EUCLID'S

ELEMENTS OF GEOMETRY:

BOOKS I. II. III. IV. VI. AND PORTIONS OF BOOKS V. AND XI.

WITH

NOTES, EXAMPLES, EXERCISES, APPENDICES, AND A COLLECTION OF EXAMINATION PAPERS

ARRANGED BY

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PREFACE.

In this edition of those parts of Euclid's Elements most commonly read in schools and required for examinations, no fundamental changes have been made, the axioms and enunciations remaining the same as in Simson's edition. Minor changes have, however, been frequently introduced, when it appeared that, either by the insertion of an additional step or reference, the path of the beginner could be made smoother, or conciseness and clearness gained by the removal of redundant and antiquated phrases. In some cases, moreover, Euclid's proofs have been displaced in favour of shorter ones based on his methods. Well-known abbreviations of words have been freely employed throughout, but the symbols used in the text are only such as are allowed in all the principal public examinations, and these are employed from the very first in the belief that their use is easily acquired, and that, by enabling the proofs to be presented in an attractive form, they help rather than hinder the beginner. In some of the examples a few other symbols in common use in modern geometry have, on account of their great convenience, been employed.

The text is so arranged that the enunciation, figure, and proof of each proposition are all in view together. Notes and easy exercises are also directly appended to the propositions to which they refer, and numerous examples, fully worked out, are inserted at intervals as models; while at the end of the different books additional propositions connected with those books are given, followed by collections of miscellaneous exercises.

Great care has been taken to provide large and clear diagrams, and, by the use of varied lines, to mark the distinction between given lines and lines of construction, as well as between real and hypothetical constructions.

Appendices containing some of the simpler theorems of modern geometry, and a few alternative proofs of propositions on methods somewhat different from Euclid's, and for this reason not inserted in the body of the work, have been added, together with a collection of the questions lately set in various public examinations.

A. E. L.

INTRODUCTORY NOTE.

Geometry, as the name signifies, probably had its origin in land-surveying, of which the Egyptians had some knowledge as far back as 2000 B.C., and it was from Egypt that the Greeks, to whom the great development of the science which occurred between 600 B.C. and 400 A.D. is due, obtained their first knowledge of it.

One of the most celebrated of the early Greek mathematicians was Euclid, who was born about 330 B.C., and educated, probably, at Athens, but who afterwards settled at Alexandria, where he gained a great reputation as a teacher of geometry. Little is known of his life beyond a few sayings which tradition ascribes to him; among these is his reply to the Egyptian king that there was no royal road to geometry.

Euclid was the author of several works on mathematics, of which the "Elements" is the most important. It at once became the standard text-book on elementary mathematics, a position which, so far as the geometrical part of it is concerned, it has now held for more than 2000 years. The first part of the work was compiled from the writings of the earlier Greek mathematicians, Pythagoras and Hippocrates. The Greek text, on which our English ones are based, is that of an edition prepared by Theon (the father of Hypatia) about 380 A.D.

Books I., II., III., IV. and VI. of Euclid's Elements treat of *Plane* Geometry, Book V. of Proportion, Books VII., VIII. and IX. of rational numbers, Book X. of surd numbers, Books XI. and XII. of *Solid* Geometry, and Book XIII. contains additional propositions relating to the other books.

Most modern English editions of Euclid follow a translation of the Greek text made by Robert Simson, who was born in 1687, and was professor of mathematics at Glasgow University.

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BOOK XI.

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EXAMINATION PAPERS.

ABBREVIATIONS OF WORDS.

adj.	*for*	adjacent.
alt.	,,	altitude.
altern.	,,	alternando.
ax.	,,	axiom.
cent.	,,	centre.
chd.	,,	chord.
circ^{d}	,,	circumscribed.
com.	,,	common.
compo.	,,	componendo.
comp^{t}	,,	complement.
const.	,,	constant.
constr.	,,	construction.
conv.	,,	converse.
cont^{d}	,,	contained.
cor.	,,	corollary.
def.	,,	definition.
desc.	,,	describe.
diag.	,,	diagonal.
diam.	,,	diameter.
diff.	,,	difference.
dist.	,,	distance.
dup.	,,	duplicate.
equiang.	,,	equiangular.
equidist.	,,	equidistant.
equilat.	,,	equilateral.
equimult.	,,	equimultiple.
ext^{r}	,,	exterior.
ex. æq.	,,	ex æquali.
fig.	,,	figure.
harm.	,,	harmonic.
hyp.	,,	hypothesis.
hypot.	,,	hypotenuse.
insc^{d}	,,	inscribed.
int^{r}	,,	interior.
invert.	*for*	invertendo.
isos.	,,	isosceles.
mag.	,,	magnitude.
max.	,,	maximum.
mid.	,,	middle.
min.	,,	minimum.
mult.	,,	multiple.
no.	,,	number.
opp.	,,	opposite.
post.	,,	postulate.
prob.	,,	problem.
prod.	,,	produce.
prop.	,,	proposition.
prop^{n}	,,	proportion.
pt.	,,	point.
quad^{l}	,,	quadrilateral.
rad.	,,	radius.
rect.	,,	rectangle.
rect^{l}	,,	rectilineal.
rect^{r}	,,	rectangular.
reg.	,,	regular.
rem^{g}	,,	remaining.
rem^{r}	,,	remainder.
req^{d}	,,	required.
rt.	,,	right.
seg^{t}	,,	segment.
sim^{r}	,,	similar.
sq.	,,	square.
st.	,,	straight.
supp^{t}	,,	supplement.
tang.	,,	tangent.
tet^{n}	,,	tetrahedron.
theor.	,,	theorem.
vert^{l}	,,	vertical.

join AB *for* "draw a straight line from A to B."
rect. AB, CD ,, "the rectangle contained by AB and CD."

SYMBOLS.

The following symbols are used in the *Text*:—

$=$	*for*	any one of the phrases "is equal to," "are equal to," "be equal to," or "equal to."
$>$	"	"is greater than."
$<$	"	"is less than."
$\therefore$	"	"therefore."
$\because$	"	"because."
$\angle$	"	"angle."
$\triangle$	"	"triangle."
▱	"	"parallelogram."
$\perp$	"	"perpendicular."
$\parallel$	"	"parallel."
$::$[1]	"	"proportional."
$\odot$	"	"circle."
○ce	"	"circumference."

The following symbols are used in *Examples* only:—

AB^2	*for*	"the square described on AB."
AB.CD	"	"the rectangle contained by AB and CD."
$+$	"	"the sum of."
$-$	"	"the difference between."
$a:b::c:d$	"	"a is to b as c is to d."
$\frac{A}{B}$	"	"the ratio of A to B."

A symbol, when used in the plural, is followed by the letter s.

EUCLID'S
ELEMENTS OF GEOMETRY.

BOOK I.

DEFINITIONS.

1. A **point** is that which has no parts, or which has no magnitude.
2. A **line** is length without breadth.
3. The extremities of a line are points.
4. A **straight line** is that which lies evenly between its extreme points.
5. A **surface** (or *superficies*) is that which has only length and breadth.
6. The extremities of a surface are lines.
7. A **plane surface** is that in which any two points being taken, the straight line between them lies wholly on that surface.

NOTES.

A point has *position* but no *size*, and so cannot be divided into parts.

A straight line is sometimes called a *right* line.

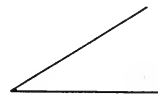
A plane surface is generally spoken of as "**a plane.**" (*e.g.*, By "a plane curve" is meant *a curved line drawn upon a plane surface.*)

EXERCISES.

1. How many *dimensions* has (i.) a solid, (ii.) a surface, (iii.) a line, (iv.) a point?
2. How many dimensions has (i.) a sheet of paper, (ii.) a shadow, (iii.) a brick, (iv.) a cricket-ball, (v.) a bit of spider's web?
3. Are the surfaces of the walls of an ordinary room plane?
4. How would you show that the surface of a cricket-ball is not plane?
5. How many plane surfaces has a well-cut square of plate-glass? How many edge-lines? How many corner-points?
6. Is it possible for a straight line to lie wholly upon the polished surface of a cedar pencil? Is this a plane surface?
7. Is a point a magnitude? Is a line a magnitude?
8. If two points were taken at random on a plane, would a straight line through those points bridge over any part of the plane?

8. A plane angle is the inclination of two lines to each other in a plane, which meet but are not in the same direction.

9. A **plane rectilineal angle** is the inclination of two straight lines to each other in a plane, which meet but are not in the same straight line.

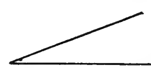


10. When a straight line standing on another straight line makes the *adjacent* angles equal, each is called a **right angle.**

The line which stands upon the other is called a **perpendicular** to it.

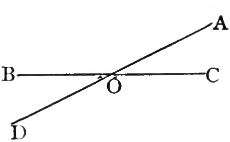
11. An **obtuse angle** is greater than a right angle.

12. An **acute angle** is less than a right angle.



13. A term, or boundary, is the extremity of anything.

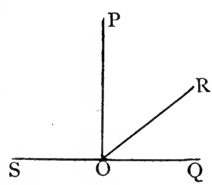
ADDITIONAL DEFINITIONS.



The *vertex* of an angle is the point at which the lines, which form the angle, meet.

Adjacent angles are such as lie next each other. Thus AOB and AOC are *adjacent* angles.

The angles AOC and BOD are called *opposite vertical* angles.



If a right angle is divided into any two parts, each part is called the *complement* of the other. Thus, if POQ is a right angle, the angle ROP is the complement of the angle ROQ.

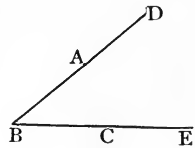
The angle ROS is called the *supplement* of ROQ.

The *bisector* of an angle is a straight line which divides it into two equal angles.

NOTES.

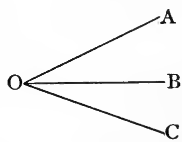
Def. 8 is of no importance, for only angles contained by *straight* lines are dealt with in Elementary Geometry, and "angle" when it occurs always means *plane rectilineal angle.*

An angle is generally denoted by three letters, *the letter which stands at the vertex of the angle being placed between the other two.* The *same* angle may be named in various ways; for instance



ABC, CBA, DBE, EBD, ABE, EBA, DBC, CBD are all names for the angle here represented.

N.B.—The *size* of an angle does not depend on the *length* of the lines.

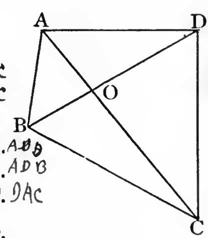


If OA, OB, OC are three straight lines which meet at O, the angle AOC is the *sum* of the angles AOB and BOC; the angle BOC is the *difference* of the angles AOC and AOB.

An angle equal to one-ninetieth part of a right angle is called a *degree* (written thus 1°).

EXERCISES.

1. Write down names for all the angles in the accompanying figure which have (i.) vertex A, (ii.) vertex B, (iii.) vertex C, (iv.) vertex D, (v.) vertex O.

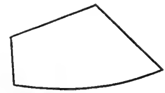


2. Name the angle which is equal to

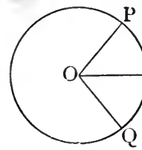
   (i.) the sum of the angles ABO and CBO.
   (ii.) the sum of the angles ABD and DBC.
   (iii.) the sum of the angles BCO and ACD.
   (iv.) the difference of the angles ADC and BDC.
   (v.) the difference of the angles CDA and CDO.
   (vi.) the difference of the angles BAD and BAC.

3. Mention a pair of adjacent angles with vertex O.
4. Mention a pair of opposite vertical angles with vertex O.
5. Mention an angle which is the supplement of the angle AOD.
6. Is an angle of 93° acute or obtuse?
7. What is the complement of an angle of 45°?
8. What is the supplement of an angle of 125°?
9. Are angles of 23° and 67° complementary?
10. Are angles of 68° and 102° supplementary?

14. A **figure** is that which is enclosed by one or more boundaries.

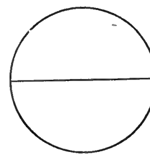


15. A **circle** is a plane figure contained by one line called the *circumference,* and is such that all straight lines drawn from a certain point within the figure to the circumference are equal to one another.

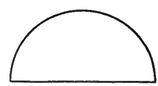


16. This point is called the **centre** of the circle.

17. A *diameter* of a circle is a straight line drawn through the centre and terminated both ways by the circumference.



18. A *semicircle* is a figure contained by a diameter and the part of the circumference that it cuts off.



19. A *segment* of a circle is a figure contained by a straight line and the part of the circumference that it cuts off.

ADDITIONAL DEFINITIONS.

A *radius* of a circle is a straight line drawn from the centre to the circumference.

(In the fig. of Def. 15, OP and OQ are radii.)

An *arc* is part of a circumference.

Concentric circles are such as have the same point as centre.

A *chord* of a circle is a straight line joining two points on its circumference.

The **perimeter** of a figure (*i.e.* the measure round it) is the sum of the lengths of its sides.

The **area** of a figure is the surface (or space) enclosed by its boundaries.

EXERCISES.

1. Is a circumference a figure?
2. Is an angle a figure?
3. Is an arc a figure?
4. Is a semicircle a plane figure?
5. Can a figure be bounded by one straight line?
6. Mention a figure which is bounded by two lines.
7. If the diameter of a circle is 3 feet long, what is the length of its radius?
8. The radius of a circle is 7 inches long; find the length of a diameter.
9. Draw a plane six-sided figure.
10. If two figures are equal in area, must they be of the same shape?
11. In Figure 1 mention by name

    (i.) a radius,
    (ii.) a diameter,
    (iii.) an arc,
    (iv.) a chord,
    (v.) a segment.

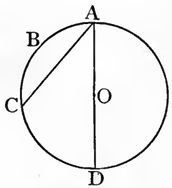


Fig. 1.

12. In Figure 2 mention by name

    (i.) each circle,
    (ii.) their common radius,
    (iii.) their common chord,
    (iv.) their centres.

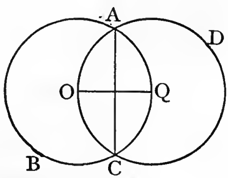


Fig. 2.

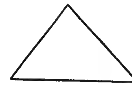
13. With a pair of compasses, describe

    (i.) three unequal circles which do not cut each other;
    (ii.) three equal circles which mutually cut each other;
    (iii.) three concentric circles.

14. Find the *locus* (or path) of a point which moves in a plane, so that its distance from a given fixed point is always the same.
15. If a line, one extremity of which is fixed, revolve in a plane, what is the locus of its other extremity?

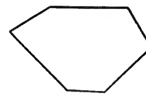
20. **Rectilineal figures** are such as are contained by straight lines.

21. A *trilateral figure*, or **triangle**, is contained by three straight lines.



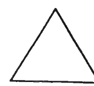
22. A **quadrilateral figure** is contained by four straight lines.

23. A *multilateral figure*, or **polygon**, is contained by more than four straight lines.



TRIANGLES.

24. An **equilateral triangle** has three equal sides.



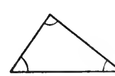
25. An **isosceles triangle** has two of its sides equal.

26. A **scalene triangle** has three unequal sides.

27. A **right-angled triangle** has a right angle.

28. An **obtuse-angled triangle** has an obtuse angle.

29. An **acute-angled triangle** has three acute angles.

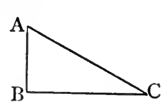


ADDITIONAL DEFINITIONS.

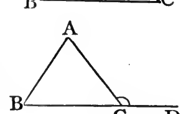
The side opposite to any angle of a triangle is said to *subtend* that angle.

In a *right-angled* triangle, the side which subtends the right angle is called the **hypotenuse**.

(For example, if ABC is the right angle, AC is the hypotenuse.)



If a side BC of a triangle be produced to D, the angle ACD is called an *exterior* angle of the triangle.



NOTES.

In *Defs.* 24, 25, 26, triangles are classed according to *sides;* in *Defs.* 27, 28, 29, according to *angles.*

It is often convenient to distinguish one particular side of a triangle by calling it the *base.* The angular point opposite to the base is called the *vertex* of the triangle. In the case of an isosceles triangle, the side which is not equal to either of the others is the base.

EXERCISES.

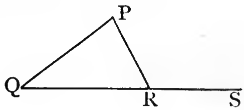
1. Is a semicircle a rectilineal figure?

2. The side QR of the triangle PQR is produced to S; mention

   (i.) an exterior angle of the triangle?

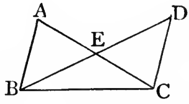
   (ii.) its adjacent interior angle?

   (iii.) the two opposite interior angles?



3. Which side subtends the angle PRQ?

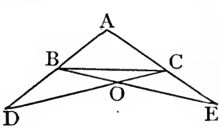
4. Write down names for the five triangles in the accompanying figure.



5. Draw two triangles standing on the same base and on the same side of it, with the vertex of each outside the other.

6. Mention names for the eight triangles in the accompanying figure.

7. Mention exterior angles of the triangle BDO.



8. What angles does OC subtend?

9. Mention a pair of vertically opposite angles with vertex O.

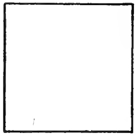
10. If one side of an equilateral triangle is 3 feet long, find the perimeter of the triangle.

11. One of the equal sides of an isosceles triangle is 3 feet long, and the base is 2 feet; find its perimeter.

12. What is the perimeter of an equilateral polygon of five sides, if the length of one side is 2 feet?

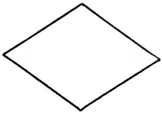
QUADRILATERALS.

30. A **square** is a four-sided figure with all its sides equal, and its angles right angles.



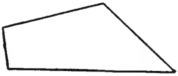
31. An *oblong* is a four-sided figure with all its angles right angles, but only its opposite sides equal.

32. A **rhombus** is a four-sided figure with all its sides equal, but its angles not right angles.



33. A *rhomboid* is a four-sided figure with its opposite sides only equal, and its angles not right angles.

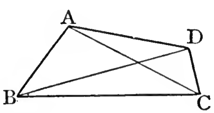
34. All other four-sided figures are called *trapeziums.*



ADDITIONAL DEFINITIONS.

A **diagonal** of a quadrilateral is a straight line joining two opposite angles.

(AC and BD are the diagonals of the quadrilateral ABCD.)



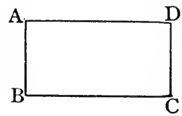
To **bisect** is to divide into two *equal* parts.

To *trisect* is to divide into three equal parts.

NOTES.

The names oblong, rhomboid, and trapezium are seldom used. Practically, an oblong is spoken of as "a *rectangle*," or "a *rectangular figure;*" a rhomboid as "a *parallelogram;*" and a trapezium as "a *quadrilateral.*" It will be shown hereafter (Prop. 34) that the class of figures defined as rectangles includes the oblong, and the class defined as parallelograms includes the rhomboid.

A four-sided figure is often denoted by *two* letters which stand at opposite corners. Thus the rectangle ABCD may be referred to as "rectangle AC," or as "rectangle BD."



EXERCISES.

1. What is meant by an equiangular figure?
2. What by a rectangular figure?
3. What by an equilateral figure?
4. What by a rectilineal figure?
5. Is a rhombus equilateral?
6. Is a square rectangular?
7. Show how to draw a line which shall divide a given square into

   (i.) two right-angled triangles;

   (ii.) two trapeziums.

8. If the length of one side of a rhombus is 2 feet, what is its perimeter?
9. Mention a dozen common objects which have a rectangular surface.
10. How many rectangular surfaces has a brick?
11. The diagonals of a quadrilateral ABCD cut at O. Write down names for all the different triangles formed.
12. If two right-angled triangles stand on opposite sides of a line which is their common hypotenuse, is the figure formed necessarily an oblong?

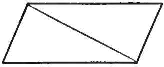
35. **Parallel straight lines** are such as lie in the same plane, and which, being produced ever so far both ways, do not meet.

ADDITIONAL DEFINITIONS.

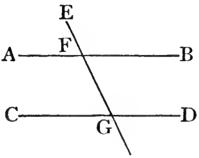
A **parallelogram** is a four-sided figure with its opposite sides parallel.

A **rectangle** is a right-angled parallelogram.

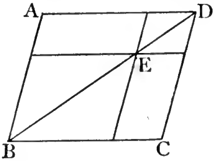
A *diameter of a parallelogram* is a straight line joining two opposite angles (*i.e.* a *diagonal*).



If a straight line EFG meets two parallel straight lines AB and CD, such a pair of angles as AFG and FGD are called **alternate angles**; such an angle as EFA is called an *exterior angle*, and FGC is the angle *interior and opposite* to EFA.



If, through any point in a diameter of a parallelogram, straight lines are drawn parallel to the sides, the figure is divided into four parallelograms. Of these four, the two through which the diameter passes, are called *parallelograms about the diameter;* and the other two, which complete the whole figure, are called the **complements**.



(For example, AE and EC are complements.)

A figure which has all its sides equal, and all its angles equal, is called a **regular** figure.

A quadrilateral with two sides parallel, and two not parallel, is sometimes called a *trapezoid.*

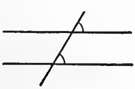
NOTE.

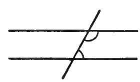
Pentagon, hexagon, octagon, decagon, dodecagon, quindecagon are names used for polygons of 5, 6, 8, 10, 12, and 15 sides respeetively.

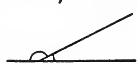
EXERCISES.

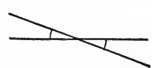
1. If two straight lines, in different planes, when produced ever so far both ways, did not meet, would they be necessarily parallel?
2. Show how to divide a quadrilateral into four triangles by lines drawn from a point within the figure.
3. Divide a pentagon into as many triangles as the figure has sides, by lines drawn from a point within it.
4. Define a regular hexagon. Make a sketch of the figure.
5. Define a regular decagon. Make a sketch of the figure.
6. If one side of a regular octagon is 3 inches long, what is its perimeter?
7. What are the marked angles called in the following diagrams?

(i) 

(ii) 

(iii) 

(iv) 

(v) 

8. Give the meaning of the words, *magnitude, rectilineal, acute, equilateral, isosceles, complement.*
9. Draw two straight lines which cut.
10. Draw two straight lines which meet.
11. Class triangles according to their sides.
12. Make a sketch of a trapezoid.
13. Make a sketch of a right-angled isosceles triangle.
14. Has every triangle a hypotenuse?
15. Has every figure an area? Is an angle a figure?
16. Define a plane.
17. Does the magnitude of an angle depend on the length of its arms?
18. Define a superficies.

POSTULATES.

Let it be granted:

1. That a straight line may be drawn from any one point to any other point.
2. That a terminated straight line may be produced to any length in a straight line.
3. That a circle may be described from any centre, at any distance from that centre.

AXIOMS.

1. Things that are equal to the same thing are equal to one another.
2. If equals be added to equals the wholes are equal.
3. If equals be taken from equals the remainders are equal.
4. If equals be added to unequals the wholes are unequal.
5. If equals be taken from unequals the remainders are unequal.
6. Things that are double of the same thing are equal.
7. Things that are halves of the same thing are equal.
8. Magnitudes which *coincide* with one another are equal.
9. The whole is greater than its part.
10. Two straight lines cannot enclose a space.
11. All right angles are equal to one another.
12. If a straight line meet two straight lines so as to make the two interior angles on the same side of it together less than two right angles, these two straight lines will, if produced, meet on that side on which the angles are less than two right angles.

NOTES.

The postulates are three problems which are assumed to be possible. They amount to a statement that certain instruments are necessary, and sufficient, for constructing the figures of the propositions. These instruments are:—

(i) A *ruler* for drawing straight lines, but *not* graduated for measuring their lengths.

(ii) *Compasses* for describing circles, but *not* to be used for carrying measurements from one place to another.

The Axioms are theorems the truth of which is assumed to be evident without proof. Euclid calls them *Common Notions*.

Axioms 1 to 7, and 9 are *general*, that is, they do not apply to geometrical magnitudes only.

Axioms 8, 10, 11, and 12 are *geometrical*.

The order of the earlier axioms is easily learned by noticing that it is the same as in the early rules of arithmetic:—

Axioms 2 and 3—*Addition* and *Subtraction* of equals.

Axioms 4 and 5—*Addition* and *Subtraction* of unequals.

Axiom 6—*Multiplication*.

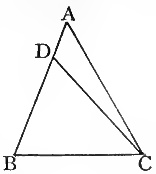
Axiom 7—*Division*.

Axiom 8 has been called Euclid's test of equality. It states that two magnitudes (whether *lines*, *angles*, or *figures*) are equal, if one can be so placed on the other that all their parts agree; if, in short, they *coincide*.

Axiom 12 is not required before the 29th Proposition; and the beginner may postpone the consideration of it until that proposition is reached.

EXERCISES.

1. How many straight lines can be drawn between two given points?
2. ABC, DEF are two equal straight lines; the part AB is equal to the part DE; is BC equal to EF?
3. State the *converse* of Axiom 8. Is this always true?
4. Are the areas of two pages of the same book equal?
5. Is the area of the triangle ABC equal to the area of the triangle DBC in the accompanying figure?



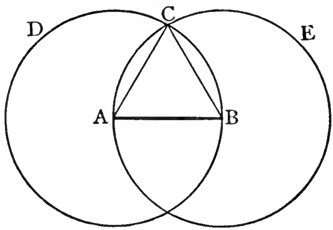
6. What is the least number of straight lines that can form a figure?
7. Can two straight lines cut in more than one point?
8. AB and CD are equal straight lines; AB is bisected at E, CD is bisected at F; is AE equal to CF?

PROPOSITION I. Problem.

To describe an equilateral triangle on a given finite straight line.

Let AB be the given st. line.

It is req[d] *to desc. an equilat.* △ *on AB.*



With cent. A and rad. AB desc. ⊙ BCD..................Post. 3.

With cent. B and rad. AB desc. ⊙ ACE.....................Post. 3.

From pt. C, where the ⊙s cut, draw st. lines CA, CB.....Post. 1.

Then shall △ ABC be equilat.

For, since A is cent. of ⊙BCD

∴ AB=AC..........................Def. 15.

and, since B is cent. of ⊙ACE

∴ AB=BC..........................Def. 15.

Hence, since AC and BC are each of them equal to AB

∴ AC=BC..........................Ax. 1.

and AB, AC, BC are all equal.

Wherefore, *the △ ABC is equilat. and has been described on AB.* Q.E.F.

NOTES.

The letters Q.E.F. which stand at the end of a problem, are the initials of the words Quod erat faciendum (*which was to be made*).

In succeeding propositions when I. 1 occurs among the references printed on the right-hand side of the page, it stands for "Book I., Proposition 1".

The propositions are either Problems or Theorems.

In a *problem* something has to be made, or done, with the ruler and compasses of the postulates.

In a *theorem* some geometrical truth, or property of a figure, has to be proved, or deduced, from the axioms.

The statement at the head of a proposition is called its *enunciation.*

The condition, subject to which any proposed problem is to be solved, or theorem proved, is called the *hypothesis.** (For example, the hypothesis of Prop. I. is that the equilat. triangle must be constructed "*on a given finite st. line*," *i.e.* on a st. line whose position and length have both been fixed beforehand.)

The beginner should carefully notice the three distinct parts of the proposition; they are—

1st. The *Particular Enunciation*, or re-statement of the general enunciation with reference to a figure.

2nd. The *Construction*, or directions for drawing such lines and circles as are needed to solve the problem.

3rd. The *Proof* that the figure made is really such as was required.

He should also compare the statement "then *shall* △ABC be equilat." at the beginning of the proof with "wherefore △ABC *is* equilat." at the end. And, in writing out a proposition, he should on no account attempt to draw the figure first, but should allow it to grow, step by step, as he writes down the construction.

The letters Q.E.D., at the end of a theorem, stand for Quod erat demonstrandum (*which was to be proved*).

EXERCISES.

(Exs. 1 to 5 refer to the figure of Prop I.)

1. If F be the other point of intersection of the circles, prove that FAB is an equilateral triangle.
2. Prove that FBCA is an equilateral figure.
3. Find a point P in AB produced such that AP is double of AB.
4. On AB as base describe an isosceles triangle with its sides each double of AB.
5. If with centre A and radius less than AB a circle be described cutting AB in P and AC in Q, prove that PB is equal to QC.
6. P and Q are fixed points; it is required to find a point which shall be equidistant from P and Q.
7. Show how to obtain a straight line which shall be treble of a given finite straight line.
8. Two concentric circles have centre O; radii OA, OB of the smaller circle are produced to meet the circumference of the larger circle in C and D respectively; prove that AC is equal to BD.

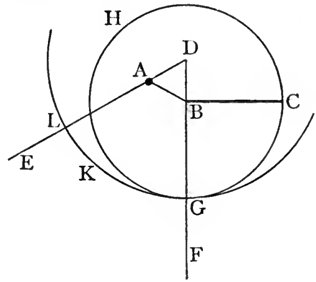
* In a problem the given conditions are also called the *data*.

PROPOSITION II. PROBLEM.

From a given point to draw a straight line equal to a given straight line.

Let A be the given pt., and BC the given st. line.

It is req^d to draw from A a st. line=BC.



Join AB ....Post. 1.
On AB desc. an equilat.
△DAB........................I. 1.
Produce DA, DB to E, F......Post. 2.
With cent. B, rad. BC desc.
⊙CGH cutting DF at G...Post. 3.
With cent. D, rad. DG desc.
⊙GKL cutting DE at L...Post. 3.

Then shall AL=BC.

For, since B is cent. of ⊙CGH
∴ BC=BG ....Def. 15.
and, since D is cent. of ⊙GKL
∴ DL=DG ....Def. 15.
but part DA=part DB........................Constr.
∴ rem^r AL=rem^r BG........................Ax. 3.
Hence, AL and BC are each of them equal to BG
∴ AL=BC...............................Ax. 1.

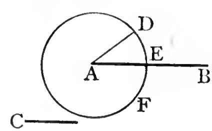
Wherefore, *from the given pt. A* &c. Q.E.F.

PROPOSITION III. PROBLEM.

From the greater of two given straight lines to cut off a part equal to the less.

Let AB and C be the two given st. lines, AB being the greater.

It is req^d to cut off from AB a part=C.



From A draw AD=C...................I. 2.
With cent. A, rad. AD, desc.
⊙DEF cutting AB in E...................Post. 3.

Then shall AE=C.

For, since A is cent. of ⊙DEF
∴ AE=AD..............................Def. 15.
but AD=C..............................Constr.
∴ AE=C...............................Ax. 1.

Wherefore, *from the given st. line AB* &c. Q.E.F.

NOTES.

The complex figure of Prop. 2. is caused by the restrictions in the postulates. (See note on Post. 3.)

Prop. 2 should be practised with varied positions of the given line and point, and with different sets of letters for the figures.

Since *either end* of the given line might be joined to the given point; the equilateral triangle described on *either side* of this line; and the sides of the triangle produced in *either direction*, it follows that, for any given

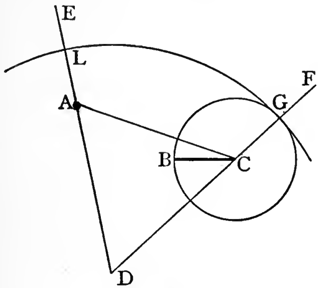


Fig. 1.

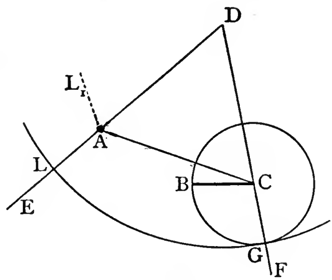


Fig. 2.

positions of the line BC and the point A, eight lines each equal to BC can, by varying the figure, be obtained at A. Fig. 1, Fig. 2 are examples of such varieties.

EXERCISES.

1. Draw the six other figures of the set to which Fig. 1 and Fig. 2 belong.
2. Make the figure and give the proof of Prop. 2 for the special case when the given point is *in* the given line.
3. In what special cases is it unnecessary to produce the sides of the equilateral triangle in Prop. 2?
4. In the figure of I. 2 AD, BD are produced to meet the circumference of the larger circle in P and Q respectively; prove that AP is equal to BQ.
5. On a given straight line as base describe an isosceles triangle having each of its other sides equal to a given straight line.

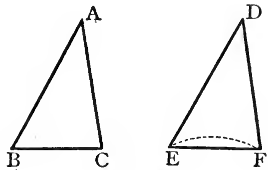
   In what case is this impossible?
6. In the figure of I. 3, from AB, or AB produced, cut off a part equal to three times C.
7. On a given straight line as base construct a triangle having its other sides equal to two given straight lines.

   When is this impossible?
8. Equal circles, whose centres are O and Q, cut the line OQ at A and B respectively. Prove that OB is equal to AQ.

PROPOSITION IV. Theorem.

If two triangles have two sides of the one equal to two sides of the other, each to each, and have also the angles, contained by these sides, equal; they shall have their bases, or third sides, equal, and the two triangles shall be equal, and their other angles shall be equal, each to each, namely, those to which the equal sides are opposite.

Let ABC, DEF be two $\triangle$s,
having AB=DE
AC=DF
$\angle$BAC=$\angle$EDF.



Then shall BC=EF
$\triangle$ABC=$\triangle$DEF
$\angle$ABC=$\angle$DEF
$\angle$ACB=$\angle$DFE.

For, if $\triangle$ABC be applied to $\triangle$DEF, with pt. A on D, and AB lying along DE, then

pt. B must fall on E, ∵ AB=DE..........Hyp.
AC must lie along DF, ∵ $\angle$BAC=$\angle$EDF......Hyp.
pt. C must fall on F, ∵ AC=DF..........Hyp.

Now, since B falls on E, and C on F,
BC must *coincide* with EF,
or two st. lines would enclose a space
which is impossible ....Ax. 10.

Hence, since BC coincides with EF,
∴ BC=EF..........................Ax. 8.

and, since the $\triangle$s coincide,

∴ { $\triangle$ABC=$\triangle$DEF, $\angle$ABC=$\angle$DEF, $\angle$ACB=$\angle$DFE } ....Ax. 8.

Wherefore, *if two triangles* &c. Q.E.D.

NOTES.

This proposition, the first of Euclid's theorems, is of great importance, and the beginner should not attempt to proceed further until he has thoroughly digested it, and can apply its results in examples.

A triangle has *seven* parts: 3 sides, 3 angles, and an *area*.

When all the parts of one triangle are respectively equal to all the part of another triangle, the triangles are said to be "equal in all respects."

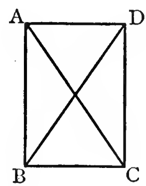
The 4th Prop. may be enunciated shortly thus: *If two sides and the included angle of one triangle are known to be equal to two sides and the included angle of another, these triangles must be equal in all respects.*

Prop. 4 is proved by "the method of superposition," *i.e.* it is shown that one of the triangles could be so placed on the other as exactly to cover it without overlapping, and then, since all their parts would *coincide*, the truth of the proposition follows from Axiom 8.

In this, and all succeeding theorems, the particular enunciation is arranged thus: the known facts, which form the *hypothesis*, stand first; then follow, printed in dark type, those which must not be taken for granted, but are to be proved. This part is called the *conclusion*.

The proof of the following theorem is given as an example of the way in which the results of Prop. 4 may be used:

THEOREM. *The diagonals of an oblong are equal.*



Let ABCD be an oblong, AC and BD its diagonals;
then shall AC=BD.

For in △s ABC, DBC,

∵ { AB=DC .... Def. 31.
BC is common to both △s
rt. ∠ABC=rt. ∠DCB .... Ax. 11. }

∴ AC=BD .... by Prop. 4.

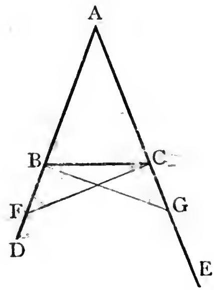
Q.E.D.

EXERCISES.

1. If two triangles have two sides of one equal to two sides of the other, must the triangles be equal in all respects?
2. AB is a straight line; D is its middle point; DC is a line at right angles to AB; prove that CA is equal to CB.
3. The diagonals of a square are equal.
4. ABC is an isosceles triangle having AB equal to AC; AD bisects the vertical angle BAC and meets the base BC at D; prove that BC is bisected at D.
5. The bisector of the vertical angle of any isosceles triangle is at right angles to the base.
6. D and E are the middle points of the equal sides AB, AC of an isosceles triangle; prove that the triangles ABE and ACD are equal in all respects.
7. In the figure of I. 2, if AG and BL be joined, AG is equal to BL.
8. In the figure of I. 2, if AG and BL be joined, prove that the angle AGD is equal to the angle BLD.
9. In the figure of I. 3, if with centre A and radius AB a circle be described, and AD be produced to meet this circle in G, the triangles AGE and ABD are equal in all respects.
10. ABCDE is a regular pentagon; prove that BD is equal to CE.

PROPOSITION V. Theorem.

The angles at the base of an isosceles triangle are equal to one another; and, if the equal sides be produced, the angles on the other side of the base are also equal.



Let ABC be an isos. △ having AB=AC,
and let AB, AC be prod[d] to D, E.

Then shall (i) ∠ABC=∠ACB,
(ii) ∠DBC=∠ECB.

In BD take any pt. F.
From AE cut off AG=AF ....I. 3.
Join BG, CF,........................Post. 1.

Then, in the △s AFC, AGB,

∵ { AF=AG.....................Constr.
AC=AB.....................Hyp.;
∠BAC is com.:

∴ { FC=BG
∠AFC=∠AGB*
∠ACF=∠ABG } ....I. 4

Again, the whole AF=whole AG ....Constr.
and part AB=part AC ....Hyp.
∴ rem[r] BF=rem[r] CG..............Ax. 3.

Hence, in △s FBC, GCB,

∵ { BF=CG
FC=BG
∠BFC=∠CGB* } ....proved above.

∴ { ∠FBC=∠GCB
∠FCB=∠GBC } ....I. 4.

Now, the whole ∠ABG=whole ∠ACF,
and part ∠GBC= part ∠FCB; } ......proved above.
∴ rem[r] ∠ABC=rem[r] ∠ACB ....Ax. 3.
which are the angles at the base,
and, from above, ∠FBC=∠GCB
which are the angles on the other side the base.

Wherefore, *the angles at the base* &c. Q.E.D.

Cor. An equilat[l] △ must also be equiang[r].

---

* *N.B.*—AGB and CGB are both names for the *same angle.*

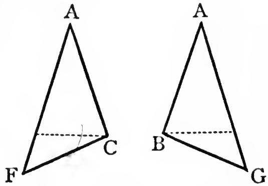
NOTES.

The 5th Prop. is proved by a double application of the 4th, and so offers little difficulty to one who has thoroughly understood the 4th. It may, however, help the beginner if he draws separate figures for the pairs of triangles dealt with. Thus, in the first pair of triangles:—

AF is known to be equal to AG (constr.).

AC is known to be equal to AB (given).

FAC, BAG are both names of the same angle BAC, which is common to both triangles.



Hence, since we find two sides, and the angle contained by them, of △FAC equal to the same three parts of △BAG, we know, from Prop. 4, that *all* the parts of these two triangles are equal.

N.B.—In writing out a proof of the equality of two triangles, all parts of the left-hand figure should stand on the *left*, and those of the right-hand figure on the *right*, of the sign of equality. Also, in written work clearness is gained by writing the name of the angle partly within the angle symbol; *e.g.* ∠ABC.

The first part of Prop. 5 may be enunciated thus—*If two sides of a triangle are equal, two of its angles must also be equal.*

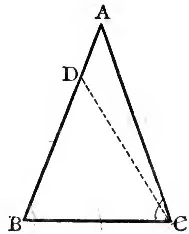
A *corollary* is the statement of some geometrical fact not mentioned in the enunciation, but the truth of which may be inferred from the proof of the proposition.

EXERCISES.

1. In the figure of I. 3 if DE be joined ∠ADE = ∠AED.
2. In the figure of I. 5 prove that ∠FCG = ∠FBG.
3. If, in the figure of I. 5, FG be joined, prove that the △FBG is equal in area to the △FCG.
4. If, in the figure of I. 5, FC and BG meet at H, prove that △s BFH, CGH are equal in area.
5. In the figure of I. 1, if the circles cut at C and D, prove that the angle CAD is equal to the angle CBD.
6. If the middle point of the base of an isosceles triangle be joined to the vertex, the figure is divided into two right-angled triangles.
7. ABC is an isosceles triangle. On the side of AB remote from C, any point P is taken and joined to C and B. Prove that angle PBC is greater than angle PCB.
8. AOB is a diameter of a circle whose centre is O. C is a point on the circumference. Prove that the angle ACB is equal to the sum of the angles CAB and CBA.
9. The opposite angles of a rhombus are equal.

PROPOSITION VI. Theorem.

If two angles of a triangle are equal to one another, the sides also which subtend the equal angles, are equal.



Let ABC be a △
having ∠ABC = ∠ACB.

Then shall AB = AC.

For, if AB is unequal to AC,
one must be the greater.
If possible, suppose AB > AC.
From AB cut off BD = AC ....I. 3.
Join DC ....Post. 1.

Then in △s DBC, ABC,

∵ { DB = AC ....Constr.
BC is com.
∠DBC = ∠ACB ....Hyp. }

∴ △DBC = △ABC ....I. 4.
i.e. the part = the whole,
which is absurd ....Ax. 9.

Hence, AB cannot be unequal to AC,

i.e. AB = AC.

Wherefore, *if two angles* &c. Q.E.D.

Cor. An equiang[r] △ must also be equilat[l].

NOTES.

Prop. 6 is the *converse* of the first part of Prop. 5; *i.e.* the hypothesis of Prop. 5 forms the conclusion of Prop. 6, and *vice versà*.

These two propositions may be stated thus:—

Prop. 5 (i). If a triangle is isosceles, ...Hyp.
two of its angles must be equal ....Conclusion.

Prop. 6. If a triangle has two angles equal, ...Hyp.
it must be isosceles ....Conclusion.

Prop. 6 is proved by an *indirect* method. In order to show that AB and AC are equal, the question, Can they possibly be unequal? is first considered; and this it is shown can only be the case if the area of the whole triangle ABC is equal to the area of its part the triangle DBC, which is contrary to axiom. Thus the truth of the proposition is evident.

This method of proof is sometimes called a *reductio ad absurdum*

EXERCISES.

1. In the figure of I. 5, if FC and BG meet at O, prove that OB is equal to OC.

2. In the figure of I. 6, prove that angle DCB is less than angle DBC.

3. ABC is an isosceles triangle, with AB equal to AC. The bisectors of the angles ABC and ACB meet at O. Prove that CO=BO.

4. ABC is a triangle; BA is produced to D; AD is cut off equal to AC, and DC joined. Prove that angle BCD is greater than angle BDC.

5. If, in the figure of I. 6, any point O be taken within the triangle ABC, prove that angle OBC is less than angle ACB.

6. If, in the figure of I. 6, a point P be taken on the side of AC remote from B, the angle PCB is greater than the angle PBC.

7. ABC is an isosceles triangle, having AB equal to AC; BO, CO, the bisectors of the angles ABC and ACB, meet at O, and AO is joined. Prove that the triangles ABO and ACO are equal in all respects.

8. If, in the figure of I. 6, CD be produced to E, and EB joined, the angle EBC is greater than the angle ECB.

9. If, in the figure of Ex. 8, EB and EC are produced to G and H respectively, prove that the angle GBC is less than angle HCB.

10. If, in the figure of I. 1, the circles cut at C and D, prove that CD bisects the angle ACB.

    Prove also that CD bisects AB.

11. ABCDE is a regular pentagon; prove that the triangle ACD is isosceles.

12. Explain the words *Geometry, Proposition, Problem, Theorem, Corollary, Enunciation, Hypothesis, Postulate, Axiom, Finite, Hexagonal, Octagonal.*

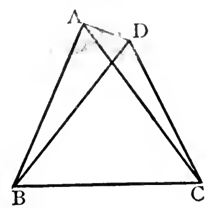
PROPOSITION VII. THEOREM.

On the same base and on the same side of it there cannot be two triangles which have their sides, terminated in one extremity of the base, equal to one another, and likewise those terminated in the other extremity.

For, if it be possible, let the $\triangle$s ABC, DBC, standing on the same side of BC, have

AB=DB
and also AC=DC.

CASE 1. When the vertex of each $\triangle$ is outside the other.



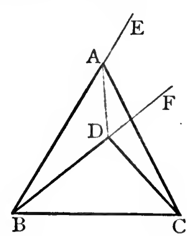
Join AD.................................Post. 1.

Then, since AB=DB..............Hyp.
∴ ∠BAD=∠BDA..........I. 5 (i).
but ∠BAD>∠CAD..........Ax. 9.
∴ ∠BDA>∠CAD
but ∠CDA>∠BDA..........Ax. 9.
much more ∴ ∠CDA>∠CAD.

Again, since DC=AC............................Hyp.
∴ ∠CDA=∠CAD.........................I. 5.

i.e. the ∠CDA is both = and > ∠CAD
which is impossible.

CASE 2. When the vertex D of one $\triangle$ is within the other.



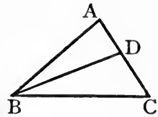
Join AD.................................Post. 1.
Produce BA to E, and BD to F......Post. 2.

Then, since AB=DB..............Hyp.
∴ ∠EAD=∠FDA..........I. 5 (ii).
but ∠EAD>∠CAD..........Ax. 9.
∴ ∠FDA>∠CAD
but ∠CDA>∠FDA..........Ax. 9.
much more ∴ ∠CDA>∠CAD.

Again, since DC=AC............................Hyp.
∴ ∠CDA=∠CAD.........................I. 5.

i.e. the ∠CDA is both = and > ∠CAD
which is impossible.

CASE 3. When the vertex of one △ is on a side of the other.



This case needs no proof.

Wherefore, *on the same base* &c. Q.E.D.

NOTE.

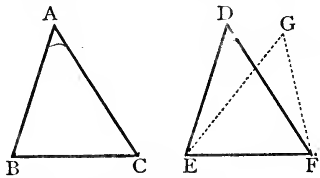
This negative theorem is only required by Euclid for the 8th Proposition. Some writers have used another proof of Prop. 8, omitting the 7th Proposition altogether. See note on Prop. 8.

EXERCISES.

1. Why does Case 3 "need no proof"?
2. Is it possible for two triangles, standing on the same side of the same base, to have their sides, which terminate in one extremity of the base, equal, and their sides which terminate in the other extremity, unequal?
3. On the same base, and on the same side of it, there can be but one equilateral triangle.
4. Two circles cannot cut in more than two points.
5. What axiom is implied in the proof of the 7th Proposition?
6. On the same base, but on opposite sides of it, are two acute-angled triangles, having their sides terminated in one extremity of the base equal, and, likewise those terminated in the other extremity; prove, by joining their vertices and using Proposition 5, that their vertical angles are equal.
7. ABC, DBC are two isosceles triangles standing on the same side of the same base BC; prove that the angle ABD is equal to the angle ACD.
8. ABC, DBC are two isosceles triangles standing on opposite sides of the same base BC; prove that the angle ABD is equal to the angle ACD.
9. AOB is an angle. OC is drawn within the angle AOB and OC is made equal to OB. If AB, AC, BC are joined prove that angle ACB is greater than angle ABC.

PROPOSITION VIII. Theorem.

If two triangles have two sides of one equal to two sides of the other, each to each, and have, likewise, their bases equal; the angle contained by the two sides of the one is equal to the angle contained by the two sides, equal to them, of the other.



Let ABC, DEF be two △s,
having AB=DE,
AC=DF,
and BC=EF.

Then shall ∠BAC=∠EDF.

For, if △ABC be applied to △DEF with pt. B on E, and BC along EF,

the pt. C must fall on F, ∵ BC=EF ....Hyp.

And if BA, AC did not fall on ED, DF, but had a different position such as EG, GF, then DEF, GEF would be two △s standing on the same side of EF and having DE=GE, and DF=GF,

which is impossible ....I. 7.

Hence BA, AC must fall on ED, DF,
and ∠BAC must coincide with ∠EDF
∴ ∠BAC=∠EDF ....Ax. 8.

Wherefore, *if two triangles* &c. Q.E.D.

NOTES.

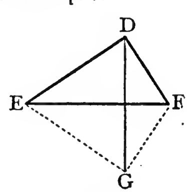
Since the triangles have been shown to coincide it follows that they are equal in *all* respects.

The 8th Proposition is a *converse* of the 4th, and, as is the case with most converse propositions, it is proved indirectly.

It should be noticed that, when the hypothesis of a proposition contains more than one condition, there will be more than one converse.

The following direct proof of Prop. 8 is independent of Prop. 7.

Apply △ABC to DEF so that their bases coincide, but with their vertices on *opposite* sides of EF. Let GEF be the new position of △ABC. Join DG.



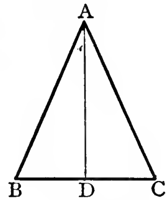
Then since ED=EG.................Hyp.
∴ ∠EDG=∠EGD ....I. 5.
Again, since FD=FG.................Hyp.
∴ ∠FDG=∠FGD ....I. 5.
Hence, the whole ∠EDF=whole ∠EGF...Ax. 2.

Q.E.D.

The following theorem illustrates the application of Prop. 8 to examples.

THEOREM. *The straight line which joins the vertex to the middle point of the base of an isosceles triangle, is at right angles to the base.*

Let ABC be an isos. △ and D the mid. pt. of base BC; then shall AD be at rt. ∠s to BC.



For, in △s ABD, ACD

∵ { BD=CD........................Hyp.
AD is com.
AB=AC........................Hyp }

∴ ∠ADB=∠ADC..................I. 8.

and these being equal adjacent ∠s, are rt. ∠s...........Def. 10.

∴ AD is at rt. ∠s to BC. Q.E.D.

EXERCISES.

1. In the figure of I. 1, if the circles cut at C and D, prove that angle ACB is equal to angle ADB.
2. In the figure of Euc. I. 5, if BG and CF cut at H, prove that—

   (i) ∠AHB=∠AHC
   (ii) ∠BHF=∠CHG.

3. ABC is an isosceles triangle; D is the middle point of the base BC; prove that AD bisects the vertical angle BAC.
4. A diagonal of a rhombus bisects the angles through which it passes.
5. Two isosceles triangles stand on the same side of the same base; prove that the line joining their vertices, when produced, meets the base at right angles.
6. Two circles, whose centres are O and Q, cut at P and R, show that the angle OPQ is equal to the angle ORQ.
7. Show, by the method of superposition, that a square is divided by its diagonal into two triangles of equal area.
8. A rhomboid is bisected by its diagonal.
9. Two isosceles triangles stand on opposite sides of the same base; show that the line joining their vertices bisects their vertical angles.
10. Every oblong is bisected by its diameter.
11. ABCD is an oblong whose diagonals cut at O; prove that triangle OAD is equal in area to triangle OBC.

PROPOSITION IX. PROBLEM.

To bisect a given rectilineal angle.

Let ABC be the given rect^{l} ∠.

It is req^{d} to bisect it.

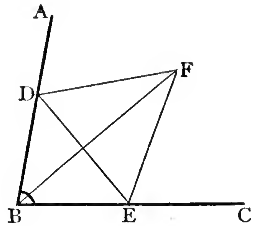
In AB take any pt. D.

From BC cut off BE=BD...I. 3.

Join DE........................Post. 1.

On the side of DE remote from B desc. an equilat. △ DEF........................I. 1.

Join BF........................Post. 1



Then shall BF bisect ∠ABC.

In △s DBF, EBF,

∵ { DB=EB..............................Constr.
BF is com.
DF=EF..................(equilat^{l} △) Constr. }

∴ ∠DBF=∠EBF ....I. 8.

Wherefore, *the given rectilineal angle* &c. Q.E.F.

EXERCISES.

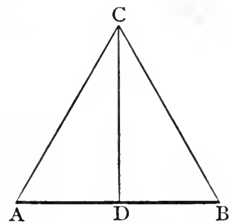
1. Why is the equilateral triangle constructed on the side of DE remote from B?
2. Divide a given rectilineal angle into four equal parts.
3. In the figure of Prop. 9, prove that angle BDF is equal to BEF.
4. Prove the following construction for bisecting a given angle POQ. With centre O and any radius describe a circle cutting OP in R, and OQ in S. With centre R and any radius greater than half RS, and with centre S and the same radius, describe circles cutting at T. Join OT.
5. ABC is an isosceles triangle; the bisectors of the base angles meet at O, and O is joined to the vertex A. Prove that OA bisects the vertical angle BAC.
6. The bisectors of the three angles of an equilateral triangle meet at a point.—[*Note.* Draw *two* bisectors, join their pt. of intersection to the remaining ∠, and then prove that this line bisects that ∠.]

PROPOSITION X. PROBLEM.

To bisect a given finite straight line.

Let AB be the given finite st. line.

It is req^d to bisect it.



On AB desc. an equilat. △CAB............................I. 1.

Bisect ∠ ACB by CD meeting AB at D..................I. 9.

Then shall AB be bisected at D.

In △s CAD, CBD,

∵ { CA=CB..................................Constr.
CD is com.
∠ ACD= ∠ BCD..............................Constr. }

∴ AD=BD..................................I. 4.

Wherefore, *the given st. line* &c. Q.E.F.

EXERCISES.

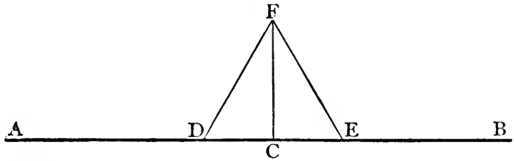
1. What is meant by a *finite* straight line?
2. Prove the following construction for bisecting a given line PQ:
   With centre P, and any radius greater than half PQ; and with centre Q and the same radius; describe arcs cutting at R and S. Join SR meeting PQ at T.
3. Show how to divide a given line into eight equal parts.
4. Prove that the common chord of two equal circles which cut one another bisects the line which joins their centres.
5. If, in the figure of Proposition 10, CB be bisected at E, and AE be joined, meeting CD at O; prove that OD is equal to OE.
6. In the figure of I. 9, if BF and DE cut at H, prove that DE is bisected at H, and that FH is at right angles to DE.
7. Prove that the common chord of two circles which cut is at right angles to the line joining their centres.

PROPOSITION XI. PROBLEM.

To draw a straight line at right angles to a given straight line, from a given point in the same.

Let AB be the st. line, and C the given pt. in it.

It is req^d^ to draw from C a st. line at rt. $\angle$*s to AB.*



In AC take any pt. D.
From CB cut off CE=CD...I. 3.
On DE desc. an equilat[l] $\triangle$ DEF...............................I. 1.
Join CF...Post. 1.

Then CF shall be at rt. $\angle$s to AB.

In $\triangle$s FDC, FEC,

$\because$ { DC=EC...Constr.
CF is com.
DF=EF..Constr. }

$\therefore$ $\angle$DCF=$\angle$ECF..................................I. 8.

and these, being equal adjacent $\angle$s, are rt. $\angle$s...............Def. 10.

Wherefore, *from the given point C* &c. Q.E.F.

EXERCISES.

1. Prove the following construction for this proposition:—With centre D and any radius greater than DC, and with centre E and the same radius, describe circles cutting at G. Join CG.
2. Find a point in a given straight line, of unlimited length, which shall be equidistant from two given external points. Draw figures for the various cases that may occur.

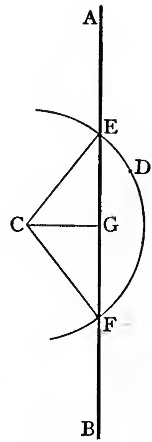
   For what special positions of the given points with respect to the given line is there—(i) no solution of the problem; (ii) an infinite number of solutions?
3. Find the locus of a point which moves in a plane so as always to be equidistant from two given fixed points in the plane.
4. Show, by a *reductio ad absurdum*, that, on the same side of the line AB, but *one* straight line at right angles to AB can be drawn from the point C.

PROPOSITION XII. PROBLEM.

To draw a straight line perpendicular to a given straight line of unlimited length, from a given point without it.

Let AB be the given st. line of unlimited length, and C the given pt. without it.

It is req^d to draw from C a st. line ⊥ to AB.



Take any pt. D on the other side of AB.
With cent. C and rad. CD desc.
a ⊙ EDF, cutting AB at E and F............Post. 3.
Bisect EF at G..................................I. 10.
Join CG ....Post. 1.

Then CG shall be ⊥ to AB.

Join CE, CF,.......................................Post. 1.

Then in △s CEG, CFG

∵ { EG=FG.........................Constr.
CG is com.
CE=CF.................(radii) Constr. }

∴ ∠CGE=∠CGF...................I. 8.

and these, being equal adj. ∠s, are rt. ∠s...Def. 10.

∴ CG is ⊥ to AB.................Def. 10.

Wherefore, *from the given point C* &c. Q.E.F.

EXERCISES.

1. Why is the given line "of unlimited length"?
2. Why is the point D taken "on the other side" of AB?
3. What distinction does Euclid make between *a line at right angles*, and *a perpendicular* to another line?
4. Construct an angle which shall be double of a given angle.
5. Prove that the perpendiculars, let fall from any point in the bisector of an angle upon the lines which contain the angle, are equal.*
6. The perpendiculars drawn from the ends of the base of an isosceles triangle to the opposite sides are equal.*
7. Find a point within an equilateral triangle which shall be equally distant from each of the angular points of the triangle.
8. P and Q are fixed points on opposite sides of a fixed straight line RS. Find a point T in RS such that the angle PTR may be equal to the angle QTR. In what case is this impossible?

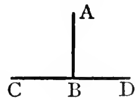
* *Note.*—Use the method of Ex. 6, p. 32, *i.e.* draw *one* ⊥.

PROPOSITION XIII. THEOREM.

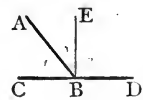
The angles which one straight line makes with another straight line, on one side of it, are either two right angles, or are together equal to two right angles.

Let the st. line AB meet CD at B.

Then shall either (i) ∠s CBA, ABD be two rt. ∠s,
or (ii) ∠s CBA, ABD together=two rt. ∠s.



(i) If ∠CBA=∠ABD,
each is a rt. ∠............................Def 10.
and ∴ ∠s CBA, ABD are two rt. ∠s.



(ii) If ∠ CBA is not=∠ABD,
from B draw BE at rt. ∠s to CD....I. 11.

Then, since ∠CBE=∠s CBA, ABE,
add ∠EBD to each,
∴ ∠s CBE, EBD=∠s CBA, ABE, EBD................Ax. 2.

Again, since ∠ABD=∠s ABE, EBD,
add ∠CBA to each,
∴ ∠s CBA, ABD=∠s CBA, ABE, EBD................Ax. 2.

Hence, ∠s CBA, ABD=∠s CBE, EBD.........................Ax. 1.

But ∠s CBE, EBD are two rt. ∠s.......................Constr.

∴ ∠s CBA, ABD are together=two rt. ∠s.

Wherefore, *the angles which* &c. Q.E.D

EXERCISES.

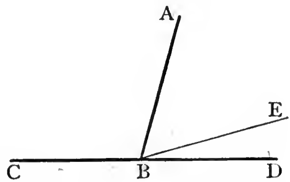
1. State the axiom assumed in the statement "∠CBE=∠s CBA, ABE."
2. AOB and COD are right angles, C lying within the angle AOB; prove that the angle AOC is equal to the angle BOD.
3. If, in the figure of Prop. 1, AB be produced both ways and meet the circle BCD in P, and the circle ACE in Q, prove that the angle CAP is equal to the angle CBQ.
4. In an isosceles triangle, the angles at the base together with the angles on the other side the base, are equal to four right angles.
5. OA, OB, OC, OD are radii of a circle ABCD. OB is at right angles to OA, and OC to OD. Prove that AC is equal to BD.
6. The bisectors of adjacent angles are at right angles.

PROPOSITION XIV. Theorem.

If, at a point in a straight line, two other straight lines, on opposite sides of it, make the adjacent angles together equal to two right angles; these two straight lines must be in one and the same straight line.

At the pt. B, in the st. line AB, let CB, BD, on opp. sides of AB, make ∠s CBA, ABD together=two rt. ∠s.

Then CB shall be in the same st. line with BD.



For, if not, if possible produce CB in some other direction BE.

Then, since AB meets st. line CBE at B

∴ ∠s CBA, ABE=two rt. ∠s ....I. 13.

But ∠s CBA, ABD=two rt. ∠s ....Hyp.

∴ ∠s CBA, ABE=∠s CBA, ABD............Ax. 1.

Take away the com. ∠ CBA,

∴ rem^{g} ∠ ABE=rem^{g} ∠ ABD...............Ax. 3.

or, the part=the whole,

which is absurd...........................Ax. 9.

Hence, CB is in the same st. line with BD.

Wherefore, *if at a point* &c. Q.E.D.

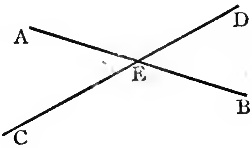
EXERCISES.

1. Of what proposition is the 14th the converse?
2. What method of proof is adopted in Proposition 14?
3. In Prop. 14 Euclid first makes use of Axiom 11. Point out where this occurs.
4. If, in the figure of Prop. 11, a second line, CG, be drawn from C at right angles to AB, but on the other side of it; then will CF and CG be in one and the same straight line.
5. If the angles on the other side of the base of a triangle, whose sides have been produced, are equal, prove that the triangle is isosceles. Of what theorem is this the converse?
6. Two squares have a corner-point common; if two of their sides are in one straight line, then two other sides will also be in one straight line.

PROPOSITION XV. Theorem.

If two straight lines cut one another, the opposite vertical angles are equal.

Let the st. lines AB and CD cut at E.



Then shall ∠AED= ∠CEB,
and ∠AEC= ∠DEB.

Since AE meets CD at E,

∴ ∠s AEC, AED=two rt. ∠s......................I. 13.

Since CE meets AB at E,

∴ ∠s AEC, CEB=two rt. ∠s......................I. 13.

Hence, ∠s AEC, AED=∠s AEC, CEB......Ax. 1. and Ax. 11.

Take away the com. ∠AEC,

∴ rem^{g} ∠ AED=rem^{g} ∠CEB...................Ax. 3.

In the same way it may be shown that ∠AEC=∠DEB.

Wherefore, *if two straight lines* &c. Q.E.D.

Cor. 1. If two straight lines cut, the four angles formed are together equal to four right angles.

Cor. 2. If any number of straight lines meet at a point, the sum of all the angles formed is equal to four right angles.

EXERCISES.

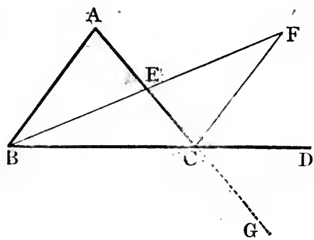
1. Write out in full the proof of the second part of this proposition, that angle AEC is equal to angle DEB.
2. AB and CD bisect each other at O; prove that the triangles BOC and AOD are equal in all respects.
3. AOB, COD are diameters of a circle; prove that AC is equal to BD.
4. Could a perfect pavement be formed entirely of equal regular hexagons whose angles are each of 120°?
5. Any point O is taken within a rectangle, and is joined to the four angular points. Prove that all the angles of all the triangles so formed are together equal to eight right angles.
6. The bisectors of opposite vertical angles are in the same straight line.

PROPOSITION XVI. Theorem.

If one side of a triangle be produced, the exterior angle is greater than either of the opposite interior angles.

Let ABC be a △ having one of its sides, BC, produced to D.

Then shall (i) ∠ACD > ∠BAC,
and (ii) ∠ACD > ∠ABC.



Bisect AC at E..............I. 10.
Join BE......................Post. 1.
Produce BE to F...........Post. 2.
Cut off EF=BE..............I. 3.
Join CF......................Post. 1.

Then, in △s ABE, ECF

∵ { AE=EC ....Constr.
BE=EF ....Constr.
∠AEB=∠CEF ....I. 15.

∴ ∠BAE=∠ECF ....I. 4.

But ∠ECD>∠ECF ....Ax. 9.

∴ ∠ECD>∠BAE

i.e. ∠ACD>∠BAC.

In the same way, if BC be bisected, and AC produced to G, it may be shown that ∠BCG (*i.e.* ∠ACD) > ∠ABC.

Wherefore, *if one side of a triangle* &c. Q.E.D.

EXERCISES.

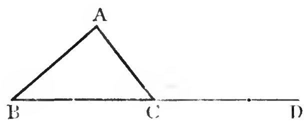
1. Write out in full the proof of the second part of this proposition.
2. If, in the figure of Prop. 16, AF be joined, then AF is equal to BC.
3. In the figure of I. 7, case 2, prove that angle ADB is greater than ACB.
4. Prove that the angles at the base of an isosceles triangle are together less than two right angles.
5. A triangle can have but one right angle.
6. In the figure of Prop. 5, if FC and BG meet in H, prove that angle FHG is greater than BAC.
7. In the figure of I. 16 prove that triangles ABC FBC are equal in area.

PROPOSITION XVII. THEOREM.

Any two angles of a triangle are together less than two right angles.

Let ABC be a △.

Then shall any two of its ∠s be together less than two rt. ∠s.



Produce BC to D..Post. 2.

Then, ext^{r} ∠ACD > opp. int^{r} ∠ABC..............I. 16.

Add ∠ACB to each

∴ ∠s ACD, ACB > ∠s ABC, ACB...............Ax. 4.

but ∠s ACD, ACB = two rt. ∠s.......................I. 13.

∴ ∠s ABC, ACB < two rt. ∠s.

In the same way it may be shown

that ∠s CBA, CAB < two rt. ∠s,

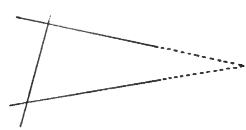
and ∠s BAC, BCA < two rt. ∠s.

Wherefore, *any two angles* &c. Q.E.D.

NOTE.

This proposition is the converse of Axiom 12.

Ax. 12 states that if two st. lines make with a third two angles on the same side together less than two right angles, those two lines will meet (*i.e.* the three will form a triangle). Prop. 17 shows that if two st. lines with a third form a triangle, the two angles formed with the third line will be together less than two right angles.



EXERCISES.

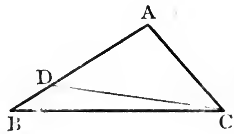
1. Write out in full the proof that angles CBA, CAB are less than two right angles.
2. Prove Prop. 17 by joining A to a point in BC, instead of producing BC.
3. From a given external point but one perpendicular can be drawn to a given straight line.
4. From a given external point only two equal straight lines can be drawn to a given straight line, one on each side of the perpendicular.
5. A triangle can have but one obtuse angle.

PROPOSITION XVIII. THEOREM. A

The greater side of every triangle is opposite to the greater angle.

Let ABC be a △ having the side AB > side AC.

Then shall ∠ACB > ∠ABC.



From AB cut off AD=AC.............I. 3.
Join DC...............................Post. 1.

Then, since AD=AC...........Constr.

∴ ∠ADC=∠ACD.......................................I. 5.

But ext^r ∠ADC>opp. int^r ∠DBC of △CBD.............I. 16.

∴ also ∠ACD>∠DBC

but ∠ACB>∠ACD.......................................Ax. 9.

much more ∴ ∠ACB>∠DBC (or ABC).

Wherefore, *the greater side* &c. Q.E.D.

NOTE.

In this proposition the *base* and *vertical angle* are not considered; the base may, of course, be the *greatest* of the three sides, as it is in the above figure.

If the two sides are equal, we know from Prop. 5 that they subtend equal angles. This proposition then is an extension of the 5th, and shows that, if the two sides are not equal, the greater of the two subtends a greater angle than the other subtends.

EXERCISES.

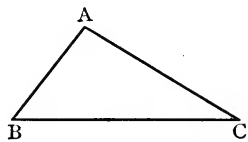
1. What is the hypothesis in this proposition?
2. In the figure of Prop. 5, show that the angle ABG is greater than the angle AGB.
3. In a scalene triangle the *greatest* side subtends the *greatest* of the three angles.
4. Enunciate the converse of the following proposition:—
   "If a triangle have two of its sides unequal, the greater of these two sides subtends an angle which is greater than that subtended by the other side."
5. ABCD is a quadrilateral. AB is equal to AD, but BC is less than DC. Prove that angle ABC is greater than angle ADC.

PROPOSITION XIX. Theorem.

The greater angle of every triangle is opposite to the greater side.

Let ABC be a △ having ∠ABC > ∠ACB.

Then shall AC > AB.



For, if AC be not > AB,
then must either (i) AC = AB,
or (ii) AC < AB.

(i) If AC = AB,
∠ACB = ∠ABC.................................I. 5.
but this, by hypothesis, is not the case.

(ii) If AC < AB,
∠ABC < ∠ACB.................................I. 18.
but this, by hypothesis, is not the case.

Hence, since AC can neither be =, nor <, AB,
∴ AC > AB.

Wherefore, *the greater angle* &c. Q.E.D.

NOTE.

To avoid confusing Prop. 18 with its converse Prop. 19, it should be noticed that the *hypothesis* stands *first* in each enunciation.

EXERCISES.

1. The shortest distance of a given point from a given line is the perpendicular. (Hence, by the *distance* of a point from a line is meant the *perpendicular distance.*)
2. The hypotenuse is the greatest side of a right-angled triangle.
3. The diagonal of a square is greater than its side.
4. The straight line joining the vertex of an isosceles triangle to any point in its base is always less than one of the equal sides of the triangle.
5. The greatest chord which can be drawn through any point in the circumference of a circle, is the diameter.
6. Prove that a circle cannot cut a straight line in more than two points.

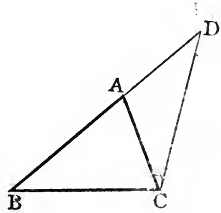
PROPOSITION XX. Theorem.

Any two sides of a triangle are together greater than the third side.

Let ABC be a △.

Then shall any two sides together be > the third side.

Produce BA to D ....Post. 2.
Cut off AD=AC..................I. 3.
Join DC ....Post. 1.



Then, since AC=AD ....Constr.
∴ ∠ACD=∠ADC.........I. 5.
But ∠BCD>∠ACD.........Ax. 9.
∴ also ∠BCD>∠ADC
Hence BD>BC..............I. 19.
i.e. BA, AC>BC ....Constr.

In the same way it may be shown
that AC, CB>BA,
and CB, BA>AC.

Wherefore, *any two sides of a triangle* &c. Q.E.D.

EXERCISES.

1. Write out in full the proof that AC, CB are greater than BA.
2. Prove, by a similar construction to that of the proposition, that the difference of any two sides of a triangle is less than the third side.
3. Any three sides of a quadrilateral are greater than the fourth side.
4. The sum of the four sides of a quadrilateral is greater than the sum of its two diagonals.
5. The sum of the lines joining any point within a triangle to its angles is greater than half the sum of its sides.
6. In the figure of Prop. 6, prove that the perimeter of the triangle ABC is greater than the perimeter of triangle DBC.
7. In the figure of I. 16, prove that BC and CF are greater than twice BE.
8. Points are taken on the circumference of a circle and joined in order, forming a polygon. Show that the perimeter of the polygon increases as the number of sides is increased.
9. Two sides of any triangle are together greater than twice the straight line joining the vertex to the middle point of the base.

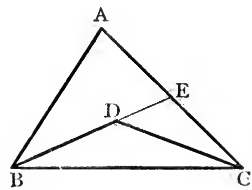
PROPOSITION XXI. Theorem.

If, from the ends of a side of a triangle, two straight lines be drawn to a point within the triangle; these two straight lines are together less than the other two sides of the triangle, but contain a greater angle.

Let ABC be a △, and, from the ends of the base BC, let BD, CD be drawn to a pt. D within the △.

Then shall (i) BD, DC < BA, AC,

(ii) ∠BDC > ∠BAC.



Produce BD to meet AC in E......Post. 2.

(i) Then, in △BAE,

BA, AE > BE........I. 20.

Add EC to each,

∴ BA, AC > BE, EC.

And, in △EDC,

DE, EC > DC..I. 20

Add BD to each,

∴ BE, EC > BD, DC.

Much more ∴ BA, AC > BD, DC.

(ii) Again, in △EDC,

ext^{r} ∠BDC > opp. int^{r} ∠DEC.............................I 16.

And, in △ABE,

ext^{r} ∠DEC > opp. int^{r} ∠BAC..................I. 16

Much more ∴ ∠BDC > ∠BAC.

Wherefore, *if from the ends of a side* &c. Q.E.D.

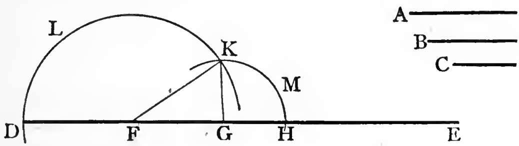
EXERCISES.

1. State an axiom which is assumed in this proposition.
2. If any point be taken within a quadrilateral and joined to two opposite angular points, the new quadrilateral thus formed will have a less perimeter than the other.
3. If a point O within a triangle ABC be joined to the angles, then OA, OB, OC are together less than the perimeter of the triangle.

PROPOSITION XXII. Problem.

To make a triangle whose sides shall be equal to three given straight lines, any two of which are greater than the third.

Let A, B, and C be the given st. lines, any two being > the third.
It is req^d to make a △ with sides respectively = A, B, and C.



Take a st. line DE terminated at D, but unlimited towards E.
From DE cut off DF=A, FG=B, GH=C........................I. 3.
With cent. F and rad. FD desc. a ⊙ DLK........................Post. 3.
With cent. G and rad. GH desc. a ⊙ HMK,......................Post. 3.
cutting the other ⊙ in K.
Join KF, KG..Post. 1.

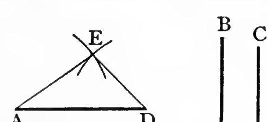
Then shall △ KFG have its sides=A, B, and C.

Since F is cent. of ⊙ DLK
∴ FK=FD................................Def. 15.
but FD=A..................................Constr.
∴ FK=A....................................Ax. 1.
Since G is cent. of ⊙ HMK
∴ GK=GH................................Def. 15.
but GH=C..................................Constr.
∴ GK=C....................................Ax. 1.
Also FG=B..................................Constr.

Wherefore, *a △KFG has been made* &c. Q.E.F.

NOTE.

In Practical Geometry, when unrestricted use of compasses can be made, a triangle with sides of given lengths would be constructed thus:—If AD, B, and C are the given lengths, extend the compasses till the distance between their feet is the length B, and then with cent. A describe an arc. In the same way with cent. D, but radius C, describe another arc, cutting the former at E. Rule lines EA and ED.



PROPOSITION XXIII. PROBLEM.

At a given point in a given straight line to make an angle equal to a given angle.

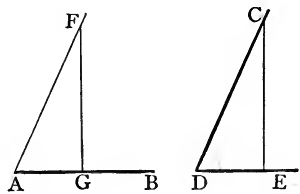
Let AB be the given st. line, A the given pt. in it, and CDE the given ∠.

It is req[d] *to make at A an ∠ = ∠ CDE.*

In DC and DE take any pts. C and E.

Join CE............................Post. 1.

On AB make a △AFG with its side FA=CD, AG=DE, GF=EC ....I. 22.



Then shall ∠FAG = ∠CDE.

In △s FAG, CDE

∵ {FA=CD, AG=DE, GF=EC} ....Constr.

∴ ∠FAG = ∠CDE..........................I. 8.

Wherefore, *at the given pt. A in the given st. line* &c. Q.E.F.

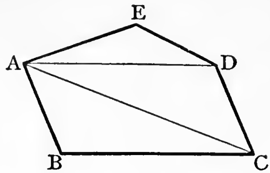
EXERCISES.

1. Prove that two triangles, satisfying the conditions of the problem, can be obtained with the construction of Prop. 22.
2. What previous problem is a special case of Prop. 22?
3. Show, by diagrams, that in Prop. 22 the restriction "any two of which are greater than the third" is necessary.
4. If, in the figure of Prop. 22, KD and KH be joined, the perimeter of KDH is more than double the sum of the lines A, B and C.
5. Construct a rhombus having given the length of a diagonal, and one of the angles through which it passes.
6. If one angle of a triangle is equal to the sum of the other two angles, the triangle can be divided into two isosceles triangles.
7. Construct a quadrilateral whose sides shall be respectively equal to those of a given quadrilateral.
8. Construct a triangle, having given the base, one of the angles at the base, and a line equal to the sum of its two sides.

EXAMPLES.

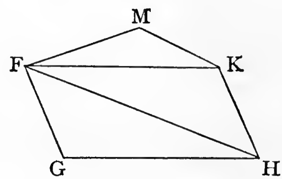
I. *To construct a rectilineal figure whose sides and angles shall be equal respectively to those of a given rectilineal figure.*

Let ABCDE be the given figure. Join AC, AD, dividing it into △s. Make a △FGH with its sides equal to those of △ABC (FG=AB &c.)...I. 22.



On FH make a △FKH with its sides equal to those of △ADC (FK=AD and HK=CD) ....I. 22.

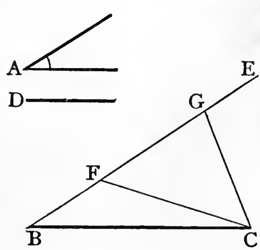
On FK make a △FMK with its sides equal to those of △AED (FM=AE, &c.)..........................I. 22.



Then fig. FGHKM has (by constr.) its sides=to those of the given figure. Also, by Euc. I. 8, the three pairs of △s have their ∠s equal, each to each, since their sides are equal. ∴ the sum of the three ∠s at F=the sum of the three ∠s at A, and so on. Hence the figure is also equiangular to ABCDE. Q.E.F.

II. *To construct a triangle, having given the base, one of the angles at the base, and the difference of the sides.*

Let A be the given ∠,
BC the given base,
and D the given difference of sides.



At B, in BC, make ∠CBE=A...I. 23.

From BE cut off BF=D...I. 3.
Join FC.

At C make ∠FCG=∠CFE......I 23.
CG meeting BE in G.

Then GBC shall be the △ req^{d}.

For, since ∠GFC=∠GCF....................................Constr.

∴ GF=GC..I. 6.

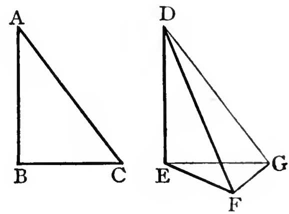
∴ the difference of GB and GC is BF,
and BF=D.

Also ∠GBC=A..Constr.

Hence, △GBC has one of its base ∠s=A, stands upon the given base BC, and has the difference of its sides=D. Q.E.F.

PROPOSITION XXIV. THEOREM.

If two triangles have two sides of the one equal to two sides of the other, each to each, but the angle contained by the two sides of the one greater than the angle contained by the two sides of the other; the base of that which has the greater angle is greater than the base of the other.



Let ABC, DEF be two △s,
having AB=DE,
AC=DF,
but ∠BAC > ∠EDF.

Then shall BC > EF.

Of the two sides DE, DF, let DE be the one which is not > the other.

At pt. D in DE, and on the same side of it as F, make the ∠EDG=∠BAC............I. 23.
Cut off DG=DF or AC............I. 3.
Join EG, GF............Post. 1.

Then, in △s ABC, DEG
∵ { AB=DE............Hyp.
AC=DG............Constr.
∠BAC=∠EDG............Constr.
∴ BC=EG............I. 4.
Again, since DG=DF............Constr.
∴ ∠DGF=∠DFG............I. 5.
But ∠DGF > ∠EGF............Ax. 9.
∴ also ∠DFG > ∠EGF.
But ∠EFG > ∠DFG............Ax. 9.
much more ∴ ∠EFG > ∠EGF.
Hence EG > EF............I. 19.
But BC=EG............Proved above.
∴ BC > EF.

Wherefore, *if two triangles* &c. Q.E.D.

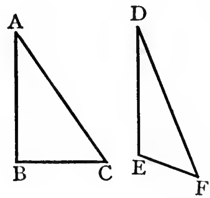
NOTE.

The shorter of the two sides DE, DF is selected in the construction (if the sides are not equal) in order to ensure that F shall lie on the side of EG remote from D. Otherwise three cases might occur:—Case i. as above; Case ii. when F fell on the same side as D (in which case produce DF and DG in order to prove EG > EF by the method of Prop. 7, Case 2); and Case iii. when F fell *on* EG.

PROPOSITION XXV. THEOREM.

If two triangles have two sides of one equal to two sides of the other, each to each, but the base of the one greater than the base of the other; the angle contained by the sides of that which has the greater base is greater than the angle contained by the sides of the other.

Let ABC, DEF be two △s,
having AB=DE,
AC=DF,
but BC>EF.



Then shall ∠BAC>∠EDF.

For, if ∠BAC is not>∠EDF,
then either (i) ∠BAC=∠EDF,
or (ii) ∠BAC<∠EDF.

(i) If ∠BAC=∠EDF
BC=EF .... I. 4.
but this, by hypothesis, is not the case.

(ii) If ∠BAC<∠EDF
BC<EF .... I. 24.
but this, by hypothesis, is not the case.

Hence, since ∠BAC can neither be =, nor <, ∠EDF,
∴ ∠BAC>∠EDF.

Wherefore, *if two triangles* &c. Q.E.D.

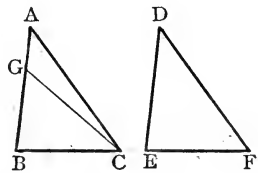
EXERCISES.

1. ABCD is a quadrilateral; the sides AD and BC are equal, but the side CD is greater than AB; prove that the angle CBD is greater than the angle ADB; also, that the angle CAD is greater than the angle ACB.
2. AB, BC, CD are three equal straight lines. The angle ABC is greater than the angle BCD. Prove that AC is greater than BD.
3. The sides AB, AC of a scalene triangle are produced to D and E. In BD any point F is taken; from CE is cut off CG, equal to BF; and FC, BG are joined. Prove that, if AB is greater than AC, FC will be greater than BG.
4. Prove Prop. 24 by using the greater of the two sides DE and DF, instead of the one which is not greater. (See note on Prop. 24.)
5. Prove that, with the construction of Prop. 24, F must lie on the side of EG remote from D.

PROPOSITION XXVI. THEOREM.

If two triangles have two angles of the one equal to two angles of the other, each to each, and one side equal to one side, namely, either the sides adjacent to equal angles in each triangle, or sides opposite to equal angles; then shall the remaining sides be equal, each to each, and the third angle of the one to the third angle of the other.

CASE I. Let the △s ABC, DEF
have ∠ABC=∠DEF,
∠ACB=∠DFE,
BC=EF.



Then shall AB=DE,
AC=DF,
∠BAC=∠EDF.

For, if AB is not=DE, one must be the greater.
If possible, suppose AB>DE.

From AB cut off BG=DE..I. 3.
Join GC..Post. 1.

Then in △s GBC, DEF,

∵ { GB=DE ....Constr.
BC=EF...............................Hyp.
∠GBC=∠DEF ....Hyp. }
∴ ∠GCB=∠DFE ....I. 4.

But ∠ACB=∠DFE ....Hyp.
∴ ∠GCB=∠ACB ....Ax. 1.
or, the part=the whole,
which is absurd..........................Ax. 9.
∴ AB cannot be unequal to DE,
i.e. AB=DE.

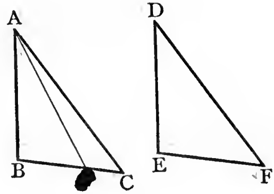
Hence, in △s ABC, DEF,

∵ { AB=DE.........................proved above.
BC=EF...............................Hyp.
∠ABC=∠DEF.........................Hyp. }

∴ { AC=DF
∠BAC=∠EDF } ....I. 4.

CASE II. Let △s ABC, DEF

have ∠ABC=∠DEF,
∠ACB=∠DFE,
AB=DE.



Then shall BC=EF,
AC=DF,
∠BAC=∠EDF.

For, if BC is not=EF, one must be the greater.

If possible, suppose BC > EF.

From BC cut off BG=EF.....................................I. 3.

Join AG...Post. 1.

Then in △s ABG, DEF,

∵ { AB=DE ....Hyp.
BG=EF........................Constr.
∠ABG=∠DEF...................Hyp.

∴ ∠AGB=∠DFE...................I. 4.

But ∠ACB=∠DFE...................Hyp.

∴ ∠AGB=∠ACB....................Ax. 1.

or, ext^{r} ∠ of △AGC=an opp. int^{r} ∠,
which is impossible.....................I. 16.

∴ BC cannot be unequal to EF,
i.e. BC=EF.

Hence, in △s ABC, DEF,

∵ { AB=DE........................Hyp.
BC=EF........................proved above.
∠ABC=∠DEF...................Hyp.

∴ { AC=DF
∠BAC=∠EDF } ....I. 4.

Wherefore, *if two triangles* &c. Q.E.D.

NOTE.

It follows from I. 4 that the *areas* of the triangles are equal. Hence Prop. 26 may be stated shortly thus:—

If two angles and a side of one triangle are equal to two angles and the corresponding side of another, the triangles are equal in all respects.

PROPOSITION XXVI. THEOREM.

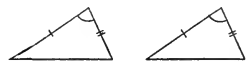
NOTES.

Prop. 26 completes one section of Book I. Among the most important results of this section are those of Props. 4, 8, and 26, which deal with four of the six cases which occur in the general question:—

If two triangles have three parts of the one known to be equal to three parts of the other, each to each, must the triangles be equal in **all** *respects?*

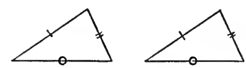
CASE I.

PROP. IV. — **Two sides and the included angle.** △s equal in all respects.



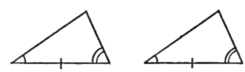
CASE II.

PROP. VIII. — **Three sides.** △s equal in all respects (See note to Prop. 8.)



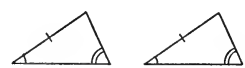
CASE III.

PROP. XXVI. (i) — **Two angles and the adjacent side.** △s equal in all respects.



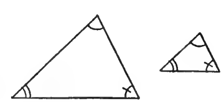
CASE IV.

PROP. XXVI. (ii) — **Two angles and an opposite side.** △s equal in all respects.



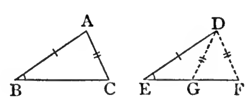
CASE V.

EXCEPTION. — **Three angles.** △s *not* necessarily equal in all respects, since two figures may be of the same shape but of different size.



CASE VI.

EXCEPTION. — **Two sides and an angle not included by them.** △s *not* necessarily equal in all respects, since there are, in general,* *two* positions for the shorter of the given sides. (See DF, DG in the figure.)



* Case vi. ceases to be ambiguous if both triangles are known to be acute-angled, or both right-angled, or both obtuse-angled.

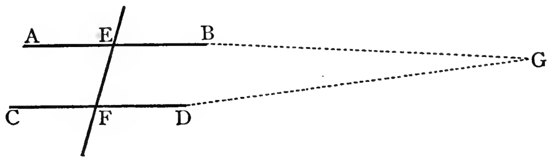
A point is said to be *equidistant* from two lines when the perpendiculars drawn from the point to the lines are equal.

EXERCISES.

1. If two triangles have three parts of one equal to three of the other, each to each, can they be proved to be equal in all respects?
2. If two triangles PQR, STV have PQ equal to ST, QR equal to TV, and angle QPR equal to angle TSV, is PR equal to SV?
3. If the perpendicular from the vertex on the base of a triangle bisect the base, the triangle is isosceles.
4. Enunciate and prove two theorems each of which is a converse of Ex. 3.
5. If any point be taken in the bisector of an angle, the perpendiculars drawn from it to the lines which contain the angle, are equal.
6. Prove that the point in which the bisector of the vertical angle of a triangle meets the base is equidistant from the sides.
7. Find a point in a given straight line, such that the perpendiculars from it to two given straight lines, may be equal.
8. Through a given point draw a straight line such that the perpendiculars on it from two given fixed points may be equal.
9. Find points equidistant from *two* given straight lines.
   Find, also, a point equidistant from *three* given straight lines.
   Examine the various cases that occur.
   In what case is there no solution?
10. If, in the figure of I. 5, FC and BG meet at H; prove that the point H is equidistant from AD and AE.
11. If, in the figure of I. 1, AB be produced both ways to meet the circles in D and E respectively; then D is the same distance from AC that E is from BC.
12. Draw a straight line through a fixed point which shall be equally inclined to two fixed straight lines.
13. The perpendiculars, drawn from the other angles to the line which joins the vertex to the middle point of the base of any triangle, are equal.
14. If two right-angled triangles have their hypotenuses equal, and also a side of one equal to a side of the other, they are equal in all respects. Prove by the method of Prop. 26.
15. If two obtuse-angled triangles have two sides of one equal to two sides of the other, each to each, and also the angles opposite to one pair of equal sides, equal; the triangles are equal in all respects.
16. In the figure of Case vi. (Prop. 26, *Note*) prove that the angles subtended by AB and DE are either equal or supplementary.

PROPOSITION XXVII. THEOREM.

If a straight line, falling on two other straight lines, make the alternate angles equal; these two straight lines shall be parallel.



Let EF, falling on AB and CD make ∠AEF=alt. ∠EFD.

Then shall AB be || to CD.

For, if AB be not || to CD they will meet, if prod[d] either towards B and D, or A and C.............................Def. 35.

If possible, suppose that they meet at G, when prod[d] towards B and D.

Then GEF is a △,
∴ ext[r] ∠AEF > opp. int[r] ∠EFG..............I. 16.
but ∠AEF = ∠EFG..........................Hyp.
i.e. ∠AEF is both >, and =, ∠EFG,
which is absurd.

Hence AB and CD cannot meet towards B and D. In the same way it may be shown that they cannot meet when prod[d] towards A and C.

∴ AB is || to CD.......................Def. 35.

Wherefore, *if a straight line* &c. Q.E.D.

EXERCISES.

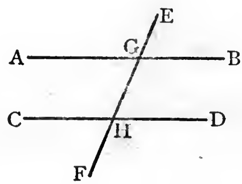
1. Show that AB and CD cannot meet towards A and C.
2. What is the hypothesis of this proposition?
3. Enunciate the converse of Prop. 27.
4. In the figure of I. 16, if AF be joined, prove that AF is parallel to BC. Prove also that AB is parallel to FC.
5. If, in the figure of Prop. 1, the circles cut at C and F; prove that AC is parallel to BF.
6. Opposite sides of an oblong are parallel.
7. Opposite sides of a rhombus are parallel.
8. If two straight lines bisect one another, the figure formed by joining their extremities is a parallelogram.
9. Every rhomboid is a parallelogram.

PROPOSITION XXVIII. Theorem.

If a straight line, falling on two other straight lines, make the exterior angle equal to the interior and opposite angle upon the same side of the line; or, make the two interior angles on the same side together equal to two right angles; the two straight lines shall be parallel.

Part I. Let st. line EF, falling on AB and CD, make ext^r ∠EGB=int^r opp. ∠GHD.

Then shall AB be ∥ to CD.



For, since ∠EGB=∠GHD ....Hyp.
and ∠EGB=∠AGH ....I. 15.
∴ ∠AGH=∠GHD ....Ax. 1.
and these being alt. ∠s,
∴ AB is ∥ to CD ....I. 27.

Part II. Let EF, falling on AB and CD, make int^r ∠s BGH, GHD together=two rt. ∠s.

Then shall AB be ∥ to CD.

For, since ∠s BGH, GHD=two rt. ∠s................Hyp.
and ∠s BGH, AGH=two rt. ∠s ....I. 13.
∴ ∠s BGH, AGH=∠s BGH, GHD.........Ax. 1.
Take away the com. ∠BGH,
∴ rem^g ∠AGH=rem^g ∠GHD ....Ax. 3.
and these being alt. ∠s,
∴ AB is ∥ to CD.....................I. 27.

Wherefore, *if a straight line* &c. Q.E.D.

EXERCISES.

1. What is meant by the words "on the same side" in the second part of the enunciation?
2. Prove Part I., assuming ext^r angle EGA equal to int^r angle GHC.
3. Prove Part II., assuming the two interior angles AGH and GHC together equal to two right angles.
4. Mention two other pairs of exterior and interior angles to be found in the figure, which have not been mentioned before.
5. Enunciate the converse of each of the two theorems contained in the enunciation of Prop. 28.
6. Prove that the opposite sides of a square are parallel.
7. Prove that straight lines at right angles to the same straight line are parallel to one another.

PROPOSITION XXIX. Theorem.

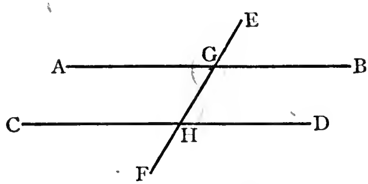
If a straight line fall on two parallel straight lines, it makes the alternate angles equal; an exterior angle equal to the interior and opposite angle upon the same side; and two interior angles on the same side together equal to two right angles.

Let AB and CD be ∥ s, and let st. line EGHF fall on them.

Then shall (i) ∠AGH=alt. ∠GHD

(ii) extr ∠EGB=intr opp. ∠GHD

(iii) two intr ∠s BGH, GHD=two rt. ∠s.



(i) For, if ∠AGH is not=∠GHD, one must be the greater. If possible, suppose ∠AGH>∠GHD

Add ∠BGH to each,

∴ ∠s AGH, BGH>∠s BGH, GHD............Ax. 4.

But ∠s AGH, BGH=two rt. ∠s....................I. 13.

∴ ∠s BGH, GHD<two rt. ∠s.

∴ AB, CD will meet, if prod^d towards B and D........Ax. 12.

which is impossible, for they are ∥ ,...............Hyp.

Hence, ∠AGH cannot be unequal to ∠GHD

i.e. ∠AGH=∠GHD.

(ii) Again, since ∠AGH=∠GHD..................proved above.

and ∠AGH=∠EGB.........................I. 15.

∴ ∠EGB=∠GHD.......................Ax. 1.

(iii) Also, since ∠EGB=∠GHD..................proved above.

Add ∠BGH to each

∴ ∠s EGB, BGH=∠s BGH, GHD.............Ax. 2.

but ∠s EGB, BGH=two rt. ∠s.....................I. 13.

∴ ∠s BGH, GHD=two rt. ∠s.....................Ax. 1.

Wherefore, *if a straight line* &c. Q.E.D.

NOTES.

Prop. 29 contains three converse propositions—those of 27, 28 part i, and 28 part ii.

It is in this proposition that Euclid first makes use of Axiom 12. Euclid's definition of parallels being a *negative* one, it became necessary to assume some *positive* fact relating to them before any of the properties of parallels could be proved. But the nature of the subject is such that there are no "*common notions*" respecting parallels. Euclid, therefore, requested that the *theorem*, known as Axiom 12 (and which he probably placed among the postulates) should be taken for granted. He proved, however, its converse. (See note on Prop. 17.)

Many attempts have been made to improve on Euclid's treatment of this part of Elementary Geometry. But, as the fundamental difficulty can never be entirely avoided, it may be doubted whether a little gain in clearness is of such value as to justify any tampering with Euclid's text; especially as a change in either definition or axiom necessitates changes, in some cases considerable, in the propositions. The following is generally regarded as the best of the various substitutes for Euclid's axiom which have been suggested. "Two straight lines which cut cannot both be parallel to the same straight line."

EXERCISES.

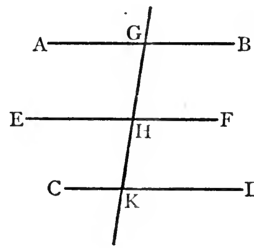
1. Prove, by a *reductio ad absurdum* that, with the hypothesis and figure of Prop. 29, angle BGH is equal to angle GHC.
2. If two straight lines which meet are both parallel to the same straight line, these two straight lines are in one and the same straight line.
3. A straight line drawn parallel to the base of an isosceles triangle makes equal angles with the sides.
4. The angle between two straight lines is equal to the angle between two others which are parallel to them respectively.
5. If a perpendicular be drawn to each of two parallels, these perpendiculars are themselves parallel.
6. A is a point equidistant from two parallel straight lines; show that the portion of any line through A, which is intercepted by the parallels, is bisected at A. Also, if two lines be drawn through A, show that the triangles formed by them with the parallels are equal in all respects.
7. The diagonals of a parallelogram bisect each other.
8. If a straight line DE be drawn parallel to the base BC of an isosceles triangle ABC, the trapezoid DBCE will have each pair of its opposite angles together equal to two right angles.

PROPOSITION XXX. Theorem.

Straight lines that are parallel to the same straight line are parallel to one another.

Let st. lines AB, CD be each || to EF.

Then shall AB be || to CD.



Draw GHK cutting the lines in G, H, and K.

Then, since AB is || to EF............Hyp.

∴ ∠AGH=alt. ∠GHF.......I. 29.

and, since EF is || to CD............Hyp.

∴ ext^{r} ∠GHF=int^{r} ∠HKD......I. 29.

Hence ∠AGH=∠HKD..........Ax. 1.

and these being alt. ∠s,

∴ AB is || to CD.........I. 27.

Wherefore, *straight lines that are* &c. Q.E.D.

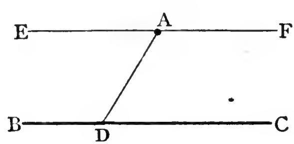
PROPOSITION XXXI. Problem.

To draw a straight line, through a given point, parallel to a given straight line.

Let A be the given point, and BC the given st. line.

It is req^{d} *to draw, through A, a st. line || to BC.*

In BC take any pt. D.



Join AD..........................Post. 1.

At A, in AD, make ∠DAE = ∠ADC, but on the opp. side of AD............................I. 23.

Produce EA to F................Post. 2.

Then shall EF be || to BC.

For, since AD, meeting two st. lines, makes ∠EAD=alt. ∠ADC...Constr.

∴ EF is || to BC...I. 27.

Wherefore, *through the given point* &c. Q.E.F.

EXERCISES.

1. Through a given point draw a straight line which shall make a given angle with a given straight line.
2. Through three given points draw three straight lines so as to form a triangle two of whose angles shall be equal to given angles.

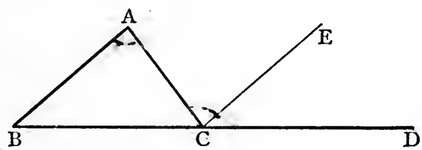
PROPOSITION XXXII. THEOREM.

If a side of a triangle be produced, the exterior angle is equal to the two opposite interior angles; and the three angles of every triangle are together equal to two right angles.

Let ABC be a △ with BC produced to D.

Then shall (i) extr ∠ACD = ∠s CAB, ABC.

(ii) ∠s CAB, ABC, BCA = two rt. ∠s.



Through C draw CE ∥ to AB..I. 31.

(i) Then since CE is ∥ to AB

∴ ∠ECA = alt. ∠CAB........................I. 29. (i)

and extr ∠ECD = intr ∠ABC........................I. 29. (ii)

∴ whole ∠ACD = ∠s CAB, ABC..................Ax. 2.

(ii) Again, since ∠ACD = ∠s CAB, ABC...............proved above.

Add ∠BCA to each,

∴ ∠s BCA, ACD = ∠s CAB, ABC, BCA..........Ax. 2.

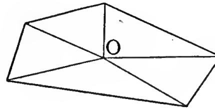
But ∠s BCA, ACD = two rt. ∠s........................I. 13.

∴ ∠s CAB, ABC, BCA = two rt. ∠s.

Wherefore, *if a side of a triangle* &c. Q.E.D.

COR. 1. *All the interior angles of a rectilineal figure, together with four right angles, are equal to twice as many right angles as the figure has sides.*

For, by joining any pt. O inside the figure to each of the angles, the figure is divided into as many △s as it has sides.



Now, the three ∠s of a △ = two rt. ∠s...I. 32.

Hence, *twice as many rt. ∠s as the fig. has sides*

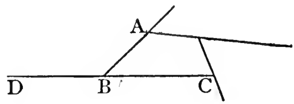
= all the ∠s of all the △s,

= all the intr ∠s of fig. with ∠s at O,

= *all the intr ∠s of fig. with 4 rt. ∠s*...........I. 15. Cor. 2.

PROPOSITION XXXII. THEOREM.

COR. 2. *All the exterior angles of a rectilineal figure are together equal to four right angles.*



For, any int^r ∠ ABC with its adj. ext^r ∠ ABD=two rt. ∠s......I. 13.
Hence, all the ext^r ∠s with all the int^r ∠s
=twice as many rt. ∠s as the fig. has sides
=all the int^r ∠s with 4 rt. ∠s..................I. 32. Cor. 1.
Take away the com. int^r ∠s,
∴ *all the ext^r ∠s=4 rt. ∠s.*

NOTE.

The exterior angles must be formed by producing the sides so that no two produced lines can cut; and the rectilineal figure must have no *re-entrant* angles, that is, all its angles must have their vertices pointing outwards.

EXAMPLES.

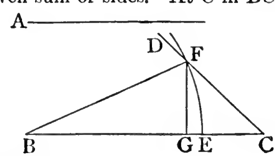
I. *Find the magnitude of an interior angle of a regular decagon.*

The 10 int^r ∠s with 4 rt. ∠s=20 rt. ∠s..................I. 32. Cor. 1.
∴ the 10 int^r ∠s=16 rt. ∠s..................Ax. 3.
∴ one int^r ∠=$\frac{16}{10}$ of a rt. ∠..............Fig. is *regular.*
=$\frac{8}{5}$ of a rt. ∠.

Hence, the no. of *degrees* in an angle of a reg. decagon is $\frac{8}{5}\times 90=144°$.

II. *Construct a right-angled triangle having given the hypotenuse and the sum of the other two sides.*

Let A be the given hypot. and BC the given sum of sides. At C in BC make an ∠BCD=to half a rt. ∠.
From BC cut off BE=A..................I. 3.
With cent. B and rad. BE desc. an arc cutting CD at F. Join BF.
Draw FG ⊥ to BC........................I. 12.
Then FBG shall be the △ req^d.



For, since ∠GCF is half a rt. ∠ and ∠FGC is a rt. ∠ } ....Constr.
∴ ∠GFC is half a rt. ∠..................I. 32.
Hence, ∠GFC=∠GCF ....Ax. 1.
and ∴ GF=GC ....I. 6.

Wherefore, △FBG has the sum of its sides=BC, has a right angle FGB, and its hypot. FB=A. Q.E.F.

EXERCISES.

1. What had been proved about the exterior angle of a triangle previous to Prop. 32?
2. If two triangles have two angles of one equal to two angles of the other, the third angle of the one is equal to the third angle of the other.
3. ABC, DEF are equilateral triangles, prove that angle ABC is equal to angle DEF.
4. Prove that all the interior angles of any quadrilateral are together equal to four right angles.
5. Show that each angle of an equilateral triangle is equal to two-thirds of a right angle.
6. Trisect a right angle.
7. One angle of a triangle is the complement of another; prove that the triangle is right-angled.
8. Through three given points draw three straight lines so as to form an equilateral triangle.
9. If two triangles have two angles of the one equal to two angles of the other, then must the third angle of the one be equal to the third angle of the other.
10. ABC is an isosceles triangle having AB equal to AC; BA is produced to D; prove that angle DAC is double of angle ABC.
11. If an isosceles triangle has each base angle double of the vertical, find their magnitude.
12. Find the size of an interior angle of a regular (i) pentagon, (ii) hexagon, (iii) octagon, (iv) quindecagon, taking the right angle as unit of measurement. Find also the magnitude of each angle in *degrees.*
13. One angle of a regular polygon contains 135°; find the number of sides.
14. Could a pavement be formed of tiles which were all equal regular (i) hexagons? (ii) pentagons?
15. Find the magnitude of each angle of a right-angled isosceles triangle.
16. Construct an isosceles triangle having the angles at the base each equal to one-sixth of the vertical angle.
17. Construct a right-angled triangle having given the hypotenuse and the difference of the other two sides. Draw both the figures which the construction yields.
18. From a point O within a triangle ABC perpendiculars OM, ON are drawn to the sides AB and AC; prove that the angles MON and MAN are together equal to two right angles.
19. ABC is a triangle and the exterior angles at B and C are bisected by BD and CD meeting at D; show that angle BDC, with half angle BAC, make up a right angle.
20. Trisect an angle equal to one-fourth of a right angle.

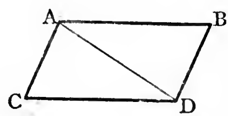
    (*The only angles which can be trisected with ruler and compasses alone are the right angle, or its half, fourth, eighth, &c. parts.*)

PROPOSITION XXXIII. THEOREM.

The straight lines which join the extremities of two equal and parallel straight lines, towards the same parts, are themselves equal and parallel.

Let AB, CD be = and ∥ st. lines joined towards the same parts by AC and BD.

Then shall AC be = and ∥ to BD.



Join AD ....Post. 1.

Then, since AB is ∥ to CD ....Hyp.

∴ ∠BAD = alt. ∠ADC ....I. 29 (i).

Hence, in △s ABD, ACD,

∵ { AB = CD ....Hyp.
AD is com.
∠BAD = ∠ADC ....Proved above. }

∴ { AC = BD
∠BDA = ∠DAC } ....I. 4.

and these being alt. ∠s,

∴ AC is ∥ to BD ....I. 27.

Wherefore, *the straight lines which join* &c. Q.E.D.

NOTE.

Prop. 27 to 33 form the 2nd section of Book I.—in which the theory of *parallels* is dealt with.

EXERCISES.

1. Explain the words "towards the same parts" in the enunciation.
2. Does the converse of this proposition need any proof?
3. A straight line joining the middle points of two opposite sides of a parallelogram, divides the figure into two parallelograms.
4. The straight line joining the middle points of two opposite sides of a rectangle is at right angles to those sides.
5. AB, CD, EF are three equal and parallel straight lines; prove that triangle ACE is equal to triangle BDF.
6. The straight lines, which join the extremities of equal and parallel straight lines towards opposite parts, bisect each other.
7. AB and CD are parallel straight lines. AB is produced both ways to E and F so that AE and BF are each equal to CD; prove that the triangle BCE is equal in all respects to the triangle ADF.
8. Find the locus of a point which moves so as always to keep at the same distance from a given straight line.

PROPOSITION XXXIV. THEOREM.

The opposite sides, and angles, of a parallelogram are equal to one another; and a diameter bisects it.

Let ABCD be a ▱, and AC a diam^{r}.

Then shall (i) AD=BC,
DC=AB;
(ii) ∠ADC=∠ABC,
∠BAD=∠BCD;
(iii) △ADC=△ABC.

Since AD is ‖ to BC .... Hyp.
∴ ∠DAC=alt. ∠ACB .... I. 29 (i).
and since DC is ‖ to AB .... Hyp.
∴ ∠BAC=alt. ∠ACD .... I. 29 (i).
Hence the whole ∠BAD=whole ∠BCD .... Ax. 2.

Now, in △s ADC, ABC,

∵ { ∠DAC=∠ACB, ∠ACD=∠BAC } .... Proved above.
AC is com.

∴ { AD=BC, DC=AB, ∠ADC=∠ABC } .... I. 26 (i).

Hence it follows that △s ADC, ABC are equal in area...I. 4.

Wherefore, *the opposite sides* &c. Q.E.D.

NOTE.

From this proposition it is manifest that every oblique parallelogram is either a rhomboid or a rhombus; and every rectangle either an oblong or a square.

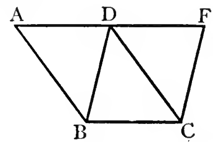
EXERCISES.

1. State and prove the converse of part i. of Prop. 34. Also, that of part ii.
2. If, in the figure of Prop. 34, BD be joined, triangles ABC and DBC are equal in area.
3. ABCD and EBCF are parallelograms on the same base; prove that AEFD is also a parallelogram.
4. If, in the figure of Prop. 34, BE be drawn parallel to AC and meeting DA produced in E, then the parallelogram EBCA will be equal in area to the parallelogram ABCD.
5. Bisect a parallelogram by a straight line drawn (i) through a given point in one of its sides; (ii) parallel to a given straight line; (iii) perpendicular to one of its sides.
6. The area of any triangle is half that of the rectangle on the same base, one of whose sides passes through the vertex of the triangle.

PROPOSITION XXXV. Theorem.

Parallelograms on the same base and between the same parallels are equal to one another.

Let ▱s ABCD, EBCF be on the same base BC, and between the same ∥s AF, BC.



Then shall ▱ABCD = ▱EBCF.

Case 1. When the pt. E coincides with D.

Then each ▱ is double of △DBC..I. 34 (iii).

∴ they are equal..............Ax. 6.

Case 2. When pt. E does not coincide with D.

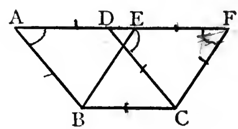


Fig. 1.

Then, since ABCD is a ▱, ...Hyp.

∴ AD=BC....................I. 34 (i).

and, since EBCF is a ▱, ...Hyp.

∴ EF=BC....................I. 34 (i).

Hence AD=EF ....Ax. 1.

To each add DE (in Fig. 1.), or take away DE (in Fig. 2.).

∴ the whole, or rem^r, AE=whole, or rem^r, DF......Ax. 2, or Ax. 3.

Hence, in △s ABE, DCF,

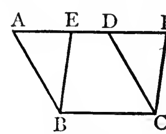


Fig. 2.

∵ { AB=DC ....I. 34.
AE=DF ....Proved above.
∠BAE=∠CDF ....I. 29 (ii).

∴ △ABE=△DCF ....I. 4.

Now, if from the whole trapezium ABCF the △ABE be taken away, the ▱EBCF is left.

Or, if from the same trapezium the △DCF be taken away, the ▱ABCD is left.

Hence these rem^{rs} are equal..................Ax. 3.

i.e. ▱ABCD=▱EBCF.

Wherefore, *parallelograms on the same base* &c. Q.E.D.

NOTES.

In the enunciations of this and succeeding propositions, the word "*equal*" must be understood to mean "*equal in area*," not "equal in all respects."

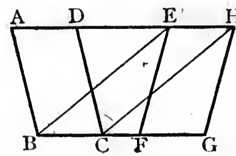
The altitude of a parallelogram is the perpendicular distance between the parallels between which it stands, and the altitude of a triangle the length of the perpendicular from its vertex on its base, or base produced.

PROPOSITION XXXVI. Theorem.

Parallelograms on equal bases, and between the same parallels, are equal to one another.

Let ▱s ABCD, EFGH be on = bases BC, FG, and between the same ∥ s AH, BG.

Then shall ▱ABCD=EFGH.



Join BE, CH ....Post. 1.

Then, since BC=FG...................Hyp.

and EH=FG...................I. 34.

∴ BC=EH...................Ax. 1.

Hence, BC and EH being = and ∥,

∴ BE is ∥ to CH ....I. 33.

and EBCH is a ▱.

Now, ▱s ABCD, EBCH are on the same base BC, and between the same ∥ s AH, BC,

∴ ▱ABCD=▱EBCH.................I. 35.

Also ▱s EBCH, EFGH are on the same base EH, and between the same ∥ s BG, EH,

∴ ▱EBCH=▱EFGH.................I. 35.

Hence ▱ABCD=▱EFGH.................Ax. 1.

Wherefore, *parallelograms on equal bases* &c. Q.E.D.

EXERCISES.

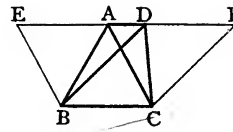
1. What special form of Axiom 3 is used in Prop. 35?
2. The perimeter of a rectangle is less than that of any other parallelogram standing on the same base and having the same area.
3. AFCD, EBGF are equal parallelograms standing on equal bases FC, EF, in the same straight line, but *not* towards the same parts. If AB, DG be joined, prove that ABGD is a parallelogram.
4. Construct a rhombus equal to a given parallelogram.
5. ABCD, EBCF are parallelograms on the same base but on opposite sides of it; prove that, if AE and DF be joined, AEFD is a parallelogram; and that its area is equal to the sum of the areas of ABCD and EBCF.
6. If two parallelograms stand on the same base, but the altitude of one is double that of the other, the area of the former is double that of the latter.
7. If, in Fig. 1 of Case 2, Prop. 35, BE and CD meet at O, prove that trapezium ABOD is equal to trapezium EOCF.

PROPOSITION XXXVII. THEOREM.

Triangles on the same base, and between the same parallels, are equal to one another.

Let △s ABC, DBC be on the same base BC, and between the same ∥ s AD, BC.

Then shall △ABC=△DBC.



Produce AD both ways................Post. 2.
Through B, draw BE ∥ to AC, ...I. 31.
and meeting AD prod^d in E.
Through C, draw CF ∥ to BD, ...I. 31.
and meeting AD prod^d in F.

Then, since EBCA, DBCF are ▱s on the same base BC, and between the same ∥ s,

∴ ▱EBCA=▱DBCF..................I. 35.
But △ABC is half ▱EBCA } ....I. 34 (iii).
and △DBC is half ▱DBCF }
∴ △ABC=△DBC......................Ax. 7.

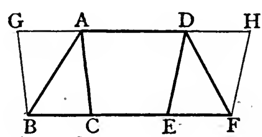
Wherefore, *triangles on the same base* &c. Q.E.D.

PROPOSITION XXXVIII. THEOREM.

Triangles on equal bases, and between the same parallels, are equal to one another.

Let △s ABC, DEF be on the equal bases BC, EF, and between the same ∥ s AD, BF.

Then shall △ABC=△DEF.



Produce AD both ways ....Post. 2.
Through B, draw BG ∥ to AC,....I. 31.
and meeting AD prod^d in G.
Through F draw FH ∥ to ED,.....I. 31.
and meeting AD prod^d in H.

Then, since GBCA, DEFH are ▱s on equal bases BC, EF, and between the same ∥ s,

∴ ▱GBCA=▱DEFH.................I. 36.
But △ABC is half ▱GBCA } ....I. 34 (iii).
and △DEF is half ▱DEFH }
∴ △ABC=△DEF.....................Ax. 7.

Wherefore, *triangles on equal bases* &c. Q.E.D.

NOTE.

It follows from Prop. 38 that triangles on equal bases and with a common vertex are equal in area. Also, that if two triangles of the same altitude are on unequal bases that which has the greater base has the greater area.

EXAMPLES.

I. *To construct a triangle equal in area to a given quadrilateral.*

Let ABCD be the given quad^{l}.

Join AC.

Through B draw BE || to AC and meeting DC prod^{d} in E..................I. 31.

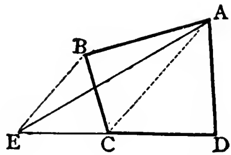
Join AE.

Then AED shall be the req^{d} $\triangle$.

For, $\triangle$AEC=$\triangle$ABC..................I. 37.

Add $\triangle$ACD to each,

∴ the whole $\triangle$AED=fig. ABCD.



Q.E.F.

II. *The straight line joining the middle points of two sides of any triangle cuts off a triangle whose area is one-fourth of that of the whole triangle.*

Let ABC be a $\triangle$, and let D and E, the mid. pts. of its sides AB, AC, be joined. Then $\triangle$ADE shall be one-fourth of $\triangle$ABC.

Join BE.

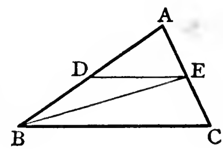
Then, $\triangle$BAE=$\triangle$BEC..............I. 38, note.

∴ $\triangle$BAE is half $\triangle$ABC,

also $\triangle$EAD=$\triangle$EDB..............I. 38, note.

∴ $\triangle$EAD is half $\triangle$BAE.

Hence $\triangle$EAD is one-fourth of $\triangle$ABC.



Q.E.D.

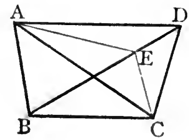
EXERCISES.

1. Given a triangle, construct (i) a right-angled triangle of the same area.
   (ii) a triangle whose area shall be double that of the given triangle.
   (iii) a triangle whose area shall be half that of the given triangle.
   (iv) a triangle equal to the given triangle in area and having one of its sides equal to a given straight line.
2. The four triangles into which a parallelogram is divided by its diagonals are equal in area.
3. In the figure of Prop. 37, prove that triangle ABD is equal to triangle ACD. And if AC, BD cut in O, prove that (i) triangle AOE is equal to triangle BOA; (ii) triangle OAB is equal to triangle OCD.
4. Triangles of equal altitude, on opposite sides of the same base, are equal.
5. If D, E, are the middle points of the sides AB, AC of a triangle ABC, and if CD and BE cut at O, prove that $\triangle$AOE=$\triangle$BOD.
6. In the figure of I. 38, prove that trapezium ABED is equal to ACFD.
7. Construct a triangle equal in area to a given rectilineal figure.
8. ABCD is a parallelogram. E is any point in the diagonal AC, or in AC produced. Prove that triangles EBC, EDC are equal in area.
9. Bisect a triangle by a straight line drawn through a given point in one of its sides.
10. Bisect a trapezium by a line drawn through one of its angles.

PROPOSITION XXXIX. THEOREM.

Equal triangles on the same base, and on the same side of it, are between the same parallels.

Let ABC, DBC be equal △s on the same base BC, and on the same side of it, and let AD be joined.



Then shall AD be || to BC.

For, if not, if possible draw AE || to BC meeting BD, or BD produced, in E........I. 31.
Join EC..Post. 1.

Then, if AE is || to BC

△ABC=△EBC........................I. 37.
but △ABC=△DBC........................Hyp.
and ∴ △EBC=△DBC........................Ax. 1.
or, the part=the whole,
which is absurd........................Ax. 9.

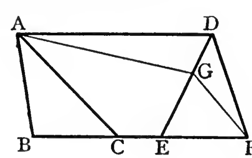
Hence, no other st. line but AD can be || to BC,
i.e. AD is || to BC.

Wherefore, *equal triangles on the same base* &c. Q.E.D.

PROPOSITION XL. THEOREM.

Equal triangles, on equal bases in the same straight line, and towards the same parts, are between the same parallels.

Let ABC, DEF, be equal △s on equal bases BC, EF in the same st. line, and towards the same parts, and let AD be joined.



Then shall AD be || to BF.

For, if not, if possible draw AG || to BF..................I. 31.
meeting DE, or DE produced, in G.
Join GF ....Post. 1.

Then, if AG is || to BF

△ABC=△GEF........................I. 38.
but △ABC=△DEF........................Hyp.
and ∴ △GEF=△DEF........................Ax. 1.
or, the part = the whole,
which is absurd........................Ax. 9.

Hence, no other st. line but AD can be || to BF,
i.e. AD is || to BF.

Wherefore, *equal triangles on equal bases* &c. Q.E.D.

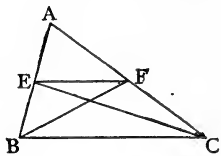
EXAMPLE.

The straight line joining the middle points of the sides of a triangle is parallel to the base.

Let E, F be the middle points of the sides AB, AC of $\triangle$ABC.

Join EF, BF, CE,

Then shall EF be || to BC.



For, since AE=EB..................Hyp.

∴ $\triangle$FAE=$\triangle$FEB..........I. 38 note.

and since AF=FC..................Hyp.

∴ $\triangle$FAE=$\triangle$FEC............I. 38.

Hence, $\triangle$FEB=$\triangle$FEC............Ax. 1.

and they are on the same base EF,

∴ BC is || to EF..................I. 39.

Q.E.D.

EXERCISES.

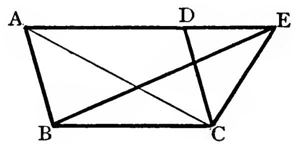
1. If, in the above figure, D be the middle point of BC, prove that DE is parallel to AC.
2. Of what propositions are the 39th and 40th converse respectively?
3. Prove Prop. 39, supposing AE to meet CD produced in E.
4. Prove Prop. 40, supposing AG to meet DF in G.
5. In the figure of I. 5, if FG be joined, prove that FG is parallel to BC.
6. The figure formed by joining the middle points of the sides of any quadrilateral is a parallelogram, and its area is half that of the quadrilateral.
7. If the middle points of the sides of a triangle be joined, the triangle is divided into four equal triangles.
8. In the figure of the above example, prove that
   (i) EF is equal to half BC.
   (ii) If BF, CE cut at O, triangles OBE, OCF are equal in area.
   (iii) Triangles OEA, OFA are equal in area.
   (iv) Triangle OBE is one-third of triangle CBE.
9. If two equal triangles have a common vertex, and their bases in a straight line, their bases are equal. Hence, show that if OC in the above figure is bisected at Q, EC will be trisected in O and Q.
10. If, of the four triangles into which the diagonals divide a quadrilateral, any two opposite ones are equal, the quadrilateral has two of its sides parallel.
11. In a right-angled triangle, the straight line joining the right angle to the middle point of the hypotenuse is half the hypotenuse.
12. What is the locus of the vertex of a triangle whose base is fixed, and whose area remains constant?

PROPOSITION XLI. THEOREM.

If a parallelogram and a triangle be on the same base, and between the same parallels, the parallelogram shall be double of the triangle.

Let ▱ABCD and △EBC be on the same base BC, and between the same ‖ s AE, BC.

Then shall ▱ABCD be double of △EBC.



Join AC..Post. 1.

Then, since AE is ‖ to BC..................Hyp.

∴ △ABC=△EBC....................I. 37.

But ▱ABCD is double of △ABC............I. 34 (iii).

∴ ▱ABCD is double of △EBC............Ax. 6.

Wherefore, *if a parallelogram* &c. Q.E.D.

EXERCISES.

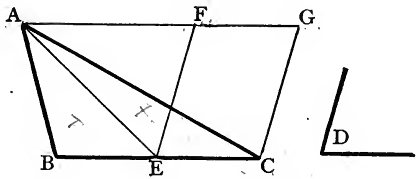
1. In what special form is Axiom 6 used in this proposition?
2. Prove that if a parallelogram and a triangle stand on equal bases and between the same parallels the parallelogram is double of the triangle.
3. Construct a rectangle which shall be double of a given triangle.
4. If any point be taken in a side of a parallelogram and joined to the opposite angles, one of the triangles so formed is equal to the sum of the other two.
5. Any point O is taken within a parallelogram ABCD; prove that the sum of the triangles OAD, OBC is half the parallelogram.
6. Any point O is taken without a parallelogram ABCD; prove that the difference of the triangles OAD, OBC is half the parallelogram.
7. Construct a triangle equal in area to a given parallelogram.
8. Construct a right-angled triangle equal in area to a given rectangle.
9. If in the figure of Prop. 41, BE and DC cut at O, and AO be joined; then triangles AOD, OCE are equal in area.

PROPOSITION XLII. PROBLEM.

To make a parallelogram equal to a given triangle, and having an angle equal to a given rectilineal angle.

Let ABC be the given △, and D the given ∠.

It is req^d to make a ▱ = △ABC, and having an ∠ = D.



Bisect BC in E..I. 10.
At E, in CE, make ∠CEF=D.......................................I. 23.
Through A draw AFG ∥ to BC,.....................................I. 31.
and meeting EF at F.
Through C draw CG ∥ to EF,......................................I. 31.
and meeting AFG at G.

Then shall FECG be the ▱ req^d.

Join AE.

Then, since BE=EC...............................Constr.
∴ △ABE=△AEC........................I. 38.
Hence △ABC is double of △AEC,
but ▱FECG is double of △AEC..........I. 41.
∴ ▱FECG=△ABC.......................Ax. 6.
and it has an ∠FEC=D..................Constr.

Wherefore, *a parallelogram has been made* &c. Q.E.F.

EXERCISES.

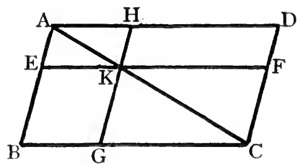
1. Construct a right-angled triangle equal to a given square.
2. Construct a rectangle equal to a given triangle.
3. Make a triangle equal to a given parallelogram and having an angle equal to a given angle.
4. Construct an isosceles triangle equal to a given square.
5. Construct a triangle equal to a given rectangle and having one of its angles equal to half a right angle.
6. Make a rectangle equal in area to a given parallelogram, having one of its angles at a given point, and one of its sides in a given direction.

PROPOSITION XLIII. THEOREM.

The complements of the parallelograms which are about the diameter of any parallelogram, are equal to one another.

Let ABCD be a ▱, AC its diam., EH, FG ▱s about AC, and BK, KD the complements.

Then shall compt BK=compt KD.



For, since EH is a ▱,

∴ △AEK=△AHK ....I. 34 (iii).

And, since GF is a ▱,

∴ △KGC=△KFC..................I. 34 (iii).

∴ △s AEK, KGC=△s AHK, KFC.........Ax. 2.

But the whole △ABC=whole △ADC...........I. 34 (iii).

∴ the rem^r BK=rem^r KD...................Ax. 3.

Wherefore, *the complements* &c. Q.E.D.

EXERCISES.

1. Define "Parallelograms about the diameter," and "Complements."
2. If a point not on a diameter were taken, and through it parallels drawn to the sides, would any of the four parallelograms formed be necessarily equal?
3. In the figure of I. 43 prove that,

   (i) ▱HB=▱ED.

   (ii) ∠AEK=∠KGC.

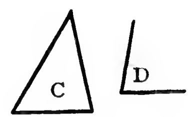
   (iii) EH is || to GF.
4. Prove that, if K is the middle point of AC, the complements are equal in all respects.
5. Prove that in the case of a rectangle the complements are rectangles.
6. Parallelograms about the diameter of a square are squares.

PROPOSITION XLIV. Problem.

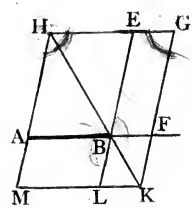
To a given straight line to apply a parallelogram equal to a given triangle and having an angle equal to a given angle.

Let AB be the given st. line, C the given $\triangle$, and D the given $\angle$.

It is req[d] *to apply to AB a* ▱ $=$ *C, and having an* $\angle = \angle$ *D.*



Make ▱ EBFG $= \triangle$ C, having $\angle$ EBF $= \angle$ D, and side BF in the direction of AB prod[d] ....I. 42.*
Through A draw AH ∥ to BE ....I. 31.
and meeting GE prod[d] in H,
Join HB..................................Post. 1.



Now, since AH is ∥ to FG ....Constr.
$\therefore$ $\angle$s AHG, HGF = two rt. $\angle$s...I. 29 (iii).
and $\therefore$ $\angle$s BHG, HGF $<$ two rt. $\angle$s.
Hence HB and GF, if prod[d] towards B and F, will meet............Ax. 12.

Produce HB and GF to meet in K......Post. 2.
Through K draw KLM ∥ to AF,.........I. 31.
meeting EB and HA prod[d] in L and M.

Then shall AMLB be the ▱ req[d].

For, since HMKG is a ▱, and HK its diam.......Constr.
$\therefore$ comp[t] MB = comp[t] BG....................I. 43.
but ▱ BG $= \triangle$ C ....Constr.
$\therefore$ ▱ MB $= \triangle$ C ....Ax. 1.
Also, since $\angle$ ABL $= \angle$ EBF.......................I. 15.
and $\angle$ EBF $= \angle$ D...........................Constr.
$\therefore$ $\angle$ ABL $= \angle$ D...........................Ax. 1.

Wherefore, *to the given st. line AB has been applied* &c. Q.E.F.

NOTE.

* This can be done thus:—On AB produced describe a $\triangle$PBQ $= \triangle$C (I. 22), and then, by I. 42, make ▱EBFG $= \triangle$PBQ &c.

EXERCISES.

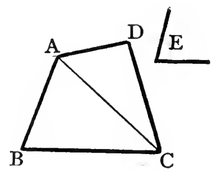
1. In the figure of I. 41, prove that AB and EC, when produced, will meet.
2. On a given base construct a rectangle equal to a given rectangle.
3. On a given finite straight line construct a rectangle equal in area to a given parallelogram.

PROPOSITION XLV. PROBLEM.

*To make a parallelogram equal to a given rectilineal figure, and having an angle equal to a given rectilineal angle.**

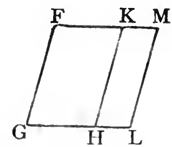
Let ABCD be the given rect[l] fig., and E the given ∠.

It is req[d] to make a ▱ = ABCD and having an ∠ = ∠ E.



Join AC.....................................Post. 1.
Make ▱ FGHK = △ ABC
and having ∠ FGH = ∠ E...............I. 42.
To HK apply ▱ HLMK = △ ADC
and having ∠ KHL = ∠ E..............I. 44.

Then FL shall be the ▱ req[d].



For, since ∠ FGH = ∠ E................Constr.
and ∠ KHL = ∠ E................Constr.
∴ ∠ FGH = ∠ KHL...........Ax. 1.
Add ∠ KHG to each
∴ ∠s FGH, KHG = ∠ KHG, KHL...Ax. 2.
but ∠s FGH, KHG = two rt. ∠s........I. 29 (iii).
∴ ∠s KHG, KHL = two rt. ∠s.......Ax. 1.
and ∴ GH, HL are in the same st. line..........................I. 14.

Again, since KM is ∥ to GL.......................Constr.
∴ ∠ GHK = alt. ∠ HKM...............I. 29 (i).
Add ∠ HKF to each
∴ ∠s GHK, HKF = ∠s HKF, HKM.........Ax. 2.
but ∠s GHK, HKF = two rt. ∠s.................I. 29 (iii).
∴ ∠s HKF, HKM = two rt. ∠s.................Ax. 1.
and ∴ FK, KM are in the same st. line.............I. 14.

Also, since FG is ∥ to KH.......................Constr.
and ML is ∥ to KH.......................Constr.
∴ FG is ∥ to ML.......................I. 30.
Hence, the fig. FGLM is a ▱.

And, since ▱ FH = △ ABC.....................Constr.
and ▱ KL = △ ADC.....................Constr.
∴ whole ▱ FL = whole fig. ABCD.........Ax. 2.
and it has ∠ FGH = ∠ E..........................Constr.

Wherefore, *a parallelogram has been made* &c. Q.E.F.

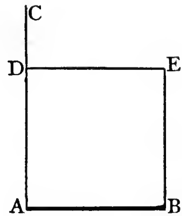
EXERCISES.

1. To a given straight line apply a rectangle equal to a given quadrilateral.
2. Prove Prop. 45 for the case of a rectilineal figure of five sides.

* This problem may be easily solved by using the method of Ex. I. p. 67.

PROPOSITION XLVI. PROBLEM.

To describe a square on a given straight line.



Let AB be the given st. line.

It is req^{d} *to desc. a sq. on AB.*

From A draw AC at rt. ∠s to AB ......I. 11.
From AC cut off AD=AB................I. 3.
Through D draw DE ∥ to AB............I. 31.
Through B draw BE ∥ to AC............I. 31.
and meeting DE in E.

Then shall DABE be the sq. req^{d}.

For, since DABE is a ▱.....................Constr.
∴ AB=DE
and AD=BE } ....I. 34 (i).
But AD=AB..........................Constr.
∴ the four sides AB, AD, DE, BE are all equal.

Again, since DE is ∥ to AB.....................Constr.
∴ ∠s BAD, ADE=two rt. ∠s..............I. 29 (iii).
But ∠BAD is a rt. ∠....................Constr.
∴ ∠ADE is also a rt. ∠.

And, since DABE is a ▱, its opp. ∠s are equal......I. 34 (ii).
∴ each of the ∠s DEB and EBA is a rt. ∠.

Hence the figure DABE is a sq................Def. 30.

Wherefore, *on the given straight line* &c. Q.E.F.

COR. Hence every ▱ that has one rt. ∠ has all its ∠s rt. ∠s.

EXERCISES.

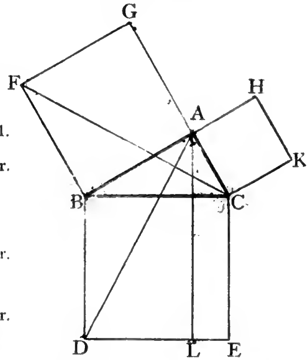
1. Prove that the squares described on equal straight lines are equal.
2. State and prove the converse of Ex. 1.
3. Prove that the perimeter of a square is less than that of any other equal parallelogram standing on the same base.
4. Show how to draw a line from A at right angles to AB without producing AB.
5. The diagonals of a square bisect each other at right angles.
6. A square and a rhombus stand on the same base; prove that the area of the square is greater than that of the rhombus.
7. Show that Euclid's definition of a square states more than is necessary.

PROPOSITION XLVII. Theorem.

In any right-angled triangle, the square described on the side subtending the right angle is equal to the squares described on the sides containing the right angle.

Let ABC be a rt. angled △, having the rt. ∠ at A.

Then shall the sq. on BC=sqs. on BA and AC.



On BC desc. sq. BDEC..........I. 46.
and on BA, AC the sqs. GB, HC...I. 46.
Through A draw AL ∥ to BD.....I. 31.
meeting DE in L.
Join AD, FC..........................Post. 1.

Then, since ∠BAG is a rt. ∠......Constr.
and ∠BAC is a rt. ∠......Hyp.
∴ GA, AC are in one st. line...I. 14.

And, since GA is ∥ to FB..........Constr.
∴ sq. GB is double of △FBC...I. 41.

Also, since AL is ∥ to BD..........Constr.
∴ ▱BL is double of △ABD..I. 41.

Again, since each of the ∠s FBA and DBC is a rt. ∠ ......Constr.
∴ ∠FBA=∠DBC......................Ax. 11.
Add ∠ABC to each.
∴ the whole ∠FBC=whole ∠ABD...............Ax. 2.

Hence, in △s FBC, ABD,

∵ { FB=AB............................Constr.
BC=BD............................Constr.
∠FBC=∠ABD....................Proved above. }

∴ △FBC=△ABD......................I. 4.

But sq. BG is double of △FBC,
and ▱ BL is double of △ABD } ....Proved above.

∴ sq. BG=▱BL........................Ax. 6.

In like manner, by joining BK, AE, it may be proved
that sq. HC=▱CL.

∴ the whole sq. BDEC=sqs. GB and HC............Ax. 1.
i.e. the sq. on BC=sqs. on BA and AC.

Wherefore, *in any right-angled triangle* &c. Q.E.D.

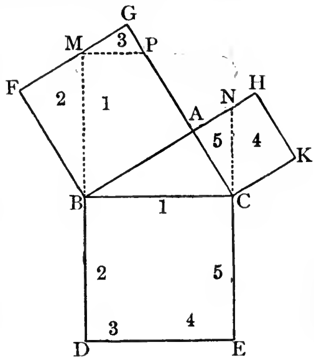
NOTES.

The result of this proposition is very important. It will be more easily remembered in the following form:—

In a right-angled triangle, the square on the hypotenuse is equal to the sum of the squares on the sides.

The theorem is supposed to have been discovered by Pythagoras more than 500 years B.C., and 200 years before the time of Euclid.

Various proofs of its truth have been given; some of them "ocular demonstrations," showing how the smaller squares may be cut up into bits which will form the larger one, or *vice versâ*. One way of doing this is given below.*



Produce DB to cut FG in M.
Draw MP parallel to BC.
Produce EC to cut AH in N.
Then the five fragments, numbered 1, 2, 3, 4, 5, can be so arranged as to cover the square BDEC.

It is left as an exercise for the student to discover how they must be arranged. The numbers on the large square show the locality of the pieces.

EXERCISES.

1. The sides of a right-angled triangle are 3 inches and 4 inches respectively; find the length of the hypotenuse.
2. The hypotenuse is 10, one side is 8, find the other side of the right-angled triangle.
3. Write out the proof that square HC is equal to parallelogram CL.
4. The sum of the squares on the diagonals of a rectangle is equal to the sum of the squares on its four sides.
5. Find a line the square on which shall be double the square on a given line.
6. In the figure of I. 47 prove that (i) F, A, K are in a straight line; (ii) GB is parallel to HC; (iii) triangles FBD, ABK are equal in area; (iv) squares on FA and AK are double the square on BC.
7. Prove Prop. 47 when the square BDEC is described on the other side of BC.
8. The sum of the squares on the sides of a rhombus is equal to the sum of the squares on its diagonals.
9. If any point P be taken inside a rectangle ABCD, the squares on PA and PC are together equal to the squares on PB and PD.
10. The sum of the squares on the sides of an equilateral triangle is equal to four times the square on the perpendicular from an angle to the opposite side.

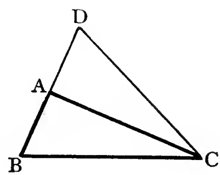
* See also Appendix II.

PROPOSITION XLVIII. THEOREM.

If the square described on one of the sides of a triangle be equal to the squares described on the other two sides, the angle contained by these two sides is a right angle.

Let the sq. on BC=sqs. on the sides BA, AC of △ABC.

Then shall ∠BAC be a rt. ∠.



From A draw AD at rt. ∠s to AC......I. 11.
Cut off AD=AB......I. 3.
Join DC......Post. 1.

Then, since DA=BA......Constr.
∴ sq. on DA=sq. on BA.
Add sq. on AC to each
∴ sqs. on DA, AC=sqs. on BA, AC......Ax. 2.

But sqs. on DA, AC=sq. on DC......I. 47.

and sqs. on BA, AC=sq. on BC......Hyp.
∴ sq. on DC=sq. on BC
and ∴ DC=BC.

Hence, in △s ABC, ADC,

∵ { AB=AD......Constr.
AC is com.
BC=DC......Proved above. }

∴ ∠BAC=∠DAC......I. 8.

But ∠DAC is a rt. ∠......Constr.
∴ ∠BAC is a rt. ∠.

Wherefore, *if the square* &c. Q.E.D.

NOTES.

The following theorem, and its converse, is assumed as an axiom in this proposition:—"The squares on equal straight lines are equal."

Prop. 48, the converse of Prop. 47, unlike most converse propositions, is proved by a *direct* method.

N.B.—In the construction *AD must be drawn at rt.* $\angle$*s to AC.* It will not do to *produce BA to D.*

Props. 34 to 48 form the 3rd section of Book I., which deals with *areas.*

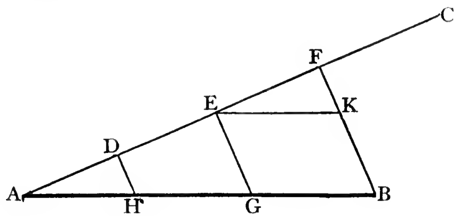
EXERCISES.

1. Is the triangle right-angled whose sides are (i) 3, 4, 5; (ii) 5, 6, 8; (iii) 6, 8, 10; (iv) 7, 9, 12; (v) 5, 12, 13?
2. Given a line one inch long, show how to obtain a line
   (i) $\sqrt{2}$ inches long.
   (ii) $\sqrt{5}$ inches long.
3. Show how to cut up the large square of Prop. 47 so as to form the smaller squares.
4. What is the distance between opposite corners of a page of a book 8 inches long and 6 inches wide?
5. If two right-angled isosceles triangles have one side common, their hypotenuses are at right angles.
6. If a straight line be divided into any two parts, the square on the whole line is greater than the sum of the squares on the two parts.
7. In any triangle, the square on a side subtending an acute angle is less than the sum of the squares on the sides which contain that angle.
8. If the square on one side of a triangle is less than the sum of the squares on the other two sides, the angle contained by these two sides is acute.
9. If the square on one side of a triangle is greater than the squares on the other two sides, the triangle is obtuse-angled.
10. If two right-angled triangles have their hypotenuses equal, and have also one side equal to one side, then their other sides are equal.

MISCELLANEOUS EXAMPLES.

1. *To divide a given straight line into any given number of equal parts.*

Let AB be the given st. line, and suppose, for example, that it is req^{d} to divide it into *three* = parts.



Through A draw any other st. line AC.
Take any pt. D in AC, and from DC cut off DE=AD,
and from EC cut off EF=AD............I. 3.
Join BF.
Through E draw EG ∥ to BF, meeting AB in G }
Through D draw DH ∥ to BF, meeting AB in H }..........I. 31.

Then shall AB be divided into three equal parts at H and G.

Through E draw EK ∥ to AB, meeting BF in K..........I. 31.

Then, since GK is a ▱,..........Constr.
∴ GB=EK..........I. 34 (i).

and, since EK is ∥ to AB..........Constr.
ext^{r} ∠KEF=int^{r} opp. ∠HAD..........I. 29 (ii).
also, since HD is ∥ to BF..........Constr.
ext^{r} ∠HDA=int^{r} opp. ∠KFE..........I. 29 (ii).

Hence in △s HAD, KEF

∵ { ∠HAD=∠KEF, ∠HDA=∠KFE }..........Proved above.
AD=EF..........Constr.

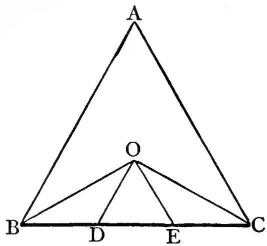
∴ AH=EK..........I. 26 (i).
But EK=GB..........Proved above.
∴ AH=GB.

Similarly it may be shown that HG=GB. Q.E.F.

A line may also be divided into *three* equal parts, thus:—

II. *To trisect a given finite straight line.*

Let BC be the given st. line.



On BC desc. an equilat. △ABC..I. 1.
Bisect the ∠s at B and C by BO and CO meeting at O.............I. 9.
Through O draw OD ∥ to AB, and OE ∥ to AC,
meeting BC in D and E..I. 31.

Then BC shall be trisected at the pts. D and E.

For, since OD is ∥ to AB,.................... ....Constr.
∴ extr ∠ODE=intr ∠ABC,..................I. 29 (ii).
and, since OE is ∥ to AC,...Constr.
∴ extr ∠OED=intr ∠ACB....................I. 29 (ii).
∴ the remg ∠DOE=remg ∠BAC...................I. 32.
Hence, △ODE has its angles equal, and is ∴ equilat...........I. 6. Cor.

Again, since OD is ∥ to AB................................ .....Constr.
∴ ∠ABO=alt. ∠BOD.....................I. 29 (i).

But ∠ABO=∠OBD.........................Constr.
∴ ∠BOD=∠OBD.........................Ax. 1.
and ∴ OD=BD.............................I. 6.

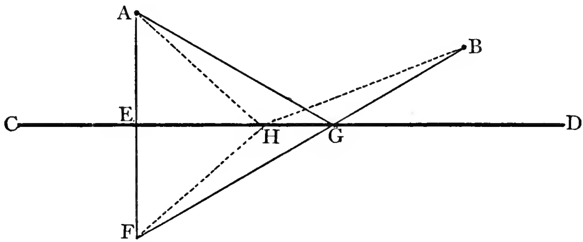
But OD=DE..........................Equilat. △
∴ BD=DE..............................Ax. 1.

Similarly it may be shown that DE=EC.

Wherefore, *BC is trisected at D and E.* Q.E.F.

III. *To find a point in a given straight line such that the sum of its distances from two given fixed points, on the same side of the line, may be the least possible.*

Let A and B be the given pts. and CD the given st. line.



Draw AE ⊥ to CD .... I. 12.
Produce AE to F,
Cut off EF=AE .... I. 3.
Join FB meeting CD at G.

Then G shall be the pt. req^d.

For. take any other pt. H, and join AH, HB, FH.

Then in △s AEG, FEG,

∵ { AE = EF .... Constr.
EG is com.
rt. ∠ AEG=rt. ∠ FEG .... Ax. 11. }

∴ AG = FG .... I. 4.

Similarly, it may be shown that AH=FH.

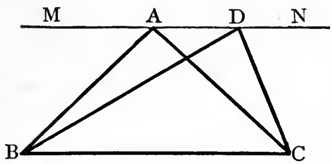
Hence AG, GB=FB
and AH, HB=FH, HB
but FB<FH, HB .... I. 20.
∴ AG, GB<AH, HB.

Q.E.F.

Cor.—The point G is such that AG, GB make *equal angles* with CD.

IV. *Of all equal triangles, standing on the same base, the isosceles has the least perimeter.*

Let ABC be an isos. △ and let DBC be any other △ on the same base BC and between the same ||s AD, BC, and having therefore the same area.



Then shall perimeter of △ABC be < that of DBC.

Produce AD both ways to M and N.

Then, since MN is || to BC,

∴ ∠MAB = alt. ∠ABC .... I. 29.

= ∠ACB .... I. 5.

= alt. ∠CAN .... I. 29.

i.e. BA, CA make equal angles with MN.

∴, by the previous corollary,

BA, AC < BD, DC.

Add BC to each,

∴ BA, AC, BC < BD, DC, BC.

i.e. perimeter of △ABC < perimeter of △DBC.

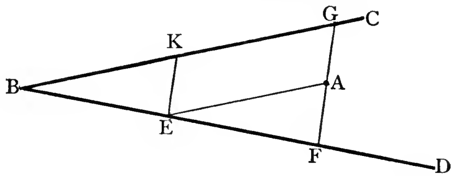
Q.E.D.

By the method used in IV. the following theorem may be easily proved:—

If a polygon be not regular, there may be found another polygon having the same number of sides and the same area, but having a less perimeter.

V. *Through a given point to draw a straight line, so that the part of it intercepted between two given straight lines, may be bisected at the given point.*

Let A be the given pt.; BC, BD the given st. lines.



Through A draw AE || to BC..I. 31.
and meeting BD at E.
From ED cut off EF=EB ....I.
Join FA and produce to meet BC in G.

FAG shall be bisected at A.

Through E draw EK || to FG, meeting BG in K......................I. 31.

Then, since KA is a ▱,...............................Constr.

∴ AG=EK

and, in △s KBE, AEF

∵ { ∠KBE=∠AEF, ∠KEB=∠AFE }....................I. 29 (ii).
BE=EF ....Constr.

∴ EK=FA.......................... I. 26 (i).

but AG=EK........................Proved above.

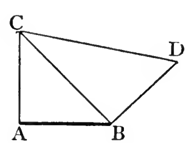
∴ FA=AG..........................Ax. 1.

Q.E.F.

VI. *To make a square which shall be any given multiple of the square on a given straight line.*

Let AB be the given straight line.

It is req[d] *to make a sq. which shall be* (suppose for example) *three times the sq. on AB.*



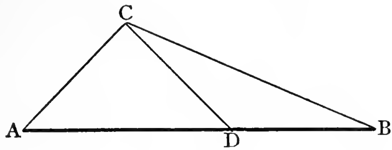
From A draw AC at rt. ∠s to AB.........I. 11.
Cut off AC=AB, and join CB................I. 3.
From B draw BD at rt. ∠s to BC...........I. 11.
Cut off BD=AB, and join CD................I. 3.

Then shall the sq. on CD be three times the sq. on AB.

For, sq. on CD=sqs. on CB, BD.........................I. 47.
=sqs. on CA, AB, BD....................I. 47.
=three times sq. on AB. Q.E.F.

VII. *To divide a given straight line into two parts, so that the square on one part may be double of the square on the other.*

Let AB be the given st. line.



At pt. A in AB make ∠BAC = half a rt. ∠ ....
At pt. B in AB make ∠ABC = a quarter of a rt. ∠ .... I. 11, I. 9, and I. 23.
At pt. C in BC make ∠BCD = a quarter of a rt. ∠
And let CD meet AB in D.

Then shall sq. on AD be double of sq. on DB.

For, ∠CDA = ∠s DCB, DBC .... I. 32 (i).
= twice ∠DCB,
= half a rt. ∠,
= ∠CAD.

Hence, since ∠s CAD, CDA are each half a rt. ∠
∴ ∠ACD is a rt. ∠ .... I. 32 (ii).

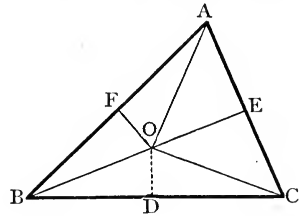
Again, since ∠CAD = ∠CDA .... Ax. 11.
∴ CA = CD .... I. 6.

And, since ∠DCB = ∠DBC .... Constr.
∴ CD = DB .... I. 6.

Hence, sq. on AD = sqs. on AC, CD .... I. 47.
= twice sq. on CD,
= twice sq. on DB.

Q.E.F.

VIII. *The straight lines drawn at right angles to the sides of a triangle, from their middle points, meet in a point.*



Let ABC be a △, and D, E, F the mid. pts. of its sides.

Draw EO, FO at rt. ∠s to AC, AB and meeting at O.
Join OD.

Then OD shall be at rt. ∠s to BC.

For, in △s OEA, OEC,

∵ { AE=EC .... Constr.
OE is com.
rt. ∠AEO=rt. ∠CEO .... Ax. 11.

∴ OA=OC .... I. 4.

Similarly it may be shown that OA=OB

∴ OB=OC .... Ax. 1.

Hence, in △s OBD, OCD,

∵ { BD=CD .... Constr.
OD is com.
OB=OC .... Proved above.

∴ ∠ODB=∠ODC .... I. 8.

Hence OD is at rt. ∠s to BC .... Def. 10.

Wherefore, *the straight lines* &c. Q.E.D.

Since OA, OB, OC are all equal, a circle described with centre O, and radius OA will pass through B and C. Such a circle is said to be circumscribed about the triangle ABC. Hence, the following constructions:—

(i) *To circumscribe a circle about a triangle.*

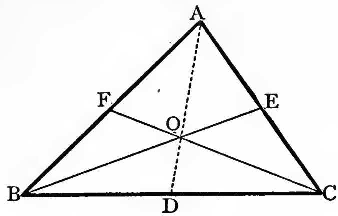
Bisect two of the sides of the triangle, and through the points of bisection draw lines at right angles to those sides; the point at which the lines cut is the centre, and the distance of this point from one of the angles is the radius.

(ii) *To find the centre of a given circle.*

From any point on the circumference of the circle, draw two chords; bisect these chords, and from the points of bisection draw lines at right angles to the chords; the point in which these lines cut is the centre.

IX.—*The straight lines drawn from the angles of a triangle to the middle points of the opposite sides,* meet at a point.*

Let D, E, F be mid. pts. of sides BC, AC, AB, of △ABC.



Join BE, CF, cutting at O.
Join AO, DO.

Then shall AO, DO be in one st. line.

For, since AE=EC.................................Hyp.
∴ △ABE=△CBE..........................I. 38.
∴ △ABE is half △ABC.
Similarly △ACF is half △ABC.
∴ △ABE=△ACF..........................Ax. 7.

Take away the com. part AFOE
∴ rem^{g} △OBF=rem^{g} △OCE..................Ax. 3.
but △OBF=△OAF..........................I. 38.
=half △OBA
and △OCE=△OAE..........................I. 38.
=half △OCA
∴ △OBA=△OCA..........................Ax. 6.
Again, since BD=CD.................................Hyp.
∴ △OBD=△OCD..........................I. 38.
Hence △s OBA, OBD=△s OCA, OCD................Ax. 2.
i.e. AO, OD bisect the △ABC.
But if AD were drawn it would bisect △ABC.........I. 38.
∴ AO, OD must coincide with st. line AD.
i.e. AD passes through O.

Wherefore, *AD, BE, CF meet at one point.* Q.E.D.

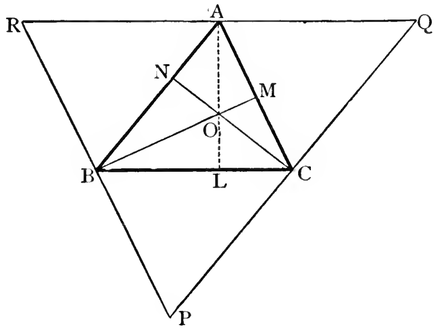
---

* These lines are sometimes called *medians;* and the proposition enunciated thus:—*The medians of a triangle are concurrent.*

X. *The perpendiculars from the angles of a triangle on the opposite sides meet at a point.*

In △ABC, let the ⊥s BM, CN meet at O.

Then AO produced shall be ⊥ to BC.



Through A draw RAQ ‖ to BC
Through B draw RBP ‖ to AC .... I. 31.
Through C draw PCQ ‖ to AB

Then, since ABPC is a ▱,

∴ BP=AC.................................I 34

and, since ARBC is a ▱,

RB=AC..............................I. 34.

∴ RB=BP.

Similarly, RA=AQ, and PC=CQ.

And, since AC is ‖ to PR, and BM ⊥ to AC,

∴ BM is also ⊥ to PR.......................I. 29 (iii).

Similarly, NC is ⊥ to PQ.

Since O is the pt. of intersection of lines from the mid. pts. of the sides PQ, PR of △PQR at rt. ∠s to those sides,

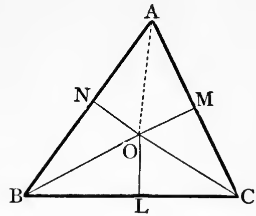
∴, by VIII., AO is at rt. ∠s to RQ.

But BC is ‖ to RQ,

and ∴ AOL is ⊥ to BC. Q.E.D.

The point O at which the perpendiculars meet is called the *Orthocentre* of the triangle.

XI. *The straight lines which bisect the angles of a triangle meet at a point.*



Let ABC be a $\triangle$.
Bisect $\angle$s ABC, ACB by BO, CO meeting at O....................I. 9.
Join OA.

Then shall OA bisect $\angle$BAC.

Draw the $\perp$s OL, OM, ON from O to the sides..............................I. 12.

Then, in $\triangle$s OBN, OBL,

$\because$ { $\angle$OBN$=\angle$OBL........................Constr.
rt. $\angle$ONB$=$rt. $\angle$OLB.....................Ax. 11.
OB is com. }

$\therefore$ ON$=$OL............................I. 26 (ii).

Similarly it may be proved that OM$=$OL

$\therefore$ ON$=$OM............................Ax. 1.

Now, since $\angle$ONA and $\angle$OMA are rt. $\angle$s

$\therefore$ sq. on OA$=$sqs. on ON, NA
and sq. on OA$=$sqs. on OM, MA } ....I. 47.

$\therefore$ sqs. on ON, NA$=$sqs. on OM, MA.

But since ON$=$OM,
sq. on ON$=$sq. on OM
$\therefore$ sq. on NA$=$sq. on MA
$\therefore$ NA$=$MA.

Hence in $\triangle$s NAO, MAO

$\because$ { NA$=$MA..........................Proved above.
AO is com.
ON$=$OM..........................Proved above. }

$\therefore$ $\angle$NAO$=\angle$MAO......................I. 8.

i.e. the bisector of $\angle$A passes through O.

Q.E.D.

Hence the following constructions:—

To find points equidistant from three given straight lines.

(i) Inside the triangle formed by the lines *one* point can be found.

Bisect two of the angles of this triangle; the point is where these bisectors cut.

(ii) Outside this triangle *three* points can be found.

Bisect the exterior angles formed by producing two of the sides; the points of intersection of these bisectors can be proved to be equidistant from the sides by the method used in XI.

MISCELLANEOUS EXERCISES.

1. Are any two angles of a triangle greater than the third? Give a reason for your answer.
2. BAC, EAD are two equal angles; if the points B, A, D be in the same straight line, the points C, A, E can also be in the same straight line.
3. If two equilateral triangles be described on opposite sides of a straight line, the line joining their vertices will bisect the line.
4. If from the vertex of a triangle a perpendicular be drawn to the base, the difference of the squares on the sides is equal to the difference of the squares on the segments of the base.
5. ABC is a right-angled triangle, C being the right angle. Prove that if P be a point on BC between C and B, AP is less than AB, but if P be in CB produced, AP is greater than AB.
6. D is any point on the side BC of the triangle ABC. If the angle ABC be an obtuse angle, prove that AD is greater than AB but less than AC.
7. If, upon the same base AB, two triangles BAC, ABD be constructed, having the angle BAC equal to ABD and ABC equal to BAD, prove that the triangles BDC, ADC are equal in all respects.
8. BD, CF bisect the equal angles of the isosceles triangle ABC, and meet the opposite sides in D and F; prove that CD, DF, and BF are all equal.
9. A point A is taken in the circumference of a circle whose centre is O and a circle is described having A as centre and meeting the first circle in B and C; prove that AO bisects the angle BAC.
10. In any right-angled triangle, the distance of the middle point of the hypotenuse from the right angle is half the length of the hypotenuse.
11. If AB, AC be equal sides of an isosceles triangle, and a circle with centre B and distance BA cut AC (or AC produced) in E, and BF be taken in AB (or AB produced, if E lies in AC produced) equal to CE, prove that the angle CFA is equal to the angle FAC.
12. Find a point D in the side BC of a triangle ABC such that AD may be half the sum of AB and AC.
13. One interior angle of a regular polygon contains 165°; find the number of sides.
14. How many acres, roods, and poles are there in a triangular field whose longest side is 27 poles, and the distance of the opposite corner from this side 18 poles?
15. ABDE, BFGC are squares on the sides AB, BC of a triangle ABC. AF, CD are joined; show that AF and CD are equal.

16. The base of a triangle, whose sides are unequal, is divided into two segments by the line bisecting the vertical angle; prove that the greater segment is adjacent to the greater side.

17. If a point P be taken inside a quadrilateral ABCD, prove that the sum of the distances of P from the angular points is the least possible when P is situated at the intersection of the diagonals.

18. ABCD is a parallelogram and CE drawn parallel to BD meets AD produced in E; prove that AD is equal to DE.

19. AOB, COD are two straight lines intersecting O; if the triangles AOC, BOD be equal in area, BC shall be parallel to AD.

20. ABC is a given triangle; construct a triangle of equal area having its base in AB and its vertex in a given straight line parallel to AB.

21. Find a square which shall be equal to three given squares.

22. Describe an isosceles right-angled triangle equal to a given square.

23. ABC is a triangle, D, E the middle points of AB, AC respectively; prove that the triangle BFD is equal to the triangle CEF, where F is the point of intersection of BE, CD.

24. The sum of the four sides of any quadrilateral is less than twice the sum of its two diagonals.

25. Find the magnitude of one angle of a regular figure of 100 sides.

26. From a point P outside an angle draw a straight line cutting the lines containing the angle in B and C such that PB shall be equal to BC.

27. Find a point in a given straight line, the difference of the distances of which from two given points on the same side of the line, shall be the greatest possible.

28. An obtuse-angled triangle ABC, having the angle ABC obtuse, is turned over about its side BC; prove that the line joining the two positions of A is perpendicular to BC produced.

29. In the base BC of a triangle ABC any point D is taken; draw a straight line such that, if the triangle ABC be folded along this line, the point A shall fall on the point D.

30. In a right-angled triangle the square whose diagonal is the side subtending the right angle is equal to the squares whose diagonals are the sides containing the right angle.

31. On a given straight line describe a triangle which shall be equal to a given parallelogram, and have one of its angles equal to a given rectilineal angle.

32. If the opposite angles of a quadrilateral are equal, it is a parallelogram.

33. If a quadrilateral be bisected by each of its diagonals, it is a parallelogram.

34. Through a given point draw a straight line which shall be equidistant from two other given points.

35. If two adjacent corners of a rhombus be fixed, the other corners lie on fixed circles; but if two opposite corners be fixed the other corners lie on a fixed straight line.

36. ABCD is a rectangle; O is any point in the diagonal BD; show that the sum of the squares on OA and OC is equal to the sum of the squares on OB and OD.

37. Any point P is taken in the line joining an angular point A of a triangle to the middle point of the opposite side BC; prove that the triangles APB, APC are equal.

38. In any triangle straight lines are drawn from each angle to any point in the opposite side; prove that the sum of the lengths of these lines is greater than the semi-perimeter but less than three times the semi-perimeter of the triangle.

39. If ABC be a triangle, in which the angle A is a right angle, and BE, CF be drawn bisecting the opposite sides respectively, show that four times the sum of the squares on BE and CF is equal to five times the square on BC.

40. Perpendiculars AD, BE, CF are drawn from the angular points A, B, C of a triangle upon the sides respectively opposite to them; prove that the sum of the squares upon BD, CE, AF is equal to the sum of the squares upon CD, AE, BF.

41. The four straight lines which bisect the successive angles of a parallelogram include a rectangle, and if the larger side of the parallelogram be twice the shorter, the diagonal of the rectangle is equal to the shorter side.

42. ABCD is an oblique parallelogram; from A a perpendicular is drawn to the side AB, from B a perpendicular is drawn to the side BC, from C a perpendicular is drawn to the side CD, from D a perpendicular is drawn to the side DA; prove that these perpendiculars will by their intersections form a parallelogram equiangular to ABCD.

43. If two sides of a triangle be given, the area will be greatest when they contain a right angle.

44. Given three sides of a quadrilateral, and the angles adjacent to the fourth side, construct the figure.

45. Construct a triangle, two of whose angles are given, which shall have its vertex at a given point and its base in a given straight line.

46. AB, AC are two straight lines; in AB three points P, Q, R are taken such that Q is equidistant from P and R, show that the perpendicular from Q on AC is equal to half the sum of the perpendiculars from P and R on AC.

47. ABC is a right-angled isosceles triangle, A being the right angle. Any line is drawn through A, and perpendiculars BM, CN from B and

C are drawn to this line. Prove that MN is equal either to the sum or difference of BM and CN.

48. The point of intersection of the diagonals of the square described on the hypotenuse of a right-angled triangle is equidistant from the two sides which contain the right angle.

49. If, in the figure of I. 35, the diagonals of the parallelograms be drawn, prove that the two triangles which have each of them two of these diagonals for two of their sides will be equal.

50. Two sides AB, AD of a quadrilateral ABCD are given in position and magnitude, and the area of the quadrilateral is given; find the locus of the middle point of AC.

51. Prove that two of the angles of an equilateral triangle are together equal to one of the angles of an equiangular six-sided polygon.

52. Construct a triangle having given the perimeter and two of its angles.

53. A chord of a circle, whose radius is 10 feet, is 16 feet long; find its distance from the centre.

54. Given a pair of compasses; find a point C to which a right line AB must be produced so that AC is equal to three times AB.

55. If two triangles have the sides of one parallel to the sides of the other, the angles of the one are equal to the angles of the other.

56. The sides of a rectangular floor are 16 feet and 12 feet long respectively, find the distance between its opposite corners.

57. Find the magnitude of one interior angle of a regular polygon of n sides.

58. If lines be drawn from an angular point of a regular hexagon, to all the other angles, prove that the angles between these lines are equal.

59. The leaf of a book is turned down, so that the corner always lies on the same line of printing; find the locus of the foot of the perpendicular from the corner on the crease.

60. Find a point at given distances from the arms of a given angle.

61. If K be the common angular point of the parallelograms about the diameter of a parallelogram, and BD the other diameter, the difference of the parallelograms is equal to twice the triangle BKD.

62. Of all parallelograms which can be formed with diameters of given lengths, the rhombus is the greatest.

63. Trisect a parallelogram by lines drawn through an angular point.

64. If AA′, BB′, CC′, DD′ be equal lengths cut off from the sides of the parallelogram ABCD taken in order, then will A′B′C′D′ be also a parallelogram.

65. Through the angular points A, B, C of a triangle are drawn three parallel straight lines meeting the opposite sides, or those sides produced, in A′, B′, C′ respectively; prove that the triangles AB′C′, BC′A′, CA′B′ are all equal.

66. If through D, the middle point of the hypotenuse BC of a right-angled triangle ABC, DE be drawn at right angles to BC and meeting AC at E, prove that the square on EC is equal to the squares on EA, AB.

67. Between two given straight lines place a straight line equal to one, and parallel to another, given straight line.

68. Find the locus of the middle point of the hypotenuse of all right-angled triangles having a hypotenuse of given length and a common rt. angle.

69. Divide a given triangle into five equal parts.

70. Find a point within a triangle which shall be equidistant from the sides of the triangle.

71. Find a point equidistant from the angles of a triangle.

72. If L, M, N be the feet of the perpendiculars, drawn to the sides of a triangle ABC, from any point P within the triangle, then the sum of the squares on AM, BN and CL is equal to the sum of the squares on AN, BL and CM.

73. In the figure of I. 47, prove that the area of the hexagon formed by a side of each square and the three straight lines which join the adjacent corners of the squares is equal to four times the area of the original triangle together with twice the square on the hypotenuse.

74. The area of a quadrilateral is equal to that of the triangle, two of whose sides are respectively equal to the diagonals of the quadrilateral, and the angle included by these sides equal to the angle between the diagonals.

75. Construct a square equal to the difference of two given squares.

76. ABC is a triangle, and from A a line AD is drawn to the base making the angle BAD equal to ACB, a second line AE is drawn to meet the base in E so that AE is equal to AD. Show that the angle CAE is equal to the angle ABC.

77. If, in a right-angled triangle, the square on one of the sides containing the right angle be three times that on the other, the angle subtended by the first is double of that subtended by the second.

78. Construct a right-angled triangle having given the hypotenuse and one side.

79. ABC is a triangle in which ABC is a right angle, AB is 3 inches long, and BC is 4 inches long. CE is drawn at right angles to AC and equal to three times BC. Find the length of AE.

80. Given one side of a triangle, and the segments into which one of the other sides is divided by a perpendicular drawn to it from the extremity of the first side, construct the triangle.

81. If the diagonals of a parallelogram are equal it is rectangular.

82. If the bisectors of two angles adjacent to one side of a parallelogram meet on the opposite side, what will be the ratio of the unequal sides?

83. If the equal sides AB, AC of an isosceles triangle be produced to D and E so that AD=2AB, and AE=2AC, and if CD, BE meet in F, show that F is one of the points of trisection of CD and BE.

84. If the interior angle at one angular point of a triangle, and the exterior angle at another be bisected by straight lines, the angle contained by these two bisectors is equal to half the third angle of the triangle.

85. Describe a parallelogram equal and equiangular to a given parallelogram, and such that two of its opposite sides shall be at a given distance from each other.

86. E, F, G, H are points in the sides AB, BC, CD, DA respectively of a parallelogram ABCD such that AH=FC and AE=CG; show that EFGH is a parallelogram.

87. The sum of the perpendiculars, from any point within an equilateral triangle, upon the sides, is constant.

88. Describe a circle which shall pass through two given points and have its centre in a given straight line. Is this always possible?

89. Describe a parallelogram equal to a given square, having an angle equal to half a right angle, and one side equal to a given straight line longer than a side of the square.

90. The area of a square is to the area of the equilateral triangle described on one of its sides in the ratio of 4 to $\sqrt{3}$.

91. ABCD is a quadrilateral. DA is produced to E, AB to F, BC to G, and CD to H. The straight lines which bisect the angles EAB, FBC, GCD, HDA are drawn. Show that any two opposite angles of the quadrilateral formed by these lines are together equal to two right angles.

92. The bisectors of the angles of a parallelogram form a rectangle whose diagonals are parallel to the sides of the parallelogram and equal to the difference between them.

93. If through the angular points of a triangle ABC there be drawn three parallel straight lines AD, BE, CF meeting the opposite sides, or those sides produced, in D, E, F, then will the area of the triangle DEF be double that of ABC.

94. A′, B′, C′ are the middle points of the sides of a triangle ABC, and through A, B, C are drawn three parallel straight lines meeting B′C′, C′A′, A′B′ in *a*, *b*, *c* respectively; prove that triangle *abc* is half triangle ABC, and that *bc* passes through A, *ca* through B, *ab* through C.

95. In a given triangle inscribe a parallelogram equal to half the triangle so that one side is in the same straight line with one side of the triangle and one extremity at a given point in that side.

96. Given a line 1 inch long, show how to construct a triangle whose base shall be 2 inches, one of its sides 3 inches, and area $1\frac{1}{2}$ square inches.

97. Prove that, if O be any point in the plane of a parallelogram ABCD, and the parallelograms OAEB, OBFC, OCGD, ODHA be completed, then EFGH will be a parallelogram whose area is double that of the parallelogram ABCD.

98. In a right-angled triangle ABC the sides AB, AC which contain the right angle are equal to one another. A second right-angled triangle is described, having the sides containing the right angle together equal to AB and AC but not equal to one another. Prove that this triangle is less than the triangle ABC.

99. The angular points of one triangle lie on the sides of another; if the latter triangle be thus divided into four equal parts, prove that the lines joining its angular points with the corresponding angular points of the former triangle will be bisected by the sides of the former.

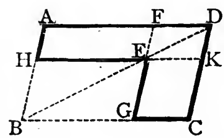
100. Trisect a triangle by lines drawn from a point in one of its sides.

BOOK II.

DEFINITIONS.

1. A rectangle is said to be **contained by** any two of its sides which meet.

2. In any parallelogram, one of the parallelograms about the diameter together with the two complements, is called a **gnomon**.



NOTES.

In Book II. rectangles are often mentioned which are not actually constructed, and such an expression as "rect. AB, CD" must often be understood to mean "the area which would be enclosed if a rectangle were constructed having two of its adjacent sides equal to AB and CD respectively."

The rectangle contained by two lines is sometimes spoken of as the rectangle *under* those lines.

A gnomon is named by three letters standing at opposite corners of the parallelograms; thus AKG, or HFC, is a name for the gnomon drawn to illustrate def. 2.

Two magnitudes are said to be commensurable when some unit of measurement can be found which is contained in both of them an exact number of times.

If the sides of a rectangle are commensurable lines its area can be *exactly* expressed arithmetically, but if the sides are incommensurable lengths (such, for instance, as the length of the side and the length of the diagonal of a square) its area can only be *approximately* expressed arithmetically.

Book II. treats, mainly, of the properties of rectangles.

EXERCISES.

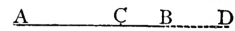
1. Prove that, if two adjacent sides of one rectangle are equal, respectively, to two adjacent sides of another, the rectangles are equal in area.
2. Why cannot *every* parallelogram be correctly said to be *contained* by two of its adjacent sides?

ADDITIONAL DEFINITIONS.

If any point be taken in a line, it is called *a point of section.*

The parts into which a line is divided at any point are called *segments of the line.*

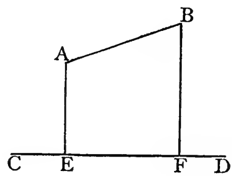
If any point be taken in a line produced, the line is said to be *divided externally.*



(For example, the line AB is divided internally at C, and externally at D; AC and CB are internal segments; AD and BD are external segments.)

The *projection of a point* on a line is the foot of the perpendicular on the line from the point.

The *projection of a line* on another line is that part of the second line which is intercepted between the feet of the perpendiculars to the second line from the extremities of the first.



(For example, EF is the projection of AB on CD.)

EXERCISES.

1. What is the area of a rectangle, one of whose sides is a feet long and another b yards?

2. Two adjacent sides of an oblique parallelogram are 3 feet and 4 feet long respectively: is its area greater than, equal to, or less than 12 square feet?

3. What is the area of a triangle standing on the same base and having the same altitude as the rectangle in Ex. 1?

4. Find approximately, to within one square inch, the area of a rectangle two of whose sides are each 3 feet long, and the other two each equal in length to the diagonal of a square whose side is 3 feet.

5. A line PQ is divided internally at O and externally at R, mention its internal and external segments.

6. Show that the length of the line PQ is equal to half the sum of its internal segments together with half the difference of its external segments.

7. ABCD is a rectangle; in AD any points E and F are taken, and through E, F are drawn EG, FH parallel to AB and meeting BC in G and H; prove that EGHF is a rectangle.

8. What is the altitude of a rectangle whose area is ab and base c?

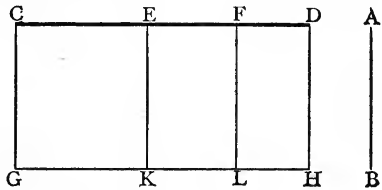
9. A line AB is 3 feet long; find the projection of AB on CD when AB is inclined to CD at an angle of (i) 60°; (ii) 45°; (iii) 30°.

PROPOSITION I. THEOREM.

If there be two straight lines, one of which is divided into any number of parts, the rectangle contained by the two straight lines is equal to the rectangles contained by the undivided line, and the several parts of the divided line.

Let AB and CD be two st. lines, and let CD be divided into any number of parts in E, F.

Then shall rect. AB, CD=rect. AB, CE; rect. AB, EF; and rect. AB, FD.



From C draw CG at rt. ∠s to CD..I. 11.
Cut off CG = AB...I. 3.
Through G draw GH ∥ to CD..I. 31.
Through E, F, D draw EK, FL, DH ∥ to CG,.........................I. 31.
and meeting GH in K, L, H.

Then, fig. CH=figs. CK, EL and FH,*
and all these figs. are rectangles............................Constr. & I. 29.
But, since CG=AB...........................Constr.
∴ fig. CH is rect. AB, CD.
And, since EK=CG...........................I. 34.
=AB
∴ fig. CK is rect. AB, CE.
Similarly, fig. EL is rect. AB, EF,
and fig. FH is rect. AB, FD.
∴ rect. AB, CD = rect. AB, CE, rect. AB, EF and rect. AB, FD.

Wherefore, *if there be two straight lines* &c. Q.E.D.

EXERCISE.

Prove that if two straight lines be each divided into any number of parts (say three), the rectangle contained by the two lines is equal to the sum of all the rectangles contained by all the parts of one taken separately with all the parts of the other.

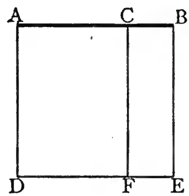
* The whole is equal to the sum of its parts.

PROPOSITION II. THEOREM.

If a straight line be divided into any two parts, the rectangles contained by the whole and each of the parts are together equal to the square on the whole line.

Let the st. line AB be divided into any two parts at C.

Then shall rect. AB, AC with rect. AB, BC = sq. on AB.



On AB desc. sq. ADEB.................... I. 46.
Through C draw CF || to AD or BE, and meeting DE at F .... I. 31.

Then, figs. AF and CE = fig. AE.
But, since AD = AB.................. Constr.
∴ fig. AF is rect. AB, AC.

And, since BE = AB.................. Constr.
∴ fig. CE is rect. AB, BC.

Also, fig. AE is sq. on AB.................................. Constr.

∴ rect. AB, AC with rect. AB, BC = sq. on AB.

Wherefore, *if a straight line* &c. Q.E.D.

NOTES.

Prop. 2 is the particular case of prop. 1 in which the two given lines are equal.

If, in prop. 1, AB contained a units of length; if CE, EF, FD contained b, c, d units respectively, and, consequently, CD contained $b+c+d$ units; the area of the rectangle CH would be $a(b+c+d)$ square units, the areas of the rectangles CK, EL, FH would be ab, ac, ad square units respectively; and the enunciation of the theorem becomes the statement of the algebraical identity $a(b+c+d\ldots)=ab+ac+ad\ldots$.

EXERCISES.

1. The square on any straight line is equal to the rectangle contained by its double and its half.
2. Construct a rectangle equal to a given rectangle in area, but having one of its sides three times the length of one of the sides of the given rectangle.
3. If the sides of the given rectangle in Ex. 2 are a and b, the difference of the perimeters of the rectangles is $4(a-\frac{b}{3})$.

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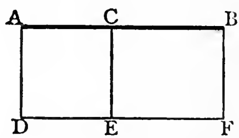
PROPOSITION III. THEOREM.

If a straight line be divided into any two parts, the rectangle contained by the whole and one of the parts, is equal to the rectangle contained by the two parts, together with the square on the aforesaid part.

Let the st. line AB be divided into any two parts at C.

Then shall rect. AB, AC=rect. AC, CB with sq. on AC.



On AC desc. the sq. ADEC.........I. 46.
Through B draw BF || to AD or CE, and meeting DE prod^{d} in F..........................I. 31.

Then, fig. AF=figs. AE and CF.

But, since AD=AC......................Constr.
∴ fig. AF is rect. AB, AC.

And since CE=AC......................Constr.
∴ fig. CF is rect. AC, CB.

Also fig. AE is sq. on AC.................Constr.

∴ rect. AB, AC=rect. AC, CB with sq. on AC.

Wherefore, *if a straight line* &c. Q.E.D.

NOTE.

This proposition is the particular case of Prop. 1, in which one part of the divided line is equal to the undivided line.

EXERCISES.

1. Which is "the aforesaid part" in the proof as it is given above?
2. Write out the proof of the proposition, making the square upon BC instead of on AC.
3. State an algebraical identity which corresponds to this proposition.
4. Enunciate a geometrical theorem corresponding to $(a+b)a-ab=a^2$.
5. Prove the following theorem:—

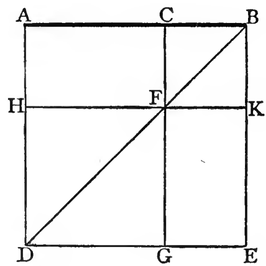
   If a straight line be produced to any point, the rectangle contained by the whole line so produced and the given line is equal to the rectangle contained by the given line and the part produced, together with the square on the former.

PROPOSITION IV. Theorem.

If a straight line be divided into any two parts, the square on the whole line is equal to the squares on the two parts, together with twice the rectangle contained by the parts.

Let the st. line AB be divided into any two parts at C.

Then shall sq. on AB=sqs. on AC and CB, with twice rect. AC, CB.



On AB desc. sq. ADEB......I. 46.
Join BD.
Through C draw CFG ∥ to AD or BE, and meeting BD in F, and DE in G......................I. 31.
Through F draw HFK ∥ to AB or DE, and meeting AD in H, and BE in K......................I. 31.

Then, ∠CFB=∠ADB,..................I. 29.
but ∠ADB=∠ABD,..................I. 5.
∴ ∠CFB=∠ABD
∴ CF=CB..........................I. 6.
But CF=BK and CB=FK.................I. 34.
∴ ▱CFKB is equilat.
And, since ∠CBK is a rt. ∠ ....Constr.
∴ all its ∠s are rt. ∠s.....................I. 46 Cor.
Hence, fig. CFKB is a sq...................Def. 30.
Similarly it may be shown that HDGF is a sq.

Now, fig. AE=figs. HG, CK, AF, and FE;
=sqs. on HF and CB, with figs. AF and FE...Above.
=sqs. on AC and CB, with figs. AF and FE...I. 34.
=sqs. on AC and CB, with twice comp^{t} AF....I. 43.
=sqs. on AC and CB, with twice rect. AC, CF;
i.e. sq. on AB=sqs. on AC and CB, with twice rect. AC, CB..Above.

Wherefore, *if a straight line* &c. Q.E.D.

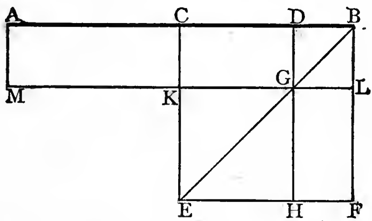
Cor.—From this prop. it is manifest that ▱s about the diam. of a sq. are likewise sqs.

PROPOSITION V. Theorem.

If a straight line be divided into two equal, and also into two unequal parts, the rectangle contained by the unequal parts together with the square on the line between the points of section, is equal to the square on half the line.

Let the st. line AB be bisected at C, and divided unequally at D.

Then shall rect. AD, DB with sq. on CD=sq. on CB.



On CB desc. sq. CEFB .... I. 46.
Join BE.
Through D draw DGH || to CE or BF, and meeting
BE in G, EF in H.. I. 31.
Through G draw KGL || to CB or EF, and meeting
CE in K, BF in L.. I. 31.
Through A draw AM || to CE or BF, meeting LK prod[d] in M... I. 31.

Then, rect. AD, DB with sq. on CD = rect. AD, DG with sq. on CD.. II. 4 Cor.
= rect. AD, DG with sq. on KG.. I. 34.
= fig. AG with fig. KH .... II. 4 Cor.
= figs. AK, CG, and KH,
= figs. AK, GF, and KH........ I. 43.
= figs. CL, GF, and KH......... I. 36.
= fig. CF,
= sq. on CB........................ Constr.

Wherefore, *if a straight line* &c. Q.E.D.

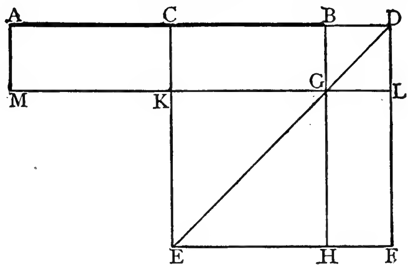
Cor.—From this prop. it is manifest that the difference of the squares on two unequal st. lines is equal to the rectangle contained by their sum and difference. (See page 106.)

PROPOSITION VI. THEOREM.

If a straight line be bisected and produced to any point, the rectangle contained by the whole line thus produced and the part of it produced, together with the square on half the line bisected, is equal to the square on the straight line which is made up of the half and the part produced.

Let the st. line AB be bisected at C and prod[d] to D.

Then shall rect. AD, DB with sq. on CB=sq. on CD.



On CD desc. sq. CEFD..I. 46.
Join DE.
Through B draw BGH || to CE or DF, and meeting DE in G, EF in H...I. 31.
Through G draw KGL || to CD or EF, and meeting CE in K, DF in L...I. 31.
Through A draw AM || to CE or DF, meeting LK prod[d] in M...I. 31.

Then, rect. AD, DB with sq. on CB=rect. AD, DL with sq. on CB..II. 4 Cor.
=rect. AD, DL with sq. on KG..I. 34.
=fig. AL with fig. KH ....II. 4 Cor.
=figs. AK and CL with fig. KH,
=figs. CG and CL with fig. KH..I. 36.
=figs. GF and CL with fig. KH..I. 43.
=fig. CF,
=sq. on CD......................Constr.

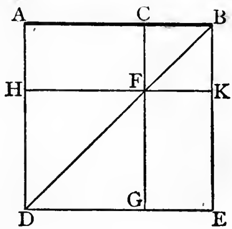
Wherefore, *if a straight line &c.* Q.E.D.

PROPOSITION VII. Theorem.

If a straight line be divided into any two parts, the squares on the whole line and on one of the parts are equal to twice the rectangle contained by the whole and that part, together with the square on the other part.

Let the st. line AB be divided into any two parts at C.

Then shall sqs. on AB and BC = twice rect. AB, BC with sq. on AC.



On AB desc. sq. ADEB, and complete the figure as in Prop 4.

Then, sqs. on AB and BC = figs. AE and CK.....................II. 4 Cor.
= figs. AK, FE, HG and CK
= figs. AK, AF, CK and HG..........I. 43.
= figs. AK, AK and HG,
= twice rect. AB, BK and sq. on AC..II. 4 Cor.
= twice rect. AB, BC and sq. on AC..II. 4 Cor.

Wherefore, *if a straight line* &c. Q.E.D.

NOTES.

From Prop 4 we know that—

I. *The square on the* SUM *of two lines is equal to the sum of the squares on those lines, together with twice the rectangle contained by them.*

By help of Prop. 7 it can be shown that—

II. *The square on the* DIFFERENCE *of two lines is less than the sum of the squares on those lines by twice the rectangle contained by them.*

For sqs. on AB and BC = twice rect. AB, BC with sq. on AC....Prop. 7.

∴ sq. on AC alone < sqs. on AB and BC by twice rect. AB, BC.

And AC is the difference of AB and BC,

∴ sq. on diff. of AB and BC < sum of sqs. on AB and BC by twice rect. AB, BC.

EXERCISES.

1. Interpret the theorems I. and II. algebraically.
2. Prove that, if AB is divided at C, the sqs. on AB and AC are equal to twice the rectangle AB, AC together with the square on BC.

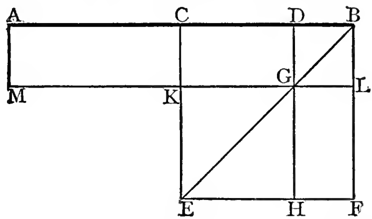
NOTES.

The following is an independent proof of the important corollary to Prop. 5:—

The difference of the squares of two unequal straight lines is equal to the rectangle contained by their sum and difference.

Let CB be the greater of the two lines, and from it cut off CD equal to the less.

(*Construction as in Prop. 5.*)



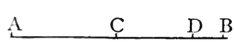
Then, diff. of sqs. on CB and CD=diff. of figs. CF and KH........II. 4.
=gnomon CLH,
=figs. CL and GF,
=figs. AK and GF.................I. 36.
=figs. AK and CG........... .....I. 43.
=fig. AG,
=rect. AD, DG,
=rect. AD, DB.....................II. 4.

Now, AD is sum of AC and CD,
=sum of CB and CD.

And DB is diff. of CB and CD.

∴ diff. of sqs. on CB and CD=rect. contained by their sum and diff.

Q.E.D.



Suppose AC to contain a units of length, and CD to contain b units; then DB contains $a-b$ units.

Then, rect. AD, DB becomes $(a+b)(a-b)$,
sq. on CD.....................b^2,
sq. on CB.....................a^2,

and the corollary of Prop. 5, expressed in algebraical form, becomes

$$a^2-b^2=(a+b)(a-b).$$

NOTES.

Props. 5 and 6 may be included in one enunciation, thus:—

If a straight line be divided into two equal segments and (either internally or externally) into two unequal segments, the rectangle contained by the unequal segments is equal to the difference of the squares on the half line and on the line between the points of section.

Prop. 6 may also be deduced from Prop. 5, thus:—

Produce BA to E, making AE = BD. Then ED is bisected at C, and divided unequally at D.

∴ rect. EB, BD with sq. on CB = sq. on CD .... II. 5.

But EA = BD .... Constr.

∴ EB = AD.

∴ rect. AD, DB with sq. on CB = sq. on CD. Q.E.D.

EXERCISES.

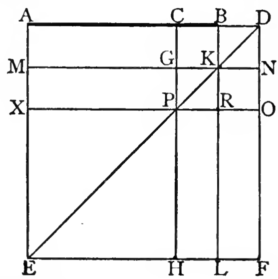
1. The square on any line is four times the square on its half.
2. Express Prop. 4 in algebraical form.
3. In the figure of Prop. 5, mention names for—
   (i) the line between the points of section;
   (ii) the two unequal segments;
   (iii) the figure which is equal to the difference between the squares on half the line and on the smaller segment.
4. In the figure of Prop. 5, prove that—
   (i) CD = half the difference of AD and DB;
   (ii) perimeter of gnomon CLH = perimeter of rect. AD, DB.
   (iii) AC = half the sum of AD and DB;
   (iv) AK = DF.
5. Prove that the *sum* of the sum and difference of two lines is double of the *greater*.
6. Prove that the *difference* of the sum and difference of two lines is double of the *less*.
7. Express Prop. 5 in algebraic form.
8. Construct a rectangle equal to the difference of two given squares.
9. If a straight line be divided into two equal and into two unequal segments, the square on the whole line is equal to four times the rectangle contained by the unequal segments, together with four times the square on the line between the points of section.
10. Divide a given straight line into two parts such that the rectangle contained by those parts shall be a maximum.
11. By help of the theorems I. and II., prove that the square on the sum of two straight lines together with the square on their difference is double of the sum of the squares on the two lines. (*Prop.* 7, *note.*)

PROPOSITION VIII. THEOREM.

If a straight line be divided into any two parts, four times the rectangle contained by the whole line and one part, together with the square on the other part, is equal to the square on the straight line which is made up of the whole and that part.

Let the st. line AB be divided into any two parts at C.

Then shall four times rect. AB, BC with sq. on AC = sq. on the sum of AB and BC.



Produce AB to D.
Cut off BD = BC....................I. 3.
On AD desc. sq. AEFD,...........I. 46.
and complete the
double fig. as in Prop. 4.

Then GP = GK...........II. 4.
= CB...........I. 34.
= BD...........Constr.
= BK...........II. 4.
= GC...........I. 34.

Hence, four times rect. AB, BC with sq. on AC
= four times rect. AB, BD with sq. on AC............Constr.
= four times fig. AK with fig. XH.......................II. 4.
= twice figs. AK, AG, CK with fig. XH,
= twice figs. AK, MP, GR with fig. XH...............I. 36.
= figs. AK, KF; MP, PL; twice fig. GR with fig. XH...I. 43.
= figs. AK, KF, MP, PL, GR, BN, XH..........I. 46. Ex. 1.
= fig. AF,
= sq. on AD...Constr.
= sq. on sum of AB and BC.

Wherefore, *if a straight line* &c. Q.E.D.

NOTES.

No use is made of this proposition by Euclid, and it is of very little importance.

The following is another form of the enunciation:—

The square on the sum of two lines exceeds the square on their difference by four times the rectangle contained by the lines.

EXERCISES.

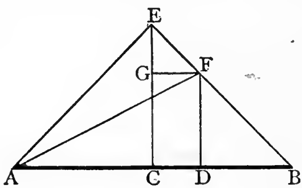
1. In the figure of Prop. 8, prove (i) BN = KO; (ii) AN = CL; (iii) CO = four times BN; (iv) PL = RF.
2. Express the enunciation of Prop. 8, as given in the note, algebraically.

PROPOSITION IX. Theorem.

If a straight line be divided into two equal and also into two unequal parts, the squares on the two unequal parts are together double of the square on half the line and of the square on the line between the points of section.

Let the st. line AB be bisected at C, and divided unequally at D.

Then shall sqs. on AD, DB=twice sqs. on AC, CD.



From C draw CE at rt. ∠s to AB......................I. 11.
Cut off CE=AC..................I. 3.
Join AE, EB.
Through D draw DF ‖ to CE, meeting EB in F..................I. 31.
Through F draw FG ‖ to AB, meeting CE in G..................I. 31.
Join AF.

Then, since AC=CE..............................Constr.
∴ ∠CAE=∠CEA........................I. 5.
but ∠ACE is a rt. ∠......................Constr.
∴ ∠s CAE, CEA together=a rt. ∠.............I. 32.
Hence ∠CAE and ∠CEA each=half a rt. ∠.

Similarly it may be shown that ∠CBE and ∠CEB each=half a rt. ∠.
∴ whole ∠AEB is a rt. ∠.

Again, since GF is ‖ to AB..................Constr.
∴ ∠EGF=∠ECB........................I. 29.
but ∠ECB is a rt. ∠........................Constr.
∴ ∠EGF is a rt. ∠.
But ∠GEF is half a rt. ∠................Above.
∴ ∠GFE is half a rt. ∠..................I. 32.
Hence ∠GFE=∠GEF,
∴ GF=GE............................I. 6.

Similarly it may be shown that ∠FDB is a rt. ∠, and that DF=DB.

Now, sqs. on AD and DB=sqs. on AD and DF..................Above.
=sq. on AF................................I. 47.
=sqs. on AE and EF...................I. 47.
=sqs. on AC, CE and on EG, GF....I. 47.
=twice sqs. on AC and GF...........Above.
=twice sqs. on AC and CD...........I. 34.

Wherefore, *if a straight line* &c. Q.E.D.

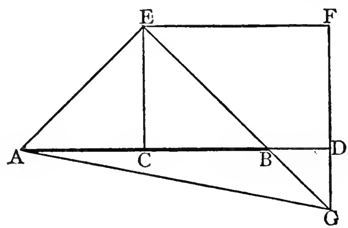
PROPOSITION X. Theorem.

If a straight line be bisected and produced to any point, the square on the whole line thus produced and the square on the part of it produced are together double of the square on half the line bisected, and of the square on the line made up of the half and the part produced.

Let the st. line AB be bisected at C and prod[d] to D.

Then shall sqs. on AD, DB=twice the sqs. on AC, CD.

From C draw CE at rt. ∠s to AB............I. 11.
Cut off CE=AC...........I. 3.
Join AE, EB.
Through E draw EF ‖ to AD.................I. 31.
Through D draw DF ‖ to CE and meeting EF at F...........I. 31.



Since CE is ‖ to DF........................Constr.
∴ ∠s CEF, EFD=two rt. ∠s..................I. 29.
∴ ∠s BEF, EFD<two rt. ∠s.

Hence EB, FD will meet if prod[d] towards B and D......Ax. 12.

Produce EB, FD to meet at G. Join AG.

Then, each of the ∠s CAE, CEA, CEB, CBE is half a rt. ∠ and ∠AEB is a rt. ∠ ....} ...II. 9.
But ∠GBD=∠CBE............................I. 15.
∴ ∠GBD is half a rt. ∠.
Also ∠BDG=alt. ∠BCE......................I. 29.
∴ ∠BDG is a rt. ∠.
Hence ∠BGD is half a rt. ∠.....................I. 32.
∴ ∠GBD=∠BGD
∴ BD=DG................................I. 6.

Similarly, it may be shown that EF=FG.

Now, sqs. on AD and DB=sqs. on AD and DG ....Above.
=sq. on AG................................I. 47.
=sqs. on AE and EG....................I. 47.
=sqs. on AC, CE and on EF, FG....I. 47.
=twice sqs. on AC and EF...........Above.
=twice sqs. on AC and CD...........I. 34.

Wherefore, *if a straight line* &c. Q.E.D.

NOTES.

The following proof of Prop. 10, which is almost as old, is often substituted for Euclid's:—

Let AB be bisected at C and prod[d] to D.

Then shall sqs. on AD and DB=twice sqs. on AC and CD.

Produce DA to E.
Cut off AE=BD .... I. 3.

Then ED is bisected at C and divided into two unequal parts at A.

∴ sqs. on AD and AE=twice sqs. on AC and CD........... II. 9.

i.e. sqs. on AD and BD=twice sqs. on AC and CD. Q.E.D.

Prop. 9 can be easily deduced from Props. 4 and 7, thus:—

Let AB be bisected at C, and divided unequally at D.

Then shall sqs. on AD and DB=twice sqs. on AC and CD.

For, sq. on AD=sqs. on AC and CD with twice rect. AC, CD........... II. 4.

Add sq. on DB to each.

∴ sqs. on AD and DB=sqs. on AC, CD, DB with twice rect. AC, CD
=sqs. on BC, CD, DB with twice rect. BC, CD
=sqs. on BC, CD with sqs. on BC, CD.......... II. 7.
=twice sqs. on BC and CD. Q.E.D.

Props. 9 and 10 can be combined in one enunciation, thus:—

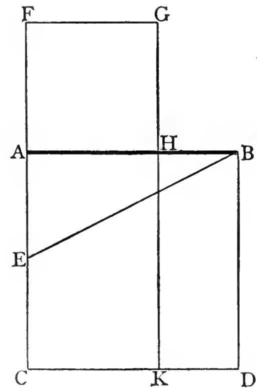
If a straight line be bisected, and be also divided (either internally or externally) *into any other two segments, the sum of the squares on the two unequal segments is equal to the square on half the line together with the square on the line between the points of section.*

EXERCISES.

1. In the figure of Prop 9 give the name of "the line between the points of section."
2. In the figure of Prop. 10 give the names of "the line bisected," and "the line made up of the half and the part produced."
3. Enunciate the geometrical proposition which can be expressed algebraically thus: $(a+b)^2+(a-b)^2=2(a^2+b^2)$.
4. Interpret Prop. 10 algebraically.
5. Translate the enunciation of Props. 9 and 10 combined, into algebra.
6. Prove (i) geometrically, (ii) algebraically, that the sum of the squares on any two lines is equal to twice the square on half their sum together with twice the square on half their difference.
7. Divide a given straight line into two segments so that the sum of the squares on the segments may be the least possible.
8. In II. 9, show how to cut up the squares on the two unequal parts into bits which form the other squares.

PROPOSITION XI. PROBLEM.

To divide a straight line into two parts so that the rectangle contained by the whole and one of the parts shall be equal to the square on the other part.



Let AB be the given st. line.

It is req[d] *to divide AB into two parts such that the rect. contained by AB and one part may be = the square on the other.*

On AB desc. sq. ACDB.................I. 46.
Bisect AC at E.........................I. 10.
Join BE.
Produce CA to F.
Cut off EF=EB.........................I. 3.
On AF desc. sq. AFGH................I. 46.

Then shall rect. AB, BH=sq. on AH.

Produce GH to meet CD in K.

Then, since AC is bisected at E and prod[d] to F,
∴ rect. CF, FA with sq. on AE=sq. on EF...................II. 6.
=sq. on EB...................Constr.
=sqs. on AE and AB.......I. 47.

Take away the com. sq. on AE,
∴ sq. on AB=rect. CF, FA,
=rect. CF, FG...............Constr.
i.e. fig. AD=fig. FK.

Take away the com. part AK,
∴ rem[g] fig. HD=rem[g] fig. FH,
i.e. rect. DB, BH=sq. on AH.

But DB=AB...........................Constr.
∴ rect. AB, BH=sq. on AH.

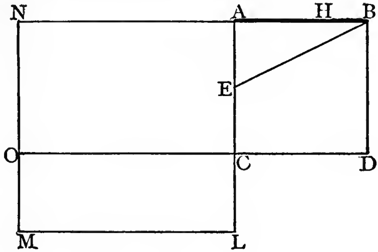
Wherefore, *the given straight line* &c. Q.E.F.

NOTES.

Euclid's 11th prop. is the solution of one case of the following problem:—

To divide a straight line, either internally or externally, into two segments so that the rectangle contained by the whole line and one segment may be equal to the square on the other.

CASE 2.—To divide the line externally.



On AB desc. sq. ACDB .... I. 46.
Bisect AC at E .... I. 10.
Join EB.
Produce AC to L.
Cut off EL=EB .... I. 3.
On AL desc. sq. ALMN .... I. 46.

Then shall rect. AB, BN=sq. on AN.

Produce DC to meet MN at O.

Then, since AC is bisected at E and prod^d to L,
∴ rect. CL, LA with sq. on EC=sq. on EL........................ II. 6.
=sq. on EB........................ Constr.
=sqs. on EA and AB............ I. 47.
=sqs. on EC and AB...... ..... Constr.

Take away the com. sq. on EC,
∴ sq. on AB=rect. CL, LA,
i.e. fig. AD=fig. CM.

Add fig. CN to each,
∴ fig. ND=fig. AM,
i.e. rect. AB, BN=sq. on AN........................ Constr.

Wherefore, *AB has been divided externally at N* &c. Q.E.F.

It should be noticed that, in the external solution of the problem, the construction corresponds to that for the internal solution, but that AC is produced in the *opposite direction.*

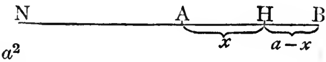
A line divided as in prop. 11 is said to be divided *in medial section.*

ALGEBRAIC SOLUTION OF PROP. 11.

Suppose AB to contain a units of length, and let x be the distance of the req[d] pt. from A, so that BH is $a-x$; then the enunciation of the problem takes the form of the quadratic

$$a\,(a-x)=x^2$$

$$\text{or,}\quad x^2+ax=a^2,$$



$$\therefore\ x^2+ax+\left(\frac{a}{2}\right)^2=a^2+\frac{a^2}{4},$$

$$=\frac{5a^2}{4},$$

$$\therefore\ x+\frac{a}{2}=\pm\frac{a\sqrt{5}}{2},$$

$$\text{and } x=\frac{a(\sqrt{5}-1)}{2},\quad\text{or}\quad -\frac{a(\sqrt{5}+1)}{2}$$

The first of these values of x is *positive*, and gives the distance of the point H from A: the second is *negative* and gives the distance of the point N from A; the difference in sign denoting that AH and AN are measured in opposite directions from A.

From the surd form of the roots of this quadratic we see that AH and AN are incommensurable with AB.

EXERCISES.

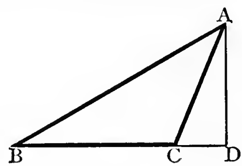
1. In the figure of Prop. 11, prove that—
   (i) KG and CF are each divided in the same way as AB.
   (ii) square on EF is equal to five times the square on EC.
   (iii) AH is greater than HB.
   (iv) if CH and FB be joined, CH produced is perpendicular to FB.
   (v) the ratio of AH to HB is that of $\sqrt{5}-1$ to $3-\sqrt{5}$.
   (vi) the squares on AB and BH = three times the square on AH.
2. In the figure of Prop. 11 Note, prove that—
   (i) square on EL is equal to five times the square on EA.
   (ii) if NC and BL be joined, NC produced is perpendicular to BL.
   (iii) the ratio of AN to BN is that of $\sqrt{5}+1$ to $3+\sqrt{5}$.
3. If AB is divided at C so that the rectangle contained by the whole line and one part is equal to the square on the other part; and if AC is the greater part, and from AC CD is cut off equal to CB; prove that AC is divided at D in a similar manner.
4. Produce a line so that the sum of the squares on the whole line thus produced and on the part produced may be equal to three times the square on the given line.
5. Find two lines such that the rectangle contained by them shall be equal to the rectangle contained by their sum and difference.
6. If in the figure of Prop. 11, FD be joined, meeting AB in P, and HK in Q, then will FP be equal to QD.

PROPOSITION XII. THEOREM.

In obtuse-angled triangles if a perpendicular be drawn from either of the acute angles to the opposite side produced, the square on the side subtending the obtuse angle is greater than the squares on the sides containing the obtuse angle by twice the rectangle contained by the side upon which, when produced, the perpendicular falls, and the straight line intercepted, without the triangle, between the perpendicular and the obtuse angle.

Let ABC be an obtuse-angled △, having the obtuse ∠ at C, and let AD be drawn ⊥ to BC prod[d].

Then shall sq. on AB > sqs. on BC and AC by twice the rect. BC, CD.



For, sq. on BD = sqs. on BC and CD with twice rect. BC, CD................II. 4.

Add sq. on AD to each,

∴ sqs. on AD and BD = sqs. on BC, CD and AD with twice rect. BC, CD.

But, sqs. on AD and BD = sq. on AB..................................I. 47.

And, sqs. on AD and CD = sq. on AC..................................I. 47.

Hence, sq. on AB = sqs. on BC and AC with twice rect. BC, CD.

i.e. sq. on AB > sqs. on BC and AC by twice rect. BC, CD.

Wherefore, *in obtuse-angled triangles* &c. Q.E.D.

NOTE.

Since CD is the projection of AC on BC, Prop. 12 may be stated thus:

In an obtuse-angled triangle, the square on the side opposite to the obtuse angle is greater than the squares on the sides which contain it, by twice the rectangle contained by one of these two sides and the projection of the other upon it.

EXERCISES.

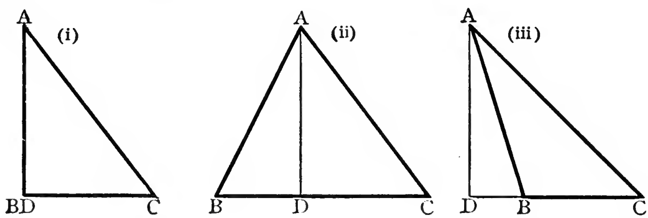
1. If BE be perpendicular to AC produced, prove that the square on AB is equal to the sqs. on BC and AC with twice rect. AC, CE.
2. Two right-angled triangles ACB, ADB stand on the same side of the same hypotenuse AB; if AD and BC cut at O, prove that rect. AO, OD = rect. BO, OC.
3. The sides of a triangle are 4, 5, and 7; is it obtuse-angled?
4. The sides of a triangle are 4, 5, and 7, find the length of the projection of the side 4 upon the side 5.
5. Find the ratio of CD to AC when ∠ACB is—(i) 120°; (ii) 135°; (iii) 150°.
6. If AC and BC are each 3 feet long, and the angle ACB is 120°, find the length of AB to within an eighth of an inch.

PROPOSITION XIII. THEOREM.

In every triangle the square on the side subtending an acute angle is less than the squares on the sides containing that angle by twice the rectangle contained by either of these sides, and the straight line intercepted between the perpendicular let fall upon it from the opposite angle and the acute angle.

Let ABC be any △, having an acute ∠ at C, and let AD be drawn ⊥ to BC, or BC prod[d].

Then shall sq. on AB < sqs. on BC and AC by twice the rect. BC, CD.



CASE 1.—If the ∠ at B is a rt. ∠, D coincides with B, and the rect. BC, CD becomes the sq. on BC, and the truth of the prop. is evident....................I. 47.

CASE 2.—If the ∠ at B is not a rt. ∠.

Since, in fig. ii., BC is divided at D,
and, in fig. iii., DC is divided at B,

∴, in both figs.,

sqs. on BC and CD = twice rect. BC, CD with sq. on BD...II. 7.

Add sq. on AD to each,

∴ sqs. on BC, CD, AD = twice rect. BC, CD with sqs. on BD, AD.

∴ sqs. on BC and AC = twice rect. BC, CD with sq. on AB....I. 47.

i.e. sq. on AB < sqs. on BC and AC by twice rect. BC, CD.

Wherefore, *in every triangle* &c. Q.E.D.

NOTE.

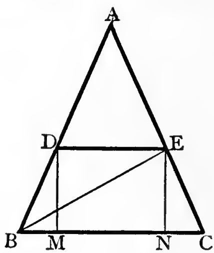
Since BD is the projection of AB on BC, the prop. may be stated thus:—

In any triangle the square on a side opposite to an acute angle is less than the sum of the squares on the sides which contain the acute angle, by twice the rectangle contained by one of these sides and the projection of the other upon it.

EXAMPLE.

If DE be drawn parallel to the base BC of an isosceles triangle ABC, the square on BE is equal to the square on CE with the rectangle contained by BC and DE.

(In this Example the notation AB^2 for *sq. on AB*, BC.CD for *rect. BC, CD*, and the algebraical symbols + and − have been used, as they are often very convenient in dealing with deductions on Book II., and are generally allowed for such purposes, but it must be remembered that they *should not be used* in writing out *Book-work.*)



Join BE.

Draw DM, EN ⊥s to BC........................I. 12.

Then in △s DBM, ECN,

∵ { DM = EN.................I. 34.
∠DBM = ∠ECN...........I. 5.
∠DMB = ∠ENC...........Constr }

∴ BM = CN.................I. 26.

Now, $BE^2 = EC^2 + BC^2 - 2\, BC.CN$................II. 13.

$= EC^2 + BC.(BC - 2\, CN)$

$= EC^2 + BC.(BC - CN - BM)$.........Above.

$= EC^2 + BC.MN$..........................II. 1.

$= EC^2 + BC.DE$..........................I. 34.

Q.E.D.

EXERCISES.

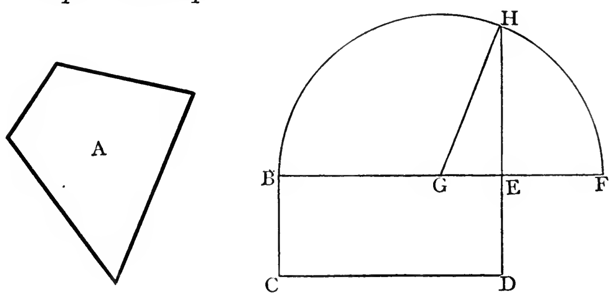
1. What line, in fig. iii., is the intercept between the foot of the perpendicular and the acute angle?
2. If BE be drawn perpendicular to AC, in fig. of Prop. 13, prove that the rect. AC, CE = rect. BC, CD.
3. The sides of a triangle are 4, 5, 6; is it acute-angled?
4. The sides of a triangle are 4, 5, 6; find the length of the projection of the side 4 upon the side 5.
5. Find the ratio of CD to AC when the angle ACB is (i) 30°, (ii) 45°, (iii) 60°.
6. State and prove the converse of Prop. 13.
7. Express the results of Props. 12 and 13 algebraically.

PROPOSITION XIV. PROBLEM.

To describe a square that shall be equal to a given rectilineal figure.

Let A be the given rect[l] fig.

It is req[d] to desc. a sq.=A.



Make the rectangle BCDE=A.................................I. 45.

Then, if BE=ED, the fig. BCDE is the req[d] sq.
But if not, produce BE to F.
Cut off EF=ED..I. 3.
Bisect BF at G...I. 10.
With cent. G and rad. GF desc. a semicircle BHF.
Produce DE to meet the circumference in H.

Then shall sq. on EH=A.

Join GH.

Then, since BF is bisected at G, and divided unequally at E,

∴ rect. BE, EF with sq. on GE=sq. on GF..................II. 5.
=sq. on GH.................Constr.
=sqs. on GE, EH..........I. 47.

Take away the com. sq. on GE,

∴ sq. on EH=rect. BE, EF,
=rect. BE, ED...................Constr.
=fig. BCDE,
=A....................................Constr.

Wherefore, *has been described* &c. Q.E.F.

EXERCISES.

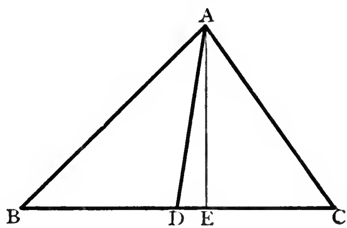
1. Construct a square equal in area to a given parallelogram.
2. Construct a square equal in area to a given triangle.
3. Construct a rectangle equal in area to a given square, and having the sum of two of its adjacent sides equal to a given straight line.
4. Construct a parallelogram equal in area to a given square, having the sum of its altitude and base equal to a given straight line and having an angle equal to a given angle.
5. In what cases are Exs. 3 and 4 impossible?
6. Construct a rectangle having given its area and perimeter.
7. Produce a given straight line so that the rectangle contained by the given line and the part produced may be equal to the square on another given line.
8. What is the length of the side of a square equal in area to a rhombus whose side is 3 inches and one of its angles 30°?
9. If in II. 14 the given rectilineal figure be that of I. 47, show how to determine the required square graphically.
10. The side of an equilateral triangle is a; find the side of a square equal in area to the triangle.
11. Write down a quadratic equation of which Prop. 14 is a geometrical solution.
12. What is meant by the *quadrature* of a given rectilineal figure, and how would you proceed practically in the case of a pentagon?

MISCELLANEOUS EXAMPLES.

I. *The sum of the squares on the sides of any triangle is equal to twice the square on half the base, together with twice the square on the line joining the vertex to the middle point of the base.*

Let ABC be any △, and D the mid. pt. of BC.

Then shall sqs. on AB and AC = twice sqs. on BD and AD.



Draw AE ⊥ to BC, or BC prod^{d} .... I. 12.

Then, if ADB is the obtuse ∠,

$AB^2 = AD^2 + BD^2 + 2BD.\ DE$ .... II. 12.

and $AC^2 = AD^2 + CD^2 - 2CD.\ DE$ .... II. 13.

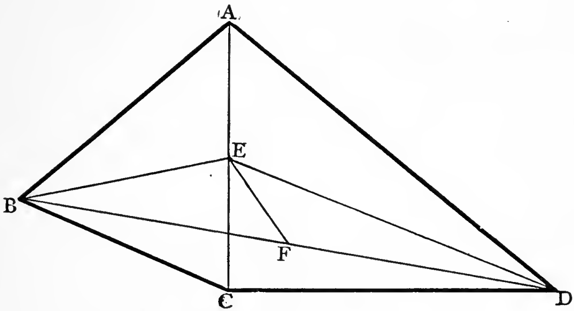
$= AD^2 + BD^2 - 2BD.\ DE$ .... Hyp.

$\therefore\ AB^2 + AC^2 = 2AD^2 + 2BD^2$.

Q.E.D.

N.B.—This result is very important.

II. *The sum of the squares on the four sides of any quadrilateral is equal to the sum of the squares on its diagonals, together with four times the square on the line joining the middle points of its diagonals.*



Let ABCD be a quad[l], AC, BD its diags., and E, F the mid. pts. of AC and BD.

Then shall sqs. on AB, BC, CD, DA = sqs. on AC and BD with four times sq. on EF.

Join BE, ED.

Then, since E is mid. pt. of AC,

∴ sqs. on AB, BC = twice sqs. on BE, AE............Ex. I.

and sqs. on CD, DA = twice sqs. on DE, AE...... .....Ex. I.

∴ sqs. on AB, BC, CD, DA

= twice sqs. on BE, DE and four times sq. on AE.

= four times sqs. on BF, EF and AE......................Ex. 1.

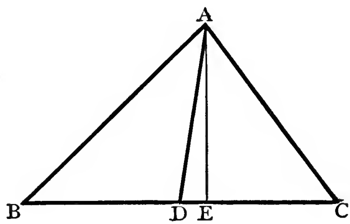
= four times sq. on EF, with sqs. on AC, BD..........II. 4 Ex. 1.

Q.E.D.

III. *The difference of the squares on the sides of any triangle is equal to twice the rectangle contained by the base and the line intercepted between the middle point of the base and the foot of the perpendicular from the vertex to the base, or the base produced.*

Let ABC be any $\triangle$, D the mid. pt. of the base, E the foot of the $\perp$ from A on BC, or BC prod[d].

Then shall diff. of sqs. on AB and AC = twice rect. BC, DE.



Join AD.

Then, if $\angle$ADC is acute,

$$AC^2 = AD^2 + DC^2 - 2\ CD\,.\,DE \quad \text{II. 13.}$$
$$= AD^2 + BD^2 - 2\ CD\,.\,DE \quad \text{Hyp.}$$

$$\text{And } AB^2 = AD^2 + BD^2 + 2\ BD\,.\,DE \quad \text{II. 12.}$$

$$\therefore AB^2 - AC^2 = 2\ BD\,.\,DE + 2\ CD\,.\,DE \quad \text{Ax. 3.}$$
$$= 2\ (BD + CD)\,.\,DE \quad \text{II. 1.}$$
$$= 2\ BC\,.\,DE.$$

Q.E.D.

EXERCISES.

1. Write out the proof when the angle ADB is acute.
2. Prove the theorem without using either Props. 12 or 13.
3. Examine the case when ADC is a right angle.

IV. *To divide a st. line into two parts so that the rectangle contained by the parts may be a maximum.*

Let AB be the given st. line.

Bisect AB at C.

Then shall rect. AC, CB be a *maximum.*

For, take any other pt. D in AB.

Then, since AB is bisected at C, and divided unequally at D,

∴ rect. AD, DB with sq. on CD=sq. on CB................II. 5.

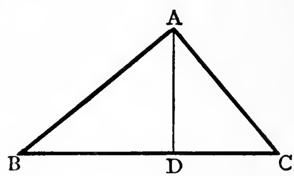
=rect. AC, CB,

i.e. rect. AD, DB alone < rect. AC, CB.

Q.E.F.

V. *In a right-angled triangle, the square on the perpendicular drawn from the right angle to the hypotenuse is equal to the rectangle contained by the segments into which it divides the hypotenuse.*

Let ABC be a rt.-angled △, having the rt. ∠ at A, and let AD be the ⊥ to BC.



Then shall sq. on AD=rect. BD, DC.

For, since BC is divided at D,

∴ sq. on BC=sqs. on BD, DC with twice rect. BD, DC.......II. 4.

But sq. on BC=sqs. on BA, AC.......................................I. 47.

=sqs. on AD, BD and on AD, DC.................I. 47.

=sqs. on BD, DC and twice sq. on AD.

∴ sqs. on BD, DC and twice sq. on AD

=sqs. on BD, DC with twice rect. BD, DC......Ax. 1.

Take away the com. sqs. on BD, DC,

∴ twice sq. on AD=twice rect. BD, DC,

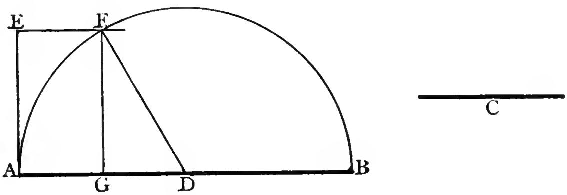
∴ sq. on AD=rect. BD, DC.

Q.E.D.

VI. *To divide a straight line into two parts so that their rectangle may be equal to the square on a given straight line which is not greater than half the former line.*

Let AB and C be the given st. lines.

It is req[d] to divide AB into two parts so that the rect. contained by the parts may be equal to the sq. on C.



Bisect AB at D..I. 10.
With cent. D and rad. DA desc. a semicircle.
From A draw AE at rt. ∠s to AB..I. 11.
Cut off AE=C..I. 3.
Through E draw EF || to AB and cutting the semicircle at F..........I. 31.
Draw FG || to AE and meeting AB at G......................................I. 31.

Then shall rect. AG, GB = sq. on C.

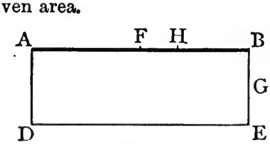
(*The proof is similar to that of Prop. 14.*)

VII. *To divide a given straight line into two parts so that the difference of the squares on the parts may be equal to a given area.*

Let AB be the given st. line, and C the given area.

To AB apply a rect. ADEB=C..........I. 45.
Bisect AB at F, and BE at G ....I. 10.
From FB cut off FH=BG................I. 3.

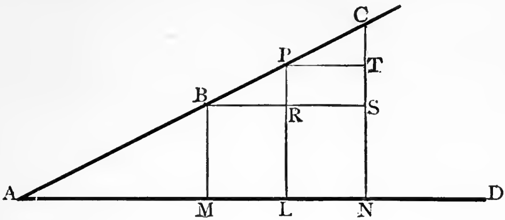
Then H shall be the req[d] pt.



For, $AH^2 - HB^2 = (AH + HB) . (AH - HB)$........II. 5 Cor.
$= AB . (AF + FH - HB)$
$= AB . (BF + FH - HB)$
$= AB . 2FH$..........................Constr.
$= AB . BE$..........................Constr.
$= C$..........................Constr.

Q.E.F.

VIII. *If ABC and AD are two straight lines, and P is the middle point of BC, then the projection of AP on AD is half the sum of the projections of AB and AC on AD.*



From B, P, and C draw ⊥s BM, PL and CN to AD……………………I. 12.
Through B draw BRS ∥ to AD, meeting PL in R, and CN in S……I. 31.
Through P draw PT ∥ to AD, meeting CN in T……………………I. 31.

Then, since PT and BS are ∥ to AD……………………Constr.
∴ ext^{r} ∠CPT = ∠CAD……………………I. 29.
= ∠PBR……………………I. 29.

Also ext^{r} ∠CTP = ∠CNA……………………I. 29.
= a rt. ∠……………………Constr.
and ext^{r} ∠PRB = ∠PLA……………………I. 29.
= a rt. ∠……………………Constr.

Hence, in △s CPT, PBR,

∵ { ∠CPT = ∠PBR……………………Above.
rt. ∠CTP = rt. ∠PRB……………………Above.
CP = PB……………………Hyp. }

∴ PT = BR……………………I. 26.

But PT = LN, and BR = ML……………………I. 34.
∴ LN = ML.

Now, projection of AP on AD = AL,
= AM and ML,
= half of 2 AM and 2 ML,
= half the sum of 2 AM and MN,
= half the sum of AM and AN,
= half sum of projections of AB and AC on AD.

Q.E.D.

EXERCISES.

1. Prove that PL is half the sum of BM and CN in the above figure.
2. If A were taken between B and C, prove that PL would be equal to half the difference of BM and CN.

MISCELLANEOUS EXERCISES.

1. Divide a straight line into two parts so that the rectangle contained by the whole line and one part may be double of the square on the other part.
2. Enunciate a geometrical theorem of which the algebraical expression is $(a+2b)^2=a^2+4ab+4b^2$.
3. If a straight line be trisected, the square on the whole line is equal to nine times the square on one of the parts.
4. If in the figure of II. 6 the produced part be equal to the original line, then the square on half the line bisected is one-eleventh of the whole figure.
5. Produce a given straight line to a point, so that the rectangle contained by the whole line thus produced and the part produced shall be equal to the square on the original line.
6. Any rectangle is half the rectangle contained by the diameters of the squares on two of its adjacent sides.
7. ABCDE is a straight line, C being the middle point of BD. Prove that the square on AC together with the rectangle BE, DE is equal to the square on EC together with the rectangle AB, AD.
8. On AB a square ABCD is described, and the angles ACE, ACF are made each equal to half the angle of an equilateral triangle, thus inscribing an equilateral triangle CEF in the square. Prove that AB is divided at E so that the square on one part is double the rectangle contained by the whole and the other part.
9. If a straight line is divided so that the sum of the squares on the whole and on one part is equal to three times the square on the other part, the line is divided in medial section.
10. AB is a diameter of a circle and AC is a chord; if CD be drawn perpendicular to AB, then the square on AC is equal to the rectangle contained by AB and AD.
11. A straight line AB is bisected in C, and from A a straight line AD is drawn at right angles to AB and equal to AC. A point E is taken in AB produced such that CE is equal to BD. Prove that the rectangle contained by AE and EB is equal to the square on AB.
12. The sum of the squares on the sides of any parallelogram is equal to the sum of the squares on its diameters.
13. If a straight line AD be divided at B and C, the rectangle contained by AC and BD is equal to the sum of the rectangles contained by AB and CD, and by BC and AD.
14. Divide a straight line into two parts so that the sum of the squares on the parts may be a minimum.
15. The rectangle contained by the sum and difference of the sides of a triangle is equal to twice the rectangle contained by the base and the line intercepted between the middle point of the base and the foot of the perpendicular on the base from the vertex.

16. If the square on the perpendicular from the vertex to the base of a triangle be equal to the rectangle contained by the segments of the base, the vertical angle is a right angle.

17. ACDB is a straight line and D bisects CB; prove that the square on AC is less than the sum of the squares on AD and DB by twice the rectangle AD, DB.

18. ABC is an equilateral triangle and D is any point in BC. Prove that the square on BC is equal to the rectangle contained by BD and DC together with the square on AD.

19. AB is divided into any two parts in C, and AC and BC are bisected at D and E; show that the square on AE with three times the square on BE is equal to the square on BD with three times the square on AD.

20. If EF is parallel to BC the base of an isosceles triangle ABC, prove that the difference of the squares on BF and BE is equal to the rectangle contained by EF and BC.

21. In II. 11, show that the rectangle contained by the sum and difference of the parts is equal to the rectangle contained by the parts.

22. If a straight line AB be bisected at C and produced to D so that the square on AD is twice the square on CD, prove that the square on AB will be twice the square on BD.

23. If a straight line AB be bisected at C and produced to D so that the square on AD is three times the square on CD, and if CB be bisected at E, show that the square on ED is three times the square on EB.

24. ABC is a triangle, and on the side of BC remote from A a square BDEC is described; prove that the difference of the squares on AB and AC is equal to the difference of the squares on AD and AE.

25. Find the locus of a point which moves so that the sum of the squares of its distances from two fixed points is constant.

26. The square on a straight line drawn from the vertex of an isosceles triangle to any point in the base, is less than the square on a side by the rectangle contained by the segments of the base.

27. Construct a rectangle which shall be equal in area to a given square, and the sum of whose sides shall be of given length.

28. The sum of the squares on the diagonals of any quadrilateral is equal to twice the sum of the squares on the lines joining the middle points of opposite sides.

29. The squares on the diagonals of a four-sided figure having two parallel sides are equal to the squares on its two non-parallel sides together with twice the rectangle contained by the parallel sides.

30. In any triangle, three times the sum of the squares on the sides is equal to four times the sum of the squares on the lines drawn from the angles to the middle points of the opposite sides.

31. If a straight line be divided into two pairs of unequal parts, the squares on the greatest and least of the four parts are together greater than the squares on the other two.

32. The base AB of triangle ABC is bisected in D, and in DB a point E is taken such that DC is equal to DE; show that the squares on AC and CB are together equal to the squares on AE and EB.

33. Given a square and one side of a rectangle equal in area to the square, find the other side.

34. Construct a rectangle equal to a given square and having the difference of its sides equal to a given line.

35. AB is bisected at C and produced to D; prove that the rectangle AC, AD is equal to the rectangle BC, BD together with twice the square on BC.

36. ABCD is a rectangle, and O is any point; prove that the sum of the squares on OA and OC is equal to the sum of the squares on OB and OD.

37. The diagonals of a quadrilateral meet in E, and F is the middle point of the straight line joining the middle points of the diagonals; prove that the sum of the squares on the straight lines joining E to the angular points of the quadrilateral is greater than the sum of the squares on the straight lines joining F to the same points by four times the square on EF.

38. Produce a given straight line so that the square on the whole line thus produced may be double the square on the part produced.

39. Show that the locus of the vertex of a triangle whose base is fixed, and the difference of the squares on whose sides is constant, is one or other of two straight lines.

40. If with the middle point of the line joining the points of bisection of the diagonals of any quadrilateral as centre and any radius a circle be described, prove that the sum of the squares of the distances of any point on the circumference from the angles of the quadrilateral, is constant.

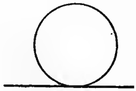
BOOK III.

DEFINITIONS.

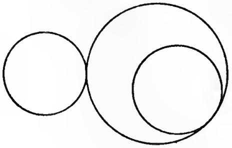
1. **Equal circles** are those of which the diameters (or the radii) are equal.

[This is not a definition but a theorem: for, if the circles be applied to one another so that their centres coincide, their circumferences must also coincide, sinee their radii are equal.]

2. A straight line is said to **touch** a circle when it meets the circle but, being produced, does not *cut* it.

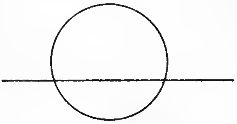


3. Circles are said to **touch** one another which *meet but do not cut* one another.



ADDITIONAL DEFINITIONS.

A straight line which *cuts* a circle is called a *secant*.

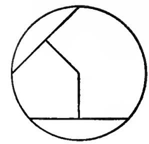


A straight line which meets but, being produced, does not cut the circle is called a *tangent*.

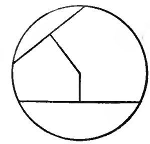
When two circles touch, and one is entirely within the other, they are said to have *internal contact*.

When two circles touch, and one is entirely without the other, they are said to have *external contact*.

4. Straight lines are said to be **equally distant from the centre** of a circle, when the perpendiculars drawn to them from the centre are equal.



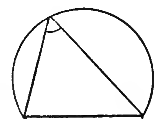
5. The chord on which the greater perpendicular falls is said to be *further from the centre.*



6. (See Book I., def. 19.)

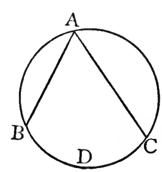
7. *The angle of a segment* is contained by the straight line and the circumference.

8. **The angle in a segment** is the angle contained by two straight lines drawn from any point in the circumference to the extremities of the straight line which is the base of the segment.



9. An angle is said to *stand upon* that part of the circumference intercepted between the straight lines which contain the angle.

(For example, angle BAC stands upon the arc BDC.)



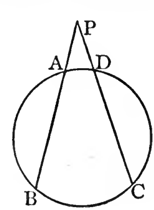
ADDITIONAL DEFINITIONS.

(For defs. of **arc, chord,** &c., see Book I., page 8.)

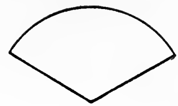
If the circumference of a circle be divided into two unequal parts the greater part is called the *major arc,* and the less the *minor arc.*

An arc is said to be *convex* or *concave* with respect to a point, according as the lines drawn from the point to the ends of the arc lie altogether without the circle or not.

(For example, arc AD is convex, and arc BC concave with respect to the point P.)



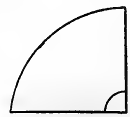
10. A **sector** of a circle is the figure contained by two straight lines drawn from the centre and the part of the circumference between them.



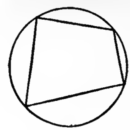
11. Similar segments of circles are those which contain equal angles.

ADDITIONAL DEFINITIONS.

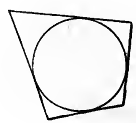
A sector, the bounding radii of which are at right angles to one another, is called a *quadrant*.



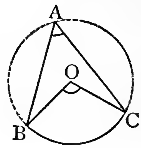
A figure is said to be **inscribed in a circle,** when each of its angular points is on the circumference.



A figure is said to be **described about** (or to *circumscribe*) **a circle,** when each of its sides *touches* the circumference.



If O be the centre of a circle ABC, the angle BOC which stands upon the arc BC is called *the angle at the centre;* and the angle BAC is called *the angle at the circumference.*



A line drawn at right angles to a tangent from its point of contact is called a *normal.*

Circles are said to cut at right angles when the tangents at the point where they cut are at right angles.

NOTE.

In Book III. Euclid deals with the simpler properties of circles.

PROPOSITION I. Problem.

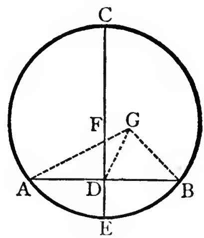
To find the centre of a given circle.

Let ABC be the given ⊙.

It is req^{d} *to find its cent.*

Draw any chd. AB.
Bisect AB at D..I. 10.
Through D draw CDE at rt. ∠s to AB
meeting the ○ce at C and E...I. 11.
Bisect CE at F...I. 10.

Then shall F be the cent.



For, if not, if possible, let some pt. G outside the line CE be the cent. Join GA, GB, GD.

Then, in △s GAD, GBD,

∵ { AD=BD..............................Constr.
GD is com.
GA=GB..............................Radii. }

∴ ∠GDA=∠GDB.......................I. 8.

∴ ∠GDA is a rt. ∠.

But ∠FDA is a rt. ∠.......................Constr.

∴ ∠GDA=∠FDA.

i.e. the whole=its part,
which is absurd.

Hence the cent. cannot be outside the line CE.

∴ F, the mid. pt. of CE, is the cent.

Wherefore, *has been found* &c. Q.E.F.

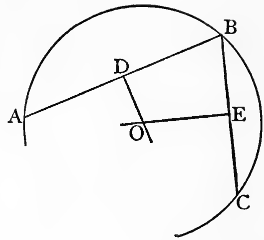
Cor.—If, in a circle, a straight line bisect a chord and be also at right angles to it, the centre of the circle is in that straight line.

NOTE.

The following extension of Prop. 1 is important:—

To find the centre of a given arc, or segment.

Let A, B, C be three pts. on the ○ce.



Join AB, BC.
Bisect AB, BC at D, E....................I. 10.
Through D, E draw lines
at rt. ∠s to AB, BC........................I. 11.
Then these lines will, if prod[d.] meet;
(for, if not they are || and then
AB, BC at rt. ∠s to them would
also be ||. See Ex. 5 on Prop. 29, Bk. I.)
Let them, when prod[d.] meet at O.

Then shall O be the cent.

For, since DO is drawn at rt. ∠s to the chd. AB from its mid. pt. D,

∴ DO passes through the cent...................III. 1. cor.

And, since EO is drawn at rt. ∠s to the chd. BC from its mid. pt. E,

∴ EO passes through the cent....................III. 1. cor.

∴ O, the pt. at which DO and EO cut, is the cent.

Q.E.F.

EXERCISES.

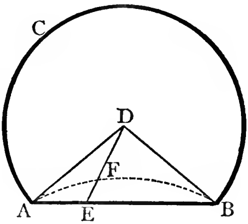
1. Find, when possible, a point equidistant from three given points.
2. AB, AC are equal chords of a circle; prove that the bisector of the angle BAC is a diameter.
3. Straight lines drawn at right angles to the sides of any rectilineal figure inscribed in a circle through their middle points all pass through a fixed point.
4. Through a given point within a circle draw a chord which shall be bisected by the given point.
5. Given two chords of a circle in magnitude and position; describe the circle.
6. To describe a circle of given radius which shall pass through two given fixed points. When is this impossible?
7. Describe a circle with a given point as centre which shall bisect the circumference of a given circle.

PROPOSITION II. THEOREM.

If any two points be taken in the circumference of a circle, the straight line which joins them must fall within the circle.

Let ABC be a ⊙, A and B any pts. in its ○ce, and let AB be joined.

Then shall AB fall within ⊙ABC.



For, if not, let it, if possible, fall without the ⊙.
Find D the cent. of ⊙ABC.......................................III. 1.
Take any pt. F in the arc AB.
Join DF and prod. DF to meet AB in E.
Join DA, DB.

Then, since DA=DB..............................Radii.
∴ ∠DAE=∠DBE.........................I. 5.

But ∠DEB>∠DAE.........................I. 16.
∴ ∠DEB>∠DBE,
∴ DB>DE..............................I. 19.

But DB=DF..............................Radii.
∴ DF>DE,
i.e. the part>the whole,
which is absurd.

Hence, the pt. E cannot fall without the ⊙ABC.

Similarly, it may be shown that E cannot fall on the ○ce,
∴ the pt. E falls within the ⊙.

But E is *any* pt. in AB,
∴ AB lies entirely within the ⊙.

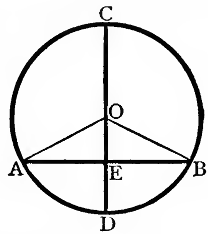
Wherefore, *if any two points* &c. Q.E.D.

PROPOSITION III. THEOREM.

If a straight line drawn through the centre of a circle bisect a straight line in it which does not pass through the centre, it shall cut it at right angles; and, conversely, if it cut it at right angles it shall bisect it.

PART I.—In the ⊙ABC let the diam. CD bisect the chd. AB at E.

Then shall CD be at rt. ∠s to AB.



Find O the cent. .... I. 10.
Join OA, OB.

Then in △s OAE, OBE,

∵ { AE=EB .... Hyp.
OE is com.
OA=OB .... Radii. }

∴ ∠OEA=∠OEB .... I. 8.

∴ CE is at rt. ∠s to AB.

PART II.—In the ⊙ABC, let the diam. CD cut the chord AB at rt. ∠s at E.

Then shall CD bisect AB at E.

Find O the cent. .... I. 10.
Join OA, OB.

Then, since OA=OB .... Radii.

∴ ∠OAE=∠OBE .... I. 5.

Hence, in △s OAE, OBE,

∵ { ∠OAE=∠OBE .... Above.
rt. ∠OEA=rt ∠OEB .... Hyp.
OE is com. }

∴ AE=BE .... I. 26 (i.)

i.e. AB is bisected at E.

Wherefore, *if a straight line* &c. Q.E.F.

NOTES.

The 2nd part of Prop. 3 is the converse of the 1st part, and they are both converse of the corollary to Prop. 1. These three results are important.

Prop. I. cor.—*A st. line drawn at rt. ∠s to a chord of a ⊙ through its mid. pt. passes through the cent.*

Prop. III. part i.—*A st. line from the cent. to the mid. pt. of a chord of a ⊙ is at rt. ∠s to the chord.*

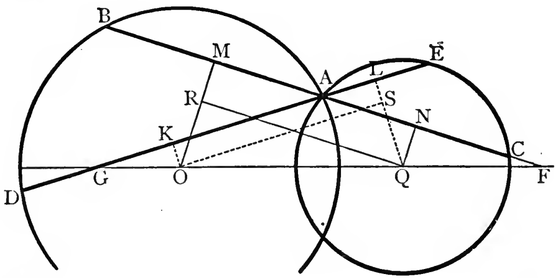
Prop. III. part ii.—*A ⊥ from the cent. of a ⊙ on any chord bisects that chord.*

EXAMPLES.

I. *If two circles cut one another, and through one of their points of intersection straight lines be drawn equally inclined to the line joining their centres and terminated by the circumferences, these straight lines will be equal.*

Let A be a pt. of intersection of the ⊙s ABD, AEC, and let BAC DAE be equally inclined to OQ the line joining their cents.

Then shall BC=DE.



Draw OM, QN ⊥s to BC .... I. 12.
Draw QR || to BC, meeting OM at R.............................. I. 31.

Then, since ∠s at M and N are rt. ∠s,

∴ OM is || to QN........................ I. 29 (iii.)

Hence RMNQ is a ▱,

∴ QR=MN............................ I. 34.

But chd. AB is bisected at M, and AC at N......... III. 3 (ii.)

Hence MN is half BC.

∴ QR is half BC.

And ∠ at R=∠ at M=a rt. ∠ .... I. 29 (ii.)

Similarly, if ⊥s OK, QL and || OS be drawn to DE, it may be shown that OS is half DE, and that ∠ at S is a rt. ∠.

Also, extr ∠RQO=intr ∠BFQ .. .. I. 29 (ii.)

=∠EGO.......................... Hyp.

=extr ∠SOQ ... . I. 29 (ii.)

Hence, in $\triangle$s ROQ, SOQ,

$$\because \begin{cases} \angle ORQ = \angle OSQ, \\ \angle RQO = \angle SOQ, \\ OQ \text{ is com:} \end{cases}$$

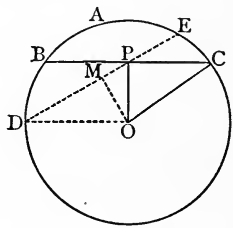
∴ RQ=OS.............................. I. 26.

Hence BC=DE.............................. Ax. 6.

Q.E.D.

II. *Through a given point within a circle to draw the shortest possible chord.*

Let ABC be the ⊙, and P the given pt.



Find O the cent.............................. III. 1.

Join OP.

Through P draw chd. BC at rt. $\angle$s to OP.............................. I. 11.

Then BC shall be < any other chd. through P, such as DE.

Draw OM ⊥ to DE.................... I. 12.

Join OC, OD.

Then, since $\angle$OMP is a rt. $\angle$.............................. Constr.

∴ $\angle$OPM is < a rt. $\angle$ .... I. 17.

∴ OM < OP.............................. I. 19.

Now, sq. on OC=sqs. on OP, PC.............................. I. 47.

and sq. on OD=sqs. on OM, MD.............................. I. 47.

Hence sqs. on OP, PC=sqs. on OM, MD.

But sq. on OP > sq. on OM,

∴ sq. on PC < sq. on MD.

∴ PC < MD.

But PC is half BC, and MD is half DE.............................. III. 3. (ii.)

∴ BC < DE.

Q.E.D.

EXERCISES.

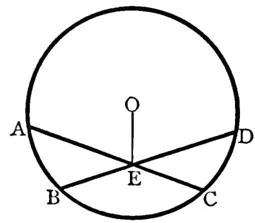
1. Prove part ii. of Prop. 3, using I. 47 instead of I. 26.
2. If from the centre of a circle a perpendicular be drawn to the nearer of two parallel chords, it will, if produced, bisect the other.
3. Find the locus of the middle points of a system of parallel chords of a circle.
4. The portions of a common chord of two concentric circles, which are intercepted between the circumferences, are equal.
5. If two circles cut, any two parallel straight lines drawn through the points of intersection and terminated by the circumferences, are equal.
6. Through a point of intersection of two circles which cut one another, draw the greatest possible straight line terminated by their circumferences.

PROPOSITION IV. THEOREM.

If, in a circle, two straight lines cut one another, which do not both pass through the centre, they do not bisect one another.

Let ABCD be a ⊙; AC and BD two chords which cut at E and which do not both pass through the cent.

Then AC and BD shall not bisect each other.



CASE 1.—If one pass through the cent. it is a diam. and is bisected by the cent.

∴ it cannot be bisected by the other which does not.

CASE 2.—If neither pass through the cent., if possible, suppose

AE=EC, and BE=ED.

Find O the cent..III. 1.
Join OE.

Then, if E is the mid. pt. of AC,

∠OEA is a rt. ∠ ....III. 3 (i.)

And, if E is the mid. pt. of BD,

∠OEB is a rt. ∠ ....III. 3 (i.)

∴ ∠OEA=∠OEB.
i.e. the part=the whole,
which is absurd.

Hence, AC and BD cannot bisect each other.

Wherefore, *if in a circle* &c. Q.E.D.

EXERCISES.

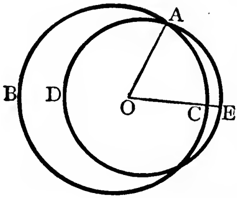
1. Draw two straight lines in a circle which shall bisect each other.
2. Draw two chords in a circle, neither of which pass through the centre, and one of which is bisected by the other.
3. Prove that, in a circle, if a chord which does not pass through the centre bisect another chord it cannot cut it at right angles.
4. Prove the converse of Ex. 3.
5. If a parallelogram be inscribed in a circle it must be rectangular.

PROPOSITION V. THEOREM.

If two circles cut one another they cannot have the same centre.

Let the ⊙s ABC, ADE cut at A.

Then they shall not have a com. cent.



For, if possible, let O be their cent.

Join OA.

Draw OCE meeting the ○ces at C and E.

Then, since O is cent. of ⊙ABC,

∴ OA=OC.

And, since O is cent. of ⊙ADE,

∴ OA=OE.

Hence, OC=OE,

i.e. the part=the whole,

which is absurd.

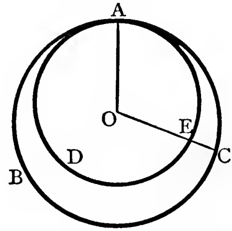
Wherefore, *if two circles cut* &c. Q.E.D.

PROPOSITION VI. THEOREM.

If two circles touch one another internally they cannot have the same centre.

Let the ⊙s ABC, ADE touch internally at A.

Then they shall not have a com. cent.



For, if possible, let O be their cent.

Join OA.

Draw OEC meeting the ○ces at E and C.

Then, since O is cent. of ⊙ABC,

∴ OA=OC.

And, since O is cent. of ⊙ADE,

∴ OA=OE.

Hence, OC=OE,

i.e. the whole=its part,

which is absurd.

Wherefore, *if two circles touch* &c. Q.E.D.

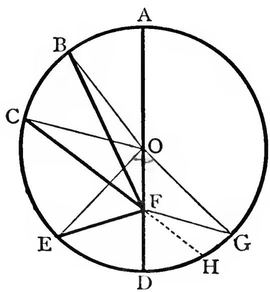
EXERCISES.

1. Prove that two concentric circles cannot cut.
2. Prove that two concentric circles cannot touch.
3. Of what propositions are Exs. 1 and 2 respectively converse?

PROPOSITION VII. Theorem.

If any point be taken in the diameter of a circle which is not the centre, of all the straight lines which can be drawn from this point to the circumference the greatest is that in which the centre is, and the other part of the diameter is the least; and, of any others, that which is nearer to the straight line which passes through the centre is always greater than one more remote; also, from the same point there can be drawn to the circumference only two straight lines equal to one another, one on each side of the shortest line.

Let ABCD be a ⊙, and O its cent.;
let F be any other pt. within it;
let AOFD be the diam. through F;
and let FB, FC, FE be any other st. lines from F to the ○ce.



Then (i) FA shall be the greatest line;
(ii) FB shall be > FC, FC > FE;
(iii) FD shall be the least line;
and **(iv) from F but one st. line = FE can be drawn to the ○ce.**

Join OB, OC, OE.
At O in OF make ∠FOG = ∠FOE ....I. 23.
Join FG.

Then (i), in △BOF,

FO, OB > FB ....I. 20.
But OB = OA ....Rad.
∴ FA > FB.
i.e. FA is the greatest line.

And (ii), in $\triangle$s BOF, COF,

∵ { OB=OC .... Rad.
OF is com.
∠FOB > ∠FOC .... Ax. 9.

∴ FB > FC .... I. 24.

Similarly it may be shown
that FC > FE.

Also (iii), in $\triangle$OFE,

OF, FE > OE .... I. 20.
But OD=OE .... Rad.

∴ OF, FE > OD.

Take away the com. part OF,

∴ rem^r FE > rem^r FD.

i.e. FD is the least line.

Again (iv), in $\triangle$s OFG, OFE,

∵ { OG=OE .... Rad.
OF is com.
∠FOG=∠FOE .... Constr.

∴ FG=FE .... I. 4.

And, from F no other line can be drawn to the ○ce=FE.

For, if possible, suppose FH=FE.

Then, since FG=FE .... Above.

∴ FG=FH.

i.e. a line nearer to FA=one more remote,
which is impossible .... Part (iii).

∴ but one st. line can be drawn to the ○ce from F=FE.

Wherefore, *if any point* &c. Q.E.D.

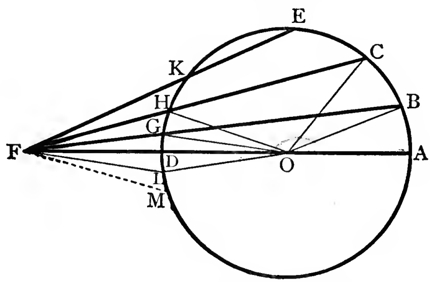
EXERCISES.

1. If from the end of a diameter any number of chords be drawn in a circle—
   (i) the diameter itself is the greatest of these chords;
   (ii) a chord nearer to the diameter is greater than one more remote;
   (iii) there can be drawn but two equal chords from the point, one on each side of the diameter.
2. Write out in full a proof that FC is greater than FE.
3. From F draw a straight line to the circumference which shall be equal to FB.

PROPOSITION VIII. Theorem.

If any point be taken without a circle and straight lines be drawn from it to the circumference, of those which fall on the concave part of the circumference, the greatest is that which passes through the centre, and one nearer to that which passes through the centre is always greater than one more remote; but of those which fall on the convex part of the circumference, the least is that which, when produced, passes through the centre, and one nearer to it is always less than one more remote; also from the same point there can be drawn to the circumference only two straight lines equal to one another, one on each side of the shortest line.

Let ABCD be a ⊙, and O its cent.;
let F be any pt. without it;
let FDOA be drawn from A through the cent.;
and let FGB, FHC, FKE be any other lines from F to the ○ce.



Then (i) FA shall be the greatest line;
(ii) FB shall be > FC, FC > FE;
(iii) FD shall be the least line;
(iv) FG shall be < FH, FH < FK;
and (v) from F but one st. line = FG can be drawn to the ○ce.

Join OB, OC, OG, OH.
At O in OF, make ∠FOL = ∠FOG .... I. 23.
Join FL.

Then (i) in △BOF,

FO, OB > FB .... I. 20.
But OB = OA .... Rad.
∴ FA > FB.

i.e. FA is the greatest line.

And (ii) in $\triangle$s BOF, COF,

$\because \begin{cases} OB=OC \ldots\ldots \text{Rad.} \\ OF \text{ is com.} \\ \angle FOB > \angle FOC \ldots\ldots \text{Ax. 9.} \end{cases}$

$\therefore$ FB$>$FC .... I. 24.

Similarly it may be shown

that FC$>$FE.

Also (iii) in $\triangle$OFG,

FG, OG$>$FO .... I. 20.

But OG$=$OD .... Rad.

$\therefore$ rem[r] FG$>$rem[r] FD.

i.e. FD is the least line.

And (iv) in $\triangle$OFH,

OG, FG$<$OH, FH .... I. 21.

But OG$=$OH .... Rad.

$\therefore$ rem[r] FG$<$rem[r] FH.

Similarly it may be shown

that FH$<$FK.

Again (v) in $\triangle$s OFL, OFG,

$\because \begin{cases} OL=OG \ldots\ldots \text{Rad.} \\ OF \text{ is com.} \\ \angle FOL = \angle FOG \ldots\ldots \text{Constr.} \end{cases}$

$\therefore$ FL$=$FG .... I. 4.

And from F no other line can be drawn to the ○ce$=$FG.

For, if possible, suppose FM$=$FG.

Then, since FL$=$FG .... Above.

$\therefore$ FL$=$FM.

i.e. a line nearer to FD$=$one more remote,

which is impossible .... Part (iv).

$\therefore$ but one st. line can be drawn to the ○ce from F$=$FG.

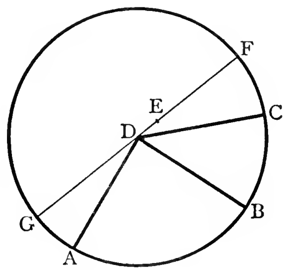
Wherefore, *if any point* &c. Q.E.D.

EXERCISES.

1. Prove that (i) AD is greater than BG; (ii) BG is greater than CH.
2. Prove that one and only one straight line can be drawn from F to the circumference equal to FB.
3. If the secant FKE be made to revolve about F, and the points K, E to approach each other until they become indefinitely near each other, what does the secant ultimately become?

PROPOSITION IX. Theorem.

If a point be taken within a circle from which more than two equal straight lines can be drawn to the circumference, that point is the centre.



Let ABC be a ⊙, and from the pt. D within it let three st. lines DA, DB, DC be equal to one another.

Then shall D be the cent.

For if not, if possible, suppose E the cent.
Join DE, and prod. to meet the ○ce in F, G.

Then since D is a pt., not the cent., in a diam. GF,

∴ DC, nearer to DF, > DB, DB > DA ......III. 7 (ii).

But this is contrary to hypothesis.

Hence, D must be the cent.

Wherefore, *if a point* &c. Q.E.D.

NOTES.

Euclid gave also a *direct* proof of this prop. depending on the cor. to Prop. 1.

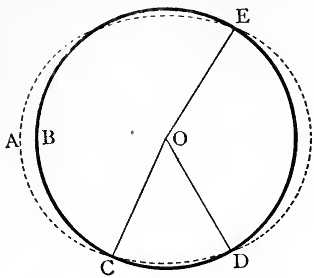
It should be noticed that the point E might be supposed to fall within the angle CDB, or within BAD.

EXERCISES.

1. Prove this prop. directly, by joining AB, BC, bisecting AB and BC at E and F, and proving that DE, DF are at right angles to AB and AC.
2. Write out an indirect proof of the prop., supposing E, the centre, to fall within the angle CDB.

PROPOSITION X. THEOREM.

One circumference of a circle cannot cut another at more than two points.



For, if it be possible, let ⊙ACDE cut ⊙BCDE at the pts. C, D, E.
Find O the cent. of ⊙ACDE.......................................III. 1.
Join OC, OD, OE.

Then, since O is cent. of ⊙ACDE,

∴ OC=OD=OE ....Radii.

And, since from a pt. O within ⊙BCDE three=st. lines are drawn to the ○ce,

∴ O is cent. of ⊙BCDE..................III. 9.

i.e., pt. O is the com. cent. of two ⊙s which cut,

which is impossible.......................III. 5.

Wherefore, *one circumference* &c. Q.E.D.

NOTE.

Euclid gives two demonstrations of this prop. also, the other being similar to that indicated in the note to Prop. 9.

It should be noticed in the above proof that the centre of one circle might be supposed to fall *on* or *outside* the circumference of the other circle.

EXERCISES.

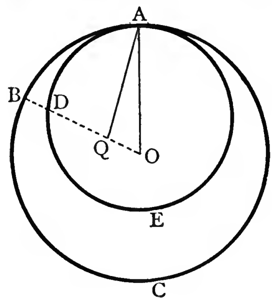
1. Write out another proof of Prop. 10, depending on III. 1, cor.
2. Complete the proof given above, by considering the cases mentioned in the note.
3. Two circumferences cannot have more than two points in common.

PROPOSITION XI. THEOREM.

If two circles touch one another internally, the straight line which joins their centres, being produced, shall pass through the point of contact.

Let ⊙ABC touch ⊙ADE internally at A; and let O and Q be their cents.

Then shall OQ prod^{d} pass through A.



For if not, let it, if possible, pass otherwise as OQDB.

Join AO, AQ.

Then, AQ, QO > AO .... I. 20.

But AO = OB .... Radii.

∴ AQ, QO > OB.

Take away the com. part OQ,

∴ rem^{r} AQ > QB.

But AQ = QD .... Radii.

∴ QD > QB,

i.e. the less > the greater,

which is absurd.

Hence, OQ when prod^{d} must pass through A.

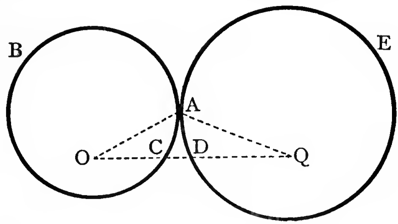
Wherefore, *if two circles* &c. Q.E.D.

PROPOSITION XII. THEOREM.

If two circles touch one another externally, the straight line which joins their centres shall pass through the point of contact.

Let ⊙ABC touch ⊙ADE externally at A, and let O and Q be their cents.

Then shall st. line OQ pass through A.



For if not, let it, if possible, pass otherwise as OCDQ.
Join AO AQ.

Then, in △AOQ,

OA, QA > OCDQ .... I. 20.

But OA = OC } and QA = QD } .... Rad.

∴ OC, QD > OCDQ.

i.e. part > the whole,

which is absurd.

Hence, st. line OQ must pass through A.

Wherefore, *if two circles* &c. Q.E.D.

NOTE.

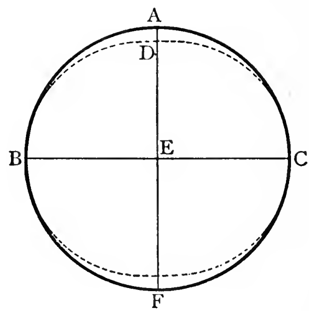
The words "*the* point of contact" in Props. 9 and 10 appear to assume that the circles can touch in but *one* point, although this is not proved until the succeeding proposition is reached. But, if the words are taken in the sense "the point of contact under consideration," it follows from the proofs that, if there were a second point of contact, the line joining the centres would, if produced, pass through it also.

EXERCISE.

State and prove the converse of (i) Prop. 11; (ii) Prop. 12.

PROPOSITION XIII. THEOREM.

One circle cannot touch another at more than one point, whether on the inside or outside.



For, if it be possible, let ⊙ABC touch ⊙DBC at the pts. B and C.

(i) Internally.

Join BC.

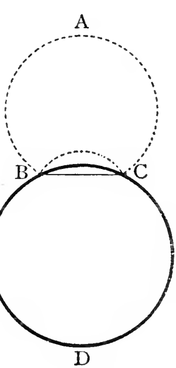
Bisect BC at E....................I. 10.

Through E draw
ADEF at rt. ∠s to BC..........I. 11.

Then, since B and C are pts. in the ○ces of both ⊙s,
∴ BC falls within both ⊙s................III. 2.

And, since AF bisects BC at rt. ∠s,
∴ the cents. of both ⊙s are in AF...........III. 1 cor.
∴ AF passes through a pt. of contact..........III. 11.

i.e. AF passes through the extremity of a line which it bisects at rt. ∠s,
which is absurd.



(ii) Externally.

Join BC.

Then, since B and C are pts. in the ○ce of ⊙ABC,

∴ BC falls within ⊙ABC...............III 2.

And, since B and C are pts. in the ○ce of ⊙DBC,

∴ BC falls within ⊙DBC...............III. 2.

i.e. the chd. BC is within each of two ⊙s which are without each other,
which is absurd.

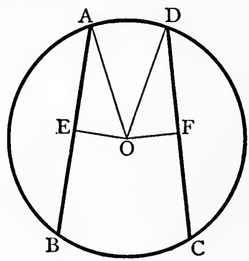
Wherefore, *one circle cannot* &c. Q.E.D.

PROPOSITION XIV. Theorem.

Equal straight lines in a circle are equally distant from the centre; and, conversely, those which are equally distant from the centre are equal to one another.

Part I.—Let AB, CD be equal chds. of ⊙ABCD.

Then shall AB, CD be equidist. from the cent.



Find O the cent.........................III. 1.
Draw OE, OF ⊥s to AB, CD........I. 12.
Join OA, OD.

Then, since OE, OF are ⊥ to AB, CD,
∴ AB, CD are bisected at E, F.....III. 3.

But AB=CD ....Hyp.
∴ AE=DF,
∴ sq. on AE=sq. on DF.

Also, since OA=OD ....Radii.
∴ sq. on OA=sq. on OD.

Hence, sqs. on AE, OE=sqs. on DF, OF ....I. 47.

But sq. on AE=sq. on DF.....................Above.
∴ rem^g sq. on OE=rem^g sq. on OF.
∴ OE=OF,

i.e. AB and CD are equidist. from O.

Part II.—Let the ⊥s OE, OF be equal.

Then shall chd. AB=chd. CD.

For, since OE=OF.............................Hyp.
∴ sq. on OE=sq. on OF.

And, since OA=OD.............................Radii.
∴ sq. on OA=sq. on OD.

Hence, sqs. on OE, AE=sqs. on OF, DF..............I. 47.

But sq. on OE=sq. on OF ....Above.
∴ rem^g sq. on AE=rem^g sq. on DF.
∴ AE=DF.

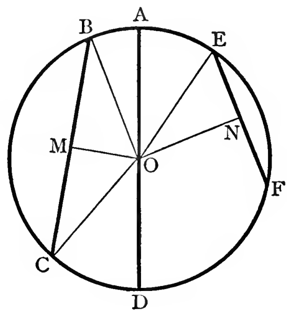
But, since OE, OF are ⊥ to AB, CD,
∴ E, F are mid. pts. of AB, CD...............III. 3.

Hence, AB=CD.............................Ax. 6.

Wherefore, *equal straight lines* &c. Q.E.D.

PROPOSITION XV. THEOREM.

The diameter is the greatest straight line in a circle; and, of all others, that which is nearer to the centre is always greater than one more remote; and, conversely, the greater is nearer to the centre than the less.



PART I.—Let ABCD be a ⊙, AD a diam., O the cent., and let the chd. BC be nearer to the cent. than EF,

Then shall (i) AD be the greatest line;
(ii) BC>EF.

From O draw ⊥s OM, ON to BC, EF....................................I. 12.
Join OB, OC, OE.

Then (i) OB, OC>BC....................................I. 20.
But OB=OA, and OC=OD....................................Radii.
∴ OA, OD>BC,
i.e. AD is the greatest line.

And (ii) since OM<ON....................................Hyp.
∴ sq. on OM<sq. on ON.
But, since OB=OE....................................Radii.
∴ sq. on OB=sq. on OE.
Hence, sqs. on BM, OM=sqs. on EN, ON....................................I. 47.
But sq. on OM<sq. on ON....................................Above.
∴ rem^{g} sq. on BM>rem^{g} sq. on EN,
∴ BM>EN.

But, since OM, ON are ⊥s to BC, EF....................................Constr.
∴ M, N are mid. pts. of BC, EF....................................III. 3.
Hence, BC>EF....................................Ax. 6.

PART II.—Let chd. BC > chd. EF, and let ⊥s OM, ON be drawn from the cent. O to these chds.

Then shall OM < ON.

Join OB, OE.

Then, since OM, ON are ⊥s to BC, EF ....Hyp.
∴ M, N are mid. pts. of BC, EF...............III. 3.
and, since BC > EF..............................Hyp.
∴ BM > EN.
∴ sq. on BM > sq. on EN.
But, since OB = OE..............................Radii.
∴ sq. on OB = sq. on OE.
Hence, sqs. on BM, OM = sqs. on EN, ON..............I. 47.
But sq. on BM > sq. on EN ....Above.
∴ rem^g sq. on OM < rem^g sq. on ON.
∴ OM < ON,
i.e. the greater chd. BC is nearer to the cent.

Wherefore, *the diameter is the greatest* &c. Q.E.D.

NOTE.

The proofs of Props. 14 and 15 depend on the following important theorem:—

If two right-angled triangles have their hypotenuses equal, and also one other side of the one equal to one other side of the other, their remaining sides must be equal.

EXERCISES.

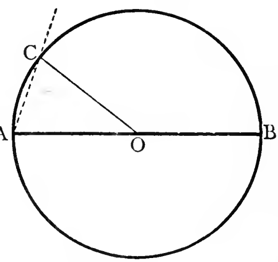
1. Write out a proof of the above theorem.
2. If the radius of a circle is 5 inches long, and a chord in it is 8 inches long, find the distance of the chord from the centre.
3. If the diameter be 20, and the distance of a chord from the centre be 8, what is the length of the chord?
4. The radius of a circle is 13 inches, the length of a chord is 2 feet; find its distance from the centre.
5. The length of a chord is 6 inches, its distance from the centre is 2 inches; find the radius of the circle.
6. Equal chords in a circle subtend equal angles at the centre.
7. In a circle the greater chord subtends a greater angle at the centre than the less.
8. Find the *locus* of the middle points of equal chords of a circle.
9. Draw a chord of a circle which shall be double of its distance from the centre.

PROPOSITION XVI. THEOREM.

The straight line drawn at right angles to the diameter of a circle, from its extremity, falls without the circle; and no straight line can be drawn from the extremity, between that straight line and the circumference, so as not to cut the circle.

PART I.—Let ABC be a ⊙, O its cent., AB a diam.

Then shall the st. line drawn from A at rt. ∠s to AB fall without the ⊙.



For, if not, if possible, let it fall within and meet the ○ce again at C.

Join OC.

Then, since OC=OA............Radii.

∴ ∠OCA=∠OAC.......I. 5.

But ∠OAC is a rt. ∠......Hyp.

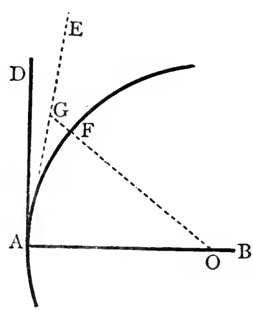
∴ ∠OCA is also a rt. ∠.

i.e. two angles of △OAC together=two rt. ∠s, which is impossible........................I. 17.

Hence the line must fall without the ⊙.

PART II.—Let AD be at rt. ∠s to the diam. AOB.

Then between AD and the ○ce no st. line can be drawn which does not cut the ⊙.



For, if it be possible, let AE be drawn between AD and the ○ce so as not to cut the ⊙.

From O draw OG ⊥ to AE and meeting the ○ce in F.

Then, since ∠OGA is a rt. ∠ ....Constr.

∴ ∠OAG is < a rt. ∠..........I. 17.

Hence, OG < OA...............I. 19.

But OF=OA...............Radii.

∴ OG<OF.

i.e. the whole<its part, which is absurd.

Wherefore, *the straight line* &c. Q.E.D.

Cor.—From this prop. it is manifest that the st. line drawn at rt. ∠s to the diam. of a ⊙, from its extremity, *touches* the circle....................................III. def. 2.

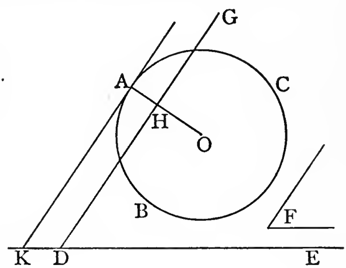
Also, that it can touch it at *one* pt. only......................III. 2.

And, that there can be but *one* tangent to a circle at any one point..Part II. above.

EXAMPLE.

To a given circle to draw a tangent which shall make a given angle with a given straight line.

Let ABC be the given ⊙, DE the given st. line, and F the given ∠.



At any pt. D in DE make ∠EDG=∠F................I. 23.

Find O the cent. of ⊙ABC......III. 1.

From O draw OH ⊥ to DG......I. 12. and let OH, prod^d if necessary, meet the ○ce in A.

From A draw AK at rt. ∠s to OA and meeting DE in K...I. 11.

AK shall be the req^d tang.

For, since ∠OAK is a rt. ∠,

∴ AK touches the ⊙......... ....III. 16. Cor.

And, since ∠OHD is a rt. ∠,

∴ ext^r ∠OHD=int^r ∠OAK..................Ax. 11.

∴ AK is || to HD........................ I. 28. (i).

Hence, ∠AKE=∠HDE........................I. 29. (ii).

=∠F.............................Constr.

Q.E.F.

EXERCISES.

1. State the converse of the first part of the corollary to Prop. 16.
2. Show that all chords of a circle which touch a concentric circle are equal.
3. Draw a tangent to a circle which shall be parallel to a given straight line.
4. Draw a tangent to a circle which shall be perpendicular to a given straight line.
5. Find the locus of the centres of all circles which touch each other at the same point.
6. Two circles cut at right angles; prove that the normals from their points of contact will be at right angles.

PROPOSITION XVII. PROBLEM.

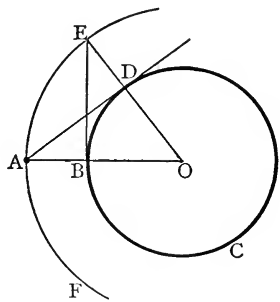
To draw a straight line from a given point, either in or without the circumference, which shall touch a given circle.

Let A be the given pt., and BCD the given ⊙.

It is req[d] *to draw from A a tang. to the ⊙BCD.*

CASE 1.—If the pt. A be in the ○ce, draw from A a line at rt. ∠s to the rad. at A; this line will touch the ⊙........III. 16. Cor.

CASE 2.—If the pt. A be without the ⊙,



Find O the cent...III. 1.
Join OA, cutting the ○ce at B.
With cent. O and rad. OA desc. a ⊙EAF.
From B draw BE at rt. ∠s to OA, meeting ⊙EAF at E...I. 11.
Join OE, cutting ⊙BCD at D.
Join AD.

Then shall AD be the tangent req[d].

For, in △s ADO, EBO,

∵ { OA=OE...........................Radii.
OD=OB...........................Radii.
∠AOE is com. }

∴ ∠ODA=∠OBE.....................I. 4.

But ∠OBE is a rt. ∠ ....Constr.

∴ ∠ODA is a rt. ∠,

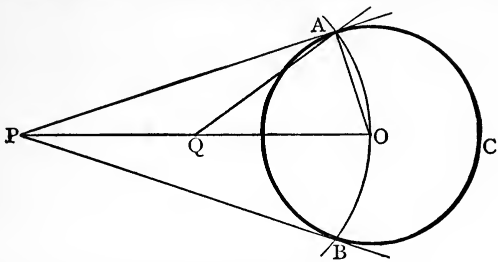
∴ AD touches the ⊙BCD.................III. 16. Cor.

Wherefore, *from the given pt.* &c. Q.E.F.

NOTE.

The following is another solution of the problem:—

From a given external point to draw a pair of tangents to a given circle.



Let ABC be the given ⊙, and P the given external pt.

Find O the cent .... III. 1.

Join OP.

Bisect OP at Q.. I. 10.

With cent. Q and rad. QO desc. an arc cutting ⊙ABC at A, B.

Join PA, PB.

Then PA, PB shall be the tangs.

Join QA, OA.

Then, since QP=QA .... Radii.

∴ ∠QPA=∠QAP .... I. 5.

And, since QO=QA............................ Radii.

∴ ∠QOA=∠QAO........................ I. 5.

∴ ∠s QPA, QOA=whole ∠OAP.

Hence ∠OAP is a rt. ∠ .... I. 32.

But OA is a rad.

∴ PA touches the ⊙..................... III. 16. Cor.

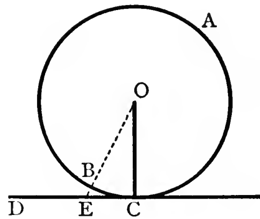
Similarly it may be shown that PB touches the ⊙. Q.E.F.

EXERCISES.

1. Prove that PA=PB, *i.e.* that two tangents to a circle from an external point are equal.
2. Prove that the line joining the given point to the centre of the given circle bisects the angle included by the tangents.
3. If the distance of the given point from the centre be 13 inches and the radius of the circle be 5 inches, find the length of the tangents.
4. If a quadrilateral circumscribe a circle, one pair of its opposite sides are together equal to the other pair.
5. No parallelogram can be described about a circle except a rhombus.

PROPOSITION XVIII. Theorem.

If a straight line touch a circle, the straight line drawn from the centre to the point of contact shall be perpendicular to the line touching the circle.



Let st. line DC touch ⊙ABC at C, and let OC be the radius to the pt. of contact C.

Then shall OC be ⊥ to DC.

For, if not, if possible, let OE be drawn ⊥ to DC meeting the ○ce in B.

Then, if ∠OEC is a rt. ∠,

∠OCE is < a rt. ∠ .... I. 17.

∴ ∠OEC > ∠OCE,

∴ OC > OE .... I. 19.

But OC = OB .... Radii.

∴ OB > OE.

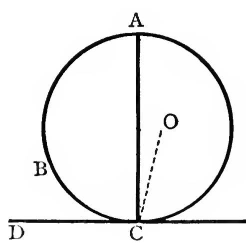
i.e. the part > the whole,

which is absurd.

Wherefore, *if a straight line* &c. Q.E.D.

PROPOSITION XIX. Theorem.

If a straight line touch a circle, and from the point of contact a straight line be drawn at right angles to the touching line, the centre of the circle shall be in that line.



Let st. line DC touch ⊙ABC at C, and let CA be drawn at rt. ∠s to DC.

Then shall CA pass through the cent.

For, if not, if possible, let O the cent. be without AC.

Join OC.

Then, since DC is a tang. and OC the rad. to the pt. of contact,

∴ ∠DCO is a rt. ∠ .... III. 18.

But ∠DCA is a rt. ∠ .... Hyp.

∴ ∠DCA = ∠DCO,

i.e. the part = the whole,

which is absurd.

Wherefore, *if a straight line* &c. Q.E.D.

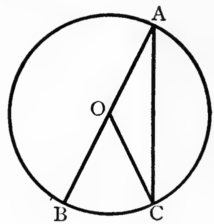
PROPOSITION XX. THEOREM.

The angle at the centre of a circle is double of the angle at the circumference standing on the same base, that is, on the same arc.

Let ABC be a ⊙, and let BOC, BAC be ∠s at the cent. and ○ce, having the same arc BC for base.

Then shall ∠BOC be double of ∠BAC.

CASE 1.—When the cent. O is on BA, or AC.



In △OAC,

extr ∠BOC = ∠s OAC, OCA... I. 32.

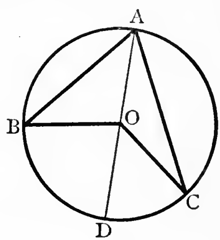
But, since OA = OC .... Radii.

∴ ∠OAC = ∠OCA.

Hence, ∠BOC is double of ∠OAC.

CASE 2.—When the cent. O is within the ∠BAC.

Join AO, and prod. AO to meet the ○ce in D.



Then, as in Case 1, it may be shown that

∠BOD is double of ∠OAB,

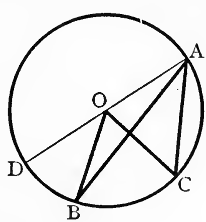
and that

∠COD is double of ∠OAC.

∴ whole ∠BOC is double of ∠BAC.

CASE 3.—When the cent. O is without the ∠BAC.

Join AO, and prod. AO to meet the ○ce in D.



Then, as in Case 1, it may be shown that

∠COD is double of ∠OAC,

and that

∠BOD is double of ∠OAB.

∴ remg ∠BOC is double of ∠BAC.

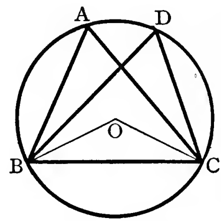
Wherefore, *the angle at the centre* &c. Q.E.D.

PROPOSITION XXI. THEOREM.

The angles in the same segment of a circle are equal to one another.

Let ABCD be a ⊙, and let BAC, BDC be ∠s in the same seg^t BAC.

Then shall ∠BAC=∠BDC.



CASE 1.—When the seg^t is > a semicircle.

Find O the cent.............................III. 1.

Join OB, OC.

Then, since ∠BOC at the cent. and ∠BAC at the ○ce are on the same base,

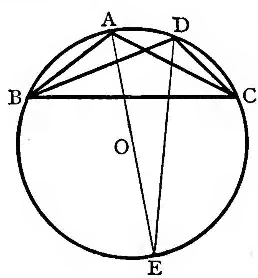
∴ ∠BOC is double of ∠BAC...............III. 20.

And, for the same reason,

∠BOC is double of ∠BDC.

∴ ∠BAC=∠BDC.

CASE 2.—When the seg^t is not > a semicircle.



Find O the cent.........................III. 1.

Join AO.

Produce AO to meet the ○ce in E.

Join DE.

Then, the seg^t BACE is > a semicircle,

∴ ∠BAE=∠BDE.....Case 1, above.

Also, the seg^t EBDC is > a semicircle,

∴ ∠EAC=∠EDC.........Case 1.

Hence, the whole ∠BAC=whole ∠BDC.........Ax. 2.

Wherefore, *the angles in the same segment* &c. Q.E.D.

EXERCISES.

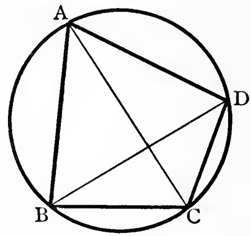
1. ABC is a triangle inscribed in a circle whose centre is O. OD is perpendicular to BC. Prove that angle BOD is equal to BAC.
2. If, in either figure of Prop. 21, AC and BD cut at F, then the triangle ABF is equiangular to DCF.
3. State and prove, by a *reductio ad absurdum*, the converse of Prop. 21.
4. In the figure of Ex. X. page 88, prove, by help of Ex. 3, that a circle will go round B, C, M, N.
5. Divide a given circle into two segments so that the angle in one segment may be double that in the other.

PROPOSITION XXII. THEOREM.

The opposite angles of any quadrilateral figure inscribed in a circle are together equal to two right angles.

Let ABCD be a quad[l] insc[d] in ⊙ABCD.

Then shall ∠s BAD, BCD together = two rt. ∠s.



Join AC, BD.

Then, since ∠s BAC, BDC are in the same seg[t],

∴ ∠BAC = ∠BDC.........III. 21.

And, for a like reason,

∠DAC = ∠DBC.........III. 21.

∴ ∠s BAC, DAC = ∠s BDC, DBC,

i.e. ∠BAD = ∠s BDC, DBC.

To each add ∠BCD,

∴ ∠s BAD, BCD = ∠s BDC, DBC, BCD.

= two rt. ∠s.......................I. 32.

Similarly, it may be shown that

∠s ADC, ABC = two rt. ∠s.

Wherefore, *the opposite angles* &c. Q.E.D.

NOTE.

Props. 21, 22 are very important, as are also the converse props. (See p. 177.)

EXERCISES.

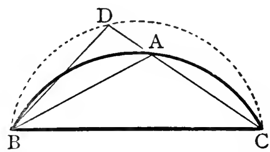
1. In the figure of Prop. 22, if BC be produced to E, angle DCE is equal to angle BAD.
2. If, in Prop. 22, the diagonals cut at F, the triangles ADF and BCF are equiangular.
3. If, in the figure of Prop. 22, AD and BC be produced to meet at G, the angle AGB is equal to the difference of the angles ACB and DBC.
4. In the figure of Ex. X., page 88, prove that (assuming the converse of III. 22),
   (i) a circle can be described about each of the quadrilaterals ANOM, BNOL, CLOM.
   (ii) a circle can be made to pass through the points P, B, O, C.
   (iii) if LN be joined, angle OBN is equal to angle OLN.
   (iv) if LM be joined, angle ACN is equal to angle ALM.

PROPOSITION XXIII. THEOREM.

On the same straight line, and on the same side of it, there cannot be two similar segments of circles not coinciding with one another.

For, if it be possible, on the same chd. BC, and on the same side of it, let there be two sim^r seg^{ts} BAC, BDC, not coinciding.

Then, since the ⊙s cut at B and C,
they cannot cut at any other pt. ....III. 10.
∴ one seg^t falls within the other.



Draw a chd. CA of the inner seg^t and prod. it to meet the outer seg^t in D.
Join BA, BD.

Then, since BAC, BDC are sim^r seg^{ts},
∴ ∠BAC = ∠BDC ....III. Def. 11.
i.e. the ext^r ∠ of a △ = int^r opp. ∠,
which is impossible ....I. 16.

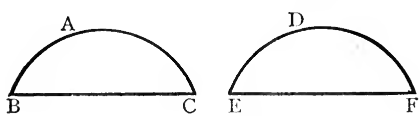
Wherefore, *on the same base* &c. Q.E.D.

PROPOSITION XXIV. THEOREM.

Similar segments of circles on equal straight lines are equal.

Let BAC, EDF be sim^r seg^{ts} of ⊙s on = st. lines BC, EF.

Then shall seg^t BAC = seg^t EDF.



For, if seg^t BAC be applied to EDF so that pt. B fall on E and st. line BC lie along EF,

the pt. C must fall on F, ∵ BC = EF ....Hyp.
And, since BC coincides with EF,
∴ seg^t BAC must coincide with EDF ....III. 23.
∴ seg^t BAC = seg^t EDF ....Ax. 8.

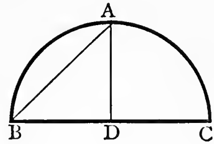
Wherefore, *similar segments* &c. Q.E.D.

PROPOSITION XXV. Problem.

A segment of a circle being given, to describe the circle of which it is a segment.

Let ABC be the given seg^{t}.

It is req^{d} to complete the ⊙.



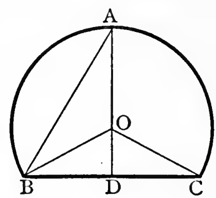
Bisect BC at D............................I. 10.
From D draw DA at rt. ∠s to BC......I. 11.
Join AB.

Then (i) if ∠DBA=∠DAB,

DB=DA..I. 6.
=DC..Constr.
i.e. DA, DB, DC are all equal,
∴ D is the cent..........................III. 9.

And, with D as cent., DB as rad., the ⊙ can be completed.

Again, (ii) if ∠DBA is not=∠DAB,
at pt. B in AB make ∠ABO=∠DAB....I. 23.
Let BO meet AD, or AD prod^{d} at O.
Join OC.

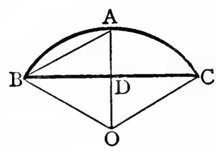


Then in △s OBD, OCD,

∵ { BD=CD ....Constr.
OD is com.
rt. ∠BDO=rt. ∠CDO. }

∴ OB=OC..................I. 4.

And, since ∠ABO=∠OAB............Constr.
∴ OB=OA...................I. 6.



Hence, OA, OB, OC are all equal,
∴ O is cent..............III. 9.

And, with O as cent., rad. OB, the ⊙ can be completed.

Wherefore, *a segment of a circle* &c. Q.E.F.

Cor.—If ∠DBA=∠DAB, the cent. lies in BC and the given seg^{t} is a semicircle.

If ∠DBA < ∠DAB, the cent. lies without the seg^{t}, which is less than a semicircle.

If ∠DBA > ∠DAB, the cent. lies within the seg^{t}, which is greater than a semicircle.

NOTE.

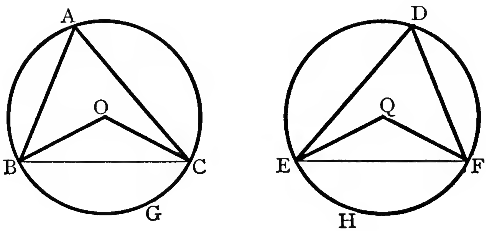
The example to Prop. 1 gives a solution of this problem which is practically simpler than the above, and is often substituted for it.

PROPOSITION XXVI. THEOREM.

In equal circles, equal angles stand on equal arcs, whether at the centres or circumferences.

Let ABC, DEF be equal ⊙s, let BOC, EQF be equal ∠s at the cents., and BAC, EDF at the ○ces.

Then shall arc BGC=arc EHF.



Join BC, EF.

Then in △s OBC, QEF,

∵ { OB=QE..........................III. Def. 1.
OC=QF..........................III. Def. 1.
∠BOC=∠EQF..........................Hyp. }

∴ BC=EF..........................I. 4.

And, since ∠BAC=∠EDF..........................Hyp.

∴ seg^t BAC is sim^r to seg^t EDF..........................III. Def. 11.

But the st. lines BC, EF on which they stand are equal......Above.

∴ seg^t BAC=seg^t EDF..........................III. 24.

And the whole ⊙ABC=whole ⊙DEF..........................Hyp.

∴ rem^g seg^t BGC=rem^g seg^t EHF.

Hence, arc BGC=arc EHF.

Wherefore, *in equal circles* &c. Q.E.D.

NOTES.

The following is another form of the enunciation of this proposition:—
In equal circles, the arcs, on which equal angles stand, must be equal.

Beginners are apt to confuse Props. 26, 27, 28, 29 with one another.

In Prop. 26, prove *arcs* equal. Prop. 27 is converse of 26.

In Prop. 28, prove *arcs* equal. Prop. 29 is converse of 28.

The results proved in these four propositions for *equal* circles hold good for *the same* circle.

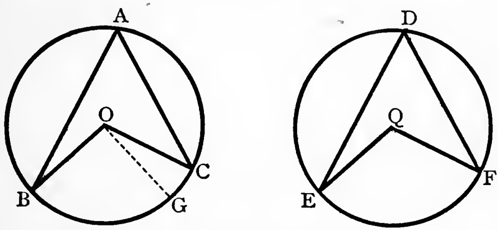
It follows from this prop. that in any circle, or in equal circles, the greater angle stands upon the greater arc.

PROPOSITION XXVII. THEOREM.

In equal circles the angles which stand on equal arcs are equal, whether at the centres or circumferences.

Let ABC, DEF be equal ⊙s, and BC, EF equal arcs.

Then shall ∠BOC=∠EQF at the cents.
and ∠BAC=∠EDF at the ○ces.



For, if ∠BOC is not=∠EQF, one must be >.
If possible, suppose ∠BOC> ∠EQF.
At O in OB make ∠BOG=∠EQF.................................I. 23.

Then, if ∠BOG=∠EQF,
arc BG=arc EF........................III. 26.

But arc BC=arc EF........................Hyp.
∴ arc BG=arc BC.
i.e. the part=the whole,
which is absurd.

Hence, ∠BOC cannot be unequal to ∠EQF,
i.e. ∠BOC=∠EQF.
∴ also ∠BAC=∠EDF........................III. 20.

Wherefore, *in equal circles* &c. Q.E.D.

EXERCISES.

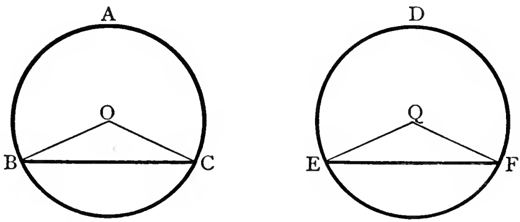
1. Prove Prop. 26 by the method of superposition.
2. Prove Prop. 27 by the method of superposition.
3. AB, AC, AD are chords of a circle; if the angle BAC is the supplement of the angle BAD, then the sum of the arcs on which they stand is equal to the whole circumference.

PROPOSITION XXVIII. THEOREM.

In equal circles, equal straight lines cut off equal arcs, the greater equal to the greater, and the less to the less.

Let ABC, DEF be equal ⊙s, and BC, EF equal chds.

Then shall the major arc BAC=the major arc EDF, and the minor arc BC=the minor arc EF.



Find O and Q the cents. of the ⊙s............................III. 1.
Join OB, OC, QE, QF.

Then, in △s OBC, QEF,

∵ { OB=QE..............................III. Def. 1.
OC=QF..............................III. Def. 1.
BC=EF..............................Hyp. }

∴ ∠BOC=∠EQF.........................I. 8.

Hence, arc BC=arc EF.........................III. 26.

But, whole ○ce ABC=whole ○ce DEF.............Hyp.

∴ rem^{g} arc BAC=rem^{g} arc EDF.

Wherefore, *in equal circles* &c. Q.E.D.

NOTE.

The enunciation may be stated thus:—

In any circle, or in two equal circles, the arcs cut off by equal chords must be respectively equal to one another: the major to the major, and the minor to the minor.

EXERCISES.

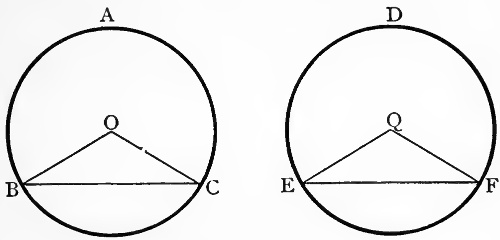
1. Prove Prop. 28 by superposition.
2. APC, BPD are equal chords of a circle ABCD; prove that the arc AB is equal to the arc CD.
3. The arc AB is equal to the arc CD in the circle ABCD; prove that the chord AD is parallel to BC.

PROPOSITION XXIX. THEOREM.

In equal circles, equal arcs are subtended by equal straight lines.

Let ABC, DEF be equal ⊙s, and BC, EF equal arcs, and let BC, EF be joined.

Then shall chd. BC=chd. EF.



Find O and Q the cents. of the ⊙s.............................III. 1.
Join OB, OC, QE, QF.

Then, since arc BC=arc EF.........................Hyp.

∴ ∠BOC=∠EQF.........................III. 27.

Hence, in △s OBC, QEF,

∵ { OB=QE...........................III. Def. 1.
OC=QF...........................III. Def. 1.
∠BOC=∠EQF.........................Above. }

∴ BC=EF............................I. 4.

Wherefore, *in equal circles* &c. Q.E.D.

NOTE.

The enunciation might be stated thus:—*In any circle, or in equal circles, the chords, which cut off equal arcs, must be equal.*

EXERCISES.

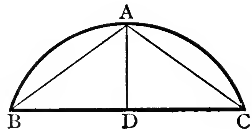
1. Prove this prop. by superposition.
2. The straight lines which join the extremities of two parallel chords of a circle are equal to one another.
3. In the circle ACBD, the arc ACB is equal to the arc CBD; prove that the triangle ACB is equal to the triangle CBD.

PROPOSITION XXX. PROBLEM.

To bisect a given arc of a circle.

Let BAC be the given arc.

It is req^d to bisect it.



Join BC.

Bisect BC at D.........................I. 10.

From D draw DA at rt. $\angle$s to BC, and meeting the ○ce at A...I. 11.

Then shall arc BAC be bisected at pt. A.

Join AB, AC.

Then in △s ABD, ACD,

$\because$ { BD=CD...........................Constr.
AD is com.
rt. $\angle$ADB=rt. $\angle$ADC.

$\therefore$ chd. AB=chd. AC....................I. 4.

Now AD, if prod^d is a diam.................III. 1. Cor.

Hence AB and AC are both minor arcs.

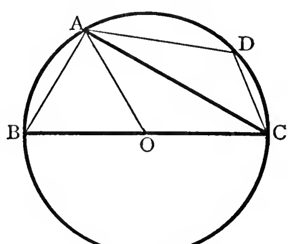
$\therefore$ arc AB=arc AC......................III. 28.

Wherefore, *the given arc* &c. Q.E.F.

PROPOSITION XXXI. THEOREM.

In a circle, the angle in a semicircle is a right angle; the angle in a segment greater than a semicircle is less than a right angle; and the angle in a segment less than a semicircle is greater than a right angle.

Let ABCD be a ⊙, BC a diam. and AC a chd.



Then shall (i) $\angle$ **in semicircle BAC be a rt.** $\angle$;
(ii) $\angle$ **in > seg^t ABC be < a rt.** $\angle$;
(iii) $\angle$ **in < seg^t ADC be > a rt.** $\angle$.

Find O the cent...I. 10.
Join BA, AD, DC, AO.

Then (i) since OA=OB.................................Rad.
∴ ∠OAB=∠OBA.............................I. 5.
And, since OA=OC.................................Rad.
∴ ∠OAC=∠OCA.............................I. 5.
∴ ∠s OAB, OAC=∠s OBA, OCA.
i.e. ∠BAC=∠s OBA, OCA.
But ∠s BAC, OBA, OCA=two rt. ∠s.................I. 32.
∴ ∠BAC is a rt. ∠.

Again, (ii) ∠s BAC, ABC < two rt. ∠s....................I. 17.
But ∠BAC is a rt. ∠..........................Above.
∴ ∠ABC < a rt. ∠.

Also, (iii) since ABCD is a quad[1] inscribed in the ⊙,
∴ ∠s ABC, ADC=two rt. ∠s....................III. 22.
But ∠ABC < a rt. ∠..........................Above.
∴ ∠ADC > a rt. ∠.

Wherefore, *the angle in a semicircle* &c. Q.E.D.

COR.—If one ∠ of a △ be equal to the other two, it is a rt. ∠.

EXERCISES.

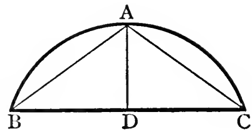
1. If, in a circle, one angle is the complement of another, the sum of the arcs on which they stand is half the circumference.
2. Prove that, if any other point E be taken in the arc BAC (prop. 30), the area of the triangle EBC is less than the area of the triangle ABC.
3. Prove, by a *reductio ad absurdum*, that if a semicircle be described on the hypotenuse of a right-angled triangle as diameter, it will pass through the right angle.
4. What is the *locus* of the vertices of all right-angled triangles described on the same hypotenuse?
5. ABC is a triangle; D is the middle point of BC; L, M, N are the feet of the perpendiculars from A, B, C on the sides; prove that
   (i) the circle, centre D and radius DB, will pass through M and N;
   (ii) the triangle LMN has its angles each double of the complements of those of ABC respectively.
6. The greatest rectangle that can be inscribed in a circle is a square.
7. Draw a straight line at right angles to another straight line, from its extremity, without producing the given line.

PROPOSITION XXX. Problem.

To bisect a given arc of a circle.

Let BAC be the given arc.

It is req[d] to bisect it.



Join BC.

Bisect BC at D.........................I. 10.

From D draw DA at rt. ∠s to BC, and meeting the ⊙ce at A...I. 11.

Then shall arc BAC be bisected at pt. A.

Join AB, AC.

Then in △s ABD, ACD,

∵ { BD=CD..........................Constr.
AD is com.
rt. ∠ADB=rt. ∠ADC.

∴ chd. AB=chd. AC..................I. 4.

Now AD, if prod[d] is a diam................III. 1. Cor.

Hence AB and AC are both minor arcs.

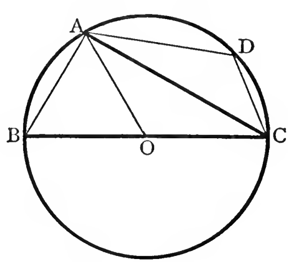
∴ arc AB=arc AC....................III. 28.

Wherefore, *the given arc* &c. Q.E.F.

PROPOSITION XXXI. Theorem.

In a circle, the angle in a semicircle is a right angle; the angle in a segment greater than a semicircle is less than a right angle; and the angle in a segment less than a semicircle is greater than a right angle.

Let ABCD be a ⊙, BC a diam. and AC a chd.



Then shall (i) ∠ in semicircle BAC be a rt. ∠;

(ii) ∠ in > seg[t] ABC be < a rt. ∠;

(iii) ∠ in < seg[t] ADC be > a rt. ∠.

Find O the cent. .... I. 10.
Join BA, AD, DC, AO.

Then (i) since OA=OB .... Rad.
∴ ∠OAB=∠OBA .... I. 5.
And, since OA=OC .... Rad.
∴ ∠OAC=∠OCA .... I. 5.

∴ ∠s OAB, OAC=∠s OBA, OCA.
i.e. ∠BAC=∠s OBA, OCA.

But ∠s BAC, OBA, OCA=two rt. ∠s .... I. 32.
∴ ∠BAC is a rt. ∠.

Again, (ii) ∠s BAC, ABC < two rt. ∠s .... I. 17.
But ∠BAC is a rt. ∠ .... Above.
∴ ∠ABC < a rt. ∠.

Also, (iii) since ABCD is a quad[1] inscribed in the ⊙,
∴ ∠s ABC, ADC=two rt. ∠s .... III. 22.
But ∠ABC < a rt. ∠ .... Above.
∴ ∠ADC > a rt. ∠.

Wherefore, *the angle in a semicircle* &c. Q.E.D.

Cor.—If one ∠ of a △ be equal to the other two, it is a rt. ∠.

EXERCISES.

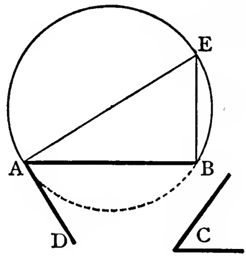
1. If, in a circle, one angle is the complement of another, the sum of the arcs on which they stand is half the circumference.
2. Prove that, if any other point E be taken in the arc BAC (prop. 30), the area of the triangle EBC is less than the area of the triangle ABC.
3. Prove, by a *reductio ad absurdum*, that if a semicircle be described on the hypotenuse of a right-angled triangle as diameter, it will pass through the right angle.
4. What is the *locus* of the vertices of all right-angled triangles described on the same hypotenuse?
5. ABC is a triangle; D is the middle point of BC; L, M, N are the feet of the perpendiculars from A, B, C on the sides; prove that
   (i) the circle, centre D and radius DB, will pass through M and N;
   (ii) the triangle LMN has its angles each double of the complements of those of ABC respectively.
6. The greatest rectangle that can be inscribed in a circle is a square.
7. Draw a straight line at right angles to another straight line, from its extremity, without producing the given line.

NOTES.

The following construction for Prop. 33 is practically simpler than that in the text:—

Let AB be the given line, C the given ∠.

At A make ∠BAD=∠C .... I. 23.
Draw AE at rt. ∠s to AD................. I. 11.
From B draw BE at rt. ∠s to AB, meeting AE at E................. I. 11.
On AE as diam. desc. a ⊙.

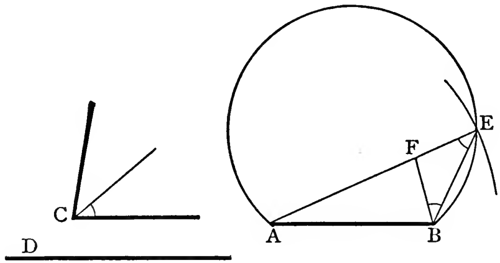


Then it can be proved, as in Ex. 3 on III. 31, that B is on the ○ce, and the rest of the proof follows as before.

EXAMPLE.

To construct a triangle, having given the base, the vertical angle, and the sum of the sides.

Let AB be the given base, C the given vertical ∠, D the given sum of sides.



Bisect ∠C.. I. 9.
On AB desc. a seg[t] containing an ∠=half ∠C....................... III. 33.
With cent. A and rad.=D, desc. an arc cutting the seg[t] at E.
Join AE, EB.
At B make ∠EBF=∠AEB, and let BF meet AE at F.

Then FAB shall be the △ req[d].

EXERCISES.

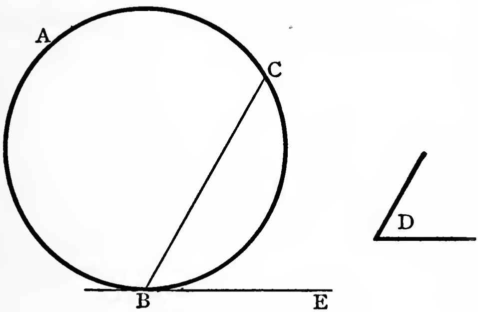
1. Write out the proof of the example given above.
2. Construct a triangle, having given the base, the vertical angle, and the length of the line joining the vertex to the middle point of the base.
3. Construct a triangle having given the base, the vertical angle, and the altitude of the triangle.
4. Construct a triangle having given the base, the vertical angle, and the difference of its sides.

PROPOSITION XXXIV. Problem.

From a given circle to cut off a segment containing an angle equal to a given rectilineal angle.

Let ABC be the given ⊙, and D the given ∠.

It is req^d^ to cut off from ⊙ ABC a seg^t^ containing an ∠ = ∠ D.



At any pt. B on the ○ce draw the tang. BE....................III. 17.

At pt. B in BE make ∠ EBC = ∠ D ....I. 23.

Then shall BAC be the seg^t^ req^d^.

For, since BC is a chd. drawn from the pt. of contact of tang. BE,

∴ ∠ EBC = ∠ in alt. seg^t^ BAC.........III. 32.

But ∠ EBC = ∠ D ....Constr.

∴ ∠ in seg^t^ BAC = ∠ D.

Wherefore, *from the given circle* &c. Q.E.F.

EXERCISES.

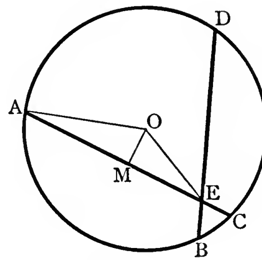
1. From a given circle cut off a segment similar to a given segment.
2. Through a given point within a circle, draw a chord which shall cut off a segment containing a given angle.
3. Through a given point without a circle draw a secant which shall cut off a segment similar to a given segment.
4. State and prove the converse of Prop. 32.

PROPOSITION XXXV. Theorem.

If two straight lines cut one another within a circle, the rectangle contained by the segments of one of them shall be equal to the rectangle contained by the segments of the other.

In the ⊙ ABCD let the chds. AC, BD cut at E.

Then shall rect. AE, EC=rect. BE, ED.



Find O the cent. of the ⊙ ....III. 1.

From O draw OM ⊥ to AC.......I. 12.

Join OA, OE.

Then, since OM is drawn from the cent. ⊥ to AC,

∴ M is mid. pt. of AC.....III. 3.

And, since AC is bisected at M, and divided unequally at E,

∴ rect. AE, EC with sq. on EM=sq. on AM............II. 5.

To each add the sq. on OM,

∴ rect. AE, EC with sqs. on EM, OM=sqs. on AM, OM.

i.e. rect. AE, EC with sq. on OE=sq. on OA ....I. 47.

=sq. on a rad.

In the same way, if a ⊥ from O be drawn to BD, it may be shown that

rect. BE, ED with sq. on OE=sq. on a rad.

Hence, rect. AE, EC with sq. on OE=rect. BE, ED with sq. on OE.

Take away the com. sq. on OE,

∴ rect. AE, EC=rect. BE, ED.

Wherefore, *if two straight lines* &c. Q.E.D.

NOTES.

Four cases are dealt with by Euclid in this prop.:—

Case i.—When both lines pass through the centre;

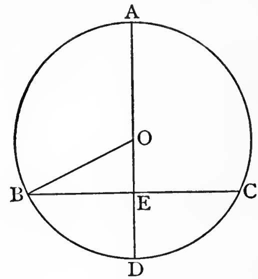
Case ii.—When one passes through the centre and cuts the other at rt. ∠s;

Case iii.—When one passes through the centre and cuts the other, but not at right angles;

Case iv.—When the position of the chords is entirely unrestricted, as above.

The last and most general case only has been inserted in the text, for it includes the special cases, while they add greatly to the length of the demonstration. The following separate proof of Case ii. is, however, given here as an example:—

If, in a circle, a chord be bisected at right angles by a diameter, the square on half the chord is equal to the rectangle contained by the segments into which it divides the diameter.



In the ⊙ABC let the chd. BC be bisected at rt. ∠s in E by the diam. AD.

Then shall sq. on BE = rect. AE, ED.

Find O the cent. .... I. 10.
Join OB.

Then, since AD is bisected at O, and divided unequally at E,

∴ rect. AE, ED with sq. on OE = sq. on OD .... II. 5.
= sq. on OB .... Rad.
= sqs. on OE, BE .... I. 47.

Take away the com. sq. on OE,

∴ rect. AE, ED = sq. on BE. Q.E.D.

EXERCISES.

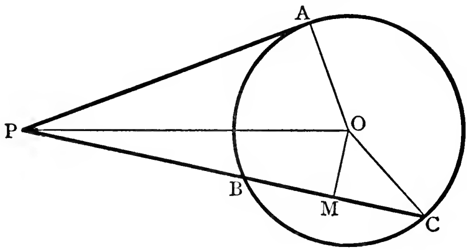
1. Write out an independent proof of Case iii. as stated in the note.
2. In the figure of Ex. X. page 88, prove that rect. AO, OL = rect. BO, OM.
3. If a chord of a circle when produced meet a diameter produced at a point outside the circle, prove that the rectangle contained by the segments of the chord is equal to that contained by the segments of the diameter.
4. Prove the same property when two chords, being produced, intersect outside a circle.
5. If, through any point in the common chord of two circles which intersect, there be drawn two other chords, one in each circle, the four ends of these latter chords will lie on the circumference of a circle.
6. If a diameter of a circle be produced to any point, the rectangle contained by the whole line so produced, and the part of it without the circle, shall be equal to the square on the tangent drawn from the point to the circle.
7. State, and prove by a *reductio ad absurdum*, the converse of Prop. 35.
8. In the above figure, prove that if DC be joined, $DC^2 = DE \cdot DA$.

PROPOSITION XXXVI. THEOREM.

If from a point without a circle two straight lines be drawn, one of which cuts the circle and the other touches it; the rectangle contained by the whole line which cuts the circle and the part of it without the circle, shall be equal to the square on the line which touches it.

Let ABC be a ⊙, and from the external pt. P let the tang. PA, and the secant PBC, be drawn.

Then shall rect. PB, PC=sq. on PA.



Find O the cent. .... III. 1.
From O draw OM ⊥ to BC .... I. 12.
Join OP, OA, OC.

Then, since OM is a ⊥ from the cent.

∴ M is mid. pt. of BC .... III. 3.

And, since CB is bisected at M and prod[d] to P,

∴ rect. PB, PC with sq. on CM=sq. on PM .... II. 6.

Add sq. on OM to each,

∴ rect. PB, PC with sqs. on CM, OM=sqs. on PM, OM.

i.e. rect. PB, PC with sq. on OC=sq. on OP .... I. 47.

=sqs. on PA, OA .... I. 47.

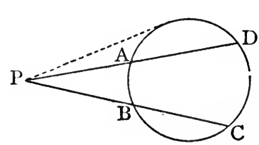
But sq. on OC=sq. on OA .... Rad.

∴ rect. PB, PC=sq. on PA .... Ax. 3.

Wherefore, *if from a point* &c. Q.E.D.

COR.—If, from a point without a circle. two straight lines be drawn cutting the circle, the rectangles contained by their segments are equal,

i.e. rect PA, PD=rect. PB, PC, for each is equal to the sq. on the tang. from P to the ⊙.

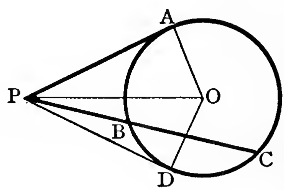


PROPOSITION XXXVII. Theorem.

If from a point without a circle there be drawn two straight lines one of which cuts the circle and the other meets it, and if the rectangle contained by the whole line which cuts the circle and the part of it without the circle be equal to the square on the line which meets the circle, the line which meets shall touch the circle.

Let ABC be a ⊙, and, from the external pt. P, let PBC be drawn cutting the ⊙, and PA meeting it, and let the rect. PB, PC be equal to sq. on PA.

Then shall PA be a tang.



From P draw the tang. PD.....III. 17.
Find O the cent....................III. 1.
Join OA, OP, OD.

Then, since OD is drawn from the cent. to the pt. of contact,

∴ ODP is a rt. ∠...........III. 18.

And, since PD is a tang.

∴ rect. PB, PC=sq. on PD....................III. 36.
But rect. PB, PC=sq. on PA....................Hyp.
∴ sq. on PD=sq. on PA.
∴ PD=PA.

Hence, in △s OPD, OPA,

∵ { OD=OA............................Rad.
PD=PA............................Above.
OP is com. }

∴ ∠ODP=∠OAP.......................I. 8.
But ∠ODP is a rt. ∠...........................Above.
∴ ∠OAP is a rt. ∠,
∴ PA touches the ⊙.......................III. 16. Cor.

Wherefore, *if from a point* &c. Q.E.D.

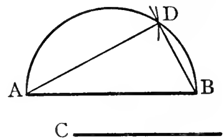
EXERCISES.

1. State and prove the converse of the corollary to Prop. 36.
2. Prove that, if two circles cut, their common chord produced bisects their common tangents.
3. Tangents drawn from any point in their common chord produced to two circles which intersect, are equal.

MISCELLANEOUS EXAMPLES.

I. *To find a square which shall be equal to the difference of two given squares.*

Let AB be a side of the greater sq., and C a side of the less. On AB desc. a semicircle. With cent. A and rad.=C desc. an arc cutting the semicircle in D. Join DA, DB.



Then ∠ADB in a semicircle, is a rt. ∠...III. 31.

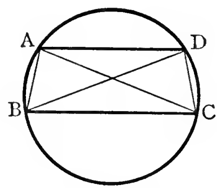
∴ sq. on DB=diff. of sqs. on AB and AD (or C)............I. 47.

Q.E.F.

II. *In a circle, the arcs intercepted by parallel chords are equal.*

Let AD, BC be || chds. of ⊙ABCD.

Join AB, AC, DB, DC.



Then, ∠BAC=∠BDC.........III. 21.

And ∠CAD=∠CBD.........III. 21.

=∠BDA.........I. 29 (i).

∴ ∠s BAC, CAD=∠s BDC, BDA.

i.e. ∠BAD=∠CDA.

∴ arc BCD=arc ABC........III. 26.

Take away the com. arc BC.

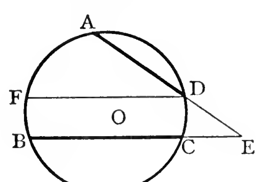
∴ arc DC=arc AB.

Q.E.D.

III. *If two chords of a circle be produced to meet without the circle, the angle they contain is equal to half that subtended at the centre by the difference of the intercepted arcs.*

Let the chds. AD, BC of ⊙ABCD be prod[d] to meet at E.

Through D draw DF || to BC.



Then, since arc FB=arc DC.........Ex. II.

∴ arc AF is diff. of arcs AB and DC.

And, since DF is || to BC,

∴ ∠ADF=∠AEB......... I. 29.

But ∠ADF at the ○ce is half ∠AOF at the cent.........III. 20.

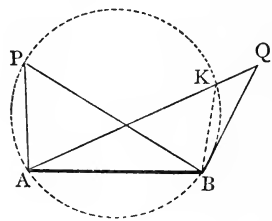
∴ ∠AEB is half ∠AOF at the cent. which stands on the diff. of the intercepted arcs AB and DC.

Q.E.D.

IV. (i) *If a straight line subtend equal angles at any points, a circle can be described the circumference of which will pass through the extremities of the line and through each of the points.*

Let AB be the line, and P, Q, pts. such that ∠APB=∠AQB.

Then shall the ⊙ desc^d through A, B, P pass through Q.



For, if not, if possible, let the ⊙ round A, B, P not pass through Q, but cut AQ, or AQ prod^d, in K, and join KB.

Then, since APB, AKB are ∠s in the same segt.,

∴ ∠APB=∠AKB ......III. 21.

But ∠APB=∠AQB........Hyp.

∴ ∠AKB=∠AQB.

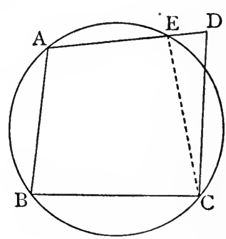
i.e. ext^r ∠ of a △ = int^r opp. ∠

which is impossible............................I. 16.

Hence Q must lie on the ○ce of the ⊙ through A, B, P.

Q.E.D.

IV. (ii) *If two opposite angles of a quadrilateral are together equal to two right angles, a circle can be described about the figure.*



In quad^l ABCD let ∠s ABC, ADC=two rt. ∠s.

Desc. a ⊙ through pts. A, B, C.........Ex. 8. p. 86.

If possible, suppose this ⊙ does not pass through D, but cuts AD (or AD prod^d) at E. Join EC.

Then ∠s ABC, AEC=two rt. ∠s.......III. 22.

But ∠s ABC, ADC=two rt. ∠s..... Hyp.

∴ ∠s ABC, AEC=∠s ABC, ADC.

Take away the com. ∠ABC,

∴ rem^g ∠AEC=rem^g ∠ADC,

which is impossible.. ....I. 16.

∴ D must lie on the ○ce of the ⊙ through A, B, C. Q.E.D.

IV. (i), which is the converse of III. 21, and IV. (ii), the converse of III. 22, are very important. IV. (i) may be enunciated in various ways, of which the following are examples:—

The locus of the vertices of all triangles on the same base, and having equal vertical angles, is a circle.

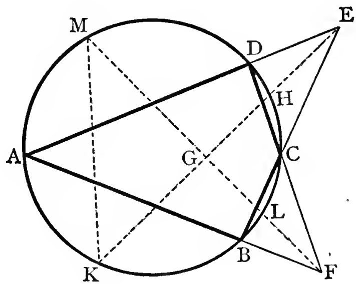
If four points be so situated that the line joining any two subtends equal angles at the other two, the four points lie on a circle.

V. *If the opposite pairs of sides of a quadrilateral in a circle be produced to meet and the angles so formed be bisected, the bisectors are at right angles to one another.*

Let ABCD be a quad^{l} in a ⊙.

Let AD and BC meet at E, and AB and DC at F.

Let the bisectors of the angles at E, F cut one another at G, and the ○ce at H, K, L, M.



Then ∠AEK=∠ at ○ce which stands on diff. of arcs AK and DH...Ex. III.

And ∠BEK=∠ at ○ce which stands on diff. of arcs BK and CH...Ex. III.

But ∠AEK=∠BEK............................Hyp.

∴ diff. of arcs AK and DH=diff. of arcs BK and CH.......III. 26.

Hence, sum of arcs AK and CH=sum of arcs BK and DH.

Similarly, it may be shown that

sum of arcs AM and CL=sum of arcs DM and BL.

∴ sum of arcs AK, CH, AM, CL=sum of arcs BK, DH, DM, BL..Ax. 2.

i.e. sum of arcs MK, HL=sum of arcs MH, KL.

Hence, sum of arcs MH, KL=half the ○ce of the ⊙.

∴ sum of the ∠s at the ○ce which stand on them=a rt. ∠...III. 31.

i.e. ∠s MKH, KML=a rt. ∠.

But ∠MGE=∠s MKH, KML...............I. 32.

∴ ∠MGE is a rt. ∠.

i.e. EG, FG cut at rt. ∠s.

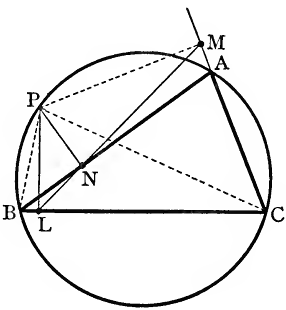
Q.E.D.

VI. *If, from any point on the circumference of the circle circumscribing a triangle, perpendiculars be drawn to the sides, the feet of these perpendiculars lie in a straight line.*

Let P be any pt. in the ○ce of the ⊙ circumscribing the △ABC.

From P draw PL ⊥ to BC and PN ⊥ to AB.
Join LN, and prod. LN to meet AC prod^d at M.

Then PM shall be ⊥ to AC.



Since PNB, PLB are rt. ∠s,

∴ a ⊙ can be desc^d round PBLN .... Ex. IV.

∴ ∠PBN = ∠PLN .... III. 21.

i.e. ∠PBA = ∠PLM.

But ∠PBA = ∠PCM .... III. 21.

∴ ∠PLM = ∠PCM,

∴ the pts. P, L, C, M lie on a ⊙ .... Ex. IV.

And, since PLC is a rt. ∠,

∴ PMC is also a rt. ∠ .... III. 22.

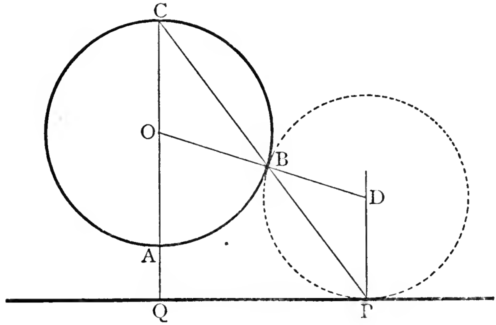
Hence L, M, N, the feet of the ⊥s, lie in a st. line.

Q.E.D.

The above theorem is sometimes referred to as *Simson's Theorem*, and the line LNM as *Simson's line*.

VII. *To describe a circle which shall touch a given circle, and a given straight line at a given point.*

Let P be the given pt., PQ the given st. line, and ABC the given ⊙, cent. O.



Draw OQ ⊥ to PQ, and let OQ prod^{d} meet the ○ce in C.
Join CP cutting the ○ce in B.
Join OB, and from P draw PD at rt. ∠s to PQ and meeting OB prod^{d} in D.

Then D shall be the cent. of the ⊙ req^{d}.

For, since the ∠s at P, Q are rt. ∠s,

∴ PD is ∥ to QC..........................I. 28.
∴ ∠DPB=alt. ∠OCB..........................I. 29.
=∠OBC ....I. 5.
=∠DBP ....I. 15.
∴ DP=DB..........................I. 6.

Hence, the ⊙ desc^{d} with cent. D and rad. DP will pass through B.

Also, since PQ is at rt. ∠s to rad. DP.................Constr.
∴ PQ touches this ⊙.................III. 16, Cor.

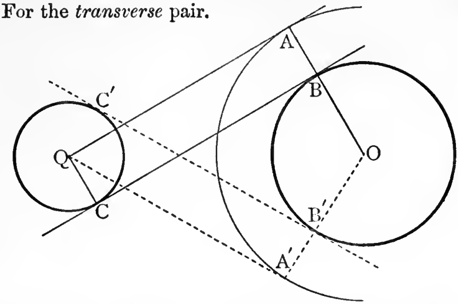
And, since OD joining the cents. passes through B.................Constr.
∴ ⊙ABC touches this ⊙.................III. 12.

Q.E.F.

VIII. *To draw common tangents to two given circles.*

Let O be the cent. of the >, Q of the < ⊙.

Case 1. For the *transverse* pair.



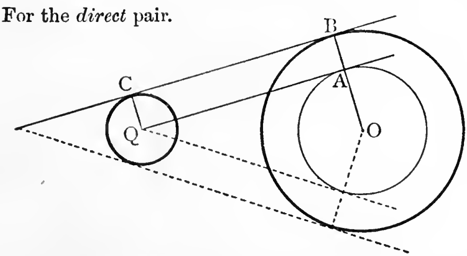
With cent. O, and rad.=*sum* of radii of ⊙s O and Q, desc. a ⊙.
From Q draw QA a tang. to this ⊙.
Join OA, cutting the ○ce of the given ⊙ in B.
From Q draw QC at rt. ∠s to QA, and join BC.

BC shall be a com. tang.

For, since QA is a tang. to ⊙AA′,

∴ QAO is a rt. ∠ .... III. 16, Cor.
But AQC is also a rt. ∠ .... Constr.
∴ AB is ∥ to QC .... I. 28.
Also AB=QC .... Constr.
∴ QA is=and ∥ to BC .... I. 33.
i.e. QCBA is a ▱,
∴ ∠QCB=∠QAB .... I. 34.
=a rt. ∠ .... Above.
and ∠OBC=∠QAB .... I. 29.
=a rt. ∠,
∴ BC is a tang. to both ⊙s .... III. 16, Cor.

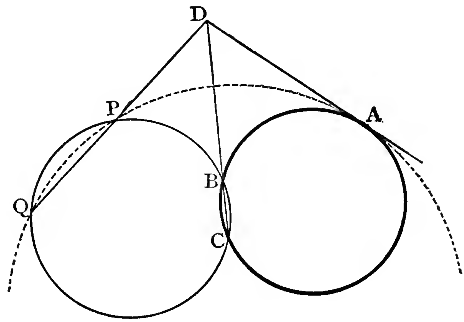
Case 2. For the *direct* pair.



With cent. O, and rad.=*diff.* of radii of ⊙s O and Q, desc. a ⊙, and complete the construction and proof as before.

IX. *To describe a circle which shall pass through two given points and touch a given circle.*

Let P, Q be the given pts., ABC the given ⊙.



Through P, Q describe any ⊙ cutting ABC in B, C.

Join PQ, BC and let them be prod[d] to meet at D.

From D draw DA a tang. to ⊙ABC....................................II. 17.

Desc. a ⊙ to pass through P, Q, A.................................... III. 1, Ex.

PQA shall be the reqd ⊙.

For, since DPQ, DBC cut ⊙QPBC,

∴ rect. DP, DQ=rect. DB, DC..............III. 36, Cor.

And, since DBC cuts, and DA touches ⊙ABC,

∴ rect. DB, DC=sq. on DA.................III. 36.

∴ rect. DP, DQ=sq. on DA.

∴ DA touches ⊙ QPA at pt. A................III. 37.

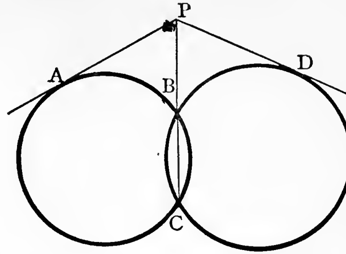
Hence, since ⊙s QPA, ABC have a com. tang. at A,

∴ the ⊙s touch at A.

Q.E.F.

X. *To find the locus of a point from which the tangents to two circles are equal.*

CASE 1. When the ⊙s intersect.



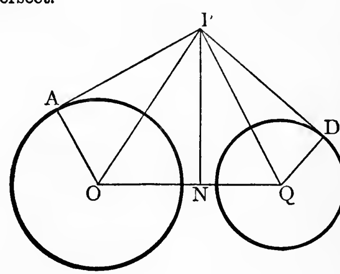
Let ABC, DBC be the ⊙s.
Take any pt. P in their com. chd. BC prod^{d}, and draw PA, PD tangs. to the ⊙s.

Then, in ⊙ABC,
rect. PB, PC=sq. on PA...III. 36.

And, in ⊙DBC,
rect. PB, PC=sq. on PD...III. 36.

∴ sq. on PA=sq. on PD.
∴ PA=PD.
Hence the prod^{d} com. chd. is the req^{d} locus.

CASE 2. When the ⊙s do not intersect.



Find O, Q the cents...........III. 1.
Join OQ.
Divide OQ at N so that the diff. of the sqs. on ON and QN may=diff. of sqs. on the radii. (See Ex. VII. p. 124, and Ex. I. p. 176.)
Draw NP at rt. ∠s to OQ......I. 11.

PN shall be the req^{d} locus.

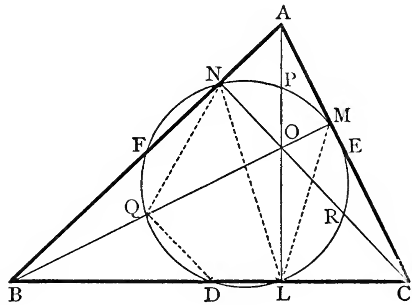
Take any pt. P in PN,
Draw PA, PD tangs. to the ⊙s...III. 17.
Join OP, OA, QP, QD.

Then, $PA^2=OP^2-OA^2$......................I. 47.
$=PN^2+ON^2-OA^2$.............I. 47.
And $PD^2=QP^2-QD^2$......................I. 47.
$=PN^2+QN^2-QD^2$.............I. 47.
But $ON^2-QN^2=OA^2-QD^2$......................Constr.
∴ $ON^2-OA^2=QN^2-QD^2$.
∴ $PN^2+ON^2-OA^2=PN^2+QN^2-QD^2$,
i.e. $PA^2=PD^2$...............................Above.
∴ PA=PD.
Hence the st. line PN is the req^{d} locus.

Q.E.F.

This st. line is called the *Radical Axis* of the circles.

XI. *In any triangle ABC, if O be the orthocentre, and L, M, N the feet of the perpendiculars, the circle described through L, M, N will* (i) *bisect OA, OB, OC, and will also* (ii) *pass through the middle points D, E, F, of the sides of the triangle.*



For, (i) since OMC, OLC are rt. ∠s,

∴ a ⊙ will go round O, M, C, L..................Ex. IV.

∴ ∠OCM=∠OLM..........................III. 21.

And, since ONB, OLB are rt. ∠s,

∴ a ⊙ will go round O, N, B, L................ ..Ex. IV

∴ ∠OBN=∠OLN..........................III. 21.

Also, since BNC, BMC are rt. ∠s,

∴ a ⊙ will go round B, N, M, C..................Ex. IV.

∴ ∠OBN=∠OCM.. ....III. 21.

Hence, ∠OLM=∠OLN.

Now, OB is diam. of ⊙ round O, N, B, L...........III. 31.

and Q, the mid. pt. of OB, is its cent.

∴ ∠OQN=twice ∠OBN.....................III. 20.

=twice ∠OLN.....................Above.

=∠NLM..................... ...Above.

∴ Q is a pt. on the ⊙ through L, M, N .. ..Ex. IV

Similarly it may be shown that the middle pts. P, R of OA, OC lie on this circle.

Again (ii), if QD be joined, since Q is mid. pt. of OB, and D is mid. pt. of BC,

∴ QD is ∥ to OC .... Ex. p. 69.

∴ ∠QDB = ∠OCL .... I. 29.

= ∠OML (or QML) .... III. 21.

Add ∠QDL to each,

∴ ∠s QDB, QDL = ∠s QML, QDL.

But ∠s QDB, QDL = two rt. ∠s .... I. 13.

∴ ∠s QML, QDL = two rt. ∠s.

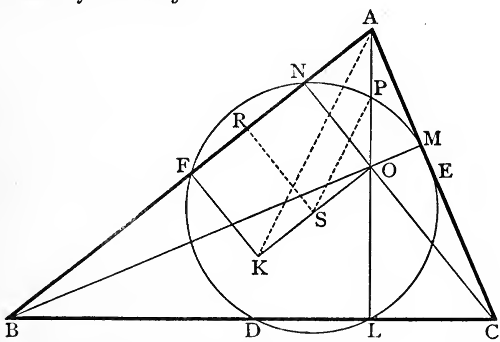
∴ D is on the ○ce of the ⊙ through LMQ..... ..... Ex. IV.

Similarly it may be shown that E, F lie on this ⊙.

Q.E.D.

This circle, which passes through the points L, M, N; P, Q, R; D, E, F, is called the *Nine-points circle.*

XII. *The centre of the Nine-points circle is on the line joining the orthocentre to the centre of the circle circumscribing the triangle, and bisects this line; and the radius of the Nine-points circle is half the radius of the circumscribed circle of the triangle.*



(i) From E, F draw EK, FK at rt. ∠s to AC, AB,

then K is cent. of ⊙ circumscribing △ABC................ Ex. VIII. p. 86.

And cent. of Nine-points ⊙ is found by bisecting its chds. FN, EM and drawing ⊥s to these chds.................... III. 1, Ex.

Hence, if RS be one of these ⊥s, since RS is ∥ to ON and KF,

∴ RS bisects OK at S .... Ex. VIII. p. 125.

Similarly, the other ⊥ will also bisect OK,

∴ they meet in OK at S its mid. pt.

Again (ii), if SP, KA be joined,

since S is mid. pt. of OK, and P of OA...... Ex. XI.

∴ SP = half of KA.................... Ex. VIII. (i) p. 69.

Q.E.D.

MISCELLANEOUS EXERCISES.

1. Prove that two circles cannot have a common arc.
2. Show that straight lines drawn from the same point to touch two circles having the same centre cannot be equal.
3. Through a given point within a circle draw two equal chords at right angles to each other.
4. Two points are taken on the circumference of a circle at equal distances from one extremity of a given diameter. Show that they are equidistant from the other extremity.
5. Two circles touch one another internally at A; AB, AC are chords of one of them meeting the other in D, E. Prove that DE is parallel to BC.
6. From a given point as centre describe a circle which shall cut a given circle at right angles. When is this impossible?
7. Show that the centre of a circle which cuts two given circles at right angles lies on a fixed straight line.
8. Describe a circle which shall have a given radius and touch two given straight lines. In what case is this impossible?
9. Describe a circle to pass through a given point and touch a given circle at a given point.
10. Describe a circle to touch a given circle, and a given straight line at a given point.
11. Of all triangles on the same base and having equal vertical angles, the isosceles has the greatest area.
12. Given the hypotenuse and the length of the perpendicular from the right angle upon it, construct the right-angled triangle.
13. A circle is described on the radius of another circle as diameter. Show that any chord of the greater through the point of contact is bisected by the circumference of the less.
14. Two circles intersect in P and Q. Any line through P cuts the circles in R and S. Show that the angle RQS is constant.
15. Given the base BC of a triangle, the length of the side AC, and the length of the perpendicular from B on AC. Construct the triangle.
16. If two straight lines AEB, CED in a circle intersect in E, the angles subtended by AC and BD at the centre are together double of the angle AEC.
17. AB, CD are chords of a circle at right angles to one another. Prove that the sum of the arcs AC, BD is equal to the sum of the arcs AD, BC.

18. If two straight lines intersect each other in a circle, the sum of the arcs cut off between their extremities is the same as that cut off by any two lines respectively parallel to them, and intersecting each other within the circle.

19. Describe, when possible, a circle to touch three given straight lines.

20. Find the locus of the middle points of all chords of a circle which pass through a fixed point.

21. Two equal circles intersect at right angles. Show that the length of their common chord bears to their radius the ratio of $\sqrt{2}$ to 1.

22. From a point P tangents PA, PB are drawn to a circle ABC; DCE is any other tangent meeting PA, PB in D and E. Prove that the perimeter of the triangle PDE is constant whatever be the position of the point C.

23. If two circles touch, either internally or externally, any straight line through the point of contact cuts off similar segments.

24. Two circles PAB, CABD intersect in the points, A, B. PAC, PBD are straight lines drawn from any point P on the circumference of the first circle to meet the second in the points C, D. Prove that the arc CD is of constant length.

25. AA′, BB′, CC′ are parallel chords of a circle. Prove that the triangles ABC, A′B′C′ are equal.

26. If a straight line be drawn cutting any number of concentric circles, the segments so cut off are not similar.

27. Find a point in a given straight line from which the tangent drawn to a given circle shall be of given length.

28. The lines which bisect the vertical angles of all triangles on the same base, and having the same vertical angle, intersect at a point.

29. If two opposite sides of a quadrilateral figure inscribed in a circle be equal, the other two are parallel.

30. The angles subtended at the centre of a circle by two opposite sides of a quadrilateral figure circumscribed about it, are together equal to two right angles.

31. Two circles intersect. Draw through one of the points of intersection a secant of both circles which shall be bisected at the point.

32. AB is a diameter of a circle, CD a chord perpendicular to it. A straight line through A cuts the circle in E, and CD produced, in F. Prove that the angles AEC and DEF are equal.

33. Two circles intersect at A and B, a common tangent meets them in C and D. Prove that the bisectors of the angles ACB and ADB are at right angles to each other.

34. One circle is wholly within another and contains the centres of both. Find the greatest and least chords of the outer circle which touch the inner circle.

35. A straight line intersects one circle in P and Q, and a second circle in R and S. If the tangents at P and R are parallel, the tangents at Q and S are also parallel.

36. Two circles intersect at A and B. In the circumference of one of the circles ABC any point P is taken, and the straight lines PA, PB, produced if necessary, meet the circumference of the other circle at Q, R. Show that the chord QR is of constant length whatever may be the position of the point P.

37. In a given straight line determine a point at which two straight lines drawn to it from given fixed points, both on the same side of the given line, shall contain the greatest angle.

38. If two triangles have their bases, areas, and vertical angles equal, they are equal in all respects.

39. Two circles AOB, COD touch externally at O; two straight lines AOC, BOD are drawn cutting the circles. Prove that AB, CD are parallel.

40. Two given circles intersect in O, draw through O a chord to meet the circles in A, B. Find when AB is greatest.

41. A regular pentagon is inscribed in a circle, and every second angular point joined by a straight line. Prove that these lines will form by their intersections an equiangular pentagon.

42. Describe a circle to pass through two given points and touch a given straight line.

43. Describe a circle which shall cut off three chords, each equal to a given straight line, from the sides of a given triangle.

44. If two circles touch one another externally, and through the point of contact any two straight lines be drawn cutting the circles, the tangents at the points of section form a parallelogram.

45. AB is the diameter of a circle, C is any point on the circumference; AC, BC, produced, meet the tangents at B, A in D, E, and the tangent at C meets the tangents at B, A in F, G. Prove that FG is half the sum of BD and AE.

46. A, B are the points of intersection of two circles. Through any point C on the common chord AB between A and B, a straight line is drawn cutting one of the circles in D and E; and through the same point C is drawn a straight line FG cutting the other circle in F and G. Show that the angles FDG and FEG are together equal to two right angles.

47. Given three points in a plane; show how, with only a ruler and a square, to find any number of points on the circumference of the circle which could be described through these points; and find the centre of this circle.

48. AB, CD are two chords of a circle intersecting at right angles in E; from C a line CF is drawn perpendicular to AD, and cutting AB at G. Prove that GE is equal to EB.

49. AB is a straight line divided in two equal parts in C and two unequal parts in D; at C and D circles, whose radii are equal to CD, DB respectively, touch the straight line AB. Show that, if O and Q be their respective centres, then $AQ^2 = 2AO^2$.

50. DEF is a circle which touches the side AD of the parallelogram ABCD; DC produced meets the circle in E, BE produced meets the circle in F, and BC produced cuts DF in G. Prove that G, C, E, F lie on a circle.

51. If two circles touch one another at A, and BC is a common tangent touching them in B and C, show that the circle described on BC as diameter touches at A the line joining their centres.

52. ABCD is a quadrilateral inscribed in a circle, AB, DC produced meet in E, and a circle is described round the triangle AED. Show that the tangent at E to this circle is parallel to BC.

53. From one of the points of intersection of two equal circles each of which passes through the centre of the other, a line is drawn to intersect the circles in two other points. Prove that these points form an equilateral triangle with the other point of intersection of the circles.

54. Through each of the points of intersection of two circles any straight line is drawn. If these lines cut one circle in A and B and the other in C and D, then AB is parallel to CD.

55. Two points D, E are taken in the base BC of a triangle ABC, so that the tangents from B and C to the circle circumscribing the triangle ADE are equal. Show that D and E must be equidistant from B and C respectively.

56. ABC is a triangle inscribed in a circle, and the arcs AB, AC are bisected at G and H; if GH cuts AB in D and AC in E, then will ADE be an isosceles triangle.

57. Two circles cut at A and B. At A tangents AC, AD are drawn to each circle and terminated by the circumference of the other. If BC, BD be joined, show that AB, or AB produced, bisects the angle CBD.

58. The locus of the point of intersection of tangents to a circle at the extremities of a chord through a fixed point is a straight line parallel to the shortest of the chords.

59. Given the base, vertical angle, and radius of the inscribed circle, construct the triangle.

60. If two chords produced intersect at right angles without a circle, the sum of the squares on the four segments shall be equal to the square on the diameter.

61. If from two fixed points on the circumference of a circle straight lines be drawn intercepting an arc of given length and meeting without the circle, the locus of their point of intersection is a circle.

62. The point, in which the bisector of an exterior angle of a triangle again cuts the circumscribed circle, is equidistant from the other two angular points of the triangle.

63. Two equal circles touch one another externally, and through the point of contact chords are drawn, one to each circle, at right angles to each other. Prove that the straight line joining the other extremities of these chords is equal and parallel to the straight line joining the centres of the circles.

64. From a point P outside a circle a perpendicular PN is drawn to a diameter AB, such that AN is equal to the tangent from P to the circle, and BA is produced to N′ so that AN′ is equal to AN. If a circle be described on BN′ as diameter and a perpendicular to AB be drawn from A cutting this circle in K, prove that AK and PN are equal.

65. With a given radius, to describe a circle touching two given circles.

66. If two circles touch each other, and parallel diameters be drawn, then lines joining the opposite extremities of these diameters will pass through the point of contact.

67. Find a point within an acute-angled triangle from which, if straight lines be drawn to the angles of the triangle, they shall make equal angles with one another.

68. ACDB is a semicircle whose diameter is AB, and AD, BC are any chords intersecting at P. Prove that the sum of the rectangles DA, AP and CB, BP$=AB^2$.

69. A flag-staff of given height is erected on a tower whose height is also given. Find the distance from the foot of the tower at which the flag-staff subtends the greatest angle.

70. The four common tangents to two circles which do not meet intersect, two and two, on the line which joins the centres.

71. The radii of two circles which do not cut are 13 and 5 inches respectively. Draw a straight line to cut the circles so that the chords intercepted shall be 10 and 6 inches long respectively.

72. Draw a straight line to touch one given circle, so that the part of it contained by another given circle which is wholly without the former may be equal to a given straight line, not greater than the diameter of this latter circle.

73. Two circles intersect in A, B: PAP′, QAQ′ are drawn equally inclined to AB to meet the circles in P, P′, Q, Q′. Prove that PP′ is equal to QQ′.

74. Find the distance of a point from the centre of a given circle, so that if tangents be drawn from it to the circle, the concave part of the circumference may be double of the convex.

75. The foot of a ladder leaning against a wall slips, and the ladder slides down to the ground. Find the locus of its middle point as it descends.

76. Find a point in a diameter of a given circle produced such that the tangent drawn from it to the circle may be equal to a given straight line.

77. A straight line and two circles are given; find the point in the straight line from which two tangents drawn to the circles shall be equal.

78. The centre C of a circle BPQ lies on another circle APQ. Any chord through P meets them in A, B. Show that BQ, CA are perpendicular.

79. A given straight line is drawn at right angles to the straight line joining the centres of two given circles. Prove that the difference between the squares on two tangents drawn, one to each circle, from any point on the given straight line, is constant.

80. An acute-angled triangle is inscribed in a circle, and the paper is folded along each of the sides of the triangle. Show that the circumferences of the three segments will pass through the same point.

81. Describe a circle to touch a given line, and a given circle at a given point.

82. Describe through two given points a circle such that the chord intercepted by it on a given unlimited straight line may be of given length.

83. Through a given point without a circle draw a chord such that the difference of the angles in the two segments into which it divides the circle may be equal to a given angle.

84. Through a point within a circle draw a chord, such that the rectangle contained by the whole chord and one part may be equal to a given square. Determine the necessary limits to the magnitude of this square.

85. AB, CD are parallel diameters of two circles and AC cuts the circles in P, Q. Prove that the tangents to the circles at P, Q are parallel.

86. Produce a given line so that the rectangle contained by the whole line produced, and the part produced, shall be equal to the square of a given line.

87. Two circles with centres A and B cut at right angles, and their common chord meets AB in C. DE is any chord of the first circle which, when produced, passes through the centre B, of the second circle. Show that D, A, E, C lie on the circumference of a circle.

88. If from any point in a given circular arc, perpendiculars be drawn to its bounding radii, the distance between their feet is invariable.

89. Describe three circles, having given centres, to touch one another.

90. E, F, G, H are the middle points of the arcs AB, BC, CD, DA which subtend the sides of a quadrilateral inscribed in the circle ABCD. Prove that GE and HF are at right angles to each other.

91. APQB is half and PQ the fourth part of the circumference of a circle. If the chords AQ, BP intersect in C, prove that the difference between the squares on AQ, QC is double of the rectangle under BP, PC.

92. From a point L in the diameter AB of a circle produced a perpendicular to the diameter is drawn, and any point E is taken in it, from which a line is drawn cutting the circle at C and D. Prove that the rectangle under EC and ED is greater than that under LA and LB by the square on LE.

93. If three circles touch, two and two, the three tangents at the points of contact meet at a point.

94. Two equal semicircles are described on the diameter of a semicircle so as to touch the semicircle and each other, and a circle, inscribed in the space between them, touches the three circumferences. Show that its diameter is one-third of the diameter of the circle.

95. From the obtuse angle of an obtuse-angled triangle draw a straight line to the base, the square on which shall be equal to the rectangle contained by the segments into which it divides the base.

96. If a quadrilateral be inscribed in a circle, and the middle points of the arcs subtended by its sides be joined to make another quadrilateral, and so on; show that these quadrilaterals tend to become squares.

97. If perpendiculars be drawn from the extremities of the diameter of a circle upon any chord, or chord produced, the rectangle under the perpendiculars is equal to that under the segments between the feet of the perpendiculars and either extremity of the chord.

98. If a triangle ABC be inscribed in a circle, and AA′, BB′, CC′ be drawn parallel to BC, CA, AB meeting the circle in A′, B′, C′, show that B′C′, C′A′, A′B′ are parallel to the tangents at A, B, C.

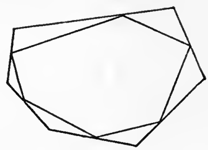
99. Two circles touch externally. In them place a line of given length so that it shall pass through the point of contact and have its extremities on the circumferences of the circles.

100. AB is the diameter of a semicircle, P, Q, R…K are any number of points on the circumference taken in order from A; show that the square upon AB is not less than the sum of the squares on AP, PQ, QR, … KB.

BOOK IV.

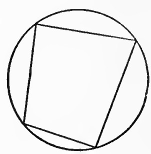
DEFINITIONS.

1. A rectilineal figure (or polygon) is said to be inscribed in another, when all the angles of the first are on the sides of the second, each on each.

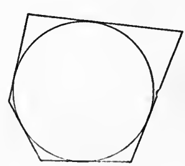


2. A rectilineal figure (or polygon) is said to be described about another, when all the sides of the first pass through the angular points of the second, each through each.

3. A rectilineal figure is said to be *inscribed* in a circle, when all its angles lie on the circumference of the circle.

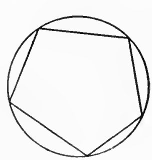


4. A rectilineal figure is said to be *described about* (or to circumscribe) a circle, when each of its sides *touches* the circumference of the circle.



5. A circle is said to be *inscribed* in a rectilineal figure, when the circumference of the circle *touches* each side of the figure.

6. A circle is said to be *described about* (or to circumscribe) a rectilineal figure, when its circumference passes through all the angular points of the figure.



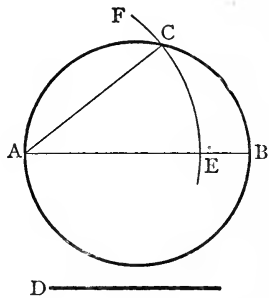
7. A straight line is said to be *placed in a circle,* when its extremities are on the circumference of the circle.

PROPOSITION I. PROBLEM.

In a given circle to place a straight line equal to a given straight line which is not greater than the diameter of the circle.

Let ABC be the given ⊙, and D the given st. line not > a diam^r.

It is req^d to place in ⊙ABC a chd.=D.



Draw any diam. AB................III. 1.

Then, if AB=D,
what was req^d is done.

If not, from AB cut off AE=D...I. 3.

With cent. A, rad. AE, desc.
a ⊙ ECF cutting ⊙ ABC at C.

Join AC.

Then shall AC=D.

For, since A is cent. of ⊙ECF,

∴ AC=AE..................Rad.

=D ....Constr.

Wherefore, *in the given circle* &c. Q.E.F.

NOTES.

Book IV. consists entirely of problems connected with the circle. Props. 1 to 5 deal with triangles; props. 6 to 9 with squares; and props. 11 to 16 with the inscription and circumscription of regular polygons in and about a circle, and *vice versâ*. The polygons dealt with by Euclid are the pentagon, hexagon, quindecagon. The octagon, nonagon, decagon, dodecagon, &c., may also be dealt with by Euclid's methods; but it is not possible, with ruler and compasses alone, to inscribe or describe a regular seven-sided, or eleven-sided, polygon in or about a circle.

EXERCISES.

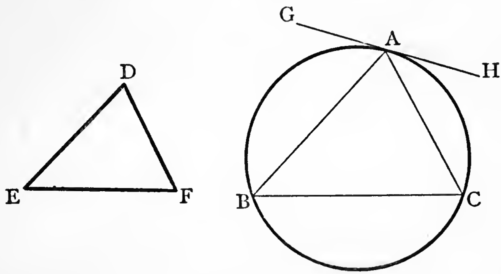
1. Construct a right-angled triangle having given the hypotenuse and one side.
2. About a given straight line describe the smallest possible circle.
3. Construct a right-angled triangle whose hypotenuse shall bear to one of its sides the ratio of 3 to 1.
4. The radius of a circle is 2 inches; how many lines, each 2 inches long, can be placed in succession in the circle?
5. Through a given point within a circle draw a chord of given length.

PROPOSITION II. PROBLEM.

In a given circle to inscribe a triangle equiangular to a given triangle.

Let ABC be the given ⊙, and DEF the given △.

It is req[d] *to inscribe in ⊙ABC a △ equiang*[r] *to DEF.*



Draw GAH to touch the ⊙ at any pt. A.........................III. 17.
At pt. A in AG make ∠GAB=∠DFE.........................I. 23.
At pt. A in HA make ∠HAC=∠DEF.........................I. 23.
Join BC.

Then shall ABC be the △ req[d].

For, since GAH is a tang., and AB a chd. from its pt. of contact A.........................Constr.

∴ ∠GAB=∠ACB in the alt. seg[t] ....III. 32.

But ∠GAB=∠DFE.........................Constr.

∴ ∠ACB=∠DFE.

Similarly, it may be shown that

∠ABC=∠DEF.

Hence, rem[g] ∠BAC=rem[g] ∠EDF.....................I. 32.

Wherefore, *in the given circle* &c. Q.E.F.

EXERCISES.

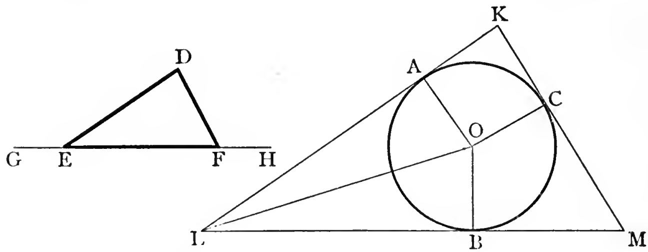
1. In a given circle inscribe
   (i) an equilateral triangle;
   (ii) a right-angled isosceles triangle;
   (iii) an isosceles triangle which shall have each of the angles at the base equal to one-sixth of the vertical angle.
2. An equilateral triangle is inscribed in a circle whose radius is 3 inches; find the length of its sides.

PROPOSITION III. PROBLEM.

About a given circle to describe a triangle equiangular to a given triangle.

Let ABC be the given ⊙, and DEF the given △.

It is req^{d} *to describe about ⊙ABC a △ equiang*^{r} *to DEF.*



Produce EF both ways to G, H.
Find O the cent. of ⊙ABC...III. 1.
Draw any rad. OB.
At pt. O in OB make ∠BOA = ∠DEG,
and on the other side of OB make ∠BOC = ∠DFH...........I. 23.
At pts. A, B, C draw tangs. to the ⊙ ....III. 17.

(These tangs. will, if prod^{d}, meet one another; for, if the tangs. at A and B do not meet they are ||, and then OA, OB at rt. ∠s to them are || ....I. 29 (Ex. 5). which is impossible, since they cut at O.)

Let K, L, M be the pts. at which the tangs. meet.

Then shall KLM be the △ req^{d}.

Join OL.

Then, in △OAL,

∠s ALO, AOL, OAL = two rt. ∠s ....I. 32.
But OAL is a rt. ∠ ....Constr.
∴ ∠s ALO, AOL = a rt. ∠.
Similarly, ∠s BLO, BOL = a rt. ∠.
Hence, ∠s ALB, AOB = two rt. ∠s..................Ax. 2.
But ∠s DEF, DEG = two rt. ∠s..................I. 13.
∴ ∠s ALB, AOB = ∠s DEF, DEG.
But ∠AOB = ∠DEG ....Constr.
∴ rem^{g} ∠ALB = rem^{g} ∠DEF.
Similarly, it may be shown that ∠BMC = ∠DFE.
∴ the third ∠AKC = the third ∠EDF..........I. 32.

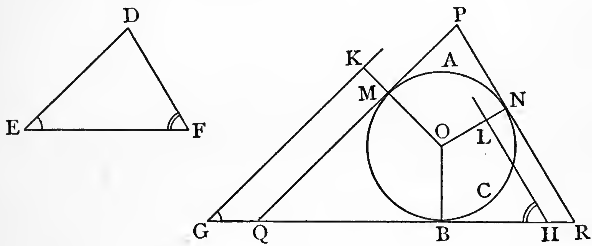
Wherefore, *about the given circle* &c. Q.E.F.

NOTES.

The proof that the tangents will, if produced, meet is not given by Euclid.

The following is another construction for the problem:—

Let ABC be the ⊙, DEF the △.
Draw any tang. GBH to the ⊙ .... III. 17.
At any pt. G in BG make $\angle$BGK $=\angle$DEF .... I. 23.
At any pt. H in BH make $\angle$BHL $=\angle$DFE .... I. 23.
From O, the cent., draw $\perp$s OK, OL to GK, HL .... I. 12.
and let these $\perp$s, prod[d] if necessary, cut the ○ce in M, N.



Through M, N draw tangs. to the ⊙ and let these tangs. cut one another at P and the tang. at B in Q, R.

Then PQR shall be the reqd. △.

EXERCISES.

1. Prove the construction given above.
2. Prove the following construction for Prop. 3:—

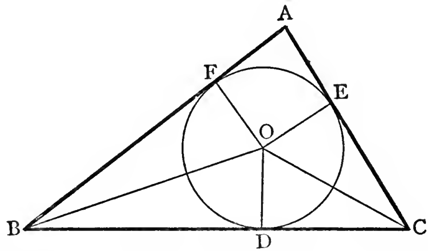
   In the circle inscribe a triangle equiangular to the given triangle (Prop. 2), and from the centre draw radii perpendicular to the sides of the inscribed triangle; the tangents at the ends of these radii will, by their intersections, form the required triangle.
3. About a given circle circumscribe
   (i) an equilateral triangle;
   (ii) a right-angled isosceles triangle.
4. If an equilateral triangle be circumscribed about a circle, the radius of which is 3 inches, find the length of its sides.
5. About a circle describe a trapezium equiangular to a given trapezium.
6. An equilateral triangle whose side is *a* circumscribes a circle; find the length of the radius.

PROPOSITION IV. PROBLEM.

To inscribe a circle in a given triangle.

Let ABC be the given △.

It is req[d] *to inscribe a* ⊙ *in it.*



Bisect the ∠s ABC, ACB by BO, CO meeting at O..........I. 9.
From O draw OD, OE, OF ⊥s to BC, AC, AB..................I. 12.

Then, in △s OBD, OBF,

∵ { ∠ OBD = ∠ OBF........................Constr.
rt. ∠ ODB = rt. ∠ OFB,
OB is com.

∴ OD = OF..........................I. 26 (ii).

Similarly, it may be proved that
OD = OE.

Hence OD, OE, OF are all equal.

∴ the ⊙ desc[d] with cent. O and rad. OD will pass through E and F; let this ⊙ be desc[d].

Then, since the ∠s at D, E, F are rt. ∠s,
∴ the ⊙ touches BC, AC, AB................................III. 16 Cor.

Wherefore, *a circle has been inscribed* &c. Q.E.F.

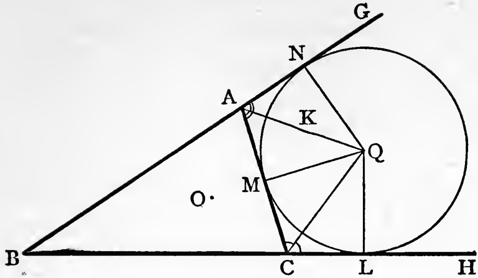
EXAMPLES.

I. *To describe a circle which shall touch one side of a triangle and the other two sides produced.*

Let the sides BA, BC of △ ABC be prod[d] to G, H.

Bisect ∠s CAG, ACH by AK, CQ.

Then, since ∠s BAC, GAC = two rt. ∠s......................I. 13.
and ∠s BCA, HCA = two rt. ∠s......................I. 13.
∴ ∠s BAC, GAC, BCA, HCA = four rt. ∠s.
∴ ∠s GAC, HCA < four rt. ∠s.



Hence, $\angle$s KAC, QCA $<$ two rt. $\angle$s.

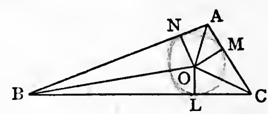
∴ AK, CQ will meet if prod[d] .... Ax. 12.

Let AK, CQ, when prod[d], meet at Q.

Then Q shall be the cent. of the req[d] ⊙.

This circle is called an escribed circle of the triangle.

II. *The area of any triangle is equal to that of the rectangle contained by the semi-perimeter and the radius of the inscribed circle.*



Let O be cent. of ⊙ insc[d] in △ ABC.
Draw ⊥s OL, OM, ON to the sides.
Join OA, OB, OC.

Then, area of △OBC = half the rect. OL, BC .... I 41.

= rect. cont[d] by OL and half BC.

And, area of △OAC = rect. cont[d] by OM and half AC.

= rect. cont[d] by OL and half AC .... Hyp.

Similarly, area of △OAB = rect. cont[d] by OL and half AB.

∴ area of whole △ABC = sum of rects. cont[d] by OL and half BC, OL and half AC, OL and half AB.

= rect. cont[d] by OL and half the perimeter ... II. 1.

Q.E.D.

EXERCISES.

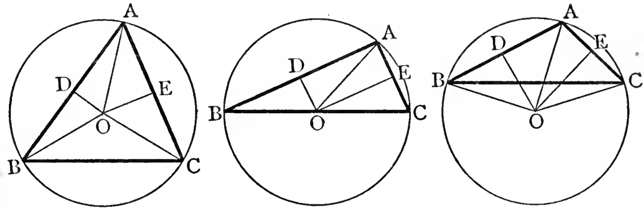
1. Prove the construction given above for the escribed circle.
2. In Prop. 4 prove that the bisectors of the angles B and C must meet.
3. In the figure above, prove that B, O, Q are in one straight line.
4. Prove that the line joining the centres of any two escribed circles passes through an angular point of the triangle.
5. With three given points as centres describe circles mutually touching.
6. Describe a circle to cut off equal chords from the sides of a triangle.
7. How many circles can, in general, be drawn to touch three given straight lines? What are the exceptional cases?
8. Prove that the sum of the tangents drawn to an escribed circle from the remote angle of the triangle is equal to the perimeter of the triangle.

PROPOSITION V. Problem.

To describe a circle about a given triangle.

Let ABC be the given △.

It is req[d] *to describe a* ⊙ *about it.*



Bisect AB, AC at D, E..I. 10.

From D, E draw st. lines at rt. ∠s to AB, AC...................I. 11.

(These lines will, if prod[d], meet; for, if not, they would be ‖, and then AB, AC at rt. ∠s to them would be ‖ ....I. 29 (Ex. 5).

which is impossible, since AB, AC meet at A.)

Let them meet at O.

Join OA, and, if O be not in BC, join OB, OC.

Then, in △s AOD, BOD,

∵ { AD = BD..........................Constr.
OD is com.
rt. ∠ADO = rt. ∠BDO..................Constr. }

∴ OA = OB..............................I. 4.

Similarly, it may be shown that

OA = OC.

Hence, OA, OB, OC are all equal.

With cent. O and rad. OA desc. a ⊙; this ⊙ passes through B and C and circumscribes the △.

Wherefore, *has been described* &c. Q.E.F.

Cor.—When the cent. falls within the △, each of the ∠s of the △ is in a seg[t] > a semicircle, and is < a rt. ∠......III. 31.

When the cent. falls on a side of the △, the ∠ opp. to this side is in a semicircle, and is a rt. ∠....................III. 31.

When the cent. falls without the △, the ∠ opp. to the side beyond which the cent. lies, is in a seg[t] < a semicircle, and is > a rt. ∠......................................III. 31.

And, conversely, if the △ is acute-angled, the cent. of the ⊙ falls within it;

if the △ is right-angled, the cent. falls on the hypot.;

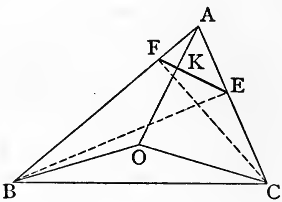
if the △ is obtuse-angled, the cent. falls beyond the side which subtends the obtuse angle.

NOTE.

Props. 4 and 5, and Example I. on Prop. 4, are very important. For particular cases of these props. see pages 86, 89, 94 (*Ex.* 71), 135 (*Note and Ex.* 1).

EXAMPLE.

If O is the centre of the circle circumscribing the triangle ABC, and if E, F are the feet of the perpendiculars from B, C on the opposite sides, then OA cuts EF at right angles.



Let OA and EF cut at K.

Then shall AKE be a rt. ∠.

For, since OA=OC....................Rad.

∴ ∠OAE=∠OCA ....I. 5.

∴ ∠AOC, and twice ∠OAE =two rt. ∠s............I. 32.

But ∠AOC at cent. is double of ∠ABC at ⊙ce,

∴ twice ∠s ABC, OAE=two rt. ∠s.

∴ ∠s ABC, OAE=a rt. ∠.

But, since each of the ∠s BFC, BEC is a rt. ∠,

∴ a ⊙ will go round B, F, E, C............................Ex. IV. page 177.

∴ ∠AEF=∠FBC................................Ex. I. page 159.

Hence ∠s AEF, OAE=a rt. ∠.

∴ the rem^{g} ∠AKE of △AKE is a rt. ∠ ....I. 32.

Q.E.D.

EXERCISES.

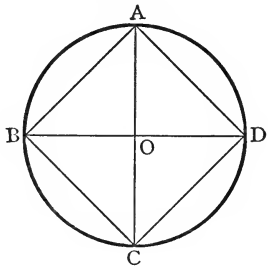
1. If the centres of the inscribed and circumscribed circles of a triangle coincide, the triangle is equilateral.
2. The diameter of the circle circumscribing an equilateral triangle is double of the diameter of the inscribed circle.
3. If, in the figure of IV. 5, OF be drawn perpendicular to BC, then BC is bisected at F.
4. If, in the figure of IV. 5, OD be produced to meet the circumference in G, then the angle GCB is half of the angle ACB.
5. If the perpendiculars from the angles A, B, C of a triangle ABC to the opposite sides be produced to meet the circumference of the circumscribed circle in D, E, F respectively, then the arcs EF, FD, DE are bisected at A, B, C respectively.
6. In the figure of the example above, prove that, if D be the foot of the perpendicular from A on BC, then ED is at right angles to OC.

PROPOSITION VI. PROBLEM.

To inscribe a square in a given circle.

Let ABCD be the given ⊙.

It is req[d] to inscribe a sq. in it.



Find O the cent. .... III. 1.

Draw the diams. AOC, BOD at rt. ∠s to each other.......... I. 11.

Join AB, BC, CD, DA.

Then ABCD shall be the sq. req[d].

For, in △s AOB, COB,

∵ { AO=CO............................ Rad.
OB is com.
rt. ∠ AOB=rt. ∠ COB .... Constr. }

∴ AB=CB............................ I. 4.

Similarly, BC=CD, CD=DA.

Hence, fig. ABCD is equilat.

And, since AC is a diam.

∴ ∠ABC in a semicircle is a rt. ∠ .... III. 31.

Hence, fig. ABCD is a sq.................. I. 46 Cor.

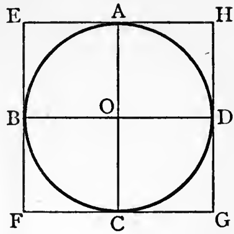
Wherefore, *has been inscribed* &c. Q.E.F.

EXERCISES.

1. The area of the inscribed square is double of the square on the radius of the circle.
2. If any points E, F, G, H be taken in the arcs AB, BC, CD, DA respectively, the sum of the angles AEB, BFC, CGD, DHA is six right angles.
3. If the radius of the circle be 5 inches, find the length of the side of the inscribed square.
4. In a given circle inscribe a regular octagon.
5. If a regular octagon be inscribed in a circle of 3 inches radius, find the length of one of its sides.

PROPOSITION VII. PROBLEM.

To describe a square about a given circle.



Let ABCD be the given ⊙.

It is req^d to desc. a sq. about it.

Find O the cent........................III. 1.
Draw the diams. AOC, BOD
at rt. ∠s to each other...............I. 11.
At A, B, C, D draw tangs. to
the ⊙, cutting at E, F, G, H........III. 17.

Then EFGH shall be the req^d sq.

For, since EF and HG are tangs.

∴ ∠s at B and D are rt. ∠s................III. 18.
But ∠s at O are also rt. ∠s...............Constr.

Hence, EF, AC, HG are ∥ ....I. 29.

Similarly, EH, BD, FG are ∥ .

∴ all the quad^ls in the fig. are ▱s.

Hence, EH=FG, and EF=HG........................I. 34.

Also, EH=BD..................................I. 34.
=AC,
=EF..................................I. 34.

∴ the fig. EFGH is equilat.

And, since EBOA is a ▱,

∴ ∠AEB=∠AOB............................I. 34.
=a rt. ∠.............................Constr.

∴ the fig. EFGH is rectangular.....................I. 46 Cor.

Hence EFGH is a sq.

Wherefore, *about the given circle* &c. Q.E.F.

EXERCISES.

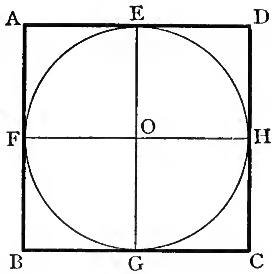
1. Prove that the tangents at A and D must meet each other.
2. The square described about, is double that inscribed in, a circle.
3. About a given circle describe a rhombus two of the sides of which shall include a given angle.
4. If, in the figure of this proposition, HF be joined, prove that HF passes through O.

PROPOSITION VIII. PROBLEM.

To inscribe a circle in a given square.

Let ABCD be the given sq.

It is req^d to inscribe a $\odot$ *in it.*



Bisect AD, AB at E, F ....I. 10.

Through E, F draw EG, FH || to AB, AD and cutting at O...I. 31.

Then all the quad^{ls} in the fig. are ▱s.

Hence AE=FO, and ED=OH..............I. 34.

But AE=ED ....Constr.

$\therefore$ FO=OH.

Similarly, EO=OG.

And, since AD=AB ....Hyp.

and E, F are their mid. pts................Constr.

$\therefore$ AE=AF.

Hence, OE, OF, OG, OH are all equal.

With cent. O, rad. OE, desc. a $\odot$; this will pass through the pts. F, G, H.

And, since EG is || to AB,

$\therefore$ $\angle$DEO=$\angle$DAB......................I. 29.

=a rt. $\angle$. ....Constr.

Similarly, the $\angle$s at F, G, H are rt. $\angle$s.

$\therefore$ the $\odot$ EFGH touches the sides of the sq.

Wherefore, *in the given sq.* &c. Q.E.F.

EXERCISES.

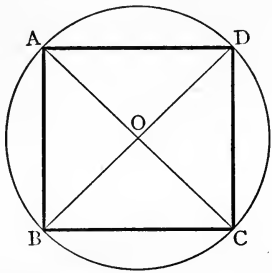
1. Inscribe a circle in a given rhombus.
2. Prove that, if in the figure above, AO, DO be joined, AOD is a right angle.
3. Can a circle be inscribed in a rhomboid?

PROPOSITION IX. PROBLEM.

To describe a circle about a given square.

Let ABCD be the given sq.

It is req^d^ to desc. a ⊙ about it.



Join AC, BD, cutting at O.

Then, in △s ABC, ADC,

∵ { AB=AD .... Hyp.
AC is com.
BC=DC............................ Hyp. }

∴ ∠BAC=∠DAC........................ I. 8.
=half ∠BAD,
=half a rt. ∠.................... Hyp.

Similarly, ∠ABD is half a rt. ∠.

∴ ∠BAO=∠ABO.

∴ AO=BO............................ I. 6.

Similarly, BO=CO, CO=DO, DO=AO.

Hence, AO, BO, CO, DO are all equal.

With cent. O, rad. OA, desc. a ⊙; this will pass through the pts. B, C, D, and circumscribe the sq.

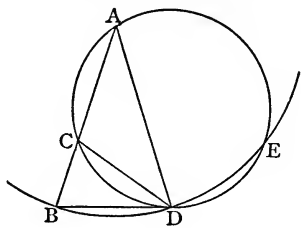
Wherefore, *about the given sq.* &c. Q.E.F.

EXERCISES.

1. Describe a circle about a given rectangle.
2. Describe, when possible, a circle about a given quadrilateral. When is this impossible?
3. If the side of a square be 3 inches long, find the length of the radius of the circumscribed circle.

PROPOSITION X. PROBLEM.

To describe an isosceles triangle having each of the angles at the base double of the third angle.



Take any st. line AB.
Divide AB at C so that rect. AB, BC=sq. on AC................II. 11.
With cent. A and rad. AB desc. a ⊙BDE.
In ⊙BDE place chd. BD=AC..IV. 1.
Join AD.

Then ABD shall be the △ req^d.

Join DC.
About △ACD desc. a ⊙ACD..IV. 5.
Then, since rect. AB, BC=sq. on AC......................Constr.
=sq. on BD ....Constr.
∴ BD touches the ⊙ACD.................................III. 37.
And, since DC is drawn from the pt. of contact D,
∴ ∠BDC=∠DAC in alt. segt..........III. 32.
Add ∠CDA to each,
∴ whole ∠BDA=∠s DAC, CDA.
But ∠DCB=∠s DAC, CDA.............I. 32.
∴ ∠BDA=∠DCB.
But, since AD=AB...............................Rad.
∴ ∠BDA=∠ABD ....I. 5.
Hence ∠DCB=∠ABD.
∴ CD=BD.............................I. 6.
=AC.............................Constr.
∴ ∠CDA=∠DAC ....I. 5.
Hence ∠DCB is double of ∠DAC.
But ∠BDA and ∠ABD each=∠DCB.........................Above.
∴ ∠BDA and ∠ABD are each double of ∠DAC.

Wherefore, *has been described* &c. Q.E.F.

N.B.—This prop. is a very important one.

EXAMPLES.

I. *Find the number of degrees in each of the angles of the triangle in IV. 10.*

Since the three ∠s of △ABD = two rt. ∠s.....................I. 32.
and each of the ∠s at B and D is double of ∠A,
∴ ∠A is one-fifth of two rt. ∠s,
or ∠A = $\frac{2}{5}$ rt. ∠ = $\frac{2}{5} \times 90° = 36°$.
Also, ∠B and ∠D each = $\frac{4}{5}$ rt. ∠ = 72°.

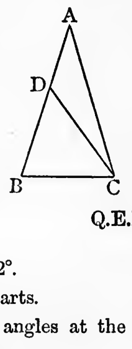
Q.E.D.

II. *Describe an isosceles triangle having each of the angles at the base equal to three-quarters of the vertical angle.*

[Since the three ∠s of a △ = 2 rt. ∠s, the req^d △ will be such that its vert^l ∠ with twice $\frac{3}{4}$ of its vert^l ∠ = 2 rt. ∠s,
i.e. $\frac{10}{4}$ of its vert^l ∠ = 2 rt. ∠s.
∴ its vert^l ∠ = $\frac{4}{5}$ of a rt. ∠,
which is the magnitude of one of the base ∠s in IV. 10. From the above *analysis* (as it is called) of the problem we arrive at the following construction:—]

Make an isos. △ABC having each of the ∠s at B and C double of the ∠ at A...IV. 10.
From BA cut off BD = BC...................I. 3.
Join DC.

Then BCD shall be the req^d △.



For, since BC = BD..............................Constr.
∴ ∠BCD = ∠BDC......................I. 5.
But ∠DBC = $\frac{4}{5}$ rt. ∠ = $\frac{2}{5}$ of 2 rt. ∠s......Ex. I. above.
∴ ∠s BCD, BDC together = $\frac{3}{5}$ of 2 rt. ∠s...I. 32.
∴ ∠BCD = $\frac{3}{5}$ of a rt. ∠ = $\frac{3}{4}$ of ∠DBC.

Q.E.D.

EXERCISES.

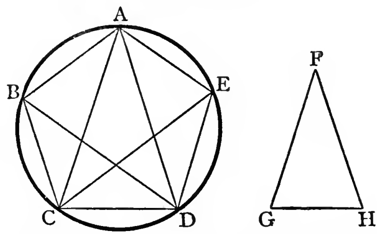
1. Construct angles of (i) 18°, (ii) 54°, (iii) 126°, (iv) 12°.
2. Divide a right angle into (i) five, (ii) fifteen, equal parts.
3. Describe an isosceles triangle having each of the angles at the base equal to one-eighth of the vertical angle.
4. In the figure of IV. 10, (i) If DE be joined DE = BD.
   (ii) What portion of the circumference of the smaller circle does the larger circle intercept?
   (iii) If O be centre of circle ACD and the diameter COF be drawn, prove that the triangle CEF has one of its angles four times the size of another.
   (iv) If AE meets BD produced in F, prove that FAB is another isosceles triangle of the same kind.
   (v) Arc BD is one-tenth of the whole circumference.
   (vi) If AE meets BD produced at F, then CDFE is a parallelogram.

PROPOSITION XI. PROBLEM.

To inscribe an equilateral and equiangular pentagon in a given circle.

Let ABCDE be the given ⊙.

It is req^d to insc. a reg. pentagon in it.



Make an isos. △FGH having ∠s at G, H each double of ∠F...IV. 10.
In ⊙ABCDE insc. a △ACD equiang^r to △FGH..............IV. 2.
having ∠ at A=∠ at F.
Bisect ∠s ACD, ADC by CE, BD meeting the ○ce at B, E....I. 9.
Join AB, BC, DE, EA.

Then ABCDE shall be the pentagon req^d.

For, since ∠s ACD, ADC are each double of ∠CAD ....Constr.
and are bisected by CE, BD..Constr
∴ the ∠s ADB, BDC, CAD, DCE, ECA are all equal.
∴ the arcs AB, BC, CD, DE, EA are all equal....................III. 26.
∴ the chds. AB, BC, CD, DE, EA are all equal....................III. 29.
∴ pentagon ABCDE is equilat^l.
Again, arc AB=arc DE..........................Above.
Add arc BCD to each,
∴ whole arc ABCD=whole arc BCDE.
∴ ∠AED which stands on arc ABCD=∠BAE on arc BCDE...III. 27.
Similarly, each of the ∠s ABC, BCD, CDE=∠BAE.
∴ pentagon ABCDE is equiang^r.
Hence, ABCDE is a reg. pentagon.

Wherefore, *an equilateral* &c. Q.E.F.

EXERCISES.

1. Find, in degrees, the magnitude of the angle subtended at the centre by a side of the regular pentagon.
2. If all the diagonals of the pentagon be drawn, prove that the diagonals form, by their intersections, another regular pentagon.

PROPOSITION XII. Problem.

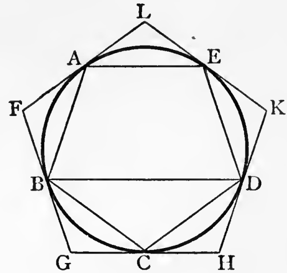
To describe an equilateral and equiangular pentagon about a given circle.

Let ABCDE be the given ⊙.

It is req^{d} to desc. a reg. pentagon about it.

In ⊙ABCDE inscribe a reg. pentagon ABCDE..................IV. 11.

At the pts. A, B, C, D, E draw tangs. to the ⊙ cutting at F, G, H, K, L.......................III. 17.



Then FGHKL shall be the pentagon req^{d}.

Join BD.

Then, since CB=CD ....Constr.

∴ ∠CBD=∠CDB ....I. 5.

And, since GH is a tang., and CD a chd. from the pt. of contact C,

∴ ∠HCD=∠CBD in alt. seg^{t}..........III. 32.

Also, since HK is a tang., and DC a chd. from the pt. of contact D,

∴ ∠CDH=∠GBD in alt. seg^{t}..........III. 32.

∴ ∠HCD=∠HDC.

∴ HC=HD.............................I. 6.

Similarly, GC=GB.

Again, since GH is a tang. and CB a chd. from the pt. of contact C,

∴ ∠GCB=∠CDB in alt. seg^{t}..........III. 32.

=∠CBD ....Above.

=∠HCD........................Above.

Hence, in △s BGC, DHC,

∵ { ∠GBC=∠HDC........................Above.
∠GCB=∠HCD........................Above.
CB =CD...............................Constr. }

∴ CG=CH
and ∠ at G=∠ at H } ....I. 26.

Similarly it may be shown that any other side FG is bisected at the point of contact B, and that ∠ at F=∠ at G, and so on.

Hence pentagon FGHKL is equiang^{r}.

Also, since GC=GB...............................Above.

∴ GH=GF ....Ax. 6.

Similarly it may be shown that GF=FL, and so on.

Hence pentagon FGHKL is equilat^{l}.

∴ FGHKL is a reg. pentagon.

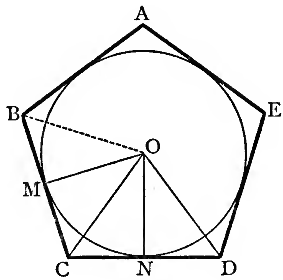
Wherefore, *an equilateral* &c. Q.E.F.

PROPOSITION XIII. PROBLEM.

To inscribe a circle in a given equilateral and equiangular pentagon.

Let ABCDE be the given reg. pentagon.

It is req^{d} to insc. a $\odot$ in it.



Bisect $\angle$s BCD, CDE by CO, DO meeting at O.................I. 9.
Join OB.
From O draw $\perp$s OM, ON to BC, CD..............................I. 12.

Then, in $\triangle$s OBC, ODC,

$\because$ { BC=CD..............................Hyp.
OC is com.
$\angle$BCO=$\angle$DCO........................Constr. }

$\therefore$ $\angle$CBO=$\angle$CDO.......................I. 4.
=half $\angle$CDE.................Constr.
=half $\angle$CBA.................Hyp.

Similarly, if OA, OE be drawn, it may be shown that they bisect the $\angle$s at A and E.

Again, in $\triangle$s OCM, OCN,

$\because$ { $\angle$OCM=$\angle$OCN........................Constr.
$\angle$OMC=$\angle$ONC........................Ax. 11.
OC is com. }

$\therefore$ OM=ON............................I. 26.

Similarly, if $\perp$s be drawn from O to BA, AE, ED they may be shown to be each equal to OM.

With cent. O and rad. OM desc. a $\odot$; then since this $\odot$ passes through the feet of the other $\perp$s,

$\therefore$ it *touches* the sides of the pentagon...........III. 16.
i.e. a $\odot$ is inscribed in the pentagon...........IV. def. 5.

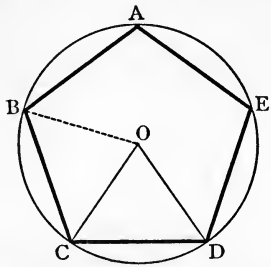
Wherefore, *a circle* &c. Q.E.F.

PROPOSITION XIV. PROBLEM.

To describe a circle about a given equilateral and equiangular pentagon.

Let ABCDE be the given reg. pentagon.

It is req^d to desc. a ⊙ about it.



Bisect ∠s BCD, CDE by CO, DO meeting at O..................I. 9.

Join OB.

Then, in △s OBC, ODC,

∵ { BC =CD..............................Hyp.
OC is com.
∠BCO=∠DCO ....Constr.

∴ ∠CBO=∠CDO ....I. 4.
=half ∠CDE..................Constr.
=half ∠CBA..................Hyp.

Similarly, if OA, OE be drawn, it may be shown that they bisect the ∠s at A and E.

And, since ∠CBO=half one of the ∠s of the pentagon...Above.
and ∠BCO=half one of the ∠s of the pentagon...Constr.
∴ ∠CBO=∠BCO ....Ax. 7.
∴ OB=OC...I. 6.

Similarly it may be shown that OA, OE, OD are each=OC.

With cent. O, and rad. OC desc. a ⊙; then this ⊙ passes through B, A, E, D, and ∴ circumscribes the pentagon ....IV. def. 6.

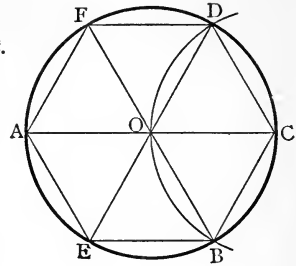
Wherefore, *a circle* &c. Q.E.F.

EXERCISES.

1. Prove that a circle can be described about any regular polygon.
2. Prove that a circle can be inscribed in any regular polygon.
3. If any regular polygon be inscribed in a circle, the tangents drawn to the circle at its points of contact form a regular polygon.

PROPOSITION XV. PROBLEM.

To inscribe an equilateral and equiangular hexagon in a given circle.



Let ABC be the given ⊙.

It is req[d] *to insc. a reg. hexagon in it.*

Find O the cent....................III. 1.
Draw the diam. AOC.
With cent. C, rad. CO desc. ⊙BOD cutting ⊙ABC at B, D...........
Draw the diams. BOF, DOE.
Join AE, EB, BC, CD, DF, FA.

Then AEBCDF shall be the hexagon req[d].

For, since △s DOC, BOC are equilat.... ....Constr.
∴ each of the ∠s DOC, BOC=one-third of two rt. ∠s.....I. 32.
∴ whole ∠DOB=two-thirds of two rt. ∠s.
∴ rem[g] ∠DOF=one-third of two rt. ∠s.....I. 13.
Hence, ∠s FOA, AOE, EOB each=one-third of two rt. ∠s.....I. 15.
i.e. the six ∠s at O are all equal.
∴ the six arcs on which they stand are all equal.....III. 26.
∴ the six chds. AE, EB, BC, CD, DF, FA are all equal...III. 29.
i.e. the hexagon is equilat.

Again, since the sum of the four arcs AE, EB, BC, CD
=the sum of the four arcs EB, BC, CD, DF..............Ax. 2.
∴ ∠DFA=∠FAE.......................III. 27.
Similarly, ∠FAE=∠EBC, and so on.
i.e. the hexagon is equiang.

Wherefore, *AEBCDF is a reg. hexagon.* Q.E.F.

COR.—The side of a reg. hexagon insc[d]. in a ⊙ is equal to the rad.

EXERCISES.

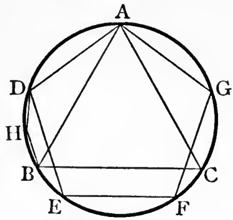
1. Opposite sides of a regular hexagon are parallel.
2. Inscribe a circle in a given regular hexagon.
3. If the radius of the circle be 3 inches, find the length of the chord AB.
4. If the radius of the circle be a, find the length of the perpendicular from the centre on a side of the hexagon.
5. If the radius be a feet, find the area of the hexagon.
6. The difference between the areas of the inscribed hexagon and dodecagon in a circle of radius 2 inches, is $6(2-\sqrt{3})$ square inches.

PROPOSITION XVI. PROBLEM.

To inscribe an equilateral and equiangular quindecagon in a given circle.

Let ABC be the given ⊙.

It is req^d to inscribe a reg. quindecagon in it.



Inscribe an equilat. △ABC in the ⊙ .... IV. 2.
and a reg. pentagon ADEFG .... IV. 11.

Then, of fifteen equal parts into which the whole ○ce is to be divided,

the arc AB contains five,
and the arc AD contains three:
∴ the rem^g arc DB contains two.

Bisect the arc DB at H.. III. 30.

∴ arcs DH, HB are each one-fifteenth of the whole ○ce.

Join DH, HB, and in the ⊙ place, in succession, chds. equal to either of these lines.

These chds. will form a quindecagon, which may be proved to be regular by the method used in the preceding props.

Wherefore, *has been inscribed* &c. Q.E.F.

NOTE.

A regular hexagon, or quindecagon, may be described about a circle by drawing tangents to the circle through the angular points of the inscribed figure, as was shown in the case of the pentagon. And, by the same method as was used for the pentagon, a circle may be inscribed in, or circumscribed about, a regular hexagon, or quindecagon.

Also, by bisecting the arcs which are cut off by the sides of any one of the figures dealt with in Book IV., a regular figure of twice that number of sides may be inscribed in the circle. Hence, by the methods used in Book IV., regular polygons of 3, 6, 12, 24 &c.; 4, 8, 16, 32 &c.; 5, 10, 20, 40 &c.; 15, 30, 60, 120 &c., sides may be inscribed in, or described about, a circle.

MISCELLANEOUS EXERCISES.

1. Find the magnitude of the angle subtended at the centre by the part of any tangent intercepted by the square circumscribing the circle.
2. ABC is a triangle; if any tangent to that part of the circumference of the inscribed circle which is convex to the point A, meet AB, AC in D, E, then the difference of the perimeters of the triangles ABC, ADE is twice BC.
3. Any equilateral figure inscribed in a circle is also equiangular.
4. The area of a regular octagon inscribed in a circle is equal to that of the rectangle contained by the sides of the inscribed and circumscribed squares.
5. Inscribe a circle in a given quadrant.
6. Find the locus of the centres of the circles inscribed in all right-angled triangles standing on the same hypotenuse as base.
7. Circles are described, each touching one side of the triangle and the other two sides produced. Show that the straight line joining any two of the centres of these circles passes through an angular point of the triangle, and that a circle can be described passing through these two centres and the other two angular points of the triangle.
8. The perpendicular from A, upon BC, meets the circumference of the circumscribed circle in G. If P be the point in which the perpendiculars from the angles upon the opposite sides intersect, prove that PG is bisected by the side BC of the triangle ABC.
9. If the circle inscribed in a triangle ABC touch the sides AB, AC in the points D, E, and a straight line be drawn from A to the centre of the circle, meeting the circumference in G, show that the point G is the centre of the circle inscribed in the triangle ADE.
10. If it be possible to describe a triangle with its angular points on the outer of two concentric circles and its sides tangents to the inner, the radius of one circle must be double that of the other, and the triangle so described must be equilateral.
11. P is a point on the circumference of the circle circumscribing a given triangle ABC. The sides of a triangle DEF are parallel to the straight lines PA, PB, PC. Prove that the triangle DEF is equiangular to the triangle ABC.
12. If with one of the angular points of a regular pentagon as centre and one of its diagonals as radius a circle be described; a side of the pentagon will be equal to a side of the regular decagon inscribed in the circle.

13. If I, O be the centres of the inscribed and circumscribed circles of the triangle ABC, and if AI be produced to meet the circumscribed circle in D, prove that OD bisects BC.

14. If ABC be a triangle, show that the circle through B, C, and the centre of the escribed circle touching BC, passes through the centre of the inscribed circle.

15. A, B, C, D, E, F are successive angular points of a regular decagon inscribed in a circle of which O is the centre. OC cuts AD in G. Prove that AE bisects OG at right angles.

16. If the line bisecting the angle A of a triangle ABC meet the lines bisecting internally and externally the angle C in E and F, and the circle described about the triangle ABC in O, then EO = FO.

17. Describe an isosceles triangle having each of the angles at the base one-third of the vertical angle.

18. If one side of a regular pentagon be produced, trisect the external angle.

19. If ABCDE be an equilateral and equiangular pentagon inscribed in a circle, and if P be the middle point of the arc AB, prove that AP together with the radius of the circle is equal to PC.

20. Show that the circles, each of which touches two sides of an equilateral and equiangular pentagon inscribed in a circle at the extremities of a third, meet in a point.

21. In a given circle inscribe a triangle of given area having its vertex at a fixed point in the circumference and its vertical angle equal to a given rectilineal angle.

22. If DA be one side of a regular hexagon inscribed in a circle, AB a tangent equal in length to AD and making an obtuse angle with it, C the centre of the circle, and if BD meet the circle in E and BC meet the nearer part of the circumference in F, prove that AE and EF are equal to sides of regular polygons in the circle of twelve and twenty-four sides respectively.

23. Two equilateral triangles are described about the same circle; show that the intersections will form a hexagon, equilateral but not generally equiangular.

24. If a regular pentagon, hexagon, and decagon are inscribed in the same circle, the square of a side of the pentagon is equal to the square of a side of the hexagon together with the square of a side of the decagon.

25. Triangles are constructed on the same base, with equal vertical angles; prove that the locus of the centres of the escribed circles, each of which touches one of the sides externally and the other side and base produced, is an arc of a circle, the centre of which is on the circumference of the circle circumscribing the triangles.

26. Describe a circle touching the side BC of the triangle ABC and the other two sides produced, and prove that the distance between the points of contact of the side BC with the inscribed circle, and the latter circle, is equal to the difference between the sides AB and AC.

27. Having given an angular point of a triangle, the circumscribed circle, and the centre of the inscribed circle, construct the triangle.

28. A line drawn parallel to the base BC of a triangle ABC meets the other sides in D and E respectively. Show that the circles circumscribing the triangles ABC, ADE have a common tangent.

29. If two equiangular triangles be circumscribed about the same circle, a circle will pass through any two corresponding angular points and the intersections of the lines containing them.

30. ABCDE is a regular pentagon. F is the middle point of the side CD Show that the pentagon is equal in area to a rectangle, one of whose sides is AF, and the other the excess of AC over CF.

31. Having given the length, but not the position, of one side of a triangle the centres of the inscribed circle and of the circle which touches the given side and the other two sides produced, and the position of the vertex opposite to the given side, construct the triangle.

32. If, in a triangle ABC, straight lines from B and C, perpendicular to the opposite sides, meet in L, and B′, C′ be the centres of the circles described round the triangles CLA, ALB, then B′C′ will be equal and parallel to BC.

33. If O be the centre of the circle inscribed in the triangle ABC and AO, BO be produced to meet the opposite sides in E, F; prove that, if a circle can be described round the quadrilateral CEOF, then the angle C must be equal to one-third of two right angles.

34. A circle B passes through the centre of another circle A; a triangle is described round A having two of its angular points on B; prove that the third angular point is on the line of centres.

35. The perpendiculars from the centres of the escribed circles of a triangle upon the corresponding sides meet at a point when produced.

36. If the inscribed circle touch the sides of the triangle in D, E, F, and the diameter which passes through A meet FD in M and DE in N; show that CM, BN, EF are parallel.

37. A triangle is inscribed in a given circle so as to have its centre of perpendiculars at a given point; prove that the middle points of its sides lie on a fixed circle.

38. Given the vertical angle, and the radii of the inscribed and circumscribed circles, construct the triangle.

BOOK V.

DEFINITIONS.

1. A less magnitude is said to be **a part** of a greater when the less measures the greater; that is, when the less is contained a certain number of times exactly in the greater.

2. A greater magnitude is said to be **a multiple** of a less when the greater is measured by the less; that is, when the greater contains the less a certain number of times exactly.

3. **Ratio** is the mutual relation of two magnitudes of the same kind to one another in respect of quantity.

NOTES.

Book V. treats of Proportion.

The first four Books deal with the absolute equality or inequality of Geometrical magnitudes; in Book VI., however, *relative* greatness is considered; this necessitates a definition of Proportion holding good for all Geometrical magnitudes, whether commensurable or incommensurable, and either the proof, or assumption, of the principles of the theory. Only those parts of Book V. which supply this necessary introduction to Book VI. are given here, and then the enunciations only of the propositions. The proofs are excluded because Book V. seldom forms part of a course of elementary Geometry, and the theory of Proportion is now generally studied in Algebra. But it should be remembered that Algebraical proofs must not be regarded as complete when applied to Geometrical magnitudes, since they assume the commensurability of those magnitudes.

In def. 1, Euclid uses the word *part* in the restricted sense *aliquot part*, or *sub-multiple;* i.e. a part which is contained an exact number of times in the whole.

In def. 3, the word *quantity* must be understood to mean *the number of times the one contains the other;* this is sometimes expressed by the word *quantuplicity.*

5. Equal ratios.

The first of four magnitudes is said to have the same ratio to the second that the third has to the fourth, when, any *equimultiples* whatever of the *first* and *third* being taken, and any equimultiples whatever of the second and fourth, the multiple of the *first* is greater than, equal to, or less than that of the *second*, according as the multiple of the third is greater than, equal to, or less than that of the fourth.

And the four magnitudes are then called **proportionals.**

[For example: if A, B, C, D be four magnitudes,

mA, mC equimultiples of A, C,
nB, nD equimultiples of B, D,

and if, whatever integral values be given to m, n,

mC is always $> nD$, when $mA > nB$,
mC is always $= nD$, when $mA = nB$,
mC is always $< nD$, when $mA < nB$,

then the ratio of A to B is equal to the ratio of C to D, and A, B, C, D are proportionals.

This is expressed by saying "A *is to* B *as* C *is to* D."]

10. When three magnitudes are proportionals the first is said to have to the third the **duplicate ratio** of that which it has to the second.

And the second is said to be a **mean proportional** between the first and third.

11. When four magnitudes are proportionals the first is said to have to the fourth the triplicate ratio of that which it has to the second; and so on.

Compound Ratio.

When there are any number of magnitudes of the same kind, the first is said to have to the last the ratio which is compounded of the ratios of the first to the second, the second to the third, and so on.

[For example: if A, B, C, D are four magnitudes, the ratio A to D is compounded of the ratios of A to B, B to C, and C to D.]

12. In proportionals, the *antecedent terms* are said to be **homologous** to one another; as also the *consequents* to one another.

ADDITIONAL DEFINITIONS.

Equimultiples of magnitudes contain them the same number of times.

When the two terms of a ratio are equal it is called a *ratio of equality*. When the first term of the ratio is greater than the second it is called *a ratio of greater inequality*. When the first term of the ratio is less than the second it is called *a ratio of less inequality*.

The first term of a ratio is called the **antecedent**, and the second term the **consequent**.

If A, B, C, D are proportionals, A and D, the first and last terms, are called the **extremes**; and B and C, the second and third terms, are called the **means**.

Three magnitudes are said to be **in continued proportion** when the first is to the second as the second is to the third.

The ratio of B to A is called the *reciprocal* of the ratio of A to B.

NOTES.

The definition of proportion is of the greatest importance, and should be carefully studied by the student before he proceeds further.

The statement of the proportion *A is to B as C is to D* is often expressed thus:—

$$A : B :: C : D,$$

or thus,

$$A : B = C : D.$$

The Algebraic method of expressing a ratio, $\frac{A}{B}$, being a very convenient one, will also be found in the Examples, where it should be regarded as a symbol for the words *the ratio of A to B*, and not as implying the operation of division; it should not be used for *book-work* (*see Preface*).

In the "book-work" of this edition none of these forms have been used, as the words themselves are so short that they scarcely seem to need abbreviating.

Euclid's definitions of duplicate and triplicate ratio can be easily shown to agree with those of Algebra; for instance, if a, b, c are in continued proportion, $a : b = b : c$, or $\frac{a}{b} = \frac{b}{c}$, $\therefore \frac{a}{b} \cdot \frac{a}{b} = \frac{a}{b} \cdot \frac{b}{c}$, or $\frac{a^2}{b^2} = \frac{a}{c}$, which shows that Euclid's definition of duplicate ratio is in agreement with the Algebraic one that $\frac{a^2}{b^2}$ is the duplicate ratio of $\frac{a}{b}$.

Theorems of Book V referred to in Book VI.

PROPOSITION B.

If four magnitudes are proportionals, they are proportionals when taken inversely.

[*i.e.*, If A : B : : C : D;
Then B : A : : D : C.] **Invertendo.**

PROPOSITION D.

If the first term of a proportion be a multiple, or a part, of the second, the third is the same multiple, or part, of the fourth.

[*i.e.*, If A : B : : C : D,
and if $A=mB$; then $C=mD$.]

PROPOSITION VII.

Equal magnitudes have the same ratio to the same magnitude; and conversely.

[*i.e.*, If A=B; then A : C : : B : C,
and C : A : : C : B.]

PROPOSITION IX.

Magnitudes which have the same ratio to the same magnitude are equal to one another; and conversely.

[*i.e.*, If A : C : : B : C,
or, if C : A : : C : B; then A=B.]

PROPOSITION XI.

Ratios that are equal to the same ratio are equal to one another.

[*i.e.*, If A : B : : C : D,
and if E : F : : C : D;
then A : B : : E : F.]

PROPOSITION XII.

If any number of magnitudes be proportionals, as one of the antecedents is to its consequent, so is the sum of all the antecedents to the sum of all the consequents.

[*i.e.*, If A : B : : C : D,
and C : D : : E : F;
then A : B : : A+C+E : B+D+F.]

PROPOSITION XIV.

If the first term of a proportion is greater than the third, the second is greater than the fourth; if equal, equal; and if less, less.

[*i.e.*, If A : B : : C : D,
and if $A>C$; then $B>D$; if $A=C$; then $B=D$; if $A<C$; then $B<D$.]

PROPOSITION XV.

Magnitudes have the same ratio to one another that their equimultiples have. [*i.e.*, A : B : : mA : mB.]

PROPOSITION XVI.

If four magnitudes are proportionals, they are also proportionals when taken alternately. [*i.e.*, If A : B : : C : D;
then A : C : : B : D.] **Alternando.**

PROPOSITION XVII.

If magnitudes taken jointly are proportionals, they are also proportionals when taken separately.

[*i.e.*, If A+B : B : : C+D : D;
then A : B : : C : D.] **Dividendo.**

PROPOSITION XVIII.

If magnitudes taken separately are proportionals, they are also proportionals when taken jointly.

[*i.e.*, If A : B : : C : D;
then A+B : B : : C+D : D.] **Componendo.**

PROPOSITION XXII.

If there be any number of magnitudes, and as many others, which have the same ratio, taken two and two in order, then the first has to the last of the first set the same ratio which the first has to the last of the second set. [*i.e.*, If A, B, C and D, E, F be the two sets of magnitudes,
and if A : B : : D : E,
and B : C : : E : F;
then A : C : : D : F.] **Ex æquali.**

PROPOSITION XXIV.

If the first magnitude have to the second the same ratio which the third has to the fourth, and the fifth have to the second the same ratio which the sixth has to the fourth, then the first and fifth together have to the second the same ratio which the third and sixth together have to the fourth. [*i.e.*, If A : B : : C : D,
and E : B : : F : D;
then A+E : B : : C+F : D.]

NOTE.

The terms *Invertendo, Alternando, Dividendo, Componendo, Ex æquali* are used when reference is made to props. B, 16. 17, 18, 22, of Bk. V.

The propositions "B" and "D" were inserted by Simson.

BOOK VI.

DEFINITIONS.

1. **Similar rectilineal figures** are such as have their angles equal, each to each, and their sides, taken in order, about their equal angles proportionals.

2. Two triangles, or parallelograms, are *reciprocal* when their sides about two angles are proportionals in such a way that a side of the first figure is to a side of the second as the remaining side of the second is to the remaining side of the first.

3. A straight line is said to be cut *in extreme and mean ratio*, when the whole is to the greater segment as the greater segment is to the less.

4. The *altitude* of a figure is the straight line drawn from the vertex perpendicular to the base.

ADDITIONAL DEFINITIONS.

Similar figures are said to be *similarly situated* when each pair of homologous sides are either parallel or inclined at the same angle, and are drawn towards the same parts.

Three magnitudes are said to be in *Harmonic* proportion when the first is to the third as the difference between the first and second is to the difference between the second and third.

(For example, if A, B, C are in Harmonic proportion, then A is to C as $A \sim B$ is to $B \sim C$.)

Three magnitudes are in *Geometric* proportion when the second is a mean proportional between the first and third.

NOTES.

No use is made by Euclid of def. 2 as it stands, but the *sides* of such figures are said to be *reciprocally proportional.*

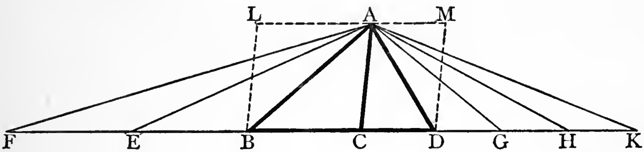
In this book, references to the postulates, axioms, and propositions of Book I. will not be inserted in the *constructions.*

PROPOSITION I. THEOREM.

Triangles and parallelograms of the same altitude are to one another as their bases.

Let ABC, ACD be two △s having the same alt.

Then shall △ABC be to △ACD as BC to CD.



Produce BD both ways to F, K.
From BF cut off any no. of parts BE, EF each=BC.
From DK cut off any other no. of parts DG, GH, HK each=CD.
Join AE, AF, AG, AH, AK.

Then, since FE, EB, BC are all equal,
∴ △s AFE, AEB, ABC are all equal.................I. 38, note.
∴ △AFC is the same mult. of △ABC that FC is of BC.
Similarly, it may be shown that
△ACK is the same mult. of △ACD that CK is of CD.

Hence, △AFC and FC are equimults. of △ABC and BC,
the first and third,
and △ACK and CK are equimults. of △ACD and CD,
the second and fourth,
of the four mags. △ABC, △ACD, base BC, base CD;
also, if FC=CK, △AFC=△ACK..........I. 38.
if FC>CK, △AFC>△ACK,
if FC<CK, △AFC<△ACK, } ......I. 38, note.
∴ △ABC is to △ACD as BC is to CD......V. Def. 5.

Again, if ▱s LBCA, ACDM be completed,
then ▱LC is double of △ABC,
and ▱CM is double of △ACD, } ....I. 41.
But △ABC is to △ACD as BC is to CD...Above.
∴ twice △ABC is to twice △ACD as BC is to CD...V. 15.
i.e. ▱CL is to ▱CM as BC is to CD.

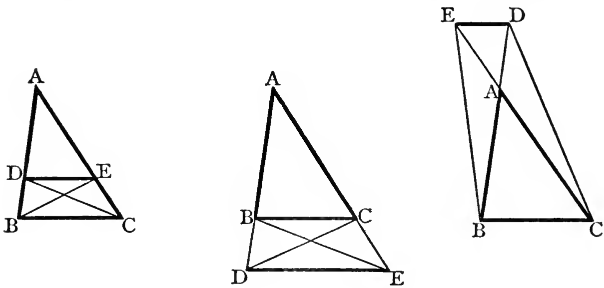
Wherefore, *triangles* &c. Q.E.D.

PROPOSITION II. THEOREM.

If a straight line be drawn parallel to one of the sides of a triangle it shall cut the other sides, or those sides produced, proportionally; and, conversely, if the sides, or the sides produced, be cut proportionally, the straight line which joins the points of section shall be parallel to the remaining side of the triangle.

PART I.—Let DE be || to BC, a side of $\triangle$ABC;

Then shall BD be to DA as CE to EA.



Join BE, CD.

Then, since DE is || to BC .... Hyp.
∴ $\triangle$BDE = $\triangle$CDE .... I. 37.
∴ $\triangle$BDE is to $\triangle$ADE as $\triangle$CDE is to $\triangle$ADE ... V. 7.
But $\triangle$BDE is to $\triangle$ADE as BD is to DA ...... VI. 1.
And $\triangle$CDE is to $\triangle$ADE as CE is to EA ..... VI. 1.
∴ BD is to DA as CE is to EA ..... V. 11.

PART II.—Let BD be to DA as CE to EA;

Then shall DE be || to BC.

Join BE, CD.

Then, $\triangle$BDE is to $\triangle$ADE as BD is to DA ..... VI. 1.
And $\triangle$CDE is to $\triangle$ADE as CE is to EA ..... VI. 1.
But BD is to DA as CE is to EA ..... Hyp.
∴ $\triangle$BDE is to $\triangle$ADE as $\triangle$CDE is to $\triangle$ADE ... V. 11.
∴ $\triangle$BDE = $\triangle$CDE .... V. 9.
and they stand upon the same base DE,
∴ DE is || to BC .... I. 39.

Wherefore, *if a straight line* &c. Q.E.D.

NOTE.

Since BD is to DA as CE is to EA.

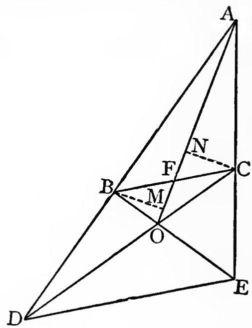
∴ sum of BD, DA is to DA as sum of CE, EA is to EA......Compo.

i.e., AB is to AD as AC is to AE.

A form of the result which is often useful.

EXAMPLE.

If, in the figure of VI. 2, BE and CD cut at O, then AO, or AO produced, will bisect BC.



Let AO meet BC at F.

Then, △DBC=△EBC...............I. 37.

Take away the com. △OBC,

∴ △ODB=△OEC.

But $\frac{\triangle ODB}{\triangle OBA}=\frac{DB}{BA}$ ....VI. 1.

$=\frac{EC}{CA}$ ....VI. 2.

$=\frac{\triangle OEC}{\triangle OCA}$ ....VI. 1.

$=\frac{\triangle ODB}{\triangle OCA}$ ....Above.

∴ △OBA=△OCA ....V. 9.

and they stand on the same base OA,

∴ their alts. are equal.......................Ex. 4, p. 67.

i.e., ⊥ BM=⊥ CN.

Hence in △s BFM, CFN,

$$\because \begin{cases} \angle BFM=\angle CFN, \\ \angle BMF=\angle CNF, \\ BM=CN. \end{cases}$$

∴ BF=CF.............................I. 26.

Q.E.D.

EXERCISES.

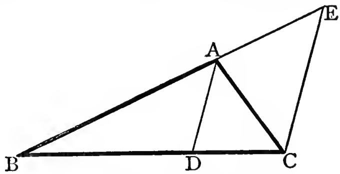
1. If two straight lines are cut by three parallels, they are cut proportionally. Prove this and the converse.
2. If a quadrilateral have two of its sides parallel, the line joining the middle points of these sides will pass through the point of intersection of its diagonals.
3. If O is a fixed point, and the ratio OP : OQ is constant, then if the locus of P is a straight line, so also is that of Q.
4. If from any point O in the diagonal AC of a quadrilateral ABCD, OE, OF be drawn parallel to AB, AD and meeting BC, CD in E, F; then EF shall be parallel to BD.
5. From a given point P in the side AB of a triangle ABC, draw a straight line to AC produced which shall be bisected by BC.

PROPOSITION III. THEOREM.

If the vertical angle of a triangle be bisected by a straight line which also cuts the base, the segments of the base shall have the same ratio which the other sides of the triangle have to one another; and, conversely, if the segments of the base have the same ratio which the other sides of the triangle have to one another, the straight line drawn from the vertex to the point of section shall bisect the vertical angle.

PART I.—Let ∠BAC of △ABC be bisected by AD meeting BC at D.

Then shall BD be to DC as BA to AC.



Through C draw CE ∥ to AD, meeting BA prod[d] at E.

Then since AD, CE are ∥,

∴ ∠BAD = ∠AEC .... I. 29, ii.
and ∠DAC = ∠ACE .... I. 29, i.
But ∠BAD = ∠DAC .... Hyp.
∴ ∠AEC = ∠ACE.
∴ AE = AC .... I. 6.
But BD is to DC as BA is to AE .... VI. 2.
∴ BD is to DC as BA is to AC .... V. 7.

PART II.—Let BD be to DC as BA to AC;

Then shall AD bisect ∠BAC.

Through C draw CE ∥ to AD, meeting BA prod[d] at E

Then, since BD is to DC as BA is to AC .... Hyp.
and BD is to DC as BA is to AE .... VI. 2.
∴ BA is to AC as BA is to AE .... V. 11.
∴ AC = AE .... V. 9.
∴ ∠ACE = ∠AEC .... I. 5.
But ∠DAC = ∠ACE ....
and ∠BAD = ∠AEC .... } I. 29.
∴ ∠BAD = ∠DAC.
i.e., AD bisects ∠BAC.

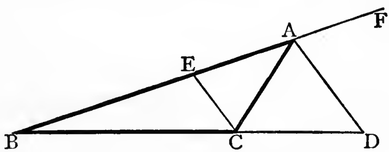
Wherefore, *if the vertical* &c. Q.E.D.

PROPOSITION A, THEOREM.

If the exterior angle of a triangle, made by producing one of its sides, be bisected by a straight line which also cuts the base produced, the segments between the dividing line and the extremities of the base shall have the same ratio which the other sides of the triangle have to one another; and, conversely, if the segments of the base produced have the same ratio which the other sides of the triangle have, the straight line drawn from the vertex to the point of section shall bisect the exterior angle of the triangle.

PART I.—Let ext^{r} ∠CAF of △ABC be bisected by AD meeting BC prod^{d} at D;

Then shall BD be to DC as BA to AC.



Through C draw CE ∥ to AD, meeting AB at E.

Then, since AD, EC are ∥,

∴ ∠FAD = ∠AEC .... I. 29, ii.
and ∠CAD = ∠ACE .... I. 29, i.
But ∠FAD = ∠CAD .... Hyp.
∴ ∠AEC = ∠ACE.
∴ AE = AC .... I. 6.
But BD is to DC as BA is to AE .... VI. 2.
∴ BD is to DC as BA is to AC .... V. 7.

PART II.—Let BD be to DC as BA to AC;

Then shall AD bisect ∠CAF.

Through C draw CE ∥ to AD, meeting AB at E.

Then, since BD is to DC as BA is to AC .... Hyp.
and BD is to DC as BA is to AE .... VI. 2.
∴ BA is to AC as BA is to AE .... V. 11.
∴ AC = AE .... V. 9.
∴ ∠ACE = ∠AEC .... I. 5.
But ∠ACE = ∠CAD .... } I. 29.
and ∠AEC = ∠FAD .... }
∴ ∠CAD = ∠FAD.
i.e., AD bisects ∠CAF.

Wherefore, *if the exterior* &c. Q.E.D.

NOTES.

Prop. A is not Euclid's; it was added by Simson.

Props. III. and A might be included in one enunciation, thus:—

If the interior and exterior angles at the vertex of any triangle be bisected by straight lines which meet the base and the base produced, they divide the base, internally and externally, into segments which have the same ratio as the sides of the triangle; and, conversely.

EXAMPLES.

I. *If the vertical angle of a triangle be bisected, both internally and externally, the bisectors divide the base harmonically.*

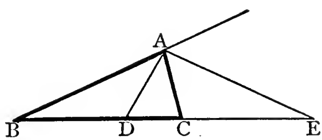
Let the bisectors meet the base in D, E.

Then $\frac{BD}{DC}=\frac{BA}{AC}$ ....VI. 3.

$=\frac{BE}{EC}$ ....VI. A.

$\therefore \frac{BD}{BE}=\frac{DC}{EC}$ ....Altern.

$=\frac{BC-BD}{BE-BC}$.

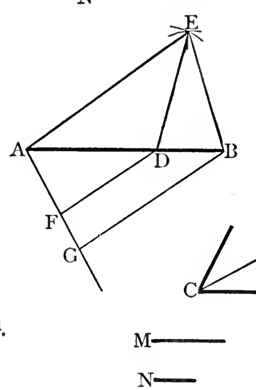


∴ BD, BC, BE are in harmonic proportion.......def. p. 222. Q.E.D.

II. *Construct a triangle, having given the base, the vertical angle and the ratio of the sides.*

Let AB be the given base, C the given vert[l] ∠, $\frac{M}{N}$ the given ratio of the sides.

Bisect ∠ C.
From A draw a st. line making any ∠ with AB, and from it cut off AF=M, FG=N.
Join GB.
Draw FD ‖ to GB and meeting AB at D.
On AD desc. a seg[t] of a ⊙ containing an $\angle=\frac{1}{2}$ C................................III. 33.
On DB desc. a seg[t] of a ⊙ containing an $\angle=\frac{1}{2}$ C, meeting the other seg[t] at E.
Join AE, EB. Then AEB shall be the △ req[d].
For, since $\angle$AED$=\frac{1}{2}$ C$=\angle$DEB......Constr.
∴ whole ∠AEB=C,

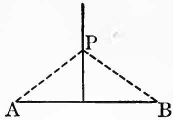


Also, AE is to EB as AD is to DB... ....VI. 3.
as AF is to FG................ ....VI. 2.
as M is to N ... .Constr.

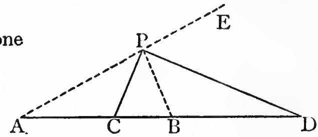
Q.E.F.

III. *Find the locus of a point which moves so that the ratio of its distances from two fixed points is constant.*

Let A, B be the fixed pts. and P the moving pt.



Case i. If the const. ratio is one of equality, *i.e.*, if, PA=PB, the locus is evidently a st. line ⊥ to AB through its mid. pt.



Case ii. If the const. ratio $\frac{M}{N}$ is not one of equality;

Divide AB at C in the given ratio (see preceding Ex.).

Join PC.

Draw PD at rt. ∠s to PC meeting AB prod^{d} at D.

Then, since AC is to CB as M is to N ....Constr.

as AP is to PB ....Hyp.

∴ PC bisects ∠APB..................................VI. 3.

and ∴PD at rt. ∠s to PC bisects the ext^{r} ∠BPE..................Ex. 6, p. 36.

∴ AD is to DB as M is to N..................VI. A.

∴ C and D are both fixed pts. for all positions of P. Hence the locus of P is that of the vertex of a rt. ∠^{d} △CPD whose hypot. CD is fixed, and this is a ⊙ on CD as diam..................Ex. 4, p. 167.

EXERCISES.

1. ABC is a triangle with the base BC bisected at D. DE, DF bisect the angles ADC, ADB, meeting AC, AB at E, F. Prove that EF is parallel to BC.

2. Trisect a line by the help of vi. 3.

3. The sides of a triangle are 3, 4, and 5 inches; find the lengths of the segments into which the bisector of the opposite angle divides the longest side.

4. A, B, C are points in a line and D is a point at which AB, BC subtend equal angles; show that the locus of D is a circle.

5. The ∠C of a △ABC is bisected by CF meeting AB at F. ∠B is bisected by BG meeting CF at G. Prove that AF : FG : : AC : CG.

6. Two circles touch internally at O. A straight line touches the inner circle at C and meets the outer circle at A, B; and OA, OB meet the inner circle at P, Q; prove that OP : OQ=AC : CB.

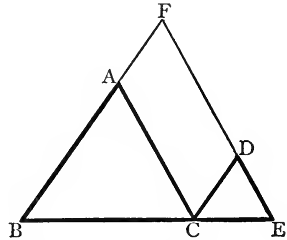
7. Interpret the result when the two sides in VI. A. are equal.

8. The straight lines bisecting one interior and two exterior angles of a triangle are concurrent.

9. If, in the figure of Example I., O is the middle point of BC, then OB is a mean proportional between OD and OE.

PROPOSITION IV. Theorem.

The sides about the equal angles of triangles which are equiangular to one another are proportionals; and those which are opposite to the equal angles are homologous sides, that is, are the antecedents or the consequents of the ratios.



Let $\triangle$s ABC, DCE
have $\angle$ABC$=\angle$DCE,
$\angle$ACB$=\angle$DEC,
and $\angle$BAC$=\angle$CDE;

Then shall BA be to BC as CD to CE,
BC be to AC as CE to DE,
and BA be to AC as CD to DE.

Let the $\triangle$s be so placed that their bases BC, CE lie in one continuous st. line with their vertices A, D on the same side of it.

Then, since $\angle$ACB$=\angle$DEC......................Hyp.
and $\angle$s ABC, ACB$<$two rt. $\angle$s..................I. 17.
$\therefore$ $\angle$s ABC, DEC$<$two rt. $\angle$s.
$\therefore$ BA, ED, if prod[d], will meet..............Ax. 12.
Prod. BA, ED to meet at F.
Then, since $\angle$ACB $=\angle$ DEC.....................Hyp.
$\therefore$AC is || to EF.......................I. 28.
and since $\angle$ABC $=\angle$ DCE.....................Hyp.
$\therefore$CD is || to BF.......................I. 28.
Hence ACDF is a ▱,
$\therefore$ AC$=$FD and CD$=$AF...................I. 34.
Again, since AC is || to EF,..............................Above.
$\therefore$ BA is to AF as BC is to CE.........VI. 2.
or BA is to CD as BC is to CE.........Above.
$\therefore$ BA is to BC as CD is to CE.........Altern.
And, since CD is || to BF................................Above.
$\therefore$ BC is to CE as FD is to DE.........VI. 2.
or BC is to CE as AC is to DE.........Above.
$\therefore$ BC is to AC as CE is to DE.........Altern.
Hence BA is to AC as CD is to DE.........Ex. æq.
Wherefore, *the sides about* &c. Q.E.D.

NOTES.

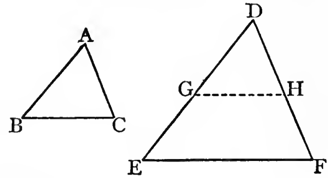
The enunciation of Prop. 4 may, by def. 1, be expressed thus:

Triangles which are equiangular to one another are **similar.**

The proposition may be easily proved by the method of superposition, thus—

Let △s ABC, DEF,
have ∠A = ∠D,
∠B = ∠E,
∠C = ∠F;

Then the △s shall be similar.



Apply △ABC to DEF, with pt. A on D and AB along DE,
Then, since ∠A = ∠D, AC must lie along DF.
Let G, H be the positions of the pts. B, C.
Join GH.

Then, since ∠G = ∠B = ∠E,
∴ GH is ∥ to EF .... I. 28.
∴ DG is to EG as DH is to HF............ VI. 2.
∴ DG is to DE as DH is to DF .... Compo.
or DG is to DH as DE is to DF............ Altern.
i.e. AB is to AC as DE is to DF.

Similarly, by applying the △s with pt. B on E, or pt. C on F, it may be shown that the sides about these ∠s are ∷ ls.

∴ the △s are similar...................... VI. def. 1.

Q.E.D.

EXERCISES.

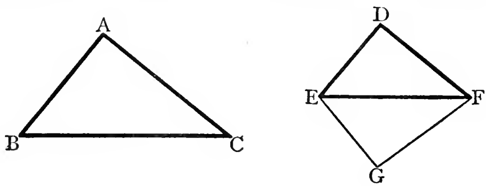
1. Show that the line joining the middle points of the sides of a triangle is half the base.
2. The shadow of a stick 3 feet long is 5 feet when the shadow of a tower is 40 yards; find the height of the tower.
3. In the figure of VI. 2, F is the middle point of BC, show that AF bisects DE.
4. AB, CD are two parallel straight lines; E is the middle point of CD; AC and BE meet at F, and AE and BD at G: show that FG is parallel to AB.
5. ABC is a triangle with angle ACB double of ABC; the bisector of angle ACB meets AB at D; prove that AC is a mean proportional between AD and AB.
6. The diagonals of a trapezium, two of whose sides are parallel and one of these double the other, cut one another at a point of trisection.
7. In the figure of I. 43, show that GE, FH will meet, if produced, on CA produced.

PROPOSITION V. Theorem.

If the sides of two triangles about each of their angles, be proportionals, the triangles shall be equiangular to one another, and shall have those angles equal which are opposite to the homologous sides.

Let the △s ABC, DEF have their sides : :ls,
namely, AB to BC as DE to EF,
BC to AC as EF to DF,
and AB to AC as DE to DF;

Then shall △ ABC be equiang^r to △DEF.



At pt. E in EF, make ∠FEG=∠ABC.
At pt. F in EF, make ∠EFG=∠ACB.
∴ rem^g ∠EGF=rem^g ∠BAC......................I. 32.
i.e. △ABC is equiang^r to △GEF.
∴ AB is to BC as GE is to EF .... I. 4.
But AB is to BC as DE is to EF ....Hyp.
∴ GE is to EF as DE is to EF ....V. 11.
∴ GE=DE..................................V. 9.

Similarly it may be shown that GF=DF.
Hence in △s GEF, DEF,

∵ { GE=DE,
GF=DF,
EF is com.

∴ ∠EGF=∠EDF ....I. 8.

Similarly ∠GEF=∠DEF,
and ∠GFE=∠DFE.
i.e. △GEF is equiang^r to △DEF,
But △ABC is equiang^r to △GEF..................Above.
∴ △ABC is equiang^r to △DEF.

Wherefore, *if the sides* &c. Q.E.D.

NOTE.

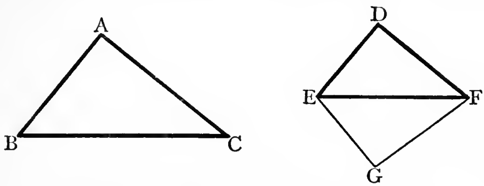
It follows from def. 1, that the △s are *similar*.
This proposition is the converse of Prop. 4.

PROPOSITION VI. Theorem.

If two triangles have one angle of the one equal to one angle of the other and the sides about the equal angles proportionals, the triangles shall be equiangular to one another, and shall have those angles equal which are opposite to the homologous sides.

Let $\triangle$s ABC, DEF have $\angle$ABC $=$ $\angle$DEF,
and let AB be to BC as DE to EF;

Then shall $\triangle$ABC be equiang^r to $\triangle$DEF.



At pt. E in EF make $\angle$FEG $=$ $\angle$ABC, or DEF.
At pt. F in EF make $\angle$EFG $=$ $\angle$ACB.
∴ rem^g $\angle$EGF $=$ rem^g $\angle$BAC .... I. 32.
i.e. $\triangle$ABC is equiang^r to $\triangle$GEF.

∴ AB is to BC as GE is to EF.................. VI. 4.
But AB is to BC as DE is to EF .... Hyp.
∴ GE is to EF as DE is to EF.................. V. 11.
∴ GE $=$ DE................................. V. 9.

Hence in $\triangle$s GEF, DEF,

∵ { GE $=$ DE
EF is com.
$\angle$GEF $=$ $\angle$DEF.

∴ $\triangle$GEF $=$ $\triangle$DEF in all respects...... I. 4.
But $\triangle$ABC is equiang^r to $\triangle$GEF............ Above.
∴ $\triangle$ABC is equiang^r to $\triangle$DEF.

Wherefore, *if two triangles* &c. Q.E.D.

NOTE.

It follows from prop. 4, since the $\triangle$s are equiang^r to one another, that their sides about their equal angles are proportionals, and consequently, from def. 1, that the triangles are *similar*.

Props. 5 and 6 can, like prop. 4, be easily proved by superposition.

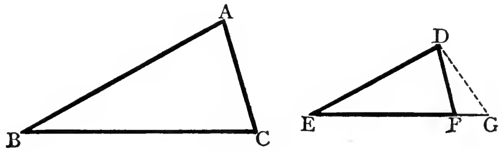
PROPOSITION VII. THEOREM.

If two triangles have one angle of the one equal to one angle of the other, and the sides about two other angles proportionals; then, if each of the remaining angles be either less, or not less, than a right angle, or if one of them be a right angle, the triangles shall be equiangular to one another, and shall have those angles equal about which the sides are proportionals.

Let △s ABC, DEF have ∠ABC=∠DEF,
and let BA be to AC as ED to DF,
also, (i) let ∠s at C and F be either both acute or both obtuse;
or, (ii) let ∠ at C be a rt. ∠;

Then shall △ ABC be equiang^r to △ DEF.

(i) For, if ∠BAC be not=∠EDF, one must be the >.
If possible, suppose ∠BAC > ∠EDF.
At pt. D in ED make ∠EDG=∠BAC.
and let DG meet EF prod^d at G.



Then, since ∠ABC=∠DEG................Hyp.
and ∠BAC=∠EDG................Constr.
∴ rem^g ∠ACB=rem^g ∠DGE..........I. 32.
i.e. △ ABC is equiang^r to △ DEG.

∴ BA is to AC as ED is to DG.........VI. 4.
But BA is to AC as ED is to DF.........Hyp.
∴ ED is to DG as ED is to DF.........V. 11.
∴ DG=DF............................V. 9.
∴ ∠DGF=∠DFG......................I. 5.
But ∠DGF=∠ACB......................Above.
∴ ∠DFG=∠ACB.
But ∠s DFE, DFG=two rt. ∠s..................I. 13.
∴ ∠s DFE, ACB=two rt. ∠s,
which is impossible,
since they are, by hyp., either both <, or both >, a rt. ∠.

Hence, ∠ BAC is not unequal to ∠ EDF,
i.e. ∠ BAC = ∠ EDF.
But ∠ ABC = ∠ DEF ....Hyp.
∴ rem^{g} ∠ ACB = rem^{g} ∠ DFE ....I. 32.
i.e. △ ABC is equiang^{r} to △ DEF.

(ii) Again, if the ∠ at C be a rt. ∠,
then it may be shown, as in the preceding case,
that ∠s ACB, DFE together = two rt. ∠s.
and ∴, in this case, ∠ DFE must also be a rt. ∠,
and, consequently, △ ABC be equiang^{r} to △ DEF.

Wherefore, *if two triangles* &c. Q.E.D.

NOTES.

From this proposition it is manifest that—

If two triangles have one angle of the one equal to one angle of the other, and the sides, taken in order, about one other angle in each proportionals; then the third angles are either equal, or supplementary; and that, in the first case, the triangles are similar.

The enunciations of VI. 5 and VI. 7 are faulty:—

In VI. 5 it should be stated that the sides must be *taken in order* about the angles; and, in VI. 7, that, of the sides which are proportionals, those *which subtend the equal angles must be homologous.*

EXERCISES.

1. A point D is taken within a triangle ABC; on BC, without the triangle, is constructed a triangle BEC similar to BDA; prove that the triangle DBE is similar to ABC.

2. The straight lines which join corresponding angular points of two similar triangles whose homologous sides are parallel will, if produced, meet in one point.

3. Distinguish between "an equiangular triangle" and "a triangle equiangular to another."

4. ABCD is a parallelogram; P, Q are points in a straight line parallel to AB; PA and QB meet at R, and PD, QC at S. Show that RS is parallel to AD.

5. In the sides AB, AC of a triangle ABC two points D, E are taken such that BD = CE; DE, BC are produced to meet at F, show that AB : AC : : EF : DF.

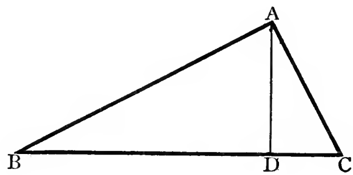
6. ABC is a common tangent to two circles whose centres are O, Q. ABC meets OQ at A, and through A a secant is drawn cutting the circles; prove that the radii to corresponding points of section are parallel.

PROPOSITION VIII. Theorem.

In a right-angled triangle, if a perpendicular be drawn from the right angle to the base, the triangles on each side of it are similar to the whole triangle and to one another.

Let ABC be a rt.-angled △ having the rt. ∠ at A, and let AD be drawn ⊥ to BC;

Then shall △s DBA, DAC be sim^r to △ABC and to each other.



For, in △s DBA, ABC,

since rt. ∠BDA=rt. ∠BAC,

and ∠ at B is com.:

∴ rem^g ∠DAB=rem^g ∠ACB .... I. 32.

i.e. △DBA is equiang^r to △ABC,

∴ these △s are sim^r .... VI. 4.

In like manner it may be shown that

△s DAC, ABC are equiang^r to one another,

and ∴ sim^r.

Hence, since △s DBA, DAC are each equiang^r to △ABC,

∴ △DBA is equiang^r to △DAC.

∴ these △s are sim^r .... VI. 4.

Wherefore, *in a right-angled triangle* &c. Q.E.D.

Cor.—From this prop. it is manifest

(i) that the ⊥ from the rt. ∠ to the hypot. is a mean : :[1] between the seg^ts of the base.

For, since △s DBA, DAC are sim^r,

∴ BD is to DA as DA is to DC............. VI. def. 1.

(ii) that each side is a mean : :[1] between the base and the seg^t of the base adj. to that side.

For, since △s DBA, ABC are sim^r,

∴ BD is to BA as BA is to BC;

and, since △s DAC, ABC are sim^r,

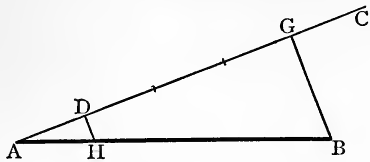
∴ DC is to CA as CA is to CB.

PROPOSITION IX. Problem.

From a given straight line to cut off any part required.

Let AB be the given st. line:

It is req^d to cut off from it a certain aliquot part.



From A draw a st. line AC making any ∠ with AB.
In AC take any pt. D.
From DC cut off, in succession, parts each=AD until the whole AG contains AD as many times as AB contains the req^d part.
Join GB.
Through D draw DH ∥ to GB, meeting AB at H.

Then shall AH be the part req^d.

For, since DH and GB are ∥,

∴ AD is to DG as AH is to HB ....VI. 2.
∴ AD is to AG as AH is to AB ....Compo.
But AG is a mult. of AD..................Constr.
∴ AB is the same mult. of AH.................V. D.
i.e. AH is the req^d part of AB.

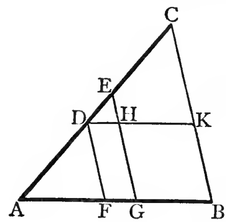
Wherefore, *from a given* &c. Q.E.F.

EXERCISES.

1. The sides of a right-angled triangle are 3 and 4 inches; find (i) the length of the perpendicular to the hypotenuse, (ii) the length of the segments of the hypotenuse.
2. Prove that the three sides and the perpendicular are four proportionals.
3. Prove that the segments of the base are to one another in the duplicate ratio of the sides, assuming the theorem on p. 249.
4. The radius of a circle is a mean proportional between the segments into which the part of any tangent intercepted by two other parallel tangents, is divided at its point of contact.
5. Trisect a given straight line. Indicate three other constructions by which this problem may be solved.
6. Given a line $\frac{4}{5}$ of an inch long, draw a line 1 inch long.
7. Cut off from a triangle one-seventh of its area.

PROPOSITION X. PROBLEM.

To divide a straight line similarly to a given divided straight line; that is, into parts which shall have the same ratios to one another that the parts of the given divided line have.



Let AB, AC be the given st. lines, AC being divided at D, E;

It is req[d] *to divide AB similarly to AC.*

Let AB, AC be so placed as to contain any ∠.

Join BC.

Through D, E, draw DF, EG ∥ to CB meeting AB in F, G.

Then AB shall be divided similarly to AC at pts. F, G.

Through D draw DHK ∥ to AB.

Then DG and HB are ▱s.

∴ DH=FG, and HK=GB ....I. 34.

But, since EH, CK are ∥ ....Constr.

∴ DH is to HK as DE is to EC.........VI. 2.

i.e. FG is to GB as DE is to EC.........Above.

Also, since DF, EG are ∥ ....Constr.

∴ AF is to FG as AD is to DE.........VI. 2.

Hence, AF is to GB as AD is to EC.........Ex. æq.

Wherefore, *the given straight line* &c. Q.E.F.

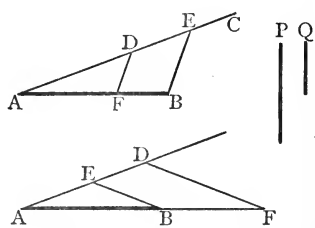
NOTE.

The following is an important special case of the problem.

To divide a given straight line into two parts which shall be in a given ratio.

Let AB be the given st. line, and P to Q the given ratio.

From A draw any st. line AC. From AC cut off AD=P.



CASE i: *Internally;*

From DC cut off DE=Q. Join EB.

Through D draw DF ∥ to EB.

CASE ii: *Externally;*

From DA cut off DE=Q. Join EB.

Through D draw DF ∥ to EB, meeting AB prod[d] at F.

Then, in both cases, since EB, DF are ∥,

∴ AF is to FB as AD is to DE.......................VI. 2.

as P is to Q ....Constr.

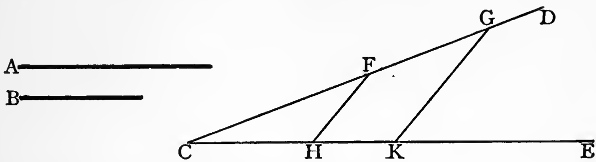
Q.E.F.

PROPOSITION XI. PROBLEM.

To find a third proportional to two given straight lines.

Let A, B, be the two given st. lines.

It is req^{d} to find a third : :^{l} to A and B.



Draw two st. lines CD, CE containing any ∠.
From CD cut off CF=A, FG=B.
From CE cut off CH=B.
Join FH.
Through G draw GK ∥ to FH, meeting CE at K.

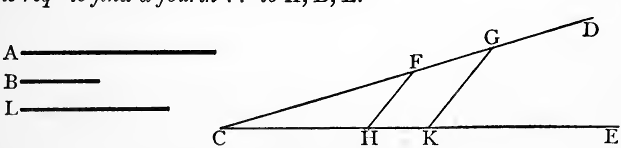
Then shall HK be the req^{d} third : :^{l}.

For, since FH, GK are ∥ ....Constr.
∴ CF is to FG as CH is to HK....................VI. 2.
i.e., A is to B as B is to HK....................Constr.
Wherefore, *has been found* &c. Q.E.F.

PROPOSITION XII. PROBLEM.

To find a fourth proportional to three given straight lines.

Let A, B, L be the three given st lines;
It is req^{d} to find a fourth : :^{l} to A, B, L.



Draw two st. lines CD, CE containing any ∠.
From CD cut off CF=A, FG=B.
From CE cut off CH=L.
Join FH.
Through G draw GK ∥ to FH, meeting CE at K.

Then shall HK be the req^{d} fourth : :^{l}.

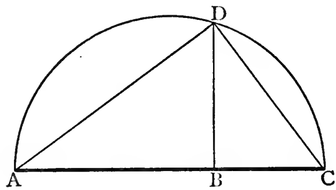
For, since FH, GK are ∥ ....Constr.
∴ CF is to FG as CH is to HK ....VI. 2.
i.e., A is to B as L is to HK ....Constr.
Wherefore, *has been found* &c. Q.E.F.

PROPOSITION XIII. PROBLEM.

To find a mean proportional between two given straight lines.

Let AB, BC be the two given st. lines;

It is required to find a mean : :[l] between them.



Place AB, BC in a st. line.

On AC desc. a semicircle ADC.

From B draw BD at rt. ∠s to AC.

Then BD shall be the req[d] mean : :[l].

Join AD, DC.

Then ∠ ADC, in a semicircle, is a rt. ∠ .... III. 31.

And, since DB is the ⊥ from the rt. ∠ to the base,

∴ DB is a mean : :[l] between AB, BC, the seg[ts] of the base .... VI. 8, cor.

Wherefore, *has been found* &c. Q.E.F.

EXERCISES.

1. The perpendicular let fall from any point in the circumference of a circle on any diameter is a mean proportional between the perpendiculars let fall from the point on the tangents at the extremities of that diameter.
2. ABC is a triangle. At A a straight line AD is drawn, making the angle CAD equal to CBA, and at C a straight line CD is drawn making the angle ACD equal to BAC. Show that AD is a fourth proportional to AB, BC, CA.
3. OAB is a triangle. Any straight line through O cuts AB at G, and a parallel to OB, drawn through A, at X, and a parallel to OA, drawn through B, at Y. Show that GO is a mean proportional between GX and GY.
4. If two circles touch externally and also touch a straight line, the part of the line intercepted between the points of contact is a mean proportional between the diameters.
5. A, B, C are three points in a straight line: find a point P in the line such that PB may be a mean proportional between PA and PC.

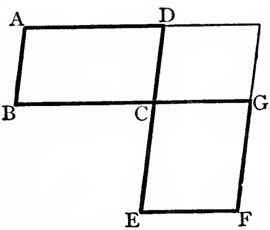
PROPOSITION XIV. Theorem.

Equal parallelograms which have one angle of the one equal to one angle of the other, have their sides about the equal angles reciprocally proportional; and, conversely, parallelograms which have one angle of the one equal to one angle of the other, and their sides about the equal angles reciprocally proportional, are equal to one another.

Part I.—Let AC, CF be = ▱s, having ∠ BCD = ∠ ECG;

Then shall BC be to CG as EC to CD.

Place the ▱s with their sides BC, CG in a st. line,
the ▱s standing on opp. sides of it.
Then EC, CD are also in a st. line .... I. 14.
Complete ▱ DG.



Then, since ▱ AC = ▱ CF .... Hyp.
∴ ▱ AC is to ▱ DG as ▱ CF is to ▱ DG .... V. 7.
But ▱ AC is to ▱ DG as BC is to CG .... VI. 1.
And ▱ CF is to ▱ DG as EC is to CD .... VI. 1.
∴ BC is to CG as EC is to CD .... V. 11.

Part II.—Let ∠ BCD = ∠ ECG,
and let BC be to CG as EC to CD;

Then shall ▱ AC = ▱ CF.

Place the ▱s with their sides BC, CG in a st. line,
and the ▱s on opp. sides of this line.
Then EC, CD are also in a st. line .... I. 14.
Complete ▱ DG.

Then BC is to CG as EC is to CD .... Hyp.
But BC is to CG as ▱ AC is to ▱ DG .... VI. 1.
and EC is to CD as ▱ CF is to ▱ DG .... V. 1.
∴ ▱ AC is to ▱ DG as ▱ CF is to ▱ DG .... V. 11.
∴ ▱ AC = ▱ CF .... V. 9.

Wherefore, *equal parallelograms* &c. Q.E.D.

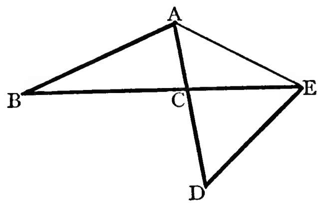
PROPOSITION XV. Theorem.

Equal triangles which have one angle of the one equal to one angle of the other, have their sides about the equal angles reciprocally proportional; and, conversely, triangles which have one angle of the one equal to one angle of the other, and their sides about the equal angles reciprocally proportional, are equal to one another.

Part I.—Let ABC, CDE be $=\triangle$s, having $\angle$BCA$=\angle$DCE;

Then shall BC be to CE as DC to CA.

Place the $\triangle$s with their sides BC, CE in a st. line, and their vertices A, D, on opp. sides of this line.
Then DC, CA are also a st. line............I. 14.
Join AE.



Then, since $\triangle$ABC$=\triangle$CDE ....Hyp.
$\therefore$ $\triangle$ABC is to $\triangle$ACE as $\triangle$CDE is to $\triangle$ACE ...V. 7.
But $\triangle$ABC is to $\triangle$ACE as BC is to CE......VI. 1.
And $\triangle$CDE is to $\triangle$ACE as DC is to CA......VI. 1.
$\therefore$ BC is to CE as DC is to CA......V. 11.

Part II.—Let $\triangle$s ABC, CDE have $\angle$BCA$=\angle$DCE,
and let BC be to CE as DC to CA;

Then shall $\triangle$ABC$=\triangle$CDE.

Place the $\triangle$s with their sides BC, CE in a st. line, and their vertices A, D on opp. sides of this line.
Then DC, CA are also in a st. line....................................I. 14.
Join AE.

Then BC is to CE as DC is to CA..........Hyp.
But BC is to CE as $\triangle$ABC is to $\triangle$ACE........VI. 1.
and DC is to CA as $\triangle$CDE is to $\triangle$ACE........VI. 1.
$\therefore$ $\triangle$ABC is to $\triangle$ACE as $\triangle$CDE is to $\triangle$ACE........V. 11.
$\therefore$ $\triangle$ABC$=\triangle$CDE........................V. 9.

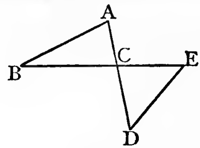
Wherefore, *equal triangles &c.* Q.E.D.

NOTE.

Prop. 14 might have been combined with prop. 15, and the truth of the prop. for parallelograms deduced from that for triangles, as in prop. 1.

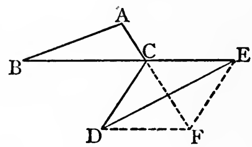
Both props. have a second converse not dealt with by Euclid; the following is a proof of this converse of prop. 15.

Equal triangles which have the sides about a pair of angles reciprocally proportional, have that pair of angles either equal, or supplementary.



Let ABC, DCE be = △s having BC to CE as DC to CA.

Then shall (i) either ∠ACB = ∠DCE, or
(ii) ∠s ACE, DCB = two rt. ∠s.



Place the △s with BC, CE in a st. line and their vertices on opp. sides of this line,

Then (i.) if DC, CA are in a st. line
∠ACB = ∠DCE .... I. 15.

But (ii.) if not, produce AC to F, make CF = CD, and join EF, DF.

Then, since BC is to CE as DC is to CA .... Hyp.
∴ BC is to CE as CF is to CA .... Above.
Also ∠FCE = ∠ACB. .... I. 15.
∴ △FCE = △ABC .... VI. 15.
But △DCE = △ABC .... Hyp.
∴ △FCE = △DCE.
∴ DF is ∥ to BE .... I. 39.
∴ ∠ACB = ∠CFD .... I. 29.
= ∠CDF .... I. 6.
∴ ∠s ACB, DCE = ∠CDF, DCE,
= two rt. ∠s .... I. 29.

Q.E.D.

EXERCISES.

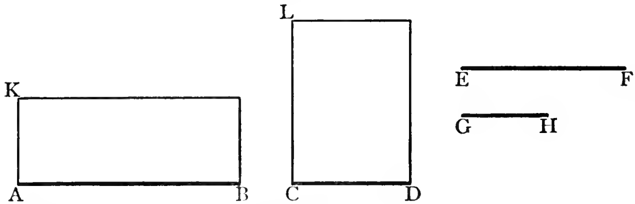
1. In the figure of VI. 15, prove that the triangles BCD, ACE are similar.
2. If, in the figure of VI. 15, BA, DE are produced and meet at F, then CF will when produced bisect BD.
3. If from the ends A, B, of the hypotenuse of a right-angled triangle, perpendiculars AE, BD be drawn meeting BC, AC produced at E, D, then triangle ACB is equal to triangle ECD.
4. Construct an isosceles triangle, the area and vertical angle being given.
5. P is any point on the side AC of the triangle ABC; CQ is parallel to BP and meets AB produced at Q; AN, AM are mean proportionals between AB, AQ and AC, AP. Prove that the triangles ANM, ABC are equal.

PROPOSITION XVI. THEOREM.

If four straight lines be proportionals, the rectangle contained by the extremes is equal to the rectangle contained by the means; and, conversely, if the rectangle contained by the extremes be equal to the rectangle contained by the means, the four straight lines are proportionals.

PART I.—Let the four st. lines AB, CD, EF, GH be $::^{ls}$,
so that AB is to CD as EF to GH;

Then shall rect. AB, GH=rect. CD, EF.



Draw AK, CL at rt. ∠s to AB, CD.
Cut off AK=GH, and CL=EF.
Complete the rectangles KB, LD.

Then, since AB is to CD as EF is to GH............Hyp.
∴ AB is to CD as CL is to AK............V. 7.
i.e. the sides about the = ∠s at A and C of the ▱s KB, LD are reciprocally $::^{l}$,
∴ ▱ KB=▱ LD........................VI. 14.
i.e. rect. AB, AK=rect. CD, CL.................II. def. 1.
∴ rect. AB, GH=rect. CD, EF...............Above.

PART II.—Let rect. AB, GH=rect. CD, EF;

Then shall AB be to CD as EF to GH.

Draw AK, CL at rt. ∠s to AB, CD.
Cut off AK=GH, and CL=EF.
Complete the rectangles KB, LD.

Then, since rect. AB, GH=rect. CD, EF.....................Hyp.
∴ rect. AB, AK=rect. CD, CL.....................Above.
i.e. ▱ KB=▱ LD.
∴ the sides about the = ∠s at A, C are reciprocally $::^{l}$,
or, AB is to CD as CL is to AK............VI. 14.
i.e. AB is to CD as EF is to GH............V. 7.

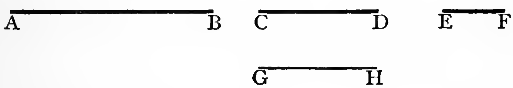
Wherefore, *if four straight lines* &c. Q.E.D.

PROPOSITION XVII. Theorem.

If three straight lines be proportionals, the rectangle contained by the extremes is equal to the square on the mean; and, conversely, if the rectangle contained by the extremes be equal to the square on the mean, the three straight lines are proportionals.

Part I.—Let the three st. lines AB, CD, EF be $: :^{ls}$,
so that AB is to CD as CD to EF;

Then shall rect. AB, EF=sq. on CD.



Draw GH=CD.

Then, since AB is to CD as CD is to EF..............Hyp.
∴ AB is to CD as GH is to EF..............V. 7.
∴ rect. AB, EF=rect. CD, GH................VI. 16.
=rect. CD, CD................Above.
=sq. on CD.

Part II.—Let rect. AB, EF=sq. on CD,

Then shall AB be to CD as CD to EF.

Draw GH=CD.

Then, since rect. AB, EF=sq. on CD....................Hyp.
=rect. CD, GH................Above.
∴ AB is to CD as GH is to EF.........VI. 16.
or AB is to CD as CD is to EF.........V. 7.

Wherefore, *if three straight lines* &c. Q.E.D.

NOTE.

VI. 17 is merely one special case of VI. 16, that in which the two means are equal.

EXERCISES.

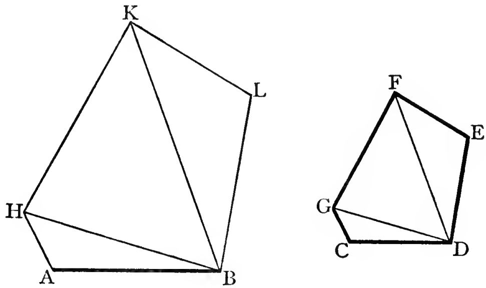
1. If the perpendicular from the right angle in a right-angled triangle divide the hypotenuse so that one of its segments is equal to one of the sides of the triangle, the hypotenuse is divided in medial section.
2. ABCD is a parallelogram; from B a straight line is drawn cutting the diagonal AC at F, the side DC at G, and the side AD produced at E; show that the rectangle EF, FG is equal to the square on BF.

PROPOSITION XVIII. PROBLEM.

On a given straight line to describe a rectilineal figure similar and similarly situated to a given rectilineal figure.

Let AB be the given st. line and CDEFG the given fig.

It is req^{d} to desc. on AB a fig. sim^{r} and similarly situated to CDEFG.



Join FD, GD.
At pt. A in AB make $\angle$BAH = $\angle$DCG.
At pt. B in AB make $\angle$ABH = $\angle$CDG.

∴ rem^{g} $\angle$AHB = rem^{g} $\angle$CGD .... I. 32.
and $\triangle$ HAB is equiang^{r} to $\triangle$ GCD.

At pt. H in HB make $\angle$BHK = $\angle$DGF.
At pt. B in HB make $\angle$HBK = $\angle$GDF.

∴ rem^{g} $\angle$HKB = rem^{g} $\angle$GFD.................... I. 32.
and $\triangle$ KHB is equiang^{r} to $\triangle$ FGD.

At pt. K in KB make $\angle$BKL = $\angle$DFE.
At pt. B in KB make $\angle$KBL = $\angle$FDE.

∴ rem^{g} $\angle$KLB = rem^{g} $\angle$FED.................... I. 32.
and $\triangle$ LKB is equiang^{r} to $\triangle$ EFB.

Let all these $\angle$s be drawn towards the same parts as the corresponding $\angle$s in the given fig.

Then shall fig. ABLKH be sim^{r} and similarly situated to CDEFG.

For, since $\angle$AHB = $\angle$CGD }
and $\angle$BHK = $\angle$DGF } .... Constr.
∴ whole $\angle$AHK = whole $\angle$CGF.

Similarly, ∠HKL = ∠GFE,
and ∠LBA = ∠EDC.

Hence, fig. ABLKH is equiang^r to CDEFG.

Again, since each pair of △s in the fig. are respectively equiang^r to one another,

∴ AB is to AH as CD is to CG..............VI. 4.

i.e. the sides about the ∠s at A, C are : :^ls.

Also AH is to BH as CG is to DG..............VI. 4.
and BH is to KH as DG is to FG.... ....VI. 4.
∴ AH is to KH as CG is to FG..............Ex. æq.

i.e. the sides about the ∠s at H, G are : :^ls.

Similarly the sides about the ∠s at K, F, and L, E, are : :^ls.

Lastly, since LB is to KB as ED is to FD.............VI. 4.
and KB is to HB as FD is to GD.............VI. 4.
∴ LB is to HB as ED is to GD.............Ex. æq.

But HB is to AB as GD is to CD.............VI. 4.
∴ LB is to AB as ED is to CD.............Ex. æq.

i.e. the sides about the ∠s at B, D, are : :^ls.

∴ fig. ABLKH is sim^r to CDEFG.

Wherefore, *on the given straight line* &c. Q.E.F.

NOTE.

The problem has been solved for the case of a five-sided figure; this includes the case of a triangle or quadrilateral, and might, obviously, be extended for a figure of six, or any number of sides.

EXERCISES.

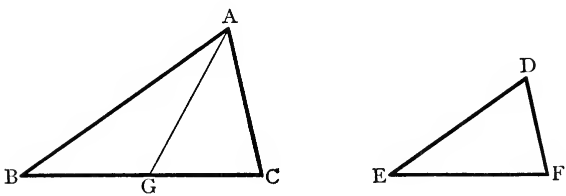
1. If HL, GE be joined, prove that the triangles KHL, FGE are similar.
2. On a given straight line AB, describe a quadrilateral similar but *oppositely* situated to the given quadrilateral CDEF.
3. In the figure of VI. 18, AB, CD, being parallel, prove that HG, KF, LC, if joined and produced, meet in one point.
4. If the middle points of adjacent sides of the figures ABLKH, CDEFG be joined, the figures so formed are similar.

PROPOSITION XIX. Theorem.

Similar triangles are to one another in the duplicate ratio of their homologous sides.

Let ABC, DEF be sim^{r} △s, having ∠B = ∠E
and sides BC, EF homologous:

Then shall △ABC be to △DEF in the dup. ratio of BC to EF.



From BC cut off BG a third : :^{l} to BC and EF................VI. 11.
Join AG.

Then, since △s ABC, DEF are sim^{r} ....Hyp.
∴ AB is to BC as DE is to EF..................VI. def. 1.
or, AB is to DE as BC is to EF..................Altern.
But BC is to EF as EF is to BG..................Constr.
∴ AB is to DE as EF is to BG..................V. 11.
i.e., the sides about the = ∠s at B, E,
of △s ABG, DEF, are reciprocally : :^{l},
∴ △ABG = △DEF ....VI. 15.

Again, since BC is to EF as EF is to BG..............Constr.
∴ BC is to BG in the dup. ratio of BC to EF.........V. def. 10.
But △ABC is to △ABG as BC is to BG.........VI. 1.
∴ △ABC is to △ABG in the dup. ratio of BC to EF.....V. 11.
∴ △ABC is to △DEF in the dup. ratio of BC to EF.....V. 7.

Wherefore, *similar triangles* &c. Q.E.D.

Cor. From this it is manifest that, if three st. lines be : :^{ls}, as the first is to the third, so is any △ desc^{d} on the first to a sim^{r}, and similarly desc^{d} △ on the second.

NOTE.

Prop. 19 is of great importance.

The following theorem will be required for some of the exercises.

If two ratios are equal, their duplicates are also equal.

Let A, B, C, D be : :[ls], such that A is to B as C is to D;

Then shall the dup. ratio of A to B=dup. ratio of C to D.

To A, B, take a third : :[l] M, and to C, D, a third : :[l] N......VI. 11.

Then, A is to B as B is to M ....Constr.
and C is to D as D is to N ....Constr.
But A is to B as C is to D ....Hyp.
∴ B is to M as D is to N ....V. 11.
Hence, A is to M as C is to N ....Ex æq.

But, since A, B, M, are in cont[d] prop[n],
∴ A is to M in dup. ratio of A to B ....V. def. 10.

and, since C, D, N, are in cont[d] prop[n],
∴ C is to N in dup. ratio of C to D ....V. def. 10.

∴ the dup. ratio of A to B=the dup. ratio of C to D.

Q.E.D.

EXERCISES.

1. State and prove the converse of the theorem given above.
2. Prove that similar triangles are to one another in the duplicate ratio of
   (i) their altitudes;
   (ii) their corresponding *medians;*
   (iii) the radii of their inscribed circles;
   (iv) the radii of their circumscribed circles.
3. ABC is a triangle with angle A greater than B : if D be taken in BC such that angle CAD=B, then CD : CB in the duplicate ratio of AD to AB.
4. Any two equilateral triangles, described on the sides of an acute-angled triangle, are together greater than the third.
5. ABC is a given triangle: construct a similar triangle of double the area.
6. ABC is a triangle, AB is produced to E, AD is a line meeting BC in D, BF is parallel to ED and meets AD in F; determine a triangle similar to ABC and equal to AEF.
7. ABC is a triangle inscribed in a circle, AD, AE, are drawn parallel to the tangents at B, C, to meet the base in D, E. Prove that BD is to CE in the duplicate ratio of AB to AC.

PROPOSITION XX. THEOREM.

Similar polygons may be divided into the same number of similar triangles, having the same ratio to one another that the polygons have; and the polygons are to one another in the duplicate ratio of their homologous sides.

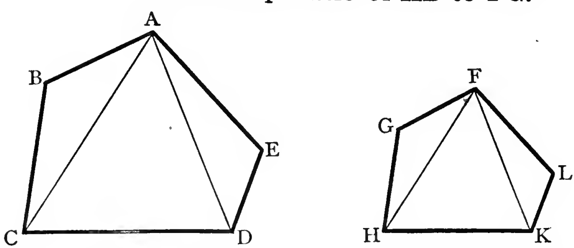
Let ABCDE, FGHKL, be sim^r polygons, with the side AB homologous to FG.

Join AC, AD, FH, FK.

Then shall (i) the polygons be divided into the same number of sim^r △s;

(ii) each pair of △s have the same ratio to one another that the polygons have;

(iii) polygon ABCDE be to polygon FGHKL in the dup. ratio of AB to FG.



(i) For, since polygon ABCDE is sim^r to polygon FGHKL...Hyp.

∴ ∠ABC=∠FGH,
and AB is to BC as FG is to GH } ....VI. def. 1.

∴ △ABC is sim^r to △FGH ....VI. 6.

∴ ∠BCA=∠GHF ....VI. def. 1.

But ∠BCD=∠GHK....................Hyp.

∴ rem^g ∠ACD=rem^g ∠FHK.

And, since △ABC is sim^r to △FGH.........Above.

∴ AC is to BC as FH is to GH..........VI. def. 1.

But BC is to CD as GH is to HK ....Hyp.

∴ AC is to CD as FH is to HK.........Ex æq.

i.e. sides about=∠s ACD, FHK, of △s ACD, FHK are : :^s,

∴ △ACD is sim^r to △FHK..............VI. 6.

Similarly, it may be shown that

△ADE is sim^r to △FKL.

(ii) Again, since △ABC is sim^r to △FGH....................Above.

∴ △ABC is to △FGH in the dup. ratio of AC to FH...VI. 19.

And, since $\triangle$ACD is sim^r to $\triangle$FHK.........Above.
∴ $\triangle$ACD is to $\triangle$FHK in the dup. ratio of AC to FH...VI. 19.
∴ $\triangle$ABC is to $\triangle$FGH as $\triangle$ACD is to $\triangle$FHK........V. 11.

Similarly, it may be shown that
$\triangle$ACD is to $\triangle$FHK as $\triangle$ADE is to $\triangle$FKL.

Hence, $\triangle$ABC is to $\triangle$FGH as sum of $\triangle$s ABC, ACD, ADE is to sum of $\triangle$s FGH, FHK, FKL......V. 12.

i.e., $\triangle$ABC is to $\triangle$FGH as fig. ABCDE is to fig. FGHKL.

(iii) Also, since $\triangle$ABC is sim^r to $\triangle$FGH.....................Above.
∴ $\triangle$ABC is to $\triangle$FGH in the dup. ratio of AB to FG...VI. 19.
Hence, polygon ABCDE is to polygon FGHKL in the dup. ratio of AB to FG...V. 11.
Wherefore, *similar polygons* &c. Q.E.D.

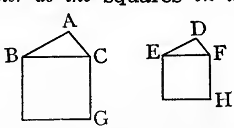
Cor. To AB, FG let a third : :^{l} MN be taken.
Then AB is to MN in the dup. ratio of AB to FG..........V. def. 10.
But, any polygon on AB is to the sim^r and similarly desc^d fig. on FG in the dup. ratio of AB to FG............Above.
∴ AB is to MN as fig. on AB is to fig. on FG.....V. 11.

i.e., If three st. lines be : :^{ls}, as the first is to the third, so is the rect^l fig. desc^d on the first to a sim^r and similarly desc^d fig. on the second.

NOTES.

By the help of this proposition it can be shown that

(i) *Similar triangles are to one another as the* squares *on their corresponding sides.*



For, let ABC, DEF be sim^r $\triangle$s and on BC, EF let sqs. BG, EH be desc^d.

Then, $\triangle$ABC is to $\triangle$DEF in the dup. ratio of BC to EF......VI. 19.

Also, since the sqs. are sim^r figs.,
∴ sq. BG is to sq. EH in the dup. ratio of BC to EF..........VI. 20.
∴ $\triangle$ABC is to $\triangle$DEF as sq. on BC is to sq. on EF.........V. 11.

In the same way it may be shown that

(ii) *Similar polygons are to one another in the ratio of the squares on their corresponding sides.*

Also, since $\triangle$ABC is to $\triangle$DEF in the dup. ratio of BC to EF, and $\triangle$ABC is to $\triangle$DEF as sq. on BC is to sq. on EF, it follows that

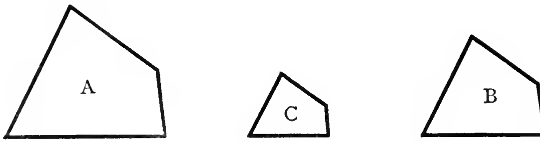
(iii) *The duplicate ratio of BC to EF is the same as the ratio of the squares on BC and EF.* (See note on page 219.)

PROPOSITION XXI. Theorem.

Rectilineal figures which are similar to the same rectilineal figure, are also similar to each other.

Let each of the rect^l figs. A and B be sim^r to the rect^l fig. C.

Then shall A be sim^r to B.



For, since A is sim^r to C .... Hyp.

∴ A is equiang. to C,

and the sides about their $=\angle$s are : :^{ls} .... VI. def. 1.

And, since B is sim^r to C.. Hyp.

∴ B is equiang. to C,

and the sides about their $=\angle$s are : :^{ls} .... VI. def. 1.

Hence, A is equiang. to B,

and the sides about their $=\angle$s are : :^{ls}................ V. 11.

∴ A is sim^r to B...................... VI. def. 1.

Wherefore, *rectilineal figures* &c. Q.E.D.

PROPOSITION XXII. Theorem.

If four straight lines be proportionals, the similar rectilineal figures similarly described on them shall also be proportionals; and, conversely, if the similar rectilineal figures similarly described on four straight lines be proportionals, those straight lines shall be proportionals.

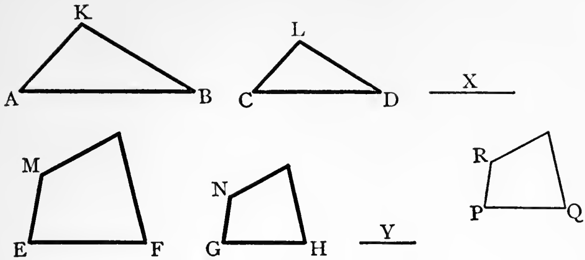
Part I.—Let AB, CD, EF, GH be four st. lines, such that AB is to CD as EF is to GH,

and on AB, CD, let the sim^r rect^l figs. KAB, LCD, be similarly desc^d,

and on EF, GH, let the sim^r rect^l figs. MF, NH, be similarly desc^d.

Then shall fig. KAB be to fig. LCD as fig. MF to fig. NH.

To AB and CD find a third : :^{l} X, and
to EF and GH find a third : :^{l} Y........................... VI. 11.



Then, since AB is to CD as CD is to X } ....Constr.
and EF is to GH as GH is to Y }

Also, AB is to CD as EF is to GH................Hyp.

∴ CD is to X as GH is to Y ....V. 11.

Hence, AB is to X as EF is to Y ....Ex æq.

But AB is to X as fig. KAB is to fig. LCD } VI. 20. cor.
and EF is to Y as fig. MF is to fig. NH }

∴ fig. KAB is to fig. LCD as fig. MF is to fig. NH....V. 11.

Part II.—Let fig. KAB be to fig. LCD as fig. MF is to fig. NH.

Then shall AB be to CD as EF to GH.

To AB, CD, EF find a fourth : :[1] PQ..............................VI. 12.

On PQ desc. a fig. RQ sim^{r} and similarly situated to MF, or NH..VI. 18.

Then, since AB is to CD as EF is to PQ......................Constr.

∴ fig. KAB is to fig. LCD as fig. MF is to fig. RQ...Part 1.

But fig. KAB is to fig. LCD as fig. MF is to fig. NH..Hyp.

∴ fig. RQ = fig. NH...................V. 9.

and RQ, NH are sim^{r} figs. similarly situated,

∴ PQ = GH.*

But AB is to CD as EF is to PQ........Constr.

∴ AB is to CD as EF is to GH........V. 7.

Wherefore, *if four straight lines* &c. Q.E.D.

NOTE.

*In this prop. Euclid assumes that—

Equal, similar and similarly described rectilineal figures have their homologous sides equal.

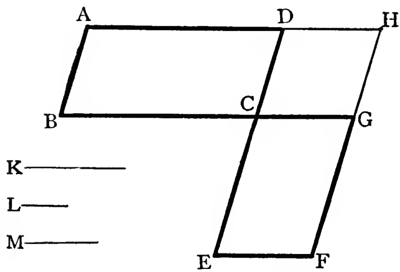
This is easily proved; for, by prop. 20, the figures are to one another in the dup. ratio of their homologous sides;
that is, in the ratio of the squares on their homologous sides.......VI. 20, note.
and, if the squares on them are equal, the sides are themselves equal.

PROPOSITION XXIII. THEOREM.

Parallelograms which are equiangular to one another have to one another the ratio which is compounded of the ratios of their sides.

Let ▱AC be equiang. to ▱CF, having ∠BCD = ∠ECG.

Then shall ▱AC be to ▱CF in the ratio compounded of the ratios of their sides.



Let the ▱s be so placed that BC, CG are in a st. line and the ▱s on opp. sides of this line.

∴ DC, CE are in a st. line................I. 14.

Complete ▱DG.

Take any st. line K.

To BC, CG, and K find a fourth ::[1] L..........................VI. 12.

To DC, CE, and L find a fourth ::[1] M..........................VI. 12.

Then BC is to CG as K is to L }
and DC is to CE as L is to M } ....Constr.

But K is to M in the ratio compounded of the ratios of K to L, and L to M........V. def. 11.

∴ K is to M in the ratio compounded of the ratios of BC to CG, and DC to CE.

But ▱AC is to ▱DG as BC is to CG................VI. 1.
as K is to L.................Constr.

and ▱DG is to ▱CF as DC is to CE...............VI. 1.
as L is to M................Constr.

∴ ▱AC is to ▱CF as K is to M................Ex æq.

∴ ▱AC is to ▱CF in the ratio compounded of the ratios of BC to CG, and DC to CE....Above.

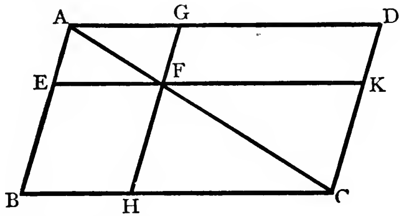
Wherefore, *parallelograms* &c. Q.E.D.

PROPOSITION XXIV. THEOREM.

Parallelograms about the diameter of any parallelogram are similar to the whole parallelogram and to one another.

Let ABCD be a ▱, AC a diam., and EG, HK, ▱s about the diam.

Then shall ▱s EG, HK, be sim^r to ▱ABCD, and to one another.



For, since DC is ‖ to GF...Hyp.

∴ ∠ADC=∠AGF.......................I. 29.

and, since BC is ‖ to EF ....Hyp.

∴ ∠ABC=∠AEF ....I. 29.

also, ∠BCD, and ∠EFG, are each=∠BAD......................I. 34.

∴ ∠BCD=∠EFG.

Hence ▱ABCD is equiang. to ▱AEFG.

Again, in △s ABC, AEF,

since ∠ABC=∠AEF.....................Above.

and ∠BAC is com.

∴ △ABC is equiang. to △AEF ....I. 32.

∴ AB is to BC as AE is to EF...............VI. 4.

But BC=AD, and EF=AG ....I. 34.

Hence, AB is to AD as AE is to AG...............V. 7.

Also DC=AB, and GF=AE ....I. 34.

Hence, DC is to BC as GF is to EF }
and DC is to AD as GF is to AG } ....V. 7.

i.e. sides about =∠s of ▱s ABCD, AEFG are : :^{ls},

∴ ▱ABCD is sim^r to ▱AEFG............VI. def. 1.

In the same way it may be shown that

▱ABCD is sim^r to ▱FHCK.

∴ ▱AEFG is sim^r to ▱FHCK............VI. 21.

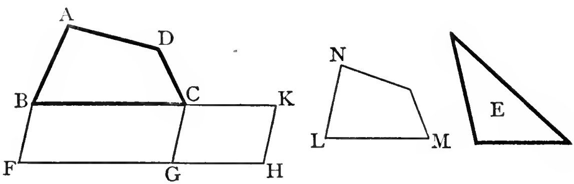
Wherefore, *parallelograms* &c. Q.E.D.

PROPOSITION XXV. PROBLEM.

To describe a rectilineal figure which shall be similar to one, and equal to another given rectilineal figure.

Let ABCD be one, and E the other, given rect^{l} fig.

It is req^{d} to desc. a rect^{l} fig. sim^{r} to ABCD and = E.



To BC apply a ▱BFGC=fig. ABCD.
To GC apply a ▱CGHK=fig. E, having ∠GCK=∠FBC.

Then BC and CK are in one st. line }
and FG and GH are in one st. line } ....I. 45.

Between BC and CK find a mean : :[1] LM ....VI. 13.

On LM desc. a rect^{l} fig. NM sim^{r} and similarly situated to fig. ABCD ....VI. 18.

Then NM shall be the fig. req^{d}.

For, since BC is to LM as LM is to CK ......Constr.
∴ B C is to CK as fig. AC is to fig. NM......VI. 20, cor.
But BC is to CK as ▱BG is to ▱CH ......VI. 1.
∴ fig. AC is to fig. NM as ▱BG is to ▱CH ......V. 11.
But fig. AC =▱BG......................Constr.
∴ fig. NM=▱CH ....V. 9.
=fig. E ....Constr.
And fig. NM is also sim^{r} to fig. AC ....Constr.

Wherefore, *has been described* &c. Q.E.F.

NOTE.

This prop., inserted as it is between VI. 24 and its converse VI. 26, appears somewhat out of place; it might well have followed prop. 26.

It may be thus enunciated:—

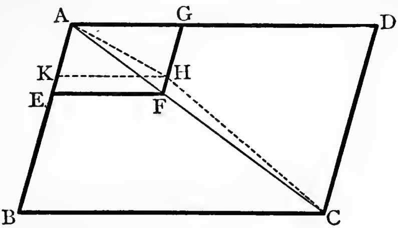
To make a rectilineal figure having the *size* of one, and the *shape* of another, given rectilineal figure.

PROPOSITION XXVI. THEOREM.

If two similar parallelograms have a common angle, and be similarly situated, they are about the same diameter.

Let the ▱s ABCD, AEFG be sim^r and similarly situated, and have the com. ∠BAD.

Then shall ▱s ABCD, AEFG, be about the same diam.



Join AC.

Then, if the diam. AC of ▱ABCD do not pass through F,
let it, if possible, cut FG, or FG prod^d, at H.
Through H draw HK || to AD, or BC, and meeting AB at K.

Then, since ▱s ABCD, AKHG are about the same diam.
∴ ▱ABCD is sim^r to ▱AKHG..........VI. 24.
∴ BA is to AD as KA is to AG...........VI.def.1.
But, since ▱ABCD is sim^r to ▱AEFG....................Hyp.
∴ BA is to AD as EA is to AG...........VI.def.1.
Hence KA is to AG as EA is to AG...........V. 11.
∴ KA=EA ....V. 9.
which is absurd ....Ax. 9.
∴ AC cannot pass otherwise than through F;
i.e. ▱s ABCD, AEFG are about the same diam.

Wherefore, *if two similar* &c. Q.E.D.

NOTE.

Propositions 27, 28, 29 are of no importance, and their proofs have therefore, in accordance with almost universal custom, been omitted. Their enunciations are the following:—

PROP. 27.—Of all parallelograms applied to the same straight line, and deficient by parallelograms, similar and similarly situated to that which is described upon the half of the line; that which is applied to the half, and is similar to its defect, is the greatest.

NOTE.

Prop. 28.—To a given straight line to apply a parallelogram equal to a given rectilineal figure, and deficient by a parallelogram similar to a given parallelogram; but the given rectilineal figure to which the parallelogram to be applied is to be equal, must not be greater than the parallelogram applied to half of the given line, having its defect similar to the defect of that which is to be applied; that is, to the given parallelogram.

Prop. 29.—To a given straight line to apply a parallelogram equal to a given rectilineal figure, exceeding by a parallelogram similar to another given.

PROPOSITION XXX. Problem.

To cut a given straight line in extreme and mean ratio.

Let AB be the given st. line.

It is req^d to divide it in extreme and mean ratio.

Divide AB at pt. C so that rect. AB, BC=sq. on AC ......II. 11.

Then shall C be the pt. req^d.

For, since rect. AB, BC=sq. on AC..........................Constr.

∴ AB is to AC as AC is to BC ....VI. 17.

Wherefore, *the given straight line* &c. Q.E.F.

NOTE.

To divide a line in extreme and mean ratio is equivalent to dividing it *in medial section.*

EXERCISES.

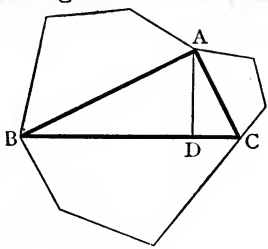
1. If a semicircle be described on AB, and CD be drawn at right angles to AC and meeting the circumference at D, then will AC be equal to BD.
2. Any two intersecting diagonals of a regular pentagon cut one another in extreme and mean ratio.
3. If the radius of a circle be cut in extreme and mean ratio, the greater segment will be equal to a side of a regular decagon inscribed in the circle.
4. Construct a quadrilateral similar to a given quadrilateral, but one-third of its area.
5. Construct a regular pentagon equal in area to a given regular hexagon.

PROPOSITION XXXI. THEOREM.

In any right-angled triangle, any rectilineal figure described on the side subtending the right angle is equal to the similar and similarly described figures on the sides containing the right angle.

Let ABC be a rt. ∠ᵈ △, having the rt. ∠ at A, and having simʳ rectˡ figs. similarly descᵈ on its sides.

Then shall fig. on BC = sum of simʳ and similarly descᵈ figs. on BA and AC.



From A draw AD ⊥ to BC.

Then, since AD is the ⊥ from the rt. ∠ on the hypot.,

∴ △ABC is simʳ to △ABD ....VI. 8.

∴ BC is to BA as BA is to BD...............VI. def. 1.

∴ BC is to BD as fig. on BC is to simʳ fig. on BA...VI.20. cor.

Hence BD is to BC as fig. on BA is to simʳ fig. on BC....invert.

Similarly, it may be shown that

CD is to BC as fig. on AC is to fig. on BC.

Hence, sum of BD, CD is to BC

as sum of figs. on BA, AC is to fig. on BC...V. 24.

But, sum of BD, CD = BC.

∴ sum of figs. on BA, AC = fig. on BC.

Wherefore, *in any right-angled triangle* &c. Q.E.D.

NOTES.

This important prop. is an extension of prop. 47 of Book I. It might be stated thus:—

In any right-angled triangle, the polygon described on the hypotenuse is equal to the sum of the similar and similarly described polygons on the other two sides.

Prop. 32 is omitted for reasons similar to those stated in the note on props. 27, 28, 29. The enunciation is as follows:—

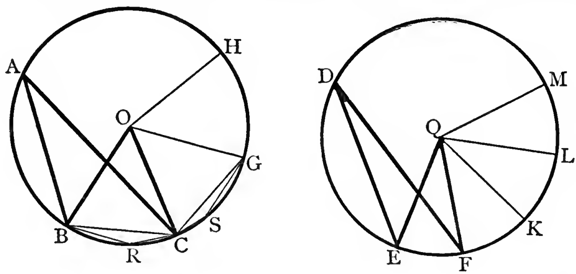
If two triangles which have two sides of the one proportional to two sides of the other, be joined at one angle so as to have their homologous sides parallel to one another, the remaining sides shall be in a straight line.

PROPOSITION XXXIII. Theorem.

In equal circles, angles, whether at the centres or at the circumferences, have the same ratio which the arcs on which they stand have to one another; so also have the sectors.

Let ABC, DEF be = ⊙s; let BOC, EQF be ∠s at their cents. O, Q; and BAC, EDF ∠s at their ○ces.

Then shall (i) ∠BOC be to ∠EQF as arc BC to arc EF;
(ii) ∠BAC be to ∠EDF as arc BC to arc EF;
(iii) sect. OBC be to sect. QEF as arc BC to arc EF.



From the ○ce ABC cut off any no. of arcs CG, GH, each=BC.
From the ○ce DEF cut off any no. of arcs FK, KL, LM, each=EF.
Join OG, OH, QK, QL, QM.

Then, (i) since arcs BC, CG, GH are all equal,
∴ ∠s BOC, COG, GOH are all equal...........III. 27.
∴ ∠BOH is the same mult. of ∠BOC that arc BH is of BC.
Similarly, it may be shown that
∠EQM is the same mult. of ∠EQF that arc EM is of EF.

Hence, ∠BOH, arc BH, are equimults. of ∠BOC, arc BC, the first and third,
and ∠EQM, arc EM, are equimults. of ∠EQF, arc EF, the second and fourth,
of the four mags. ∠BOC, ∠EQF, arc BC, arc EF;

also, if arc BH=arc EM, ∠BOH = ∠EQM ......III. 27.
if arc BH > arc EM, ∠BOH > ∠EQM,
if arc BH < arc EM, ∠BOH < ∠EQM. } III. 27, Note.

∴ ∠BOC is to ∠EQF as arc BC is to arc EF...........V def. 5.

And, (ii) since ∠BOC is double of ∠BAC
and ∠EQF double of ∠EDF } ....III. 20.

∴ ∠BAC is to ∠EDF as arc BC is to arc EF...........V. 15.

Again, (iii) Join BC, CG.
In arcs BC, CG, take any pts. R, S.
Join BR, RC, CS, SG.

Then, in △s OBC, OCG,

∵ { OB=OG .... Radii.
OC is com.:
∠BOC=∠COG .... Above.

∴ BC=CG,
and △OBC=△OCG, } .... I. 4.

Now, since arc BC=arc CG .... Constr.
and the whole ○ces are equal .... Hyp.
∴ rem^{g} arc BAC=rem^{g} arc CAG .... Ax. 3.
∴ ∠BRC=∠CSG .... III. 27.
Hence, seg^{t} BRC is sim^{r} to seg^{t} CSG .... III. def. 11.
and they are on equal st. lines BC, CG .... Above.
∴ seg^{t} BRC=seg^{t} CSG .... III. 24.
But △OBC=△OCG .... Above.
∴ sect. OBC=sect. OCG .... Ax. 2.

Similarly it may be shown that
sect. OGH=sect. OBC, or OCG;
and that sects. QEF, QFK, QKL, QLM are all equal.

∴ sect. OBH is the same mult. of sect. OBC that arc BH is of BC,
and sect. QEM is the same mult. of sect. QEF that arc EM is of EF.

Hence, sect. BOH, arc BH, are equimults. of sect. BOC, arc BC,
the first and third,
and sect. EQM, arc EM, are equimults. of sect. QEF, arc EF,
the second and fourth,
of the four mags. sect. OBC, sect. QEF, arc BC, arc EF;
also, if arc BH=arc EM, sect. OBH=sect. QEM,
if arc BH>arc EM, sect. OBH>sect. QEM,
if arc BH<arc EM. sect. OBH<sect. QEM.

∴ sect. OBC is to sect. QEF as arc BC is to arc EF ...... V. def. 5.

Wherefore, *in equal circles* &c. Q.E.D.

NOTE.

In this proposition Euclid appears to have set aside any restriction as to the size of an angle being limited by two right angles; for the angle BOH may be *any multiple* of the angle BOC.

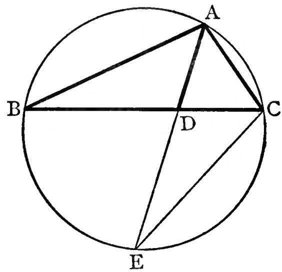
This prop. depends on none of the preceding props. of Bk. VI., and so might have been placed anywhere in the book.

PROPOSITION B. THEOREM.

If the vertical angle of a triangle be bisected by a straight line which likewise cuts the base, the rectangle contained by the sides of the triangle is equal to the rectangle contained by the segments of the base, together with the square on the straight line which bisects the angle.

Let ABC be a △, with ∠BAC bisected by AD.

Then shall rect. BA, AC=rect. BD, DC with sq. on AD.



Desc. a ⊙ about △ABC...IV. 5.
Prod. AD to meet the Oce in E.
Join EC.

Then, since ∠BAD=∠EAC.................................Hyp.
and ∠ABD=∠AEC in same seg^{t}..................III. 21.
∴ △ABD is equiang^{r} to △AEC.........................I. 32.
∴ BA is to AD as EA is to AC.........................VI. 4.
∴ rect. BA, AC=rect. EA, AD.........................VI. 16.
=rect. ED, DA, with sq. on AD ......II. 3.
=rect. BD, DC, with sq. on AD ......III. 35.

Wherefore, *if the vertical angle* &c. Q.E.D.

NOTE.

Props. B, C, D were inserted by Simson.

EXERCISES.

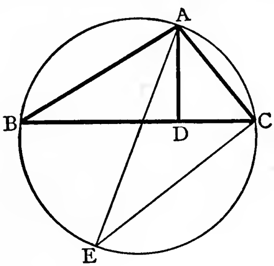
1. State and prove the converse of Prop. B.
2. Prove that, if the exterior angle at A be bisected by AD meeting the base produced at D, then rect. AB, AC = rect. BD, DC – AD^2.
3. Prove that in equal circles equal sectors stand on equal arcs.
4. Prove, what is assumed in Prop. 33, that
   (i) in equal circles the greater angle stands upon the greater arc.
   (ii) in equal circles the greater sector stands upon the greater arc.

PROPOSITION C. THEOREM.

If from the vertical angle of a triangle a straight line be drawn perpendicular to the base, the rectangle contained by the sides of the triangle is equal to the rectangle contained by the perpendicular and the diameter of the circle described about the triangle.

Let ABC be a △, with AD ⊥ to BC.

Then shall rect. BA, AC = rect. cont^{d} by AD and the diam^{r} of the ⊙ desc^{d} about the △.



Desc. a ⊙ about △ABC ....IV. 5.
Draw diam^{r} AE.
Join EC.

Then, ∠ECA in a semicircle is a rt. ∠ ....III. 31.
And, since rt. ∠BDA = rt. ∠ECA,
and ∠ABD = ∠AEC in same seg^{t} ....III. 21.
∴ △ABD is equiang^{r} to △AEC.....................I. 32.
∴ BA is to AD as EA is to AC ....VI. 4.
∴ rect. BA, AC = rect. EA, AD ....VI. 16.

Wherefore, *if from the vertical* &c. Q.E.D.

EXERCISES.

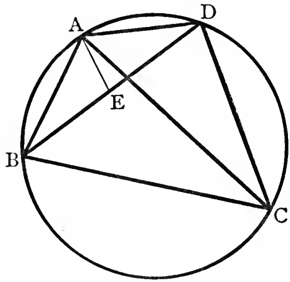
1. State and prove the converse of prop. C.
2. If AB and AC are equal, what form does the proposition assume?
3. Construct a triangle having given the base, the radius of the circumscribed circle, and the rectangle contained by the sides.

PROPOSITION D. THEOREM.

The rectangle contained by the diagonals of a quadrilateral figure inscribed in a circle is equal to both the rectangles contained by its opposite sides.

Let ABCD be a quad^{l} insc^{d} in a ⊙, and let AC, BD be its diags.

Then shall rect. BD, CA=sum of rects. AB, CD and BC, DA.



At A in BA make ∠BAE=∠DAC, and let AE meet BD at E.

Then, since ∠BAE=∠DAC..........................Constr.

Add ∠EAC to each.

∴ ∠BAC=∠EAD.

But ∠BCA=∠EDA in same seg^{t} ....III. 21.

Hence, △ABC is equiang^{r} to △AED ....I. 32.

∴ BC is to CA as ED is to DA..................VI. 4.

∴ rect. BC, DA=rect. ED, CA....................VI. 16.

Again, since ∠BAE=∠DAC..........................Constr.

and ∠ABE=∠ACD in same seg^{t} ....III. 21.

∴ △ABE is equiang^{r} to △ACD ....I. 32.

∴ AB is to BE as AC is to CD..................VI. 4.

∴ rect. AB, CD=rect. BE, CA...................VI. 16.

But rect. BC, DA=rect. ED, CA ....Above.

∴ sum of rects. AB, CD and BC, DA

=sum of rects. BE, CA and ED, CA.

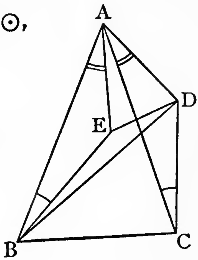
=rect. BD, CA ....II. 1.

Wherefore, *the rectangle contained* &c. Q.E.D.

NOTES.

The important proposition D is sometimes referred to as "Ptolemy's Theorem," as it occurs in his work on Astronomy. (Ptolemy lived at Alexandria about A.D. 70.) It is the special case of the following theorem—

The rectangle contained by the diagonals of any quadrilateral is less than the sum of the rectangles contained by its opposite sides, unless the quadrilateral can be inscribed in a circle.



Let ABCD be a quad^{l} which cannot be insc^{d} in a $\odot$,
i.e. having $\angle$ABD not$=\angle$ACD.
At B in AB make $\angle$ABE$=\angle$ACD.
At A in AB make $\angle$BAE$=\angle$CAD. Join ED.

Then, since $\triangle$ABE is equiang^{r} to $\triangle$ACD...Constr.
$\therefore$ AB is to BE as AC is to CD..........VI. 4.
$\therefore$ rect. AB, CD$=$rect. BE, AC........VI. 16.
Again, since $\angle$BAE$=\angle$CAD..............Constr.
Add $\angle$EAC to each,
$\therefore$ $\angle$BAC$=\angle$EAD.

Also, since $\triangle$ABE is equiang^{r} to $\triangle$ACD........................Above.
$\therefore$ BA is to AE as CA is to AD ....VI. 4.
or, BA is to CA as AE is to AD .. ..Altern.
Hence $\triangle$ABC is equiang^{r} to $\triangle$AED...............VI. 6
$\therefore$ BC is to CA as ED is to DA..................VI. 4.
$\therefore$ rect. BC, DA$=$rect. ED, AC ....VI. 16.

Hence, sum of rects. AB, CD and BC, DA
$=$sum of rects. BE, AC, and ED, AC
$=$rect. cont^{d} by AC and sum of BE, ED
$>$rect. cont^{d} by AC and BD..........................I. 20.

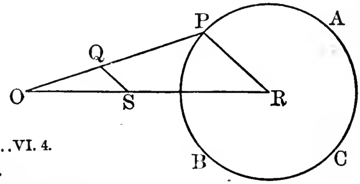
Q.E.D.

EXERCISES.

1. From the ends B, C of the base of an isosceles triangle ABC, straight lines are drawn at right angles to AB, AC, and meeting at D: show that the rectangle BC, AD is double of the rectangle AB, DB.
2. The rectangle contained by two parallel chords AB, DC of a circle ABCD is equal to the difference of the squares on AC and AD.
3. A circle is described about an equilateral triangle and, from any point in the circumference, lines are drawn to the angular points of the triangle; show that one of these lines is equal to the sum of the other two.
4. Prove that if the rectangle contained by the diagonals is equal to the sum of the rectangles contained by the opposite sides, a circle can be described about the quadrilateral. (Converse of VI. D.)

MISCELLANEOUS EXAMPLES.

I. *From a fixed point O any straight line OP is drawn cutting a fixed circle ABC at P, and in OP a point Q is taken such that the ratio of OP to OQ is constant; prove that the locus of Q is a circle.*



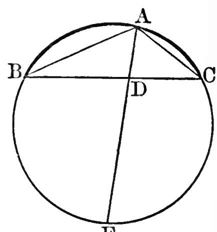
Find R the cent. of ⊙ABC.
Join OR, PR.
Through Q draw QS ∥ to PR.

Then △s OPR, OQS are sim^{r},
∴ OR is to OS as OP is to OQ...VI. 4.
i.e., OR is to OS in a const. ratio.
But, since pts. O and R are fixed, OR is const.
∴ OS is also const., and S is a fixed pt.

Again, since △s OPR, OQS are sim^{r},
∴ QS is to PR as OS is to OR.
i.e., QS is to PR in a const. ratio.
But PR is const., being the rad. of a fixed circle,
∴ QS is const.

Hence, the locus of Q is a circle having the fixed pt. S as cent. Q.E.D.

II. *To divide a given arc into two parts whose chords shall be in a given ratio.*



Let BAC be the given arc.
Join BC.
Divide BC at D in the given ratio....................VI. 10, Note.
Complete ⊙BACE ....III. 1, Note.
Bisect arc BEC at E.. ....III. 30.
Join ED.
Prod. ED to meet ○ce at A.
Join BA, AC.

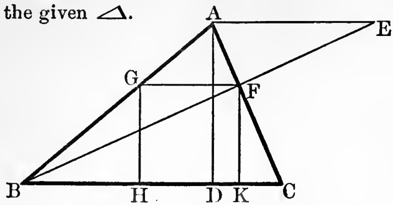
Then shall BA be to AC in the given ratio.

For, since arc BE=arc EC ....Constr.
∴ ∠BAE = ∠EAC.......................III. 26.
∴ BA is to AC as BD is to DC..............VI. 3.
i.e., BA is to AC in the given ratio.

Q.E.F.

III. *To inscribe a square in a given triangle.*

Let ABC be the given △.



Draw AD ⊥ to BC.
From A draw AE at rt. ∠s to AD, and make AE=AD.
Join BE, cutting AC at F.
Draw FG ∥ to BC or AE, and GH, FK ∥ to AD.

Then shall GHKF be the req^{d} sq.

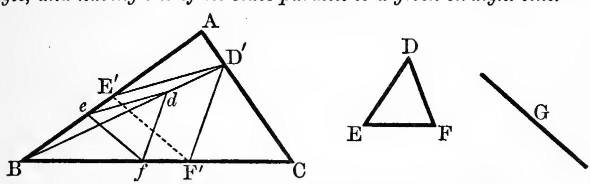
For $\frac{GF}{AE} = \frac{BG}{BA} = \frac{GH}{AD}$ ....VI. 4.

But AE = AD....................................Constr.

∴ GF = GH.......................................V. 9.

And GHKF is, by constr., a rectangle.
∴ since two of its adj. sides are equal, it is a sq.

IV. *In a given triangle to inscribe a triangle similar to another given triangle, and having one of its sides parallel to a given straight line.*



Let ABC, DEF be the given △s, and G the given st. line.
Draw any line ef ∥ to G and terminated by two sides AB, BC of the △ABC.
On ef desc. a △ def sim^{r} to DEF, so placing it that the pt. d falls within ∠ABC, and is on the side of ef remote from B.
Join Bd and prod. to meet AC at D′.
Through D′ draw D′E′ ∥ to de, and D′F′ ∥ to df.
Join E′F′.

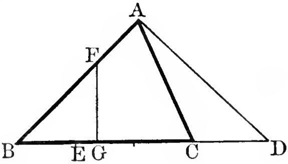
Then E′F′ shall be ∥ to ef, and, consequently, △D′E′F′ sim^{r} to △DEF.

EXERCISES.

1. In the figure of Ex. IV. prove that E′F′ is parallel to ef.
2. Inscribe a square in a given (i) sector; (ii) segment; (iii) regular pentagon.

V. *To cut off from a given triangle any part required by a line perpendicular to the base.*

Let ABC be the given $\triangle$; let it be req[d] to cut off one-third part. From A draw AD at rt. $\angle$s to BA, meeting BC prod[d] at D.



From BC cut off BE=one-third BC...VI. 9.
From BA cut off BF a mean : :[l] to BE, BD..................VI. 13.
From F draw FG $\perp$ to BC.

Then since $\angle$s FGB, DAB are rt. $\angle$s,
$\therefore$ $\triangle$s FGB, DAB are sim[r].

$$\therefore \frac{\triangle BFG}{\triangle DAB} = \frac{BF^2}{BD^2} \quad \text{..VI. 20, Note.}$$

$$= \frac{BE \cdot BD}{BD^2} \quad \text{...Constr.}$$

$$= \frac{BE}{BD} \quad \text{...VI. 1.}$$

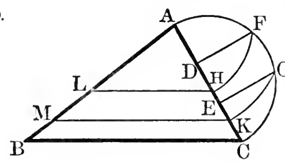
$$\text{But } \frac{\triangle ABC}{\triangle DAB} = \frac{BC}{BD} \quad \text{...VI. 1.}$$

$$\therefore \frac{\triangle BFG}{\triangle ABC} = \frac{BE}{BC} \quad \text{...V. 11.}$$

i.e. $\triangle$BFG is one-third of $\triangle$ABC. Q.E.F.

VI. *To divide a given triangle into any number of equal parts by straight lines parallel to the base.*

Let ABC be the given $\triangle$; let it be req[d] to divide it into three = parts.



Trisect AC at D, E.......................VI. 9.
On AC as diam. desc. a semicircle.
Draw DF, EG at rt. $\angle$s to AC, meeting the $\bigcirc$ce at FG.
With cent. A and radii AF, AG, desc. arcs cutting AC at H, K.
Through H, K, draw HL, KM || to BC.

$$\text{Then } \frac{\triangle ALH}{\triangle AMK} = \frac{AH^2}{AK^2} \quad \text{...VI. 20, Note.}$$

$$= \frac{AF^2}{AG^2} \quad \text{...Radii.}$$

$$= \frac{AC \cdot AD}{AC \cdot AE} \quad \text{...VI. 8, Cor. and VI. 17.}$$

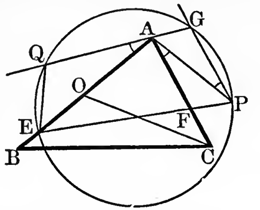
$$= \frac{AD}{AE} \quad \text{...VI. 1.}$$

i.e., $\triangle$ALH = $\frac{1}{2}$ $\triangle$AMK. $\therefore$ $\triangle$ALH = fig. LMKH.

Similarly it may be shown that fig. LMKH=fig. MBCK. Q.E.F.

VII. *To bisect a triangle by a line drawn through a given point without it.*

Let ABC be the given $\triangle$, and P the given pt.
Suppose AC to be a side adjacent to P.



Join PA. Bisect AB at O. Draw PG || to AC.
At pt. A in BA, on the side remote from P, make $\angle$BAQ $=$ $\angle$CAP.
Prod. QA to meet PG at G.
Cut off AQ a fourth : :[1] to AP, AC, AO ....VI. 12
Through the pts. P, G, Q, desc. a $\odot$ cutting AB at E.
Join EP, cutting AC at F.

Then shall EP bisect $\triangle$ABC.

Join QE, OC.

For $\angle$AQE $=$ supp[t] of $\angle$GPF ....III. 22.
$=$ $\angle$AFP ....I. 32.
Also, $\angle$QAE $=$ $\angle$FAP................................Constr.
Hence $\triangle$s AQE, AFP are equiang[r] ....I. 32.

$$\therefore \frac{AE}{AQ} = \frac{AP}{AF} \quad \text{.................................VI. 4.}$$

or, AE . AF $=$ AP . AQ ....VI. 16.

$$\text{But } \frac{AP}{AC} = \frac{AO}{AQ} \quad \text{.................................Constr.}$$

or, AP . AQ $=$ AC . AO .... VI. 16.
Hence, AE . AF $=$ AC . AO.

$$\text{or, } \frac{AE}{AC} = \frac{AO}{AF} \quad \text{.................................VI. 16.}$$

i.e., the sides about the com. $\angle$ at A of $\triangle$s AEF, AOC, are reciprocally : :[1],

$\therefore$ $\triangle$AEF $=$ $\triangle$AOC ....VI. 15.
$=$ $\frac{1}{2}$ $\triangle$ABC ....I. 38.

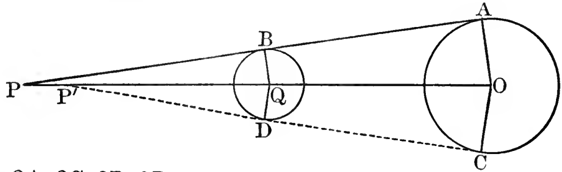
Q.E.F.

EXERCISES.

1. Bisect a triangle by a line (i) perpendicular, (ii) parallel, to the base.
2. Bisect a triangle by a line through a given point within it.

VIII. *Pairs of common tangents to two circles intersect on the line joining their centres.*

If not, if possible, let the tangs. at A, B, and at C, D, to the ⊙s, cents. O, Q, cut OQ prod^{d} at P, P′.



Join OA, OC, QB, QD.

Then, since ∠s at A, B are rt. ∠s,

∴ △s OAP, QBP are sim^{r}.

∴ PO is to AO as PQ is to BQ .... VI. 4.

or, PO is to PQ as AO is to BQ .... Altern.

as CO is to DQ .... Radii.

as P′O is to P′Q .... VI. 4.

i.e., PQ + OQ is to PQ as P′Q + OQ is to P′Q.

∴ OQ is to PQ as OQ is to P′Q .... Divid.

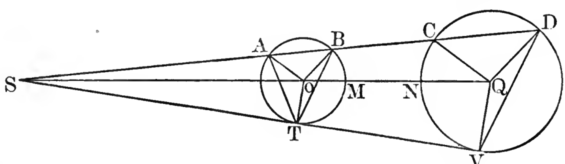
∴ PQ = P′Q .... V. 9.

i.e., P, P′ coincide. Q.E.D.

Cor.—The point of intersection of the common tangents divides the line joining the centres either externally or internally, in the ratio of the radii.

Def.—A point which divides the line joining the centres of two circles in the ratio of their radii is called a *Centre of Similitude* of the circles.

IX. *If, through a centre of similitude, S, of two circles a secant SABCD be drawn, the radii drawn to corresponding points are parallel.*



For, if OA, QC be radii to corresponding pts. A, C,
∠s OAS, QCS are both obtuse.

And OS is to QS in the ratio of the radii .... Def.

i.e., OS is to QS as OA is to QC.

Also, the ∠ at S is com.

∴ △OAS is sim^{r} to △QCS .... VI. 7.

Hence OA is ∥ to QC.

Similarly it may be shown that OB is ∥ to QD. Q.E.D.

X. *If through a centre of similitude a secant of both circles be drawn, the rectangle contained by the distances of that centre of similitude from two non-corresponding points is equal to the rectangle contained by its distances from the other two points of section.*

Let S be a cent. of simil., O, Q the cents. of the ⊙s, and SABCD the secant. (See fig. of Ex. IX.)

Join OA, OB, QC, QD.

Then, since OA, QC are ||, and OB, QD are || .... Ex. IX.

$$\therefore \frac{SA}{SC} = \frac{OA}{QC} = \frac{OB}{QD} = \frac{SB}{SD}.$$

$$\therefore SA \,.\, SD = SB \,.\, SC \text{...................... VI. 16.}$$

Q.E.D.

COR. 1.—If STV be a com. tang., and OT, QV, AT, BT, DV be joined,

then, by simr △s, $\frac{SB}{SD} = \frac{SO}{SQ} = \frac{ST}{SV}$,

∴ BT is || to DV.

Also ∠ATS = ∠ABT in alt. segt,

= ∠CDV.

∴ by simr △s, $\frac{SA}{ST} = \frac{SV}{SD}$,

or SA. SD = ST. SV, *a const. rect.*

COR. 2.—If OQ cuts the ⊙s in M, N, SMN is one special position of the secant, and, as before, SM . SN = ST . SV the const. rect.

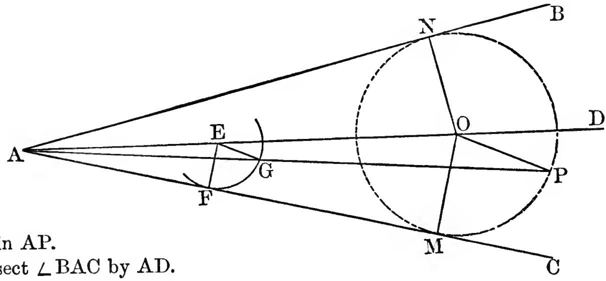
∴ SB . SC = SM . SN.

EXERCISES.

1. Prove Ex. VIII. for a pair of transverse common tangents.
2. Where are the two centres of similitude when the circles touch internally?
3. If S, S′ be the two centres of similitude of the circles whose centres are O, Q, then SOSQ′ is divided harmonically.
4. If two similar and similarly situated figures have their homologous sides parallel, the lines joining corresponding points will all meet at one point. (This point is called a *centre of similarity* of the figures.)
5. Prove Ex. 5 for figures *oppositely* situated.
6. If, through a centre of similarity O, any line OMN is drawn cutting the sides of the figures in M, N; then OM is to ON in a constant ratio.
7. In the figure of Ex. IX. prove that, if a second secant SA′B′C′D′ be drawn,
   (i) the tangs. at corresponding points A, C, are parallel;
   (ii) the chords AA′, CC′ of corresponding pairs of points are parallel;
   (iii) the chords BB′, CC′ of non-corresponding pairs of points meet on the radical axis of the circles.

XI. *To describe a circle which shall pass through a given point, and touch two given straight lines.*

If the lines are not ||, let AB, AC be the given lines and P the given pt.



Join AP.
Bisect ∠BAC by AD.
Take any pt. E in AD, and draw EF ⊥ to AC.
With cent. E, rad. EF, desc. an arc cutting AP at G. Join EG.
Through P draw PO || to GE, meeting AD at O.

Then shall O be cent. of reqd ⊙.

Draw OM, ON, ⊥s to AC, AB.

Then, since OP is || to EG .... Constr.
∴ △AOP is equiang. to △AEG .... I. 29.
∴ OP is to EG as AO is to AE .... VI. 4.
But, since △AOM is equiang. to △AEF .... Constr.
∴ OM is to EF as AO is to AE .... VI. 4.
Hence, OP is to EG as OM is to EF .... V. 11.
But EG = EF .... Radii.
∴ OP = OM .... V. 9.

Hence the ⊙ descd with cent. O and rad. OP will pass through the feet M, N of the = ⊥s OM, ON .... Ex. 6, p. 53.
and will touch AB, AC .... III. 16, Cor.

Q.E.F.

EXERCISES.

1. Solve the above problem for the case in which the given lines are parallel.

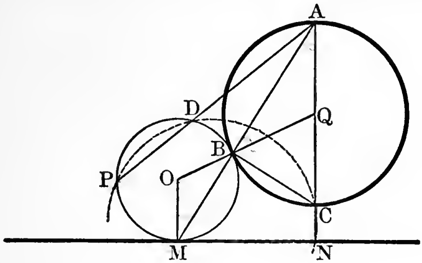
2. Inscribe in the angle BAC a circle which shall touch AB, AC and the circle MNP.

3. Prove the following construction for Ex. XI.:—
Draw PK ⊥ to AD and produce to meet AB at L; cut off KP′=KP, and in LA take LN a mean : :[1] to LP, LP′; describe a ⊙ through the pts. N, P, P′.

4. Prove that lines joining opposite extremities of parallel diameters of two circles pass through a centre of similitude.

XII. *To describe a circle which shall pass through a given point, and touch a given straight line and a given circle.*

Let ABC be the given ⊙, MN the given st. line, and P the given pt.



Find Q the cent. of ⊙ABC.
Draw QN ⊥ to MN, and prod. to meet the ○ce at A, C.
Join AP.
Desc. a ⊙ through the pts. N, C, P, cutting AP at D..................IV. 5.
Desc. a ⊙ through the pts. P, D, to touch MN at M...............Ex. 42, p. 188.

Then shall PDM be the reqd ⊙.

Join AM, cutting ⊙ABC at B. Join BC.

Then ∠ABC in a semicircle is a rt. ∠ ....III. 31.

Hence △s ABC, ANM are simr.

∴ AB is to AC as AN is to AM...............VI. 4.

∴ rect. AB, AM = rect. AC, AN ....VI. 16.

= rect. AD, AP ....III. 36, Cor.

Hence B is a pt. on ⊙PDM..........................Conv. of III. 36, Cor.

Find O the cent. of ⊙PDM, and join OM, OB, BQ.

Then, since OM, AN are ⊥s to MN,

∴ OM is ∥ to AN ....I. 28.

Hence ∠OBM = ∠OMB ....I. 6.

= ∠QAB ....I. 29.

= ∠QBA ....I. 6.

But ABM is a st. line,

∴ OBQ is also a st. line ....I. 14.

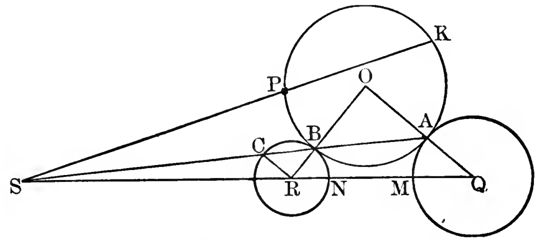
∴ ⊙PDM touches ⊙ABC at B...........Conv. of III. 12.

Q.E.F.

Since two circles can be described through P, D to touch MN, two solutions of the problem can be obtained by joining P to A, and two others by joining P to C, the other extremity of the diameter.

XIII. *To describe a circle to touch two given circles and pass through a given point.*

Let P be the given pt. and Q, R the cents. of the given ⊙s.



Join QR, cutting the ⊙s at M, N.
In QR prod^{d} find the cent. of similitude S .... Ex. VIII., p. 121.
Join SP.
From SP prod^{d} cut off SK a fourth : :^{l} to SP, SN, SM .... VI. 12.
Through P, K, desc. a ⊙, cent. O, to touch ⊙, cent. R, at B ... Ex. IX., p. 182.
Join SB, and prod. to cut the given ⊙s at A, C.

Then, since SP is to SN as SM is to SK .... Constr.
∴ SP . SK = SM . SN .... VI. 16.
= SB . SA .... Ex. X., Cor. 2.

∴ A is a pt. on the ⊙ through P, B, K .... Conv. of III. 36, Cor.

Join OA, OB, QA, RB, RC.

Then OBR is a st. line........................ III. 12.

And, since QA is || to RC........................... Ex IX.
∴ ∠QAS = ∠RCS .... I. 29.

Hence, ∠s OAB, QAS = ∠s OBA, QAS .... I. 6.
= ∠s OBA, RCS .... Above.
= ∠s RBC, RCS .... I. 15.
= ∠s RCB, RCS .... I. 6.
= two rt. ∠s..................... I. 13.
∴ OAQ is a st. line .... I. 14.

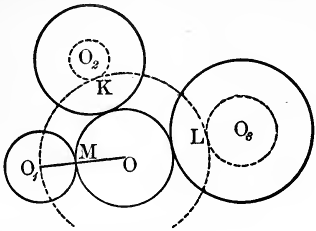
Hence ⊙ PBK touches the ⊙ whose cent. is Q, at A...... Conv. of III. 12

Q.E.F.

Since two circles can be drawn through P, K to touch the circle centre R (*i.e.*, a second touching internally on the opposite side to B), the point S gives two solutions of the problem; and by taking the other centre of similitude, which lies between Q, R, two more solutions can be obtained.

XIV. *To describe a circle which shall touch three given circles.*

Let O_1, O_2, O_3 be the cents. of the three given ⊙s, and r_1, r_2, r_3 their radii, r_1 being that which is not > either r_2 or r_3.



With cent. O_2 and rad. $= r_2 - r_1$ desc. a ⊙.
With cent. O_3, and rad. $= r_3 - r_1$ desc. a ⊙.
Desc. a ⊙ to pass through pt. O_1 and touch these two ⊙s at K, L...Ex. XIII.
Find O its cent.
Join OO_1 cutting the ○ce in M.
With cent. O rad. OM desc. a ⊙; this will be the ⊙ req[d].

NOTE.

Examples XI., XII., XIII., XIV. belong to a class of problems known as *The Tangencies; i.e.*, to describe a circle which shall, in the case of a point, pass through, and, in the case of a line or circle, touch any given three of the following nine—three points, three lines, three circles. These can be arranged in ten different sets; of which four are the examples mentioned above, two others are Exs. 4, 5 below, and for the remaining four see page 198, Prop. 4; page 200, Prop. 5; page 182, Ex. IX.; page 188, Ex. 42.

EXERCISES.

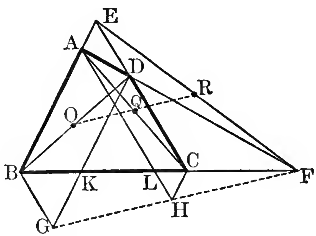
1. Draw the three other figures which fulfil the conditions of the problem (i) in Ex. XII.; (ii) in Ex. XIII.
2. Prove Ex. XIV.
3. In the figure of Ex. XIV. the circle has been described to touch all the given circles externally; how many possible solutions are there if the circle touch any one or more of the given circles internally?
4. Describe a circle which shall touch two given circles and a given straight line.
5. Describe a circle to touch two given straight lines and a given circle.
6. The Tangency solved in Ex. XII. may be indicated shortly thus:—"*point, line, circle;*" indicate the remaining nine in a similar way.

DEF.—If opposite pairs of sides of a quadrilateral be produced to meet, and their points of intersection be joined, the line which joins these points is called the *third diagonal* of the quadrilateral; and the figure thus formed is called a *complete quadrilateral.*

XV. *The middle points of the three diagonals of a complete quadrilateral lie in one straight line.*

Let ABCD be a quad[l] with BA, CD, prod[d] to meet at E, and AD, BC, prod[d] to meet at F, and let the three diags. AC, BD, EF be bisected at O, Q, R.

Then shall OQR be a st. line.



Complete ▱s EBGD, AHCE, and let DG, AH meet BC at K, L.

Then $\frac{FC}{CL} = \frac{FD}{DA} = \frac{FK}{KB}$ ....VI. 2.

$\therefore \frac{FC}{FK} = \frac{CL}{KB} = \frac{CH}{KG}$ ....VI. 4.

i.e., the sides about the $=\angle$s at C, K, of $\triangle$s FCH, FKG are : :[ls],

$\therefore$ $\triangle$FGH is equiang. to $\triangle$FKG ....VI. 6.

Hence, the side GF of $\triangle$FKG must pass through H,

i.e., GHF is a st. line.

But, since the diags. of a ▱ bisect each other..................Ex. 7, p. 57.

$\therefore$ the diag. EG of ▱ EBGD is bisected at O,

and diag. EH of ▱ EAHC is bisected at Q.

$\therefore$ OQ is $\parallel$ to GH ....Ex. p. 69.

Similarly, QR is $\parallel$ to HF.

But GHF is a st. line..........................Above.

$\therefore$ OQR is also a st. line. Q.E.D.

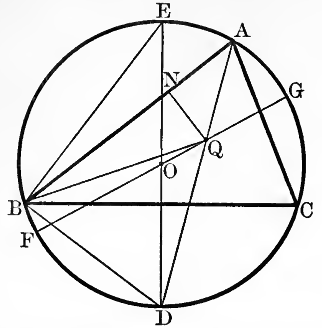
EXERCISES.

1. The circles described on the three diagonals of a complete quadrilateral have the same radical axis.

2. The four circles round the four triangles formed by four straight lines have a common point.

The method of proof used in the above prop. is due to the Rev. C. Taylor, D.D., and is inserted here with his permission.

XVI. *If O be the centre, and R the radius, of the circumscribed circle of a triangle ABC; Q the centre, and r the radius of the inscribed circle, then* $OQ^2 = R^2 - 2Rr$.



Join AQ, and prod. AQ to meet the circumscribed ⊙ at D.
Join DO, and prod. DO to meet the same ⊙ at E.
Join BD, BE, BQ, OQ, and prod. OQ to meet ○ce at F, G.
Draw QN ⊥ to AB.

Then ∠BED = ∠NAQ in same seg^t .... III. 21.
and rt. ∠ANQ = rt. ∠EBD in semicircle EBD.................. III. 31.

Hence △s EBD, ANQ are sim^r.

∴ ED is to DB as AQ is to QN .... VI. 4.
∴ ED . QN = DB . AQ .... VI. 16.
i.e., 2 R . *r* = DB . AQ.

But ∠DQB = ∠s QBA + BAQ .... I. 32.
= ½ ∠s A + B.
And ∠DBQ = ∠s DBC + CBQ.
= ∠s DAC + CBQ.................. III. 21.
= ½ ∠s A + B.
∴ DB = QD .... I. 6.

Hence, 2 R*r* = AQ . QD.
= GQ . QF .... III. 35.
= (OG − OQ) . (OF + OQ).
= (R − OQ) . (R + OQ).
= $R^2 - OQ^2$ .... II. 5, Cor.

i.e., $OQ^2 = R^2 - 2Rr$.

Q.E.D.

EXERCISE.

Prove that if S be the centre and r_a the radius of the escribed circle to the side BC, then $OS^2 = R^2 + 2Rr_a$.

The above theorem is known as *Euler's Theorem*.

MISCELLANEOUS EXERCISES.

1. Triangles on equal bases are to one another as their altitudes.

2. If ACB, BCD be equal angles and BD be perpendicular to BC and BA to AC, prove that the triangle DBC is to the triangle ABC as DC is to CA.

3. D is a point in the side AC of a triangle ABC, E a point in AB. If BD, CE divide each other into parts in the ratio 4 : 1, then D, E divide CA, BA in the ratio 3 : 1.

4. Through a given point O two straight lines are drawn meeting two fixed lines which intersect in A; one of the lines BC is bisected in O, and the other DE makes equal angles with the fixed lines: prove that AB + AC = AD + AE.

5. Enunciate the propositions which prove that in the case of triangles the conditions of similarity are not independent.

6. The side BC of a triangle ABC is produced to D, so that the triangles ABD, ACD are similar. Prove that AD touches the circle described about the triangle ABC.

7. If A′B′C′ are respectively the feet of the perpendiculars from A, B, C on the sides of the triangle ABC, show that the triangle AB′C′ is similar to ABC.

8. If D be the middle point of the side BC of the triangle ABC and if any straight line be drawn through C, meeting AD in E and AB in F, then the ratio of AE to ED will be double of the ratio of AF to FB.

9. If the vertical angle of a triangle be bisected, and if two lines be drawn through the vertex equally inclined to the bisector, one of which meets the base and the other the circumscribing circle of the triangle, the rectangle contained by them is constant.

10. ABCD is a parallelogram, and APQ is drawn cutting BC and DC produced in P and Q. If the angle ABP′ be made equal to the angle ADQ′, BP′=BP, and DQ′=DQ, show that the angles PBP′, QDQ′, P′AQ′ will be equal, and that the ratio of AP′ to AQ′ will be equal to the ratio of AP to AQ.

11. ABC is a triangle; the angle A is bisected by AD meeting BC at D; in AC a point E is taken such that AE is a third proportional to AB, AD: prove that angle CDE is half of angle BAC.

12. The angle B of a triangle ABC is bisected. Lines AD, CE are drawn from A, C at right angles to the bisector. Prove that rect. AD, BE = rect. BD, CE.

13. Draw a straight line such that the perpendiculars let fall from any point in it on two given straight lines may be in a given ratio.

14. In a triangle ABC, BC is bisected in D, AD is bisected in E, and BE is produced to meet AC in F: show that AF$=\frac{1}{3}$ AC, and EF$=\frac{1}{4}$ BF.

15. From two fixed points A, B perpendiculars AC, BD are drawn on a fixed line CD. Find, when possible, a point P in CD such that the rectangle CP, PD may be equal to the rectangle AC, BD.

16. ABC is a triangle having the angle ACB double of the angle ABC; the bisector of the angle ACB meets AB in D: prove that the square on AC is equal to the rectangle AD, AB.

17. ACB is a diameter of a circle, CD a radius perpendicular to it. The chord AD is bisected in E; BE meets CD in F and the circle in G. Show that three times the rectangle contained by BF, EG is equal to the square on the radius.

18. A tangent to a circle at the point A, intersects two parallel tangents in B, C, the points of contact of which with the circle are D, E respectively: show that if BE, CD intersect in F, AF is parallel to the tangents BD, CE.

19. Find a point in the side of a triangle from which two lines, drawn one to the opposite angle, and the other parallel to the base, shall cut off, towards the vertex and towards the base, equal triangles.

20. In the triangle ABC, AC=2 BC, CD, CE bisect angle C, and the exterior angle formed by producing AC: prove that the triangles CBD, ACD, ABC, CDE have their areas as 1 : 2 : 3 : 4.

21. If a point O be taken within a parallelogram ABCD such that the angles OBA, ODA are equal, prove that the angles OAD, OCD are equal.

22. The opposite sides BA, CD of a quadrilateral ABCD, which can be inscribed in a circle, meet, when produced, at E; F is the point of intersection of the diagonals, and EF meets AD at G: prove that the rectangle EA, AB is to the rectangle ED, DC as AG is to GD.

23. From the angular points of a parallelogram ABCD perpendiculars are drawn on the diagonals meeting them in E, F, G, and H respectively; prove that EFGH is a parallelogram similar to ABCD.

24. In a given square inscribe a square of which the area shall be equal to three-fourths of that of the given square.

25. Given the base and the ratio of the sides of a triangle, find the locus of the vertex.

26. From a given point without a circle draw a straight line cutting the circle, and dividing the diameter perpendicular to it in a given ratio.

27. AB is a diameter, and P any point in the circumference of a circle; AP and BP are joined and produced, if necessary; if from any point C of AB a perpendicular be drawn to AB, meeting AP and BP in points D and E respectively, and the circumference of the circle in a point F, show that CD is a third proportional to CE and CF.

28. In a given circle place a straight line parallel to a given straight line and having a given ratio to it; the ratio being not greater than that of the diameter to the given line.

29. ABC being a given triangle, show how to construct a similar triangle which has double the area of ABC.

30. EA, EA′ are diameters of two circles touching each other externally at E; a chord AB of the former circle when produced touches the latter at C′, while a chord A′B′ of the latter touches the former at C: prove that the rectangle contained by AB, A′B′ is four times as great as that contained by BC′, B′C.

31. Three lines AA′, BB′, CC′ drawn from the angles of a triangle to meet the opposite sides in A′, B′, C′ meet in a point P such that the ratios AP : PA′, BP : PB′, and CP : PC′ are all equal: find the position of P.

32. Draw through a given point a straight line, so that the part of it intercepted between a given straight line and a given circle may be divided at the given point in a given ratio. Between what limits must the ratio lie in order that a solution may be possible?

33. AB is a fixed chord of a circle; PQ is any chord bisected by AB: prove that the locus of the point of intersection of the tangents at P and Q is a circle.

34. Two circles touch one another at C. Any double chord PCQ is drawn through C. If P′ be the other extremity of the diameter through P, show that P′Q passes through a fixed point.

35. A, B, C, D are pts. in a st. line; AC is the harmonic mean between AB, AD. Prove that if AC is bisected in O, OC is the geometric mean between OB, OD.

36. Construct an equilateral triangle equal in area to a given triangle.

37. From a given point outside two given circles, which do not meet, to draw a straight line such that the portions of it intercepted by each circle shall be respectively proportional to their radii.

38. The diagonals AC, BD of a quadrilateral figure inscribed in a circle cut at E: show that rect. AB, BC : rect. AD, DC : : BE : ED.

39. A and B are fixed points on the circumference of a circle, and C is the middle point of the arc AB; show that, if D be any other point on the circumference, then DA + DB is to DC in a constant ratio.

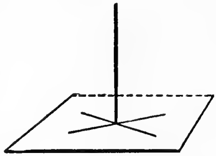
40. The straight lines EAB, EDC and FDA, FCB form four triangles in one plane; O is the common point of intersection of the circles circumscribing these triangles: prove that the rectangle contained by OA and OC is equal to the rectangle contained by OE and OF.

BOOK XI.

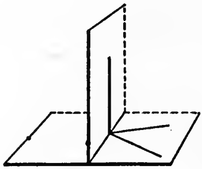
DEFINITIONS.

1. A **solid** is that which has length, breadth, and thickness.

2. That which bounds a solid is a surface.



3. A **straight line is perpendicular to a plane** when it is perpendicular to *every* straight line which can be drawn in that plane to meet it.



4. A **plane is perpendicular to another plane** when straight lines, drawn in the one plane perpendicular to the *common section* of the two planes, are also perpendicular to the other plane.

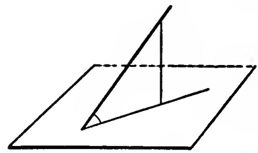
NOTES.

The first six Books of Euclid treat of *Plane* Geometry; in other words, all the lines composing the figures, the properties of which are to be considered, are supposed to lie in one plane; consequently, the figures can be easily represented on paper. But Book XI. treats of *Solid* Geometry, or Geometry in Space, and, the lines of the figures lying in different planes, these can only be represented on paper by *perspective sketches.* This constitutes the chief difficulty of the beginner in Solid Geometry, (especially if he has had no preliminary training in Drawing), as equal lines or angles are no longer necessarily represented by equal lengths or angles on the paper, as was the case in Plane Geometry. To represent a figure in Solid Geometry with the same degreee of correctness as those of Plane Geometry, it would be necessary to construct models with sheets of card-board for the different planes and wires for the lines.

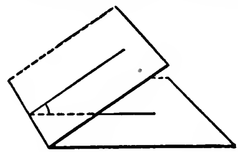
In the diagrams which follow, the lines used to indicate the position of the different planes are varied, a darker line being used to denote the edge of any plane, or figure, nearest to the observer.

Def. 1 amounts to the statement that *space* has *three dimensions.*

5. The **inclination of a straight line to a plane** is the acute angle between that line and the line joining the point at which it meets the plane to the point at which the perpendicular, drawn from any point in it to the plane, meets the plane.



6. The **inclination of a plane to a plane** is the acute angle between two straight lines drawn at right angles to the *common section* of the planes, from any point in it, one in one plane and one in the other.



7. Two planes are said to have *the same inclination* to one another which two other planes have, when their angles of inclination are equal.

8. **Parallel planes** are such as do not meet when produced.

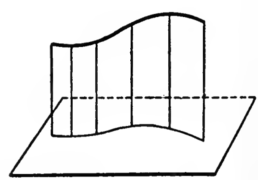
ADDITIONAL DEFINITIONS.

The line in which two planes cut one another is called their *common section.*

The point at which a perpendicular, drawn to a plane, meets the plane, is called its *foot.*

The *projection of a point on a plane* is the foot of the perpendicular drawn from the point to the plane.

The *projection of a line on a plane* is the locus of the feet of the perpendiculars drawn from all points in the line to the plane.

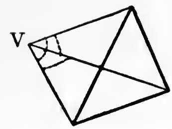


Hence, *the inclination of a line to a plane* may be defined as the angle between the line and its projection on the plane.

The inclination of a plane to a plane is called a *dihedral angle.*

A line drawn at right angles to a plane is said to be *normal* to the plane.

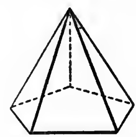
9. A **solid angle** is contained by three, or more, plane angles, which have a common vertex (V), but are not in the same plane.



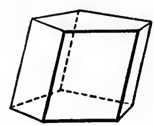
10. Equal and similar solid figures are such as are contained by similar planes equal in number and magnitude.

11. Similar solid figures are such as have all their solid angles equal, and are contained by the same number of similar planes.

12. A **pyramid** is a solid figure contained by planes, of which all but one (the *base*) pass through the same point (the *vertex*).



13. A **prism** is a solid figure contained by plane figures, two of which (the *bases*) are equal, similar, and similarly situated; and the others, consequently, are parallelograms.



ADDITIONAL DEFINITIONS.

A *trihedral angle* is a solid angle contained by *three* plane angles.

A *polyhedral angle* is a solid angle contained by more than three plane angles.

The plane surfaces of a solid figure are called its *faces;* the straight lines in which these surfaces intersect, its *edges;* the solid angles its *corners.*

A pyramid on a triangular base is called a *tetrahedron.*

A prism, the sides of which are rectangles, is called a *right prism.*

The *altitude* of a pyramid is the perpendicular drawn from its vertex to its base; and the altitude of a prism is the perpendicular distance between its bases.

NOTE.

The solid figures dealt with by Euclid are all *convex* figures; *i.e.* they have no *re-entrant* angles (see p. 60, note). Without this restriction, Def. 10 would not hold good.

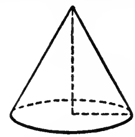
14. A **sphere** is a solid figure described by the revolution of a semi-circle about its diameter, which remains fixed.

15. The **axis** of a sphere is the fixed straight line about which the semi-circle revolves.

16. The *centre* of a sphere is that of the semi-circle.

17. A *diameter* of a sphere is a straight line drawn through the centre, terminated both ways by the surface of the sphere.

18. A **cone** is a solid figure described by the revolution of a right-angled triangle about one of the sides containing the right angle, which side remains fixed.



The cone is called right-angled, obtuse-angled, or acute-angled, according as the triangle has its fixed side equal to, less than, or greater than the other of the two sides which contain the right angle.

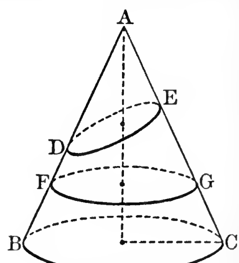
19. The **axis** of a cone is the fixed side about which the triangle revolves.

20. The **base** of a cone is the circle described by that side, (of the two containing the right angle), which revolves.

ADDITIONAL DEFINITIONS.

A *radius* of a sphere is a straight line drawn from the centre to any point in the surface of the sphere.

The solid figure of Def. 18 is often called a *right* cone; and a portion of it, ADE, cut off by a plane not parallel to the base, an *oblique* cone.



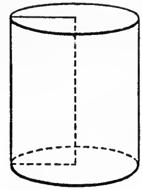
The point A is called the *vertex*.

The portion FBCG included between two parallel planes is called a *frustum*.

The *altitude* of a cone is the perpendicular distance of the vertex from the plane of the base.

The hypotenuse AB of the right-angled triangle is called a *generating line* of the cone, or its slant side.

DEFINITIONS.

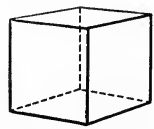


21. A **cylinder** is a solid figure described by the revolution of a rectangle about one of its sides, which remains fixed.

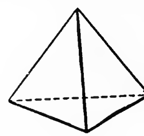
22. The *axis* of a cylinder is the fixed side about which the rectangle revolves.

23. The *bases* of a cylinder are the circles described by those opposite sides of the rectangle which revolve.

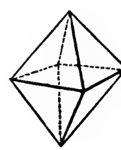
24. *Similar cones,* and *cylinders,* are those which have their axes, and the diameters of their bases, proportionals.



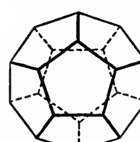
25. A **cube** is a solid figure, contained by six equal squares.



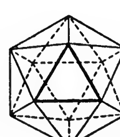
26. A **regular tetrahedron** is a solid figure contained by *four equal, equilateral triangles.*



27. A **regular octahedron** is a solid figure contained by *eight equal, equilateral triangles.*



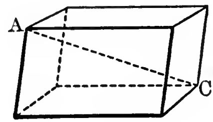
28. A **regular dodecahedron** is a solid figure contained by *twelve equal, regular pentagons.*



29. A **regular icosahedron** is a solid figure contained by *twenty equal, equilateral triangles.*

ADDITIONAL DEFINITIONS.

A **parallelepiped** is a solid figure contained by six quadrilaterals of which each opposite pair are parallel.



A *rectangular parallelepiped* has all its faces rectangles.

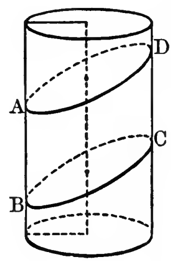
A line, AC, joining opposite corners of a parallelepiped is called a *diagonal.*

A **polyhedron** is any solid figure bounded by planes.

A *regular polyhedron* has all its faces equal and similar regular polygons.

Hexahedron, octahedron, &c., are names for polyphedra of 6, 8, &c., faces.

The solid figure of Def. 21 is often called a *right* cylinder; and a portion of it, ABCD, cut off by parallel planes, not parallel to its bases, is called an *oblique* cylinder.

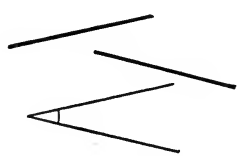


The altitude of a cylinder is the perpendicular distance between its bases.

The side of the rectangle parallel to the fixed side is called a generating line of the cylinder.

A straight line is said to be parallel to a plane when they do not meet if produced.

The angle between two lines which do not meet is the angle between two other lines parallel to them respectively, which meet at any point.



The **volume** of a solid figure is the space enclosed by its surfaces.

NOTES.

The regular tetrahedron, cube, regular octahedron, regular dodecahedron, and regular icosahedron are called *the five regular solids*. It is shown hereafter (see page 316) that there can be but five.

The sphere, cone, and cylinder are called *solids of revolution.*

The parallelepiped is a prism whose bases are parallelograms, and the cube a parallelepiped whose faces are squares.

A plane can be supposed to revolve round the straight line joining any two points in it while this line remains unmoved, and can thus occupy any position in space—hence we see that an infinite number of planes can be drawn through *two* fixed points, but only one plane through *three* points which are not all in a straight line.

EXERCISES.

1. How many faces, edges, corners, has

   (i) a tetrahedron,
   (ii) a cube,
   (iii) a hexahedron,
   (iv) an octahedron,
   (v) a dodecahedron,
   (vi) an icosahedron?

2. How many diagonals can be drawn in

   (i) a tetrahedron; (ii) a parallelepiped?

3. What is the least number of faces that

   (i) a pyramid; (ii) a prism, can have?

4. Find the length of a generating line of a cone whose altitude is 5 inches, and the diameter of its base 2 feet.

5. A line, one extremity of which is fixed, moves freely in space; what is the locus of the other extremity?

6. A line 3 inches long is inclined to a plane at an angle of (i) 30°, (ii) 45°, (iii) 60°; find its projection on the plane.

7. Find the height of a frustum of a right cone, the radii of the ends of which are 11 and 17 inches, and the slant side of which is 10 inches.

8. The height of a frustum of a right cone is 8 inches, and the radii of its ends are 9 and 12 inches; find the height of the cone.

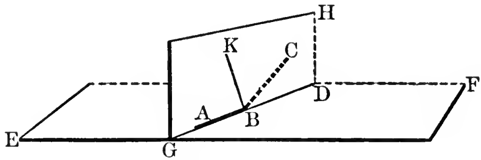
9. How many plane angles form each solid angle of an icosahedron?

10. Find the height of a pyramid on a square base, whose sides are equilateral triangles the sides of which are each 3 inches long.

PROPOSITION I. Theorem.

One part of a straight line cannot be in a plane and another part without it.

If it be possible, let the part AB, of the st. line ABC, lie in the plane EF, and the part BC without it.



Then, since the st. line AB is in the plane EF, it can be prod[d] in that plane.*

Prod. AB to any pt. D in the plane EF.

Let a plane GH, which passes through ABD, be turned about ABD until it passes through the pt. C.

Then, since B and C are pts. in this plane,

∴ the st. line BC lies in it, ... I. def. 7.

At B in this plane draw any st. line BK.

Then, since ABC is a st. line,.............................. Hyp.

∴ ∠s ABK, KBC=two rt. ∠s,.................. I. 13.

And, since ABD is a st. line................................ Constr.

∴ ∠s ABK, KBD=two rt. ∠s.................... I. 13.

∴ ∠s ABK, KBC = ∠s ABK, KBD.

Take away the com. ∠ ABK.

∴ rem[g] ∠ KBC=rem[g] ∠ KBD.

or, the part=the whole,

which is absurd.

Wherefore, *one part of a straight line* &c. Q.E.D.

NOTE.

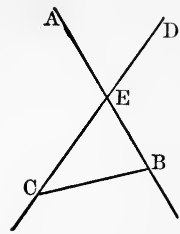
The proof of this proposition is made to depend upon that of the following: *Two straight lines cannot have a common part, or segment,* a proof of which was inserted by Simson as a corollary to Prop. 11 of Book I., but of which he made no use before Prop. 1 of Book XI. It hardly seems to need formal proof, and any necessity might be removed by some such extension of Axiom 10 as follows:—"Two straight lines cannot enclose a space; and if two st. lines meet in more than one point, they must coincide throughout their length; also, if produced, they will continue to be coincident."

* This follows from Post. 2 and the definition of a plane, and is tacitly assumed all through the first Six Books.

PROPOSITION II. THEOREM.

Two straight lines which cut one another are in one plane; and three straight lines which meet one another are in one plane.

Let the st. lines AB, CD cut at E, and in AB, CD let any pts. B, C be joined.



Then shall

(i) AB, CD be in one plane;

(ii) AB, BC, CD be in one plane.

Let any plane through AB be turned about AB until it passes through C.

Then (i), since the pts. C and E are in this plane, ∴ the whole st. line CED is in it.................. XI. 1.

i.e., AB and CD are in one plane.

Again, (ii), since the pts. B and C are in this plane,

∴ st. line BC is in it .... I., def. 7.

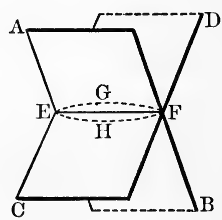
i.e., AB, BC, CD are in one plane.

Wherefore, *two straight lines* &c. Q.E.D.

PROPOSITION III. THEOREM.

If two planes cut one another their common section is a straight line.

Let the planes AB, CD cut one another in the line EF.



Then shall EF be a st. line.

For, if not, if possible, draw the st. line EGF in plane AB, and the st. line EHF in plane CD.

Then the st. lines EGF, EHF, meeting at E and F, enclose a space, which is impossible.

Hence EF cannot be other than a st. line.

Wherefore, *if two planes cut* &c. Q.E.D.

EXERCISES.

Show that but *one* plane can be made to pass through

(i) two straight lines which cut;

(ii) the sides of a triangle;

(iii) three points not *collinear* (*i.e.*, not in the same straight line);

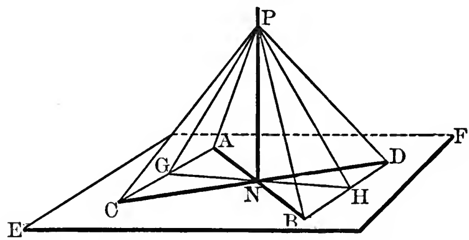
(iv) two parallel straight lines.

PROPOSITION IV. Theorem.

If a straight line be perpendicular to each of two straight lines at their point of intersection, it is perpendicular to the plane which passes through them, that is, in which they are.

Let the st. line PN be ⊥ to AB and CD at their pt. of intersection N.

Then shall PN be ⊥ to plane EF in which AB, CD lie.



Cut off NA, NB, NC, ND all equal to one another.
Join AC, BD.
Through N draw any st. line GNH, cutting AC, BD, at G, H.
Take any pt. P in PN.
Join PA, PB, PC, PD, PG, PH.

Then in △s ACN, BDN,

∵ { AN=BN .... Constr.
CN=DN .... Constr.
∠ANC=∠BND .... I. 15.

∴ AC=BD, and ∠CAN=∠DBN .... I. 4.

Hence, in △s AGN, BHN,

∵ { ∠GAN=∠HBN .... Above.
∠GNA=∠HNB .... I. 15.
AN=BN .... Constr.

∴ AG=BH, and GN=HN .... I. 26.

Again, in △s PAN, PBN,

∵ { AN=BN .... Constr.
PN is com.
rt. ∠PNA=rt. ∠PNB .... Hyp.

∴ PA=PB .... I. 4.

Similarly, PC=PD.

Hence, in $\triangle$s PAC, PBD,

$$\because \left\{ \begin{array}{l} AC = BD \\ PA = PB \\ PC = PD \end{array} \right\} \text{..................... Above.}$$

$$\therefore \angle PAC = \angle PBD \text{..................... I. 8.}$$

And, in $\triangle$s PAG, PBH,

$$\because \left\{ \begin{array}{c} PA = PB \\ AG = BH \\ \angle PAG = \angle PBH \end{array} \right\} \text{..................... Above.}$$

$$\therefore PG = PH \text{..................... I. 4.}$$

Now in $\triangle$s PGN, PHN,

$$\because \left\{ \begin{array}{l} GN = HN \text{ .... Above.} \\ PN \text{ is com.} \\ PG = PH \text{ .... Above.} \end{array} \right.$$

$$\therefore \angle PNG = \angle PNH \text{..................... I. 8.}$$

$$\therefore PN \text{ is } \perp \text{ to } GH \text{..................... I. def. 10.}$$

Similarly it may be shown that PN is $\perp$ to every st. line through N in the plane EF.

$$\therefore PN \text{ is } \perp \text{ to the plane EF .... XI. def. 3.}$$

Wherefore, *if a straight line* &c. Q.E.D.

NOTE.

The above is Euclid's proof of this important proposition; a shorter proof will be found on page 328.

EXERCISES.

1. If a line be drawn through the centre of a circle perpendicular to the plane of the circle, any point in this line is equidistant from all points in the circumference.

2. At a given point in a given plane but one straight line can be drawn at right angles to the plane.

3. The edges of a rectangular parallelepiped are 3, 4, and 12 inches long respectively; find the length of a diagonal.

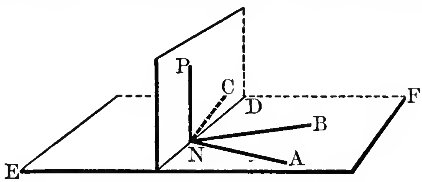
4. Prove XI. 4 from the following construction:—Let PN be perpendicular to AN, BN; in any line NC in the plane of AN, BN take a point C, and through C draw ACB such that AB is bisected at C; join PA, PB, PC. (Apply the theorem given on page 120.)

PROPOSITION V. THEOREM.

If three straight lines meet at one point, and a straight line stand at right angles to each of them at that point, the three straight lines shall be in one and the same plane.

Let PN be at rt. ∠s to AN, BN, and CN, at their pt. of intersection N.

Then shall AN, BN, and CN be in the same plane.



For if not, if possible, suppose CN to lie without the plane EF in which AB and CD are.

Let a plane through PN be turned about PN until it passes through the pt. C.

This plane will intersect the plane EF in a st. line ND.........XI. 3.

And, since PN is ⊥ to AN and BN,

∴ PN is ⊥ to the st. line DN in the same plane......XI. 4.

i.e. ∠PND is a rt. ∠.

But ∠PNC is a rt. ∠ ....Hyp.

∴ ∠PND=∠PNC,

or, the whole=its part,

which is absurd.

Hence, CN cannot lie without the plane EF.

i.e. AN, BN, CN are in the same plane.

Wherefore, *if three straight lines* &c. Q.E.D.

PROPOSITION VI. THEOREM.

If two straight lines be at right lines to the same plane, they shall be parallel to one another.

Let AB and CD be each ⊥ to the plane EF.

Then shall AB be ∥ to CD.

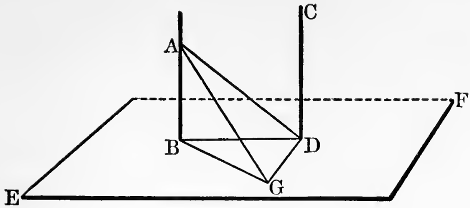
Let AB, CD meet the plane EF at the pts. B, D.

Join BD.

In plane EF draw DG at rt. ∠s to DB.

Cut off DG=AB.

Join BG, AG, AD.



Then, since AB is ⊥ to plane EF................................Hyp.

∴ each of the ∠s ABD, ABG is a rt. ∠...................XI. def. 3.

Similarly, each of the ∠s CDB, CDG is a rt. ∠.

Hence, in △s ABD, GDB,

∵ { AB=GD ....Constr.
BD is com.
rt. ∠ABD=rt. ∠GDB. }

∴ AD=BG ....I. 4.

And, in △s ABG, GDA,

∵ { AB=GD ....Constr.
BG=DA ....Above.
AG is com. }

∴ ∠ABG=∠GDA.......................I. 8.

But ∠ABG is a rt. ∠......................Above.

∴ ∠GDA is a rt. ∠.

Hence GD is at rt. ∠s to each of the st. lines BD, AD, CD, at their pt. of intersection D.

∴ BD, AD, CD are in the same plane..............XI. 5.

But AB lies in the same plane with BD and AD...........XI. 2.

∴ AB and CD are in the same plane.

Also, ∠s ABD, CDB are two rt. ∠s.......................Hyp.

∴ AB is ∥ to CD............................I. 28.

Wherefore, *if two straight lines* &c. Q.E.D.

NOTE.

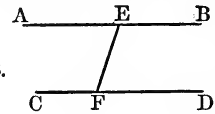
It is important to notice that AB and CD must be shown to lie in the same plane before I. 28 can be applied to prove them parallel.

PROPOSITION VII. THEOREM.

If two straight lines be parallel, the straight line drawn from any point in one to any point in the other, is in the same plane with the parallels.

Let AB be ∥ to CD, and let E, F be any pts. in them.

Then shall st. line EF lie in the plane of AB, CD.



Join EF.

Then, since AB is ∥ to CD.................Hyp.

∴ AB and CD are in the same plane...I. def. 35.

But, E and F are pts. in this plane,

∴ the st. line EF lies wholly in this plane...................I. def. 7.

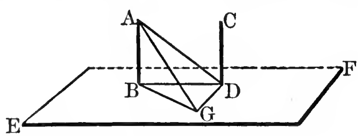
Wherefore, *if two straight lines* &c. Q.E.D.

PROPOSITION VIII. THEOREM.

If two straight lines be parallel, and one of them be at right angles to a plane, the other shall also be at right angles to the same plane.

Let AB be ∥ to CD, and let AB be ⊥ to the plane EF.

Then shall CD be also ⊥ to the plane EF.



Let B, D, be the pts. in which AB, CD, meet plane EF.

Join BD.

In plane EF, draw DG at rt. ∠s to BD.

Cut off DG=AB.

Join BG, AG, AD.

Then, since AB is ⊥ to plane EF............................Hyp.

∴ each of the ∠s ABD, ABG is a rt. ∠...............XI. def. 3.

And, since AB is ∥ to CD.....................................Hyp.

∴ ∠s ABD, CDB=two rt. ∠s.....................I. 29.

But ∠ABD is a rt. ∠ ....Above.

∴ ∠CDB is also a rt. ∠.

Again, in △s ABD, GDB,

∵ { AB=GD............................Constr.
BD is com.
rt. ∠ABD=rt. ∠GDB, }

∴ AD=GB.............................I. 4.

And, in $\triangle$s ABG, GDA,

$\because$ { AB=GD ....Constr.
GB=AD ....Above.
AG is com.

$\therefore$ $\angle$ABG=$\angle$GDA.......................I. 8.

But $\angle$ABG is a rt. $\angle$.....................Above.

$\therefore$ $\angle$GDA is also a rt. $\angle$.

Hence, since each of the $\angle$s GDB, GDA is a rt. $\angle$,

$\therefore$ GD is $\perp$ to the plane in which BD, AD lie..........XI. 4.

But, BD, AD are both in the plane of the $\parallel$s....................XI. 7.

$\therefore$ $\angle$CDG is a rt. $\angle$.......................XI. def. 1.

But $\angle$CDB is a rt. $\angle$ ....Above.

i.e. CD is $\perp$ to both BD and GD,

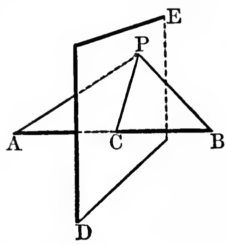
$\therefore$ CD is $\perp$ to the plane EF, in which they are....XI. 4.

Wherefore, *if two straight lines* &c. Q.E.D.

EXAMPLE.

The locus of points equidistant from two fixed points is the plane which bisects at right angles the line joining the two points.

Let A, B be the fixed pts., C the mid. pt. of AB, DE the plane bisecting AB at rt. $\angle$s.
Take any pt. P in plane DE.
Join PA, PB, PC.



Then, in $\triangle$s PAC, PBC,

$\because$ { AC=BC ....Hyp.
PC is com.
rt. $\angle$PCA=rt. $\angle$PCB.....XI. def. 3.

$\therefore$ PA=PB............I. 4.

i.e. any pt. in plane DE is equidist. from A and B.

$\therefore$ plane DE is the *locus* of such pts. Q.E.D.

EXERCISES.

1. Find the locus of straight lines which are at right angles to a fixed straight line at a fixed point.

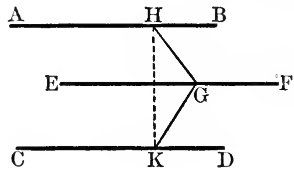
2. The perpendicular is the shortest line that can be drawn to a plane from a given external point; and of any others, those which make equal angles with the perpendicular are equal; and conversely.

PROPOSITION IX. Theorem.

Two straight lines, which are each of them parallel to a third straight line not in the same plane with them, are parallel to one another.

Let AB and CD be each || to EF.

Then shall AB be || to CD.



In EF take any pt. G.

In the plane containing AB and EF, draw GH at rt. ∠s to EF, meeting AB at H.

In the plane containing CD and EF, draw GK at rt. ∠s to EF, meeting CD at K.

Then, since EF is at rt. ∠s to GH and GK................Constr.

∴ EF is ⊥ to the plane HKG ....XI. 4.

But AB is || to EF..Hyp.

∴ AB is also ⊥ to the plane HKG............XI. 8.

Similarly, CD is ⊥ to the plane HKG.

i.e., AB and CD are both ⊥ to the plane HKG,

∴ AB is || to CD.......................XI. 6.

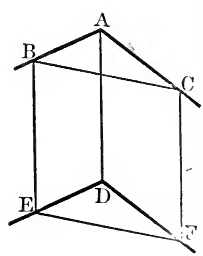
Wherefore, *two straight lines* &c. Q.E.D.

PROPOSITION X. Theorem.

If two straight lines meeting one another be parallel to two others that meet but are not in the same plane with the first two, the first two and the other two shall contain equal angles.

Let AB, AC be || to DE, DF respectively.

Then shall ∠ BAC = ∠ EDF.



Cut off AB, AC, DE, DF all equal.

Join AD, BE, CF, BC, EF.

Then, since AB is = and || to DE.....Constr.

∴ BE is = and || to AD....I. 33.

Similarly, CF is = and || to AD.

Hence BE is = and || to CF... { Ax. 1 & XI. 9.

∴ BC = EF ....I. 33.

Hence, in △s ABC, DEF,

∵ { AB = DE, AC = DF } ....Constr.

BC = EF...........................Above.

∴ ∠ BAC = ∠ EDF ....I. 8.

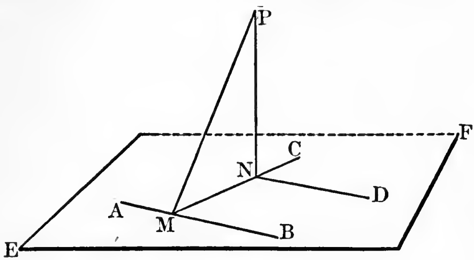
Wherefore, *if two straight lines* &c. Q.E.D.

PROPOSITION XI. PROBLEM.

To draw a straight line perpendicular to a given plane from a given point without it.

Let EF be the given plane, and P the given external pt.

It is req[d] *to draw from* P *a st. line* ⊥ *to the plane* EF.



Draw any st. line AB in the plane EF.
From P draw PM ⊥ to AB .... I. 12.

Then if PM is ⊥ to plane EF, what was req[d] is done.

But, if not, from M, in plane EF, draw MC at rt. ∠s to AB... I. 11.
From P draw PN ⊥ to MC .... I. 12.

Then shall PN be ⊥ to plane EF.

Through N, in plane EF, draw ND ∥ to AB .... I. 31.

Then, since each of the ∠s BMP, BMN is a rt. ∠ .... Constr.
∴ BM is ⊥ to the plane PMN .... XI. 4.
But DN is ∥ to BM .... Constr.
∴ DN is also ⊥ to the plane PMN .... XI. 8.
∴ ∠PND is a rt. ∠ .... XI. def. 1.
But ∠PNM is a rt. ∠ .... Constr.

i.e., PN is at rt. ∠s to DN and MN in the plane EF,

∴ PN is ⊥ to the plane EF .... XI. 4.

Wherefore, *from the given point* &c. Q.E.F.

EXERCISES.

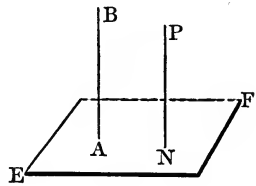
1. The projection of a straight line on a plane is a straight line.
2. If two equal straight lines are equally inclined to a plane their projections on the plane are equal.

PROPOSITION XII. Problem.

To draw a straight line at right angles to a given plane from a given point in the plane.

Let EF be the given plane, and A the given pt. in it.

It is required to draw, from A, *a st. line at rt.* ∠*s to the plane* EF.



From any external pt. P, draw
PN ⊥ to the plane ....XI. 11.
From A draw AB ∥ to PN...........I. 31.

Then shall AB be at rt. ∠s to plane EF.

For, since PN is ⊥ to plane EF,
and AB is ∥ to PN ....Constr.
∴ AB is also ⊥ to plane EF......XI. 8.

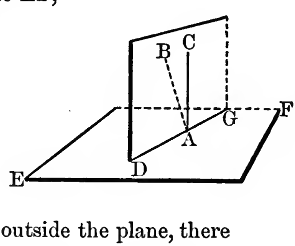
Wherefore *a straight line* &c. Q.E.F.

PROPOSITION XIII. Theorem.

From the same point, in a given plane, there cannot be two straight lines at right angles to the plane, on the same side of it; and there can be but one perpendicular to a given plane from a given point without it.

Part I.—If it be possible, at pt. A, let AB, AC be both at rt. ∠s to the plane EF, on the same side of it, and let DG be the com. section of the plane EF with the plane containing AB and AC.

Then DAG is a st. line ....XI. 3.
And, since AB is at rt. ∠s to the plane EF,
∴ ∠BAD is a rt. ∠......XI. def. 3.



Similarly ∠CAD is a rt. ∠.

∴ ∠BAD=∠CAD, in the same plane,
or, the part=the whole,
which is absurd.

Part II.—Also, from the same pt., outside the plane, there can be drawn but one st. line ⊥ to the plane,

For if there could be two, they would be ∥ ....XI. 6.
which is absurd.......................I. def. 35.

Wherefore, *from the same point* &c. Q.E.D.

PROPOSITION XIV. Theorem.

Planes to which the same straight line is perpendicular are parallel to one another.

Let AB be ⊥ to each of the planes CD and EF.

Then shall plane CD be || to plane EF.

For, if not, they will meet, when prod[d] ....XI. def. 8.

If possible, let them meet, and let GH be their com. section.

Then GH is a st. line..........XI. 3.

In GH take any pt. K.

Join KA, KB.

Then, since BA is ⊥ to plane CD......Hyp.

∴ ∠BAK is a rt. ∠ ....XI. def. 3.

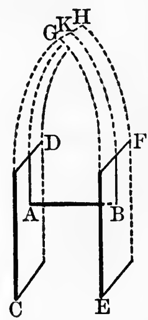
Similarly, ∠ABK is a rt. ∠.

Hence, ∠s BAK, ABK of △KAB=two rt. ∠s,

which is impossible ....I. 17.

∴ the planes CD, EF cannot meet when prod[d].

i.e. plane CD is || to EF.



Wherefore, *planes* &c. Q.E.D.

EXAMPLE.

Find the locus of a point equidistant from three given points which are not in the same straight line.

Let A, B, C be the given pts. Join AB, BC, CA.

Find O, the cent. of ⊙ circumscribing △ABC ....IV. 5.

From O draw OP at rt. ∠s to plane ABC............ ....XI. 12.

Take any pt. P in OP, and join PA, PB, OA, OB.

Then, in △s PAO, PBO,

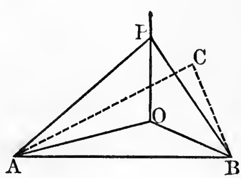
∵ { AO=BO...........Constr.
OP is com.
rt. ∠AOP=rt. ∠BOP....XI. def. 3. }

∴ PA=PB...........I. 4.

Similarly, PA=PC.

i.e., any pt. in OP is equidist. from A, B and C.

∴ OP is the req[d] locus.



EXERCISES.

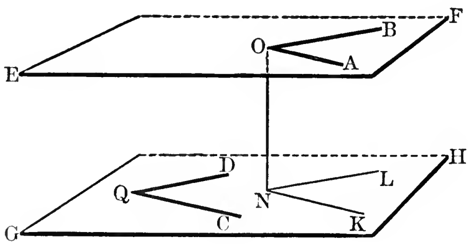
1. State and prove the converse of Prop. XIV.
2. The parts of parallel lines intercepted between parallel planes are equal.
3. If two parallel planes be cut by two other parallel planes, the dihedral angles are equal.

PROPOSITION XV. THEOREM.

If two straight lines which meet one another be parallel to two others which meet one another, but are not in the same plane with the first two, the plane passing through the first two shall be parallel to the plane passing through the other two.

Let OA, OB, in plane EF, be respectively || to QC, QD, in plane GH.

Then shall plane EF be || to plane GH.



From O draw ON ⊥ to plane GH .... XI. 11.

Draw NK || to QC, and NL || to QD .... I. 31.

Then, since ON is ⊥ to plane GH .... Constr.

∴ each of the ∠s ONK, ONL is a rt. ∠ .... XI. def. 3.

Also, since OA is || to QC, and NK is || to QC .... Hyp. and Constr.

∴ OA is || to NK .... XI. 9.

∴ ∠s AON, ONK together = two rt. ∠s .... I. 29.

But ∠ONK is a rt. ∠ .... Above.

∴ ∠AON is also a rt. ∠.

Similarly it may be shown that ∠BON is a rt. ∠.

i.e. ON is at rt. ∠s to the two st. lines OA, OB in plane EF.

∴ ON is ⊥ to plane EF .... XI. 4.

But ON is ⊥ to plane GH .... Constr.

∴ plane EF is || to plane GH .... XI. 14.

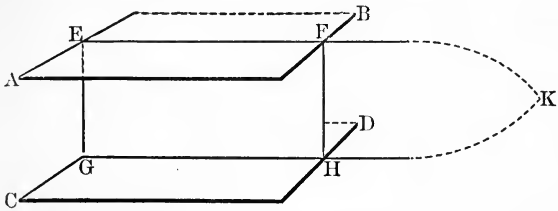
Wherefore, *if two straight lines* &c. Q.E.D.

PROPOSITION XVI. THEOREM.

If two parallel planes be cut by another plane, their common sections with it shall be parallel.

Let the || planes AB, CD be cut by the plane EH in the st. lines EF, GH.

Then shall the st. line EF be || to GH.



For, if not, the st. lines EF, GH in the same plane will meet if prod[d] either towards E, G, or F, H........................I. def. 35.
Let them, if possible, meet when prod[d] towards F, H, at the pt. K.

Then, since EF, a part of st. line EFK, is in plane AB......Hyp.
∴ K is in plane AB..........................XI. 1.

Similarly it may be shown that K is in plane CD.
∴ the planes AB and CD, if prod[d], will meet,
which is impossible, for they are || ....Hyp.

Hence, EF, GH cannot meet, when prod[d] towards F, H.

Similarly it may be shown that they cannot meet, when prod[d] towards E, G.
∴ EF is || to GH..........................I. def. 35.

Wherefore, *if two parallel planes* &c. Q.E.D.

NOTE.

The following are other forms of the enunciation of Proposition XV.:—

If one pair of intersecting straight lines be parallel to another pair in a different plane, the plane of the first pair shall be parallel to the plane of the second pair;

or,

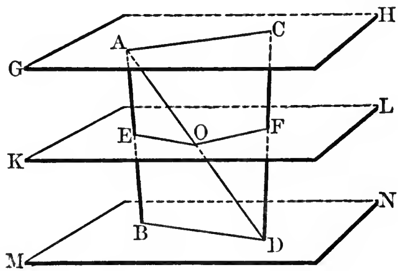
If two angles, in different planes, have their arms respectively parallel, the planes are also parallel.

PROPOSITION XVII. THEOREM.

If two straight lines be cut by parallel planes, they shall be cut in the same ratio.

Let the st. lines AB, CD be cut by the || planes GH, KL, MN, at the pts. A, E, B and C, F, D.

Then shall AE be to EB as CF to FD.



Join AC, AD, BD.
Let AD cut the plane KL at O.
Join EO, OF.

Then, since the plane ABD cuts the || planes KL, MN,

∴ the com. sections EO, BD are || .... XI. 16.

And, since the plane DAC cuts the || planes GH, KL,

∴ the com. sections OF, AC are || .... XI. 16.

Hence, since EO is || to BD, a side of △ABD,

∴ AE is to EB as AO is to OD .... VI. 2.

And, since OF is || to AC, a side of △DAC,

∴ AO is to OD as CF is to FD .... VI. 2.

Hence AE is to EB as CF is to FD .... V. 11.

Wherefore, *if two straight lines* &c. Q.E.D.

NOTE.

A more perfect form of the enunciation is the following:—

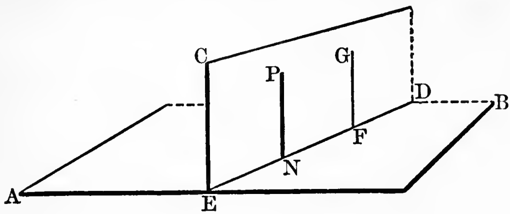
If two straight lines be cut by three parallel planes they shall be cut proportionally, and those segments shall be homologous which are intercepted between the same planes.

PROPOSITION XVIII. Theorem.

If a straight line be at right angles to a plane, every plane which passes through it shall be at right angles to that plane.

Let PN be ⊥ to the plane AB.

Then shall every plane through PN be ⊥ to plane AB.



Let CD be a plane passing through PN and having a com. section ED with the plane AB.
Take any pt. F in ED, and draw FG, in plane CD, at rt. ∠s to ED..I. 11.

Then, since PN is ⊥ to plane AB ....Hyp.

∴ ∠PNF is a rt. ∠ ....XI. def. 3.

But ∠GFN is a rt. ∠ ....Constr.

∴ ∠s PNF, GFN=two rt. ∠s.

∴ GF is || to PN ....I. 28.

But PN is ⊥ to plane AB ....Hyp.

∴ GF is also ⊥ to plane AB ....XI. 8.

Hence, since any st. line GF, in plane CD, drawn at rt. ∠s to their com. section ED, is ⊥ to plane AB,

∴ plane CD is ⊥ to plane AB....................XI. def. 4.

Wherefore, *if a straight line* &c. Q.E.D.

EXERCISES.

1. If one plane is perpendicular to a second, the second is perpendicular to the first.

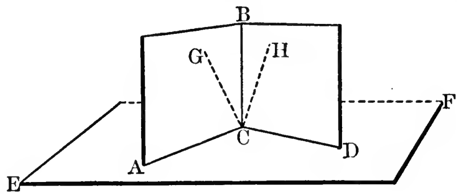
2. Prove the converse of Prop. XVIII.,—that if one plane be perpendicular to another, the perpendicular to the second, drawn from any point in their common section, lies in the first plane.

PROPOSITION XIX. Theorem.

If two planes which cut one another be each of them perpendicular to a third plane, their common section shall be perpendicular to the same plane.

Let the planes AB and BD be each ⊥ to plane EF.

Then shall their com. section BC be ⊥ to plane EF.



Let AC, CD be the com. sections of planes AB, BD with plane EF.

Then, if BC be not ⊥ to plane EF, if possible, draw, in plane AB, from pt. C, the st. line CG at rt. ∠s to AC; and, in the plane BD, from pt. C, the st. line CH at rt. ∠s to CD ....I. 11.

Then, since plane AB is ⊥ to plane EF ....Hyp.
and CG is drawn in plane AB ⊥ to their com. section.........Constr.

∴ CG is ⊥ to plane EF ....XI. def. 4.

Similarly, CH is ⊥ to plane EF.

i.e., from the same pt. C two straight lines have been drawn, each ⊥ to plane EF,

which is impossible......................XI. 13.

Hence BC cannot be otherwise than ⊥ to plane EF.

Wherefore, *if two planes* &c. Q.E.D.

PROPOSITION XX. Theorem.

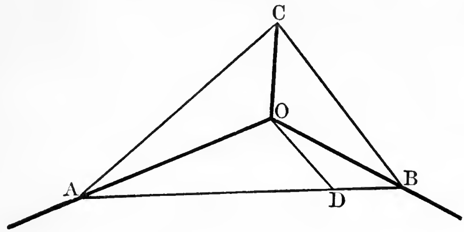
If a solid angle be contained by three plane angles, any two of them are together greater than the third.

Let the solid ∠ at O be cont[d] by the three plane ∠s AOB, BOC, COA.

Then shall any two of these ∠s be together > the third.

CASE 1. If ∠s AOB, BOC, COA are all equal, it is evident that any two are > the third.

CASE 2. But, if ∠s AOB, BOC, COA are not all equal, let ∠ AOB be that which is not < either of the others.



At pt. O, in the plane AOB, make ∠ AOD = ∠ COA .... I. 23.
Cut off OD=OC.
Through D draw st. line ADB, in the plane AOB, meeting OA, OB at A, B.
Join CA CB.

Then, in △s AOC, AOD,

∵ { OC=OD .... Constr.
OA is com.
∠ COA = ∠ DOA .... Constr. }

∴ AC=AD .... I. 4.

But AC, CB > AB .... I. 20.

∴ CB > DB .... Ax. 3.

Hence, in △s BOC, BOD,

∵ { OC=OD .... Constr.
OB is com.
CB > DB .... Above. }

∴ ∠ BOC > ∠ BOD .... I. 25.

But ∠ COA = ∠ AOD .... Constr.

∴ ∠s BOC, COA > ∠s BOD, AOD.

i.e. ∠s BOC, COA > ∠ AOB.

Also, since ∠ AOB is not < either of the others,

∴ ∠ AOB, together with either of the others, > the third.

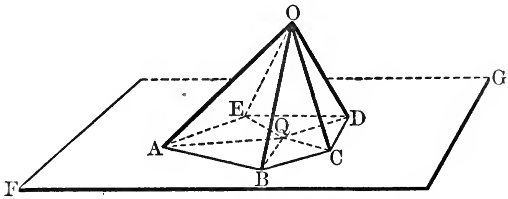
Wherefore, *if a solid angle* &c. Q.E.D.

PROPOSITION XXI. THEOREM.

The plane angles, which contain a solid angle, are together less than four right angles.

Let the solid angle at O be cont^d by the plane ∠s AOB, BOC, COD, DOE, EOA.

Then shall these ∠s be together < four rt. ∠s.



Let a plane FG make the com. sections AB, BC, CD, DE, EA, with the planes in which the ∠s are.
Take any pt. Q within the polygon ABCDE.
Join QA, QB, QC, QD, QE, dividing the polygon into as many △s as it has sides, that is, as there are plane ∠s forming the solid ∠ at O.

Then, since the solid ∠ at B is cont^d by three plane ∠s,

∴ ∠s OBA, OBC > ∠ABC........................XI. 20.

i.e. ∠s OBA, OBC > ∠s QBA, QBC,

and so on, for the other corners C, D &c.

Hence, *all the base* ∠s of all the △s with vertex O
> all the base ∠s of all the △s with vertex Q.

But *all* the ∠s of all the △s with vertex O
= all the ∠s of all the △s with vertex Q.........I. 32.

∴ *all the vertical* ∠s of all the △s with vertex O
< all the vert^l ∠s of all the △s with vertex Q.

But the ∠s at Q=four rt. ∠s.......................................I. 15 cor.

∴ the ∠s at O < four rt. ∠s.

Wherefore, *the plane angles* &c. Q.E.D.

NOTE.

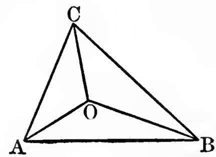
This proposition was only proved for a trihedral angle by Euclid. Simson added to Euclid's proof a second case dealing with a polyhedral angle. As, however, the above proof is quite independent of the number of angles which form the solid angle, the first case has not been inserted in the text.

NOTES.

The following was Euclid's proof;

Let the solid angle at O be cont^d by the three plane ∠s AOB, BOC, COA.
Then shall these three ∠s be together < four rt. ∠s.

Take any pts. A, B, C in OA, OB, OC.
Join AB, BC, CA.



Then since solid ∠ A is cont^d by three plane ∠s,

∴ ∠s BAO, OAC > ∠BAC.....XI. 20.

Similarly ∠s ABO, OBC > ∠ABC,

and ∠s BCO, OCA > ∠BCA.

∴ the six ∠s BAO, OAC, ABO, OBC, BCO, OCA > ∠s BAC, ABC, BCA.

But ∠s BAC, ABC, BCA = two rt. ∠s................................I. 32.

∴ the six ∠s BAO, OAC, ABO, OBC, BCO, OCA > two rt. ∠s.

But the three ∠s of each of the △s OAB, OBC, OCA = two rt. ∠s...I. 32.

∴ the nine ∠s BAO, ABO, AOB, OBC, BCO, BOC, OCA, OAC, COA = six rt. ∠s.

But, of these, the six BAO, OAC, ABO, OBC, BCO, OCA > two rt. ∠s..........................Above.

∴ the rem^g three ∠s AOB, BOC, COA < four rt. ∠s.

It should be noticed that Prop. XXI. only holds good for a *convex* solid angle, *i.e.* the polygon ABCDE in the figure must have no *re-entrant* angle.

[It is not customary to read the remaining nineteen propositions of the Eleventh Book, or those of Book XII. These contain some very important results concerning the volumes of solids, but the proofs are often long and difficult, and the modern methods by which these results are obtained are beyond the scope of the present work.]

EXERCISES.

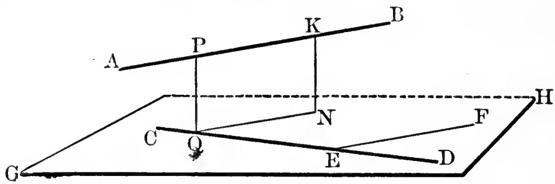
1. If in the fig. of Prop. XX., OQ be drawn within the solid angle at O and not in the same plane with any two of the lines OA, OB, OC, then
   (i) the sum of the ∠s AOQ, BOQ, COQ > half the sum of the ∠s AOB, BOC, COA.
   (ii) the sum of the ∠s AOQ, BOQ, COQ < sum of the ∠s AOB, BOC, COA.

2. If, in the figure of Prop. XXI., OQ be joined, prove that the sum of the angles AOQ, BOQ &c. is greater than half the sum of the angles AOB, BOC &c.

MISCELLANEOUS EXAMPLES.

I. *To draw a straight line perpendicular to each of two given straight lines which are not in the same plane.*

Let AB and CD be the two given st. lines.



In CD take any pt. E.
Through E draw EF ∥ to AB.
Let GH be the plane containing CD and EF.
Take any pt. K in AB.
From K draw KN ⊥ to plane GH....XI. 11.
Through N draw NQ ∥ to EF, meeting CD at Q....I. 31.

Then, since QN is ∥ to EF, and EF is ∥ to AB....Constr.
∴ QN is ∥ to AB....XI. 9.
and, AB, QN, KN are in the same plane....XI. 7.

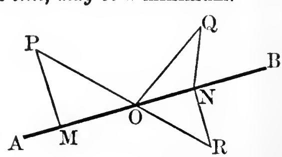
Through Q draw, in this plane, QP ∥ to KN, meeting AB at P....I. 31.

Then shall PQ be ⊥ to both AB and CD.

For, since KN is ⊥ to plane GH, and PQ is ∥ to KN....Constr.
∴ PQ is ⊥ to plane GH....XI. 8.
∴ PQ is ⊥ to CD in that plane....XI. def. 1.

Also, since PQNK is a ▱....Constr.
∴ ∠KPQ = ∠KNQ....I. 34.
But ∠KNQ is a rt. ∠....XI. def. 1.
∴ ∠KPQ is also a rt. ∠
i.e. PQ is ⊥ to AB. Q.E.F.

II. *Find a point in a given straight line such that the sum of its distances from two given points, not in the plane of the line, may be a* **minimum.**



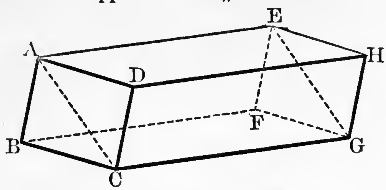
Let P, Q be the pts., AB the line,
Draw PM, QN ⊥s to AB....XI. 11.
In plane PMN draw NR
at rt. ∠s to AB, and = QN.
Join PR cutting AB at O, and join OQ.

Then PO + OQ shall be a min.

(The proof is similar that of Ex. III. page 82.)

III. *The faces of a parallelepiped are parallelograms, and opposite faces are equal.*

Let ABCD, EFGH be opp. faces of a || [ped.]



Then, (i) since plane AH is || to BG .... Def. p. 286.
and plane AC meets them,
∴ the com. sections AD, BC are || .... XI. 16.
Similarly, AB, DC are ||.
Hence, ABCD is a ▱ .... Def. p. 14.
In the same way it may be shown that the other faces are ▱s.

Again, (ii) Join AC, EG.
Then, since ABFE is a ▱ .... Above.
∴ AB=EF .... I. 34.
Similarly, DC=HG.
Also, since AD is || to EH, and DC is || to HG .... Above.
∴ ∠ADC=∠EHG .... XI. 10.

Hence, in △s ADC, EHG,

∵ { AD=EH, DC=HG, ∠ADC=∠EHG } .... Above.

∴ △ADC=△EHG .... I. 4.
Hence, ▱ABCD=▱EFGH .... I. 34.

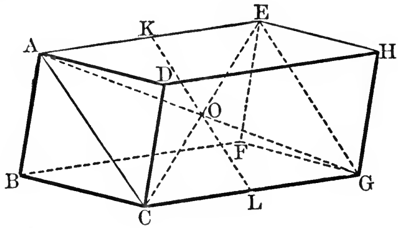
In the same way it may shown that the other pairs of opp. faces are equal. Q.E.D.

EXERCISES.

1. In the figure of Ex. 1 prove that—
   (i) PQ is the shortest distance between the lines;
   (ii) no other line but PQ can be perpendicular to both lines.
2. In the figure of Ex. III., prove that—
   (i) the sum of the squares on the diagonals AG, EC is double the sum of the squares on AC, CG;
   (ii) if BD, AC, cut at Q, and EG, FH at R, then QR is equal and parallel to CG.
3. Points are taken one on each of two adjacent walls of a room; find the shortest line that can be drawn on the walls between the points.

IV. *The diagonals of a parallelepiped meet at a point, and bisect one another.*

Let ACGE be a $\parallel^{\text{ped}}$.
Join AC, EG.



Then, since AE is=and || to DH
and DH is=and || to CG } ....Ex. II.

∴ AE is=and || to CG..................Ax. 1 and XI. 9.

∴ ACGE is a ▱ ....I. 33.

Hence its diags. AG, CE bisect one another at O ...Ex. 7, p. 57.

Similarly, DCFE is a ▱, and its diags. DF, CE bisect one another.
i.e. the diag. DF also passes through O.

Similarly, ABGH is a ▱, and its diags. AG, BH bisect one another at O.

i.e. the four diags. of the $\parallel^{\text{ped}}$ AG, BH, CE, DF, are *concurrent* and bisect each other. Q.E.D.

V. *Straight lines joining the middle points of opposite edges of a parallelepiped are bisected by the point of intersection of its diagonals.*

For if K be mid. pt. of AE, in fig. of Ex. IV., and KO be joined and prod[d] it will meet CG, which lies in the same plane AG, at some pt. L.

Then, in △s AKO, GLO,

∵ { AO=OG ....Ex. IV.
∠KAO=∠LGO.................Ex. IV. and I. 29.
∠AOK=∠GOL................I. 15. }

∴ AK=LG, and KO=OL ....I. 26.

But AK=half AE
=half CG ....Ex. IV.

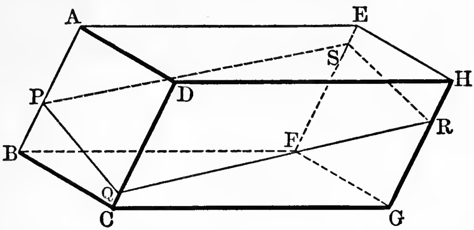
∴ LG=half CG.

i.e. KOL joins the mid. pts. of opp. sides, and is bisected at O.
Similarly for any other pair of sides.

Q.E.D.

VI. *The section of a parallelepided by any plane which cuts two pairs of opposite edges, but does not cut the remaining pair, is a parallelogram.*

Let PQRS be such a section of the || ped ACGE.



Then since plane PR cuts the || planes AC, EG,

∴ the com. sections PQ and SR are || ....XI. 16.

Similarly, PS is || to QR.

∴ fig. PQRS is a ▱. Q.E.D.

VII. *The square on a diagonal of a rectangular parallelepiped is equal to the sum of the squares on three adjacent edges.*

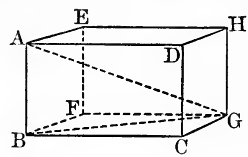
Let ACGE be a rectr || ped.

Join BG, AG.

Then, since each of the ∠s ABC, ABF is a rt. ∠ ....Def. p. 286.

∴ AB is ⊥ to plane BG......XI. 4.

∴ ∠ABG is a rt. ∠.



Hence, $AG^2 = AB^2 + BG^2$............................I. 47.

$= AB^2 + BC^2 + CG^2$...................I. 47.

$= AB^2 + BC^2 + BF^2$...................I. 34.

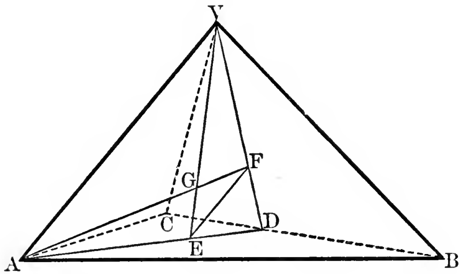
Q.E.D.

EXERCISES.

1. The diagonals of a rectangular parallelepiped are equal to one another.
2. Cut a cube by a plane so that the section may be
   (i) an equilateral triangle;
   (ii) a trapezium;
   (iii) a regular hexagon.
3. The section of a parallelepiped by a plane parallel to an edge is a parallelogram.
4. The sum of the squares on the four diagonals of any parallelepiped is equal to four times the sum of the squares on three adjacent edges.

VIII. *The straight lines joining the vertices of a tetrahedron to the points of intersection of the* **medians** *of the opposite faces meet at a point, and divide each other in the ratio of* 3 *to* 1.

Let VABC be any tet^n.



Bisect BC at D. Join AD, VD. From DA cut off DE=⅓ of AD. From DV cut off DF=⅓ of VD. Join VE, AF, EF .... VI. 9.

Then, since VE, AF lie in one plane VAD .... XI. 2.

∴ these lines cut at some pt. G.

And, since $\frac{AE}{ED}=\frac{2}{1}=\frac{VF}{FD}$ .... Constr.

∴ FE is ∥ to VA .... VI. 2 (ii.)

Hence, $\frac{VG}{GE}=\frac{AG}{GF}$ .... VI. 2 (i)

But, $\frac{VG}{GE}=\frac{VA}{EF}$ .... (by sim^r △s VAG, GFE) } VI. 4

$=\frac{VD}{FD}$ .... (by sim^r △s VAD, FED) }

$=\frac{3}{1}$ .... Constr.

i.e. VE, AF cut at G, and divide one another in the ratio of 3 to 1. Similarly, it may be shown that VE, BK, and VE, CL (where K, L, are the points of intersection of the medians of △s VAC, VAB), cut at the same point G. Q.E.D.

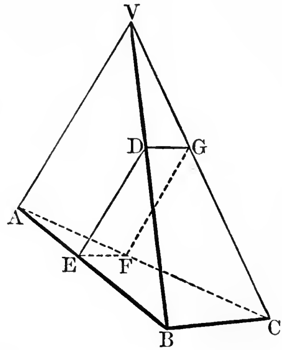
EXERCISES.

1. In any tetrahedron the straight lines joining the middle points of opposite edges meet at a point.
2. If *abc* be a section of the tetrahedron by a plane parallel to ABC, then VE passes through the intersection of the medians of *abc*.

IX. *The section of a tetrahedron by a plane parallel to each of two opposite edges is a parallelogram.*

Let VABC be any tet^n, and let DEFG be the section made by a plane DF || to the opp. edges VA and BC.

Then shall DEFG be a ▱.



For, since plane DF is || to BC............................Hyp.

∴ they cannot meet if prod^d....Def. p. 286.

∴ No st. line in plane DF can meet BC.

∴ EF cannot meet BC.

But EF and BC are in the same plane ABC.

∴ EF is || to BC..................I. def. 35

Similarly, DG is || to BC.

∴ DG is || to EF..............................XI. 9.

In the same way it may be shown that DE is || to GF.

∴ DEFG is a ▱..........................Def. p. 14.

Q.E.D.

EXERCISES.

1. In a regular tetrahedron

   (i) The foot of perpendicular from the vertex on the base is the centre of the circle circumscribing the base.

   (ii) The square of this perpendicular is $\frac{2}{3}$ of the square on an edge.

   (iii) The sum of the perpendiculars on the faces from any point in the base is equal to the altitude.

   (iv) The lines joining the middle points of opposite sides meet at a point, are at right angles to one another, and bisect each other.

   (v) Find the height when the edges are each 3 inches long.

   (vi) Find the height, and area of surface, when the edge is a.

2. Cut a regular tetrahedron so that the section may be a square.

3. Find the locus of a point equidistant from the angles of an equilateral triangle.

X. *Sections of a pyramid made by planes parallel to the base are similar.*

Let VABCDE be a pyr., and *abcde* a section || to the base.

Then shall abcde be sim^r to ABCDE.

For since plane *ad* is || to plane AD, and plane VBC meets them,

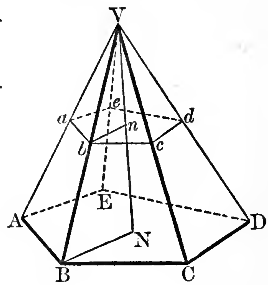
∴ the com. sections *bc*, BC are || ....XI. 16.

Similarly, *cd* is || to CD.

Hence $\angle bcd = \angle BCD$................XI. 10.

Also, $\frac{bc}{BC} = \frac{Vc}{VC}$ .... } VI. 4.

$= \frac{cd}{CD}$ .... }



i.e. the $\angle$s at *c*, C are =, and the sides about these $\angle$s are : :^{ls}.
Similarly, for the other $\angle$s of the polygons.

∴ the polygons *abcde*, ABCDE are sim^r........................XI. def. 1.

Q.E.D

XI. *The areas of the sections of a pyramid, made by planes parallel to the base, are to one another in the duplicate ratio of their altitudes.*

Join BN, *bn*. Draw VN ⊥ to plane AD, meeting plane *ad* at *n*...XI. 11.

Then, since VN is ⊥ to plane AD,

∴ it is ⊥ to the || plane *ad*................XI. 14 (conv.).

And, since the plain VBN meets the || planes *ad*, AD,

∴ the com. sections *bn*, BN are || ....XI. 16.

Hence, $\triangle$s V*bn*, VBN are sim^r.

And, since fig. *abcde* is sim^r to fig. ABCDE..........................Ex. IX.

$$\therefore \frac{\text{fig. } abcde}{\text{fig. ABCDE}} = \frac{bc^2}{BC^2} = \frac{Vb^2}{VB^2} = \frac{Vn^2}{VN^2} \text{......VI. 20, note, and VI. 4.}$$

Q.E.D.

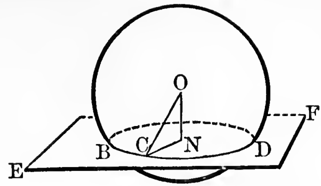
XII. *Every plane section of a sphere is a circle.*

Let the plane EF cut the sphere of rad. *r*, and whose cent. is O.

Then shall their com. section BCD be a ⊙.

From O draw ON ⊥ to plane EF.

Take any pt. C in BCD and join OC, CN.



Then, since ON is ⊥ to plane EF,

∴ ∠ONC is a rt. ∠.........XI. def. 3.

∴ $CN^2 = OC^2 - ON^2$......I. 47.

$= r^2 - ON^2$

i.e. any pt. C in the curve BCD is at a *constant distance* from the fixed pt. N.

∴ locus of C is a ⊙. Q.E.D.

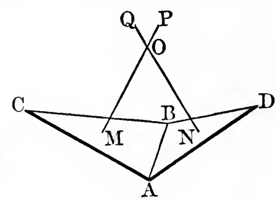
XIII. *To find the centre of a sphere which shall pass through four given points not in the same plane.*

Let A, B, C, D, be the four pts.

Find M, N, the cents. of the ⊙s circumscribing △s ABC, ABD...IV. 5.

Draw MP at rt. ∠s to plane ABC,

and NQ at rt. ∠s to plane ABD ...XI. 12.



Then, since MP is a ⊥ to plane ABC drawn from cent. of circumscribing ⊙...............................Constr.

∴ any point in MP is equidist. from A and B...........................Ex. p. 299.

Similarly, any pt. in NQ is equidist. from A, B.

∴ MP, NQ, both lie in the plane which bisects AB at rt. ∠s...Ex. p. 295.

But MP, NQ, being ⊥ to different planes, are not ||.

∴ they will meet at some pt. O.

Then, since O is equidist. from A, B, C, and also from A, B, D,

∴ O is equidist. from the four pts. A, B, C, D, and is the req^{d} cent.

Q.E.F.

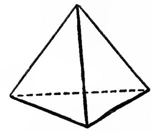
NOTE.

If the plane EF passes through the centre of the sphere, the section is called a *great circle* of the sphere; if not, a *small circle.*

EXERCISES.

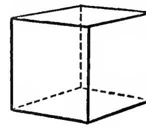
1. The common section of two spheres is a circle, and its plane is perpendicular to the line joining their centres, which it divides into parts the diff. of whose squares is equal to the difference of the squares of the radii. (*Note.* This is called the *radical plane* of the spheres.)
2. All tangents to a sphere from a given external point are equal.
3. The planes of small circles of equal radii are equidistant from the centre of the sphere.
4. Find the radii of the inscribed and circumscribed spheres of a regular tetrahedron whose edge is *a*.

XIV. *There can be but five regular solids.*



(i)

For, since the sum of the plane $\angle$s forming any solid $\angle$ must be $<$ 360°............XI. 21.
and an $\angle$ of an equilat. △ is 60°,

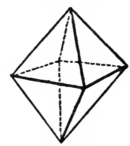


(ii)

∴ *three, four,* or *five,* but not more, equilat. △s can form a solid $\angle$.

Also, an $\angle$ of a sq. is 90°,

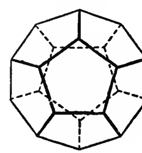
∴ *three,* but not more, sqs. can form a solid $\angle$.



(iii)

And an $\angle$ of a reg. pentagon is 108°.........I. 32 cor.

∴ *three,* but not more, reg. pentagons can form a solid $\angle$.

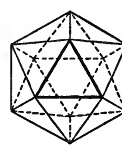


(iv)

Again, an $\angle$ of a reg. hexagon is 120°, and three such $\angle$s are *not* $<$ 360°,

∴ three, or more, reg. hexagons *cannot* form a solid $\angle$.

Neither, for like reasons, can reg. heptagons, octagons, &c.



(v)

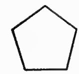
Hence there can be but *five* reg. solids, namely—

(i)	the *reg. tetrahedron,*	having each solid $\angle$	formed by			3 equilat. △s;
(ii)	,, *hexahedron* (or cube),	,,	,,	,,		3 squares;
(iii)	,, *octahedron,*	,,	,,	,,		4 equilat. △s;
(iv)	,, *dodecahedron,*	,,	,,	,,		3 reg. pentagons;
(v)	,, *icosahedron*	,,	,,	,,		5 equilat. △s.

XV. *If* E *be the number of edges,* F *the number of faces, and* S *the number of solid angles of any polyhedron, then* $S+F=E+2$.

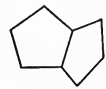
Suppose the polyhedron to be gradually built up by fitting together n polygons, taken one at a time; and let E, E_2, E_3, &c., stand for the number of edges when the first, second, third, &c. polygon is added on; and so on.

Then, for the **first** polygon, there are as many edges as corners.



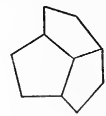
$$\therefore E_1 = S_1, \qquad \text{or} \qquad E_1 = S_1 + (\mathbf{1} - 1).$$

And, when the **second** polygon is added, *one edge*, and *two corners* will coincide with those of the first polygon,



$$\therefore E_2 = S_2 + 1, \qquad \text{or} \qquad E_2 = S_2 + (\mathbf{2} - 1).$$

Also, when the **third** polygon is added, *two edges*, and *three corners* will coincide with those of the first and second polygons,



$$\therefore E_3 = S_3 + 1 + 1, \quad \text{or} \quad E_3 = S_3 + (\mathbf{3} - 1).$$

And so on.

Hence, when the **last but one** of the polygons is added,

$$E_{n-1} = S_{n-1} + (\overline{n-1} - 1).$$

But the *last* polygon adds *no* fresh edges, nor corners,

i.e. E is E_{n-1}, S is S_{n-1}, and F is n.

$$\therefore E = S + (\overline{F-1} - 1),$$

$$\text{or } E + 2 = S + F.$$

MISCELLANEOUS EXERCISES.

1. The area of the surface of a cube is double of the square on one of its diagonals.

2. Prove that if every straight line which can be drawn meeting two given straight lines meet a third given straight line, the three given lines all lie in one plane.

3. If two straight lines are parallel, their projections on any plane are also parallel.

4. From a point P, outside a given plane, two perpendiculars are drawn, one to the plane, and the other to a given straight line in the plane; prove that the line joining their feet is at right angles to the given line.

5. Two perpendiculars are let fall from any point on two given planes: show that the angle between the perpendiculars is equal to the angle of inclination of the planes to one another.

6. Three lines, OP, OQ, OR, intersect at right angles in the point O, and OS is drawn perpendicular to QR. Show that, if PS is joined, PS will be perpendicular to QR.

7. Find the locus of the foot of the perpendicular from a given point upon a plane which passes through a given straight line.

8. AB, CD are two straight lines, of which AB lies in a plane to which CD is perpendicular. Show that the perpendiculars drawn to AB from the different points of CD all pass through a fixed point.

9. The projections of parallel straight lines on any plane are proportional to the lines.

10. ABCD is a regular tetrahedron, and, from the vertex A, a perpendicular is drawn to the base BCD, meeting it in O: show that three times the square on AO is equal to twice the square on AB.

11. The angle which a line makes with its projection on a plane, is less than that which it makes with any other line in that plane.

12. Draw a straight line, which shall be equally inclined to three straight lines, which meet at a point, but are not in the same plane.

13. If a prism be cut by parallel planes the sections will be equal.

14. Show that the shortest distance between two opposite edges of a regular tetrahedron is equal to half the diagonal of the square described on an edge.

15. Straight lines are drawn from two points, not in that plane, to meet each other in a given plane. Find when their sum is a *minimum.*

16. OA, OB, OC are three straight lines, not in the same plane, planes through OB, OC perpendicular to OBC intersect in Oa, planes through OC, OA perpendicular to OCA intersect in Ob, and planes through OA, OB perpendicular to OAB intersect in Oc; prove that OA, OB, OC are perpendicular to the planes Obc, Oca, Oab respectively.

17. If two straight lines in one plane, be equally inclined to another plane, they will be equally inclined to the common section of the two planes.

18. If in the three edges, which meet at one angle of a cube, three points A, B, C be taken at equal distances from the angle, the area of the triangle ABC formed by joining these points with each other is $\frac{\sqrt{3}}{2}a^2$, where a is the distance OA.

19. OA and OB are two intersecting straight lines. OC is any straight line through their point of intersection, but not in their plane, which is such that the angle COA is equal to the angle COB. Show that the projection of the straight line OC on the plane AOB bisects the angle AOB.

20. A pyramid is constructed on a square base, having all its edges equal to one another; find the inclination of two of the triangular faces to one another.

21. Of all the planes which can be drawn through a given line, find that which has least inclination to a given plane.

22. On a given equilateral triangle as base, describe a regular tetrahedron.

23. If any point be taken within a cube, the square of its distance from a corner of the cube is equal to the sum of the squares of the perpendiculars from the point on the three faces containing that corner.

24. The lines joining the middle points of adjacent sides of a quadrilateral, the sides of which are not all in the same plane, form a parallelogram. (*Note.*—Such a quadrilateral is called *gauche.*)

25. Find the locus of a point in space equidistant from three fixed points.

26. From the centre of the circle circumscribing a triangle ABC, a perpendicular to its plane is drawn of length equal to the side of the square inscribed in that circle; show that the radius of the sphere which passes through A, B, C and the extremity of the perpendicular is three-fourths of the perpendicular.

27. If P be a point equidistant from the angles A, B, C of a right-angled triangle of which A is the right angle and D the middle point of BC, prove that PD is at right angles to the plane of ABC.

28. A pyramid on a square base is cut by planes parallel to the base; show that the points of intersection of the diagonals of the sections all lie in a straight line.

29. The sum of the squares on the four diagonals of an oblique parallelepiped is equal to the sum of the squares on its twelve edges.

30. Given three straight lines, draw through a given point a straight line equally inclined to the three.

31. If a regular tetrahedron be cut by a plane parallel to two edges which do not meet, the perimeter of the parallelogram in which it is cut shall be double of an edge of the tetrahedron.

32. The angles of inclination of two faces of a regular tetrahedron and regular octahedron are supplementary.

33. If the middle points of the edges of a regular tetrahedron be joined, the figure formed is a regular octahedron.

34. Make a trihedral angle equal to a given trihedral angle.

35. Find the locus of points in space the difference of the squares of the distances of which from two given points is constant.

APPENDIX I.

TRANSVERSALS; HARMONIC SECTION; POLE AND POLAR.

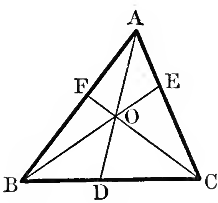
NOTE.—In modern geometry many important results are obtained by combining Euclid's methods with those of arithmetic and algebra.

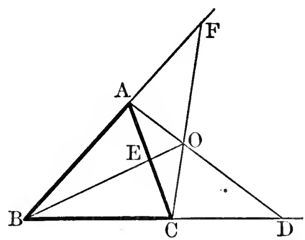
DEF.—A line which cuts a system of lines is called a **transversal.**

I. *If three concurrent straight lines, drawn through the angles of a triangle, meet the sides, or sides produced, the products of alternate segments of the sides, taken in order, are equal.*

Let ABC be a $\triangle$, and AOD, BOE, COF be concurrent lines meeting the sides or sides prod[d] at D, E, F.

Then shall **BD . CE . AF=DC . EA . FB.**





For, $\dfrac{\triangle BOA}{\triangle BOD}=\dfrac{OA}{OD}=\dfrac{\triangle COA}{\triangle COD}$ ....VI. 1.

$\therefore \dfrac{\triangle BOA}{\triangle COA}=\dfrac{\triangle BOD}{\triangle COD}$ ....Altern.

$=\dfrac{BD}{DC}$ ....VI. 1.

Similarly, $\dfrac{\triangle BOC}{\triangle BOA}=\dfrac{CE}{EA}$,

and, $\dfrac{\triangle COA}{\triangle BOC}=\dfrac{AF}{FB}$.

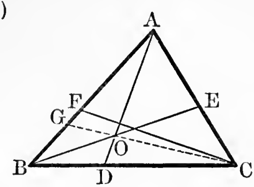
Hence, multiplying,

$$\frac{\triangle BOA}{\triangle COA}\cdot\frac{\triangle BOC}{\triangle BOA}\cdot\frac{\triangle COA}{\triangle BOC}=\frac{BD}{DC}\cdot\frac{CE}{EA}\cdot\frac{AF}{FB}.$$

$$\text{or, } 1=\frac{BD}{DC}\cdot\frac{CE}{EA}\cdot\frac{AF}{FB}.$$

i.e. DC . EA . FB=BD . CE . AF.

II. *If in a triangle ABC, BD . CE . AF=DC . EA . FB, then AD, BE, and CF are concurrent.* (Converse of Theor. I.)



For, if not, suppose, if possible, that CF does not pass through O, the pt. of intersection of AD and BE.

Join CO. Prod. CO to meet AB at G.

Then, since AD, BE, CG are concurrent,

$\therefore$ BD . CE . AG=DC . EA . GB ....Theor. I.

But BD . CE . AF=DC . EA, FB.......................Hyp.

$\therefore$ dividing, $\frac{AG}{AF}=\frac{GB}{FB}$.

or, a ratio of greater inequality=one of less inequality, which is absurd.

Hence CF cannot pass otherwise than through O.

III. *If a transversal cut the sides, or sides produced, of a triangle, the products of the alternate segments, taken in order, are equal.*

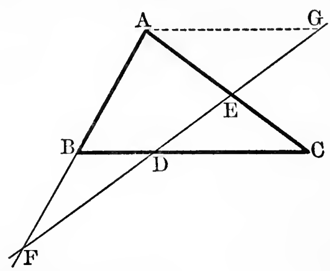
Let a transversal meet the sides BC, CA, AB of $\triangle$ ABC at D, E, F.

Then shall BD . CE . AF=DC . EA . FB.

Through A, draw AG ∥ to BC, meeting the transversal at G.

Then, by sim^r^ $\triangle$s FBD, FAG,

$$\frac{BD}{FB}=\frac{AG}{AF};$$



And, by sim^r^ $\triangle$s EDC, EGA,

$$\frac{CE}{DC}=\frac{EA}{AG};$$

$\therefore$, mult^g^, $\frac{BD}{FB}\cdot\frac{CE}{DC}=\frac{AG}{AF}\cdot\frac{EA}{AG}$,

$$=\frac{EA}{AF},$$

or, BD . CE . AF=DC . EA . FB.

EXERCISES.

1. State, and prove by a *reductio ad absurdum*, the converse of Theor. III.
2. Use Theor. II. to prove Examples VIII., IX., X., XI. on pages 86, 87, 88, 89.
3. Prove that the lines joining the vertices of a triangle to the points of contact of the inscribed circle are concurrent.
4. Prove that lines drawn at right angles to the sides of a triangle through their middle points are concurrent.

DEFS.—A system of straight lines drawn through a point is called a **pencil.**

The point is called the *vertex* of the pencil.

Any one of the lines is called a *ray.*

If the four points, in which a pencil of four rays meets a transversal, divide that transversal in harmonic proportion, the four points are called a **harmonic range.**

The four rays are called a **harmonic pencil.**

If a line AB is divided internally at C, and externally at D, in the same ratio, we know (see page 228) that ACBD is divided harmonically. The points C and D are called *harmonic conjugates* of A and B.

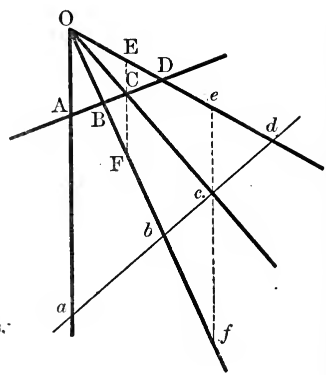
The ray OB is said to be the *conjugate* of OA, and OD of OC.

IV. *If a pencil divide one transversal harmonically, it divides all transversals harmonically.*

Let the pencil, vertex O, divide the transversal ABCD harmonically, and let *abcd* be any other transversal.

Then shall abcd be a harm. range.

Draw ECF, *ecf* || to OA.



Then, by sim^r △s OBA, FBC,

$$\frac{AB}{BC}=\frac{OA}{CF},$$

And, by sim^r △s ODA, EDC,

$$\frac{AD}{DC}=\frac{OA}{EC}.$$

But, $\frac{AB}{BC}=\frac{AD}{DC}$ ....Hyp.

$$\therefore \frac{OA}{CF}=\frac{OA}{EC}.$$

$\therefore$ EC=CF.

Again, $\frac{ec}{EC}=\frac{Oc}{OC}=\frac{cf}{CF}$ ......sim^r △s.

$\therefore ec=cf,$

$$\therefore \frac{Oa}{ec}=\frac{Oa}{cf}.$$

But, $\frac{Oa}{ec}=\frac{ad}{dc}$, and $\frac{Oa}{cf}=\frac{ab}{bc}$ ....by sim^r △s

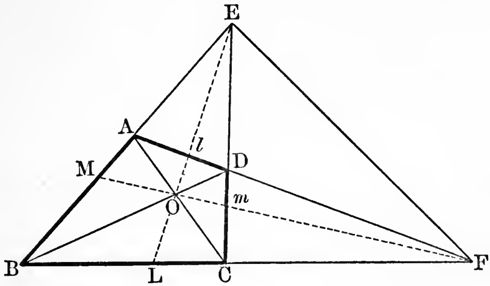
$$\therefore \frac{ab}{bc}=\frac{ad}{dc}.$$

i.e. abcd is divided harmonically.

COR.—A transversal drawn parallel to any ray is bisected by the conjugate ray.

V. *If the extremities of the third diagonal of a complete quadrilateral be joined to the point of intersection of the other two diagonals, all the pencils are harmonic.*

Let ABCD be a quad^{l}, EF the third diag., let AC, BD cut at O, and EO, FO meet BC, AB, at L, M, and AD, DC, at *l*, *m*.,



Then, since EL, BD, CA, in △ EBC, are concurrent at O,

∴ BL . CD . EA = LC . DE . AB ....Theor. I.

And, since the transversal FDA meets the sides of △ EBC,

∴ BF . CD . EA = FC . DE . AB .... Theor. III.

Hence, dividing, $\frac{BL}{BF} = \frac{LC}{FC}$,

or, $\frac{BL}{LC} = \frac{BF}{FC}$.

i.e. E(BLCF) is a harmonic pencil.

Hence E(MO*m*F), and E(A*l*DF) are also harm. pencils..............Theor. IV.

Similarly it may be shown that all other pencils in the figure are harmonic.

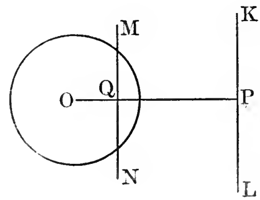
EXERCISES.

1. If BD meet EF at K, BODK is a harmonic range.
2. Find the arithmetic, geometric, and harmonic means between two given straight lines.
3. If a straight line AB be divided internally at C, and externally at D, in the same ratio, then $\frac{1}{AB} - \frac{1}{AC} = \frac{1}{AD} - \frac{1}{AB}$.
4. The internal and external bisectors of the vertical angle of a triangle form with the sides a harmonic pencil.
5. The line joining the centres of similitude of two circles is divided harmonically by their circumferences.
6. The centres of similitude of three circles lie, three by three, on four straight lines.

Defs.—If from the cent. O of a circle, of radius r, a line be drawn, and on it two points P, Q, be taken such that OP . OQ$=r^2$, then P is called the **inverse** of Q, and Q the inverse of P.

The **polar of a point with respect to a circle** is the straight line drawn through the inverse of the point at right angles to the radius which contains the point.

(For example: If OP. OQ$=r^2$, and KL, MN be ⊥s to OP through P, Q, then MN is the polar of P, and KL the polar of Q.)



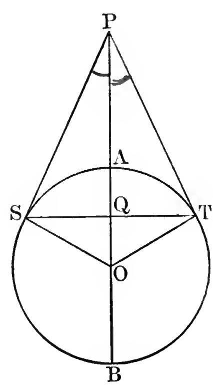
A triangle is called **self-conjugate** when each of its sides is the polar of the opposite angular point with respect to some circle.

A triangle is said to be of **given species** when its angles, and the ratios of its sides are given.

VI. *The polar of an external point with respect to a circle is the chord of contact of the tangents drawn to the circle from that point.*

Let PS, PT be the tangs. from P to a circle with cent. O, and let ST cut PAOB at Q.

Then, since PS^2=PA, PB.................. .... III. 37.
=PT^2.
∴ PS=PT.



And, in △s OSP, OTP,

∵ { PS=PT,
OP is com.
OS=OT,

∴ ∠SPO=∠TPO.

Hence in △s SPQ, TPQ,

∵ { SP=TP,
∠PSQ=∠PTQ,
∠SPQ=∠TPQ.

∴ ∠SQP=∠TQP,
i.e. ST is at rt. ∠s to OP.

Also, since OSP is a rt. ∠,

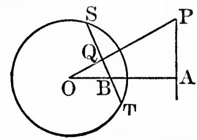
∴ OQ . OP=OS^2.........................VI. 8.

Hence SQT is the polar of P.

VII. *The pole of a line drawn through a fixed point lies on the polar of that fixed point; and conversely.*

Let P be the fixed pt., PA any line through P, Q the inverse of P with respect to the ⊙, cent. O, rad. $=r$, and SQT the polar of P.

Then shall the pole of AP lie on SQT.



Draw OBA ⊥ to PA and meeting SQT at B.

Then, since BQP and PAB are rt. ∠s, a ⊙ will go round P, Q, B, A,

∴ OB . OA=OQ . OP.........................III. 36, cor.

$=r^2$.

Hence B is the pole of AP.

The converse is evident, for, if B be a fixed pt. on SQT, AP is the polar of B, and AP passes through P the pole of SQT.

VIII. *Any secant of a circle drawn through a fixed point is divided harmonically by the fixed point, its polar with respect to the circle, and the circumference of the circle.*

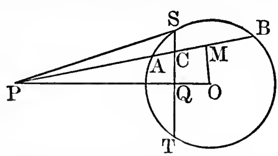
Let the secant PAB be cut by SQT, the polar of P, at C.

Then shall PACB be a harm. range.

Join PS. Then PS is the tang. at S....................................Theor. VI.

Draw OM ⊥ to PAB.

Then since OMC, OQC are rt. ∠s, a ⊙ will go round C, Q, O, M.



And PA . PB=PS^2.................................III. 36.

$=OP^2-OS^2$.......................I. 47.

$=OP^2-OP . OQ$................Hyp.

$=OP . (OP-OQ)$

$=OP . PQ$

$=PM . PC$ ....III. 36, cor.

∴ 2 PA . PB=2 PM . PC

$=(PA+PB) . PC$

$=PA . PC+PB . PC$.

∴ PA . PB − PA . PC=PB . PC − PA . PB.

or, PA . (PB − PC)=PB . (PC − PA).

$$\text{or, } \frac{PA}{PB}=\frac{PC-PA}{PB-PC};$$

i.e., PA, PC, PB are in harm. : : ⁿ.

MISCELLANEOUS EXERCISES.

1. Prove Theorem VIII. for an internal point.
2. If a triangle is self-conjugate with respect to a circle, the orthocentre of the triangle is at the centre of the circle.
3. If G be the point of intersection of the diagonals of a quadrilateral inscribed in a circle, and EF be the third diagonal of the complete quadrilateral, the triangle EFG is self-conjugate with respect to the circle.
4. If A, B, C, D be any four points on a straight line, then, having regard to sign as well as magnitude, AB . CD + AC . DB + AD . BC = 0
5. Given three rays of a harmonic pencil; find the fourth ray.
6. ABCD and A*bcd* are harmonic ranges; prove that B*b*, C*c*, D*d* are concurrent.
7. If a circle cut the sides BC, CA, AB of a triangle ABC at D, *d*; E, *e*; F, *f*; then will AF . A*f*. BD . B*d* . CE . C*e* = FB . *f*B . DC . *d*C . EA . *e*A.
8. If a triangle of given species have one angle fixed and another lies on a fixed circle, the locus of the third angle is a circle.
9. If a, b, c, be the lengths of the sides of any triangle, then
   (i) (area of △)$^2 = s(s-a)(s-b)(s-c)$, where s is half the sum of a, b, c.
   (ii) (area of △)$^2 = r \cdot r_a \cdot r_b \cdot r_c$, where r, r_a, r_b, r_c are the radii of the inscribed and escribed circles of the triangle.

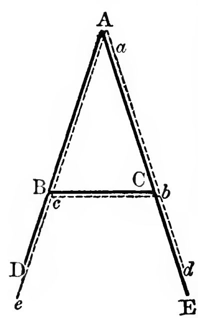
APPENDIX II.

ALTERNATIVE PROOFS OF PROPOSITIONS.

Book I. Prop. 5.

Let ABC be an isos. △, with AB=AC, and AB, AC prod[d] to D, E.

Suppose the figure to be taken up, *turned over*, and then superposed upon its former position.



Let a, b, c denote the ∠s A, B, C after reversal.

Then, since a lies on A, and ae along AD,

the pt. c must fall on B, for ac=AB...Hyp.

the line ad must lie along AE, for ∠s cad, DAE are identical,

the pt. b must fall on C, for ab=AC...Hyp.

And, since c falls on B, and b on C,

∴ cb coincides with BC..........Ax. 10.

Hence, ∠acb coincides with ∠ABC

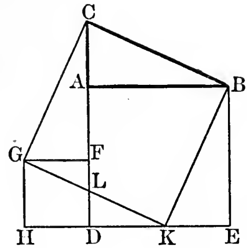
∴ ∠acb = ∠ABC..............Ax. 8.

i.e., ∠ACB = ∠ABC.

Similarly, ∠DBC = ∠ECB.

Book I. Prop. 47.

Let ABC be a rt. ∠[d] △, A being the rt. ∠.
On AB desc. sq. ADEB.
From DA cut off DF=AC.
On DF desc. a sq. FGHD external to sq. AE.
From ED cut off EK=AC.
Join CG, BK, and GK, cutting AD at L.
Then, since each of the ∠s FDH, FDK is a rt. ∠.



∴ HDK is a st. line.
And, since AC=DF
Add AF to each.
∴ CF=AD=AB.
Also GF=FD=AC.
And ∠GFC=∠HDF=∠CAB.
∴ △s GFC, CAB are equal in all respects.

Similarly it may be shown that each of the △s GHK and KEB is equal to CAB or GFC, and, consequently, the four △s are equal.

Hence, fig. CGKB is equilat[l].

And, since ∠CBA=∠KBE,
Add ∠ABK to each.
∴ ∠CBK=∠ABE.
i.e., ∠CBK is a rt. ∠.
Hence fig. CGKB is a square.

Now, sq. on BC=fig. CGKB.
=figs. CAB, GFC, GFL, ALKB.
=figs. KEB, GHK, GFL, ALKB.
=figs. GHDF and ADEB.
=sqs. on HD, AB.
=sqs. on CA, AB.

NOTE.

The above provides the following easy method of cutting up the smaller squares into bits which form the larger: Set the small squares GD, AE side by side, cut off EK=DH, and join GK, KB.

Book II. Prop. 4.

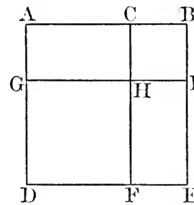
Euclid's proof depends upon the theorem *Parallelograms about the diameter of a square are squares*, which, not having been previously proved, he demonstrates in the proof of this proposition. In other words, he makes use of a *Lemma*. This may be avoided by deducing Prop. 4 from previous propositions of Book II. thus:—

(i) Let AB be divided into any two parts at C.

Then, sq. on AB = rect. AB, AC, with rect. AB, CB................II. 2.
= sq. on AC and rect. AC, CB,
with rect. AC, CB and sq. on CB.............II. 3.
= sqs. on AC, CB with twice rect. AC, CB;

or, by the following construction:—

(ii) On AB desc. a sq. ADEB.
Through C draw CF = to AD or BE.
From AD cut off AG = BC.
Draw GHK ∥ to AB or DE, meeting CF at H.

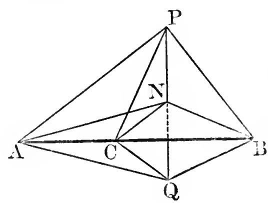


Then all the figs. are, by constr., rectangles.
∴ CH = BK = AG = CB = HK = FE.
Hence CHKB is the sq. on CB.
Also, since AD = AB, and AG = FE
∴ GD = DF.
Hence GDFH is the sq. on GH (or AC),
Now sq. on AB = figs. AH, GF, CK, HE.
= rect. AC, CH, sq. on GH, sq. on CB, rect. HK, KE,
= rect. AC, CB, sqs. on AC and CB, rect. AC, CB,
= sqs. on AC, CB with twice rect. AC, CB.

Similarly Props. 5, 6, 7 of Bk. II. may be proved.

Book XI. Prop. 4.

Let PN be ⊥ to each of the lines AN, BN at their pt. of intersection N,
In plane ABN draw any line NC cutting AB at C.
Prod. PN to Q, making NQ = PN.
Join PA, PB, PC, QA, QB, QC.



Then in △s APN, AQN,
∵ { PN = QN
AN is com.
rt. ∠ ANP = ∠ ANQ }
∴ AP = AQ
Similarly BP = BQ.
Hence △s PAB, QAB are equal in all respects.

And, if △ PAB be supposed to revolve about the side AB, the point P will eventually coincide with Q, and PC with QC.
∴ PC = QC.
Hence, in △s CNP, CNQ,
∵ { PN = QN
CN is com.
PC = QC. }
∴ ∠ PNC = ∠ QNC, and each is a rt. ∠.
∴ PN is ⊥ to plane ABN.....................XI. def. 3.

EXAMINATION PAPERS.

[The following specimens of Examination Papers lately set by some of the chief public examining bodies are arranged in order of difficulty, as far as is possible without separating the sets. They will, it is thought, be found useful as tests of the student's progress as well as interesting to those preparing for any of these examinations.]

COLLEGE OF PRECEPTORS.

CERTIFICATE EXAMINATION—Christmas, 1889.

THIRD CLASS.

Book I. Prop. 1–26.

1. Define an *angle*, *adjacent angles*, a *right angle*.

2. If two triangles have two sides of the one equal to two sides of the other, and the angle contained by the two sides of the one equal to that contained by the two sides of the other: prove that the third sides also are equal.

Hence show that the lines joining the opposite angles of a square are equal.

3. If two angles of a triangle are equal, the sides also opposite to them will be equal.

4. Bisect a given finite straight line.

5. If two straight lines cut one another, the vertically opposite angles are equal.

And, if at a point in a straight line two straight lines on opposite sides of it make the vertically opposite angles equal, these two straight lines shall be in one and the same straight line.

6. Any two sides of a triangle are together greater than the third side.

7. At a given point in a given straight line make an angle equal to a given rectilineal angle.

Having given two finite straight lines and an angle, show how you could construct a triangle having an angle equal to the given angle, and the sides containing it equal to the two given straight lines.

SECOND CLASS.

Book I.

1. Define *point*, *perpendicular*, *circumference*, *axiom*, *theorem*. Name and define the various rectilinear quadrilateral figures.

2. After the manner of Euclid's sixth proposition, prove that an equiangular triangle is also equilateral.

3. If two triangles have two sides of the one equal to two sides of the other, each to each, and have likewise their bases equal, the angle which is contained by the two sides of the one shall be equal to the angle which is contained by the two sides, equal to them, of the other.

Show also that the two triangles are equal in every respect.

4. If, at a point in a straight line, two other straight lines, on the opposite sides of it, make the adjacent angles together equal to two right angles, these two straight lines shall be in one and the same straight line.

State also the *converse* of this proposition.

5. The greater angle of every triangle has the greater side opposite to it.

In addition, show from this that the perpendicular is the shortest straight line that can be drawn from a given point to a given straight line.

6. If a straight line falling on two other straight lines make the exterior angle equal to the interior and opposite angle on the same side of the line, the two straight lines shall be parallel to each other.

Prove that the opposite sides of a square are parallel to one another.

7. Distinguish between *equal* triangles and triangles which are *equal in all respects.*

Construct an isosceles triangle which shall be equal in area to a given scalene triangle.

Either,

8. If the diagonals of a quadrilateral bisect each other, it is a parallelogram.

Or,

9. The middle points of the adjacent sides of a square are joined. Find the sum of the squares on the four straight lines thus formed.

FIRST CLASS.

Books I.–IV.

1. Define a *triangle*, an *equilateral triangle*, and an *isosceles triangle.*

Bisect a given rectilineal angle. Can this be done without drawing an equilateral triangle?

2. Prove that, if one side of a triangle be produced, the exterior angle will be greater than either of the interior opposite angles.

The perpendiculars from two of the angular points of a triangle on the opposite sides intersect *within* the triangle. Prove that the triangle is acute-angled.

3. At a point in a given straight line make an angle equal to a given rectilineal angle.

Construct a triangle, having its sides respectively equal to two given straight lines, and one of the angles at the base equal to a given rectilineal angle.

4. Define *parallel straight lines.*

Prove that straight lines which are parallel to the same straight line are parallel to one another.

5. Prove that the straight lines bisecting the four angles of a parallelogram will either form a rectangle or pass through a single point.

6. If a straight line be divided into two equal, and also into two unequal parts, the squares on the two unequal parts will be together double of the square on half the line, and of the square on the line between the points of section.

Either, demonstrate this proposition by means of the First Book of Euclid;

Or, deduce it from previous propositions of the Second Book.

7. Draw a tangent to a given circle from a given point without it.

What is the length of a tangent to a circle, whose diameter is 6 inches, drawn from a point 5 inches from the centre?

8. Prove that similar segments of circles on equal straight lines are equal to one another.

9. Describe a circle about a given equilateral and equiangular pentagon.

Show that the radius of the circle is less than a side of the pentagon.

PUPILS' EXAMINATION.—Christmas, 1888.

THIRD CLASS.

Book I. Prop. 1–26.

1. Define a *point*, a *line*, a *surface*, a *straight line*, a *plane surface*, a *rectilineal angle*.

When are (i) two straight lines, and when are (ii) two angles, said to be equal?

2. From a given point draw a straight line equal to a given straight line.

3. If two triangles have the three sides of one equal to the three sides of the other, each to each, the triangles shall be equal in all their parts.

By means of this proposition, show that the line drawn from the vertex of an isosceles triangle to the middle point of the base cuts the base at right angles.

4. Draw a straight line perpendicular to a given straight line from a given point without it.

5. The side AC of a triangle is greater than the side AB; show that the angle ABC is greater than the angle ACB.

6. If from the ends of one side of a triangle two straight lines are drawn to a point within the triangle, they shall be less than the other two sides of the triangle.

Within the triangle ABC are two points D and E, and the lines AD, DE, EB are drawn: show that these three together are less than AC, CB together.

7. If two triangles have two sides of the one equal to two sides of the other, but the angle contained by the two sides of the one greater than the angle contained by the two sides equal to them of the other, the base of that which has the greater angle is greater than the base of the other.

State the converse of this proposition.

SECOND CLASS.

Book I.

1. Define the terms *surface*, *plane surface*, *angle*, *figure*, *parallelogram*, *acute-angled triangle*.

2. To bisect a given rectilineal angle; that is, to divide it into two equal parts.

3. The angles which one straight line makes with another on one side of it are together equal to two right angles. Having proved this proposition, prove also its corollaries.

4. If one side of a triangle be produced, then the exterior angle shall be greater than either of the two interior and opposite angles.

5. If a straight line falling on two other straight lines make the alternate angles equal, then these straight lines shall be parallel.

6. (i) Write the enunciation (*without the proof*) of the converse of the proposition set for Question 5 in this paper.

(ii) If from points A and C the parallel straight lines AB and CD are drawn towards the same parts; and if also in the plane of AB and CD the straight lines AP and CQ are drawn inclined to AB and CD and towards the same parts, so that the acute angles BAP and DCQ are equal, prove that AP and CQ are parallel.

7. The opposite sides and angles of a parallelogram are equal to one another, and the diagonal of a parallelogram bisects it.

8. Equal triangles on equal bases in the same straight line and on the same side of it are between the same parallels.

9. For any right-angled triangle, the squares on the sides containing the right angle are together equal to the square on the side opposite to the right angle.

10. To what angular magnitude are the three interior angles of every triangle together equal?

If BDEC be the square on side BC opposite to the right angle BAC of the right-angled triangle ABC, and AD be joined, prove that the angles BAD and BDA are together equal to the angle BCA.

FIRST CLASS.

Books I.–IV.

1. If two triangles have two sides of the one equal to two sides of the other, each to each, and have also the angles contained by those sides equal, prove that the triangles are equal in all respects.

Prove also that, if ABC, DEF are two triangles such that AB, AC are equal to DE, DF, each to each, and the angles ABC, DEF are right angles, then the triangles will be equal in all respects.

2. Draw a straight line perpendicular to a given straight line of unlimited length, from a given point without it.

Prove that the perpendicular is the shortest straight line which can be drawn from the given point to the given line.

3. Prove that equal triangles on the same base, and on the same side of it, are between the same parallels.

AEKG, KHCF are parallelograms about the diameter AC of a parallelogram ABCD; show that EG is parallel to HF.

4. Describe a square on a given straight line.

Show that, if two squares are equal, the straight lines on which they stand are equal.

5. If a straight line is divided into any two parts, the rectangle contained by the whole line and one of the parts is equal to the square on that part, together with the rectangle contained by the two parts.

6. Divide a straight line into two parts so that the rectangle contained by the whole line and one of the parts may be equal to the square on the other part.

Does the figure used in the construction contain any other straight lines which are divided in this manner?

7. Prove *one, but not both* of the following propositions:—

(*a*) If ABCD is a trapezium such that AB is parallel to DC, and AD equal but not parallel to BC, then the square on AC will be equal to the square on BC together with the rectangle contained by AB and CD.

(*b*) If from the extremities of the diameter of a circle tangents are drawn to meet in A and B the tangent at any point P of the circle, and if O is the centre of the circle, then the angle AOB will be a right angle.

8. *Either*,

Show that, if from a point within a circle more than two equal straight lines can be drawn to the circumference, that point is the centre of the circle.

Or, Show that the opposite angles of any quadrilateral figure inscribed in a circle are together equal to two right angles.

9. Describe a circle about a given triangle.

CIVIL SERVICE.

APPRENTICES IN H.M. DOCKYARDS.—April, 1889.

Books I. II. III.

[*Ordinary abbreviations may be used, but the method of proof must be geometrical.*]

1. The angles at the base of an isosceles triangle are equal, and, if the equal sides be produced, the angles on the other side of the base will be equal.

2. Make a square equal to a given rectilineal figure.

3. O is a point in the diameter ACOB of a circle whose centre is C. Show that OA is greater, and OB less, than any other straight line which can be drawn to the circumference from O. Also, if R and S be two points on the circumference such that the angle ACR is greater than the angle ACS, prove that OS is greater than OR.

4. PQRS is a quadrilateral. Find a point O within it such that the sum of the distances OP, OQ, OR, and OS may be the least possible.

5. O is a point in a straight line PQ. Prove that the greatest value of the rectangle PO, OQ is one-fourth of the square on PQ.

6. One circle touches another internally. XY and ZV are equal chords of the outer circle, which touch the inner circle at P and Q. Show that, if XY be not parallel to ZV, their point of intersection must lie on the straight line (produced if necessary) which joins the centre of the two circles.

ENGINEER STUDENTS.—April, 1889.

Books I.–IV. and VI., with Definitions of Book V.

1. The side BC of the triangle BCD is greater than the side BD. Prove that the angle BCD is less than the angle BDC.

2. Define a *parallelogram*, and show that the opposite sides and angles of a parallelogram are equal to one another.

If two sides of a quadrilateral are parallel, and the other two equal, the angles will be equal, two and two.

3. If a straight line be divided into any two parts, the squares on the whole line and on one of the parts are equal to twice the rectangle contained by the whole and that part, together with the square on the other part.

4. Find the points on the circumference of a circle which are at the greatest and least distances from a given point, either within or without the circle; and verify your constructions.

Also through a point within a circle draw the longest and shortest chords.

5. If a straight line touch a circle, and from the point of contact a straight line be drawn cutting the circle, the angles which this line makes with the line touching the circle shall be equal to the angles which are in the alternate segments of the circle.

6. Inscribe a square in a given circle.

ABCD is a square inscribed in a circle whose centre is O, and BD is produced to P so that PD=AB. Prove that PB is equal to the diagonal of the square described on PO.

7. Give Euclid's test of proportion; and show that, if four straight lines be proportionals, the rectangle contained by the extremes is equal to the rectangle contained by the means.

The straight line drawn perpendicular to the side CA of an isosceles triangle through its middle point meets the base AB produced in D. Prove that CA touches the circle circumscribing the triangle BCD.

ASSISTANT CLERKSHIPS IN THE ROYAL NAVY.—June, 1889.

Euclid, Books I. II. III.

[*Ordinary abbreviations may be employed, but the method of proof must be geometrical. Proofs other than Euclid's must observe Euclid's sequence of propositions.*]

1. If from the ends of the side of a triangle there be drawn two straight lines to a point within the triangle, these shall be less than the other two sides of the triangle, but shall contain a greater angle.

2. If any side of a triangle be produced, the exterior angle is equal to the two interior opposite angles, and the three interior angles of every triangle are equal to two right angles.

D is a point on the side BC of a triangle ABC. If the angles ADC, ADB are respectively double of the angles ABC, ACB, show that the triangle ABC is right-angled.

3. If a straight line be bisected and produced to any point, the rectangle contained by the whole line thus produced and the part of it produced, together with the square on half the line bisected, is equal to the square on the line made up of the half and the part produced.

ABC is a triangle, right-angled at C. Points D and E are taken in AB and in AB produced, such that BD=BE=BC. Show that the rectangle contained by AD and AE is equal to the square on AC.

4. The diameter is the greatest straight line in a circle; and, of all others, that which is nearer to the centre is always greater than one more remote; and the greater is nearer to the centre than the less.

5. On a given straight line describe a segment of a circle containing an angle equal to a given rectilineal angle.

Find a point P within a triangle ABC such that if AP, BP, CP be joined, the angles PAB, PBC, PCA shall be all equal to one another.

ASSISTANT CLERKSHIPS IN THE ROYAL NAVY.—Nov. 1889.

Books I. II. III.

1. Draw a perpendicular to a given straight line from a given point in the line.

2. If a side of a triangle be produced, the exterior angle is equal to the two interior opposite angles; and the three interior angles of any triangle are together equal to two right angles.

ABC is a triangle. Through the point B, BD is drawn perpendicular to BC, and through the point A, AD perpendicular to BA; these straight lines meet at D. Prove that the angle ADB is equal to the angle ABC.

3. Divide a straight line into two parts so that the rectangle contained by the whole and one part may be equal to the square on the other part.

4. If a straight line drawn through the centre of a circle bisect a straight line in it which does not pass through the centre, it shall also cut it at right angles.

Prove that the straight line joining the middle points of two parallel chords of a circle is at right angles to the chords.

5. If two straight lines cut one another within a circle, the rectangle contained by the segments of one of them shall be equal to the rectangle contained by the segments of the other.

6. C is the centre of a circle, and CA, CB are two fixed radii; if, from any point P on the arc AB perpendiculars PX, PY are drawn to CA and CB, show that the distance XY is constant.

DEPARTMENT OF SCIENCE AND ART.

May, 1889.

FIRST STAGE.

1. If you had a ruler and a pair of compasses, how would you bisect a given finite straight line?

Prove that the method you employ is correct.

In the triangle ABC the side AB is greater than AC; find a point D in AC produced, such that AD may be as much greater than AC as AB is greater than AD.

2. Show that any two sides of a triangle are together greater than the third side.

In the triangle ABC the side AB is greater than AC, produce BA to D, making AD equal to AC, and join DC; show that the angle DCB is greater than a right angle.

3. Show that the straight lines, which join the extremities of two equal and parallel straight lines, towards the same parts, are themselves equal and parallel.

In the triangles ABC, DEF, the side AB is equal and parallel to DE, and BC is equal and parallel to EF; show that AC is equal and parallel to DF.

4. Show how to describe a parallelogram, which shall be equal to a given triangle, and have an angle equal to a given angle.

A triangle ABC is equal to a rectangle DEFG; the side DE equals half the base BC; show that the perimeter of the rectangle is less than that of the triangle.

5. Define a square, a rhombus, and a quadrilateral.

If a quadrilateral has its diagonals at right angles to each other, show that the sum of the squares on two opposite sides equals the sum of the squares on the other two opposite sides.

6. In the triangles ABC the angles at A and C are each half a right angle; AD is drawn bisecting the angle at A and meeting BC in D; show that the square on DC is double the square on BD.

SECOND STAGE.

1. If a straight line is bisected and produced to any point, show that the rectangle contained by the whole line thus produced, and the part produced, together with the square on half the line, is equal to the square on the line made up of the half and the part produced.

In a right-angled triangle show that the square on the perpendicular equals the rectangle under the lines which are severally equal to the sum and difference of the hypotenuse and base.

2. If equal straight lines (AP, AQ) be drawn from a point A to a given straight line of indefinite length, and if a third straight line AR be drawn from A to a point R in the given straight line, show that AR is shorter or longer than AP, according as R is between P and Q or not.

What property of the circle follows from this theorem?

3. Show that the opposite angles of any quadrilateral inscribed in a circle are together equal to two right angles.

Two given circles intersect in A and B; on the circumference of one of them take any two points C and D, and let CA, BD, produced if necessary, cut the other circle again in E and F. Show that the lines CD and EF are parallel.

4. Show that the lines, drawn from a point outside of a circle to touch the circle, are equal.

Show that the hypotenuse of a right-angled triangle, together with the diameter of the inscribed circle, equals the sum of the sides containing the right angle.

5. P is any point in the diameter AB of a given circle; bisect AP in D, PB in E, and DE in C; with centre C and radius CD describe a circle; through P draw any line PQR cutting this circle in Q and the given circle in R. Show that PQ is equal to QR, and that the tangent at Q to the inner circle is parallel to the tangent at R to the outer circle.

6. Show how to construct a rectangle which shall be equal to a given square, and have the difference between two adjacent sides equal to a given straight line.

THIRD STAGE.

1. C is the point of contact of two circles, which touch each other internally, and CPQ is a straight line, cutting the inner circle at P and the outer circle at Q; the tangent to the inner circle at P cuts the outer circle at A and B; show that the arc AQB is bisected at Q.

2. Two unequal circles intersect at A; show how to draw a straight line PAQ, cutting the one circumference in P and the other in Q, so that PA may equal AQ.

HONOURS.

1. Take any point E and F in the sides AC, AB of a triangle ABC; join EF, and draw from A a straight line, passing through the middle point of EF and cutting BC in D: show that

$$CA.AF.BD=BA.AE.CD.$$

2. C is the centre and AB the diameter of a semicircle ADEB; the tangent at D intersects the tangents at A and E in F and G, respectively; AE and CG intersect in H; show that FH is perpendicnlar to CG.

3. ABC is an equilateral triangle; a circle intersects AB in D, AC in E, and BC produced in F; it also intersects AC, AB, CB, all produced in G, H, and K, respectively; show that

$$AE+BK+CG=AD+BH+CF.$$

Find the corresponding result, when all the vertices of an equilateral triangle are inside a circle, which intersects the sides of the triangle produced both ways.

FOURTH STAGE.

1. Inscribe a regular pentagon in a given circle, and in the pentagon inscribe a regular figure of ten sides, having its alternate sides coincident with the sides of the pentagon.

2. In the triangles ABC, DEF, the angle ABC is equal to the angle DEF and the sides about these angles are proportional: show that the angle BAC is equal to one of the two angles EDF, DFE, and give the ratio of the third sides of the two triangles.

MPQ, NPR, are two intersecting circles such that the sum of the squares on their radii is equal to the square on the distance between their centres. Show how to draw a straight line MPN such that the rectangle MP, PN shall be equal to a given square.

3. When are quantities said to be in continued proportion? Divide a given finite straight line into two parts such that the whole line and the two parts shall be in continued proportion.

The perpendicular let fall from the right angle to the hypotenuse of a right-angled triangle divides the hypotenuse in mean and extreme ratio: show that the three sides of the triangle are in continued proportion.

4. When is a line said to be perpendicular to a plane, and when parallel to the plane?

From a given point, how many lines can be drawn perpendicular to a given plane, and how many parallel to that plane?

The angle B of the triangle ABC is a right angle: P is a point, not in the plane of the triangle, and equidistant from A, B, and C: if a straight line PD bisect AC, prove that PD is perpendicular to the plane ABC.

May, 1888.

FIRST STAGE.

1. Two angles of a triangle are equal; show that the sides opposite to these angles are equal.

In the equal sides AB, AC of an isosceles triangle ABC points D and E are taken, so that AD is equal to AE, and CD and BE are drawn intersecting in F; show that the triangle BFC is isosceles.

2. Show how to divide a given angle into two equal parts.

If a diagonal AC of a quadrilateral ABCD bisects the angles at A and C, show that it is at right angles to the other diagonal BD.

3. If one angle of a triangle is greater than another, show that the side which is opposite the greater angle is longer than the side which is opposite the less angle.

ABC is a triangle having an acute angle at B, which is greater than the angle at A; the side AB is produced to D, and BE is drawn to meet AC produced in E in such a way that the angle DBE is equal to the angle ABC; show that BE is longer than BC.

4. Show that the complements of the parallelograms which are about the diagonals of any parallelogram are equal to one another.

If one of the parallelograms about the diagonal of any parallelogram is equal to half one of the complements, show that the complement is equal to half the other parallelogram.

5. ABC is a given triangle and P a given point; show how to draw through P a straight line to cut AB and AC (or those sides either or both produced) in Q and R, so that AQR may be an isosceles triangle.

Within what space must P be situated if Q and R are on the sides (not the sides produced) respectively?

6. In the triangle ABC the angles at A and C are each one-fourth of a right angle; CD is drawn cutting AB produced at right angles in D; show that the square on AB is double the square on BD.

SECOND STAGE.

1. Show that if a straight line be divided into any two parts, the squares on the whole line and on one of the parts are equal to twice the rectangle

contained by the whole line and that part together with the square on the other part.

Let ABC be a triangle having a right angle at C; from D, any point in AC, draw DE at right angles to AB; without using any property of the circle, show that the rectangle CA.AD is equal to the rectangle BA.AE.

2. AB is a chord of a circle; C the middle point of AB; show that the straight line drawn through C perpendicular to AB passes through the centre of the circle.

Show that all circles whose centres lie on a given straight line and whose circumferences pass through a given point have a common chord of intersection.

3. Define similar segments of a circle, and show that on the same straight line and on the same side of it there cannot be two similar segments of circles not coinciding with each other.

A straight line is drawn through the point of contact of two circles touching each other internally; show that the segments cut off on the same side of the line are similar.

4. Show that in equal circles the arcs which subtend equal angles, whether at the centre or circumference, are equal.

ABCD is a quadrilateral inscribed in a circle whose centre is O; if the angles BAD and BOD are together equal to two right angles, show that the arc BAD is double the arc BCD.

5. Show that a tangent to a circle makes with a chord through the point of contact angles equal to those in the alternate segments of the circle.

In the hypotenuse AB of a right-angled triangle ABC a point E is taken, so that AE equals AC; CD is drawn to meet AB at right angles in D; show that the line joining C and E bisects the angle BCD.

6. Given a circle and two parallel chords, show how to draw a circle to touch both chords, and the circle internally.

THIRD STAGE.

1. A line of given length moves between two lines at right angles to each other, having one of its ends on each of them; find when its centre is at a minimum distance from a given line.

2. A, B, C are three points on the circumference of a circle; D is the middle point of the arc AB, and E is the middle point of the arc BC; draw the chord CE; join AE and CD intersecting in F; show that CE is equal to EF.

HONOURS.

1. A line AB is divided in C so that CB is double AC, and circles are described on AC and CB as diameters; show how to draw through A a line such that the chords intercepted by the two circles may be equal.

2. Show how to describe the greatest equilateral triangle, each side of which passes through a vertex of a given triangle.

3. Through C, the middle point of the arc ACB of a given circle, any chord CD is drawn cutting the straight line AB in E; find the locus of the centre of the circle passing through B, D and E.

FOURTH STAGE.

1. Give the necessary and sufficient conditions that two triangles may be similar: also that two polygons may be similar.

Show that the areas of similar triangles are as the squares on corresponding sides.

A triangle and a parallelogram have a common angle, and the area of the triangle is three times that of the parallelogram; give the relation connecting those sides of the two figures which contain the common angle.

2. Show how to trisect the arc of a circle whose chord is the side of a regular pentagon inscribed in the circle.

If regular figures of five, six, and ten sides respectively be inscribed in the same circle, show that the square on a side of the hexagon and the square on a side of the decagon are together equal to the square on a side of the pentagon.

3. If four parallel lines are given in position, show that they cut off on any transverse line segments which are in a constant ratio.

Similar and similarly situated polygons are described on AB and CD, which are given unequal parallel lines. Show that all the lines joining corresponding corners meet at a point.

4. If two lines which intersect are respectively parallel to two other intersecting lines, show that the plane of the former pair is parallel to the plane of the latter pair, or coincident with it.

A pyramid on a four-sided base is cut by any number of parallel planes; show that the sections are similar figures, and that the points of intersection of the diagonals of all the sections are in a straight line.

LONDON UNIVERSITY MATRICULATION EXAMINATION.

JANUARY, 1889.

EUCLID, BOOKS I.–IV.

1. Prove that, if two triangles have the sides of the one respectively equal to the sides of the other, they are equal in all respects.

2. Two triangles on the same base are such that one lies wholly inside the other; prove that the inner one has the smaller perimeter.

Extend this proposition to two polygons on the same base, of which the inner one has no re-entrant angle.

3. Prove that the sum of the angles of any triangle is equal to two right angles.

4. Show that the middle points of the sides of any quadrilateral are the vertices of a parallelogram.

Prove that the area of this parallelogram is half the area of the quadrilateral.

5. Prove by a geometrical construction that, if a straight line is divided into two segments, the square described on the whole line is equal to the squares described on the segments together with twice the rectangle contained by the segments.

6. Show that, if two circles touch each other, the line joining their centres passes through the point of contact.

7. A segment of a circle is described on a straight line AB, at any point P on it the tangent PT is drawn meeting AB produced in T; prove that the angle which PT makes with AB is equal to the difference of the angles PAB and PBA.

8. Chords of a circle are drawn through a given point inside it; prove that the rectangles contained by their segments are all equal.

Investigate also for what other points these rectangles have the same area as for the given point.

9. A number of triangles with equal vertical angles are inscribed in the same circle; show that their bases are all tangents to a circle.

10. Show how to inscribe a regular polygon of fifteen sides in a given circle.

June, 1889.

1. In the triangles ABC, DEF, it is given that AB = DE, the angle ABC = the angle DEF, and the angle BCA = the angle EFD. Prove that the triangles are equal in all respects.

2. Prove that the complements of the parallelograms about the diagonal of a parallelogram are equal.

3. Show how to divide a given straight line into five equal parts.

4. In an obtuse-angled triangle, if a perpendicular be drawn from either of the acute angles to the opposite side produced, the square on the side subtending the obtuse angle exceeds the sum of the squares on the sides containing the obtuse angle by twice the rectangle contained by the side on which, when produced, the perpendicular falls, and the straight line intercepted without the triangle, between the perpendicular and the obtuse angle.

5. Let A and B be two fixed points, and CD a straight line in the same plane as A, B. Find the position of the point P on the straight line CD, which is such that the sum of the squares on PA, PB is least.

6. Prove that two similar segments of circles which do not coincide cannot be constructed on the same chord, and on the same side of that chord.

7. Let A and B be two points on a circle, ADB, whose centre is C. Let an arc of a circle be described through ACB: let any straight line APQ be drawn cutting the arc ACB at P, and ADB at Q. Then prove that PB = PQ.

8. If AB be the diameter of a circle, CD a fixed straight line perpendicular to AB, then, if AQP be any straight line through A, cutting the circle at Q and CD at P, prove that the rectangle contained by AQ and AP is constant.

9. Show how to inscribe, in a given circle, a triangle equiangular to a given one.

10. A polygon having all its sides equal is inscribed in a circle, prove that all its angles are equal.

EDUCATION DEPARTMENT.

EXAMINATION FOR ADMISSION INTO TRAINING COLLEGES.

Books I. II.—Midsummer, 1889.

[*All generally understood abbreviations for* words *may be used, but no symbols of* operations (*such as* −, +, ×) *are admissible.*
Capital letters, not numbers, must be used in the diagrams.]

1. If two triangles have two angles of the one equal to two angles of the other, each to each, and one side equal to one side, these sides being opposite to equal angles in each, then shall the other sides be equal, each to each, and also the third angle of the one equal to the third angle of the other.

A, B, C are three given points. Through A draw a line such that the perpendiculars upon it from B, C may be equal.

2. Define parallel straight lines.

To draw a straight line through a given point parallel to a given straight line.

A is a point without the angle CBD. Draw from A a straight line, AEF, cutting BC in E, BD in F, so that AE may be equal to EF.

3. If the square described on one of the sides of a triangle be equal to the squares described on the other two sides of it, the angle contained by these two sides is a right angle.

PQRS is a rectangle, T any point, show by Bk. I. Prop. 47, that the squares on TP, TR are together equal to the squares on TQ, TS.

4. What is the objection to the use of algebraical processes in demonstrating the propositions of the Second Book?

In every triangle, the square on the side subtending an acute angle, is less than the squares on the sides containing that angle, by twice the rectangle contained by either of these sides, and the straight line intercepted between the perpendicular let fall on it from the opposite angle, and the acute angle.

EXAMINATION OF STUDENTS IN TRAINING COLLEGES.

1st Year.

Books I., II., III.—Christmas, 1889.

[*Capital letters* (*excluding A,B,C,D,E,F*), *not numbers, must be used in the diagrams.*
All generally understood abbreviations and symbols for words *may be used, but not symbols of* operations, *such as* −, +, ×, *PQ*², *PQ, RS.*]

1. Define *plane superficies, plane angle, semicircle, gnomon, angle of a segment, sector of a circle.*

If a triangle PQR be turned over about its side PQ, show by Prop. 4, Bk. I., that the line joining the two positions of R is perpendicular to PQ.

2. To draw a straight line perpendicular to a given straight line of an unlimited length, from a given point without it.

From a given point draw a straight line, making equal angles with two given straight lines. (I. 26 may be employed.)

3. All the exterior angles of any rectilineal figure, made by producing the sides successively in the same direction, are together equal to four right angles.

PQR is an equilateral triangle; H, K are points in QR, PR, such that QH=RK; QK, PH intersect at S. Show that the angle PSQ is equal to the sum of two angles of the equilateral triangle.

4. Equal triangles upon the same base and upon the same side of it are between the same parallels.

Bisect a given triangle by a straight line drawn from a given point in one of its sides. (I. 37, 38.)

5. Show that Propositions 2 and 3 of the Second Book are special cases of Proposition 1. What are the corresponding algebraical formulæ for these three propositions?

6. If a straight line be divided into two equal and also into two unequal parts; the squares on the two unequal parts are together double of the square on half the line and of the square on the line between the points of section.

In PR, a diagonal of the square PQRS, a point T is taken. Show that the triangle whose sides are equal to PT, TR and the diagonal of a square described on QT will be right-angled.

7. If a straight line drawn through the centre of a circle, bisect a straight line in it which does not pass through the centre, it shall cut it at right angles.

If a straight line be drawn intersecting two concentric circles, prove that the portions of the straight line, intercepted between the two circles, are equal.

8. If two circles touch one another internally, the straight line which joins their centres, being produced, shall pass through the point of contact.

Prove also that if a straight line be drawn through their point of contact, cutting the circumferences, two radii drawn to the points of intersection are parallel.

9. The angle at the centre of a circle is double of the angle at the circumference on the same base, that is, on the same arc.

GH and KL are two chords in a circle which intersect when produced in the point O without the circle; prove that the difference of the angles subtended at the centre by the arcs GK and HL is double of the angle GOK. (Apply I. 32.)

10. If from any point without a circle there be drawn two straight lines, one of which cuts the circle and the other meets it, and if the rectangle contained by the whole line which cuts the circle, and the part of it without the circle, be equal to the square on the line which meets the circle, the line which meets the circle shall touch it.

From a point T two tangents are drawn to a circle whose centre is U, and TU meets the chord of contact at S; show that the rectangle contained by US, UT is equal to the square on the radius.

2ND YEAR.

BOOKS I., II., III., IV. and VI. (PROPS. 1–17). CHRISTMAS, 1889.

1. Construct a triangle whose sides are equal to three given straight lines, any two of which are together greater than the third.

Show by diagrams what will happen if you attempt to construct triangles whose sides contain units in the following relations:—(1.) 2, 3, 4; (2.) 2, 3, 5; (3.) 2, 3, 6.

2. If a straight line falling on two other straight lines make the alternate angles equal to one another, or make the interior angles on one side together equal to two right angles, these two straight lines shall be parallel.

Show that a four-sided figure, which has all its sides equal and one angle a right angle is a square.

3. To divide a given straight line into two parts, so that the rectangle contained by the whole and one of the parts may be equal to the square of the other part.

Show how to produce a straight line, so that the rectangle contained by the given line and the line made up of the given line and the part produced may be equal to the square of the part produced.

4. In any acute-angled triangle the squares of the sides containing the acute angle are together greater than the square of the side subtending the acute angle by twice the rectangle contained by one of the sides and the straight line intercepted between the acute angle and the perpendicular let fall on it from the opposite angle.

The straight line PQ is divided in R, so that the rectangle PQ, QR is equal to the square of PR. Circles with centres Q,R, and radii equal to PR intersect in S. Show that PQS is an isosceles triangle, having each of the angles PQS, PSQ double of QPS.

5. The angle at the centre of a circle is double of the angle at the circumference upon the same base, that is, upon the same part of the circumference.

Two equal circles intersect in P, Q. Any straight line is drawn through P meeting the circles again in R, S. Show that if QR and QS be joined, QR is equal to QS.

6. If from any point without a circle two straight lines be drawn, one of which cuts the circle, and the other touches it; the rectangle contained by the whole line which cuts the circle, and the part of it without the circle, shall be equal to the square on the line which touches it.

Two circles intersect in P, Q. If tangents be drawn to the circles from any point R in PQ produced they shall be equal.

7. In a circle, the angle in a semicircle is a right angle; but the angle in a segment greater than a semicircle is less than a right angle; and the angle in a segment less than a semicircle is greater than a right angle.

PQR is a triangle, PS, RT, meeting in V, are drawn perpendicular to the opposite sides. QV is produced to meet PR in W. Show that QW is perpendicular to PR.

8. To inscribe a circle in a given triangle.

M is the centre of the circle inscribed in the triangle PQR; and N is the centre of the circle which touches PQ and PR produced and QR. Show that MN is the diameter of a circle which passes through Q and R.

9. To inscribe an equilateral and equiangular hexagon in a given circle.

Compare the areas of regular hexagons inscribed in and described about a given circle.

10. If a straight line be drawn parallel to one of the sides of a triangle it shall cut the other sides, or these produced, proportionally; and conversely, if the sides, or the sides produced, be cut proportionally, the straight line which joins the points of section shall be parallel to the remaining side of the triangle.

MNP, MNQ are equal triangles. MQ and NP intersect in R. Through R a straight line is drawn parallel to MN meeting MP in T and NQ in V. Show that TR is equal to RV.

11. Equal triangles which have one angle of the one equal to one angle of the other, have their sides about the equal angles reciprocally proportional; and conversely, triangles which have one angle of the one equal to one angle in the other, and their sides about the equal angles reciprocally proportional, are equal to one another.

Prove that the equilateral triangle described on the hypotenuse of a right-angled triangle is equal to the sum of the equilateral triangles described on the other two sides. (VI. 16 may be used.)

SCOTCH EDUCATION DEPARTMENT.

LEAVING CERTIFICATE, 1889.

GEOMETRY—Second (or Lower) Grade.

[*Candidates are not expected to attempt more than about three-fourths of this paper. But any omissions, whether of reasoning, explanation, or calculation, will be treated as errors. All ordinary contractions may be used. Additional marks will be given for neatness and good style.*]

1. Define a straight line, a right angle, a square, and a parallelogram.

The angles which one straight line makes with another straight line on one side of it are either two right angles or are together equal to two right angles.

2. If a side of any triangle be produced, the exterior angle is equal to the two interior and opposite angles, and the three interior angles of every triangle are equal to two right angles.

ABCD is a square. On CD an equilateral triangle CDE is described, so that E lies within the square. AE is joined. Find what part of a right angle the angle EAB is.

3. Parallelograms on the same base and between the same parallels are equal.

ABC is a triangle, and AB is bisected in D. From D, DE is drawn parallel to BC, meeting AC in E. Prove that AE = EC.

4. If a straight line be bisected and produced to any point, the rectangle contained by the whole line thus produced and the part produced, together with the square on half the line bisected, is equal to the square on the line made up of the half and the part produced.

Show that the preceding proposition is equivalent to the following:

The rectangle contained by the sum and difference of two straight lines is equal to the difference of their squares.

5. Draw a straight line from an external point to touch a given circle.

Show that your construction enables two tangents to be drawn, and that these tangents are equal in length.

6. The angle at the centre of a circle is double of the angle at the circumference subtended by the same arc.

From a point P outside a circle, two straight lines PQR, PST are drawn, cutting the circle. Show that the difference between the angles which RT and QS subtend at the centre of the circle is equal to twice the angle QPS.

7. If two chords of a circle intersect, the rectangle contained by the segments of the one shall be equal to the rectangle contained by the segments of the other.

AB is a diameter of a circle, and C is a point in AB produced. Through C, CD is drawn perpendicular to AB. If through any point P in CD a straight line PBQ be drawn meeting the circle in Q, the rectangle PB, BQ is constant.

8. What is meant by a locus?

A and B are two fixed points, and the area of the triangle ABC is constant; find the locus of C.

Higher Grade and Honours.

1. If a straight line AB is bisected in C and produced to D, prove that the rectangle contained by AD and DB together with the square on AC is equal to the square on CD.

ABC are three given points on a straight line. Find a point D in the line produced such that rect. AD, AB + sq. CD = sq. AC.

2. Prove that in every circle angles at the circumference which stand on the same arc are equal.

Prove that if the angles ABC and ADC are equal and B and D are on the same side of AC a circle will pass through the four points ABCD.

AB is bisected in O and P is any point in OB; OR is drawn perpendicular to AB and equal to OP; AR is joined. From R is drawn RS perpendicular to AR towards AB and equal to AR. Prove that SP is perpendicular to AB.

3. Describe a circle which shall touch one side of a triangle and the other two sides produced.

What is meant by the locus of a point? If BC is fixed and A moves on the circumference of a fixed circle passing through B and C, find the locus of the centre of the circle escribed to ABC and opposite to A.

4. When is A to B in the duplicate ratio of C to D?

Two similar parallelograms OABC and Oabc are similarly placed so that the angles AOC and aOc coincide. cb and AB, produced if necessary, meet in D. Prove Oabc : OADc : : OADc : OABC; and thence show that similar parallelograms are to one another in the duplicate ratio of their homologous sides.

5. Prove that in equal circles angles at the centres are proportional to the arcs on which they stand.

6. When is a straight line harmonically divided?

Show that a straight line which is bisected may be looked on as harmonically divided.

Prove that any diagonal of a quadrilateral is harmonically divided by the corners of the quadrilateral through which it passes, and the points where it meets the other two diagonals.

7. From a given point outside a plane draw a perpendicular to the plane.

O is a point outside a plane. OA is perpendicular to it. BC is a straight line in the plane and AD is perpendicular to it. Prove that OD is also perpendicular to BC.

8. If transversals through the angular points A, B, C of a triangle are concurrent, and intersect the opposite sides in D, E, F respectively, then

$$BD \cdot CE \cdot AF = DC \cdot EA \cdot FB.$$

ABC is a triangle and any straight line CF is drawn meeting AB in F. The angles BFC, AFC are bisected by FD, FE meeting the opposite sides in D, E. Show that AD, BE, CF are concurrent.

OXFORD LOCAL EXAMINATIONS.

June, 1889.

JUNIORS.—Books I.-VI.

[*Euclid's axioms will be required, and no proof of any proposition will be admitted which assumes the proof of anything not proved in preceding propositions of Euclid.*]

1. Define a right angle, a rhombus, a parallelogram.

2. If two triangles have two sides of the one equal to two sides of the other, each to each, and have also the angles contained by those sides equal to one another, they shall also have their bases or third sides equal; and the two triangles shall be equal, and their other angles shall be equal, each to each, namely those to which the equal sides are opposite.

If the diagonals of a quadrilateral bisect one another at right angles the quadrilateral is a rhombus or a square.

3. The straight lines which join the extremities of two equal and parallel straight lines towards the same parts are also themselves equal and parallel.

4. Show that every rhomboid is a parallelogram.

5. In any right-angled triangle, the square which is described on the side subtending the right angle is equal to the squares described on the sides which contain the right angle.

6. If a straight line be divided into any two parts, the square on the whole line is equal to the squares on the two parts, together with twice the rectangle contained by the two parts.

7. If ABC be a triangle obtuse-angled at C, and AD be drawn perpendicular to BC produced, prove that the square on AC is less than the squares on AB, BC by twice the rectangle DB, BC.

8. If two circles touch one another internally, the straight line joining their centres being produced passes through the point of contact.

9. If a straight line touch a circle, and from the point of contact a straight line be drawn cutting the circle, the angles which this line makes with the line touching the circle shall be equal to the angles in the alternate segments of the circle.

AB, AC are tangents to a circle, B, C being the points of contact; BD is a chord through B parallel to AC: show that the arcs CB, CD are equal.

10. Describe a circle about a given triangle.

11. Equal parallelograms which have an angle of the one equal to an angle of the other have their sides about the equal angles reciprocally proportional; and, conversely, parallelograms which have one angle of the one equal to one angle of the other and their sides about the equal angles reciprocally proportional are equal to one another.

Construct a rhombus equal to a given parallelogram and equiangular with it.

12. In equal circles, angles, whether at the centre or at the circumferences, have the same ratio which the arcs on which they stand have to one another.

SENIORS.—Books I.–VI. and XI. (1–21).

1. The angles which one straight line makes with another upon one side of it are either two right angles or are together equal to two right angles.

2. If ABC is a triangle such that the square on BC is equal to the squares on AB and AC together, the angle BAC will be a right angle.

If the square on BC (on the side remote from A) is BDEC, show that the perpendicular from D on AC (produced if necessary) is equal to AB and AC together.

3. Describe a square which shall be equal to a given rectilineal figure.

Of all rectangles having a given area the square is that the sum of the lengths of whose sides is least.

4. If two circles touch one another externally, the straight line which joins their centres will pass through the point of contact.

5. Define a segment of a circle, and the angle in a segment.

On the same chord and on the same side of it there cannot be two similar segments of circles, not coinciding with one another.

6. Inscribe a square in a given circle.

If a parallelogram can have a circle inscribed in it, it must be equilateral.

7. If the vertical angle of a triangle be bisected by a straight line which also cuts the base, the segments of the base will have the same ratio which the other sides of the triangle have to one another.

8. If four straight lines are proportionals, the rectangle contained by the extremes is equal to the rectangle contained by the means.

On a given base describe an isosceles triangle equal in area to a given rectangle.

9. If two straight lines meeting one another are parallel to two others that meet one another, and are not in the same plane with the first two, the first two and the other two will contain equal angles.

OXFORD UNIVERSITY RESPONSIONS.

Books I., II.

1. Define—superficies, centre of a circle, parallel straight lines, plane angle, rhomboid.

2. On the same base, and on the same side of it, there cannot be two triangles having their sides which are terminated at one extremity of the base equal to one another, and likewise those which are terminated at the other extremity.

3. Make a triangle of which the sides shall be equal to three given straight lines, but any two whatever of these must be greater than the third.

4. If two triangles have two angles of the one equal to two angles of the other, each to each, and one side equal to one side, namely, either the sides adjacent to the equal angles, or sides which are opposite to equal angles in each, then shall the other sides be equal, each to each, and also the third angle of the one equal to the third angle of the other.

5. To a given straight line apply a parallelogram, which shall be equal to a given triangle, and have one of its angles equal to a given rectilineal angle.

6. In any right-angled triangle, the square which is described on the side subtending the right angle is equal to the squares described on the sides which contain the right angle.

7. If a straight line be divided into any two parts, the squares on the whole line, and on one of the parts, are equal to twice the rectangle contained by the whole and that part, together with the square on the other part.

8. If a straight line be bisected, and produced to any point, the square on the whole line thus produced, and the square on the part of it produced, are together double of the square on half the line bisected and of the square on the line made up of the half and the part produced.

9. Divide a given straight line into two parts, so that the rectangle contained by the whole and one of the parts may be equal to the square on the other part.

CAMBRIDGE LOCAL EXAMINATIONS.

1889.

JUNIORS.—Books I., II., III., IV., VI.

[*The only abbreviation admitted for "the square on AB" is "sq. on AB," and for "the rectangle contained by AB and CD," "rect. AB, CD." All generally understood abbreviations or symbols for* words *may be used, but not symbols of* operations *such as* −, +, ×.]

A 1. Define a superficies, a circle, and parallel straight lines.
Give one of Euclid's postulates.

A 2. If two angles of a triangle be equal to one another, the sides also which are opposite to the equal angles shall be equal to one another.

> ABC is a triangle in which the sides AB, AC are equal to each other: equilateral triangles ADB, AEC are described on AB, AC, outside the triangle ABC: BE and CD intersect in O; prove that OD and OE are equal.

A 3. If two triangles have two angles of the one equal to two angles of the other, each to each; and one side equal to one side, namely sides which are opposite to equal angles in each; then shall the other sides be equal, each to each; and also the third angle of the one equal to the third angle of the other.

> AB, AC are two given straight lines. Show how to draw through B a line BPQ cutting AC in P and such that, if produced to Q so that PQ is equal to PA, the angle QCA may be equal to the angle QBA.

A 4. If a side of any triangle be produced, the exterior angle is equal to the two interior and opposite angles.

> A point O is taken within a triangle ABC such that the angles AOB, AOC are equal to the exterior angles of the triangle at C and B, prove that the angle BOC is equal to the exterior angle at A.

A 5. From a point in one of the lines containing a given angle draw a line which, with the lines containing the angle, shall include a given area.

A 6. In any right-angled triangle, the square which is described on the side subtending the right angle is equal to the squares described on the sides containing the right angle.

> If squares be described on the sides of any triangle ABC as in this proposition, prove that the perpendicular from A on BC divides the square on BC into two parts which differ from the squares on AB and AC by equal areas.

A 7. If there be two straight lines, one of which is divided into any number of parts, the rectangle contained by the two straight lines is equal to the rectangles contained by the undivided line and the several parts of the divided line.

A 8. In every triangle, the square on the side subtending an acute angle is less than the squares on the sides containing that angle, by twice the rectangle contained by either of these sides, and the straight line intercepted between the perpendicular let fall on it from the opposite angle and the acute angle.

> The perpendicular from A meets the base of the triangle ABC in D, and E is the middle point of BC, prove that the difference of the squares on AB and AC is equal to twice the rectangle contained by BC and DE.

B 1. Draw a straight line from a given point, without the circumference of a given circle, which shall touch the circle.

> From a point without a circle draw a line such that the part of it included within the circle may be of a given length less than the diameter of the circle.

B 2. If a straight line touch a circle, and from the point of contact a straight line be drawn cutting the circle, the angles which this line makes with the line touching the circle shall be equal to the angles which are in the alternate segments of the circle.

A straight line touches a circle at the point P, and QR is a chord of a second circle parallel to this tangent. PQ, PR cut the first circle in S, T, and the second circle in U, V; prove that ST and UV are parallel.

B 3. Inscribe a circle in a given triangle.

B 4. If two triangles have one angle of the one equal to one angle of the other, and the sides about the equal angles proportionals, the triangles shall be equiangular to one another, and shall have those angles equal which are opposite to the homologous sides. If a point be taken within a parallelogram such that the line joining it to one of the angular points subtends equal angles at the two adjacent angular points, prove that the lines joining it to any angular point will subtend equal angles at the two angular points adjacent to that angular point.

B 5. If any segment of a circle described on the side BC of a triangle ABC cut BA, CA, produced if necessary, in P and Q, prove that PQ is always parallel to a fixed straight line.

SENIORS.—Books I., II., III., IV., VI. and XI. (1–21).

1. If a side of any triangle be produced, the exterior angle is equal to the two interior and opposite angles.

Squares ABDE and ACFG are described on the sides AB, AC of a triangle ABC external to the triangle: prove that the lines CE and BG are at right angles.

2. If a parallelogram and a triangle be on the same base and between the same parallels, the parallelogram is double of the triangle.

HF and EG are the parallelograms about the diagonal AC of a parallelogram ABCD. Show that the sum of the areas of the triangles AEG and AHF is equal to the sum of the areas of CHF and CEG.

3. If a straight line be divided into two equal, and also into two unequal parts, the squares on the two unequal parts are together double of the square on half the line and of the square on the line between the points of section.

The triangle ABC is equilateral and AD is drawn meeting the base BC at right angles in D. In CB a part CE is taken equal to AD. Prove that the square on ED is equal to the rectangle contained by BE and BC.

4. If a straight line touch a circle, and from the point of contact a straight line be drawn cutting the circle, the angles which this line makes with the line touching the circle shall be equal to the angles which are in the alternate segments of the circle.

A line AD is drawn bisecting the angle A of a triangle ABC and meeting the side BC in D. Find a point E in BC produced either way such that the square on ED may be equal to the rectangle contained by EB and EC.

5. Describe a circle about a given triangle.

A triangle ABC is inscribed in a circle and B′, C′ are the middle points of the sides AC and AB. The perpendiculars from B and C on the opposite sides meet at P and PB′, PC′ meet the circle again in E and F respectively; prove that EF is equal and parallel to BC.

6. The sides about the equal angles of triangles which are equiangular to one another are proportionals; and those which are opposite to the equal angles are homologous sides, that is, are the antecedents or consequents of the ratios.

Construct a triangle which shall have one angular point at a given point, the other angular points on two fixed straight lines respectively, and its sides proportional to those of a given triangle.

7. If a solid angle be contained by three plane angles, any two of them are together greater than the third.

CAMBRIDGE HIGHER LOCAL EXAMINATIONS.

JUNE, 1889.—BOOKS I.–IV., VI. and XI. (1–21).

1. If a straight line fall on two parallel straight lines, it makes the alternate angles equal to one another, and the exterior angle equal to the interior and opposite angle on the same side.

Prove that the straight lines which bisect two opposite angles of a parallelogram either coincide or are parallel to one another.

2. In any right-angled triangle, the square which is described on the side subtending the right angle is equal to the sum of the squares described on the sides which contain the right angle.

If ABC be a right-angled triangle having A for the right angle, and squares be described on the lines BC, CA, AB on the sides opposite to the angles A, B, C, then will the diagonal of the square on AB passing through B be parallel to the diagonal of the square on AC passing through C.

3. If a straight line be divided into any two parts, the squares on the whole line and on one of the parts are together equal to twice the rectangle contained by the whole and that part together with the square on the other part.

A straight line AB is divided in C and D so that AC is equal to DB: prove that the squares on AB, CD are together double of the sum of the squares on AC and CB.

4. Draw a tangent to a circle from a given point without it.

If three circles touch, two and two, the tangents at the points of contact meet at a point and are equal, or are parallel.

5. Inscribe a circle in a given triangle.

6. The sides about the equal angles of triangles which are equiangular to one another are proportionals; and those sides which are opposite to the equal angles are homologous.

Two circles ABC, ABD intersect in A and B. Through A a line CAD is drawn cutting the circles in C and D respectively, and the tangent at A to the circle ABD cuts the circle ABC in E. Prove that the chord DA is to the chord CE in the ratio of the radii of the circles.

7. If two straight lines be cut by parallel planes, they are cut in the same ratio.

CAMBRIDGE UNIVERSITY PREVIOUS EXAMINATION.

JUNE, 1889.—BOOKS I., II., III. and VI. (PROPS. 1–19).

1. Define a triangle, a circle, and a square.

   What is the difference between a postulate and an axiom?

2. Any two sides of a triangle are together greater than the third side.

3. Construct a triangle, the sides of which shall be respectively equal to three given straight lines, any two of these lines being greater than the third.

4. Describe a parallelogram equal to a given rectilineal figure, and having an angle equal to a given rectilineal angle.

5. If a straight line be divided into any two parts, the squares of the whole line and of one of the parts are equal to twice the rectangle contained by the whole line and that part together with the square of the other part.

6. If a straight line be divided into two equal, and also into two unequal parts, the squares of the two unequal parts are together double of the squares of half the line, and of the line between the points of section.

7. If one circle touch another internally, the straight line which joins their centres, being produced, shall pass through the point of contact.

8. If a straight line touch a circle, and from the point of contact a straight line be drawn cutting the circle, the angles made by this line with the line touching the circle shall be equal to the angles which are in the alternate segments of the circle.

9. If the sides of two triangles about each of their angles be proportionals, the triangles shall be equiangular.

10. The bisector of the exterior angle A of a triangle ABC meets the side BC produced in D. Prove that the perpendiculars drawn from D to the sides AB, AC produced are equal to each other.

11. In a right-angled triangle prove that the line drawn from the right angle to the middle point of the base is equal to half the base.

12. Construct an isosceles triangle in which the vertical angle shall be equal to four times each angle at the base.

CAMBRIDGE MATHEMATICAL TRIPOS.

1888.

1. Parallelograms on the same base, and between the same parallels, are equal to one another.

   A straight line DE is drawn to cut the base BC of a triangle ABC and is terminated by the sides AB, AC produced if necessary; prove that, if the quadrilateral BDCE be of constant area, the middle point of DE lies on one of two fixed straight lines.

2. Prove that, if from any point without a circle two straight lines be drawn, one of which cuts the circle and the other touches it, the rectangle contained by the whole line which cuts the circle and the part of it without the circle shall be equal to the square on the line which touches it.

Squares are described on the sides of a triangle ABC, namely BCDE, CAFG, ABHK, and A_1, B_1, C_1 are the intersections of BF and CK, CH and AE, BG and AD, respectively; prove that AA_1, BB_1, CC_1 meet in a point.

Prove that a similar theorem is true if the intersections of BG and CH, CK and AD, AE and BF, be taken.

3. If the vertical angle of a triangle be bisected by a straight line which also cuts the base, the segments of the base have the same ratio which the other sides of the triangle have to one another: and if the segments of the base have the same ratio which the other sides of the triangle have to one another, the straight line drawn from the vertex to the point of section bisects the vertical angle.

Prove that, if a point on the internal bisector of the angle of a triangle be joined to the two other vertices and the joining lines be produced to intersect the external bisector of the angle, the straight lines joining the points of intersection to the two vertices meet in a second point on the internal bisector.

4. Prove that the sides about the equal angles of triangles which are equiangular to one another are proportionals; and those sides which are opposite to the equal angles are homologous, that is, are the antecedents or the consequents of the ratios.

Construct a rhombus of which two sides lie along two given parallel straight lines while the other two pass each through a fixed point. Of how many solutions does the problem admit?

5. Prove that the locus of a point which is such that its distances from two fixed points are in a constant ratio is a circle.

Prove that there are two points, each of which has the property that its distances from the angular points of a triangle are proportional to the opposite sides, and that the straight line joining them passes through the centre of the circumscribed circle.

6. Prove that, if two straight lines be cut by parallel planes, they are cut in the same ratio.

Prove that the straight lines which intersect three given non-intersecting lines that are parallel to the same plane are also all parallel to a plane.

1889.

1. Parallelograms on the same base and between the same parallels are equal to one another.

Show how in the three cases of the proposition to cut up one parallelogram so that the parts when properly fitted together will form the other parallelogram; and show also how to do the same for any two equal triangles.

2. In every triangle the square on the side subtending an acute angle is less than the squares on the sides containing it by twice the rectangle contained by either of these sides and the straight line intercepted between the perpendicular let fall on it from the opposite angle and the acute angle.

A point P is taken within a triangle ABC such that when perpendiculars PM, PN are let fall on AB, AC respectively the rectangles CN.AC and BM.AB are equal. Prove that P lies on a fixed straight line.

3. Prove that the feet of the perpendiculars drawn to the sides of a triangle from any point on the circumscribing circle are collinear.

A crossed quadrilateral whose opposite sides are equal is inscribed in a circle. Prove that the feet of the perpendiculars drawn to the sides from any point on the circumference lie on a circle the locus of whose centre is a straight line.

4. Inscribe a circle in a given triangle.

A triangle is formed by the centres of the three circles which pass each through two angular points of a given triangle and cut its inscribed circle orthogonally. Show that (1) its sides are parallel to the lines joining the points of contact with the sides of the inscribed circle of the given triangle, (2) the centres of the circumscribing circles of the two triangles coincide, (3) the radii of these circles differ by one half of the radius of the inscribed circle of the given triangle.

5. If two triangles are equiangular the sides about their equal angles are proportional.

A straight line moves so that it is divided in a constant ratio by the sides of a triangle. Prove that the locus of a point which divides one of the segments in a constant ratio is a straight line.

6. If a straight line stand at right angles to each of two straight lines at their point of intersection, it is at right angles to the plane which passes through them.

Two tetrahedra ABCD, $\alpha\beta\gamma\delta$ are such that the five edges BC, CA, AB, DA, DB of the first are at right angles to the five edges $\delta\alpha$, $\delta\beta$, $\delta\gamma$, $\beta\gamma$, $\gamma\alpha$ of the second. Prove that the remaining edges are also at right angles and show that the perpendiculars from the angular points of the one on the corresponding faces of the other are concurrent.

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