, frequently used by Prof. Nagaoka[1] in the study of the structure of mercury lines and other works, was used in the present research also. The constants of this instrument are as follows :—

Thickness of plate	9·350 mm.,
Number of plates	35,
Steps	1·0 mm.,
Length	32·73 cm.,
$A=1{\cdot}555055$,	
$B=5{\cdot}9595\times10^{5}$,	
$C=1{\cdot}9514\times10^{12}$.	

where A, B, C are the constants in Cauchy's formula for the index of refraction

$$\mu=A+\frac{B}{\lambda^2}+\frac{C}{\lambda^4},$$

λ being expressed in Å.U.

The wave number intervals corresponding to the distance of two successive orders were calculated from the usual formula

$$\frac{\delta\lambda_{max}}{\lambda^2}=\frac{1}{t\left\{(\mu-1)-\lambda\frac{d\mu}{d\lambda}\right\}}$$

for the wave lengths of constant intervals of 50 Å.U., and their values are given in Table I.

1) Nagaoka and Takamine, Memoir of the Imperial Academy, Sec. II, Vol. I., No. 1, 1913.

TABLE I.

$\lambda \times 10^8$	$\frac{\partial \lambda_{max}}{\lambda^2}$	$\lambda \times 10^8$	$\frac{\partial \lambda_{max}}{\lambda^2}$	$\lambda \times 10^8$	$\frac{\partial \lambda_{max}}{\lambda^2}$
3800	1·4739	4550	1·6103	5300	1·6948
3850	1·4853	4600	1·6172	5350	1·6992
3900	1·4963	4650	1·6239	5400	1·7035
3950	1·5070	4700	1·6304	5450	1·7077
4000	1·5172	4750	1·6367	5500	1·7117
4050	1·5272	4800	1·6428	5550	1·7157
4100	1·5368	4850	1·6487	5600	1·7195
4150	1·5460	4900	1·6545	5650	1·7233
4200	1·5551	4950	1·6600	5700	1 7269
4250	1·5637	5000	1·6655	5750	1·7305
4300	1·5722	5050	1·6707	5800	1·7339
4350	1·5803	5100	1·6758	5850	1·7373
4400	1·5882	5150	1 6808	5900	1·7406
4450	1·5958	5200	1·6856	5950	1·7438
4500	1·6031	5250	1·6902	6000	1·7469

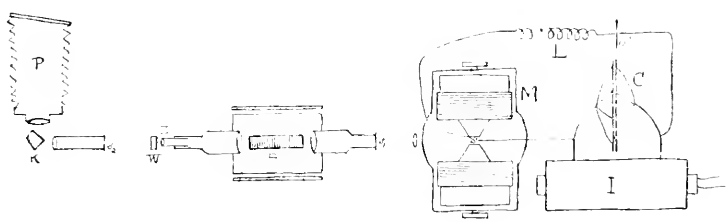
Actual values of $\frac{\partial \lambda_{max}}{\lambda^2}$ for iron lines were obtained by interpolation, and, if desired we can get the wave length intervals by multiplying by λ^2; this is unnecessary, as it is more rational to express the Zeeman separation in change of frequency than in wave length.

The chief advantage of using the echelon spectroscope consists in its high resolving power, the great intensity of light and the comparatively large value of $\frac{\partial \lambda_{max}}{\lambda^2}$. The first allows us to observe fine separations in weak fields, the second is convenient especially in investigations with strong fields, and the last is important in the investigation of spark spectra, for the lines in such spectra have considerable widths. Fabry-Perot's interferometer and Lummer Gehrcke's plate satisfy the first and the second, but not the third condition. Even with the echelon grating $\frac{\partial \lambda_{max}}{\lambda^2}$ is not sufficiently large to enable us to measure the separation in some fields when one component falls upon or close by the other. Complex separations giving more than four-components with the same polarization were not measured in different fields. Though the appearance of the components belonging to the next order is troublesome in the measurement of complex separations, it has some advantage. Irregular contraction of the gelatine film of a photographic plate may cause some displacement of the silver deposit, and the error coming from this may be proportional to the separation, giving a large error for a strong field. With echelon spectra we need not measure a separation larger than $\frac{\partial \lambda_{max}}{\lambda^2}$, but may simply add the multiple of this quantity to the measured fraction, and can thus get an accurate value for the separation, provided that $\frac{\partial \lambda_{max}}{\lambda^2}$ is calculated with sufficient accuracy.

Apparatus and Method of the Experiment.

The spark placed between the magnet poles was focused by a lens on the horizontal slit s_1 of the echelon spectroscope, and the image formed by the echelon was projected on the vertical slit s_2 of a Hilger constant deviation spectroscope by means of a Rudolph's planar of 5 cm. focus. The telescope of this spectroscope was removed and its place was taken by a photographic camera with an objective of 65 cm. focus. The rough sketch is given in Fig. 1.

Fig. 1.



I induction coil	M electromagnet	s_2 vertical slit
C Leyden jars	s_1 horizontal slit	K constant deviation prism
L adjustable inductance	E echelon grating	P photographic camera
S spark under examination	Z Rudolph's planar	
S' auxiliary spark gap	W Wollaston prism	

When the horizontal slit was opened wide only the prism spectrum was observed at the camera, and then the slit was slowly closed until the vertical lines contracted to dots of echelon spectra. Thus many lines were photographed at a juxtaposition, admitting exact comparison of the magnetic separations among these lines. Proper adjustment of the inclination and position of the photographic plate and the position of the planar behind the echelon spectroscope made it possible to photograph green lines in one and the same exposure with violet lines, if we allowed the disturbing of the sharpness of the prism spectrum, but the difference in the intensity and the photographic sensitivity made it convenient to separate the exposure into several steps, each lasting 10 to 360 minutes according to the magnetic field and the lines to be photographed. In some plates I have photographed the echelon spectra of the whole region in one focus with disturbed prism spectrum, and in other plates with sharp prism spectrum, changing the position of the planar from exposure to exposure. For the larger part of the experiment a Wollaston prism W was inserted between the planar and the vertical slit, by which it was possible to photograph both the parallel and perpendicular Zeeman components at once. For some exposures the Wollaston prism was removed and a nicol was placed in front of the horizontal slit. Some photos were taken without separating the polarized components to see if any dissymmetry of the resolution exists, but without any conclusive result. The others were photographed without the planar to shorten the time of exposure at the cost of small magnification.

In order to make the apparatus free from mechanical and thermal disturbances, the whole arrangement was placed in a cellar, and the box containing the echelon grating was protected by cork plates. On fine days, dry air was allowed to enter the room through the windows; then shutting the windows and the door tightly, the temperature in the room was kept as constant as possible, and the image of the echelon spectra did not suffer any sensible disturbance even if the exposure was continued during the whole day.

In order to eliminate errors arising from irregular contraction of the gelatine film of the photographic plate and other sources, photos were taken repeatedly, increasing and decreasing the magnetic field. The photographic plates mostly used were the panchromatic and process plates by Wratten & Wainwright. Wratten double instantaneous and Ilford proces plates were used occasionally for the stronger and the weaker fields respectively.

Result.

In the earlier course of the experiment it was my chief object to study the behavior of the nine strong lines in the violet region λλ 4415·13, 4404·75, 4383·55, 4325·78, 4307·92, 4271·75, 4071·75, 4063·61, 4045·82[1] over a wide range of magnetic fields. Three of these lines are arranged in a particular spacing, thus, || |, the distance between the second and the third lines being twice as large as that between the first and the second. Another aim of the investigation was to observe the separations of the principal lines in the green region, as former results in this region showed some discrepancies, which may have been due partly to experimental errors and partly to the difference of the magnetic fields. In the course of the experiment it seemed to me important to study the separations of weak lines, for a weaker line may be affected more than a stronger by mutual action, though we do not know the mechanism of radiation in the atom. Thus the resolutions of the weak lines, so far as observed with the present instrument, are also here given. Many weak lines lost their intensity with the increase of the magnetic field and became very diffuse, some showing complicated resolutions. Especially in the green region the increase of the intensity of the back ground disturbed the echelon spectra so much that they appeared to melt into it,— this phenomenon is somewhat due to the character of these lines, whose components are diffuse or complex and one falling close by

1) The wave lengths of these lines and the others given in the present report were first identified in the map of the iron spectrum by Buisson and Fabry in "Recueil de Constantes Physiques," and then the exact numbers were taken from the international in Kayser's „Handbuch der Spectroscopie."

the other in the increased field—, and the separations of these lines were not measured for strong fields.

As 4404·75 came out most sharply in many plates, its separation was taken as the standard, and the separations of the other lines were compared with it, without any anticipation of the absolute value of the magnetic field. The magnetic fields given in the following were calculated from the separation of the above line assuming the specific separation

$$\frac{\Delta\lambda}{\lambda^2 H} = 1{\cdot}064 \times 10^{-4},$$

which was borrowed from the result of Mr. Yamada[1] who found that the separation is proportional to the magnetic field by comparing the separation of this line with that of the zinc line 4680. As my observation extends both in weak and strong fields beyond the limits in his experiment, it may be objected that the calculation of the field for these portions is an extrapolation, but the linear relation between the magnetic field and the separations of many sharp lines shows that the separation of 4404·75 is proportional to the magnetic field.

I. Nine Strong Lines in the Violet Region.

λ **4415·13,** at the end of the first three lines toward the red, appears as a triplet in weak fields. The intensity of this line is comparable with that of 4404·75 in the spark spectrum without or with a weak magnetic field, but, when the field is increased, all the three components become diffuse and weak, so that it is difficult to study the behavior fully in strong fields with the echelon spectroscope. On Wratten double instantaneous plates photographed with fields of 35650 and 37230 gauss I have observed its fine resolution, giving at least 3*p*-and 4*n*-components. King[2] also assumes the line to be septuple from the broadening of the

1) Yamada, Jour. Sci. Coll., Imperial University, Tokyo, Vol. XLI., Art. 7.

2) King, The Influence of a Magnetic Field upon the Spark Spectra of Iron and Titanium, Papers of the Mt. Wilson Solar Observatory, Vol. II., Part I.

components, though van Bilderbeek-van Meurs[1] and Graftdijk[2] found this to be triple even in a field as strong as 32040 gauss.

Table II.[3]

$\lambda = 4415 \cdot 13$, $3p$-and $4n$-comps. or more.

H	$\frac{\Delta\lambda}{\lambda^2} \times 10^3$	$\frac{\Delta\lambda}{\lambda^2 H} \times 10^4$	H	$\frac{\Delta\lambda}{\lambda^2} \times 10^3$	$\frac{\Delta\lambda}{\lambda^2 H} \times 10^4$
	n 4730 (outer)	1·270 = 5 × 0·254	7690	754	0·980
	n 3780 (inner)	1·015 = 4 × 0·254			
37230	p 980	0·263	7580	807	1·064
35650	n 3660 (inner)	1·027 = 4 × 0·257	6880	725	1·053
	p 928	0·261	6850	764	1·115
20100	2280	1·134	6820	716	1·050
18590	2104	1·126	,,	715	1·049
18070	2072	1·147	6770	693	1·023
10980	1251	1·140	6750	700	1·037
10510	1155	1·099	6670	715	1·071
10200	1039	1·019	6500	649	0·998
9130	979	1·071	6450	678	1·051
9060	964	1·064	6300	664	1·054
8830	884	1·001	5910	666	1·128
8800	878	0·998	5670	592	1·044
8580	940	1·095	5590	572	1·023
8410	935	1·111	5260	552	1·050
8350	870	1·042	5000	556	1·112
8230	873	1·061	4730	484	1·022
8180	855	1·045	4650	502	1·080
7810	810	1·038	4350	474	1·090
7740	792	1·023	4110	417	1·015

1) Van Bilderbeek-van Meuss, Arch. Néerl., (2), **15** (1911) p. 353.

2) Graftdijk, Arch. Néerl., (3), **2** (1912) p. 192.

3) In this and the following tables the separation is for the pair of components symmetrically situated about the initial line. In the case of an ordinary triplet no remark with regard to polarization is given.

Table III.[1]

Comparison with former results.

	H	$\frac{\Delta\lambda}{\lambda^2 H}\times 10^4$
Takahashi	H<11000	1·056
van Bilderbeek-van Meurs	32040	1·06
Graftdijk	,,	1·095
King (7 comps. ?)	16000	1·084
Hartmann		1·048

In Table II. and Fig. 2 it can be seen that the specific separation measured as a triplet increases with the magnetic field, which may be attributed to the complicated separation.

λ **4404·75,** the middle line of the first group, is resolved into a sharp triplet. The separation of this line, being measured most accurately, is taken as the standard of the separation, so that neither table nor diagram of the separation is given for this line. But the linear relation between the separations of the principal lines in the following and the magnetic field—separation of 4404·75—shows that the separation of this line is exactly proportional to the magnetic field. According to Mr. Yamada, the separation of this line is given by

$$\frac{\Delta\lambda}{\lambda^2 H}=1{\cdot}064\times 10^{-4}.$$

As the absolute magnitude of the specific separation is not the chief object of the present research, I have calculated the magnetic field from the separation of this line by means of the above value for the specific separation, without any absolute determination of the magnetic field.

1) The magnetic fields applied by former investigators are not clear, as the separations of some lines, being measured at different fields, are reduced to correspond to their standard fields under the assumption that the separation is proportional to the magnetic field. But, as the difference between the standard and the observed fields may probably not be large, the standard fields are here quoted. As Hartmann's original paper is inaccessible, his result is taken from Lütig's paper in Ann. d. Physik, **38** (1912) p. 43 ; the magnetic field is therefore unknown.

The mean specific separation is here obtained from

$$\text{mean}\ \frac{\Delta\lambda}{\lambda^2 H}=\sum\frac{\Delta\lambda}{\lambda^2}\Big/\sum H.$$

λ 4383·55, at the end of the first group toward the violet, is resolved into a sharp triplet. The result is given in Table IV. and Fig. 3.

TABLE IV.

λ=4383·55, triple.

H	$\frac{\Delta\lambda}{\lambda^2}\times 10^4$	$\frac{\Delta\lambda}{\lambda^2 H}\times 10^4$	H	$\frac{\Delta\lambda}{\lambda^2}\times 10^4$	$\frac{\Delta\lambda}{\lambda^2 H}\times 10^4$
37230	3982	1·070	9010	965	1·071
36330	3879	1·068	8590	919	1·069
35650	3830	1·074	8580	913	1·064
,,	3814	1·070	8520	900	1·055
34830	3696	1·061	8410	910	1·081
34490	3669	1·064	8350	895	1·071
25960	2781	1·071	8270	880	1·064
25350	2745	1·083	8260	884	1·070
25140	2688	1·069	8250	888	1·076
24870	2692	1·082	8180	875	1·069
24350	2639	1·084	8040	867	1·079
24040	2580	1·073	7830	827	1·056
23840	2560	1·074	7810	838	1·073
23600	2541	1·077	7740	815	1·052
23160	2498	1·079	7690	830	1·079
22660	2451	1·082	7580	849	1·120
22090	2355	1·066	7170	795	1·109
21940	2366	1·079	7150	774	1·081
21360	2303	1·078	6850	730	1·066
20880	2246	1·075	6820	729	1·069
20760	2229	1·074	6770	754	1·114
20550	2208	1·074	6700	710	1·060
19860	2147	1·081	,,	704	1·051
19510	2134	1·094	6670	716	1·074
18860	2032	1·077	6560	702	1·071
18800	2019	1·074	6470	675	1·043
18690	2030	1·086	6300	674	1·070
18540	1990	1·073	5910	623	1·054
18070	1960	1·085	5890	637	1·081
11050	1170	1·059	5670	601	1·060
10640	1143	1·075	5330	575	1·078
10600	1143	1·079	5250	562	1·070
10580	1127	1·065	5000	562	1·124
10330	1130	1·093	4730	480	1·014
10180	1108	1·089	4650	515	1·108
9730	1046	1·075	4590	493	1·074
9620	1005	1·045	4520	492	1·088
9590	1011	1·055	4350	507	1·165
9420	992	1·053	4020	467	1·161
9130	978	1·071	3900	463	1·187

TABLE V.

Comparison with former results.

	H	$\frac{\Delta\lambda}{\lambda^2 H}\times 10^4$
Takahashi	mean	1·075
van Bilderbeek-van Meurs	32040	1·110
Graftdijk	,,	1·095
King	16000	1·078
Hartmann		0·998

It can be observed that the specific separation is a little larger at about 25000 gauss and a little smaller in the strongest fields than the mean value, but the discrepancy is within the limits of experimental errors, and we may consider the separation of this line to be exactly proportional to the magnetic field within the range between 3900 and 37230 gauss.

λ **4325·78,** at the end of the second group toward the red, appears as a sharp triplet in weak fields and becomes diffuse with the increase of field. Though the broadening of the n-components is not so remarkable, the p-component becomes so broad in strong fields that we may expect some fine resolution of this component. This resolution was not actually observed, but the distance between the two successive apparent maxima, when the echelon grating was adjusted in its double order position for this component, was much shortened and the decrement in the field of 36330 gauss was nearly $\frac{1}{4}\delta\lambda_{max}$ corresponding to the separation of 0·068 Å.U.; moreover this component was almost resolved in a field of 37230 gauss. Judged from appearance, this line seems to be split into a quintuplet (or septuplet) with $3p$-and 2 (or 4) n-components. The result obtained with this line is given in Table VI. and Fig. 4.

Table VI.

$\lambda = 4325 \cdot 78$, $3p$-, $2n$-comps. ?

H	$\frac{\Delta\lambda}{\lambda^2} \times 10^3$	$\frac{\Delta\lambda}{\lambda^2 H} \times 10^4$	H	$\frac{\Delta\lambda}{\lambda^2} \times 10^3$	$\frac{\Delta\lambda}{\lambda^2 H} \times 10^4$
37230	3500	0·940	8750	806	0·921
36330	3384	0·931	8580	746	0·870
34830	3230	0·927	8410	706	0·840
34490	3279	0·951	8350	749	0·896
33440	3142	0·940	8270	708	0·856
25960	2330	0·898	8260	706	0·855
24870	2232	0·897	8250	690	0·836
24040	2130	0·886	8180	684	0·836
11120	990	0·890	8040	720	0·895
11050	970	0·877	7810	675	0·865
10980	962	0·876	7740	707	0·914
10660	944	0·885	7710	651	0·845
10640	944	0·887	7170	594	0·828
10630	942	0·886	6850	616	0·899
10600	956	0·902	6810	578	0·849
10580	924	0·873	6800	573	0·842
10510	899	0·855	6770	616	0·910
10330	922	0·892	6750	583	0·864
10200	836	0·820	6700	580	0·866
10180	876	0·860	,,	571	0·852
9900	881	0·890	6300	542	0·860
9760	863	0·884	5910	530	0·897
9620	812	0·844	5890	486	0·825
9610	812	0·845	5670	488	0·861
9590	828	0·863	5260	457	0·869
9420	754	0·800	5230	445	0·850
9280	765	0·824	4650	398	0·855
9250	781	0·844	4610	378	0·820
9130	810	0·887	4590	389	0·846
9010	746	0·828	4520	349	0·772
8830	787	0·891	4020	386	0·960
8800	778	0·884	3900	304	0·779

TABLE VII.

Comparison with former results.

	H	$\frac{\Delta\lambda}{\lambda^2 H} \times 10^4$
Takahashi	H > 11120	0·865
van Bilderbeek-van Meurs	32040	0·873
Graftdijk	,,	0·874
King	16000	0·817
Hartmann		0·862

As may be seen in the above table and in Fig. 4, the separation is proportional to the magnetic field below 11000 gauss, showing good agreement with Hartmann's result. Though King gives a considerably smaller value of the specific separation, the result of van Bilderbeek-van Meurs, of Graftdijk and of the writer show that the specific separation is larger in the stronger field. The separation of the n-components in the strongest field obtained by the writer is equal to that of the normal triplet, though the measurement is not accurate. The character of this line somewhat resembles that of 4415·13.

λ **4307·92,** middle line of the second group, is resolved into a sharp triplet.

TABLE VIII.

$\lambda = 4307·92$, triple.

H	$\frac{\Delta\lambda}{\lambda^2} \times 10^3$	$\frac{\Delta\lambda}{\lambda^2 H} \times 10^4$	H	$\frac{\Delta\lambda}{\lambda^2} \times 10^3$	$\frac{\Delta\lambda}{\lambda^2 H} \times 10^4$
37230	3958	1·063	25350	2667	1·052
36330	3869	1·065	25140	2599	1·034
35650	3744	1·050	24870	2630	1·057
,,	3729	1·046	24350	2558	1·051
34830	3640	1·045	24040	2536	1·055
34490	3605	1·045	23840	2510	1·053
26230	2742	1·045	23600	2467	1·045
25960	2690	1·036	23160	2443	1·055

H	$\frac{\Delta\lambda}{\lambda^2} \times 10^3$	$\frac{\Delta\lambda}{\lambda^2 H} \times 10^4$	H	$\frac{\Delta\lambda}{\lambda^2} \times 10^3$	$\frac{\Delta\lambda}{\lambda^2 H} \times 10^4$
22660	2410	1·064	8250	859	1·041
22220	2340	1·053	8180	867	1·060
22090	2308	1·045	,,	866	1·059
21960	2304	1·049	8040	868	1·079
21940	2308	1·052	7830	844	1·077
21600	2261	1·047	7810	818	1·048
20880	2206	1·057	7740	818	1·056
20760	2173	1·047	7710	800	1·038
20550	2132	1·038	7690	839	1·090
19860	2109	1·062	7580	813	1·072
19640	2060	1·049	7300	760	1·041
19510	2061	1·056	7170	758	1·057
18860	2020	1·071	7150	774	1·082
18800	2017	1·073	6850	706	1·030
18690	1983	1·061	6820	697	1·022
18540	1981	1·069	6800	695	1·022
11120	1176	1·058	6770	732	1·081
11050	1161	1·051	6750	723	1·071
10640	1137	1·069	6700	709	1·058
10600	1094	1·032	,,	706	1·054
10580	1130	1·068	6670	700	1·050
10510	1114	1·060	6650	728	1·094
10330	1111	1·075	6560	670	1·021
10200	1088	1·066	6300	667	1·059
10180	1083	1·064	,,	655	1·040
9730	1019	1·047	5910	622	1·052
9620	1001	1·042	5890	620	1·052
9590	988	1·030	5670	592	1·044
9420	982	1·042	5590	584	1·044
9130	956	1·047	5260	565	1·075
9010	965	1·070	5230	558	1·067
8830	907	1·027	5000	561	1·122
8800	924	1·050	4790	536	1·119
8590	897	1·044	4730	477	1·007
8580	876	1·021	4650	492	1·058
8490	912	1·074	4610	469	1·018
8480	871	1·027	4590	485	1·055
8410	876	1·042	4520	470	1·040
8350	896	1·073	4350	447	1·028
8270	866	1·047	4020	442	1·099
8260	880	1·065	3900	452	1·159

TABLE IX.

Comparison with former results.

	H	$\frac{\Delta\lambda}{\lambda^2 H}\times 10^4$
Takahashi	mean	1·054
van Bilderbeek-van Meurs	32040	1·080
Graftdijk	,,	1·081
King	16000	1·078
Hartmann		1·012

Table VIII. and Fig. 5 show that the separation is proportional to the magnetic field.

λ **4271·75,** at the end of the second group toward the violet, is resolved into a sharp triplet, the specific separation being the largest of the nine lines.

TABLE X.

$\lambda=4271{\cdot}75$, triple.

H	$\frac{\Delta\lambda}{\lambda^2}\times 10^3$	$\frac{\Delta\lambda}{\lambda^2 H}\times 10^4$	H	$\frac{\Delta\lambda}{\lambda^2}\times 10^3$	$\frac{\Delta\lambda}{\lambda^2 H}\times 10^4$
35650	4173	1·171	19640	2268	1·155
34830	4065	1·167	19510	2252	1·154
34490	4030	1·168	18860	2188	1·160
34110	3973	1·165	18800	2160	1·149
33440	3910	1·169	18660	2192	1·175
32800	3838	1·170	18540	2200	1·187
22660	2668	1·177	18070	2080	1·151
22090	2593	1·174	18000	2076	1·153
21940	2602	1·186	17420	2049	1·176
21360	2480	1·161	17340	1994	1·150
20880	2418	1·158	16910	1980	1·171
20760	2396	1·154	10640	1239	1·164
20550	2316	1·127	10630	1233	1·160
19860	2315	1·166	10600	1249	1·178

H	$\frac{\Delta\lambda}{\lambda^2}\times10^3$	$\frac{\Delta\lambda}{\lambda^2 H}\times10^4$	H	$\frac{\Delta\lambda}{\lambda^2}\times10^3$	$\frac{\Delta\lambda}{\lambda^2 H}\times10^4$
10200	1190	1·167	6770	804	1·188
9730	1130	1·161	6750	806	1·193
9590	1112	1·160	6700	754	1·125
9420	1088	1·155	6670	774	1·160
9130	1070	1·171	6560	739	1·126
9010	1042	1·157	6470	760	1·175
8800	1046	1·189	6300	729	1·157
8590	966	1·124	,,	722	1·146
8580	989	1·151	5910	699	1·182
8520	1001	1·175	5890	678	1·151
8490	1000	1·178	5670	652	1·151
8350	962	1·151	5590	651	1·164
8270	985	1·190	5230	614	1·173
8260	959	1·161	5000	610	1·219
8250	961	1·165	4790	582	1·214
8180	967	1·181	4730	546	1·154
8040	970	1·206	4650	548	1·179
7810	899	1·151	4610	540	1·171
7300	825	1·130	4590	530	1·154
7170	832	1·160	4520	489	1·082
6850	793	1·157	4350	539	1·239
6820	786	1·152	4020	468	1·165
6800	764	1·123	3900	469	1·202

Table XI.

Comparison with former results

	H	$\frac{\Delta\lambda}{\lambda^2 H}\times10^4$
Takahashi	mean	1·164
van Bilderbeck-van Meurs	32040	1·193
King	16000	1·168
Hartmann		1·088

Table X. and Fig. 6 show that the separation is proportional to the magnetic field.

λ **4071·75,** end line of the third group toward the red, is resolved into a sharp triplet, the specific separation being the smallest of the nine lines.

TABLE XII.

λ=4071·75, triple.

H	$\frac{\Delta\lambda}{\lambda^2}\times 10^3$	$\frac{\Delta\lambda}{\lambda^2 H}\times 10^4$	H	$\frac{\Delta\lambda}{\lambda^2}\times 10^3$	$\frac{\Delta\lambda}{\lambda^2 H}\times 10^4$
37230	2288	0·615	10580	666	0·629
34890	2156	0·618	10330	648	0·627
34110	2120	0·622	9420	587	0·623
18860	1185	0·628	,,	584	0·620
18000	1142	0·634	8260	521	0·631
17420	1072	0·615	8040	500	0·621
17340	1086	0·626	7160	468	0·654
16910	1061	0·627	6820	430	0·631
15120	935	0·618	,,	424	0·622
14770	909	0·615	6700	430	0·642

TABLE XIII.

Comparison with former results.

	H	$\frac{\Delta\lambda}{\lambda^2 H}\times 10^4$
Takahashi	mean	0·623
van Bilderbeek-van Meurs	32040	0·645
King	16000	0·641

In Table XII. and Fig. 7 some systematic deviation of the specific separation may be observed, but it lies within the errors of experiment, and the separation may be considered proportional to the magnetic field.

λ **4063 61,** middle line of the third group, is resolved into a sharp triplet, and the separation is proportional to the magnetic field as may be seen in Table XIV. and Fig 8.

Table XIV.

$\lambda = 4063{\cdot}61$. triple.

H	$\frac{\Delta\lambda}{\lambda^2}\times 10^3$	$\frac{\Delta\lambda}{\lambda^2 H}\times 10^4$	H	$\frac{\Delta\lambda}{\lambda^2}\times 10^3$	$\frac{\Delta\lambda}{\lambda^2 H}\times 10^4$
37230	3796	1·020	10640	1118	1·051
34890	3538	1·014	10630	1090	1·025
26230	2659	1·014	10600	1098	1·035
25960	2588	0·997	10330	1046	1·013
25350	2540	1·002	9420	945	1·003
25140	2504	0·996	,,	944	1·002
24870	2520	1·013	8590	907	1·056
24350	2467	1·013	8260	849	1·028
24040	2480	1·032	,,	840	1·017
23840	2400	1·007	8250	825	1·000
23600	2380	1·008	8180	845	1·032
23160	2340	1·010	8040	821	1·021
22660	2340	1·033	7300	763	1·045
22220	2286	1·029	7170	762	1·063
22090	2256	1·021	6820	685	1·004
21960	2276	1·036	,,	678	0·994
21940	2244	1·023	6700	699	1·043
21600	2152	0·996	,,	698	1·042
21360	2148	1·006	6300	640	1·015
20880	2101	1·006	5890	589	1·000
20550	2084	1·014	5670	545	0·961
20100	2048	1·019	5590	576	1·030
19860	1990	1·002	5230	535	1·022
19660	1986	1·010	4590	436	0·949
18800	1950	1·037	4520	423	0·936
18540	1916	1·033	3900	437	1·120

Table XV.

Comparison with former results.

	H	$\frac{\Delta\lambda}{\lambda^2 H}\times 10^4$
Takahashi	mean	1·016
van Bilderbeek-van Meurs	32040	1·037
King	16000	1·017

λ **4045·82,** end line of the third group toward the violet, is resolved into a sharp triplet, the separation being proportional to the magnetic field as may be seen in Table XVI. and Fig. 9.

Table XVI.

λ=4045·82, triple.

H	$\frac{\Delta\lambda}{\lambda^2}\times10^3$	$\frac{\Delta\lambda}{\lambda^2 H}\times10^4$	H	$\frac{\Delta\lambda}{\lambda^2}\times10^3$	$\frac{\Delta\lambda}{\lambda^2 H}\times10^4$
34890	4015	1·151	8590	968	1·127
34110	3912	1·147	8260	947	1·147
33440	3900	1·166	,,	929	1·124
32800	3770	1·149	8250	985	1·193
22090	2580	1·168	8180	929	1·135
21360	2500	1·170	8040	950	1·181
20880	2415	1·157	7300	838	1·148
20550	2339	1·138	7170	891	1·242
20100	2316	1·152	6820	767	1·125
19860	2359	1·188	,,	766	1·124
19640	2299	1·171	6700	771	1·151
18800	2174	1·156	6300	719	1·141
18660	2179	1·168	5890	675	1·145
18540	2138	1·153	5670	636	1·122
18000	2052	1·140	5590	654	1·170
17420	2022	1·161	5230	612	1·170
17340	2008	1·158	4590	492	1·071
16910	1975	1·168	4520	481	1·064
10640	1228	1·154	3900	486	1·246
9420	1088	1·154			

Table XVII.

Comparison with former results.

	H	$\frac{\Delta\lambda}{\lambda^2 \Delta}\times10^4$
Takahashi	mean	1·156
van Bilderbeek-van Meurs	32040	1·196
King	16000	1·138

II. Other less Strong Lines.

λ **5615·661** is separated into a sharp triplet.

TABLE XVIII.

$\lambda=5615{\cdot}661$, triple.

H	$\frac{\Delta\lambda}{\lambda^2}\times 10^5$	$\frac{\Delta\lambda}{\lambda^2 H}\times 10^4$	H	$\frac{\Delta\lambda}{\lambda^2}\times 10^5$	$\frac{\Delta\lambda}{\lambda^2 H}\times 10^4$
8350	944	1·130	6300	721	1·144
8230	904	1·098	4730	534	1·128
7560	849	1·123	4240	478	1·127
6880	753	1·094			

TABLE XIX.

Comparison with former results.

	H	$\frac{\Delta\lambda}{\lambda^2 H}\times 10^4$
Takahashi	mean	1·120
Graftdijk	32040	1·170
King	16000	1·161
Hartmann		1·052

King's value of the specific separation is a little larger than the writer's, Graftdijk's being still larger: the discrepancy may be due to the difference of the magnetic field applied. If so, the curve in Fig. 10 must turn upwards, but, so far as my experiment goes, the separation is proportional to the magnetic field as shown in Table XVIII and Fig. 10. The above discrepancy is probably due to experimental errors.

λ **5586·772** appears as a diffuse triplet.

TABLE XX.

λ=5586·772, triple.

H	$\frac{\Delta\lambda}{\lambda^2}\times10^3$	$\frac{\Delta\lambda}{\lambda^2 H}\times10^4$
8230	862	1·047
6300	677	1·074
4240	465	1·096

TABLE XXI.

Comparison with former results.

	H	$\frac{\Delta\lambda}{\lambda^2 H}\times10^4$
Takahashi	mean	1·068
Graftdijk	32040	1·19
King (7 comps. ?)	16000	1·021
Hartmann		0·901

As may be seen in Table XX., XXI. and Fig. 11, my points lie with King's on a straight line, which intersects the line of separation slightly above the origin, but the deviation is too small to assume that this is not due to experimental errors. Graftdijk gives a much larger specific separation.

λ **5572·86** appears as a diffuse triplet.

TABLE XXII.

λ=5572·86, triple ?

H	$\frac{\Delta\lambda}{\lambda^2}\times10^3$	$\frac{\Delta\lambda}{\lambda^2 H}\times10^4$
8350	588	0·704
7560	541	0·716
4730	382	0·807

TABLE XXIII.

Comparison with former results.

	H	$\frac{\Delta\lambda}{\lambda^2 H}\times 10^4$
Takahashi	mean	0·732
King (7 comps. ?)	16000	0·945

The points by the writer lie on a straight line as may be seen in Fig. 12. King's value of the separation being much larger.

λ **5455·614** is separated into $2p$- and $3n$-components.

TABLE XXIV.[1]

$\lambda=5455{\cdot}614$, $2p$-, $3n$-comps.

H	$\frac{\Delta\lambda}{\lambda^2}\times 10^4$		$\frac{\Delta\lambda}{\lambda^2 H}\times 10^4$	
	n	p	n	p
9690	1608		1·659	
8350	1148	1145	1·374	1·371
8230	1160	1170	1·409	1·421
6880	910	932	1·322	1·355
6300	884	853	1·402	1·353
4730	724	638	1·530	1·350
4450	712	619	1·600	1·391
4240		584		1·377

TABLE XXV.

Comparison with former results.

	H	$\frac{\Delta\lambda}{\lambda^2 H}\times 10^4$	
		n	p
Takahashi	mean	1·469	1·376
King	16000	1·452	1·429

1) The separation of the outer pair is given for the n-components.

In Table XXIV. and Fig. 13 it may be observed that the separation of the *p*-components is proportional to the magnetic field, though that by King is a little larger. One half of the separation of the outer *n*-components—the separation between the undisturbed and one of the displaced components—is given in Fig. 14 to prevent confusion, which shows that the separation curve is concave upward between 5000 and 9000 gauss, but, if we take King's point into consideration, it must go down again, showing a wavy form.

λ **5446·92** appears as a quadruplet in weak fields. The *n*-components become very diffuse when the magnetic field is increased, which may be considered as the result of complex separation as observed by King. King gives 4 *p*-components in his table, but only two sharp *p*-components are observed on my plates, the specific separation being a little smaller between 5500 and 8000 gauss and a little larger above 10000 gauss than King's for the outer pair, as may be seen in Table XXVI. and Fig. 14.

TABLE XXVI.

λ=5446·92, 2*p*-, 2*n*-comps.

H	$\frac{\Delta\lambda}{\lambda^2}\times 10^3$ *p*	$\frac{\Delta\lambda}{\lambda^2 H}\times 10^4$ *p*	H	$\frac{\Delta\lambda}{\lambda^2}\times 10^3$ *p*	$\frac{\Delta\lambda}{\lambda^2 H}\times 10^4$ *p*
10760	1109	1·030	6400	545	0·851
10510	1072	1·020	,,	531	0·830
9880	953	0·965	6300	561	0·890
,,	928	0·939	5520	510	0·924
9690	941	0·971	,,	484	0·877
9390	878	0·935	4940	446	0·901
,,	870	0·926	4730	428	0·905
8860	778	0·878	4480	412	0·919
8350	773	0·926	4450	383	0·861
8230	775	0·941	,,	*n* 567	*n* 1·274
7560	682	0·903	4240	391	0·921
6880	561	0·815	4110	374	0·910
6690	573	0·856	,,	361	0·879

Table XXVII.

Comparison with former results.

	H	$\frac{\Delta\lambda}{\lambda^2 H}\times 10^4$	
		n	p
Takahashi	mean	1·274	0·921
King ($8n$- $4p$-comps.)	16000	1·841 1·480 1·004 0·461	0·942 0·476

λ **5429·70** appears as a quadruplet in weak fields.

Table XXVIII.

$\lambda=5429{\cdot}70$, $2p$-, $2n$-comps.

H	$\frac{\Delta\lambda}{\lambda^2}\times 10^3$		$\frac{\Delta\lambda}{\lambda^2 H}\times 10^4$	
	n	p	n	p
11630		850		0·731
10760		803		0·746
10510		789		0·750
9690		727		0·750
9390		591		0·629
8860		558		0·630
8350		508		0·608
8230		520		0·631
7560		501		0·663
6880	774	429	1·125	0·624
6670		449		0·674
6400		425		0·664
6300		414		0·657
4940	520		1·052	
4730	515		1·089	
4480	560	301	1·249	0·671

Table XXIX.

Comparison with former results.

	H	$\frac{\Delta\lambda}{\lambda^2 H}\times 10^4$	
		s	p
Takahashi	$H<9500$		0·744
	$H>9500$	1·126	0·645
Graftdijk	32040	1·54	0·0
King (6 or 8 n-, 4 p-comps.?)	16000	1·286	0·636
Hartmann		0·981	0·0

The separation of the n-components was not measured accurately owing to the broadening, but the discrepancy among the above four results in Table XXIX seems to be too large, giving greater specific separation for the stronger field, to attribute it to an error of measurement. This may be due to the complicated resolution, though Graftdijk has not observed any further resolution with a field of 32040 gauss. The separation of the p-components increases suddenly at about 9500 gauss, for each side of this field the observed points in Fig. 15 lie on a separate line which does not pass through the origin.

λ **5397·12** appears as a quadruplet.

Table XXX.

$\lambda=5397{\cdot}12$, $2p$-, $2n$-comps.

H	$\frac{\Delta\lambda}{\lambda^2}\times 10^3$		$\frac{\Delta\lambda}{\lambda^2 H}\times 10^4$	
	n	p	n	p
11630		675		0·580
10760		588		0·546
10510		621		0·590
9880		478		0·484
8230	1120	451	1·360	0·548
6880	833		1·210	
4730	575		1·215	
4450	601		1·351	

TABLE XXXI.

Comparison with former results.

	H	$\frac{\Delta\lambda}{\lambda^2 H}\times10^4$	
		n	p
Takahashi	mean	1·288	0·551
Graftdijk	32040	1·46	0·6
King (6 comps.?)	16000	1·352	0·476

The measurement is not quite accurate, but the separation of the n-components agrees with that of King. The p-components indicate a wavy fluctuation of the specific separation, though the data are too scanty for minute discussion.

λ **5371·495** is separated into a sharp triplet.

TABLE XXXII.

λ=5371·495, triple.

H	$\frac{\Delta\lambda}{\lambda^2}\times10^3$	$\frac{\Delta\lambda}{\lambda^2 H}\times10^4$	H	$\frac{\Delta\lambda}{\lambda^2}\times10^3$	$\frac{\Delta\lambda}{\lambda^2 H}\times10^4$
9880	898	0·909	6500	555	0·854
9690	885	0·913	6400	581	0·908
9610	903	0·940	6300	575	0·912
9390	787	0·838	5520	501	0·908
9250	825	0·891	5490	496	0·903
8860	723	0·816	5080	435	0·856
8410	664	0·790	4940	396	0·801
8350	724	0·866	4730	392	0·829
8230	708	0·860	4650	350	0·753
7830	649	0·829	4480	372	0·830
7690	649	0·844	4450	426	0·957
7560	607	0·803	4240	362	0·853
6880	576	0·838	4110	365	0·889
6690	553	0·826	3580	282	0·788
6670	517	0·775	2480	222	0·895
6560	543	0·828			

Table XXXIII.

Comparison with former results.

	H	$\frac{\Delta\lambda}{\lambda^2 H}\times 10^4$
Takahashi	mean	0·855
Graftdijk (p-comp. prob. decomposed)	32040	0·80
King (9 comps. ?)	16000	0·890

Fig. 17 shows that the specific separation suddenly falls at about 7000 gauss.

λ **5328·06** appears as a sharp triplet.

Table XXXIV.

$\lambda = 5328{\cdot}06$, triple.

H	$\frac{\Delta\lambda}{\lambda^2}\times 10^3$	$\frac{\Delta\lambda}{\lambda^2 H}\times 10^4$	H	$\frac{\Delta\lambda}{\lambda^2}\times 10^3$	$\frac{\Delta\lambda}{\lambda^2 H}\times 10^4$
9880	1093	1·106	6670	645	0·967
9830	1060	1·079	6650	700	1·052
9690	1009	1·040	6560	687	1·048
9610	994	1·034	6500	659	1·013
9390	990	1·054	6450	646	1·001
9280	1022	1·101	6400	698	1·091
9250	989	1·069	6300	735	1·166
9060	876	0·968	5520	576	1·043
8860	904	1·021	5490	580	1·056
8790	994	1·130	5060	496	0·980
8480	936	1·103	4940	490	0·991
8410	869	1·033	4790	511	1·065
8350	910	1·090	4730	521	1·101
8230	863	1·048	4480	496	1·105
7830	820	1·046	4450	513	1·153
7690	811	1·055	4240	428	1·010
7670	840	1·095	4110	460	1·120
7560	805	1·065	3920	415	1·059
7150	752	1·051	3580	352	0·983
6880	677	0·984	2950	280	0·949
6690	722	1·079	2480	282	1·137

Table XXXV.

Comparison with former results.

	H	$\frac{\Delta\lambda}{\lambda^2 H}\times 10^4$
Takahashi	mean	1·057
Graftdijk (p-comp. prob. decomposed)	32040	1·03
King (9 comps. ?)	16000	1·034

The separation is proportional to the magnetic field as may be seen in Fig. 18.

λ **5269·53** is separated into a sharp triplet.

Table XXXVI.

λ=5269·53, triple.

H	$\frac{\Delta\lambda}{\lambda^2}\times 10^3$	$\frac{\Delta\lambda}{\lambda^2 H}\times 10^4$	H	$\frac{\Delta\lambda}{\lambda^2}\times 10^3$	$\frac{\Delta\lambda}{\lambda^2 H}\times 10^4$
9880	1130	1·144	6670	677	1·015
,,	1125	1·139	6650	698	1·050
9830	1104	1·123	6540	781	1·194
9690	1093	1·128	6500	655	1·007
9610	1101	1·147	6450	706	1·095
,,	1040	1·082	6400	700	1·093
9390	1057	1·126	6300	726	1·152
9280	1060	1·142	5680	676	1·190
9250	1033	1·117	5520	584	1·058
8860	980	1·106	5490	615	1·119
8790	991	1·128	5060	535	1·058
8490	915	1·077	4960	545	1·099
8480	906	1·069	4790	517	1·079
8410	916	1·089	4730	544	1·150
,,	915	1·088	4650	447	0·961
8350	939	1·124	4480	505	1·127
8230	910	1·105	4450	514	1·155
7830	829	1·058	4240	458	1·080
7690	837	1·088	4110	475	1·156
7670	826	1·077	3920	453	1·155
7560	786	1·040	3580	388	1·084
7150	746	1·043	2950	342	1·159
6880	708	1·029	2480	316	1·274
6690	700	1·045			

Table XXXVII.

Comparison with former results.

	H	$\frac{\Delta\lambda}{\lambda^2 H} \times 10^4$
Takahashi	mean	1·101
Graftdijk (p-comp. prob. decomposed)	32040	1·166
King	16000	1·128

It may be observed in Table XXXVI. and Fig. 19 that the specific separation suddenly falls at about 6500 gauss, which may be the result of the superposition of the faint component of λ 5270·35 on the measured one. This line appears in the natural state as if it were the satellite of the line in question, and, in a magnetic field, one of the n-components is masked by a component of the line in question and the other n-component, which is expected to appear, is not found, though the p-component is clearly observed. These n-components seem to be very faint in a weak field, but one of them can be observed in my photograms taken with sufficiently long exposure applying a field near 10000 gauss. If we assume the result of King, the separation of this line is nearly one half that of 5269·53, and the red component of the former just overlaps with the violet component of the latter at 4800 gauss, so that the separation of the latter must appear too large below the field and too small above it. The first appearance of the fall at 6500 gauss seems to be too late to be considered merely as the effect of the superposed line, and it comes too suddenly compared with the slow recovery in the stronger field, the reverse being expected in the above case. It seems necessary to study with an instrument of different constants to decide the behavior of the line 5269·53.

λ **5232·957** appears as a triplet.

Table XXXVIII.

$\lambda = 5232{\cdot}957$, triple.

H	$\frac{\Delta\lambda}{\lambda^2}\times 10^3$	$\frac{\Delta\lambda}{\lambda^2 H}\times 10^4$	H	$\frac{\Delta\lambda}{\lambda^2}\times 10^3$	$\frac{\Delta\lambda}{\lambda^2 H}\times 10^4$
8350	1046	1·252	6300	758	1·203
8230	985	1·196	4730	528	1·116
7560	859	1·136	4450	539	1·211

Table XXXIX.

Comparison with former results.

	H	$\frac{\Delta\lambda}{\lambda^2 H}\times 10^4$
Takahashi	mean	1·190
Graftdijk	32040	1·24
King (7 comps. ?)	16000	1·158
Hartmann		1·106

Though the measurement is not accurate, it may be seen that the separation is approximately proportional to the field.

λ **5227·20** appears as a triplet.

Table XL.

$\lambda = 5227{\cdot}20$, triple.

H	$\frac{\Delta\lambda}{\lambda^2}\times 10^3$	$\frac{\Delta\lambda}{\lambda^2 H}\times 10^4$	H	$\frac{\Delta\lambda}{\lambda^2}\times 10^3$	$\frac{\Delta\lambda}{\lambda^2 H}\times 10^4$
10760	1117	1·038	6880	591	0·859
10510	1149	1·092	6690	595	0·889
9880	962	0·974	6400	576	0·900
,,	945	0·956	6300	621	0·986
9690	959	0·990	5520	562	1·019
9390	868	0·924	4940	465	0·941
9250	826	0·893	4730	470	0·993
9060	765	0·845	4480	394	0·879
8860	772	0·871	4450	425	0·955
8350	765	0·916	4240	368	0·867
8230	761	0·925	4110	398	0·969
7560	752	0·995			

TABLE XLI.

Comparison with former results.

	H	$\frac{\Delta\lambda}{\lambda^2 H}\times 10^4$
Takahashi	mean	0·947
Graftdijk (p-comp. decomposed)	32040	1·01
King	16000	0·946
Hartmann		1·202

As may be seen in Fig. 21, the separation curve is concave upward between 8000 and 10000 gauss. Though King's result coincides with the mean value of the writer, Graftdijk and Hartmann give larger values.

λ **5167·492** appears as a triplet.

TABLE XLII.

λ=5167·492, triple.

H	$\frac{\Delta\lambda}{\lambda^2}\times 10^3$	$\frac{\Delta\lambda}{\lambda^2 H}\times 10^4$	H	$\frac{\Delta\lambda}{\lambda^2}\times 10^3$	$\frac{\Delta\lambda}{\lambda^2 H}\times 10^4$
10760	1119	1·040	6650	725	1·090
9880	1042	1·055	6450	725	1·124
,,	997	1·009	6400	675	1·054
9610	1006	1·047	6300	701	1·112
9390	940	1·000	4940	504	1·020
9060	906	1·000	4730	509	1·076
8350	849	1·016	4480	480	1·070
8230	869	1·055	4450	531	1·192
7560	767	1·015	4240	457	1·077
6880	689	1·001	4110	457	1·112

TABLE XLIII.

Comparison with former results.

	H	$\frac{\Delta\lambda}{\lambda^2 H}\times 10^4$
Takahashi	mean	1·050
Graftdijk (p-comp. prob. decomposed)	32040	1·027
King	16000	1·081

It will be observed that the specific separation slightly falls at 7000 gauss, the points on each side of this field lie on a different line passing through the origin, but the amount of the fall is not distinct enough to show it.

λ **5018·45** appears as a triplet.

TABLE XLIV.

λ=5018·45, triple.

H	$\frac{\Delta\lambda}{\lambda^2}\times 10^3$	$\frac{\Delta\lambda}{\lambda^2 H}\times 10^4$	H	$\frac{\Delta\lambda}{\lambda^2}\times 10^3$	$\frac{\Delta\lambda}{\lambda^2 H}\times 10^4$
6300	1148	1·822	4240	732	1·725
4730	803	1·696	4110	684	1·664
4450	741	1·665			

TABLE XLV.

Comparison with former results.

	H	$\frac{\Delta\lambda}{\lambda^2 H}\times 10^4$
Takahashi	mean	1·724
Graftdijk	32040	1·833
King (4 comps. ?)	16000	1·840

It may be seen in Fig. 23 that the points obtained by the writer lie on a straight line which does not pass through the origin, and the point corresponding to the strongest field lies on the straight line connecting those given by King and Graftdijk with the origin. We may consider that the separation is proportional to the magnetic field above 6000 gauss and curves downwards in the vicinity.

λ **4957·62** is resolved into a triplet.

Table XLVI.

$\lambda = 4957{\cdot}62$, triple.

H	$\frac{\Delta\lambda}{\lambda^2} \times 10^3$	$\frac{\Delta\lambda}{\lambda^2 H} \times 10^4$	H	$\frac{\Delta\lambda}{\lambda^2} \times 10^3$	$\frac{\Delta\lambda}{\lambda^2 H} \times 10^4$
9880	1071	1·085	6690	773	1·155
9610	1027	1·069	6670	734	1·100
9390	1022	1·089	6650	714	1·073
9280	975	1·050	6500	677	1·041
9250	999	1·080	6450	669	1·037
9060	930	1·027	6400	727	1·135
8860	924	1·042	6300	716	1·137
8410	893	1·061	5520	645	1·169
8350	924	1·106	5490	610	1·110
8230	943	1·145	5060	514	1·015
7830	878	1·121	4730	526	1·111
7670	862	1·124	4550	456	1·001
7560	827	1·094	4480	490	1·091
6880	765	1·111	4450	506	1·138
6750	757	1·121	4240	481	1·133

Table XLVII.

Comparison with former results.

	H	$\frac{\Delta\lambda}{\lambda^2 H} \times 10^4$
Takahashi	mean	1·091
Graftdijk	32040	1·144
King	16000	1·441
Hartmann		1·112

The apparent separation may be affected by the neighbouring line 4957·31, but the results of Graftdijk and Hartmann agree with the mean value here obtained, though King gives a much larger separation.

λ **4920·52** appears as a triplet.

TABLE XLVIII.

$\lambda = 4920 \cdot 52$, triple.

H	$\frac{\Delta\lambda}{\lambda^2} \times 10^3$	$\frac{\Delta\lambda}{\lambda^2 H} \times 10^4$	H	$\frac{\Delta\lambda}{\lambda^2} \times 10^3$	$\frac{\Delta\lambda}{\lambda^2 H} \times 10^4$
9880	1148	1·161	6670	667	1·000
9390	1019	1·084	6540	679	1·038
9060	910	1·004	6400	671	1·049
8860	913	1·031	6300	674	1·070
8410	882	1·049	5680	587	1·032
8350	889	1·064	5490	581	1·058
8230	930	1·129	4550	428	0·942
7690	787	1·023	4450	463	1·041
7560	749	0·991	4240	437	1·030
6750	672	0·995			

TABLE XLIX.

Comparison with former results.

	H	$\frac{\Delta\lambda}{\lambda^2 H} \times 10^4$
Takahashi	mean	1·047
Graftdijk	32040	1·065
King (prob. complex)	16000	1·154

The apparent separation may be affected by λ 4919·007, though the mean value here obtained agrees with Graftdijk's result. If we accept King's result, the specific separation in a field stronger than 9000 gauss is greater than that in a weaker field.

λ **4891·51** appears as a triplet.

Table L.

$\lambda=4891{\cdot}51$, triple.

H	$\frac{\Delta\lambda}{\lambda^2}\times10^3$	$\frac{\Delta\lambda}{\lambda^2 H}\times10^4$	H	$\frac{\Delta\lambda}{\lambda^2}\times10^3$	$\frac{\Delta\lambda}{\lambda^2 H}\times10^4$
9880	877	0·888	6690	582	0·870
9390	892	0·950	6670	618	0·926
9280	821	0·884	6400	618	0·966
8860	808	0·911	6300	568	0·902
8410	780	0·927	5520	540	0·978
8350	783	0·937	4240	392	0·924
8230	771	0·936	4110	392	0·954
7690	740	0·962			

Table LI.

Comparison with former results.

	H	$\frac{\Delta\lambda}{\lambda^2 H}\times10^4$
Takahashi	mean	0·925
Graftdijk	32040	1·06
King	16000	1·012

The apparent separation may be affected by the neighbouring line 4891·78.

λ **4583·83** is separated into a triplet.

Table LII.

$\lambda=4583{\cdot}83$, triple.

H	$\frac{\Delta\lambda}{\lambda^2}\times10^3$	$\frac{\Delta\lambda}{\lambda^2 H}\times10^4$	H	$\frac{\Delta\lambda}{\lambda^2}\times10^3$	$\frac{\Delta\lambda}{\lambda^2 H}\times10^4$
8230	954	1·159	6450	716	1·110
6880	751	1·091	4450	523	1·175
6820	725	1·063			

Table LIII.

Comparison with former results.

	H	$\frac{\Delta\lambda}{\lambda^2 H} \times 10^4$
Takahashi	mean	1·118
Graftdijk	32040	1·168
King	16000	1·118

The separation is approximately proportional to the magnetic field.

λ **4528·622** appears as a triplet.

Table LIV.

$\lambda = 4528{\cdot}622$, triple.

H	$\frac{\Delta\lambda}{\lambda^2} \times 10^3$	$\frac{\Delta\lambda}{\lambda^2 H} \times 10^4$	H	$\frac{\Delta\lambda}{\lambda^2} \times 10^3$	$\frac{\Delta\lambda}{\lambda^2 H} \times 10^4$
9280	994	1·070	6540	742	1·135
9060	924	1·020	6450	733	1·136
8860	974	1·099	6400	747	1·167
8410	915	1·087	6300	761	1·208
8350	902	1·080	5590	683	1·221
8230	916	1·113	5520	643	1·164
8180	880	1·075	4730	568	1·200
7690	857	1·115	4650	580	1·247
6880	765	1·111	4480	510	1·138
6820	773	1·134	4450	533	1·198
6750	813	1·204	4240	499	1·176
6670	715	1·072	4110	458	1·115

Table LV.

Comparison with former results.

	H	$\frac{\Delta\lambda}{\lambda^2 H} \times 10^4$
Takahashi	mean	1·127
van Bilderbeck-van Meurs	32040	1·16
Graftdijk	,,	1·213
King (7 comps. ?)	16000	1·090

Fig. 28 shows that the separation curve is convex upward between 4000 and 8000 gauss. If we accept the results of van Bilderbeek-van Meurs and Graftdijk, it must turn upwards at a still stronger field showing a wavy form.

λ **4494·572** appears as a triplet.

TABLE LVI.

λ=4494·572, triple.

H	$\frac{\Delta\lambda}{\lambda^2}\times10^3$	$\frac{\Delta\lambda}{\lambda^2 H}\times10^4$	H	$\frac{\Delta\lambda}{\lambda^2}\times10^3$	$\frac{\Delta\lambda}{\lambda^2 H}\times10^4$
9880	980	0·991	6450	663	1·028
9060	816	0·901	6300	725	1·150
8230	805	0·978	4730	545	1·151
8180	811	0·991	4650	546	1·175
7690	785	1·020	4450	466	1·047
6880	682	0·991	4240	466	1·099
6750	695	1·030	4110	430	1·047
6540	714	1·091			

TABLE LVII.

Comparison with former results.

	H	$\frac{\Delta\lambda}{\lambda^2 H}\times10^4$
Takahashi	mean	1·032
Graftdijk	32040	1·28
King (7 comps. ?)	16000	0·935

The separation curve of this line quite resembles that of λ 4528·63.

λ **4476·03** appears as a triplet.

Table LVIII.

$\lambda=4476{\cdot}03$, triple.

H	$\frac{\Delta\lambda}{\lambda^2}\times10^3$	$\frac{\Delta\lambda}{\lambda^2 H}\times10^4$	H	$\frac{\Delta\lambda}{\lambda^2}\times10^3$	$\frac{\Delta\lambda}{\lambda^2 H}\times10^4$
9880	891	0·902	6540	640	0·979
8230	818	0·994	6450	661	1·025
8180	798	0·976	4450	435	0·978
6880	648	0·942	4250	375	0·882

Table LIX.

Comparison with former results.

	H	$\frac{\Delta\lambda}{\lambda^2 H}\times10^4$
Takahashi	mean	0·960
Graftdijk	32040	0·950
King (7 comps. ?)	16000	0·955

It may be seen in Fig. 30 that the separation curve is slightly convex upward between 4000 and 10000 gauss, but the deviation is too small to warrant the drawing of any definite conclusion, and the separation is nearly proportional to the magnetic field

λ **4466·556** appears as a triplet.

Table LX.

$\lambda=4466{\cdot}556$, triple.

H	$\frac{\Delta\lambda}{\lambda^2}\times10^3$	$\frac{\Delta\lambda}{\lambda^2 H}\times10^4$	H	$\frac{\Delta\lambda}{\lambda^2}\times10^3$	$\frac{\Delta\lambda}{\lambda^2 H}\times10^4$
9060	916	1·011	6450	748	1·160
8230	925	1·123	5490	593	1·080
8180	858	1·049	4650	497	1·069
7690	838	1·090	4450	536	1·204
6880	758	1·101	4240	516	1·216
6820	717	1·051	4110	503	1·224
6750	734	1·087			

Table LXI.

Comparison with former results.

	H	$\frac{\Delta\lambda}{\lambda^2 H}\times 10^4$
Takahashi	mean	1·101
Graftdijk	32040	1·189
King (6 comps. ?)	16000	1·074

As the points in Fig. 31 are scattered, we can say nothing about this line except that the separation is approximately proportional to the magnetic field.

λ **4447·72** seems to be separated into a sextuplet, but, the n-components being too diffuse to be separated sufficiently, this line was measured as a quadruplet.

Table LXII.

$\lambda=4447{\cdot}72$, $2p$-, $4n$-comps., measured as a quadruplet.

H	$\frac{\Delta\lambda}{\lambda^2}\times 10^3$		$\frac{\Delta\lambda}{\lambda^2 H}\times 10^4$	
	n	p	n	p
9690		993		1·024
8180	1890	798	2·308	0·976
7530	1584		2·102	
6880	1205	637	1·751	0·926
6020	1108		1·840	
4730	743	502	1·570	1·060
4450	677	413	1·521	0·929

Table LXIII.

Comparison with former results.

	H	$\frac{\Delta\lambda}{\lambda^2 H}\times 10^4$	
		n	p
Takahashi	mean	1·907	0·985
King (6 comps.)	16000	2·280 1·419	0·974

Fig 32 shows that the separation of the p-components is approximately proportional to the magnetic field, but that of the n-components is concave upward. Judged from the broadening, further resolution of these n-components is probable, and, if we accept King's result, the measured mean position moves from the inner component to the outer with the increased field. Though the curvature is clear, it can be accounted for, if we assume that the relative intensity of the components changes with the magnetic field.

λ **4442·34** appears as a quadruplet.

TABLE LXIV.

$\lambda=4442{\cdot}34$, $2p$-, $2n$-comps.

H	$\frac{\Delta\lambda}{\lambda^2}\times10^3$		$\frac{\Delta\lambda}{\lambda^2 H}\times10^4$	
	n	p	n	p
11630		827		0·711
10760		808		0·750
10510		745		0·708
9690		681		0·703
8230		597		0·726
8180	1325		1·620	
7530		538		0·715
4730	811	401	1·713	0·847
4450	812		1·824	
4240	721		1·700	

TABLE LXV.

Comparison with former results.

	H	$\frac{\Delta\lambda}{\lambda^2 H}\times10^4$	
		n	p
Takahashi	mean	1·699	0·729
King ($6n$-, $4p$-comps. ?)	16000	1·533	0·583

In Fig. 33 it may be observed that the separation of the p-components is proportional to the magnetic field, but, if we take King's result into consideration, the specific separation must be smaller for the stronger field. The straight line connecting King's point for the n-component with those of the writer does not pass through the origin. The apparent separation may be affected by the neighbouring line 4443·19.

λ **4427·314** appears as a triplet.

TABLE LXVI.

$\lambda=4427{\cdot}314$, triple.

H	$\frac{\Delta\lambda}{\lambda^2}\times10^3$	$\frac{\Delta\lambda}{\lambda^2 H}\times10^4$	H	$\frac{\Delta\lambda}{\lambda^2}\times10^3$	$\frac{\Delta\lambda}{\lambda^2 H}\times10^4$
8230	1190	1·445	6020	879	1·459
7530	1155	1·533	4730	685	1·447
6880	910	1·322	4650	661	1·421
6750	930	1·377	4450	661	1·485

TABLE LXVII.

Comparison with former results.

	H	$\frac{\Delta\lambda}{\lambda^2 H}\times10^4$
Takahashi	mean	1·436
King	16000	1·371

It may be seen that the separation is approximately proportional to the magnetic field.

λ **4375·934** is resolved into a triplet.

TABLE LXVIII.

$\lambda=4375{\cdot}934$, triple.

H	$\frac{\Delta\lambda}{\lambda^2}\times10^3$	$\frac{\Delta\lambda}{\lambda^2 H}\times10^4$	H	$\frac{\Delta\lambda}{\lambda^2}\times10^3$	$\frac{\Delta\lambda}{\lambda^2 H}\times10^4$
8230	1165	1·415	6020	825	1·370
7530	1068	1·416	4650	684	1·470
6820	1008	1·477			

Table LXIX.

Comparison with former results.

	H	$\frac{\Delta\lambda}{\lambda^2}\times 10^4$
Takahashi	mean	1·429
King	16000	1·384

The separation is proportional to the magnetic field.

λ **4337·04** is resolved into a quadruplet.

Table LXX.

λ=4337·04, 2p-, 2n-comps.

H	$\frac{\Delta\lambda}{\lambda^2}\times 10^3$		$\frac{\Delta\lambda}{\lambda^2 H}\times 10^4$	
	n	p	n	p
10510	984		0·935	
8230	825		1·002	
6820	668	388	0·980	0·569
4650	413		0·888	

Table LXXI.

Comparison with former results.

	H	$\frac{\Delta\lambda}{\lambda^2 H}\times 10^4$	
		n	p
Takahashi	mean	0·957	0·569
King	16000	0·878	0·512

Though the number of the observed points is not sufficient to infer anything from them, the separation curve of the n-components seems to turn downwards at about 8000 gauss to reach the point given by King, and the line connecting the lower 3 points does not pass through the origin, as shown in Fig. 36.

λ **4315·089** is resolved into a quadruplet.

Table LXXII.

$\lambda=4315{\cdot}089$, $2p$-, $2n$-comps.

H	$\frac{\Delta\lambda}{\lambda^2}\times10^3$	$\frac{\Delta\lambda}{\lambda^2 H}\times10^4$	H	$\frac{\Delta\lambda}{\lambda^2}\times10^3$	$\frac{\Delta\lambda}{\lambda^2 H}\times10^4$
	n	n		n	n
21960	p 557	p 0·254	5590	964	1·722
20880	p 540	p 0·258	4650	835	1·795
7530	1397	1·852	,,	805	1·730
6850	1112	1·623	4460	776	1·740
6820	1162	1·704	4350	760	1·747
6020	1074	1·783	3580	626	1·750

Table LXXIII.

Comparison with former results.

	H	$\frac{\Delta\lambda}{\lambda^2 H}\times10^4$	
		n	p
Takahashi	mean	1·745	0·256
van Bilderbeek- van Meurs	32040	1·77	0·0
King	16000	1·735	0·302

The separation of the n-components is proportional to the magnetic field. The straight line connecting the points for the p-components does not pass through the origin, but we cannot say much about this.

λ **4299·26** is resolved into a triplet.

Table LXXIV.

$\lambda=4299{\cdot}26$, triple.

H	$\frac{\Delta\lambda}{\lambda^2}\times10^3$	$\frac{\Delta\lambda}{\lambda^2 H}\times10^4$	H	$\frac{\Delta\lambda}{\lambda^2}\times10^3$	$\frac{\Delta\lambda}{\lambda^2 H}\times10^4$
11120	1525	1·371	6300	835	1·325
8350	1064	1·274	6020	844	1·401
6880	945	1·372	5910	785	1·329
6850	909	1·327	5590	757	1·352
6820	949	1·390	5490	692	1·259
,,	932	1·367	4730	621	1·312
6750	965	1·429	4650	687	1·478
6540	848	1·296	4360	600	1·377

Table LXXV.

Comparison with former results.

	H	$\frac{\Delta\lambda}{\lambda^2 H} \times 10^4$
Takahashi	mean	1·353
van Bilderbeek-van Meurs	32040	1·37
King	16000	1·372

The separation is proportional to the magnetic field.

λ **4294·13** appears as a quadruplet.

Table LXXVI.

$\lambda=4294·13$, $2p$-, $2n$-comps.

H	$\frac{\Delta\lambda}{\lambda^2} \times 10^3$ n	$\frac{\Delta\lambda}{\lambda^2 H} \times 10^4$ n	H	$\frac{\Delta\lambda}{\lambda^2} \times 10^3$ p	$\frac{\Delta\lambda}{\lambda^2 H} \times 10^4$ p
9150	1058	1·156	21960	1083	0·493
8580	907	1·056	20880	979	0·468
8350	999	1·196	18690	850	0·455
8230	872	1·060	18660	953	0·511
7690	826	1·074	18070	785	0·434
6880	707	1·027	11120	533	0·479
6850	754	1·100	10980	561	0·511
6820	708	1·039	10660	498	0·467
6750	753	1·115	10200	495	0·485
6670	777	1·165	9730	490	0·503
6540	761	1·163	9150	439	0·480
6300	723	1·147	8800	408	0·464
5910	669	1·132	8580	387	0·451
5590	608	1·087	8350	369	0·442
5490	655	1·192			
4730	536	1·133			
4350	543	1·249			

Table LXXVII.

Comparison with former results.

	H	$\frac{\Delta\lambda}{\lambda^2 H}\times 10^4$	
		n	p
Takahashi	mean	1·119	0·475
van Bilderbeek-van Meurs	32040	1·15	decomposed
King (6 comps. ?)	16000	1·081	0·468

The n-components become diffuse when the magnetic field is increased, indicating further resolution; the separation of the p-components shows slightly wavy fluctuation.

λ **4282·408** appears as a triplet.

Table LXXVIII.

$\lambda=4282·408$, triple.

H	$\frac{\Delta\lambda}{\lambda^2}\times 10^3$	$\frac{\Delta\lambda}{\lambda^2 H}\times 10^4$	H	$\frac{\Delta\lambda}{\lambda^2}\times 10^3$	$\frac{\Delta\lambda}{\lambda^2 H}\times 10^4$
7690	855	1·111	5910	740	1·252
6880	820	1·191	5680	701	1·234
6850	817	1·192	5590	664	1·187
6820	788	1·155	4730	559	1·181
„	783	1·149	4650	616	1·325
6750	749	1·110	4460	509	1·141
6540	774	1·182	4350	524	1·204

Table LXXIX.

Comparison with former results.

	H	$\frac{\Delta\lambda}{\lambda^2 H}\times 10^4$
Takahashi	mean	1·182
van Bilderbeek-van Meurs	32040	1·173
King (7 comps. ?)	16000	1·058

Fig. 40 shows that the separation curve is slightly convex upward between 4000 and 8000 gauss.

λ **4260·48** is resolved into a triplet.

TABLE LXXX.

λ=4260·48, triple.

H	$\frac{\Delta\lambda}{\lambda^2}\times10^3$	$\frac{\Delta\lambda}{\lambda^2 H}\times10^4$	H	$\frac{\Delta\lambda}{\lambda^2}\times10^3$	$\frac{\Delta\lambda}{\lambda^2 H}\times10^4$
8580	1316	1·532	5590	815	1·456
8180	1208	1·475	5230	752	1·437
6850	1000	1·460	5060	745	1·472
6820	1000	1·466	4730	686	1·450
6750	1002	1·482	4650	689	1·481
6670	1020	1·529	4520	614	1·359
6300	965	1·531	4350	605	1·390
,,	935	1·483	4020	598	1·488

TABLE LXXXI.

Comparison with former results.

	H	$\frac{\Delta\lambda}{\lambda^2 H}\times10^4$
Takahashi	mean	1·475
van Bilderbeek-van Meurs	32040	1·526
King	16000	1·458
Hartmann		1·436

The separation is proportional to the magnetic field.

λ **4250·79** appears as a quadruplet.

TABLE LXXXII.

λ=4250·79, $2p$-, $2n$-comps.

H	$\frac{\Delta\lambda}{\lambda^2}\times10^3$ p	$\frac{\Delta\lambda}{\lambda^2 H}\times10^4$ p	H	$\frac{\Delta\lambda}{\lambda^2}\times10^3$ p	$\frac{\Delta\lambda}{\lambda^2 H}\times10^4$ p
10980	879	0·800	6850	616	0·899
10660	816	0·765	6300	532	0·845
9130	735	0·805	5910	n 690	n 1·168
8350	646	0·774			

TABLE LXXXIII.

Comparison with former results.

	H	$\frac{\Delta\lambda}{\lambda^2 H}\times 10^4$	
		n	p
Takahashi	mean	1·168	0·808
van Bilderbeek-van Meurs	32040	0·97	0·713
King (12 comps. ?)	16000	0·851	0·730

The n-components are too diffuse to be measured accurately. The points for the p-components lie on a straight line, which does not pass through the origin. The apparent separation may be affected by the neighbouring line λ 4250·15.

λ **4235·94** is resolved into a triplet.

TABLE LXXXIV.

λ=4235·94, triple.

H	$\frac{\Delta\lambda}{\lambda^2}\times 10^3$	$\frac{\Delta\lambda}{\lambda^2 H}\times 10^4$	H	$\frac{\Delta\lambda}{\lambda^2}\times 10^3$	$\frac{\Delta\lambda}{\lambda^2 H}\times 10^4$
6300	965	1·531	4730	693	1·464
5590	853	1·525	4650	696	1·496

TABLE LXXXV.

Comparison with former results.

	H	$\frac{\Delta\lambda}{\lambda^2 H}\times 10^4$
Takahashi	mean	1·508
van Bilderbeek-van Meurs	32040	1·62
King	16000	1·572

The straight line connecting the observed points with King's does not pass through the origin, but we may consider the separation to be proportional to the magnetic field, as the deviation is very small.

λ **4219·36** is resolved into a triplet.

Table LXXXVI.

$\lambda = 4219{\cdot}36$, triple.

H	$\frac{\Delta\lambda}{\lambda^2} \times 10^3$	$\frac{\Delta\lambda}{\lambda^2 H} \times 10^4$	H	$\frac{\Delta\lambda}{\lambda^2} \times 10^3$	$\frac{\Delta\lambda}{\lambda^2 H} \times 10^4$
21960	2150	0·979	6820	715	1·048
10660	1049	0·984	,,	,,	,,
8180	790	0·966	5590	503	0.900

Table LXXXVII.

Comparison with former results.

	H	$\frac{\Delta\lambda}{\lambda^2 H} \times 10^4$
Takahashi	mean	0·987
van Bilderbeek-van Meurs	32040	0·959
King	16000	0·996

The separation is proportional to the magnetic field.

λ **4202·04** appears as a quadruplet.

Table LXXXVIII.

$\lambda = 4202{\cdot}04$, $2p$-, $2n$-comps.

H	$\frac{\Delta\lambda}{\lambda^2} \times 10^3$ n	$\frac{\Delta\lambda}{\lambda^2 H} \times 10^4$ n	H	$\frac{\Delta\lambda}{\lambda^2} \times 10^3$ p	$\frac{\Delta\lambda}{\lambda^2 H} \times 10^4$ p
8350	941	1·126	20880	964	0·461
8040	918	1·141	20100	916	0·456
6820	767	1·124	10660	488	0·458
,,	760	1·114	10330	461	0·449
6700	751	1·121	8350	410	0·491
6300	731	1·161	8040	377	0·469
,,	703	1·116			
5590	617	1·103			
5230	564	1·078			
4730	507	1·071			
4520	440	0·973			
4020	413	1·028			

TABLE LXXXIX.

Comparison with former results.

	H	$\frac{\Delta\lambda}{\lambda^2 H}\times 10^4$	
		n	p
Takahashi	mean	1·105	0·462
van Bilderbeek-van Meurs	32040	1·098	0·383
King (10 comps. ?)	16000	1·142	0·52

The separation of the n-components and that of the p-components are both proportional to the magnetic field above 6000 gauss, while the diagram for the former components curves downwards at the weaker field, as can be seen in Fig. 45.

λ **4199·09** is resolved into a triplet.

TABLE XC.

λ=4199·09, triple.

H	$\frac{\Delta\lambda}{\lambda^2}\times 10^3$	$\frac{\Delta\lambda}{\lambda^2 H}\times 10^4$	H	$\frac{\Delta\lambda}{\lambda^2}\times 10^3$	$\frac{\Delta\lambda}{\lambda^2 H}\times 10^4$
21960	2168	0·988	6300	599	0·950
10660	1050	0·985	,,	590	0·936
8350	808	0·967	5590	558	0·999
8180	770	0·941	5230	497	0·950
8040	799	0·994	4730	419	0·886
6820	662	0·971	4520	443	0·980
6700	675	1·008			

TABLE XCI.

Comparison with former results.

	H	$\frac{\Delta\lambda}{\lambda^2 H}\times 10^4$
Takahashi	mean	0·971
van Bilderbeek-van Meurs	32040	1·010
King	16000	0·978

The separation is proportional to the magnetic field.

λ **4181·76** is resolved into a triplet.

Table XCII.

λ=4181·76, triple.

H	$\frac{\Delta\lambda}{\lambda^2}\times10^3$	$\frac{\Delta\lambda}{\lambda^2 H}\times10^4$	H	$\frac{\Delta\lambda}{\lambda^2}\times10^3$	$\frac{\Delta\lambda}{\lambda^2 H}\times10^4$
21960	2612	1·189	8180	982	1·200
20880	2470	1·182	6820	807	1·183
20100	2348	1·168	5590	680	1·215

Table XCIII.

Comparison with former results.

	H	$\frac{\Delta\lambda}{\lambda^2 H}\times10^4$
Takahashi	mean	1·185
van Bilderbeek-van Meurs	32040	1·225
King	16000	1·210

The separation is proportional to the magnetic field.

λ **4143·88** has at least $3p$-and $4n$-components.

Table XCIV.

λ=4143·88, at least $3p$-, $4n$-comps.

H	$\frac{\Delta\lambda}{\lambda^2}\times10^3$	$\frac{\Delta\lambda}{\lambda^2 H}\times10^4$	H	$\frac{\Delta\lambda}{\lambda^2}\times10^3$	$\frac{\Delta\lambda}{\lambda^2 H}\times10^4$
21960	n {3820 (2380) 3103 p 692	n {1·739=5×0·348 1·413=4×0·353 p 0·315	5590	768	1·373
			5230	673	1·287
6810	915	1·343	4730	662	1·400
6700	1004	1·499	4520	588	1·301
6300	905	1·436	4020	542	1·349
,,	856	1·360			

Table XCV.

Comparison with former results.

	H	$\frac{\Delta\lambda}{\lambda^2 H}\times 10^4$
Takahashi	$H<7000$	1·377
van Bilderbeck-van Meurs	32040	1·435
King (7 comps. ?)	16000	1·43

Appearing as a triplet in a weak field the separation is approximately proportional to the magnetic field. When the magnetic field is increased, both p-and n-components become diffuse, indicating further resolution. On a few plates photographed with the field above 22000 gauss, $3p$-and $4n$-components were observed, though it was difficult to measure them accurately, as they were faint and diffuse.

λ **4132·08** has at least $5p$-and $4n$-components.

Table XCVI.[1]

$\lambda=4132{\cdot}08$, at least $5p$-, $4n$-comps.

H	$\frac{\Delta\lambda}{\lambda^2}\times 10^3$	$\frac{\Delta\lambda}{\lambda^2 H}\times 10^4$	H	$\frac{\Delta\lambda}{\lambda^2}\times 10^3$	$\frac{\Delta\lambda}{\lambda^2 H}\times 10^4$
21960	n 4000	n 1·821 = 5 × 0·364	6820	1000	1·466
	n 3085	n 1·404 = 4 × 0·352			
	p 390	p 0·178 = ½ × 0·356	5590	824	1·473
20880	n 3620	n 1·732 = 5 × 0·346	5230	713	1·363
	n 3000	n 1·436 = 4 × 0·360			
	p 387	p 0·185 = ½ × 0·370	4020	610	1·518
20100	n 3450	n 1·716 = 5 × 0·343			
	n 2735	n 1·360 = 4 × 0·340			
	p 390	p 0·194 = ½ × 0·388			

1) The interval between two adjoining p-components is given in the table.

Table XCVII.

Comparison with former results.

	H	$\frac{\Delta\lambda}{\lambda^2 H}\times 10^4$
Takahashi	mean $H<7000$ " $H>7000$	n 1·453 n $\begin{cases}1·760=5\times0·351\\1·400=4\times0·351\end{cases}$ p $0·186=\frac{1}{2}\times0·371$
van Bilderbeek-van Meurs	32040	1·62
King (13 comps. ?)	16000	1·869 (outer)

Though this line appears as a triplet in a weak field, both p-and n-components become diffuse when the magnetic field is increased, and finally they are resolved into many fine components. I have measured 5 p-components in equal spacing and 4 (or more?) n-components.

λ **4118·552** is resolved into a triplet.

Table XCVIII.

$\lambda=4118·552$, triple.

H	$\frac{\Delta\lambda}{\lambda^2}\times 10^3$	$\frac{\Delta\lambda}{\lambda^2 H}\times 10^4$	H	$\frac{\Delta\lambda}{\lambda^2}\times 10^3$	$\frac{\Delta\lambda}{\lambda^2 H}\times 10^4$
21960	2174	0·989	8180	793	0·969
20880	2056	0·984	6820	634	0·930
10660	984	0 922	5590	533	0·953

Table XCIX.

Comparison with former results.

	H	$\frac{\Delta\lambda}{\lambda^2 H}\times 10^4$
Takahashi	mean	0·968
van Bilderbeek-van Meurs	32040	1·024
King	16000	0·998

The separation is proportional to the magnetic field.

λ 4005·26.

Table C.[1]

λ=4005·26.

H	$\frac{\Delta\lambda}{\lambda^2}\times 10^3$	$\frac{\Delta\lambda}{\lambda^2 H}\times 10^4$	H	$\frac{\Delta\lambda}{\lambda^2}\times 10^3$	$\frac{\Delta\lambda}{\lambda_2 H}\times 10^4$
20250	*p* 303	0·150	5670	*n* 772	1·362
6820	*n* 1000	1·467	5590	*n* 771	1·379

Table CI.

Comparison with former results.

	H	$\frac{\Delta\lambda}{\lambda^2 H}\times 10^4$
Takahashi	mean	*n* 1·407 *p* 0·150
van Bilderbeek-van Meurs	32040	1·55
King	16000	1·797

Though this line appears as a triplet in a weak field, the components become diffuse when the magnetic field is increased until they are resolved into many components. 5 *p*-components in equal spacing were measured, though the corresponding *n*-components were not measured. The resolution of this line quite resembles that of λ 4132·08 as King has noted in his report.

Those lines for which either the measurements were not accurate or the photos were taken with only one or two fields are given in the following table. The fields applied are between 5000 and 1000 gauss, most of the photos being taken with 8200 gauss. The specific separations reduced from King's results are given in the same table for comparison.

1) The interval between two adjoining *p*-components is given in the table.

Table CII.

λ	$\frac{\Delta\lambda}{\lambda^2 H} \times 10^4$		
	Takahashi	King	Graftdijk van Bilderbeek-van Meurs
5434·527	unaffected	unaffected	
5341·03	n diffuse $2p$ 0·86	n 1·703 p 0·939	
5192·363	1·57	n 1·738 p 0·494	
5169·03	1·44	1·317	
4736·786	1·15	1·185	
4647·439	1·09	1·139	
4549·47	1·01	0·964	0·974
4482·27	4 n { 1·56 0·49 2 p 0·97	n 0·433 p 0·712	
4469·39	1·01	1·37	
4461·65	1·31	1·366	
4459·12	4 n { 1·67 1·30 2 p 0·44	n 1·410 p 0·397	
4422·57	2 n 1·40 2 p 0·86	n { 1·380 0·492 p 0·895	
4369·77	0·93	0·923	
4352·741	1·58	n 1·371 p 0·466	
4233·615	6 n { 3·04 2·05 0·94 3 p 1·04	n { 2·720 1·921 0·923 p 0·974	
4227·44	1·01	1·080	1·120
4210·36	2·81	2·841	
4191·442	4 n { 1·91 0·89 3 p 1·04	n { 1·921 0·940 p 0·989	
4175·64	1·04	1·06	
4172·13	1·10	1·101	
4156·81	1·14	n 1·328 p 0·437	
4134·685	1·08	1·109	
4021·872	1·05	1·051	
4014·53	0·98	0·970	
3997·41	1·04	1·04	
3969·26	1·37	1·405	1·472
3956·67	1·08	1·154	
3930·30	1·41	1·424	1·429
3927·94	1·40	1·428	1·427
3922·92	1·39	1·428	1·412
3920·26	1·34	1·420	1·444

III. Nickel Lines.

As the source of light was the spark between nickel-steel terminals, some nickel lines were photographed on the same plates with iron lines.

λ **5477·12** is resolved into a triplet.

TABLE CIII.

$\lambda=5477{\cdot}12$, triple.

H	$\frac{\Delta\lambda}{\lambda^2}\times10^3$	$\frac{\Delta\lambda}{\lambda^2 H}\times10^4$	H	$\frac{\Delta\lambda}{\lambda^2}\times10^3$	$\frac{\Delta\lambda}{\lambda^2 H}\times10^4$
10760	1181	1·097	6400	607	0·949
10510	1141	1·086	6300	600	0·952
9880	1036	1·049	5520	521	0·944
9690	985	1·016	4940	421	0·852
9390	862	0·918	4730	432	0·913
8860	798	0·901	4450	444	0·998
8350	806	0·965	4240	409	0·965
8230	820	0·996	4110	410	0·998
7560	690	0·913	mean		0·976
6880	599	0·871	$H<9400$		0·936

The specific separation increases with the magnetic field. We must not, however, forget that there is a weak iron line 5476·57, which is expected to appear 0·08 $\delta\lambda_{max}$ apart from the nickel line, though only a sharp line is observed in my photogram. In Fig. 51, it may be seen that the separation curve turns towards the point given by King[1] for the above iron line, and the lower portion agrees approximately with Hartmann's result for the same line. But, as Hartmann[2] used nickel-steel terminals, we cannot say that the line observed by him is not the nickel line. On the other hand the separation given by Graftdijk[3] for the nickel line agrees quite well with the mean value by the writer.

1) l. c.

2) l. c.

3) l. c.

Moreover the observed line is too strong to be taken for the iron line. I think that the line, which I have measured, is the nickel line, and the apparent separation may be somewhat disturbed by the superposition of the iron components.

λ **4714·68** appears as a triplet.

TABLE CIV.

λ=4714·68, triple.

H	$\frac{\Delta\lambda}{\lambda^2}\times 10^3$	$\frac{\Delta\lambda}{\lambda^2 H}\times 10^4$	H	$\frac{\Delta\lambda}{\lambda^2}\times 10^3$	$\frac{\Delta\lambda}{\lambda^2 H}\times 10^4$
9880	1120	1·134	6400	749	1·170
9690	1120	1·156	6300	785	1·246
9060	932	1·029	5520	614	1·112
8860	911	1·029	4730	541	1·144
8350	920	1·101	4480	515	1·148
8230	902	1·096	4450	501	1·125
7560	772	1·021	4240	480	1·131
6880	718	1·043	4110	515	1·254
6670	767	1·150	Mean		1·114

It may be seen in Fig. 52 that the specific separation suddenly falls at 7000 gauss.

λ **4401·77** is resolved into a triplet.

TABLE CV.

λ=4401·77, triple.

H	$\frac{\Delta\lambda}{\lambda^2}\times 10^3$	$\frac{\Delta\lambda}{\lambda^2 H}\times 10^4$	H	$\frac{\Delta\lambda}{\lambda^2}\times 10^3$	$\frac{\Delta\lambda}{\lambda^2 H}\times 10^4$
9880	1214	1·229	4450	440	0·988
8230	941	1·144	mean		1·133
6820	735	1·078			

Fig. 53 shows that the observed points lie on a straight line which does not pass through the origin. We can see a slight curvature of the separation curve, but it is within the errors of experiment.

Discussion of the Results.

Examining the preceding results, we find some similarity and regularity among many lines. Of the nine strong violet lines, the specific separation increases in each group with decreasing wave length,[1] the rate of the increase being larger for the more refrangible group, and the separation of the first line larger for the less refrangible. The lines 4415·13 and 4325·78, forming the first of the first and the second group respectively, give diffuse components, and the separations are not proportional to the magnetic field, but the broadening and the loss of intensity in strong fields are more conspicuous for the former line. On the plates photographed with the strongest fields $3p$-and $4n$-components are found for 4415·13, and the p-component of 4325·78 indicates some fine resolution which, I think, consists of three components, while 4071·75, the first line of the third group, appears as a sharp triplet of the separation proportional to the magnetic field. These lines seem to have 7, 5 (or 7) and 3 components respectively. The rest of these nine lines give sharp triplets, the separations being exactly proportional to the magnetic field.

Of the nine strong lines, seven, which were measured most accurately, give the separations proportional to the magnetic field as if no mutual influence existed between the radiating electrons, yet the divergent values of the specific separations show that Runge's rule does not hold among these lines. The ratios of these separations to the normal value are given in the following table.

1) For the line 4415·13, the specific separation of the n-components in a weak field is taken.

Table CVI.

λ	Δλ / normal Δλ [1]	p / q	Δλ / Δλ of 4404·75	p / q
4404·75	1·132 (1·135)	8/7 = 1·143 17/15 = 1·133 9/8 = 1·125	1·000	1
4383·55	1·144 (1·147)	8/7 = 1·143	1·010	91/90 = 1·011 101/100 = 1·010 111/110 = 1·009
4307·92	1·122 (1·125)	9/8 = 1·125	0·991	120/121 = 0·992 110/111 = 0·991 100/101 = 0·990
4271·75	1·239 (1·243)	5/4 = 1·250 21/17 = 1·235	1·094	11/10 = 1·100 23/21 = 1·095 12/11 = 1·091
4071·75	0·663 (0·665)	2/3 = 0·667 53/80 = 0·663	0·586	10/17 = 0·588 17/29 = 0·586 7/12 = 0·583
4063·61	1·081 (1·084)	13/12 = 1·083 27/25 = 1·080	0·955	22/23 = 0·957 21/22 = 0·955 20/21 = 0·952
4045·82	1·230 (1·234)	5/4 = 1·250 21/17 = 1·235 43/35 = 1·229	1·086	12/11 = 1·091 25/23 = 1·086 13/12 = 1·083

1) The values given in brackets are those obtained by assuming the specific separation of λ4383·55 is $1{\cdot}078\times10^{4}$, as given by Mr. Yamada, instead of taking that of λ4404·75.

We find in the table that the ratios cannot be expressed in simple fractions. p and q are not small enough except for the case of 4071·75, and we can not find the unit of separation common to these lines. When the ratios between one of these separations and the rests are taken, the fractions are less simple as shown in the last two columns.

The fact that iron and other elements rich in lines give divergent values of the specific separations seems to require some consideration. That $\frac{e}{m}$ is different for electrons radiating lines of different wave lengths is not plausible, at least for those which emit radiations of nearly equal wave lengths. I think it is more reasonable to attribute the divergent values of the specific separations to the change of the atomic field. If we accept the saturnian atom, it may be considered that the strong magnetic field, which is usually supposed to be present in the atom, originates in the electrons in their orbital motions. When the external magnetic field is applied, the orbit may be changed, giving rise to the change of the atomic field, which possibly amounts to a magnitude comparable with the external field. This change of the atomic field may affect the magnetic separation, and divergent values of the separations proportional to the magnetic field may appear, if the change of the atomic field is proportional to the external field applied. Those elements rich in lines must be rich in electrons with orbital motions, whose radii are considerably large even in the non-radiating state. The peculiar positions occupied by them in the periodic table and the low electro-potentials among others seem to support the assumption. It is natural that the change of the atomic field is large for those elements rich in electrons with orbital motions, and the divergent values of the specific separations obtained with the elements rich in lines can be explained. Besides the atomic magnetic field, the change of the electric field caused by the change of the orbits may play an important rôle.

For the other lines the measurements are not so accurate as for the seven lines above related, but many of them give separations proportional to the magnetic field so far as the present experiment goes, and some others, giving complicated or diffuse resolutions, show deviations from the proportionality.

5615·661, 5328·06, 4375·934, 4299·26, 4260·48, 4235·94, 4219·36, 4199·09, 4181·76, 4118·552 appear as sharp triplets, the separations being proportional to the magnetic field, but the values of the specific separations are different. The results with 5232·957, 4957·62, 4920·52, 4891·51, 4583·83, 4476·03, 4466·556, 4427·314, are not as exact as for the lines mentioned above, but the separations are approximately proportional to the magnetic field notwithstanding some possible disturbances of the neighbouring lines, or probably complicated separations measured as triplets may be expected for some lines.

5371·495, 5269·53 and 5167·492 show some sudden fall of the specific separation at about 7000 gauss. Though the fall of the first is considerable, that of the last is only slightly noticeable. The fall of 5269·53 may be caused by the disturbance of the neighbouring line 5270·35, so that it needs more experiments with different instruments to decide its character.

5586·772, 5572·86, 4143·88, 4132·08 and 4005·26, appear as diffuse triplets in weak fields. In the field above 20000 gauss, three of them are resolved into many components, the type of resolution of 4132·08 and 4005·26 being similar, though the latter was not measured exactly. Judged from their similar appearance, the first two lines also may give some complicated separations in strong fields.

The broadening of the components of 4528·622, 4494·572 and 4282·408 also seems to indicate some complicated resolutions. The curvatures in the separation diagrams may be due to the complicated resolutions, but the result of van Bilderbeek-van Meurs and of Graftdijk show that they are triplets even in the field of 32040 gauss.

5455·614 (quintuplet) and 5227·20 (triplet) show wavy fluctuations in the specific separations of the n-components, the separation of the p-components of the former being proportional to the magnetic field. p-components of 5446·92 and 5397·12 also show wavy fluctuations of the separations, though the measurement of the last is not sufficient.

5429·70 gives *p*-components whose separation suddenly increases at about 9500 gauss and the observed points on each side of the field lie on a separate straight line as if the measurements were made for different components. King is of opinion that they are possibly four *p*-components, but the specific separation given by him for the outer pair is nearly equal to that at the weaker field as obtained by the writer. The separation curve of this component somewhat resembles that of the *n*-component of 4325·78.

The separations of the *n*-components of 5018·45 (triplet), 4337·04 and 4202·04 (both quadruplet) curve more or less downwards at weaker fields.

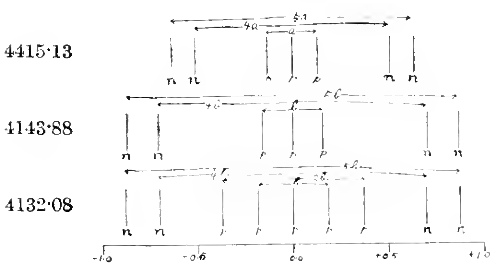
p-components of 4315·089 and 4250·79 indicate that the separations do not tend to zero when the magnetic field vanishes, though the *n*-components of the former give a separation proportional to the magnetic field. The separation of the *n*-components of 4442·34 also does not seem to vanish when the magnetic field vanishes, but that of the *p*-components is proportional to the field so far as my experiment goes.

The scattered points at the stronger fields in the separation curve of 4294·13 indicate that further resolution is possible if we increase the magnetic field or the resolving power of the instrument.

It seems that many iron lines are resolved in complicated manner, the components being spaced not in equal intervals, but they crowd together at their mean positions, so that we cannot resolve them without a strong magnetic field and an instrument of high resolving power. 4415·13 was resolved into 7-components with the strongest fields applied, but the *p*-component of 4325·78 was not resolved into fine components even with 37230 gauss, though its broadening and fringed appearance indicate the existence of a complicated resolution. 4132·08 and 4005·26 were observed to be resolvable at least into 11-components in a field stronger than 20000 gauss, and the diffuse components of many other lines also indicate further resolutions. Especially in the green region, many lines became so diffuse, when the magnetic field was increased, that I could not observe the separations in a strong field, owing partly to the increase of the luminosity of the continuous spectrum in the back ground and the decrease of the intensities of the iron lines, and partly to disturbances during the long exposure, but it seems to me that the complicated resolutions of many lines are the principal cause of failure.

The specific separation of 4415·13 seems to have a unit $a=0{\cdot}255\times10^{-4}$; the distance between the outer n-components and that between the inner are $5a$ and $4a$ respectively, and the distance between the outer pair of the p-components is a. The specific separations of 4143·88 and 4132·08 seem to have their common unit, namely $b=0{\cdot}351\times10^{-4}$, the distances between the n-components being $5b$ and $4b$ for both lines. Though we find considerable deviations for the separations of the p-components, *i. e.* the distance between the outer p-components of the former line is $0{\cdot}315\times10^{-4}$, and those between the outer and inner pairs of the latter are $2\times0{\cdot}371\times10^{-4}$, and $0{\cdot}371\times10^{-4}$, respectively, the discrepancy can be accounted for by the errors of measurements, as these separations are too small to be measured accurately. Hence we may take $0{\cdot}351\times10^{-4}$ as the unit of the separations of these two lines. But we can find no relation between a and b.

Fig. 54.



As for the nickel lines, the curvature in the separation curve of 5477·12 is clearly marked, but we must study the separation with the spark between pure nickel terminals or with concave grating to be able to fix it definitely. 4714·68 shows a separation curve similar to the iron line 5371·495, but the measurement is not so accurate. 4401·77 shows a smooth curvature in the separation curve, but the observed points are not numerous enough and the deviations are too small to speak of its existence; in fact, we can connect them by a straight line which does not pass through the origin.

Of the separations which are not proportional to the magnetic field, some may be caused by true coupling between radiating or non-radiating electrons, and others by experimental errors. Among the conditions giving rise to the appearance of false coupling, we may count the following :—

1. Errors in the determination of the magnetic field.
2. Apparent displacement of the mean position of a broad line in the double order position.
3. Errors in measuring the broad line.
4. Disturbance by other coincident or neighbouring lines known or unknown.
5. Complicated separation whose components are not thoroughly resolved.
6. Irregular contraction of the photographic film.

Those lines photographed on one plate with the same exposure as the standard line are free from the first error, as the standard line was measured most accurately. For those which were not photographed with the same exposure as the standard line, we can determine the field by measuring the photos of the standard line taken before and after the exposure for those in question without breaking the current through the electromagnet and carefully observing the constancy of the current. The separation of the standard line measured on the former agreed well with that on the latter in almost all cases, so that we may rely on the determination of the relative field.

The second is important for broad lines. This can be eliminated by taking the photos in different positions of the echelon spectra, bringing the undisturbed line in single order for some exposures and in double order for others.

The third is unavoidable, but the use of different types of cross wire somewhat improved the result. The cross wire of the micrometer used by the writer was constructed as shown in Fig. 55. *a* was mostly used for sharp lines to bisect the line in the

Fig. 55.

mean position. *b* was used for broad lines bringing the point of intersection of the wire on the mean position, and *c* was used for those which were photographed with the narrow second slit s_2 bringing the spot in the middle of the two adjacent wires.

The fourth is unavoidable, unless we use other instruments with different constants, or by crossing them. Plane grating crossed with echelon or other interferometer seems to be most convenient for such an element rich in lines as iron, though the experinent failed in the present work owing to insufficient intensity. While 4957·62, 4920·52 and 4891·51, which may be disturbed more or less by neighbouring lines, show separations approximately proportional to the field, 5477·12 (Ni), 5269·53, 4442·34 and 4250·79 give separations which are not proportional to the field with some indication of the disturbance caused by the neighbouring line.

In order to overcome the fifth difficulty, an instrument of high resolving power with considerably large value of $\delta\lambda_{max}$ is necessary, for we have to resolve the components in different positions. If $\delta\lambda_{max}$ is small, many components fall close to the others, and it is difficult to study their behavior with a sufficient number of fields. It is also necessary to apply a strong magnetic field. In the present experiment 4415·13, giving a separation which is not proportional to the magnetic field in weak fields, is resolved into seven components in fields above 35000 gauss. The separation of the *n*-components of 4447·72, given in Fig. 32, also shows false curvature, the mean position of the two components moving from the inner to the outer. The separations of other diffuse components may be subject to the same error, if they are formed of assemblages of several lines, and the measured positions correspond to the means.

To overcome the last difficulty it is necessary to measure the separation on different plates and increase the number of the observed points. In the present investigation I took great care to avoid these errors.

In conclusion the writer wishes to tender his coordial thanks to Prof. Nagaoka, at whose suggestion the present experiment was undertaken.

Summary.

1. The light emitted by the spark between non-magnetic nickel-steel terminals has been examined in different magnetic fields.

2. Many lines photographed at a juxtaposition with a prism spectrograph placed behind an echelon spectroscope have enabled us to compare their relative separations exactly.

3. The separations of the lines photographed were compared with that of the line 4404·75, and, assuming the latter to be proportional to the magnetic field, the behavior of the others was studied. The assumption is verified by the fact that the separations of many sharp lines are proportional to the separation of the standard line.

4. The magnetic fields are calculated assuming the specific separation of the standard line 4404·75 given by Mr. Yamada.

5. Nine strong lines in the violet region being studied carefully, the separations of seven of them were found to be exactly proportional to the magnetic field, while the others gave larger specific separations for stronger fields, the components appearing more and more diffuse with the increase of the field, and the line 4415·13 was resolved into seven components in strong fields.

6. Other less strong lines were also studied. Some of them give different types of separations, which are not proportional to the magnetic field, and the others show that their separations are proportional to the field. Specific separations are given for those measured with less accuracy or with only one or two fields.

7. Nickel lines photographed on the same plate as the iron lines are also given.

8. Many sharp lines give separations proportional to the magnetic field as if there was no mutual influence between the

radiating electrons, but the divergent values of the specific separations seem to indicate the existence of coupling. Moreover the separations of a number of lines which are not proportional to the magnetic field, seem to indicate the existence of some mutual influence, though the measurements are not sufficiently accurate.

9. Many iron lines apparently show simple resolutions into diffuse components, which are likely to be resolved again into fine components, if they are examined in stronger magnetic fields with an instrument of higher resolving power.

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Fig. 2.

$\lambda = 4415{\cdot}13$, $3p$-, $4n$-comps.

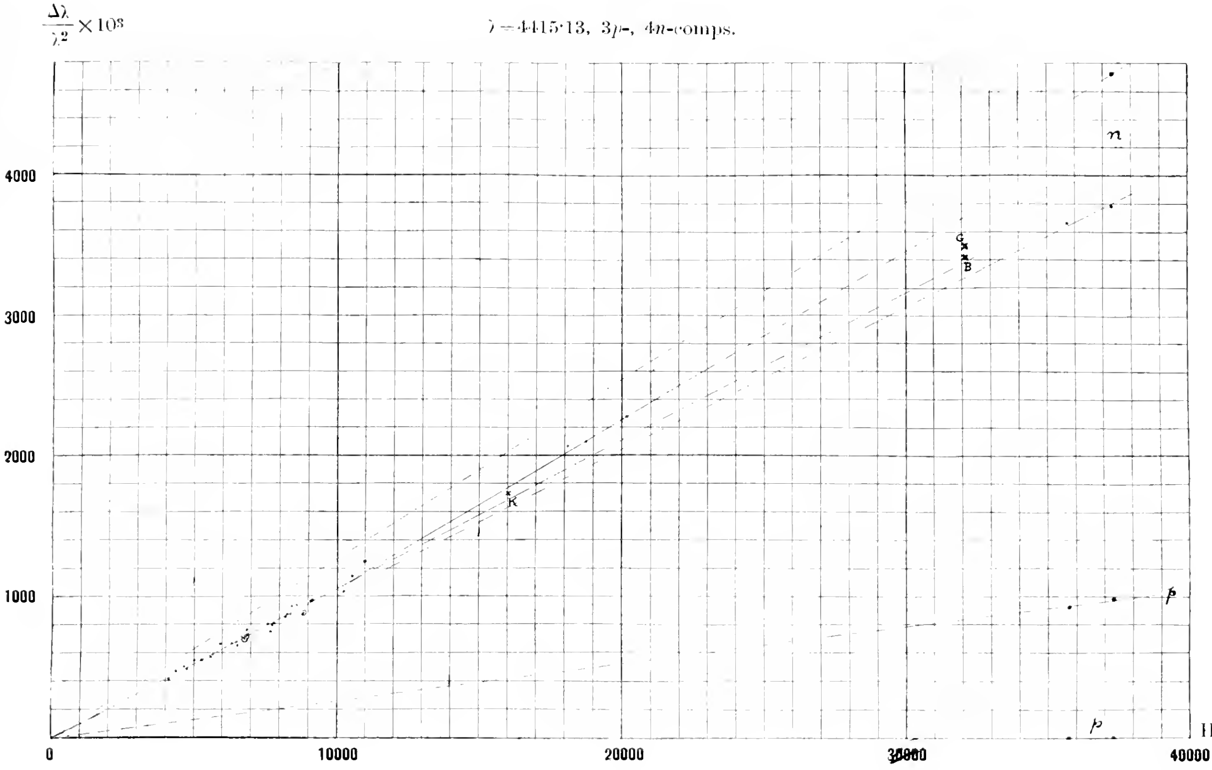
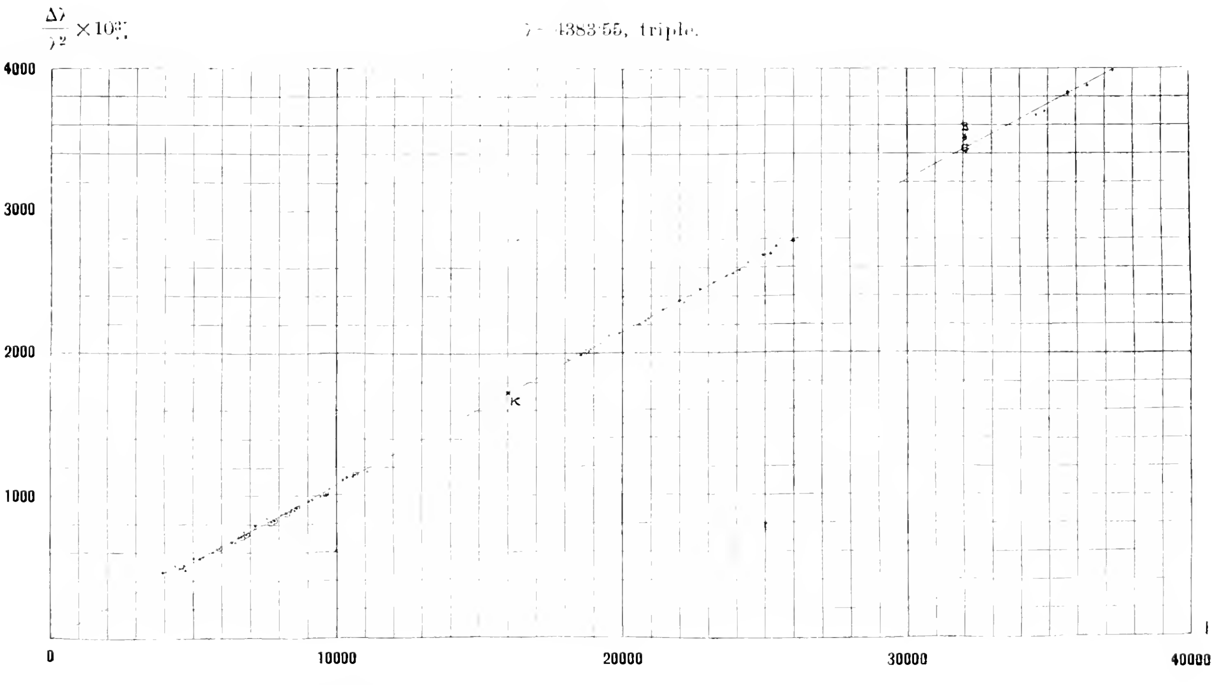


Fig. 3.

$\lambda = 4383{\cdot}55$, triple.



⦿ denotes an assemblage of two or more points.
B, *G* and *K* denote the points taken from the results of van Bilderbeek-van Meurs, Graftdijk and King respectively.

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Fig. 4.

$\lambda = 4325{\cdot}78$, quintuple?

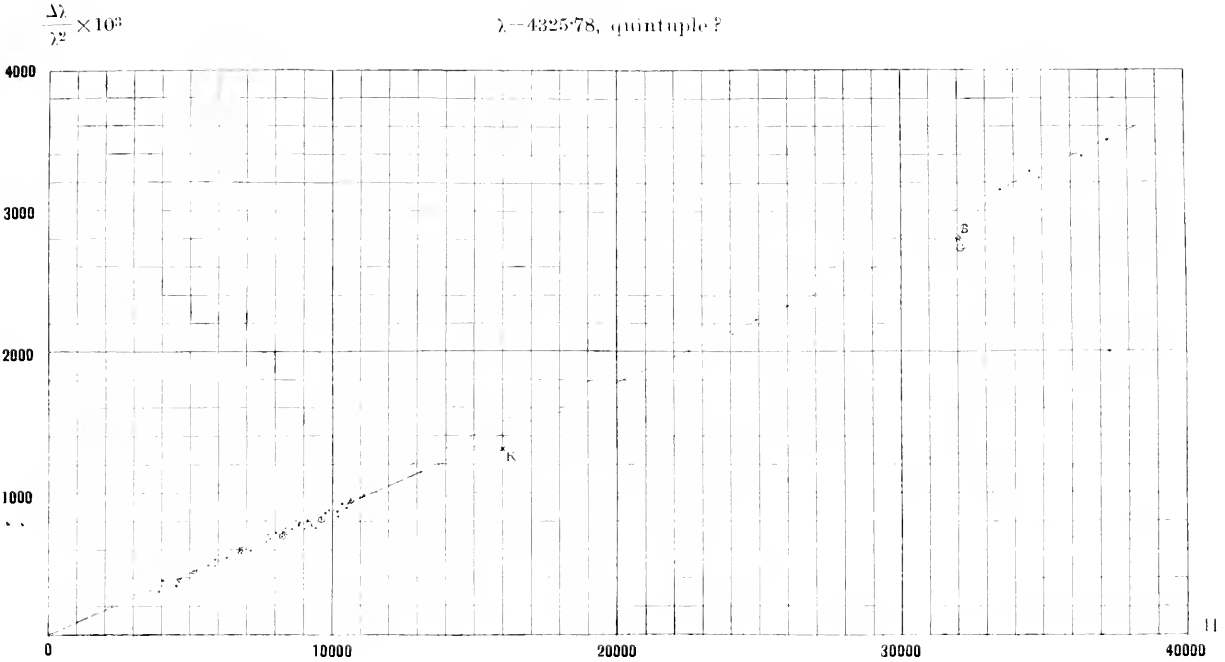

Fig. 5.

$\lambda = 4307{\cdot}92$, triple.

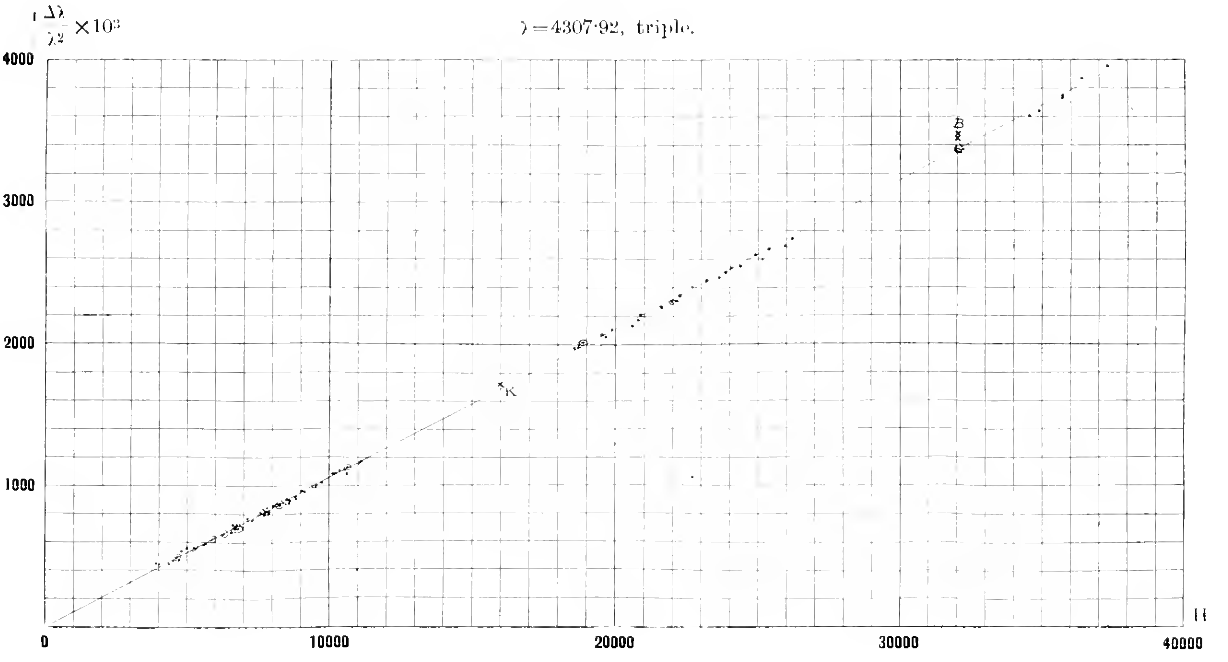

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Fig. 6.

$\lambda = 4271{\cdot}75$, triple.

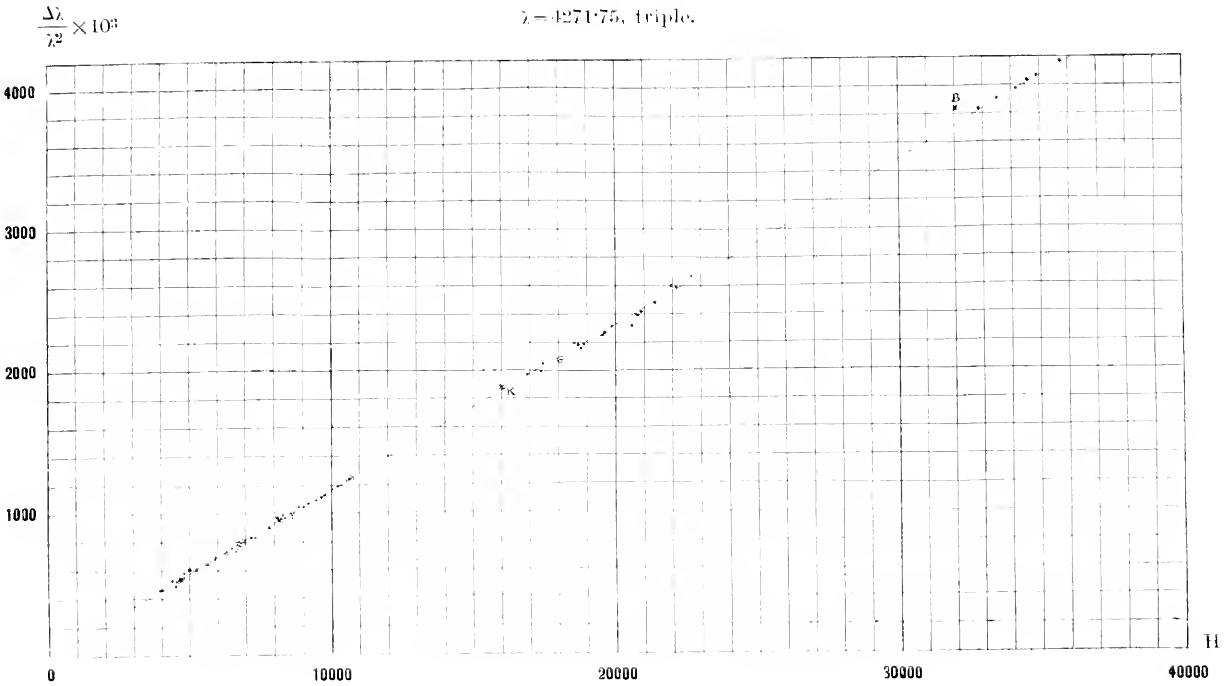

Fig. 7.

$\lambda = 4071{\cdot}75$, triple.

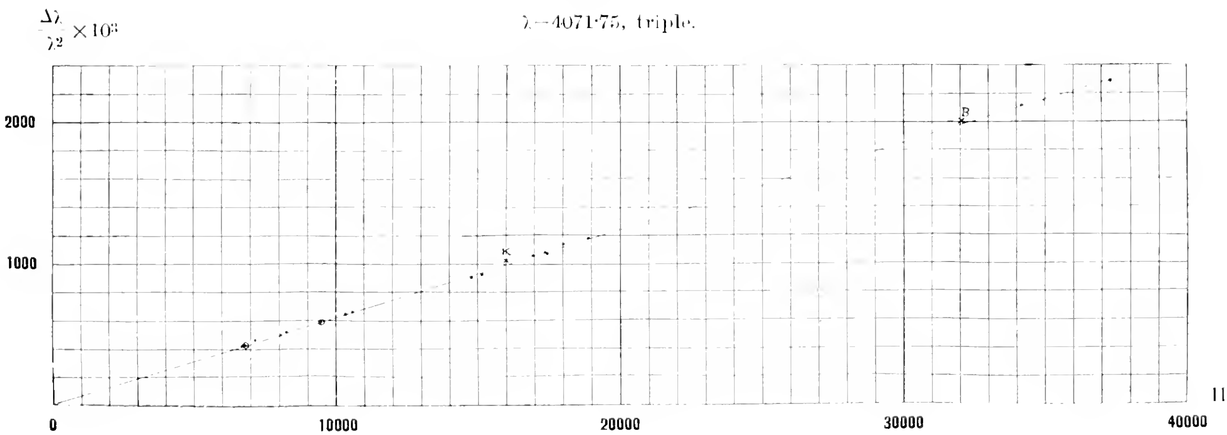

Fig. 8.

$\lambda = 4063{\cdot}61$, triple.

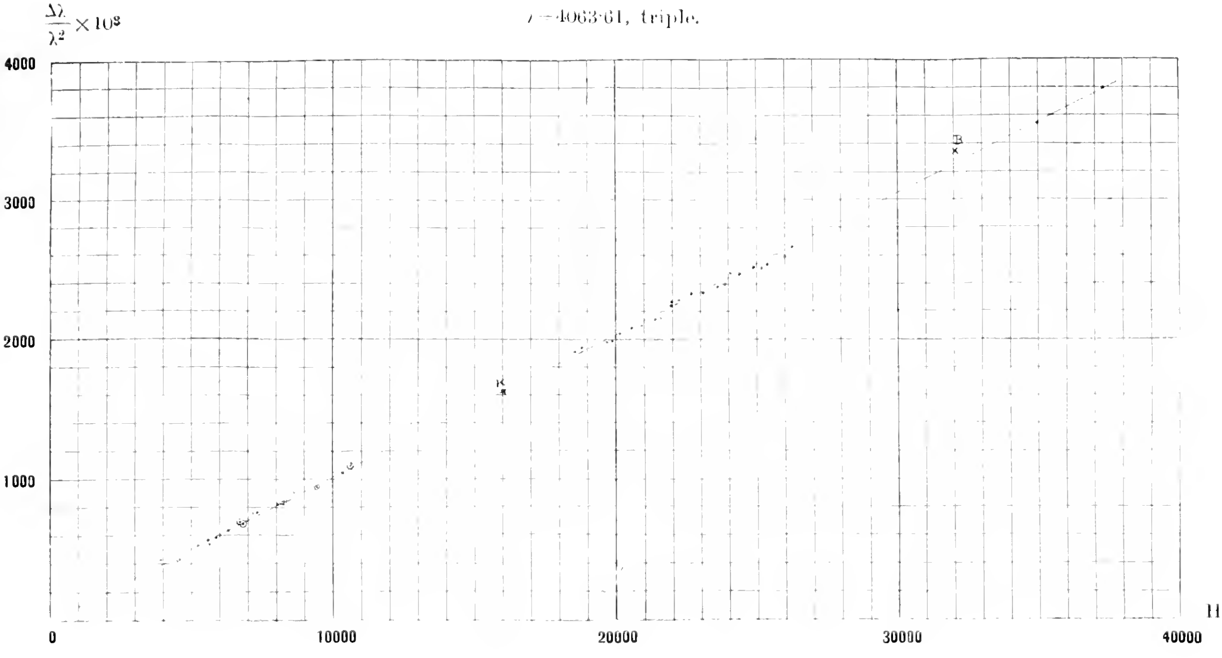

Fig. 9.

$\lambda = 4045{\cdot}82$, triple.

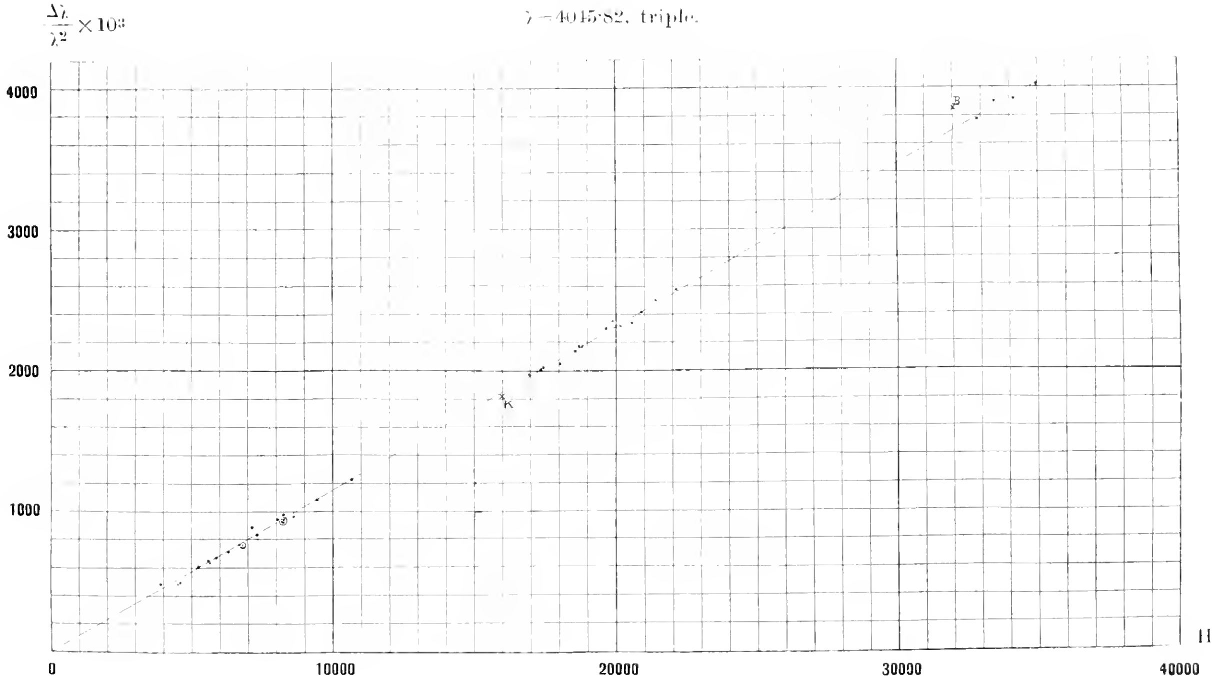

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Fig. 10.

λ=5615·661, triple.

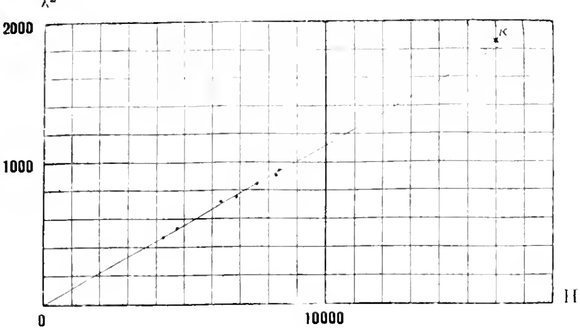

Fig. 13.

λ=5455·614, 2*p*-, 3*n*-comps.

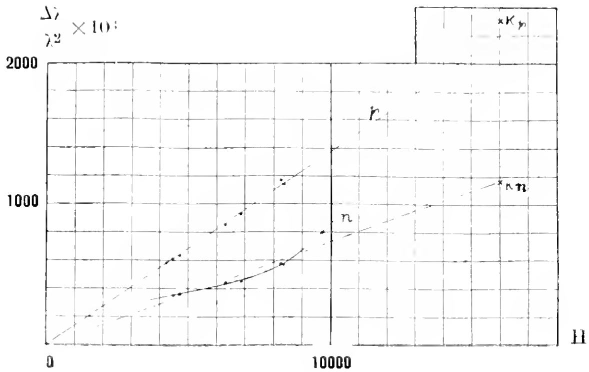

Fig. 11.

λ=5586·772, triple?

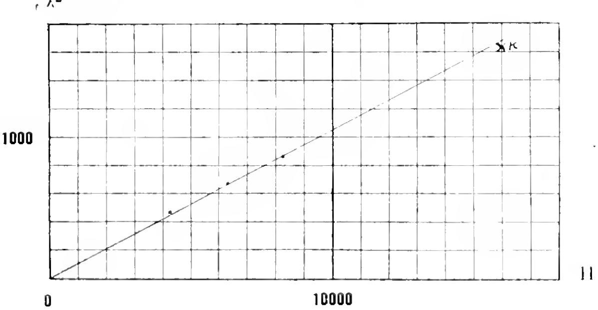

Fig. 14.

λ=5446·92, 2*p*-, 2*n*-comps.

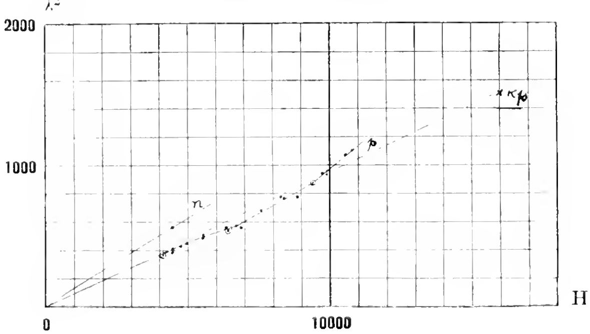

Fig. 12.

λ=5572·86, triple?

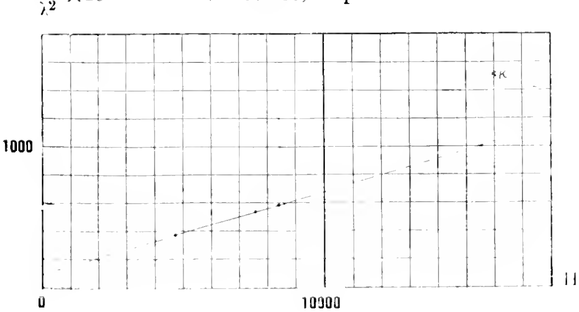

Fig. 15.

λ=5429·70, 2*p*-, 2*n*-comps.

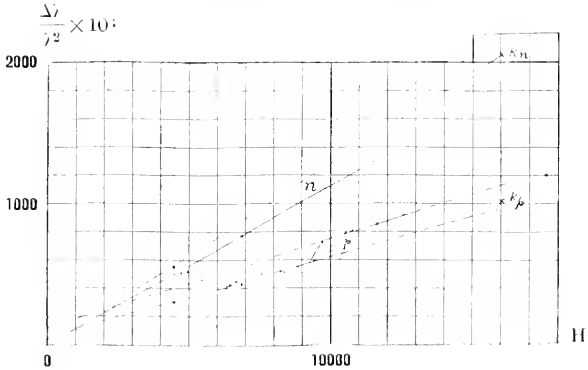

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Fig. 16.

$\lambda = 5397{\cdot}12$, $2p$-, $2n$-comps.

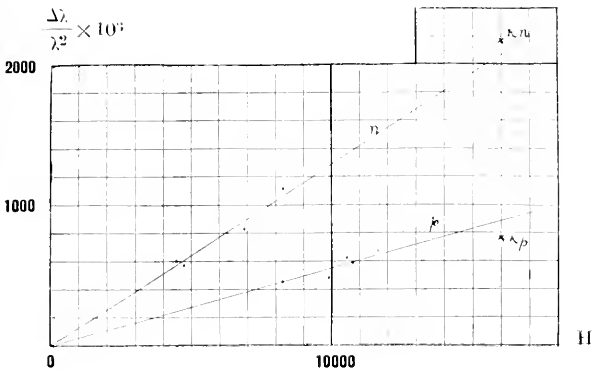

Fig. 19.

$\lambda = 5269{\cdot}53$, triple.

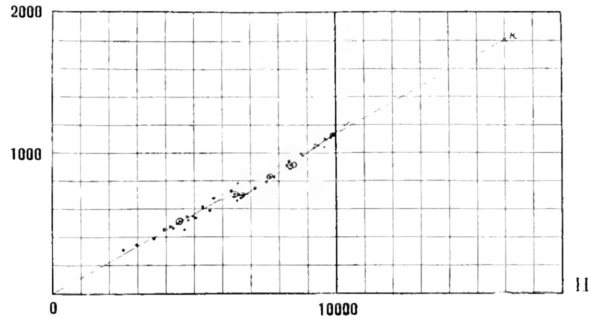

Fig. 17.

$\lambda = 5371{\cdot}495$, triple.

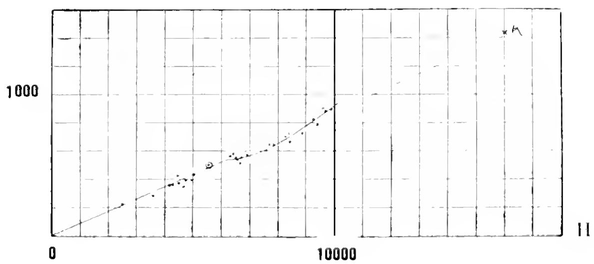

Fig. 20.

$\lambda = 5232{\cdot}957$, triple.

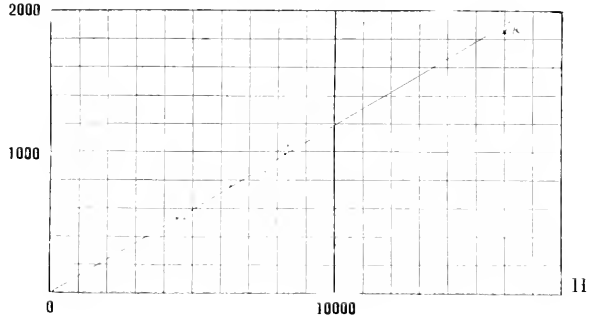

Fig. 18.

$\lambda = 5328{\cdot}06$, triple.

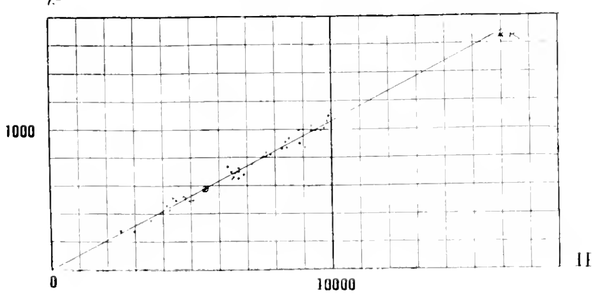

Fig. 21.

$\lambda = 5227{\cdot}20$, triple.

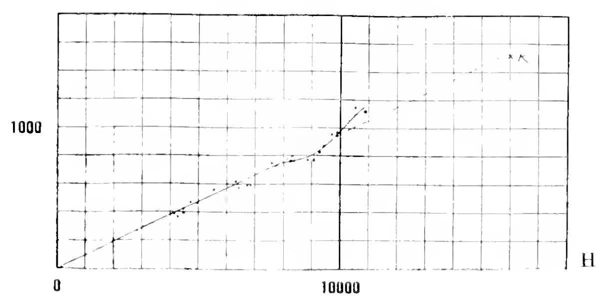

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Fig. 22.

$\lambda=5167{\cdot}492$, triple.

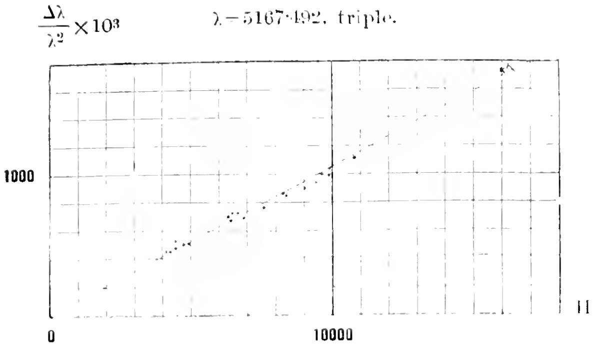

Fig. 23.

$\lambda=5018{\cdot}45$, triple.

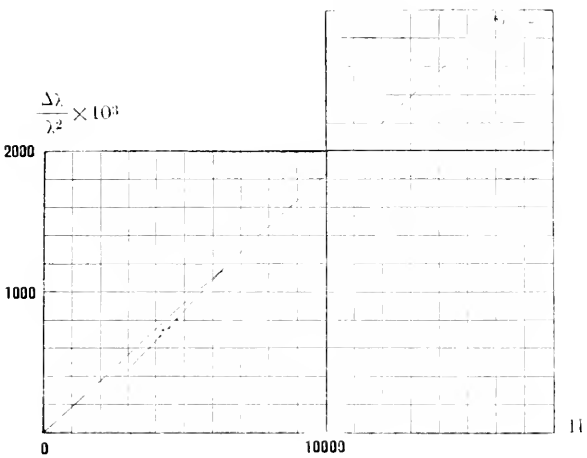

Fig. 24.

$\lambda=4957{\cdot}62$, triple.

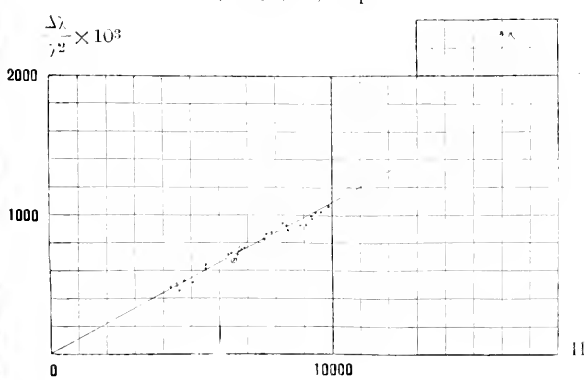

Fig. 25.

$\lambda=4920{\cdot}52$, triple.

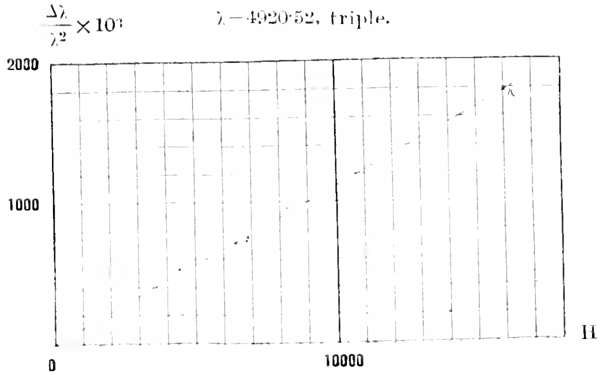

Fig. 26.

$\lambda=4891{\cdot}51$, triple.

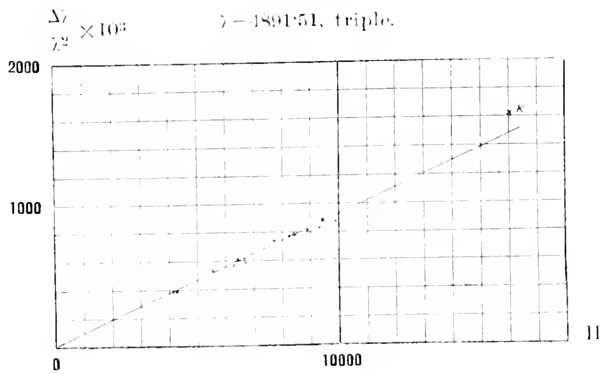

Fig. 27.

$\lambda=4583{\cdot}83$, triple.

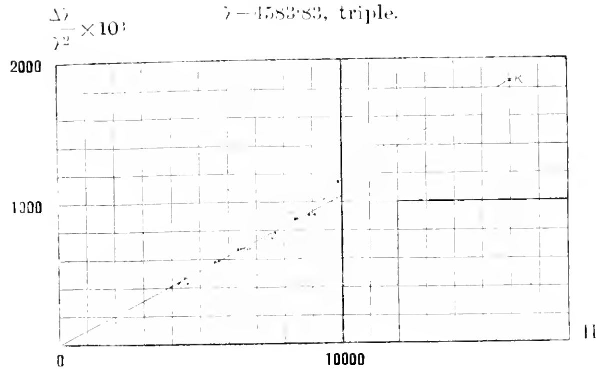

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Fig. 28.

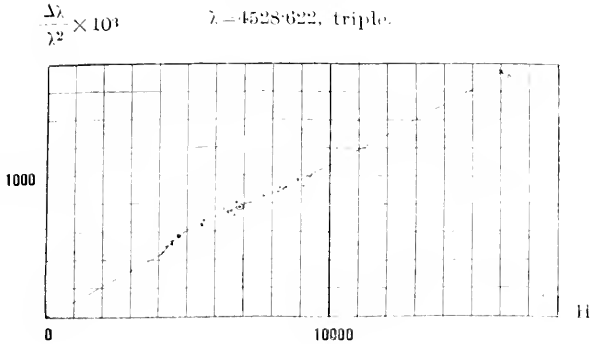


Fig. 29.

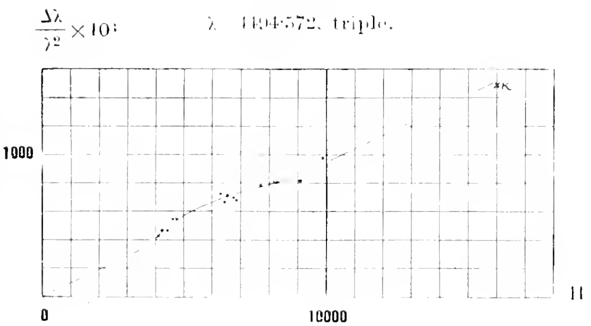


Fig. 30.

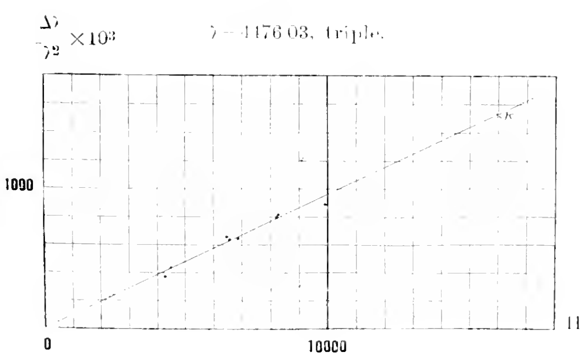


Fig. 31.

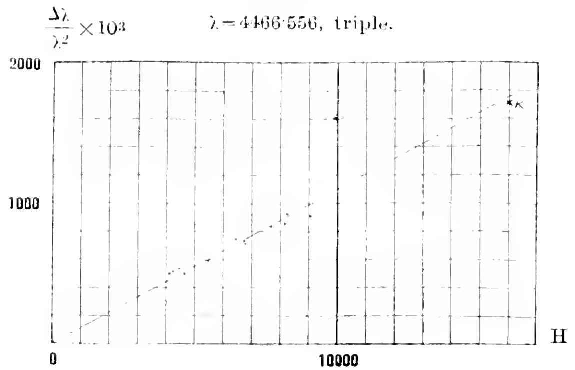


Fig. 32.

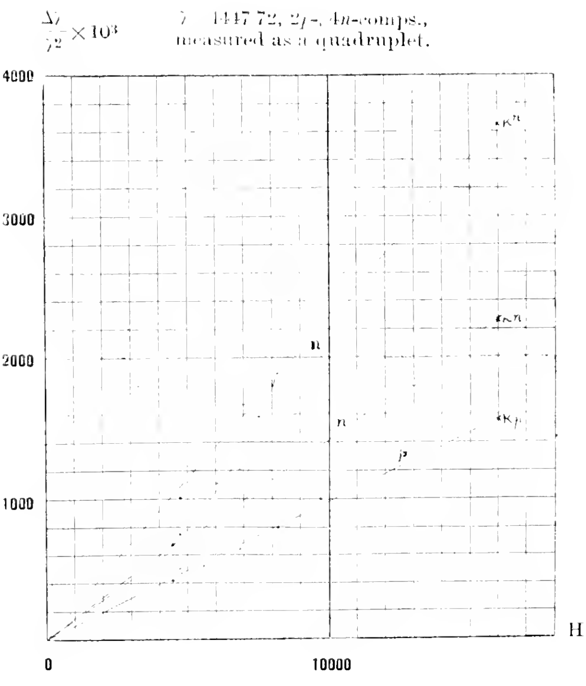


Fig. 33.

λ=4442·34. 2p-, 2n-comps.

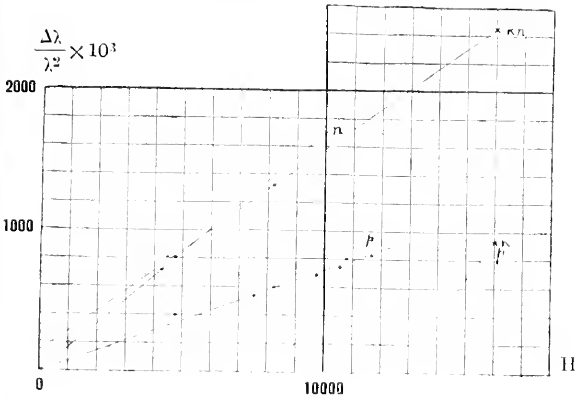


Fig. 34.

λ=4427·314, triple.

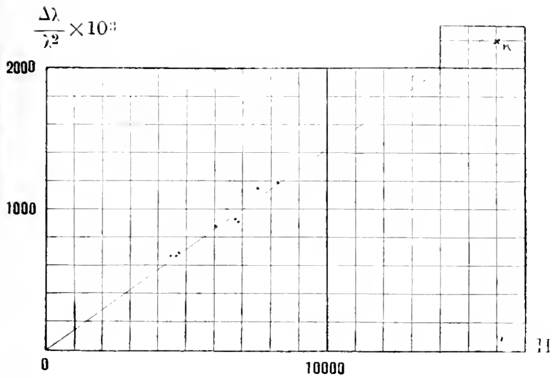


Fig. 35.

λ=4375·934, triple.

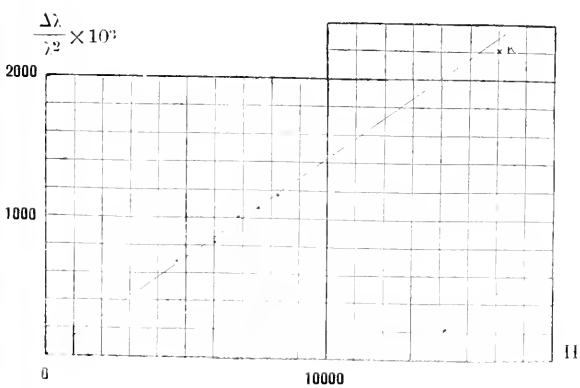


Fig. 36.

$\frac{\Delta\lambda}{\lambda^2}\times 10^3$ λ=4337·04, 2p-, 2n-comps.

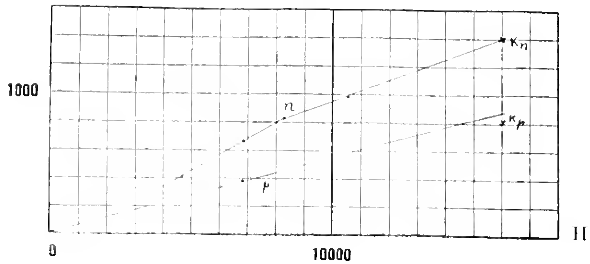


Fig. 37.

λ=4315·089, 2p-, 2n-comps.

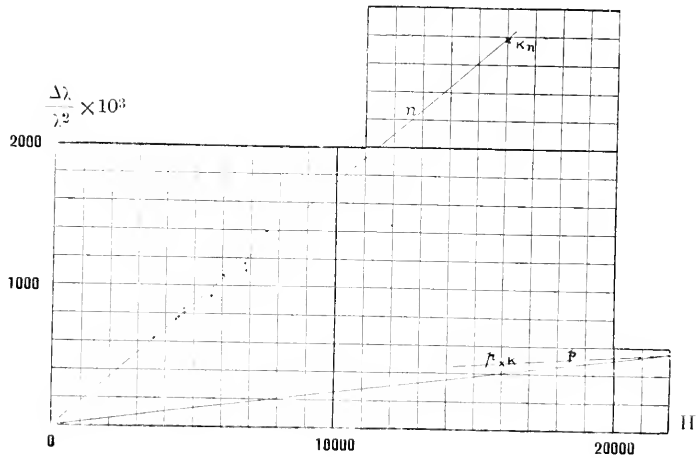


Fig. 38.

λ=4299·26, triple.

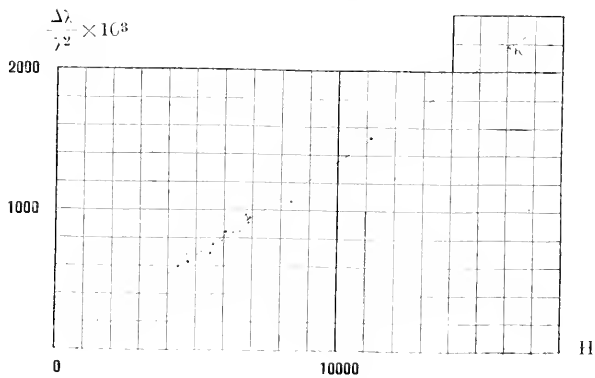


Fig. 39.

$\frac{\Delta\lambda}{\lambda^2} \times 10^3$ λ=4294·13, 2*p*-, 2*n*-comps.

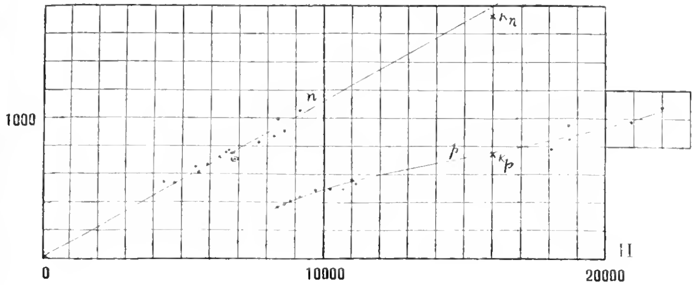


Fig. 42.

$\frac{\Delta\lambda}{\lambda^2} \times 10^3$ λ=4250·79, 2*p*-, 2*n*-comps.

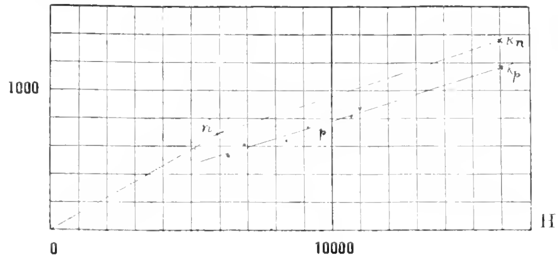


Fig. 40.

$\frac{\Delta\lambda}{\lambda^2} \times 10^3$ λ=4282·408, triple.

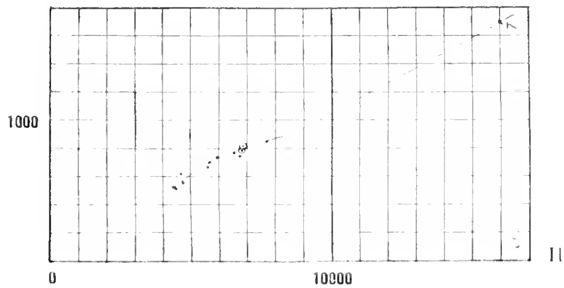


Fig. 43.

λ=4235·94, triple.

$\frac{\Delta\lambda}{\lambda^2} \times 10^3$

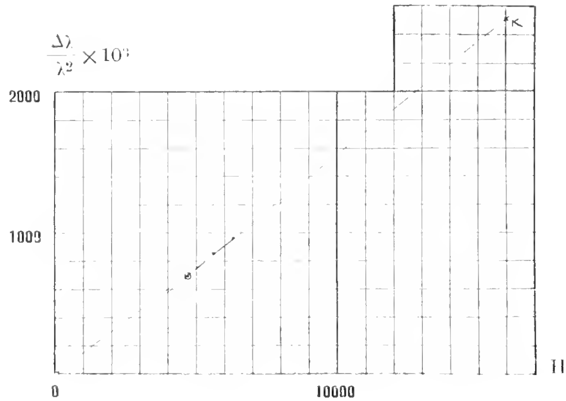


Fig. 41.

λ=4260·48, triple.

$\frac{\Delta\lambda}{\lambda^2} \times 10^3$

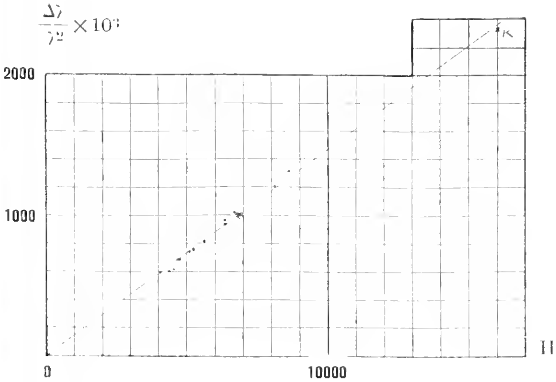


Fig. 44.

$\frac{\Delta\lambda}{\lambda^2} \times 10^3$ λ=4219·36, triple.

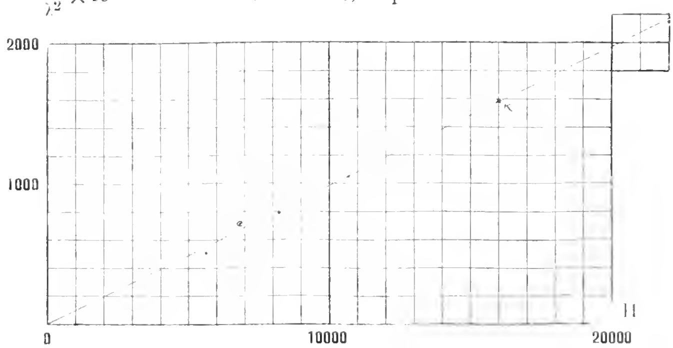


Fig. 45.

$\lambda = 4202 \cdot 04$, $2p$-, $2n$-comps.

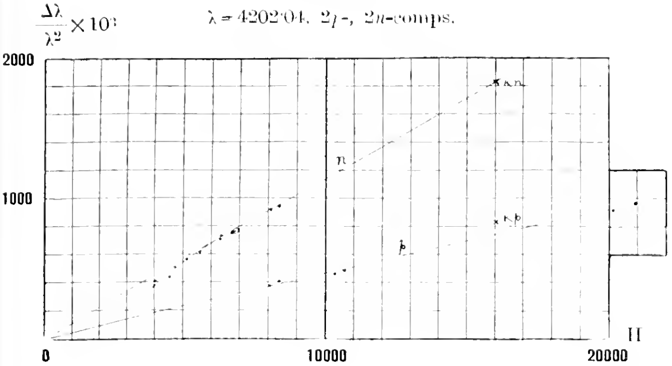


Fig. 48.

$\lambda = 4143 \cdot 88$, at least $3p$-, $2n$-comps.
The separation at 2196 gauss is not given in the figure.

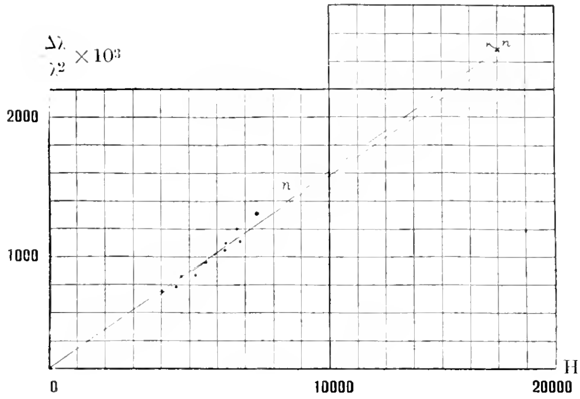


Fig. 46.

$\lambda = 4199 \cdot 09$, triple.

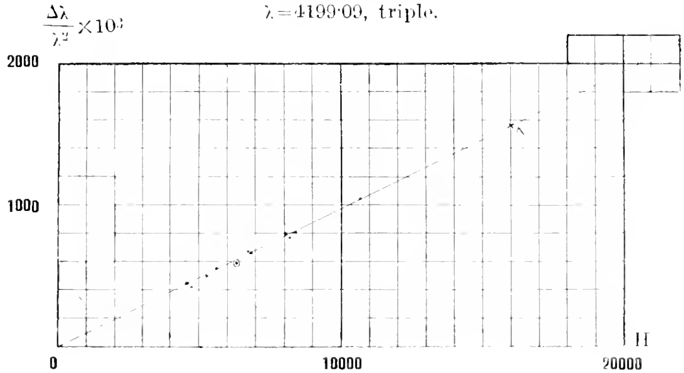


Fig. 47.

$\lambda = 4181 \cdot 76$, triple.

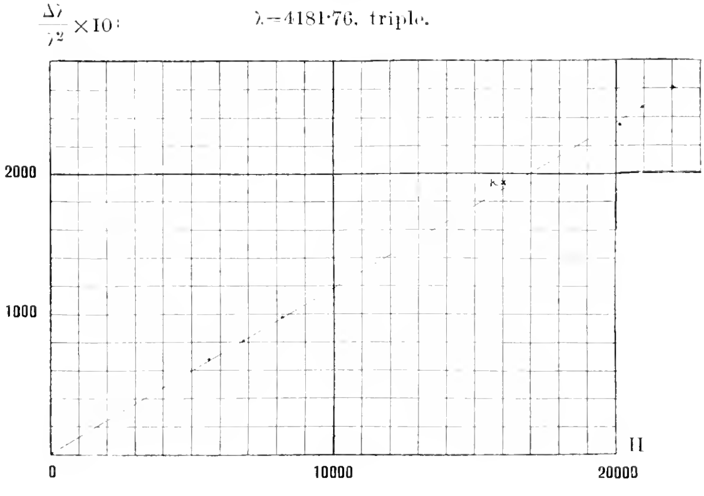


Fig. 49.

$\lambda=4132{\cdot}08$, at least $5p$-, $4n$-comps.

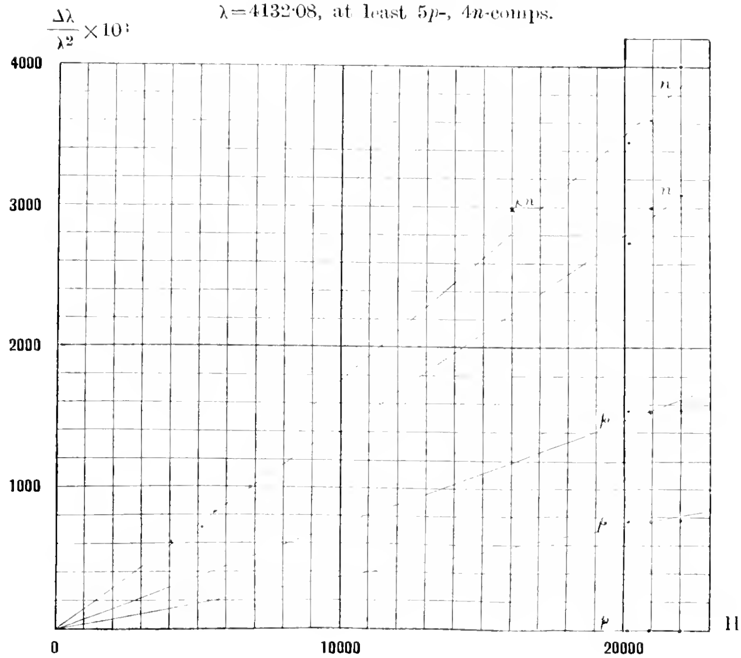


Fig. 50.

$\lambda=4118{\cdot}552$, triple.

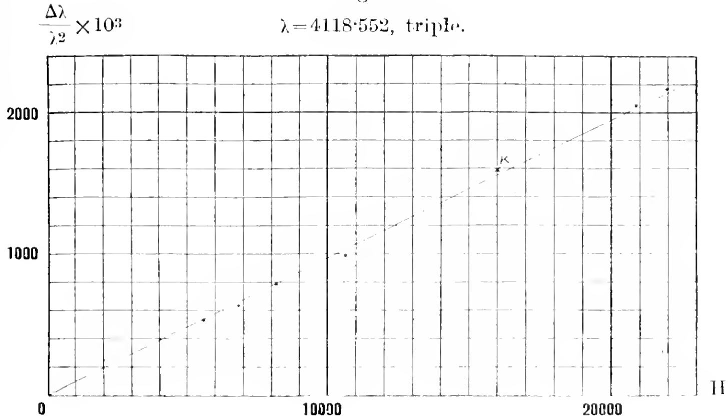


Fig. 52.

$\lambda=4714{\cdot}68$ (Ni), triple.

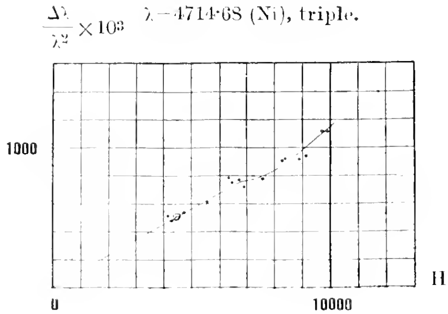


Fig. 51.

$\lambda=5477{\cdot}12$ (Ni), triple.

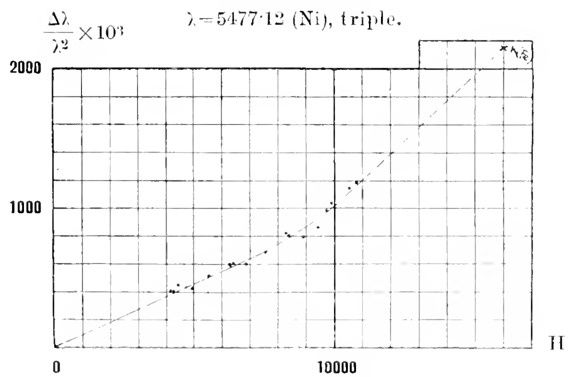
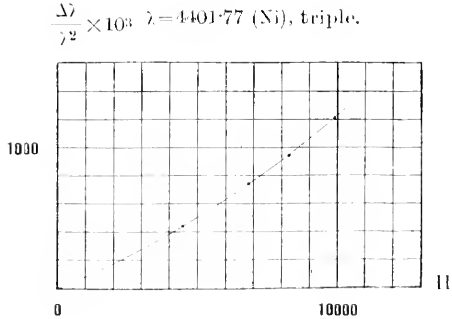
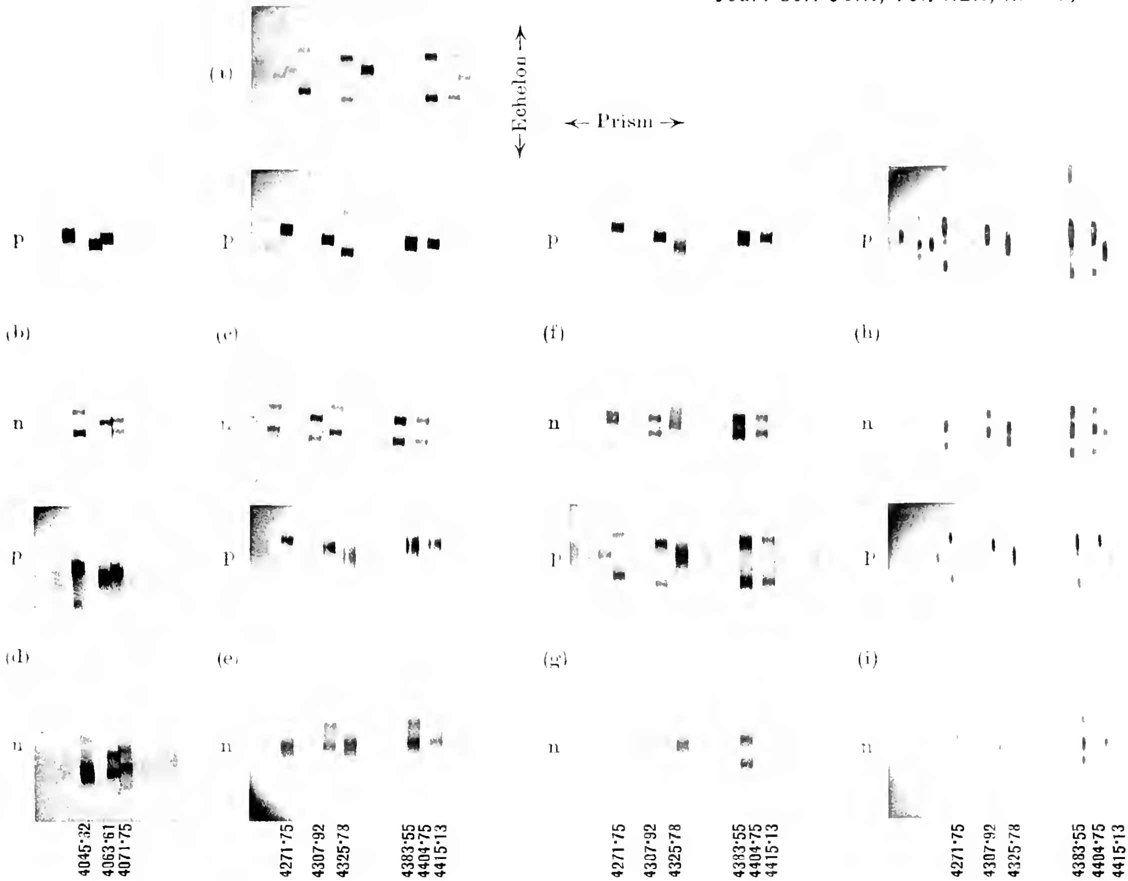


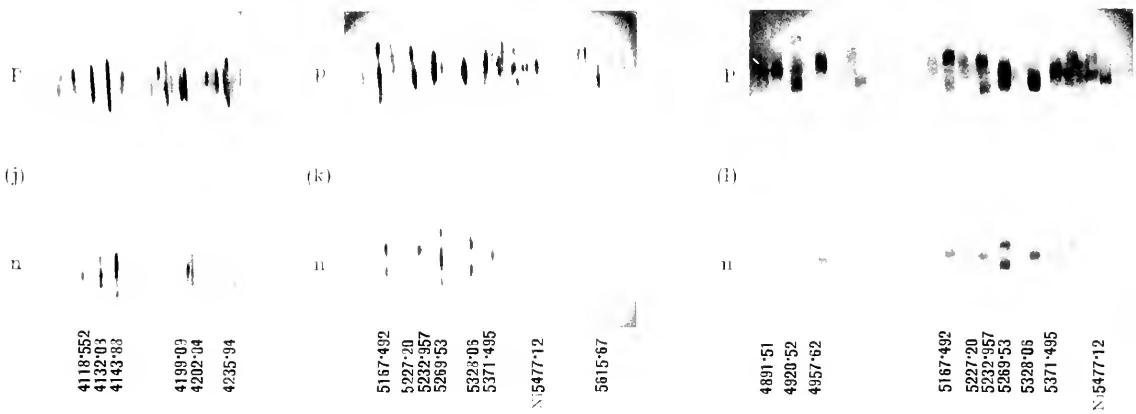
Fig. 53.

$\lambda=4401{\cdot}77$ (Ni), triple.





(a), H=0.
(b), (c), H=7170, each line is resolved into a sharp triplet.
(d), (e), H=37230, both p-and n-comps. of 4415·13 are resolved into 3 and 4 lines respectively. p-comp. of 4325·78 is fringed.
(f), H=24040, p-comp. of 4325·78 is diffuse.
(g), H=35650, both p-and n-comps. of 4415·13 are resolved. p-comp. of 4325·78 is diffuse.
(h), H=6820, both p-and n-comps. of 4415·13 are sharp.
(i), H=20880, ,, ,, ,, diffuse.



(j), H=21960, p-and n-comps. of 4143·88 and 4132·08 are resolved into many lines.
(k), H=8230.
(l), H=6490.

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JOURNAL OF THE COLLEGE OF SCIENCE, TOKYO IMPERIAL UNIVERSITY.
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Ueber eine Theorie des relativ Abel'schen Zahlkörpers.

Von

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Der vorliegende Aufsatz ist die ausführliche Darlegung einer Theorie des relativ Abel'schen Zahlkörpers, deren Umriss vor einigen Jahren in den *Proceedings* der hiesigen Mathematisch-Physikalischen Gesellschaft sehr knapp und mangelhaft skizzirt worden ist.

Diese Theorie stützt sich auf den verallgemeinerten Begriff der Idealclassen, welcher sich in der modernen Theorie der algebraischen Zahlen allmählich entwickelt, und durch Heinrich Weber eine explicite Formulirung in der sehr allgemeinen Form gefunden hat. Es werden danach zwei Ideale eines algebraischen Körpers nur dann als aequivalent betrachtet und in dieselbe Idealclasse gerechnet, wenn ihr Quotient durch eine Zahl dargestellt werden kann, welche gewisser Congruenzbedingung nach einem vorgeschriebenen Idealmodul des Körpers genügt. Es existirt alsdann zu einem beliebigen algebraischen Zahlkörper ein bestimmter relativ Abel'scher Oberkörper von der folgenden Beschaffenheit:

1) Die Relativdiscriminante des Oberkörpers enthält die und nur die Primideale als Factor, welche in den Idealmodul des Grundkörpers aufgehen, der der Classeneinteilung in demselben zu Grunde gelegt wird.

2) Die Galois'sche Gruppe des Oberkörpers in Bezug auf den Grundkörper ist holoedrisch isomorph mit der Classengruppe (im verallgemeinerten Sinne) des Grundkörpers.

3) Diejenigen Primideale des Grundkörpers, welche der Hauptclasse (im verallgemeinerten Sinne) angehören und nur diese erfahren im Oberkörper eine Zerlegung in die Primfactoren der ersten Relativgrade; allgemeiner hängt die weitere Zerlegung der Primideale des Grundkörpers in dem Oberkörper nur von der Classe ab, der die Primideale im Grundkörper angehören.

Es ist dies eine naturgemässe Verallgemeinerung der Grundeigenschaften des Classenkörpers, welcher zuerst von D. Hilbert eingeführt wurde und die Theorie desselben von Ph. Furtwängler weiter fortgeführt worden ist. Jener Oberkörper sei daher als der allgemeine Classenkörper für die zugehörigen Idealengruppe des Grundkörpers bezeichnet, welche Gruppe die Hauptclasse (im verallgemeinerten Sinne) des Grundkörpers bildet.

Eine wichtige Tatsache in der Theorie des relativ Abel'schen Zahlkörpers ist nun die, dass umgekehrt zu jedem relativ Abel'schen Oberkörper eine bestimmte Classengruppe nach einem geeignet gewählten Idealmodul in dem Grundkörper existirt, welcher jener Oberkörper als Classenkörper zugeordnet ist, so dass die relativ Abel'schen Oberkörper einerseits und die Idealengruppen in dem Grundkörper anderseits einander characterisirend in wechselseitig eindeutiger Beziehung stehen.

Ich habe so weit als möglich diese Theorie ohne die üblichen Voraussetzung entwickelt, dass der Grundkörper die Einheitswurzeln enthalte; hierbei haben sich die von Hilbert eingeführten, einem Primideal in relativ normalen Körper zugehörigen Körper, welche die weitere Zerlegung des Primideals des Grundkörpers behersschen, als ein sehr nützliches Hülfsmittel erwiesen.

Unter den Anwendungen dieser Theorie sei der Existenzbeweis für die unendlichvielen Primideale ersten Grades in jeder Classe (im verallgemeinerten Sinne) eines beliebigen algebraischen Zahlkörpers hervorgehoben; es ist dies eine schöne Verallgemeinerung des classischen Dirichlet'schen Satzes über die Primzahlen in einer arithmetischen Reihe.

Als ein Beispiel und eine naheliegende Anwendung der allgemeinen Theorie habe ich die der relativ Abel'schen Körper in Bezug auf einen imaginären quadratischen Körper in einem besonderen Capitel behandelt. Es gelang die Bestätigung der berühmten Kronecker'schen Vermutung über die aus der Theorie der complexen Multiplication der elliptischen Functionen entspringenden Körper vollständig durchzuführen, was durch H. Weber und R. Fueter (in der unten citirten Abhandlung) nur zum Teil geschehen ist.

In Verzicht auf die vollständige Litteraturangabe seien die folgenden Werke angeführt, die, sei es als Grundlage, sei es als Anregung, für diese Untersuchung von Wichtigkeit gewesen sind:

H. Weber, Ueber Zahlengruppen in algebraischen Körpern. Math. Ann. 48, 49, 50. (1897–1898).

H. Weber, Lehrbuch der Algebra. III. (1908).

D. Hilbert, Die Theorie der algebraischen Zahlkörper. Bericht, erstattet der Deutschen Mathematiker-Vereinigung, 1897.

D. Hilbert, Ueber die Theorie des relativ quadratischen Zahlkörpers, Math. Ann. 51 (1898).

D. Hilbert, Ueber die Theorie der relativ Abel'schen Zahlkörper. Nachrichten von der Kgl. Gesellschaft der Wissenschaften in Göttingen, 1898.

Ph. Furtwängler, Allgemeiner Existenzbeweis für den Classenkörper eines beliebigen algebraischen Zahlkörpers. Math. Ann. 63 (1907).

R. Fueter, Abel'sche Gleichungen in quadratisch-imaginären Zahlkörpern, Math. Ann. 75 (1914).

CAPITEL I.

Der allgemeine Classenkörper.

§. 1.

Verallgemeinerung des Classenbegriffs.

Bekanntlich heissen zwei Ideale $\mathfrak{a}$, $\mathfrak{b}$ in einem algebraischen Körper k äquivalent, wenn es in k eine ganze oder gebrochene Zahl $\varkappa$ gibt, so dass die Gleichheit besteht:

$$\mathfrak{a} = \varkappa\, \mathfrak{b}.$$

Die Gesamtheit aller Ideale, welche einem gegebenen aequivalent sind, fassen wir in eine Idealclasse zusammen. Dann ist die Anzahl h der Idealclassen im Körper k endlich. Diese Classen lassen sich durch Multiplication zusammensetzen: sind nämlich A, B irgend zwei Classen, $\mathfrak{a}$, $\mathfrak{b}$ beliebige Ideale dieser Classen, dann gehört das Product $\mathfrak{a}\mathfrak{b}$ einer durch die Classen A, B eindeutig bestimmte, von der Wahl der Representanten $\mathfrak{a}$, $\mathfrak{b}$ unabhängige Classe AB. Die h Classen bilden in der Tat eine Abel'sche Gruppe, in welcher die Multiplication als die Regel der Zusammensetzung gilt, und die Hauptclasse die Stelle des Hauptelementes einnimmt.

Man kann auch die Gesamtheit der ganzen und gebrochenen Ideale des Körpers k als eine (unendliche) Abel'sche Gruppe G auffassen, indem wir die Ideale durch Multiplication zusammensetzen. Dann bilden eben die Gesamtheit der ganzen oder gebrochenen Hauptideale eine Untergruppe O vom Index h; sind $\mathfrak{a}_1, \mathfrak{a}_2, \ldots \mathfrak{a}_h$ ein System der Representanten der h Classen, dann ist in einer, in der Gruppentheorie üblichen, Bezeichnungsweise:

$$G = O\,\mathfrak{a}_1 + O\,\mathfrak{a}_2 + \cdots + O\,\mathfrak{a}_h. \tag{1}$$

Eine engere Fassung des Classenbegriffs hat sich bei den verschiedenen Problemen als von Nutzen erwiesen. Es werden die Ideale $\mathfrak{a}$, $\mathfrak{b}$ nur dann als aequivalent aufgefasst und in eine und dieselbe Classe gerechnet, wenn ihr Quotient einem Hauptideale $(\varkappa)$ gleich ist, wo $\varkappa$ gewisser Bedingungen betreffs des Vorzeichens unterworfen ist. Es ist zum Beispiel verlangt, dass $\varkappa$ positive Norm habe[1], oder dass $\varkappa$ total positiv sei,[2] d.h. die mit $\varkappa$ conjugirten Zahlen in den sämtlichen mit k conjugirten reellen Körpern $k_1, k_2 \ldots k_r$ positiv seien. Solche Vorzeichenbedingungen lassen sich in allgemeinster Weise wie folgt auffassen: Das System der Vorzeichen, welche die mit $\varkappa$ conjugirten Zahlen in $k_1\ k_2 \ldots k_r$ aufweisen, sei mit

$$(\varepsilon_1, \varepsilon_2, \cdots\cdots \varepsilon_r)$$

bezeichnet, wo $\varepsilon = \pm 1$ ist; wir wollen es kurz die *Vorzeichencom-*

1) Vgl. Hilbert, Bericht, § 24.
2) Hilbert, Relativ Abel. Zahlkörper, § 5.

bination der Zahl $\varkappa$ nennen. Dann bilden die 2^r möglichen Vorzeichencombinationen eine Gruppe nach Multiplication, welche mit der Gruppe der entsprechenden Zahlen homomorph ist, d.h., ist

$$(\varepsilon_1', \varepsilon_2', \cdots\cdots \varepsilon_r')$$

die Vorzeichencombination von $\varkappa'$, dann ist die Vorzeichencombination der Zahl $\varkappa\varkappa'$ das *Product*

$$(\varepsilon_1\varepsilon_1', \varepsilon_2\varepsilon_2', \cdots\cdots \varepsilon_r\varepsilon_r').$$

Sei nun $\mathfrak{H}$ eine Untergruppe dieser Gruppe der sämtlichen 2^r Vorzeichencombinationen, und verlangt man, dass die Idealquotient $\varkappa$ eine Vorzeichencombination dieser Gruppe $\mathfrak{H}$ haben soll, dann ist damit ein engerer Classenbegriff definirt, wobei die Hauptclasse diejenige Untergruppe $\mathfrak{o}'$ der Gruppe $\mathfrak{o}$ der sämtlichen Hauptideale des Körpers k ist, welche nur die Hauptideale $(\varkappa)$ enthält, welche durch die Zahlen $\varkappa$ mit den Vorzeichencombinationen von $\mathfrak{H}$ erzeugt werden. An Stelle von (1) hat man nunmehr die neue Classeneinteilung:

$$\mathfrak{G} = \mathfrak{o}'\mathfrak{a}_1 + \mathfrak{o}'\mathfrak{a}_2 + \cdots\cdots \mathfrak{o}'\mathfrak{a}_{h'},$$

wo h' die Classenzahl von k im neuen, engeren Sinne ist, und es zerfällt jede Classe $\mathfrak{o}\mathfrak{a}$ im alten, weiteren Sinne in eine dieselbe Anzahl $\frac{h'}{h}$ von den Classen $\mathfrak{o}'\mathfrak{a}$ im engeren Sinne, wo die Zahl $\frac{h'}{h}$ offenbar ein Teiler von dem Index der Gruppe $\mathfrak{H}$, d.h. von 2^{r-r_0} ist, wenn 2^{r_0} die Ordnung der Gruppe $\mathfrak{H}$ ist.

Eine andere Erweiterung des Classenbegriffs erblicken wir in die sogenannten Ringclassen.[1] Es sei $\mathfrak{R}$ ein Zahlring im Körper k, $\mathfrak{f}$ der Führer desselben. Zwei zum Führer $\mathfrak{f}$ relativ prime Ringideale $\mathfrak{a}_R$ und $\mathfrak{b}_R$ werden dann aequivalent genannt, und danach die Ringclassen definirt, wenn

$$\mathfrak{a}_R = \varkappa\, \mathfrak{b}_R,$$

wo $\varkappa$ eine Körperzahl ist, mit oder ohne Vorzeichenbedingung.

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1) Vgl. Hilbert, Bericht, §§ 33, 34.

Ist nun α eine Zahl in $\mathfrak{a}_R$, dann muss in $\mathfrak{b}_R$ eine Zahl β geben, derart, dass

$$\alpha = \varkappa\beta, \quad \text{oder} \quad \varkappa = \frac{\alpha}{\beta};$$

so erscheint $\varkappa$ als Quotient zweier zu $\mathfrak{f}$ primen Ringzahlen. Wenn umgekehrt $\mathfrak{a}$, $\mathfrak{b}$ zwei zu $\mathfrak{f}$ prime Körperideale sind, und besteht zwischen ihnen die Gleichung:

$$\mathfrak{a} = \varkappa\,\mathfrak{b},$$

wo $\varkappa$ ein Quotient der Ringzahlen ist, dann besteht für die zugeordneten Ringideale die Relation:

$$\mathfrak{a}_R = \varkappa\,\mathfrak{b}_R.$$

Es kommt daher auf dasselbe hinaus, wenn man unter G die Gesamtheit der zu $\mathfrak{f}$ primen ganzen oder gebrochenen Körperideale versteht, unter $\mathfrak{o}$ die Gesamtheit der Hauptideale, welche durch die Quotienten der zu $\mathfrak{f}$ primen Ringzahlen, eventuell mit Vorzeichenbedingungen, erzeugt werden, und die Gruppe G nach dieser Untergruppe $\mathfrak{o}$ in die Complexe der Form $\mathfrak{o}\mathfrak{a}$ zerlegt: die Ringideale einer und derselben Ringclasse werden den Körperidealen eines und desselben Complexes $\mathfrak{o}\mathfrak{a}$ zugeordnet, und umgekehrt.

Ein weiterer Schritt wurde durch Heinrich Weber[1] getan. Wir betrachten nach ihm die Gruppe G der sämtlichen Ideale des Körpers k, welche (in Zähler und Nenner) zu einem gegebenen Ideal $\mathfrak{m}$, dem Exkludenten, relativ prim sind. Ist dann H eine beliebige Untergruppe von G vom endlichen Index h, und zerlegen wir G in die h Complexe der Form $H\mathfrak{a}$, dann sollen die Ideale eines und desselben Complexes in eine Classe, speciell die der Gruppe H selbst in die Hauptclasse, gerechnet werden; zwei Ideale von G sind demnach aequivalent nach H genannt, wenn ihr Quotient der Idealengruppe H angehört. Offenbar ist der Classenbegriff im gewöhnlichen, *absoluten* Sinne ein sehr specieller Fall dieses *allgemeinen* Classenbegriffs.

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1) H. Weber. Ueber Zahlengruppen in algebraischen Körpern, Math. Ann. 48–50. Lehrbuch der Algebra, III., § 161.

Die Hauptideale, welche in H enthalten sind, bilden für sich eine Gruppe H_0, offenbar vom endlichen Index. Definiren wir dann die Classen nach H_0, so sind die Classen nach H nichts anders als die Zusammenfassung einer gleichen Anzahl der Classen nach H_0; mit anderen Worten, die Classengruppe nach H ist die complementäre Gruppe G/H, wenn die Classen nach H_0 zu Grunde gelegt werden.

Jedem Hauptideal (α) von H_0 entspricht nun ein System von associrten Zahlen $\varepsilon\alpha$, wo ε Einheiten von k bedeutet. Betrachten wir nun diese Zahlen einzeln für sich, dann bilden sie in ihrer Gesamtheit eine unendliche Abel'sche Gruppe, deren Elemente einzelne Zahlen sind, und in welcher die Multiplication die Compositionsregel abgibt. Daher kann man mit Weber zur Definition des Classenbegriffs eine *Zahlengruppe* zu Grunde legen.

Die Gesamtheit z der ganzen und gebrochenen, zu dem gegebenen Ideal $\mathfrak{m}$ primen Zahlen des Körpers k ist eine Gruppe; es sei o eine Untergruppe derselben, von welcher der Index (z: o) endlich ist. Jede Zahl von o definirt ein zu $\mathfrak{m}$ primes Hauptideal, die Gesamtheit desselben ist dann eine Idealengruppe, die wir vorübergehend mit $\bar{o}$ bezeichnen wollen. Dann bilden nach Weber die Ideale eines Complexes $\bar{o}\mathfrak{a}$ eine Idealclasse nach o, also speciell die Ideale von $\bar{o}$ die Hauptclasse.

So werden die sämtlichen zu $\mathfrak{m}$ primen Idealen von k in Classen verteilt. Die Beschränkung, dass nur die zu $\mathfrak{m}$ primen Ideale in Betracht gezogen werden, ist für die Classeneinteilung ohne Belang, denn jede Idealclasse im absoluten Sinne enthält die zu $\mathfrak{m}$ primen Ideale. Erst durch die Einführung der Zahlengruppe o wird jede absolute Idealclasse in eine dieselbe Anzahl d von den Classen nach o zerlegt. Diese Anzahl d bestimmt sich nach Weber durch die Formel[1]

$$d = \frac{(z : o)}{(E : E_0)},$$

wenn E die Gruppe der sämtlichen Einheiten in k, E_0 diejenige der Einheiten in o, und allgemein (A : B) den Gruppenindex bedeutet.

1) H. Weber, Math. Ann. Bd. 48, S. 443. Lehrbuch, III, S. 598.

§. 2.

Congruenz-classengruppen.

Von einer besonderen Wichtigkeit ist nun der Fall, wo die Zahlengruppe $\mathfrak{o}$ die folgende Bedingung erfüllt:[1]

Es sei $\mathfrak{a}$ ein beliebiges ganzes Ideal in $\mathfrak{a}$, und $T(t)$ die Anzahl der in $\bar{\mathfrak{o}}$ enthaltenen durch $\mathfrak{a}$ teilbaren ganzen Hauptideale, deren Norm nicht grösser als die positive Grösse t ist. Dann soll

$$T = \frac{gt}{\mathrm{N}(\mathfrak{a})} + Mt^{1-\delta},$$

und folglich

$$\lim_{t=\infty} \frac{T}{t} = \frac{g}{\mathrm{N}(\mathfrak{a})}$$

sein, worin g eine endliche von Null verschiedene positive Grösse ist, die nur von den Gruppen $\mathfrak{a}$ und $\mathfrak{o}$, aber nicht von t und von der Wahl des Ideals $\mathfrak{a}$ abhängt, während M eine Function von t ist, welche mit unendlich wachsendem t nicht unendlich wird, und δ endlich eine nur von dem Körper k abhängende positive Grösse bedeutet, die kleiner als 1 ist.

Unter dieser Voraussetzung folgt, wenn für ein variables $s>1$

$$A(s) = \Sigma' \frac{1}{\mathrm{N}(\mathfrak{j})^s}$$

gesetzt wird, worin $\mathfrak{j}$ die sämtlichen ganzen Ideale einer Classe A nach $\mathfrak{o}$ durchläuft,

$$A(s) = \frac{g}{s-1} + G(s),$$

wo $G(s)$ eine Function ist, welche für $s=1$ in einen endlichen Grenzwert übergeht.[2]

Hieraus folgt zunächst, dass die Classenzahl nach $\mathfrak{o}$ endlich ist.[3]

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1) H. Weber, Ueber die Zahlengruppen usw., Math. Ann. 49., S. 84.

2) Do. S. 85.

3) Die Voraussetzung 2. bei Weber, a. a. O. ist in der Voraussetzung 3. enthalten.

Es sei nun H eine Untergruppe der Classengruppe nach G vom Index h. Dann gibt es bekanntlich h Systeme der Gruppencharactere

$$\chi_1,\ \chi_2, \cdots\cdots \chi_h,$$

welche für die Classen in H den Wert 1 haben. Dementsprechend definiren wir nach Weber die h Functionen $Q_i(s)$ durch die unendlichen Reihen:

$$Q_i(s) = \sum^{A} \chi_i(\mathrm{A}) A(s) = \sum^{\mathfrak{j}} \frac{\chi_i(\mathfrak{j})}{\mathrm{N}(\mathfrak{j})^s}, \quad (i=1, 2, \cdots\cdots h)$$

wo sich die erste Summe auf die h Classen A, die zweite auf die sämtlichen ganzen Ideale von G erstreckt. Diese Reihen convergiren absolut wenn $s>1$. Ist χ_1 der Hauptcharacter, dann geht für $s=1$

$$(s-1)\, Q_1(s)$$

in den endlichen von Null verschiedenen Grenzwert gh über: für die $h-1$ anderen Charactere gehen die Functionen

$$Q_i(s) \qquad (i=2, 3, \cdots\cdots h)$$

gleichfalls für $s=1$ in die endliche Grenzwerte über, die jedoch auch verschwinden können.

Die Functionen $Q_i(s)$ lassen sich, so lange $s>1$, in unendliche Producte entwickeln:

$$Q_i(s) = \prod^{\mathfrak{p}} \frac{1}{1 - \dfrac{\chi_i(\mathfrak{p})}{\mathrm{N}(\mathfrak{p})^s}},$$

wo $\mathfrak{p}$ die sämtlichen Primideale von G durchläuft.

Definiren wir demnach die Function $\log Q_i(s)$ durch die ebenfalls für $s>1$ unbedingt convergente Reihe:

$$\log Q_i(s) = -\sum^{\mathfrak{p}} \log\left(1 - \frac{\chi_i(\mathfrak{p})}{\mathrm{N}(\mathfrak{p})^s}\right)$$

$$= \sum^{\mathfrak{p}} \frac{\chi_i(\mathfrak{p})}{\mathrm{N}(\mathfrak{p})^s} + \frac{1}{2} \sum^{\mathfrak{p}} \frac{\chi_i(\mathfrak{p})^2}{\mathrm{N}(\mathfrak{p})^{2s}} + \cdots\cdots,$$

so erhalten wir, indem wir nach i summiren

$$\log \overset{i}{\Pi} Q_i(s) = h\sum \frac{1}{\mathrm{N}(\mathfrak{p}_1)^s} + \frac{h}{2}\sum \frac{1}{\mathrm{N}(\mathfrak{p}_2)^{2s}} + \cdots\cdots,$$

wo links unter log. der reelle Wert des Logarithmus zu verstehen ist, und wo die erste Summe rechts sich auf die sämtlichen in $\mathfrak{H}$ enthaltenen Primideale $\mathfrak{p}_1$ erstreckt, während sich die zweite Summe auf alle Primideale $\mathfrak{p}_2$ erstreckt, von welchen erst die zweite Potenz in $\mathfrak{H}$ enthalten sind, usw.

Da nun $(s-1)\,\Pi\, Q_i(s)$ für $s=1$ endlich ist, so erhalten wir die für $s>1$ geltende fundamentale Beziehung

$$\sum \frac{1}{\mathrm{N}(\mathfrak{p})^s} = \frac{1}{h}\log\frac{1}{s-1} + f(s), \tag{1}$$

wo sich die unendliche Summe auf die sämtlichen in $\mathfrak{H}$ enthaltene Primideale $\mathfrak{p}$ erstreckt, und wo $f(s)$ eine Function von s ist, welche für $s=1$ nicht positiv unendlich wird.[1]

Die oben für die Zahlengruppe $\omicron$ gestellte Forderung wird erfüllt, wenn $\omicron$ die Gruppe der zu $\mathfrak{m}$ primen Zahlclassen nach dem Modul $\mathfrak{m}$ ist, mit oder ohne Vorzeichenbedingung von der in § 1 erwähnten Art, und dementsprechend $\mathfrak{a}$ die Gesamsheit der zu $\mathfrak{m}$ primen Ideale des Körpers ist. In dem Falle, wo $\omicron$ die Gruppe der sämtlichen Zahlen α ist, welche die Congruenz

$$\alpha \equiv 1, \quad (\mathfrak{m})$$

befriedigen, also aus einer einzigen Zahlclasse mod. $\mathfrak{m}$ besteht, dem Falle, worauf es im Wesentlichen ankommt, bestätigt man durch die bekannte Methode der Volumenbestimmung,[2] dass

$$g = \frac{2^{\nu}\pi^{n-\nu}L}{w\mathrm{N}(\mathfrak{m})\,|\sqrt{d}\,|}, \qquad \delta = 1 - \frac{1}{n},$$

wo n den Grad des Körpers k, ν die Anzahl der Paare conjugirt imaginären unter den mit k conjugirten Körpern, d die

1) Diese Schlüsse bleibt offenbar gültig, wenn nur die Primideale ersten Grades in die Summe aufgenommen werden.

2) Vgl. H. Weber, Lehrbuch der Algebra, II. 20 und 21 Absch., auch Zahlengruppen, Math. Ann. 49 S. 90–94.

Discriminante des Körpers k, N(m) die Norm des Ideals m im Körper k, w die Anzahl der Einheitswenzeln in o, L den Regulator[1] des Systems der Fundamentaleinheiten in o bedeuten; es ist vorausgesetzt, dass für die Zahlen in o alle Vorzeichencombinationen zugelassen werden.

Eine Idealclasse nach o, d.h. die Gesamtheit der Ideale

$$\alpha \mathfrak{j},$$

wo j ein gegebenes zu m primes Ideal, α eine ganze oder gebrochene zu m prime Körperzahl ist, derart, dass

$$\alpha \equiv 1, \quad (\mathfrak{m})$$

nennen wir eine *Congruenzclasse* nach dem Modul m, ein System solcher Classen, welche sich durch Multiplication und Division reproduciren eine *Congruenzclassengruppe*.

Jedoch sind wir berechtigt, auch eine beliebige Congruenzclassengruppe H einfach als eine Classe, als die *Hauptclasse*, zu betrachten, und demnach den Classencomplex Hc als eine *Classe* zu bezeichnen. Diese Erweiterung des Classenbegriffs ist besonders von Statten, wenn H aus lauter Hauptidealen besteht; es kommt dann auf dasselbe hinaus, wie wenn in der Zahlengruppe o mehrere Zahlclassen nach m aufgenommen werden. Zum Beispiel sind die Ringclassen Congruenzclassen in dem erweiterten Sinne, wenn für den Modul der Führer des Ringes angenommen wird. Wenn m das Einheitsideal (1) ist, dann fallen wir in den Klassenbegriff im absoluten Sinne zurück. Da in der Folge ausschliesslich von den Congruenzclassen die Rede sein wird, lassen wir den Zusatz „Congruenz" weg.

Die in der Formel (1) ausgedrückte Tatsache formuliren wir als

Satz 1. *Ist* H *eine Classengruppe vom Index* h[2] *in einem Körper* k, *und durchläuft* $\mathfrak{p}$ *die sämtlichen in* H *enthaltenen Primideale (vom ersten Grade) des Körpers* k, *dann ist für* $s>1$

$$\Sigma' \frac{1}{N(\mathfrak{p})^s} = \frac{1}{h} \log \frac{1}{s-1} + f(s),$$

1) Dirichlet-Dedekind, Vorlesungen über Zahlentheorie, 4. Aufl. S. 597.

2) Gemeint ist der Index von H in Bezug auf die Gruppe der sämtlichen Classen von k, eine abkürzende Bezeichnung, die in den folgenden durchgehend beibehalten wird.

wo $f(s)$ eine Function der reellen Veränderlichen s ist, welche nicht positiv unendlich wird, wenn sich s abnehmend der Grenze 1 nähert.

Ist nun $\mathfrak{a}$ ein zu $\mathfrak{m}$ relativ primes Ideal, dann gibt es in der Zahlengruppe $\mathfrak{o}$ eine durch $\mathfrak{a}$ teilbare ganze Zahl α von der Art, dass $\alpha:\mathfrak{a}$ relativ prim zu einem beliebig vorgeschriebenen Ideal $\mathfrak{c}$ ausfällt. Denn sind $\mathfrak{q}, \mathfrak{q}', \ldots$ die von einander verschiedenen Primfactoren von $\mathfrak{c}$, welche nicht in $\mathfrak{m}$ aufgehen, dann gibt es bekanntlich eine durch $\mathfrak{a}$ teilbare ganze Zahl α_0 derart dass $\alpha_0:\mathfrak{a}$ durch keines der Ideale $\mathfrak{q}, \mathfrak{q}', \ldots$ teilbar sind. Bestimmt man dann α aus den Congruenzen

$$\left.\begin{array}{ll} \alpha \equiv \alpha_0, & (\mathfrak{a}\mathfrak{q}\mathfrak{q}'\cdots), \\ \alpha \equiv \rho, & (\mathfrak{m}), \end{array}\right\}$$

wo ρ eine in $\mathfrak{o}$ enthaltene, folglich zu $\mathfrak{m}$ prime Zahl bedeutet, dann befriedigt α die gestellten Forderungen.

Aus dieser Tatsache folgt unmittelbar, dass jedes zu $\mathfrak{m}$ prime Ideal $\mathfrak{a}$ als den grössten gemeinsamen Divisor zweier in $\mathfrak{o}$ enthaltenen ganzen Zahlen $\varkappa$, ρ dargestellt werden kann. Ist nämlich $\varkappa$ eine durch $\mathfrak{a}$ teilbare Zahl in $\mathfrak{o}$, ρ ebenfalls eine solche Zahl, dass jedoch $\rho:\mathfrak{a}$ prim zu $\varkappa:\mathfrak{a}$ ausfällt, dann ist in der Tat

$$\mathfrak{a} = (\varkappa, \rho).$$

Ferner folgern wir noch die folgende wichtige Tatsache:

Satz 2. *In jeder Classe A nach $\mathfrak{o}$ gibt es Ideale, die zu einem beliebig gegebenen Ideale $\mathfrak{c}$ relativ prim sind.*

Beweis. Sei $\mathfrak{a}$ ein beliebiges Ideal in der zu A reciproke Classe A^{-1}, α eine durch $\mathfrak{a}$ teilbare Zahl in $\mathfrak{o}$:

$$\alpha = \mathfrak{a}\,\mathfrak{b},$$

derart, dass $\mathfrak{b}$ prim zu $\mathfrak{c}$ ausfällt. Da dann $\mathfrak{b}$ der Classe A angehört, so ist der Satz bewiesen.

Wenn daher von den Idealen jeder Classe einer Classengruppe H nach dem Modul $\mathfrak{m}$, nur die beibehalten werden, welche relativ prim zu einem beliebigen Ideal $\mathfrak{c}$ sind, dann bleiben die Classenzahl ungeändert. Eine solche Classengruppe kann aber auch aufgefasst werden, als eine Classengruppe nach dem Modul

$\mathfrak{m}'$, wo $\mathfrak{m}'$ das durch $\mathfrak{m}$ teilbare Ideal bedeutet, welches dadurch aus $\mathfrak{m}$ entsteht, wenn demselben alle in $\mathfrak{c}$ enthaltenen Primideale als Factoren hinzugefügt werden, die nicht in $\mathfrak{m}$ enthalten waren. In diesem Sinne ist eine Classengruppe nach dem Modul $\mathfrak{m}$ zugleich eine Classengruppe nach jedem durch $\mathfrak{m}$ teilbaren Modul $\mathfrak{m}'$; nur spielen dabei einige Factoren von $\mathfrak{m}'$ die Rolle der zur Classeneinteilung unwesentlichen *Excludenten*.

Ist allgemein $\mathfrak{H}$ eine Classengruppe sowohl nach dem Modul $\mathfrak{m}_1$ als nach $\mathfrak{m}_2$, und ist $\mathfrak{m}$ der grösste gemeinsame Divisor von $\mathfrak{m}_1$ und $\mathfrak{m}_2$, dann ist $\mathfrak{H}$ eine Classengruppe nach $\mathfrak{m}$. Denn sei α_0 eine zu $\mathfrak{m}_1$ und $\mathfrak{m}_2$ prime Zahl, die der Congruenz:

$$\alpha_0 \equiv 1, \quad (\mathfrak{m}) \tag{2}$$

genügt, also

$$\alpha_0 = 1 + \mu,$$

wo μ durch $\mathfrak{m}$ teilbar, folglich in der Form darstellbar ist:

$$\mu = \gamma_1 + \gamma_2,$$

wenn mit γ_1 und γ_2 bez. durch $\mathfrak{m}_1$ und $\mathfrak{m}_2$ teilbare Zahlen bezeichnet werden. Setzt man daher

$$\alpha = 1 + \gamma_2,$$

dann bestehen die Congruenzen

$$\alpha \equiv \alpha_0, \ (\mathfrak{m}_1)\ ; \quad \alpha \equiv 1, \ (\mathfrak{m}_2)\ ;$$

folglich ist α prim zu $\mathfrak{m}_1$ und zu $\mathfrak{m}_2$. Nach der zweiten Congruenz ist das Ideal (α) gewiss in $\mathfrak{H}$ enthalten, und weil $\mathfrak{H}$ auch eine Classengruppe nach dem Modul $\mathfrak{m}_1$ ist, so folgt aus der ersten Congruenz, dass (α_0) in $\mathfrak{H}$ enthalten sein muss. Da aber α_0 eine beliebige der Congruenz (2) genügende Zahl ist, so ist unsere Behauptung nachgewiesen.

Demnach gibt es unter allen Moduln $\mathfrak{m}$, die dieselbe Classengruppe $\mathfrak{H}$ definiren, einen bestimmten von kleinster Norm. Derselbe nennen wir der **Führer der Classengruppe** $\mathfrak{H}$.

§. 3.

Ein Fundamentalsatz über die relativ normalen Körper.

Satz 3. *Wenn* K *ein relativ normaler Körper vom Relativgrade* n *in Bezug auf dem Körper* k *ist, und wenn* $\mathfrak{p}_1$ *alle Primideale vom Grundkörper* k *durchläuft, welche in* K *in die von einander verschiedenen Primideale des ersten Relativgrades zerfallen, dann ist für* $s>1$

$$\sum^{\mathfrak{p}_1} \frac{1}{N(\mathfrak{p}_1)^s} = \frac{1}{n} \log \frac{1}{s-1} + F(s),$$

wo $F(s)$ *eine Function des reellen Veränderlichen* s *ist, die endlich bleibt, wenn sich* s *abnehmend der Grenze 1 nähert.*[1]

Beweis. Das für $s>1$ absolut convergente, auf alle Primideale $\mathfrak{P}$ von K mit Ausschluss von den endlichvielen, in die Relativdifferente von K/k aufgehenden, zu erstreckende, unendliche Product

$$\prod^{\mathfrak{P}} \frac{1}{1-N_K(\mathfrak{P})^{-s}},$$

wo N_K die Norm im Körper K bezeichnet, lässt sich wie folgt umformen:

$$\prod^{\mathfrak{P}} \frac{1}{1-N_K(\mathfrak{P})^{-s}} = \left(\prod^{\mathfrak{p}_1} \frac{1}{1-N(\mathfrak{p}_1)^{-s}}\right)^n \prod^{f} \left(\prod^{\mathfrak{p}_f} \frac{1}{1-N(\mathfrak{p}_f)^{-s}}\right)^e,$$

wo sich das erste Product rechts auf alle Primideale $\mathfrak{p}_1$ von k, das Product $\prod^{\mathfrak{p}_f}$ auf alle Primideale $\mathfrak{p}_f$ von k, welche in K in e von einander verschiedene Primideale des $f$ten Relativgrades zerfallen, wo $f=\frac{n}{e}>1$, endlich das Product $\prod^{f}$ sich auf alle von 1 verschiedenen Teiler f von n erstreckt. Geht man in die Logarithmus über, so erhält man

$$\log \prod \frac{1}{1-N_K(\mathfrak{P})^{-s}} = n \sum^{\mathfrak{p}_1} \frac{1}{N(\mathfrak{p}_1)^s} + S,$$

1) Für den absolut normalen Körper, vgl. Hilbert, Bericht, S. 265 (Satz 84). Dieser Satz bleibt auch gültig, wenn nur die Primideale $\mathfrak{p}_1$ vom ersten (absoluten) Grade in die Summe aufgenommen werden, worauf es im wesentlichen ankommt; vgl. die Fussnote [1]) auf S. 10.

wo

$$S = n\left(\frac{1}{2}\sum' \frac{1}{N(\mathfrak{p}_1)^{2s}} + \frac{1}{3}\sum' \frac{1}{N(\mathfrak{p}_1)^{3s}} + \cdots\cdots\right)$$
$$+ \sum^{f} e\left(\sum' \frac{1}{N(\mathfrak{p}_f)^{fs}} + \frac{1}{2}\sum' \frac{1}{N(\mathfrak{p}_f)^{2fs}} + \cdots\cdots\right)$$
$$< n\left(\sum^{\mathfrak{j}}{}' \frac{1}{N(\mathfrak{j})^{2s}} + \sum^{\mathfrak{j}}{}' \frac{1}{N(\mathfrak{j})^{3s}} + \cdots\cdots\right)$$
$$= n \sum^{\mathfrak{j}}{}' \frac{1}{N(\mathfrak{j})^{s}\{N(\mathfrak{j})^{s}-1\}} < 2n \sum^{\mathfrak{j}}{}' \frac{1}{N(\mathfrak{j})^{2s}},$$

wenn $\sum'$ eine über alle von dem Einheitsideal verschiedenen ganzen Ideale von k zu erstreckende Summe bedeutet. S ist also eine für $s > \frac{1}{2}$ absolut convergente Dirichlet'sche Reihe, und geht für $s = 1$ in einen endlichen Grenzwert über.

Da bekanntlich

$$\lim_{s=1+0}\left\{\log \prod \frac{1}{1 - N_K(\mathfrak{P})^{-s}} - \log \frac{1}{s-1}\right\}$$

endlich ist, so ist unser Satz bewiesen.

Von diesem Satz machen wir eine Anwendung auf einen Specialfall, um eine Tatsache herzuleiten, die wir später einmal benutzen werden.

Sei K relativ Abel'sch über k vom Relativgrade l^t, welcher aus t von einander unabhängigen relativ cyclischen Körpern vom Primzahlgrade l zusammengesetzt ist.

Sehen wir von den in einer endlichen Anzahl vorhandenen, in die Relativdiscriminante aufgehenden Primidealen ab, dann zerfällt ein Primideal von k in K entweder in l^t von einander verschiedenen Primideale vom ersten Relativgrade oder in l^{t-1} vom l ten Relativgrade; dieses letztere zerfällt dann in einem Unterkörper K' vom Relativgrade l^{t-1} in die Primideale vom ersten Relativgrade; es ist nämlich K' der Zerlegungskörper für jedes der l^{t-1} relativconjugirten Primideale von K (K muss relativ cyclisch in Bezug auf K', also hier vom Relativgrade l sein).

Bezeichnen wir die Primideale der ersten Art durchweg mit $\mathfrak{p}_1$, die der zweiten Art, welche einem bestimmten Körper K' entsprechen, mit $\mathfrak{p}_2$, dann folgt aus Satz 3, angewandt auf K und K′, dass

$$\sum^{\mathfrak{p}_1}\frac{1}{N(\mathfrak{p}_1)^s}-\frac{1}{l}\log\frac{1}{s-1},$$

$$\left(\sum^{\mathfrak{p}_1}\frac{1}{N(\mathfrak{p}_1)^s}+\sum^{\mathfrak{p}_2}\frac{1}{N(\mathfrak{p}_2)^s}\right)-\frac{1}{l^{t-1}}\log\frac{1}{s-1},$$

folglich auch

$$\sum^{\mathfrak{p}_2}\frac{1}{N(\mathfrak{p}_2)^s}-\frac{l-1}{l^t}\log\frac{1}{s-1}$$

endlich bleiben, wenn sich der reelle Veränderliche s abnehmend der Grenze 1 nähert. Die Primideale $\mathfrak{p}_1$ sowie $\mathfrak{p}_2$ sind daher in unbegrenzter Anzahl vorhanden.

Enthält k die primitive l^{te} Einheitswurzel, dann lässt sich dieses Ergebnis wie folgt ausdrücken:

Es seien $\alpha_1, \alpha_2, \ldots\ldots \alpha_t$ ganze Zahlen des Körpers k, welche die primitive l^{te} Einheitswurzel enthält, wo l eine natürliche Primzahl ist, von der Art, dass keine der l^t-1 Producte

$$\alpha_1^{m_1}\alpha_2^{m_2}\ldots\ldots\ldots\alpha_t^{m_t},$$

die man erhält, wenn man jeden der Exponenten die Werte 0, 1, 2, ...$l-1$ durchlaufen lässt, mit Ausschluss eines Wertsystems $m_1=m_2=\ldots=m_t=0$, die l^{te} Potenz einer Zahl in k wird. Sind dann $\xi_1, \xi_2, \ldots\ldots \xi_t$ beliebig vorgeschriebene l^{te} Einheitswurzeln, dann gibt es in k stets unendlichviele Primideale $\mathfrak{p}$ vom ersten Grade, für welche

$$\left(\frac{\alpha_1}{\mathfrak{p}}\right)=\xi_1^e, \left(\frac{\alpha_2}{\mathfrak{p}}\right)=\xi_2^e, \ldots\ldots \left(\frac{\alpha_t}{\mathfrak{p}}\right)=\xi_t^e,$$

wo $\left(\frac{\alpha}{\mathfrak{p}}\right)$ den l^{ten} Potenzcharacter und e eine gewisse von $\mathfrak{p}$ abhängige nicht durch l teilbare ganze rationale Zahl ist.[1)]

In der Tat, wenn zunächst $\xi_1, \xi_2, \ldots\ldots \xi_t$ sämtlich gleich 1 sind, werden durch die gestellte Forderung diejenige Primideale von k characterisirt, die im relativ Abel'schen Oberkörper $K=k(\sqrt[l]{\alpha_1}, \sqrt[l]{\alpha_2}, \ldots\ldots \sqrt[l]{\alpha_t})$ vom Relativgrade l^t in die Primideale vom ersten

1) Vgl. Hilbert, Bericht, Satz 152.

Relativgrade zerfallen. Ist dagegen etwa $\xi_1 \neq 1$, dann bestimme man $t-1$ ganze rationale Zahlen $n_2, \ldots\ldots n_t$ so, dass

$$\xi_1^{n_2}\xi_2=1, \cdots\cdots \xi_1^{n_t}\xi_t=1,$$

und setze dementsprechend

$$\alpha_1^{n_2}\alpha_2=\beta_2, \cdots\cdots \alpha_1^{n_t}\alpha_t=\beta_t.$$

Dann lässt sich die gestellte Forderung umformen in:

$$\left(\frac{\alpha_1}{\mathfrak{p}}\right)\neq 1, \quad \left(\frac{\beta_2}{\mathfrak{p}}\right)=1, \cdots\cdots \left(\frac{\beta_t}{\mathfrak{p}}\right)=1.$$

Sie werden durch diejenige Primideale $\mathfrak{p}$ von k erfüllt, welche in dem relativ Abel'schen Körper $K'=k(\sqrt[l]{\beta_2}, \ldots\ldots \sqrt[l]{\beta_t})$ vom Relativgrade l^{t-1}, nicht aber in K, in die Primideale vom ersten Relativgrade zerfallen. Die über diese Primideale erstreckte Summe $\sum \frac{1}{N(\mathfrak{p})^s}$ wird daher nach Satz 3, für $s=1$ unendlich wie

$$\left(\frac{1}{l^{t-1}}-\frac{1}{l^t}\right)\log\frac{1}{s-1},$$

womit unsere Behauptung bestätigt wird.

§. 4.

Der Classenkörper.

Es sei K ein relativ normaler Oberkörper von k vom Relativgrade n; die Idealclassen in k seien nach dem Modul $\mathfrak{m}$ definirt. Die Gesamtheit derjenigen Classen von k, welche Relativnormen der zu $\mathfrak{m}$ primen Ideale des Oberkörpers K enthalten, bildet dann eine Classengruppe, die wir mit $\mathfrak{H}$ bezeichnen, und es sei h der Index von $\mathfrak{H}$ in Bezug auf die vollständige Classengruppe von k. Der Körper K und die Classengruppe $\mathfrak{H}$ bezeichnen wir als einander *zugeordnet*.

Die zu $\mathfrak{m}$ primen Primideale von k, welche in K in die Primideale des ersten Relativgrades zerfallen, sind demnach sämtlich in den Classen von $\mathfrak{H}$ enthalten, womit nicht gesagt wird, dass umgekehrt jedes in einer Classe von $\mathfrak{H}$ enthaltene Primideal von k in die Primideale des ersten Relativgrades in K zerfällt.

Wenn der Relativgrad des relativ normalen Körpers K und der Index der zugeordneten Classengruppe H von k einander gleich sind, dann soll K der **Classenkörper für die Classengruppe** H genannt werden.

Mit Hülfe der Sätze 1 und 3 folgt aus der obigen Definition der folgende Satz, welcher in der Folge von einer fundamentaler Bedeutung ist.

Satz 4. *Der Relativgrad des relativ normalen Körpers ist niemals klenier als der Index der zugeordneten Classengruppe des Grundkörpers.*

Beweis. Nach Satz 1 ist, wenn $\mathfrak{p}$ die sämtlichen in der Classengruppe H enthaltenen Primideale von k durchläuft,

$$\sum^{\mathfrak{p}} \frac{1}{N(\mathfrak{p})^s} = \frac{1}{h} \log \frac{1}{s-1} + f(s), \qquad (s>1)$$

wo h der Index der Classengruppe H ist, und $f(s)$ eine Function der reellen Veränderlichen s, welche für $s=1$ unter einer endlichen positiven Schranke bleibt. Die sämtlichen zu $\mathfrak{m}$ primen Primideale von k, welche in K in die von einander verschiedenen Primideale vom ersten Relativgrade zerfallen, die wir durchweg mit $\mathfrak{p}_1$ bezeichnen, sind in H enthalten; wir bezeichnen die übrigen in H enthaltenen Primideale durchweg mit $\mathfrak{p}'$. Dann ist

$$\sum^{\mathfrak{p}} \frac{1}{N(\mathfrak{p})^s} = \sum^{\mathfrak{p}_1} \frac{1}{N(\mathfrak{p}_1)^s} + \sum^{\mathfrak{p}'} \frac{1}{N(\mathfrak{p}')^s},$$

und nach Satz 3

$$\sum^{\mathfrak{p}_1} \frac{1}{N(\mathfrak{p}_1)^s} = \frac{1}{n} \log \frac{1}{s-1} + F(s), \qquad (s>1)$$

wo n der Relativgrad von K/k ist und $F(s)$ eine Function von s, die für $s=1$ endlich bleibt.

Demnach hat man

$$\sum^{\mathfrak{p}'} \frac{1}{N(\mathfrak{p}')^s} = \left(\frac{1}{h} - \frac{1}{n} \right) \log \frac{1}{s-1} + f(s) - F(s) \geqq 0$$

für $s>1$. Da $f(s)-F(s)$ nicht positiv unendlich wird, wenn sich s abnehmend der Grenze 1 nähert, so folgt hieraus

$$\frac{1}{h} - \frac{1}{n} \geqq 0,$$

oder

$$n \geqq h,$$

womit der Satz bewiesen ist.

Dieser Schluss bleibt, wie man sofort erkennt, auch dann gültig, wenn nur vorausgesetzt wird, dass die in $\mathfrak{H}$ enthaltenen Primideale vom (absolut) ersten Grade in die Primideale vom ersten Grade in K zerfallen, sogar mit einer endlichen Anzahl Ausnahme, oder unendlichvielen, wenn nur die über diese Ausnahme-ideale erstreckte Summe $\sum' \frac{1}{N(\mathfrak{p})^s}$ für $s=1$ endlich bleibt.

Eine wichtige Folgerung des obigen Beweises ist die, dass, wenn $n=h$, also wenn K Classenkörper für die Classengruppe $\mathfrak{H}$ ist, die Function $f(s)$ notwendig für $s=1$ endlich bleibt. Dann sind die Grenzwerte für $s=1$ von den Reihen

$$Q_i(s) \qquad (i=2, 3, \cdots h)$$

(§ 2, S. 9) von Null verschieden, und hieraus folgt, die folgende wichtige Tatsache[1]:

Satz 5. *In einem beliebigen algebraischen Körper existirt in jeder Classe nach dem Modul* $\mathfrak{m}$ *eine unbegrenzte, asymptotisch gleiche,[2] Anzahl von Primidealen ersten Grades; speciell existiren, wenn* μ *eine beliebige,* α *eine zu* μ *prime, ganze Zahl des Körpers ist, unendlichviele ganze Zahlen* ϖ *in dem Körper, die der Congruenz*

$$\varpi \equiv \alpha, \qquad (\mu)$$

genügen, und von der Art sind, dass (ϖ) *unendlichviele Primideale des ersten Grades darstellen;*

(dies unter der vorläufigen Annahme, dass es für jede Classengruppe $\mathfrak{H}$ eines beliebigen Körpers einen entsprechenden Classenkörper gebe, was tatsächlich der Fall ist, wie in der Folge bewiesen werden wird).

Wir fügen hier noch einen Hülfssatz hinzu, den wir später nicht wohl entbehren können.

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1) H. Weber, Zahlengruppen, Math. Ann. 49, S. 89.

2) E. Landau, Ueber die Verteilung der Primideale in den Idealklassen eines algebraischen Zahlkörpers, Math. Ann. 63, S. 196-197.

Hülfssatz. Sei K/k ein relativ normaler Körper vom Relativgrade n, H eine Classengruppe in k vom Index h, welche nicht dem Körper K zugeordnet zu sein braucht. Dann gibt es in k unendlichviele Primideale (ersten Grades), die nicht einer Classe vom H angehören, und auch nicht in K in die Primideale vom ersten Relativgrade zerfallen.[1]

Beweis. Wir beweisen diesen Satz nur in dem Falle, wo $h>2$, weil wir ihn später nur für eine Classengruppe eines ungeraden Primzahlindex anwenden werden. Nach Satz 1 gilt für die über alle nicht in H enthaltene Primideale erstrecke Summe

$$\sum \frac{1}{N(\mathfrak{p})^s} = \frac{h-1}{h} \log \frac{1}{s-1} + \Phi(s),$$

wo $\Phi(s)$ für $s=1$ endlich oder *positiv* unendlich wird. Anderseits ist

$$\sum \frac{1}{N(\mathfrak{p}_1)^s} = \frac{1}{n} \log \frac{1}{s-1} + F(s),$$

wo $F(s)$ für $s=1$ in einen endlichen Grenzwert übergeht, wenn die Summe auf alle Primideale $\mathfrak{p}_1$ erstreckt wird, die in K in die Primideale des ersten Relativgrades zerfallen.

Wenn nun $h>2$, dann ist jedenfalls

$$\frac{h-1}{h} > \frac{1}{n},$$

woraus der Satz folgt.

§. 5.

Eindeutigkeit des Classenkörpers.

Satz 6. *Seien* H, H′ *Classengruppen in* k; K, K′ *bez. die Klassenkörper für dieselben. Ist dann* H′ *Untergruppe von* H, *dann ist* K′ *Oberkörper von* K. *Für eine Classengruppe kann es daher nicht mehr als einen Classenkörper geben.*

Beweis. Seien K/k, K′/k bez. vom Relativgrade n, n'; der

1) Vgl. Ph. Furtwängler, Math. Annalen 63, S. 23.

aus K und K′ zusammengesetzte Körper K* ist dann wieder relativ normal, er sei vom Relativgrade n^*.

Seien ferner S_1, S_2, S_3 die auf die Primideale $\mathfrak{p}$ von k erstreckten Summen

$$\sum \frac{1}{N(\mathfrak{p})^s},$$

und zwar erstrecke sich S_1 auf die sämtlichen Primideale, die sowohl in K als auch in K′, folglich in K* in die Primideale des ersten Relativgrades, S_2 auf die, welche in K aber nicht in K′, S_3 auf die, welche in K′ aber nicht in K, in die Primideale des ersten Relativgrades zerfallen. Dann ist nach Satz 3

$$S_1 = \frac{1}{n^*} \log \frac{1}{s-1} + F_1(s),$$

$$S_1 + S_2 = \frac{1}{n} \log \frac{1}{s-1} + F_2(s),$$

$$S_1 + S_3 = \frac{1}{n'} \log \frac{1}{s-1} + F_3(s),$$

wo die Functionen $F(s)$ für $s=1$ endlich bleiben. Hieraus erhält man

$$S_1 + S_2 + S_3 = \left(\frac{1}{n} + \frac{1}{n'} - \frac{1}{n^*}\right) \log \frac{1}{s-1} + G(s), \tag{1}$$

wo auch $G(s)$ für $s=1$ endlich ist.

Anderseits ist, nach Annahme, die Classengruppe $\mathfrak{u}$ vom Index n; ferner soll $\mathfrak{u}$ alle oben in die Summen S_1, S_2, S_3 aufgenommenen Primideale, und möglicherweise noch die anderen, enthalten, von welchen letzteren auf einer ähnlichen Weise die Summe S' gebildet sein möge. Alsdann ist nach Satz 1.

$$S_1 + S_2 + S_3 + S' = \frac{1}{n} \log \frac{1}{s-1} + f(s), \tag{2}$$

wo $f(s)$ eine Function von s ist, welche unterhalb einer endlichen positiven Schranke bleibt, wenn s abnehmend der Grenze 1 zustrebt.

Aus (1) und (2) folgt, für $s>1$

$$S'=\left(\frac{1}{n^*}-\frac{1}{n'}\right)\log\frac{1}{s-1}+f(s)-G(s)\geqq 0,$$

woraus zu schliessen ist, dass

$$\frac{1}{n^*}-\frac{1}{n'}\geqq 0,$$

oder

$$n'\geqq n.^*$$

Da aber $n^*\geqq n'$, so erhält man

$$n^*=n'.$$

Also fällt der Körper K^* mit K' zusammen, d. h. K ist in K' enthalten.

Wenn nun K' auch der Classenkörper für $\mathfrak{H}$ ist, dann muss nach dem eben bewiesenen K' in K enthalten sein. Daher fällt K' mit K zusammen: es kann daher nicht mehr als einen Classenkörper für $\mathfrak{H}$ geben.

Wir bemerken noch, dass die obigen Schlüsse gültig bleiben, wenn nur vorausgesetzt wird, dass die Primideale von k, welche bez. in den relativ normalen Körpern K und K' in die Primideale vom ersten Relativgrade zerfallen *mit endlicher Anzahl Ausnahme* bez. in $\mathfrak{H}$ und $\mathfrak{H}'$ enthalten sind. Dasselle gilt auch dann noch, wenn nur die Primideale *ersten Grades* von k in Betracht gezogen werden.

CAPITEL II.

Die Geschlechter im relativ cyclischen Körper vom Primzahlgrade.

§ 6.

Einige allgemeine Sätze über die relativ Abel'schen Zahlkörper.

In diesem Artikel fassen wir einige Sätze über die relativ Abel'schen Körper zusammen, die wir in der Folge wiederholt anzuwenden haben. Es sind die Sätze, welche die Zerlegungs- Trägheits- und Verzweigungs-körper eines Primideals betreffen, die zuerst von D. Hilbert[1] für die absolut normalen (Galois'schen) Körper aufgestellt, und von H. Weber[2] für die relativ normalen Körper verallgemeinert worden sind, und die wir hier für die relativ Abel'schen Körper specializiren werden.

Sei K/k relativ Abel'sch vom Relativgrade n. Ein Primideal $\mathfrak{p}$ vom Grundkörper k wird in K auf einer folgenden Weise in die Primfactoren zerlegt:

$$\mathfrak{p}=(\mathfrak{P}_1\,\mathfrak{P}_2\cdots\cdots\mathfrak{P}_g)^e,$$

wo

$$n=e\,g\,f',$$

und f' der Relativgrad[3] von jedem der relativ conjugirten Ideale $\mathfrak{P}_1, \mathfrak{P}_2,\ldots\ldots\mathfrak{P}_g$ von K in Bezug auf k ist.

Die Zerlegungskörper von diesen relativ conjugirten Primidealen in Bezug auf k sind, wenn K relativ Abel'sch ist, ein und derselbe Oberkörper von k, so dass wir berechtigt sind, ihn als der Zerlegungskörper für das Primideal $\mathfrak{p}$ im Oberkörper K zu bezeichnen. Gleiches gilt für den Trägheits-, und Verzweigungs-körper.

Der Zerlegungskörper K_z für $\mathfrak{p}$ ist vom Relativgrade e in Bezug auf k, er ist der grösste in K enthaltene Oberkörper von k, in welchem $\mathfrak{p}$ in die von einander verschiedenen Primideale des ersten Relativgrades zerfällt.

Der Trägheitskörper K_t für $\mathfrak{p}$ ist vom Relativgrade ef' in Bezug auf k, und relativ cyclisch vom Grade f' in Bezug auf den Zerlegungskörper K_z. Er ist der grösste in K enthaltene Oberkörper von k, dessen Relativdiscriminante prim zu $\mathfrak{p}$ ausfällt.

Der Verzweigungskörper K_v für $\mathfrak{p}$ ist relativ cyclisch in Bezug auf den Trägheitskörper K_t, dessen Relativgrad ein Teiler von $p^{ff'}-1$ ist, wo p^f die Norm von $\mathfrak{p}$ in k, also $p^{ff'}$ die Norm von $\mathfrak{P}$ in K ist; dieser Relativgrad ist als der grösste Teiler von g bestimmt, welcher

1) D. Hilbert, Grundzüge einer Theorie des Galois'schen Zahlkörpers, Göttinger Nachrichten, 1894; vgl. auch Bericht, §§ 39–47.

2) H. Weber, Lehrbuch der Algebra, II. (2 Aufl.) 19. Abschnitt.

3) H. Weber, l. c. S. 645.

prim zu p ist. Wenn g durch p teilbar ist, dann sind zwischen K_e und K die Verzweigungskörper höheren Grades K'_e, K''_e,......einzuschalten: die Relativkörper K'_e/K_e, K''_e/K'_e,...... sind relativ Abel'sch und aus nicht mehr als ff' von einander unabhängigen relativ cyclischen Körpern p^{ten} Grades zusammengesetzt. Es ist $\mathfrak{P}^g$ ein Primideal in K_t, dasselbe wird in K/K_t in die g te Potenz eines Primideals $\mathfrak{P}$ zerlegt, welches vom ersten Relativgrade in Bezug auf K_t ist. Wir heben speciell die folgenden Sätzen hervor.

Satz 7. *Ist* K/k *relativ cyclisch vom Primzahlpotenzgrade* l^h, *und geht ein zu* l *primes Primideal* $\mathfrak{p}$ *von* k *in die Relativdiscriminante des in* K *enthaltenen relativ cyclischen Oberkörper von* k *vom Relativgrade* l *auf, dann ist die Relativdiscriminante von* K/k *genau durch die* l^h-1^{te} *Potenz von* $\mathfrak{p}$ *teilbar; ferner ist*

$$N(\mathfrak{p}) \equiv 1, \qquad (l^h),$$

wo N *die in* k *genommene Norm bedeutet.*

Satz 8. *Es sei* K/k *relativ cyclisch vom Primzahlgrade* l, *ferner sei* $\mathfrak{l}$ *ein in* l *aufgehendes Primideal von* k. *Wenn dann die Relativdiscriminante von* K/k *durch* $\mathfrak{l}$ *teilbar, dann ist sie genau durch die* $(v+1)(l-1)$ *te Potenz von* $\mathfrak{l}$ *teilbar, wo* $v > 0$. *Die Zahl* v *ist dadurch characterisirt, dass für jede ganze Zahl* A *von* K *die Congruenz besteht:*

$$sA \equiv A, \qquad (\mathfrak{L}^{v+1})$$

wo s *eine erzeugende Substitution der Galois'schen Gruppe des Relativkörpers* K/k, sA *die relativ conjugirte Zahl von* A, *und* $\mathfrak{L}$ *das in* $\mathfrak{l}$ *aufgehendes Primideal von* K *bedeutet. Speciell ist, wenn* A *genau durch die erste Potenz von* $\mathfrak{L}$ *teilbar ist,* $sA-A$ *genau durch die* $v+1^{\text{te}}$ *Potenz von* $\mathfrak{L}$ *teilbar.*[1]

Für die Zahl v *gilt die Beziehung*

$$\frac{s\,l}{l-1} \geq v \geq 1,$$

wenn s *der Exponent der höchsten in* l *aufgehenden Potenz von* $\mathfrak{l}$ *ist. Ferner ist* v *nur dann durch* l *teilbar, wenn*

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1) Hilbert, Bericht, § 44, 47; es ist $v+1$ der dort mit L bezeichnete Exponent.

$$v = \frac{sl}{l-1},$$

(also wenn s durch $l-1$ teilbar ist).

Beweis. Es genügt, den zweiten Teil des Satzes zu beweisen. Sei Λ eine genau durch die erste Potenz von $\mathfrak{L}$ teilbare Zahl von K. Ist dann A genau durch $\mathfrak{L}^e$ teilbar, dann kann man eine zu $\mathfrak{L}$ prime Zahl B so bestimmen, dass

$$A \equiv B\Lambda^e, \qquad (\mathfrak{L}^u), \tag{1}$$

wo u ein beliebig grosser Exponent sein kann. Ist nun $e \not\equiv 0$, (l), dann ist $s\Lambda^e - \Lambda^e$ genau durch $\mathfrak{L}^{v+e}$ teilbar, daher auch

$$sA - A \equiv B(s\Lambda^e - \Lambda^e) + (sB - B)s\Lambda^e, \qquad (\mathfrak{L}^u)$$

genau durch $\mathfrak{L}^{v+e}$ teilbar, weil das zweite Glied rechts wenigstens durch $\mathfrak{L}^{v+e+1}$ teilbar und nach Annahme $u > v+e$ ist. Ist aber $e \equiv 0$, (l), dann kann man in (1) Λ^e durch eine Zahl λ von k ersetzen, welche genau durch die $e:l^{\text{te}}$ Potenz von $\mathfrak{l}$ teilbar ist: man erhält dann

$$sA - A \equiv (sB - B)\lambda, \qquad (\mathfrak{L}^u),$$

folglich ist $sA - A$ gewiss durch eine höhere als die $v+e^{\text{te}}$ Potenz von $\mathfrak{L}$ teilbar.

Bildet man daher aus der Zahl $A_1 = sA - A$ wieder die Zahl $A_2 = sA_1 - A_1$, und so fort, bis man erhält $A_n = sA_{n-1} - A_{n-1}$, welche letztere Zahl A_n symbolisch mit

$$(s-1)^n A$$

bezeichnet sein möge, dann ist dieselbe genau durch die $e+nv^{\text{te}}$ Potenz von $\mathfrak{L}$ teilbar, wenn keine der n Zahlen e, $e+v$, $e+2v$, ... $e+(n-1)v$ durch l teilbar ist, andernfalls aber gewiss durch eine höhere als die $e+nv^{\text{te}}$ Potenz von $\mathfrak{L}$ teilbar.

Vermöge der Identität

$$1 + x + x^2 + \cdots\cdots + x^{l-1} = l + \binom{l}{2}(x-1) + \binom{l}{3}(x-1)^2 + \cdots\cdots$$
$$+ l(x-1)^{l-2} + (x-1)^{l-1}$$

schreiben wir nun die Relativspur von A in der Form:

$$S(A)=(1+s+s^2+\cdots\cdots+s^{l-1})A$$
$$=lA+\binom{l}{2}(s-1)A+\binom{l}{3}(s-1)^2A+\cdots\cdots+(s-1)^{l-1}A.$$

Das erste Glied auf der rechten Seite ist genau durch die $sl+1^{\text{te}}$, alle folgenden Glieder bis auf das letzte durch höhere Potenzen von $\mathfrak{l}$ teilbar; das letzte Glied aber möge genau durch $\mathfrak{l}^a$ teilbar sein. Nach dem vorhin Bemerkten ist dann

$$a>1+r(l-1),$$

ausser wenn $r\equiv 0$ oder $\equiv 1$ (l). Da der Exponent der höchsten in $S(A)$ aufgehenden Potenz von $\mathfrak{l}$ durch l teilbar sein muss, so ist jedenfalls

$$sl+1\geqq a. \tag{2}$$

Hieraus folgt für $r\not\equiv 0,\ \not\equiv 1,\ (l)$,

$$sl>r(l-1).$$

Dasselbe muss aber auch für $r\equiv 1,\ (l)$ gelten, weil dann

$$a=1+r(l-1)$$

durch l teilbar, folglich das Gleichheitszeichen in (2) ausgeschlossen ist. Wenn endlich $r\equiv 0,\ (l)$, so ist $a=1+r(l-1)$ nicht durch l teilbar, daher muss in (2) notwendig das Gleichheitszeichen gelten, also

$$sl=r(l-1),$$

womit der Satz bewiesen ist.

Wenn k die primitive l^{te} Einheitswurzel ζ enthält, und wenn ein Primideal $\mathfrak{l}$ genau zur σ^{ten} Potenz in $(1-\zeta)$ aufgeht, dann ist $s=\sigma(l-1)$. Ist dann μ eine Zahl von k die genau durch eine Potenz von $\mathfrak{l}$ teilbar ist, deren Exponent zu l prim ist, dann geht $\mathfrak{l}$ in die Relativdiscriminante des relativ cyclischen Körpers $K=k(\sqrt[l]{\mu})$ auf, und die entsprechende Zahl r nimmt den grösstmöglichen Wert σl an. Wenn dagegen μ nicht durch l teilbar ist und m der höchste Exponent bedeutet, für den es eine Zahl α in k gibt, so dass $\mu\equiv\alpha^l\ (\mathfrak{l}^m)$, dann ist die Relativdiscriminante von $K=k(\sqrt[l]{\mu})$ nur dann durch $\mathfrak{l}$ teilbar, wenn $m<\sigma l$. In diesem

Falle ist aber m notwendig prim zu l. Für die entsprechende Zahl v erhält man den Wert $v=\sigma l-m$. Denn die Zahl $A=\alpha-\sqrt[l]{\mu}$ von K ist genau durch $\mathfrak{L}^m$, und $sA-A=(1-\zeta)\sqrt[l]{\mu}$ genau durch $\mathfrak{L}^{\sigma l}$ teilbar, so dass $\sigma l=m+v$.[1]

Endlich sei noch das folgende bemerkt: Ist K/k relativ cyclisch vom Grade l^h, und wird mit $K^{(\nu)}$ der in K enthaltene relativ cyclische Oberkörper von k vom Relativgrade l^ν $(\nu=1, 2,\cdots\cdots h)$ bezeichnet, und geht $\mathfrak{l}$ in die Relativdiscriminante von $K^{(1)}/k$ auf, dann zerfällt $\mathfrak{l}$ in K in die l^hte Potenz eines Primideals; die Relativdiscriminante von K/k enthält dann $\mathfrak{l}$ genau zu der Potenz mit dem Exponenten:

$$\begin{aligned} l^{h-1}(v+1)(l-1)+l^{h-2}(v_1+1)(l-1)+\cdots\cdots+(v_{h-1}+1)(l-1) \\ =l^h-1+(l-1)\{vl^{h-1}+v_1 l^{h-2}+\cdots\cdots+v_{h-1}\}, \end{aligned}$$

wo v_1, v_2,...... dieselbe Bedeutung für $K^{(2)}/K^{(1)}$, $K^{(3)}/K^{(2)}$,...... haben, wie v für $K^{(1)}/k$, und es ist

$$1\leqq v<v_1<v_2<\cdots\cdots<v_{h-1}\leqq\frac{sl^h}{l-1}.$$

§. 7.

Über die Normenreste des relativ cyclischen Körpers vom Primzahlgrade.

Es sei k ein beliebiger algebraischer Körper, K ein relativ cyclischer Oberkörper von k vom Relativgrade l, wo l eine gerade oder ungerade natürliche Primzahl ist. Eine Zahl α in k heisse dann ein **Normenrest** des Relativkörpers K nach einem Idealmodul $\mathfrak{j}$ in k, wenn es eine Zahl A in K gibt, für die

$$N(A)\equiv\alpha,\quad (\mathfrak{j}),$$

wo mit N die Relativnorm in Bezug auf k bezeichnet wird.

---

1) Vgl. Hilbert, Bericht, Satz 148, wo die hier angedeuteten Tatsachen für den Kreiskörper k bewiesen wird; dieser Beweis ist leicht auf den allgemeinen Körper k zu übertragen.

Ueber die Normenreste nach Primidealpotenzen in k gilt der folgende fundamentale Satz.

Satz. 9. *Es sei* K/k *relativ cyclisch vom Primzahlgrade* l. (I) *Wenn dann* $\mathfrak{p}$ *ein Primideal in* k *ist, welches nicht in die Relativdiscriminante von* K/k *aufgeht, dann ist jede zu* $\mathfrak{p}$ *prime Zahl in* k *Normenrest des Körpers* K *nach jeder Potenz von* $\mathfrak{p}$. (II) *Wenn dagegen* $\mathfrak{p}$ *in die Relativdiscriminante aufgeht, jedoch* $\mathfrak{p}$ *prim zu* l *ist, dann ist, von allen zu* $\mathfrak{p}$ *primen und einander nach* $\mathfrak{p}$ *incongruenten Zahlen in* k *genau der* l^{te} *Teil Normenreste nach* $\mathfrak{p}$, *hier bedeutet* e *eine beliebige natürliche Zahl.* (III) *Dasselbe gilt auch für die Potenz* $\mathfrak{l}^e$ *eines in* l *aufgehendes Primideals* $\mathfrak{l}$ *von* k, *falls* $\mathfrak{l}$ *zur Potenz* $\mathfrak{l}^{(v+1)(l-1)}$ *in die Relativdiscriminante aufgeht, und* $e>v$ *ist. Dagegen ist jede zu* $\mathfrak{l}$ *prime Zahl in* k *Normenrest nach* $\mathfrak{l}^e$, *wenn* $e \leqq v$ *ist. Hier hat die Zahl* v *die in Satz 8 angegebene Bedeutung.*[1]

Beweis (I). Wir unterscheiden vier Fälle, jenachdem $\mathfrak{p}$ in l aufgeht oder nicht, und $\mathfrak{p}$ in K zerfällt oder nicht.

Zunächst sei $\mathfrak{p}$ prim zu l, und es zerfalle $\mathfrak{p}$ in K in l von einander verschiedene Primideale:

$$\mathfrak{p}=\mathfrak{P}\mathfrak{P}'\cdots\cdots\mathfrak{P}^{(l-1)}$$

Sei f der Grad des Primideals $\mathfrak{p}$ in k, also auch der Primideale $\mathfrak{P}, \mathfrak{P}',\ldots\ldots$ in K, und es sei ρ eine Primitivzahl nach $\mathfrak{p}$. Jede Zahl α in k, die zu $\mathfrak{p}$ prim ist, genügt dann offenbar einer Congruenz der Form

$$\alpha \equiv \alpha_0 \rho^n, \quad (\mathfrak{p}^e),$$

wo n eine Zahl aus der Reihe $0, 1, 2, \ldots\ldots p^f-2$, und α_0 eine ganze Zahl in k ist, welche die Congruenz

$$\alpha_0 \equiv 1, \quad (\mathfrak{p})$$

befriedigt. Demnach ist α_0 ein l ter Potenzrest nach $\mathfrak{p}^e$:

$$\alpha_0 \equiv \gamma^l, \quad (\mathfrak{p}^e).$$

Ferner sei eine Zahl P in K so bestimmt, dass

---

1) Vgl. Hilbert, Bericht, §130, wo der Satz für den Kreiskörper der $l$ten Einheitswurzeln aufgestellt ist, allerdings ohne genaue Angabe des critischen Wertes des Exponenten e in (III.)

$$P \equiv \rho, \quad (\mathfrak{P}^e), \quad \equiv 1, \quad (\mathfrak{P}'^e \mathfrak{P}''^e \cdots\cdots);$$

dann ist

$$\mathrm{N}(P) \equiv \rho, \quad (\mathfrak{p}^e),$$

demnach

$$\alpha \equiv \mathrm{N}(\gamma P^n), \quad (\mathfrak{p}^e),$$

womit der Satz im vorliegenden Falle bewiesen ist.

Zweitens sei $\mathfrak{p}$ prim zu l, und es bleibe $\mathfrak{p} = \mathfrak{P}$ prim in K. Ist dann P eine Primitivzahl nach $\mathfrak{P}$ in K, dann ist

$$\rho = \mathrm{N}(P) \equiv P^{1+p^f+p^{2f}+\cdots\cdots+p^{(l-1)f}} \qquad (\mathfrak{P})$$

offenbar eine Primitivzahl nach $\mathfrak{p}$ in k. Da jede zu $\mathfrak{p}$ prime Zahl α in k einer Congruenz der Form

$$\alpha \equiv \alpha_0 \rho^n, \quad (\mathfrak{p}^e)$$

genügt, wo $\alpha_0 \equiv 1$ $(\mathfrak{p})$ und folglich $\alpha_0 \equiv \gamma^l$, $(\mathfrak{p}^e)$, in k, so ist auch in diesem Falle

$$\alpha \equiv \mathrm{N}(\gamma P^n), \quad (\mathfrak{p}^e).$$

Drittens, sei $\mathfrak{p} = \mathfrak{l}$ ein in l aufgehehendes Primideal von k, welches in K in l von einander verschiedene Primideale zerfällt,

$$\mathfrak{l} = \mathfrak{L}\mathfrak{L}'\mathfrak{L}''\cdots\cdots\mathfrak{L}^{(l-1)}.$$

Da jede zu $\mathfrak{l}$ prime Zahl in k offenbar l^{ter} Potenzrest von $\mathfrak{l}$ ist, so ist unser Satz richtig für die erste Potenz von $\mathfrak{l}$.

Angenommen nun, es sei eine zu $\mathfrak{l}$ prime Zahl α Normenrest nach $\mathfrak{l}^e$. Wir setzen

$$\mathrm{N}(A) \equiv \alpha + \beta\lambda^e, \quad (\mathfrak{l}^{e+1}),$$

wo λ eine genau durch die erste Potenz von $\mathfrak{l}$ teilbare Zahl in k ist. Bestimmt man dann eine Zahl θ in K gemäss den Congruenzen:

$$\theta \equiv 1, \quad (\mathfrak{L}), \quad \equiv 0, \quad (\mathfrak{L}'\mathfrak{L}''\cdots\cdots),$$

so dass für die Relativspur von θ gilt:

$$\mathrm{S}(\theta) \equiv 1, \quad (\mathfrak{l}),$$

dann ist

$$\mathrm{N}(1 + \xi\theta\lambda^e) \equiv 1 + \xi\lambda^e, \quad (\mathfrak{l}^{e+1}),$$

wenn ξ eine beliebige Zahl in k ist.

Demnach hat man

$$\mathrm{N}\{A(1+\xi\theta\lambda^e)\}\equiv\alpha+(\beta+\alpha\xi)\lambda^e, \quad (\mathfrak{l}^{e+1}).$$

Da man nun ξ gemäss der Bedingung

$$\beta+\alpha\xi\equiv 0, \quad (\mathfrak{l})$$

bestimmen kann, so ist erwiesen, dass α Normenrest nach der höheren Potenz $\mathfrak{l}^{e+1}$ von $\mathfrak{l}$ ist, und hiermit ist der Satz bewiesen.

Zuletzt, sei $\mathfrak{l}$ ein Primfactor von l in k, und $\mathfrak{l}=\mathfrak{L}$ prim in K. Der Beweis verlaüft genau wie im vorhergehenden Falle; nur muss die Existenz einer Zahl θ in K, deren Relativspur prim zu $\mathfrak{l}$ ausfällt, besonders bewiesen werden. Sei also P eine Primitivzahl nach $\mathfrak{L}$ und

$$P^l+\alpha_1 P^{l-1}+\cdots\cdots+\alpha_l=0$$

die Gleichung l^{ten} Grades in k, welche durch P befriedigt wird. Wäre nun $\mathrm{S}\,(P^n)$ für $n=1, 2, \cdots\cdots l-1$ durch $\mathfrak{l}$ teilbar, dann musste, nach der Newton'schen Formel für die Potenzsummen, die Coefficienten $\alpha_1, \alpha_2, \cdots\cdots \alpha_{l-1}$ durch $\mathfrak{l}$ teilbar sein, also

$$P^l\equiv \mathrm{N}(P), \quad (\mathfrak{l}).$$

Alsdann wäre

$$P^l\equiv P^{1+l^f+l^{2f}+\cdots\cdots+l^{(l-1)f}}, \quad (\mathfrak{L}),$$

wo f der Grad von $\mathfrak{l}$ in k, also lf der Grad von $\mathfrak{L}$ in K ist, folglich

$$l\equiv 1+l^f+l^{2f}+\cdots\cdots+l^{(l-1)f}, \quad (l^{lf}-1),$$

was offenbar unmöglich ist. Daher gibt es in der Tat eine Zahl θ in K derart, dass

$$\mathrm{S}\,(\theta)\equiv 1, \quad (\mathfrak{l}).$$

Hiermit ist der Teil (I) unseres Satzes vollständig bewiesen.

Beweis (II.) Sei $\mathfrak{p}$ ein zu l primer Primfactor der Relativdiscriminante. Dann ist

$$\mathfrak{p}=\mathfrak{P}^l,$$

wo $\mathfrak{P}$ ein Primideal in K, und

$$\Phi(\mathfrak{P}^e)=\varphi(\mathfrak{p}^e)=p^{(e-1)f}(p^f-1),$$

wenn Φ bez. φ die Euler'sche Function bez. in K und k, und f der Grad des Primideals $\mathfrak{p}$ in k ist. Daher ist jede zu $\mathfrak{p}$ prime Zahl A in K nach jeder Potenz von $\mathfrak{P}$ einer Zahl α in k congruent,

$$A\equiv\alpha, \quad (\mathfrak{P}^{el}),$$

woraus

$$N(A)\equiv\alpha^l, \quad (\mathfrak{p}^e),$$

d. h. jeder Normenrest nach $\mathfrak{p}^e$ ist ein l^{ter} Potenzrest von $\mathfrak{p}^e$, und umgekehrt.

Ist nun ρ eine Primitivzahl nach $\mathfrak{p}$, dann ist für jede zu $\mathfrak{p}$ prime Zahl α in k

$$\alpha\equiv\alpha_0\rho^n, \quad (\mathfrak{p}^e),$$

wo $\alpha_0\equiv 1, (\mathfrak{p})$, und n eine Zahl aus der Reihe $0, 1, 2, \ldots\ldots p^f-2$ ist. Es ist nun α_0 offenbar ein l^{ter} Rest von $\mathfrak{p}^e$. Da nach Satz 7. $p^f-1\equiv 0, (l)$, so ist ρ^n dann und nur dann ein l^{ter} Rest nach $\mathfrak{p}^e$, wenn n durch l teilbar ist. Hiermit ist der Teil (II) unseres Satzes bewiesen.

Beweis (III). Sei $\mathfrak{l}$ ein Primfactor von l in k, welcher zur $(v+1)(l-1)$ten Potenz in die Relativdiscriminante von K/k aufgeht, ferner sei $\mathfrak{l}=\mathfrak{L}^l$, wo $\mathfrak{L}$ Primideal in K ist. Wir bezeichnen in den Folgenden durchweg mit λ und A_e eine genau durch die e te Potenz bez. von $\mathfrak{l}$ und $\mathfrak{L}$ teilbare Zahl von k und K. Für die Relativspur von A_e erhält man dann, wie beim Beweise des Satzes 8

$$S(A_e)=lA_e+\binom{l}{2}(s-1)A_e+\binom{l}{3}(s-1)^2A_e+\cdots\cdots+(s-1)^{l-1}A_e.$$

Das erste Glied rechts ist nun genau durch die $sl+e^{\text{te}}$ Potenz von $\mathfrak{L}$, alle folgende Glieder bis auf das letzte durch die höheren Potenzen, das letzte Glied aber wenigstens durch die $e+v(l-1)^{\text{te}}$ Potenz von $\mathfrak{L}$ teilbar. Daher erhält man, wenn man die Relation:

$$sl\geqq v(l-1)$$

berücksichtigt (Satz 8).

$$\left.\begin{array}{lll} \text{wenn} \quad e<v: & S(A_e)\equiv 0, & (\mathfrak{l}^{e+1}), \\ & S(A_v)\equiv 0, & (\mathfrak{l}^{v}), \\ \phantom{\text{wenn}} \quad e>v: & S(A_e)\equiv 0, & (\mathfrak{l}^{v+1}). \end{array}\right\} \qquad (1)$$

Hieraus ist nun auf das folgende zu schliessen:

$$\text{wenn} \quad e<v: \qquad N(1+A_e)=1+\lambda_e, \qquad (2)$$

$$N(1+A_v)\equiv 1+S(A_v)+N(A_v), \quad (\mathfrak{l}^{v+1}), \qquad (3)$$

$$\text{wenn} \quad e>v: \qquad N(1+A_e)\equiv 1, \quad (\mathfrak{l}^{v+1}). \qquad (4)$$

Dies geschieht am einfachsten dadurch, dass man mit Hülfe der Newton'schen Formel über die Potenzsummen die Teilbarkeit der elementarsymmetrischen Functionen von $A_e, sA_e, \cdots\cdots s^{l-1}A_e$ durch die entsprechenden Potenzen von $\mathfrak{l}$ nach (1) bestätigt. Nach (2) und (3) folgt nun, dass

$$N(1+A_e)\equiv 1, \quad (\mathfrak{l}^{v}),$$

dann und nur dann, wenn

$$e \geqq v,$$

woraus weiter, dass für zwei zu $\mathfrak{L}$ prime Zahlen A, B

$$N(A)\equiv N(B), \quad (\mathfrak{l}^{v}),$$

dann und nur dann, wenn

$$A \equiv B, \quad (\mathfrak{L}^{v}).$$

Berücksichtigt man daher die Relation

$$\Phi(\mathfrak{L}^{v})=\varphi(\mathfrak{l}^{v}),$$

dann ersieht man, dass jede zu $\mathfrak{l}$ prime Zahl in k, Normenrest nach $\mathfrak{l}^{v}$ und folglich nach jeder niederen Potenz von $\mathfrak{l}$ ist.

Da, nach (4), auch für den Modul $\mathfrak{l}^{v+1}$, aus der Congruenz

$$A \equiv B, \quad (\mathfrak{L}^{v+1}),$$

die andere:

$$N(A)\equiv N(B), \quad (\mathfrak{l}^{v+1})$$

zu folgern ist, so wird unser Satz für $\mathfrak{l}^{v+1}$ bewiesen sein, wenn nachgewiesen wird, dass die Bedingung

$$N(A) \equiv 1, \quad (\mathfrak{l}^{v+1}) \tag{5}$$

durch genau l einander nach $\mathfrak{l}^{v+1}$ incongruenten Zahlen befriedigt wird. Nach (2) kommt hierzu nur die Zahlen von der Form

$$1 + A_v \tag{6}$$

in Frage. Es gibt nun in der Tat eine Zahl von dieser Gestalt, welche der Congruenz (5) genügt. Es ist nämlich $A_1 - \mathrm{s}\,A_1$ genau durch $\mathfrak{L}^{v+1}$ teilbar. Bringt man daher den Bruch $\mathrm{s}\,A_1 : A_1$ in die Gestalt

$$\frac{\mathrm{s}\,A_1}{A_1} = \frac{A_0}{\alpha},$$

wo α und A_0 zu $\mathfrak{L}$ prime ganze Zahlen bez. in k und K sind, und worin $\alpha \equiv 1$ nach einer beliebig hohen Potenz von $\mathfrak{l}$ angenommen werden kann, dann ist

$$N(A_0) = \alpha^l \equiv 1, \quad (\mathfrak{l}^{v+1}).$$

Anderseits folgt aus

$$\alpha \mathrm{s}\,A_1 = A_0\,A_1,$$

oder

$$A_1\,(A_0 - \alpha) = \alpha\,(\mathrm{s}\,A_1 - A_1),$$

dass $A_0 - \alpha$ genau durch $\mathfrak{L}^v$ teilbar ist, demnach nach Annahme über α

$$A_0 = 1 + A_v^{(0)}.$$

Nach (3) genügt diese besondere Zahl $A_v^{(0)}$ der Congruenz

$$S\,(A_v^{(0)}) + N\,(A_v^{(0)}) \equiv 0, \quad (\mathfrak{l}^{v+1}). \tag{7}$$

Für jede Zahl A von der Form (6) gilt nun

$$A \equiv 1 + \rho\,A_v^{(0)}, \quad (\mathfrak{L}^{v+1}), \tag{8}$$

also nach (4) und (3)

$$N(A) \equiv 1 + \rho\,S(A_v^{(0)}) + \rho^l\,N(A_v^{(0)}), \quad (\mathfrak{l}^{v+1}).$$

Daher ist

$$\mathrm{N}(A)\equiv 1, \quad (\mathfrak{l}^{v+1})$$

dann und nur dann, wenn

$$\rho\, \mathrm{S}(A_v^{(0)})+\rho^l\, \mathrm{N}(A_v^{(0)})\equiv 0, \quad (\mathfrak{l}^{v+1}),$$

oder nach (7), wenn

$$\rho\,(\rho^{l-1}-1)\, \mathrm{N}(A_v^{(0)})\equiv 0, \quad (\mathfrak{l}^{v+1}),$$

oder

$$\rho\,(\rho^{l-1}-1)\equiv 0, \quad (\mathfrak{l}),$$

was dann und nur dann der Fall ist, wenn ρ einer rationalen Zahl nach $\mathfrak{l}$ congruent ist. Die Congruenz (5) wird daher genau durch l nach $\mathfrak{l}^{v+1}$ incongruente Zahlen befriedigt, die man erhält, wenn in (8) $\rho=0, 1, 2, \ldots\ldots l-1$ gesetzt wird, wie zu beweisen war.

Ferner ist, wenn t eine positive ganze rationale Zahl, ρ eine zu $\mathfrak{l}$ prime Zahl in k ist,

$$\mathrm{N}(1+\rho\lambda_t A_v)\equiv 1+\rho\lambda_t\, \mathrm{S}(A_v), \quad (\mathfrak{l}^{v+t+1}),$$

also, da nach (7), $\mathrm{S}(A_v)$ genau durch $\mathfrak{l}^v$ teilbar ist,

$$\mathrm{N}\,(1+\rho\lambda_t A_v)=1+\rho\lambda_{v+t}. \qquad (9)$$

Ist also α Normenrest nach $\mathfrak{l}^{v+1}$, und zwar

$$\mathrm{N}(A)\equiv \alpha+\beta\lambda_{v+t} \quad (\mathfrak{l}^{v+t+1}),$$

wo β zu $\mathfrak{l}$ prim, und für λ_{v+t} dieselbe Zahl wie in (9) angenommen wird, dann ist

$$\mathrm{N}\{A(1+\rho\lambda_{t+v})\}\equiv \alpha+(\alpha\rho+\beta)\lambda_{v+t}, \quad (\mathfrak{l}^{v+t+1}).$$

Da man ρ aus

$$\alpha\rho+\beta\equiv 0, \quad (\mathfrak{l})$$

bestimmen kann, so ist α Normenrest nach $\mathfrak{l}^{v+t+1}$. Jeder Normenrest nach $\mathfrak{l}^{v+t}$ ist daher Normenrest nach jeder höheren Potenz von $\mathfrak{l}$, und weil jeder Normennichtrest nach $\mathfrak{l}^{v+1}$ umsomehr Normennichtrest nach jeder höheren Potenz von $\mathfrak{l}$ ist, so ist hiermit unser Satz in allen seinen Teilen vollständig bewiesen.

In der Folge benutzen wir den Satz 9 in der folgenden verallgemeinerten Form:

Satz 10. *Sei* K/k *relativ cyclisch vom Primzahlgrade* l, *die Relativdiscriminante* $\mathfrak{d}$ *von* K/k *enthalte* d *von einander verschiedene Primideale von* k *als Factor, derart, dass*

$$\mathfrak{d}=\mathfrak{f}^{l-1}, \quad \mathfrak{f}=\Pi\,\mathfrak{p}.\ \Pi\,\mathfrak{l}^{v+1},$$

wo die Producte $\Pi\,\mathfrak{p}$, $\Pi\,\mathfrak{l}^{v+1}$ *bez. auf die zu* l *primen und in* l *aufgehenden Primfactoren von* $\mathfrak{d}$ *zu erstrecken sind. Ist dann* $\mathfrak{m}$ *ein beliebiges durch* $\mathfrak{f}$ *teilbares Ideal von* k, *dann ist, von allen zu* $\mathfrak{m}$ *primen und einander nach* $\mathfrak{m}$ *incongruenten Zahlen von* k, *genau der* l^d *te Teil Normenrest des Körpers* K *nach dem Modul* $\mathfrak{m}$.

§. 8.

Einheiten im relativ cyclischen Körper.

Im relativ cyclischen Körper K/k vom Primzahlgrade l, sei eine Zahlengruppe O vorgelegt, welche eine Congruenzgruppe ist mit oder ohne Vorzeichenbedingung, und welche gegenüber der Substitution s des Relativkörpers invariant ist, d.h. von der Art, dass mit einer Zahl A zugleich die relativ conjugirte A^s darin enthalten ist. Die Gesamtheit der Zahlen von O, welche im Grundkörper k enthalten sind, bildet dann eine Zahlengruppe o in k, welche auch eine Congruenzgruppe ist.

Wenn mit R, r bez. die Anzahl der Grundeinheiten in K, k also auch in O, o bezeichnet wird, dann ist, wenn l ungerade ist

$$R-r=(l-1)(r+1), \tag{1}$$

dagegen, wenn $l=2$, also $K=k(\sqrt{\mu})$ relativ quadratisch ist,

$$R-r=r+1-\nu, \tag{2}$$

wenn ν die Anzahl derjenigen reellen mit k conjugirten Körper bedeutet, worin die mit μ conjugirten Zahlen negativ ausfallen.

Satz 11. *In der Zahlengruppe* O *lassen sich stets ein System von* n *Einheiten* $H_1, H_2, \ldots H_n$ *finden, derart, dass sich jede Einheit* E *in* O *in der Form:*

$$E=H_1^{u_1}H_2^{u_2}\cdots\cdots H_n^{u_n}H^{1-s}[\xi] \qquad (3)$$

darstellen lässt, wo die Exponenten $u_1, u_2, \ldots$ *Zahlen aus der Reihe* $0, 1, 2, \ldots l-1$ *sind,* H *eine Einheit in* O, $[\xi]$ *eine Einheit in* o *oder aber eine Einheit in* O, *deren* l^{te} *Potenz in* o *liegt, bedeutet. Die Einheiten* $H_1, H_2, \ldots H_n$ *sind in dem Sinne von einander unabhängig, dass eine Einheit von der Gestalt* (3) *nur dann gleich* 1 *sein kann, wenn* $u_1=u_2=\ldots\ldots=u_n=0$.

Die Zahl n *hat den folgenden Wert:*

$n=r+1$, *wenn* l *ungerade ist,*

$n=r+1-\nu$, *wenn* $l=2$.

Beweis. Die Einheiten

$$H^{1-s}[\xi]$$

bilden in ihrer Gesamtheit eine Untergruppe der Gruppe der sämtlichen Einheiten in O, von einem endlichen Index l^n, weil die l^{te} Potenz jeder Einheit in O darin enthalten ist. Letzteres folgt unmittelbar aus der Identität:

$$1+s+s^2+\cdots\cdots+s^{l-1}=l+(1-s)\,Q(s), \qquad (4)$$

wo

$$Q(s)=(1-s)^{l-2}-l\,(1-s)^{l-3}+\cdots\cdots+\binom{l}{3}(1-s)-\binom{l}{2},$$

speciell

$$Q(s)=-1, \quad \text{wenn } l=2.$$

Hiernach ist die Existenz eines Systems von Einheiten mit der im Satz angegebenen Eigenschaften ohne weiteres klar; es handelt sich nur noch darum, die Anzahl n dieser Einheiten zu finden, was auf der folgenden Weise geschieht.

Da sich die Einheit H auf der rechten Seite von (3) wieder in der Form:

$$H=H_1^{u_1'}H_2^{u_2'}\cdots\cdots H_n^{u_n'}H'^{1-s}[\xi']$$

darstellen lässt, so kann man setzen

$$E=H_1^{u_1+u_1'(1-s)}\cdots\cdots H_n^{u_n+u_n'(1-s)}H'^{(1-s)^2}[\xi],$$

wenn man sich bedenkt, dass $[\xi']^{1-s}=1$ oder $=\zeta$, wo ζ eine primitive l^{te} Einheitswurzel bedeutet, letzteres nur dann, wenn

$$\mathrm{K}=\mathrm{k}([\xi']),$$

und folglich $\zeta=[\xi']^{1-s}$ in $\mathfrak{o}$ enthalten ist.

Indem wir auf diese Weise fortfahren, erhalten wir

$$E=H_1^{\mathrm{F}_1(s)}\cdots\cdots H_n^{\mathrm{F}_n(s)}H^{(1-s)^{l-1}}[\xi], \tag{5}$$

wo

$$\mathrm{F}_1(s)=u_1+u_1'(1-s)+\cdots\cdots+u_1^{(l-2)}(1-s)^{l-2},\cdots\cdots$$

und die Coefficienten u_1, u_1'.....sämtlich Zahlen aus der Reihe: 0, 1, 2,......$l-1$ sind.

Wir untersuchen nun die Annahme: es sei

$$1=H_1^{\mathrm{F}_1(s)}\cdots\cdots H_n^{\mathrm{F}_n(s)}H^{(1-s)^{l-1}}[\xi]. \tag{6}$$

Aus der Bedeutung des Einheitensystems H_1, H_2,......H_n folgt zunächst

$$u_1=u_2=\cdots\cdots=u_n=0,$$

so dass, für $l=2$, schon

$$\mathrm{F}_1(s)=0,\cdots\cdots\mathrm{F}_n(s)=0,$$

und für ein ungerades l,

$$1=\left(H_1^{\mathrm{G}_1(s)}\cdots\cdots H_n^{\mathrm{G}_n(s)}H^{(1-s)^{l-2}}\right)^{1-s}[\xi], \tag{7}$$

wo

$$\mathrm{G}_1(s)=u_1'+u_1''(1-s)+\cdots\cdots+u_1^{(l-2)}(1-s)^{l-3},\cdots\cdots$$

Eine Relation von der Gestalt

$$H^{1-s}=[\xi],$$

wo H eine Einheit in $\mathfrak{O}$ bedeutet, ist aber offenbar nur dann möglich, wenn $\mathrm{N}([\xi])=[\xi]^l=1$, so dass $[\xi]$ eine l^{te} Einheitswurzel ist. Ist $[\xi]=1$, dann ist H selbst, ist aber $[\xi]=\zeta$ eine primitive l^{te} Einheitswurzel, H' eine Einheit in $\mathfrak{o}$; jedenfalls ist H selbst eine Einheit, die wir mit $[\xi]$ bezeichnen können. Demnach kann man statt (7) einfach setzen:

$$1=H_1^{G_1(s)}\ldots\ldots H_n^{G_n(s)}H^{(1-s)^{l-2}}[\xi].$$

Die Einheit H auf der rechten Seite bringen wir wieder auf die Form

$$H=H_1^{v_1}\cdots\cdots H_n^{v_n}H'^{1-s}[\xi'],$$

so dass wir erhalten

$$1=H_1^{F_1'(s)}\ldots\ldots H_n^{F_n'(s)}H^{(1-s)^{l-1}}[\xi],$$

wo

$$F_1'(s)=u_1'+u_1''(1-s)+\cdots\cdots+u_1^{(l-2)}(1-s)^{l-3}+v_1(1-s)^{l-2},\cdots\cdots$$

ähnliche Bedeutung wie $F_1(s)$......haben. Daher folgt weiter

$$u_1'=u_2'=\cdots\cdots\cdots u_n'=0.$$

So fortfahrend sieht man ein, dass, auch für ungerades l, aus (6) notwendig folgt:

$$F_1(s)=0,\cdots\cdots\cdots F_n(s)=0.$$

Daher sind die $n(l-1)$ Einheiten

$$H_1,\ H_1^{1-s},\cdots\cdots H_1^{(1-s)^{l-2}};\cdots\cdots H_n,\ H_n^{1-s},\cdots\cdots H_n^{(1-s)^{l-2}}$$

unabhängig in Bezug auf die Gruppe der Einheiten:

$$H^{(1-s)^{l-1}}[\xi].$$

Diese Gruppe ist aber identisch mit der Gruppe der Einheiten:

$$H^l[\xi],$$

weil

$$(1-s)^{l-1}\equiv 1+s+\cdots\cdots\cdots+s^{l-1},\ (l),$$

und anderseits

$$(1-s)^{l-1}\varphi(s)+(1+s+\cdots\cdots+s^{l-1})\psi(s)=l,$$

wo $\varphi(s)$, $\psi(s)$ ganzzahlige ganze rationale Functionen von s sind.[1]

1) Für $\psi(s)$ kann man die $l-1$ ersten Glieder der formalen Entwickelung von $l/(1+s+\cdots\cdots+s^{l-1})$ nach steigenden Potenzen von $1-s$ nehmen.

Daher lässt sich jede Einheit E von O in der Gestalt

$$E=H_1^{F_1(s)}\cdots\cdots H_n^{F_n(s)}H^l\,[\zeta]$$

darstellen, wo $F_1(s),\ldots\ldots F_n(s)$ mit E eindeutig bestimmt sind.

Da die sämtlichen Einheiten in O und die Einheiten $[\zeta]\ H^l$ bez l^{R+1} und l^{r+1}, oder l^R und l^r Einheitenverbände[1] in O ausmachen, jenachdem eine Einheitswurzel, deren Ordnung eine Potenz von l ist, in O vorkommt oder nicht, so ergibt sich

$$R-r=n\,(l-1).$$

Wenn man hierin den Wert von $R-r$ nach (2) oder (3) einträgt, so erhält man den im Satz angegebenen Wert von n.

Satz 12. *Machen die Relativnormen sämtlicher Einheiten in O l^{v_0} Einheitenverbände in o aus, dann gibt es in O ρ Einheiten E_1, $E_2,\ldots\ldots E_\rho$ mit der Relativnorm 1, von der Beschaffenheit, dass jede Einheit in O mit der Relativnorm 1 in der Form:*

$$E_1^{u_1}\,E_2^{u_2}\cdots\cdots\cdots E_\rho^{u_\rho}\,H^{1-s} \tag{8}$$

darstellbar ist, wo $u_1,\ u_2,\ldots\ldots u_\rho$ Zahlen aus der Reihe 0, 1, 2,…… $l-1$ sind, und H eine Einheit in O bedeutet: diese Einheiten E_1, $E_2,\ldots\ldots E_\rho$ sind in dem Sinne von einander unabhängig, dass eine Einheit der Form (8) nur dann gleich 1 sein kann, wenn $u_1=u_2=\ldots\ldots=u_n=0$.

Die Zahl ρ hat den Wert:

$$\rho=r+1+\delta-v_0,\qquad \textit{wenn } l \textit{ ungerade ist,}$$
$$\rho=r+1+\delta-\nu-v_0,\qquad \textit{wenn } l=2,$$

wo $\delta=1$ oder $\delta=0$ zu setzen ist, jenachdem die primitive l^{te} Einheitswurzel in o vorkommt oder nicht.

Beweis. Hier wiederum handelt es sich nur um die Bestätigung des für ρ angegebenen Wertes, da die Existenz des Einheitensystems $E_1,\ E_2,\ldots\ldots E_\rho$ mit der im Satz angegebenen

[1] Unter einem Einheitenverband in O verstehen wir ein System der Einheiten in O von der Form EH^l, wo E eine gegebene Einheit in O ist, und H alle Einheiten von O durchläuft. Vgl. Hilbert, Math. Ann. 51, S. 21.

Eigenschaften ohne weiteres klar ist. Wir unterscheiden nun drei Fälle:

Erstens sei vorausgesetzt: die primitive l^{te} Einheitswurzel ζ kommt nicht in o vor. Dann kann die Einheit $[\xi]$ in (3) nur die Einheiten in o bedeuten, und weil es keine Einheit in o gibt, ausser der Einheit 1, mit der Relativnorm 1, so kann man E_1, $E_2, \ldots\ldots E_\rho$ für ρ der Einheiten H_1, $H_2, \ldots\ldots H_n$ in (3) nehmen, es seien diese $H_{n-\rho+1}, \ldots\ldots H_n$, sodass jede Einheit in O in der Form:

$$E = H_1^{u_1} \cdots\cdots H_{n-\rho}^{u_{n-\rho}} E_1^{v_1} \cdots\cdots E_\rho^{v_\rho} H^{1-s}[\xi] \qquad (0 \leq u, v < l)$$

darstellbar ist, und zwar so, dass die Relativnorm der Einheit E nicht gleich 1 sein kann, ausser wenn $u_1 = u_2 = \cdots\cdots = u_{n-\rho} = 0$. Setzt man daher

$$\eta_1 = \mathrm{N}(H_1), \cdots\cdots\cdots \eta_{n-\rho} = \mathrm{N}(H_{n-\rho}),$$

dann ergibt sich für jede Einheit E in O

$$\mathrm{N}(E) = \eta_1^{u_1} \cdots\cdots \eta_{n-\rho}^{u_{n-\rho}} \xi^l, \qquad (0 \leq u < l)$$

und somit

$$v_0 = n - \rho,$$

woraus nach Einsetzen des im Satz 11 angegebenen Wertes von n und Berücksichtigung von $\delta = 0$ der gesuchte Wert von ρ sich ergibt.

Zweitens sei vorausgesetzt: es komme ζ in o vor, jedoch sei K nicht durch die l^{te} Wurzel einer Einheit in o erzeugt. Hier ist wieder die Einheit $[\xi]$ in (3) die Einheit in o, und es ist ζ in dem System der Einheiten $[\xi]$, nicht aber in H^{1-s} enthalten. Wir setzen demnach

$$E_1 = H_{n-\rho+2}, \cdots\cdots\cdots E_{\rho-1} = H_{n-1}; \quad E_\rho = \zeta,$$

sodass jede Einheit E in O sich in der Form

$$E = H_1^{u_1} \cdots\cdots H_{n-\rho+1}^{u_{n-\rho+1}} E_1^{v_1} \cdots\cdots E_\rho^{v_\rho} H^{1-s}[\xi] \qquad (0 \leq u, v < l)$$

darstellen lässt, wo für jedes E das System der Exponenten u, v eindeutig bestimmt ist. Folglich

$$N(E)=\eta_1^{u_1}\cdots\cdots\eta_{n-\rho+1}^{u_{n-\rho+1}}\xi^l,$$

woraus

$$v_0=n-\rho+1.$$

Da hier $\delta=1$ zu setzen ist, so ist in diesem Falle unser Satz bewiesen.

Zuletzt sei vorausgesetzt: es komme ζ in o vor, und $K=k(\sqrt[l]{\eta_0})$, wo η_0 eine Einheit in o ist. Setzt man nun

$$H_0=\sqrt[l]{\eta_0},$$

dann kann in (3) die Einheiten $[\xi]$ durch H_0^u ersetzt werden, wenn u_0 eine Zahl aus der Reihe: 0, 1, 2,......$l-1$ und ξ eine Einheit in o bedeutet. Ferner ist ζ in dem System H^{1-S} enthalten, es ist nämlich $\zeta=H_0^{1-S}$. Demnach kann man setzen

$$E_1=H_{n-\rho+1},\cdots\cdots\cdots E_\rho=H_n,$$

so dass jede Einheit E in O auf einer einzigen Weise in der Form:

$$E=H_0^{u_0}H_1^{u_1}\cdots\cdots H_{n-\rho}^{u_{n-\rho}}E_1^{v_1}\cdots\cdots E_\rho^{v_\rho}\xi \qquad (0\leqq u,\, v<l)$$

darstellbar ist; und es ist

$$N(E)=\eta_0^{u_0}\eta_1^{u_1}\cdots\cdots\cdots\eta_{n-\rho}^{u_{n-\rho}}\xi^l.^{1)}$$

Daher ist

$$v_0=n-\rho+1,$$

woraus mit $\delta=1$ der gesuchte Wert von ρ sich ergibt.

§. 9.

Formulirung eines Fundamentalsatzes.

Nachdem in den vorhergehenden die vorbereitenden Sätze erledigt worden, sind wir nun im Stande, einen Fundamentalsatz zu formuliren, dessen Beweis das Hauptzweck der nachfolgenden Paragraphen dieses Capitels sein soll.

---

1) Wenn $l=2$, ist η_0 durch $-\eta_0$ zu ersetzen.

Satz 13. *Die Relativdiscriminante des relativ cyclischen Körpers* K/k *vom ungeraden Primzahlgrade* l *sei* $\mathfrak{d}=\mathfrak{f}^{l-1}$, *wo*

$$\mathfrak{f}=\Pi\mathfrak{p}.\ \Pi\mathfrak{l}^{e+1},$$

wo $\mathfrak{p}$ *ein zu* l *primes, und* $\mathfrak{l}$ *ein in* l *aufgehendes Primideal von* k *bedeutet. Die Idealclassen von* k *seien nach einer Zahlengruppe* $\omicron$ *definirt, welche aus den Zahlen* α *besteht, die der Congruenz:*

$$\alpha\equiv 1,\ (\mathfrak{m})$$

genügen, wo der Modul $\mathfrak{m}$ *ein beliebiges durch* $\mathfrak{f}$ *teilbares Ideal von* k *ist. Dann sind die Relativnormen aller zu* $\mathfrak{m}$ *primen Ideale von* K *in einer Classengruppe vom Index* l *in* k *enthalten.*

Dasselbe gilt auch für den relativ quadratischen Körper $K=k(\sqrt{\mu})$, *wenn an Stelle von* $\omicron$ *eine Zahlengruppe* $\bar{\omicron}$ *mit gewisser Vorzeichenbedingung angenommen wird. Es soll nämlich nur diejenigen Zahlen von* $\omicron$ *in* $\bar{\omicron}$ *aufgenommen werden, welche wenigstens in allen denjenigen mit* k *conjugirten reellen Körpern, worin* μ *negativ ausfällt, positiv sind.*[1]

Mit andern Worten:

Jeder relativ Abel'sche Körper vom Primzahlgrade l *mit der Relativdiscriminante* $\mathfrak{f}^{l-1}$ *ist der Classenkörper für eine Classengruppe nach dem Modul* $\mathfrak{f}$.[2]

§. 10.

Die Anzahl der ambigen Classen im relativ cyclischen Körper eines ungeraden Primzahlgrades.

Es sei K/k ein relativ cyclischer Körper von einem ungeraden Primzahlgrade l, und es sei s eine erzeugende Substitution der Galois'schen Gruppe des Relativkörpers K/k. Eine Idealclasse C des Körpers K heisst **ambig**, wenn sie mit der relativ conjugirten Classe sC identisch ist; im Zeichen:

$$C^{1-s}=1.$$

1) Wenn k_1 ein mit k conjugirter reeller Körper ist, dann soll eine Zahl α von k abkürzend als „positiv oder negativ in k_1" bezeichnet werden, wenn die mit α conjugirte Zahl in k_1 positiv bez. negativ ausfällt, ungeachtet des Vorzeichens von α selbst oder auch wenn α selbst imaginär ist; diese Abkürzung wird in den folgenden durchgehend beibehalten werden.

2) Vgl. § 4.

Eine Classe ist ambig, wenn sie ein Ideal des Grundkörpers k, oder ein ambiges Ideal des Relativkörpers K/k, oder aber ein Product eines ambigen Ideals und eines Ideals in k enthält, nicht aber umgekehrt.

Ueber die Anzahl der ambigen Classen im Körper K gibt der folgende Satz Aufschluss.

Satz 14. *Wenn*

h *die Classenzahl des Körpers* k,

r *die Anzahl der Grundeinheiten in* k,

δ *die Zahl* 1 *oder* 0, *jenachdem* k *die primitive* l^{te} *Einheitswurzel* ζ *enthält oder nicht,*

d *die Anzahl der von einander verschiedenen ambigen Primideale des Körpers* K/k,

l^{γ} *die Anzahl der Einheitenverbände in* k, *die durch die Relativnormen von Einheiten und von gebrochenen Zahlen des Körpers* K *gebildet sind,*

a *die Anzahl der ambigen Classen des Körpers* K *ist, dann wird*

$$a = h l^{d - r - (\gamma + 1 + \delta)}$$

In diesem Satze sollen die Idealclassen der Körper K *und* k *im absoluten Sinne genommen werden.*

Beweis. Wir zählen zunächst diejenigen ambigen Classen des Körpers K ab, welche durch die ambigen Ideale von K/k und die Ideale von k erzeugt werden. Die Ideale

$$\mathfrak{D}\mathfrak{j},$$

wo $\mathfrak{D}$ ein ambiges Ideal von K/k (oder das Ideal 1) und $\mathfrak{j}$ ein Ideal in k bedeutet, bilden, weil $\mathfrak{D}^l$ ein Ideal in k ist, in ihrer Gesamtheit eine Gruppe der Ordnung $l^d h$, worin der Inbegriff der ganzen und gebrochenen monomischen (Haupt-) Ideale von k das Hauptelement der Gruppe ist. Diese Gruppe sei mit D bezeichnet. Diejenigen der Elemente dieser Gruppe, welche in K in die Hauptclasse übergehen, bilden dann eine Untergruppe D_0 von D. Dann ist offenbar die Anzahl a_0 der aus $\mathfrak{D}$ und $\mathfrak{j}$ entspringenden ambigen Classen von K gleich dem Gruppenindex $(D : D_0)$.

Es seien nun, wie in Satz 12, wo jetzt O und o sämtliche Zahlen des Körpers K bez. k umfassen sollen,

$$E_1, E_2, \cdots E_\rho$$

die Einheiten des Körpers K mit der Relativnorm 1, von der folgenden Beschaffenheit:

1°. Jede Einheit E von K mit der Relativnorm 1 ist in der Form darstellbar:

$$E=E_1^{u_1}\,E_2^{u_2}\cdots\cdots E_\rho^{u_\rho}\,H^{1-s},$$

wo $u_1,\ u_2,\ldots\ldots u_\rho$ Zahlen aus der Reihe: 0, 1, 2,......$l-1$ sind, und H eine Einheit von K bedeutet.

2°. Diese ρ Einheiten sind in dem Sinne von einander unabhängig, dass niemals eine Beziehung von der Form

$$1=E_1^{u_1}\,E_2^{u_2}\cdots\cdots E_\rho^{u_\rho}\,H^{1-s}\qquad(0\leqq u<l)$$

bestehen kann, ausser wenn $u_1=u_2=\ldots\ldots=u_\rho=0$.

Da $\mathrm{N}(E_i)=1$ ist, so gibt es ganze Zahlen A_i in K, derart, dass[1)]

$$E_i=A_i^{1-s}\qquad(i=1,\ 2,\cdots\cdots\rho)$$

und zwar ist nach 2° A_i nicht eine Einheit in K. Das Hauptideal (A_i) ist daher von der Form $\mathfrak{D}\mathfrak{j}$ und es ist $(\mathfrak{D}\mathfrak{j})^l=\mathrm{N}(A_i)$ ein Hauptideal in k.

Da eine Beziehung von der Form:

$$A_1^{u_1}\,A_2^{u_2}\cdots\cdots A_\rho^{u_\rho}=H\alpha,\qquad(0\leqq u<l)$$

wo H eine Einheit in K, α eine Zahl in k bedeutet, die andere:

$$E_1^{u_1}\,E_2^{u_2}\cdots\cdots E_\rho^{u_\rho}=H^{1-s}$$

nach sich zieht, so bedingt sie, dass die Exponenten $u_1,\ u_2,\ldots\ldots u_\rho$ sämtlich verschwinden. Setzt man also

$$(A_i)=\mathfrak{D}_i\mathfrak{j}_i\qquad(i=1,\ 2,\cdots\cdots\rho)$$

so erzeugen diese Ideale genau l^ρ Elemente der Gruppe D_0.

Ist aber umgekehrt

$$\mathfrak{D}\mathfrak{j}=(A)$$

ein Hauptideal in K, so ist

$$A^{1-s}=E,$$

1) Vgl. Hilbert, Bericht, Satz 90.

wo E eine Einheit in K ist, für welche

$$N(E)=1$$

ausfällt. Daher ist nach 1°

$$E=E_1^{u_1}\,E_2^{u_2}\cdots\cdots E_\rho^{u_\rho}\,H^{1-s}, \qquad (0\leqq u<l)$$

wo H eine Einheit in K ist, oder

$$A^{1-s}=(A_1^{u_1}\,A_2^{u_2}\cdots\cdots A_\rho^{u_\rho}\,H)^{1-s},$$

folglich

$$A=A_1^{u_1}\,A_2^{u_2}\cdots\cdots A_\rho^{u_\rho}\,H\alpha,$$

wo α eine Zahl in k bedeutet. Das Ideal $\mathfrak{D}$j ist daher unter den oben erwähnten l^ρ Elementen der Gruppe $\mathfrak{D}_0$ enthalten.

Hiermit ist nachgewiesen, dass die Gruppe $\mathfrak{D}_0$ von der Ordnung l^ρ ist: für den Gruppenindex $a_0=(\mathfrak{D}:\mathfrak{D}_0)$ ergibt sich daher

$$a_0=hl^{d-\rho} \tag{1}$$

Wenn mit v_0 die Anzahl der Einheitenverbände in k, die aus den sämtlichen Relativnormen der Einheiten von K bestehen, bezeichnet wird, dann gibt es nach Annahme noch $r-v_0$ unabhängigen Einheiten in k, welche Relativnormen der gebrochenen Zahlen von K sind:

$$\varepsilon_1=N(\theta_1),\cdots\cdots\varepsilon_{r-v_0}=N(\theta_{r-v_0}),$$

von der Art, dass jede Einheit ε von k, welche Relativnorm einer gebrochenen Zahl von K ist, in der Form darstellbar ist:

$$\varepsilon=\varepsilon_1^{u_1}\,\varepsilon_2^{u_2}\cdots\cdots N(H), \qquad (0\leqq u<l)$$

wo H eine Einheit in K bedeutet, und dass eine Beziehung

$$1=\varepsilon_1^{u_1}\,\varepsilon_2^{u_2}\cdots\cdots N(H) \qquad (0\leqq u<l)$$

niemals bestehen kann, ausser wenn die $r-v_0$ Exponenten $u_1, u_2, \ldots\ldots$ sämtlich verschwinden.

Sei nun

$$\theta_1=H\mathfrak{p}^{F(s)}$$

die Zerlegung der gebrochenen Zahl θ_1 in die Primideale von K, wo also der Exponent $F(s)$ der symbolischen Potenz eine ganzzahlige ganze rationale Function vom Grade $l-1$ in s bedeutet, und das Product auf alle mit Θ_1 verwandten, einander nicht relativ conjugirten Primideale $\mathfrak{p}$ erstreckt werden soll. Da aber $N(\Theta_1)$ gleich einer Einheit ist, so folgt, dass

$$F(s)(1+s+s^2+\cdots\cdots+s^{l-1})$$

durch $1-s^l$, folglich $F(s)$ selbst durch $1-s$ teilbar ist. Wir können demnach setzen:

$$\Theta_i=\mathfrak{A}_i^{1-s} \qquad (i=1, 2, \cdots\cdots r-v_0)$$

wo $\mathfrak{A}_i$ ein ganzes oder gebrochenes Ideal von K ist. Die l^{te} Potenz dieses Ideals $\mathfrak{A}_i$ ist in K mit einem Ideal $\mathfrak{a}_i$ von k, nämlich der Relativnorm von $\mathfrak{A}_i$ aequivalent:

$$\mathfrak{A}_i^l=N(\mathfrak{A}_i)\mathfrak{A}_i^{(1-s)Q(s)}=\Theta_i^{Q(s)}\mathfrak{a}_i,$$

wo $Q(s)$ die bekannte Bedeutung hat.[1] Es kann aber eine Beziehung von der Form

$$\mathfrak{A}_1^{u_1}\,\mathfrak{A}_2^{u_2}\cdots\cdots=\mathfrak{D}\mathfrak{j}A, \qquad (0\leq u<l)$$

wo A eine Zahl von K bedeutet, niemals bestehen, ausser wenn die $r-r_0$ Exponenten u_1, u_2,...... sämtlich verschwinden; denn aus dieser Idealgleichheit folgt, durch das Erheben in die symbolische $1-s^{\text{te}}$ Potenz,

$$\Theta_1^{u_1}\,\Theta_2^{u_2}\cdots\cdots=HA^{1-s},$$

wo H eine Einheit in K ist, und daraus ferner, indem wir in die Relativnorm übergehen,

$$\varepsilon_1^{u_1}\,\varepsilon_2^{u_2}\cdots\cdots=N(H),$$

was das Verschwinden der Exponenten u_1, u_2,......bedingt.

Mit anderen Worten: die Ideale $\mathfrak{A}_1$, $\mathfrak{A}_2$,...... erzeugen $r-v_0$ ambigen Classen von K, die sowohl von einander als von den durch die Ideale $\mathfrak{D}\mathfrak{j}$ erzeugten unabhängig sind.

Anderseits ist jedes Ideal $\mathfrak{A}$ aus einer ambigen Classe von K in der Form darstellbar:

1) Vgl. S. 36.

$$\mathfrak{A}=\mathfrak{A}_1^{u_1}\,\mathfrak{A}_2^{u_2}\cdots\cdots\mathfrak{D}\mathfrak{j}.A, \qquad (0\leqq u<l)$$

wo A eine Zahl von K bedeutet.

Denn aus $\mathfrak{A}^{1-s}=\Theta$, folgt $\mathrm{N}(\Theta)=\varepsilon$, wo ε eine Einheit in k ist, und hieraus der Reihe nach:

$$\varepsilon=\varepsilon_1^{u_1}\,\varepsilon_2^{u_2}\cdots\cdots\mathrm{N}(H), \qquad \text{wo } H \text{ eine Einheit in K ist,}$$

$$\mathrm{N}(\Theta)=\mathrm{N}(\Theta_1^{u_1}\,\Theta_2^{u_2}\cdots\cdots H),$$

$$\Theta=\Theta_1^{u_1}\,\Theta_2^{u_2}\cdots\cdots H A^{1-s}, \qquad \text{wo } A \text{ eine Zahl von K ist,}$$

$$\mathfrak{A}^{1-s}=(\mathfrak{A}_1^{u_1}\,\mathfrak{A}_2^{u_2}\cdots\cdots A)^{1-s},$$

$$\mathfrak{A}=\mathfrak{A}_1^{u_1}\,\mathfrak{A}_2^{u_2}\cdots\cdots A\mathfrak{D}\mathfrak{j}.$$

Demnach ist

$$a=a_0\, l^{v-v_0},$$

also nach (1)

$$a=h\,l^{d+v-(\rho-v_0)}$$

Da nach Satz 12

$$\rho=r+1+\delta-v_0,$$

so ist

$$a=h\,l^{d+v-(r+1+\delta)}$$

wie zu beweisen war.

§. 11.

Die Anzahl der ambigen Classen im relativ quadratischen Körper.

Satz 15. *Wenn* $\mathrm{K}=\mathrm{k}(\sqrt{\mu})$ *relativ quadratisch in Bezug auf* k *ist, und wenn* ν *die Anzahl derjenigen mit* k *conjugirten reellen Körper ist, worin die Conjugirten von* μ *negativ ausfallen, dann ist, unter Beibehaltung der übrigen Bezeichnungsweise von Satz 14.*

$$a=h\,l^{d+v+\nu-(r+2)}$$

Die Classen in K *wie in* k *sollen wiederum im absoluten Sinne genommen werden.*

Der Beweis verläuft genau wie bei Satz 14; nur soll am Schlusse für die Zahl ρ der im gegenwärtigen Falle gültige Wert:

$$r+2-\nu-v_0$$

eingesetzt werden (vgl. Satz 12).

Es sei noch bemerkt, dass im Falle, wo die mit k conjugirten Körper sämtlich imaginär sind, dieser Satz genau mit Satz 14 zusammenfällt, weil dann $\nu=0$ und die Zahl δ in Satz 14 gleich 1 zu setzen ist, da die Einheitswurzel -1 in k vorkommt.

§. 12.

Die Geschlechter im relativ cyclischen Körper eines ungeraden Primzahlgrades.

Es sei K/k relativ cyclisch vom ungeraden Primzahlgrade l. $\mathfrak{d}=\mathfrak{f}^{l-1}$ die Relativdiscriminante desselben, $\mathfrak{o}$ die Zahlengruppe in k, die aus der Gesamtheit der zu $\mathfrak{f}$ primen *Normenreste des Körpers* K/k *nach* $\mathfrak{f}$ besteht. Die Idealclassen in k seien nach $\mathfrak{o}$ definirt, so dass zwei Ideale $\mathfrak{j}_1$ und $\mathfrak{j}_2$ in k dann und nur dann aequivalent sind, wenn die Idealgleichheit besteht:

$$\mathfrak{j}_1=\mathfrak{j}_2\alpha \text{ und } \alpha\equiv N(A), (\mathfrak{f}),$$

wo α und A zu $\mathfrak{f}$ prime ganze oder gebrochene Zahlen von k bez. K sind.

Wenn dann zwei Ideale $\mathfrak{J}_1$ und $\mathfrak{J}_2$ von K im absoluten Sinne aequivalent sind, und einer Classe (im absoluten Sinne) C von K angehören, dann fallen die Relativnormen dieser Ideale in eine und dieselbe Classe $\mathfrak{c}$ nach $\mathfrak{o}$ hinein; diese Classe $\mathfrak{c}$ heisse die Relativnorm der Classe C; im Zeichen

$$\mathfrak{c}=N(C).$$

Da $\mathfrak{o}$ eine Congruenzgruppe nach dem Modul $\mathfrak{f}$ ist, so ist Satz 4 anwendbar, demzufolge die Classengruppe von k, welche sämtliche Relativnormen der Classen von K enthält, von einem Index i sein muss, welcher den Relativgrad l des Relativkörpers K/k nicht übertreffen kann:

$$i\leqq l. \tag{1}$$

Sei G die Gruppe der sämtlichen Classen von K, H die Untergruppe von G, welche aus der Gesamtheit derjenigen Classen von K besteht, deren Relativnormen die Hauptclasse nach $\mathfrak{o}$ sind. Dann ist der Gruppenindex (G: H) offenbar gleich der Ordnung derjenigen Classengruppe von k nach $\mathfrak{o}$, welche aus der sämtlichen Relativnormen der Classen von K besteht. Daher folgt aus (1)

$$(G : H) = \frac{h'}{i} \geqq \frac{h'}{l}, \tag{2}$$

wenn h' die Classenzahl von k nach $\mathfrak{o}$ bedeutet.

Ferner sei H_0 die Gruppe der Classen von K, welche symbolische $1-s^{te}$ Potenzen der Classen von K sind, so dass der Gruppenindex

$$(G : H_0) = a,$$

der Anzahl der ambigen Classen von K ist.

Da offenbar H_0 eine Untergruppe von H ist, so folgt nach (2)

$$a = (G : H_0) \geqq (G : H) \geqq \frac{h'}{l}. \tag{3}$$

Nach Satz 14 ist nun[1]

$$a = hl^{d+r-(t+1+\delta)} \tag{4}$$

wenn h die Classenzahl von k im absoluten Sinne bedeutet.

Anderseits ist, wenn $\mathfrak{o}'$ die Gruppe der sämtlichen zu $\mathfrak{f}$ primen Zahlen in k bedeutet, nach Satz 10

$$(\mathfrak{o}' : \mathfrak{o}) = l^d, \tag{5}$$

wo d die Anzahl der von einander verschiedenen in $\mathfrak{f}$ aufgehenden Primideale in k ist, also dieselbe Bedeutung hat, wie in (4).

Ist ferner E' die Gruppe der sämtlichen Einheiten in k, und E die der Einheiten in $\mathfrak{o}$, dann ist offenbar

$$(\mathrm{E}' : \mathrm{E}) = l^{r+\delta-n}, \tag{6}$$

wenn r und δ dieselbe Bedeutung haben, wie in (4), und l^n die Anzahl der Einheitenverbände in $\mathfrak{o}$ ist.

Demnach ist[2] nach (5) und (6)

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1) Die Beschränkung, dass wir hier nur die zu $\mathfrak{f}$ primen Ideale von K in Betracht ziehen, hat keinen Einfluss auf die Anzahl a gewisser Classen von K, die ja im absoluten Sinne genommen wird, vgl. Satz 2.

2) Vgl. § 1, S. 7.

$$h' = h\frac{(\mathfrak{o}':\mathfrak{o})}{(\mathrm{E}':\mathrm{E})} = hl^{l+n-(r+\delta)} \qquad (7)$$

Aus (3), (4), und (7) folgt

$$d+v-(r+1+\delta) \geqq d+n-(r+\delta)-1,$$

oder

$$0 \geqq n-v.$$

Da offenbar $n-v \geqq 0$, so erhält man

$$n=v. \qquad (8)$$

Dies hat zur Folge, dass in (3) und somit auch in (2) und (1) notwendig das Gleichheitszeichen gelten muss. Demnach ergibt sich

$$a = \frac{h'}{l}, \qquad (9)$$

$$\mathrm{H} = \mathrm{H}_0, \qquad (10)$$

$$i = l. \qquad (11)$$

Hiermit ist der Fundamentalsatz 13 für einen relativ cyclischen Körper vom ungeraden Primzahlgrade bewiesen, denn wenn die Classen von k nach einem beliebigen durch $\mathfrak{f}$ teilbaren Ideale $\mathfrak{m}$ definirt werden, so mag sich jede Classe nach $\mathfrak{o}$ in gleichviele Classen nach $\mathfrak{m}$ auflösen, jedoch ohne dass der *Index* einer Classengruppe verändert wird.

Aus dem vorhergehenden Beweis von Satz 13 ziehen wir noch einige wichtige Schlüsse:

Alle diejenige Classe von K, deren Relativnorm eine und dieselbe Classe von k nach der Gruppe $\mathfrak{o}$ der Normenreste nach $\mathfrak{f}$ ist, fassen wir in ein **Geschlecht** zusammen, und definiren speciell das *Hauptgeschlecht* als den Inbegriff derjenigen Classen von K, deren Relativnormen die Hauptclasse von k nach $\mathfrak{o}$ sind. Das Hauptgeschlecht ist also die Classengruppe H, und das Geschlecht, welchem eine Classe C angehört der Classencomplex HC. Also folgt aus (9) und (10):

Satz 16. *Die Anzahl der Geschlechter in* K *ist gleich dem* l^{ten} *Teil der Classenzahl von* k *nach* $\mathfrak{o}$.

Satz 17. *Jede Classe des Hauptgeschlechts in* K *ist die symbolische* $1-S$ *Potenz einer Classe von* K.

Ferner gilt

Satz 18. *Wenn eine Einheit in* k, *oder eine Zahl in* k, *die* l^{te} *Potenz eines Ideals von* k *ist, Normenrest des Körpers* K/k *nach dem Ideale* $\mathfrak{f}$ *ist, dann ist sie wirkliche Relativnorm einer ganzen oder gebrochenen Zahl von* K.

Beweis. Was die Einheiten betrifft ist dieser Satz schon in (8) enthalten. Sei also $(\nu)=\mathfrak{j}^l$, und ν Normenrest des Körpers K/k nach $\mathfrak{f}$. Da $N(\mathfrak{j})=\mathfrak{j}^l=(\nu)$, und ν der Zahlengruppe $\mathfrak{o}$ angehört, so ist das Ideal $\mathfrak{j}$ in einer Classe des Hauptgeschlechts von K enthalten. Daher ist nach Satz 17

$$\mathfrak{j}=\mathfrak{J}^{1-S}\Theta,$$

wo $\mathfrak{J}$ ein Ideal, Θ eine Zahl in K bedeutet. Bildet man beiderseits die Relativnormen, so erhält man

$$\nu=\varepsilon N(\Theta),$$

wo ε eine Einheit in k ist. Da nun ν Normenrest nach $\mathfrak{f}$ ist, so gilt dasselbe auch von ε; folglich ist nach (8) ε eine wirkliche Relativnorm. Setzt man daher

$$\varepsilon=N(A),$$

dann folgt

$$\nu=N(A\Theta).$$

§. 13.

Die Geschlechter im relativ quadratischen Körper.

Wenn $K=k(\sqrt{\mu})$ relativ quadratisch in Bezug auf k ist, und wenn ν die Anzahl derjenigen mit k conjugirten reellen Körper ist, worin die Conjugirten von μ negativ ausfallen, dann rechnen wir nur diejenigen Normenreste nach $\mathfrak{f}$, welche in diesen ν Körpern *positiv* ausfallen, in die Zahlengruppe $\mathfrak{o}^+$, und legen dieselbe der Classeneinteilung in k zu Grunde.

Da die Relativnormen der zu $\mathfrak{f}$ primen Zahlen von K offenbar der Zahlengruppe $\mathfrak{o}^+$ angehören, so fallen die Relativnormen aller

Ideale einer Classe (im absoluten Sinne) C von K in eine und dieselbe Classe c von k nach $\mathfrak{o}^+$; dieselbe nennen wir demnach die Relativnorm der Classe C von K: $c=\mathrm{N}(C)$.

Die Betrachtungen, die wir im vorhergehenden Paragraphen angestellt haben, werden mit geringen Modificationen auch im gegenwärtigen Falle genau dieselben Resultaten ergeben. Indem wir uns durchweg die Bezeichnungsweise des vorigen Artikels bedienen, ist zunächst h' die Classenzahl von k nach $\mathfrak{o}^+$, so dass

$$h'=h\frac{(\mathfrak{o}':\mathfrak{o}^+)}{(\mathrm{E}':\mathrm{E}^+)},$$

wo E^+ die Gruppe der Einheiten in $\mathfrak{o}^+$ bedeutet. Es ist also nach Satz 10

$$(\mathfrak{o}':\mathfrak{o}^+)=2^{d+\nu},$$

weil die Congruenzbedingung, Normenrest nach $\mathfrak{f}$ zu sein, welcher eine Zahl von k zu genügen hat, unabhängig ist von der Vorzeichencombination dieser Zahl in den ν oben specifirten Körpern.

Sodann ist

$$(\mathrm{E}':\mathrm{E})=2^{r+1-n},$$

wenn 2^n die Anzahl der Einheitenverbände in $\mathfrak{o}^+$ bedeutet.

Daher ist

$$h'=h.\ 2^{d+\nu+n-(r+1)}.$$

Die Bedingung

$$a\geqq\frac{h'}{2}$$

ergibt, wenn man darin für a den in Satz 15 angegebenen Wert:

$$a=h\,2^{d+r+\nu-(r+2)}$$

einsetzt,

$$d+r+\nu-(r+2)\geqq d+\nu+n-(r+1)-1,$$

oder

$$0\geqq n-r,$$

woraus wie vorhin

$$n = r,$$

und folglich

$$a = \frac{h'}{2},$$

$$H = H_0,$$

$$i = 2.$$

Die letzte Gleichheit beweist Satz 13 für einen relativ quadratischen Körper.

Wenn die Gesamtheit derjenigen Classen von K, deren Relativnormen eine und dieselbe Classe von k nach $\mathfrak{o}^+$ sind, in ein **Geschlecht**, diejenigen, deren Relativnormen die Hauptclasse von k nach $\mathfrak{o}^+$ sind, in das *Hauptgeschlecht* gerechnet werden, dann gelten die Sätze:

Satz 19. *Die Anzahl der Geschlechter in einem relativ quadratischen Körper ist gleich der Hälfte der Classenzahl von* k *nach* $\mathfrak{o}^+$.

Satz 20. *Jede Classe des Hauptgeschlechts in einem relativ quadratischen Körper ist die symbolische* $1-s^{te}$ *Potenz einer Classe von* K.

Ferner gilt.

Satz 21. *Wenn eine Einheit von* k *oder eine Zahl von* k, *welche Idealquadrat in* k *ist, positiv in den mit* k *conjugirten reellen Körpern, worin die Zahl* μ *negativ ausfällt*[1] *und Normenrest des relativquadratischen Körpers* $K = k(\sqrt{\mu})$ *nach dem Ideal* $\mathfrak{f}$ $(=\mathfrak{d})$ *ist, dann ist sie wirklich Relativnorm einer Zahl von* K.

§. 14.

Eine Verallgemeinerung des Geschlechterbegriffs.

Es sei $\mathfrak{f}^{l-1}$ die Relativdiscriminante des relativ cyclischen Körpers K/k vom Primzahlgrade l, m ein beliebiges durch $\mathfrak{f}$ teilbares Ideal in k, $\mathfrak{o}$ die Zahlengruppe in k, welche aus der Gesamtheit derjenigen Zahlen ω in k besteht, die der Congruenz:

$$\omega \equiv 1, \quad (\mathrm{m})$$

1) Vgl. Fussnote 1), S. 42.

genügen, und im Falle: $l=2$, überdies total positiv sind.

Wir legen diese Zahlengruppe o der Classeneinteilung im Grundkörper k zu Grunde, und verallgemeinern den Begriff der Geschlechter in K dahin, dass die Ideale in K nur dann in ein Geschlecht gerechnet werden, wenn ihre Relativnormen in eine und dieselbe Classe nach o hineinfallen. Insbesondere ist demnach das Hauptgeschlecht die Gesamtheit der Ideale $\mathfrak{J}$ in K, deren Relativnormen in der Hauptclasse nach o liegen, d. h.

$$\mathrm{N}(\mathfrak{J})=(\omega), \quad \text{wo} \quad \omega\equiv 1, \quad (\mathfrak{m}),$$

und, wenn $l=2$, überdies noch ω total positiv ist.

Dass die Anzahl der Geschlechter gleich dem l^{ten} Teil der Classenzahl nach o ist, dass also die Sätze 16 und 19 auch für die Geschlechter im verallgemeinerten Sinne gelten, ist einleuchtend, nach einer Bemerkung in § 12 (S. 50). Zweck dieses Artikels ist es nun, nachzuweisen, dass es möglich ist, eine geeignete Zahlengruppe O in K so zu bestimmen dass, wenn die Classen in K nach derselben definirt werden, jede Classe des Hauptgeschlechtes in K die symbolische $(1-s)^{\text{te}}$ Potenz einer Classe von K wird, dass also auch die Sätze 17 und 20 ihre Gültigkeit beibehalten werden. Wir müssen uns aber zunächst mit einigen Hulfssätzen beschäftigen.

Hülfssatz 1. *Ist* $\mathfrak{q}$ *ein Primideal in* k, *welches nicht in die Relativdiscriminante des relativ cyclischen Körpers* K/k *vom Primzahlgrade* l *aufgeht,* θ *eine Zahl in* K, *welche der Bedingung*

$$\mathrm{N}(\theta)\equiv 1, \quad (\mathfrak{q}^e) \tag{1}$$

genügt, wo e *ein beliebiger positiver Exponent ist, dann gibt es in* K *eine zu* $\mathfrak{q}$ *prime Zahl* A, *derart, dass*

$$\theta\equiv A^{1-s} \quad (\mathfrak{q}^e).$$

Beweis. Wir bedienen uns auch hier mit Vorteil des Gruppenbegriffs. Sei G die Gruppe der sämtlichen zu $\mathfrak{q}$ primen Zahlclassen von K nach dem Modul $\mathfrak{q}^e$, H diejenige der Zahlclassen, deren

Zahlen die Bedingung (1) befriedigen, endlich H_0 die der Zahlclassen, welche durch die Zahlen A^{1-s} representirt werden. Es ist dann zu beweisen, dass

$$H = H_0.$$

Da offenbar H_0 eine Untergruppe von H ist, so gilt für die Gruppenindices

$$(G : H) \leqq (G : H_0).$$

Berücksichtigt man nun, dass, wenn $\theta \equiv 1$, $(\mathfrak{q}^e)$, offenbar $N(\theta) \equiv 1$, $(\mathfrak{q}^e)$ ist, so sieht man ein, dass (G:H) gleich der Anzahl der Normenrestclassen in k nach $\mathfrak{q}^e$, also nach Satz 9

$$(G : H) = \varphi(\mathfrak{q}^e).$$

Anderseits ist $(G : H_0)$ offenbar gleich der Anzahl der Zahlclassen, deren Zahlen der Bedingung

$$A^{1-s} \equiv 1, \quad (\mathfrak{q}^e) \qquad (2)$$

genügt. Unser Satz wird daher bewiesen sein, wenn gezeigt wird, dass jede Zahl A, welche der Congruenz (2) genügt, notwendig congruent einer Zahl in k nach dem Modul $\mathfrak{q}^e$ ausfallen muss.

Dies ist einleuchtend, wenn $\mathfrak{q}$ prim zu l ist, denn aus (2) folgt

$$A \equiv A^s \equiv A^{s^2} \cdots \equiv A^{s^{l-1}} \quad (\mathfrak{q}^e),$$

daher

$$lA \equiv S(A), \quad (\mathfrak{q}^e),$$

wo $S(A)$ die Relativspur von A, also eine Zahl in k ist. Da l prim zu $\mathfrak{q}$ ist, so folgt hieraus das Gesagte.

Wenn $\mathfrak{q} = \mathfrak{l}$ ein in l aufgehendes Primideal ist, unterscheiden wir zwei Fälle, jenachdem $\mathfrak{l}$ in K in l von einander verschiedene Primideale zerfällt, oder prim bleibt.

Im ersten Falle, sei

$$\mathfrak{l} = \mathfrak{L}(s\mathfrak{L})(s^2\mathfrak{L})\cdots\cdots$$

Da dann $\Phi(\mathfrak{L}^e)=\Phi(s\mathfrak{L}^e)=\cdots\cdots=\varphi(\mathfrak{l}^e)$ so ist für jede Zahl A in K

$$A\equiv\alpha,\ (\mathfrak{L}^e),\ \equiv\alpha',\ (s\mathfrak{L}^e)\cdots\cdots$$

wo $\alpha, \alpha',\cdots\cdots$ Zahlen in k sind. Ist also $A\equiv A^s$, $(\mathfrak{l}^e)$, dann muss, da $A^s\equiv\alpha$, $(s\mathfrak{L}^e),\cdots\cdots$ $\alpha\equiv\alpha'$, $(s\mathfrak{L}^e)$, also $\alpha\equiv\alpha'$, $(\mathfrak{l}^e)$; ebenso $\alpha'\equiv\alpha''$, $(\mathfrak{l}^e)$, usw. Folglich ist

$$A\equiv\alpha,\quad (\mathfrak{l}^e).$$

Zweitens sei $\mathfrak{l}=\mathfrak{L}$ prim in K. Ist dann $\mathfrak{l}$ Primideal f^{ten} Grades in k, also lf ten Grades in K, dann ist bekanntlich für jede zahl A in K

$$A^s\equiv A^{l^f}\quad (\mathfrak{L}).$$

Ist daher $A\equiv A^s$ $(\mathfrak{L})$, dann ist

$$A\equiv A^{l^f}\quad (\mathfrak{L}),$$

also, wenn A prim zu $\mathfrak{l}$ ist,

$$A^{l^f-1}\equiv 1,\quad (\mathfrak{L}).$$

Dies ist aber das Kriterium dafür, dass A einer Zahl α in k nach $\mathfrak{L}$ congruent sein soll.

Sei ferner

$$A\equiv A^s\quad (\mathfrak{L}^2),$$

dann kann man setzen

$$A\equiv\alpha+\lambda B,\quad (\mathfrak{L}^2),$$

wo α eine Zahl in k, λ eine durch die erste Potenz von $\mathfrak{l}$ teilbare Zahl in k, und B eine Zahl in K ist. Dann folgt

$$B\equiv B^s\quad (\mathfrak{L}).$$

also nach dem vorhergehenden

$$B\equiv\beta,\quad (\mathfrak{L}),$$

wo β eine Zahl in k ist. Es ist also auch

$$A\equiv\alpha',\quad (\mathfrak{L}^2),$$

wo α' eine Zahl in k ist. So fortfahrend beweist man den Satz für jede beliebige Potenz von $\mathfrak{l}$ als Modul.

Es sei noch bemerkt, dass dieser Beweis für das Primideal $\mathfrak{l}$ auch für die zu l primen Primideale $\mathfrak{p}$ seine Gültigkeit beibehält.

Hülfssatz 2. *Es sei $\mathfrak{p}$ ein zu l primes Primideal in* k, *welches in die Relativdiscriminante des relativ cyclischen Körpers* K/k *vom Primzahlgrade l aufgeht, so dass $\mathfrak{p}$ die l^{te} Potenz eines Primideals $\mathfrak{P}$ in* K *ist. Ferner sei Θ eine Zahl in* K, *welche der Bedingung*

$$N(\Theta) \equiv 1, \quad (\mathfrak{p}^e)$$

genügt, wo e ein beliebiger positiver Exponent ist. Dann gibt es eine Zahl A in K, *derart, dass*

$$\Theta \equiv A^{1-S} \quad (\mathfrak{P}^{(e-1)l+1}),$$

wenn für A auch eine durch $\mathfrak{P}$ teilbare Zahl zugelassen wird.

Dasselbe gilt auch dann, wenn $\mathfrak{p} = \mathfrak{l}$ in l aufgeht, vorausgesetzt, dass für den Modul der ersten Congruenz $\mathfrak{l}^{v+n}$, für den der zweiten $\mathfrak{L}^{v+l}$ angenommen wird, wo n eine beliebige positive ganze rationale Zahl ist, und v die bisherige Bedeutung für das Primideal $\mathfrak{l} = \mathfrak{L}^l$ hat.[1]

Beweis. Es habe G, H, H_0 dieselbe Bedeutung wie bei dem Beweis des vorhergehenden Hülfssatzes. Wir bemerken zuvörderst, dass, wenn Π (bez. A) eine genau durch die erste Potenz von $\mathfrak{P}$ (bez. $\mathfrak{L}$) teilbare Zahl in K ist, Π^{1-S} (bez. A^{1-S}) offenbar eine Zahl ist, die der Gruppe H angehört, von der aber erst die l^{te} Potenz der Gruppe H_0 angehören kann, weil eine Congruenz

$$\Pi^{1-S} \equiv A^{1-S} \ (\mathfrak{P}), \quad \text{bez.} \quad A^{1-S} \equiv A^{1-S} \quad (\mathfrak{L}^{v+1})$$

unmöglich ist, wenn A prim zu $\mathfrak{P}$ (bez. $\mathfrak{L}$) sein soll.

Daher ist der Gruppenindex

$$(H : H_0) \geqq l.$$

Anderseits ist, weil aus $\Theta \equiv 1$, $(\mathfrak{P}^{(e-1)l+1})$ bez. $(\mathfrak{L}^{v+nl})$ offenbar folgt: $N(\Theta) \equiv 1$ $(\mathfrak{p}^e)$ bez. $(\mathfrak{l}^{v+n})$[2], der Gruppenindex (G : H) gleich der Anzahl der Normenrestclassen in k nach $\mathfrak{p}^e$ bez. $\mathfrak{l}^{v+n}$, also nach Satz 9

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1) Vgl. Satz 8, S. 24.

2) Vgl. S. 34, Gl. (9).

$$(\mathrm{G}:\mathrm{H}) = \frac{1}{l}\,\varphi(\mathfrak{p}^e) \text{ bez. } \frac{1}{l}\,\varphi(\mathfrak{l}^{v+n}).$$

Unser Satz wird daher bewiesen sein, wenn gezeigt wird, dass $(\mathrm{G}:\mathrm{H}_0)$ oder, was dasselbe ist, die Anzahl der zu $\mathfrak{P}$ bez. $\mathfrak{L}$ primen Zahlclassen nach dem Modul $\mathfrak{P}^{(e-1)l+1}$ bez. $\mathfrak{L}^{v+nl}$, deren Zahlen A der Congruenz

$$A \equiv A^S, \quad (\mathfrak{P}^{(e-1)l+1}),$$

bez. $$A \equiv A^S, \quad (\mathfrak{L}^{v+nl}) \qquad (3)$$

genügen, genau $\varphi(\mathfrak{p}^e)$ bez. $\varphi(\mathfrak{l}^{v+n})$ beträgt.

Für das zu l primes $\mathfrak{P}$ ist dies einleuchtend, wie beim Beweis des vorhergehenden Hülfssatzes. Um den Satz für das Primideal $\mathfrak{L}$ zu beweisen, sei A_t eine genau durch die t^{te} Potenz von $\mathfrak{L}$ teilbare Zahl. Setzt man eine Zahl A in der Form an:

$$A \equiv \alpha + A_t,$$

wo α eine Zahl in k ist, und t für gegebenes A den möglichst grossen Wert haben soll, so dass t nicht durch l teilbar ist, dann genügt A dann und nur dann der Congruenz (3), wenn

$$t \geqq nl.$$

Diese Zahlen A werden also durch

$$A \equiv \alpha_0 + \beta_1 A_{nl+1} + \cdots\cdots + \beta_{v-1} A_{v+nl-1}, \quad (\mathfrak{L}^{v+nl}).$$

gegeben, wenn für α_0 die $\varphi(\mathfrak{l}^{n+1})$ einander nach $\mathfrak{l}^{n+1}$ incongruenten zu $\mathfrak{l}$ prime Zahlen in k, für $\beta_1, \cdots\cdots \beta_{v-1}$ je ein System der l^f einander nach $\mathfrak{l}$ incongruenten Zahlen in k gesetzt werden. Es ergibt sich also für die Anzahl in Frage der Wert

$$\varphi(\mathfrak{l}^{n+1}).\, l^{f(v-1)} = \varphi(\mathfrak{l}^{n+v}).$$

wie nachzuweisen war.[1]

[1] Ohne Satz 9 zu benutzen, zeigt man leicht, wie aus der vorhergehenden Beweise einzusehen ist, dass der Normenrest nach $\mathfrak{p}^e$ bez. $\mathfrak{l}^{v+n}$ *höchstens* den l^{ten} Teil der sämtlichen Zahlclassen nach $\mathfrak{p}^e$ bez. $\mathfrak{l}^{v+n}$ ausmachen kann. Mit dieser Obergrenze für die Anzahl der Normenreste kommt man aber beim Beweis des Satzes 13 in §12 aus. Denn alsdann ist auf der rechten Seite von (5) (S. 49) $d+x$ statt d zu setzen, wo $x \geqq 0$. Dann erhält man zunächst $0 \geqq n-r+x$, woraus notwendig $n-r=0$ und $x=0$ folgt. So wäre der Satz 10 auf diesem Umwege von neuem bewiesen sein. Diese Bemerkung füge ich zu, als eine Verificirung des Satzes 10.

In den Folgenden benutzen wir die Hülfssätze 1, 2 in der verallgemeinerten Fassung, die wie folgt lautet.

Hülfssatz 3. *Es sei $\mathfrak{f}^{l-1}$ die Relativdiscriminante des relativ cyclischen Körpers* K/k *vom Primzahlgrade* l, $\mathfrak{m}=\mathfrak{f}\mathfrak{a}$ *ein beliebiges durch* $\mathfrak{f}$ *teilbares Ideal in* k. *Entsprechend seien*

$$\mathfrak{F}=\Pi\mathfrak{P}.\,\Pi\mathfrak{L}^{v+l},\quad \mathfrak{M}=\mathfrak{F}\mathfrak{a}$$

Ideale in K, *wo das Product* $\Pi\mathfrak{P}$ *auf alle von einander verschiedenen in* $\mathfrak{f}$ *aufgehenden und zu* l *primen Primideale von* K, *und das Product* $\Pi\mathfrak{L}^{v+l}$ *auf alle diejenigen, welche in* l *aufgehen, zu erstrecken ist. Ist dann* Θ *eine zu* $\mathfrak{M}$ *prime Zahl in* K, *welche der Bedingung*

$$N(\Theta)\equiv 1,\quad (\mathfrak{m})$$

genügt, dann gibt es in K *eine Zahl* A, *derart, dass*

$$\Theta\equiv A^{1-s},\quad (\mathfrak{M})$$

wird. Die Zahl A *ist unter Umständen nicht prim zu* $\mathfrak{M}$, *ist aber von der Art, dass*

$$(A)^{1-s}=\mathfrak{A}^{1-s},$$

wo $\mathfrak{A}$ *ein zu* $\mathfrak{M}$ *primes Ideal in* K *ist, dass ferner, wenn* $l=2$, A *eine beliebig vorgeschriebene Vorzeichencombination in den mit* K *conjugirten Körpern haben kann.*

Beweis. Setzt man

$$\mathfrak{m}=\Pi\mathfrak{p}^{e}\,\Pi\mathfrak{l}^{v+n}\,\Pi\mathfrak{q}^{e'},$$

dann ist, nach Annahme

$$\mathfrak{M}=\Pi\mathfrak{P}^{(e-1)l+1}\,\Pi\mathfrak{L}^{v+nl}\,\Pi\mathfrak{q}^{e'},$$

wo das erste und das zweite Product bei $\mathfrak{m}$ sowie bei $\mathfrak{M}$ die bekannte Bedeutung haben und das dritte Product auf alle in $\mathfrak{m}$ enthaltenen, zu $\mathfrak{f}$ primen Primideale zu erstrecken ist. Nach Hülfssätzen 1 und 2 ergibt daher für Θ die Congruenzen

$$\Theta \equiv (\Pi^{\alpha} A_1)^{1-s}, \qquad (\mathfrak{P}^{(e-1)l+1}), \cdots\cdots$$
$$\equiv (\Lambda^{\beta} A_2)^{1-s}, \qquad (\mathfrak{Q}^{v+nl}), \cdots\cdots$$
$$\equiv A_3^{1-s}, \qquad (\mathfrak{q}^{e'}), \cdots\cdots$$

wo Π bez. Λ genau durch die erste Potenz von $\mathfrak{P}$ bez. $\mathfrak{Q}$ teilbar, und A_1, A_2, A_3......bez. zu $\mathfrak{P}$, $\mathfrak{Q}$, $\mathfrak{q}$......prim, und ausserdem Π und $A_1 \equiv 1$, $(\mathfrak{M} : \mathfrak{P}^{(e-1)l+1})$, Λ und $A_2 \equiv 1$, $(\mathfrak{M} : \mathfrak{Q}^{v+nl})$, $A_3 \equiv 1$, $(\mathfrak{M} : \mathfrak{q}^{e})$, usw. angenommen sind, und die Exponenten $\alpha, \beta, \cdots\cdots$ Zahlen aus der Reihe $0, 1, 2, \ldots\ldots l-1$ bedeuten.

Daher ist

$$\Theta \equiv A^{1-s}, \qquad (\mathfrak{M}),$$

wenn

$$A = \Pi^{\alpha} \Lambda^{\beta} A_1 A_2 A_3 \cdots\cdots$$

gesetzt wird. Da $\mathfrak{M} = \mathfrak{M}^s$, so wird, wenn $A \equiv B$, $(\mathfrak{M})$, offenbar $A^{1-s} \equiv B^{1-s}$, $(\mathfrak{M})$. Ersetzt man daher nach Bedarf B durch

$$A^* = A + m\Gamma,$$

wo Γ für $l=2$, eine Zahl in K mit einer vorgeschriebene Vorzeichencombination, und m eine durch $\mathfrak{M}$ teilbare positive rationale Zahl bedeutet, die hinlänglich gross angenommen werden mag, so dass A^* dieselbe Vorzeichencombination wie Γ hat, dann wird die Forderung betreffs der Vorzeichen erfüllt. Endlich ist, wenn

$$\Pi = \mathfrak{P}\mathfrak{A}_1, \cdots\cdots$$
$$\Lambda = \mathfrak{Q}\mathfrak{A}_2, \cdots\cdots$$

gesetzt wird, nach Annahme, $\mathfrak{A}_1$, $\mathfrak{A}_2$, ...prim zu $\mathfrak{M}$. Daher ist

$$(A^{1-s}) = \mathfrak{A}^{1-s},$$

wo $\mathfrak{A} = \mathfrak{A}_1^{\alpha} \mathfrak{A}_2^{\beta} A_1 A_2 A_3 \cdots\cdots$ ein zu $\mathfrak{M}$ primes Ideal ist. Somit ist Hülfssatz 3 in allen seinen Teilen bewiesen.

Wir gehen jetzt zum Beweis des am Anfang dieses Artikels angedeuteten Satzes über, den wir wie folgt formuliren wollen:

Satz 22. *Es sei* K/k *relativ cyclisch vom Primzahlgrade l, es habe die Ideale* m, $\mathfrak{M}$ *die im Hülfssatz 3 erklärte Bedeutung; ferner seien* o *und* O *die Zahlengruppen in* k *bez.* K, *welche aus den Zahlen* ω *bez.* Ω *bestehen, welche bez. die Congruenzen:*

$$\omega \equiv 1, \quad (\mathfrak{m}); \qquad \Omega \equiv 1, \quad (\mathfrak{M})$$

befriedigen, und überdies, wenn $l=2$, *total positiv in Bezug auf* k *bez.* K *sind.*

Werden alsdann die Idealclassen in k *und* K *bez. nach* o *und* O *definirt, dann ist jede Classe des Hauptgeschlechts in* K *nach* O *eine symbolische* $(1-s)^{\text{te}}$ *Potenz einer Classe in* K *nach* O.

Beweis. Greifen wir zum Beweise des Satzes 13 in §12 und §13 zurück, so sehen wir ein, dass jener Satz gültig bleibt, wenn man in k die Classen nach der Gruppe der Normenreste nach m definiren, in K aber die Classen im absoluten Sinne annehmen (nur sollen die nicht zu m primen Ideale ausser Betracht gelassen sein, was der Classeneinteilung nicht beeinflusst). Demnach genügt es nachzuweisen, dass jedes Ideal der Form $\mathfrak{J}^{1-s}\Theta$ in K, für welches

$$N(\mathfrak{J}^{1-s}\Theta)=(\omega) \tag{4}$$

ausfällt, notwendig von der Form $\mathfrak{J}'^{1-s}\Omega$ sein muss; hier bedeutet Θ eine beliebige zu $\mathfrak{M}$ fremde Zahl in K, ω und Ω dagegen Zahlen bez. in o und O.

Aus (4) folgt nun

$$N(\Theta)=\varepsilon\omega, \tag{5}$$

wo ε eine Einheit in k ist, welche, weil $\omega \equiv 1$, $(\mathfrak{f})$, Normenrest nach $\mathfrak{f}$, folglich nach Satz 18 sich als eine wirkliche Zahlennorm erweist.

Sei also

$$N(B)=\varepsilon, \quad \text{demnach} \quad (B)=\mathfrak{B}^{1-s}, \tag{6}$$

woraus

$$N\left(\frac{\Theta}{B}\right)=\omega. \tag{7}$$

Daher ist nach Hülfssatz 3,

$$\frac{\Theta}{B}=A^{1-s}\Omega, \tag{8}$$

wo

$$(A^{1-s})=\mathfrak{A}^{1-s}.$$

Demnach folgt nach (6) und (8)

$$\Theta=(\mathfrak{A}\mathfrak{B})^{1-s}\Omega,$$

woraus, in der Tat,

$$\mathfrak{J}^{1-s}\Theta=\mathfrak{J}'^{1-s}\Omega,$$

wenn $\mathfrak{J}'=\mathfrak{A}\mathfrak{B}\mathfrak{J}$ gesetzt wird.

Wir haben oben die Vorzeichenbedingungen ausser Acht gelassen. Ist nun $K=k(\sqrt{\mu})$ relativ quadratisch, dann ist in (5) ω total positiv, also ε positiv in jedem mit k conjugirten reellen Körpern, worin μ negativ ausfällt. Daher gilt nach Satz 21 die Gleichheit (6) auch in diesem Falle. Da ferner nach (7), die Zahl $\frac{\Theta}{B}$ dieselbe Vorzeichen in jedem Paare zu K conjugirten Oberkörpern von k′ hat, wo k′ ein beliebiger zu k conjugirter reeller Körper ist, in welcher μ positiv ausfällt, und weil A in (8), nach Hülfssatz 3, beliebig vorgeschriebene Vorzeichencombination haben kann, so kann man A so wählen, dass A^{1-s} dieselbe Vorzeichencombination wie $\frac{\Theta}{B}$ bekommt, so dass die Zahl Ω in (8) total positiv in Bezug auf K wird. Hiermit ist unser Satz in allen seinen Teilen vollständig bewiesen.

CAPITEL III.

Existenzbeweis für den allgemeinen Classenkörper.

§. 15.

Formulirung des Existenzsatzes.

Satz 23. *In einem algebraischen Körper* k *sei eine Classengruppe* H *nach dem Modul* m *mit oder ohne Vorzeichenbedingung vorgelegt. Dann existirt stets ein Classenkörper* K *für diese Classengruppe* H, *welcher die folgenden Eigenschaften besitzt:*

1) K *ist relativ Abel'sch in Bezug auf* k.
2) *Die Galois'sche Gruppe des Relativkörpers* K/k *ist holoedrisch isomorph mit der complementären Gruppe* G/H, *wo* G *die Gruppe der sämtlichen Classen von* k *bedeutet.*
3) *Die Relativdiscriminante von* K/k *enthält kein Primideal als Factor, welches nicht in den Modul* m *aufgeht.*

Dieser Satz ist die naturgemässe Verallgemeinerung des zuerst von D. Hilbert[1] für den Fall: m=1, also für den Classenkörper im absoluten Sinne ausgesprochenen Satzes, welcher von ihm in den einfachsten Specialfällen, dann später von Ph. Furtwängler[2] für beliebige Grundkörper k bewiesen worden ist. Der Beweis des oben aufgestellten Existenzsatzes für den *allgemeinen* Classenkörper gelingt durch die gehörige Erweiterung der Hilbert'schen Methode; eine grosse Erleichterung erzielen wir aber durch Zuhülfenahme des Fundamentalsatzes 13.

§. 16.

Rang der Gruppe der Zahlclassen.

Es sei l eine gerade oder ungerade natürliche Primzahl, $\mathfrak{l}$ ein Primideal des Körpers k, welches zur $s$ten Potenz in l aufgeht, und vom f^{ten} Grade ist. Es existirt alsdann in k ein System von ρ Zahlen $\gamma_1, \gamma_2, \ldots\ldots \gamma_\rho$, welche sämtlich $\equiv 1$, $(\mathfrak{l})$, und so beschaffen sind, dass für jede zu $\mathfrak{l}$ prime Zahl γ von k eine Relation von der Form

$$\gamma \equiv \gamma_1^{u_1}\, \gamma_2^{u_2} \cdots\cdots \gamma_\rho^{u_\rho}\, \xi^l, \quad (\mathfrak{l}^g),$$

besteht, wo g eine beliebige natürliche Zahl ist, und die Exponenten $u_1, u_2, \ldots\ldots u_\rho$ für gegebenes γ eindeutig bestimmte Zahlen aus der Reihe: $0, 1, 2, \ldots\ldots l-1$ sind.

Die Zahl ρ ist der Rang von der Abel'schen Gruppe, der Ordnung $l^{(g-1)f}$, deren Elemente diejenigen Zahlclassen nach dem

1) D. Hilbert, Ueber die Theorie der relativ Abel'schen Zahlkörper, Göttinger Nachrichten, 1898.

2) Ph. Furtwängler, Allgemeiner Existenzbeweis für den Klassenkörper usw. Math. Ann. 63.

Modul $\mathfrak{l}^g$ sind, die aus den Zahlen $\equiv 1$, $(\mathfrak{l})$ bestehen. Daher bestimmt sich ρ daraus, dass l^ρ die Anzahl der einander nach $\mathfrak{l}^g$ incongruenten Lösungen der Congruenz:

$$\xi^l \equiv 1, \quad (\mathfrak{l}^g) \qquad (1)$$

ist.

Hülfssatz.[1] *Es ist*

$$\rho = \left[g - \frac{g}{l}\right] f, \quad \textit{wenn} \quad \frac{sl}{l-1} \geqq g > 0;$$

$$\rho = sf + e, \quad \textit{wenn} \quad g > \frac{sl}{l-1},$$

(speciell $\rho=0$, wenn $g=1$), wo $e=1$, oder $=0$, jenachdem die Congruenz

$$l + \xi^{l-1} \equiv 0, \quad (\mathfrak{l}^{s+1})$$

in k *lösbar ist, oder nicht; das Zeichen $[x]$ hat die gewöhnliche Bedeutung der grössten ganzen rationalen Zahl, die x nicht übertrifft.*

Der Fall $e=1$ ist nur dann möglich, wenn

$$s = \sigma(l-1)$$

durch $l-1$ teilbar ist. Speciell ist $e=1$, wenn der Körper k *die primitive l^{te} Einheitswurzel ζ enthält, also stets, wenn $l=2$.*

Beweis. Bezeichnet man allgemein mit λ_n eine genau durch die n^{te} Potenz von $\mathfrak{l}$ teilbare Zahl von k, dann ist, wie leicht nachzuweisen ist,

$$\text{wenn} \quad n < \frac{s}{l-1}, \quad (1+\lambda_n)^l = 1 + \lambda_{nl}, \qquad (2)$$

$$\text{wenn} \quad n > \frac{s}{l-1}, \quad (1+\lambda_n)^l = 1 + \lambda_{s+n}, \qquad (3)$$

$$\text{und wenn} \quad \frac{s}{l-1} = \sigma, \quad (1+\lambda_\sigma)^l \equiv 1 + l\lambda_\sigma + \lambda_\sigma^l, \quad (\mathfrak{l}^{\sigma l+1}). \qquad (4)$$

Ist also

$$\frac{sl}{l-1} \geqq g > 0,$$

[1] Vgl. T. Takenouchi, diese Journal, vol. 36, Art. 1.

dann ist, nach (2)

$$(1+\lambda_n)^l \equiv 1, \qquad (\mathfrak{l}^g),$$

dann und nur dann, wenn

$$nl \geqq g, \qquad \text{oder} \qquad n \geqq g_0,$$

wo g_0 die kleinste natürliche Zahl ist, die noch $\geqq \frac{g}{l}$ ist. Die Lösungen der Congruenz (1) sind daher die Zahlen:

$$\xi \equiv 1, \qquad (\mathfrak{l}^{g_0}),$$

welche nach dem Modul $\mathfrak{l}^g$ genau $l^{(g-g_0)f}$ incongruenten Zahlen abgeben. Daher ist in diesem Falle

$$\rho = (g-g_0)f = \left[g - \frac{g}{l}\right] f.$$

Ist zweitens

$$g > \frac{sl}{l-1},$$

aber s nicht durch $l-1$ teilbar, dann sind nach (3) die Lösungen der Congruenz (1) die Zahlen:

$$\xi \equiv 1, \qquad (\mathfrak{l}^{s+1}). \qquad (5)$$

Daher ist in diesem Falle

$$\rho = s f.$$

Wenn aber s durch $l-1$ teilbar, also

$$g > \sigma l,$$

dann kommen nach (4) ausserdem noch die Zahlen von der Form $1+\lambda_\sigma$ in Betracht, wenn für dieselben

$$l \lambda_\sigma + \lambda_\sigma^l \equiv 0, \qquad (\mathfrak{l}^{\sigma l+1}),$$

oder

$$l + \lambda_\sigma^{l-1} \equiv 0, \qquad (\mathfrak{l}^{s+1})$$

ausfällt. Ist nun für eine dieser speciellen λ_σ

$$\alpha = 1 + \lambda_\sigma, \qquad \alpha^l \equiv 1, \qquad (\mathfrak{l}^{\sigma l+1}),$$

so kann man, wie leicht ersichtlich, in

$$\alpha' = \alpha\,(1 + \eta\,\lambda_{\sigma+1})$$

η so bestimmen, dass $\alpha'^{l} \equiv 1\,(\mathfrak{l}^{\sigma l+2})$ wird. So fortfahrend erhält man eine Zahl

$$\beta_0 = 1 + \lambda_\sigma^{(0)},$$

welche für beliebig grosses g der Congruenz (1) genügt. Jede Zahl β von der Form $1+\lambda_\sigma$ kann aber in der Form dargestellt werden:

$$\beta \equiv 1 + \gamma\,\lambda_\sigma^{(0)}, \qquad (\mathfrak{l}').$$

Soll diese Zahl die Congruenz (1) befriedigen, so muss jedenfalls

$$l + (\gamma\,\lambda_\sigma^{(0)})^{l-1} \equiv 0, \qquad (\mathfrak{l}^{s+1}),$$

weil aber auch

$$l + \lambda_\sigma^{(0)\,l-1} \equiv 0, \quad (\mathfrak{l}^{s+1}),$$

so ist notwendig

$$\gamma^{l-1} \equiv 1, \qquad (\mathfrak{l}),$$

folglich

$$\gamma \equiv c, \qquad (\mathfrak{l}),$$

wo c eine zu l prime ganze rationale Zahl ist. Demnach ist

$$\beta \equiv \beta_0^c \qquad (\mathfrak{l}^{\sigma+1}),$$

also

$$\beta \equiv \beta_0^c\,(1 + \lambda_n) \qquad (\mathfrak{l}'),$$

wo $n > \sigma$. Damit diese Zahl β der Congruenz (1) genüge, ist aber nach (3) notwendig und hinreichend, dass

$$n \geqq g - s.$$

Man sieht, dass im gegenwärtigen Falle, alle Lösungen von (1) durch die Producte der Zahlen (5) mit einer der l Zahlen

$$1,\ \beta_0,\ \beta_0^2,\ \cdots\cdots\ \beta_0^{l-1}$$

gegeben werden. Es ergibt sich also

$$\rho = s f + 1.$$

Wenn die primitive l^{te} Einheitswurzel ζ in k vorkommt, dann wird die Congruenz

$$l+\xi^{l-1}\equiv 0 \qquad (\mathfrak{l}^{e+1})$$

durch $\xi=1-\zeta$ (wenn $l=2$, durch $\xi=2$) befriedigt, weil

$$\frac{l}{(1-\zeta)^{l-1}}=\frac{(1-\zeta)(1-\zeta^2)\cdots(1-\zeta^{l-1})}{(1-\zeta)^{l-1}}=(1+\zeta)\cdots(1+\zeta+\cdots+\zeta^{l-2})$$
$$\equiv \underline{l-1}\equiv -1, \qquad (\text{mod. } 1-\zeta).$$

In diesem Falle ist daher stets $e=1$.

§. 17.

Rang der Classengruppe.

Wenn die Idealclassen des Körpers k nach der Zahlengruppe o der Zahlen $\equiv 1$ (m) definirt werden, und ist die Ordnung der Gruppe G der sämtlichen Classen von k, d.h. die Classenzahl von k nach o genau durch die h^{te} Potenz einer geraden oder ungeraden Primzahl l teilbar, dann bezeichnen wir mit G_0 die Untergruppe von G von der Ordnung l^h, und mit D den Inbegriff aller Classen, deren Ordnungen prim zu l sind, so dass

$$G=G_0 D$$

das directe Product der beiden Gruppen G_0 und D ist. Im folgenden spielt der Rang dieser Gruppe G_0 eine fundamentale Rolle.

Satz 24. *Sei t die Anzahl der in G_0 enthaltenen unabhängigen Idealclassen im absoluten Sinne: $\mathfrak{r}_1, \mathfrak{r}_2, \ldots\ldots \mathfrak{r}_t$ ein System der Representanten dieser Classen, die prim zu* m *sind; $\rho_1, \rho_2, \ldots\ldots \rho_t$ die niedrigsten Potenzen dieser Ideale, welche monomisch sind; $\varepsilon_1, \varepsilon_2, \ldots\ldots \varepsilon_{r+\delta}$ ein System der Grundeinheiten von* k, *zu welchem wir eine derjenigen Einheitswurzeln mitrechnen, deren Ordnung eine Potenz von l, und zwar die höchste in* k, *ist, so dass $\delta=1$ oder $\delta=0$, jenachdem die primitive l^{te} Einheitswurzel in* k *vorhanden ist oder nicht, und es sei l^n die Anzahl der l^{ten} Potenzreste nach* m, *welche in dem System von $l^{r+\delta+t}$ Zahlen:*

$$\varepsilon_1^{u_1}\cdots\cdots\varepsilon_{r+\delta}^{u_{r+\delta}}\rho_1^{v_1}\cdots\cdots\rho_t^{v_t} \qquad (1)$$

$$(0\leq u,\ v<l)$$

enthalten sind. Dann ist der Rang der Classengruppe C_0

$$\bar{t}=d+\Sigma R(g)+n-(r+\delta), \qquad (2)$$

wo d die Anzahl der in $\mathfrak{m}$ *aufgehenden, von einander verschiedenen zu l primen Primideale* $\mathfrak{p}$, *für welche* $\varphi(\mathfrak{p})$ *durch l teilbar ist,* $R(g)$ *der im Hülfssatz des* §16 *angegebene Rang der Zahlengruppe nach dem Modul* $\mathfrak{l}^g$ *ist, und die Summation über alle in* $\mathfrak{m}$ *aufgehenden Potenzen* $\mathfrak{l}^g$ *erstreckt werden soll.*

Beweis. Da nach Voraussetzung

$$N=d+\Sigma R(g) \qquad (3)$$

unabhängige l^{te} Nichtreste nach $\mathfrak{m}$ gibt und

$$N'=r+\delta+t-n \qquad (4)$$

von denselben durch die Zahlen des Systems (1) gegeben werden, so lässt sich ein System von N Zahlen

$$\gamma_1,\ \gamma_2,\cdots\cdots\gamma_{N-N'},\ \eta_1,\ \eta_2,\cdots\cdots\eta_{N'}$$

aufstellen, von denen die N' letzten aus dem System (1) entnommen werden sollen, derart, dass sich jede zu $\mathfrak{m}$ prime Zahl γ von k in der Form darstellen lässt:

$$\gamma\equiv\gamma_1^{x_1}\cdots\cdots\gamma_{N-N'}^{x_{N-N'}}\eta_1^{y_1}\cdots\cdots\eta_{N'}^{y_{N'}}\xi^l, \quad (\mathfrak{m})$$

oder

$$\gamma=\gamma_1^{x_1}\cdots\cdots\gamma_{N-N'}^{x_{N-N'}}\eta_1^{y_1}\cdots\cdots\eta_{N'}^{y_{N'}}\alpha\,\xi^l, \qquad (5)$$

wo die Exponenten x, y für jedes gegebene γ eindeutig bestimmte Zahlen aus der Reihe: 0, 1, 2,......$l-1$ sind, und α eine Zahl in $\mathfrak{o}$ bedeutet: $\alpha\equiv 1$, $(\mathfrak{m})$.

Ist daher $\mathfrak{r}$ ein beliebiges zu $\mathfrak{m}$ primes Ideal von k, dann besteht eine Idealgleichheit von der Form

$$\mathfrak{r}=\mathfrak{r}_1^{a_1}\cdots\cdots\mathfrak{r}_t^{a_t}\gamma_1^{b_1}\cdots\cdots\gamma_{N-N'}^{b_{N-N'}}\alpha\,\mathfrak{j}^l, \qquad (6)$$

$$(0 \leqq a,\ b < l)$$

wo j ein zu m primes ganzes oder gebrochenes Ideal von k bedeutet.

Ein Ideal von der Form (6) ist aber nur dann gleich 1, wenn die Exponenten $a_1, \ldots\ldots a_t$ sämtlich verschwinden, also eine Zahlengleichheit von der Form besteht:

$$1 = \gamma_1^{b_1} \cdots\cdots \gamma_{N-N'}^{b_{N-N'}}\ \alpha\, [\varepsilon, \rho]\, \xi^l,$$

oder

$$1 \equiv \gamma_1^{b_1} \cdots\cdots \gamma_{N-N'}^{b_{N-N'}}\ [\varepsilon, \rho]\, \xi^l, \quad (\mathfrak{m}),$$

wo mit $[\varepsilon, \rho]$ eine Zahl des Systems (1) bezeichnet wird. Da nun $\gamma_1, \gamma_2, \cdots\cdots \gamma_{N-N'}$ sowohl von einander als von $[\varepsilon, \rho]$ unabhängige Nichtreste sind, so bedingt diese Congruenz, dass auch die Exponenten $b_1, \ldots\ldots b_{N-N'}$ sämtlich verschwinden.

Hiermit ist gezeigt, dass für jedes gegebene Ideal $\mathfrak{r}$, die Exponenten a, b auf der rechten Seite von (6) eindeutig bestimmt sind, dass daher der gesuchte Rang der Gruppe G_0 gleich

$$\bar{t} = t + N - N'.$$

Wenn man hierin für N und N' die Werte (3) und (4) einsetzt, so erhält man die Formel (2).

Da offenbar $N > N'$, so ist stets $\bar{t} > t$, wie es sein musste.

Zusatz. *Wenn* v *eine beliebig vorgeschriebene Gruppe der Vorzeichencombinationen*[1] *ist, und werden die Zahlen von* o *mit den Vorzeichencombinationen dieser Gruppe* v *in eine engere Zahlengruppe* o′ *zusammengefasst, nach welcher nun die Classen von* k *zu definiren sind, dann wächst für $l=2$ der Rang der Classengruppe* G_0 *um*

$$p = (r_1 - r_0) - (n - n_0), \tag{7}$$

so dass an Stelle von (2)

$$\bar{t} = d + \Sigma R(g) + n_0 + r_1 - (r + r_0 + 1) \tag{8}$$

1) Vgl. § 1. S. 4.

zu setzen ist; hierbei ist r_1 die Anzahl der mit $\mathfrak{k}$ *conjugirten reellen Körper, 2^{r_0} die Anzahl der Vorzeichencombinationen von* $\mathfrak{v}$, *endlich bestimmt sich die Zahl n_0 dadurch, dass von den 2^n im System* (1) *enthaltenen quadratischen Reste nach* $\mathfrak{m}$ *genau 2^{n_0} die Vorzeichencombinationen von* $\mathfrak{v}$ *besitzen.*

Denn nach Annahme lässt sich ein System der r_1-r_0 quadratischen Reste nach $\mathfrak{m}$:

$$\alpha_1, \cdots\cdots \alpha_p, \; \eta_1'', \cdots\cdots \eta''_{n-n_0}$$

aufstellen, welche die sämtlichen r_1-r_0 von $\mathfrak{v}$ unabhängigen Vorzeichencombinationen aufweisen, und von denen die $n-n_0$ letzten dem System (1) angehören. Daher lässt sich der Ausdruck $\alpha\xi^{2l}$ auf der rechten Seite von (5) durch den folgenden ersetzen:

$$\alpha_1^{c_1} \cdots\cdots \alpha_p^{c_p} \, \eta_1''^{d_1} \cdots\cdots \eta''^{d_{n-n_0}}_{n-n_0} \, \alpha' \xi^{2l'}$$

wo die Exponenten c, d die Zahlen 0 oder 1 sind, und α' eine Zahl in $\mathfrak{o}'$ bedeutet. An Stelle von (6) kann man demnach setzen:

$$\mathfrak{r} = \mathfrak{r}_1^{a_1} \cdots\cdots \gamma_1^{b_1} \cdots\cdots \alpha_1^{c_1} \cdots\cdots \alpha_p^{c_p} \, \alpha' \mathfrak{j}^2,$$

$$(0 \leqq a, \; b, \; c < 2)$$

und es kann ein Ideal dieser Form nur dann gleich 1 sein, wenn wie vorhin die sämtlichen Exponenten a, b verschwinden, und

$$1 = \alpha_1^{c_1} \cdots\cdots \alpha_p^{c_p} \; [\varepsilon, \rho] \; \alpha' \xi^2,$$

wo $[\varepsilon, \rho]$ ein quadratischer Rest nach $\mathfrak{m}$ bedeutet, welcher dem System (1) angehört. Da nach der Voraussetzung die Zahlen $\alpha_1, \cdots\cdots \alpha_p$, $[\varepsilon, \rho]$, α' von einander unabhängige Vorzeichencombinationen besitzen, so müssen auch alle Exponenten $c_1 \cdots\cdots c_p$ verschwinden.

Daher ergibt sich für den Rang von $\mathfrak{c}_0$ der Wert

$$\bar{t} = t + N - N' + p,$$

wie zu beweisen war.

§ 18.

Existenzbeweis des Classenkörpers vom ungeraden Primzahlgrade.

Wir beschäftigen uns nun mit demjenigen Falle des in § 15 aufgestellten Existenzsatzes, in welchem der Index l der Classengruppe H eine ungerade Primzahl ist, und der Grundkörper k die primitive l^{te} Einheitswurzel ζ enthält. Unter Beibehaltung der in den beiden vorhergehenden Artikeln benutzten Bezeichnungsweise, ist zunächst

$$\delta=1\,;$$ [1]

sodann, wenn m der Grad des Körpers k ist,

$$m=2\,(r+1),$$

ferner ist für jedes Primideal $\mathfrak{l}$,

$$e=1,$$ [2]

und

$$s=\sigma(l-1)$$ [2]

durch $l-1$ teilbar.

Der Modul $\mathfrak{m}$ enthalte d von einander verschiedene zu l prime Primideale: $\mathfrak{p}, \mathfrak{p}', \cdots\cdots \mathfrak{p}^{(d-1)}$ als Factoren, für jedes derselben $\varphi(\mathfrak{p})$ durch l teilbar ist.[3] Von den in l aufgehenden Primidealen seien diejenigen, die in $\mathfrak{m}$ aufgehen, deren Anzahl d' (mit Einschluss des Wertes: $d'=0$) sei, durchweg mit $\mathfrak{l}$, die übrigen mit $\mathfrak{l}'$ bezeichnet.

Einfachheitshalber wollen wir zunächst annehmen, dass jedes Primideal $\mathfrak{l}$, wenn überhaupt, wenigstens zur $\sigma l+1^{\text{ten}}$ Potenz in $\mathfrak{m}$ aufgehe, so dass in der Formel (2), § 17 für den Rang der Classengruppe G_0

$$R(g)=sf+1$$

1) Vgl. Formel (2) §17.

2) Vgl. Hülfssatz, §16.

3) Dies folgt aus der Tatsache, dass die Norm jedes Primideals in dem durch ζ erzeugten Kreiskörper congruent 1 nach l ist; vgl. Hilbert, Bericht, Satz 119.

zu setzen ist. Dieselbe Formel lautet daher im gegenwärtigen Falle:

$$\bar{t}=d+d'+\Sigma sf+n-(r+1), \tag{1}$$

wo die Summation auf alle in $\mathfrak{m}$ aufgehenden Primideale $\mathfrak{l}$ zu erstrecken ist. Die l^n in dem System

$$\varepsilon_1^{x_1}\cdots\cdots\varepsilon_{r+1}^{x_{r+1}}\,\rho_1^{y_1}\cdots\cdots\rho_t^{y_t}\qquad(0\leqq x,y<l)$$

enthaltenen l^{ten} Potenzreste nach $\mathfrak{m}$ seien mit

$$\alpha^e\,\alpha'^{e'}\cdots\cdots\alpha^{(n-1)e^{(n-1)}}\qquad(0\leqq e<l)$$

bezeichnet. Ich führe dann nach Hilbert n Primideale

$$\mathfrak{q},\ \mathfrak{q}',\cdots\cdots\mathfrak{q}^{(n-1)}$$

ein, die zu $\mathfrak{m}$, $\mathfrak{l}$, und $\mathfrak{r}_1, \mathfrak{r}_2,\cdots\cdots\mathfrak{r}_t$ prim sind, von der Art, dass

$$\left(\frac{\alpha^{(i)}}{\mathfrak{q}^{(i)}}\right)\neq 1,\quad\left(\frac{\alpha^{(i)}}{\mathfrak{q}^{(j)}}\right)=1,\quad(i\neq j) \tag{2}$$

und setze

$$\bar{\mathfrak{m}}=\mathfrak{m}\mathfrak{q}\mathfrak{q}'\cdots\cdots\mathfrak{q}^{(n-1)}$$

Dann ist in dem Ausdruck (1) für den Rang der entsprechenden Gruppe $G_{\bar{m}}$ für den Modul $\bar{\mathfrak{m}}$, d durch $d+n$, dagegen n durch 0 zu ersetzen, so dass $\bar{t}$ unverändert bleibt. Dies hat zur Folge, dass jede der $l^t-1:l-1$ Classengruppen vom Index l nach dem Modul $\mathfrak{m}$ auch Classengruppen nach $\bar{\mathfrak{m}}$ ist, wobei die Primideale $\mathfrak{q}, \mathfrak{q}'\cdots\cdots\mathfrak{q}^{(n-1)}$ als unwesentlicher Excludenten auftreten.[1]

Nunmehr sei

$$\mathfrak{p}\,\mathfrak{r}_1^{a_1}\,\mathfrak{r}_2^{a_2}\cdots\cdots\mathfrak{r}_t^{a_t}\,\mathfrak{j}^l=(\varpi),$$

$$\mathfrak{l}\,\mathfrak{r}_1^{b_1}\,\mathfrak{r}_2^{b_2}\cdots\cdots\mathfrak{r}_t^{b_t}\,\mathfrak{j}'^l=(\lambda),$$

$$\mathfrak{q}\,\mathfrak{r}_1^{c_1}\,\mathfrak{r}_2^{c_2}\cdots\cdots\mathfrak{r}_t^{c_t}\,\mathfrak{j}''^l=(\kappa),$$

1) Vgl. § 2. S. 13.

wo $\mathfrak{j}, \mathfrak{j}', \mathfrak{j}'' \cdots\cdots$ Ideale in k; $\varpi, \lambda, \varkappa, \cdots\cdots d+d'+n$ Zahlen in k, die wir prim zu jedem Primideale $\mathfrak{l}'$ annehmen können, und die Exponenten $a, b, c \ldots\ldots$ Zahlen aus der Reihe: $0, 1, 2, \ldots\ldots l-1$ bedeuten. Wir betrachten dann das System der

$$l^{r+1+t+d+d+n}$$

Zahlen:

$$\underbrace{\varepsilon_1^{x_1}\cdots\cdots}_{r+1}\underbrace{\rho_1^{y_1}\cdots\cdots}_{t}\underbrace{\varpi^{u}\cdots\cdots}_{d}\underbrace{\lambda^{v}\cdots\cdots}_{d}\underbrace{\varkappa^{w}\cdots\cdots}_{n} \qquad (3)$$

$$(0 \leqq x,\ y,\ u,\ v,\ w < l)$$

Wir unterwerfen diese Zahlen (3) zunächst der Bedingung, dass sie jedes der t Ideale $\mathfrak{r}_1, \mathfrak{r}_2, \cdots\cdots \mathfrak{r}_t$ zu einer Potenz mit einem durch l teilbaren Exponenten als Factor enthalten sollen, so dass die Exponenten x, y, u, v, w den t linearen Congruenzen:

$$\left.\begin{array}{l} a_1 u + \cdots + b_1 v + \cdots\cdots + c_1 w + \cdots\cdots \equiv 0, \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ a_t u \cdots\cdots + b_t v + \cdots\cdots + c_t w + \cdots\cdots \equiv 0. \end{array}\right\} \quad (l)$$

zu genügen haben.

Wir verlangen sodann, dass die Zahlen (3) noch *primär* in Bezug auf jedes $\mathfrak{l}'$ d.h. l^{te} *Reste nach der* $\sigma' l^{\text{en}}$ *Potenz von* $\mathfrak{l}'$ sein sollen. Bezeichnen wir für einen Augenblick mit $\nu = s'f'$ den Rang der Zahlengruppe, mit $\gamma_1, \gamma_2, \cdots\cdots \gamma_\nu$ ein System der unabhängigen l^{ten} Nichtreste nach dem Modul $\mathfrak{l}'^{\sigma' l}$, und ist demnach

$$\left.\begin{array}{l} \varepsilon_1 \equiv \gamma_1^{e_1} \cdots\cdots \gamma_\nu^{e_\nu} \alpha^l, \cdots\cdots \\ \rho_1 \equiv \gamma_1^{r_1} \cdots\cdots \gamma_\nu^{r_\nu} \beta^l, \cdots\cdots \\ \varpi \equiv \gamma_1^{p_1} \cdots\cdots \gamma_\nu^{p_\nu} \gamma^l, \cdots\cdots \\ \lambda \equiv \gamma_1^{l_1} \cdots\cdots \gamma_\nu^{l_\nu} \delta^l, \cdots\cdots \\ \varkappa \equiv \gamma_1^{k_1} \cdots\cdots \gamma_\nu^{k_\nu} \varepsilon^l, \cdots\cdots \end{array}\right\} (\mathfrak{l}'^{\sigma' l})$$

so ist eine Zahl (3) dann und nur dann ein l^{ter} Rest nach $\mathfrak{l}^{el}$, wenn die Exponenten $x, y, u, v, w,$ dem System von ν linearen Congruenzen:

$$\left.\begin{array}{l} e_1 x_1 + \cdots \; r_1 y_1 + \cdots + p_1 u \cdots\cdots + l_1 v + \cdots + k_1 w + \cdots\cdots \equiv 0, \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ e_\nu x_1 + \cdots\cdots t_\nu y_1 + \cdots\cdots p_\nu u + \cdots\cdots l_\nu v + \cdots\cdots k_\nu w + \cdots\cdots \equiv 0, \end{array}\right\} (l)$$

genügen.

Unter den Zahlen (3) gebe es nun t' Zahlen, welche diese $t + \Sigma s'f'$ Bedingungen genügen, die wir dann in der Form

$$\mu_1^{e_1} \mu_2^{e_2} \cdots\cdots \mu_{t'}^{e_{t'}} \qquad (0 \leqq e < l) \tag{4}$$

darstellen können, wobei die Zahlen $\mu_1, \mu_2 \cdots\cdots \mu_{t'}$ in dem Sinne von einander unabhängig sind, dass eine Zahl (4) nur dann l^{e} Potenz einer Zahl in k sein kann, wenn die sämtlichen Exponenten $e_1, e_2, \ldots\ldots e_{t'}$ verschwinden: und es ist

$$t' \geqq r + 1 + t + d + d' + n - (t + \Sigma s'f''), \tag{5}$$

woraus folgt

$$t' \geqq \overline{t}. \tag{6}$$

In der Tat: nach (1) und (5)

$$t' - \overline{t} \geqq 2(r+1) - (\Sigma sf + \Sigma s'f'') = 2(r+1) - m = 0.$$

Adjungirt man nun dem Körper k die l^{e} Wurzel einer der Zahlen (4), die wir durchweg mit μ bezeichnen wollen, so erhalten wir also wenigstens $\overline{t}$ von einander unabhängige relativ cyclische Körper

$$\mathrm{K} = \mathrm{k}(\sqrt[l]{\mu})$$

vom l^{ten} Grade in Bezug auf k. Die Relativdiscriminante dieser Körper ist durch kein Primideal teilbar, welches nicht in $\mathfrak{m}$

aufgeht. weil jedes der Ideale $\mathfrak{r}_1, \mathfrak{r}_2, \ldots \mathfrak{r}_t$ genau zu einer Potenz in μ aufgeht, deren Exponent Vielfaches von l ist. und überdies μ l^{er} Potenzrest nach jedem $\mathfrak{l}^{\sigma l}$ ist. Da ferner für jedes wirklich in die Relativdiscriminante aufgehende Primideal $\mathfrak{l}$ die entsprechende Zahl $r \leqq \sigma l$ (Satz 8), und anderseits nach Voraussetzung der Modul $\mathfrak{m}$ dasselbe Ideal $\mathfrak{l}$ wenigstens zur $\sigma l+1^{\text{ten}}$ Potenz als Factor enthält, so ist Satz 13 auf den Körper K anwendbar, demzufolge K Classenkörper für eine der Classengruppen $\mathfrak{n}$ sein muss.

Da es genau $l-1 : l-1$ Classengruppen $\mathfrak{n}$, und nach (6) wenigstens ebensoviele Körper K gibt, da ferner nach Satz 6 für jede Classengruppe nicht mehr als ein Classenkörper existiren kann, so folgt, dass jeder Classengruppe $\mathfrak{n}$ ein Classenkörper K zugeordnet sein muss.

Es bleibt noch übrig, nachzuweisen, dass die Relativdiscriminante des Körpers K durch keines der Primideale $\mathfrak{q}, \mathfrak{q}', \ldots \mathfrak{q}^{(n-1)}$ teilbar ist. Wäre aber der Gegenteil der Fall, so wähle man ein zweites, vollständig vom ersten verschiedenes System der n Primideale $\bar{\mathfrak{q}}, \bar{\mathfrak{q}}', \ldots \bar{\mathfrak{q}}^{(n-1)}$, welche den Bedingungen (2) genügen, und bilde darauf die entsprechenden Körper $\bar{\text{K}}$, deren Discriminanten dann sicher nicht durch $\mathfrak{q}, \mathfrak{q}', \ldots \mathfrak{q}^{(n-1)}$ teilbar sind, und folglich notwendig von K verschieden sein mussten. Da auch diese Körper $\bar{\text{K}}$ Classenkörper für je eine der nämlichen Gruppen $\mathfrak{n}$ sein müssen, so führt die Annahme zu einem Widerspruch gegen Satz 6.

Hiermit ist im gegenwärtigen Falle unser Existenzsatz bewiesen.

Wir haben zu Beginn dieses Beweises angenommen, dass jedes in l aufgehende Primideal $\mathfrak{l}$ entweder gar nicht oder wenigstens zur $\sigma l+1^{\text{ten}}$ Potenz in $\mathfrak{m}$ als Factor enthalten sein soll. Es ist nun leicht, diese Beschränkung aufzuheben. Es sei nämlich $\mathfrak{m}_0$ ein Teiler vom $\mathfrak{m}$ derart, dass $\mathfrak{m}_0$ genau durch $\mathfrak{l}^g$ teilbar ist, wo $g \leqq \sigma l$, und es sei $\bar{t}_0$ der Rang der entsprechenden Classengruppe G_0 für den Modul $\mathfrak{m}_0$, so dass offenbar $\bar{t}_0 \leqq \bar{t}$. Fällt nun $\bar{t}_0 < \bar{t}$ aus, so gibt es unter den $l^{\bar{t}}-1 : l-1$ Classengruppen $\mathfrak{n}$ nach $\mathfrak{m}$ genau $l^{\bar{t}_0}-1 : l-1$, welche Classengruppen nach $\mathfrak{m}_0$ sind. Den letzteren müssen nun genau ebensoviele unter den vorhin aufgestellten Körper K als Classenkörper zugeordnet sein. Denn, andernfalls mussten für die übrigbleibenden $l^{r-t_0}:l-1$ Gruppen $\mathfrak{H}$ insgesamt eine grössere Anzahl der zugeordneten Körper K vorhanden sein, was einen Verstoss gegen Satz 6 nach sich ziehen würde.

Jedoch konnten wir auch ohne die beschränkende Annahme über $\mathfrak{m}$ direct den Nachweis des Existenzsatzes führen, wozu eine sehr geringe Modification der vorhin benutzten Methode hinreichen würde.

Ausser den d' Primidealpotenzen $\mathfrak{l}^g$, für welche $g>\sigma l$, mögen noch gewisse andere, sie seien durchweg mit $\mathfrak{l}_1^{g_1}$ bezeichnet, wo $g_1 \leqq \sigma_1 l$, in $\mathfrak{m}$ aufgehen; die übrigbleibenden in l aufgehenden Primideale seien, wie vorhin, durchweg mit $\mathfrak{l}'$ bezeichnet. Indem wir die sonstingen Bezeichnungsweise des vorhergehenden Beweises beibehalten, ist nun nach Satz 25

$$\bar{t}=d+d'+\Sigma sf+\Sigma R(g_1)+n-(r+1),$$

wo $R(g_1)$ die in Satz 25 erläuterte Bedeutung hat, und die Summation $\Sigma R(g_1)$ auf alle Primideale $\mathfrak{l}_1$ zu erstrecken ist. Wir unterwerfen dann die Zahlen μ ausser den $t+\Sigma s'f'$ früheren Bedingungen noch den, dass für jedes Primideal $\mathfrak{l}_1$ die Congruenz

$$\xi^l \equiv \mu, \quad (\mathfrak{l}_1^{\sigma_1 l-g_1+1}) \tag{7}$$

in k möglich sein sollen. Da diese offenbar $\Sigma R(\sigma_1 l-g_1+1)$ neue lineare Congruenzbedingungen für die Exponenten x, y, u, v, w involviren, so bleiben nun

$$t' \geqq r+1+d+d'+n-\{t+\Sigma s'f'+\Sigma R(\sigma_1 l-g_1+1)\}$$

unabhängige Zahlen μ, welche ebensoviele unabhängige Körper

$$\mathrm{K}=\mathrm{k}(\sqrt[l]{\mu})$$

hervorbringen werden. Da nach Hülfssatz des § 16

$$R(g_1)+R(\sigma_1 l-g_1+1)=\left\{\left[g_1-\frac{g_1}{l}\right]+\left[\sigma_1 l-g_1+1-\sigma_1+\frac{g_1-1}{l}\right]\right\}f_1$$

$$=\sigma_1(l-1)f_1=s_1 f_1,$$

so ist auch in diesem Falle noch

$$t'-\bar{t} \geqq 2(r+1)-(\Sigma s f+\Sigma s_1 f_1+\Sigma s' f')=2(r+1)-m=0.$$

Enthält nun die Relativdiscriminante eines dieser Körper K den Primfactor $\mathfrak{l}_1$ dann ist wegen (7) die entsprechende Zahl $r_1 \leq g_1-1$. Daher ist Satz 13 noch anwendbar, und es folgt, genau wie vorhin, die Existenz der $\bar{t}$ unabhängigen Classenkörper für die Gruppen H, welche alle Forderungen des Existenzsatzes befriedigen.

Das Ergebnis dieser Betrachtungen sprechen wir in den folgenden Satz aus:

Satz 25. *Geht ein in l aufgehendes Primideal $\mathfrak{l}$ zur g^{ten} Potenz in $\mathfrak{m}$ auf, wo $g \leq \sigma l$, und ist die Relativdiscriminante eines Classenkörpers für eine Classengruppe vom Index l nach dem Modul $\mathfrak{m}$ durch dieses Primideal $\mathfrak{l}$ teilbar, dann ist die entsprechende Zahl r kleiner als g.*[1]

Dasselbe gilt offenbar auch, wenn $g=\sigma l+1$. Ferner ist, wenn $g=1$, die Relativdiscriminante des Classenkörpers prim zu $\mathfrak{l}$. Ein einfacher Factor $\mathfrak{l}$ von $\mathfrak{m}$ macht keinen Beitrag zu der Rangzahl von G_0

Ferner gilt[1]

Satz 26. *Hat der relativ cyclische Körper K/k vom Primzahlgrade l die Relativdiscriminante $\mathfrak{d}=\mathfrak{f}^{l-1}$, dann ist $\mathfrak{f}$ der Führer*[2] *der zugeordneten Classengruppe vom Index l im Grundkörper* k.

Das soll heissen: Um die Relativnormen aller zu $\mathfrak{d}$ primen Ideale von K in eine Classengruppe vom Index l in k einzuschliessen, genügt es nach Satz 13, die Classen von k nach einem durch $\mathfrak{f}$ teilbaren Modul $\mathfrak{m}$ zu definiren. In Satz 26 wird nun umgekehrt behauptet, dass es auch notwendig ist, dass $\mathfrak{m}$ alle Primfactoren von $\mathfrak{f}$, speciell jeden Factor $\mathfrak{l}$ wenigsten zur $r+1^{\text{ten}}$ Potenz, als Factor enthalte.

Beweis. Betreffs eines zu l primen Primfactor $\mathfrak{p}$ von $\mathfrak{f}$ ist dies evident; denn wäre H eine Classengruppe vom Index l nach einem zu $\mathfrak{p}$ primen Modul $\mathfrak{m}$, dann musste nach dem vorhergehen-

1) Dies zunächst unter der Annahme, dass l ungerade ist, und k die l^{te} primitive Einheitswurzel enthält; diese Beschränkung wird später aufgehoben werden. Vgl. § 19.

2) Vgl. § 2, S 13.

den Beweis ein Classenkörper K′ für $\mathfrak{n}$ existiren, dessen Relativdiscriminante zu $\mathfrak{p}$ prim ist, der folglich gewiss von K verschieden ist. Enthalte aber $\mathfrak{m}$ einen Primfactor $\mathfrak{l}$ zu einer Potenz, deren Exponent kleiner als $v+1$ ist, dann musste nach Satz 25 ein Classenkörper K′ für $\mathfrak{n}$ existiren, für welchen die entsprechende Zahl $v'<v$ ausfällt, oder dessen Relativdiscriminante prim zu $\mathfrak{l}$ ist, welcher also jedenfalls von K verschieden wäre. Beide Annahme führen somit zu einem Widerspruch gegen Satz 6.

§. 19.

Fortsetzung des vorhergehenden Artikels.

In dieser Fortsetzung des vorhergehenden Artikels behandle ich denjenigen Fall des Existenzsatzes, wo der Index der Classengruppe $\mathfrak{n}$ eine ungerade Primzahl l ist, aber der Grundkörper nicht die primitive l^{te} Einheitswurzel ζ enthält. Für den Fall, wo $\mathfrak{m}=1$, also für den absoluten Classenkörper hat Herr Ph. Furtwängler[1] den Existenzbeweis dadurch geführt, dass er zunächst dem Körper k die l^{te} Einheitswurzel ζ adjungirte, dann einen geeigneten Oberkörper zu den so erweiterten Grundkörper k′ construirte; sodann zeigte er, dass dieser Oberkörper den gesuchten Körper als Unterkörper enthalten muss. Diese Beweismethode bewährt sich auf in unserem Falle. Indem ich hier dieselbe Methode anwende, schicke ich einen Hülfssatz voran, welcher eine gewisse Vereinfachung des Beweises bewirken wird.

Hülfssatz. *Es sei* k′ *relativ cyclisch vom Relativgrade n in Bezug auf* k, K *relativ cyclisch vom Grade l in Bezug auf* k′, *und relativ normal aber nicht relativ Abel'sch in Bezug auf* k: *und es sei l eine Primzahl, die nicht in n aufgeht. Wenn dann ein Primideal von* k, *welches nicht in die Relativdiscriminante von* K/k *aufgeht, in weniger als n Primfactoren in* k′ *zerfällt, dann zerfällt jedes dieser Primideale von* k′ *in l Primfactoren in* K.

Beweis. Sei $\mathfrak{G}$ die Galois'sche Gruppe des relativ normalen Körpers K/k, von der Ordnung nl, $\mathfrak{H}$ die invariante Untergruppe

1) Math. Ann. 63.

von $\mathfrak{G}$, welche den Körper k'/k unverändert lässt, so dass nach Voraussetzung $\mathfrak{H}$ cyclisch von der Ordnung l, und die complementäre Gruppe $\mathfrak{G}/\mathfrak{H}$ ebenfalls cyclisch von der Ordnung n ist. Daher gibt es in $\mathfrak{G}$ eine Substitution T, von der Art, dass die Zerlegung gilt:

$$\mathfrak{G} = \mathfrak{H} + \mathfrak{H}T + \mathfrak{H}T^2 + \cdots\cdots + \mathfrak{H}T^{n-1}.$$

Die Ordnung der Substitution T, welche durch n teilbar und in nl aufgeht, muss notwendig gleich n sein, weil die Gruppe $\mathfrak{G}$ nicht cyclisch sein soll. Ist aber H eine beliebige Substitution von $\mathfrak{H}$, dann ist HT auch von der Ordnung n, weil HT in der oben angegebenen Zerlegung von $\mathfrak{G}$ an Stelle von T treten kann. Ebenso folgert man, dass die Ordnung jeder nicht in $\mathfrak{H}$ enthaltenen Substitution ein Teiler von n sein muss.

Sei nun $\mathfrak{p}$ ein Primideal von k, welches die Voraussetzung des Satzes genügt, und es gelte in K die Zerlegung

$$\mathfrak{p} = \mathfrak{P}_1 \mathfrak{P}_2 \cdots\cdots \mathfrak{P}_\nu,$$

so dass

$$nl = \nu f,$$

wenn f der Relativgrad der Primideale $\mathfrak{P}_1, \mathfrak{P}_2, \cdots\cdots$ in Bezug auf k ist. Nach Voraussetzung ist also $f > 1$. Die Zerlegungsgruppe des Primideals $\mathfrak{P}_1$, welche von der Ordnung f ist, muss hier eine cyclische Gruppe sein, weil die Trägheitsgruppe die identische ist. Nach dem vorhin bewiesenen, muss daher f ein Teiler von n, oder gleich l sein. Die letzte Eventualität ist aber ausgeschlossen, weil alsdann $\mathfrak{H}$ die Zerlegungsgruppe ist und folglich $\mathfrak{p}$ in n Factoren in k' zerlegt werden muss. Da also f ein Teiler von n ist, so muss ν durch l teilbar sein, womit der Satz bewiesen ist.

Wir gehen nunmehr zum Beweis des Existenzsatzes über, unter der Voraussetzung, dass der Grundkörper k nicht die primitive l^{te} Einheitswurzel enthält. Durch Adjunction derselben erweitern wir k zum Körper k', welcher relativ cyclisch über k von einer Ordnung n ist, wo n ein Teiler von $l-1$, folglich prim zu l ist. Die Idealclassen von k seien nach der Zahlengruppe o der Zahlen $\equiv 1$ ($\mathfrak{m}$) definirt, wo der Modul $\mathfrak{m}$ ein jedes in l aufgehendes Primideal $\mathfrak{l}$ mindestens zur ersten Potenz als Factor enthalten soll, eine Annahme, die ohne Schaden der Allgemeinheit geschieht, weil die Hinzunahme eines einfachen Factors $\mathfrak{l}$ zu $\mathfrak{m}$, falls $\mathfrak{m}$ nicht durch $\mathfrak{l}$ teilbar sein sollte, offenbar den Rang $\bar{l}$ der vollständigen Classengruppe G_0 von k nicht beeinflussen wird.[1] Legt man dann der Classeneinteilung in k′ die Zahlengruppe o′ der Zahlen $\equiv 1$ ($\mathfrak{m}$) zu Grunde, dann fallen die Relativnormen der Ideale einer Classe nach o′ in eine und dieselbe Classe nach o in k hinein, so dass wir berechtigt sind, von den Relativnormen der Classen von k′ zu sprechen. Dasselbe gilt offenbar auch für jedem in k′ enthaltenen Oberkörper von k.

Sei nun in leicht verständlicher Bezeichnungsweise

$$G = \{C_1, C_2, \cdots\cdots C_{\bar{l}}; D\} \tag{1}$$

die vollständige Classengruppe von k, wo D wie in § 17 die Gruppe der Classen, deren Ordnung zu l prim sind, und $C_1, C_2, \ldots\ldots$ ein System der Basisclassen der Gruppe G_0 bedeuten, die so gewählt sind, dass eine gegebene Untergruppe H von G vom Index l in der Form dargestellt werden kann:

$$H = \{C_1^l, C_2, \cdots\cdots C_{\bar{l}}; D\}. \tag{2}$$

Anderseits bezeichnen wir mit Γ_0 diejenige Untergruppe von D, welche aus allen Relativnormen der Classen von k′ besteht. Obgleich es sich später herausstellen wird, dass der Gruppenindex (D: Γ_0) gleich n ist, sind wir in dem gegenwärtigen Stadium nicht berechtfertigt, dies vorauszusetzen, weil Satz 13 nur für einen Oberkörper vom Primzahlgrade bewiesen worden ist. Wir wissen aber, dass gewiss (D: Γ_0)>1, also Γ_0 nicht mit D zusammenfällt, eben zufolge jenes Satzes, weil derselbe auf jeden Unterkorper k_0' von k′ angewandt werden kann, welcher von einem Primzahlgrade im Bezug auf k ist, da $\mathfrak{m}$ jedes in l aufgehendes Primideal als Factor enthält.

1) Vgl. Hülfssatz in § 16 und Satz 24.

Bezeichnen wir ferner mit D' die Gruppe derjenigen Classen von k', deren Relativnormen in D_0 hineinfallen, dann lässt sich die vollständige Classengruppe G' von k' in der Form darstellen:

$$G' = \{C_1, C_2, \cdots\cdots D'\}. \tag{3}$$

Denn, ist C' eine beliebige Classe in k' und

$$n(C') = C_1^{e_1} C_2^{e_2} \cdots\cdots [D_0],$$

wo n die im Relativkörper k'/k genommene Relativnorm bezeichnet, und $[D_0]$ eine Classe in der Classengruppe D_0 bedeutet. Setzt man dann

$$C' = C_1^{x_1} C_2^{x_2} \cdots\cdots U, \tag{4}$$

so dass

$$n(C') = C_1^{nx_1} C_2^{nx_2} \cdots\cdots n(U).$$

Bestimmt man dann $x_1, x_2, \ldots\ldots$ so, dass

$$C_1^{nx_1} = C_1^{e_1}, \quad C_2^{nx_2} = C_2^{e_2}, \cdots\cdots.$$

was ja möglich ist, weil n prim zu l ist, dann folgt

$$n(U) = [D_0],$$

also, dass in (4) U der Gruppe D' angehört.

Zugleich sieht man ein, dass die Classen $C_1, C_2, \ldots\ldots$ in k' unabhängig in Bezug auf der Gruppe D' sind. Denn die Annahme

$$C_1^{x_1} C_2^{x_2} \cdots\cdots [D'] = 1,$$

wo mit $[D']$ eine Classe der Classengruppe D' bezeichnet wird, bedingt, dass

$$C_1^{nx_1} C_2^{nx_2} \cdots\cdots [D_0] = 1,$$

was nur dann der Fall ist, wenn

$$C_1^{nx_1} = 1, \quad C_2^{nx_2} = 1, \cdots\cdots ; \ [D_0] = 1$$

in k; also, weil n prim zu l ist, wenn

$$C_1^{x_1} = 1, \quad C_2^{x_2} = 1, \cdots\cdots$$

Die Classen $c_1, c_2, \ldots\ldots$ erleiden also in k′ weder die Verlust der Unabhängigkeit noch die Erniederung der Ordnungen.

Demnach wird durch

$$H' = \{C_1^l, C_2, \cdots\cdots D'\} \tag{5}$$

eine Classengruppe vom Index l in k′ definirt.

Für diese Classengruppe H′ existirt nun nach dem vorhergehenden Artikel ein zugeordneter Classenkörper K vom Relativgrade l in Bezug auf k′, weil k′ die primitive l^{te} Einheitswurzel enthält.

Weil aber die Classengruppe H′ gegenüber der Substitutionen des Relativkörpers k′/k invariant ist, so fallen die in Bezug auf k mit K relativ conjugirten Körper als Classenkörper von H′ nach Satz 6 mit K zusammen; also ist K relativ normal in Bezug auf k. Aus der Tatsache, dass die Relativnormen der Idealen von K in Bezug auf k in die Classengruppe

$$\{C_1^l, C_2, \cdots\cdots;\ D_0\} \tag{6}$$

hineinfallen, ist aber zu schliessen, dass K relativ Abel'sch in Bezug auf k sein muss.

In der Tat, sei $\mathfrak{p}$ ein Primideal von k, welches nicht in die Primideale des ersten Relativgrades in k′ zerfällt, und zugleich in einer Classencomplex

$$C_1^{e_1} C_2^{e_2} \cdots\cdots D \quad \text{mit} \quad e_1 \not\equiv 0, \quad (l) \tag{7}$$

enthalten ist: die Existenz solcher Primideale folgt aus dem Hülfssatze des § 4. Wäre nun K nicht relativ Abel'sch in Bezug auf k, dann musste jedes in $\mathfrak{p}$ aufgehende Primideal $\mathfrak{p}'$ von k′, nach dem vorhin bewiesenen Hülfssatze in l von einander verschiedene Primfactoren in K zerfallen. Folglich musste $\mathfrak{p}'$ einer Classe der Gruppe (5) angehören, und infolgedessen $n(\mathfrak{p}') = \mathfrak{p}^f$ in eine Classe der Gruppe (6) in k hineinfallen. Da aber $\mathfrak{p}$ der Classe (7) angehört, und da f als Teiler von n prim zu l ist, so ist dies unmöglich.

Da also K relativ Abel'sch vom Grade nl in Bezug auf k ist, so enthält K einen Unterkörper K_0, welcher relativ cyclisch vom Grade l in Bezug auf k ist. Dieser Körper K_0 muss nach Satz 13 (vgl. weiter unten) einer Classengruppe in k vom Index l als Classenkörper zugeordnet sein. Diese Classengruppe muss aber, da K_0 in K enthalten ist, offenbar die Classengruppe (6) enthalten, kann also keine andere sein als die vorgelegte Gruppe $\mathfrak{H}$.

Die Relativdiscriminante des Körpers K_0/k enthält offenbar kein Primideal als Factor, welches nicht in $\mathfrak{m}$ aufgeht. Geht insbesondere ein Primideal $\mathfrak{l}$ genau zur g^{ten} Potenz in $\mathfrak{m}$ auf, wo $1<g\leqq\sigma l$, dann ist in k' der Modul $\mathfrak{m}$ genau durch die gn^{ten} Potenz von dem entsprechenden Primideal $\mathfrak{l}'$ $(\mathfrak{l}=\mathfrak{l}'^{n})$ teilbar. Geht daher dieses Primideal $\mathfrak{l}'$ in die Relativdiscriminante des Körpers K/k' auf, dann ist nach Satz 26 die entsprechende Zahl $r'<gn$. Hieraus ist aber zu schliessen, dass $\mathfrak{l}$ in die Relativdiscriminante von K_0/k aufgehen muss (ausser wenn $g=1$), und zwar ist die entsprechende Zahl $r<g$. Denn setzt man $\mathfrak{l}=\mathfrak{L}^{nl}$, wo $\mathfrak{L}$ Primideal in K bedeutet, dann folgt, wenn man die Relativdifferente des Körpers K/k einmal als Product der Relativdifferenten von K/K_0 und von K_0/k, das andere Mal als Product der Relativdifferenten von K/k' und von k'/k darstellt,

$$\mathfrak{L}^{n-1}\cdot\mathfrak{L}^{n(r+1)(l-1)}=\mathfrak{L}^{(r'+1)(l-1)}\,\mathfrak{L}^{l(n-1)},$$

folglich

$$r=\frac{r'}{n},$$

woraus das Gesagte folgt. Wenn aber $g\geqq\sigma l+1$, dann ist die Beziehung $r<g$ selbstverständlich.[1]

Ist dagegen $\mathfrak{m}$ nur durch die erste Potenz von $\mathfrak{l}$ teilbar $(g=1)$, dann ist die Relativdiscriminante von K_0/k prim zu $\mathfrak{l}$. Denn andernfalls würde, wie oben, aus $r'<n$ die unmögliche Beziehung $r<1$ folgen.

1) Die Beziehung $r<g$ rechtfertigt die Anwendung von Satz 13 auf den Körper K_0/k (vgl. oben).

§ 20.

Relativ quadratische Classenkörper.

Um den Beweis unseres Existenzsatzes für den Fall durchzuführen, wo der Index der vorgelegten Classengruppe gleich 2 ist, sprechen wir ihn in precisirter Fassung wie folgt aus.

Satz 27. *In einem algebraischen Körper* k *vom Grade m sei die Idealclassen nach der Zahlengruppe* o *derjenigen Zahlen* α *definirt, welche die Bedingung:* $\alpha \equiv 1\ (\mathfrak{m})$ *befriedigen, jedoch ohne irgendwelcher Vorzeichenbeschränkung unterworfen zu sein. Dann existirt für jede vorgelegte Classengruppe vom Index* 2 *ein relativ quadratischer Oberkörper* K *von* k, *welcher derselben als Classenkörper zugeordnet ist und von der Art, dass unter den mit* K *conjugirten* $2m$ *Körpern doppelt so viel reelle Körper als unter den mit* k *conjugirten* m *Körpern vorhanden sind.*

Oder allgemeiner:

Wenn von den r_1 *reellen mit* k *conjugirten Körpern eine beliebige Anzahl* ν : *es seien diese* $k_1, k_2, \ldots\ldots k_\nu$ *ausgewählt wird, und wenn in die Zahlengruppe* o^+ *nur diejenigen Zahlen von* o *aufgenommen werden, welche positiv in* $k_1, k_2, \ldots\ldots k_\nu$ *ausfallen, dann existirt für jede Classengruppe vom Index* 2 *nach* o^+, *ein Classenkörper* K, *welcher unter den* $2m$ *conjugirten Körpern wenigstens* $2(r_1-\nu)$ *reelle aufweist.*

Natürlich soll K die in Satz 23 und Satz 25 ausgesprochenen Bedingungen in Bezug auf die Relativdiscriminante befriedigen.

Beweis. Es genügt, den Satz in der im zweiten Teil ausgesprochenen allgemeineren Form zu beweisen; dies geschieht auf derselben Weise wie in § 18. Nur soll im gegenwärtigen Falle für die Zahl $\bar{t}$ der Wert[1)]

$$\bar{t} = d + d' + \Sigma sf + \Sigma R(g_1) + \nu + n - (r+1) \qquad (a)$$

angenommen werden, wo 2^n die Anzahl der quadratischen Reste nach $\mathfrak{m}$ ist, welche in dem System der Zahlen

$$(-1)^{u_0} \varepsilon_1^{u_1} \cdots\cdots \varepsilon_r^{u_r} \rho_1^{v_1} \cdots\cdots \rho_t^{v_t} \qquad (0 \leqq u, v < 2)$$

1) Vgl. Formel (8) in § 17, wo jetzt an Stelle von $\Sigma R(g)$ und $r_1 - r_0$ bez. $\Sigma(sf+1) + \Sigma R(g_1)$ und ν gesetzt werden müssen.

enthalten sind, und in $k_1, k_2, \ldots k_\nu$ positiv ausfallen. Ferner sollen die Zahlen des Systems (3) in § 18, ausser den dort erklärten $t+\Sigma s'f'+\Sigma R(2s_1-g_1+1)$ Bedingungen (vgl. S. 76), noch $r_1-\nu$ weiteren unterworfen sein, in den von $k_1, k_2, \ldots k_\nu$ verschiedenen $r_1-\nu$ reellen mit k conjugirten Körpern positiv zu sein. Da diese letzteren Bedingungen $r_1-\nu$ lineare Congruenzen mod. 2 involviren, welche die Exponenten $x, y, u, v, w, \ldots$ der Zahlen (3) in § 18 zu befriedigen haben, so haben wir jetzt an Stelle von (5) in § 18,

$$t' \geqq r+1+t+d+d'+n-\{t+\Sigma s'f'+\Sigma R(2s_1-g_1+1)+r_1-\nu\}. \quad (b)$$

Man erhält aus (a) und (b)

$$t'-\bar{t} \geqq 2(r+1)-r_1-m,$$

woraus, weil bekanntlich

$$r+1=\frac{m+r_1}{2},$$

noch immer

$$t' \geqq \bar{t}.$$

Da die Zahl μ (vgl. (4), § 18) nun höchstens in den ν Körpern $k_1, k_2, \ldots k_\nu$ negativ ausfallen kann, so ist die Anwendbarkeit von Satz 13 gesichert, und man überzeugt sich wie in § 18 von der Richtigkeit des zu beweisenden Satzes.

Der obige Beweis bleibt gültig, wenn $\nu=0$, was den ersten Teil unseres Satzes bestätigt.

Man erhält alle für ein gegebenes m überhaupt möglichen relativ quadratischen Classenkörper, wenn man in o^+ nur die total positiven Zahlen von o zulässt, und für jede Classengruppe vom Index 2 nach o^+ den entsprechenden Classenkörper construirt.

§ 21.

Relativ cyclische Classenkörper vom Primzahlpotenzgrade.

Wir wollen nunmehr den Existenzsatz in dem Falle beweisen, wo H eine Classengruppe nach dem Modul m von einem geraden oder ungeraden Primzahlpotenzindex l^h und die complementäre Gruppe G/H cyclisch ist.

Es sei also C eine solche Classe in k, dass erst die l^h te Potenz von C in H enthalten ist; ferner sei G_0 die Classengruppe vom Index l, welche H in sich enthält, so dass in einer wiederholt angewandten Bezeichnungsweise

$$G=\{C, H\}, \quad G_0=\{C^l, H\}.$$

Es existirt alsdann ein relativ cyclischer Körper k′/k vom Grade l, welcher Classenkörper für G_0 ist. Nach Satz 22 ist es nun möglich, ein in m aufgehendes invariantes Ideal m′ so zu wählen, dass die Relativnormen aller Ideale einer Classe nach m′ in k′ einer und derselben Classe nach m in k angehören werden. Demnach ist die Gruppe der sämtlichen Classen von k′ in der Form darstellbar:

$$\{C, H'\},$$

wo H′ die Gruppe derjenigen Classen von k′ bedeutet, deren Relativnormen in die Classengruppe H von k hineinfallen, und C diejenige Classe von k′, welche die Ideale von C in k enthält. Von den Potenzen dieser Classe C von k′ ist erst die l^{h-1} te in H′ enthalten.

Wir nehmen nun an: der Existenzsatz sei bewiesen für den Index l^{h-1}. Demnach existirt ein relativ cyclischer Körper K vom Relativgrade l^{h-1} in Bezug auf k′, welcher Classenkörper für die Classengruppe H′ von k′ ist. Da H′ offenbar gegenüber der erzeugenden Substitution s der Galois'schen Gruppe des Relativkörpers k′/k invariant ist, so folgt nach Satz 6, dass K relativ normal in Bezug auf k ist. Weil aber K der Classengruppe H von k zugeordnet ist, so folgt, dass K keinen Unterkörper ausser k′ enthält, welcher relativ cyclisch vom Relativgrade l in Bezug auf k ist. Denn ein solcher müsste als Classenkörper einer Classengruppe vom Index l in k zugehören, welche notwendig H in sich enthalte. Ausser G_0, welcher der Classenkörper k′ zugeordnet ist, gibt es aber keine solche Classengruppe in k.

Bedeutet daher $\mathfrak{G}$ die Galois'sche Gruppe des relativ normalen Körpers K/k, dann ist $\mathfrak{G}$ von der Ordnung l^h, und es enthält $\mathfrak{G}$ eine einzige Untergruppe $\mathfrak{G}_0$ vom Index l (welche die Zahlen von k' unverändert lässt), und es ist

$$\mathfrak{G}=\mathfrak{G}_0+\mathfrak{G}_0 s+\cdots\cdots+\mathfrak{G}_0 s^{l-1}.$$

Hieraus ist aber zu schliessen, dass $\mathfrak{G}$ cyclisch, also s von der Ordnung l^h sein muss. Denn widrigenfalls musste es bekanntlich[1] in $\mathfrak{G}$ eine Untergruppe der Ordnung l^{h-1} geben, welche s enthält und folglich von $\mathfrak{G}_0$ verschieden wäre. Daher ist K relativ cyclisch in Bezug auf k.

Wenn die Relativdiscriminanten der Relativkörper k'/k und K/k' bez. mit $\mathfrak{d}$ und $\mathfrak{d}'$ bezeichnet werden, dann ist die Relativdiscriminante von K/k gleich $\mathfrak{d}\mathfrak{d}'^l$. Da nach Annahme jeder Primfactor von $\mathfrak{d}'$ in $\mathfrak{m}'$ also auch in $\mathfrak{m}$ aufgeht, und da dasselbe ebenfalls von $\mathfrak{d}$ gilt, so ist die Relativdiscriminante von K/k durch kein Primideal teilbar, welches nicht in $\mathfrak{m}$ aufgeht.

Oben haben wir die Vorzeichenbedingung für die Classengruppe H ausser Betracht gelassen. Um unseren Existenzbeweis für $l=2$ allgemein zu führen, haben wir die Classen von k nach total positiver Zahlengruppe zu definiren, und demgemäss nach Satz 22 die Classen von k' einer entsprechenden Vorzeichenbedingung zu unterwerfen. Tatsächlich ist aber, wenn der Index von H grösser als 2 ist, und G/H cyclisch, jede Vorzeichenbedingung für die Gruppe $G_0=\{C^2, H\}$ ohne Belang, so dass k' ein relativ quadratischer Körper von der im ersten Teil des Satzes 27 erläuterten Art ist. Es musste dies so sein, wenn überhaupt ein relativ cyclischer Körper vom 2^hten Grade existiren soll, welcher den Körper k' als Unterkörper enthält. Denn für einen reellen Grundkörper muss jeder relativ cyclische Körper vom Grade 2^h notwendig den reellen Unterkörper vom Relativgrade 2^{h-1} enthalten.

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1) Vgl. z.B. H. Weber, Lehrbuch der Algebra, II (2te Aufl., Braunschweig, 1899) S. 140. Der hier benutzte Gruppensatz ist ein specieller Fall eines allgemeinen Satzes von W. Burnside: vgl. dessen Theory of finite groups (2. ed. Cambridge. 1911.) p. 131–132.

§ 22.

Existenzbeweis im allgemeinen Falle.

Nachdem im Vorhergehenden unser Existenzsatz in allen denjenigen Fällen bewiesen worden ist, wo die complementäre Gruppe der gegebenen Classengruppe cyclisch von einer Primzahlkotenzordnung ist, können wir nun den allgemeinen Fall rasch erledigen. Sei also H eine Classengruppe von einem beliebigen Index n nach dem Modul $\mathfrak{m}$ mit oder ohne Vorzeichenbeschränkung. Es seien ferner $C_1, C_2, \ldots C_r$ ein System der Basisclassen von den Primzahlpotenzordnungen $l_1^{h_1}, l_2^{h_2}, \ldots l_r^{h_r}$ der vollständigen Classengruppe G in Bezug auf H, derart dass

$$G = \{C_1, C_2, \cdots\cdots C_r, H\},$$

$$n = l_1^{h_1} l_2^{h_2} \cdots\cdots l_r^{h_r}.$$

Diejenige Untergruppe von G, welche ausser H noch die sämtlichen Basisclassen mit alleiniger Ausnahme von C_ρ enthält, sei mit H_ρ bezeichnet, so dass G/H_ρ cyclisch von der Ordnung $l_\rho^{h_\rho}$ ist. Ist dann K_ρ der Classenkörper für H_ρ, deren Existenz in den vorhergenden Artikeln bewiesen worden ist, dann entsteht durch Zusammensetzung der r Körper $K_1, K_2, \ldots K_r$ ein relativ Abel'scher Körper K, welcher der gesuchte Classenkörper für die Classengruppe H sein wird.

Denn da die Relativnormen der Ideale des zusammengesetzten Körpers $K_1 K_2$ offenbar sowohl der Classengruppe H_1 als auch H_2, folglich der Classengruppe $\{C_3, \ldots H\}$ vom Index $l_1^{h_1} l_2^{h_2}$ angehören, so folgt nach Satz 4, dass der Relativgrad von $K_1 K_2$ wenigstens gleich $l_1^{h_1} l_2^{h_2}$ sein muss. Anderseits kann aber dieser Relativgrad höchstens gleich dem Product der Relativgrade der beiden Körper K_1 und K_2 sein; also ist er genau gleich $l_1^{h_1} l_2^{h_2}$. Mit andern Worten: $K_1 K_2$ ist der Classenkörper für die Classengruppe $\{C_3, \ldots H\}$. So fortfahrend überzeugt man sich davon, dass der Körper $K = K_1 K_2 \ldots K_r$ in der Tat der Classenkörper für die Classengruppe H ist.

Hieraus folgt aber weiter, dass K vom Relativgrade n ist und dass die Galois'sche Gruppe des Relativkörpers K/k mit der complementären Gruppe G/H holoedrisch isomorph ist.

Da endlich die Relativdiscriminante jedes der Körper K_1, K_2, ...K_r kein Primideal als Factor enthält, welches nicht in den Modul $\mathfrak{m}$ aufgeht, so gilt dassalbe auch von der Relativdiscriminante von K.

Wie man sieht, erfüllt der Körper K alle Forderungen des zu Beginn dieses Capitels aufgestellten Satzes 23, welcher nunmehr in allen seinen Teilen vollständig bewiesen worden ist.

CAPITEL IV.

Weitere allgemeine Sätze.

§ 23.

Der Vollständigkeitssatz.

Ist $\mathfrak{m}$ ein beliebiges Ideal im Grundkörper k, $\mathfrak{o}^+$ die Zahlengruppe der *total positiven* Zahlen α, welche die Bedingung: $\alpha \equiv 1$. ($\mathfrak{m}$) erfüllen, dann ist die Classenzahl nach $\mathfrak{o}^+$ durch die Formel gegeben:

$$h(\mathfrak{m}) = h_0 \frac{\varphi(\mathfrak{m})}{e},$$

wo $h_0 = h(1)$ die Classenzahl im absoluten Sinne, φ die Euler'sche Function, und

$$e = (\mathrm{E} : \mathrm{E}_0)$$

der Gruppenindex ist, wobei E die Gruppe der sämtlichen Einheiten in k, die Einheitswurzeln mitgerechnet, und E_0 die Gruppe der Einheiten in $\mathfrak{o}^+$ bedeutet.

Dann existirt nach Satz 23 ein relativ Abel'scher Körper vom Grade $h(\mathfrak{m})$ in Bezug auf k, welcher Classenkörper für die durch $\mathfrak{o}^+$ erzeugte Idealengruppe in k ist. Derselbe sei mit K ($\mathfrak{m}$) bezeichnet.

Dieser Körper K ($\mathfrak{m}$) soll der **vollständige Classenkörper nach dem Modul** $\mathfrak{m}$ genannt werden. Für $\mathfrak{m} = (1)$ ist der Körper K (1) der zuerst von D. Hilbert eingeführte Classenkörper von k, den wir als den *absoluten Classenkörper* bezeichnen wollen. Ist ferner $\mathfrak{m}$ der Führer eines Ringes in k, dann ist derjenige Körper, welcher Hilbert gelegentlich[1] als einen Ringclassenkörper bezeichnet hat, als einen Unterkörper in K ($\mathfrak{m}$) enthalten, wie überhaupt jeder Classenkörper für irgend eine Classengruppe nach dem Modul $\mathfrak{m}$.

Eine wichtige Frage ist nun, ob auch umgekehrt jeder relativ Abel'sche Körper in Bezug auf k als Classenkörper einer Classengruppe nach einem geeignet gewählten Modul $\mathfrak{m}$ in k zugeordnet ist? Diese Frage ist im bejahenden Sinne zu beantworten:

Satz 28. *Alle relativ Abel'schen Körper in Bezug auf einen beliebigen algebraischen Körper werden durch die Classenkörper nach den Idealmoduln in demselben erschöpft.*

Es genügt, diesen Satz für die relativ cyclischen Oberkörper vom Primzahlpotenzgrade zu beweisen.

Denn aus solchen lässt sich jeder relativ Abel'sche Körper zusammensetzen. Anderseits seien K, K' relativ Abel'sch von den Relativgraden n, n' in Bezug auf k und bez. den Classengruppen $\mathfrak{H}$, $\mathfrak{H}'$ nach den Moduln $\mathfrak{m}$, $\mathfrak{m}'$ als Classenkörper zugeordnet. Ist $\mathfrak{m}_0$ das kleinste gemeinsame Vielfaches von $\mathfrak{m}$ und $\mathfrak{m}'$, dann sind $\mathfrak{H}$, $\mathfrak{H}'$ als Classengruppen nach dem Modul $\mathfrak{m}_0$ aufzufassen. Unter der Voraussetzung, dass K, K' keinen gemeinsamen Unterkörper über k enthalten, folgt, dass die Gruppe $\{\mathfrak{H}, \mathfrak{H}'\}$ mit der vollständigen Classengrappe G von k zusammenfällt, weil sonst der zu der Classengruppe $\{\mathfrak{H}, \mathfrak{H}'\}$ gehörige Classenkörper nach Satz 6 notwendig sowohl in K als auch in K' enthalten sein musste. Da nun $\mathfrak{H}$, $\mathfrak{H}'$ bez. vom Index n, n' sind, und $\{\mathfrak{H}, \mathfrak{H}'\}=G$, so muss die grösste gemeinsame Untergruppe $\mathfrak{H}_0$ von $\mathfrak{H}$ und $\mathfrak{H}'$ notwendig vom Index nn' sein. Weil aber die Relativnormen der Ideale von dem zusammengesetzten Körper KK' vom Relativgrade nn' sämtlich in $\mathfrak{H}_0$ enthalten sind, so folgt, dass KK' der Classenkörper für die Classengruppe $\mathfrak{H}_0$ ist. Da dasselbe auch von mehreren relativ Abel'schen Körpern K, K,' K''...gilt, so folgt das Gesagte.

1) D. Hilbert, über die Theorie der relativ Abel'schen Körper. Göttinger Nachr. 1898.

Da Satz 28 schon für die relativ cyclischen Körper vom Primzahlgrade in Satz 13 bewiesen worden ist, so handelt es sich jetzt darum, den letzteren auf die relativ cyclischen Körper vom Primzahlpotenzgrade zu verallgemeinern.

§. 24.

Ueber die Geschlechter im relativ cyclischen Körper eines Primzahlpotenzgrades.

Um am Ende des vorigen Artikels angezeigten Beweis durchzuführen, stellen wir den folgenden Satz auf.

Satz 29. *Es sei* K/k *ein relativ cyclischer Körper vom Primzahlpotenzgrade* l^h. *Dann gibt es stets ein Ideal* m *in* k, *welches jedes in die Relativdiscriminante von* K *aufgehende Primideal von* k *als Factor, und zwar solches, welches zu* l *prim ist, zur ersten, dagegen solches, welches in* l *aufgeht, zu einer hinreichend hohen Potenz, enthält, von der Art, dass* K *Classenkörper für eine Classengruppe vom Index* l^h *nach dem Modul* m *ist.*

Ferner lässt sich im Oberkörper K *ein in* m *aufgehendes Ideal* 𝔐 *auffinden, derart, dass, wenn die Classen in* K *und* k *nach den Zahlengruppen definirt werden, die aus den Zahlen dieser Körper bestehen, welche bez. nach den Moduln* 𝔐, m *mit* 1 *congruent sind, jede Classe von* K, *deren Relativnorm die Hauptclasse von* k *ist, die symbolische* $1-s^{\text{te}}$ *Potenz einer Classe von* K *wird, wenn* s *eine erzeugende Substitution der Galois'schen Gruppe des Relativkörpers* K/k *bedeutet.*

Beweis. Zunächst sei das Folgende bemerkt: Wenn es nachgewiesen wird, dass der Körper K einer Classengruppe н vom Index l^h als Classenkörper zugeordnet ist, dann ist klar, dass die complementäre Gruppe G/н notwendig cyclisch sein muss, wo G, wie immer, die vollständige Classengruppe von k bedeutet. Denn andernfalls musste es mehr als eine Classengruppe vom Index l geben, welche н enthält, und jeder derselben nach Satz 23 ein relativ cyclischer Körper vom Relativgrade l als Classenkörper zugeordnet sein muss. Diese Körper mussten aber nach Satz 6 sämtlich in K enthalten sein, was unmöglich ist, weil K relativ cyclisch sein sollte.

Um nun unseren Satz durch vollständige Induction zu beweisen, werde angenommen: der Satz sei für den in K enthaltenen relativ cyclischen Körper K′/k vom Grade l^{h-1} richtig. Hierunter ist genauer folgendes zu verstehen: Die in die Relativdiscriminante von K aufgehenden, zu l primen, und in l aufgehenden Primideale von k seien bez. durchweg mit $\mathfrak{p}$ und $\mathfrak{l}$, die in sie aufgenden Primideale von K bez. K′ durchweg mit $\mathfrak{P}$ und $\mathfrak{L}$, bez. $\mathfrak{P}'$ und $\mathfrak{L}'$ bezeichnet, so dass

$$\mathfrak{P}'=\mathfrak{P}^l, \quad \mathfrak{L}'=\mathfrak{L}^l.$$ [1]

Man setze

$$\mathfrak{m}=\Pi\mathfrak{p}\Pi\mathfrak{l}^u, \quad \mathfrak{M}=\Pi\mathfrak{P}.\ \Pi\mathfrak{L}^U, \quad \mathfrak{M}'=\Pi\mathfrak{P}'.\ \Pi\mathfrak{L}'^{U'} \qquad (1)$$

Es soll dann angenommen werden, dass sobald U und U' bez. grösser als gewisse näherzubestimmende feste Grössen sind, die Relativnormen der Classen (nach $\mathfrak{M}'$) von K′ eine Classengruppe $\mathfrak{n}'$ vom Index l^{h-1} in k (nach $\mathfrak{m}$) ausmachen, dass ferner die Classen von K,′ deren Relativnormen Hauptclasse von k sind, durch die symbolischen $1-s^{\text{ten}}$ Potenzen in K′ erschöpft werden.

Um nun auf Grund dieser Annahme die entsprechende Tatsache für den Körper K nachzuweisen, machen wir die erlaubte Annahme, dass

$$U'>v, \qquad (2)$$

und setzen

$$U=(U'-v)l+v, \qquad (3)$$

wo v die mehrmals erklärte Bedeutung in Bezug auf das Primideal $\mathfrak{L}$ und den Relativkörper K/K′ besitzt: es ist die Relativdifferente von K/K′ genau durch die $(v+1)(l-1)^{\text{te}}$ Potenz von $\mathfrak{L}$ teilbar. (Ist also $U'=v+n$, dann ist $U=v+nl$, wo $n>0$).

Wir setzen diesen Wert von U in den Ausdruck von $\mathfrak{M}$ in (1) ein, und definiren die Classen von K nach diesem Modul $\mathfrak{M}$. Dann kommt Satz 22 in Anwendung, demzufolge die Relativnormen

1) Hiermit ist nicht gesagt, dass jedes $\mathfrak{p}$ und jedes $\mathfrak{l}$ schon in die Relativdiscriminante von K′ aufgeht. Auch sollen, wenn mehrere von einander verschiedene $\mathfrak{P}'$ in ein $\mathfrak{p}$ aufgehen, das Product $\Pi\mathfrak{P}'$ und $\Pi\mathfrak{P}$ in (1) auf alle diese Primfactoren von $\mathfrak{p}$ erstreckt werden; gleiches gilt für die Primideale $\mathfrak{l}$.

der Classen von K in Bezug auf K′ eine Classengruppe H′ vom Index l nach $\mathfrak{M}'$ ausmachen, und speciell die Classen von K, deren Relativnormen die Hauptclasse nach $\mathfrak{M}'$ sind, symbolische $1-s^{l^{h-1}}$ te Potenzen der Classen von K sein müssen.

Da nun die Classengruppe H′ ihrer Bedeutung nach offenbar gegenüber s invariant ist, so ist zu schliessen, dass die $(1-s)$ te Potenz jeder Classe von K′ notwendig in H′ enthalten sein muss. In der Tat: sei C eine nicht in H′ enthaltene Classe von K′, so dass auch C^s nicht in H′, folglich in einem Classencomplex $H'C^{a}$ enthalten sein muss, wo a eine Zahl aus der Reihe: $1, 2, \ldots l-1$ bedeutet. Da dann C^{s^n} in $H'C^{a^n}$ enthalten ist, so folgt, wenn man $n=l^{h-1}$ macht, dass die $(1-a^n)^{\text{te}}$ Potenz von C in H′ enthalten ist, d. h. es ist

$$a^{l^{h-1}} \equiv 1, \quad (l),$$

woraus folgt, dass $a\equiv 1$, (l), also $a=1$ sein muss. Es ist daher C^{1-s} in H′ enthalten, wie behauptet wird.

Demnach folgt, nach Annahme, dass alle Classen von K′, deren Relativnormen in Bezug auf k die Hauptclasse nach m in k sind, in H′ enthalten, folglich, da H′ nur den l^{ten} Teil der sämtlichen Classen von K′ ausmacht, dass die Relativnormen aller Classen von K in Bezug auf k eine Classengruppe H vom Index l^h in k ausmachen, welche in der Classengruppe H′ enthalten ist.

Nunmehr ist noch zu zeigen, dass die Classen von K, deren Relativnormen in Bezug auf k die Hauptclasse in k sind, notwendig die symbolischen $1-s^{\text{ten}}$ Potenzen in K sein müssen. Da, wie vorhin bemerkt, die complementäre Gruppe G/H cyclisch ist, so kann man in k eine Basisclasse c angeben, deren Ordnung eine Potenz von l, und von der erst die l^h te Potenz in H enthalten ist. Demnach hat man, in einer leicht verständlichen Bezeichnungsweise

$$G=\{C, D\}, \quad H=\{C^{l^h}, D\}, \quad H'=\{C^{l^{h-1}}, D\};$$

dementsprechend lässt sich die vollständige Classengruppe von K′ in der Form darstellen:

$$\{C, D'\}, \tag{4}$$

wo D' den Inbegriff der Classen von K' bedeutet, deren Relativnormen in k in $\mathfrak{d}$ hineinfallen; so dass jede Classe in D' Relativnorm einer Classe von K in Bezug auf K' ist.

Sei nun C eine Classe von K, deren Relativnorm die Hauptclasse in k ist. Dann ist, nach Annahme

$$\mathfrak{N}(C)=C'^{1-s}, \tag{5}$$

wo $\mathfrak{N}$ die in K genommene Relativnorm in Bezug auf K', und C' eine Classe von K' bedeutet. Da aber nach (4)

$$C'=c^{u}\,[D'] \tag{6}$$

wo mit $[D']$ eine Classe von K' bezeichnet wird, welche der Gruppe D' angehört, folglich Relativnorm einer Classe D von K ist:

$$D'=\mathfrak{N}(D). \tag{7}$$

Aus (5), (6), (7) folgt

$$\mathfrak{N}(C)=\mathfrak{N}(D^{1-s}).$$

Setzt man daher

$$C=D^{1-s}A, \tag{8}$$

so ist A eine solche Classe von K, dass $\mathfrak{N}(A)$ die Hauptclasse von K' ist. Folglich ist A eine $1-s'^{\,h-1}$ te Potenz, also auch eine $1-s$^te^ Potenz einer Classe von K. Dasselbe gilt daher nach (8) auch von C selbst, wie zu beweisen war.

Um eine untere Grenze für den Exponenten u zu bestimmen, sei angenommen, dass das Primideal $\mathfrak{L}$ von K genau zur l^g ten Potenz in $\mathfrak{f}$ aufgeht, sodass die Verzweigung von $\mathfrak{f}$ erst in dem Unterkörper von K vom Relativgrade l^{h-g-1} über k beginnt. Indem wir allgemein mit K_γ den in K enthaltenen relativ cyclischen Oberkörper vom Grade l^γ über k bezeichnen, seien $v_1, v_2, \ldots v_g$ die Zahlen, die mehrmals erklärte Bedeutung[1] in Bezug auf die Relativkörper K_{h-g+1}/K_{h-g}, $K_{h-g+2}/K_{h-g+1}, \ldots\ldots K_h/K_{h-1}$ haben, so dass bekanntlich

$$1\leqq v_1<v_2<\cdots\cdots<v_g \tag{9}$$

1) Es ist v_g die oben (S. 92) mit v bezeichnete Zahl.

Für den Modul $\mathfrak{M}_\gamma$ in K_γ genügt es, den entsprechenden Exponenten U_γ so zu bestimmen, dass

$$u = U_1 = U_2 = \cdots\cdots = U_{h-g},$$

und, gemäss (2) und (3), für $\gamma = h-g,\ h-g+1, \ldots\ldots h$,

$$U_\gamma > v_{\gamma-(h-g)+1},$$

$$U_{\gamma+1} = lU_\gamma - (l-1)v_{\gamma-(h-g)+1}.$$

Diese Bedingungen werden erfüllt, wie man leicht mit Hülfe von (9) bestätigt, wenn u so gross genommen wird, dass

$$U = U_h = ul^g - (l-1)\{v_g + v_{g-1}l + \cdots\cdots + v_1 l^{g-1}\} > v_g. \qquad (10)$$

In die Relativdiscriminante von K/k geht $\mathfrak{l}$ genau zur $\delta(l-1)$ ten Potenz auf, wo[1)]

$$\delta = \{(v_g+1) + (v_{g-1}+1)l + \cdots\cdots (v_1+1)l^{g-1}\} l^{h-g}$$

$$= \left\{v_g + v_{g-1}l + \cdots\cdots + v_1 l^{g-1} + \frac{l^g-1}{l-1}\right\} l^{h-g}.$$

Nach (10) kann man daher einen Wert von u finden, derart, dass

$$\delta > u,$$

ausser wenn $h = g = 1$, wo notwendig $\delta = u = v_1 + 1$.

Ohne nähere Kenntnis über die Zahlen $v_1, v_2, \ldots\ldots v_g$ kann man eine untere Grenze für u angeben, welche sich für alle Fälle bewähren wird: nämlich

$$u > gs + \frac{s}{l-1}, \qquad (11)$$

wo s der Exponent der höchsten in l aufgehenden Potenz von $\mathfrak{l}$ bedeutet. Denn es ist nach Satz 8

$$\frac{sl}{l-1} \geqq v_1, \quad \frac{sl^2}{l-1} > v_2, \cdots\cdots \frac{sl^g}{l-1} \geqq v_g,$$

so dass aus (11) folgt

$$ul^g > gsl^g + \frac{sl^g}{l-1} \geqq (l-1)\{v_g + v_{g-1}l + \cdots\cdots + v_1 l^{g-1}\} + v_g,$$

wodurch (10) befriedigt wird.

Wir haben bisher den Fall ausser Betracht gelassen, wo $l = 2$ und unter den mit k conjugirten Körpern reelle vorhanden sind,

---

1) Vgl. Hilbert, Bericht, Satz 79.

wo also unter Umständen eine Vorzeichenbedingung für die Classeneinteilung in k unentbehrlich werden kann. Gebe es nun in diesem Falle einen mit k conjugirten reellen Körper k^*, für welchen der entsprechende mit K conjugirte Körper K^* imaginär ausfällt, dann ist notwendig der in K^* enthaltene mit K' conjugirte Körper K'^* vom Relativgrade 2^{h-1} reell. Es ist daher leicht, in Bezugnahme auf Satz 22 einzusehen, dass unser Beweis seine Gültigkeit beibehält, wenn in der Zahlengruppe, welche der Classeneinteilung in k zu Grunde gelegt wird, nur diejenigen Zahlen, die in allen vorhandenen Körpern k^* positiv ausfallen, umsomehr also, wenn nur die total positiven Zahlen zugelassen werden.

Durch das Vorhergende ist, nach der Bemerkung am Ende des § 23, Satz 28 allgemein bewiesen worden. Es ist jeder relativ Abel'sche Körper K/k Classenkörper für eine Classengruppe H in k, deren Führer jedenfalls ein Teiler der Relativdiscriminante von K/k ist, wie man sich auf Grund des vorhergehenden Beweises leicht überzeugt. Nach Satz 23 ist die Galois'sche Gruppe des Relativkörpers K/k holoedrisch isomorph mit der complementären Gruppe G/H. Allgemeiner ist *jeder Unterkörper* K'/k *von* K/k *als Classenkörper einer Classengruppe* H' *zugeordnet, welche* H *in sich enthält und umgekehrt: es ist dabei die Galois'sche Gruppe des relativ Abel'schen Körpers* K/K' *holoedrisch isomorph mit der complementären Gruppe* H'/H.

§ 25.

Der Zerlegungssatz.

Wenn K der Classenkörper für die Classengruppe H des Grundkörpers k ist, dann ist jedes zum Führer der Classengruppe relativ prime Primideal von k, welches in K in die Primideale des ersten Relativgrades zerfällt, in einer Classe von H enthalten. Umgekehrt gilt der folgende sehr wichtige Satz.

Satz 30. (*Der Zerlegungssatz*). *Jedes in einer Classengruppe eines beliebigen Körpers enthaltene Primideal zerfällt in die von* *einander verschiedenen Primideale des ersten Relativgrades in dem Classenkörper für diese Classengruppe.*

Beweis. Es genügt, diesen Satz für den Fall zu beweisen, wo der Oberkörper relativ cyclisch von einem Primzahlpotenzgrade ist. Denn die dem Oberkörper K zugehörige Classengruppe H ist die grösste gemeinsame Untergruppe der Classengruppen, welche den relativ cyclischen Körpern von den Primzahlpotenzgraden zugeordnet sind, aus welchen K zusammengesetzt wird. Zerfällt anderseits ein Primideal des Grundkörpers in allen jenen Körpern in die von einander verschiedenen Primideale des ersten Relativgrades, so muss dasselbe auch in dem zusammengesetzten Körper K gelten, wie leicht einzusehen ist.

Sei also K relativ cyclisch vom Relativgrade l^n in Bezug auf k, H die zugehörige und G die vollständige Classengruppe von k, so dass die complementäre Gruppe G/H cyclisch von der Ordnung l^n ist. Wir setzen

$$G = \sum^{\alpha} HA^{\alpha}, \qquad (0 \leq \alpha < l^n)$$

wo A eine Classe bedeutet, von welcher erst die l^nte Potenz in H enthalten ist. Sei ferner C eine Classe in H, und $\mathfrak{p}$ ein Primideal der Classe C. Wir nehmen zunächst an, es sei C nicht l^{te} Potenz einer Classe von k. Dann gibt es offenbar eine Classengruppe H′ vom Index l, welche nicht die Classe C enthält, und es ist

$$G = \sum H' C^{\beta}. \qquad (0 \leq \beta < l).$$

Ist dann H_0 die Durchschnitt der beiden Gruppen H und H′, dann ist H_0 vom Index l^{n+1}, und man hat

$$\begin{aligned} H &= \sum H_0 C^{\beta}, & (0 \leq \beta < l) \\ H' &= \sum H_0 A^{\alpha}, & (0 \leq \alpha < l^n) \\ G &= \sum H_0 A^{\alpha} C^{\beta}. \end{aligned}$$

Der relativ cyclische Körper l^{ten} Grades über k, welcher der Classengruppe H′ zugeordnet ist, sei mit K′ bezeichnet. Dann ist der zusammengesetzte Körper KK′ vom Relativgrade l^{n+1} der Classenkörper für H_0.

Angenommen nun, das Primideal $\mathfrak{p}$ zerfalle in K in ein Product von e von einander verschiedenen Primidealen, und $e < l^n$. Da $\mathfrak{p}$ nicht in $\mathfrak{n}'$ enthalten ist, so bleibt $\mathfrak{p}$ prim in K'. Wir betrachten nun den Zerlegungskörper K_z für $\mathfrak{p}$ in dem Körper KK'. Da $\mathfrak{p}$ nicht in die Relativdiscriminante von KK' aufgeht, muss KK' relativ cyclisch in Bezug auf K_z sein. Weil aber K_z nach Annahme nicht K und auch nicht K' enthält, ist dies nur so möglich, dass $K_z K$ mit KK' zusammenfällt. Daher ist K_z nicht in K enthalten. Diesem Körper K_z muss daher eine Classengruppe zugeordnet sein, welche $\mathfrak{n}_{\mathfrak{m}}$, aber nicht $\mathfrak{n}$ enthält, folglich gewiss nicht die Classe $\mathfrak{c}$ enthalten kann. Dann könnte aber das in $\mathfrak{c}$ enthaltene Primideal $\mathfrak{p}$ nicht in die Primideale vom ersten Relativgrade in K_z zerfallen, was ein Widerspruch ist. Es ist daher unsere Annahme zu verwerfen: $\mathfrak{p}$ muss notwendig in l^n von einander verschiedene Primideale in K zerfallen. Somit ist der Satz im gegenwärtigen Falle bewiesen.

Wir gehen nun zu dem Falle über, wo $\mathfrak{c}$ l^{te} Potenz einer Classe in k ist. Dann muss es eine Zahl ϖ in der Zahlengruppe $\mathfrak{o}$ geben, die der Classeneinteilung in k zu Grunde gelegt ist, von der Art, dass

$$\mathfrak{p}\,\mathfrak{j}^l = (\varpi),$$

wo $\mathfrak{j}$ ein gewisses Ideal von k bedeutet. Zum Modul $\mathfrak{m}$ der Zahlengruppe $\mathfrak{o}$ sei alsdann ein Primfactor $\mathfrak{q}$ hinzugefügt, von der Beschaffenheit, dass jede Einheit ε und jede Zahl ρ, welche l^{te} Potenz eines Ideals von k ist, l^{er} Potenzrest nach $\mathfrak{q}$, dagegen die Zahl ϖ ein l^{ter} Nichtrest nach $\mathfrak{q}$ ist. Definirt man dann die Classen von k nach dem Modul $\overline{\mathfrak{m}} = \mathfrak{m}\mathfrak{q}$, dann wird das Primideal $\mathfrak{p}$ gewiss in einer Classe enthalten sein, welche nicht die l^{te} Potenz einer Classe ist, und wir können den Beweis des Satzes genau wie oben durchführen.

Es kommt also darauf an, die Existenz des Primideals $\mathfrak{q}$ nachzuweisen. Enthält k die primitive l^{te} Einheitswurzel, dann ist dies evident, weil eine Gleichung von der Gestalt

$$\varpi = \varepsilon^{u} \cdots \rho^{v} \cdots \xi \qquad (0 \leqq u, v < l) \qquad (1)$$

offenbar nicht durch eine Zahl ξ von $\mathfrak{k}$ zu befriedigen ist.[1] Enthält aber $\mathfrak{k}$ nicht die primitive l^{te} Einheitswurzel, dann adjungire man dieselbe dem Körper $\mathfrak{k}$, und erweitre ihn zu $\mathfrak{k}'$. Da der Relativgrad von $\mathfrak{k}'/\mathfrak{k}$ prim zu l ist, so kann eine Relation von der Form (1) auch nicht in $\mathfrak{k}'$ bestehen. Daher gibt es in $\mathfrak{k}'$ ein Primideal ersten Grades $\mathfrak{q}'$, für welches

$$\left(\frac{\varpi}{\mathfrak{q}'}\right) \neq 1, \qquad \left(\frac{\varepsilon}{\mathfrak{q}'}\right) = 1, \ldots\ldots \left(\frac{\rho}{\mathfrak{q}'}\right) = 1, \ldots\ldots$$

Ist dann $\mathfrak{q}$ das durch $\mathfrak{q}'$ teilbare Primideal von $\mathfrak{k}$, dann ist offenbar $\mathfrak{q}$ ein Primideal von der geforderten Beschaffenheit.

Nur scheinbar allgemeiner als der vorhergehende ist

Satz 31. *Ist* K *der Classenkörper für die Classengruppe* $\mathfrak{H}$ *von* $\mathfrak{k}$, *dann werden die Primideale von* $\mathfrak{k}$, *welche einem und demselben Classencomplex* $\mathfrak{H}\mathfrak{c}$ *angehören, in* K *auf derselben Weise zerlegt, d. h. sie erfahren in* K *eine Zerlegung in dieselbe Anzahl von Primidealen derselben Relativgrade.*

Beweis. Ist $\mathfrak{p}$ ein Primideal, welches der Classe $\mathfrak{c}$ oder einer Classe des Complexes $\mathfrak{H}\mathfrak{c}$ angehört, dann ist der Zerlegungskörper für $\mathfrak{p}$ in K der umfassendste in K enthaltene Oberkörper von $\mathfrak{k}$, in welchem $\mathfrak{p}$ in die Primideale des ersten Relativgrades zerfällt. Dieser Körper ist daher der kleinsten Classengruppe in $\mathfrak{k}$ zugeordnet, welche $\mathfrak{H}$ und $\mathfrak{c}$ enthält, d. h. der Classengruppe $\{\mathfrak{H}, \mathfrak{c}\}$. Ist daher n der Relativgrad des Körpers K/$\mathfrak{k}$, also der Index der Classengruppe $\mathfrak{H}$, und ist f der kleinste positive Exponent, für welchen $\mathfrak{c}^f$ in $\mathfrak{H}$ enthalten ist, dann ist der Index der Classengruppe $\{\mathfrak{H}, \mathfrak{c}\}$, und demnach auch der Relativgrad des Zerlegungskörpers für $\mathfrak{p}$ gleich $e = \frac{n}{f}$; und das Primideal $\mathfrak{p}$ zerfällt in K in e von einander verschiedene Primideale vom f^{ten} Relativgrade.

Wir erläutern noch kurz das Zerlegungsgesetz für das in die Relativdiscriminante aufgehende Primideal. Das Gesetz ist besonders einfach für den vollständigen Classenkörper K($\mathfrak{m}$) nach dem Modul $\mathfrak{m}$. Sei $\mathfrak{l}$ ein genau zur u^{ten} Potenz in $\mathfrak{m}$ aufgehendes Primideal, so dass

1) Vgl. § 4. S. 16.

$$\mathfrak{m}=\mathfrak{l}^n\mathfrak{m}_0,$$

wo $\mathfrak{m}_0$ prim zu $\mathfrak{l}$ ist. Der Körper $K(\mathfrak{m})$ enthält eine Reihe von Unterkörpern, von welcher der erste der absolute Classenkörper und der letzte der Körper $K(\mathfrak{m})$ selbst ist:

$$K(1),\quad K(\mathfrak{m}_0),\quad K(\mathfrak{l}\mathfrak{m}_0),\quad K(\mathfrak{l}^2\mathfrak{m}_0),\cdots\cdots K(\mathfrak{l}^n\mathfrak{m}_0).$$

Die Relativgrade dieser Körper werden durch die entsprechende Zerlegung des Relativgrades von $K(\mathfrak{m})$ klargestellt:

$$h\times\frac{\varphi(\mathfrak{m}_0)}{(\mathrm{E}:\mathrm{E}_0)}\times\frac{l^f-1}{(\mathrm{E}_0:\mathrm{E}_1)}\times\frac{l^f}{(\mathrm{E}_1:\mathrm{E}_2)}\times\cdots\cdots\frac{l^f}{(\mathrm{E}_{n-1}:\mathrm{E}_n)};$$

hierbei bedeutet h die Classenzahl des Grundkörpers k im absoluten (sogenannten engeren) Sinne; f der absolute Grad des Primideals $\mathfrak{l}$ in k. E die Gruppe der sämtlichen total positiven Einheiten in k, E_e für $e\geqq 0$ die Gruppe derjenigen, welche nach dem Modul $\mathfrak{l}^e\mathfrak{m}_0$ mit 1 congruent sind, und das Zeichen $(\mathrm{A}:\mathrm{B})$ wie bisher den Gruppenindex.

Bezeichnen wir ferner mit G die vollständige Classengruppe von k, mit G_{-1} die durch die sämtlichen total positiven Zahlen von k definirte Idealgruppe, und allgemein mit G_e $(e\geqq 0)$ die Idealgruppe, welche durch die total positiven Zahlen α erzeugt wird, die der Congruenz

$$\alpha\equiv 1,\quad(\mathfrak{l}^e\mathfrak{m}_0)$$

genügen, dann sind die oben angegebenen Körper der Reihe nach den Classengruppen zugeordnet:

$$\mathrm{G}_{-1},\quad \mathrm{G}_0,\quad \mathrm{G}_1,\quad \mathrm{G}_2,\cdots\cdots \mathrm{G}_n.$$

Es ist die complementäre Gruppe $\mathrm{G}_1/\mathrm{G}_0$ und dementsprechend der Relativkörper $K(\mathfrak{l}\mathfrak{m}_0)/K(\mathfrak{m}_0)$ cyclisch, dagegen $\mathrm{G}_2/\mathrm{G}_1,\ldots\ldots \mathrm{G}_{n-1}/\mathrm{G}_n$ und entsprechend $K(\mathfrak{l}^2\mathfrak{m}_0)/K(\mathfrak{l}\mathfrak{m}_0),\ldots\ldots K(\mathfrak{m})/K(\mathfrak{l}^{n-1}\mathfrak{m}_0)$ Abel'sch vom Typus $(l, l,\ldots\ldots l)$, wo der Rang nicht grösser als f ist. Es ist $K(\mathfrak{m}_0)$ der Trägheitskörper, $K(\mathfrak{l}\mathfrak{m}_0)$ der Verzweigungskörper, $K(\mathfrak{l}^2\mathfrak{m}_0),\ldots\ldots K(\mathfrak{m})$ die Verzweigungskörper höheren Grades für $\mathfrak{l}$ in $K(\mathfrak{m})$. Das Primideal $\mathfrak{l}$ wird in $K(\mathfrak{m})$ die Potenz mit dem Exponenten:

$$\frac{\varphi(\mathfrak{l}^n)}{(\mathrm{E}_0:\mathrm{E}_n)}=\frac{l^f-1}{(\mathrm{E}_0:\mathrm{E}_1)}\cdot\frac{l^f}{(\mathrm{E}_1:\mathrm{E}_2)}\cdots\cdots\frac{l^f}{(\mathrm{E}_{n-1}:\mathrm{E}_n)}$$

eines Ideals, welches ein Product von einer gewissen Anzahl von einander verschiedener Primideale in $K(\mathfrak{m})$ ist. Diese Anzahl und der Relativgrad dieser Primideale werden gefunden, indem man die Zerlegung von $\mathfrak{l}$ in dem Trägheitskörper $K(\mathfrak{m}_0)$ nach Satz 31 bestimmt.

Wenn allgemein K der Classenkörper für die Classengruppe H nach dem Modul $\mathfrak{m}$ ist, dann ist der Trägheitskörper $K_{\mathfrak{l}}$ für $\mathfrak{l}$ in K der grösste gemeinsame Unterkörper von K und $K(\mathfrak{m}_0)$, also der Classengruppe

$$\{H, G_0\}=HG_0$$

zugeordnet; sie ist eine Classengruppe nach dem Modul $\mathfrak{m}_0$. Ist

$$\mathfrak{l}=(\mathfrak{L}_1\cdots\cdots\mathfrak{L}_e)^g$$

die Zerlegung von $\mathfrak{l}$ in K, dann ist g gleich dem Relativgrade von $K/K_{\mathfrak{l}}$ also gleich dem Gruppenindex $(HG_0:H)=(G_0:H_0)$, wenn H_0 die Durchschnitt von G_0 und H bedeutet.

Das in diesem Artikel auseinandergesetzte Zerlegungssatz ist die naturgemässe Verallgemeinerung des Gesetzes, welches die Zerlegung der natürlichen Primzahlen in dem Kreisteilungskörper regeln. Der durch die primitiven m^{ten} Einheitswurzeln definirte Kreisteilungskörper $\varphi(m)^{\text{ten}}$ Grades ist der vollständige Classenkörper $K(m)$, wenn der Grundkörper k der natürliche ist. Ist p eine nicht in m aufgehende rationale Primzahl, dann ist die in Satz 31 mit f bezeichnete Zahl der kleinste positive Exponent, für welchen $p^f\equiv 1,\ (m)$ ausfällt; p zerfällt daher in $K(m)$ in $e=\varphi(m):f$ von einander verschiedene Primideale. Ist ferner l eine genau zur n^{ten} Potenz in $m=l^n m_0$ aufgehende natürliche Primzahl, dann ist der Trägheitskörper für l in $K(m)$ der Körper $K(m_0)$, d. h. der durch die m_0^{te} primitiven Einheitswurzeln definirte Körper. In $K(m)$ zerfällt l in ein Product von $\varphi(l^n)$ ten Potenzen der e von einander verschiedenen Primideale, wo e genau wie oben zu bestimmen ist, indem man m_0 an Stelle von m setzt.[1]

1) Vgl. Hilbert, Bericht, Satz 125.

Als ein weiteres Beispiel sei der Teilungskörper der lemniskatischen Function sn $(u; i)$ angeführt. Der Grundkörper k ist der Gauss'sche; sei $\mathfrak{l}=(1+i)$, und $\mathfrak{m}=(\mu)$ ein ungerades Primideal in k. Der Teilungskörper zum Divisor $\mathfrak{m}$[1] ist dann der Classenkörper $K(\mathfrak{l}^3\mathfrak{m})$ vom Relativgrade $\varphi(\mathfrak{m})=N(\mathfrak{m})-1$. Der Trägheitskörper für $\mathfrak{l}$ ist $K(\mathfrak{m})$ vom Relativgrade $\varphi(\mathfrak{m}):4$. Ist also f der kleinste positive Exponent, für welchen

$$(1+i)^f \equiv i,^a \quad (\mathfrak{m})$$

ausfällt, wo i^a die Einheiten von k bedeutet, und setzt man

$$\frac{\varphi(\mathfrak{m})}{4}=ef,$$

dann zerfällt $\mathfrak{l}$ in $K(\mathfrak{l}^3\mathfrak{m})$ in ein Product von 4^{ten} Potenzen der e von einander verschiedenen Primideale.[2]

§ 26.

Ein Criterium für den relativ Abel'schen Zahlkörper.

H. Weber[3] hat den Classenkörper durch die folgende Definition eingeführt, welche, offenbar auf der Analogie mit gewissen in der Theorie der complexen Multiplication der elliptischen Functionen vorkommenden Körpern beruhend, von der unsrigen gründlich verschieden ist.

Es sei im Grundkörper k eine Zahlengruppe $\mathfrak{n}$ nach dem Modul $\mathfrak{m}$ vorgelegt, welche eine Idealengruppe vom Index h erzeugen möge; ferner sei $\mathfrak{K}$ ein Oberkörper von k vom Relativgrade n, welcher aber nicht als relativ normal vorausgesetzt wird. Dann heisst $\mathfrak{K}$ nach Weber Classenkörper für die Zahlengruppe $\mathfrak{n}$, wenn die folgenden Bedingungen erfüllt sind:

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1) Vgl. weiter unten, § 32.

2) Dieses Ergebnis ist durch direkte Rechnung hergeleitet in der Abhandlung: T. Takagi, Über die im Bereiche der rationalen complexen Zahlen Abel'schen Zahlkörper, diese Journal, vol. 19. (1903) Vgl. daselbst S. 25, wo jedoch ein Fehler zu corrigiren ist: es soll statt $(1+i)^f\equiv 1$, die richtige: $(1+i)^f\equiv i^a$ zu setzen.

3) Lehrbuch der Algebra, III. S. 607; Vgl. auch Über Zahlengruppen usw. Math. Ann. Bd. 49, S. 87.

1) *Alle Primideale ersten Grades von* k, *die in* H *enthalten sind, zerfallen in* $\mathfrak{K}$ *in ein Produkt von lauter Primidealen ersten Grades.*

2) *Kein Primideal ersten Grades von* $\mathfrak{K}$ *geht in ein Primideal von* k *auf, welches nicht in* H *enthalten ist.*

In den beiden Forderungen 1) und 2) wird eine endliche Anzahl Ausnahme zugelassen, die dann als Factor in den Modul m hingenommen werden, weil von den in den Modul aufgehenden Primidealen von k überhaupt abgesehen werden.

Auf dieser Definition gestützt, beweist Weber[1] die folgenden Tatsachen:

3) Es ist $n \geqq h$.

4) Für ein gegebenes H, kann es nicht mehr als einen Classenkörper $\mathfrak{K}$ geben.

5) $\mathfrak{K}$ ist relativ normal in Bezug auf k.

Ferner spricht er die Vermutung aus:

6) Für jeden Classenkörper $\mathfrak{K}$ ist $n = h$.[2]

Es sei nun K der Classenkörper (in unserem Sinne) für die Classengruppe H. Da die Forderung 2) in unserer Definition des Classenkörpers enthalten ist, und da nach Satz 30 auch 1) erfüllt ist, so ist die Existenzfrage[3] für den Körper $\mathfrak{K}$ nach Satz 23 gelöst, und zwar wie aus 4) folgt, mit der Eindeutigkeit der Lösung. Ferner ist die Vermutung 6) bestätigt, und das Prädicat in 5) zu „*relativ Abelsch*" precisirt. Nachträglich folgt noch aus Satz 30, dass für alle nicht in dem Führer der Classengruppe H aufgehenden Primideale von k ohne Ausnahme die Bedingung 1) erfüllt sind.

In der Weber'schen Definition des Classenkörpers $\mathfrak{K}$ ist die Forderung versteckt, dass $\mathfrak{K}$ relativ normal in Bezug auf k ist, eine Forderung, die von der Classeneinteilung in k unabhängig ist. In der Tat, besagen 1) und 2), dass überhaupt jedes Primideal ersten Grades von k, welches bei der Zerlegung in $\mathfrak{K}$ ein

1) Lehrbuch, III. S. 607–611.

2) In der S. 102 citirten Abhandlung, nimmt Weber diese Beziehung als eine Forderung in der Definition des Classenkörpers auf.

3) Eine Frage, in die Weber nicht eingeht, indem er sich nur mit den von der Theorie der elliptischen Functionen gelieferten actuell vorhandenen Körpern beschäftigt.

Primideal ersten Grades von $\mathfrak{K}$ unter den Factoren aufweist. notwendig in lauter Primideale ersten Grades von $\mathfrak{K}$ zerfallen muss; dies ist aber ein Criterium dafür, dass $\mathfrak{K}$ relativ normal in Bezug auf k ist. Denn aus dieser Voraussetzung folgt

$$\prod^{\mathfrak{P}} \frac{1}{1-\mathrm{N}(\mathfrak{P})^{-s}} = \left(\prod^{\mathfrak{p}_0} \frac{1}{1-\mathrm{N}(\mathfrak{p}_0)^{-s}}\right)^n,$$

wo das Product links auf alle Primideale ersten Grades von $\mathfrak{K}$, und das Product rechts auf alle Primideale ersten Grades von k, welche in n Primideale ersten Grades von $\mathfrak{K}$ zerfallen, erstreckt wird, und wo N das Zeichen für die in dem bezüglichen Körper genommene absolute Norm ist. Daher ist, wenn

$$P_0(s) = \prod^{\mathfrak{p}_0} \frac{1}{1-\mathrm{N}(\mathfrak{p}_0)^{-s}}$$

gesetzt wird,

$$\mathrm{Lim}\,(s-1)^{\frac{1}{n}}\; P_0(s) = a \qquad (1)$$

endlich und von Null verschieden, wenn sich der reelle Veränderliche s abnehmend der Grenze 1 zustrebt.

Ist nun $\mathfrak{K}'$ ein beliebiger mit $\mathfrak{K}$ relativ conjugirter Körper, dann zerfallen alle Primideale $\mathfrak{p}_0$ in lauter Primideale ersten Grades in $\mathfrak{K}'$, welche letztere mit einer endlichen Anzahl Ausnahmen alle Primideale ersten Grades von $\mathfrak{K}'$ erschöpfen. Gleiches gilt daher von dem zusammengesetzten Körper $\mathfrak{K}\mathfrak{K}'$, für welchen also die entsprechende Relation (1) bestehen muss, demzufolge der Relativgrad von $\mathfrak{K}\mathfrak{K}'$ notwendig gleich n ist. Daher fällt $\mathfrak{K}$ mit $\mathfrak{K}'$ zusammen, ist folglich relativ normal in Bezug auf k.

Unter Hervorhebung dieser Forderung kommt die Weber'sche Definition des Körpers $\mathfrak{K}$ auf das folgende hinaus: Ein relativ normaler Körper $\mathfrak{K}$/k soll in dem zu Beginn des § 4. erläuterten Sinne der Classengruppe H zugeordnet sein (Weber'sche Bedingung 2); in Bezug auf diesen Körper $\mathfrak{K}$ und diese Classengruppe H soll das in Satz 30 ausgesprochene Zerlegungsgesetz gelten (Weber'sche Bedingung 1). Wie oben bemerkt, folgt aus diesen Bedingungen die Uebereinstimmung des Körpers $\mathfrak{K}$ mit unseren Classenkörper für $\mathfrak{n}$. Wir sind aber umgekehrt aus dem Zusammenfallen des Körpergrades und des Gruppenindex als Definition des Classenkörpers ausgegangen und durch eine Reihe von Schlüssen an den Zerlegungssatz gelangt. Für den Existenzbeweis hat dieser Weg als eine grosse Erleichterung erwiesen. Immerhin gibt die Weber'sche Definition ein Criterium für den relativ Abel'schen Körper, welches sich für die Anwendung auf die Theorie der complexen Multiplication besonders eignet. Wir wollen dieses Criterium noch als einen Satz aussprechen.

Satz 32. *Wenn* K *relativ normal in Bezug auf* k *ist, und wenn alle in einer Classengruppe* $\mathfrak{n}$ *von* k *enthaltenen Primideale ersten Grades, und nur diese, wieder in die Primideale ersten Grades in* K *zerfallen, dann ist* K *relativ Abel'sch in Bezug auf* k, *und der Relativgrad von* K *stimmt mit dem Index der Classengruppe* $\mathfrak{n}$ *überein.*

Zum Schluss sei noch das folgende bemerkt. Sei immer K/k relativ normal vom Relativgrade n, und die Classen von k nach einem Modul $\mathfrak{m}$ definirt. Der Inbegriff aller Classe von k, die ein Primideal enthält, welches in die Primideale ersten Grades in K zerfällt, bildet eine Classengruppe $\mathfrak{n}$. Denn sind c und c' zwei beliebige dieser Classen, dann enthält die Classe cc' gewiss ein Ideal $\mathfrak{j}$, welches Relativnorm eines Ideals $\mathfrak{J}$ von K ist. Denkt man sich nun die Classen von K auch nach dem Modul $\mathfrak{m}$ definirt, so enthält die Classe von K, welche eben das Ideal $\mathfrak{J}$ enthält, nach Satz 5, ein Primideal ersten Grades $\mathfrak{P}$, derart, dass $\mathfrak{P}=A\mathfrak{J}$, wo $A\equiv 1$, ($\mathfrak{m}$). Hieraus folgt: $\mathfrak{j}=\mathfrak{N}(A\mathfrak{P})=\alpha\mathfrak{p}$, wo $\alpha=\mathfrak{N}(A)\equiv 1$, ($\mathfrak{m}$), $\mathfrak{p}=\mathfrak{N}(\mathfrak{P})$, wenn $\mathfrak{N}$ die Relativnorm in Bezug auf k bedeutet. Daher enthält die Classe cc' auch ein Primideal ersten Grades von k.

Der Index h dieser Classengruppe $\mathfrak{n}$ hängt von der Wahl des Moduls $\mathfrak{m}$; nur bleibt stets $h\leqq n$ (Satz 4). Erreicht nun h für einen gewissen Modul $\mathfrak{m}$ die obere Grenze n, dann ist K relativ Abel'sch, er ist der Classenkörper für $\mathfrak{n}$. Wird dagegen die obere Grenze n nie erreicht, dann sei $\mathfrak{n}$ diejenige offenbar eindeutig bestimmte Classengruppe, bei der der Index h den möglichst grossen Wert hat. Dann ist der Classenkörper für $\mathfrak{n}$ der grösste relativ Abel'sche Körper, welcher in K enthalten ist; K selbst ist folglich nicht relativ Abel'sch. In diesem Falle müssen also in п unendlichviele Primideale enthalten sein, welche in K nicht in die Primideale ersten Grades zerfallen. Mit andern Worten, *die Primideale von* k, *welche in einem relativ normalen aber nicht relativ Abel'schen Oberkörper in die Primideale des ersten Grades zerfallen, lassen sich nicht durch eine Congruenzbedingung characterisiren, wie sie in unseren bisherigen Betrachtungen zu Grunde gelegt worden ist.*

CAPITEL V.

Anwendung auf die Theorie der complexen Multiplication der elliptischen Functionen.

§. 27.

Absolut Abel'scher Zahlkörper.

Wenn der Grundkörper k der natürliche ist, dann ist der vollständige Classenkörper K(m) derjenige Zahlengruppe $o(m)$ zugeordnet, welche aus den positiven Zahlen a besteht, die der Congruenz

$$a\equiv 1, \ (m)$$

genügen. Er ist also von der Ordnung $\varphi(m)$. Der Führer für den Körper K(m) ist m, ausgenommen der Fall, wo $m=2m'$, und m' ungerade ist, wo K(m)=K(m'), und der Führer für denselben gleich m' ist.

Der Körper K(m) ist der Kreisteilungskörper, welcher durch die primitive m^{te} Einheitswurzel erzeugt wird. Denn sei ζ eine solche, und $\mathfrak{P}$ ein Primideal erstens Grades des Kreisteilungskörpers, welches in die rationale Primzahl p aufgehen mag. Dann ist notwendig

$$\zeta^p\equiv\zeta, \ (\mathfrak{P}),$$

also, von einer endlichen Anzahl der in die Zahlen der Form $1-\zeta^a$ aufgehenden $\mathfrak{P}$ abgesehen,

$$p\equiv 1, \ \ (m).$$

Der Kreisteilungskörper ist daher der durch die Zahlengruppe $\mathfrak{o}(m)$ definirten Idealengruppe von k zugeordnet. Berücksichtigt man daher nur die Tatsache, dass der Kreisteilungskörper höchstens von der Ordnung $\varphi(m)$ sein kann, so folgt hieraus nach § 26. die Übereinstimmung desselben mit $K(m)$ und somit auch die Irreducibilität der Kreisteilungsgleichung $\varphi(m)^{\text{ten}}$ Grades für ζ.[1)]

Wenn a, b relativ prime ganze rationale Zahlen sind, dann ist $K(ab)$ aus $K(a)$ und $K(b)$ zusammengesetzt:

$$K(ab) = K(a).K(b), \tag{1}$$

weil die Gruppe $\mathfrak{o}(ab)$ die Durchschnitt der Gruppen $\mathfrak{o}(a)$, $\mathfrak{o}(b)$ ist. Daher lassen sich alle Abel'sche Körper auf die Körper

$$K(p^n)$$

zurückführen, wenn p natürliche Primzahlen, und n positive ganzzahlige Exponenten bedeutet.

Wenn von den Zahlengruppen $\mathfrak{o}(m)$ die Vorzeichenbedingung aufgehoben wird, dann erhält man eine Idealengruppe vom Index $\frac{1}{2}\varphi(m)$. Daher ist $K(m)$ imaginär, enthält aber einen reellen Körper vom halben Grade, welcher durch $\cos\frac{2\pi}{m}$ erzeugt wird. Bezeichnen wir denselben mit $K_0(m)$, dann gelten für diesen das Compositionsgesetz (1) nicht mehr. Denn der zusammengesetzte Körper $K_0(a).K_0(b)$ ist der Zahlengruppe zugeordnet, deren Zahlen den Congruenzen genügen:

$$x \equiv 1, \quad (a), \quad \equiv \pm 1, \ (b),$$

oder

$$x \equiv -1, \ (a), \quad \equiv \pm 1, \ (b).$$

Diese Gruppe ist daher als eine Untergruppe vom Index 2 in der Zahlengruppe für $K_0(ab)$ enthalten, welcher folglich relativ quad-

1) Vgl. H. Weber, Lehrbuch, II. (2. Aufl.) S. 728.

ratisch in Bezug auf $K_0(a).K_0(b)$ ist.[1] Sind aber a, b ungerade und relativ prim, dann ist, wie man leicht einsieht,

$$K_0(4ab)=K_0(4a).K_0(4b).$$

Alle reelle Abel'sche Körper lassen sich daher auf die Körper

$$K_0(2^n), \qquad K_0(4p^n)$$

zurückführen. Diese sind cyclisch vom Grade 2^{n-2}, $\varphi(p^n)$, und bez. durch $\sin\frac{2\pi}{2^n}$, und $\sin\frac{2\pi}{p^n}$ erzeugt.

Ich habe diese an sich triviale Tatsache erwähnt, weil sie ein gewisses Analogon in der Theorie der complexen Multiplication der elliptischen Function hat, welches dort eine bedeutende Rolle spielen wird.

§. 28.

Relativ Abel'sche Oberkörper eines imaginären quadratischen Körpers.

Nebst dem Körper der rationalen Zahlen zeichnen sich die imaginären quadratischen dadurch aus, dass sich die relativ Abel'schen Oberkörper derselben auf gewisse von den Primidealpotenzen im Grundkörper abhängende elementare Körper zurückführen lassen.

Es sei k ein imaginärer quadratischer Körper von der Discriminante Δ, $\mathfrak{m}$ ein beliebiges Ideal in k, dann ist der vollständige Classenkörper $K(\mathfrak{m})$ zum Modul $\mathfrak{m}$ vom Relativgrade

$$\frac{\Phi(\mathfrak{m})h}{w},$$

wo h die Classenzahl von k im absoluten Sinne, Φ die Euler'sche Function in k, und w die in § 23 mit $(\text{E}:\text{E}_0)$ bezeichnete Zahl, hier also die Anzahl der nach $\mathfrak{m}$ incongruenten Einheiten von k bedeutet; es ist demnach,

1) Ausgenommen der Fall, wo a oder $b=2$ ist.

wenn $\Delta < -4$,

$w=2$, im allgemeinen,

$w=1$, wenn $\mathfrak{m}$ in 2 aufgeht;

wenn $\Delta = -4$,

$w=4$, im allgemeinen,

$=2$, wenn $\mathfrak{m}=(2)$,

$=1$, ween $\mathfrak{m}=(1+i)$ oder $\mathfrak{m}=(1)$;

wenn $\Delta = -3$,

$w=6$, im allgemeinen,

$=3$, wenn $\mathfrak{m}=(2)$,

$=2$, wenn $\mathfrak{m}=(\sqrt{-3})$,

$=1$, wenn $\mathfrak{m}=(1)$.

Das Ideal $\mathfrak{m}$ ist nicht notwendigerweise der Führer für die Classengruppe, welche dem Körper $K(\mathfrak{m})$ zugeordnet ist. Ist aber $\mathfrak{f}$ der Führer, so muss, weil $\mathfrak{f}$ Teiler von $\mathfrak{m}$ ist, $K(\mathfrak{f})$ in $K(\mathfrak{m})$ enthalten sein, und da $K(\mathfrak{f})$ der umfassendste Classenkörper für den Modul $\mathfrak{f}$ ist, so ist notwendig $K(\mathfrak{f})=K(\mathfrak{m})$: also

$$\frac{\Phi(\mathfrak{m})}{\Phi(\mathfrak{f})}=\frac{w_{\mathfrak{m}}}{w_{\mathfrak{f}}},$$

wo $w_{\mathfrak{m}}$ die oben angegebene Bedeutung hat. Im folgenden geben wir die Tabelle für sämtliche Fälle, wo $\mathfrak{m}$ nicht mit $\mathfrak{f}$ zusammenfällt. Darin werden mit $\mathfrak{p}$, $\mathfrak{p}'$ und $\mathfrak{q}$, $\mathfrak{q}'$ die Primideale ersten Grades von k bezeichnet, welche bez. in 2 und 3 aufgehen, so dass

$$(2)=\mathfrak{p}^2 \text{ oder} =\mathfrak{p}\mathfrak{p}'; \quad (3)=\mathfrak{q}^2 \text{ oder} =\mathfrak{q}\mathfrak{q}'.$$

$\Delta\equiv 0,\ (4)$: $(2)=\mathfrak{p}^2$.

$K(\mathfrak{m})=K(\mathfrak{p}\mathfrak{m})$, wenn $\mathfrak{m}$ ungerade ist.

$k=K(1)=K(\mathfrak{p})=K(\mathfrak{q})=K(\mathfrak{p}\mathfrak{q})$

$K(2)=K(2\mathfrak{p})=K(2\mathfrak{q})$.

$\Delta\equiv 1,\ (8)$: $(2)=\mathfrak{p}\mathfrak{p}'$.

$K(\mathfrak{m})=K(\mathfrak{p}\mathfrak{m}),=K(\mathfrak{p}'\mathfrak{m}),=K(2\mathfrak{m})$,

wo $\mathfrak{m}$ prim bez. zu $\mathfrak{p}$, $\mathfrak{p}'$, 2 ist.

$k=K(1)=K(P)=K(\mathfrak{q})=K(\mathfrak{p}\mathfrak{q})=K(\mathfrak{p}'\mathfrak{q})=K(2\mathfrak{q})$,

wo P die 7 eigentlichen Teiler von 4 bedeutet.

$\Delta\equiv 5,\ (8)$:

$$k=K(1)=K(\mathfrak{q}).$$
$$K(2)=K(2\mathfrak{q}).$$

$\Delta=-4$: $\mathfrak{p}=(1+i)$

$$K(\mathfrak{m})=K(\mathfrak{p}\mathfrak{m}),\qquad \mathfrak{m},\ \text{ungerade}$$
$$k=K(1)=K(\mathfrak{p}^n)=K(1\pm 2i),\qquad 0\leqq n\leqq 3.$$

$\Delta=-3$: $\mathfrak{q}=(\sqrt{-3})$

$$k=K(1)=K(\mathfrak{q})=K(\mathfrak{q}^2)=K(2)=K(2\mathfrak{q})=K(2\pm\sqrt{-3})$$

Sind nun $\mathfrak{a}, \mathfrak{b}$ relativ prim, dann ist der aus $K(\mathfrak{a})$ und $K(\mathfrak{b})$ zusammengesetzte Körper der Classengruppe in k zugeordnet, welche aus den monomischen Idealen (α) besteht, wo für α eine Zahl gesetzt werden kann, die den Bedingungen

$$\left.\begin{aligned}\alpha&\equiv\varepsilon_1,\ (\mathfrak{a}),\\ &\equiv\varepsilon_2,\ (\mathfrak{b}),\end{aligned}\right\}\quad\text{oder}\quad\left.\begin{aligned}\alpha&\equiv 1,\ (\mathfrak{a}),\\ &\equiv\varepsilon,\ \ (\mathfrak{b})\end{aligned}\right\}$$

genügen, wenn mit ε, ε_1, ε_2 beliebige Einheiten von k bezeichnet werden. Für den Körper $K(\mathfrak{ab})$ dagegen müssen $\varepsilon_1\equiv\varepsilon_2$ oder $1\equiv\varepsilon$ sein. Es gilt demnach

Satz 33. *Wenn* $\mathfrak{a}, \mathfrak{b}$ *relativ prime Ideale in einem imaginären quadratischen Körper sind, dann ist, abgesehen von gewissen trivialen speciellen Fällen, der aus* $K(\mathfrak{a})$ *und* $K(\mathfrak{b})$ *zusammengesetzte Körper als echter Unterkörper in* $K(\mathfrak{ab})$ *enthalten. Der Relativgrad von* $K(\mathfrak{ab})$ *in Bezug auf* $K(\mathfrak{a})K(\mathfrak{b})$ *ist* $\frac{w_{\mathfrak{a}}w_{\mathfrak{b}}}{w_{\mathfrak{ab}}}$, *wo* $w_{\mathfrak{m}}$ *die in S 109 erläuterte Bedeutung hat.* (Wenn von den speciellen Fällen: $\Delta=-4$ und $\Delta=-3$ abgesehen wird, ist dieser Relativgrad gleich 2, ausser wenn $\mathfrak{a}$ oder $\mathfrak{b}$ in 2 aufgeht).

Dagegen ist, wenn $\mathfrak{a}, \mathfrak{b}, \mathfrak{c}$ relativ prim sind und $\mathfrak{a}$ nicht in 2 aufgeht (für $\Delta=-3$, auch noch nicht gleich $(\sqrt{-3})$ ist)

$$K(\mathfrak{abc})=K(\mathfrak{ab})\,K(\mathfrak{ac}).$$

Denn der zusammengesetzte Körper $K(\mathfrak{ab})$. $K(\mathfrak{ac})$ ist der Zahlengruppe zugeordnet, welche durch das Congruenzensystem

$$\alpha\equiv 1,\ (\mathfrak{ab}),\qquad \equiv\varepsilon,\ (\mathfrak{ac})$$

definirt wird, wo ε eine Einheit von $\mathfrak{k}$ bedeutet. Es muss daher

$$1\equiv\varepsilon,\quad(\mathfrak{a}),$$

und wegen der dem Ideale $\mathfrak{a}$ auferlegten Beschränkung

$$\varepsilon=1,$$

folglich

$$\alpha\equiv 1,\quad(\mathfrak{a}\mathfrak{b}\mathfrak{c}).$$

Um mich bestimmt auszudrücken und in Hinsicht auf die Beziehung auf die Theorie der complexen Multiplication der elliptischen Functionen, setze ich $\mathfrak{a}=\mathfrak{l}$, wo $\mathfrak{l}$ ein *in 2 aufgehendes Primideal von* $\mathfrak{k}$ bedeutet, und

$$e=3\quad\text{oder}\quad 2,$$

jenachdem

$$\Delta\equiv 0\quad\text{oder}\quad 1,\ (4),$$

so dass $\mathfrak{l}^e$ nicht in 2 aufgeht. Dann ist, wenn $\mathfrak{l}, \mathfrak{m}, \mathfrak{m}_1, \mathfrak{m}_2,\ldots\ldots$ zu je zweien relativ prim sind

$$K(\mathfrak{m}_1\mathfrak{m}_2\cdots)<K(\mathfrak{l}^e\mathfrak{m}_1).\,K(\mathfrak{l}^e\mathfrak{m}_2)\cdots\cdots=K(\mathfrak{l}^e\mathfrak{m}_1\mathfrak{m}_2\cdots),$$

$$K(\mathfrak{l}^n\mathfrak{m})=K(\mathfrak{l}^n).\,K(\mathfrak{l}^e\mathfrak{m}),\qquad(n\geqq e)$$

wo zur Abkürzung mit $K<K'$ das „Enthaltensein von K als echtem Teil in K'" angedeutet wird. Daher folgt

Satz 34. *Jeder relativ Abel'sche Oberkörper von* $\mathfrak{k}$ *lässt sich zurückführen auf die Classenkörper* $K(\mathfrak{l}^n)$, $K(\mathfrak{l}^e\mathfrak{m})$, *wo* $\mathfrak{m}$ *Potenz eines von* $\mathfrak{l}$ *verschiedenen Primideals bedeutet.*

Bedeutet p eine ungerade rationale Primzahl, dann kann man auch mit den Classenkörpern der folgenden Typen auskommen:

$$K(2^n),\qquad K(p^n),\qquad K(4p),$$

wie man leicht einsehen wird, wenn man sich erwägt, dass

$$K(4p^n)=K(p^n).K(4p).$$

§. 29.

Der durch den singulären Wert der elliptischen Modulfunction erzeugte Ordnungskörper.

Ist ω eine quadratische Irrationalzahl von k, welche der primitiven quadratischen Gleichung von der Discriminante $D=\Delta m^2$:

$$A\omega^2+B\omega+C=0,$$

$$(D=\Delta m^2=B^2-4AC)$$

genügt, also eine ganze oder gebrochene Zahl des Ringes mit dem Führer m, dann entsteht, wenn dem Grundkörper k ein singulärer Wert der Modulfunction: $J(\omega)$ adjungirt wird, ein relativ Abel'scher Körper in Bezug auf k, welcher nach H. Weber der *Ordnungskörper für den Führer m* genannt wird. Wir wollen ihn mit $\mathrm{M}(m)$ bezeichnen. Derselbe ist der Classenkörper für die Idealengruppe, welche durch die Zahlen α erzeugt wird, die nach dem Modul m mit rationalen Zahlen r congruent sind:[1]

$$\alpha\equiv r, \quad (m).$$

Daher ist $\mathrm{M}(1)$ der Classenkörper im absoluten Sinne; allgemein ist $\mathrm{M}(m)$ der Ringclassenkörper für den Ring mit dem Führer m.

Der Körper $\mathrm{M}(m)$ ist vom Relativgrade

$$\frac{\psi(m)}{w_0}h,$$

wo

h die Classenzahl von k im absoluten Sinne,

$\psi(m)=\dfrac{\Phi(m)}{\varphi(m)}=m\overset{p}{\Pi}\left(1-\dfrac{\left(\dfrac{\Delta}{p}\right)}{p}\right)$, wenn mit Φ, φ die Euler'schen Functionen bez. in k und im Körper der rationalen Zahlen bezeichnet werden, und das Product $\overset{p}{\Pi}$ auf alle in m aufgehenden natürlichen Primzahlen erstreckt wird,

1) H. Weber, Lehrbuch, III. §. 122.

$$\begin{aligned} w_0 &= 1, \quad \text{im allgemeinen,} \\ &= 2, \quad \text{wenn} \quad \varDelta = -4, \\ &= 3, \quad \text{wenn} \quad \varDelta = -3. \end{aligned}$$ [1]

Der Führer m des Ringes ist begrifflich verschieden von dem Führer der Classengruppe, welcher der Körper $\mathrm{K}(m)$ zugeordnet ist, wie wir ihn in § 2 definirt haben. Diesen letzteren bezeichnen wir mit $\mathfrak{f}$. Es ist wichtig, denselben für $\mathrm{M}(m)$ zu bestimmen.

Da $\mathrm{M}(m)$ jedenfalls Classenkörper nach dem Modul m ist, so ist $\mathfrak{f}$ ein Teiler von m, und wie aus der Natur der zugehörigen Classengruppe ersichtlich, ein invariantes Ideal von k. Wir setzen

$$m = \mathfrak{f}\mathfrak{a} = fa, \tag{1}$$

wo f die kleinste durch $\mathfrak{f}$ teilbare natürliche Zahl bedeutet. Dann muss durch jede Zahl γ von k, die der Congruenz

$$\gamma \equiv 1, \quad (\mathfrak{f}) \tag{2}$$

genügt, auch die andere:

$$\gamma \equiv r\varepsilon, \quad (m) \tag{3}$$

befriedigt werden, wenn r eine rationale Zahl und ε eine Einheit von k ist. Da im allgemeinen $\varepsilon = \pm 1$, so ersetzen wir (3) durch

$$\gamma \equiv r, \quad (m). \tag{4}$$

Vergleicht man die Anzahlen der nach m incongruenten Lösungen von (2) und (4) mit einander, so erhält man

$$\mathrm{N}(\mathfrak{a}) = a.$$

Da aber nach (1) $\mathfrak{a}$ durch a teilbar ist, so folgt hieraus $\mathfrak{a} = 1$, also ist im allgemeinen $\mathfrak{f} = m$.

In dem speciellen Falle: $\varDelta = -4$, sind noch in (3) die Werte $\varepsilon = \pm i$ zu berücksichtigen; weil aber nach (2), (3) $1 \equiv r\varepsilon$ $(\mathfrak{f})$ so kommen nur die Möglichkeiten: $\mathfrak{f} = (1)$ und $\mathfrak{f} = (1+i)$ in Betracht.

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1) H. Weber, Lehrbuch, III. S. 366. Für $m=1$ ist der Relativgrad immer gleich h, also ist $w_0 = 1$ zu setzen.

Da K(1+i)=K(1), so kann (1+i) überhaupt nicht als ein Führer der Classengruppe auftreten. Daher bleibt nur noch ein Fall: $\mathfrak{f}=(1)$ zu untersuchen übrig. In diesem Falle, muss offenbar M(m)=K(1)=k, also

$$\psi(m)=2,$$

woraus als der einzig mögliche Fall, $m=2$ sich ergibt.

In dem zweiten speciellen Falle: $\Delta=-3$, erhält man durch genau dieselbe Überlegung die Bedingung: $\mathfrak{f}=1$. M(m)=k, woraus

$$\psi(m)=3,$$

so dass man erhält: $m=2$ oder $m=3$.

Daher haben wir nach § 24

Satz 35. *In die Relativdiscriminante von* M(m) *gehen alle und nur die Primideale von* k *auf, welche in* m *aufgehen: ausgenommen sind nur die drei Fälle, wo* M(m) *mit dem Grundkörper* k *zusammenfällt:*

$$\Delta=-4, \quad m=2;$$

$$\Delta=-3, \quad m=2 \text{ oder } 3.$$

Als ein Beispiel für die am Ende des § 26 gemachten Bemerkung behandeln wir noch kurz eine von H. Weber gelöste Aufgabe:

Alle in M(f) *enthaltenen absolut Abel'schen Körper zu finden.*

Es handelt sich darum, den grössten Abel'schen Körper zu bestimmen, welcher in dem (absolut) normalen Körper M(f) enthalten ist, der daher nach § 26 Classenkörper für die dort mit $\mathfrak{n}$ bezeichnete Gruppe in dem absoluten Rationalitätsbereich ist. Diese Classengruppe $\mathfrak{n}$ ist aber offenbar durch die rationalen Zahlen a definirt, welche Normen der Zahlen α von k sind, die nach f mit rationalen Zahlen r congruent ausfallen: also

$$\begin{cases} a>0, \\ a\equiv r^2, \quad (f) \\ a=\text{Normenrest nach } \Delta. \end{cases}$$ [1]

[1] Vgl. § 7.

Ist daher f_0 das kleinste gemeinsame Vielfache von f und Δ, dann soll a zunächst quadratischer Rest nach jeder in f_0 aufgehenden ungeraden Primzahl sein, und ausserdem noch in Bezug auf die in f_0 aufgehende Potenz von 2 die folgenden Bedingungen befriedigen:

1) wenn, $f_0 \equiv 4,\ (8)$, $\qquad a \equiv 1,\ (4)$;

2) wenn, $f_0 \equiv 0,\ (8)$, aber $f \not\equiv 0,\ (4)$, folglich $\Delta \equiv 0,\ (8)$,

$a =$ Normenrest nach 8,

$\equiv \pm 1,\ (8)$, wenn $\dfrac{\Delta}{4} \equiv 2,\ (8)$,

$\equiv 1, 3,\ (8)$, wenn $\dfrac{\Delta}{4} \equiv -2,\ (8)$;

3) wenn $f_0 \equiv 0,\ (8)$, und f wenigstens durch 4 teilbar,[1]

$a \equiv 1,\ (8)$.

4) wenn f_0 nur durch 2 teilbar ist, so ist a nur der irrelevanten Beschränkung unterworfen, ungerade zu sein.

Der gesuchte Abel'sche Körper ist demnach zusammengesetzt aus den unabhängigen quadratischen Körpern, die durch die folgenden Zahlen erzeugt werden können:[2]

$\sqrt{(-1)^{\frac{p-1}{2}}\, p}$, wo p die in f_0 aufgehenden ungeraden Primzahlen sind; und

1) $\sqrt{-1}$, wenn $f_0 \equiv 4,\ (8)$;

2) $\sqrt{\pm 2}$, wenn $f \not\equiv 0,\ (4)$ und $\Delta \equiv 0,\ (8)$, jenachdem $\dfrac{\Delta}{4} \equiv \pm 2,\ (8)$;

3) $\sqrt{-1}$ und $\sqrt{2}$, wenn $f \equiv 0,\ (8)$,

oder $f \equiv 4,\ (8)$ und $\Delta \equiv 0,\ (8)$.

1) Wenn f nur durch 4 teilbar ist, dann soll $a \equiv 1\ (4)$, und Normenrest nach 8 sein, sodass $a \equiv 5\ (8)$ ausgeschlossen ist.

2) Vgl. H. Weber, Lehrbuch, III. S. 619. R. Fueter, Math. Ann. 75, S. 183.

Endlich seien die folgenden den Modul der Jacobi'schen Functionen betreffenden Tatsachen angeführt, weil wir sie später einmal benutzen müssen.

Es sei ω eine quadratische Irrationalzahl des Körpers k, mit der zugehörigen Discriminante D, d.h. ω genüge einer primitiven quadratischen Gleichung mit ganzen rationalen Coefficienten

$$A\omega^2+B\omega+C=0, \tag{5}$$

wo

$$D=B^2-4AC=f^2\Delta,$$

wenn Δ die Discriminante des Körpers k bedeutet. (Demnach ist ω ein Quotient zweier Zahlen des Ringes mit dem Führer f, speciell ist $A\omega$ eine Zahl, die mit 1 eine Basis des Ringes bildet). Wir wollen die Wurzel der Gleichung (5) mit dem positiven imaginären Teil mit

$$\omega=\{A, B, C\}$$

bezeichnen; dann ist

$$\frac{\omega}{2}=\left\{4A,\ 2B,\ C\right\},\quad \left\{2A,\ B,\ \frac{C}{2}\right\},\quad \text{oder}\quad \left\{A,\ \frac{B}{2},\ \frac{C}{4}\right\}$$

also der Discriminante, $4D$, D, oder $\frac{D}{4}$ zugehörig, jenachdem $C\equiv 1$, (2), $C\equiv 2$, (4), oder $D\equiv 0$, $C\equiv 0$, (4).

Ist dann $\varkappa(\omega)$ der Modul der Jacobi'schen Function, und adjungirt man dem Körper k $\varkappa^2(\omega)$ oder $\varkappa(\omega)$, so ist nach Weber[1]

$$\mathrm{k}[\varkappa^2(\omega)]=\mathrm{M}(2f), \tag{6}$$

ferner ist

$$\mathrm{k}[\varkappa(\omega)]=\mathrm{M}(2f)\quad \text{oder}\quad \mathrm{M}(4f) \tag{7}$$

jenachdem C gerade oder ungerade ist.

Wendet man dieses Resultat auf $\varkappa\left(\frac{\omega}{2}\right)$, dann folgt mit Hülfe der Formel (der Gauss'schen Transformation)

$$\varkappa\left(\frac{\omega}{2}\right)=\frac{2\sqrt{\varkappa(\omega)}}{1+\varkappa(\omega)}$$

1) H. Weber, Lehrbuch, III, S. 505–507.

$$\mathrm{k}[\sqrt{\varkappa(\omega)}] = \mathrm{M}(2f), \quad \mathrm{M}(4f), \quad \mathrm{M}(8f), \tag{8}$$

jenachdem

$$C \equiv 0, (4), \qquad C \equiv 2, (4), \qquad C \equiv 1, (2).$$

Nun sind, wenn $D \equiv 5$, (8), A, C notwendig ungerade, in anderen Fällen kann man stets ein ω so bestimmen, das A ungerade und C gerade und zwar $C \equiv 0$, (4) wird, ausgenommen der Fall: $\Delta \equiv 0$, (4) und $f \equiv 1$, (2), wo notwendig $C \equiv 2$, (4) ausfällt. Unter dieser Voraussetzung folgt aus (6), (7), (8):

wenn $f \equiv 1$, (2), $\quad \Delta \equiv 0$, (4),

$$\mathrm{k}[\varkappa^2(\omega)] = \mathrm{k}[\varkappa(\omega)] = \mathrm{M}(2f); \quad \mathrm{k}[\sqrt{\varkappa}] = \mathrm{M}(4f); \tag{9}$$

wenn $f \equiv 1$, (2), $\quad \Delta \equiv 5$, (8),

$$\mathrm{k}[\varkappa^2] = \mathrm{M}(2f), \quad \mathrm{k}[\varkappa] = \mathrm{M}(4f); \quad \mathrm{k}[\sqrt{\varkappa}] = \mathrm{M}(8f); \tag{10}$$

wenn $f \equiv 0$, (2), $\quad \Delta \equiv 0$, (4) oder $\Delta \equiv 5$, (8),

oder wenn $\Delta \equiv 1$, (8), für beliebiges f,

$$\mathrm{k}[\varkappa^2] = \mathrm{k}[\varkappa] = \mathrm{k}[\sqrt{\varkappa}] = \mathrm{M}(2f). \tag{11}$$

§ 30.

Gleichzeitige Adjunction der singulären Moduln und der Einheitswurzeln.

Wenn der Ordnungskörper $\mathrm{M}(m)$ durch die Adjunction der primitiven m^{ten} Einheitswurzeln erweitert wird, so entsteht ein relativ Abel'scher Körper über k, den wir mit

$$\mathbf{M}(m)$$

bezeichnen wollen. Da $\mathrm{M}(m')$ in $\mathrm{M}(m)$ enthalten ist, wenn m' in m aufgeht, und ähnliches für die Kreisteilungskörper gilt, so ist das Gleichsetzen von dem Führer des Ordnungskörpers und dem Grad der zu adjungirenden Einheitswurzel offenbar keine wesentliche Beschränkung.

Der Körper $\mathbf{M}(m)$ ist der Classenkörper für die Idealengruppe, welche durch die Zahlen α definirt wird, die der Congruenz

$$\alpha \equiv r_0, \quad (m) \tag{1}$$

genügen, wo r_0 eine rationale Zahl bedeutet, derart, dass

$$r_0^2 \equiv 1, \quad (m). \tag{2}$$

Wenn von den in Satz 35 angegebenen drei trivialen Fällen abgesehen wird, ist m der Führer für den Classenkörper $\mathbf{M}(m)$.[1]

Der Relativgrad von $\mathbf{M}(m)$ ist, in der Bezeichnungsweise des § 29,

$$\frac{\Phi(m).h}{w.2^\rho}, \tag{3}$$

wo 2^ρ die Anzahl der nach m incongruenten Lösungen der Congruenz (2) bedeutet.

Wenn $m=p^n$ eine ungerade Primzahlpotenz ist, dann ist in (1) $r_0=\pm1$ zu setzen, so dass

$$\mathbf{M}(p^n)=\mathrm{K}(p^n). \tag{4}$$

Ebenso ist

$$\mathbf{M}(4)=\mathrm{K}(4); \tag{5}$$

dagegen ist, wenn $n\geqq3$

$$\mathbf{M}(2^n)<\mathrm{K}(2^n)<\mathbf{M}(2^{n+1}), \tag{6}$$

da dann noch die Werte $r_0=\pm1+2^{n-1}$ auftreten.

Wenn ferner a, b zwei beliebige relativ prime ganze rationale Zahlen sind, abgesehen von den Specialfällen $\varDelta=-4$, $\varDelta=-3$,

$$\mathbf{M}(ab)=\mathbf{M}(a).\ \mathbf{M}(b),$$

also insbesondere, wenn p eine ungerade Primzahl ist, nach (4) und (5)

$$\mathbf{M}(4p)=\mathbf{M}(4).\mathbf{M}(p)=\mathrm{K}(4)\mathrm{K}(p)<\mathrm{K}(4p),$$

und zwar gelangt man von $\mathbf{M}(4p)$ aus erst durch die Adjunction einer Quadratwurzel an $\mathrm{K}(4p)$, eine Tatsache, welche auch in den Speialfällen: $\varDelta=-4$, $\varDelta=-3$, ihre Geltung beibehält; in der Tat,

1) In Nichtübereinstimmung mit R. Fueter, vgl. Math. Ann, 75, S. 239. Vgl. auch T. Takenouchi, On the relatively Abelian corpus with respect to the corpus defined by a primitive cube root of unity, diese Journal, vol. 37. Art. 5 (S. 70). 1916.

$\mathsf{M}(4p)$ ist allgemein der Classengruppe zugeordnet, die durch die Zahlen α definirt ist, welche der Congruenz

$$\alpha \equiv 1,\ 1+2p,\quad (4p)$$

genügen.

Da anderseits $\mathsf{M}(m)$ nur dann $4p$ zum Führer hat, wenn $m=4p$, so ist $K(4p)$ niemals in einem Körper $\mathsf{M}(m)$ enthalten.

Man sieht hieraus, dass, von den in Satz 34 angegebenen elementaren Körpern, die beiden ersten Typen $K(2^n)$ und $K(p^n)$, nicht aber der letzte $K(4p)$ durch die singulären Moduln und die Einheitswurzeln zu erzeugen sind, dass um $K(4p)$ zu erhalten, weitere Ausziehung einer Quadratwurzel unumwendbar notwendig ist.[1]

Allgemeiner ist, wenn m (>2) eine ganze rationale Zahl ist, $K(m)$ Oberkörper von $\mathsf{M}(m)$ vom Relativgrade $2^{\rho-1}$, welche aus $\rho-1$ unabhängigen relativ quadratischen Körpern über $\mathsf{M}(m)$ zusammengesetzt werden kann; hierbei hat die Zahl ρ dieselbe Bedeutung wie oben in (3).

Das Ergebnis dieser Betrachtungen formuliren wir als

Satz 36. *Jeder in Bezug auf einen imaginären quadratischen relativ Abel'sche Zahlkörper vom ungeraden Relativgrade lässt sich durch Einheitswurzeln und singuläre Werte der Modulfunction $J(\tau)$ erzeugen. Gleiches gilt auch im Falle eines geraden Relativgrades, wenn die Relativdiscriminante keine anderen Primfactoren enthält, als solche, die in eine und dieselbe natürliche Primzahl aufgehen; im gegenteiligen Falle aber kann noch die Adjunction gewisser Quadratwurzeln notwendig werden, deren Anzahl im äussersten Falle bis zu der Anzahl der von einander verschiedenen, durch die Primfactoren der Relativdiscriminante teilbaren, rationalen Primzahlen ansteigt.*

Wie in den folgenden Paragraphen nachgewiesen werden soll, können alle relativ Abel'sche Oberkörper erzeugt werden, wenn man noch die Teilwerte der Perioden der Jacobi'schen Function $\mathrm{sn}(u)$ zu Hülfe nimmt.

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1) Eine zuerst von R. Fueter entdeckte Tatsache; vgl. Math. Ann. 75.

§ 31.

Ueber die complexe Multiplication der Jacobi'schen Function.

Um die zuletzt erwähnte Frage zu erledigen, betrachten wir die Teilungsgleichung der Jacobi'schen Function $\mathrm{sn}(u)$ mit einem singulären Modul $\varkappa(\omega)$ durch ein ungerades Ideal. Da es aber nicht in unserer Absicht liegt, die Theorie des Teilungskörpers für sich ausführlich zu entwickeln, so begnügen wir uns damit, nachzuweisen, dass der Elementarkörper $\mathrm{K}(4p)$ oder $\mathrm{K}(\mathfrak{f}^e\mathfrak{m})$ (vgl. §28) durch die Teilwerte von $\mathrm{sn}(u)$ erzeugt wird, indem wir das hierzu nötige Material aus dem Weber'schen Buche[1] entnehmen.

Sei

$$\omega = \{A,\ B,\ C\} \tag{1}$$

eine zur Stammdiscriminante Δ gehörige Irrationalzahl von k, so dass

$$\Delta = B^2 - 4AC,$$

und $[1, A\omega]$ eine Basis des Körpers k bildet.

Für die Function

$$S(v) = \sqrt{\varkappa}\,\mathrm{sn}(2Kv,\ \varkappa) = \frac{\vartheta_1(v \mid \omega)}{\vartheta_0(v \mid \omega)}$$

und einen ungeraden complexen Multiplicator μ, welcher dem Ringe mit dem Führer 2 angehört, also

$$\mu = a + b\omega, \tag{2}$$

wo a eine ungerade und b eine durch $2A$ teilbare ganze rationale Zahl bedeutet, besteht die folgende Multiplicationsformel:

$$\varepsilon S(\mu v) = \frac{\mathrm{A}(S)}{\mathrm{D}(S)}, \tag{3}$$

wo

$$S = S(v),$$

1) H. Weber, III, 23. Abschnitt, vgl. insbesondere S. 576–596.

und

$$\left.\begin{aligned}\mathrm{A}(S)&=A_1S+A_3S^3+\cdots\cdots+A_{m-2}+S^{m-2}+S^m,\\ \mathrm{D}(S)&=A_1S^{m-1}+A_3S^{m-3}+\cdots\cdots+A_{m-2}S+1\end{aligned}\right\}\qquad(4)$$

ganze ganzzahlige Functionen im Körper $k'=k(\varkappa)$ sind, und

$$m=\mathrm{N}(\mu)=\mu\mu',$$

ferner

$$\varepsilon=\pm1 \text{ oder } \pm i,$$

je nach der Beschaffenheit von μ nach dem Modul 4.

Es ist

$$\mathrm{A}(x)=\prod^{\rho}\left\{x-S\left(\frac{2\rho}{\mu}\right)\right\}=0$$

die *Teilungsgleichung zum Divisor* μ, deren Wurzeln die m Teilwerte

$$S\left(\frac{2\rho}{\mu}\right)\qquad(5)$$

sind, wo ρ ein vollständiges Restsystem nach μ durchläuft, allerdings unter der Voraussetzung, dass der Coefficient A in (1) ungerade und prim zu μ ist.[1)]

Es ist nun für unseren Zweck unerlässlich, den Coefficienten ε in der Weber'schen Formel (3) genau zu bestimmen, was wir dadurch erreichen, dass die Function $\mathrm{A}(S)$ durch die Thetafunction dargestellt wird.

Ist μ eine beliebige ganze Zahl von k, dann kann man setzen

$$\left.\begin{aligned}\mu&=a+b\omega,\\ \mu\omega&=c+d\omega,\end{aligned}\right\}\qquad(6)$$

wo a, b, c, d ganze rationale Zahlen sind, so dass

1) Für unseren Zweck genügt es schon, wenn wir ein für allemal annehmen: $A=1$.

$$\begin{vmatrix} a-\mu & b \\ c & d-\mu \end{vmatrix} = \mu^2-(a+d)\mu+ad-bc=0,$$

$$m=N(\mu)=\mu\bar{\mu}=ad-bc. \qquad (7)$$

Für die conjugirte Zahl $\bar{\mu}$ ergibt dann

$$\left.\begin{aligned} \bar{\mu}&=d-b\omega, \\ \bar{\mu}\omega&=-c+a\omega. \end{aligned}\right\} \qquad (8)$$

Ich setze nun

$$\Phi(v)=\alpha e^{\pi i b\mu v^2}\frac{\vartheta_1(\mu v)}{\vartheta_0(v)^m} \qquad (9)$$

wo für den constanten Coefficienten α noch zu verfügen ist. Für diese Function ergibt sich

$$\frac{\Phi(v+1)}{\Phi(v)}=(-1)^{a+b}e^{\pi i b(\mu-b\omega)}=(-1)^{a+b+ab},$$

$$\frac{\Phi(v+\omega)}{\Phi(v)}=e^{2\pi i v(b\mu\omega-d\mu+m)}\times(-1)^{c+d+m}e^{\pi i\omega(b\mu\omega-d^2+m)}$$

Nun ist nach (6), (7), (8)

$$b\mu\omega-d\mu+m=\mu(b\omega-d+\bar{\mu})=0,$$

$$\omega(b\mu\omega-d^2+m)=\omega(d\mu-d^2)=d(\mu\omega-d\omega)=cd,$$

so dass

$$\frac{\Phi(v+\omega)}{\Phi(v)}=(-1)^{c+d+cd+m}$$

So weit gilt unsere Formel für jede ganze Zahl μ von k. Ist nun μ wie in (2) eine ungerade Zahl aus dem Ringe mit dem Führer 2, dann ist

$$\left.\begin{aligned} a\equiv d\equiv 1, \quad b\equiv c\equiv 0, \quad (2), \\ m\equiv ad \quad (4), \end{aligned}\right\} \qquad (10)$$

und

$$\Phi(v+1) = -\Phi(v), \quad \Phi(v+\omega) = \Phi(v).$$

Demnach ist $\Phi(v)$ eine ganze Function von $S(v)$, und da sie dieselben Nullstellen (5) hat wie $S(\mu v)$, so kann man den constanten Factor α in (9) so bestimmen, dass

$$\mathrm{A}(S) = \Phi(v)$$

wird. Setzen wir $v=0$ und $v=\frac{\omega}{2}$, so erhalten wir nacheinander

$$A_1 = \frac{\alpha\mu}{\vartheta_0^{m-1}},$$

$$1 = \alpha e^{\frac{\pi i b\mu\omega^2}{4}} \left(\frac{\vartheta_1(\mu v)}{\vartheta_1'(v)^m}\right)_{v=\frac{\omega}{2}} = \frac{\alpha}{\vartheta_0^{m-1}} \times i^{c+d-m} e^{\frac{\pi i\omega}{4}(b\mu\omega - d^2 + m)}$$

$$= \frac{\alpha}{\vartheta_0^{m-1}} i^{c+d-m+\frac{cd}{2}}$$

Daher ist

$$A_1 = \mu i^{m-c-d-\frac{cd}{2}}$$

und für ε in (3) erhalten wir, indem wir $v=0$ setzen,

$$\varepsilon = i^{m-c-d-\frac{cd}{2}},$$

oder nach (10)

$$\varepsilon = (-1)^{\frac{a-1}{2}} i^{-\frac{c}{2}(d+2)}, \tag{11}$$

und speciell,

$$\text{wenn} \quad c \equiv 0, \ (4), \qquad \varepsilon = (-1)^{\frac{a-1}{2} + \frac{c}{4}}. \qquad (12$$

Da nach (6)

$$b\omega^2 + (a-d)\omega - c = 0,$$

so folgt aus (1)

$$\frac{b}{A}=\frac{a-d}{B}=\frac{-c}{C}=2b',$$

wo b' eine ganze Zahl ist, weil nach (2) b durch $2A$ teilbar ist.

Der in (12) angegebene Fall tritt daher ein, wenn für $\Delta\equiv 0, (4)$ und $\Delta\equiv 1, (8)$, ω so angenommen wird, dass C gerade ausfällt, was stets angeht, oder wenn für $\Delta\equiv 5, (8)$ die Zahl μ dem Ringe mit dem Führer 4 angehört, so dass b' gerade wird; in beiden Fällen ist

$$\varepsilon=(-1)^{\frac{a-1}{2}+\frac{b'c}{2}} \tag{13}$$

§ 32.

Ueber die arithmetische Natur des Teilungskörpers.

Es sei

$$\omega=\{A,\ B,\ C\} \tag{1}$$

eine zur Stammdiscriminante Δ gehörige Irrationalzahl von k, von der wir annehmen, dass A ungerade ist und C gerade, wenn $\Delta\equiv 0, (4)$ oder $\Delta\equiv 1, (8)$, so dass, wenn $\varkappa=\varkappa(\omega)$, $\mathrm{k}'=\mathrm{k}[\varkappa]$ gesetzt wird, nach § 29

$$\left.\begin{array}{lll} \text{(I)} & \mathrm{k}'=\mathrm{M}(2)=\mathrm{K}(2), & \text{wenn } \Delta\equiv 0,\ (4), \\ \text{(II)} & \mathrm{k}'=\mathrm{M}(2)=\mathrm{K}(1), & \quad ,, \quad\ \Delta\equiv 1,\ (8), \\ \text{(III)} & \mathrm{k}'=\mathrm{M}(4)=\mathrm{K}(4), & \quad ,, \quad\ \Delta\equiv 5,\ (8) \end{array}\right\} \tag{2}$$

und folglich k′ der Ringclassenkörper für den Ring

$$\text{R mit dem Führer 2, 1, 4 im Falle (I), (II), (III)} \tag{3}$$

ist.

Ferner sei $\mathfrak{m}$ ein beliebiges ungerades Ideal von k, $T(\mathfrak{m})$ der Teilungskörper, welcher entsteht, wenn dem Ordnungskörper k′ ein eigentlicher $\mathfrak{m}^{\text{ter}}$ Teilwert von $S(v)=\sqrt{\varkappa}\,\mathrm{sn}(u,\varkappa)$ adjungirt wird, und welcher relativ Abel'sch in Bezug auf k′ ist, von einem

Relativgrade, welcher höchstens gleich $\Phi(\mathfrak{m})$ ist. Es handelt sich darum, nachzuweisen, dass $T'(\mathfrak{m})$ auch relativ Abel'sch in Bezug auf k selbst ist, und vor allem die Classengruppe in k zu bestimmen, welcher $T'(\mathfrak{m})$ zugeordnet ist.

Wir bezeichnen durchweg mit ϖ eine ungerade Zahl vom Ringe $\mathfrak{R}$ in (3), welche ein Primideal ersten Grades von k erzeugt, mit Ausschluss einer endlichen Anzahl, die in $\mathfrak{m}$ oder in die Discriminante der $\mathfrak{m}$-teilungsgleichung von $S(v)$ in k' aufgehen, und wir setzen

$$p = N(\varpi).$$

Dann ist nach (3), (4), § 31

$$\varepsilon S(\varpi v) = \frac{A_1 S + A_3 S^3 + \cdots\cdots + A_{p-2} S^{p-2} + S^p}{A_1 S^{p-1} + A_3 S^{p-3} + \cdots\cdots + A_{p-2} S^2 + 1}, \tag{4}$$

wo ε die in (13), § 31 angegebene Bedeutung für $\mu = \varpi$ hat, und die Coefficienten $A_1, A_3, \ldots\ldots A_{p-2}$ durch ϖ teilbar sind.[1] Versteht man daher unter v in (4) einen eigentlichen $\mathfrak{m}^{\text{ten}}$ Teil der Periode von $S(v)$, so sind $S(v)$ und $S(\varpi v)$ Wurzel der $\mathfrak{m}$-teilungsgleichung, wenn, wie vorausgesetzt, ϖ nicht in $\mathfrak{m}$ aufgeht, und es folgt

$$\varepsilon S(\varpi v) \equiv S(v)^p, \quad (\varpi). \tag{5}$$

Wenn nun $\mathfrak{P}$ ein Primideal ersten Grades in $T'(\mathfrak{m})$ ist, welches mit einer endlichen Anzahl Ausnahme in ein ϖ aufgeht, so muss

$$S(v)^p \equiv S(v), \quad (\mathfrak{P}), \tag{6}$$

so dass nach (5)

$$\varepsilon S(\varpi v) \equiv S(v), \quad (\mathfrak{P}). \tag{7}$$

Da nach Voraussetzung $\mathfrak{P}$ nicht in die Discriminante der Teilungsgleichung aufgeht, so ist dies nur dann möglich, wenn

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1) H. Weber, l. c. S. 594; vgl. auch T. Takagi, On a fundamental property of the equation of division etc. Proceedings of the Tōkyō Math. Physical Soc., Ser. 2, vol. 7. S. 414.

$$\varepsilon S(\varpi v)=S(v) \tag{8}$$

d.h., wenn

$$\left.\begin{array}{lll}\varpi\equiv 1, & (\mathfrak{m}), & \varepsilon=1,\\ \varpi\equiv -1, & (\mathfrak{m}), & \varepsilon=-1.\end{array}\right\} \tag{9}$$

oder

Umgekehrt, wenn eine Zahl ϖ die Bedingung (9) erfüllt, und ist $\mathfrak{P}$ ein Primideal von $\mathrm{T}'(\mathfrak{m})$, welches in ϖ aufgeht, dann folgt nach (5), da (8) und somit (7) besteht, die Relation (6). Weil aber $S(v)$ den Relativkörper $\mathrm{T}'(\mathfrak{m})/\mathrm{k}'$ erzeugt, und für jede Zahl α in k'

$$\alpha^p\equiv\alpha, \quad (\varpi),$$

so ist für jede Zahl A von $\mathrm{T}'(\mathfrak{m})$

$$A^p\equiv A, \quad (\mathfrak{P}),$$

demnach ist $\mathfrak{P}$ ein Primideal ersten Grades in $\mathrm{T}'(\mathfrak{m})$.

Da $\varepsilon=\pm 1$ eine Congruenzbedingung für die Zahl ϖ nach einer Potenz von 2 als Modul bedeutet, so ist hiermit nach § 26 dargetan, dass der Körper $\mathrm{T}'(\mathfrak{m})$ relativ Abel'sch in Bezug auf k, und zwar derjenige Idealengruppe zugeordnet ist, welche durch die Zahlen α des Ringes $\mathfrak{r}$ erzeugt wird, die der Congruenzbedingung (9) genügen:

$$\left.\begin{array}{ll}\alpha\equiv\pm 1, & (\mathfrak{m})\\ \varepsilon=\pm 1, & \end{array}\right\} \tag{10}$$

Es ist nunmehr unser Ziel, diese Idealengruppe näher zu untersuchen; wie es sich herausstellen wird, ist der Index derselben gleich $\Phi(\mathfrak{m})h'$, wenn h' der Relativgrad von k'/k bedeutet, so dass sich nebenbei ergibt, dass die $\mathfrak{m}$-teilungsgleichung in k' irreducibel ist. Wir müssen aber fernerhin die zu Beginn des Artikels unterschiedenen drei Fälle einzeln in Betracht ziehen.

(1) $\Delta\equiv 0$, (4).

In diesem Falle, ist in (1) A ungerade, C gerade, folglich

$$C \equiv 2, \ (4).$$

Setzt man

$$\theta = A\omega,$$

so ist in k

$$(2) = \mathfrak{l}^2, \text{ wo } \mathfrak{l} = [2, \ \theta].$$

Für eine ungerade Zahl α im Ringe $\mathfrak{R}$ mit dem Führer 2:

$$\alpha = a + b\omega = a + 2b'\theta$$

wird nach (13), § 31, da $\frac{C}{2}$ ungerade ist

$$\varepsilon = (-1)^{\frac{a-1}{2} + b'}, \tag{11}$$

also $\varepsilon = 1$, dann und nur dann, wenn

$$a \equiv 1, \quad (4), \quad b' \equiv 0, \ (2),$$

oder

$$a \equiv -1, \ (4), \quad b' \equiv 1, \ (2),$$

Nach (10) kommt daher die Zahlengruppe

$$\left.\begin{aligned} &\alpha \equiv 1, \ (\mathfrak{m}), \\ &\alpha \equiv 1 \text{ oder } -1 + 2\theta, \ (\mathfrak{l}^4) \end{aligned}\right\} \tag{12}$$

in Betracht. Man sieht daher ein, dass

$$K(\mathfrak{l}^2\mathfrak{m}) < T'(\mathfrak{m}) < K(\mathfrak{l}^4\mathfrak{m}), \tag{13}$$

ohne dass $T'(\mathfrak{m})$ mit $K(\mathfrak{l}^3\mathfrak{m})$ zusammenfällt, welcher letztere der Zahlengruppe

$$\alpha \equiv 1, \ (\mathfrak{m}), \quad \alpha \equiv 1, \ 1 + 2\theta, \ (\mathfrak{l}^4)$$

zugeordnet ist.

Bezeichnet man nun mit $T_0(\mathfrak{m})$ denjenigen Körper, welcher aus k′ entsteht durch Adjunction der Quadrat $S(v)^2$ des mten Teilwertes von $S(v)$, oder, was auf dasselbe hinauskommt, von der Quadrat $\mathrm{sn}^2(u)$ des $\mathfrak{m}^{\text{ten}}$ Teilwertes von der Function $\mathrm{sn}(u)$ selbst, dann ist

$$\mathrm{T}_0(\mathfrak{m})=\mathrm{K}(\mathfrak{f}^2\mathfrak{m}), \tag{14}$$

weil für diesen die Bedingung $\varepsilon=1$ wegfällt.[1]

Um aber den Körper $\mathrm{K}(4\mathfrak{m})=\mathrm{K}(\mathfrak{f}^4\mathfrak{m})$ zu erhalten, hat man dem Körper $\mathrm{T}'(\mathfrak{m})$ noch $\sqrt{\varkappa}$ zu adjungiren, weil nach § 29, $\mathrm{k}[\sqrt{\varkappa}]=\mathrm{M}(4)$ der Zahlengruppe: $\alpha\equiv\pm1,\ (4)$ zugeordnet ist.

Der Körper $\mathrm{K}(4\mathfrak{m})$ ist relativ biquadratisch in Bezug auf $\mathrm{K}(2\mathfrak{m})$; er lässt sich zusammensetzen aus zwei relativ quadratischen Körpern über $\mathrm{K}(2\mathfrak{m})$, enthält folglich drei von einander verschiedenen relativ quadratischen Körper über $\mathrm{K}(2\mathfrak{m})$, welche bez. den Zahlengruppen

$$\begin{array}{llll} \alpha\equiv1,\ (\mathfrak{m}), & \alpha\equiv1, & -1+2\theta, & (\mathfrak{f}^4), \\ \alpha\equiv1,\ (\mathfrak{m}), & \alpha\equiv1, & -1, & (\mathfrak{f}^4), \\ \alpha\equiv1,\ (\mathfrak{m}), & \alpha\equiv1, & 1+2\theta, & (\mathfrak{f}^4) \end{array}$$

zugeordnet sind. Der erste ist $\mathrm{T}'(\mathfrak{m})$, der zweite entsteht aus $\mathrm{T}_0(\mathfrak{m})$ durch Adjunction von $\sqrt{\varkappa}$: der dritte, welcher $\mathrm{K}(\mathfrak{f}^5\mathfrak{m})$ ist, muss daher notwendig derjenige Körper $\mathrm{T}(\mathfrak{m})$ sein, welcher durch die Adjunction von dem *Teilwerte von* sn (u) *selbst*, (d.h $S(v)/\sqrt{\varkappa}$) entsteht:

$$\mathrm{T}(\mathfrak{m})=\mathrm{K}(\mathfrak{f}^5\mathfrak{m}). \tag{15}$$

Dieses merwürdige Ergebnis wollen wir noch auf einem directeren Weg herleiten. Da nach § 29

$$\mathrm{k}[\sqrt{\varkappa}]=\mathrm{M}(4),$$

so zerfällt ein Primideal (ϖ) von k, wo

$$\varpi=a+2b'\theta,$$

dann und nur dann in die Primideale ersten Grades in $\mathrm{k}(\sqrt{\varkappa})$, wenn

$$b'\equiv0,\ (2).$$

Hieraus ist aber zu schliessen, dass[2]

1) Vgl. Weber, l. c. S. 596.

2) Da sowohl $4\varkappa$ als auch $\frac{4}{\varkappa}$ ganze Zahlen sind, so enthält $\varkappa$ im Zähler und Nenner keinen ungeraden Idealfactor, vgl. Weber, l. c. S. 581.

$$\kappa^{\frac{p-1}{2}} \equiv (-1)^{\nu}, \quad (\varpi).$$

Daher folgt aus (5) und (11)

$$(-1)^{\frac{a-1}{2}} \operatorname{sn}(\varpi u) \equiv \operatorname{sn}(u)^{p}, \quad (\mathfrak{P}),$$

sodass nun für ϖ die Bedingung erhalten wird:

$$\left.\begin{aligned} \varpi = a + 2b'\theta &\equiv 1 \quad (\mathfrak{m}) \\ a &\equiv 1 \quad (4) \end{aligned}\right\}.$$

Da b' beliebig ist, so wird für die zugeordnete Zahlengruppe

$$\left.\begin{aligned} \alpha &\equiv 1, \quad (\mathfrak{m}), \\ \alpha &\equiv 1, \quad (\mathfrak{l}^3), \end{aligned}\right\}$$

wie zu beweisen war.

(II) $\Delta \equiv 1, \quad (8).$

Es empfiehlt sich in diesem Falle A ungerade und

$$C \equiv 0, \quad (4)$$

anzunehmen, was erreicht wird, wenn man nötigenfalls ω durch $\omega + 2$ ersetzt. Dann ist in k

$$(2) = \mathfrak{l}\mathfrak{l}', \text{ wo } \mathfrak{l} = [2, \theta], \quad \mathfrak{l}^2 = [4, \theta], \quad \mathfrak{l}' = [2, 1+\theta].$$

Es ist hier $k' = K(1)$, aber wenn verlangt wird, dass α ungerade, also prim zu $\mathfrak{l}$ und $\mathfrak{l}'$ sein soll, so ist

$$\alpha = a + b\omega = a + 2b'\theta,$$

demnach kommt nach (13) § 31, da $C \equiv 0$, (4),

$$\varepsilon = (-1)^{\frac{a-1}{2}}.$$

Daher ist $\varepsilon = 1$, dann und nur dann, wenn $a \equiv 1$, (4), d. h. aber, wenn

$$\alpha \equiv 1 \quad (\mathfrak{l}^2 \mathfrak{l}').$$

Man erhält somit

$$T'(\mathfrak{m}) = K(\mathfrak{l}^2 \mathfrak{l}' \mathfrak{m}) = K(\mathfrak{l}^2 \mathfrak{m}),^{(1)}$$

(1) Vgl. § 28.

und weil nach § 29 $\mathrm{k}[\sqrt{\varkappa}]=\mathrm{k}[\varkappa]$, so ist hier

$$\mathrm{T}(\mathfrak{m})=\mathrm{T}'(\mathfrak{m})=\mathrm{K}(\mathfrak{f}^2\mathfrak{m}). \tag{16}$$

Für $\mathrm{T}_0(\mathfrak{m})$ fällt die Bedingung: $\varepsilon=1$ weg, sodass $\mathrm{T}_0(\mathfrak{m})$ gleich $\mathrm{K}(2\mathfrak{m})$, folglich[1]

$$\mathrm{T}_0(\mathfrak{m})=\mathrm{K}(\mathfrak{m}). \tag{17}$$

(III) $\Delta\equiv 5,\ (8)$.

In diesem Falle sind die Coefficienten A, B, C ungerade, und 2 bleibt prim in k. Für die Zahl α im Ringe $\mathfrak{R}$ mit dem Führer 4

$$\alpha=a+b\omega=a+4b'\theta$$

erhält man nach (13) § 31, da C ungerade ist,

$$\varepsilon=(-1)^{\frac{a-1}{2}+b'}, \tag{18}$$

also $\varepsilon=1$, dann und nur dann, wenn

$$a\equiv 1\quad (4),\quad b'\equiv 0\ (2),$$

oder

$$a\equiv -1\ (4),\quad b'\equiv 1\ (2).$$

Die Zahlengruppe wird folglich durch die folgenden Congruenzen definirt:

$$\left.\begin{array}{l}\alpha\equiv 1,\quad (\mathfrak{m}),\\ \alpha\equiv 1,\ 5,\ -1+4\theta,\ -5+4\theta,\ (8),\end{array}\right\}$$

woraus einzusehen ist, dass $\mathrm{T}'(\mathfrak{m})$ in $\mathrm{K}(8\mathfrak{m})$ enthalten ist, ohne aber mit $\mathrm{K}(4\mathfrak{m})$ zusammenzufallen.

Nun ist im gegenwärtigen Falle $\mathrm{k}[\sqrt{\varkappa}]=\mathrm{M}(8)$, sodass

$$\sqrt{\varkappa}^{\,p-1}=\varkappa^{\frac{p-1}{2}}\equiv 1,\quad (\varpi),$$

dann und nur dann, wenn $\varpi=c+4b'\theta$, und $b'\equiv 0\ (2)$; also

$$\varkappa^{\frac{p-1}{2}}\equiv(-1)^{b'},\quad (\varpi).$$

---

(1) Vgl. § 28.

Nach (5) und (18) erhält man daher

$$(-1)^{\frac{a-1}{2}} \operatorname{sn}(\varpi u) \equiv \operatorname{sn}(u)^p, \qquad (\mathfrak{P}),$$

sodass für den Teilungskörper der Function sn, die Zahlengruppe:

$$\left.\begin{array}{ll} a \equiv 1, & (\mathfrak{m}), \\ a \equiv 1, & (4) \end{array}\right\}$$

auftritt, d.h. es ist

$$T(\mathfrak{m}) = K(4\mathfrak{m}). \tag{19}$$

Für den Körper $T_0(\mathfrak{m})$ erhält man, da die Bedingung $\varepsilon = 1$ wegfällt, die Zahlengruppe:

$$\left.\begin{array}{ll} a \equiv 1, & (\mathfrak{m}), \\ a \equiv \pm 1, & (4). \end{array}\right\}$$

Abgesehen von dem Falle $\Delta = -3$, kann man daher setzen

$$T_0(\mathfrak{m}) = K(4) K(\mathfrak{m}). \tag{20}$$

In allen Fällen hat sich somit ergeben, dass bei der geeigneten Wahl von ω im imaginären quadratischen Körper k, der Teilungskörper $T(\mathfrak{m})$ der Jacobi'schen Function $\operatorname{sn}(u, \omega)$ für einen ungeraden Divisor $\mathfrak{m}$ mit dem Elementarkörper $K(\mathfrak{f}\mathfrak{m})$ des § 28 übereinstimmt. Mit Rücksicht auf Satz 36 erhalten wir daher in

Bestätigung der Kronecker'schen Vermutung

Satz 37. *Alle relativ Abel'sche Oberkörper eines imaginären quadratischen Körper werden durch die Einheitswurzeln, die singulären Moduln und die Teilwerte der Jacobi'schen Function erzeugt.*

Abgeschlossen im Februar, 1920.

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On the Stereochemical Configuration of the Aquotriammine and Diammine Cobalt Complex Salts.

By

Kichimatsu MATSUNO, *Rigakushi.*

Introduction.

The present investigation was undertaken to determine the stereochemical configuration of the aquo-triammine and diammine cobalt complex salts, by observing the effects on the absorption spectra exercised by water molecules which coördinate in place of ammonia in cobaltammines in their aqueous solutions. The aquocobaltammines are, in general, more soluble in water than other cobalt-ammines and they are of various colours and different solubilities, according to the number of water molecules coördinated in them. The reason why such cobaltammines as $[Co(NH_3)_5NO_3]X_2$, $[Co(NH_3)_4(NO_3)_2]X$, $[Co(NH_3)_3(NO_3)_3]$, $[Co(NH_3)_5Cl]X_2$, $[Co(NH_3)_4H_2O\,Cl]X_2$ and $[Co(NH_3)_5Br]X_2$ etc., which are not soluble in cold water become readily soluble in warm water, or slightly acidic or alkaline solution, is based upon the facts that the nitrate groups, and halogen atoms are substituted by water; thus the cobaltammines change into $\left[Co(NH_3)_5H_2O\right]^{NO_3}_{X_2}$, $\left[Co(NH_3)_4(H_2O)_2\right]^{(NO_3)_2}_{X}$, $\left[Co(NH_3)_3(H_2O)_3\right](NO_3)_3$, $\left[Co(NH_3)_5H_2O\right]^{Cl}_{X_2}$, $\left[Co(NH_3)_4(H_2O)_2\right]^{X_2}_{Cl}$ and $\left[Co(NH_3)_5H_2O\right]^{Br}_{X_2}$ etc. . . respectively. Therefore the study of the relation of water molecules with cobaltammines appears to be of some importance. Up to the present, the following series of the aquocobaltammines are known. X denoting a monovalent anion. en is $\begin{matrix} CH_2NH_2 \\ | \\ CH_2NH_2 \end{matrix}$.

(1)	$[Co(NH_3)_5H_2O]X_3$	Roseo salt.	Brick red colour
(2)	$[Co(NH_3)_4(H_2O)_2]X_3$	Diaquotetrammine cobaltic salt.	Dark red
a)	$\left[Co \begin{matrix} H_2O\,(1) \\ H_2O\,(2) \end{matrix} en_2\right]X_3$	Cis-diaquodiethylenediamine cobaltic salt.	Brownish red
b)	$\left[Co \begin{matrix} H_2O\,(1) \\ H_2O\,(6) \end{matrix} en_2\right]X_3$	Trans. do	Greyish brown
(3)	$[Co(NH_3)_3(H_2O)_3]X_3$	Triaquotriammine cobaltic salt (Prepared from $[Co(NH_3)_3(NO_3)_3]$).	Reddish violet
(4)	$[Co(NH_3)_2(H_2O)_4]X_3$	Tetraquodiammine cobaltic salt, Aqueous solution only is known.	
?	$[Co(NH_3)(H_2O)_5]X_3$	Unknown.	
(5)	$[Co(H_2O)_6]X_2$	Normal cobaltous salt.	Rose colour
(6)	$[Co\,Cl\,H_2O(NH_3)_4]X_2$	Chloraquotetrammine cobaltic salt.	Reddish violet
(7)	$[Co\,Cl(H_2O)_2(NH_3)_3]X_2$	Chlorodiaquotriammine cobaltic salt. Two isomers, one is violet and the other greyish indigo.	
(8)	$[Co\,Cl(H_2O)_3(NH_3)_2]X_2$	Chlorotriaquodiammine cobaltic salt.	Indigo blue
(9)	$[Co\,Cl_2H_2O(NH_3)_3]X$	Dichloraquotriammine cobaltic salt (dichro-salt).	

According to the manner in which the chlorine atoms coördinate to the central cobalt atom, there are two isomers in this salt, one is green and the other grey.

(10)	$[Co\,Cl_2(H_2O)_2(NH_3)_2]X$	Dichlorodiaquodiammine cobaltic salt.	

In the compound (10) three isomers may be theoretically produced, but only one is known, which is green.

Among the above ten series of complex salts, the absorption spectra of (1), (2), (5) and (6) having already been studied in aqueous solution,[1] I have now measured the absorption spectra of the following eight salts :

(1) $\left[Co \begin{smallmatrix} H_2O_{(1)} \\ H_2O_{(2)} \end{smallmatrix} en_2\right] Cl_3 + 2H_2O$

(2) $\left[Co \begin{smallmatrix} H_2O_{(1)} \\ H_2O_{(6)} \end{smallmatrix} en_2\right] Cl_3 + 2H_2O$

(3) $[Co\,Cl(H_2O)_2\,(NH_3)_3]\,SO_4$ (Violet)

(4) $[Co\,Cl_2\,H_2O\,(NH_3)_3]\,Cl$ (Green)

(5) $[Co\,Cl_2\,(H_2O)_2\,(NH_3)_2]\,SO_4H$

(6) $[Co\,Cl\,(H_2O)_3\,(NH_3)_2]\,SO_4 + H_2O$

(7) $[Co\,(H_2O)_3\,(NH_3)_3]\,Cl_3$ Prepared from $[Co\,(NH_3)_3\,(NO_3)_3]$.

(8) $[Co\,(H_2O)_3\,(NH_3)_3]\,Cl_3$ Prepared from $[Co\,Cl_2\,H_2O\,(NH_3)_3]\,Cl$.

In order to determine whether the coördinated chlorine atoms of these salts dissociate in water or not, I have also measured the conductivities of the following six complex salts :

$[Co\,Cl\,(H_2O)_3\,(NH_3)_2]\,SO_4.\,H_2O$, $[Co\,Cl\,H_2O\,(NH_3)_4]\,SO_4$,

$[Co\,Cl\,(H_2O)_2\,(NH_3)_3]\,SO_4$, $[Co\,Cl\,(NH_3)_5]\,SO_4$,

$[Co\,CO_3\,(NH_3)_4]_2SO_4$, $[Co\,H_2O\,(NH_3)_5]_2\,(SO_4)_3$.

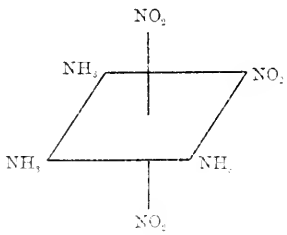
It was found, as the result, that the chlorine atoms (or halogen atoms generally) and the radicals of the strong acids, which coördinate to the central cobalt atom, are apt to dissociate, when they are dissolved in water and are sometimes in equilibrium

(1) Y. Shibata, Journ. College of Science, Imp. Univ. Tokyo Vol. XXXVII, Art. 2, 1915.

as described below. In this investigation I was able to establish the structural formulae of all the cobalt complex salts, mentioned above, by the study of their absorption spectra as well as their conductivities and I have explained clearly the behaviour in aqueous solution of such cobaltammines as have strong acid radicals in their complex nucleus.

Theoretical Part.

The configuration of the trinitrotriammine cobaltic salt, $[Co(NO_2)_3(NH_3)_3]$, a starting substance of many triammine cobalt complex salts, has already been determined by Y. Shibata.(1) as follows :



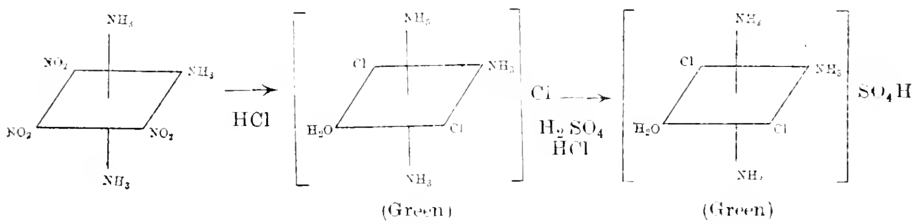
If this salt is treated with conc. hydrochloric acid $[Co\,Cl_2\,H_2O\,(NH_3)_3]Cl$ (9), the so-called dichro-salt, which is of a green colour is obtained. On treating the dichro-salt with a cold mixture of 2 Vol. of strong sulphuric acid and 1 Vol. of conc. hydrochloric acid, the greenish grey $[Co\,Cl_2\,H_2O\,(NH_3)_3]\,SO_4H$ is produced. On dissolving this last named substance in water and then adding an equal volume of alcohol, $[Co\,Cl\,(H_2O)_2\,(NH_3)_3]\,SO_4$, an indigo blue coloured powder, is precipitated. When the indigo-bluish $[Co\,Cl\,(H_2O)_2\,(NH_3)_3]\,SO_4$ is ground with strong hydrochloric acid, it changes into the grey $[Co\,Cl_2\,H_2O\,(NH_3)_3]\,Cl$ (2) which is the isomer of the green $[Co\,Cl_2\,H_2O\,(NH_3)_3]\,Cl$. If the aqueous solution of $[Co\,Cl_2\,H_2O\,(NH_3)_3]\,SO_4H$ is warmed on the water bath for a little while, the colour of the solution is changed into dark violet and, when it is evaporated in vacuo, a violet crystalline precipitate,

(1) Journ. College, Science, Imp. Univ. Tokyo, Vol. XXXVII, Art. 2, 1915.

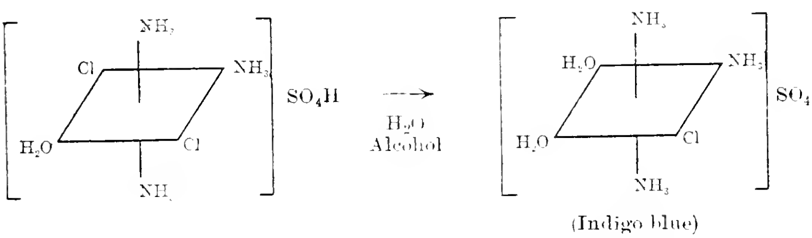
(2) Werner: Zeit. anorg. Chem., **15**, 143, 1897.

$[Co\,Cl\,(H_2O)_2\,(NH_3)_3]\,SO_4H_2O$ (7)[1], the isomer of the above indigo bluish salt, remains.

Judging from the above reactions, we can build up their stereochemical structures as below. As regards the configuration of the dichloraquotriammine cobaltic salt, the two chlorine atoms must, as described by Y. Shibata, be coördinated in the trans-position analogous to those of the diethylenediamine praseo cobaltic salt, in which they are in the trans-position and show green colour. The process of formation will be as follows :

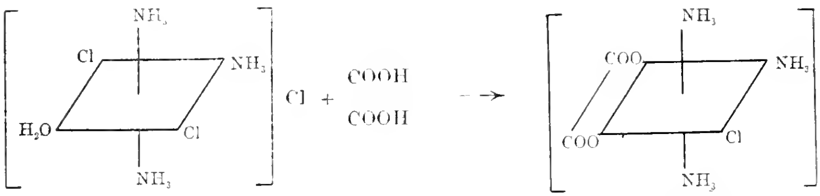


Two isomers of $[Co\,Cl\,(H_2O)_2\,(NH_3)_3]\,SO_4$[2] can be derived from $[Co\,Cl_2\,H_2O\,(NH_3)_3]\,SO_4H$. The indigo blue coloured one (which is easily obtained by the action of water upon $[Co\,Cl_2\,H_2O\,(NH_3)_3]SO_4H$ and by adding an equal volume of alcohol to it) has the following configuration :



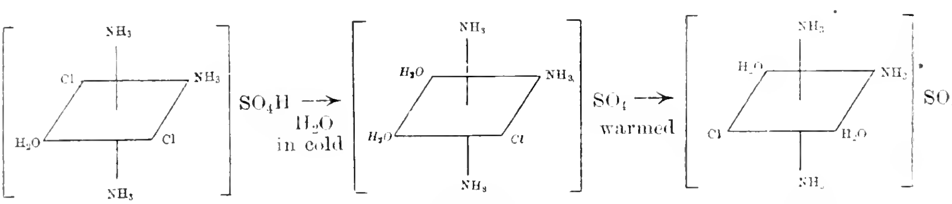
This reaction is analogous to that of the production of the chloroxalatotriammine cobaltic salt, $[Co\,Cl\,C_2O_4\,(NH_3)_3]$,[3] from the dichrochloride. As to the configuration of this salt, only one form can be deduced theoretically, as the following equation shows :

(1) Werner : Ibid.
(2) " "
(3) Werner and Miolati : Zeit. physik. Chem., **12**, 35 1893.



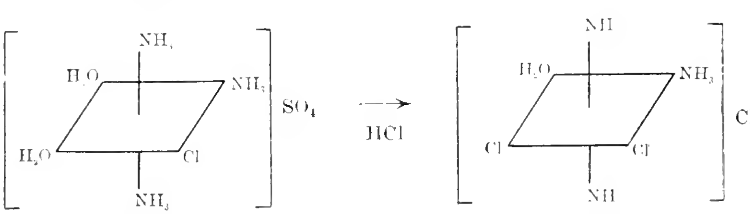
(Indigo blue)

If we assume that the two water molecules are placed in the cis-position in the indigo blue $[Co\,Cl\,(H_2O)_2\,(NH_3)_3]\,SO_4$, they must necessarily be situated in the trans-position in the other isomer, i.e. in the violet $[Co\,Cl\,(H_2O)_2\,(NH_3)_3]\,SO_4$. This presumption was clearly verified by the study of the absorption spectra (cf. the part on the absorption spectra). Therefore in the dichro-salt the two chlorine atoms must coördinate in the trans-position. These relations are shown as follows :



(Indigo blue) (Violet)

If the indigo blue compound $[Co\,Cl\,(H_2O)_2\,(NH_3)_3]\,SO_4$ is ground up with concentrated hydrochloric acid, a grey coloured powder is obtained, analysis of which gives the formula $[Co\,Cl_2\,H_2O\,(NH_3)_3]Cl$[1]; this substance must be considered as the isomer of the greenish dichro-salt of the following configuration :



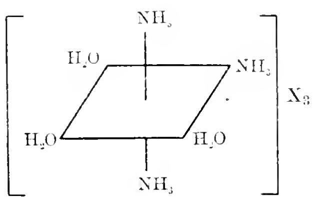
(Indigo blue) (Grey)

Having come to this conclusion, we have no need for further

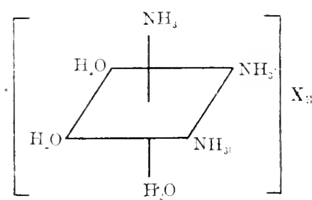
(1) Werner: Zeit. anorg. Chem., 15, 144, 1897.

explanations to prove the fact that the two chlorine atoms of the dichro-salt are coördinated in the trans-position.

With regard to the configurations of the triaquotriammine cobaltic salts, we can theoretically establish two isomers: one in which all the three water molecules are coördinated in the cis and consecutive positions and the other in which two of them are in the trans-position as follows:



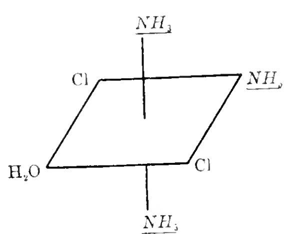
(I)



(II)

On addition of hydrochloric acid saturated at 0°C to the acetic acid solution of $[Co(NO_2)_3(NH_3)_3]$, which is kept over night, $[Co(H_2O)_3(NH_3)_3]Cl_3$ [1] separates in the shape of reddish violet crystals. The stereochemical configuration of this salt, though it cannot be determined by means of ordinary chemical analysis, can be ascertained by the study of its absorption spectra. With the help of this recently well-developed spectrochemical method, I was able to prove that in the above triaquotriammine cobaltic salt, the three water molecules are coördinated in the cis and consecutive positions as indicated by (II). Accordingly, there must exist another isomer of the type (I).

The position of the three ammonia molecules of the dichro salt has already been proved to be as follows:



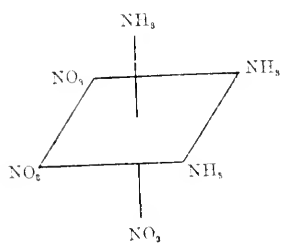
(1) Werner: Ber., **39**, 2678, 1906.

If the two chlorine atoms of this compound are replaced by water, we are able to produce the triaquotriammine cobaltic salt (I). Thus, when the dichro-salt is dissolved in a considerable quantity of water, acidified with hydrochloric acid, and evaporated in vacuo, dark violet crystals separate, analysis of which satisfies the formula $[Co(NH_3)_3(H_2O)_3]Cl_3$ not $[Co(NH_3)_3(H_2O)_2Cl]Cl_2 H_2O$. Thus, 0.2032 gr. of the substance gives 0.1138 gr. $CoSO_4$; 0.1353 gr. of the salt gives 0.2150 gr. AgCl.

	Calculated	Obtained
Co	21,3 %	21,8 %
Cl	39,31%	39,32%

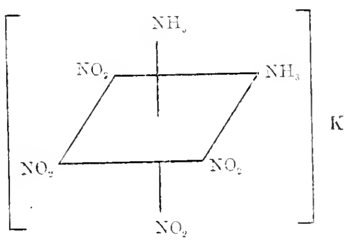
When the salt is heated at a temperature of 100-110° C for one hour it does not change in weight, which suggests that there is no water of crystallization. The absorption spectra of this salt were compared with those of the other isomer; the great divergence between them indicates the different configurations.

Thus I was able to obtain one of the isomers of triaquotriammine cobaltic salts, verifying also both of their configurations. As the cis and consecutive triaquotriammine cobaltic salt was prepared by the action of water upon $[Co(NO_2)_3(NH_3)_3]$, the constitution of the latter must be as follows:



The trans-triaquotriammine cobaltic salt is not stable in water; at first the solution is bluish violet in colour, but afterwards it shows exactly the same colour as the solution of the cis and consecutive, i.e. reddish violet. The same reaction takes place in the case of the diaquodiethylenediamine cobaltic salts. When the trans-salt stands for a long time in water or when it is warmed for a short time, it produces readily the stable cis-salt. Accordingly, one is much inclined to the conclusion that the cis-aquocompounds are as a rule more stable than the corresponding trans-compounds in aqueous solution.

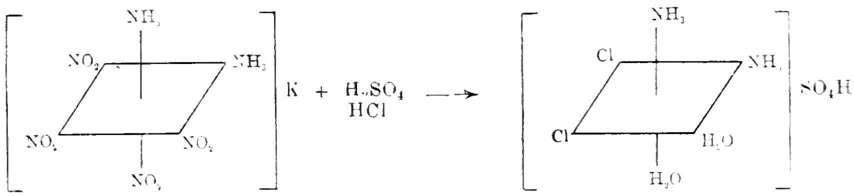
Now that I have described the structures of the cobalt complex salts of the triammine series, I shall discuss the configurations of those of the diammine series. As the stereochemical structure of one of these salts, i.e. the tetranitrodiammine cobaltic salt $[Co(NO_2)_4(NH_3)_2]K$, a starting substance of these series, has been thoroughly investigated by Y. Shibata[(1)] and the late T. Maruki, no further confirmation as to it is necessary. In this salt the two ammonia molecules are coördinated in the cis-position as follows :



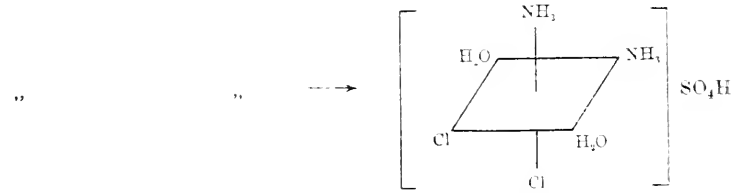
When $[Co(NO_2)_4(NH_3)_2]K$ [(2)] is treated with a cold mixture of strong hydrochloric acid and sulphuric acid, a malachite green salt $[CoCl_2(H_2O)_2(NH_3)_2]SO_4H$ is produced, and on this salt being dissolved in water acidified with H_2SO_4 and evaporated in the desiccator over sulphuric acid, indigo blue crystals of $[CoCl(H_2O)_3(NH_3)_2]SO_4 \cdot H_2O$ separate out. If the latter is acted upon by strong hydrochloric acid, it changes into a greenish powder which corresponds to the formula $[CoCl_2(H_2O)_2(NH_3)_2]Cl$ [(3)].

From these reactions, we can now deduce their constitutions as below. With respect to $[CoCl_2(H_2O)_2(NH_3)_2]X$, we are able theoretically to derive the following three configurations :—

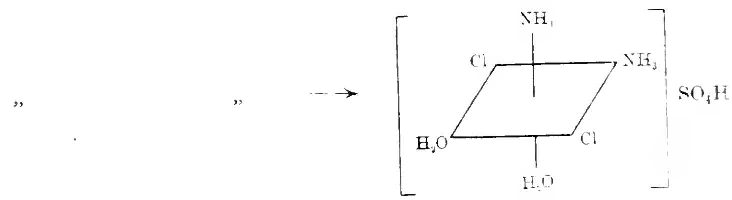
(1) Journ. College of Science, Imp. Univ. Tokyo Vol. XLI., Art. 2, 1917.
(2) Jörgensen : J. prakt. Chem., [2] **23**, 249, 1881.
(3) Werner : Zeit. anorg. Chem., **15**, 165, 1897.



(I)



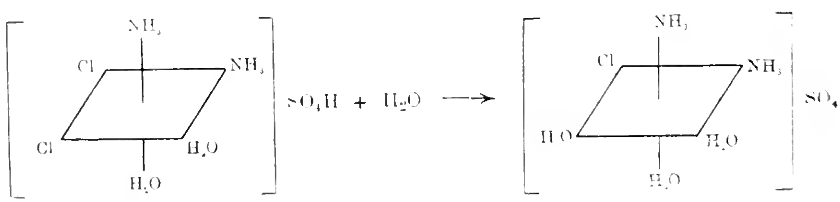
(II)



(III)

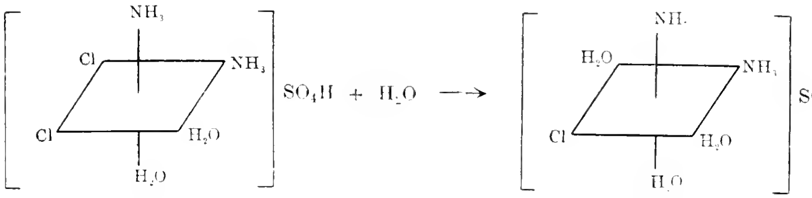
On the action of water upon $[Co(NH_3)_2(H_2O)_2Cl_2]SO_4H$, we obtain the indigo blue salt $[Co(NH_3)_2(H_2O)_3Cl]SO_4 \cdot H_2O$ (6), the configuration of which must be, as the following equations illustrate, either of (A) or (B) (below).

Whichever of the formulae (I), (II) and (III) one may assume for the configuration of $[Co(NH_3)_2(H_2O)_2Cl_2]SO_4H$, that of $[CoCl(H_2O)_3(NH_3)_2]SO_4$ will be as shown by either of the following two formulae (A) or (B), thus :

(1) 

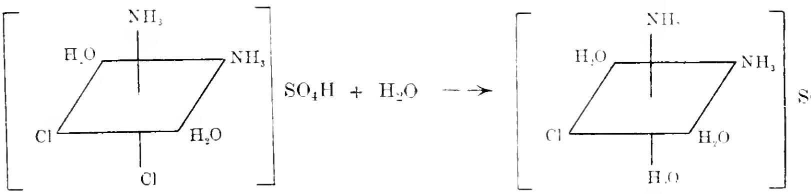
(I) (A)

(2)



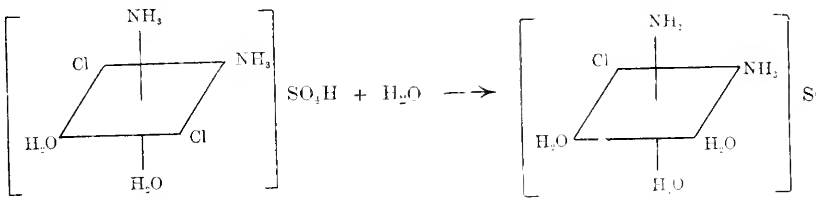
(I) (B)

(3)



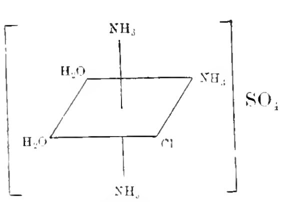
(II) (B)

(4)

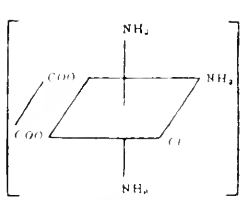


(III) (A)

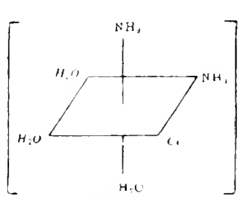
By the study of the absorption spectra, it is confirmed that the above triaquochlorodiammine cobaltic salt. $[Co(NH_3)_2Cl(H_2O)_3]SO_4$ has the configuration shown under (A). (For full explanations see under the section on absorption). It is naturally to be expected that some relations exist between the configuration and the colour of these chemical compounds; examples in which the salts have analogous configurations and the same colour are shown below:



(Indigo blue)

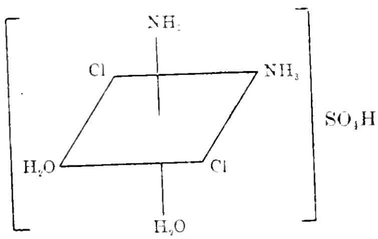


(Indigo blue)

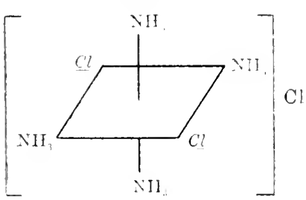


(Indigo blue)

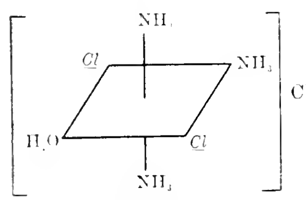
As we easily obtain from $[Co Cl_2 H_2O)_2(NH_3)_2]SO_4H$, $[Co Cl(H_2O)_3(NH_3)_2]SO_4$ as above mentioned, which has the structure shown by (A), the configuration of the former is not as in (II): if it were so, $[Co\,Cl\,(H_2O)_3\,(NH_3)_2]\,SO_4$ should have the configuration of the type (B) as indicated by the above equation (3). However this is not in keeping with experimental results, therefore it must be either as shown in (I) or (III). If it were like (I), then since the possibility of producing the chlorotriaquodiammine cobaltic salts shown by (A) and (B) should be exactly equal, the product obtained from it would be a mixture of these two isomers. But the compound thus obtained, i.e. $[Co\,Cl\,(H_2O)_3\,(NH_3)_2]\,SO_4$ crystallizes homogeneously and the absorption curve thereof suggests to us that the three water molecules are in the cis and consecutive position as shown by (A). Accordingly the above dichlorodiaquodiammine cobaltic salt has the constitution of the type (III), thus:—



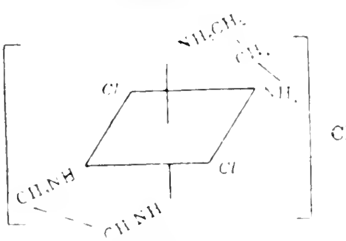
And also from the consideration of the colour, we can confirm this configuration, comparing it with the compounds which have analogous configurations and the same colour, thus:—



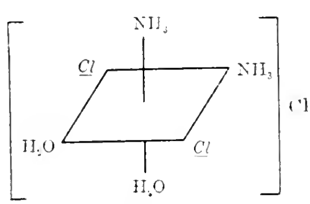
(green)



(green)



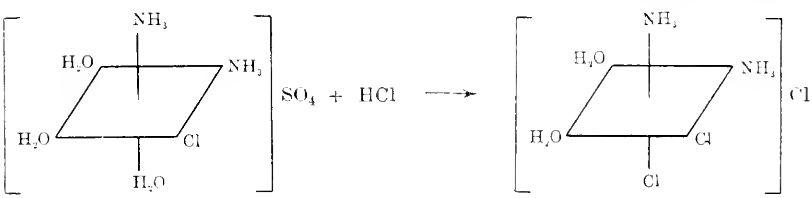
(green)



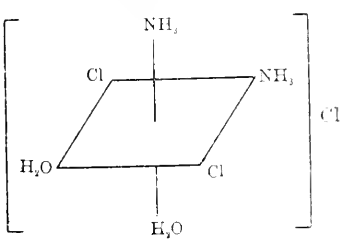
(green)

From this, it seems to be a rule that the cobaltammine in which the two chlorine atoms are coördinated in the trans-position is green in colour.

When one of the water molecules is replaced by a chlorine atom by the direct action of concentrated hydrochloric acid upon $[Co\,Cl\,(H_2O)_3\,(NH_3)_2]\,SO_4$, a green salt is produced which corresponds to the formula $[Co\,Cl_2\,(H_2O)_2\,(NH_3)_2]\,Cl$. This salt is soluble in water, thereby changing its colour to violet. Werner [1] supposed this salt to be the isomer of the above $[Co\,Cl_2\,(H_2O)_2\,(NH_3)_2]\,X$ (III). If that be correct, we are inclined to the view that the two chlorine atoms of the salt coördinate in the cis-position as shown:



In supporting this statement, this salt should be regarded as one of the exceptions to the general rule that the cobaltammines in which the two chlorine atoms coördinate in the trans, are, as already described, green coloured. But is it not more probable that this salt is identical with $[Co\,Cl_2\,(H_2O)_2\,(NH_3)_2]X$ shown by (III) and that the two chlorine atoms are situated in the trans-position? So I suggest that the formula of the salt may be :



Up to this point, I have determined the configurations of the isomers of $[Co\,(NH_3)_3\,H_2O\,Cl_2]\,X$, $[Co\,(NH_3)_3\,(H_2O)_2\,Cl]\,X_2$ and $[Co\,(NH_3)_3\,(H_2O)_3]\,X_3$ respectively and $[Co\,(NH_3)_2\,(H_2O)_2\,Cl_2]\,X$, $[Co\,(NH_3)_2\,(H_2O)_3\,Cl]\,X_2$ and $[Co\,(NH_3)_3\,(NO_2)_3]$, but this is not sufficient without an explanation of the study of the absorption spectra.

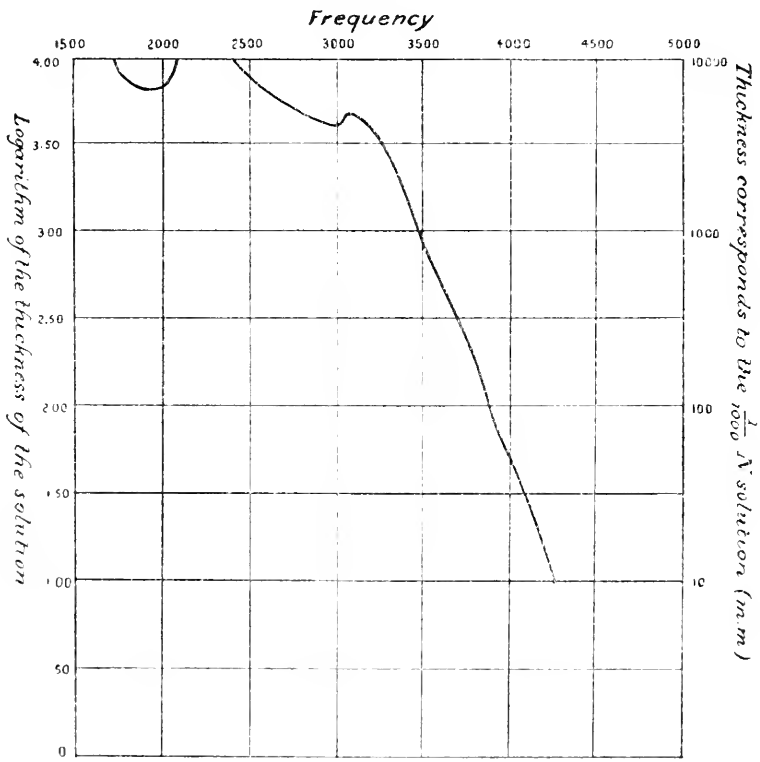
(1) Zeit. anorg. Chem., **15**, 172, 1897.

Experemental Part:

The Measurement of the Absorption Spectra.

The absorption spectra of these salts were measured by means of the quartz spectrograph of Adam Hilger & Co, London. The concentrations of the solutions were varied from 1/100 N to 1/1000. The salts used were specially purified by recrystallization, the formulae of some of which were ascertained by analysis. We might go into discussions from the standpoint of the absorption curves, which were drawn by the method suggested by Hartley & Baly. As the result, the eight compounds can be grouped into two classes: to the first class belong

Fig. 1.

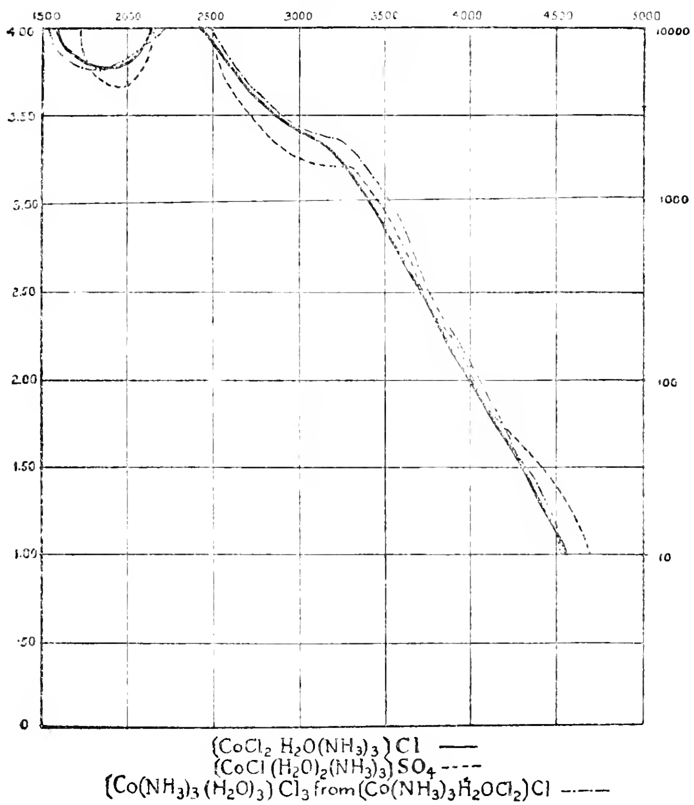


$\left(\begin{smallmatrix}(1)\,H_2O \\ (6)\,H_2O\end{smallmatrix} Co\,en_2\right) Cl_3$

$\left[Co\begin{smallmatrix}H_2O_{(1)}\\H_2O_{(6)}\end{smallmatrix}en_2\right]Cl_3$ (fig. I), green $[CoCl_2H_2O(NH_3)_3]Cl$ (fig. II), violet $[Co(NH_3)_3(H_2O)_2Cl]SO_4$ (fig. II) and $[Co(NH_3)_3(H_2O)_3]Cl_3$ (fig. II) which are prepared from the green $[Co(NH_3)_3Cl_2H_2O]Cl$ and the second class includes the following four salts :

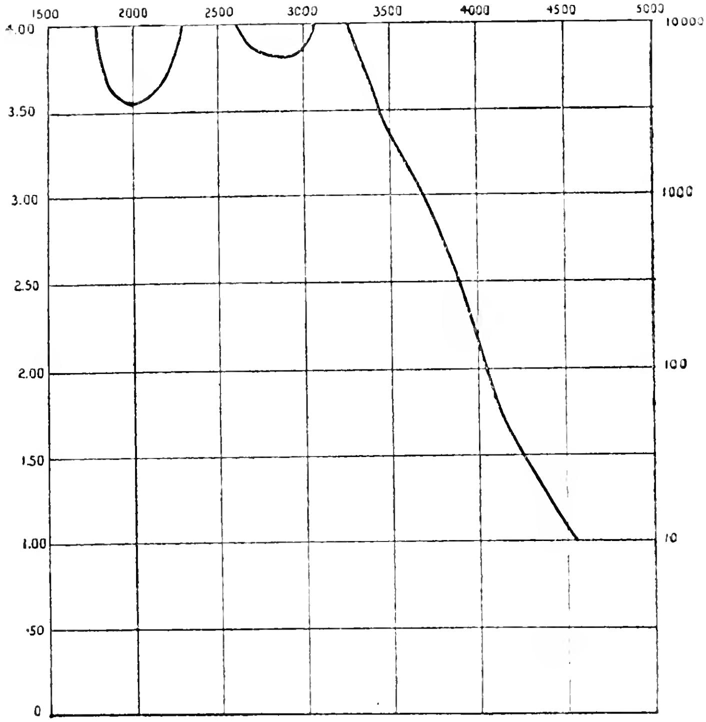
$\left[Co\begin{smallmatrix}H_2O_{(1)}\\H_2O_{(2)}\end{smallmatrix}en_2\right]Cl_3$ (fig. III), $[Co(NH_3)_2(H_2O)_2Cl_2]SO_4H$ (fig. IV) $[Co(NH_3)_3(H_2O)_3]Cl_3$ (fig. IV) which are prepared from $[Co(NH_3)_3(NO_3)_3]$ and $[Co(NH_3)_2(H_2O)_3Cl]SO_4$ (fig. IV).

Fig. 2.



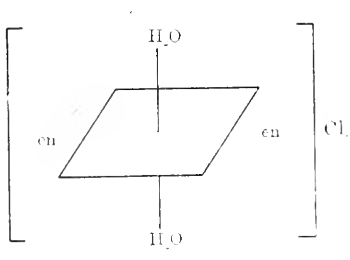
In the first class as well as in the second, all the absorption curves in their own groups are remarkably similar to each other, although they differ much from those in the other class, which

Fig. 3.



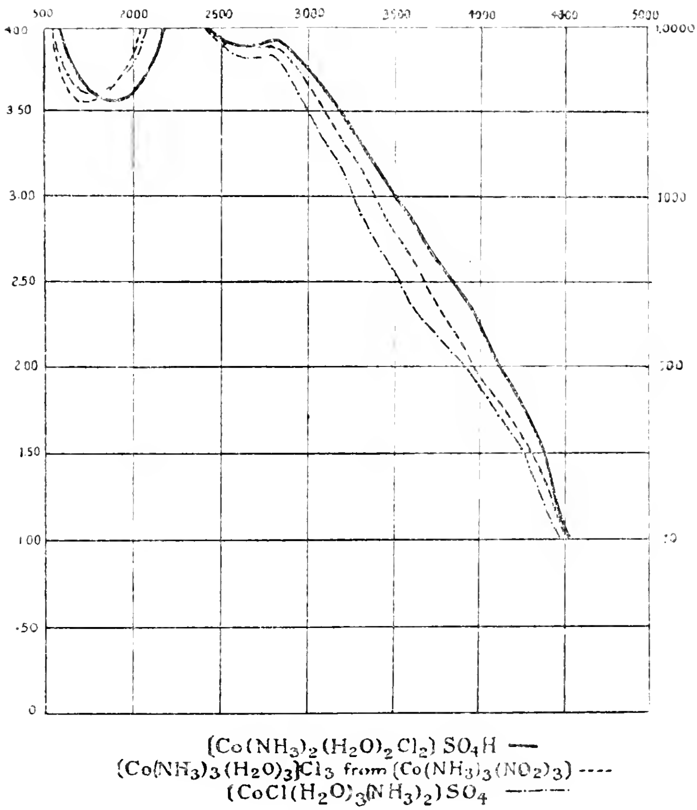
$\left[\begin{matrix}(1)\ H_2O \\ (2)\ H_2O\end{matrix} Co\ en_2\right] Cl_3$

proves that their constitutions are also identical in their groups. The structural formula of $\left[Co \begin{matrix} H_2O\ (1) \\ H_2O\ (6) \end{matrix} en_2\right] Cl_3$ was previously determined by Werner[1] as below, the two water molecules of which are in the trans-position thus:

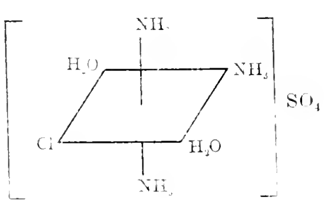


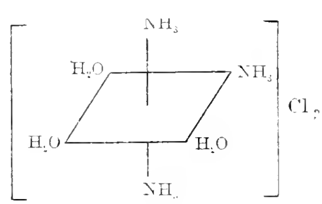
(1) Ber., 40. 285. 1907

Fig. 4.

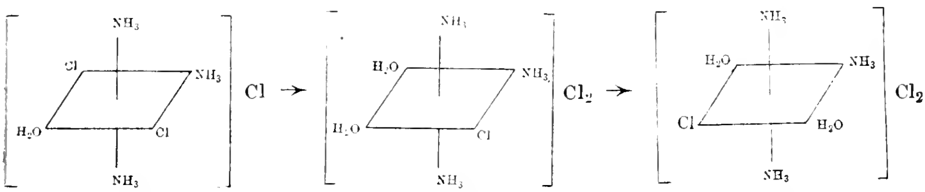


As the absorption curve of $\left[Co \begin{matrix} H_2O\ (1) \\ H_2O\ (6) \end{matrix} en_2\right] Cl_3$ is the fundamental one of the first class, the configuration of the other three compounds must be analogous to that of this salt; in other words, the two water molecules in them coördinate in the trans-position. So $[Co(NH_3)_2(H_2O)_2Cl]SO_4$ and $[Co(NH_3)_3(H_2O)_3]Cl_3$ (8) have the following configurations respectively:





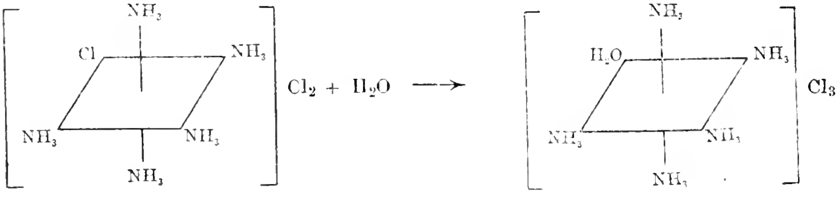
That the dichro-salt gives the same curve (fig. II) as those of the rest of the first class is to be explained by the fact that although at first while the colour of its solution is yet green, it has the configuration already mentioned, with the gradual transformation of the colour to indigo and at last to violet, the conversion of the structure must take place as follows :



And $[Co(NH_3)_3(H_2O)_2Cl]Cl_2$ will be also in equilibrium with $[Co(NH_3)_3(H_2O)_3]Cl_3$ in the water solution thus:

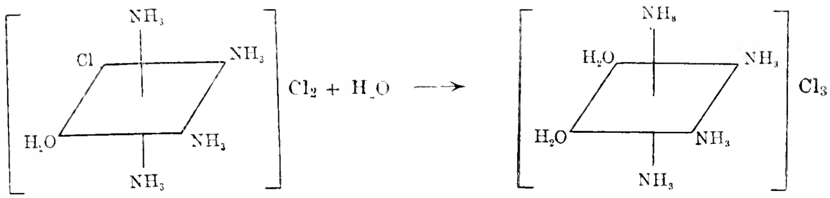
$$[Co(NH_3)_3(H_2O)_2Cl]Cl_2 + H_2O \rightleftarrows [Co(NH_3)_3(H_2O)_3]Cl_3$$

As regards this point, it will be fully discussed later. There are many examples where, when the chlorocobaltammines are dissolved in water their chlorine atoms are easily replaced by water molecules. The absorption spectra[1] of $[Co(NH_3)_5Cl]Cl_2$ and $[Co(NH_3)_5H_2O]Cl_3$ in aqueous solution are identical, which proves that the following substitution occurs to a certain extent:—

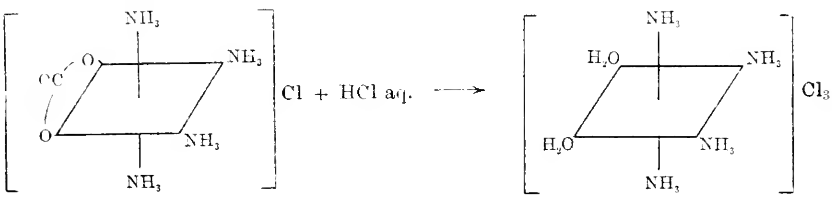


$[CoClH_2O(NH_3)_4]Cl_2$ and $[Co(H_2O)_2(NH_3)_4]Cl_3$ have also the same absorption curves, which are confirmed as follows:

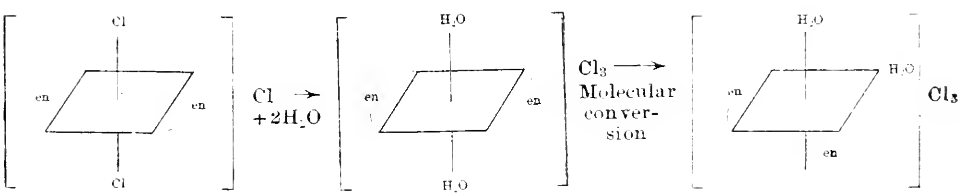
(1) Y. Shibata, Journ. College of Science, Imp. Univ., Tokyo, Vol. XXXVII, Art. 2, 1915.



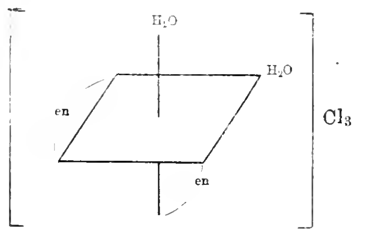
The fact that the two water molecules of $[Co(H_2O)_2(NH_3)_4]Cl_3$ coördinate in the cis-position can easily be recognized from the absorption curve, which is very similar to that of the second class, and also by the process of formation as follows :



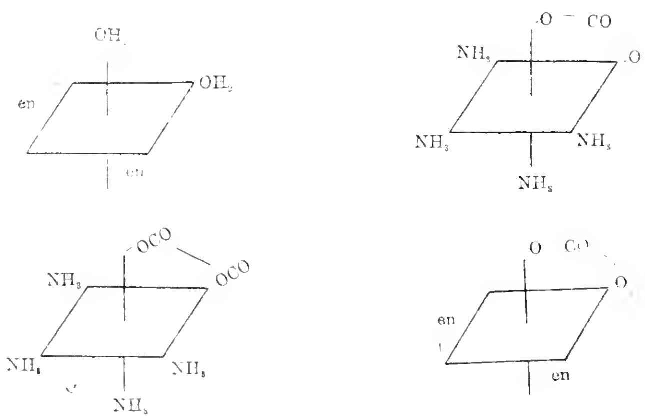
When $\left[Co\begin{smallmatrix}Cl\ (1)\\ Cl\ (6)\end{smallmatrix}Co\ en_2\right]Cl$ is kept overnight in aqueous solution, its colour changes to carmine red and then it shows the same absorption curve as that of the cis-diaquodiethylenediamine cobaltic chloride, $\left[Co\begin{smallmatrix}H_2O\ (1)\\ H_2O\ (6)\end{smallmatrix}en_2\right]Cl_3$. This will be verified below. At first its two chlorine atoms are replaced by water, producing the trans-diaquodiethylenediamine cobaltic chloride, $\left[Co\begin{smallmatrix}H_2O\ (1)\\ H_2O\ (6)\end{smallmatrix}en_2\right]Cl_3$ and then it is converted into the more stable cis-salt, i.e. $\left[Co\begin{smallmatrix}H_2O\ (1)\\ H_2O\ (2)\end{smallmatrix}en_2\right]Cl_3$ thus :



As to the complex salts of the second class, discussions follow quite in the same way as in the case of the first class. Werner has already studied the constitution of $\left[Co \begin{matrix} H_2O\ (1) \\ H_2O\ (2) \end{matrix} en_2\right] Cl_3$ [1], two water molecules of which are coördinated in the cis-position, thus :



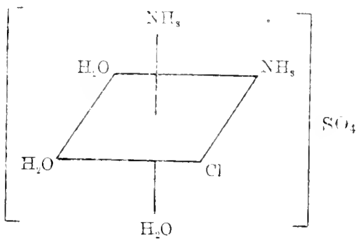
The compounds belonging to this series are generally more stable than those of the first class, while the latter seems to have a great tendency to be converted into the former. From the fact that the absorption curve of the cis-diaquodiethylenediamine cobaltic chloride (fig. III) is analogous to the curves of $[Co\ CO_3\ (NH_3)_4]\ Cl$, $[Co\ C_2O_4\ (NH_3)_4]\ Cl$ and $[Co\ CO_3\ en_2]\ Cl$, we can affirm that, when the water molecules coördinate to a cobalt atom to form a complex salt, then unite with the central atom by the residual affinity of the oxygen of water as illustrated below :



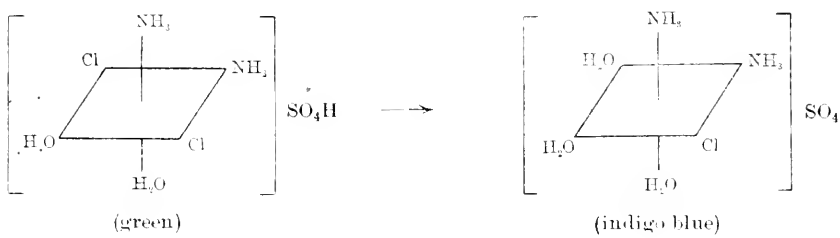
The chlorotriaquodiammine cobaltic sulphate must have the following configuration. For its absorption curve (fig. IV) is very similar to that of the cis-diaquodiethylenediamine cobaltic chloride,

(1) Ber., **40**, 285, 1907.

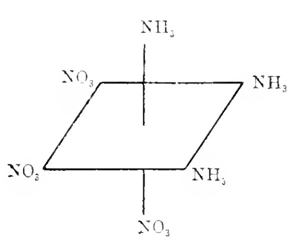
which suggests that its three water molecules coördinate in the cis and consecutive position, thus:—



As a consequence, the structure of $[Co\,Cl_2(H_2O)_2(NH_3)_2]\,SO_4H$ could also be determined, as has been already mentioned. The compound $[Co\,Cl_2(H_2O)_2(NH_3)_2]\,SO_4H$ has at first a green colour in aqueous solution, but this changes gradually until its colour becomes indigo blue. This change of colour is a proof that the following substitution takes place :



The curve of $[Co(NH_3)_3(H_2O)_3]\,Cl_3$ (7) is very different from that of $[Co(NH_3)_3(H_2O)_3]\,Cl_3$ (8), which is the best verification that the two salts are isomeric; and the former is very stable in water. It is prepared from $[Co(NH_3)_3(NO_3)_3]$ the configuration of the latter will be as follows :



General Consideration of the Absorption Spectra.

The aquocobaltammines in which the water molecules coördinate in the cis-position tend remarkably with an increase in the number of water molecules, to produce the bathochromic phenomenon and their absorption spectra differ greatly from those of the trans-salts. All the aquocobaltammines, according to the number of the coördinated water molecules, are apt to diminish the second absorption band, which affords an additional proof that the second absorption band of these salts is, as Prof. Y. Shibata has suggested, due to the coördinated ammonia.

Measurements of Electrolytic Conductivity.

Although I have in the previous chapter demonstrated by means of the absorption apectra that in some of the chlorocobaltammines the chlorine atoms are replaced by water molecules in aqueous solution, the confirmation will not be satisfactory without another verification. Therefore I have measured the conductivities of solutions of these salts in order to confirm the above opinion. The method used was that devised by Ostwald. The data are as follows :

Table I.

$[CoCl(NH_3)_5]SO_4$		$[CoCl\,H_2O(NH_3)_4]SO_4$		$[CoCl(H_2O)_2(NH_3)_3]SO_4$	
v	μ	v	μ	v	μ
100	200,3	......	......	......	......
200	231,4	.....	......	......	......
400	263,4	400	190,1	......	......
800	300,4	800	218,7	800	259,2
1600	343,9	1600	259,2	1600	462,9

$[CoCl(H_2O)_3(NH_3)_2]SO_4$		$[CoCO_3(NH_3)_4]_2SO_4$		$[CoH_2O(NH_3)_5]_2(SO_4)_3$	
100	184,2	100	78,83	100	117,0
200	222,3	200	87,18	200	151,8
400	267,6	400	92,13	400	182,2
800	332,7	800	99,13	800	217,9
1600	412,3	1600	106,2	1600	265,1

Where O is the dilution, U the molecular conductivity. Measurements were made at the temperature of 25°C. The following table shows the variation of the conductivity with time.

Table 2.

$[Co\,H_2O\,(NH_3)_5]_2\,(SO_4)_3$

O	Time (Min,)	U
100	0	117,0
,,	5	122,3
,,	10	124,3
,,	20	129,3
,,	30	133,6
,,	40	137,9
,,	50	141,3
,,	60	143,5

Summary of the data at the dilutions of 800 & 1600 :—

Table 3.

O	$\left[Co \begin{smallmatrix} CO_3 \\ (NH_3)_4 \end{smallmatrix}\right]_2 SO_4$	$\left[Co \begin{smallmatrix} H_2O \\ (NH_3)_5 \end{smallmatrix}\right]_2 (SO_4)_3$	$\left[Co \begin{smallmatrix} Cl \\ (NH_3)_5 \end{smallmatrix}\right] SO_4$
800	99,13	217,9	300,4
1600	106,2	265,1	343,9

O	$\left[Co \begin{smallmatrix} Cl\,H_2O \\ (NH_3)_4 \end{smallmatrix}\right] SO_4$	$\left[Co \begin{smallmatrix} Cl(H_2O)_2 \\ (NH_3)_3 \end{smallmatrix}\right] SO_4$	$\left[Co \begin{smallmatrix} Cl(H_2O)_3 \\ (NH_3)_2 \end{smallmatrix}\right] SO_4$
800	218,7	259,2	332,7
1600	259,2	462,9	412,3

As seen in the above tables the carbonatotetrammine cobaltic sulphate is stable in water and shows little variation of conductivity with the dilution. The molecular conductivity at the dilution of 800–1600 is about 100. The conductivity of the aquopentammine cobaltic sulphate varies more with its dilution than that of the carbonatotetrammine cobaltic salt. The dilution from 800 to that of 1600 causes an increase of about 20%, giving a mean value of about 250. The effect of time will be seen in table 2. It increases 11,5% during one hour. Werner and his co-workers[1] had previously expressed both the chemical and electrochemical behaviour of the cobaltammine chlorides by the graphic representation of their molecular conductivities and gave the following data:

Number of ions	Molecular conductivity at the dilution of 1000
2	Approx. 100
3	,, 250
4	,, 480

It shows that the number of ions of the cobaltammines is proportional to the conductivity of the salts. While $[Co\,CO_3(NH_3)_4]_2\,SO_4$ and $[Co\,H_2O\,(NH_3)_5]_2\,(SO_4)_3$ have three and five ions, as well as the molecular conductivities of about 100 and 250 respectively, the chlorocobaltammines mentioned above should have molecular conductivities of less than 100, if there occurs no transformation in water because they have two ions, judging from the molecular formula only. The results of the experiments are contrary to this supposition and show two or three times the required value for the molecular conductivities. They vary very much with the dilution, especially in the triammine and diammine complex salts, which have values approximating to the maximum conductivity of hydrochloric acid, i.e. 403, at a dilution of 1600. The fact that strong acid radicals such as Cl or NO_3 when coördinated in the cobalt complex nucleus to form a complex salt sometimes show ionic reactions, in other words, dissociate in aqueous solution, was demonstrated by Werner. For example, the purpureo cobaltammine chloride, if dissolved in water, undergoes to a certain extent the following substitution:

$$[Co\,Cl\,(NH_3)_5]\,Cl_2 + H_2O \longrightarrow [Co\,H_2O\,(NH_3)_5]\,Cl_3$$

The above system reacts sometimes acidic when the conversion takes place, thus:

(1) Werner: New Ideas of Inorganic Chemistry, P 157 & 184.

$$[Co(H_2O)(NH_3)_5]Cl_3 = [Co(HO)(NH_3)_5]Cl_2 + HCl$$

According to Urbain's[1] opinion, such reactions are not so simple and it is more probable that they are in equilibrium under definite surrounding conditions. No doubt the equilibrium can be affected by the atoms, atom groups and molecules coördinated around the central metallic atom, as well as by the atoms and atom groups outside the complex nucleus. One of such complicated examples will be shown below :

$$[Co(NH_3)_5X]X_2 + H_2O \rightleftarrows [CoH_2O(NH_3)_5]X_3 \rightleftarrows [Co(OH)(NH_3)_5]X_2 + HX$$

$$[Co(NH_3)_5(OH)]X_2 + H_2O \rightleftarrows [Co(NH_3)_5H_2O]\begin{matrix}OH\\X_2\end{matrix}$$

The above assumption is exactly in keeping with my experiments. In the case of the diammine and triammine cobalt complex salts the substitutions are apt to take place when the dilution is large. It seems that the absorption spectra of $[Co(NH_3)_3(H_2O)_2Cl]SO_4$ and $[Co(NH_3)_2(H_2O)_3Cl]SO_4$ confirm the reverse of the conclusion that the chlorine atom is not separated from the complex nucleus, but it can be soon restored, re-establishing the equilibrium.

Thus, I have given a clear explanation, by means of absorption spectra and conductivity measurements, of the substitution phenomena in aqueous solutions of the cobaltammines, in which the strong acid radicals coördinate.

In conclusion, I will describe the complex salt which is obtained by the action of silver nitrate upon $[CoClC_2O_4(NH_3)_3]$. Werner[2] gave to it the molecular formula of $\left[Co\begin{matrix}(H_2O)_2\\C_2O_4\\(NH_3)_3\end{matrix}\right]NO_3$, but I also prepared this salt in the same way and after analysis found it must be amended to $[CoNO_3C_2O_4(NH_3)_3]\cdot H_2O$. The following data will confirm this :

0,1023 gr. and 0,1090 gr. of the salt gave 0,0620 gr. and 0,0642 gr. $CoSO_4$. After heating for one hour and a half at the temperature of 100°–115°C. it loses the weight of one molecule of water.

(1) Introduction à la Chimie des Complexes (P, 178)

(2) Zeit. anorg. Chem., **15**, 162, 1897.

	Theoretical	Observed		
H_2O	6,92 %	7,29 %	6,15 %	6,42 %
Co	22,68 %	22,65 %	22,41 %	

The molecular conductivity of this salt was also measured.

∅	U
500	99,8
1000	105,8

which corresponds to the data given for the salts which have two ions, as the following examples indicate :

∅	$[Co(NO_2)_2(NH_3)_4]Cl$	$[Co(NO_2)_4(NH_3)_2]K$	$[PtCl_3(NH_3)_3]Cl$	$[PtCl_5NH_3]Cl$
1000	98,35	99,29	96,75	108,5

If the compound $[Co(NH_3)_3C_2O_4NO_3]\cdot H_2O$ does not change its constitution in aqueous solution, the conductivity should be zero. Therefore, there is no doubt that the following substitution occurs and gives two ions thus :

$$[Co(NH_3)_3C_2O_4\cdot NO_3] + H_2O = [Co(NH_3)_3\,C_2O_4\cdot H_2O]\,NO_3$$

The aqueous solution of the salt shows the reactions of NO_3, which confirms the above assumption. The fact that the oxalate radical is firmly fixed to the central atom is the parallel phenomenon of the carbonatotetrammine cobaltic salt, which in water is stable. Consequently, it can be generalised that in the cobaltammines the radicals of the strong acids have the less coördinating power, whilst, on the contrary, the radicals of the weak acids, molecules, such as NH_3 and its derivatives and water have the greater coördinating power.

Summary.

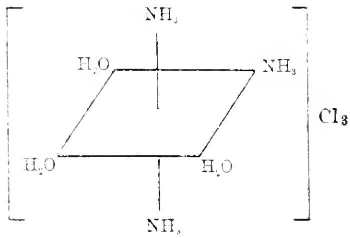
1.—I have measured the absorption spectra of the aquocobaltammines and thereby demonstrated the influence of the coördinated water molecules.

2.—The stereochemical configurations of the following cobaltammines were thoroughly determined.

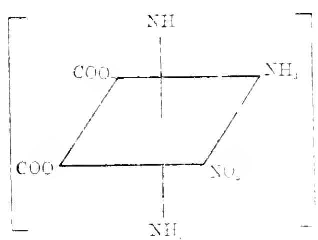
$[Co(NH_3)_3Cl_2H_2O]X$, $[Co(NH_3)_3Cl(H_2O)_2]X_2$, $[Co(NH_3)_3(H_2O)_3]X_3$,
$[Co(NH_3)_2Cl_2(H_2O)_2]X$, $[Co(NH_3)_2(H_2O)_3Cl]X_2$, $[Co(NH_3)_3(NO_3)_3]$,
$[Co(NH_3)_4(H_2O)_2]X_3$, $[Co(NH_3)_4ClH_2O]X_2$, $[Co(NH_3)_3C_2O_4.NO_3]$

3.—The following two new cobalt complex salts were produced.

(1) Trans $[Co(NH_3)_3(H_2O)_3]Cl_3$



(2) $[Co(NH_3)_3C_2O_4.NO_3]$



4.—The formula $[Co(NH_3)_3C_2O_4(H_2O)_2]NO_3$, which was given by Werner, must be amended to $[Co(NH_3)_3C_2O_4.NO_3].H_2O$.

5.—With regard to the substitution reactions of the cobaltammines which have radicals of the strong acids in the complex nucleus, a clear explanation has been given by the study of absorption of light and electrolytic conductivity.

In conclusion, the author tenders his sincere thanks to Prof. Y. Shibata for his kind guidance and suggestions.

(The Laboratory of Inorganic Chemistry, the College of Science, Imperial University of Tokyo).

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The Coagulation of Arsenious Sulphide Sol by Cobaltic Complexes.

By

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Introduction

The valency of inorganic complex ions is usually determined by measuring their electric conductivities in solutions [WERNER and MIOLATI: Zs. physik. Chem., **12**, 35: **14**, 506 (1894)].

This method is dependent to a certain extent on the migration velocity of the ions. It is well known that the valency of a coagulating ion has a great effect in determining its coagulating power. Galecki [Zs. Elek. chem, **14**, 767, (1908)] utilised this fact in determining the valency of beryllium by coagulation experiments with arsenious sulphide sol. Freundlich [Zs physik. Chem., **89**, 564, (1912)] has shown that it is possible to follow the change in the valency of the cation of a cobaltic complex by coagulation experiments with arsenious sulphide sol. But such experiments with cations of a valency greater than three have not been tried so far. It is now generally accepted that the so-called valency law of Schulze is a rough generalisation [Bancroft, J. Physic. Chem., **19**, 348, (1915): Brit. As. Rep. (1918). Wo. OSTWALD; Koll. Zeitsch., **26** (1920), 69]. Experiments with complex cations having a valency up to six is of interest as a test of so-called valency law. It will be seen from the sequel that the valency of the cation has a predominant effect and that the valencies determined by coagulation experiments agree perfectly with the usual formulae given to these salts.

Experimental

The limiting concentration of a salt which just failed to produce any perceptible change in the sol after an interval of five minutes, was taken to be the measure of its coagulating power. This method is more sensitive than other methods [Mukherjee, J. Amer. Chem. Soc., **37**, 2024 (1915)]. The coagulating power of an electrolyte is dependent on the quality of the sol, the sulphide content, its age and the temperature [Mukherjee and Sen, Journ. Chem. Soc., **115**, 462 (1919); **117**, 350 (1920)].

The present investigation was carried on with the same sample of sol in order that the data can be strictly comparative. To minimise the effect of "aging", the sol was allowed to stand for one year before the experiments were made [Compare Pauli and Matula, (Koll. Zeitsch, 1, 1917) who took the same precaution with ferric hydroxide sols]. The sol contained 37.6 milimoles of arsenious sulphide per litre.

One c.c. of the sol was placed in each of ten carefully cleaned test-tubes of Jena Glass. Two c.c. of solution of a salt of different

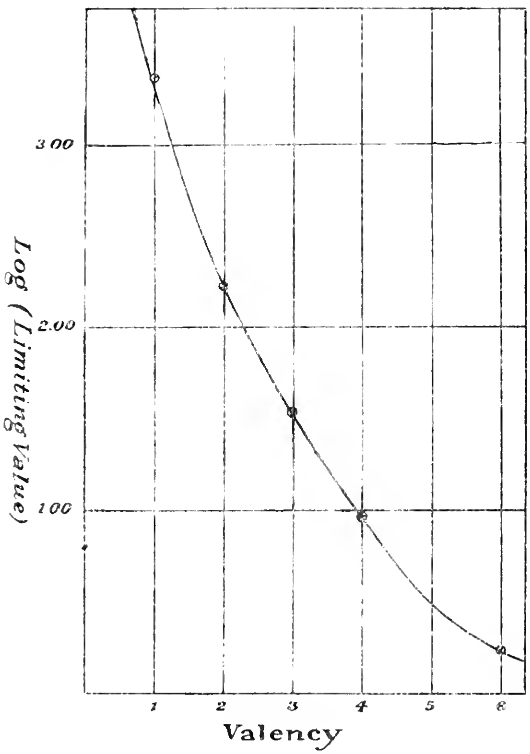
Table I.

	Cobaltammine	Limiting concentration (Eq. Mol)
1	$[Co(NH_3)_4C_2O_4]Cl$	$\frac{10}{1500}$
2	$[Co(NH_3)_4(NO_2)_2\,{}^{(1)}_{(6)}]_3[Co(NO_2)_6]$	$\frac{9}{1500}$
3	$[Co(NH_3)_4(NO_2)_2\,{}^{(1)}_{(2)}]Cl$	$\frac{8}{1500}$
4	$[Co(NH_3)_4(CO_3)]_2SO_4$	$\frac{8}{1500}$
5	$[Co(NH_3)_4(NO_2)_2\,{}^{(1)}_{(2)}][Co(NH_3)_2(NO_2)_4]$	$\frac{8}{1500}$
6	$[Co(NH_3)_4(NO_2)_2\,{}^{(1)}_{(6)}]Cl$	$\frac{7}{1500}$
7	$[Co(NH_3)_4CO_3]NO_3\,\frac{1}{2}H_2O$	$\frac{\cdot 7}{1500}$
	Mean	$\frac{8}{1500}$

1	$[Co(NH_3)_5NO_2]Cl_2$		$\frac{1}{2500}$
2	$[Co(NH_3)_5NO_2][Co(NH_3)_2(NO_2)_4]_2$		$\frac{1}{2500}$
3	$[Co(NH_3)_5SCN]Cl_2$		$\frac{1}{3000}$
4	$[Co(NH_3)_5Cl]Cl_2$		$\frac{1}{3000}$
5	$[Co(NH_3)_5Cl]_2(SO_4H)_2 \cdot SO_4$		$\frac{1}{3000}$
		Mean	$\frac{1}{3000}$
1	$[Co\,en_3]Cl_3$		$\frac{1}{12500}$
2	$[Co(NH_3)_5H_2O]_2(SO_4)_3\,3H_2O$		$\frac{1}{12500}$
3	$[Co(NH_3)_6]Cl_3$		$\frac{1}{15000}$
4	$[Co(NH_3)_6](NO_3)_3$		$\frac{1}{15000}$
6	$\left[(NH_3)_3Co{<}\begin{matrix}OH\\OH\\OH\end{matrix}{>}Co(NH_3)_3\right]Cl_3,\,H_2O$		$\frac{1}{18750}$
7	$\left[(NH)_3\,Co{<}\begin{matrix}OH\\OH\\OH\end{matrix}{>}Co(NH_3)_3\right](HSO_4)_3$		$\frac{1}{18750}$
		Mean	$\frac{1}{15000}$
1	$[(NH_3)_5Co—NH—Co(NH_3)_5]Cl_4$		$\frac{5}{240000}$
2	$[(NH_3)_4Co{<}\begin{matrix}NH_2\\OH\end{matrix}{>}Co(NH_3)_4]Cl_4\,4H_2O$		$\frac{4 \cdot 5}{240000}$
3	$\left[\begin{matrix}(NH_3)_4\\H_2O\end{matrix}{>}Co—NH_2—Co{<}\begin{matrix}Cl\\(NH_3)_4\end{matrix}\right]Cl_4$		$\frac{4 \cdot 5}{240000}$
4	$[(NH_3)_4Co{<}\begin{matrix}OH\\OH\end{matrix}{>}Co(NH_3)_4]Cl_4$		$\frac{4}{210000}$
		Mean	$\frac{4 \cdot 5}{240000}$
	$\left\{Co\left[\begin{matrix}HO\\HO\end{matrix}Co(NH_3)_4\right]_3\right\}Cl_6$		$\frac{1}{240000}$

concentrations were added to each. During addition the tubes were vigorously shaken to ensure thorough mixing. In this way two such concentrations are obtained that the lower does not show any perceptible change in five minutes whereas the higher does. This concentration is then more carefully examined and the limiting concentration determined. In Table I the limiting concentrations corresponding to the different salts are given. For comparison the limiting concentrations of a number of simple electrolytes were determined with the same sol. The results are given in Table IV. (p. 8)

FIG. 1



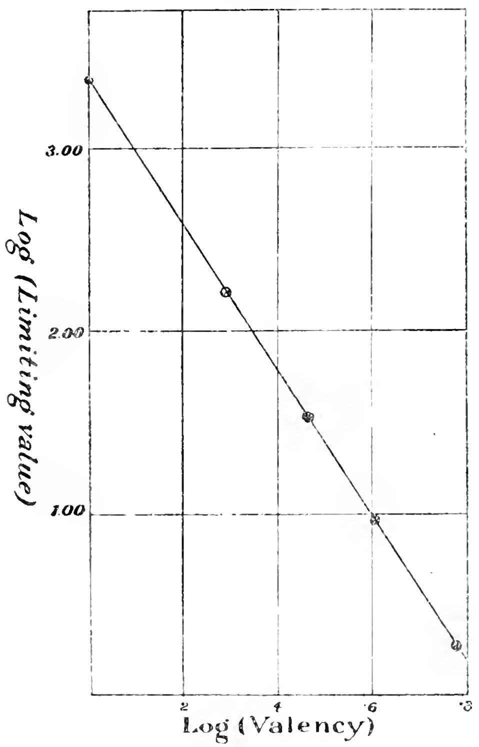
The Valency Rule on Coagulation

As regards the quantitative study of the effect of the valency on the coagulation of hydrosols, Whetham [Phil. Mag., V. **48**, 474, (1899)] deduced the following equation on considerations of probability of collision:

$$C_1 : C_2 : C_3 = K^3 : K^2 : K^1,$$

where C_1, C_2 and C_3 are the molecular concentrations of coagulation of mono-, di- and trivalent ions respectively, K is a constant. If this be true, it must be a straight line when plotted, taking the logarithm of the concentrations of coagulation and the valency of the ions as its axes. This did not, however, hold good in the present experiment. So far as the present investigation was concerned, it gave a curve, as shown by Fig. 1, in place of a straight line. Freundlich [Zs. physik. Chem., **73**, 385, (1910)] calculated approximately the precipitating concentrations by utilizing the adsorption curve and by assuming that the neutral salts in their equivalent solutions are, in despite of their valencies, adsorbed in the same manner. His idea can be expressed also as follows: Since the electric charges of mono-, di- and trivalent ions are in the ratio of 1 : 2 : 3, the amounts which should be adsorbed causing a complete coagulation should be in the ratio of 3 : 1·5 : 1, and if we plot the logarithm of the precipitating concentrations against those of the numbers , i.e. 3, 1·5 and 1 for mono-, di- and trivalent ions respectively it would give a straight line.

FIG 2



It seems that this idea of Freundlich was clearly verified by the present investigation. Taking the logarithm of the valencies of the cobaltammines as abscissa and those of the limiting values (for convenience the author took the values given in the third row of the table III) measured by the method already described, as ordinate, a straight line, as shown by Fig. 2, was obtained. From the diagram the author was able to deduce the following equations:

$$\text{Log } S_N + \beta \text{ Log } N + \alpha = 0 \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad (1)$$

S_N is the limiting value of a N-valent complex ion, while β and α are constants. Putting N=1 in the equation (1), we obtain,

$$\text{Log } S_1 + \alpha = 0$$

or

$$-\text{Log } S_1 = \alpha$$

and the equation (1) will now be,

$$\text{Log } S_N + \beta \text{ Log } N - \text{Log } S_1 = 0 \quad \ldots\ldots\ldots\ldots (2)$$

or

$$\text{Log } S_N = \text{Log } S_1 - \beta \text{ Log } N$$

or

$$S_N = S_1 \times \frac{1}{N^\beta} \quad \ldots\ldots\ldots\ldots\ldots\ldots (3)$$

Put into words, the limiting value of a N-valent cobalt complex ion is equal to one N^βth of that of the monovalent ion. The value of β in the above equation can be calculated from the equation, $\beta = \frac{\text{Log } S_1 - \text{Log } S_N}{\text{Log } N}$ substituting the experimental data for S_1 and S_N. The author obtained $\beta = 4$ approximately as shown by the table II.

TABLE II.

3·973	For divalent ions
3·989	,, tri- ,, ,,
4·076	,, tetra- ,, ,,
3·993	,, hexa- ,, ,,

Accordingly, putting $\beta = 4$ in the equation (3), the equation was deduced :

$$S_N = S_1 \times \frac{1}{N^4} \quad \ldots\ldots\ldots\ldots\ldots\ldots (4)$$

Namely, the limiting value of a N-valent cobalt complex ion is equal to one N^4th of that of monovalent complex ion so far as the cobaltammines and the arsenious sulphide sol are concerned.

The equation (3) could be also deduced theoretically as follows. In the adsorption isotherm, i.e. $\frac{x}{m} = \alpha C^{\frac{1}{n}}$, where α and n are constants, x, m, and C are respectively the quantity adsorbed, the quantity of the adsorbent, and the equilibrium concentration. As a first approximation, we may take $S-\frac{x}{m}$, or S in the place of C of the above equation [Freundlich, loc. cit.]. If we replace S with C, we obtain the next equation:

$$\frac{x}{m}=\alpha S^{\frac{1}{n}} \quad \ldots\ldots (5)$$

And comparing the case of the N-valent, and the monovalent, we can easily deduce the following equation:

$$\frac{x_N}{m}=\alpha S_N^{\frac{1}{n}} \quad \text{and} \quad \frac{x_1}{m}=\alpha S_1^{\frac{1}{n}}$$

$$\frac{x_N}{x_1}=\left(\frac{S_N}{S_1}\right)^{\frac{1}{n}} \quad \text{or} \quad \left(\frac{x_N}{x_1}\right)^n=\frac{S_N}{S_1} \quad \ldots\ldots (6)$$

According to the opinion of Freundlich,

$$\frac{x_N}{x_1}=\frac{1}{N}$$

Therefore the equation (6) may change to the following:

$$\left(\frac{1}{N}\right)^n=\frac{S_N}{S_1} \quad \ldots\ldots (7)$$

Thus it is clear that β in the equation (3) is identical with one of the constants of the adsorption isotherm. β has been found to be equal to 4 in the present experiments. It is quite possible in the case of the cobaltammines that the irregularity observed in coagulation experiments with the normal salts would be absent, as the cobaltammines used had all their co-ordination valencies satisfied, so that they had no further tendency for complex formation in aqueous solution, the adsorbability should, therefore, be the same for ions with different valencies and hence the effect of valency is likely to be the sole factor. As shown by the table III the limiting values calculated from the equation (4) are in fair agreement with those found by the experiment.

TABLE III.

Valency	1	2	3	4	5	6
Limiting value (Eq. Mol) (Observed)	$\frac{1}{187.5}$	$\frac{1}{3000}$	$\frac{1}{15000}$	$\frac{1}{53330}$	—	$\frac{1}{240000}$
or	$1{\cdot}280 \times \frac{1}{240000}$	80 ,,	16 ,,	4·5 ,,	—	1 ,,
,, (Calculated)	—	$\frac{1}{3000}$	$\frac{1}{15190}$	$\frac{1}{47990}$	—	$\frac{1}{243000}$

The figures of the second row of the above table are the mean of the limiting values obtained from seven monovalent, six divalent, seven trivalent, four tetravalent and one haxavalent cobaltammines as shown by table I. Unfortunately, as the author had not prepared a pentavalent cobaltammine, the figure is lacking. But it is very easy to predict the limiting value of the pentavalent complex ions. By using the above equation, it should be $\frac{1}{100000}$ for the same sol.

The valency effect for the normal salts.

In order to compare the results obtained by the cobaltammines with those of the normal salts, the experiment was undertaken by using the following salts and the same sol. Although the salts were not widely selected, there was much deviation from the theory as shown by the table IV, V, and Fig. 3.

TABLE IV

Electrolyte		Limiting value
1 KCL		$\frac{8}{150}$
2 KBr		,,
3 KNO_3		,,
4 NaCl		,,
5 $NaC_6H_{11}O_3$ (Sod. Leuciate)		$\frac{10}{150}$
	Mean	$\frac{8}{150}$

1	$BaCl_2 2H_2O$		$\frac{1}{1500}$
2	$NiSO_4 7H_2O$		,,
3	$Sr(NO_3)_2$		,,
4	$CaCl_2$		,,
5	$CoCl_2 6H_2O$		$\frac{4}{5000}$
6	$FeSO_4 7H_2O$		,,
7	$ZnSO_4 7H_2O$		,,
		Mean	$\frac{11}{15000}$
8	$CuSO_4 5H_2O$		$\frac{10}{5000}$
9	$Zn(CH_3COO)_2$		$\frac{1}{1071}$
1	$Ce_2(SO_4)_3 8H_2O$		$\frac{1.6}{30000}$
2	$Al_2(SO_4)_3 18H_2O$		$\frac{1}{30000}$
3	$Cr_2(SO_4)_3 18H_2O$		,,
1	$ZrCl_4$		$\frac{1}{10000}$
2	$Th(NO_3)_4 4H_2O$		$\frac{1}{24000}$

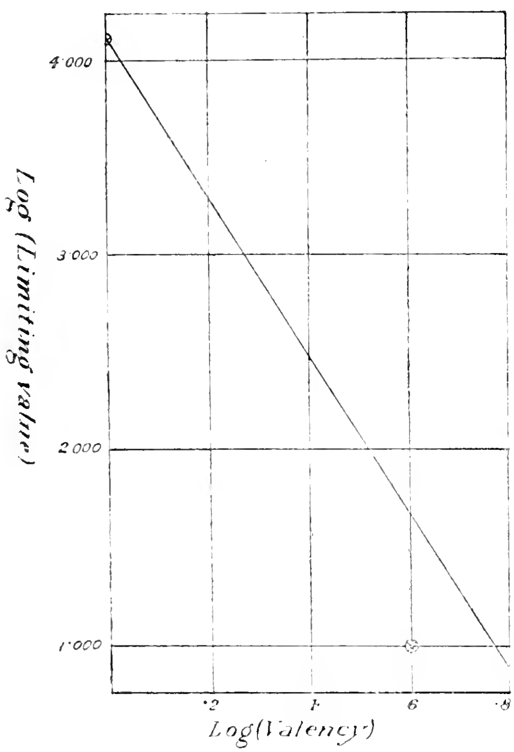
TABLE V.

Valency	1	2	3	4
Limiting value	$\frac{8}{150}$	$\frac{11}{15000}$	$\frac{1.6}{30000}$	$\frac{1}{24000}$
or	$12800 \times \frac{1}{240000}$	176 ,,	12·8 ,,	10 ,,

Fig. 3 was drawn in the same manner as the Fig. 2 was graphed.

For the monovalent ions, sodium chloride, potassium chloride, potassium bromide, potassium nitrate and sodium leuciate were used, the same limiting concentrations were obtained except the last one. For the divalent, barium chloride, nickel sulphate, strontium nitrate, calcium chloride, cobaltous chloride, ferrous sulphate, zinc sulphate, copper sulphate and zinc acetate were tested, the values of the last two salts were very different to those of the others, and even comparing the values of $Zn^{\cdot\cdot}$ both obtained by using zinc sulphate and zinc acetate, a difference was recognized. In the case of copper sulphate it is clear that the ionisation is not so simple as we consider, namely, $CuSO_4 aq. = Cu^{\cdot\cdot} + SO_4''$. It is possibly a complicated one. Considering the fact that an old aqueous solution of copper sulphate reacts acidic, the idea of the following partial hydrolysis is reasonable

FIG. 3



$$2CuSO_4 + 2H_2O = Cu_2(OH)_2SO_4 + H_2SO_4$$

But the author thinks that complex such as $Cu(SO_4)x(H_2O)y$ is possible. For the trivalent, cerium sulphate, aluminium sulphate and chromic sulphate were used and the last two gave the same value, but the first one was different. As tetravalent ions zirconium chloride and thorium nitrate were used. The limiting values obtained by these two salts were not only in disagreement with themselves, they were not suitable for the tetravalent ions.

In the case of these salts the following hydrolytic reactions are most probable:

$$Th(NO_3)_4 + H_2O = ThO(NO_3)_2 + 2HNO_3$$

$$ZrCl_4 + H_2O = ZrOCl_2 + 2HCl$$

Chemical Changes of the Cobaltammines in their Aqueous Solutions.

The chemical changes in the aqueous solutions of cobaltammines with co-ordinated radicals of strong acids were clearly explained by the author in his previous paper [This Journ., the same Vol. Art. 10]. He has been able to confirm it by the study of the limiting values as follow:

(A) $[Co(NH_3)_4NO_2Cl]Cl \cdot H_2O$

If the above salt dissociates in the aqueous solution, thus,

$$[Co(NH_3)_4NO_2Cl]Cl\,aq. \rightleftarrows [Co(NH_3)_4NO_2Cl]^{\cdot} + Cl'$$

the limiting value obtained by it will be $\frac{1}{187 \cdot 5}$ (see the chapter of the valency rule). But the experimental data did not agree therewith and it gave $\frac{1}{1250}$ in a fresh solution and $\frac{1}{2500}$ when 30 min. have elapsed. The latter figure corresponded with that of a divalent ion. It proves the truth of the author's suggestion that this kind of salt undergoes a further substitution as below:

$$[Co(NH_3)_4NO_2Cl]Cl + H_2O \rightarrow [Co(NH_3)_4NO_2H_2O]Cl_2$$

(B) $\left[Co\,en_2 \begin{matrix} Cl(1) \\ Cl(6) \end{matrix}\right] Cl$, $\left[Co(NH_3)_4 \begin{matrix} Cl(1) \\ Cl(6) \end{matrix}\right] Cl$ and $\left[Co\,en_2 \begin{matrix} Br(1) \\ Br(2) \end{matrix}\right] Br$.

It is a well known fact that the above mentioned salts undergo colour changes when they are dissolved in water. The author proved in his previous paper [loc. cit.] the following substitutions would take place:

$$\left[Co\,en_2\begin{matrix}Cl\,(1)\\ Cl\,(6)\end{matrix}\right]Cl:\ (1)\ \left[Co\,en_2\begin{matrix}Cl\,(1)\\ Cl\,(6)\end{matrix}\right]Cl + H_2O \rightarrow \left[Co\,en_2\begin{matrix}Cl\ \ (1)\\ H_2O\,(6)\end{matrix}\right]Cl_2$$

$$(2)\ \left[Co\,en_2\begin{matrix}Cl\ \ (1)\\ H_2O\,(6)\end{matrix}\right]Cl_2 + H_2O \rightarrow \left[Co\,en_2\begin{matrix}H_2O\,(1)\\ H_2O\,(2)\end{matrix}\right]Cl_3$$

$$\left[Co(NH_3)_4\begin{matrix}Cl(1)\\ Cl(6)\end{matrix}\right]Cl:\ (1)\ \left[Co(NH_3)_4\begin{matrix}Cl(1)\\ Cl(6)\end{matrix}\right]Cl + H_2O \rightarrow \left[Co(NH_3)_4\begin{matrix}Cl\ \ (1)\\ H_2O(6)\end{matrix}\right]Cl_2$$

$$(2)\ \left[Co(NH_3)_4\begin{matrix}Cl\ \ (1)\\ H_2O(6)\end{matrix}\right]Cl_2 + H_2O \rightarrow \left[Co(NH_3)_4\begin{matrix}H_2O(1)\\ H_2O(2)\end{matrix}\right]Cl_3$$

$$\left[Co\,en_2\begin{matrix}Br\,(1)\\ Br\,(2)\end{matrix}\right]Br:\ (1)\ \left[Co\,en_2\begin{matrix}Br\,(1)\\ Br\,(2)\end{matrix}\right]Br + H_2O \rightarrow \left[Co\,en_2\begin{matrix}Br\ \ (1)\\ H_2O\,(2)\end{matrix}\right]Br_2$$

$$(2)\ \left[Co\,en_2\begin{matrix}Br\ \ (1)\\ H_2O\,(2)\end{matrix}\right]Br_2 + H_2O \rightarrow \left[Co\,en_2\begin{matrix}H_2O\,(1)\\ H_2O\,(2)\end{matrix}\right]Br_3$$

With these substitutions, the valency of the salts varies also. The limiting concentration of $\left[Co\,en_2\begin{matrix}Cl\,(1)\\ Cl\,(6)\end{matrix}\right]Cl$ changed from $\frac{14}{15000}$ to $\frac{1}{3000}$ during three hours and after 24 hours it gave the value $\frac{1}{15000}$, i.e. value for the trivalent. It is obvious that the first substitution took place within half an hour and the second within 24 hours. As to $\left[Co(NH_3)_4\begin{matrix}Cl\,(1)\\ Cl\,(6)\end{matrix}\right]Cl$ and $\left[Co\,en_2\begin{matrix}Br\,(1)\\ Br\,(2)\end{matrix}\right]Br$, it seemed that the first substitution took place very rapidly, for the limiting values of both these fresh solutions were $\frac{1}{7500}$ and after 30 min. they gave the value $\frac{1}{15000}$ corresponding to the trivalent ion.

(C) $[Co(NH_3)_4H_2O\,Cl]SO_4$, $[Co(NH_3)_4H_2O\,Cl]Cl_2$

Each of the above mentioned salts gave the limiting value of a trivalent ion, i.e. $\frac{1}{15000}$ instead of that of a divalent ion which proved the following changes occurred in their aqueous solutions respectively:

$$[Co(NH_3)_4H_2OCl]SO_4 + H_2O \rightarrow \left[Co(NH_3)_4(H_2O)_2\right]\begin{matrix}Cl\\SO_4\end{matrix}$$

$$[Co(NH_3)_4H_2OCl]Cl_2 + H_2O \rightarrow [Co(NH_3)_4(H_2O)_2]Cl_3$$

(D) $[Co(NH_3)_5Cl]X_2$, $[Co(NH_3)_2(H_2O)_2Cl_2]SO_4H$,

$[Co(HN_3)_3(H_2O)_2Cl]SO_4$ and $[Co(NH_3)_3H_2OCl_2]Cl$

In the previous paper [loc. cit.] the author discussed the equilibrium of the aqueous solution of some cobaltammines and showed that a complicated equilibrium results. It was also confirmed by coagulation experiments. Freundlich [loc. cit.] expressed the same opinion regarding the salt $[Co(NH_3)_5Cl]Cl_2$. If in these salts the substitution occurs completely as follows:

$$[Co(NH_3)_5Cl)X_2 + H_2O \rightarrow \left[Co(NH_3)_5H_2O\right]\begin{matrix}X_2\\Cl\end{matrix}$$

$$[Co(NH_3)_2(H_2O)_2Cl_2]SO_4H + 2H_2O \rightarrow \left[Co(NH_3)_2(H_2O)_4\right]\begin{matrix}Cl_2\\SO_4H\end{matrix}$$

$$[Co(NH_3)_3(H_2O)_2Cl]SO_4 + H_2O \rightarrow \left[Co(NH_3)_3(H_2O)_3\right]\begin{matrix}SO_4\\Cl\end{matrix}$$

$$[Co(NH_3)_3H_2OCl_2]Cl + H_2O \rightarrow [Co(NH_3)_3(H_2O)_3]Cl_3$$

the limiting concentrations of these salts will have a value near about $\frac{1}{15000}$, since the resulting salts are all trivalent. But the experiments show that the reactions as indicated above are not complete. $[Co(NH_3)_5Cl]Cl_2$ gave $\frac{1}{6000}$ in a solution which stood for over a night. $[Co(NH_3)_2(H_2O)_2Cl]SO_4H$ gave $\frac{1}{3000}$ in a fresh solution and gave $\frac{1}{7500}$ after three hours and there was no change up to the next day. It shows that the following successive reactions occur:

(1) $[Co(NH_3)_2(H_2O)_2Cl_2]SO_4H + H_2O \rightarrow \left[Co(NH_3)_2(H_2O)_3Cl\right]\begin{matrix}Cl\\SO_4H\end{matrix}$

(2) $\left[Co(NH_3)_2(H_2O)_3Cl\right]\begin{matrix}Cl\\SO_4H\end{matrix} + H_2O \rightleftarrows \left[Co(NH_3)_2(H_2O)_4\right]\begin{matrix}Cl_2\\SO_4H\end{matrix}$

$[Co(NH_3)_3(H_2O)_2Cl]SO_4$ was rather stable in aqueous solution and gave the limiting concentration of $\frac{1}{7500}$ instead of $\frac{1}{15000}$ which verifies that there is an equilibrium as indicated below:

$$[Co(NH_3)_3(H_2O)_2Cl]SO_4 + H_2O \rightleftarrows \left[Co(NH_3)_3(H_2O)_3\right]\begin{matrix}Cl\\SO_4\end{matrix}$$

$[Co(NH_3)_3H_2OCl_2]Cl$ gave also the value of $\frac{1}{7500}$ in place of $\frac{1}{15000}$.

(E) $[Co(NH_3)_3(H_2O)_3\overset{(1)}{\underset{(3)}{(2)}}]Cl_3$ and $[Co(NH_3)_3(H_2O)_3\overset{(1)}{\underset{(6)}{(2)}}]Cl_3$

If the above two salts dissociate as $[Co(NH_3)_3(H_2O)_3]Cl_3aq. \rightleftarrows [Co(NH_3)_3(H_2O)_3]^{\cdots} + 3Cl'$ in their aqueous solutions, the limiting concentrations of these salts must be $\frac{1}{15000}$, but in fact both of them gave the value of $\frac{1}{7500}$ which is the same as those of

$$[Co(NH_3)_3(H_2O)_2Cl]SO_4, \quad [Co(NH_3)_3H_2OCl_2]Cl$$
$$\text{and} \quad [Co(NH_3)_2(H_2O)_2Cl_2]SO_4H.$$

This is the best confirmation that the following reversible reaction takes place to a certain extent:

$$[Co(NH_3)_3(H_2O)_3]Cl_3\,aq. \rightleftarrows [Co(NH_3)_3(H_2O)_2Cl]Cl_2\,aq.$$

Summary:

1.—The relation of the valency of the cobaltammines to their coagulating power on arsenious sulphide sol has been studied and it has shown that the limiting concentration can be expressed by the following formula,

$$S_N = S_1 \times \frac{1}{N^4}$$

where S_N is the limiting concentration (Eq. Mol) of a N-valent complex ion. The equation can be deduced from Freundlich's absorption theory.

2.—The valency of many simple and complicated cobaltammines was determined by means of the limiting concentration.

3.—By utilising the valency effect on the sol, the chemical changes in aqueous solutions of some cobaltammines has been followed. The results are in agreement with those obtained from conductivity measurements and absorption of light.

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