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CONDUCTED BY

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AND

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"Nec aranearum sane textus ideo melior quia ex se fila gignunt, nec noster vilior quia ex alienis libamus ut apes." JUST. LIPS. *Polit.* lib. i. cap. 1. Not.

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"Meditationis est perscrutari occulta; contemplationis est admirari perspicua . . . . Admiratio generat quæstionem, quæstio investigationem, investigatio inventionem."—*Hugo de S. Victore.*

——"Cur spirent venti, cur terra dehiscat,
Cur mare turgescat, pelago cur tantus amaror,
Cur caput obscura Phœbus ferrugine condat,
Quid toties diros cogat flagrare cometas,
Quid pariat nubes, veniant cur fulmina cœlo,
Quo micet igne Iris, superos quis conciat orbes
Tam vario motu."

J. B. Pinelli ad Mazonium.

ALERE FLAMMAM.

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[SIXTH SERIES.]

JULY 1918.

I. *Scientific Signalling and Safety at Sea.*

By Prof. JOHN JOLY, *M.A., D.Sc., F.R.S., F.G.S.**

[Plate I.]

I. APPROACHING THE COAST.

THE most common-place and often one of the most urgent of the problems which confront the sailor is the determination of his position upon near approach to the coast. We may, indeed, say that the determination of latitude and longitude at any time is solely in preparation for that stage of the voyage when the ship draws near the land. The special difficulties sometimes attending the solution of this problem are known only to those who have endeavoured to make a landfall or pick up a lightship in wild or thick weather or in the calm obscurity of a fog.

In our Admiralty Sailing Directions or Pilots we read that there is no help for the sailor in a fog save unreliable fog-signals and the use of the lead. The whole passage as ordinarily given is intimately connected with our subject and highly instructive. "Sound is conveyed in a very capricious way through the atmosphere. Apart from wind, large areas of silence have been found in different directions and at different distances from the fog-signal station, in some instances even when in close proximity to it.

* Communicated by the Author. Being the Tyndall Lectures delivered at the Royal Institution, April 1918.

"The Mariner should not assume—

a. That because he fails to hear the sound he is out of hearing distance.

b. That, because he hears a fog-signal faintly, he is at a great distance from it.

c. That because he hears the sound plainly he is near it.

d. That, because he does not hear it, even when in close proximity, the fog-signal has ceased sounding.

e. That the distance from and the intensity of the sound on any one occasion, are a guide to him for any future occasion.

Taken together, these facts should induce the utmost caution in closing the land in fogs. The lead is generally the only safe guide."

It would be, of course, entirely wrong to conclude that such drastic warnings are intended to imply the general worthlessness of aerial sound signals. It is probable that the disuse of such signals would not find favour. Our present purpose is rather to consider additional aids to navigation whereby the sailor escapes the special dangers arising from the failure of aerial fog-signals, and is supplied with other signals at once more reliable, heard at greater distances, and giving him information beyond the power of aerial fog-signals to convey. Such modern methods of signalling are based on recent advances in science.

We shall consider first what may be called "synchronous signalling," that is the use of signals propagated in different media but timed so as to start at the same instant.

The principles of synchronous signalling have for long formed a part of familiar household science. When timid people see the flash of lightning and hear the crash of thunder they feel reassured when they perceive an appreciable interval separating the one phenomenon from the other. On the other hand, when both occur together they infer, and rightly so, that there is more danger. And most people are aware of the principle underlying this inference. If the seat of the electric discharge, the flash itself, in fact, is remote, the sound originated by it, *i.e.* the thunder, takes an appreciable time to reach the ear. Travelling nearly 1100 feet in a second, this time interval may amount to several seconds. On the other hand, the velocity of propagation of light is so enormous that we may consider that we see the flash at the very instant of its occurrence however remote it may be placed. Hence if one second intervenes between the moment of seeing the flicker of the lightning and hearing the thunder, the scene of the discharge must be about 1100 feet distant. If two seconds elapse the distance is 2200 feet, and so on. The whole theory of synchronous signalling is involved in this time-honoured chapter of domestic science.

For suppose a gun to be fired from a lightship guarding some peril of the coast, and simultaneously with the explosion a light be flashed from the lantern of the lightship ; a vessel afar off sees first the flash—at the very instant of its occurrence—and later she receives the sound of the gun. For every second of interval between the seeing of the flash and hearing of the gun about 1100 feet may be allowed. If the interval was $3\frac{1}{2}$ seconds then the ship is $1100 \times 3\frac{1}{2} = 3850$ feet distant from the lightship. But this is just the information which the mariner approaching at night values above all other and which is most conducive to his safety. It gives him the means of determining not only his distance from the danger guarded by the lightship but also it gives him his actual position.

On the existing system of coast signals the mariner is given the light and the sound in no way co-ordinated one with another. Each of these signals, therefore, is aimed at accomplishing the same thing, *i.e.* telling the sailor the *direction* in which the danger lies. They give him, also, some idea of his distance as being within the limits of visibility or audibility of the one or other of the signals. But the inference of distance is so affected by weather conditions as to be uncertain and even deceptive in character. It is possible to hear the gun of the lightship and to think it sometimes close by and again far off, and for the direction of the sound to remain quite uncertain. The bearing of the light is indeed certain when it is visible. Our coast signals, as at present ordered, therefore, give the mariner at best the bearing of the danger and but a rough and uncertain indication of distance. But the synchronized signals we have described give him not only the bearing but a determination of distance sufficiently accurate to enable him to fix his position.

In order to understand this clearly let l (Pl. I. fig. 1) mark the position of the lighthouse. The circle struck round it is to the radius d, which is the distance as determined by the synchronous signal. The ship must be located somewhere on this circle. If now the bearing of l from the ship is S.W., the ship is at ×. It cannot be anywhere else. And evidently the bearing of the light and the distance must, similarly, in every case give the sailor his position.

There are objections, as we have seen, to the use of sound signals propagated through the atmosphere. They are difficult to pick up in stormy weather ; more especially when the wind is blowing from the ship towards the source of sound. The sound appears to be weakened by these conditions, and if the wind is making much noise on the ship it may not be heard at a sufficient distance. It has to be listened for in the open. A more serious objection is the strange and, fortunately, not very common phenomenon of silent areas as referred to in the Admiralty Sailing Directions quoted above. A ship may be well within the range of audition ; the weather may be calm, even windless, and the sound may be mute over certain areas. The phenomenon is a remarkable one, and a full explanation cannot be said to exist. It has been investigated by Tyndall and by Lord Rayleigh. The nature of the sound seems without influence. Even the very beautiful elliptic trumpet of Lord Rayleigh, whereby the sound is caused to spread laterally and its vertical dissipation prevented, cannot counteract the evil when the necessary conditions prevail. There appears to be such a deviation of the sound as causes it to rise and arch over certain areas. It may be heard ten miles from the source and be entirely mute close to it. Or, again, when approaching the source we may find more than one silent area as if the sound waves followed a sinuous path, rising and again sinking to the surface of the sea. In such cases the value of the synchronization may be lessened in another way. The sound which is heard outside a mute area will not have travelled directly from its source. The question is : how much has its journey been lengthened? Probably the increase of distance is not much. Nevertheless there may be appreciable error here. Again, the use of light-flash has, of course, the drawback of being invisible in fog or thick weather. Hence only under certain conditions and at certain times is the combination of synchronized light and sound signals of value.

Notwithstanding these limitations such a system would undoubtedly prove very useful. To condemn it in advance is as senseless as to condemn all our lighthouses and fog-signal stations because conditions arise when they are useless. And it should be considered by all responsible authorities if, for the general use of small craft—fishing boats, small coasters, and the like—a system of buoyage based on light-flash and bell-stroke would not be valuable. We may profitably consider here, before going further, how such a system may be worked so as to meet the requirements of untrained observers.

Automatic bell-buoys are common around our coasts at the present time; and buoys which show an occulting light are also common. Very often both functions are performed by the one buoy. The bell-stroke is operated by energy derived from the motion of the waves. There is, even under conditions of apparent calm, considerable energy available from this source. It is not necessary for the bell to be struck at regular intervals, but it is of importance that the blow upon the bell should always be given with the same force, so that the sound emitted should be of uniform loudness. We may suppose, then, that the up and down movements of the buoy, however gentle and slow, are resisted by a horizontal vane immersed in the water beneath. This vane, as it oscillates respecting the buoy with the rise and fall of the latter, compresses, by means of a ratchet, a spring which when stressed to a certain degree is released and its stored energy expended in actuating the hammer. As we shall see later very similar mechanism is in frequent and successful use. We have, then, a bell-stroke in air, at intervals, and made with a certain constant force. It is matter of observation that even in calm weather three or more strokes will be given per minute. We would require, in fact, a certain controlling mechanism limiting the number of strokes to, say, 3 per minute.

I assume now that a light-flashing system is also installed upon this buoy similar to many of the blinking or occulting lights marking sand-bank or other danger close to the shore. A connexion between the mechanism actuating the hammer and that causing the occultation of the light is arranged, of such a nature that simultaneously with the stroke of the bell there is a sudden flare-up of the light, or sudden luminous flash, followed by a succession of flashes spaced at short regulated intervals.

We can so order the signals that the sailor making harbour requires no stop-watch to measure the lag of the sound upon the light signal. The light flashes, repeated at regular intervals, themselves afford the measure of the lag of the sound waves. For suppose 20 successive light flashes spaced at such an interval of time as the sound takes to travel one-tenth of a nautical mile—that is one cable. Flashes so timed are easily counted, this interval ($0''\cdot53$) being very little over one-half second. Then if the first flash is emitted 0·53 second later than the instant of the first bell-stroke, when the first flash reaches the ship the sound has already travelled one cable, and if the sailor is at the distance of one cable he hears the stroke of the bell at the instant at which he sees the first light-flash. If the sound comes to him along with the 2nd flash he must be 2 cables distant, and if with the 10th flash he is 10 cables from the buoy. Thus he has only to count up the flashes till he hears the bell, and the result is the number of cables which separate him from the buoy.

The value of this buoy to small craft is more especially evident when we remember that such craft are to a great extent debarred from the use of wireless and submarine signals. Expert knowledge denied to the humble skipper is required for the care and use of the former; and the small draught reduces seriously the efficiency of the latter.

We can picture now the working of this simple and inexpensive substitute for the lightship on dark and wild nights. When the sailor picks up the light he is, maybe, some three or four miles away. It may be of serious importance to determine his distance: either for laying his course along the coast or the making of harbour. He sees the distant flash and he knows it is safe to stand in till he hears the bell. Presently he picks this up. He now waits for the next group of flashes and he counts them as they come in:— one, two, three . . . . till he hears the clang of the bell. It may come with the 15th flash. If so he knows he is 15 cables or $1\frac{1}{2}$ mile distant. Nothing can be simpler. As mere indicator of direction the light-and-bell buoys of our coasts possess nothing like the value of this synchronized light-and-bell buoy. The first cost would be small and the cost of upkeep, compared with that of a lightship, trifling.

Modern advance has given us signals of other kinds which —as all know—have already afforded invaluable help to the sailor. Wireless is a sort of light signal against which fog and snow and thick weather are powerless. Its velocity of propagation is practically instantaneous. Submarine signalling utilizes the propagation of sound through water, and this may be regarded as furnishing a sound signal which also is unaffected by weather conditions. The sound of a bell-stroke beneath the water travels at about 4800 feet (1463 metres) per second. Hence the submarine bell-stroke lags behind the wireless "dot" by 1·2 seconds for each nautical mile traversed, if both signals are started together. If an air-sound and a water-sound be started synchronously from the same point, the lag of the atmospheric sound on the submarine is 4·3 seconds for each nautical mile traversed.

For the benefit of those unacquainted with recent advances in this branch of applied science a word may be said here about the advent of submarine signalling. The idea is really an old one. It has long been known that sound travels under water with remarkably little loss of clearness and intensity. In 1826 Collodon and Sturm carried out their well-known experiment on the Lake of Geneva. The object of this experiment was to ascertain the velocity of sound in water. A submerged bell was used. The hammer which struck the bell was so connected with a trigger above the water that a charge of gunpowder was ignited at the instant of the striking of the bell. An observer in a boat at a certain measured distance listened at a hearing trumpet immersed in the lake. He heard the sound of the bell as propagated through the water and saw the flash of the explosion as propagated through the æther. Assuming the velocity of the latter to be comparatively infinite, the interval between the seeing of the flash and the hearing of the bell affords the velocity of sound in water. Obviously we can reverse the objective of this experiment. Knowing the velocity of sound in water and measuring the interval of time elapsing between the flash and the sound, we can determine the distance over which the latter has travelled.

The fact of the easy propagation of sound through water is an old discovery of the diver. The perfect audibility is even startling. It is said that a lost watch, which being watertight continued to go, was recovered by a diver tracing the tick of the watch to its source *.

Many years ago I experimented on the audibility of explosive sound signals beneath water. The object in view was to test a method of determining the depth beneath a ship travelling at full speed, by the dropping of a sinker which would detonate a small charge of explosive on contact with the bottom. The time interval between the moment of releasing the sinker and hearing the explosion, knowing the rate of descent of the sinker, gives the depth with sufficient accuracy. In order to test the distance to which the

* We may note parenthetically the curious fact that marine animals do not seem to avail themselves of this property of the medium in which they live to the extent we might have expected. The organism appears to be ever ready to avail itself of every advantage which the nature of the medium offers it. In this case evidence that it does so seems wanting. True it may develop listening orgaus but, whether it seeks to preserve the secrecy which is the chief protection of the submarine, or whether its silence benefits it in some other way, the fact remains that the sounds emitted by rattlesnake or cricket do not appear to be emulated by fish or crustacean. Our sensitive microphones must have discovered the existence of any such devices. The matter deserves further investigation.

sound of the explosion would be propagated, small metal cartridges containing about half an ounce of gunpowder were exploded at the bottom off the coast of Dublin. The explosion was heard with astonishing distinctness at least a mile away in boats unprovided with any form of sound-receiving apparatus. The sound was perceived in an open boat as an apparent blow or percussion against the bottom of the boat.*

The apparatus both for sending out and receiving submarine signals has been developed to a high pitch of reliability, largely due to American initiative and to the scientific methods of the Submarine Signal Company. As may be imagined, a long period of suggestions, initial experiments, and abortive patents preceded the existing apparatus. The submarine bell has taken its place as a standard means of sound-production, although invention in other directions has produced wonderful results as we shall see. Repeated trials of various types of bell have resulted in a pattern weighing 220 lb., made of bronze, and with a period of 1215 vibrations in water. This bell is now doing duty in every part of the world: on lightships; bell buoys; on the bottom of the sea; at the pier-head or on ships.

The striking mechanism is contained in a cylindrical bronze case attached above the bell (Pl. I. fig. 2). The striking is generally operated pneumatically. A twin rubber-hose pipe connects the bell, which is suspended by a chain at a depth of about 18 feet, with a reservoir of compressed air on the lightship, or shore station. This reservoir is kept pumped full of compressed air by means of a small oil or steam engine. The mechanism for operating the bell-stroke is simple. An air-driven code regulating valve forms part of the over-water plant and determines the frequency and character of the submarine signal. Some 30 or 35 strokes may be struck per minute. In 1906 the United States Government tested five of these bells for 51 days; the ringing being continuous, six seconds between the blows. Their introduction into England was slower than in the States. The British Admiralty tested the system later that same year and reported as follows: " . . . . The submarine bell increases the range at which the fog signal can be heard by a vessel, until it approximates to the range of a light-vessel's light in clear weather, and moreover its bearing can be determined

* A patent was obtained at the time (1890) for this form of sounding machine. Failing any encouragement from the Admiralty, it was abandoned. Some years later the method was independently re-patented by an officer in H.M. Navy.

with quite sufficient accuracy for safe navigation in fog, from distances far beyond the range of aerial fog signals if the vessel is equipped with receivers. If the light-vessels round the coast were fitted with submarine bells it would be possible for ships fitted with receiving apparatus to navigate in fog with almost as great certainty as in clear weather." In spite of this report the multiplication of submarine bell stations was slow in England. The first was installed off the Mersey in December, 1906. The Irish Lights Commissioners placed a submarine bell on the Kish Bank Lightship in 1909.

The submarine bell is in some cases operated electrically; more especially for use off the coast on the floor of the sea. A power station on the coast supplies the requisite current. The bell is suspended from the apex of a steel tripod about 25 feet high and weighing 3 tons, a cable being taken ashore. The depth varies down to 25 fathoms. Such a bell is located off the Stack Lighthouse, Holyhead. The frequency of the bell-stroke is controlled by rotary time switches. In the United States this system came into use as early as 1901.

Finally the submarine bell-buoy claims our attention. This is a simple and effective signalling machine and one which may be maintained at small annual cost. The buoy carries the bell and its simple mechanical mechanism housed beneath it in a boiler-plate receptacle which is open below, the bell alone partly protruding. Thus the mooring chain cannot foul the bell or its operating mechanism. The motive power is entirely derived from the wave energy of the sea. The mechanism is such that the energy imparted by the rising and falling of the buoy to a hinged vane immersed beneath is accumulative. A spring is compressed by the movement of the buoy, whether this be up or down. Each oscillation thus compresses the spring a little more till when a certain compression is produced the spring is released and in the act of release causes the hammer to strike the bell. The uniform intensity of the blows is thus secured. The frequency of the strokes depends on the state of the sea, but, as already mentioned, is never less than three or four strokes per minute. To secure the mechanism against the rusting effects of sea-water the chamber holding it is filled with oil; any leakage of which is made good from a small tank above.

The recognition of sound by those on the vessel presents a problem of equal importance with that which we have been considering. It is requisite not only to receive the sound, but to receive it in such a manner as to enable the sailor to determine the direction from whence it proceeds.

The earlier attempts were directed mainly to develop listening devices which could be towed astern of the ship or could be attached without to her sides. It was expected that the noises on the ship would render any other mode of listening ineffective. Later results showed that listening to such acoustic vibrations as the walls of the ship pick up from the sea is the most effective method. This method, too, permits of determining the direction whence the sounds proceed. It was ascertained that the deeper the listening device was located in the ship the better. A small vessel is thus at a disadvantage in hearing the bell, and over-board receivers will not do.

As finally worked out the listening arrangements are simple. A small cast-iron tank is screwed on to the inner wall of the ship, being open against the ship's plates. This tank is filled with water. In it two microphones are immersed near each other, but one forward, the other more aft. One such tank holding two microphones is fixed to starboard, another to port. The sound gathered by the iron walls of the vessel passes directly to the water in these tanks, and this in turn conveys it to the microphones. The best position for the tanks is well forward, nearly in the bow, this being the most frequent presentation to the source of sound. The best position of the tank is found by direct trial and varies with various peculiarities of the particular vessel.

Leads from the microphones pass upwards to the bridge. There two telephone receivers are used for listening: one being applied to each ear. One of these telephone receivers goes to the forward, the other to the after microphone in the one tank. A switch enables either the port or starboard tank to be put on to the telephones. A semaphore tells the sailor to which side he is listening. The operator listens alternately to the sound received on port and starboard. If the signal station lies to port the telephones when switched on to the microphones on that side are loud while the starboard microphones are mute. If the bell is right ahead both microphones speak equally loudly. For obtaining an accurate bearing of the bell it is usual to swing the ship till she is bow on to the bell as judged by the equality of the sound in the microphones. The course of the vessel is then the bearing of the bell.

The conditions which are most favourable to the receipt of the sounds involve the presentation of the surface of the ship where the tanks are placed towards the source of the sound. It follows that the loudness of the sound and the distance at which the bell is heard depend on the bearing of the bell from the ship. If the bell is right aft no sound is heard save at close quarters. In this case the stoppage of the sound is assisted by the action of the propellers in breaking up the medium. A bearing right abeam is, generally, the best. The sounds weaken when well forward. Sounds coming from the opposite side of the vessel are not heard save at small distances from the bell.

While weather conditions affect to some extent the picking up of the signals—chiefly owing to noises developed by the pitching of the vessel—the signals are recognizable at long distances in any weather.

It is evident that of all modes of synchronous signalling which may be suggested, the combination of under-water sounds and wireless dot is the most free from liability to failure. True the sensitiveness is not so great as we obtain by other combinations. But facilities for receiving such signals are confined—it may be said—to the larger vessels, and these approach at such speeds that they obtain all they require if they can determine their distance from the shore to an accuracy of one quarter of a mile or even of half a mile. They should be able to effect the more accurate determination by this combination. If the radio dot is sent out at intervals of about 0·6 second the submarine bell-stroke lags the interval between two dots for each half sea-mile traversed. If the ship is 5 miles off the coast the sound lags 10 such intervals, and the bell comes in with the 10th dot, supposing that the first dot is emitted 0·6 second later than the first bell-stroke. In this case the sailor counts up the dots, and so obtains the number of half sea-miles separating his ship from the signal station. As it is quite possible to tell when the bell-stroke falls somewhere between two consecutive radio dots, estimation to the $\frac{1}{4}$ sea-mile is feasible.

In these operations the receipt of the signals is effected by listening with one ear to the bell sounds and with the other to the radio sounds,—a telephone receiver covering each ear of the operator.

In September 1911, the United States Hydrographic Department undertook an experiment on the use of synchronized signals in air, water, and æther. The signals were sent out from the Nantucket Lightship near New York (see fig. 3, Pl. I.). The aerial sound signals were created by blast from a steam whistle and those in water by submarine bell. At the instant the whistle blew, a wireless tick of two or three seconds' duration was sent out, and simultaneously with the making of the contact the valve of the striking mechanism of the submarine bell was tripped and a stroke was given to the bell. The coincidence of all three signals was tested by observation close to the lightship.

On board the U.S.S. 'Washington' the interval between the arrival of the aerial sounds and the wireless tick was read and recorded to one half second; and that between the bell-stroke and the wireless to tenths of seconds. It was assumed that the velocity of sound in air at the prevailing temperature (68°·5 F.) was 1132 feet per second, and in water (at 66° F.) 4794 feet per second. The weather was calm and hazy.

The course steered was first due West from the lightship for a distance of 8 miles, then turning and heading E.S.E., the lightship being passed on the port beam at a distance of 3450 yards. Standing on for 8 miles further she turned to the N.W., passing the lightship on the port beam at about 4600 yards; and thence back to the lightship on a S.E. course.

At starting the whistle and bell were right astern. The bell was lost at a distance of about 2 miles, and the whistle at, it is stated, about half a mile. The loss of the sound of the whistle can only be ascribed to the phenomenon of silent areas. The loss of the bell is a consequence of the defective presentation of the receiving tanks towards sounds coming from right astern. As might be expected, the bell was not again picked up till the 'Washington' turned to go eastward. It was then picked up at a distance of about 7·6 miles, the lag of the sounds on the radio dots being 9·5 seconds. The sounds were then reaching her on the port bow. The whistle was not recovered till the 'Washington' had approached much nearer to the lightship—a distance of about 4 miles. Bell and whistle were held on this course till the lightship was passed and left well astern, the bell sound being lost when the distance from the lightship was about $5\frac{1}{4}$ miles, and the angle of approach of the sounds was 19° with the course and approaching on the stern of the 'Washington.' This, again, is to be expected as a consequence of the lessening presentation of the receiving tanks. The whistle was held on this E.S.E. course till the 'Washington' was about $7\frac{1}{2}$ miles from the lightship. Here the whistle had the advantage. The bell was recovered immediately on turning N.W., and when the distance was 8·6 miles. The whistle was picked up on the N.W. course when 6 miles from the lightship. The real superiority of the submarine transmission of sound is here plainly shown. Both sounds were then held till the finish.

The course of the ship and each observation are recorded on a chart: a zig-zag red line connecting the determinations of distance by the bell and a similar blue line those by the whistle. Both these lines cross and recross the true course, which appears as a straight black line. The true course was found by range-finder and compass.

The experiment is highly instructive. The outstanding features are (*a*) the fact that both bell and whistle when heard suffice to determine the distance with fair accuracy: (*b*) the failure of the aerial sounds when the 'Washington' was quite close to the lightship, the bell being still audible in spite of the unfavourable presentation: (*c*) the distance of hearing the bell being cut down to 2 miles owing to sternward presentation: (*d*) its audibility on favourable bearings over a wide angle to $8\frac{1}{2}$ miles, and (*e*) its audibility at $5\frac{1}{4}$ miles when the approach of the sound was sternward at 19° with the course: (*f*) the maximum carriage of the aerial sounds—$7\frac{1}{2}$ miles—is exceeded by that of the submarine bell. The ultimate limit of audibility of the latter was not reached.

The general conclusion must be that with favourable presentation the submarine sound affords a more reliable signal than aerial sound. The causes of its failure can be foretold and are not capricious. It is certain that if at any time the 'Washington' had been swung into a more favourable course the sounds would again have been heard. On the other hand the loss of aerial signals is capricious and cannot be anticipated, and swinging the ship must fail to recover them.

That the U.S. Government were satisfied with the results of this experiment is shown by the recent establishment on Fire Island Lightship, off New York Harbour, of a synchronized signal station, involving the emission of submarine bell-sounds and wireless dots.

Synchronous signalling is, therefore, in practical use. This first installation professes to be in a sense experimental, "although this station has proved accurate on test." Ship captains are asked to report their experience to the Hydrographic Department. The British Board of Trade has recently issued to mariners the requisite notice and descriptive particulars. "The apparatus will be in operation during fog, mist, rain or falling snow. The range of this apparatus is limited to the receiving range of the submarine bell receiving equipment employed on shipboard, and in all practical cases this is within six or seven miles. The submarine bell strikes six strokes, pause, then eight strokes once every 38 seconds." The series of radio signals begins about $\frac{1}{2}$ second after the first of the group of six bell-strokes. The distance is determined by counting up these radio dots until the first stroke of the six submarine signals is received. The number of dots thus determined gives the distance in half sea-miles. It will here be seen that two unsymmetrical groups of bell-strokes are emitted. This appears to have in view the certain identification of the station and of rendering it easier to first pick up the signals. The number of the lightship is stated to be spelled out by the signals. Numerical examples are added showing readings of distance to the quarter mile. In this installation the size of the radio antenna is designed to send out signals which will not be heard much beyond the range of the submarine bell, in order to avoid unnecessary interference with near-by radio stations.

We may now picture to ourselves the practical application of this system of synchronized submarine bell-strokes and radio dots installed at Fire Island. We are on board a liner going westward and—we will suppose—are deep in a fog bank. Our whistle emits prolonged and far sounding blasts every two minutes. In former years, when the present writer experienced just such conditions approaching New York, frequent determination of depth was the only means available for fixing with any approach to accuracy the position of the ship, and hours were thus wasted, gradually stealing closer to the land. Let us now, however, imagine the little instruments on Fire Island busy tapping out to the mariner the knowledge he so anxiously desires to obtain. The speed of the ship is but little reduced, for ample warning by wireless dots and submarine bell stroke may be counted on. And now upon our ship the wireless operator reports to the bridge the first wireless dots. Then the bell-strokes are picked up. There are the six—pause—eight strokes once every 40 seconds. There can be no doubt as to what he is listening to. He waits for the first of the group of radio dots and counts them up till he hears a bell-stroke. He finds that the bell-stroke falls—say—just between the 12th and 13th radio dot; that is to say he must divide $12\frac{1}{2}$ by 2 for the distance in knots. He reports, accordingly, $6\frac{1}{4}$ miles from Fire Island Lightship, the signals being heard on the port bow. The land fall is made.

That radio signals and submarine bell could be worked reliably from a buoy, and, if desired, in combination with light flashes, seems very probable. The emission of the instantaneously propagated signals would be started by the bell-stroke. The wireless would have a range comparable with the carriage of the bell-sound ; the light flashes being probably of lesser range and for the use of smaller vessels which can generally approach with safety nearer to the coast owing to shallow draught. The emission of wireless signals at intervals of one or two minutes would be quite adequate, and economy of current would be attained.

The Fessenden Oscillator has long been a potential rival to the submarine bell as a means of generating sound waves in water. But lately such developments of the Oscillator have been made that it seems highly probable that on the more important ships it will take the place of the bell. Professor Fessenden has, in short, by his recent improvements rendered possible uses of submarine signalling almost unhoped for, although often wished for in the past. On the results of the experiments claim has been made to—

(*a*) Increased radius of audibility up to 30 miles or even more.
(*b*) The easy signalling by Morse code over these great distances by an ordinary telegraph key.
(*c*) The receipt and emission of the signals by one and the same apparatus located in the ship or lowered overboard.

To these may be added the following, provisionally on further experiments proving as successful as those already made :—

(*d*) The determination of depth beneath the moving vessel by echo from the bottom.
(*e*) The location of icebergs by reflected sound from the submerged part of the berg.

The transmission of speech over short but useful distances is, in addition to the claims founded on experiments, a highly probable development. What these claims involve may not at first be fully realized. Even if we accept the first three only we approach the consideration of the instrument on which these are founded with considerable interest.

The new Oscillator is not in principle different from the earlier invention of Professor Fessenden. The sound generated in the water originates in the rapid in-and-out vibration of a metallic diaphragm. This diaphragm may form part of the side of the ship. Now, obviously, the difficulty to be overcome in making such an apparatus successful is to generate and apply a force of sufficient intensity to overcome the inertia of the diaphragm and other moving parts (weighing in point of fact over 100 lb.) as well as that of the water, in a space of time measured in hundredths or even thousandths of a second. In order to transmit 20 words per minute by code about 100 compressional waves are required per minute, and to transmit speech several thousands of waves.

This power is, of course, applied electrically, an armature being excited by a powerful alternating current having a frequency of about 500 per second.

It is remarkable that this instrument, in spite of the great inertia and accelerations involved, can act as a receiver to sound waves reaching the ship through the water; functionating then as a generator. Hence it is only necessary, when the oscillator is being used both for the transmission and reception of sound, to set over a switch with each change in the nature of the operations required. The sounds may also be received by ordinary microphone as fitted for the submarine bell. (An interesting account of the improved oscillator is issued by the Submarine Signal Company.)

In an early test the oscillator was lowered 12 feet off the Boston lightship. The signals were plainly heard by microphone 31 miles away. They have been emitted also from moving ships and heard more than 20 miles away. It is evident that on vessels and in situations where an alternating current of sufficient power is available the use of this new device possesses great advantages. For not only is the range of the sound greatly increased over that claimed for the bell, but code signals can be easily transmitted. And there are also new possibilities as regards synchronous signalling. A vessel moving at the high speed of 25 knots may learn her distance from the land, the bearing of the signal station, and hence the correct course to steer, more than an hour before she makes her harbour. Remember, too, that this information comes in in any weather. It has not to be listened for in the open but is quietly whispered in the cabin.

Of great interest, too, are the applications of the oscillator as a depth-finder and as a protection against icebergs. In both cases the reflexion of the sound and its return to the observer are used.

The depth-finder is admirably simple. Imagine a commutator-wheel with one conducting segment leading to the armature of the oscillator. Two brushes touch this wheel, one connected to the alternating current generator, and the other to the telephone-receiver. As the wheel is rotated the oscillator is excited while the brush connected with the source of current is passing over the conducting segment. Excitation then ceases and the sound from the oscillator travels to the bottom of the sea, comes back by reflexion, and acting on the oscillator generates a current in it. This will be heard in the telephone receiver if the brush connected to the telephone is in contact with the segment just at the instant when the reflected sound impulse reaches the ship. The setting of the telephone brush will, therefore, determine the depth. In a depth of 8 fathoms beneath the oscillator the time for the sound to travel to the bottom and back will be about the one-fortieth of a second. The echo, according to the results of a trial made from a U.S. Revenue Cutter, may be heard in the ship without the use of the receiver. A stop-watch used to determine the interval between the start and return of the sound afforded a good approximation to the depth.

Experimenting from the same vessel, the distance of an iceberg 450 feet long and 130 feet high was determined by echo from the submerged part of the berg at various distances from one-half mile to two and one-half miles. The echoes were not only heard in the oscillator receiver, but in the officers' wardroom and elsewhere in the ship. The distances agreed with those determined by the range-finder. The prosecution of the experiments was hindered by rough weather, the oscillator not being permanently installed but lowered overboard. The echoes were loud at $2\frac{1}{2}$ miles. It is stated that as regards the intensity of this underwater echo, it made no difference whether the face of the berg was presented to the ship or otherwise. It must be remembered that the immersed volume of the berg was some ten times as bulky as that presented to view.

Marvellous as all this undoubtedly is, the purely sensational part of it is surpassed by the achievements of wireless telephony. The wireless telephone can speak in plain words to the sailor, telling him the name of the signal station he is approaching and warning him of his danger if he comes too close. The speaker is a machine, a dead thing, and æther waves carry the energy, translated out of its rightful medium, through miles of wild weather, to the ship labouring far off the coast, and there it is again restored to the medium, whereby it reaches the sailor's cognizance. He listens at a telephone in his cabin or wireless room and hears the words reiterated over and over again by the machine in the lighthouse. At this latest achievement of Science, we feel inclined to say: "Hold! Enough!"

The wireless telephone is no very recent achievement. Speech has been transmitted by its means from stations in the United States to the Eiffel Tower. The system of Dr. de Forrest is, I believe, used in the maritime application of which I am speaking. The installation is at Point Judith at the western approach to Narraganset Bay. Here, in outline, is how this marvel is accomplished :—

A phonograph speaks the words. It cries the name of the lighthouse or lightship into the transmitter. The system is entirely automatic. The movement of a switch starts the phonograph into operation. The voice, translated into æther waves, reaches the antenna on the ship and is there retranslated to the spoken words by a detector and telephone. No training in Morse signals is required. The sailor hears the words just as the householder hears the message in his telephone. It is stated in an account of this system kindly sent to me by the Submarine Signal Company :—"The receiving apparatus is so small and requires so little tuning that for small ships with no operator, the Captain with a few minutes' instruction could pick up and use the signals." At Point Judith "the intensity of the sound and radiation of the transmitter are so designed that ships equipped with the ordinary antenna will hear the signals the same approximate distance that the light would be seen in clear weather." There is heard first a voice which cries the name of the Station every five seconds. After every third repetition of the name of the Station a much feebler voice speaks the warning "You are getting closer; keep off." This signal the sailor will only hear when close in to the lighthouse. The instrument accomplishing this marvel has been called the Radiophone. It is intended to set up Radiophones at several stations on the Atlantic and Pacific coasts.

When in addition to this instrument you fit the vessel with the wireless compass or goniometer—an instrument whereby the directions from which wireless messages are approaching the ship may be approximately determined,—you have an equipment which replaces the use of the lighthouse in fog or thick weather. It must, however, be remembered that this system, interesting and wonderful as it is, possesses some of the defects of the light signals. Even if the wireless goniometer gave him his angles as accurately as he obtains them by station pointer in clear weather—which is very doubtful—the distance indications can only depend on the strength of the wireless signal. But here the influence of atmospheric conditions in affecting the amount of absorption of the transmitted energy must introduce capricious variations. Position cannot be fixed without reliable distance determinations. With stations so distributed as to give simultaneous readings of angles by wireless goniometer, the seaman can, indeed, proceed by wireless alone from headland to headland, hearing the name of each proclaimed in plain language and laying his course by the use of the radio-goniometer.

It is evident that the last few years have opened up wonderful prospects to coastal navigation. And surely those voices, crying to the Mariner through darkness and storm, reassuring him and guiding him on his way, captivate the imagination beyond any other of the marvels of applied science in our time.

II. Avoiding Collision.

The existing rules for avoiding collision at sea have been in force for more than one generation and, it is needless to say, have done inestimable service. They date from a period when the resources of science were much less than they now are. Wireless telegraphy was unthought of, and submarine signalling, if occasionally mooted as a possibility, had not been put to any practical trial. These time-honoured rules tell the sailor what he is to do when he sights another ship with which collision may occur. In general one only of the ships may alter course, and their relative position decides which of them is to do so. The compulsory use of certain regulation lights on vessels enables these rules to apply also to night time in clear weather.

When the weather gets thick, or fog or snow comes on, it is assumed that all methods save those of whistling and listening fail. A prolonged blast must be emitted at intervals of not more than two minutes. The ship must slow down to "a moderate speed." The great problems then confronting the sailor are to hear the sound on the other ship in good time; to locate it; and then do the right thing; at the same time letting the other ship know what he has done. The trouble is, mainly, that the relative position of the ships is difficult to determine. Sound directions are liable to deceive and in very wild weather to carry badly, or to be inaudible owing to the noise and uproar upon and around the ship. For a happy issue out of all these afflictions the mariner can only trust to his vigilance, to his presence of mind, and to a considerable measure of luck. These failing him his own ship or the other ship may be lost. The circumstance may be such that the last factor only—luck—determines the result.

The time has come when a complete re-consideration of the whole matter in the light of modern advances in signalling is desirable if not indeed imperatively necessary. Not that existing rules need be abrogated. In close traffic these may well prove essential, especially when small coasting craft are concerned. Nor are the modern methods and the older ones mutually exclusive. The chief danger, however, is really in the ocean routes where high speeds must be maintained and the risk taken. There is little doubt that with the compulsory use of such methods of signalling as are now available high speeds could be maintained and very little risk remain. It is irrational to suppose that educated officers who have been trained in far more difficult navigational methods could not use the methods we have to consider. The actual taking in of the signal will probably always fall to the wireless operators on board: men who hold certificates of proficiency in dealing with such matters. Alone the interpretation of the signals lies with the Officer of the Watch. And as a fundamental regulation the Board of Trade would require continuous watch in the wireless room on *all* ships during thick weather.

With the advent of wireless telegraphy at sea—due in the first instance to Admiral Sir Henry Jackson—the sailor inherited a means of speech which is available in almost every state of the weather. And in submarine signalling yet another mode of communication is open to him, whereby ship may speak with ship over distances from 6 to 20 or more miles in all weathers. Directions may be determined approximately by both these methods of intercommunication. But when the problem of avoiding collision in all circumstances is fully considered it will, I believe, be recognized that determination of distance—that is, the distance between the vessels—is an essential element for safety.

And this brings us back to synchronous signalling as the only means whereby distance from ship to ship can be safely determined.

There is no doubt that the combination of submarine sound signal and wireless signal is the most reliable one available. True, the sensitiveness is no more than will determine the distance to the half-knot although the quarter-knot may be estimated. Practice would do a great deal in such a matter, as everyone who has observed small time intervals in the laboratory soon finds. And it is also to be remembered that we are dealing with ships moving at considerable speeds; so that they may cover a half-knot in $1\frac{1}{2}$ minutes, or even in one minute when the relative velocities of the vessels are taken into account. As the mutual avoidance of such vessels cannot safely be left to less than the last minute, the sensitiveness of the method is quite adequate to the use required of it.

It will not, probably, be superfluous to say something as to the available means of submarine signalling between ship and ship as apart from the mere reception on the ship of a submarine signal sent from the shore. The latter subject we have already considered, but nothing has been said as to the emission of submarine signals from the ship.

There is first of all the use of the bell. The sound of the bell may normally be taken as carrying 7 or, at least, 5 miles. It gives a sharp, unmistakable sound; and the apparatus concerned has the advantages of compactness and simplicity of construction. Its application to ships would appear to involve the provision of a recess somewhere in the ship's bottom. The bottom is assuredly the best position; for the radiation of the sound is then not interfered with in any direction by the ship herself. The provision of the recess is a protection to the bell, which is supposed to be raised into the recess and housed therein when not required for use. This construction has, I understand, already been applied to submarines.

The rival sound-signalling machine is the Fessenden Oscillator. This is an instrument for which a much greater range is claimed, and is, in addition, highly adapted for transmission of code signals.

Whether bell or oscillator are employed we may suppose the signals completely controlled from the room of the wireless operator and the easy possibility of securing mechanical control of the signals, so that by clock-work their emission may be accurately regulated and timed to the signals sent out by wireless in the ætherial medium. We may, in short, discuss the use of synchronous signalling in avoiding collision, with our minds at ease as to the complete practical possibility of putting the method into operation. Both the bell and the oscillator have, in fact, already been applied to moving vessels.

We shall assume that ships navigating in fog or thick weather are required by (future) Board of Trade regulations to emit a certain low-power wireless signal at intervals, say, of 5 minutes, and that when two ships become aware of each other's signals they may, if they deem it necessary, exchange the usual code signals giving course and speed. The communication of these data is a simple matter. It forms a familiar part of the preliminary correspondence of a ship station with a coast station after the latter has been called. Both are numbers. The course is given in degrees from 0° to 360° reckoned from North round by E. S. and W.; *i. e.*, clockwise. The speed is signalled in nautical miles per hour. The signals are emitted at the rate of some 20 words per minute, or a word of five letters in 3 seconds. Thus, to fix our ideas, we may suppose that ship A learns that ship B is proceeding South (180°) at $16\frac{1}{2}$ knots, and B learns that A is holding a course NE. $\frac{3}{4}$ E. (say 53°) at a speed of 11 knots.

Additional to these means of dealing with the problems presented by methods of averting collision, it must be recalled that in the radio-goniometer or wireless compass and in submarine signals the mariner possesses a means of finding the bearing of another ship with approximate accuracy.

Now there are four criteria which enable the sailor to say in advance whether a particular ship in his locality, but assumed to be quite invisible to him, is moving so as to collide with his ship or whether she is not. Let us first write down what these criteria are.

If two ships, A and B, are moving so as to collide:—

(1) *The mutual bearings of the ships are determined by and deducible from the courses and speeds of the vessels.*

(2) *The rate of mutual approach of A and B*—i. e., *the relative velocity—is fixed and determined by the courses and speeds and is the maximum possible for these courses aad speeds.*

(3) *The bearing of ship from ship is constant and invariable up to the moment of collision.*

(4) *The rate of mutual approach remains constant up to the moment of collision.*

It is convenient to first consider those two criteria which are dependent upon and deducible from the prevailing courses and speeds; *i. e.*, the *bearing* which indicates threatened collision and the *relative velocity* which indicates threatened collision. How may these two criteria be used by the mariner? The matter may be stated thus:—

With the knowledge in his possession of the course and speed of each ship the navigator, by simple methods to be presently described, determines what the bearing of the ships from each other must be *if collision is threatened* and what the relative velocity or rate of mutual approach of the vessels must be *if collision is threatened*. These criteria are decisive on the question of collision or no collision. By comparing what may be called the "danger bearing" and the "danger rate of approach" with the actual bearing and actual rate of approach he obtains complete assurance on the serious issue before him. Thus, for example, with the courses and speeds cited above the navigator finds that B will bear NNE. from A, and A will bear SSW. from B, and the rate of approach of the ships towards each other will be 24 knots per hour: *if collision is threatened and only if collision is threatened.*

Accordingly, the navigator can tell whether collision is threatened or not (*a*) by observing the actual bearing of the other ship or (*b*) by determining the rate of approach of the ships.

A very large percentage of cases presented to him may be at once dismissed by determination of bearing. The bearing of the other ship may be determined by radio-goniometer or by submarine signalling. I do not think the accuracy of such determinations will suffice for all cases of threatened collision. This point must be further considered later. But fairly accurate determination of bearing would suffice to rule out many cases. Suppose, for instance, in the case of A and B above, that the bearing of B from A is observed to be more than a point divergent from NNE., *i. e.* from the danger bearing; it is then certain that collision will not occur. The Officers on A and B might exchange a short code signal expressing understanding on this point. And, of course, both know that the existing courses and speeds which insure safety must be carefully held and maintained.

But if there is any close approximation of the observed bearing to the pre-determined danger bearing, then the determination of the rate of approach of the vessels would be entered on at once. For one thing there is nothing in the observation of bearing to tell the navigator the *distance* of the other ship. Guessing this distance by the strength or intensity of the signals may prove seriously inaccurate. And with no knowledge of distance the sailor is placed in a very anxious position, and one which may compel him to alter course quite needlessly—as we shall see.

The determination of the rate of approach is got by successive determinations of the distance separating the vessels. Now, knowing the danger rate, the sailor can tell from the very first determination of distance *when* collision would be due, supposing it to be threatened. And this tells him whether he must act at once or whether he has plenty of time. He knows, in fact, "there are so many minutes to go before collision can occur." The second determination of distance compared with the first tells him whether the actual rate of approach approximates to the danger rate of approach. If it does, and continues to do so on a few more observations by synchronous signal, one of the ships must give way in good time.

In short, the procedure whereby collision may be averted in all weather involves :—(*a*) exchange of signals, wireless or submarine, giving courses and speeds : (*b*) finding from these data the "danger bearing" and "danger rate of approach" : (*c*) ascertainment of the actual bearing and rate of approach.

These operations, even if called for in their completeness, are simple and easily carried out : characteristics of value under conditions which may involve hurry and anxiety. It is necessary now to consider the successive steps more in detail and to enter briefly on the principles upon which the operations are based.

What are the conditions determining collision? Suppose ships A and B are moving on paths which intersect. Then the conditions for collision involve that A and B are, at a given instant, at distances A O and B O from the point of intersection, O, such that their speeds will carry each ship over the respective distances A O and B O in the same interval of time. In other words courses, speeds, and positions are involved. When these three factors are such as to lead to collision then is the following important condition fulfilled :—the direct distance between the ships will decrease at the maximum rate possible for the given courses and speeds. In other words, the relative velocity is a maximum for the courses and speeds. This is evident, for its entire velocity is then carrying each ship directly towards the other ship at the only point where they can meet : that is, the point of intersection of their courses.

We may reverse the steps of our reasoning and say, if the relative velocity of the vessels is the maximum for the courses and speeds, then is collision sure to occur, and if it is not the maximum, collision cannot occur : the ships will pass clear.

In order to find the maximum relative velocity or "danger rate of approach" of the ships, knowing the courses and speeds we may construct a triangle of velocities. Two of the sides of the triangle are parallel with the courses ; and the lengths of these sides are proportional to the speeds of the ships. The third side, completing the triangle, gives us, now, by its direction the bearing of A from B and of B from

A, and by its length the relative velocity of A and B. The conditions are now the same as if A was at rest and B moving with this velocity towards A. Evidently the distance between them will diminish at the maximum rate, for B is moving *straight* towards A. But this procedure for finding the maximum rate of approach is one which we cannot expect the seaman to carry out in the urgent circumstances of his position. We suppose, instead, that he is provided with a simple instrument which may be named a Collision Predictor.

This instrument (Pl. I. fig. 4) consists of a circle upon which compass bearings and angles measured from N, clockwise, are engraved. It carries two limbs, *a* and *b*, which rotate independently about the centre; which are divided to read speeds in knots per unit time; and which can be clamped in any position. An arm, *c*, is pivoted upon a sliding piece or cursor, which can be slipped along the limb *a*. This arm carries centrally a transparent divided scale, as shown.

When the sailor is given the courses and speeds he proceeds as follows:—One limb, say *a*, he sets round to the course of his own ship A. The other, *b*, he sets to the course of the other ship B. He then slides the cursor along *a* till it reads on *a* a number which is proportional to the velocity of his own ship A. He next inflects the arm *c*, so that it intersects the limb *b* at a distance from the centre proportional to the speed of the other ship B. He has now constructed his triangle of velocities, and he reads on the transparent scale of the arm *c* the relative velocity he seeks: that is, he reads on it the relative velocity *when collision is threatened*: which, as we have seen, is the maximum for the courses and speeds.

It is convenient to read on *c* the rate of approach or relative velocity in terms of the amount by which the direct distance separating the vessels diminishes in two minutes; or one minute, according to the interval separating the observations of distance by synchronous signalling. I assume that this is 2 minutes. Then, reverting to our example, having set the limb *a* NE.$\frac{3}{4}$E., and the limb *b* due South and slipping the cursor along *a* till it reads the speed of A—*i. e.*, 11 knots—and inflecting *c* to read on *b* 16$\frac{1}{2}$ knots (*i. e.*, the speed of B), the navigator finds that the scale on *c* is cut by its intersection with *b*, at the reading 0·8. What is this? It is the distance in knots by which the ships A and B, in our example, must approach towards one another in two minutes if collision is threatened, *i. e.*, eight-tenths of a knot. This is *the danger rate of approach*; and is the same as 24 knots per hour as given above.

We have incidentally found also *the danger bearing.* The arm *c* in fact points from A to B; that is, if we moved it parallel to itself till it crossed the centre of the divided compass circle, then it would read the bearings required. It is easy to arrange for the attachment of a parallel motion to *c*, which can be moved so as to cross the centre of the divided circle and read the bearings. In the example chosen, for instance, it shows that B bears NNE. from A, and A SSW. from B if collision is threatened. We here use the instrument as a means of constructing a triangle of displacements rather than of velocities: which is obviously legitimate. Thus the navigator by merely setting the limbs on this instrument to the courses, and setting the arm *c* to the speeds, obtains the danger rate of approach and the danger bearing.

It may be helpful to some to consider a quite simple case. Suppose the courses of the ships are directed exactly opposite. Suppose X is going due south and Y is going due north. Let the speeds be 20 and 10 knots respectively.

Now the fact of the courses being opposed does not involve collision. The bearing of X from Y or of Y from X may be anything at all so far as courses are concerned. For instance, the ships might be passing abeam of one another. But there is one particular bearing of X from Y and one of Y from X which denotes collision:—when X bears north from Y and Y bears south from X. The ships are then approaching end on and collision is threatened. These are in this case the danger bearings of ship from ship.

Again, the rate of approach of X and Y may be anything at all, within certain limits, so far as courses and speeds are concerned. Thus the distance between the vessels would be shortening quite slowly supposing Y bore somewhere forward of the beam of X; or, it might be, actually increasing if Y bore aft of the beam of X. It is evident that only when the vessels are approaching end on is the distance decreasing at the maximum rate: that is, the relative velocity is a maximum; and it must amount to $20+10=30$ knots. This is the danger rate of approach, and it is evidently associated with collision. If observations of distance are taken every two minutes this rate would involve the distance between the vessels diminishing at each observation by one knot.

We see then that there is a danger bearing and a danger rate of approach peculiar to collision. The Collision Predictor applied to the above case would give the danger bearings as north and south and the danger rate as one knot.

As we have already seen, the knowledge of the danger bearing may rule out the possibility of collision. For if the actual bearing is decisively different from the danger bearing there cannot be collision. By radio-goniometer—an instrument we must refer to again later on—the bearing may be found, at least approximately. Or by submarine sounds an approximate bearing may be taken. If the Fessenden Oscillator is carried on the ships the bearing could be determined while yet 10 or more miles separated the vessels. If the bell is carried the bearing could be found some 6 or 7 miles away under normal conditions. The determination of the bearing will, probably, be the first procedure. But we will suppose that the bearing as observed — however determined—is doubtful; that it cannot be safely discriminated from the danger bearing. And as the sailor knows not with any certainty at this stage the distance separating the vessels, he regards it as unsafe to undertake prolonged observations of the bearing, and decides on finding the distance and the rate of approach.

When he determines on this he makes a code signal announcing to the other ship that he is about to find the distance by synchronous signals. If this is announced by A, B prepares to listen. When at length B picks up the oscillator or bell she tells A the distance as so many miles. This first observation of distance assures to the sailor a knowledge of the time at his disposal. For suppose—reverting to our example—that B says "our distance is 5 miles." Then both on A and B it is known that collision cannot occur sooner than 12 minutes from that instant. For on A and B it is already known that the danger rate or maximum rate of approach for the courses and speeds is 0·8 knot in 2 minutes, or 0·4 knot in 1 minute. Hence we have to divide 5 by 0·4 to get the interval in minutes before collision can occur; and this gives 12½ minutes. In this way the sailor knows at the earliest moment at which the submarine signals are audible from ship to ship how much time is available for further observation.

Two minutes after the first observation of distance, a second synchronous signal is sent out—say from A—and B says "our distance is 4 miles." This looks like danger. For there is some error certainly seeing that the approach cannot be so much as 1 mile in 2 minutes. There is now 10 minutes to go and there is no reason why several more distance determinations should not be made. The emission of the signals is automatic. On receiving the third signal B says "our distance is $3\frac{1}{2}$ miles." This also is evidently quite in keeping with the danger rate. There is now 8 minutes to go. On the 4th signal the distance falls, we will suppose, to 2·5 miles with 6 minutes to go. The danger may now be regarded as established. But there is no reason why further signals should not be exchanged, before B gives way to A.

The successive observations of distance may be recorded on paper as they come in, and be compared with the successive danger distances written down upon the finding of the initial distance. Or they may be observed and followed one by one on the Collision Predictor. Taking the former method first we may suppose a ruled sheet with columns set out to take the figures thus :—

Course A 53°. Course B 180°.
Speed A 11. Speed B $16\frac{1}{2}$.
Danger Bearing NNE. Obs. Bearing NNE.
Danger Rate 0·8.

Sig.		Danger Distance.		Observed Distance.		T.
1	...		...	5	...	12
2	...	4·2	...	4	...	10
3	...	3·4	...	3·5	...	8
4	...	2·6	...	2·5	...	6
5	...	1·8	...	2·0	...	4
6	...	1·0	...	1·0	...	2
7	...	0·2	...		...	0
8						
9						
10						

In the two minutes interval between the 1st and 2nd observation of distance the Officer of the Watch fills in columns 2 and 4.

We assume here readings typical of threatened collision. If collision is not going to occur the observed distances will disagree with the danger distances already written down. In what way will they differ? We have seen that the danger rate is the maximum for the courses and speeds. If then collision is not threatened the vessels will be approaching more slowly than if collision is threatened. The difference between the rates will give rise to a cumulative increase in the distance separating the vessels. *As the observations progress it will be found that danger distances and safety distances differ more and more widely.*

And there is another reason why this difference between danger entries and safety entries in the table will increase. We must anticipate here the reasons for the 4th criterion denoting collision. It is easy to see that only in the case of coming collision is the velocity of approach constant. If we place A in the centre of a series of concentric circles described, say, at radial distances of one knot; then if collision is coming B is traversing these circles radially in its advance towards A and her velocity of approach is constant. But if B is not aiming for A she is not advancing radially. Her velocity of approach towards A cannot be constant. When she is far from A her rate of approach to A is greater than when she draws near. When she gets abeam of A there is a moment when there is no further diminution of distance. The relative velocity is then zero. After this B begins to recede from A.

Now this must come out in the successive observations of distance between A and B and will tend to further accentuate the difference between danger and safety readings of distance. This cumulative and increasing distinction in the character of the two sets of figures—those which have been written down on the assumption that collision is threatened and those which come in if it is not threatened—is a feature of much value. It tends to redress the want of sensitiveness of the readings and to distinguish true from fictitious safety. Danger readings cannot be confounded with safety readings as observations multiply.

There are advantages in keeping the observations on paper as described above. There must be ample time for several observations if a good look-out has been kept. The observations are actually made by the wireless operator. The Officer of the Watch has only to write them down. His own ship's course and speed are already entered. The fact that the signals are sent out and read by one individual, who has no responsibility beyond reporting them, and that they are interpreted by another, is an element of safety, for it leaves each operator free to give his attention to one matter only.

The Collision Predictor is intended to give the navigator the means of following the approximation of the two ships step by step as the readings of distance come in, and permitting him to appreciate the imminence of danger in case of threatened collision by merely looking at the instrument.

These functions it accomplishes in virtue of the fact that if the cursor is slipped along the limb *a*, keeping the arm *c* clamped on the cursor at the angle determined by the first setting of *c*, the gradual mutual approach of the ships is traced out on the scale carried by the arm *c* at its intersection with the scale on *b*. Thus, suppose when the second reading is made we move the cursor along *a* a total distance corresponding to the run of A in 4 minutes, then the arm *c* shifts on the limb *b* to the corresponding run of B (as it moves parallel to itself) and twice the danger distance is read on *c* *. Now if we have a mark on the arm *c* at the reading of the initial distance separating the vessels—*i.e.* 5 miles—then as we shift forward the cursor at each synchronous signal, that is to say at intervals of two minutes, we can by simply looking at the reading on *c* at its intersection with *b*, observe upon it the total distance which has so far been run and the distance which remains to be run *if collision is really threatened.* For example, if collision is actually approaching, in the first position of *c* we read on it 0·8 knots as run and 4·2 knots still to run. In the second position 1·6 knots run and 3·4 knots to run, and so on. The navigator compares these readings one by one with the actual readings of distance coming in. If they show a tendency to sustained agreement he knows for certain there is danger, and by a glance at the scale on *c* he infers how much time remains before collision can occur. He has, in the still remaining length of the scale on *c*, a visible and tangible indication of the time left to him for action; and with each reading a simple and definite mechanical operation has to be effected which necessitates his attention being fixed on the lie of the two ships relatively to each other. *If he holds the Collision Predictor in its true compass position he sees at once the direction in which the other ship is approaching, this being the direction in which the arm* c *is pointing, and at the same time he has indicated on the arm* c *at once the distance separating the ships and the time taken to cover this distance. It is like as if he followed the approach of the vessels upon a chart.*

Incidentally we may observe here that the Collision Predictor finds a use even in clear weather for averting collision. For suppose two ships sighting each other and, uncertain as to the risk, exchange courses and speeds. Then the Predictor tells immediately the danger bearing. If the ships show that bearing towards one another then is collision threatened. Otherwise there is safety.

The foregoing description, embodying as it does an account

* This is the position of *c* as shown in the figure.

of an unfamiliar instrument, may create the impression that the proposed system of averting collision is one involving rather complicated operations ; and these too at a time when simplicity and clearness of procedure are above all desirable. In order to show how unfounded this impression is we shall now follow the necessary steps as they would be taken on an adequately equipped ship in order to ascertain if there was risk of collision with another vessel. We may continue to call our own vessel A and the other vessel B.

The scene is once more in the North Atlantic. The weather is thick and night falling. Vision is restricted to a couple of ship's lengths. The Officer of the Watch has at hand a Predictor which is already set to the course and speed of his own ship.

The operator in charge of the wireless room has started into action a clockwork contact-maker which automatically sends out, periodically, the wireless fog signals required by (future) Board of Trade rules.

Presently he hears the characteristic radio signals of another vessel. He calls the other ship ; communicates his own course and speed and learns hers in exchange. These he reports at once to the Officer of the Watch who adjusts the other limb of the Predictor accordingly, and reads forthwith the danger rate and danger bearing proper to the courses and speeds of the two vessels.

Meanwhile the wireless operator has signalled to the other ship asking for signals whereby the bearings may be determined. He makes an estimate of the bearing; informs the other ship "You bear NNE."; and then reports this bearing to the Bridge.

The Officer of the Watch is now able to say right off that the signals must be continued. For on comparing this bearing with the danger bearing he perceives that they are indistinguishable. On board the other ship the same decision is come to.

Our ship now signals to B, by code, "Bearing dangerous : prepare to receive distance signals." Then when B acknowledges this the wireless operator on our ship sets in operation the automatic emission of synchronous signals. These continue to be emitted till B reports "Distance 5 miles."

When this is reported to the Bridge the Officer marks the arm *c* of his Predictor at the reading 5 miles, and makes in his own mind an estimate of the minutes still to run before collision can occur. He finds there are 12 minutes. He may note the time or start a stop-watch.

After 2 minutes B gets the second distance signal and tells A "Distance 4 miles." And the Officer on A sees that this also points to danger, for the distance of approach, as observed, obviously has an error of excess and may easily represent the danger distance of 0·8 mile. And in this way he continues to compare the successive announcements of distance with the readings he gets on his Predictor by setting forward at each announcement the cursor on *a* the distance which A travels in 2 minutes.

All this time he has a clear mental picture of where B is. He can point to her compass direction and say she is there at such a distance. Finally B alters her course so as to pass clear of A and lets A know what she has done. The danger is over.

It is not improbable that the future extension to large numbers of vessels of the powers conferred by recent improvements in submarine signalling will result in a considerable amount of the signalling between vessels being carried out by this means: in this way reducing the number of wireless messages required.

We shall now consider the use of the 3rd criterion—that which seeks to foretell threatened collision by observation of the constancy of bearing. This criterion does not involve a knowledge of the courses and speeds of the vessels.

The wireless goniometer—which in late years has been much improved—is claimed to afford a means of finding the direction whence a wireless message proceeds, with much accuracy; and this even with the use of considerable wavelengths. It is difficult to discuss these claims or to pronounce on how far they may extend to conditions of hurry, anxiety, and bad weather. Readings to a single degree have been claimed. The polar diagram of the energy transmitted by the Bellini-Tosi Directive Transmitter shows that some 10 or 11 degrees from the maximum of 88 microvolts the reading may be still 86 microvolts. Taking the distribution of intensity, when the system is used as receiver, to be similar, it would appear that the loudness of the sounds must vary but slowly near the maximum. The setting depends entirely on the discrimination of the direction of loudest telephonic sound. Most of us find difficulty in determining maxima when the senses, either of hearing or sight, are appealed to quantitatively and by consecutive impressions.

There must apparently exist a similar bar to attaining reliability of goniometric readings in the case of finding bearings by submarine signals. And the fact that an altered course must, in general, be held while the bearing is being watched is a serious objection to the use of submarine signalling for the purpose now in view. I refer to these difficulties because *prima facie* one might think that by merely determining the constancy of bearings by wireless goniometer or submarine signals the danger of collision can be always foretold and averted.

The principle involved in this method is familiar to every sailor. If the bearing of B from A is steady, collision is threatened. Represent the courses of A and B by inclined lines, intersecting at O. Now suppose that A runs the distance A O in the time that B runs the distance B O: then collision must occur. And it follows that when A has run half the distance A O, B must have run half the distance B O. If we join the simultaneous positions of the two ships we get, therefore, for the bearings lines which are parallel. That is, the bearing of ship from ship remains constant.

Even when used in clear daylight it is often objected to this method:—"While you are looking for a change of bearing the ships may collide." And the answer sometimes given is "there is no other method." Now, whatever may be said against this objection when the sailor can guess his distance from the other vessel by visual observation, it must be remembered that the proposed use in this way of the wireless goniometer leaves the sailor without any reliable estimate of distance. For the strength of the wireless signals must afford but a doubtful estimate of distance. They suffer not alone from instrumental sources of variation but from variations, due to atmospheric causes, quite out of the control of the operator to alter or to predict. See on this subject, more especially, the observations of Admiral Sir Henry Jackson which showed a capricious reduction of carrying power from 65 to 22 miles; or again, those of Captain Wildman who obtained frequent successive variations in audibility of 1 to 5 and rising to 1 to 10 or even more. The cause of these variations may most probably be traced to changes in the conductivity of the atmosphere due to mist, spray, etc., whereby variations in the absorption of the energy of the electric waves are brought about. Judging from these observations any dependence on the strength of wireless signals as giving an estimate of distance must be attended with danger; just as much as judging the distance of a light by its brightness.

The wireless compass may indeed under certain conditions, in addition to other inestimable benefits conferred by it upon the sailor, be a valuable means of discriminating between safety and danger. For if there is a rapid alteration of bearing there must be safety. The question is: can it always be reliably employed? It is a fact that just when the courses of the vessels are so directed as to cause the ships to rapidly approach one another, there may be only a small change of bearing in a given time; and for the sailor to spend valuable minutes in setting the wireless compass to the loudest sound in the telephone when the ships are rapidly approximating one towards the other, and when it is not known for certain whether 10 miles or 3 miles separate them, is not a safe procedure. The complete and final solution of the problem of collision would appear to involve the determination of the distance separating the vessels. Otherwise the navigator must make his observations under conditions of anxiety; and in the fear lest worse should happen must frequently be driven to alter needlessly the course of his ship.

The 4th criterion—that foretelling collision by constancy in the rate of approach, as in the case of the method by constancy of bearings, does not involve knowledge of courses and speeds. But it possesses the very great advantage of keeping the sailor informed all the time of the distance separating the vessels; obviously an important element of safety and conducive to the peace of mind of all concerned.

The application of this method involves simply the periodic emission from one of the ships of synchronous signals, the other ship receiving them and reporting the distance. Or each ship may alternately emit the signals and make its own observations of distance. When A hears the regulation fog signals of B the synchronous signalling is commenced and the signals sent out, say, at intervals of two minutes. Or the method might be applied under Board of Trade regulations enjoining the periodic emission of synchronous signals in thick or foggy weather. When the vessels come within hearing of the submarine sounds the Officer writes down the distances as his wireless operator reports them to him. If they continue to show a constant rate of approach even when the vessels draw near one another there is danger. If the rate of approach diminishes there is safety. If the vessels are approaching so as to pass close to one another the change of rate will be very marked as the distance diminishes. The most valuable feature of this method is that the sailor knows all along what time must remain for action supposing collision is really threatened. A rough determination of bearing completes his observations and tells him how he must act.

Trial of all these available methods should be made by the

Authorities and rules framed according to the results of experience so obtained, dictating to the mariner when and under what conditions one or the other method is best employed.

The problem presented when three vessels are in the area of audition requires careful consideration and would involve special Board of Trade regulations. Generally we must suppose the entry of a third ship, C, into the area occupied by A and B and while the latter were exchanging signals. It would seem reasonable for C, under these circumstances, to have to keep out till A and B are clear of one another. Again, while C is waiting for the decision between A and B, she must be restricted as to wireless and submarine signals if confusion is to be avoided. She might be restricted to the emission of the aerial sound signals now in vogue. These could not create confusion with the other signals. And when, finally, she enters on an exchange of signals with the other ships, it should be a rule that each vessel emits her distinguishing signal in conjunction with her other signals, so as to avoid confusion as to the origin of the signals.

If such general rules were observed there does not seem to be any serious difficulty in conducting the signalling between all three vessels. After exchanging courses and speeds C sends out "bearing" signals and A reports thus: "C bears from A . . . ." and B reports: "C bears from B . . . ." The Officer on C now finds, using two Predictors, that he is clear of A but not certainly of B. This is also known on A and B. C makes sure of this by saying "C is clear of A," and A replies "A is clear of C." The signals are now restricted to C and B. After this all proceeds as before. When the time for giving way arrives the ship which is giving way takes account of the position of A. The latter is probably quite clear, but whether she is or not her position would be sufficiently known to B or C to avoid risk attending alteration of course or speed.

The congress of more than three vessels must be rare. The difficulties involved can only be met by clear and definite rules framed by experienced seamen.

One thing stands out clearly from our review of available methods: Modern improvements in signalling may be utilized to reduce the risks of collision enormously, no matter what the circumstances. Future developments of wireless telephony may even further simplify the sending and receipt of signals. A vessel's distinguishing signal may be her own name dictated from the phonograph. In fog and thick weather she proceeds over the ocean calling her own name at regulated intervals; and at longer intervals she cries, to all whom it may concern, her course and speed. All possible courses and speeds might be dictated from two phonographs. Timed with these wireless announcements she sends out submarine signals also which enable ships 10 or 20 miles away to estimate her distance. She is as it were surrounded by an aureole of radiations transmitted both in matter and æther, proclaiming to such other vessels as enter that aureole what she is ; how far off she is ; where she is going to ; and at what speed she is approaching or receding.

This is no fancy picture—It could be made reality to-day. And doubtless the time will come when it will be made reality. The fabled wonders of Jason's Argo fade to commonplace compared with the accomplished wonders of our day. The miraculous gifts of Lynceus were not so marvellous as those powers of vision and audition which Science confers upon the sailor.

II. *Variably-Coupled Vibrations* : III. *Both Masses and Periods Unequal.* *By* Edwin H. Barton, *D.Sc., F.R.S., Professor of Physics, and* H. Mary Browning, *B.Sc., Lecturer and Demonstrator in Physics, University College, Nottingham* *.

[Plates II.–V.]

Contents.

I. Introduction.

In the work described in previous papers †, the double-cord pendulum was experimented with : (1) when the masses of the bobs and the periods of vibration of the

* Communicated by the Authors.

† Phil. Mag. [6] vol. xxxiv. no. 202, pp. 245–270 (Oct. 1917); Phil. Mag. [6] vol. xxxv. no. 205, pp. 62–79 (Jan. 1918).

separate pendulums were equal; (2) when either the masses of the bobs or the separate periods were unequal. The present paper deals with the cases where the masses of the bobs and the periods of the separate vibrations are both unequal.

This mechanical case may be regarded as somewhat analogous to the electrical case of coupled circuits in which the inductions and periods are both unequal.

A series of photographs was taken from sand traces obtained when the masses were 20 : 1 and the length of the pendulum with the heavier bob as 4 : 3 of that with the lighter bob. The ratio of the frequencies then slightly exceeds 8 : 7, *i.e.*, to put the matter in acoustical terms, the pendulums are out of tune by 248 logarithmic cents or approximately a tone and a quarter.

Other photographs were taken with masses 20 : 1 and pendulum lengths as 9 : 8, the lighter bob still being on the shorter pendulum. The ratio of the frequencies slightly exceeds 21 : 20, *i. e.*, the pendulums are out of tune by 102 logarithmic cents or approximately an equal-tempered semitone.

In both cases it was noticeable for small couplings that very little of the energy of the heavy bob was required to build up in the lesser bob an amplitude nearly equal to that of the heavier bob. Further, that for couplings about 30 per cent. the curves obtained were almost identical in the two cases and almost indistinguishable from that of 30 per cent. coupling shown in Paper II. for masses 20 : 1 and lengths equal.

The pendulum with the heavy bob was altered in length until it was 3 : 4 times as long as that with the light bob, the masses remaining as 20 : 1. The results of theory and experiment were rather striking. The ratio of the frequencies of the separate pendulums slightly exceeds 8 : 7. As the coupling was increased from one to about six per cent. the ratio of the frequencies diminished to about 13 : 12, and the two pendulums had greater action and reaction on one another. When the coupling was further increased, the ratio increased to 2 :1 at coupling about 30 per cent. as in the other cases.

Quenched Spark.—Prof. J. A. Fleming has pointed out that by means of a rapidly damped spark discharge in a primary circuit a slowly damped electrical vibration may be produced in the secondary or antenna. In this paper a photograph of a mechanical analogue of such a discharge is reproduced from a sand trace.

II. Theory of General Case.

Equations of Motion and Coupling.—The double-cord pendulum was shown in figs. 1 and 2, p. 253, of the first paper (Phil. Mag., Oct. 1917). The equations of motion and coupling were there given as equations (25), (26), and (29) (pp. 253–254). They may now be rewritten as follows:—

$$\mathrm{P}\frac{d^2y}{dt^2}+\mathrm{P}g\theta=0, \quad \ldots \ldots \quad (1)$$

$$\mathrm{Q}\frac{d^2z}{dt^2}+\mathrm{Q}g\psi=0, \quad \ldots \ldots \quad (2)$$

$$\gamma^2=\frac{\beta\mathrm{PQ}}{(\mathrm{P}+\mathrm{Q}+\beta\mathrm{Q})(\mathrm{P}+\beta\mathrm{P}+\mathrm{Q})}. \quad \ldots \quad (3)$$

The ratio of the masses of the bobs may be expressed by $\rho=\mathrm{Q}/\mathrm{P}$ and the lengths of the suspensions for the y and z vibrations by ηl and l respectively, the droop of each bridle being βl.

Then

$$\theta=\frac{y-\beta l\omega}{\eta l}; \quad \psi=\frac{z-\beta l\omega}{l}. \quad \ldots \quad (4)$$

Further, neglecting masses of suspensions, connector, and bridles, ω must satisfy

$$\left.\begin{aligned}\mathrm{Q}g(\psi-\omega)&=\mathrm{P}g(\omega-\theta)\\ \text{or} \quad \rho g(\psi-\omega)&=g(\omega-\theta).\end{aligned}\right\} \quad \ldots \quad (5)$$

Then (4) in (5) gives

$$\omega l=\frac{y+\eta\rho z}{\beta+\eta+\eta\rho+\beta\eta\rho}. \quad \ldots \ldots \quad (6)$$

And (6) in (4) gives

$$\theta=\frac{(1+\rho+\beta\rho)y-\beta\rho z}{l(\beta+\eta+\eta\rho+\beta\rho\eta)}, \quad \ldots \quad (7)$$

$$\psi=\frac{(\beta+\eta+\eta\rho)z-\beta y}{l(\beta+\eta+\eta\rho+\beta\rho\eta)}. \quad \ldots \quad (8)$$

Inserting frictional term $2k\mathrm{P}dy/dt$ in (1), putting $g/l=m^2$ and dividing (1) by P and (2) by Q, then (7) and (8) in (1) and (2) give

$$\frac{d^2y}{dt^2}+2k\frac{dy}{dt}+\frac{1+\rho+\beta\rho}{\beta+\eta+\eta\rho+\beta\eta\rho}m^2y=\frac{\beta\rho m^2}{\beta+\eta+\eta\rho+\beta\eta\rho}z, \quad (9)$$

$$\frac{d^2z}{dt^2}+\frac{\beta+\eta+\eta\rho}{\beta+\eta+\eta\rho+\beta\eta\rho}m^2z=\frac{\beta m^2}{\beta+\eta+\eta\rho+\beta\eta\rho}y. \quad \ldots \quad (10)$$

Further, the coupling may be written

$$\gamma^2=\frac{\beta^2\rho}{(1+\rho+\beta\rho)(\beta+\eta+\eta\rho)}. \quad . \quad . \quad . \quad (11)$$

To simplify, (9) and (10) may be abbreviated thus:

$$\frac{d^2y}{dt^2}+2k\frac{dy}{dt}+ay=\rho bz, \quad . \quad . \quad . \quad . \quad (12)$$

$$\frac{d^2z}{dt^2}+\quad cz \quad =by. \quad . \quad . \quad . \quad . \quad (13)$$

Where

$$a=\frac{1+\rho+\beta\rho}{\beta+\eta+\eta\rho+\beta\eta\rho}m^2; \quad b=\frac{\beta m^2}{\beta+\eta+\eta\rho+\beta\eta\rho};$$

$$\text{and } c=\frac{\beta+\eta+\eta\rho}{\beta+\eta+\eta\rho+\beta\eta\rho}m^2. \quad . \quad (14)$$

Equations (12) and (13) are the same as (6) and (7) of Paper II., but the values of a, b, and c are different.

Solution and Frequencies.—To solve (12) and (13) we write

$$z=e^{xt}.$$

Then

$$y=\left(\frac{x^2+c}{b}\right)e^{xt} \quad . \quad . \quad . \quad . \quad . \quad (15)$$

From equation (11) of Paper II., we see that the values of x may be written

$$-r\pm ip \quad \text{and} \quad -s\pm iq. \quad . \quad . \quad . \quad . \quad (16)$$

Hence, omitting small quantities, we have

$$\left.\begin{aligned}2p^2&=[c+a+\sqrt{\{(a-c)^2+4\rho b^2\}}]\\2q^2&=[c+a-\sqrt{\{(a-c)^2+4\rho b^2\}}]\end{aligned}\right\} \quad . \quad . \quad . \quad (17)$$

$$\left.\begin{aligned}r&=\frac{a-c+\sqrt{\{(a-c)^2+4\rho b^2\}}}{2\sqrt{\{(a-c)^2+4\rho b^2\}}}k\\s&=\frac{c-a+\sqrt{\{(a-c)^2+4\rho b^2\}}}{2\sqrt{\{(a-c)^2+4\rho b^2\}}}k\end{aligned}\right\} \quad . \quad . \quad . \quad (18)$$

Thus using (15) and (16) and introducing the usual constants, the general solution may be written in the form

$$z=e^{-rt}(\mathrm{A}e^{pit}+\mathrm{B}e^{-pit})+e^{-st}(\mathrm{C}e^{qit}+\mathrm{D}e^{-qit}), \quad . \quad (19)$$

and

$$y=\frac{(-p^2+c)}{b}e^{-rt}(\mathrm{A}e^{pit}+\mathrm{B}e^{-pit})+\frac{(-q^2+c)}{b}e^{-st}(\mathrm{C}e^{qit}+\mathrm{D}e^{-qit})$$

$$+\frac{2pri}{b}e^{-rt}(-\mathrm{A}e^{pit}+\mathrm{B}e^{-pit})+\frac{2qsi}{b}e^{-st}(\mathrm{C}e^{qit}+\mathrm{D}e^{-qit}). \quad . \quad (20)$$

When small quantities are further neglected, these will simplify to

$$z = \mathrm{E}e^{-rt}\sin(pt+\epsilon) + \mathrm{F}e^{-st}\sin(qt+\phi), \quad . \quad (21)$$

and

$$y = \mathrm{G}e^{-rt}\sin(pt+\epsilon) + \mathrm{H}e^{-st}\sin(qt+\phi). \quad . \quad (22)$$

(21) and (22) are each equations of two superposed vibrations, of which the frequency ratio is

$$\frac{p}{q} = \left[\frac{c+a+\sqrt{\{(a-c)^2+4\rho b^2\}}}{c+a-\sqrt{\{(a-c)^2+4\rho b^2\}}}\right]^{1/2}, \quad . \quad . \quad . \quad (23)$$

where a, b, and c are given by (14).

Initial Conditions.—Let the heavy bob of mass Q on the pendulum of length l be pulled aside a distance f by a horizontal force, the light bob on the other pendulum hanging freely at rest. The displacement of the light bob can then be written at once from equation (31) page 67 of the January paper, since the length of this pendulum makes no difference to the quantity in question. Hence we have

For $t=0$,

$$\left.\begin{aligned} &z=f, \quad y=\frac{\beta\rho f}{1+\rho+\beta\rho}=\frac{\beta}{1+\beta}f \text{ nearly for } \rho=20 \\ &\frac{dz}{dt}=0, \qquad \text{and} \qquad \frac{dy}{dt}=0. \end{aligned}\right\} \quad . \quad (24)$$

Then, introducing these conditions in (21) and (22) and into the differentiations of these with respect to the time and, as before, omitting small quantities, we find

$$\left.\begin{aligned} f &= \mathrm{E}\sin\epsilon + \mathrm{F}\sin\phi, \\ \frac{\beta f}{1+\beta} &= \mathrm{G}\sin\epsilon + \mathrm{H}\sin\phi. \end{aligned}\right\} \quad . \quad . \quad . \quad (25)$$

$$\left.\begin{aligned} 0 &= \mathrm{E}p\cos\epsilon + \mathrm{F}q\cos\phi, \\ 0 &= \mathrm{G}p\cos\epsilon + \mathrm{H}q\cos\phi. \end{aligned}\right\} \quad . \quad . \quad (26)$$

Equations (26) are satisfied by

$$\epsilon = \frac{\pi}{2}, \qquad \phi = \frac{\pi}{2}. \quad . \quad . \quad . \quad . \quad . \quad . \quad (27)$$

From (17), (20), and (22) we have

$$\left.\begin{aligned} \mathrm{G} &= \frac{-p^2+c}{b}\mathrm{E} = \frac{c-a-\sqrt{\{(a-c)^2+4\rho b^2\}}}{2b}\mathrm{E}, \\ \mathrm{H} &= \frac{-q^2+c}{b}\mathrm{F} = \frac{c-a+\sqrt{\{(a-c)^2+4\rho b^2\}}}{2b}\mathrm{F}. \end{aligned}\right\} \quad . \quad (28)$$

Equations (27) and (28) in (25) give

$$f = E + F, \quad \ldots \ldots \quad (29)$$

and

$$\frac{\beta f}{1+\beta} = \frac{c-a-\sqrt{\{(a-c)^2+4\rho b^2\}}}{2b}E + \frac{c-a+\sqrt{\{(a-c)^2+4\rho b^2\}}}{2b}F; \quad (30)$$

whence

$$\frac{E}{F} = \frac{(c-a+\delta)(1+\beta)-2b\beta}{(-c+a+\delta)(1+\beta)+2b\beta}, \quad \ldots \quad (31)$$

and

$$\frac{G}{H} = -\frac{4\rho b^2(1+\beta)+2b\beta(c-a-\delta)}{4\rho b^2(1+\beta)+2b\beta(c-a+\delta)}, \quad \ldots \quad (32)$$

where

$$\delta^2 = (c-a)^2 + 4\rho b^2. \quad \ldots \ldots \quad (33)$$

These give the values of the *ratios* of the constants determined by the initial conditions in question, and this is all that we need to check the records experimentally obtained.

III. Experimental Results.

Masses 20 : 1, *Lengths* 4 : 3 ($\eta = 3 : 4$).—The relations were calculated from the theory given so as to obtain any desired values of the coupling and frequencies, the results are shown in Table I. For the longer pendulum the sum of pendulum length and droop of bridle was 229 cm., and it had the heavier bob.

Table I.—Masses 20 : 1, Lengths 4 : 3 ($\eta = 3 : 4$).

Coupling $=\gamma$.	$\frac{\text{Bridle Droop}}{\text{Long Pendulum Length.}} = \beta$	Frequency Ratio $p:q$.
Per cent.		
0	0	1·154
4·245	0·2	1·255
9·96	0·5	1·403
17·07	1	1·62
28	2·12	2
34·43	3	2·29

Figures 1–5 (Pl. II.) show photographic reproductions of the double sand-traces simultaneously obtained, with masses 20 : 1, *i. e.*, $\rho = 20$ and the length of the pendulum carrying the lighter bob 3 : 4 of that with the heavier bob. The first

four photographs (with couplings 4 per cent. to 28 per cent.) were obtained by drawing aside the heavy bob and allowing the lighter one to settle in its more or less displaced position, according as the coupling was tight or loose. The fifth (with coupling 28 per cent.) was obtained by holding the light bob in its undisplaced position and pulling the heavy one aside.

In all the curves it is noticeable that there is very little fluctuation of the amplitude of the heavier bob, although the amplitude of the lighter one waxes and wanes considerably. Comparing fig. 1 of this paper with figs. 6 and 21 of the January paper, it is seen that the amplitude of the lighter bob in fig. 6, Paper II., is much greater than that attained when the lengths are unequal as well as the masses. But the shorter pendulum in fig. 21, Paper II., has an amplitude much less than that of the shorter pendulum in the present case. Fig. 4 in this paper is almost identical with fig. 8 of Paper II., the amplitudes in the two cases are nearly the same and the couplings are almost alike. Fig. 5 of this paper is also similar to fig. 9 of Paper II.

Masses 20 : 1, *Lengths* 9 : 8 ($\eta=8:9$).—Table II. shows the frequencies for certain couplings with the masses of the bobs 20 : 1 and the lengths of the pendulums 9 : 8, the longer one having the heavier bob and being 229 cm. long if the droop of the bridle were zero.

Table II.—Masses 20 : 1, Lengths 9 : 8.

Coupling $=\gamma$.	$\dfrac{\text{Bridle Droop}}{\text{Long Pendulum Length.}}=\beta$	Frequency Ratio $p:q$.
Per cent.		
0	0	1·06
4	0·19	1·162
10	0·55	1·32
15·75	1	1·494
29·5	2·6	2

Figs. 7–12 (Pl. III.) show photographs taken with masses still 20 : 1, but the length of the pendulum with the lighter bob 8/9 that of the one with the heavier bob. Again we see very little fluctuation of the amplitude of the heavy bob throughout. Figs. 7–9 were taken with the heavy bob held aside and the light one free to hang in its more or less displaced position. Fig. 10 was taken with the light bob held

aside and the heavy one allowed to hang freely. Figs. 11 and 12 were obtained with the heavy bob drawn aside and the light one held in its zero position. It may be seen that fig. 6 in Paper II. is more like fig. 7 than like fig. 1, both of the present paper. On the other hand, fig. 21 of Paper II. is less like fig. 7 than like fig. 1, both of this paper. This is because the separate frequencies of the component pendulums are more nearly in tune with each other. Fig. 10 shows the effect of drawing aside the lighter bob, little energy is given to the heavy bob and there is but little action or reaction of the one bob on the other. Fig. 12 is almost identical with fig. 5 of the present paper and with fig. 9 of Paper II., and the couplings in all cases are very nearly the same.

Masses 20 : 1, *Lengths* 3 : 4 ($\eta=4:3$).—Table III. shows the frequencies, couplings, and bridle droops with bob masses 20 : 1 and pendulum lengths 3 : 4, the longer one having the lighter bob, and its length being 137 cm. if the bridle droop were zero.

Table III.—Masses 20 : 1, Lengths 3 : 4.

Coupling $=\gamma$.	$\frac{\text{Bridle Droop}}{\text{Short Pendulum Length.}}=\beta$	Frequency Ratio $p:q$.	Ratio of Amplitudes.	
			E : F.	G : H.
Per cent.				
0	0	1·154	∞	Indeterminate.
1·76	0·1	1·106	104·6	−0·809
3·37	0·2	1·07	12·9	
4·85	0·3	1·054	2·283	−0·893
6·3	0·4	1·065		
7·5	0·5	1·09	0·2875	−0·836
9·9	0·7	1·154		
12·97	1·0	1·243	0·0696	−0·640
31·47	4·0	1·952		

The figures illustrating the cases in Table III. were obtained with the new portable apparatus shown in fig. 13 (Pl. V.) and which is described in detail later. Figs. 14–22 (Pls. IV. & V.) show traces taken with masses 20 : 1 and the length of the pendulum with the heavy bob 3/4 of that with the light one. Figs. 14–21 were obtained by drawing aside the heavy bob and allowing the light one to rest in its more or less displaced position. In figs. 14–17 it is seen that the light bob is almost undisplaced. In fig. 22 the light bob was held undisplaced while the heavy one was drawn aside.

It is noticeable that with couplings between 2 and 13 per cent. the fluctuations of amplitude of the heavy bob are distinctly marked especially about 6 per cent. In this case the heavy bob gives up nearly all its energy to the light bob, which then attains an amplitude more than three times that with which the heavy bob started. For very small or very large couplings there is very little fluctuation of amplitude in the vibration of the heavy bob. This is seen in figs. 14, 15, and 20–22. This is in accord with the theory. For the ratio between E and F, the amplitudes of the driver's superposed vibrations have been calculated for the initial conditions in use. The results are given in Table III., which shows that E/F has values near unity for couplings about six per cent. Whereas for very small couplings much exceeding six per cent. E/F is very small. And either a large or small value of E/F means inappreciable fluctuation of the driver's resultant amplitude.

Let us now consider the question of the ratio (p/q) of frequencies of the superposed vibrations and the variation of this ratio with coupling. When the coupling is zero this ratio naturally has that value which applies to the pendulums when separate. When the bobs were equal and lengths unequal, the value of this ratio increased with the coupling until p/q almost merged into the value for equal pendulum lengths (see fig. 2, p 75, Phil. Mag. January 1918). When the bobs were unequal as well as the lengths but the heavy bob was on the long pendulum, the same behaviour was noticeable in the ratio p/q and its dependence on coupling (see Tables I. and II.).

On the other hand, when bobs are unequal as well as lengths but the heavy bob is on the short pendulum, a quite new feature is theoretically predicted (see Table III.). Thus when the coupling is gradually increased from zero, the value of p/q at first diminishes, reaches a minimum and then increases. These striking features are to a first approximation upheld by the experiments. For, as seen in passing along figs. 14–20, the number of vibrations in the beat cycle at first increases and then decreases. The maximum number of vibrations in the cycle is about 13 and occurs in fig. 17 for a coupling of 6·3 per cent. Accordingly this coupling should correspond to a minimum value of about 1·08 of the ratio p/q. From Table III., however, it is seen that the minimum value of p/q is about 1·054 and occurs for a coupling of about 5 per cent. These slight discrepancies are easily accounted for by the presence of the sand in the funnels and a possible error in estimating the lengths of the simple pendulums equivalent to those in use. Thus, if with the average amounts of sand in the funnels the masses were in the ratio 19 : 1 and if the lengths were really 11 : 16 (instead of 20 : 1 and 12 : 16 respectively), the minimum value of p/q as calculated would be in sensible agreement with that experimentally observed and would occur for practically the same coupling as that in actual use. Table IV. is calculated from the above data and is found to agree fairly well with the observations.

TABLE IV.—Masses 19 : 1, Lengths 11 : 16.

Coupling $=\gamma$.	$\frac{\text{Bridle Droop}}{\text{Short Pendulum Length.}}=\beta$	Frequency Ratio $p:q$.
Per cent.		
0	0	1·21
1·724	0·1	1·155
3·301	0·2	1·115
4·758	0·3	1·086
6·112	0·4	1·072
7·379	0·5	1·076

Figs. 21 and 22 show traces with 32 per cent. coupling, which gives a ratio of p/q almost equal to 2 : 1 or a tone and its octave. In fig. 21 the conditions of starting masked the compound character of the vibrations, but this is clearly revealed in fig. 22.

Quenched Spark.—Fig. 52, p. 714 of Professor J. A. Fleming's 'Principles of Electric Wave Telegraphy and Telephony,' 2nd ed., shows "the electrical beats produced in the primary and secondary circuits when a sustained primary spark is used and the single periodic oscillations in the secondary circuit when the Quenched Spark is employed." The mechanical analogue of beats was obtained on the double-cord pendulum (see figs. 1 and 2, Plate V., Phil. Mag. October 1917). The damping was not so marked as in Prof. Fleming's case, because our damping factor was almost negligible.

To produce the effect of the quenched spark the masses of the bobs were equal and also their separate frequencies; further their coupling was 10 per cent. One of the bobs was drawn aside and the other allowed to hang in its slightly displaced position. The bob was then freed and its oscillations were quickly diminished by the transference of its energy to the other pendulum, which in about six vibrations had attained an amplitude equal to that with which the other pendulum started. The first pendulum had at this instant lost all amplitude, and it was then suddenly raised by hand and held in this position while the other bob oscillated with the single period. Fig. 6 (Pl. II.) is a photographic reproduction of the sand traces thus obtained. The lower trace represents the quenched spark and the upper one shows the vibrations set up in the secondary circuit or antenna.

IV. Portable Apparatus.

The work with the double-cord pendulum up to fig. 12 inclusive was done with a rough apparatus suspended from beams of the roof. At this point it seemed desirable to have an apparatus that was portable and so arranged as to facilitate the various adjustments required. This was accomplished by the new apparatus shown in fig. 13 (Pl. V.).

It consists essentially of a braced framework of deal, one and a half inches square, the main rods being each six feet long. The bridles are of whipcord and fastened off on cleats fixed on the end frame. The pendulum suspensions in actual use are wires of various lengths with hooks at each end, the fine adjustment being attained by a thin cord and tightener as used for tent ropes. In the photograph these working bridles and suspensions would have been scarcely visible and so were replaced by coarse white cords. The two longitudinal rods at the base of the frame are provided with rails made of hoop-iron set edgewise and let into saw-gates along their length. These rails carry four ball-bearing sheaves, which are fixed on the under side of the board 31 by 23 inches arranged to carry the detachable cards which receive the sand traces. To draw this board along, a cord passes from the centre of one end through two tension-eyes to a bobbin on one side of the end frame. This bobbin is turned by a handle slowly or quickly as may be desired for the purpose in view.

V. Coupling Graph for Cord and Lath Pendulum.

Both in the electrical case and for the double-cord pendulum the coupling may approach but cannot reach the value unity. But in the case of the cord and lath pendulum the conditions are somewhat different (see pp. 258 and 259, Phil. Mag. October 1917). Thus we have

$$\gamma^2 = \frac{\alpha^2}{1+\alpha+\alpha^2},$$

where γ is the coupling and α is the ratio in which the second suspension divides the lath.

So that here also with increasing positive values of α, γ only approaches but never reaches unity except for $\alpha = \infty$. But for negative values of α we see that γ may reach unity for $\alpha = -1$.

This suggested plotting a graph giving γ as ordinates, α being the abscissæ. This is shown in fig. 23 (Pl. V.). The graph has a maximum and a minimum at $\alpha = -2$ and points of inflexion at $\alpha = -0{\cdot}344$ and $-2{\cdot}906$ nearly, and it also asymptotes to $\gamma = \pm 1$.

VI. Possible Further Work.

The vibrations of two coupled pendulums have hitherto been developed for their own sake and as an analogue to electrical vibrations in coupled circuits. It appears, however, that by modification of the pendulums the analogue may be usefully extended so as to illustrate phenomena in various other branches of physics.

The following have occurred to us as worthy of investigation and plans of attack of several have already been matured:—

1. Kater Pendulum for "g" and the possible disturbance of period due to vibration of bracket. Theory and experiment will enable us to evaluate the possible error and eliminate it.
2. Large vibrations with restoring forces not simply proportional to the displacement but involving its squares or cubes.
3. Such a system under double forcing.
4. Optical Dispersion.
5. Dynamical Analogies to Colour Vision and Hearing.
6. Any of the above but further specialized by damping where necessary.

Nottingham,
March 16, 1918.

III. *On Ship-Waves, and on Waves in Deep Water due to the Motion of Submerged Bodies.* By George Green, *D.Sc., Lecturer in Natural Philosophy in the University of Glasgow* *.

Note by Professor Gray.

The following paper was ready for publication at the beginning of 1916, but was put aside on account of war work. It was further deferred by Dr. Green's appointment to the Royal Engineers and his departure to France on military service. Recently, when he was in Glasgow on leave, I advised him to revise the MS. in order that, if possible, it might be published without further delay. The paper may be regarded as a continuation of Lord Kelvin's work on Waves, with which Dr. Green was associated for some time before Lord Kelvin's death in 1907.

Glasgow, Feb. 16, 1918. A. Gray.

The present paper may be regarded as a continuation of Lord Kelvin's work on Ship-Waves. It deals first with the fundamental problem of Ship-Waves, which is—to determine the wave-motion produced by any arbitrary applied surface-pressure. The method used to obtain the solution of this problem is virtually that used by Lord Kelvin in his last paper on Water-Waves,—but here extended to apply to any arbitrary conditions of applied surface-pressure. The paper then proceeds to indicate how we may use the solution given for the case of an arbitrary surface-pressure to obtain the solution of any problem involving the motion of submerged bodies; and a complete discussion is given of the wave-disturbance due to a cylinder and a sphere moving with constant velocity at a considerable depth beneath the surface.

§ 1. Arbitrary Surface Pressure.

Taking an origin in the free undisturbed surface of an infinitely extended mass of liquid, with x and y axes in the surface and the z axis drawn downwards, we can express the conditions to be fulfilled by the velocity-potential, $\phi(x, y, z, t)$, corresponding to any possible motion of the fluid by the equations—

$$\nabla^2\phi = 0, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

$$p/\rho = g(z+\zeta) - \frac{\partial\phi}{\partial t}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

* Communicated by Prof. A. Gray, F.R.S.

where p denotes the pressure, ρ the density of the fluid, at any point (x, y, z), and ζ denotes the vertical component displacement of the particle of fluid whose equilibrium position is at point (x, y, z). When the upper surface of the liquid is free from applied pressure equation (2) takes the form

$$\zeta = \frac{1}{g}\frac{\partial\phi}{\partial t}, \text{ or } \frac{\partial\phi}{\partial z} = \frac{1}{g}\frac{\partial^2\phi}{\partial t^2}, \quad . \quad . \quad . \quad . \quad . \quad (3)$$

for all points on the free surface. If the velocity-potential ϕ satisfies equation (3) at all points of the fluid, each surface which is level when the fluid is undisturbed is a surface of constant pressure in the motion corresponding to this velocity-potential. Equations (2), and (3), also each involve the assumptions that the motion is small and irrotational. The first of these requires that the squares of velocities of the fluid particles should be negligible, and the latter is evidently fulfilled in all the cases of motion to be considered, since in each case the motion is produced from rest by pressures applied to the boundary.

When a particular motion is such as could be produced from rest by impulsive pressures applied to the boundary of the fluid there is a relation between $\phi(x, y, z, t)$, the velocity-potential of the motion at any instant, and the impulsive pressure $\Pi(x, y)$ which caused the motion. This relation is expressed by the equation

$$\Pi(x, y) = -\rho\phi(x, y, z, t), \quad . \quad . \quad . \quad . \quad . \quad (4)$$

with $z=0$, and $t=0$, if we exclude from consideration pressures which are uniform over the whole free surface. An application of this relation, which is of special importance in connexion with the type of problem with which we are dealing, is to the case where a finite pressure $p(x, y)$ is applied to the surface for an infinitesimal period of time $d\tau$. The impulsive pressure is in this case $p(x, y)d\tau$, and the velocity-potential of the motion to which it gives rise is $-(1/\rho)p(x, y, z, t)d\tau$, where $p(x, y, z, t)$ must be determined to satisfy equations (1) and (3) in addition to the equation

$$p(x, y, 0, 0) = p(x, y). \quad . \quad . \quad . \quad . \quad . \quad (5)$$

As we may regard any continuous application of pressure to the surface as equivalent to a series of impulsive pressures delivered in consecutive infinitesimal intervals $d\tau_1$, $d\tau_2$, $d\tau_3$, &c., it is clear that a summation of velocity-potentials, similar to that expressed by $-(1/\rho)\,.\,p(x, y, z, t)d\tau$ for the interval

$d\tau$, can be obtained to represent the motion due to any system of applied pressure.

In working out the application of this process to ship-waves, we may, without loss of generality, take the case of a pressure symmetrical about a vertical line, represented by

$$p(x, y) = f(\varpi), \text{ where } \varpi^2 = x^2 + y^2. \quad . \quad . \quad . \quad (6)$$

Let this pressure be applied to the surface at time $t=0$, with its mid point at the origin, and let it move with uniform velocity v, in the positive direction of the x axis. At time τ from the commencement of its motion, the moving pressure has reached the point $(v\tau, 0, 0)$, and in the ensuing interval $d\tau$ it applies the impulsive pressure $f\{(x-v\tau, y\}d\tau$ to the surface. The corresponding velocity-potential at any point in the fluid, at any time t from the commencement of motion, is represented by

$$d\phi = -\frac{1}{\rho} d\tau f\{(x-v\tau), y, z, t-\tau\}, \quad . \quad . \quad (7)$$

where the complete function $f(x, y, z, t)$ is determined to satisfy equations (1) and (3). The velocity-potential of the resultant motion due to all the impulses delivered up to time t is therefore

$$\phi(x, y, z, t) = -\frac{1}{\rho}\int_0^t d\tau f\{x-v\tau, y, z, t-\tau\}. \quad . \quad (8)$$

In an exactly similar way we can make a summation of the vertical component velocities, or of the vertical component displacements, corresponding to each increment $d\phi$ of velocity-potential appearing in the above summation. From (7) above, with $d\dot{\zeta}$ used to denote the vertical component velocity corresponding to $d\phi$, we have

$$d\dot{\zeta} = -\frac{1}{\rho} d\tau \frac{\partial}{\partial z} f\{(x-v\tau), y, z, (t-\tau)\}, \quad . \quad . \quad (9)$$

or in virtue of (3) above:

$$d\dot{\zeta} = -\frac{1}{g\rho} d\tau \frac{\partial^2}{\partial t^2} f\{(x-v\tau), y, z, t-\tau\}, \quad . \quad (10)$$

and

$$d\zeta = -\frac{1}{g\rho} d\tau \frac{\partial}{\partial t} f\{(x-v\tau), y, z, t-\tau\}, \quad . \quad (11)$$

provided the function $f(x, y, z, t)$ satisfies the equation

$$\frac{\partial f}{\partial t} = 0, \text{ when } t=0. \quad . \quad . \quad . \quad . \quad . \quad . \quad (12)$$

Accordingly, in the resultant motion, the vertical velocity and vertical displacement at any point in the fluid are given by

$$\dot{\zeta} = -\frac{1}{g\rho}\int_0^t d\tau f''\{(x-v\tau), y, z, (t-\tau)\}, \quad . \quad (13)$$

and

$$\zeta = -\frac{1}{g\rho}\int_0^t d\tau f'\{(x-v\tau), y, z, (t-\tau)\}. \quad . \quad (14)$$

From the equations (8), (13), and (14), it appears that the solution of our problem is reduced to the determination of $f(x, y, z, t)$ to satisfy equations (1), (2), (3), and (12): and this determination is easily made by means of the theorem analogous to Fourier's double integral theorem, according to which

$$f(\varpi) = \frac{1}{2\pi}\int_0^\infty J_0(k\varpi)k\,dk\int_0^\infty f(\alpha)J_0(k\alpha)\alpha\,d\alpha, \quad . \quad . \quad (15)$$

and

$$f(\varpi, z, t) = \frac{1}{2\pi}\int_0^\infty e^{-kz}J_0(k\varpi)\,.\,k\cos(gkt^2)^{\frac{1}{2}}dk \int_0^\infty f(\alpha)J_0(k\alpha)\alpha\,d\alpha. \quad . \quad . \quad . \quad (16)$$

The complete solution of our problem to determine the velocity-potential and vertical component displacement, at any point in the fluid, due to moving pressure, $f(\varpi)$, which is applied at the origin at time $t=0$, is thus contained in the integrals

$$\phi(x, y, z, t) = -\frac{1}{2\pi\rho}\int_0^t d\tau\int_0^\infty e^{-kz}J_0(k\varpi')k\cos\{gk(t-\tau)^2\}^{\frac{1}{2}}dk \int_0^\infty f(a)J_0(ka)\alpha\,da, \quad . \quad . \quad . \quad . \quad . \quad (17)$$

$$\zeta(x,y,z,t) = +\frac{1}{2\pi\rho g^{\frac{1}{2}}}\int_0^t d\tau\int_0^\infty e^{-kz}J_0(k\varpi')k^{\frac{3}{2}}\sin\{gk(t-\tau)^2\}^{\frac{1}{2}}dk \int_0^\infty f(\alpha)J_0(k\alpha)a\,da. \quad . \quad . \quad . \quad . \quad . \quad (18)$$

in which $\varpi'^2 = (x-v\tau)^2 + y^2$.

When v is put equal to zero in these formulas we obtain the solution corresponding to the case where the pressure $f(\varpi)$ is applied at the origin at $t=0$ and kept applied without change of position till time t. If we indicate the velocity-potential and vertical-component-displacement for the case where the pressure is impulsively applied by $\phi_0(x, y, z, t)$ and $\zeta_0(x, y, z, t)$ respectively, the corresponding results for the same pressure in motion with velocity v are given by

$$\phi(x, y, z, t) = \int_0^t d\tau\, \phi_0\{(x - v\tau), y, z, (t-\tau)\}, \quad . \quad (19)$$

and

$$\zeta(x, y, z, t) = \int_0^t d\tau\, \zeta_0\{(x - v\tau), y, z, (t-\tau)\}. \quad . \quad (20)$$

By means of these equations the solution for any moving pressure problem may be derived from the solution of the corresponding impulsive pressure problem by a single integration. All that is required is to change x into $(x - v\tau)$ and t into $(t-\tau)$ in the expressions representing the motion due to an impulsive pressure, and then to integrate with respect to τ.

§ 2. General Treatment of Fluid Motion due to Motion of Submerged Bodies.

We now proceed to consider the application of the results contained in § 1 to the problems in which the wave-disturbance is due to the motion of a submerged solid. The velocity-potential in this case, in addition to satisfying equations (1) and (2), must satisfy

$$p = 0, \text{ at the free surface}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (21)$$

and the condition that the fluid in contact with the solid has no velocity normal to the surface of the solid,

$$\frac{\partial \phi}{\partial n} = v \cos(n, x), \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (22)$$

where v is the velocity of the solid in the direction of x.

Let us assume that the solid is moving at a uniform velocity, its centroid being at a constant depth beneath the free surface, large in comparison with the dimensions of the solid. A velocity-potential satisfying all required conditions can then be obtained by the following system of successive approximations.

(*a*) Find first the velocity-potential ϕ_1, corresponding to the motion of the given solid in an infinite mass of liquid. This fulfils required conditions at the boundary of the solid, but involves a certain impulsive pressure at the free surface when the motion of the solid commences and also a certain surface-elevation. Each of these leads to a violation of condition (21).

(*b*) Take next the velocity-potential of the image of the given solid in the free surface from that already obtained. This term, $-\phi_1'$, involves an equal and opposite impulsive pressure applied to the free surface at the commencement of motion, and an equal surface elevation subsequently. The total elevation of the surface at each point is now double that due to the velocity-potential (*a*), and in addition the term added in (*b*) violates (but to a much smaller degree) the conditions required at the surface of the solid (22).

(*c*) Find by means of (2) the pressure acting at the free surface required by the fluid motions referred to in (*a*) and (*b*). This is the pressure applied by the fluid above the surface which is to be the free surface ultimately to the fluid below it. This pressure must be applied to the surface to maintain the two motions represented by the terms in (*a*) and (*b*) when the infinite mass of fluid until now assumed to be above this surface is removed. Further, this system of pressure must move along the surface so as to accommodate itself to the motion of the solid, that is, it moves along the surface with the same velocity as the solid. The terms $\phi_1 - \phi_1'$ of the velocity potential imply that this pressure system acts on the surface. We must therefore apply to the surface a system of pressure equal and opposite to that required by the terms introduced in (*a*) and (*b*); and the surface then becomes a free surface. The motion due to this system of pressure can readily be expressed by means of the results obtained in § 1. The resultant fluid motion given by the three terms which have been indicated above fulfils all the required conditions except that contained in equation (22); which is however satisfied to a first approximation, since the solid is assumed to be at a considerable depth beneath the free surface.

(*d*) To proceed to a higher order of approximation we must add a motion giving a normal velocity at the surface of the solid equal and opposite to that given by the resultant of the terms introduced in (*b*) and (*c*), with zero velocity at infinite distance. This term in turn violates the condition for a free surface (21): the next term introduced to fulfil the requirements of a free surface violates conditions at the surface of the solid; and so on. The whole process amounts to finding a series of reflected motions founded on the motion due to the solid and its negative image in the free surface; and it may of course be applied to all cases where the solution for translational or rotational motion of a solid in an infinite liquid has been obtained. The case of viscous liquid can be treated in the same way.

§ 3. Fluid Motion due to Moving Submerged Cylinder.

As a first illustration of the above process we may write down the first three terms of the solution for the case of an infinite cylinder moving with velocity v in the direction of x positive, its axis being parallel to the y axis. The terms referred to in (a) and (b) of § 2 are readily obtained from the well known solution for the fluid motion due to the steady motion of a cylinder in an infinite liquid:—

$$\phi = -a^2 v \cos\theta/r = -a^2 vx/r^2, \quad . \quad . \quad . \quad . \quad (23)$$

in which the coordinates are referred to an origin coinciding with the instantaneous position of the axis of the cylinder. The surface elevation, ζ, required by (2) to determine the most important part of the pressure term referred to in (c) of § 2 is more readily obtained from the corresponding stream function, representing the fluid-motion relative to the cylinder,

$$\psi = -v\sin\theta(r - a^2/r) = -vz + a^2vz/r^2.$$

At an infinite distance from the cylinder each stream-line is practically horizontal and each particle traversing the line retains its initial vertical coordinate. Hence taking the particles in the plane at $z=h$ above the cylinder, we have $\psi=-vh$ as the constant value of the stream function for the particles of fluid lying in the free surface. Accordingly the equation to the surface as determined by the (a) term alone is

$$-\zeta = z - h = a^2z/r^2, \quad . \quad . \quad . \quad . \quad (24)$$

and, to the degree of accuracy we are aiming at, we may replace z by h on the right. The surface elevation due to the (a) and (b) terms together is then given by

$$\zeta = -2a^2h/(x^2+h^2). \quad . \quad . \quad . \quad (25)$$

To the same order of accuracy, the pressure which must be kept applied to the surface to maintain the motion represented by terms referred to in (a) and (b) of § 2 is given by

$$p = -2g\rho a^2h/(x^2+h^2). \quad . \quad . \quad (26)$$

This pressure must maintain a constant relation to the cylinder, that is, in (26) we may regard x as measured from an origin in the surface vertically above the instantaneous position of the axis of the moving cylinder. In order to make use of (17) and (18), however, it is convenient to use an origin in the free surface vertically above the initial position of the axis of the cylinder in expressing the motion due to the pressure equal and opposite to that given in (26). If we use $\phi_1 - \phi_1'$ to represent the solution corresponding to the motion of the solid and its negative image, and ϕ_2 to represent the motion due to the moving surface-pressure, equal and opposite to that given by (26) above, then, according to (17) and (18),

$$\phi_2(x, z, t) = -(2ga^2h/\pi)\int_0^t d\tau \int_0^\infty e^{-kz} \cos\{gk(t-\tau)^2\}^{\frac{1}{2}} dk$$

$$\int_{-\infty}^{+\infty} \frac{\cos k\{(x - v\tau) - \alpha\}}{\alpha^2 + h^2} d\alpha, \quad . \quad (27)$$

$$\zeta_2(x, z, t) = (2g^{\frac{1}{2}}a^2h/\pi)\int_0^t d\tau \int_0^\infty e^{-kz} k^{\frac{1}{2}} \sin\{gk(t-\tau)^2\}^{\frac{1}{2}} dk$$

$$\int_{-\infty}^{+\infty} \frac{\cos k\{(x - v\tau) - a\}}{a^2 + h^2} da. \quad . \quad . \quad (28)$$

The integrations with respect to a can readily be performed, and we obtain

$$\phi_2 = -2ga^2\int_0^t d\tau \int_0^\infty e^{-k(z+h)} \cos k(x - v\tau) \cos\{gk(t-\tau)\}^{\frac{1}{2}} dk.$$

$$. \quad . \quad . \quad (29)$$

$$\zeta_2 = 2g^{\frac{1}{2}}a^2 \int_0^t d\tau \int_0^\infty e^{-k(z+h)} k^{\frac{1}{2}} \cos k(x - v\tau) \sin\{gk(t-\tau)^2\}^{\frac{1}{2}} dk.$$

$$. \quad . \quad . \quad (30)$$

The motion here represented may be regarded as that reflected from the free surface when a cylinder moves steadily with velocity v at a fixed depth h. Equations (29) and (30) may be interpreted to mean that this reflected motion is the same as that produced by a line pressure acting on the surface which contains the axis of the image cylinder and coinciding at each instant with that axis. This brings our solution into line with the usual interpretation of the motion given by (23) as equivalent to a line pressure acting at each instant at the axis of the cylinder.

The integrations required in (29) and (30) can in each case be carried out by means of the stationary phase principle.

The first integration gives

$$\phi_2(x,z,t)=$$

$$-g^{\frac{3}{2}}a^2\pi^{\frac{1}{2}}\int_0^t d\tau\, e^{-\frac{g(t'-\tau)^2(h+z)}{4(x-v\tau)^2}}\cdot\frac{(t-\tau)}{(x-v\tau)^{\frac{3}{2}}}\cos\left\{\frac{g(t-\tau)^2}{4(x-v\tau)}-\frac{\pi}{4}\right\}, \quad . \; . \; . \quad (31)$$

$$\zeta_2(x,z,t)=$$

$$\frac{1}{2}g^{\frac{3}{2}}a^2\pi^{\frac{1}{2}}\int_0^t d\tau\, e^{-\frac{g(t-\tau)^2(h+z)}{4(x-v\tau)^2}}\cdot\frac{(t-\tau)^2}{(x-v\tau)^{\frac{5}{2}}}\sin\left\{\frac{g(t-\tau)^2}{4(x-v\tau)}-\frac{\pi}{4}\right\}, \quad . \; . \; . \quad (32)$$

each of these calculations being subject to the condition that $(t-\tau)/(x-v\tau)$ is large, and that $(x-v\tau)$ is large in comparison with the space over which the applied pressure is a first order effect. These conditions, which are explained in a former paper*, are fulfilled in the present case where we are considering places and times at which the motion has become steady. On proceeding to the final integration, we find that, corresponding to each value of t, only one value of τ fulfils the condition for stationary phase—that given by

$$(x-v\tau)=(vt-x). \quad . \; . \; . \; . \; . \; . \quad (33)$$

In this, values of x greater than vt are evidently inadmissible as they require that τ should exceed t, hence we may conclude that the effective part of the wave-disturbance is behind the mid line of the applied pressure at each instant, the forward part being negligible in comparison. The final evaluations of (31) and (32), obtained by means of the stationary phase principle, are

$$\phi_2(x,z,t)=-(4ga^2\pi/v)\,e^{-\frac{g}{v^2}(h+z)}\cos\left\{\frac{g}{v^2}(vt-x)\right\}, \quad . \quad (34)$$

$$\zeta_2(x,z,t)=\quad(4ga^2\pi/v^2)\,e^{-\frac{g}{v^2}(h+z)}\sin\left\{\frac{g}{v^2}(vt-x)\right\}. \quad . \quad (35)$$

These results are valid at all points well behind the moving pressure at which steady motion has been established. In the immediate neighbourhood of the mid line of the applied surface-pressure the complete solutions of (29) and (30) are required. The problem of the moving line pressure has

* See short note at end of this paper, or Proc. R.S.E. vol. xxx. p. 247.

been dealt with by Prof. Lamb*, and we can avail ourselves of his results for the surface elevation ζ_2. We can now write down the complete solution for the motion due to a moving cylinder. With $X=(vt-x)$ in the above equations, we refer the motion to an origin in the free surface, vertically above the instantaneous position of the axis of the moving cylinder. For values of X that are considerable our approximate solution is given by

$$\phi=\phi_1-\phi_1'+\phi_2$$

$$=\frac{a^2vX}{X^2+(z-h)^2}-\frac{a^2vX}{X^2+(z+h)^2}-(4ga^2\pi/v)\,e^{-\frac{g}{v^2}(h+z)}\cos\frac{g}{v^2}X \quad . \quad . \quad . \quad (36)$$

$$\zeta\dagger=\zeta_1-\zeta_1'+\zeta_2$$

$$=-\frac{2a^2h}{X^2+h^2}+(4ga^2\pi/v^2)\,e^{-\frac{g}{v^2}h}\sin\frac{g}{v^2}X. \quad . \quad . \quad . \quad (37)$$

In the region in front of the moving cylinder the train of regular waves represented by the periodic terms in these equations is absent.

§ 4. Fluid Motion due to Moving Submerged Sphere.

The corresponding investigation of the wave-disturbance due to a sphere moving with constant velocity v, at a uniform depth h beneath the surface, proceeds exactly as in the case of the cylinder. The velocity-potential ϕ_1 of the fluid motion due to a sphere moving steadily in an infinite liquid in the positive direction of x is given by

$$\phi_1=-\frac{va^3\cos\theta}{2r^2}=-\frac{va^3x}{2r^3}, \quad . \quad . \quad (38)$$

* For the moving line pressure represented in (30) above

$$\zeta_2(X,0,t)=4\kappa a\pi e^{-\kappa h}\sin\kappa X-2\kappa a^2\int_0^\infty\frac{m\cos mh-\kappa\sin mh}{m^2+\kappa^2}e^{-mX}\,dm,$$

at distance X behind the moving line pressure, and

$$\zeta_2(X,0,t)=-2\kappa a^2\int_0^\infty\frac{m\cos mh-\kappa\sin mh}{m^2+\kappa^2}e^{-mX}\,dm;$$

at distance X in front of the moving pressure; with $\kappa=(g/v^2)$: 'Hydrodynamics,' 3rd edition, p. 385.

† This result differs from that given by Prof. Lamb (*Annali di Matematica*, tome xxi, serie iii. p. 237) in the sign of the term representing the system of regular waves, but gives the same value for the wave-making resistance of the cylinder: $R=4\pi^2g\rho a^4\kappa^2e^{-2\kappa h}$, given in equation (78) of his paper.

a being the radius of the sphere, and the coordinates being referred to the instantaneous position of the centre of the sphere as origin. The corresponding stream function for the fluid motion relative to the sphere is

$$\psi_1 = -vr^2 \sin^2\theta + \frac{va^3 \sin^2\theta}{2r} \quad . \quad . \quad (39)$$

We have first to determine from this the stream surface containing the fluid particles which, when at rest, lie in the plane at vertical distance $z=h$ above the centre of the moving sphere. On transferring to polar coordinates (x, ϖ), x in the line of motion, and ϖ $(=\sqrt{y^2+z^2})$ perpendicular to the line of motion, we obtain the stream function in the form

$$\psi_1 = -v\varpi^2 + \frac{va^3\varpi^2}{2r^3} \quad . \quad . \quad . \quad . \quad . \quad (40)$$

On any stream line at infinite distance from the sphere, ϖ has its value the same for any fluid particle as when the particle is at rest. Putting $\varpi=\varpi_0$ at $r=\infty$, we have $\psi_1=-v\varpi_0^2$, an equation which enables us to write (40) in the form

$$\varpi^2-\varpi_0^2=\frac{a^3\varpi^2}{2r^3}, \quad . \quad . \quad . \quad . \quad . \quad (41)$$

If we assume, as in the case of the cylinder, that the depth of the centre of the sphere h is large compared with the radius, on this standard we may replace ϖ by ϖ_0 on the right-hand side of (41) and on the left-hand side we may put $\varpi+\varpi_0=2\varpi_0$. This leads to

$$\varpi-\varpi_0=\frac{a^3\varpi_0}{4r^3} \quad : \quad r^2=x^2+y^2+h^2, \quad . \quad . \quad (42)$$

at the plane $z=h$. The change in the z coordinate of any point on this surface, being $(\varpi-\varpi_0)h/\varpi_0$, is now easily obtained in the form

$$-\zeta=z-z_0=\frac{a^3h}{4r^3} \quad : \quad r^2=x^2+y^2+h^2. \quad . \quad . \quad (43)$$

An equal elevation of surface at each point is produced by the image sphere. This enables us now to obtain the principal term of the pressure system which must be kept applied to the upper surface to maintain the motion represented by $\phi_1-\phi_1'$ in the fluid beneath—

$$p=-\tfrac{1}{2}g\rho a^3h/r^3\,; \quad r^2=x^2+y^2+h^2. \quad . \quad . \quad . \quad (44)$$

To fulfil the condition of a free surface, we must apply a pressure equal and opposite to that given by (44)—that is, a pressure symmetrical about a point in the surface at each instant vertically above the instantaneous position of the centre of the moving sphere. Referred to an origin vertically above the initial position of the centre of the sphere, the expressions representing the fluid motion due to the moving pressure system are—

$$\phi_2(x,y,z,t) = -\frac{ga^3h}{4\pi}\int_0^t d\tau \int_0^\infty e^{-kz} J_0(k\varpi')$$

$$k \cos\{gk(t-\tau)^2\}^{\frac{1}{2}}dk\int_0^\infty \frac{J_0(k\alpha)\alpha\, d\alpha}{(\alpha^2+h^2)^{\frac{3}{2}}}. \quad . \quad . \quad . \quad (45)$$

$$\zeta_2(x,y,z,t) = \frac{g^{\frac{1}{2}}a^3h}{4\pi}\int_0^t d\tau \int_0^\infty e^{-kz} J_0(k\varpi')$$

$$k^{\frac{3}{2}} \sin\{gk(t-\tau)^2\}^{\frac{1}{2}}dk\int_0^\infty \frac{J_0(k\alpha)\alpha\, d\alpha}{(\alpha^2+h^2)^{\frac{3}{2}}}, \quad . \quad . \quad (46)$$

with $\varpi'^2 = (x-v\tau)^2+y^2$.

These become, on integration with respect to α,

$$\phi_2(x,y,z,t) = -\frac{ga^3}{4\pi}\int_0^t d\tau \int_0^\infty e^{-k(z+h)} J_0(k\varpi')$$

$$k \cos\{gk(t-\tau)^2\}^{\frac{1}{2}}dk, \quad . \quad . \quad . \quad . \quad (47)$$

$$\zeta_2(x,y,z,t) = \frac{g^{\frac{1}{2}}a^3}{4\pi}\int_0^t d\tau \int_0^\infty e^{-k(z+h)} J_0(k\varpi')$$

$$k^{\frac{3}{2}} \sin\{gk(t-\tau)^2\}^{\frac{1}{2}}dk, \quad . \quad . \quad . \quad (48)$$

which correspond to (29) and (30) above and are open to an interpretation similar to that given for the motion reflected from the free surface in the case of the moving cylinder. In this case the reflected motion is the same as would be produced by a point pressure, acting on a free surface at a height h above the free surface in our present problem, and coinciding at each instant with the centre of the image sphere. An important part of the motion can again be obtained by applying the principle of stationary phase to the integrals contained in (47) and (48)*. Corresponding to

* T. H. Havelock, Proc. R.S. 1910.

(31) and (32), we have, by one integration,

$$\phi_2(x,y,z,t) = -\frac{g^{\frac{3}{2}}a^2}{2^{\frac{7}{2}}\pi}\int_0^t d\tau\, e^{-\frac{g(t-\tau)^2(z+h)}{4\varpi'^2}} \frac{(t-\tau)^2}{\varpi'^3} . \cos\frac{g(t-\tau^2)}{4\varpi'} \quad . \quad . \quad . \quad (49)$$

$$\zeta_2(x,y,z,t) = \frac{g^2a^3}{2^{\frac{9}{2}}\pi}\int_0^t d\tau\, e^{-\frac{g(t-\tau)^2(z+h)}{4\varpi'^2}} \frac{(t-\tau)^3}{\varpi'^4} . \sin\frac{g(t-\tau)^2}{4\varpi'}. \quad . \quad . \quad . \quad (50)$$

These integrals are well known in connexion with the ship-waves problem. To obtain the motion at any point of space at any fixed time t we have now to consider for each value of t two values of τ at which the phase is stationary, namely those given by

$$x - v\tau = \tfrac{1}{2}\left[(vt-x) \pm \sqrt{(vt-x)^2 - 8y^2}\right] = \tfrac{1}{2}(X \pm R). \quad (51)$$

With τ_1 and τ_2 to represent the values of τ given by (51), our final results for the motion due to the applied surface pressure indicated in (c) of § 2 are

$$\phi_2(x,y,z,t) = \sum\binom{\tau=\tau_1}{\tau=\tau_2} - \frac{g^{\frac{3}{2}}a^3}{2^{\frac{7}{2}}\pi^{\frac{1}{2}}} e^{-\frac{g(t-\tau)^2(z+h)}{4\varpi'^2}} . \frac{(t-\tau)^2}{\varpi'^3} \times \sqrt{\frac{2!}{\frac{\partial^2\theta}{\partial\tau^2}}} . \cos\left(\theta + \frac{\pi}{4}\right), \quad . \quad . \quad . \quad . \quad (52)$$

$$\zeta_2(x,y,z,t) = \sum\binom{\tau=\tau_1}{\tau=\tau_2} \frac{g^2a^3}{g^{\frac{9}{2}}\pi^{\frac{1}{2}}} e^{-\frac{g(t-\tau)^2(z+h)}{4\varpi'^2}} . \frac{(t-\tau)^3}{\varpi'^4} \times \sqrt{\frac{2!}{\frac{\partial^2\theta}{\partial\tau^2}}} . \sin\left(\theta + \frac{\pi}{4}\right), \quad . \quad . \quad . \quad . \quad (53)$$

where $\theta = \frac{g(t-\tau)^2}{4\varpi'}$.

These results are valid in the regions well behind the mid point of the moving pressure which is included by the planes $X^2 - 8y^2 = 0$, but are not valid along, or in the immediate neighbourhood of, these planes. Along each of these planes the correct evaluations of the integrals in (49) and (50), obtained by means of the principle of stationary phase, are

$$\phi_2(x,y,z,t) = -\frac{g^{\frac{3}{2}}a^3}{2^{\frac{7}{2}}\pi} e^{-\frac{g(t-\tau)^2(z+h)}{4\varpi'^2}} \cdot \frac{(t-\tau)^2}{\varpi'^3}$$

$$\times \sqrt[\frac{1}{3}]{\frac{3\,!}{\frac{\partial^3\theta}{\partial\tau^3}}}\,\frac{\Gamma(\frac{1}{3})}{3^{\frac{1}{2}}} . \cos\theta . \quad . \quad . \quad . \quad . \quad . \quad (54)$$

$$\zeta_2(x,y,z,t) = \frac{g^2 a^3}{2^{\frac{9}{2}}\pi} e^{-\frac{g(t-\tau)^2(z+h)}{4\varpi'^2}} \cdot \frac{(t-\tau)^3}{\varpi'^4}$$

$$\times \sqrt{\frac{3\,!}{\frac{\partial^3\theta}{\partial\tau^3}}} \cdot \frac{\Gamma(\frac{1}{3})}{3^{\frac{1}{2}}} \cdot \sin\theta . \quad . \quad . \quad . \quad . \quad . \quad (55)$$

The value of τ applicable in these expressions is that for which the two values τ_1 and τ_2 coincide, namely

$$\tau = \frac{1}{2v}(3x - vt),$$

and these expressions then reduce to

$$\phi_2(X,y,z) = +\frac{3^{\frac{1}{2}}\, g^{\frac{7}{6}}a^3}{2^{\frac{3}{2}}\,\pi v^{\frac{7}{3}}} e^{-\frac{3g}{2v^2}(z+h)} \cdot \frac{1}{X^{\frac{1}{3}}} . \cos\left\{\left(\frac{27}{32}\right)^{\frac{1}{2}} \frac{g}{v^2} X\right\}, \quad . \quad . \quad . \quad (56)$$

$$\zeta_2(X,y,z) = -\frac{3}{4} \cdot \frac{g^{\frac{5}{3}}a^3}{\pi v^{\frac{10}{3}}} e^{-\frac{3g}{2v^2}(z+h)} \cdot \frac{1}{X^{\frac{1}{3}}} . \sin\left\{\left(\frac{27}{32}\right)^{\frac{1}{2}} \frac{g}{v^2} X\right\} \quad . \quad . \quad . \quad (57)$$

the motion being now referred to an origin in the free surface vertically above the instantaneous position of the centre of the moving sphere. The approximate solution we have obtained for the fluid motion due to a submerged sphere moving with velocity v is accordingly

$$\phi = \phi_1 - \phi_1' + \phi_2$$

$$= \frac{a^3 vX}{2\{X^2+y^2+(z-h)^2\}^{\frac{3}{2}}} - \frac{a^3 vX}{2\{X^2+y^2+(z+h)^2\}^{\frac{3}{2}}} + \phi_2, \quad (58)$$

$$\zeta = \zeta_1 - \zeta_1' + \zeta_2$$

$$= -\frac{a^3 h}{2\{X^2+y^2+h^2\}^{\frac{3}{2}}} + \zeta_2 ; \quad X = (vt - x) ; \quad . \quad . \quad . \quad (59)$$

where ϕ_2 and ζ_2 are given by equations (52) to (57) above.

In a later paper we hope to carry the investigation further and to calculate the wave-making resistance of the sphere.

NOTE ON THE PRINCIPLE OF STATIONARY PHASE.

The meaning of the process which we have employed to evaluate the integrals in the preceding paper is illustrated in its application to the diffraction integral of the type

$$I = \int_0^S A \sin\theta\, ds\ ,\ \theta = 2\pi\left(\frac{t}{T}-\frac{\rho}{\lambda}\right). \quad . \quad . \quad (1)$$

Here S may denote any wave-surface, and ρ the length of path from an element of surface to any point P at which the resultant disturbance due to the disturbances arriving from all the elements of S is required. By Taylor's theorem we may express θ in the form

$$\theta = \theta_0 + (s-s_0)\left(\frac{\partial\theta}{\partial s}\right)_0 + \frac{(s-s_0)^2}{2\,!}\left(\frac{\partial^2\theta}{\partial s^2}\right)_0 + \frac{(s-s_0)^3}{3\,!}\left(\frac{\partial^3\theta}{\partial s^3}\right)_0 + \text{etc.} \quad . \quad . \quad . \quad (2)$$

near any element of the surface S at which the argument has the value s_0 and the phase has the value θ_0,

The various elements of area ds in (1) are not equally effective in contributing to the resultant disturbance at P. If the phase θ varies rapidly from element to element compared with the amplitude term A, then the various elements provide contributions which are nearly equal in value but are alternately positive and negative and thus give a very small resultant effect. The effective elements in (1) are those for which the phase is stationary for small variations in s, that is, those in the neighbourhood of which $(\partial\theta/\partial s)=0$. In the optical problem this is the same as $(\partial\rho/\partial s)=0$, so that the condition of stationary phase for the effective elements is equivalent to the usual condition for a ray—that the optical path is either a maximum or a minimum. Thus the principle of Stationary Phase or Group-velocity in the wave-theory is the equivalent of the Law of Least Path for the rays, if we look at the matter from the point of view of geometrical optics.

Then again, in the neighbourhood of any value of s, say s_0, at which the stationary phase condition is fulfilled, the range of values of $s-s_0$ which provide the main contribution to the value of the integral, may be regarded as very small, so that the corresponding values of θ may be taken as sufficiently represented by

$$\theta = \theta_0 + \frac{(s-s_0)^2}{2\,!}\left(\frac{\partial^2\theta}{\partial s^2}\right) = \theta_0 + \sigma^2, \quad . \quad . \quad . \quad (3)$$

provided $(\partial^2\theta/\partial s^2)$ is large compared with the greatest range of $(s-s_0)$ to be considered. This implies a large number of oscillations from positive to negative in the sine term in (1) within the effective range of $s-s_0$ so that the limits of σ may be taken to be $\pm\infty$ with only slight error, while the amplitude term may be considered constant and may be put equal to its value at $s=s_0$. Under these conditions the integral (1) becomes

$$I=\left\{A\sqrt{\frac{2\,!}{\frac{\partial^2\theta}{\partial s^2}}}\cdot\int_{-\infty}^{+\infty}\sin(\theta_0+\sigma^2)\,d\sigma\right\}_{s=s_0}$$

$$=\left\{A\sqrt{\frac{2\,!}{\frac{\partial^2\theta}{\partial s^2}}}\cdot\sqrt{\pi}\sin\left(\theta_0+\frac{\pi}{4}\right)\right\}_{s=s_0} \quad . \quad . \quad (4)$$

The value or values of s_0 to be taken are determined by the roots of $(\partial\theta/\partial s)=0$, and, corresponding to each root, a term similar to (4) must appear in the expression for the value of the integral. When two roots of this equation coincide, then $(\partial\theta/\partial s)$ and $(\partial^2\theta/ds^2)$ vanish simultaneously and we must take, instead of (3),

$$\theta=\theta_0+\frac{(s-s_0)^3}{3\,!}\frac{\partial^3\theta}{\partial s_0{}^3}=\theta_0+\sigma^3. \quad . \quad . \quad . \quad (5)$$

This gives us finally for (1)

$$I=\left\{A\sqrt{\frac{3\,!}{\frac{\partial^3\theta}{\partial s^3}}}\cdot\int_{-\infty}^{+\infty}\sin(\theta_0+\sigma^3)\,ds\right\}_{s=s_0}$$

$$=\left\{A\sqrt{\frac{3\,!}{\frac{\partial^3\theta}{\partial s^3}}}\cdot\frac{\Gamma(\frac{1}{3})}{3^{\frac{1}{2}}}\sin\theta_0\right\}_{s=s_0} \quad . \quad . \quad . \quad . \quad (6)$$

Equation (3) cannot be employed when $(\partial^2\theta/\partial s^2)$ is small or zero. The two conditions $(\partial\theta/ds)=0=(\partial^2\theta/ds^2)$, or $(\partial\rho/\partial s)=0=(\partial^2\rho/\partial s^2)$ in the optical case, are the conditions that two rays coincide at the point P. The effective elements are then in the neighbourhood of a point of inflexion on the wave-surface S, and the point P at which the rays coincide must lie on the wave-caustic, which corresponds to a line of cusps in the wave-system.

IV. *Resonance and Ionization Potentials for Electrons in Metallic Vapours.* *By* JOHN T. TATE, *Ph.D.*, *and* PAUL D. FOOTE, *Ph.D.**

IT has been shown by Franck and Hertz and others that when electrons are accelerated through gases or vapours there are, especially for gases having small electron affinity such as the inert gases and metallic vapours, well-defined potentials at which a large transfer of energy takes place between electrons and gas atoms. The evidence for this transfer of energy is in the emission at these definite potentials of radiations having frequencies characteristic of the gas.

This absorption by the atom of the kinetic energy of the electrons only at definite velocities of the electrons is to be expected from a purely mechanical standpoint. A considerable transfer of energy from the electron to the relatively heavy atom can be explained only by assuming that there are in the gas atom certain vibrational degrees of freedom of which the characteristic period bears some simple relation to the time of encounter between the electron and atom. Evidence of the existence of these characteristic vibrations is seen in the absorption spectra of gases, and it is therefore to be expected that, when the potential accelerating the electrons reaches a value such that the time of encounter between electrons and atoms is simply related to the natural period of vibration of an absorption line of the gas in question, a loss in the kinetic energy of the electrons and an emission of a radiation of the frequency of the absorption line will be observed. The essential similarity between the phenomena of the absorption by a gas of radiant energy and of the absorption of the kinetic energy of motion of electrons should be emphasized.

The experimental results thus far obtained indicate that there are in general two types of inelastic encounter between electrons and atoms—first, those encounters which are accompanied by the emission of a radiation of a single frequency without ionization of the gas, and second, those encounters which ionize the gas and excite the radiation of a composite spectrum of frequencies. The potential giving the first type of encounter may be termed a resonance potential, that giving the second type of encounter, an ionization potential.

* Communicated by the Director, United States Bureau of Standards.

It has been shown that the resonance potentials may be calculated from the quantum relation

$$h\nu = e\mathrm{V},$$

where ν is the frequency of the radiation excited at the potential V. It is interesting to note that it is possible from the experimentally observed values of V and ν to calculate the order of magnitude of the diameter of the sphere of interaction between the electron and the gas molecule, assuming that the time of interaction is simply related to the period of vibration of the light emitted. Taking mercury as a typical case, it is observed that one of the principal resonance potentials, exciting the line 2537 Å (period (τ) $=8{\cdot}46\times10^{-16}$), is 4·90 volts. Assuming that the electron is completely stopped by the encounter, and that during the encounter it travels with an average velocity $\bar{v}=\frac{1}{2}\sqrt{2\frac{e}{m}\mathrm{V}}$, where $\mathrm{V}=4{\cdot}90$ volts, we find that in a time equal to $\tau/4$ it will have travelled a distance $\sigma=1{\cdot}4\times10^{-8}$ cm., a value comparable with the diameter of the molecule as calculated from other data. It should also be noted that, if we assume as an experimental fact the validity of the relation $h\nu=e\mathrm{V}$, we must conclude that σ is not constant but is given by an expression of the form

$$\sigma=\frac{1}{4}\sqrt{\frac{h}{2m\nu}};$$

and we are led to the conclusion that there are, for the different characteristic frequencies of vibration in the atom, differing spheres of interaction with the moving electron, a conception which might be regarded as pointing to an atomic structure such as Bohr's.

The experimental values of the ionization potentials are found also to satisfy a relation of the form $h\nu=e\mathrm{V}$, except that in this case ν is the limiting frequency of the series of lines excited at the potential V. Conversely, it may be concluded that this frequency is also the long wave-length limit of the photo-sensibility of the gas, and experiments are now in progress which, it is hoped, will determine whether or not this is true. Unfortunately the data on the photo-sensibility of gases and metallic vapours are rather unreliable on account of possible surface effects, and it is obviously not permissible to apply the results obtained on metallic surfaces to the metal in the form of vapour.

The present paper is an account of an experimental determination of the resonance and ionization potentials for electrons in cadmium, sodium, potassium, and zinc vapours. The results for cadmium and sodium have been published elsewhere*, but are included here for the sake of completeness.

Fig. 1.

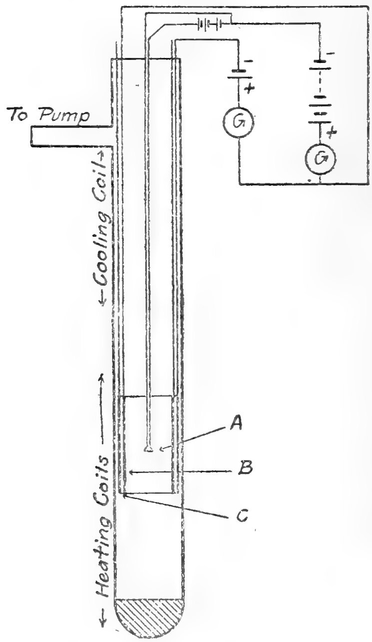


Diagram of Apparatus.

The method employed was that of Franck and Hertz† for determining the resonance potentials with the modification described by Tate‡ for determining the ionization potentials and initial potentials. The arrangement of apparatus is shown in fig. 1. The metal was vaporized at the bottom of a pyrex glass or glazed porcelain tube, and the vapour, after

* Tate and Foote: Sodium, J. Wash. Acad. Sci. vii. p. 517 (1917); Cadmium, Bulletin Bu. Standards, 1918 (in press).

† *Verh. d. Phys. Ges.* xvi. pp. 457–467 (1914).

‡ Tate, Phys. Rev. vii. p. 686 (1916); idem, x. p. 81 (1917).

passing through the superheated ionization-chamber, condensed in the upper half of the tube. The source of electrons was a hot platinum, tungsten, or molybdenum wire cathode A, of low resistance, coated with lime. Surrounding the cathode was a cylinder of nickel net B, and outside and coaxial with this a cylinder of sheet nickel C. The apparatus was evacuated by means of a Langmuir condensation-pump or by a Stimson* two-stage condensation-pump. The latter pump was very kindly made for us by Dr. Stimson of this Bureau. In general the pressure employed, as registered by a McLeod gauge, was about 10^{-5} cm. Hg. The experimental procedure consisted in applying a constant retarding potential, usually from 1 to 3 volts, depending upon the metal used, between the net and the outside cylinder and measuring both the total current from the hot wire and that portion of it which reached the outside cylinder, against the retarding field, as functions of a varying accelerating potential applied between the hot wire and net.

The results obtained with the four vapours are shown graphically in the curves of figs. 2 to 10, and the analyses of the curves are given in Tables I. to IV.

The curves showing the variation of the partial current to the outside cylinder with the accelerating potential show maxima, or at least pronounced changes in curvature, at successive points which differ in potential by a constant amount. This constant difference gives the resonance potential directly, eliminating any consideration of initial velocities.

The curves showing the variation with accelerating potential of the total current from the hot wire show a rapid increase in slope at a point for which the potential, corrected for the initial velocity of the electrons as obtained from the partial current curves, gives the ionization potential. The inelastic nature of the encounters producing ionization is shown in the case of cadmium, fig. 2 curve 7, and sodium, fig. 5 curves 5 and 6, by the occurrence of secondary maxima at potentials differing from the ionization potential by multiples of the resonance potential†.

The accelerating potential was applied at one end or the other of the hot wire, and the values given in the Tables for the initial velocities therefore include the drop in potential between the point of application of the potential and the

* Stimson, J. Wash. Acad. Sci. vii. p. 477 (1917).

† The method of interpreting the curves is discussed in more detail by the writers in J. Wash. Acad. Sci. vii. p. 517 (1917).

centre of the hot wire. The curves of fig. 7 for potassium were obtained with the potential applied at the positive end of the hot wire, while those of fig. 8 were obtained with the potential applied at the negative end. It will be seen that the initial velocity of the electrons correcting for the drop in potential along the hot wire must have been 1·6 volts, a surprisingly high value. This is interesting in connexion with the recent experiments of Wood and Okano*, who found that an applied potential of 0·5 volt was sufficient to excite the D-lines in sodium vapour, indicating, if the resonance potential exciting the D-line in sodium vapour is taken as 2·1 volts, that the initial velocity of the electrons in their experiments must have been 1·6 volts. It is our opinion that the appearance of the many-lined spectrum in metallic vapours at potentials lower than the ionizing potential is due to the presence of high-speed electrons rather than to a decrease in the value of the critical potentials. In no case was there any evidence of a decrease in the value of the resonance potential even under conditions which allowed the striking of the visible arc at applied potentials considerably lower than the ionizing potential.

The curves obtained for zinc, figs. 9 and 10, deserve special attention. It will be noted that the points *a* to *g* form a double series of points, *a*, *c*, *e*, *g*, and *b*, *d*, *f*, having a common difference in potential of 4·1 volts. This common difference is taken as the value of the resonance potential. An explanation for the double series of points is lacking, however. It is possible that there are two groups of electrons possessing differing initial velocities, or that there is a secondary resonance potential at 2·3 volts. The fact that there is no evidence of a succession of points differing in potential by 2·3 volts would indicate that the first explanation was the correct one. A consideration of the total current curves, 7 and 8, fig. 10, however, leaves the question very much in doubt. The total current shows a rapid falling off in rate of increase at the points *a* . . . *e* for which no satisfactory explanation has been found. At all events, however, the points P indicating ionization of the zinc vapour are very definitely located at an effective potential of 9·5 volts. The fact that there are not two such points P on each curve indicates that there are not present two groups of electrons having different initial velocities. Further work is being carried out on zinc vapour with a view to clearing up some of these difficulties.

* Wood and Okano, Phil. Mag. xxxiv. p. 177 (1917).

TABLE I.—Resonance and Ionization Potentials for Cadmium. Referring to figs. 2 and 3.

Applied Potentials at Resonance.						Applied Potentials at Ionization	Resonance Potential.	Weight.	Initial Potential.	Ionization Potential.	Weight.
Curve.	*a.*	*b.*	*c.*	*d.*	*e.*	*i.*					
1......	3·6	7·6	...	...	...	8·5	4·0	1	·9	8·9	1
2......	2·8	6·8	...	...	...	7·6	4·0	1	·8	8·8	1
3......	3·5	7·6	...	...	...	8·2	4·1	1	·6	8·8	1
4......	2·8	6·8	...	...	...	...	4·0	1			
5......	3·6	7·5	11·3	...	...	8·4	3·85	1	·9	8·6	1
6......	3·4	7·5	...	...	...	8·5	4·1	1	1·1	9·2	1
7......	3·6	7·7	11·2	14·4	18·4	...	3·72*	5	·06*		
7......	8·6 (*i*)	13·0 (*j*)	16·8 (*k*)	20·7 (*l*)	...	8·97†	3·92†	2	·06*	9·03†	3
8......	Total current curve ...			...	...	8·9 (P)	...	...	·06*	8·96‡	
						Weighted Mean	3·88 volts			8·92 volts	

* Least square reduction of 5 points.
† Least square reduction of 4 points.
‡ Not used in average because this point was involved in obtaining the value immediately above.

Fig. 2.—Cadmium. Total and Partial Current Curve.

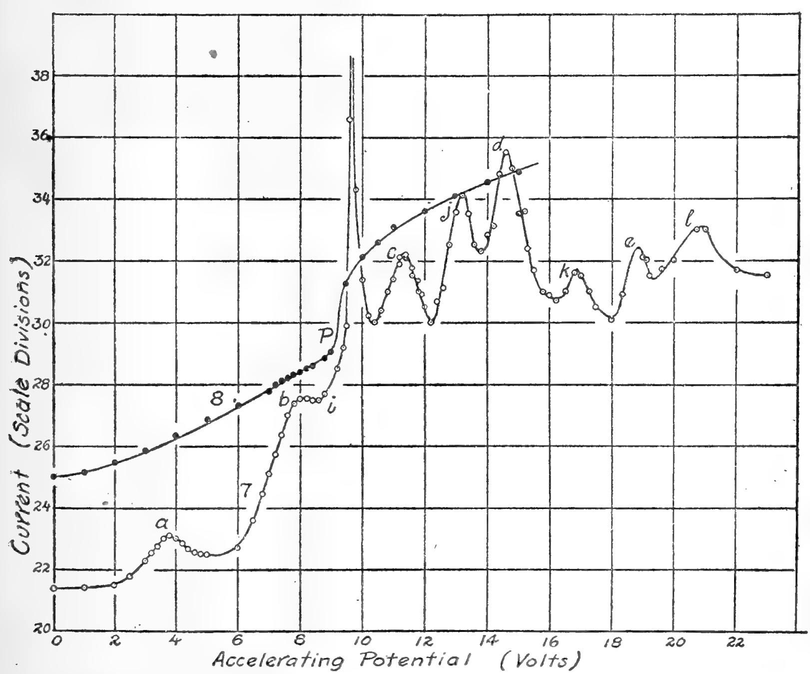


Fig. 3.—Cadmium. Partial Current Curves.

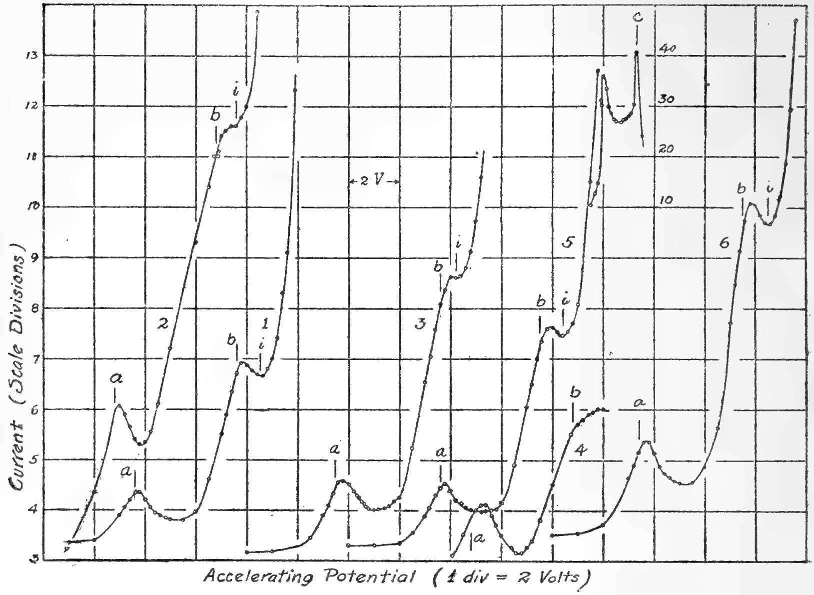


TABLE II.—Resonance and Ionization Potentials for Sodium. Referring to figs. 4, 5, and 6.

Applied Potentials at Resonance.								Resonance Potential.	Initial Potential.
Curve.	*a*.	*b*.	*c*.	*d*.	*e*.	*f*.	*g*.	*b*—*a*.	
1 ...	2·3	4·5	6·3	8·1					
2 ...	2·3	4·5	6·5	8·7					
3 ...	1·4	3·4	5·5	...	...	...	...	2·0	0·6
4 ...	1·2	3·3	5·6	...	...	...	...	2·1	0·9
5 ...	1·4	3·6	5·6	7·8	10·0	...	..	2·2	0·8
6 ...	1·3	3·4	5·9	8·0	10·2	12·2	14·5	2·1	0·8
7 ...	1·3	3·5	6·0	...	...	...	...	2·2	0·9
8 ...	...	...	...	7·8					
							Mean...	2·12 ±0·06	0·80

9*a*...	4·3	Applied potential for ionization.
10*a*...	4·3	Applied potential for ionization.
11*a*...	4·4	Applied potential for ionization.
	4·33	Mean applied potential.
	0·80	Initial potential.
	5·13	Ionization potential.

Fig. 4.—Sodium. Partial Current Curves.

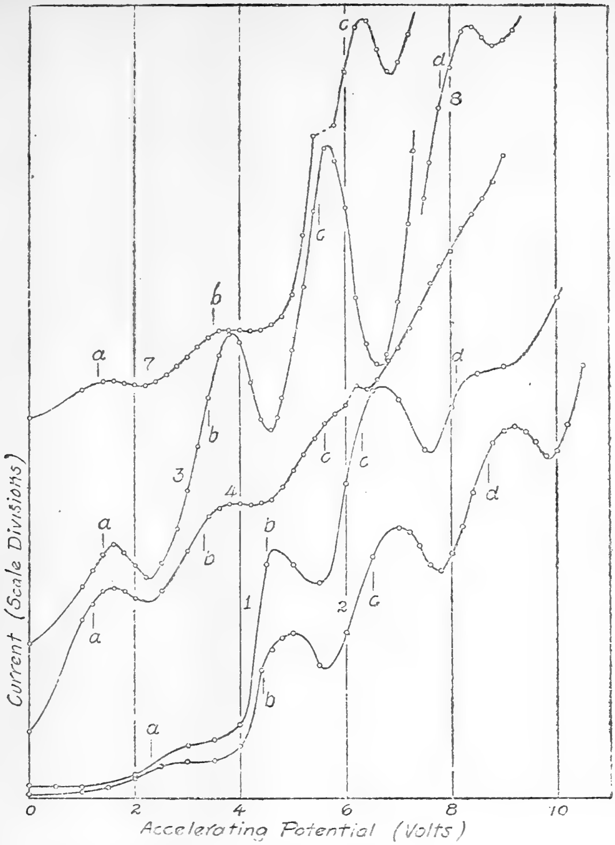


Fig. 5.—Sodium. Partial Current Curves.

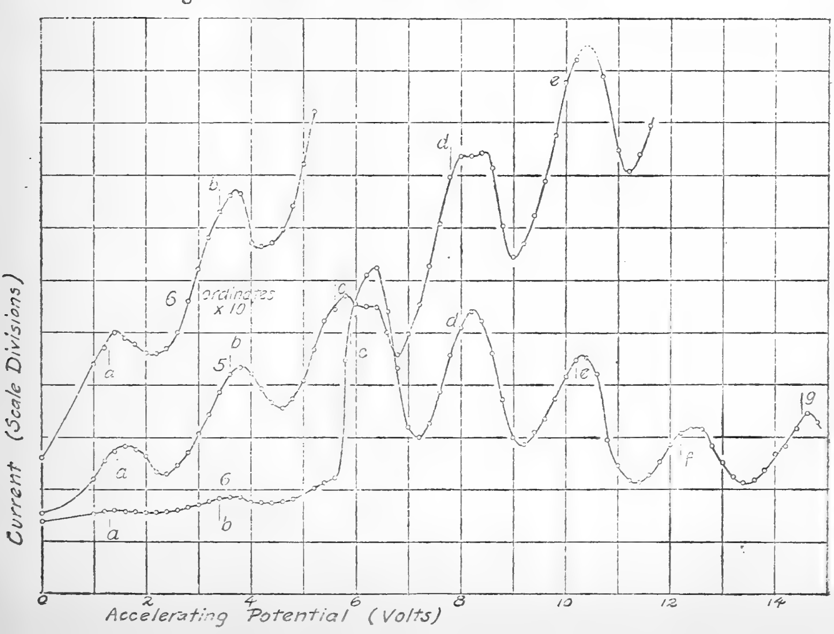
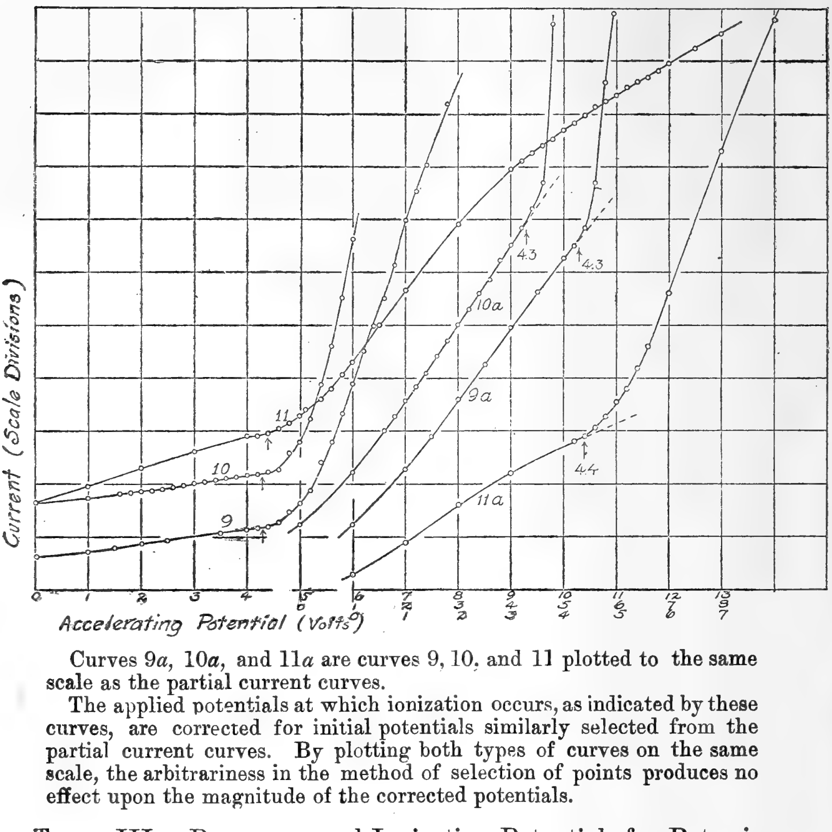


Fig. 6.—Sodium. Total Current Curves.



Curves 9*a*, 10*a*, and 11*a* are curves 9, 10, and 11 plotted to the same scale as the partial current curves.

The applied potentials at which ionization occurs, as indicated by these curves, are corrected for initial potentials similarly selected from the partial current curves. By plotting both types of curves on the same scale, the arbitrariness in the method of selection of points produces no effect upon the magnitude of the corrected potentials.

Table III.—Resonance and Ionization Potentials for Potassium. Referring to figs. 7 and 8.

Partial Current.	Applied Potential at Resonance.		Resonance Potential.	Initial Potential.	Total Current.	Applied Potential at Ionization.	Ionization Potential.
Curve.	*a*.	*b*.	*b*−*a*.		Curve.	*c*.	*c*+Initial Potential.
1		1·0		2·2	5 ...	2·0	4·2
2		1·0			6 ...	2·0	4·2
3		1·0					
4	−0·6	1·0	1·6				
7	+0·45	2·0	1·55	1·05			
8	0·5	2·05	1·55		15...	2·9	4·0
10	0·5	2·10	1·60		16...	2·9	4·0
11	0·5	2·10	1·61				
12	0·5	1·90	1·40				
13	0·6	2·2	1 6				
14	0·5	2·0	1·5				
		Mean ...	1·55			Mean ...	4·1

Fig. 7.—Potassium. Total and Partial Current Curves.

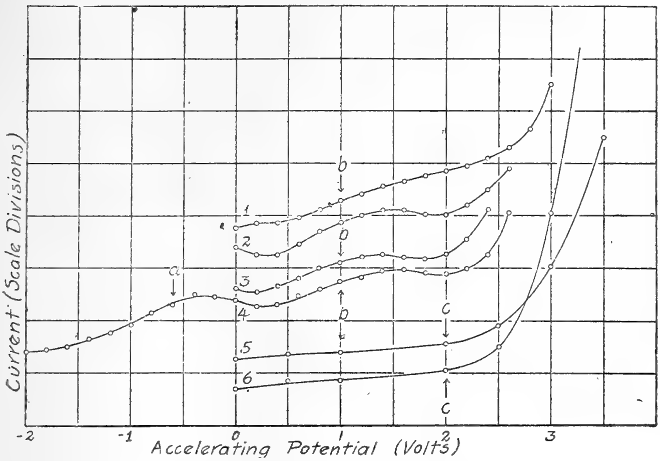


Fig. 8.—Potassium. Total and Partial Current Curves.

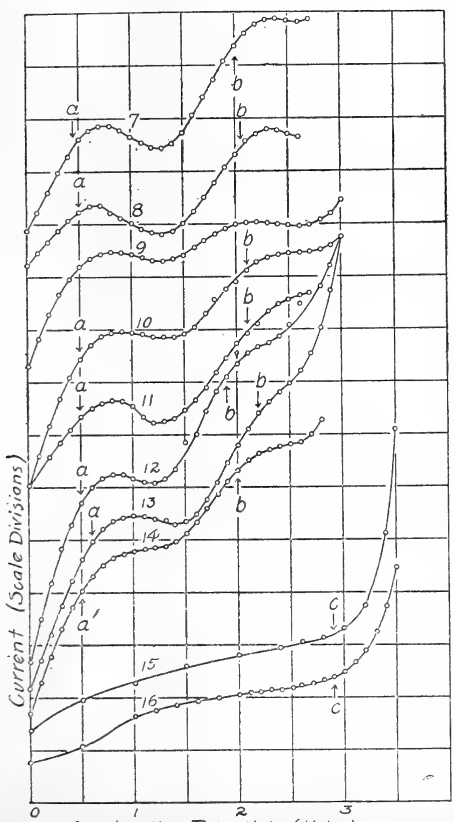


Table IV.

Resonance and Ionization Potentials for Zinc.
Referring to figs. 9 and 10.

Partial Current.	Applied Potentials at Resonance.							Resonance Potential.				Initial Potential.
Curve.	*a.*	*b.*	*c.*	*d.*	*e.*	*f.*	*g.*	*c–a.*	*d–b.*	*e–c.*	*f–d.*	
1	...	4·3	...	8·4	...	...	...	...	4·1	...	...	0
3	...	4·2	...	8·4	...	...	...	...	4·2	...	...	0
5	2·2	4·0	6·4	8·0	10·6	12·0	13·7	4·2	4·0	4·2	4·0	0
6	2·4	4·0	6·4	8·0	10·6	12·0	...	4·0	4·0	4·2	4·0	0
				Mean Resonance Potential=							4·1	

Total Current.	Ionization P.	
4	9·7	Mean Ionization Potential=9·5.
7	9·4	
8	9·4	

Fig. 9.—Zinc. Total and Partial Current Curves.

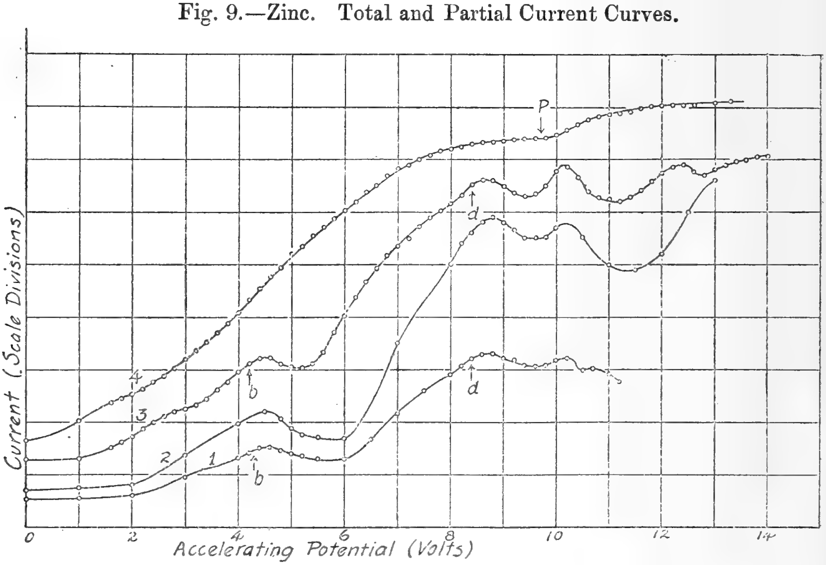
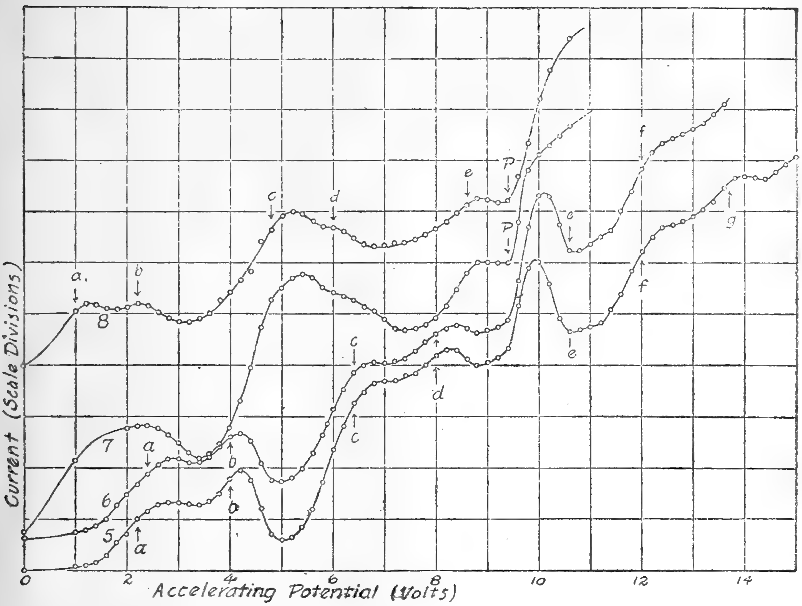


Fig. 10.—Zinc. Total and Partial Current Curves.



Conclusions.—The final results are grouped together in Table V.

TABLE V.

Summary of Resonance and Ionization Potentials.

Metal.	Resonance Potential.		Wave-length of Radiation.	Ionization Potential.		Limiting Wave-length of Series.
	Observed.	Calculated.		Observed.	Calculated.	
Cadmium ...	3·88	3·79	3260·17	8·92	8·95	1378·69
Sodium	2·12	2·10	5893·	5·13	5·13	2413·
Potassium...	1·55	1·60	7685·	4·1	4·32	2856·
Zinc	4·1	4·02	3075·99	9·5	9·35	1319·95

The agreement between the experimentally observed values and the values calculated from the relation $h\nu = eV$ is in all cases within the limits of experimental error. We have here further striking evidence of the fundamental correctness of deductions based upon Bohr's theory of atomic structure.

Bureau of Standards,
Washington, D.C.
November 24, 1917.

V. *On the Dynamics of the Electron.* *By* Megh Nad Saha, *M.Sc., Lecturer on Theoretical Physics, University College of Science, Calcutta* *.

MASS as a fundamental physical concept has been introduced into Physics by Newton's Second Law of Motion, which may be said to form the corner-stone of classical Mechanics. But in spite of its splendid success, physicists have always encountered some difficulty in realising mass as a fundamental physical concept in the same sense as the concepts of time and space. The fundamental object of mechanics is to provide a scaffolding by means of which the motion of material bodies can be surveyed and followed, when these are subjected to various disturbing influences. Some hypothesis must be introduced for taking into account the influence of these disturbing agencies. The question is: "Are Newton's Second Law of Motion and the ideas underlying it quite sufficient for all possible cases of motion, or are we to search for some more general principle?" Some physicists are in fact in favour of introducing Energy as a more fundamental physical concept than Mass, thereby basing the Science of Mechanics on various Energy-theorems.

So long as we hold to the principle of invariability of mass, there can of course be no question about the utility of the second law. But in the electron we have a physical entity which defies this limitation. If we want to survey its motion, and have no other means of doing so than classical mechanics, we must ascribe to it a certain mass, but for aught we know this mass is neither definite nor invariant during motion. Consequently the scaffolding which enables us to study and survey the motion of material (*i. e.* non-electrical) particles breaks down in this case. Some other system of Mechanics other than Newtonian must be formulated. In this attempt, we must remember that the electric charge is the only invariant physical quantity, consequently in place of mass, this quantity ought to appear in the equation of motion. We must also take cognisance of the newly discovered relations between time and space which are embodied in the Principle of Relativity.

I may be allowed to remark at this place that though the inadequacy of classical mechanics for studying the motion of electrons is now admitted on all hands, and many attempts are being made for formulating the exact dynamics of the

* Communicated by Prof. A. W. Porter, F.R.S.

electron,—the authors of many of these theories have not been able to rid themselves of the preconceived ideas of classical mechanics. I shall, in the first place, explain my own method, point out the characteristic features of my theory, and then compare it with other theories.

1.

The equations of motion of a material particle are derived from Newton's Second Law of Motion—rate of change of momentum is proportional to the force applied. Combining this principle with the principle of constancy of mass during motion, we obtain

$$m\frac{d^2x}{dt^2}=X, \quad m\frac{d^2y}{dt^2}=Y, \quad m\frac{d^2z}{dt^2}=Z.$$

The terms $m\frac{d^2x}{dt^2}$, $m\frac{d^2y}{dt^2}$, $m\frac{d^2z}{dt^2}$ are known as the components of the "Effective Force," and the law may be expressed by saying that the "Effective Force" is equivalent to the "Impressed Force."

In the case of the Electron, we hold to the axiom that the "Effective force is equivalent to the Impressed force." No *prima facie* reason can be given for the introduction of this hypothesis, just as in the case of the motion of material bodies. It is to be justified by its success in dealing with the problem at hand.

2.

The Impressed force on the electron can be easily calculated with the aid of Lorentz's Theorem of ponderomotive force. If (X, Y, Z) be the components of the electric field, (L, M, N) be the components of the magnetic field at any point, and ρ be the density of electricity, the components of the force per unit volume at the point are

$$\left.\begin{aligned} f_x&=\rho\left[X+\frac{1}{c}(v_2N-v_3M)\right] \\ f_y&=\rho\left[Y+\frac{1}{c}(v_3L-v_1N)\right] \\ f_z&=\rho\left[Z+\frac{1}{c}(v_1M-v_2L)\right] \end{aligned}\right\},$$

(v_1, v_2, v_3) being the components of the velocity with which the charge moves.

The rate at which work is done is given by the equation

$$f_t = f_x v_1 + f_y v_2 + f_z v_3$$
$$= \rho[\mathrm{X}v_1 + \mathrm{Y}v_2 + \mathrm{Z}v_3],$$

In accordance with the ideas of the Principle of Relativity we can write the components of the force-four-vector in the form

$$\left.\begin{aligned} f_x &= \rho_0[\quad\quad + f_{12}w_2 + w_3 f_{13} + w_4 f_{14}] \\ f_y &= \rho_0[f_{21}w_1 \quad\quad + w_3 f_{23} + w_4 f_{24}] \\ f_z &= \rho_0[w_1 f_{31} + w_2 f_{32} \quad\quad + w_4 f_{34}] \\ f_l &= \rho_0[w_1 f_{41} + w_2 f_{42} + w_3 f_{43} \quad\quad] \end{aligned}\right\} ; \quad . \quad . \quad (1)$$

these equations are obtained by writing *

$$\begin{aligned} f_{23}, f_{31}, f_{12} &\quad \text{for} \quad \mathrm{L, M, N}, \\ f_{14}, f_{24}, f_{34} &\quad \text{for} \quad -i(\mathrm{X, Y, Z}), \\ w_1, w_2, w_3, w_4 &\quad \text{for} \quad \frac{1}{\sqrt{1-v^2/c^2}}[v_1/c,\ v_2/c,\ v_3/c,\ i], \\ \rho_0 &\quad \text{for} \quad \rho\sqrt{1-v^2/c^2}. \end{aligned}$$

For finding out the total force on the electron, we have to integrate the above expressions for the force-four-vector over the whole volume of the electron. Supposing that the components of the electric and magnetic force do not vary throughout the volume of the electron, the force-components are obtained by writing simply (e) the invariant charge instead of (ρ_0) in equations (1).

3.

The calculation of the Effective force is a matter of some difficulty. The question is: "If an electron moves with a variable velocity, what are the terms corresponding to the quantities $\left(m\frac{d^2x}{dt^2},\ m\frac{d^2y}{dt^2},\ m\frac{d^2z}{dt^2}\right)$ in particle dynamics? Einstein solves the difficulty by saying that instead of the observer's time dt we have to introduce here the proper time

* The notation used throughout the paper is that of Minkowski, *vide Math. Ann.* vol. lxviii. p. 472 *et seq.* § 12, where this particular theorem occurs in an abbreviated form.

(Eigenzeit) of motion of the electron. This conclusion * is reached in a general way from his theory of the equivalence of the forms for equation of motion of material particles when referred to systems moving with uniform velocity past each other. Minkowski † practically uses the same hypothesis as I have done (Effective force is equivalent to the Impressed force), but in case of the electron he begins by implicitly ascribing a rest-mass to the electron. But the method adopted by me is fundamentally different, as will appear in due course. Elsewhere, Minkowski ‡ deduces it from the Principle of Least Action, combined with the principle of conservation of mass in a space perpendicular to the axis of motion. Besides, the investigation has a direct bearing on the theory of Electromagnetic momentum as developed by Lorentz and Abraham.

4.

Let us now concentrate our attention on a single electron moving with a velocity $\boldsymbol{v}$. The force components at an external point due to the motion of the electron are given by the equations (1). Generalizing, or rather recasting Maxwell's theorem of stresses into new forms, Minkowski has shown that the force components (f_x, f_y, f_z, f_l) can be put into the forms

$$\left.\begin{aligned} f_x &= \frac{\partial X_x}{\partial x} + \frac{\partial X_y}{\partial y} + \frac{\partial X_z}{\partial z} + \frac{\partial X_l}{\partial l} \\ f_y &= \frac{\partial Y_x}{\partial x} + \frac{\partial Y_y}{\partial y} + \frac{\partial Y_z}{\partial z} + \frac{\partial Y_l}{\partial l} \\ f_z &= \frac{\partial Z_x}{\partial x} + \frac{\partial Z_y}{\partial y} + \frac{\partial Z_z}{\partial z} + \frac{\partial Z_l}{\partial l} \\ f_l &= \frac{\partial L_x}{\partial x} + \frac{\partial L_y}{\partial y} + \frac{\partial L_z}{\partial z} + \frac{\partial L_l}{\partial l} \end{aligned}\right\} , \quad . \quad . \quad . \quad (2)$$

where

$$\left.\begin{aligned} X_x &= \frac{1}{8\pi}\left[f_{23}^2 + f_{34}^2 + f_{42}^2 - f_{12}^2 - f_{13}^2 - f_{14}^2\right] \\ X_y &= \frac{1}{4\pi}\left[f_{13}f_{32} + f_{14}f_{42}\right]. \end{aligned}\right\} \quad . \quad . \quad (3)$$

* A. Einstein, *Jahrbuch der Radioaktivität*, vol. iv. 1907.
† H. Minkowski, "Raum und Zeit," § iv. *Phys. Zeit.* 1911.
‡ H. Minkowski, *Math. Ann.* vol. lxviii. Appendix.

The theorem is proved by substituting, in equations (1), the values of $\rho_0 w_1$, $\rho_0 w_2$, $\rho_0 w_3$, $\rho_0 w_4$ obtained from the fundamental equation

$$\text{lor } f = 4\pi\rho_0(w_1, w_2, w_3, w_4)$$

and effecting the necessary transformations with the aid of the second fundamental equation

$$\text{lor } f^* = 0.$$

In the present case, the field is due to a single moving charge. The quantities $[X_x, X_y \ldots]$ can be easily calculated from the Potential four-vector **a**, for the six-vector f is equivalent to curl **a**.

In a paper * communicated some time ago to the Philosophical Magazine, I have shown that the Potential four-vector **a** at an external space-time point (x', y', z', l') due to the motion of a charge e occupying the point (x, y, z, l) is equivalent to $\frac{ew}{R}$, where

$$w = \text{velocity four-vector} = \left(\frac{dx}{ds}, \frac{dy}{ds}, \frac{dz}{ds}, \frac{dl}{ds}\right),$$

and R is the perpendicular distance from the external point on the line of motion of the electron. We have

$$R^2 = (x-x')^2 + (y-y')^2 + (z-z')^2 + (l-l')^2$$
$$+ [(x-x')w_1 + (y-y')w_2 + (z-z')w_3 + (l-l')w_4]^2.$$

We have now

$$f_{hk} = \frac{\partial \mathbf{a}_k}{\partial x_h} - \frac{\partial \mathbf{a}_h}{\partial x_k} \quad (h, k = 1, 2, 3, 4),$$

$$\therefore \quad f_{12} = \frac{\partial \mathbf{a}_2}{\partial x'} - \frac{\partial \mathbf{a}_1}{\partial y'} = \frac{\partial}{\partial x'}\left(\frac{ew_2}{R}\right) - \frac{\partial}{\partial y'}\left(\frac{ew_1}{R}\right)$$
$$= e(\alpha_1 w_2 - \alpha_2 w_1);$$

where

$$\alpha_1 = \frac{\partial}{\partial x'}\left(\frac{1}{R}\right), \quad \alpha_2 = \frac{\partial}{\partial y'}\left(\frac{1}{R}\right), \quad \alpha_3 = \frac{\partial}{\partial z'}\left(\frac{1}{R}\right), \quad \alpha_4 = \frac{\partial}{\partial l'}\left(\frac{1}{R}\right).$$

* It seems to have escaped the notice of investigators on this particular subject that the Potential four-vector in the form given by me is implicitly contained in a statement of Minkowski's ("Raum und Zeit," § 5). The passage came to my notice only recently when I was making a critical study of Minkowski's works.

Therefore we have

$$X_x = \frac{e^2}{8\pi}\left[(\alpha_2 w_3 - \alpha_3 w_2)^2 + (\alpha_3 w_4 - \alpha_4 w_3)^2 + (\alpha_4 w_2 - \alpha_2 w_4)^2 \right.$$
$$\left. - (\alpha_1 w_2 - \alpha_2 w_1)^2 - (\alpha_1 w_3 - \alpha_3 w_1)^2 - (\alpha_1 w_4 - \alpha_4 w_1)^2\right].$$

Now putting $\alpha^2 = \alpha_1^2 + \alpha_2^2 + \alpha_3^2 + \alpha_4^2$

and using the identity

$$\alpha_1 w_1 + \alpha_2 w_2 + \alpha_3 w_3 + \alpha_4 w_4 = 0,$$

we easily prove that

$$X_x = \frac{e^2}{8\pi}\left[-\alpha^2(1 + 2w_1^2) + 2\alpha_1^2\right].$$

Similarly

$$Y_y = \frac{e^2}{8\pi}\left[-\alpha^2(1 + 2w_2^2) + 2\alpha_2^2\right],$$

$$X_y = \frac{e^2}{4\pi}\left[-w_1 w_2 \alpha^2 + \alpha_1 \alpha_2\right], \quad \text{\&c.}$$

We shall now calculate the total force on the space exterior to the electron. According to the Principle of Relativity, this space must be uniquely defined. In our case, this space is perpendicular to the axis of motion of the electron, and is bounded on the inside by the surface of the electron. The external boundary is at an infinite distance. Let $d\Omega$ denote an element of volume of this space. Then the total force is given by

$$F_x = \int f_x d\Omega = \int \left[\frac{\partial X_x}{\partial x'} + \frac{\partial X_y}{\partial y'} + \frac{\partial X_z}{\partial z'} + \frac{\partial X_l}{\partial l'}\right] d\Omega. \quad . \ (4)$$

Now since $\mathbf{a}$, and consequently $f_{12}, f_{23} \ldots f_4 \ldots$, are functions of the relative distance $[(x-x'), (y-y'), (z-z'), (l-l')]$, we have

$$\frac{\partial X_x}{\partial x'} = -\frac{\partial X_x}{\partial x},$$

Therefore

$$F_x = -\left[\frac{\partial}{\partial x}\int X_x d\Omega + \frac{\partial}{\partial y}\int X_y d\Omega + \frac{\partial}{\partial z}\int X_z d\Omega + \frac{\partial}{\partial l}\int X_l d\Omega\right].$$

Now

$$\int X_x d\Omega = \frac{e^2}{8\pi}\int[-\alpha^2(1+2w_1^2)+2\alpha_1^2]d\Omega,$$

$$\int X_y d\Omega = \frac{e^2}{4\pi}\int[-w_1 w_2 \alpha^2 + \alpha_1\alpha_2]d\Omega, \quad \&c.$$

We have now to calculate the value of the integrals

$$\int \alpha_1^2 d\Omega, \quad \int \alpha_2^2 d\Omega, \quad \int \alpha_1\alpha_2 d\Omega, \quad \&c.$$

We have

$$\alpha_1 = \frac{\partial}{\partial x'}\left(\frac{1}{R}\right) = \frac{1}{R^3}[(x-x') + w_1\{(x-x')w_1 + (y-y')w_2 + (z-z')w_3 + (l-l')w_4\}].$$

Now let us introduce a Lorentz-transformation by means of which the axis of motion becomes the new time-axis. Let (ξ, η, ζ, ν) denote the new coordinates. We have then

$$\begin{pmatrix} (\xi-\xi') \\ (\eta-\eta') \\ (\zeta-\zeta') \\ (\nu-\nu') \end{pmatrix} = \begin{pmatrix} A_{11} & A_{12} & A_{13} & A_{14} \\ A_{21} & A_{22} & A_{23} & A_{24} \\ A_{31} & A_{32} & A_{33} & A_{34} \\ A_{41} & A_{42} & A_{43} & A_{44} \end{pmatrix} \begin{pmatrix} x-x' \\ y-y' \\ z-z' \\ l-l' \end{pmatrix},$$

where

$$\left.\begin{aligned} & A_{1k}^2 + A_{2k}^2 + A_{3k}^2 + A_{4k}^2 = 1, \\ \text{and} \quad & A_{1h}A_{1k} + A_{2h}A_{2k} + A_{3h}A_{3k} + A_{4h}A_{4k} = 0 \end{aligned}\right\}.$$

Since the line of motion is the new time-axis, we have

$$(\nu-\nu') = i[w_1(x-x') + w_2(y-y') + w_3(z-z') + w_4(l-l')];$$

we have therefore

$$A_{41} = iw_1, \quad A_{42} = iw_2, \quad A_{43} = iw_3, \quad A_{44} = iw_4.$$

Now using the above transformation, we have

$$R^2 = (\xi-\xi')^2 + (\eta-\eta')^2 + (\zeta-\zeta')^2$$

and

$$(x-x')+w_1[(x-x')w_1+(y-y')w_2+(z-z')w_3+(l-l')w_4]$$
$$=A_{11}(\xi-\xi')+A_{21}(\eta-\eta')+A_{31}(\zeta-\zeta')+A_{41}(\nu-\nu') - iw_1(\nu-\nu')$$
$$=A_{11}(\xi-\xi')+A_{21}(\eta-\eta')+A_{31}(\zeta-\zeta'),$$

for $A_{41}=iw_1$.

Then

$$\int \alpha^2 d\Omega = A_{11}^2\int\frac{(\xi-\xi')^2}{R^6}d\Omega + A_{21}^2\int\frac{(\eta-\eta')^2}{R^6}d\Omega + A_{31}^2\int\frac{(\zeta-\zeta')^2}{R^6}d\Omega + 2A_{11}A_{21}\int\frac{(\xi-\xi')(\eta-\eta')}{R^6}d\Omega + \ldots$$

Now we have, since the integration extends over the space internally bounded by the sphere

$$(\xi-\xi_0)^2+(\eta-\eta_0)^2+(\zeta-\zeta_0)^2=a^2,$$

$$\int\frac{(\xi-\xi')^2}{R^6}d\Omega=\int\frac{(\eta-\eta')^2}{R^6}d\Omega=\int\frac{(\zeta-\zeta')^2}{R^6}d\Omega=\tfrac{1}{3}\int\frac{d\Omega}{R^4},$$

and from symmetry,

$$\int\frac{(\xi-\xi')(\eta-\eta')}{R^6}d\Omega=0.$$

Now we have
$$\int_\infty^a\frac{d\Omega}{R^4}=-\frac{4\pi}{a},$$

$$\therefore \int \alpha_1^2 d\Omega = -\frac{4\pi}{3a}[A_{11}^2+A_{21}^2+A_{31}^2]=-\frac{4\pi}{3a}(1+w_1^2),$$

and
$$\int \alpha_1\alpha_2 d\Omega = -\frac{4\pi}{3a}[A_{11}A_{21}+A_{12}A_{22}+A_{13}A_{23}]$$
$$=-\frac{4\pi}{3a}[A_{14}A_{24}]=-\frac{4\pi}{3a}w_1w_2.$$

$$\therefore X_x=-\frac{e^2}{8\pi}\left[-\frac{4\pi}{a}(1+2w_1^2)+2\cdot\frac{4\pi}{3a}(1+w_1^2)\right]$$
$$=\frac{2e^2}{3a}(\tfrac{1}{4}+w_1^2),$$

and
$$X_y=-\frac{e^2}{4\pi}\left[-w_1w_2\cdot\frac{4\pi}{a}+\frac{4\pi}{3a}\cdot w_1w_2\right]=\frac{2e^2}{3a}w_1w_2.$$

Then we have similarly

$$\left.\begin{array}{ll} Y_y = \frac{2e^2}{3a}\left(\frac{1}{4}+w_2^2\right), & Z_z = \frac{2e^2}{3a}\left(\frac{1}{4}+w_3^2\right), \\ L_l = \frac{2e^2}{3a}\left(\frac{1}{4}+w_4^2\right), & X_l = \frac{2e^2}{3a} w_1 w_4, \quad \&c. \end{array}\right\} . \quad (5)$$

Now we have

$$F_x = -\frac{2e^2}{3a}\left[\frac{\partial}{\partial x}\left(\tfrac{1}{4}+w_1^2\right)+\frac{\partial}{\partial y}(w_1w_2)+\frac{\partial}{\partial z}(w_1w_3)+\frac{\partial}{\partial l}(w_1w_4)\right],$$

i. e.

$$F_x = -\frac{2e^2}{3a}\left[\left(w_1\frac{\partial}{\partial x}+w_2\frac{\partial}{\partial y}+w_3\frac{\partial}{\partial z}+w_4\frac{\partial}{\partial l}\right)w_1 \right.$$
$$\left. + w_1\left(\frac{\partial w_1}{\partial x}+\frac{\partial w_2}{\partial y}+\frac{\partial w_2}{\partial z}+\frac{\partial w_4}{\partial l}\right)\right].$$

The second term $= 0$ from the condition $\text{Div}\,\mathbf{a} = 0$, for this gives

$$\frac{\partial}{\partial x}\left(\frac{ew_1}{R}\right)+\frac{\partial}{\partial y}\left(\frac{ew_2}{R}\right)+\frac{\partial}{\partial z}\left(\frac{ew_3}{R}\right)+\frac{\partial}{\partial l}\left(\frac{ew_4}{R}\right)=0,$$

i. e. $$\frac{1}{R}\,\text{Div}\,w+(w_1\alpha_1+w_2\alpha_2+w_3\alpha_3+w_4\alpha_4)=0,$$

from which $$\text{Div}\,w = 0,$$

for the last term is identically zero.

The X-component of the force on the external space

$$= -\frac{2e^2}{3a}\frac{d^2X}{ds^2}, \text{ for } \frac{d}{ds}=w_1\frac{\partial}{\partial x}+w_2\frac{\partial}{\partial y}+w_3\frac{\partial}{\partial z}+w_4\frac{\partial}{\partial l}. \quad (6)$$

We may interpret this force as the reaction of the electron on the external space, which is supposed for purposes of substantiation to be composed of æther. The effective force on the electron is equal and opposite to this force, and has therefore the components

$$\frac{2e^2}{3a}\frac{d^2x}{ds^2}, \quad \frac{2e^2}{3a}\frac{d^2y}{ds^2}, \quad \frac{2e^2}{3a}\frac{d^2z}{ds^2}, \quad \frac{2e^2}{3a}\frac{d^2l}{ds^2},$$

(N.B.—We have for small velocities $ds = cdt$ approximately,

$$\therefore \frac{2}{3}\frac{e^2}{a}\frac{d^2x}{ds^2} = \frac{2}{3}\frac{e^2}{ac^2}\frac{d^2x}{dt^2}, \text{ \&c.}$$

We therefore observe that the quantity $\frac{2}{3}\frac{e^2}{ac^2}$ plays here the same part as the mass m_0. We can therefore call $\frac{2}{3}\frac{e^2}{ac^2}$ the rest-mass of the electron, and put it equivalent to m_0.)

5.

Now a few remarks on the equations (2). These were first introduced into Mathematical Physics by Maxwell about 1865. Ever since their introduction, various efforts have been made by different investigators for getting something out of them, and in certain cases they have yielded very valuable information, and led to many important results. We may cite for example, Maxwell's prediction of the existence of Radiation Pressure. The close analogy of the equations (2) with the equations of elasticity led Maxwell to propose his famous theory of "Stresses," *i. e.* to imagine that the electric forces are due to a distribution of the stresses (X_x, X_y ...) in æther, which behaves in this case like an elastic solid. But this theory is fraught with many difficulties, which have been pointed out from time to time by several investigators *. In a paper † communicated to the Phil. Mag., the author observed that though the forces can be well accounted for, the Energy of Electrification cannot be accounted for on Maxwell's hypothesis.

Another direction in which the equations (2) have been exploited is the subject of Electromagnetic mass of an electron. When an electron moves with a certain velocity, it creates round it an electric as well as a magnetic field. We can say with Maxwell that the energy is stored in the æther, and the electron by its motions exerts a force on every particle of æther.

If we now integrate this force over the whole space ‡

* Maxwell, 'Electricity and Magnetism,' third edition, vol. i. chap. v., footnote p. 165.

† Phil. Mag. March 1917.

‡ N.B. This space is the absolute space of the Pre-Relativity Period.

exterior to the electron, the first three terms involving $\left(\frac{d}{dx}, \frac{d}{dy}, \frac{d}{dz}\right)$ can be reduced to a surface-integral. The bounding surface is taken to be at an infinite distance, thereby the surface integrals are made to vanish. The total force on the æther thus comes out in the form

$$\frac{dM}{dl} = \frac{1}{lc}\frac{\partial M}{\partial t},$$

Now assuming that the force exerted by the æther on the electron is equal and opposite to the force exerted by the electron on the æther, the reaction of æther on the electron becomes equivalent to $-\frac{1}{lc}\frac{\partial M}{\partial t}$. In analogy with Classical Mechanics, we can call $\left(\frac{iM}{c}\right)$ a momentum.

This is, in brief, the theory of Electromagnetic momentum as developed by Abraham, Lorentz *, and others. We do not enter into a discussion of the rival theories of Lorentz and Abraham on the shape of the electron during motion. The Electromagnetic mass is obtained from either of the relations $m_t = \frac{iM}{cv}$, and $m_l = i\frac{\partial M}{c\partial v}$, m_t and m_l denoting respectively the transverse and longitudinal masses of the electron.

But several objections can be raised to this theory of Electromagnetic momentum. In the first place, the integration is extended over the space of the observer, whereas the Principle of Relativity requires that it should be extended over the space perpendicular to the axis of motion of the electron, and external to the volume occupied by the electron. This is what I have done in the foregoing, and I believe that this is quite in keeping with Minkowski's ideas of equivalence of time and space. Secondly, the volume of integration is supposed to be bounded by a sphere at an infinite distance only, and no notice is taken of the internal boundary which must coincide with the surface of the electron. In fact, it looks as if the surface-integrals had to go, because the authors wanted to get rid of them.

In the theory proposed by me, I have refrained from putting any interpretation on the quantities (X_x, X_y ...).

* Lorentz, 'Theory of Electrons,' chap. 1, § 26 *et seq.*

Taking the theorem as it is, the total effective force on the æther has been obtained by integrating f over the whole space perpendicular to the axis of motion of the electron, the space being bounded on the inside by the surface of the electron. The "Effective force" on the electron has been taken to be equal and opposite to this force.

I may be allowed to point out here that this procedure by by no means confers substantiality upon the æther. It is a fictitious creation, introduced for the sake of arriving at a result which, from its very nature, can be attempted only by indirect means.

It is remarkable that none of the quantities $\int X_x d\Omega$, &c. vanish in this case, as in the other theories. *The "Effective" force on an electron, instead of simply being the rate of change of "Momentum" becomes the sum total of the time-rate of change of the quantity* $\int \frac{\partial X_l}{\partial l} d\Omega$ *plus the space-rates of changes of the quantities* $\int \frac{\partial X_x}{\partial x} d\Omega$, $\int \frac{dX_y}{\partial y} d\Omega$, ...

These latter quantities involve "velocity" in the *second order, whereas* $\int \frac{\partial X_l}{\partial l} d\Omega$ *involves "velocity" in the first order,* so that when *the velocity is a small fraction of the velocity of light, the theorem approximates to Newton's Second Law of Motion.*

The rest-mass calculated on this basis is equivalent to $\frac{2}{3}\frac{e^2}{ac^2}$, and as such coincides with the value obtained by Sir J. J. Thomson for slow-moving electrons, and with that obtained by Lorentz and Einstein. The variation of mass with velocity is determined by the Principle of Relativity as in the theories of Lorentz and Einstein.

In conclusion, I wish to express my thanks to my friend and colleague Mr. Satyendra Nath Basu, M.Sc., for much help and useful criticism.

Calcutta University College of Science,
Physical Dept., July 10, 1917.

VI. *On the Value of the Conductivity of Sea-water for Currents of Frequencies as used in Wireless Telegraphy.* By BALTH. VAN DER POL, Jun., *Docts. Sc.* (*Utrecht*)*.

THE materials of the earth's crust, over and through which the electromagnetic waves sent out by a wireless telegraphy station travel, have an important influence on the variations of the wave amplitude with distance from the sending station. This fact was first found experimentally, but was afterwards confirmed by some theoretical investigations †.

When a plane bounding surface is assumed to exist between the air and the earth, the magnitudes of the conductivity of most materials of which the earth's crust consists are such, that for the range of wave-lengths used in wireless telegraphy and for the dielectric constants these materials possess, apart from the divergence diminution, another decrease of wave amplitude with distance, due to absorption, can in general be expected. This latter diminution is principally determined by the conductivity of the materials over which the waves travel.

When, on the other hand, the above-mentioned boundary surface, in closer approximation to the actual circumstances, is supposed to be a sphere, consisting of sea-water, and if the value of the conductivity of the latter as found under direct or slowly alternating currents is used in the calculations‡, it appears that a *greater* wave amplitude for the same sender can be expected than would be obtained if the sphere were made up of an infinitely good conducting material, though the difference is small.

The greater part of wireless traffic being conducted over sea, an exact determination of the value of the conductivity of sea-water for alternating currents of high frequencies may be of importance, especially in connexion with the divergence between the decrease of wave amplitude with distance predicted by theory and values of the latter found experimentally.

That the conductivity of all materials is independent between wide limits of the frequency of the currents in

* Communicated by Sir J. J. Thomson.

† Zenneck, *Ann. d. Phys.* Bd. xxiii. p. 846 (1907). Sommerfeld, *Ann. d. Phys.* Bd. xxviii. p. 665 (1909).

‡ Macdonald, Proc. Roy. Soc. (Ser. A). vol. xcii. p. 493 (1916). Love, Roy. Soc. Phil. Trans. (Ser. A) vol. ccxv. p. 105 (1915). See also a paper shortly to be published in the Proc. Roy. Soc. by G. N. Watson.

them cannot generally be said, though the researches of Sir J. J. Thomson on the conductivity of electrolytes under very rapidly alternating currents of frequencies up to 10^8 * would lead one to expect the conductivity of sea-water to be constant within the range of frequencies between 0 and 10^6.

Professor J. A. Fleming and Mr. G. B. Dyke, on the other hand †, found the conductivity of various materials as glass, celluloid, paraffin-wax, mica, paper, slate, and sulphur to be a function of the frequency and increasing with the latter, so for instance they found the conductivity of ebonite under 4600 cycles per second 6·4 times greater than under 920 cycles. For gutta-percha under a frequency as low as 800 a conductivity was obtained already 100,000 times greater than the value usually given for direct currents in the handbooks, while at higher frequencies this ratio still increased‡. Professor Fleming therefore suggested to me to determine the conductivity of sea-water under frequencies as used in wireless telegraphy, and to compare it with the value found under steady or slowly alternating currents.

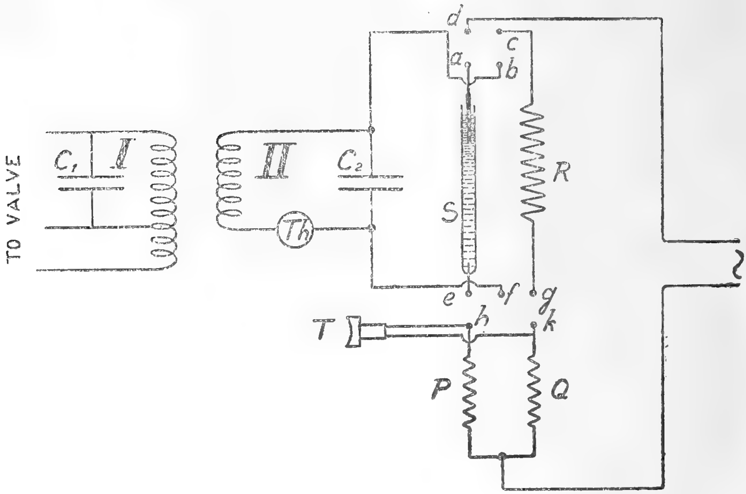
A simple calculation shows, assuming the conductivity of sea-water has its normal value $\sigma \sim 5 \,.\, 10^{-11}$, that up to the frequency 10^6 the dielectric displacement current in sea-water (the dielectric constant being assumed $\epsilon = 81$) can be neglected in comparison with the conduction current, the ratio of the two being $\frac{\epsilon p}{4\pi\sigma c^2}$, where p is the angular frequency and c the velocity of light. For the material under consideration this ratio amounts to $\frac{81 \,.\, 10^6}{4\pi \,.\, 5 \,.\, 10^{-11} \,.\, 9 \,.\, 10^{20}} = 1{\cdot}4 \,.\, 10^{-4}$; so that in order to measure the conductivity of sea-water a method can be applied in which the latter is treated as only having a resistivity.

To use a bridge method as employed by Fleming and Dyke (modified Wien bridge) did not seem advisable for high frequencies, as serious errors introduced by the inductive effects of the arms had to be avoided. The following substitution method appeared to be very reliable and exact (see fig. 1).

* J. J. Thomson, Roy. Soc. Proc. vol. xlv. p. 269 (1889).
† Journal Inst. Electr. Eng. London, vol. xlix. p. 323 (1912).
‡ See paper cited.

I is an oscillatory circuit in which high-frequency currents are generated by the aid of a three-electrode vacuum-tube. These currents induce oscillations in circuit *II* which is tuned to *I* by aid of the Duddell thermo-galvanometer Th, having a heater resistance of 3·8 Ω. By means of the paraffin switches

Fig. 1.

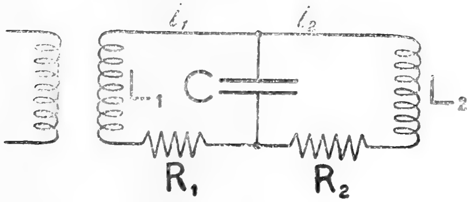


a b c d and *e f g h k* either S or R can be connected across the terminals of the condenser C_2. R consists of a fine constantan wire about 12 metres long having a diameter of about ·025 millimetre mounted zig-zag in such a way that the distances between the eight parallel parts (each of 138 centimetres length) were 26 millimetres. This form was given to the wire in order to avoid parts at appreciably different potentials being near together, so that a minimum chance was present for dielectric currents shunting parts of the wire. The other shunt S across the terminals of the condenser C_2 is made up of a glass tube (5·8 millimetres diameter and 170 centimetres long) filled with an electrolyte the conductivity of which is to be determined. Of the two platinum electrodes in S the lower one was fixed while the top one could be moved to a greater or less depth in the electrolyte.

The experiments were carried out as follows. For a constant coupling between *I* and *II* first a reading of the galvanometer Th was taken with the constantan wire resistance across the terminals of C_2 (connexions between *b*–*c* and *f*–*g*). This resistance was then replaced by the tube filled with electrolyte by connecting *a–b* and *e–f*. The length of the path of the current through the latter was then altered by moving the top electrode till an equal deflexion of Th was obtained. The high-frequency resistance of S was then equal to that of R. Immediately afterwards, in order to avoid changes in temperature, connexions were made between *d–a–c*, *e–h*, and *g–k* so that S and R formed two branches of an ordinary Wheatstone bridge, the remaining two branches being P and Q, one of which was variable. A slowly alternating current of about 90 cycles per second, drawn from a small transformer fed by the town supply, was afterwards sent through the bridge, which was balanced by aid of the telephone T. The ratio of the resistances R and S for slowly alternating currents at once gave the ratio of the resistance of the electrolyte for high-frequency currents to the resistance for low-frequency or direct currents.

A word may be said about the accuracy and the possible errors inherent to this method. The resistance of the constantan wire is assumed to be the same for high and low frequencies. The skin effect in such a fine wire of a comparatively high specific resistivity does not alter the apparent resistance for a frequency of 10^6 more than one part in a million compared with the value for steady currents, and the same reasoning can be applied to the tube filled with sea-water. It is further well known that the conductivity of metals is independent of the frequency in the range of 0 to 10^6, so that no error is introduced by assuming the resistance of the constantan wire to be constant over the range of frequencies used. The unavoidable differences of self-induction in S and R have only a very small influence on the results, as can be seen from the following consideration.

Fig. 2.



With the notation of fig. 2 the currents i_1 and i_2 in the main circuit and the shunt across the condenser C respectively have to satisfy the equations

$$\left.\begin{aligned} L_1\frac{di_1}{dt}+R_1i_1+\frac{1}{C}\int(i_1+i_2)dt&=-E\cos\omega t,\\ L_2\frac{di_2}{dt}+R_2i_2+\frac{1}{C}\int(i_1+i_2)dt&=0,\end{aligned}\right\}\quad . \quad . \quad (1)$$

where $E\cos\omega t$ is the impressed E.M.F. For simplification we take two special cases; when

A. $L_1=L_2$ and under resonance condition

$$(L_1C_1=L_2C_2=\omega^{-2})$$

the solutions are, leaving out the transients,

$$\left.\begin{aligned} i_1&=-\frac{CR_2}{R_1R_2C+L_1}\,.\,E\cos\omega t,\\ i_2&=\frac{\frac{1}{\omega}}{R_1R_2C+L_1}\,.\,E\sin\omega t.\end{aligned}\right\}\quad . \quad . \quad . \quad (2)$$

B. If, on the other hand, when the shunt has no appreciable inductance, L_2 in (1) is assumed to be zero, we get the following expression for the currents under resonance condition:

$$\left.\begin{aligned} i_1&=-E\left[R_2C\frac{(L_1+CR_1R_2)+L_1\frac{R_1}{R_2}}{\Delta}\cos\omega t+\frac{\omega CL_1^2}{\Delta}\sin\omega t\right],\\ i_2&=E\left[\frac{\frac{R_1}{\omega^2}}{\Delta}\cos\omega t+\frac{\frac{R_1R_2C+L_1}{\omega}}{\Delta}\sin\omega t\right],\\ \text{where}\quad &\Delta=(L_1+CR_1R_2)^2+\frac{R_1^2}{\omega^2}.\end{aligned}\right\}\quad (3)$$

When now the resistance R_1 in the main branch of the oscillatory circuit is decreased till it finally reaches the ideal value $R_1=0$, we see that in both cases A : $L_2=L_1$ and B : $L_2=0$ the current i_2 in the shunt circuit reaches the asymptotic value

$$i_2=\frac{E}{\omega L_1}\sin\omega t,$$

being *independent* of the resistance R_2 of the shunt across the condenser.

On the other hand, with the same ideal approximation i_1 takes the values:

A: for $L_2=L_1$,

$$i_1=-\frac{CR_2}{L_1}E\cos\omega t, \quad \ldots\ldots\ldots \quad (4)$$

and

B: for $L_2=0$,

$$i_1=-\frac{CR_2}{L_1}E\cos\omega t-\omega CE\sin\omega t. \quad \ldots \quad (5)$$

The latter expression (5) approaches the former one (4) as soon as $\frac{L_1\omega}{R_2}$ is small, this being the case in the experiments, and i_1 becomes proportional to the resistance of the shunt, the value to be measured.

This fact at once suggested the insertion of the thermo-galvanometer in the main branch of the oscillatory circuit, instead of in the shunt; and as R_1 in the experiments was kept very low, it follows at once that the deflexions of the thermo-galvanometer being proportional approximately to i_1^2, and therefore to the second power of R_2, furnish a very exact means of comparing the high-frequency resistances of different shunts. Moreover, (4) and (5) prove the approximate independency of i_1 on the value of the self-inductance L_2 of the shunt, within the limits $L_2=0$ and $L_2=L_1$, this fact being confirmed experimentally. The insertion of an extra self-inductance of a coil of six adjacent turns of diameter 15 centimetres in the shunt circuit did not alter the galvanometer deflexion, while, on the other hand, a variation of the length of the path of the current through the tube filled with the electrolyte of 1 millimetre, already gave a considerable variation of deflexion.

The results of the measurements of the conductivity of sea-water, taken from a sample obtained from Hastings, are as follows. When the conductivity for very slowly alternating currents is called σ_∞ and σ_x represents the conductivity for a current of a frequency corresponding to a wave-length of x metres, it was found (each σ being taken as the mean of several observations)

$$\sigma_{3400}=1{\cdot}001\,\sigma_\infty$$

$$\sigma_{1870}=0{\cdot}999\,\sigma_\infty$$

$$\sigma_{1070}=1{\cdot}002\,\sigma_\infty$$

$$\sigma_{600}=1{\cdot}003\,\sigma_\infty$$

$$\sigma_{275}=1{\cdot}005\,\sigma_\infty.$$

These values show that the conductivity of sea-water for all frequencies as used in wireless telegraphy is very nearly equal to the value of the same for steady currents to within less than half a percent., the small differences obtained most probably being due to a small capacity effect of parts of the shunt on other parts.

A calculation of the true direct current conductivity from the resistance and the dimensions of the tube yielded the value

$$\cdot 0377\ \Omega^{-1} \text{ per centimetre cube}$$

at 12°·5 C. corresponding to

$$\sigma = 3{\cdot}77 \,.\, 10^{-11} \text{ electromagnetic unit.}$$

As the conductivity varies very much with the temperature and the origin of the sample, a value of σ between 1 and $5 \,.\, 10^{-11}$ is therefore appropriate to form the numerical basis for the theory of propagation of electromagnetic waves over the surface of the sea.

In conclusion I wish to thank Sir J. J. Thomson for his valuable advice and kindness in putting at my disposal the instruments of the laboratory.

Cavendish Laboratory,
Cambridge.

VII. *General Relativity without the Equivalence Hypothesis.* *By* L. Silberstein, *Ph.D., Lecturer in Natural Philosophy at the University of Rome* *.

1. *Purpose and scope of the present inquiry.*

THE generalized theory of relativity as proposed by Einstein in 1912, and since that time repeatedly modified by himself and by his followers, has one very strong point, the requirement of general covariance of all physical laws, and one weak point, to wit, the originally so-called "equivalence hypothesis" which places gravitation on an entirely exceptional and privileged footing, bringing it into intimate connexion with the fundamental tensor which appears in the line-element of the world. I propose to retain the strong point and to reject the weak one, and thus to develop the implications of the general principle of relativity without the equivalence hypothesis, in fact, without privileging gravitation at all. This is the purpose of the

* Communicated by the Author.

present paper. And its scope will be a limited one, viz. to treat only some chief aspects of the physical implications of the Principle and to illustrate them to a certain extent, but by no means to try to embrace in a few generally covariant formulæ all the marvels of Nature. The strength of the "strong" point is indisputable and does not call for lengthy remarks; it amounts, in its ultimate analysis, to claiming that real, phenomenal, contents should be expressed, or expressible at least, in a way showing their independence of the particular language or scaffolding adopted. The weakness of the "weak" point, however, does require some explanations. First of all, then, independently of agreement or disagreement with experimental facts, the equivalence hypothesis is a vulnerable point because of its very special nature and of the great number of assumptions which it tacitly implies. In the next place, however, serious doubts with regard to its acceptability arise, according to my opinion, from the obstinately negative results quite recently obtained by St. John at the Mount Wilson Observatory*. The mean displacement (which according to Einstein should be about $\frac{1}{100}$ Å.U. towards the red) is at the sun's centre for 25 lines $-\cdot 001$, and for 18 lines $+\cdot 0014$ Å.U., with a mean of zero for the 43 lines in the band spectrum of nitrogen (cyanogen); again, the mean displacement at the limb is $0\cdot 000$ for 17 lines, and $+0\cdot 0063$ for 18 lines, with a mean of $+\cdot 0018$ for the 35 lines. The final conclusion is that "within the limits of error there is no evidence of a displacement to longer wave-length, either at the centre or at the limb of the sun, of the order ·008 Å." (*loc. cit.* p. 265). This negative result certainly outweighs the much more dubious and less numerous figures quoted in 1914 by E. Freundlich (*Phys. Ztschr.* xv. p. 370) in favour of Einstein's prediction. As a matter of fact, Einstein himself, while mentioning Freundlich's star-spectra testimony, is of the opinion that "a final verification is still [1916] outstanding" (*Ann. d. Physik,* xlix. p. 820). Notice also that the masses of Freundlich's stars can only be guessed in a rough manner while our sun's mass is sufficiently well known ($M/c^2 \doteqdot 1\cdot 5$ km.) to be substituted into Einstein's shift formula. Whence the obvious superiority of St. John's results. It is well-known that the predicted shift was one of the most immediate consequences of the equivalence hypothesis, even in its original form of 1911; in Einstein's recent theory

* Charles E. St. John, "The Principle of general Relativity and the Displacement of Fraunhofer Lines, etc.," Astrophys. Journ. xlvi., Nov. 1917, p. 249.

that shift or the decrease of vibration frequency is directly embodied in the tensor component g_{44}, the coefficient of c^2dt^2, produced by the gravitating centre, viz., in a first approximation,

$$g_{44}=1-\frac{2M}{c^2r}.$$

Another consequence of the theory, the bending of light rays by a gravitating mass (represented by the above in conjunction with other tensor components) still awaits its verification. Hitherto there is not the slightest evidence for the reality of such a phenomenon. It constitutes one of the chief points of the programme for the 1919 Solar Eclipse Expedition which seems particularly favourable for the proposed observations, as was pointed out by Sir Frank Dyson. Thus far the only positive and, one must confess, very conspicuous and fascinating success of Einstein's theory is the formula it gives without difficulty for the perihelion motion, amounting for instance in the case of Mercury to 43″ per century, the famous excess which has occupied the attention of astronomers since the times of Le Verrier. But it so happens that this remarkable result relating to the secular motion of the perihelion is most vitally conditioned by the same tensor component, g_{44}*, which—to everybody's true regret—has thus discredited itself at the Mount Wilson Observatory.

Such being the state of things, one is justified, if not in condemning the equivalence hypothesis, at least in doubting its validity and in not attributing to it anything like the importance one cannot help ascribing to the general principle of relativity. Whence the natural desire of the writer to draw a sharp line between the two utterly heterogeneous elements and, rejecting the former, to investigate some general physical problems from the point of view of the latter alone. It may be well to notice that the importance and utility of the requirement of general covariance has been felt, and expressed with much force, by the mathematicians many years ago and in a much wider field, viz. that of "the geometry" of manifolds of any number of dimensions as represented by quadratic differential forms,—of which the physicist's "world" or space-time is but a particular example. Is not

* The perihelion motion, as a delicate feature of planetary motion, is, of course, given by g_{44} in co-operation with the remaining coefficients. To drop the variable part of g_{44} itself, after the others have been neglected, would make it impossible to get even the ordinary Keplerian planetary motion. In short, the second term of g_{44} is the chief term involved in that motion.

this need—of studying properties intrinsically connected with the manifold itself and of developing appropriate methods—clearly and with much emphasis expressed in such purely mathematical tracts as that * of Prof. Wright? But a strong tendency of that kind, together with great expectations for the future, manifests itself even in the old book of Lamé on curvilinear coordinates (1859); see the concluding paragraph, p. 367, of these memorable Lessons.

2. *Space-time (world) of any constant curvature.*

What is here called curvature is a certain invariant of the manifold and, as such, an intrinsic property of the manifold, as real as and possibly more real than the mass of a lump of matter. Whatever its value, nil, positive, or negative, it cannot be settled either by mere reasoning or by convention, but has to be found out by experiment or observation. Being ignorant as to its sign or amount, the best way is to leave it undetermined and to develop all formulæ with the corresponding degree of generality. Its evaluation is the task of the future physicist or, more likely, the astronomer. On the other hand, the reason why it is enough to limit oneself to *constant* curvature, *i. e.* the same through all times and everywhere, will be readily seen. Again, as concerns the mathematical technicalities, it is almost as easy to study a four-manifold of any constant curvature as a non-curved or homaloidal one (from ὁμαλός = even; an old name for flat or Euclidean space, of any number of dimensions). To deprive ourselves of generality would thus be a badly compensated sacrifice.

Let x_1, x_2, x_3, x_4, the first three space-like, and the fourth time-like, be any coordinates fixing a world-point, and let the invariant line-element, determining the metric properties of the four-dimensional manifold, be given by the quadratic differential form

$$ds^2 = \Sigma\Sigma\, g_{ij} dx_i dx_j,$$

where $g_{ij} = g_{ji}$ are, in general, some functions of all the coordinates. If we pass to any other system x_i', then the new g_{ij}' will be linear homogeneous functions of the g_{ij}, viz. in the usual abbreviated notation †,

$$g_{ij}' = \frac{\partial x_\kappa}{\partial x_i'} \frac{\partial x_\lambda}{\partial x_j'} g_{\kappa\lambda}.$$

* Cambridge Tracts, No. 9; *cf.* especially pp. 3–4.

† In which the sum signs are omitted, the tacit prescription being that the sum, from 1 to 4, is to be taken over each term in which a suffix occurs at least *twice.*

The g_{ij} thus constitute what is called a (covariant) *tensor* of rank two, viz. a symmetrical one, since $g_{ij}=g_{ji}$. Any array or matrix of 4×4 constituents a_{ij} which are transformed according to the same rule is a covariant tensor of the second rank. The differentials of the four coordinates themselves, which are transformed into

$$dx_i'=\frac{\partial x_i'}{\partial x_j}\,dx_j,$$

constitute a contravariant tensor of rank one or a four-vector. The reader is supposed to be acquainted with these and higher tensors and with their transformational properties*. Here, therefore, it will be enough to recall that the importance of all tensors consists in the linearity and homogeneity of their transformation formulæ; whence, if all the constituents of a tensor vanish in one system, they will vanish also in any other system of coordinates (provided, of course, that $\partial x_\kappa/\partial x_\iota'$ are not infinite). Thus, if a physical law is written entirely in tensors, it will retain its form in passing from one system of reference to any other. Tensors, and tensors only, thus furnish the material for writing down such laws. (This does not imply, of course, that they necessarily will, but only that they may be obeyed by Nature.) The fundamental tensor, g_{ij}, will manifestly play a prominent part.

Now, to come to our subject. In Einstein's theory the tensor g_{ij} is intimately connected with gravitation so that the latter codetermines the metrical properties of the world or space-time. If there is no gravitation, or as we will say, far away from heavy masses and disregarding the feeble contribution due to electromagnetic and other energy, Einstein's world, at least that of 1916, is Euclidean or homaloidal, amounting to $ds^2=-dx_1^2-dx_2^2-dx_3^2+dx_4^2$, $x_4=ct$, or to

$$g_{11}=g_{22}=g_{33}=-1,\quad g_{44}=1 \text{ (others zero)}. \quad . \quad . \quad (a)$$

In presence of gravitation this is changed. To make things plain by an illustration, suppose there is but one conspicuous body in the universe, say, our sun of mass M, the gravitational contribution of a testing particle or "planet" being negligible. Then, far away from the sun, and the farther

* Those readers who are not familiar with the subject can inform themselves in the easiest way by reading §§ 5-13 of Einstein's paper, *Ann. d. Physik*, xlix. (1916), and Chaps. I. and II. of Wright's 'Invariants,' Cambr. Tract No. 9 (1908).

the more exactly, the tensor is as in (*a*). On approaching the sun we have instead, sensibly,

$$g_{44}=1-\frac{2\alpha}{r},\quad g_{11}=-1-\frac{2\alpha}{r}\frac{x_1^2}{r^2},\quad g_{12}=-\frac{2\alpha}{r}\frac{x_1x_2}{r^2},\ \text{etc.}^*,\ .\quad (b)$$

where $r^2=x_1^2+x_2^2+x_3^2$ and $\alpha=M/c^2$, which is about 1·5 km.; and this departure from the previous tensor cannot, of course, be transformed away; the change due to the sun is an essential one, a change of the metric properties all around that body. The eqs. of motion of a particle which in absence of the sun were given by the geodesic $\delta\int ds=0$ with the tensor (*a*), expressing uniform motion, are now again given by the geodesic $\delta\int ds=0$ with the modified tensor (*b*), however. It is this system of eqs. which yielded the remarkable result concerning the motion of the perihelion as a welcome accessory of the classical planetary motion. But what mainly interests us here is that according to Einstein's theory the tensor g_{ij} is changed not only within the sun but in all the circumjacent region of the world, the supplementary terms fading away with distance. And similarly in the presence of two or more lumps of "matter," which includes not only ordinary matter but also the electromagnetic field, for instance. The tensor components thus modified, as (*b*) for instance, are (approximate) solutions of Einstein's "field equations," certain generally covariant differential equations of the second order written down by him in terms of a tensor derived by contraction from the famous Riemann-Christoffel tensor of rank four. The particular form of his eqs. is here of no avail. It is enough to notice that, according to these eqs., within matter not only the several g_{ij} but also a certain differential invariant, the world-curvature, is changed in value, while outside of matter the modified g_{ij} are so distributed that the world-curvature remains nil as in absence of matter. To repeat it, however, even outside of matter the modification of g_{ij} is an essential one and cannot be transformed away.

To illustrate it by a bidimensional picture, imagine an ordinary surface populated by one-dimensional beings using one coordinate u for their space or extension, and another v for their time; their sun will be a line segment, Δu, and the world-tube of the sun a certain strip of the surface. Let our surface (and *their* "world"), in strict analogy to the above, be an ordinary plane in absence of the sun; then, in presence of

* In Einstein's formulæ (70), *loc. cit.* p. 819, α is a misprint for 2α, as the reader will readily convince himself.

that body, the Gaussian curvature within the strip will differ from zero' while outside it will remain nil, the neighbourhood of the solar strip being bent and possibly strained somehow but remaining developable upon a plane, as is a piece of a cylinder, say, or of a cone *. The geodesic lines of the surface, and therefore also the eqs. of motion in the vicinity of the strip, will then be changed correspondingly.

Now, what I propose is *to emancipate the fundamental tensor*, at least outside ordinary matter, *from the influence of gravitation* (as well as of any other agents), in spite of the well-known exceptional properties of gravitational fields. In other words, I propose to reject the gravitational "equivalence hypothesis," but to retain the postulate of general covariance of physical laws.

But here, at the very outset, a fundamental question presents itself. If the coefficients of the invariant line-element $ds^2 = g_{ij}dx_i dx_j$ are not manufactured or moulded by gravitating bodies, what does determine them physically? What determines the values of those tensor components, *if* in different cases they were to be essentially different, *i.e.* not reducible to one another by mere transformations of coordinates? A radical answer to this question easily suggests itself, and is already announced by having emphasized the "if." It is this:

Let the fundamental tensor g_{ij} be not different in different physical circumstances *but always, under all circumstances* (at least *in vacuo*) *essentially the same.* In other words, let us assume that ds^2 is, in vacuo, throughout the world essentially the same quadratic form, or that it is always possible to choose such coordinates x_1, x_2, x_3, $x_4 = ct$ in which ds^2 acquires a certain standard form, no matter whether suns or galaxies are near at hand or very remote. This amounts to postulating *homogeneity* of the four-manifold, which—in view of the principle of causality, in its heuristic aspect—seems to be a perfectly sound requirement.

Now, our world, as any multi-dimensional manifold, has a host of invariants, the differential invariants of various orders of its line-element. Thus, if the world is to be homogeneous (always *in vacuo*, at least), clearly all of its invariants must each have throughout one and the same numerical value; and since one of them (and even prominent amongst them) is the differential invariant of 2nd

* The idea of reducing physical phenomena to changes of curvature, especially in connexion with particles of matter, is not altogether new. It was suggested nearly fifty years ago by Clifford, with the only difference that Clifford had no opportunity of associating the time-coordinate with the remaining three. *Cf.* Clifford's 'Math. Papers,' p. 21, and his 'Common Sense of Exact Sciences,' 5th ed. p. 224 *et seq.*

order repeatedly called world-curvature, we shall claim for our world *a constant curvature.* This will henceforth be denoted by

$$k = 1/R^2.$$

Not pretending to know, or to be able to decide a priori what its sign or value might be, we shall leave them undetermined.

If k is positive, R is a real length, and if negative, then iR is a real length. If $|R| = \infty$, the world is homaloidal. Notice that, whatever the results of future observations, they can not lead to the conclusion that the world is strictly homaloidal but can give only a lower limit of $|R|$, say 10^9 astr. units or more; this under the assumption that the results of observation will be nil-effects, as in the case of Einstein's shift and of all the æther-drift observations and experiments. It may, however, happen that some observations will point to a lower *and* an upper limit of $|R|$ together with a definite sign of R^2. Then, whatever the actual sign of R^2, the result will be a very positive and an interesting one. It may run thus, for instance: $R^2 < 0$, and $10^8 < |R| < 10^{10}$, stating that the world has a negative curvature and fixing its amount between two, more or less narrow limits. I must warn the reader, however, that if he lives long enough to hear of such a result, he must not say that "the three-space" is negatively curved or hyperbolic of curvature $k \doteq -10^{-18}$ astr. un.$^{-2}$, but only that the four-manifold or the world is so. In fact, if such be *the* world, he can choose in it *a* space* just of the curvature $k = -10^{-18}$ (but not below it), as well as a linear infinity of hyperbolic spaces, the homaloidal and all positively curved spaces without upper limit. This freedom of conventional or opportunist choice, limited only at the lower end by the invariant k, is based upon a remarkable and very general theorem on manifolds of any number of dimensions proved 33 years ago by Killing† and in part before him by Beltrami, which may shortly be rendered thus:—

Every n-dimensional space of constant curvature contains in itself spherical space forms (Kugelgebilde) of less dimensions (ν) whose Riemannian curvatures form a continuous manifold having no maximum but a minimum, viz. equal to the curvature of the n-space, this minimum curvature belonging to the ν-dimensional plane.

For our case it is enough to put in this admirable theorem $n = 4$ and $\nu = 3$. After this lengthy but (in view of certain recent misunderstandings) not altogether needless digression, let us return to our subject.

Having assumed a homogeneous world we have *eo ipso* accepted one of *constant* curvature, $k = \frac{1}{R^2}$. (This being at any rate a necessary condition, it will be still incumbent to show that the line-element to be written down presently leads also to all other constant invariants,—which task may be postponed to another opportunity.) Now, to obtain the

* And a homogeneous one, or hypersphere of three dimensions.

† W. Killing, *Die Nicht-Euklidischen Raumformen*, Leipzig, 1885, pp. 79–83. This excellent old book will be helpful to every student of general relativity.

corresponding quadratic form for the line-element let us take, with Beltrami-Killing's theorem as guide, for our particular three-space just the extreme appearing in that theorem, viz. a space of constant curvature equal to that of the world. The line-element $d\lambda$ of such a space, no matter what the sign of R^2, can be written, as is well-known, in polar coordinates r, ϕ, θ, for instance,

$$d\lambda^2 = dr^2 + R^2 \sin^2 \frac{r}{R} (d\phi^2 + \sin^2 \phi d\theta^2).$$

Such being the space part of the line-element, let us use a system in which $g_{14} = g_{24} = g_{34} = 0$, which is always possible, and let us tentatively take $g_{44} = 1$. Thus, with $x_4 = ct$, the required expression for the line-element will be $dx_4^2 - d\lambda^2$, *i. e.*

$$ds^2 = c^2 dt^2 - dr^2 - R^2 \sin^2 \frac{r}{R} (d\phi^2 + \sin^2 \phi d\theta^2). \quad . \quad . \quad (1)$$

Attaching (mentally) the suffixes 1, 2, 3, 4 to the radial, the meridional, the latitudinal, and the time-direction, respectively, the equivalent fundamental tensor will conveniently be written, with $g_{ii} = g_i$,

$$g_1 = -1, \quad g_2 = -R^2 \sin^2 \frac{r}{R}, \quad g_3 = g_2 \,.\, \sin^2 \phi, \quad g_4 = 1, \quad . \quad (2)$$

all other components being zero. That the form (1) or the corresponding tensor (2) do actually express (in a particular, convenient reference system) the said four-manifold, will be seen hereafter in more than one way.

Our original assumption is now reduced to the assumption that, outside of matter, it is always possible to choose such a system of coordinates in which the line-element takes the form (1). We shall refer to such variables by the short name of *natural* coordinates.

It will be well understood, however, that we do not postulate the invariance of the particular form (1) or of the corresponding tensor (2) which, of course, could be preserved only with respect to certain very particular transformations, whereas we require all physical laws to be generally covariant. Thus, in any not "natural" system of coordinates, which we will generally denote by u_1, u_2, u_3, u_4, the line-element (1) will assume the form

$$ds^2 = g_{ij} du_i du_j, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

where g_{ij} will be some linear homogeneous functions of the natural components $g_1, \ldots g_4$ given in (2), viz., by the general transformation rule already quoted,

$$g_{ij} = \frac{\partial x_\kappa}{\partial u_i} \frac{\partial x_\kappa}{\partial u_j} (g_\kappa), \quad . \quad . \quad . \quad . \quad . \quad . \quad (4)$$

the office of ( ) being to remind us that the values (2) are meant, if x_1, x_2, x_3, x_4 stand for r, ϕ, θ, ct. Similarly we shall have for the contravariant tensor $g^{ij} = \gamma_{ij}$ in any u-system, remembering that $(g^{ii}) = 1/(g_i)$,

$$\gamma_{ij} = \frac{\partial u_i}{\partial x_\kappa} \frac{\partial u_j}{\partial x_\kappa} \frac{1}{(g_\kappa)}, \quad . \quad . \quad . \quad . \quad . \quad . \quad (4\,a)$$

to be summed over $\kappa = 1$ to 4, as before. Thus, whenever required, it will be easy to pass from the above to any other system of reference.

Even taking for granted that (1) does represent a world of constant curvature and is thus equivalent to a generally covariant way of defining that manifold, yet the reader may feel formally unsatisfied by seeing the fundamental tensor g_{ij} thus to assume a variety of forms in different reference systems. It will be well, therefore, to give here already certain properties of that tensor which do preserve even their outward form in all systems of coordinates. In fact, let, in the very old notation, $(\iota\mu\kappa\lambda)$ be the four-index symbols of Riemann belonging to the general quadratic form (3), certain differential expressions in g_{ij} to be quoted later on. These "symbols" are themselves the constituents of a tensor of rank four; in the case of n dimensions there are in the most general case only $\frac{n^2}{12}(n^2-1)$ linearly independent Riemann symbols *, which makes 20 for the four-dimensional world. Now, by a most remarkable, although half forgotten, theorem of general geometry †, the necessary and sufficient condition for a manifold to be developable upon a "sphere," *i. e.* to have constant Riemannian curvature, is that all the Riemann symbols $(\iota\mu\kappa\lambda)$ *should bear a constant ratio to the expressions* $g_{\iota\kappa} g_{\mu\lambda} - g_{\iota\lambda} g_{\mu\kappa}$; that ratio being precisely what we have called the curvature of the manifold in question.

Thus, in our case, (3) being only (1) transformed, we have in *any* reference system, natural or not,

$$(\iota\mu\kappa\lambda) = \frac{1}{R^2} (g_{\iota\kappa} g_{\mu\lambda} - g_{\iota\lambda} g_{\mu\kappa}). \quad . \quad . \quad . \quad . \quad . \quad (5)$$

* *Cf.* Wright, *l. c.* pp. 11 & 23. † Killing, *l. c.* p. 232.

Passing from a system u to any other, u', we shall have $(\iota\mu\kappa\lambda)' = \frac{1}{R^2}(g_{\iota\kappa}' g_{\mu\lambda}' g_{\iota\lambda}' g_{\mu\kappa}')$. The left-hand members being properly developed, (5) are ultimately partial differential eqs. for the g_{ij}. It is still to be proved that our above tensor components (2) do actually satisfy all these differential equations. This will be shown in the next section.

To conclude the present one, notice that by (1) the velocity of light, given by $ds=0$, becomes, in natural coordinates,

$$v = \frac{d\lambda}{dt} = c,$$

that is, constant and independent of direction, throughout the natural space (vacuum) of curvature $\frac{1}{R^2}$, whatever the value or the sign of the latter. The "rays" of light will be straight, shortest, lines in that particular space. Remember, however, that it is only *a* space, among many others at your disposal. In view of the above property we can call it visual or *optical space*. If then, by convention, we desire to choose as our reference space that among an infinity of others which has the above property, then there is certainly no objection to calling it simply *space* as a short for "optical space." And since all more remote objects are explored by optical means, such a choice will manifestly be by far the most convenient one. If $R^2 < 0$, so that

$$R \sin \frac{r}{R} = |\mathrm{R}| \sinh \frac{r}{|R|},$$

then the optical space will be hyperbolic or Lobatchewskyan, *i. e.* infinite but showing a defect in the angle sum of a triangle proportional to its area, and so on; if, in spite of the negative value of the invariant R^2, somebody would prefer to use Euclidean geometry, there would be nothing to prevent him doing so; only in that case his optics will not be so simple. Similarly, mutatis mutandis, for $k=0$ or $k>0$. To put it in a few words: The world *is* so or so (to be explored), while space—even with the requirement of homogeneity—is, in very wide limits, a matter of convention, much as was predicted years ago by Poincaré.

Positive constant world-curvature is a feature of Einstein's 1917-theory; of course, only in absence of gravitation, and with the unavoidable cooperation of a certain hypothetical "world-matter." An interesting modification of Einstein's newest theory due to de Sitter will be found in Monthly Notices R. A. S., for Nov. 1917. The latter is especially interesting because it does without the "world-matter," but has on the other hand $g_{44}=\cos^2\frac{r}{R}$, instead of 1, to suit the gravitational field equations slightly amplified by Einstein. At any rate both authors are under the strange impression that the world cannot be infinite.

Finally, notice that in the case of a homaloidal world the theorem expressed by (5) gives $(\iota\mu\kappa\lambda)=0$, as it should be, this being the well-known necessary and sufficient condition for ds^2 to be reducible to a form with all constant coefficients.

3. *Mathematical Supplement to the preceding section.*

In order to obtain the promised support for (1) as the expression for the line-element of a four-manifold of constant curvature, take $g_{14}=g_{24}=g_{34}=0$ and measure u_1, u_2, u_3 along the principal axes of the three-dimensional linear vector operator $g_{\iota\kappa}$ (1, 2, 3). This operator (which itself is no relativistic entity, of course), being self-conjugated, has always such orthogonal axes, and three corresponding principal values, say, g_1, g_2, g_3. Thus, u_1, u_2, u_3 being in general *curvilinear* coordinates, the expression for the line-element will become

$$ds^2=g_1du_1^2+g_2du_2^2+g_3du_3^2+g_4du_4^2, \quad . \quad . \quad (6)$$

and $\det. g_{ij}=g_1g_2g_3g_4$, so that the components of the contravariant tensor will be, simply,

$$\gamma_{ii}=\frac{1}{g_i}, \text{ and nil for } i\neq j.$$

As space-coordinates of this kind can be employed conveniently the polar coordinates r, ϕ, θ or any other orthogonal curvilinear coordinates known since the times of Lamé. Now, what has been repeatedly called the curvature of that world which is given by the above differential form is itself an invariant (one of many) of the differential form, viz. proportional to

$$\mathscr{C}=\gamma^{ij}B_{ij},$$

that is, in our case, to

$$\mathscr{C}=\Sigma\frac{1}{g_i}B_{ii}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (7)$$

where

$$B_{ii}=\begin{Bmatrix} i\alpha \\ \beta \end{Bmatrix}\begin{Bmatrix} i\beta \\ \alpha \end{Bmatrix}-\begin{Bmatrix} ii \\ \alpha \end{Bmatrix}\begin{Bmatrix} \alpha\beta \\ \beta \end{Bmatrix}+\frac{\partial}{\partial u_i}\begin{Bmatrix} i\alpha \\ \alpha \end{Bmatrix}-\frac{\partial}{\partial u_a}\begin{Bmatrix} ii \\ \alpha \end{Bmatrix}. \quad (7\,a)$$

These are the constituents of a covariant tensor of rank two, derived by Einstein from the Riemann-Christoffel tensor by "contraction" (equating two indices to one another and summing over them).

In $(7a)$, the general definition of the three-index symbols of Christoffel is

$$\left\{ {ij \atop k} \right\} = g^{\kappa\alpha} \begin{bmatrix} ij \\ \alpha \end{bmatrix} = \gamma_{\kappa\alpha} \begin{bmatrix} ij \\ \alpha \end{bmatrix}, \quad . \quad . \quad . \quad . \quad (8)$$

where

$$\begin{bmatrix} \mu\nu \\ \sigma \end{bmatrix} = \tfrac{1}{2} \left(\frac{\partial g_{\mu\sigma}}{\partial u_\nu} + \frac{\partial g_{\nu\sigma}}{\partial u_\mu} - \frac{\partial g_{\mu\nu}}{\partial u_\sigma} \right), \quad . \quad . \quad . \quad (9)$$

and, therefore, for the form (6), and with $u_i = x_i$, say, $x_{1,2,3,4} = r, \phi, \theta, ct$, as in the preceding section,—

$$\begin{bmatrix} ii \\ i \end{bmatrix} = \tfrac{1}{2} \frac{\partial g_i}{\partial x_i}; \quad \begin{bmatrix} ii \\ \kappa \end{bmatrix} = -\tfrac{1}{2} \frac{\partial g_i}{\partial x_\kappa} \, (i \neq \kappa); \quad \begin{bmatrix} i\kappa \\ i \end{bmatrix} = \tfrac{1}{2} \frac{\partial g_i}{\partial x_\kappa},$$

and $\begin{bmatrix} ij \\ \kappa \end{bmatrix} = 0$ when i, j, κ are all different. In our case (8) becomes

$$\left\{ {ij \atop \kappa} \right\} = \Sigma \frac{1}{g_\kappa} \begin{bmatrix} ij \\ \kappa \end{bmatrix},$$

so that finally,

$$\left\{ {ij \atop i} \right\} = \frac{1}{2g_i} \frac{\partial g_i}{\partial x_j}, \quad \left\{ {ii \atop j} \right\} = -\frac{1}{2g_j} \frac{\partial g_i}{\partial x_j} \, (i \neq j), \quad . \quad (10)$$

and $\left\{ {\iota\kappa \atop \lambda} \right\} = 0$ when all indices are different. Thus far g_1 etc. were any functions of x_1, etc. or r, ϕ, θ, ct. Henceforth it will be enough to develop the sub-case in which

$$g_1 = g_1(r), \quad g_2 = g_2(r), \quad g_3 = g_2 \sin^2\phi, \quad g_4 = g_4(r), \quad . \quad . \quad (11)$$

where, however, g_1, g_2, g_4 continue to be any functions of $r = x_1$. Then, by $(7a)$ and (10), with dashes used for derivatives, and introducing the abbreviations

$$h_1 = \log(-g_1), \quad h_2 = \log(-g_2), \quad h_4 = \log g_4, \quad h = \log \frac{g_2^2 g_4}{-g_1},$$

$$\left.\begin{aligned} B_{11} &= h_2'' + \tfrac{1}{2} h_4'' + \tfrac{1}{2}(h_2' - h_1')h_2' + \tfrac{1}{4} h_4'(h_4' - h_1') \\ B_{22} &= \frac{1}{\sin^2\phi} B_{33} = \frac{g_2}{2g_1}(h_2'' + \tfrac{1}{2} h_2' h') - 1 \\ B_{44} &= \frac{g_4}{2g_1}(h_4'' + \tfrac{1}{2} h_4' h'); \quad B_{\iota\kappa} = 0 \, (\iota \neq \kappa), \end{aligned}\right\} \quad . \quad (12)$$

and the corresponding expression for the curvature will be, by (7),

$$\mathscr{C}=\frac{1}{g_1}B_{11}+\frac{2}{g_2}B_{22}+\frac{1}{g_4}B_{44}. \quad . \quad . \quad . \quad . \qquad (13)$$

This is valid for any fundamental tensor of the form (11); our case, given by (1) or (2), is but a sub-case of (11). Taking, as in (2), $g_1=-1$, we have

$$\mathscr{C}=-\left\{2h_2''+h_4''+\tfrac{3}{2}h_2'^2+\tfrac{1}{4}h_4'^2+h_2'h_4'+\frac{2}{g_2}\right\}, \quad . \qquad (13\,a)$$

where $h_2=\log(-g_2)$, $h_4=\log g_4$ are still any functions of $x_1=r$. These more or less general formulæ have been given here since they may be useful in some other connexion. For the present, however, our purpose is only to show that the element (1) actually belongs to a world of constant curvature. Now, putting, as in (1) or (2), $g_2=-R^2\sin^2\frac{r}{R}$, we have

$$h_2'=\frac{2}{R}\cot\frac{r}{R},\quad h_2''=-\frac{2}{R^2}\operatorname{cosec}^2\frac{r}{R},$$

and, equating $\mathscr{C}$ to a constant, the differential equation for $h_4=\log g_4$ becomes

$$\frac{6}{R^2}-h_4''-h_4'h_2'-\tfrac{1}{2}h_4'^2=\mathscr{C}=\text{const.} \quad . \quad . \qquad (13\,b)$$

This equation can be satisfied by $g_4=\cos^2 ar$, $a=\text{const.}$, which would give

$$2a^2+\frac{4a}{R}\tan ar\cot\frac{r}{R}=\mathscr{C}-\frac{6}{R^2},$$

and this can be satisfied either by $a=1/R$, *i.e.* $g_4=\cos^2\frac{r}{R}$*, with $\mathscr{C}=12/R^2$, or more simply by

$$a=0,\text{ and }\mathscr{C}=\frac{6}{R^2},$$

* This is mentioned here because Prof. de Sitter's element, in absence of gravitation (M. N., Nov. 1917) has precisely $g_{44}=\cos^2\frac{r}{R}$. And this, with $g_{22}=-R^2\sin^2\frac{r}{R}$, was de Sitter's *only possibility,* since he has had to satisfy not only $\mathscr{C}=\text{const.}$ but also the four (amplified) "field equations" of Einstein, *i.e.* without "world-matter,"

$$B_{ii}=\tfrac{1}{4}\mathscr{C}g_{ii},$$

and these cannot be satisfied by $a=0$ or $g_{44}=1$. (With appropriate "world matter," as in Einstein's case, quoted also by de Sitter, the equations are so modified as to admit $g_{44}=1$.) In the theory we are proposing, it will be remembered, there are no "field equations" to satisfy; the fundamental tensor has here nothing to do with gravitation.

and, therefore, $g_4=1$, which is precisely the case of the element (1). Thus, that line-element characterizes a space-time of constant curvature, the value of $k=1/R^2$ being one-sixth of $\mathscr{E}$, the above differential invariant. Q. E. D.

Such being the case, we know beforehand that the theorem (5) will hold. Yet it may be well to verify it by calculating directly some at least of the Riemann symbols corresponding to the tensor (2), in part also to make the reader more familiar with the handling of these remarkable symbols. Now, their general definition may be put into the form

$$(ijhk)=\frac{\partial}{\partial x_k}\begin{Bmatrix} ij \\ h \end{Bmatrix}-\frac{\partial}{\partial x_j}\begin{Bmatrix} ik \\ h \end{Bmatrix}+\begin{Bmatrix} ij \\ \alpha \end{Bmatrix}\begin{Bmatrix} \alpha k \\ h \end{Bmatrix}-\begin{Bmatrix} ik \\ \alpha \end{Bmatrix}\begin{Bmatrix} \alpha j \\ h \end{Bmatrix}, \quad (14)$$

to be summed as usual. Thus, with the normal form (6), and therefore, with the values (10), we find, for instance, remembering that $g_1=\text{const.}=-1$,

$$(2112)=-\tfrac{1}{2}\frac{\partial^2 g_2}{\partial x_1^2}+\frac{1}{4g_2}\left(\frac{\partial g_2}{\partial x_1}\right)^2,$$

which becomes in the case of (2), with $x_1=r$, $(2112)=-\sin^2\frac{r}{R}$. Most symbols vanish, as for ex. all $(iiii)$, and more generally all $(ijkk)$,—this in virtue of the general property $(ijhk)=-(ijkh)$; again, many symbols that in general would not vanish do so in our case owing to $\frac{\partial}{\partial x_3}=0$, and so on. Finally we find, f. ex., another non-vanishing symbol,

$$(3113)=-\tfrac{1}{2}\frac{\partial^2 g_3}{\partial x_1^2}+\frac{1}{4g_3}\left(\frac{\partial g_3}{\partial x_1}\right)^2=-\sin^2\frac{r}{R}\cdot\sin^2\phi,$$

and so on. Thus we have

$$(2112)=\frac{1}{R^2}(g_2)\ ; \quad (3113)=\frac{1}{R^2}(g_3),$$

as it should be; for, by (5), $(2112)=\frac{1}{R^2}(g_{12}{}^2-g_{22}g_{11})$, and in our system $g_{12}=0$, $g_{11}=-1$; and so on. Having thus verified (5) in a pair of examples, we can safely apply that theorem to the tensor g_{ij} transformed from (2) to any system u_i. If we use, for instance, normal coordinates, *i. e.* such that $ds^2=\Sigma g_{ii}du_i^2$, then all Riemann symbols vanish with the exception of those of the type $(ikik)$, and these become

$$(ikik)=\frac{1}{R^2}g_{ii}g_{kk}=-(kiik), \quad . \quad . \quad . \quad (15)$$

for $i \neq k$. The most useful, of course, is the formula (5) itself, since it enables us to write at once all the constituents of the Riemann-Christoffel tensor for the assumed world in any system of reference.

4. *Natural Systems of Reference.*

In Section 2 the coordinates r, ϕ, θ, ct in which ds^2 assumes the simple form (1), and in which, therefore, light is propagated uniformly and isotropically, were called *natural* coordinates. Now, it is interesting to inquire whether, in a world with any fixed curvature, there is but one or a whole class of natural systems,—apart from such, of course, as can be derived from the original one by mere three-space transformations. From the older Relativity we know that for $R=\infty$, when ds^2 becomes

$$c^2dt^2-dr^2-r^2(d\phi^2+\sin^2\phi\, d\theta^2)=c^2dt^2-dx^2-dy^2-dz^2,$$

there is an infinity of natural systems all derivable from x, y, z, ct by the Lorentz transformation. It can be expected that something analogous will hold for any finite R, real or imaginary. Let us, therefore, try to find such natural systems. More definitely, starting from the form (1), let us ask for such transformations $r=r(r', \phi', \theta', ct')$, etc. which turn (1) into

$$c^2dt'^2-dr'^2-R^2\sin^2\frac{r'}{R}(d\phi'^2+\sin^2\phi'\, d\theta'^2).$$

Then, at least all these systems (if no others) will share with the original one the "natural" property of simple optical behaviour and other properties therewith connected. In short, let us find the generalization of the Lorentz transformation for a space-time of any constant curvature.

It will be formally convenient to introduce $|R|$ as the unit of length *, and similarly $\frac{|R|}{c}$ as the unit of time, no matter how long these units might turn out to be when compared with those usually employed by the physicist or the astronomer.

In these units, and therefore with the light velocity $c=1$, further with

$$l=it=t\sqrt{-1},$$

* With the exception, of course, of the case of $|R|=\infty$.

and with Sin written for sin or sinh according as $R^2 > 0$ or $R^2 < 0$, the original line-element becomes

$$-ds^2 = dr^2 + \mathrm{Sin}^2 r(d\phi^2 + \sin^2\phi\, d\theta^2) + dl^2, \quad . \quad . \quad (16)$$

all four variables being now pure, dimensionless, numbers; and the required *natural systems* will be defined by

$$dr^2 + \mathrm{Sin}^2 r(d\phi^2 + \sin^2\phi d\theta^2) + dl^2 = dr'^2 + \mathrm{Sin}^2 r'(d\phi'^2 + \sin^2\phi' d\theta'^2) + dl'^2.$$

In order to find them it will be most convenient to use Weierstrass coordinates, viz. to introduce a fifth, auxiliary, coordinate x_5, such that (with $x_4 = l = it$)

$$x_1^2 + x_2^2 + x_3^2 + x_4^2 + x_5^2 = R^2 = \pm 1; \quad . \quad . \quad . \quad (17)$$

then our standard form will become

$$-ds^2 = dx_1^2 + dx_2^2 + dx_3^2 + dx_4^2 + dx_5^2. \quad . \quad . \quad . \quad (18)$$

The upper sign in (17) will correspond to an elliptic, and the lower to a hyperbolic, world which thus appears, in $x_1, \ldots x_5$, l, as a four-dimensional *sphere* or *pseudosphere*, respectively *.

The well-known connexions between the x_i and r, etc. will be given presently. Whatever these are, if we require that, for the natural systems of reference,

$$dx_1^2 + \ldots + dx_5^2 = dx_1'^2 + \ldots + dx_5'^2, \quad . \quad . \quad (A)$$

and at the same time,

$$x_1^2 + \ldots + x_5^2 = x_1'^2 + \ldots + x_5'^2, \quad . \quad . \quad . \quad (B)$$

then if x_i' are retransformed into r', etc., thus getting rid of the temporary or auxiliary fifth variable, the natural form (16) will reappear in dashed letters, as is required.

Now, (A) *and* (B) can be satisfied only by taking for x_i' linear functions of the x_ι. Let us write, therefore,

$$x_1' = a_{10} + a_{11}x_1 + \ldots + a_{15}x_5, \text{ etc.}$$

or, in the usual abbreviated notation

$$x_\iota' = a_{\iota 0} + a_{\iota\kappa}x_\kappa, \quad \iota = 1, 2, \ldots 5, \quad . \quad . \quad . \quad (19)$$

where $a_{\iota\kappa}$ are thirty constant coefficients. Such, however, being the case, we have also

$$dx_\iota' = a_{\iota\kappa}dx_\kappa,$$

so that all the equations yielded by (A) are already contained

* That is to say as the four-dimensional analogy of the ordinary two-dimensional sphere or pseudosphere. In the variables x_1, etc. and t (real) equation (17) represents a one-sheeted hyperboloid for $R^2 > 0$, and a two-sheeted hyperboloid for $R^2 < 0$.

among those required by (B). These equations, common to (A) and (B), are

$$a_{1\kappa}^2+a_{2\kappa}^2+\ldots+a_{5\kappa}^2=1, \text{ five eqs.,}$$

and

$$a_{1\iota}a_{1\kappa}+a_{2\iota}a_{2\kappa}+\ldots\ldots+a_{5\iota}a_{5\kappa}=0, \text{ ten eqs.,}$$

in all 15 equations. In addition to these (B) itself gives the condition

$$a_{10}^2+a_{20}^2+\ldots\ldots+a_{50}^2=0,$$

and five more equations of the type

$$a_{10}a_{1\kappa}+a_{20}a_{2\kappa}+\ldots\ldots+a_{50}a_{5\kappa}=0. \quad . \quad . \qquad (20)$$

The latter, however, being a system of homogeneous equations for the $a_{\iota 0}$, we have either det. $(20)=0$, when (20) are reduced to four independent equations only, or det. $(20)\neq 0$, and $a_{10}=a_{20}=\ldots\ldots=a_{50}=0$. In the former case we have in all $15+1+4=20$ equations for $5\times 6=30$ coefficients, and in the latter case, the first 15 equations only for 25 coefficients. Thus, in either case the coefficients can be expressed by 10 free parameters, or the transformations in question are *ten*-parametric. Without sacrifice of generality we can take the second case, *i. e.* $a_{\iota 0}=0$, and therefore, the homogeneous transformations

$$x_\iota'=a_{\iota\kappa}x_\kappa, \quad \iota=1, 2, \ldots 5, \quad . \quad . \quad . \quad . \quad . \qquad (19\,a)$$

with 15 equations

$$\left.\begin{aligned} a_{1\kappa}^{\ 2}+a_{2\kappa}^{\ 2}+\ \ldots\ +a_{5\kappa}^{\ 2}&=1,\\ a_{1\iota}a_{1\kappa}+a_{2\iota}a_{2\kappa}+\ \ldots\ +a_{5\iota}a_{5\kappa}&=0, \end{aligned}\right\} \quad . \quad . \quad . \qquad (21)$$

for the 25 coefficients $a_{11}, a_{12}, \ldots a_{55}$. And, as every pair of transformations (19 a) can be replaced by a single transformation of the same kind, the said natural transformations constitute a *group*, *viz. a ten-parametric one.* The relations (21) are exactly of the same form as the six equations which are well-known in connexion with the ordinary transformation of Cartesian coordinates by a rotation of the system, or the 10 equations connected with the Lorentz transformation (with *fixed* origin of time and space). In fact, (21) taken by themselves would correspond to a typical orthogonal transformation in five variables. Since, however, our five coordinates are not independent but bound to one another by (17), our case is better expressed by saying that it is the four-dimensional analogy of the (rotation or) motion in itself of an ordinary two-dimensional sphere; the difference, even with $R^2 > 0$, being that the coordinate $x_4 = l = it$ is imaginary. Keeping this well in mind one can characterize the required transformations by saying that any one of them is *a rotation of the world, sphere or pseudosphere, in itself*, similarly as the Lorentz transformations were described as rotations of the Minkowskian, homaloidal world. Thus, notwithstanding the world-curvature, the said group of transformations is characterized in much the same way as the Lorentz group (with one difference to be explained presently). The details of its analytical expression will, of course, be different for non-vanishing curvature.

The result can now be stated shortly by saying that all the *natural systems* of reference are derivable from one another *by a rotation of the world in itself*, whatever its curvature. To pass from a natural system of coordinates x_1, etc., $x_4 = l$ to any non-natural, $u_1, \ldots u_4$, is to *distort* the world sphere or pseudosphere (without changing, however, its invariant curvature), while to pass from that system to any other natural system is to effect a *mere rotation* of the sphere or pseudosphere, according to the sign of R^2. The corresponding group of transformations, deriving one natural system from another, could appropriately be called *the natural group*, of which then the Lorentz group would be a particular case corresponding to $R^2 = \infty$. It must be expressly stated that I do not propose to limit the theory to the natural group; on the contrary, I require every physical law to be covariant (or contravariant) with respect to *any* transformations of the coordinates. The "natural" ones are treated here at some length only because of their eminently simple properties, as a class of reference systems among an infinity of others.

Now, as to the difference in relation to the Lorentz group, hinted at a moment ago. It is well known that the so-called general Lorentz transformations, viz. including pure space rotations, constitute a *six*-parametric group *, while our natural group is a *ten*-parametric one, since (21) are but fifteen equations for the twenty-five coefficients $a_{11}, a_{12}, \ldots a_{55}$. This, however, is only an apparent discrepancy. For the Lorentz group just mentioned relates to a *fixed origin* of x, y, z, l, or to put it shortly, to a fixed world-origin O. When we add the four degrees of freedom to choose a world-point as O, the result will precisely be $6 + 4 = 10$. That is

* The narrower or three-parametric Lorentz transformations do *not* constitute a group, although they contain the subgroups for parallel velocities. *Cf.* the author's 'Theory of Relativity,' Macmillan (1914), p. 170.

to say, including pure space-rotations *and shifts of origin,* the Lorentz group is a *ten-parametric* one, exactly as the above group. It remains only to show that the latter does in fact include free shifts of the origin of the four coordinates x_1, x_2, x_3, x_4 (the fifth being only an artifice for simplifying the investigation).

Now, return to (19 *a*) which, written out fully, are

$$\left.\begin{array}{l} x_1'=a_{11}x_1+a_{12}x_2+\ \ldots\ +a_{15}x_5 \\ \ldots\ldots\ldots\ldots \\ x_4'=a_{41}x_1+a_{42}x_2+\ \ldots\ +a_{45}x_5 \\ x_5'=a_{51}x_1+a_{52}x_2+\ \ldots\ +a_{55}x_5 \end{array}\right\} \quad . \quad . \quad . \quad (19\,a)$$

The origin O of the x-system is $x_1=x_2=x_3=x_4=0$, and by (17), $x_5^2=R^2$, say $x_5=+R=\sqrt{\pm 1}$, according as the world should be elliptic or hyperbolic. Thus the origin O' of the x'-system will be

$$x_1'=a_{15}R,\ x_2'=a_{25}R,\ \text{etc.},\ x_5'=a_{55}R,$$

satisfying (17) in virtue of the fifth of (21). Thus by ascribing appropriate values to $a_{15}, \ldots\, a_{45}$ any world-point can be made the origin of the new system. Q.E.D.

An interesting feature is that (19 *a*) with (21), although having the outward form of an ordinary rotation with "fixed origin," yet contain also *shifts* of the world in itself, unlike the Lorentz transformations if written, in four variables, $x_\iota'=a_{\iota\kappa}x_\kappa$. The simple reason is that our formulæ do express a rotation round a fixed point; a point not of the world, however, but an extraneous one, in the fifth dimension, or, to speak figuratively, "inside" the sphere whose surface represents the world. The contrast with the Lorentz six-parametric "rotation" can perhaps be best illustrated by comparing an ordinary plane with an ordinary spherical surface. And the analogy fits because the Minkowskian world is flat, while that which concerns us here is assumed to have some constant curvature.

Having thus ascertained the properties of the full, ten-parametric, group of natural transformations in their simple, "kinematical" form, it will be enough to develop the analytical expression for the sub-group only, corresponding to a fixed origin O of x_1, x_2, x_3, x_4. To obtain $O'=O$, write, in (19 *a*),

$$a_{15}=a_{25}=a_{35}=a_{45}=0,\ a_{55}=1.$$

Four of the eqs. (21) will then become $a_{51}=a_{52}=a_{53}=a_{54}=0$, so that the group ultimately becomes

$$x_\iota'=a_{\iota\kappa}x_\kappa\,(\iota,\,\kappa=1,\dots 4)\;;\;\; x_5'=x_5. \quad . \quad . \quad (19\,b)$$

There being now 10 conditions for 16 coefficients, the group with fixed origin is six-parametric. Moreover, apart from $x_5=x_5'$, the eqs. (19 *b*) are now exactly of the form of those expressing the Lorentz transformation.

In short, the natural systems, for any constant world-curvature, are obtained by subjecting the *Weierstrass coordinates* of any one of them (with $x_5'=x_5$) to a Lorentz transformation.

Of the six-parametric group the pure space-rotations are of no interest. It will thus be enough to take the case of

$$x_2'=x_2,\;\; x_3'=x_3\;;\;\; x_5'=x_5.$$

The conditions (21) then become

$$a_{11}^2+a_{41}^2=a_{14}^2+a_{44}^2=1\;;\;\; a_{11}a_{14}+a_{41}a_{44}=0,$$

and are satisfied by $a_{11}=a_{44}=\cos\tilde{\omega}$, $a_{14}=-a_{41}=\sin\tilde{\omega}$, with $\tilde{\omega}$ as the only parameter, so that the transformation ultimately becomes

$$\left.\begin{aligned} x_1'-ix_4'&=e^{i\tilde{\omega}}(x_1-ix_4),\\ x_2',\, x_3',\, x_5'&=x_2,\, x_3,\, x_5. \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad (22)$$

The first line is familiar from the older relativity, the only difference being that now it holds for the Weierstrass coordinates of the world-point. To translate this result into our original coordinates, put

$$\left.\begin{aligned} x_1,\, x_2,\, x_3&=R\sin\psi.\sin\frac{r}{R}\left[\cos\phi,\, \sin\phi\cos\theta,\, \sin\phi\sin\theta\right],\\ x_4=it&=R\cos\psi\;;\;\; x_5=R\sin\psi.\cos\frac{r}{R}, \end{aligned}\right\} . \quad (23)$$

so that
$$x_1^2+x_2^2+x_3^2+x_5^2=R^2\sin^2\psi, \text{ and } \sum_1^5 x_\iota^2=R^2,$$

identically, as required. Then we get, as a translation of (22), keeping for the moment R, to avoid confusion *, the following equations between r, ϕ, θ, ψ and their dashed

* Since $|R|$ was taken as unit length, R will here stand for 1 or $\sqrt{-1}$ according as the curvature is positive or negative.

correspondents, with $\tilde{\omega}$ as parameter,

$$\left.\begin{aligned} &\theta'=\theta\ ;\ \tan\frac{r'}{R}\sin\phi'=\tan\frac{r}{R}\sin\phi\ ;\ \sin\psi'\cos\frac{r'}{R}=\sin\psi\cos\frac{r}{R} \\ &\sin\psi'.\sin\frac{r'}{R}\cos\phi'=\cos\tilde{\omega}.(\sin\psi\sin\frac{r}{R}\cos\phi)+\sin\tilde{\omega}.\cos\psi \\ &\cos\psi'=\cos\tilde{\omega}.\cos\psi-\sin\tilde{\omega}.\left(\sin\psi.\sin\frac{r}{R}\cos\phi\right). \end{aligned}\right\} \quad (24)$$

The first of these equations, expressing axial symmetry, needs no further remarks. The third may be put aside for the moment (one of the 5 eqs. being a consequence of the others), but will be useful hereafter. The second, fourth, and fifth can be written, remembering that $\cos\psi\equiv it/R$,

$$\left.\begin{aligned} &\left(1+\frac{t^{2\prime}}{R^2}\right)^{1/2}R\sin\frac{r'}{R}\cos\phi'=\cos\tilde{\omega}\left(1+\frac{t^2}{R^2}\right)^{1/2}R\sin\frac{r}{R}\cos\phi+it\sin\tilde{\omega}, \\ &t'=t.\cos\tilde{\omega}+i\sin\tilde{\omega}.\left(1+\frac{t^2}{R^2}\right)^{1/2}R\sin\frac{r}{R}\cos\phi, \\ &R\tan\frac{r'}{R}.\sin\phi'=R\tan\frac{r}{R}.\sin\phi. \end{aligned}\right\} \quad (25)$$

These are the required transformations valid for any constant curvature $k=\frac{1}{R^2}$ of the world. If the world is elliptic, we have, with $|R|$ as unit length,

$$(1+t'^2)^{1/2}\sin r'\cos\phi'=\cos\tilde{\omega}.(1+t^2)\sin r.\cos\phi+it\sin\tilde{\omega},\text{ etc.},$$

and if hyperbolic, then

$$(1-t'^2)^{1/2}\sinh r'\cos\phi'=\cos\tilde{\omega}.(1-t^2).^{1/2}\sinh r\cos\phi+it\sin\tilde{\omega},\text{ etc.}$$

The detailed discussion of the several interesting terms may be left to the reader. If the world is homaloidal, *i. e.* Minkowskian, we have, as the simplest particular case of (25),

$$\left.\begin{aligned} &r'\sin\phi'=r\sin\phi\ ;\ r'\cos\phi'=r\cos\phi.\cos\tilde{\omega}+it\sin\tilde{\omega}, \\ &t'=t\cos\tilde{\omega}+i\sin\tilde{\omega}.r\cos\phi, \end{aligned}\right\} (25^\circ)$$

which are the well-known formulæ of the older relativity theory *; the first of (25°) expresses conservation of lateral dimensions, and the two others, with $\tilde{\omega}=\text{arc}\tan\left(\frac{iv}{c}\right)$ and

* See, for instance, my book, p. 127 *et seq.*

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$x=r\cos\phi$, are identical with Einstein's famous formulæ of 1905, for uniform relative motion with velocity v along x,

$$x'=\gamma(x-vt),\ t'=\gamma\left(t-\frac{vx}{c^2}\right);\ \gamma=\left(1-\frac{v^2}{c^2}\right)^{-1/2}.$$

It is but natural that for a non-homaloidal world the relations, as in (25), should be more complicated. It will be remembered that in appropriate, viz. Weierstrassian coordinates, the relations are as simple as in the case of a homaloidal world.

Let us once more return to the full, 10-parametric group of natural transformations. Its equations, collected from (19 *a*) and (21), are

$$\left.\begin{array}{l} x_1'=a_{11}x_1+a_{12}x_2+\ldots\ldots+a_{15}x_5 \\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot \\ x_5'=a_{51}x_1+a_{52}x_2+\ldots\ldots+a_{55}x_5 \\ \hline a_{1\kappa}^2+\ldots\ldots+a_{5\kappa}^2=1 \\ a_{1\iota}a_{1\kappa}+\ldots\ldots+a_{5\iota}a_{5\kappa}=0, \end{array}\right\} \quad . \ . \ . \quad (26)$$

the number of conditions, written for convenient reference below the equations, being 15. It will be well to rewrite also the translation of these Weierstrass coordinates into r, ϕ, θ, t, in a somewhat simpler way than in (23). Remembering that the factors of $\sin\psi$ in the expressions (23) for x_1, x_2, x_3, x_5 are the Weierstrass coordinates of a point of *space* (three-space), *i. e.* of a natural space r, θ, ϕ, call these factors $\xi_1, \xi_2, \xi_3, \xi_5$, *i. e.* put $\xi_1=R\sin\frac{r}{R}\cos\phi$, etc. Then the Weierstrass world-coordinates will be expressed by these Weierstrass space-coordinates ξ and by the time coordinate t as follows:

$$\left.\begin{array}{l} x_1,\ x_2,\ x_3,\ x_5=\left(1+\frac{t^2}{R^2}\right)^{1/2}(\xi_1,\ \xi_2,\ \xi_3,\ \xi_5) \\ x_4=l=it. \end{array}\right\} \quad . \ . \quad (27)$$

For a homaloidal world (or better, for any world, provided that t^2/R^2 is a negligible fraction) the first three x's become identical with the ξ's, and the auxiliary fifth x becomes unnecessary. With $|R|$ as unit length the factor becomes $\sqrt{1\pm t^2}$ according as the world is elliptic or hyperbolic. Everything concerning the ∞^{10} natural systems is thus conveniently expressed by (26) and (27).

The differential equations of the *geodesics* or shortest lines of the four-dimensional world are immediately seen, by (18) and (17), to assume in Weierstrass coordinates the simple form

$$\frac{d^2 x_\kappa}{ds^2} = \frac{1}{R^2} x_\kappa, \quad \kappa = 1, \dots 5, \quad . \quad . \quad . \quad . \quad (28)$$

where s is measured along the geodesic itself. Needless to say that these lines are of prime importance, first, because —owing to their definition $\delta \int ds = 0$—they are invariant, and then because they offer the first example of generally covariant laws, viz. the law of motion of a free particle. Notice in passing that, in any coordinates u the eqs. of a geodesic will be

$$\frac{d^2 u_\iota}{ds^2} + \left\{ \begin{matrix} \kappa\lambda \\ \iota \end{matrix} \right\} \frac{du_\kappa}{ds} \frac{du_\lambda}{ds} = 0,$$

the g_{ij}, entering through the Christoffel symbols, being always as in (4), since they are not moulded by gravitation or by any other agent. But let us return to the eqs. (28). Their general integrals are

$$x_\kappa = a_\kappa \cos\left(\frac{is}{R}\right) + ib_\kappa \sin\left(\frac{is}{R}\right), \quad . \quad . \quad . \quad (29)$$

where a_κ, b_κ are ten constants satisfying, by (17), the conditions

$$a_\kappa a_\kappa = -b_\kappa b_\kappa = \mathrm{R}^2, \text{ and } a_\kappa b_\kappa = 0, \quad . \quad . \quad . \quad (30)$$

to be summed according to the usual rule. These are the equations of a geodesic of the world, for any constant curvature. In natural units,

$$x_\kappa = a_\kappa \cosh s - b_\kappa \sinh s, \text{ for } R^2 > 0,$$

and

$$x_\kappa = a_\kappa \cos s + ib_\kappa \sin s, \quad \text{for } R^2 < 0.$$

In order to find the *shortest distance* s between any two world-points * whose Weierstrass coordinates are x_κ and y_κ, write (29) for the former and for the latter, multiply them in pairs and add ; then, in virtue of (30), the result will be

$$\cos \frac{is}{R} = \frac{1}{R^2} (x_\kappa y_\kappa), \quad . \quad . \quad . \quad . \quad . \quad . \quad (31)$$

a well-known formula of multidimensional non-euclidean

* Not exceeding certain obvious limits.

geometry *. According to the sign of the world's curvature we have, in natural units, $(x_\kappa y_\kappa) = \cosh s$ or $-\cos s$.

Without insisting any further on these formulæ let us only draw from the last one this simple but interesting consequence :—The shortest distance of two world-points being manifestly an invariant, so is also $x_\kappa y_\kappa$. That is to say, in passing from one to any other *natural* system (for in such only we have the above Weierstrass coordinates with all their simple properties), the sum of products of such coordinates of two world-points retains its value. This simple property, although arrived at by following upon the shortest path from x to y, is at any rate independent thereof, and belongs to that pair of world-points as such. This *restricted invariant* † $(x_\kappa y_\kappa)$ which, by (27), can be written, with $x_4 = it_x$, $y_4 = it_y$,

$$x_\kappa y_\kappa = \left(1 + \frac{t^2}{R^2}\right)^{1/2} (\xi_1\eta_1 + \xi_2\eta_2 + \xi_3\eta_3 + \xi_5\eta_5) - t_x t_y, \quad . \quad (32)$$

must, of course, follow also from the group equations (26). So, in fact, it does. As an instructive verification of the above line of reasoning write the first five eqs. (26) for the point x, and then for y, add the products of corresponding pairs of coordinates and take account of the 15 conditions between the $a_{\iota\kappa}$ immediately derivable from those given in (26). Then the result will be

$$x_\kappa' y_\kappa' = x_\kappa y_\kappa, \quad . \quad . \quad . \quad . \quad . \quad . \quad (33)$$

which was the property to be proved. This is the non-homaloidal analogy of the invariance of the "scalar product" of two four-vectors well-known from the older relativistic vector algebra. The property (33) will follow even more immediately by considering x^κ, y_κ as five-vectors in a five-space, restricted by (17) to have the fixed "size" R, and by remembering that the transformations in question are rotations of the world-sphere or pseudosphere. With the aid of (26) we can at once develop the whole vector algebra for a non-homaloidal world, as an obvious generalization of the older one. It is needless to show in detail how this is to be done. We shall construct the entities of this kind together with the rules of operating upon them every time these

* The circumstance that the usual "s" is here replaced by is is due to the negative sign in (18), which I retain for the sake of uniformity with the present notation of most authors.

† *I. e.* invariant with respect to the 10-parametric natural group of transformations.

should be particularly required. It will, of course, be always kept in mind that these entities are applicable only when the employed reference system is a natural one. But then they will offer conspicuous analytical facilities.

Having thus sufficiently explained the properties of this particular, but important, class of systems of coordinates, let us now pass to physical laws endowed with covariance for any transformations of the four variables. We shall begin with the fundamental electromagnetic laws since these embrace a vast and ever growing domain of phenomena.

5. *Electromagnetic Vacuum-Equations.*

Even before the publication of Einstein's outlines of a generalized theory of relativity, Kottler *, although confining his investigation to the Minkowskian world, has made the capital discovery that Maxwell's amplified equations, now generally known as the "vacuum-equations" or the fundamental equations of the electron theory, were *generally covariant*, i. e. with respect to any coordinate transformations. More correctly, this property belongs not to the usual four equations $\partial \mathbf{E}/\partial t+\rho\mathbf{v}=\text{curl } \mathbf{M}$, etc. containing (beside $\mathbf{v}$) only the two vectors $\mathbf{E}$, $\mathbf{M}$, but to the broader system of equations, with u_4 as time coordinate,

$$\frac{\partial \mathfrak{M}}{\partial u_4}+\text{curl } \mathbf{E}=0, \ \text{div } \mathfrak{M}=0 \ ; \quad . \quad . \quad . \quad . \quad (\text{I})$$

$$\text{curl } \mathbf{M}-\frac{\partial \mathfrak{E}}{\partial u_4}=\rho\mathbf{v}, \ \ \text{div } \mathfrak{E}=\rho, \qquad . \quad . \quad . \quad (\text{II})$$

containing four vectors which will be shortly referred to as electric and magnetic forces ($\mathbf{E}$, $\mathbf{M}$) and polarizations ($\mathfrak{E}$, $\mathfrak{M}$). The latter appear as certain linear vector functions of the former, the nature of the corresponding vector operators being dependent upon the choice of the system of the four coordinates. In the homaloidal world and in any "legitimate" or Lorentz system these operators degenerated into idemfactors, so that $\mathfrak{E}=\mathbf{E}$, $\mathfrak{M}=\mathbf{M}$, reducing (I) and (II) to their usual form. The said property is based upon the familiar assumption of the invariance of electric charge.

It will be enough to recall here Kottler's proof but briefly, giving however at the same time an explicit translation of the involved tensors into components of $\mathbf{E}$, etc. taken along

* F. Kottler, 'Raumzeitlinien der Minkowski'schen Welt,' Vienna *Sitzungsber.*, vol. cxxi. II *a*, Oct. 1912, pp. 1659–1759; see especially p. 1685 *et seq.*

any orthogonal curvilinear coordinates, such coordinates being indispensable for the treatment of a non-homaloidal world.

Let $F_{\iota\kappa}$ be a six-vector or a covariant antisymmetric tensor of rank two, so that $F_{\iota\iota}=0$, $F_{\iota\kappa}=-F_{\kappa\iota}$. Then, with any coordinates $u_1,\ldots u_4$, the four equations

$$\frac{\partial F_{\iota\kappa}}{\partial u_\lambda}+\frac{\partial F_{\kappa\lambda}}{\partial u_\iota}+\frac{\partial F_{\lambda\iota}}{\partial u_\kappa}=0 \quad . \quad . \quad . \quad . \quad . \quad . \quad (\mathrm{I}\,a)$$

are generally covariant, because their left-hand members are the (only four different) components of a tensor, to wit of an antisymmetric one of rank three. Using ordinary Cartesians, for instance, it will be seen at once that (I a), *i. e.*

$$\partial F_{23}/\partial u_4+\partial F_{34}/\partial u_2-\partial F_{24}/\partial u_3=0, \text{ etc.},$$
$$\partial F_{23}/\partial u_1+\partial F_{31}/\partial u_2+\partial F_{12}/\partial u_3=0,$$

are exactly of the form of the group (I) of electromagnetic equations. It requires, however, some care to find the correct translation of F_{23} etc. into the electro-magnetic components along coordinates of a more general kind. Such translation will be given presently. Meanwhile, to cover the group (II) of equations, consider the contravariant of $F_{\iota\kappa}$, that is, with the usual prescription as to summations,

$$F^{\iota\kappa}=g^{\iota\alpha}g^{\kappa\beta}F_{\alpha\beta}=\gamma_{\iota\alpha}\gamma_{\kappa\beta}F_{\alpha\beta}, \quad . \quad . \quad . \quad . \quad . \quad (35)$$

where $\gamma_{\iota\kappa}$ is the contravariant fundamental tensor corresponding to the chosen system u_i. Further, let σ_κ be a contravariant four-vector, embodying in its space-part the convection current, and g the determinant of the g_{ij}. Then the four equations

$$\frac{1}{\sqrt{-g}}\frac{\partial}{\partial u_\kappa}(\sqrt{-g}\,F^{\iota\kappa})=\sigma_\iota, \quad . \quad . \quad . \quad (\mathrm{II}\,a)$$

whose left hand members are the components of a contravariant four-vector, will again be generally contravariant *. That these equations are exactly of the form of the group (II) of electromagnetic equations is seen at a glance, at least for Cartesians, and with some attention, also for more general coordinates.

Thus, the eight equations contained in (I a), (II a), together with the six relations (35) express a set of electromagnetic

* Einstein usually employs a system with $g=-1$, so that (II a) are simplified to $\partial F^{\iota\kappa}/\partial u_\kappa=\sigma_\kappa$. But to fix thus the value of the fundamental determinant would hamper us unnecessarily.

vacuum-laws which are covariant with respect to any transformations of the four coordinates. It will be remembered that in Einstein's theory the linear relations (35) depend, among other things, also upon the gravitational field. But, as we have rejected his "equivalence hypothesis," our g_{ij}, and therefore also the relations between the polarizations and the forces, will depend only upon the chosen system of reference and, of course, upon the assumed fixed properties of the world.

It remains to write explicitly the components of the six-vectors $F_{\iota\kappa}$ and $F^{\iota\kappa}$ in terms of the components of $\mathfrak{M}$, etc. along the curvilinear axes of the system u_i, and thus to find also the explicit relations between the polarizations and the forces. [Then the original form (I), (II) of the Maxwellian equations, with u_4 as time, may be readopted and conveniently applied to any electromagnetic problem concerning empty space.]

It will be enough to do this for *orthogonal* curvilinear coordinates u_1, u_2, u_3, with any u_4 as time. The corresponding form of the line-element then becomes, as in (6),

$$ds^2 = g_{\kappa\kappa} du_\kappa^{\,2} = g_{11} du_1^{\,2} + g_{22} du_2^{\,2} + \ldots + g_{44} du_4^{\,2}, \quad . \quad (36)$$

and $g = g_{11} \ldots g_{44}$. In order to compare (I a) with (I), with our purpose in view, remember that, A_1, A_2, A_3 being the components of any three-vector $\mathbf{A}$ along the curvilinear coordinates in question, its divergence is

$$\text{div}\ \mathbf{A} = w_1 w_2 w_3 \left[\frac{\partial}{\partial u_1}\left(\frac{A_1}{w_2 w_3}\right) + \text{etc.}\right],$$

which covers the second of (I), and that the first of (I), with u_4 for t, splits into

$$\frac{\partial}{\partial u_4}\left(\frac{1}{w_2 w_3}\mathfrak{M}_1\right) + \frac{\partial}{\partial u_2}\left(\frac{E_3}{w_3}\right) - \frac{\partial}{\partial u_3}\left(\frac{E_2}{w_2}\right) = 0,$$

and two similar equations, where w_1, etc. are defined by $ds_\kappa = \frac{du_\kappa}{w_\kappa}$; ds_1, ds_2, ds_3 being the components of the (space) line-element. By (34), $ds_1^{\,2} = -g_{11} du_1^{\,2}$, etc., so that $\frac{1}{w_1} = \sqrt{-g_{11}}$, etc. Keeping this in mind, a glance at (I a) will suffice to see that these eqs. become identical with (I) if we put

$$F_{23} = \mathfrak{M}_1 \sqrt{g_{22} g_{33}}, \text{ etc.}; \quad F_{14} = E_1 \sqrt{-g_{11}}, \text{ etc.}$$

Again the fourth of (II a), for $\iota=4$, compared with the second of (II) gives

$$F^{14}=-\frac{1}{\sqrt{-g_{11}g_{44}}}\mathfrak{E}_1, \text{ etc.}$$

and at the same time $\sigma_4=\frac{1}{\sqrt{g_{44}}}\rho$. Finally, the first of (II), compared with (II a) for $\iota=1, 2, 3$ gives, as the remainder of the required dictionary,

$$F^{23}=M_1\sqrt{\frac{g_{11}}{g}}, \text{ etc.}; \ \sigma_1=\frac{\rho v_1}{\sqrt{-g_{11}g_{44}}}, \text{ etc.}$$

The relations between the forces and the polarizations, which are obviously of prime importance, follow at once. In fact, by (35) which for orthogonal coordinates becomes

$$F^{\iota\kappa}=F_{\iota\kappa}/g_{\iota\iota}g_{\kappa\kappa}, \quad . \quad . \quad . \quad . \quad . \quad , \qquad (35\,a)$$

we have $M_1=\mathfrak{M}_1\sqrt{g_{44}}$, $E_1=\mathfrak{E}_1\sqrt{g_{44}}$, and so on.

Thus, for *any* orthogonal system of curvilinear coordinates u^i, collecting the scattered formulæ,

$$\left.\begin{aligned} &F_{23}=\mathfrak{M}_1\sqrt{g_{22}g_{33}}, \text{ etc.}; \ F_{14}=E_1\sqrt{-g_{11}}, \text{ etc.} \\ &F^{23}=M_1\sqrt{\frac{g_{11}}{g}}, \text{ etc.}; \ F^{14}=-\mathfrak{E}_1\frac{1}{\sqrt{-g_{11}g_{44}}} \\ &\sigma_1=\frac{\rho v_1}{\sqrt{-g_{11}g_{44}}}, \text{ etc.}, \quad \sigma_4=\frac{\rho}{\sqrt{g_{44}}}, \end{aligned}\right\} . \qquad (37)$$

where v_1, v_2, v_3 are the components of $\mathbf{v}$ along the curvilinear "axes" u_1, u_2, u_3 and similarly for the remaining four vectors; and the relations between the forces and polarizations assume the remarkably simple form

$$\mathfrak{M}=\frac{1}{\sqrt{g_{44}}}\mathbf{M}, \quad \mathfrak{E}=\frac{1}{\sqrt{g_{44}}}\mathbf{E}. \quad . \quad . \quad . \qquad (38)$$

These latter are the simple relations supplementing (I) and (II). If u_4 is used as time, $\frac{1}{\sqrt{g_{44}}}$ plays the part of magnetic "permeability," and at the same time of dielectric "permittivity." Of course, according to (36), $dt=du_4\sqrt{g_{44}}$ would be the more appropriate time-element in the system under consideration. And if g_{44} is 1 or any constant, then by a mere change of the time unit the equations (I), (II) will assume their usual form $\partial \mathbf{M}/\partial t + \text{curl } \mathbf{E} = 0$, etc. This is certainly the case, without further assumptions, in any natural system, for then we have, as in (2), $g_{44} = g_4 = 1$. The dictionary, (37), is then also considerably simplified, giving $\sigma_{1,2,3} = \rho \mathbf{v}$, $\sigma_4 = \rho$, and so on. Remember that if $x_1, \ldots x_4$ be natural coordinates, say: r, ϕ, etc., then for any other system the value of g_{44} to be substituted in (38) is, by (4) and (2),

$$g_{44} = \left(\frac{\partial x_\kappa}{\partial u_4}\right)^2 g_\kappa = \left(\frac{\partial t}{\partial u_4}\right)^2 - \left(\frac{\partial r}{\partial u_4}\right)^2$$

$$-R^2 \sin^2 \frac{r}{\mathrm{R}} \left[\left(\frac{\partial \phi}{\partial u_4}\right)^2 + \sin^2\phi \left(\frac{\partial \theta}{\partial u_4}\right)^2\right] . \quad . \quad . \quad . \quad (39)$$

Thus, not only in all natural systems but also in an infinity of others, for which this expression has a constant value (say, r, ϕ, θ independent of u_4, although arbitrary functions of u_1, u_2, u_3, and $t = u_4$), the *polarizations become identical with the forces, and the Maxwellian equations retain their ordinary form*, no matter whether the world is homaloidal or curved. And for any system of orthogonal coordinates whatever the *broader* form, (I), (II) of Maxwell's equations is retained, with the reciprocal square root of (39) as permeability and permittivity.

Maxwell's equations will thus occupy a unique position in General Relativity, hitherto at least unparalleled by any other physically satisfactory laws, since they not only are generally covariant (as, for instance, Einstein's gravitational equations), but have also, unlike Einstein's recent products, the remarkable property of not being hitherto contradicted by any experiments or observations.

Questions relating to ponderomotive forces, electromagnetic energy etc. must for the present be omitted.

Here it will be enough to add that (I) and (II) give, in any natural system (owing to $\mathfrak{E} = \mathbf{E}$, $\mathfrak{M} = \mathbf{M}$) for the velocity of propagation of electromagnetic waves the constant value c, in agreement with the definition $ds = 0$ of the velocity of light (*cf.* p. 104), as it should be,—since light consists in such waves. Needless to repeat, that this vacuum velocity after our rejection of "equivalence"—is not modified by gravitation. And until any such phenomenon is discovered we are fully justified in adhering to the proposed purified theory.

6. *Dynamics of a Particle. Example of a covariant law of motion.*

Return to the general expression $ds^2 = g_{ij}du_i du_j$, where $u_1, \ldots u_4$ are any coordinates. Since du_i is a contravariant four-vector and ds an invariant, du_i/ds is again a contravariant vector; similarly d^2u_i/ds^2, and so on. Such expressions could, therefore, be employed for the construction of generally covariant or contravariant laws of motion of a particle, endowed, say, with some invariant "mass" or inertia-coefficient. It seems more convenient, however, to adopt another method.

As was already mentioned, the equations of motion of a free particle are contained in $\delta\int ds = 0$, the variational equation of the world-geodesics. And the idea easily suggests itself to derive possible laws of motion of a non-free particle (or one "acted on by external forces") from similar variational equations after an appropriate amplification of the integrand. The purpose of the present section is to give only a very simple *example* of a generally covariant law of motion obtainable by this method (but by no means to develop the general dynamics of a particle or of a system of particles).

Let Φ, a function of all the u_i, be a tensor of rank zero or what is called a *scalar*, and therefore a general invariant. Then Φds will again be invariant, and the laws of motion embodied in an equation of the form

$$\delta\int(1-2\Phi)ds = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (40)$$

will obviously retain their form in any reference system whatever, or will be generally covariant. Understanding by u_i the space-time coordinates of the particle in question, and considering u_i as fixed at the limits of the integral, develop (40) by the usual methods. Then the result will be a system of four differential equations, one of which is a consequence of the others,

$$\frac{d}{ds}\left[(1-2\Phi)\frac{\partial \dot{s}}{\partial \dot{u}_\kappa}\right] + (2\Phi - 1)\frac{\partial \dot{s}}{\partial u_\kappa} = -\frac{\partial \Phi}{\partial u_\kappa}, \quad . \quad . \quad (41)$$

where $\dot{s}^2 = g_{ij}\dot{u}_i\dot{u}_j$, the dot denoting the derivative with respect to a parameter which is ultimately made to coincide with s itself. Thus, for instance, if r, ϕ, θ, t are used as coordinates,

$$\frac{d}{ds}\left[(1-2\Phi)\frac{dr}{ds}\right]+\tfrac{1}{2}(1-2\Phi)\left[\frac{\partial g_2}{\partial r}\dot{\phi}^2+\frac{\partial g_3}{\partial r}\dot{\theta}^2\right]=\frac{\partial\Phi}{\partial r}, \text{ etc.} \quad (41a)$$

where $\dot{\phi}=d\phi/ds$, etc. The eqs. (41) can also be written without trouble in Weierstrassian or any other coordinates.

It is seen at once that the (invariant and, say, constant) mass m of the particle is replaced in the "momentum" by $(1-2\Phi)m$, and that Φ plays the part of a scalar *potential* of the "force" solliciting the particle.

Without entering, for the present, into a detailed interpretation of these equations of motion let us concentrate our attention upon the potential Φ, and let us see whether it is possible to construct a single differential equation for Φ, preferably of the second order, which would be generally covariant.

In order to obtain such an equation, start from

$$f_{\iota\kappa}=\frac{\partial^2\Phi}{\partial u_\iota\partial u_\kappa}-\begin{Bmatrix}\iota\kappa\\\lambda\end{Bmatrix}\frac{\partial\Phi}{\partial u_\lambda} \quad . \quad . \quad . \quad . \quad (42)$$

which (Φ being a scalar) is a covariant tensor of rank two, viz. a symmetrical one, $f_{\iota\kappa}=f_{\kappa\iota}$, since $\begin{Bmatrix}\iota\kappa\\\lambda\end{Bmatrix}=\begin{Bmatrix}\kappa\iota\\\lambda\end{Bmatrix}$. Equating the ten different constituents of this tensor to those of some other tensor (say, within matter) or to zero, in empty space, we should have at once a covariant system of equations for our potential. But these would be too many for our purpose, seeing that there is but one function to satisfy them. What we require is a single differential equation. In order to obtain it, the idea easily suggests itself to build up the mixed tensor of rank four $g^{\alpha\beta}f$ and to derive from it by a twofold "contraction" a tensor of rank zero or a scalar, thus :—Put $\alpha=\iota$ and sum over α from 1 to 4, so that the result will be a mixed tensor of rank two

$$H^{\beta}_{\kappa}=\sum_{\iota}g^{\iota\beta}f_{\iota\kappa}\,;$$

here put $\beta=\kappa$ and sum up over κ, obtaining the scalar

$$H^{\kappa}_{\kappa}=H=\sum_{\kappa}\sum_{\iota}g^{\iota\kappa}f_{\iota\kappa},$$

or in abbreviated notation

$$H=g^{\iota\kappa}f_{\iota\kappa}=\gamma_{\iota\kappa}f_{\iota\kappa},$$

$\gamma_{\iota\kappa}$ being, as always, the contravariant fundamental tensor.

If $f_{\iota\kappa}$ is as in (42), H is a genuine scalar or general invariant. It is in fact Beltrami's second differential parameter $\Delta_2\Phi$ whose definition can, obviously, be retained for a manifold of any number of dimensions.

Dropping the operand, Φ, we thus obtain the comparatively simple invariant differential operator

$$\Omega=\gamma_{\iota\kappa}\left[\frac{\partial^2}{\partial u_\iota \partial u_\kappa}-\begin{Bmatrix}\iota\kappa\\ \lambda\end{Bmatrix}\frac{\partial}{\partial u_\lambda}\right]; \quad . \quad . \quad . \quad (43)$$

terms to be summed up as before. Operating with this upon a scalar, as is the above "potential," and equating the result to another given scalar T, say, we shall have in

$$\Omega\Phi=T \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (44)$$

a differential equation of the second order for Φ, invariant with respect to any transformations of the coordinates. The scalar T can, for example, be zero outside of matter and, say, proportional to appropriately measured "density of mass" ρ within matter. I do not say that Φ is the gravitational potential; I am only constructing an example of an abstract generally covariant law of motion of a particle.

Having thus ascertained the general invariance of the operator Ω it is obviously interesting to see what its form is like in some simple reference system, more especially in a natural system. First of all, in any system of orthogonal coordinates we have, with $\gamma_{ii}=1/g_{ii}=1/g_i$,

$$\Omega=\frac{1}{g_i}\left(\frac{\partial^2}{\partial u_i^2}-\begin{Bmatrix}ii\\ \kappa\end{Bmatrix}\frac{\partial}{\partial u_\kappa}\right), \quad . \quad . \quad . \quad . \quad (43\,a)$$

and, in the natural system r, ϕ, etc., for instance, developing the second term of (43 a), with the tensor (2), and $c=1$,

$$\Omega=\frac{\partial^2}{\partial t^2}-\frac{1}{R^2\sin^2\frac{r}{R}\cdot\sin\phi}\left\{\frac{\partial}{\partial r}\left(R^2\sin^2\frac{r}{R}\cdot\sin\phi\frac{\partial}{\partial r}\right)\right.$$

$$\left.+\frac{\partial}{\partial\phi}\left(\sin\phi\frac{\partial}{\partial\phi}\right)+\frac{1}{\sin\phi}\frac{\partial^2}{\partial\theta^2}\right\}. \quad . \quad . \quad . \quad (43\,b)$$

In the second part of this operational equation the reader will easily recognize the Laplacian, $\nabla^2=\text{div}\,\nabla$, for a space of constant curvature $\frac{1}{R^2}$; thus, in any natural reference system the whole operator assumes the simple and familiar form $\Omega=\partial^2/\partial t^2-\nabla^2$, and the differential equation (44) becomes, with $T=4\pi\rho$,

$$\frac{\partial^2\Phi}{\partial t^2}-\nabla^2\Phi=4\pi\rho, \quad \ldots \ldots \quad (45)$$

where the scalar ρ is to be considered as some given function of the four variables. The "potential" Φ is thus propagated, in any natural system, with light velocity c, here assumed as unit velocity. If $\rho\neq 0$ within a certain region, then apart from waves (satisfying the reduced equation), Φ can be represented as the retarded potential of that distribution, or it can be treated by the well-known four-dimensional method. This completes the eqs. of motion (41) or (41 *a*). In a first approximation we should have Newtonian planetary motion, with obvious complications in higher approximations. As has already been said, the above is intended merely as an example of generally covariant laws of motion of a particle. Yet, after all, (45) or, in general coordinates, (44), with (41) may turn out to be helpful in describing gravitation. It is true that in Einstein's "Entwurf" of 1913 (§ 7) the question about the possibility of reducing the gravitational field to a scalar is answered in the negative. Einstein's objections, however, are based upon various assumptions which are by no means unavoidable. Again, his chief objection (*loc. cit.* p. 22) is based upon the restricted (Lorentzian) covariance of that reduction to a scalar which he had in mind when writing that paragraph, while our set of equations is generally covariant.

At any rate the above example has seemed sufficiently interesting and instructive to be inserted here. Notice that if the "attracting body," *i. e.* the region of $\rho\neq 0$, with all of its distributional properties, is itself at rest in a natural system, then this can with advantage be taken as the reference system, converting the retarded potential into an ordinary one. In general, however, this will not be the case, and—if higher approximations are at all contemplated—the simple potential would have to be replaced by an appropriate solution of the general equation (44), with (43) as the differential operator.

March 25, 1918.

Note, added June 16th.—If the central point-mass M is *at rest* in a natural reference system, we have, by (43 *b*),

$$\Phi=\frac{M}{R}\cot\frac{r}{R}. \quad \ldots \ldots \ldots \quad (46)$$

The equations of motion of the planet, which can easily be written down according to (41 *a*), yield a plane orbit. If this plane is taken to be $\phi=\text{const.}=\pi/2$, then, with $c=1$, and $u=\frac{1}{R}\cot\frac{r}{R}$, the differential equation of the orbit is

$$\frac{d^2u}{d\theta^2}+(1+\gamma)u=\frac{M}{h^2}, \quad \ldots \ldots \ldots \quad (47)$$

where

$$h=R^2\sin^2\frac{r}{R}\cdot\frac{d\theta}{dt}=\text{const.},$$

$$\gamma=M^2/h^2.$$

The solution is

$$u=\frac{M}{h^2(1+\gamma)}\{1+e\cos(\theta\sqrt{1+\gamma})\}, \quad \ldots \ldots \quad (48)$$

where e=const. With appropriate "initial" data this is an ellipse, with eccentricity e and with moving perihelion. The motion of the perihelion is, per period T of revolution,

$$\epsilon=-2\pi\left\{1-\frac{1}{\sqrt{1+\gamma}}\right\},$$

i. e. rejecting γ^2 and higher powers, and replacing c,

$$\epsilon=-\frac{\pi M^2}{h^2c^2}=-\frac{4\pi^3a^2}{c^2T^2(1-e^2)}, \quad \ldots \ldots \quad (49)$$

where a is the major semiaxis. Thus, the proposed equations give a *negative* (or retrograde) *secular motion of the perihelion,* equal to *minus* one sixth of that yielded by Einstein's gravitation theory.

By (49) we should have, per century, the perihelion motion multiplied by the eccentricity,

$$e\delta\varpi=-1''\cdot48 \quad \text{for Mercury,}$$

and $-0''\cdot010$, $-0''\cdot011$, $-0''\cdot021$ for Venus, Earth, and Mars respectively. According to Newcomb ('Astronomical Constants') the secular excess for Mercury, not accountable for by the perturbation due to all the other planets of our system, is $+8''\cdot48\pm0\cdot43$. Thus, if the above equations of motion are accepted, the *true excess* for Mercury would be still greater, viz.

$$e\delta\varpi=8''\cdot48+1''\cdot48=9''\cdot96 \ (\pm0\cdot43),$$

or $\delta\varpi=48''\cdot44$. I understand from a conversation with Mr. Harold Jeffreys, who has already found a satisfactory representation of Mercury's $8''\cdot48$ and of the motion of the node of Venus by means of a modification of Seeliger's zodiacal-light matter (M.N., R.A.S., December 1916, p. 112), that the above, increased, excess of about $10''$ could equally well, and possibly "more easily," be accounted for by an appropriate distribution of the said disturbing matter.

VIII. *The Practical Importance of the Confluent Hypergeometric Function.* By H. A. Webb, *M.A.*, and John R. Airey, *M.A.*, *D.Sc.**

[Plate VI.]

§ 1. *Introduction.*

IT is well-known that many physical and engineering problems depend for solution on differential equations of the type

$$\frac{d^2y}{dx^2}+f(x)\cdot\frac{dy}{dx}+\phi(x)\cdot y=0, \quad . \quad . \quad . \quad (1)$$

where $f(x)$ and $\phi(x)$ are given functions of x. For example, the investigation of the periods of lateral vibration of a flexible non-uniform rope or chain †, or the periods of vibration of a circular disk ‡, leads to an equation of this type. Again, the whirling speed of a non-cylindrical shaft, or the period of lateral vibration of a non-cylindrical bar, such as an air-screw blade, can be found, with two-figure accuracy, by the solution of such an equation§; and in fact many vibration problems in various branches of physics lead to such equations. To take another illustration, the crippling end-load of a tapered aeroplane strut, whatever law of taper is adopted, could be found if we could solve equation (1); other problems of elastic instability lead to equations of this type, and may be brought into prominence in aeronautics by the urgency of saving weight.

In structures, such as aeroplanes or bridges, the liability to secondary failure (*i. e.* elastic instability) must be foreseen and estimated, as well as the liability to primary, or stress, failure. In running machinery it is important that the period of free vibrations shall be well above, or below, the given running speed, to avoid resonance; in instruments for producing sound, on the other hand, it is required that the period of free vibrations shall have a given value, to secure resonance.

In any of these cases, the problem presents itself to the designer somewhat as follows. The main outlines of the

* Communicated by the Authors.

† Airey, "The Oscillations of Chains," Phil. Mag. June 1911.

‡ Airey, "The Vibrations of Circular Plates," Proc. Phys. Soc. April 1911.

§ Webb, "The Whirling of Shafts," Engineering, November 1917.

design, including probably the over-all dimensions, are already settled by various considerations with which we are not now concerned. But we are allowed some latitude in detail design, which we are to use to avoid elastic failure, or to avoid, or to secure, resonance, as the case may be. We want therefore to be able to calculate, *roughly but quickly*, the effect on the crippling load, or the period, of various possible alterations. We want in fact to make several trials—the more the better—and choose the one we like best. Finally, when the design is complete, we wish to check it carefully by a more accurate calculation.

The functions $f(x)$ and $\phi(x)$ in equation (1) are to be considered, for a tentative design, to be defined by their graphs, which must be represented, for the range of values of x required, by empirical formulæ, the closeness of the representation giving some idea of the accuracy to be expected in the solution. These empirical formulæ should be of the simplest type, *e. g.* polynomials, or the ratios of linear or quadratic functions of x, otherwise time is wasted in constructing them. What is required therefore is a list of suitable equations of the type (1) that are soluble in terms of tabulated functions. The two important characteristics are that $f(x)$ and $\phi(x)$ should be of a simple type, and that they should contain several arbitrary constants; we can then hope to make them fit our graphs fairly well without much trouble.

When $f(x)$ and $\phi(x)$ are constants, the solution in terms of circular and exponential functions is well-known. A useful list of equations soluble by Bessel functions, with appropriate tables, has been given by Jahnke and Emde*. It is the object of this paper to show the value, from this point of view, of the confluent hypergeometric function, tables and graphs of which are given in § 4. For quick work graphs are more convenient than tables. A list of differential equations likely to be useful to designers, and soluble by means of these tables and graphs, is given in § 3. Some properties of the functions that were used in constructing the tables, and would be useful in extending them, are given in § 2.

It may perhaps be argued that few engineers have the mathematical ability for such scientific methods of design. But it should be remembered that many engineers acquire at their technical college or university a high degree of mathematical skill; and if they lose it afterwards, it is because

* *Funktionentafeln*, Teubner, 1909.

they find mathematical works of reference rather indigestible, and gradually cease to consult them. For example, an excellent summary, from a purely mathematical point of view, of the properties of the function we are going to consider is given in Whittaker and Watson's 'Modern Analysis'*; but it would be hard reading for engineers.

Or if it is objected that the engineer can hardly be expected to be familiar with the function theory of linear differential equations and may get into trouble over singularities, he might reply, if sufficiently well read, that the equation can't have singularities in the range of x considered, unless $f(x)$ or $\phi(x)$, or both, become infinite, and he would notice that from the graphs. Or he might say that he is not looking for a rule to which there are *no* exceptions. He wants a rule that *generally* works *quickly*, and he is prepared to risk an occasional failure, because he intends to refer the finished design for a final check to an expert mathematician. Divergent series have often been used by physicists in much the same spirit, and with few, if any, failures. Finally, many expert mathematicians have come into contact with engineering work recently under war conditions; they may have opportunity and inclination to assist in design on the lines we have indicated.

§ 2. *Properties of the confluent hypergeometric function.*

We define the function $M(\alpha, \gamma, x)$ as follows:—

$$M(\alpha, \gamma, x)=1+\frac{\alpha}{1.\gamma}.x+\frac{\alpha(\alpha+1)}{1.2.\gamma(\gamma+1)}.x^2+\frac{\alpha(\alpha+1)(\alpha+2)}{1.2.3.\gamma(\gamma+1)(\gamma+2)}.x^3 + \ldots\ldots \text{ to infinity.} \quad (2)$$

The series is absolutely and uniformly convergent for all values of α, γ, and x, real or complex, except only when γ is zero or a negative integer; this case is supposed to be excluded.

The function $M(\alpha, \gamma, x)$ has been discussed under various notations by several writers †. The following is a list of such properties of the function as are of use for our purpose; most of them are easily verified from the definition (2).

* Second edition, 1915, Chapter XVI.

† For a list of references see Whittaker & Watson, *loc. cit.*

I. $y = M(\alpha, \gamma, x)$

satisfies the differential equation

$$x\frac{d^2y}{dx^2} + (\gamma - x)\frac{dy}{dx} - \alpha y = 0. \quad . \quad . \quad . \quad . \quad (3)$$

II. The complete solution of the differential equation (3) is

$$y = A\,.\,M(\alpha, \gamma, x) + B\,.\,x^{1-\gamma}.\,M(\alpha - \gamma + 1, 2 - \gamma, x), \quad (4)$$

which we shall write for brevity

$$y = \overline{M}(\alpha, \gamma, x), \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (5)$$

where A and B are arbitrary constants of integration; *except only when* γ *is a positive integer*, in which case* the coefficient of B is either infinite or identical with the coefficient of A. In this case the complete solution of (3) may be written

$$y = [A + C\log x]\,.\,M(\alpha, \gamma, x)$$
$$+ C\left[\frac{\alpha x}{\gamma}\left(\frac{1}{\alpha} - \frac{1}{\gamma} - 1\right) + \frac{\alpha(\alpha+1)}{\gamma(\gamma+1)}\cdot\frac{x^2}{1.2}\left(\frac{1}{\alpha} + \frac{1}{\alpha+1} - \frac{1}{\gamma} - \frac{1}{\gamma+1} - 1 - \frac{1}{2}\right)\right.$$
$$+ \frac{\alpha(\alpha+1)(\alpha+2)}{\gamma(\gamma+1)(\gamma+2)}\cdot\frac{x^3}{1.2.3}\left(\frac{1}{\alpha} + \frac{1}{\alpha+1} + \frac{1}{\alpha+2} - \frac{1}{\gamma} - \frac{1}{\gamma+1} - \frac{1}{\gamma+2}\right.$$
$$\left. - 1 - \frac{1}{2} - \frac{1}{3}\right)$$
$$\left. + \ldots \text{ to infinity}\right], \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (6)$$

where A and C are arbitrary constants of integration.

III. $$M(\alpha, \gamma, x) = e^x.\,M(\gamma - \alpha, \gamma, -x) \quad . \quad . \quad . \quad . \quad . \quad . \quad (7)$$

$$x^{1-\gamma}M(\alpha - \gamma + 1, 2 - \gamma, x) = e^x.\,x^{1-\gamma}.\,M(1 - \alpha, 2 - \gamma, -x). \quad (8)$$

From (7) and (8) it follows that tables will not be required for *negative* values of x, if the tables cover wide enough ranges of α and γ.

IV. The asymptotic expansion of $M(\alpha, \gamma, x)$ for large values of x is

* The situation is similar to that which arises with Bessel's equation when n is a positive integer, and a new function is required for the second solution.

$$M(\alpha, \gamma, x)$$

$$= \frac{\Gamma(\gamma)}{\Gamma(\gamma-\alpha)} \cdot (-x)^{-\alpha} \cdot \left\{ 1 - \frac{\alpha(\alpha-\gamma+1)}{1} \cdot \frac{1}{x} + \frac{\alpha(\alpha+1)(\alpha-\gamma+1)(\alpha-\gamma+2)}{1.2} \cdot \frac{1}{x^2} - \ldots \right\}$$

$$+ \frac{\Gamma(\gamma)}{\Gamma(\alpha)} \cdot e^x \cdot x^{\alpha-\gamma} \left\{ 1 + \frac{(1-\alpha)(\gamma-\alpha)}{1} \cdot \frac{1}{x} + \frac{(1-\alpha)(2-\alpha)(\gamma-\alpha)(\gamma-\alpha+1)}{1.2} \cdot \frac{1}{x^2} + \ldots \right\}. \quad (9)$$

Both these series diverge for all values of x, but they have the property that the error involved in taking the sum to n terms to be the value of the series, is less than the nth term.

V. $$\frac{d}{dx} \cdot M(\alpha, \gamma, x) = \frac{\alpha}{\gamma} \cdot M(\alpha+1, \gamma+1, x). \quad . \quad . \quad . \quad (10)$$

$$(1-\alpha) \cdot \int_{x=0} M(\alpha, \gamma, x) \cdot dx = (1-\gamma) \cdot M(\alpha-1, \gamma-1, x) + (\gamma-1). \quad . \quad . \quad . \quad (11)$$

Hence the function can easily be differentiated or integrated.

VI. The following difference relations would be useful for extending the tables :—

$$\left.\begin{aligned}
&\frac{x}{\gamma} \cdot M(\alpha+1, \gamma+1, x) = M(\alpha+1, \gamma, x) - M(\alpha, \gamma, x), \\
&\alpha \cdot M(\alpha+1, \gamma+1, x) = (\alpha-\gamma) \cdot M(\alpha, \gamma+1, x) + \gamma \cdot M(\alpha, \gamma, x), \\
&(\alpha+x) \cdot M(\alpha+1, \gamma+1, x) = (\alpha-\gamma) \cdot M(\alpha, \gamma+1, x) \\
&\qquad + \gamma \cdot M(\alpha+1, \gamma, x), \\
&\alpha\gamma \cdot M(\alpha+1, \gamma, x) = \gamma(\alpha+x) \cdot M(\alpha, \gamma, x) \\
&\qquad - x(\gamma-\alpha) \cdot M(\alpha, \gamma+1, x), \\
&\alpha \cdot M(\alpha+1, \gamma, x) = (x+2\alpha-\gamma) \cdot M(\alpha, \gamma, x) \\
&\qquad + (\gamma-\alpha) \cdot M(\alpha-1, \gamma, x), \\
&\frac{\gamma-\alpha}{\gamma} \cdot x \cdot M(\alpha, \gamma+1, x) = (x+\gamma-1) \cdot M(\alpha, \gamma, x) \\
&\qquad + (1-\gamma) \cdot M(\alpha, \gamma-1, x).
\end{aligned}\right\} (12)$$

VII. If $\alpha=\frac{1}{2}\gamma$, $M(\alpha, \gamma, x)$ can be expressed in terms of a Bessel function. In fact

$$M(\tfrac{1}{2}\gamma, \gamma, x)=2^{\gamma-1} \cdot \Gamma\left(\frac{\gamma+1}{2}\right) . e^{\frac{x}{2}} \cdot x^{\frac{1-\gamma}{2}} \cdot I_{\frac{\gamma-1}{2}}(\tfrac{1}{2}x) \quad . \quad (13)$$

VIII. The *Error Function*$=\phi(x)$

$$=\frac{2}{\sqrt{\pi}} . \int_0^x e^{-x^2} . dx$$

$$=\frac{2x}{\sqrt{\pi}} . e^{-x^2} . M(1, \tfrac{3}{2}, x^2). \quad . \quad . \quad . \quad . \quad (14)$$

The *Incomplete* γ *Function*$=\gamma(n, x)$

$$=\int_0^x e^{-t} . t^{n-1} . dt$$

$$=\frac{1}{n} . e^{-x} . x^n . M(1, n+1, x). \quad . \quad . \quad . \quad (15)$$

Sonine's Polynomial $=T_m^n(x)$

$$=\frac{(-1)^n}{m\,!\;n!} . M(-n, m+1, x). \quad . \quad . \quad . \quad (16)$$

The *Function of the Parabolic Cylinder*

$$=D_n(x)$$

$$=(-1)^n . e^{\frac{1}{4}x^2} . \frac{d^n}{dx^n}(e^{-\frac{1}{4}x^2})$$

= (if n is *even*)

$$\frac{(-2)^{\frac{n}{2}}}{\sqrt{\pi}} . \Gamma\left(\frac{n+1}{2}\right) . e^{-\frac{1}{4}x^2} . M\left(-\frac{n}{2}, \tfrac{1}{2}, \tfrac{1}{2}x^2\right), \quad (17\,a)$$

and = (if n is *odd*)

$$\frac{(-2)^{\frac{n-1}{2}}}{\sqrt{\pi}} . \Gamma\left(\frac{n}{2}\right) . e^{-\frac{1}{4}x^2} . nx . M\left(\frac{1-n}{2}, \tfrac{3}{2}, \tfrac{1}{2}x^2\right) \quad . \quad . \quad (17\,b)$$

We will also, following *Jahnke und Emde* *, define $Z_p(x)$ as

$$AJ_p(x)+BN_p(x),$$

where A and B are arbitrary constants, and $J_p(x)$, $N_p(x)$ are Bessel functions, in the usual notation. So that

$$y=Z_p(x) \quad . \quad . \quad . \quad . \quad . \quad . \quad (18)$$

is the complete solution of the differential equation

$$\frac{d^2y}{dx^2}+\frac{1}{x}\cdot\frac{dy}{dx}+\left(1-\frac{p^2}{x^2}\right)y=0. \quad . \quad . \quad . \quad (19)$$

Reference should be made to *Jahnke und Emde's* tables and graphs of these functions †, which are presented in a form convenient for engineers.

§ 3. *Soluble differential equations.*

The following differential equations are soluble by means of Bessel functions or M functions, a, b, c, α, γ, l, m, n, p, q, r, s, t being any numerical constants whatever.

(A) $$\frac{d^2y}{dx^2}+p\frac{dy}{dx}+ly=0.$$

(B) $$\frac{d^2y}{dx^2}+\frac{q}{x}\cdot\frac{dy}{dx}+\frac{m}{x}\cdot y=0.$$

(C) $$\frac{d^2y}{dx^2}+\frac{q}{x}\cdot\frac{dy}{dx}+\left(l+\frac{n}{x^2}\right)y=0.$$

(D) $$\frac{d^2y}{dx^2}+\frac{q}{x}\cdot\frac{dy}{dx}+\frac{1}{x^2}(lx^{2r}+n)y=0.$$

(E) $$\frac{d^2y}{dx^2}+(px+q)\frac{dy}{dx}+(mx+n)y=0.$$

(F) $$\frac{d^2y}{dx^2}+\left(p+\frac{q}{x}\right)\frac{dy}{dx}+\left(l+\frac{m}{x}\right)y=0.$$

(G) $$\frac{d^2y}{dx^2}+(px+q)\frac{dy}{dx}+(lx^2+mx+n)y=0.$$

(H) $$\frac{d^2y}{dx^2}+\left(p+\frac{q}{x}\right)\frac{dy}{dx}+\left(l+\frac{m}{x}+\frac{n}{x^2}\right)y=0.$$

* *Loc. cit.* p. 165.
† *Loc. cit.* pp. 106–168.

(K) $\frac{d^2y}{dx^2}+\left(px+q+\frac{s}{x}\right)\frac{dy}{dx}+\frac{1}{2}q\left(lx^2+px+n+\frac{s}{x}\right)y=0.$

(L) $\frac{d^2y}{dx^2}+\frac{1}{x}(px^r+q)\frac{dy}{dx}+\frac{1}{x^2}(lx^{2r}+mx^r+n)y=0.$

The solutions are as follows:—

(A) $\frac{d^2y}{dx^2}+p\frac{dy}{dx}+(n^2+\frac{1}{4}p^2)y=0$

$$y=e^{-\frac{1}{2}px}(\mathrm{A}\cos nx+\mathrm{B}\sin nx).$$

(B) $\frac{d^2y}{dx^2}+\frac{p+1}{x}\frac{dy}{dx}+\frac{m}{x}\cdot y=0.$

$$y=x^{-\frac{1}{2}p}\cdot \mathrm{Z}_p(2\sqrt{mx}).$$

(C) $\frac{d^2y}{dx^2}+\frac{1-2\alpha}{x}\cdot\frac{dy}{dx}+\left(\gamma^2+\frac{\alpha^2-p^2}{x^2}\right)y=0$

$$y=x^\alpha\,.\,\mathrm{Z}_p(\gamma x).$$

(D) $\frac{d^2y}{dx^2}+\frac{1-2\alpha}{x}\cdot\frac{dy}{dx}+\frac{1}{x^2}\cdot(\gamma^2r^2x^{2r}+\alpha^2-p^2r^2)y=0$

$$y=x^\alpha\,.\,\mathrm{Z}_p(\gamma x^r).$$

(E) $\frac{d^2y}{dx^2}+2(p+qx)\frac{dy}{dx}+y\,[4\alpha q+p^2-q^2m^2+2qx(p+qm)]=0$

$$y=e^{-(p+qm)x}\,.\,\overline{\mathrm{M}}[\alpha,\tfrac{1}{2},-q(x-m)^2].$$

(F) $\frac{d^2y}{dx^2}+\left(2p+\frac{\gamma}{x}\right)\frac{dy}{dx}+y[p^2-t^2+\frac{1}{x}\cdot(\gamma p+\gamma t-2\alpha t)]=0$

$$y=e^{-(p+t)x}\,.\,\overline{\mathrm{M}}(\alpha,\gamma,2t\,x).$$

(G) $\frac{d^2y}{dx^2}+2(p+qx)\frac{dy}{dx}+y[q+c(1-4\alpha)$

$$+(p+qx)^2-c^2(x-m)^2]=0$$

$$y=e^{-px-\frac{1}{2}qx^2-\frac{1}{2}c(x-m)^2}\,.\,\overline{\mathrm{M}}[\alpha,\tfrac{1}{2},c(x-m)^2].$$

(H) $\frac{d^2y}{dx^2}+\left(2p+\frac{q}{x}\right)\frac{dy}{dx}$

$$+y[p^2-t^2+\frac{1}{x}(pq+\gamma t-2\alpha t)+\frac{1}{4x^2}(\gamma-q)(2-q-\gamma)]=0$$

$$y=e^{-(p+t)x}\,.\,x^{\frac{\gamma-q}{2}}\,.\,\overline{\mathrm{M}}(\alpha,\gamma,2tx).$$

$$\text{(K)}\quad \frac{d^2y}{dx^2}+\left[\frac{2\gamma-1}{x}+2a+2(b-c)x\right]\frac{dy}{dx}$$

$$+y\left[\frac{a(2\gamma-1)}{x}+(a^2+2b\gamma-4\alpha c)+2a(b-c)x+b(b-2c)x^2\right]=0$$

$$y=e^{-ax-\frac{1}{2}bx^2}.\,\overline{\mathrm{M}}(\alpha,\gamma,cx^2).$$

$$\text{(L)}\quad \frac{d^2y}{dx^2}+\frac{1}{x}(2px^r+qr-r+1)\frac{dy}{dx}$$

$$+\frac{y}{x^2}\left[(p^2-t^2)x^{2r}+r(pq+\gamma t-2\alpha t)x^r\right.$$

$$\left.+\tfrac{1}{4}r^2(\gamma-q)(2-q-\gamma)\right]=0$$

$$y=e^{\frac{-(p+t)}{r}.x^r}.\,x^{\frac{r}{2}(\gamma-q)}.\,\overline{\mathrm{M}}\left(\alpha,\gamma,\frac{2t}{r}.\,x^r\right).$$

§ 4. *Tables and Graphs of* $\mathrm{M}(\alpha,\gamma,x)$.

The following tables of $\mathrm{M}(\alpha,\gamma,x)$ were calculated, for small values of x, from the series in ascending powers of this argument, and for large values, from the asymptotic expansions. When α and γ are positive integers, two or three values of M, for a particular value of x, are required to give the other results by means of suitable recurrence formulæ. The last two formulæ of (12) were employed to find further values of M along vertical columns and horizontal rows; the first four, to "turn the corners" and fill in the results in the rectangle of values thus obtained. When α is a negative integer, the M function is a polynomial which is easily evaluated. A similar procedure was adopted in the case of α equal to half an odd positive or negative integer, only two preliminary calculations of M being required to give the remaining 47 for each value of the argument x.

Four significant figures are given in the tables. The numbers, however, must be multiplied by the power of ten indicated by the figure after the comma. Thus,

$$\mathrm{M}(4,1,4)=2603;\ \mathrm{M}(3,2,10)=132200;$$

and $\qquad \mathrm{M}(-\tfrac{1}{2},4,10)=-3{\cdot}419.$

	α	$\gamma=1$	2	3	4	5	6	7
$x=1.$	4·0	1540, −2	5890, −3	3624, −3	2718, −3	2254, −3	1978, −3	1798, −3
	3·5	1220, −2	4920, −3	3146, −3	2423, −3	2048, −3	1823, −3	1675, −3
	3·0	9514, −3	4077, −3	2718, −3	2155, −3	1858, −3	1678, −3	1559, −3
	2·5	7279, −3	3348, −3	2338, −3	1911, −3	1683, −3	1544, −3	1450, −3
	2·0	5437, −3	2718, −3	2000, −3	1690, −3	1522, −3	1419, −3	1348, −3
	1·5	3932, −3	2179, −3	1701, −3	1490, −3	1375, −3	1302, −3	1253, −3
	1·0	2718, −3	1718, −3	1437, −3	1310, −3	1239, −3	1194, −3	1163, −3
	0·5	1753, −3	1328, −3	1204, −3	1147, −3	1114, −3	1093, −3	1079, −3
	−0·5	4252, −4	7262, −4	8218, −4	8682, −4	8955, −4	9135, −4	9262, −4
	−1·0	0000, 0	5000, −4	6667, −4	7500, −4	8000, −4	8333, −4	8571, −4
	−1·5	−3010, −4	3153, −4	5324, −4	6443, −4	7128, −4	7591, −4	7925, −4
	−2·0	−5000, −4	1667, −4	4167, −4	5500, −4	6333, −4	6905, −4	7321, −4
	−2·5	−6163, −4	4914, −5	3176, −4	4661, −4	5610, −4	6270, −4	6757, −4
	−3·0	−6667, −4	−4167, −5	2333, −4	3917, −4	4952, −4	5685, −4	6230, −4
$x=2.$	4·0	1059, −1	2709, −2	1232, −2	7389, −3	5195, −3	4027, −3	3328, −3
	3·5	7495, −2	2019, −2	9595, −3	5973, −3	4326, −3	3434, −3	2892, −3
	3·0	5172, −2	1478, −2	7389, −3	4792, −3	3584, −3	2918, −3	2508, −3
	2·5	3458, −2	1059, −2	5613, −3	3811, −3	2952, −3	2470, −3	2168, −3
	2·0	2217, −2	7389, −3	4195, −3	3000, −3	2416, −3	2082, −3	1869, −3
	1·5	1340, −2	4978, −3	3073, −3	2335, −3	1964, −3	1747, −3	1607, −3
	1·0	7389, −3	3195, −3	2195, −3	1792, −3	1584, −3	1459, −3	1377, −3
	0·5	3442, −3	1905, −3	1516, −3	1353, −3	1265, −3	1212, −3	1176, −3
	−0·5	−3690, −4	3891, −4	6145, −4	7199, −4	7805, −4	8197, −4	8471, −4
	−1·0	−1000, −3	0000, 0	3333, −4	5000, −4	6000, −4	6667, −4	7143, −4
	−1·5	−1147, −3	−2254, −4	1345, −4	3297, −4	4526, −4	5373, −4	5993, −4
	−2·0	−1000, −3	−3333, −4	0000, 0	2000, −4	3333, −4	4286, −4	5000, −4
	−2·5	−6963, −4	−3600, −4	−8524, −5	1034, −4	2377, −4	3376, −4	4146, −4
	−3·0	−3333, −4	−3333, −4	−1333, −4	3333, −5	1619, −4	2619, −4	3413, −4

	α	$\gamma=1$	2	3	4	5	6	7
$x = 3.$	**4·0**	5624, −1	1105, −1	4017, −2	2009, −2	1220, −2	8416, −3	6335, −3
	3·5	3662, −1	7542, −2	2867, −2	1492, −2	9386, −3	6670, −3	5149, −3
	3·0	2310, −1	5021, −2	2009, −2	1094, −2	7149, −3	5249, −3	4162, −3
	2·5	1399, −1	3241, −2	1375, −2	7885, −3	5383, −3	4096, −3	3345, −3
	2·0	8034, −2	2009, −2	9149, −3	5575, −3	4000, −3	3168, −3	2671, −3
	1·5	4272, −2	1178, −2	5866, −3	3847, −3	2926, −3	2424, −3	2117, −3
	1·0	2009, −2	6362, −3	3575, −3	2575, −3	2099, −3	1832, −3	1665, −3
	0·5	7380, −3	2981, −3	2019, −3	1653, −3	1471, −3	1365, −3	1297, −3
	−0·5	−1562, −3	−4756, −5	3657, −4	5496, −4	6520, −4	7169, −4	7615, −4
	−1·0	−2000, −3	−5000, −4	0000, 0	2500, −4	4000, −4	5000, −4	5714, −4
	−1·5	−1419, −3	−5961, −4	−1839, −4	6060, −5	2219, −4	3361, −4	4212, −4
	−2·0	−5000, −4	−5000, −4	−2500, −4	−5000, −5	1000, −4	2143, −4	3036, −4
	−2·5	3694, −4	−3203, −4	−2445, −4	−1058, −4	2022, −5	1255, −4	2125, −4
	−3·0	1000, −3	−1250, −4	−2000, −4	−1250, −4	−2857, −5	6250, −5	1429, −4
$x = 4.$	**4·0**	2603, 0	4186, −1	1274, −1	5460, −2	2910, −2	1800, −2	1237, −2
	3·5	1583, 0	2659, −1	8446, −2	3769, −2	2084, −2	1333, −2	9432, −3
	3·0	9282, −1	1638, −1	5460, −2	2550, −2	1470, −2	9750, −3	7124, −3
	2·5	5193, −1	9702, −2	3421, −2	1684, −2	1018, −2	7037, −3	5327, −3
	2·0	2730, −1	5460, −2	2060, −2	1080, −2	6900, −3	5000, −3	3938, −3
	1·5	1312, −1	2860, −2	1175, −2	6663, −3	4551, −3	3486, −3	2872, −3
	1·0	5460, −2	1340, −2	6200, −3	3900, −3	2900, −3	2375, −3	2062, −3
		1684, −2	5091, −3	2870, −3	2111, −3	1763, −3	1571, −3	1453, −3
	−0·5	−3519, −3	−6489, −4	5482, −5	3486, −4	5057, −4	6026, −4	6680, −4
	−1·0	−3000, −3	−1000, −3	−3333, −4	0000, 0	2000, −4	3333, −4	4286, −4
	−1·5	−9230, −4	−7586, −4	−4100, −4	−1571, −4	2365, −5	1572, −4	2594, −4
	−2·0	1000, −3	−3333, −4	−3333, −4	−2000, −4	−6667, −5	4762, −5	1429, −4
	−2·5	2111, −3	6138, −5	−2005, −4	−1808, −4	−1021, −4	−1568, −5	6522, −5
	−3·0	2333, −3	3333, −4	−6667, −5	−1333, −4	−1048, −4	−4762, −5	1587, −5

	a	$\gamma=1$	2	3	4	5	6	7
$x = 5.$	4·0	1103, 1	1509, 0	3958, −1	1484, −1	7033, −2	3929, −2	2475, −2
	3·5	6331, 0	9013, −1	2461, −1	9598, −2	4721, −2	2730, −2	1775, −2
	3·0	3488, 0	5194, −1	1484, −1	6050, −2	3104, −2	1866, −2	1257, −2
	2·5	1824, 0	2861, −1	8614, −2	3697, −2	1990, −2	1251, −2	8761, −3
	2·0	8905, −1	1484, −1	4757, −2	2171, −2	1238, −2	8189, −3	6000, −3
	1·5	3938, −1	7074, −2	2453, −2	1209, −2	7397, −3	5207, −3	4022, −3
	1·0	1484 −1	2948, −2	1139, −2	6236, −3	4189, −3	3189, −3	2626, −3
	0·5	4008, −2	9419, −3	4382, −3	2841, −3	2190, −3	1855, −3	1658, −3
	−0·5	−7014, −3	−1537, −3	−3530, −4	1033, −4	3352, −4	4734, −4	5645, −4
	−1·0	−4000, −3	−1500, −3	−6667, −4	−2500, −4	0000, 0	1667, −4	2857, −4
	−1·5	6693, −4	−6543, −4	−5252, −4	−3157, −4	−1382, −4	2959, −6	1153, −4
	−2·0	3500, −3	1667, −4	−2500, −4	−2500, −4	−1667, −4	−7143, −5	1786, −5
	−2·5	3941, −3	6586, −4	9492, −7	−1430, −4	−1411, −4	−9309, −5	−3181, −5
	−3·0	2667, −3	7917, −4	1667, −4	−4167, −5	−9524, −5	−8631, −5	−5159, −5
$x = 6.$	4·0	4397, 1	5245, 0	1210, 0	4034, −1	1718, −1	8731, −2	5061, −2
	3·5	2398, 1	2968, 0	7112, −1	2461, −1	1088, −1	5722, −2	3427, −2
	3·0	1251, 1	1614, 0	4034, −1	1457, −1	6707, −2	3671, −2	2280, −2
	2·5	6168, 0	8343, −1	2189, −1	8300, −2	4009, −2	2295, −2	1486, −2
	2·0	2824, 0	4034, −1	1121, −1	4505, −2	2302, −2	1390, −2	9451, −3
	1·5	1163, 0	1774, −1	5294, −2	2287, −2	1255, −2	8086, −3	5826, −3
	1·0	4034, −1	6707, −2	2202, −2	1051, −2	6341, −3	4451, −3	3451, −3
	0·5	9803, −2	1863, −2	7191, −3	4055, −3	2842, −3	2260, −3	1935, −3
	−0·5	1373, −2	−2946, −3	−9188, −4	−2082, −4	1307, −4	3242, −4	4482, −4
	−1·0	−5000, −3	−2000, −3	−1000, −3	−5000, −4	−2000, −4	0000, 0	1429, −4
	−1·5	3944, −3	−1900, −4	−5023, −4	−4043, −4	−2584, −4	−1239, −4	−9506, −6
	−2·0	7000, −3	1000, −3	0000, 0	−2000, −4	−2000, −4	−1429, −4	−7143, −5
	−2·5	5084, −3	1317, −3	3063, −4	−1672, −5	−1097, −4	−1144, −4	−8356, −5
	−3·0	1000, −3	1000, −3	4000, −4	1000, −4	−2857, −5	−7143, −5	−7143, −5

x	α	γ=1	2	3	4	5	6	7
x = 8.	4·0	6151, 2	5863, 1	1093, 1	2981, 0	1054, 0	4513, −1	2240, −1
	3·5	3072, 2	3025, 1	5831, 0	1645, 0	6016, −1	2664, −1	1367, −1
	3·0	1461, 2	1490, 1	2981, 0	8733, −1	3318, −1	1526, −1	8122, −2
	2·5	6524, 1	6928, 0	1444, 0	4417, −1	1753, −1	8422, −2	4676, −2
	2·0	2683, 1	2981, 0	6521, −1	2097, −1	8763, −2	4432, −2	2587, −2
	1·5	9816, 0	1150, 0	2664, −1	9110, −2	4058, −2	2187, −2	1358, −2
	1·0	2981, 0	3725, −1	9287, −2	3445, −2	1673, −2	9829, −3	6622, −3
	0·5	6171, −1	8422, −2	2348, −2	9954, −3	5579, −3	3769, −3	2877, −3
	−0·5	−5666, −2	−9700, −3	−3065, −3	−1205, −3	−4510, −4	−6737, −5	1591, −4
	−1·0	−7000, −3	−3000, −3	−1667, −3	−1000, −3	−6000, −4	−3333, −4	−1429, −4
	−1·5	2095, −2	2558, −3	1482, −4	−3028, −4	−3432, −4	−2796, −4	−1918, −4
	−2·0	1700, −2	3667, −3	1000, −3	2000, −4	−6667, −5	−1429, −4	−1429, −4
	−2·5	4828, −4	1965, −3	9556, −4	3836, −4	1041, −4	−2380, −5	−7322, −5
	−3·0	−1233, −2	−3333, −4	4667, −4	3333, −4	1619, −4	4762, −5	−1587, −4
x = 10.	4·0	7658, 3	6094, 2	9545, 1	2203, 1	6643, 0	2449, 0	1057, 0
	3·5	3554, 3	2911, 2	4697, 1	1117, 1	3475, 0	1322, 0	5888, −1
	3·0	1564, 3	1322, 2	2203, 1	5419, 0	1744, 0	6871, −1	3169, −1
	2·5	6434, 2	5628, 1	9724, 0	2484, 0	8314, −1	3407, −1	1636, −1
	2·0	2423, 2	2203, 1	3965, 0	1057, 0	3702, −1	1590, −1	7998, −2
	1·5	8059, 1	7654, 0	1445, 0	4055, −1	1499, −1	6814, −2	3634, −2
	1·0	2203, 1	2203, 0	4403, −1	1318, −1	5232, −2	2566, −2	1480, −2
	0·5	4043, 0	4310, −1	9314, −2	3067, −2	1364, −2	7580, −3	4966, −3
	−0·5	2675, −1	−3466, −2	−9101, −3	−3419, −3	−1524, −3	−6964, −4	−2609, −4
	−1·0	−9000, −3	−4000, −3	−2333, −3	−1500, −3	−1000, −3	−6667, −4	−4286, −4
	−1·5	7910, −2	1084, −2	2296, −3	3911, −4	−1312, −4	−2617, −4	−2615, −4
	−2·0	3100, −2	7667, −3	2667, −3	1000, −3	3333, −4	4762, −5	−7143, −5
	−2·5	−2934, −2	−6370, −4	9926, −4	7192, −4	3921, −4	1742, −4	4603, −5
	−3·0	−4567, −2	−5667, −3	−6667, −4	1667, −4	2381, −4	1667, −4	8730, −5

IX. *Notices respecting New Books.*

A Text Book of Physics. By J. Duncan and S. G. Starling. Pp. xxiii+1081. Macmillan & Co. Ltd., 1918. Price 15*s.*

THIS book covers, in an elementary manner, practically the whole of physics, being divided into five parts, Dynamics, Heat, Light, Sound, and Magnetism and Electricity. It is written somewhat more from an engineering standpoint than such text-books usually are, and devotes much attention to the mechanical application of physical principles, as exemplified by freezing-machines, internal combustion engines, and the like. The general arrangement of the matter follows much the usual lines: the diagrams, however, seem to be all new, and are very clearly drawn and reproduced. Mention is made of much work, recent or topical, which has not yet appeared in text books: we may instance the descriptions of Gaede's molecular pump, the periscope, and the Barr and Stroud range-finder, and the reference to the utilization of volcanic heat in Tuscany. It is a pity that no account is given of the sound phenomena accompanying moving projectiles, which are of special interest to-day, and are instructive to the student.

Unfortunately, while much care is given to description of details of machinery, the fundamental conceptions are dealt with very perfunctorily, and important phenomena (such as osmotic pressure) which present difficulties to the learner, are handled in a very superficial and unconvincing way. The part devoted to dynamics is particularly open to criticism—such definitions as "mass means quantity of matter," given without further discussion, are pernicious and unscientific. While we heartily approve the many descriptions of various machines and mechanical devices based on physical reasoning, we could wish that more space and more thinking had been devoted to indicating and explaining the nature of the general laws and basic phenomena of physics.

Applied Optics: The Computation of Optical Systems. (Steinheil and Voit.) Translated and edited by J. W. French, B.Sc. Volume I. Blackie & Son.

The Advisory Council on Scientific and Industrial Research has had under consideration a number of scientific and technical problems arising out of the war. Several recommendations were made for the improvement of the optical industry, and special attention was drawn to the urgent need of standard text-books on those parts of optics which at present are greatly neglected in this country. "In our opinion the quickest and most effective manner of dealing with this requirement is by publishing translations of existing foreign books and abstracts of foreign papers on this subject." Mr. French has given an excellent translation of 'Applied Optics' by Steinheil and Voit, and has rendered a service which will be greatly appreciated by those who are investigating the problems of the optical workshop. He has added considerably to the value of the original work, especially by the addition of diagrams elucidating the meaning of the various symbols and sign conventions adopted. The formulæ given are those due to von Seidel, which require little knowledge of mathematics beyond the elementary parts of trigonometry. Even the practical optician with a slender mathematical equipment should be able to acquire without difficulty the power to undertake computations of the kind dealt with in this volume. In any case, Mr. French has provided, under exceptional war conditions, an excellent handbook, which we commend to the serious attention of every student of technical optics.

X. *Intelligence and Miscellaneous Articles.*

ANGLE TRISECTION. BY H. R. KEMPE, M.INST.C.E.

IN the course of some quite recent geometrical investigations I evolved the following arrangement for trisecting an angle mechanically. The arrangement, so far as I am aware, is original, is geometrically correct for all angles, and is exceedingly simple.

Fig. 1.

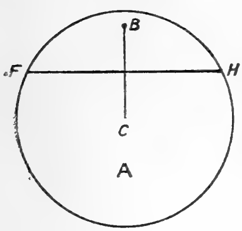


Fig. 2.

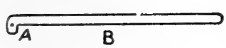
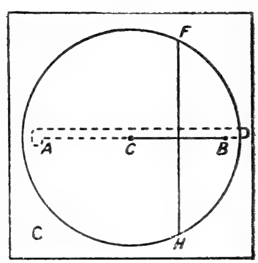


Fig. 3.



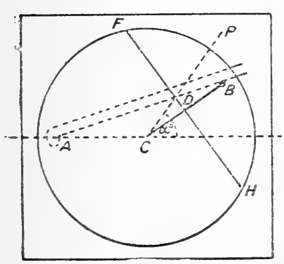


Fig. 4.

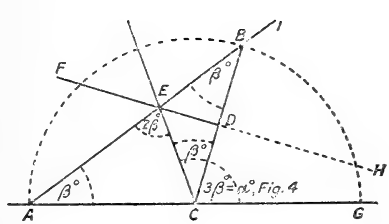


Fig. 5.

In fig. 1, **A** is a transparent disk (of celluloid), and in fig. 2, **B** is a straight-edged link or radial lever (also, conveniently, of celluloid).

On the disk **A** a line FH is scored, this line being at right angles to the second line BC, and bisecting BC.

The link **B** turns on a pivot at A, on a board **C** (fig. 3), and the disk (which is placed over the link) turns on a pin at C, AC being equal to CB. A pin B projects slightly through the under side of the disk and is fixed to the latter, so that when the disk is turned, the pin presses against the straight edge of the link and moves the latter through an angle which bears a certain proportion to the angular movement of the disk, as is obvious.

In order to trisect any angle, say the angle α°, fig. 4, the link would first be moved to the position where it banks against the pivot pin C (fig. 3), and then the disk rotated (thereby moving the link) until it is observed that the intersection of the line FH with the straight edge of the link comes on the radial line CP (fig. 4); when this is the case, then the line BC will trisect the angle α° in all cases.

The proof of the foregoing is as follows:—

From C as a centre (fig. 5), describe the semi-circle ABG.

Draw AI at any angle β°, cutting the semi-circle at B; join BC; bisect BC at D; draw DF perpendicular to BC, cutting AB at E.

Since DE is perpendicular to BC, and CD is equal to DB, the angles CBE and BCE are equal.

Also, since CA is equal to CB, the angles CAB and CBA are equal.

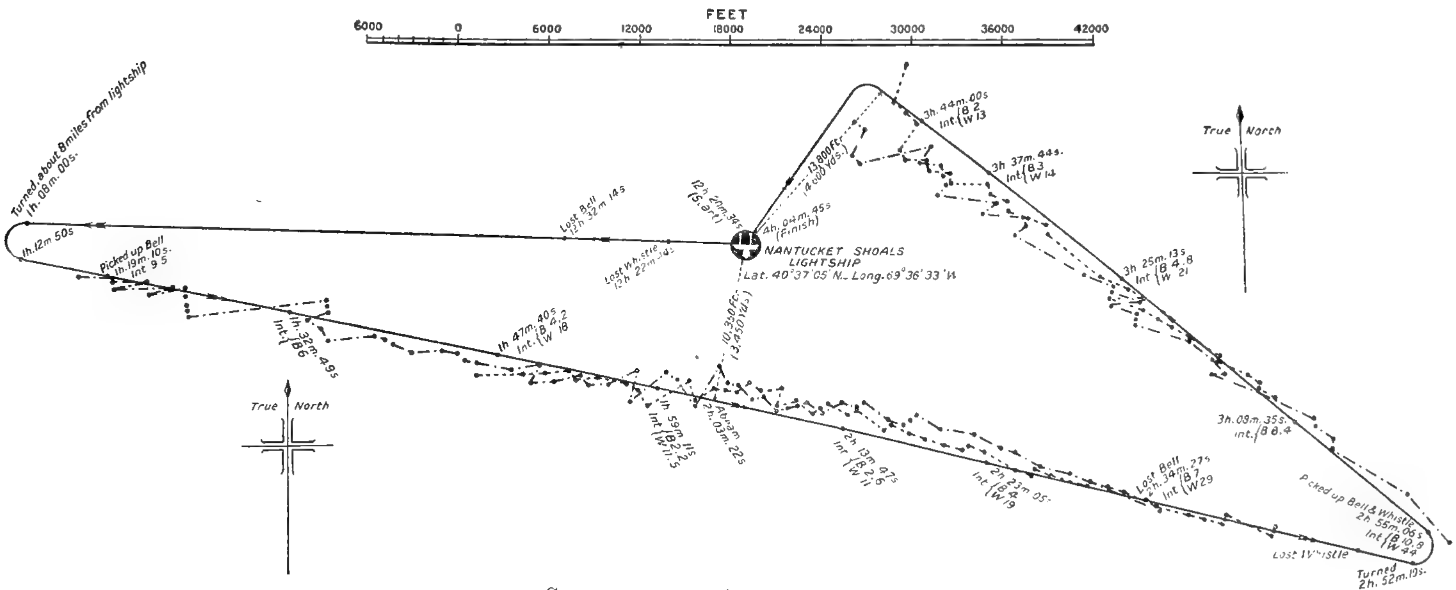
The angle AEC equals the sum of the angles EBC, ECB, *i. e.*, equals $2\beta^\circ$; and the angle ECB equals the sum of the angles AEC, EAC, *i. e.*, equals $3\beta^\circ$.

Now CB being the radius of the semi-circle, is a length of constant value, and its extremity B is in contact with the line AI under all conditions, and FD (or FD prolonged to H) cuts AI at E, and, as proved, if ECG equals $3\beta^\circ$, then ECB equals β°; hence the arrangement effects the trisection of the angle ECG.

For convenience of construction the line FD is drawn through to H, though actually the intersection of AI with FH does not extend beyond D, that is to say if the angles to be trisected do not exceed 180°; theoretically the principle involved covers all angles up to 360°.

Air Inventions Committee,
2 Clement's Inn.

Fig. 3.—Determining distance from a Wireless Station equipped with Sound Signals.



Symbols and Abbreviations.

——— track of U.S.S. 'Washington.'	B......... submarine bell.
- - - - distance determined by whistle.	W......... steam fog-whistle.
-·-·- distance determined by submarine bell.	Int........ interval.

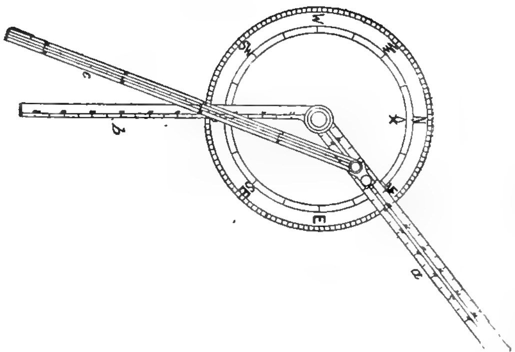


Fig. 4.

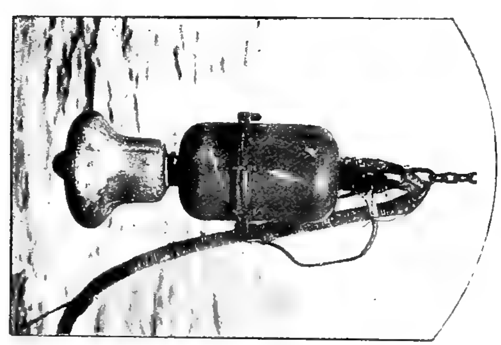


Fig. 2.

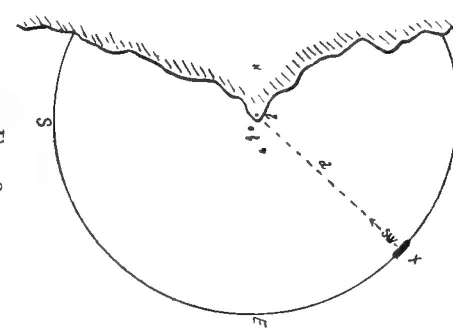
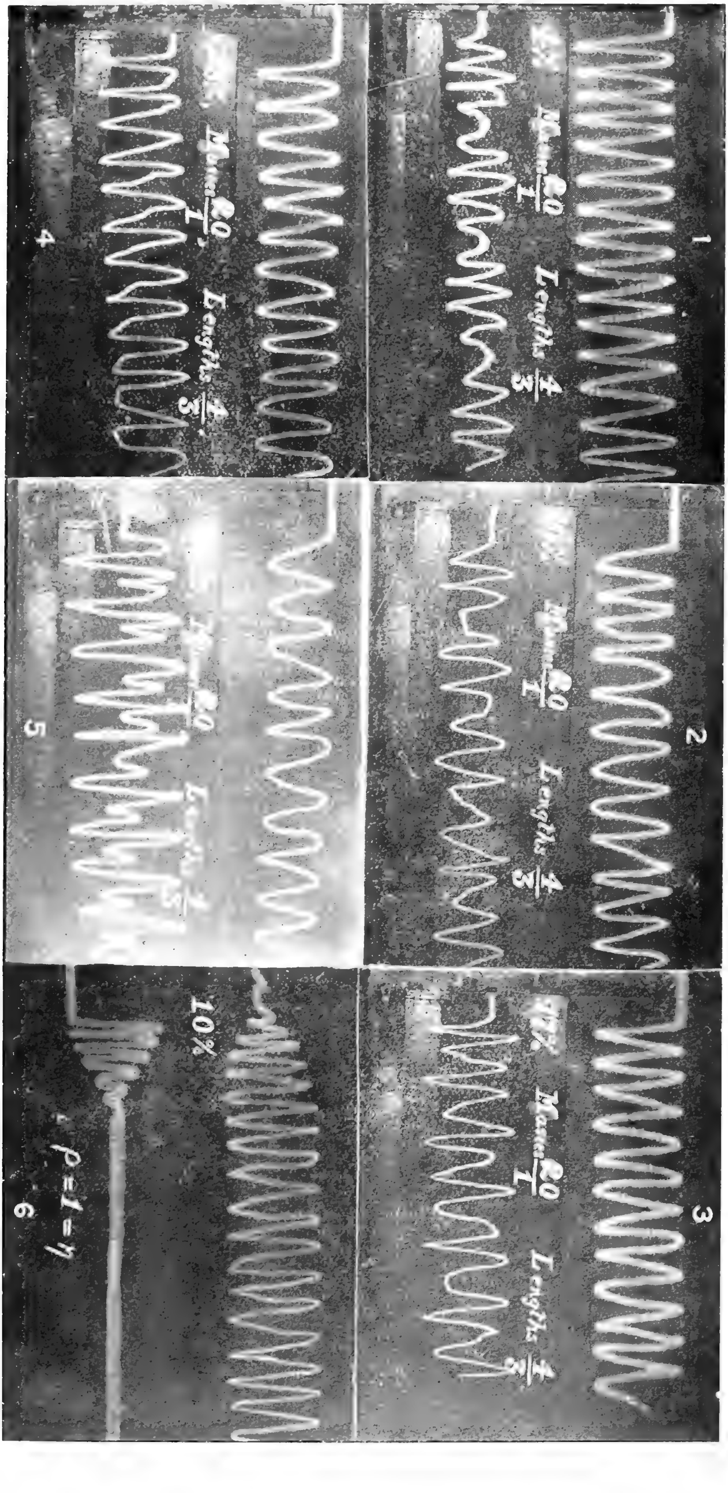
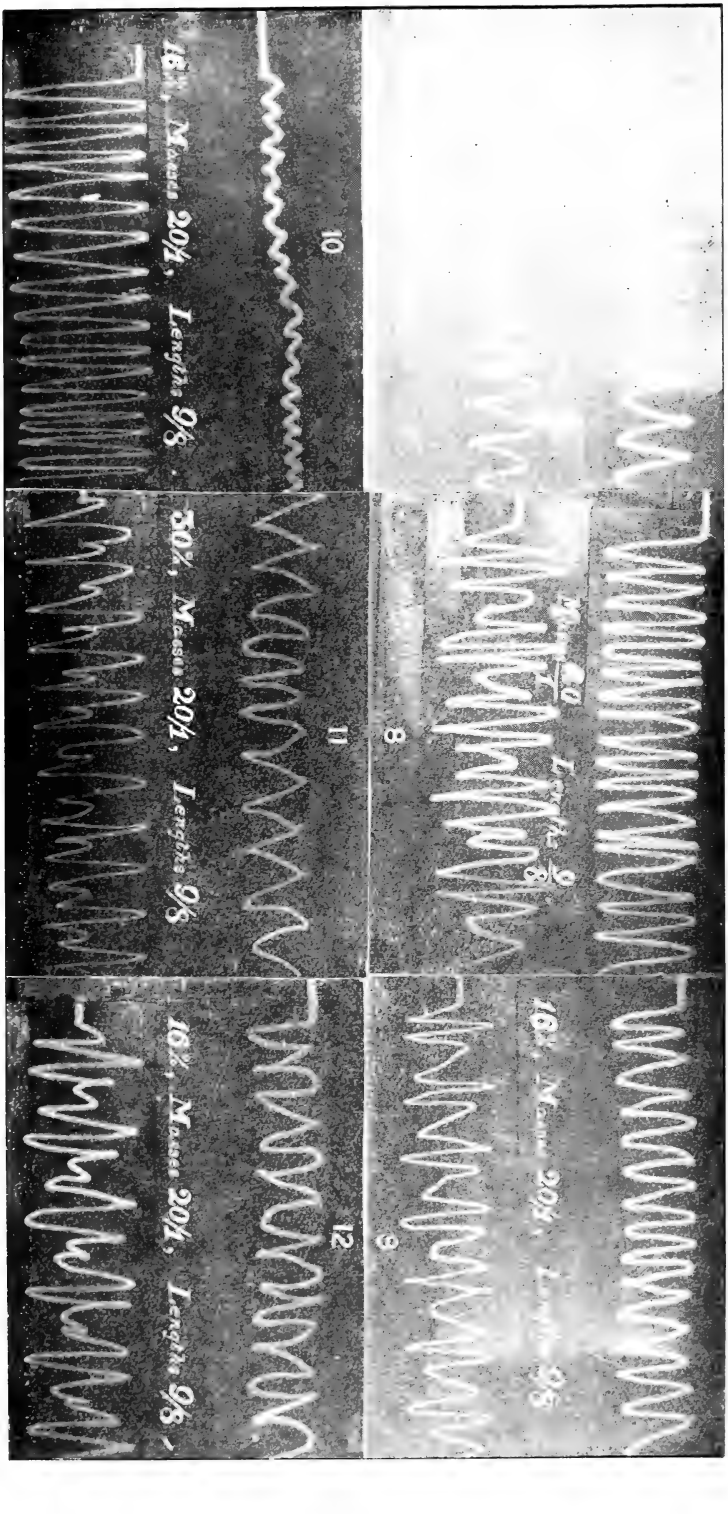


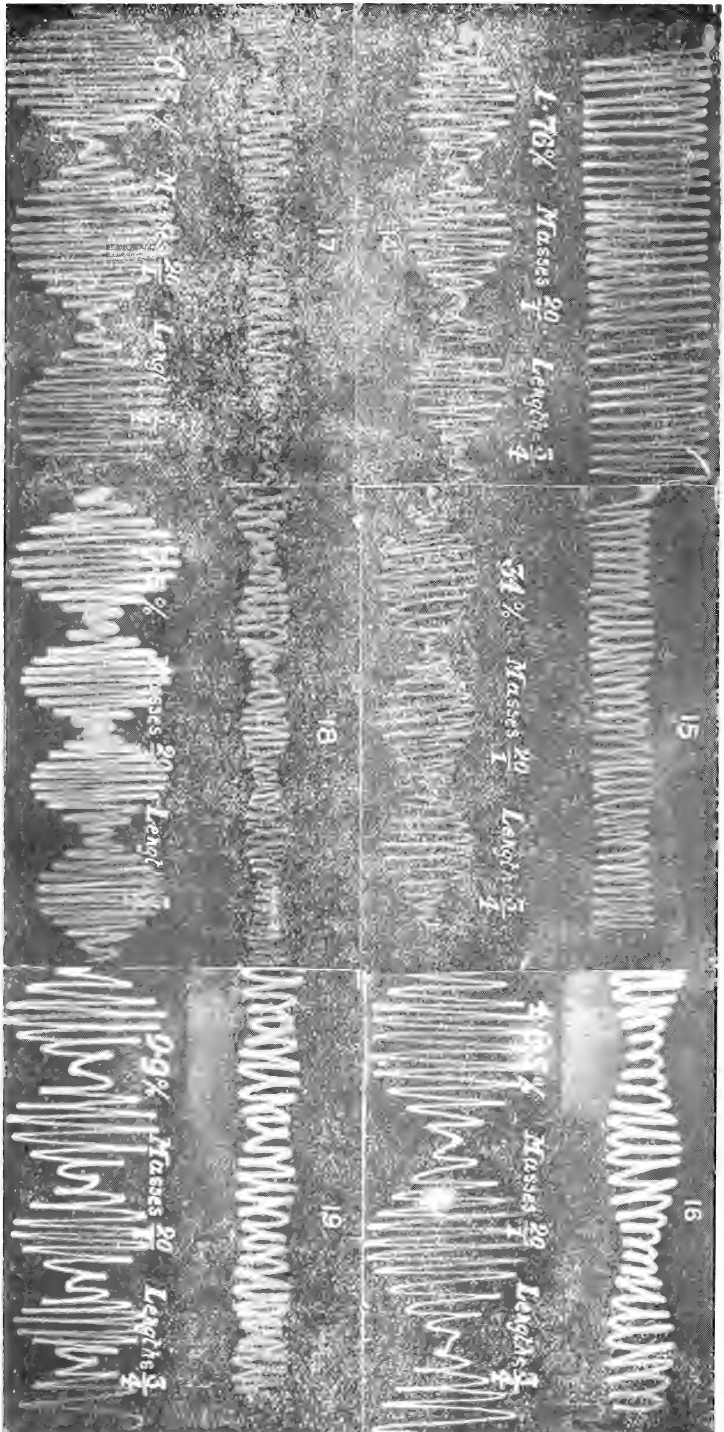
Fig. 1.

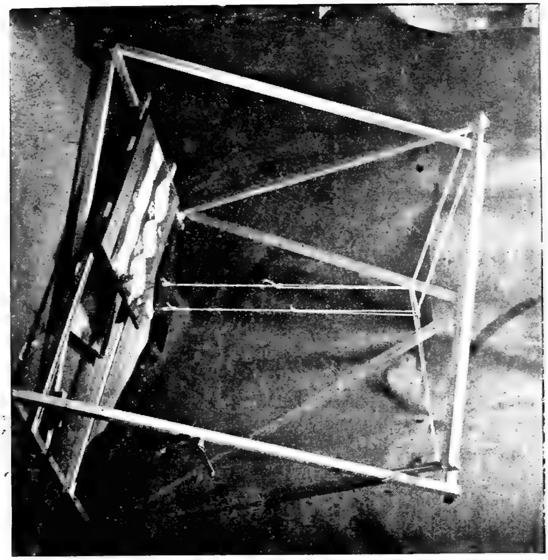
JOLY.

Phil. Mag. Ser. 6, Vol. 36, Pl. I.

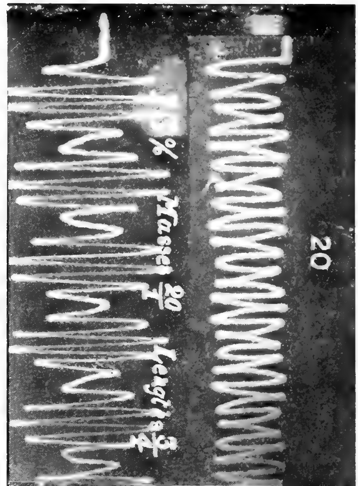

1
Mass 20/1 Lengths 4/3
2
Mass 20/1 Lengths 4/3
3
Mass 20/1 Lengths 4/3
4
Mass 20/1 Lengths 4/3
5
6
10%
ρ = 1 = η


10
16%, Masses 20%, Lengths 9/8.
11
50%, Masses 20%, Lengths 9/8
12
16%, Masses 20%, Lengths 9/8.
8
9

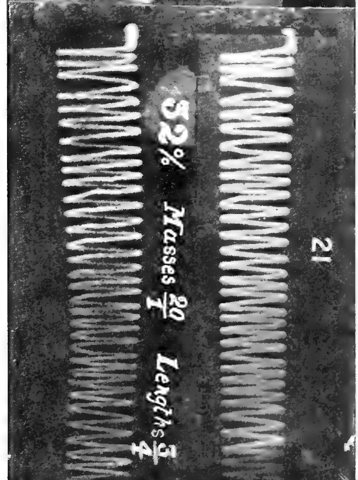

15
16
17
18
19



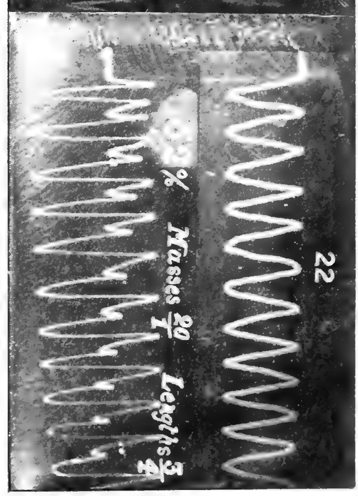
13


% Masses 20/1 Lengths 3/4

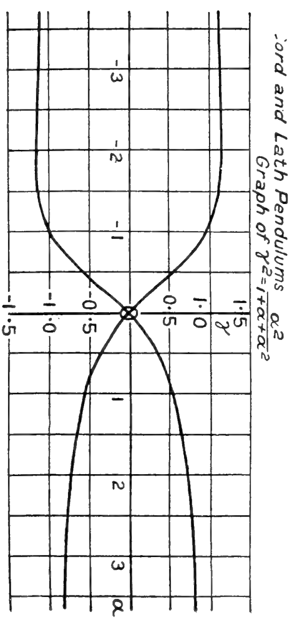
20


32% Masses 20/1 Lengths 3/4

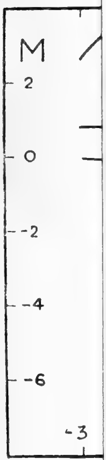
21


% Masses 20/1 Lengths 3/4

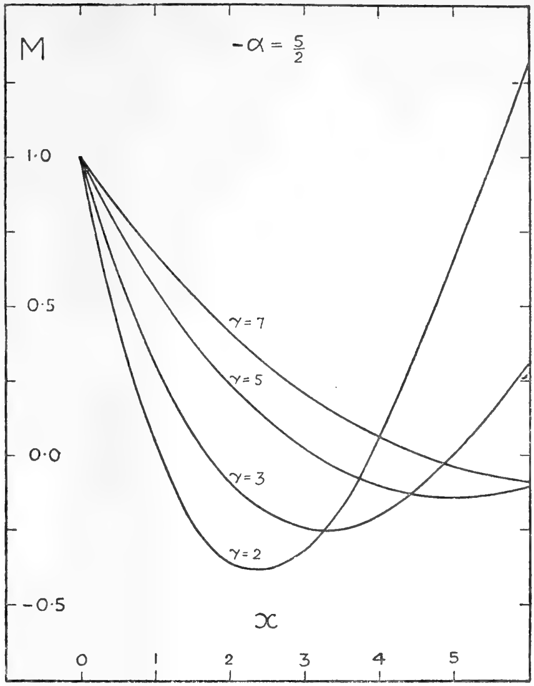
22


ord and Lath Pendulums
Graph of $\gamma^2 = \frac{\alpha^2}{1+\alpha+\alpha^2}$
γ
1·5
1·0
0·5
-0·5
-1·0
-1·5
-3
-2
-1
1
2
3
α

23

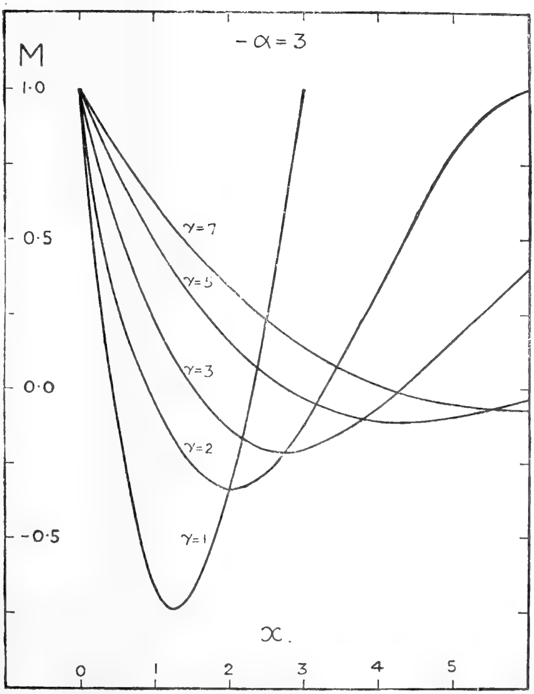


M
2
0
-2
-4
-6
-3



M
$-\alpha = \frac{5}{2}$
1·0
0·5
0·0
-0·5
γ = 7
γ = 5
γ = 3
γ = 2
x
0
1
2
3
4
5

Fig. 9.



M
$-\alpha = 3$
1·0
0·5
0·0
-0·5
γ = 7
γ = 5
γ = 3
γ = 2
γ = 1
x.
0
1
2
3
4
5

Fig. 10.

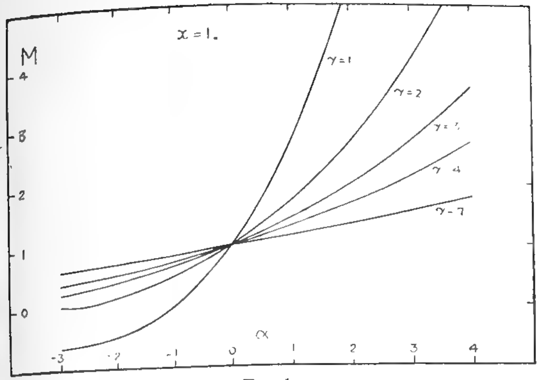


Fig. 1.

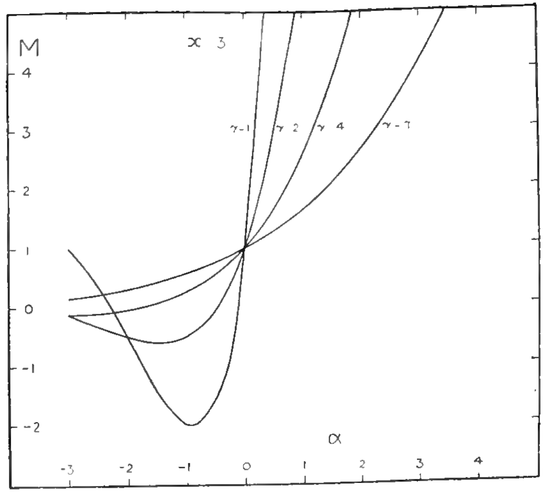


Fig. 3.

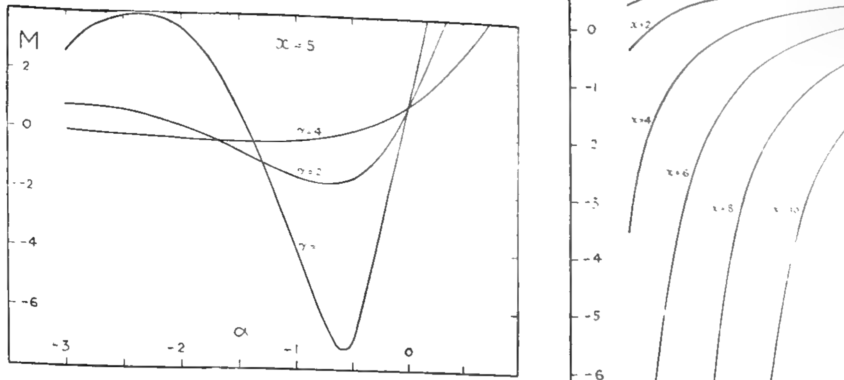


Fig. 5

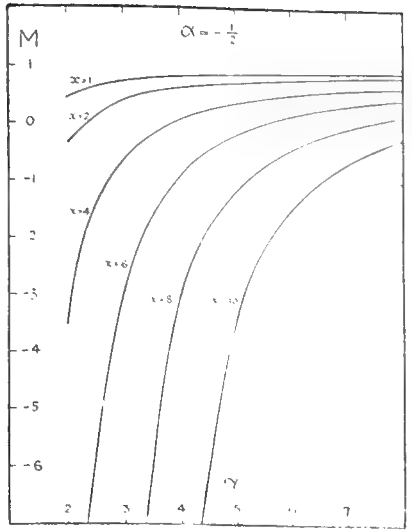


Fig. 7.

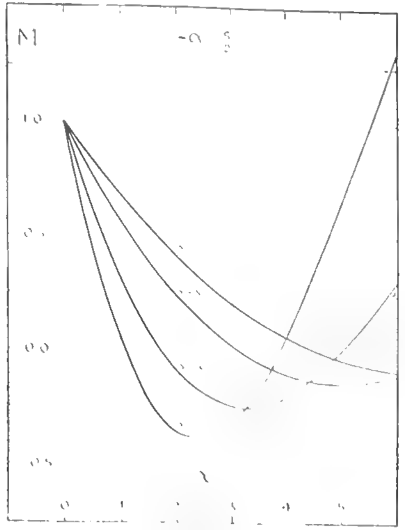


Fig. 9.

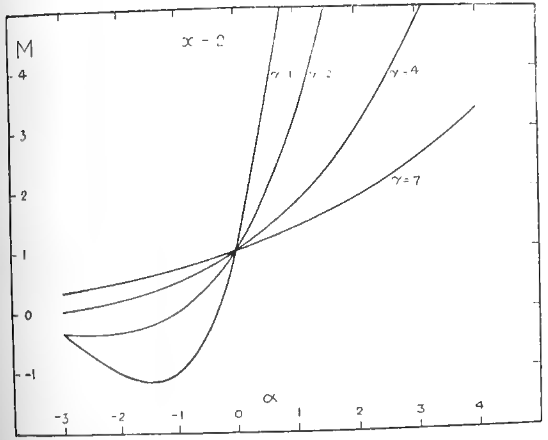


Fig. 2.

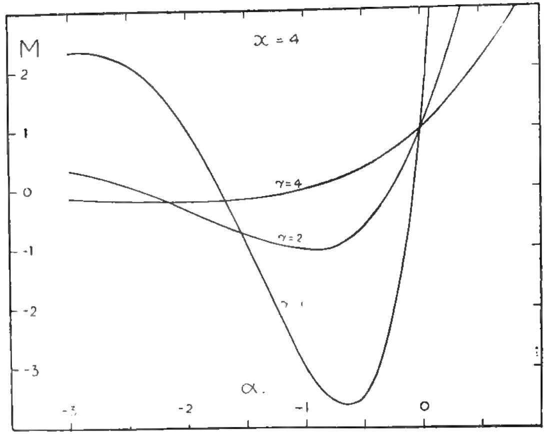


Fig. 4.

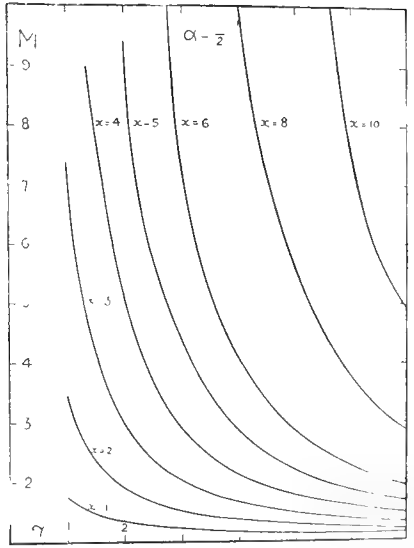


Fig. 6.

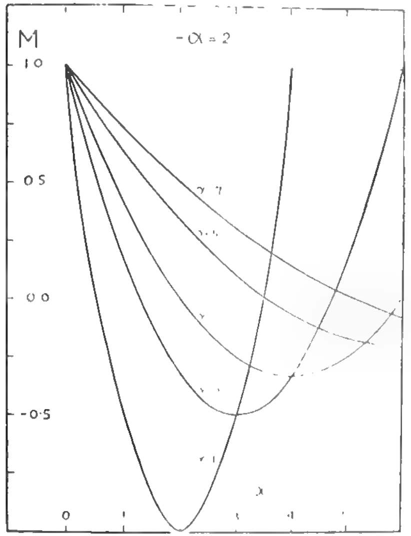


Fig. 8.

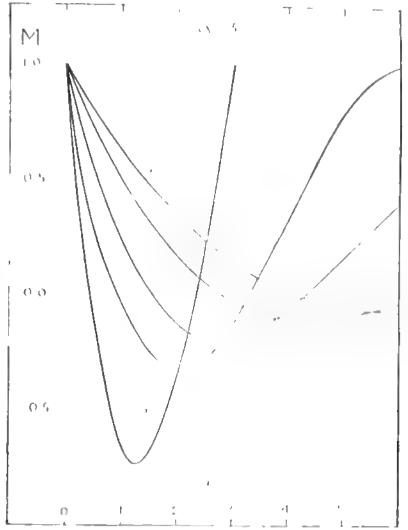


Fig. 10.

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XI. *On the Potential generated in a High-tension Magneto. By* E. TAYLOR JONES, *D.Sc., Professor of Physics in the University College of North Wales, Bangor**.

[Plate VII.]

1. *Introduction.*

THE arrangement of circuits shown diagrammatically in fig. 1 is that now usually adopted in the high-tension magneto, as used for ignition in motor-car and aeroplane engines. It consists of a primary coil P and a secondary

Fig. 1.

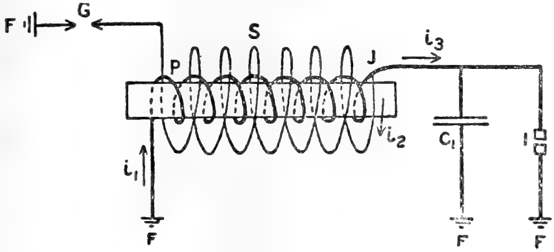


Diagram of the circuits of a high-tension magneto.

coil S, both wound on the armature core, and a contact-breaker I in parallel with which is the condenser C_1. While

* Communicated by the Author.

essentially similar to the ordinary induction-coil, it differs from it in the following respects :—

(1) One terminal of the condenser, one of the contact-pieces of I, one end of the primary winding and one side of the secondary spark-gap G, are all connected together. This is effected by connecting each of these four points with "earth," *i. e.* with the frame of the machine.

(2) The secondary wire is connected at one end with the primary, from this point of junction J being led the connexion with the contact-breaker and the condenser. The secondary coil is thus "earthed" through the primary.

(3) Instead of a battery, the rotation of the armature between the poles of the permanent magnet serves, while the contacts I are closed, to establish the primary current. At a certain point in the revolution, at or near which the primary current would have its maximum value if the contacts remained closed, the contact at I is broken ; thereupon the high-tension effect is produced in the secondary coil. The rotation has also the effect of inducing electromotive forces, in both primary and secondary coils, which are maintained after the contacts are separated.

It is generally admitted that the secondary potential causing the spark arises mainly from the interruption of the primary current, and is only contributed to in small measure by the induced E.M.F. due to rotation. Thus in any given machine the secondary potential depends mainly upon the current i_0 in the primary coil at the moment of break, and is in fact, as in the induction-coil, approximately proportional to this current.

With regard to the value of i_0, a graphical method has been given by A. P. Young * for determining this current when the open-circuit primary voltage curve, the resistance of the primary circuit, and the primary self-inductance in various positions of the armature are known. The value of i_0 determined by the graphical method is said to agree substantially with that shown by an oscillograph. We may therefore conclude that the manner of growth of the primary current after "make" is well understood, and that methods are available for determining with sufficient accuracy the value of this current at the moment of "break."

The present communication is mainly concerned with what goes on after the contacts are separated, and especially with the manner in which the secondary potential rises and in which its value depends upon the properties of the circuits. The constants and coefficients upon which the secondary potential depends may be enumerated as follows :—

* 'The Electrician,' Sept. 14, 1917, p. 923.

(1) The self-inductance of the primary coil, L_1.

(2) The self-inductance of the secondary coil, L_2. This coefficient is defined as the total induction through the secondary coil due to the current in this coil, divided by the value of the current at the end J. Owing to the fact that during the oscillations which occur after break the current is not uniformly distributed along the secondary wire, but is greatest at J and zero at the sparking-plug end (until sparking begins), the value of L_2 is somewhat smaller than that found by experimental methods in which the current is uniformly distributed.

(3) The inductance of the primary coil on the secondary, L_{21}.

(4) The inductance of the secondary on the primary, L_{12}. This coefficient is defined as the induction through the primary winding due to the secondary current, divided by the value of the latter at J. Owing to the non-uniform distribution of current in the secondary L_{12} is somewhat smaller than L_{21}.

(5) The capacity of the primary condenser, C_1.

(6) The capacity of the secondary circuit, C_2. This is defined as the charge on the secondary coil and the bodies (distributor segment, sparking-plug terminal, etc.) connected with it, divided by the difference of potential at the sparking-plug. The capacity C_2 is distributed along the secondary wire, and its value is doubtless largely influenced by the proximity of the surfaces of the pole-pieces and the core which are at zero potential.

(7) Finally there are the effective resistances R_1, R_2, of the primary and secondary coils.

Probably none of the above quantities are strictly constant during the oscillations subsequent to break. The inductances

Fig. 2.

Showing armature in position of maximum inductances.

vary with the magnetic state of the core and with the position of the armature; they are greatest in the position shown in fig. 2. The secondary capacity also varies with the position of the armature. The effective resistances include variable factors depending upon eddy currents and hysteresis in the core and pole-pieces. It is to be understood that the above symbols represent mean values of the various quantities during the short interval of time between break and the moment of maximum secondary potential.

2. *Calculation of the Secondary Potential.*

Calling the current in the primary coil i_1, the current entering the secondary coil at J i_2, the current entering the condenser i_3 (the directions being as shown in fig. 1), the potential difference of the plates of C_1 V_1', that at the sparking-plug V_2', we have the following equations :—

$$L_1\frac{di_1}{dt}+L_{12}\frac{di_2}{dt}+R_1i_1+V_1' = E, \quad . \quad . \quad . \quad . \quad . \quad (1)$$

$$L_2\frac{di_2}{dt}+L_{21}\frac{di_1}{dt}+R_2i_2+V_2'-V_1' = qE, \quad . \quad . \quad . \quad (2)$$

where E and qE are the induced E.M.F.'s in the primary and secondary coils due to the rotation. q is approximately equal to the ratio of the secondary and primary turns.

Further,
$$i_1-i_2-i_3 = 0, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

$$i_3 = C_1\frac{dV_1'}{dt}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (4)$$

$$i_2 = C_2\frac{dV_2'}{dt}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (5)$$

Thus
$$i_1 = C_1\frac{dV_1}{dt}+C_2\frac{dV_2'}{dt}. \quad . \quad . \quad . \quad (6)$$

Substituting for i_1 and i_2 in (1) and (2) we have

$$L_1C_1\frac{d^2V_1'}{dt^2}+(L_1+L_{12})C_2\frac{d^2V_2'}{dt^2}+R_1\left(C_1\frac{dV_1'}{dt}+C_2\frac{dV_2'}{dt}\right)$$
$$+V_1' = E, \quad (7)$$

$$(L_2+L_{21})C_2\frac{d^2V_2'}{dt^2}+L_{21}C_1\frac{d^2V_1'}{dt^2}+R_2C_2\frac{dV_2'}{dt}$$
$$+V_2'-V_1' = qE. \quad (8)$$

We shall now make two assumptions with the object of simplifying the calculation, viz. :—

(1) That the resistances are negligible: this may be assumed if our object is limited to the calculation of frequencies, initial amplitudes, and the determination of the

effect on the secondary potential of varying one or other of the inductances or capacities. The expressions for the damping factors are given later.

(2) That E and qE may be treated as constant during the short interval with which we are concerned. These quantities are in any case small in comparison with the values attained by V_1' and V_2', and no great error can be introduced by regarding them as constant.

Thus, omitting the resistance terms in (7) and (8), and introducing two new variables defined by $V_1=V_1'-E$ and $V_2=V_2'-(q+1)E$, we have

$$L_1C_1\frac{d^2V_1}{dt^2}+(L_1+L_{12})C_2\frac{d^2V_2}{dt^2}+V_1=0, \quad . \quad (9)$$

$$(L_2+L_{21})C_2\frac{d^2V_2}{dt^2}+L_{21}C_1\frac{d^2V_1}{dt^2}+V_2-V_1=0. \quad . \quad (10)$$

Adding (9) and (10) and writing sL_2 for the sum $L_2+L_{21}+L_{12}+L_1$, where s is a fraction not much greater than unity, we find

$$(L_1+L_{21})C_1\frac{d^2V_1}{dt^2}+sL_2C_2\frac{d^2V_2}{dt^2}+V_2=0. \quad . \quad . \quad (11)$$

The assumed solutions $V_1=Ae^{ipt}$, $V_2=Be^{ipt}$, substituted in (9) and (11) give

$$A(1-L_1C_1p^2)=B(L_1+L_{12})C_2p^2, \quad . \quad . \quad . \quad (12)$$

$$A(L_1+L_{21})C_1p^2=B(1-sL_2C_2p^2), \quad . \quad . \quad . \quad (13)$$

leading, after elimination of the ratio B/A, to the equation for p $(=2\pi n)$,

$$p^4L_1C_1L_2C_2(1-k^2)-p^2(L_1C_1+sL_2C_2)+1=0. \quad . \quad (14)$$

Here k^2 is the coupling $L_{12}L_{21}/L_1L_2$.

The system has therefore two frequencies, n_1, n_2, given by the equation

$$8\pi^2n^2(1-k^2)$$
$$=\frac{1}{L_2C_2}+\frac{s}{L_1C_1}\pm\sqrt{\left(\frac{1}{L_2C_2}+\frac{s}{L_1C_1}\right)^2-\frac{4(1-k^2)}{L_1C_1L_2C_2}}. \quad (15)$$

In the extreme case $C_1=\infty$ (primary closed) one of the frequencies is zero and the other is $n_c=1/2\pi\sqrt{L_2C_2(1-k^2)}$. In the other extreme case, $C_1=0$, one frequency is infinite and the other is given by $n_0=1/2\pi\sqrt{sL_2C_2}$, that is, it is the frequency of the primary and secondary oscillating together as one coil. The ratio of the squares of the frequencies in these cases is

$$\frac{n_0^2}{n_c^2}=\frac{1-k^2}{s}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (16)$$

a result which suggests an experimental method for determining the coupling.

In general, if we write u for the ratio L_1C_1/L_2C_2, the frequency-ratio is given by

$$\frac{n_2^2}{n_1^2}=\frac{s+u+\sqrt{(s+u)^2-4(1-k^2)u}}{s+u-\sqrt{(s+u)^2-4(1-k^2)u}}. \quad . \quad . \quad (17)$$

For any given value of k^2 the frequency-ratio is least when $u=s$, *i. e.* when $L_1C_1=sL_2C_2$.

In order to find the amplitudes multiply (11) by any factor λ and add its terms to those of (9). We then have the equation

$$\{L_1+\lambda(L_1+L_{21})\}C_1\frac{d^2V_1}{dt^2}+\{(L_1+L_{12})+\lambda sL_2\}C_2\frac{d^2V_2}{dt^2} + V_1+\lambda V_2=0.$$

If λ is so chosen that

$$(L_1+L_{12}+\lambda sL_2)C_2=\lambda\{L_1+\lambda(L_1+L_{21})\}C_1, \quad . \quad (18)$$

then

$$\{L_1+\lambda(L_1+L_{21})\}C_1\frac{d^2}{dt^2}(V_1+\lambda V_2)+(V_1+\lambda V_2)=0. \quad (19)$$

The two values of λ, viz. λ_1 and λ_2, may be calculated by (18) in terms of the coefficients of equations (9) and (11). They may also be expressed in terms of the frequencies n_1 and n_2, for, by (19),

$$\frac{1}{4\pi^2n_1^2}=\{L_1+\lambda_1(L_1+L_{21})\}C_1,$$

$$\frac{1}{4\pi^2n_2^2}=\{L_1+\lambda_2(L_1+L_{21})\}C_1.$$

Thus

$$\left.\begin{aligned}\lambda_1&=\frac{1}{(L_1+L_{21})C_1}\left(\frac{1}{4\pi^2n_1^2}-L_1C_1\right),\\ \lambda_2&=\frac{1}{(L_1+L_{21})C_1}\left(\frac{1}{4\pi^2n_2^2}-L_1C_1\right).\end{aligned}\right\} \quad . \quad . \quad (20)$$

The solution of equation (19) is represented by the two normal vibrations

$$\left.\begin{aligned}V_1+\lambda_1V_2&=A_1\sin(2\pi n_1t+\delta_1),\\ V_1+\lambda_2V_2&=A_2\sin(2\pi n_2t+\delta_2).\end{aligned}\right\} \quad . \quad . \quad . \quad (21)$$

Hence the solutions for V_2 and V_1 are

$$V_2 = \frac{A_1}{\lambda_1 - \lambda_2} \sin (2\pi n_1 t + \delta_1) - \frac{A_2}{\lambda_1 - \lambda_2} \sin (2\pi n_2 t + \delta_2), \quad (22)$$

$$V_1 = \frac{A_1 \lambda_2}{\lambda_2 - \lambda_1} \sin (2\pi n_1 t + \delta_1) - \frac{A_2 \lambda_1}{\lambda_2 - \lambda_1} \sin (2\pi n_2 t + \delta_2). \quad (23)$$

The coefficients A_1, A_2, and the phase angles δ_1, δ_2, are to be determined from the initial conditions. These express that at the moment of break

(1) the P.D. of the plates of the condenser is zero,
(2) the potential at the free secondary terminal is qE,
(3) the current in the primary coil is i_0,
(4) the current entering the secondary coil at J is zero.

The last depends of course upon the assumption already made, that during the period considered the E.M.F. due to the rotation may, owing to its comparative smallness and slow rate of variation, be regarded as constant.

Thus, at $t=0$, $V_1'=0$, $V_2'=qE$, $C_1\frac{dV_1'}{dt}=i_0$, $\frac{dV_2'}{dt}=0$, or, in terms of V_1 and V_2,

$$\left.\begin{aligned} V_1 &= -E, \\ V_2 &= -E, \\ \frac{dV_1}{dt} &= \frac{i_0}{C_1}, \\ \frac{dV_2}{dt} &= 0. \end{aligned}\right\} \quad . \; . \; . \; . \; . \; . \; . \quad (24)$$

Substituting in (21) we find

$$\left.\begin{aligned} A_1 \sin \delta_1 &= -E(1+\lambda_1), \\ A_2 \sin \delta_2 &= -E(1+\lambda_2), \\ 2\pi n_1 A_1 \cos \delta_1 &= i_0/C_1, \\ 2\pi n_2 A_2 \cos \delta_2 &= i_0/C_1. \end{aligned}\right\} \quad . \; . \; . \quad (25)$$

Consequently,

$$A_1^2 = \frac{i_0^2}{4\pi^2 n_1^2 C_1^2} + E^2(1+\lambda_1)^2,$$

$$A_2^2 = \frac{i_0^2}{4\pi^2 n_2^2 C_1^2} + E^2(1+\lambda_2)^2.$$

If we neglect the square of E in comparison with that of $i_0/2\pi n_1 C_1$ and of $i_0/2\pi n_2 C_2$, these become approximately

$$\left.\begin{aligned} A_1 &= \frac{i_0}{2\pi n_1 C_1}, \\ A_2 &= \frac{i_0}{2\pi n_2 C_2}. \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad (26)$$

Also, by (25),

$$\left.\begin{aligned} \tan\delta_1 &= -\frac{2\pi n_1 E(1+\lambda_1)C_1}{i_0}, \\ \tan\delta_2 &= -\frac{2\pi n_2 E(1+\lambda_2)C_1}{i_0}; \end{aligned}\right\} \quad . \quad . \quad (27)$$

and by (20),

$$\lambda_1 - \lambda_2 = \frac{1}{4\pi^2(L_1+L_{21})C_1} \cdot \frac{n_2^2 - n_1^2}{n_1^2 n_2^2}.$$

Equations (22) and (23) therefore give the following solutions for V_2 and V_1:—

$$\begin{aligned} V_2 = \frac{2\pi(L_1+L_{21})i_0 n_1 n_2^2}{n_2^2 - n_1^2}\sin(2\pi n_1 t + \delta_1) \\ - \frac{2\pi(L_1+L_{21})i_0 n_1^2 n_2}{n_2^2 - n_1^2}\sin(2\pi n_2 t + \delta_2), \quad . \quad (28) \end{aligned}$$

$$\begin{aligned} V_1 = -\frac{2\pi i_0}{C_1} \cdot \frac{n_1 n_2^2}{n_2^2 - n_1^2}\left(\frac{1}{4\pi^2 n_2^2} - L_1 C_1\right)\sin(2\pi n_1 t + \delta_1) \\ + \frac{2\pi i_0}{C_1} \cdot \frac{n_1^2 n_2}{n_2^2 - n_1^2}\left(\frac{1}{4\pi^2 n_1^2} - L_1 C_1\right)\sin(2\pi n_2 t + \delta_2), \quad (29) \end{aligned}$$

where

$$\left.\begin{aligned} \tan\delta_1 &= -\frac{2\pi n_1 E}{i_0} \cdot \frac{L_{21}C_1 + 1/4\pi^2 n_1^2}{L_1 + L_{21}}, \\ \tan\delta_2 &= -\frac{2\pi n_2 E}{i_0} \cdot \frac{L_{21}C_1 + 1/4\pi^2 n_2^2}{L_1 + L_{21}}. \end{aligned}\right\} \quad . \quad . \quad (30)$$

After break, therefore, there are set up in each circuit two oscillations differing in frequency, amplitude, and initial phase, and the potential at any moment in either circuit is the sum of the potentials in the two oscillations, as represented by equations (28) and (29), to which must be added the potential due to the rotation.

If the resistance terms had been retained in the equations, the expressions for the oscillations in each circuit would have contained factors $e^{-k_1 t}$ and $e^{-k_2 t}$ representing the decay of the amplitudes. It can be shown that the values of the damping factors k_1 and k_2 are

$$\left.\begin{aligned} k_1 &= 4\pi^2 n_1^2\left(\frac{\theta_1+\theta_2}{2}-\beta\right),\\ k_2 &= 4\pi^2 n_2^2\left(\frac{\theta_1+\theta_2}{2}+\beta\right), \end{aligned}\right\} \quad . \; . \; . \quad (31)$$

where

$$\theta_1 = \tfrac{1}{2}R_1C_1, \qquad \theta_2 = \tfrac{1}{2}R_2C_2, \quad . \; . \; . \; . \; . \quad (32)$$

and

$$\beta = -\frac{2\pi^2 n_1^2 n_2^2}{n_2^2-n_1^2}\{(\theta_1-\theta_2)L_1C_1+(\theta_1+\theta_2)sL_2C_2\}. \quad (33)$$

The phase angles δ_1 and δ_2 would also have been modified by the resistances. We shall, however, for the present retain the condition that the resistances are neglected, and also neglect the small angles δ_1, δ_2, given by (30).

The theory now proceeds as in the case of the induction-coil *. The greatest value of V_2 occurs when $2\pi n_1 t$ is not far from $\pi/2$, and the conditions are most favourable if positive maxima of the two oscillations represented in (28) occur simultaneously, *i. e.* if $\sin 2\pi n_1 t=1$ and $\sin 2\pi n_2 t=-1$ for the same value of t. This requires that the frequency-ratio should have one of the values given by

$$\frac{n_2}{n_1} = 3, 7, 11, 15, \ldots \quad . \; . \; . \; . \; . \; . \quad (34)$$

Assuming this condition to be fulfilled, the expression for the maximum value of V_2 is, by (28),

$$V_{2m} = 2\pi i_0(L_1+L_{21})\frac{n_1 n_2}{n_2-n_1}. \quad . \; . \; . \; . \; . \quad (35)$$

Expressed in terms of u, k^2, and s by means of (15) this becomes

$$V_{2m} = \frac{(L_1+L_{21})i_0}{\sqrt{L_2C_2}\sqrt{u+s-2\sqrt{(1-k_2)u}}}. \quad . \quad (36)$$

Let

$$U = \frac{1}{\sqrt{u+s-2\sqrt{(1-k^2)u}}}. \quad . \; . \; . \; . \; . \quad (37)$$

For given values of k^2 and s, U has a maximum value of $\dfrac{1}{\sqrt{s-1+k^2}}$ at

$$u = 1-k^2. \quad . \; . \; . \; . \; . \; . \; . \; . \quad (38)$$

* See Phil. Mag. Aug. 1915, p. 224.

The two conditions (34) and (38), determining the most effective adjustments of the system from the point of view of spark-length, are the same as those which hold in the case of an ordinary induction-coil, for which it has been shown * that they are also the conditions that the energy should exist, at the time $t=1/4n_1$, entirely in the electrostatic form in the secondary circuit. The value of U, however, in the magneto differs from that in the induction-coil problem in that in the latter s is replaced by unity.

When the value of s is known we can, by combining equations (17) and (38), find the value of k^2 corresponding with any of the values of n_2/n_1 given by (34). For example, if $s=1\cdot04$ we find for $n_2/n_1=3$, $k^2=0\cdot554$; for $n_2/n_1=7$, $k^2=0\cdot832$; for $n_2/n_1=11$, $k^2=0\cdot897$. If the coupling has one of these values, and if the capacity of the condenser is such that $L_1C_1/L_2C_2=1-k^2$, V_{2m} is then given by the equation

$$V_{2m} = i_0\sqrt{\frac{L_1}{C_2}}\sqrt{\frac{L_1+L_{21}}{L_1+L_{12}}}. \quad . \quad . \quad . \quad . \quad (39)$$

The expression on the right of (39), with $(q+1)E$ added to it, represents the greatest secondary potential attainable by any magneto in which the circuit connexions are arranged as in fig. 1.

If k^2 has not one of the above special values V_{2m} is not given by equation (39), but it can always be expressed in the form †

$$V_{2m} = \frac{(L_1+L_{21})i_0}{\sqrt{L_2C_2}}U\sin\phi, \quad . \quad . \quad . \quad . \quad (40)$$

where U is given by (37), and

$$\left.\begin{aligned} \phi &= \frac{2\pi n_1}{n_1+n_2} \text{ if } \frac{n_2}{n_1} \text{ is between 1 and 5,} \\ \phi &= \frac{4\pi n_1}{n_1+n_2} \quad ,, \quad ,, \quad ,, \quad 5 \;,,\; 9, \\ \phi &= \frac{6\pi n_1}{n_1+n_2} \quad ,, \quad ,, \quad ,, \quad 9 \;,,\; 13, \end{aligned}\right\} \quad . \quad (41)$$

and so on.

* Phil. Mag. Jan. 1915, p. 2.
† Phil. Mag. Aug. 1915, p. 226.

The equations (37) and (41) allow the optimum value of u ($=L_1C_1/L_2C_2$) to be calculated for any given values of k^2 and s, and therefore the optimum capacity of the condenser when L_1 and L_2C_2 are known. They also allow the theoretical curve to be determined showing the relation between the capacity of the condenser and the maximum secondary potential, or, which comes to the same thing, the curve of which u is the abscissa and $U \sin \phi$ the ordinate.

3. *On the Curves showing the relation between Primary Capacity and Maximum Secondary Potential.*

Examples of these curves, calculated for the case of an induction-coil ($s=1$) have been given in a former paper *. The curves consist of a series of arches which touch the curve (u, U) at points corresponding with the frequency-ratios 3, 7, 11, . . ., and intersect one another at the points for which $n_2/n_1=5, 7, 9, \ldots$ The relative proportions of the arches and the number of the one in the series which stands highest depend upon the coupling. Thus if k^2 is less than 0·71 the first arch (containing the 3/1 point of contact) stands highest; if k^2 is between 0·71 and 0·87 the second arch contains the highest point of the curve. The value of u at the summit of the highest arch determines the optimum primary capacity for any given induction-coil, and a table has been given containing the optimum values of u for various values of k^2 †.

Similar curves may be obtained experimentally by observing the spark-length of the coil for a constant current and for various values of the capacity of the primary condenser, or, better, by observing the least value of the primary current at break which will cause a spark to pass across a gap of constant width. This plan is much more convenient, and it is also more accurate, because the primary current at break is more nearly proportional to the maximum secondary potential than is the potential to the spark-length. The secondary (sparking) potential being constant, the reciprocal of the least sparking current is thus proportional to the maximum secondary potential per unit current.

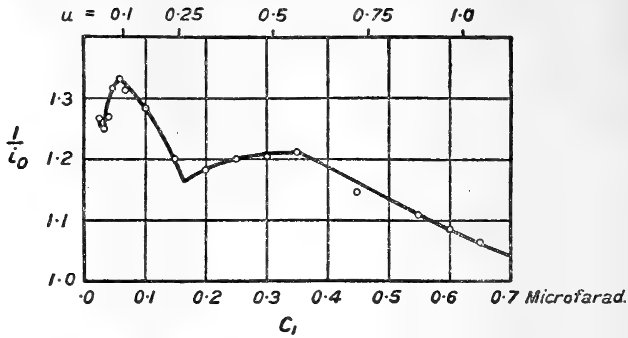
An example of a curve obtained in this way for an

* Phil. Mag. Aug. 1915, pp. 229, 230.

† *L. c.* p. 231.

induction-coil, for which k^2 was 0·768, is shown in fig. 3 (*cf. l. c.* p. 229, fig. 1). The gap used in this experiment was 0·96 cm. wide, between brass balls 2 cm. in diameter.

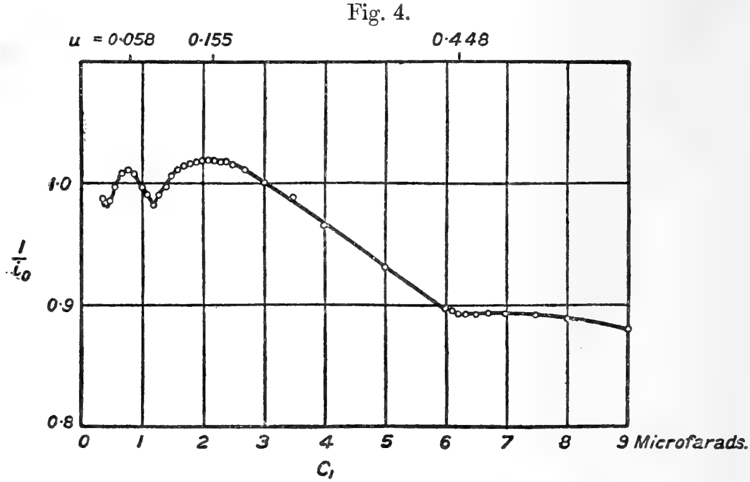
Fig. 3.



Capacity-potential curve for an induction-coil. $k^2=0·768$.

The abscissa represents the capacity of the primary condenser, the ordinate the reciprocal of the least sparking current in amperes. Values of u, found from measurements of L_1C_1 and L_2C_2, are shown above the diagram.

Fig. 4.



Capacity-potential curve for an induction coil with secondary condenser. $k^2=0·815$.

Another example is shown in fig. 4. This was obtained with the same coil when a Leyden jar (capacity 0·00104 microfarad) was connected with the secondary terminals in parallel with the gap, the gap being in this experiment 2·31 mm. wide between zinc electrodes. In this curve the first arch (counting from the right) has been much reduced in importance—a feature characteristic of increasing coupling,—its intersection with the second (at $C_1 = 6{\cdot}2$ microfarads) being, however, well marked. The curves of figs. 3 and 4, when compared, illustrate the fact that if the oscillating current in the secondary coil changes from one of non-uniform to one of uniform distribution (as it does when the secondary terminals are connected with a condenser) the coupling is increased *. In these two experiments the primary coil was in the same position within the secondary, and the currents employed were not very different. In the experiment of fig. 4 the coupling was 0·815.

A number of other such curves have been determined for an induction-coil; and in all cases in which the comparison has been made substantial agreement has been found, in regard to the form of the curves and the values of u at which the various maxima and minima occur, between the experimental curves and those calculated from the function $U \sin \phi$, although it should be remembered that in the calculated curves no account is taken of the effects of the damping resistances.

Consequently the form of the curve, when determined, enables us to estimate the coupling; and if the primary self-inductance is also known, we can calculate the oscillation constant of the secondary coil.

4. *Determination of the "Capacity-Potential" Curve for a Magneto.*

With these objects in view I have attempted to determine the capacity-potential curve for a certain H.T. magneto. The machine was of the rotating armature type, with the condenser attached to the armature and rotating with it.

The armature having been fixed so that with the timing-lever in its most advanced position—a position slightly in advance of that shown in fig. 2—the contacts were just separated, preliminary measurements by Rayleigh's method gave the following values for the self-inductances at about the same ampere-turns in both circuits:—

$$L_1 = 0{\cdot}0153 \text{ henry at } 1/50 \text{ ampere},$$
$$L_2 = 30{\cdot}7 \quad ,, \quad ,, \quad 1/2500 \quad ,,$$

The mutual inductance L_{21}, by comparison with a standard, was found by a ballistic galvanometer method to be 0·64 henry

* Compare Phil. Mag. April 1914, Table II. p. 570.

at 1/50 ampere in the primary. The mutual inductance was also determined for various other values of the primary current up to 1·5 ampere, at which value it appeared to have reached its maximum value, viz. 1·07 henry.

Thus for steady currents and with 1/50 ampere in the primary, the coupling is

$$k^2 = \frac{L_{21}^2}{L_1 L_2} = 0{\cdot}87,$$

and

$$s = 1 + \frac{2L_{21} + L_1}{L_2} = 1{\cdot}04.$$

With the lever in the most retarded position and the contact-pieces just separated—in which position of the armature the trailing horns of the core were well clear of the pole-pieces—the coupling was found in the same way, and at the same currents, to be 0·81.

The above values of the inductances are applicable when the currents are steady, and must not be assumed to hold during the rapid oscillations which take place in the magneto circuits after the interruption of the primary current.

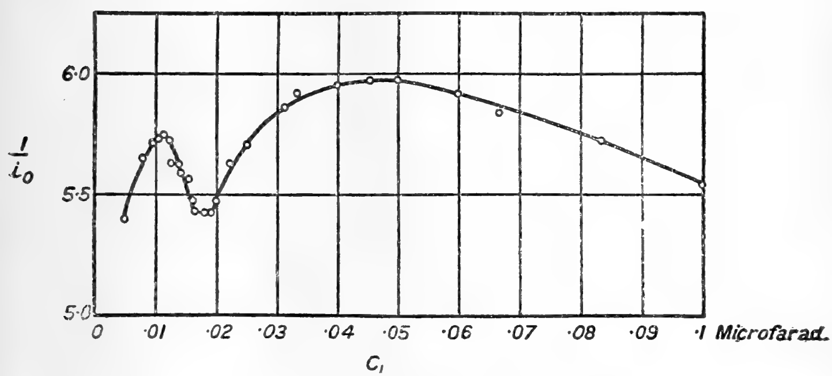
For the purpose of determining least sparking currents for various capacities, the condenser attached to the armature was disconnected from the primary coil and a separate lead connected with the wire leading to the junction J (fig. 1). A circuit was then formed including the primary coil, a battery, a rheostat, a Kelvin graded galvanometer, and a mercury-oil interrupter worked by hand, the circuit being completed through the frame of the magneto and the armature core. Connected directly in parallel with the interrupter was a mica condenser of variable capacity ranging from 0·001 to 1 microfarad.

After numerous trials it was found advisable to remove the high-tension connecting rod and its carbon collecting brush, and to replace them with an insulated copper wire pressed into contact with the collecting ring and connected with one side of the H.T. spark-gap; also to insert a connecting wire between the armature core and the frame to ensure good electrical contact. The H.T. spark electrodes consisted of two cylinders terminating in a plane and a spherical cap (radius about 8 mm.), the gap between them being finely adjustable. With these changes, and with clean mercury under paraffin oil in the interrupter, the spark could be depended upon to appear regularly—on a given occasion and with a given capacity—at practically the same value of the primary current.

Included in the primary circuit was also an air-core coil of small self-inductance (0·00099 henry). Without this it was not found possible to obtain a curve having any well-marked features; the optimum capacity was very small—apparently between 0·01 and 0·02 microfarad—and it could not be determined with any accuracy. Probably the spark method is not sufficiently delicate to enable one to detect small fluctuations in the curve when the capacity is very small.

With the series inductance, however, a curve showing clear indications of the arches was easily obtained. The curve is shown in fig. 5. In this experiment the gap was

Fig. 5.



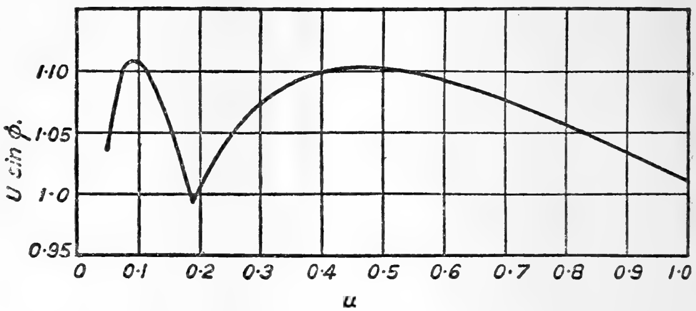
Capacity-potential curve for a magneto.

0·1 mm. wide, and the points on the curve were obtained by adjusting the current until the spark passed at one-half the number of breaks. A curve showing the same features was also obtained with a gap of 0·2 mm. The points may also, less conveniently, be determined by finding the smallest current required to produce the spark.

The curve shows two maxima, at about $C_1=0·0475$ and $C_1=0·0115$ microfarad, the minimum between them being at about $C_1=0·018$ microfarad. The rise in the curve from 0·018 to 0·0115 microfarad was verified on many occasions, and there can be no doubt that it corresponds with one of the arches of the $(u, U \sin \phi)$ curve.

In fig. 6 is shown a part of the calculated (u, U sin ϕ) curve for the coupling $k^2=0{\cdot}7$, s being taken as 1·05. The highest point of the first arch occurs at $u=0{\cdot}465$, the minimum at $u=0{\cdot}19$, and the second maximum at $u=0{\cdot}095$. So far as

Fig. 6.



Calculated (u, U sin ϕ) Curve for a Magneto.
$k^2=0{\cdot}7$. $s=1{\cdot}05$.

the proportions of the first arch (on the right) are concerned, the curves of figs. 5 and 6 are very similar : there is much the same percentage drop in both from, say, the maximum to a point of double or one-half its abscissa, and the ratio of the abscissæ at which the maximum and the minimum occur is not very different in the two curves. The chief point of difference between the curves is in the ratio of the two maxima. In this respect the calculated curve for $k^2=0{\cdot}66$ would agree much better with the experimental curve, but it would show considerably greater deviations from it in other features, especially in the ratio of the capacities at which the two maxima, and at which the first maximum and the minimum, occur.

On the whole, the curve for the coupling $k^2=0{\cdot}7$ (fig. 6) appears to be in closest agreement with the experimental curve ; and the relative smallness of the second maximum in the latter may arise from several causes which would operate more strongly at the higher frequencies—*e. g.*, the "lag" of the spark, the influence of frequency on the inductances, and the difficulty of securing good interruptions when the capacity in parallel with the break is very small.

Additional evidence in support of the view that the coupling—with the small series inductance in the primary circuit—is not far from 0·7 is found in the manner in which the optimum capacity varies when the series inductance is altered. If this inductance (L_3) is increased the optimum capacity C_0 diminishes, and the product $(L_1+L_3)C_0$ shows little variation over a wide range. For example, with the small coil the optimum was 0·0475 microfarad, and the product $(L_1+L_3)C_0$ was $0{\cdot}689 . 10^{-9}$; with a series inductance of 0·00527 henry the optimum was 0·0325 mfd., and $(L_1+L_3)C_0 = 0{\cdot}611 . 10^{-9}$; with $L_3 = 0{\cdot}0104$ henry C_0 was 0·0225, and $(L_1+L_3)C_0=0{\cdot}538 . 10^{-9}$ c.g.s. But when the series inductance was omitted altogether the optimum fell to below 0·02 mfd., and the product of the primary self-inductance and the capacity to less than $0{\cdot}27 . 10^{-9}$. Thus a marked diminution occurs in the product of primary self-inductance and optimum capacity when the small series coil is omitted; and this is precisely what we should expect when the optimum changes—owing to the shrinking of the first arch with increasing coupling—from the first to the second arch of the curve, which change occurs at about $k^2=0{\cdot}7$. That the change is from the first arch to the second, and not, for example, from the second to the third, is shown by the comparatively small variation in the product $(L_1+L_3)C_0$ as the series inductance is largely increased (*cf.* Phil. Mag. Aug. 1915, p. 231, Table I.) *.

We shall therefore take 0·7 as the coupling with the small auxiliary coil in the primary circuit; and on comparing the abscissæ of the highest point of the first arch in figs. 5 and 6, we find

$$\frac{(L_1+L_3)C_0}{L_2C_2} = 0{\cdot}465.$$

Thus the product of self-inductance and capacity for the secondary coil is

$$L_2C_2 = \frac{0{\cdot}689 . 10^{-9}}{0{\cdot}465} = 1{\cdot}48 . 10^{-9} \text{ c.g.s.}$$

5. *Measurement of the Inductances, the Coupling, and the Effective Resistances by Oscillation Methods.*

When the magneto was connected with the electrostatic oscillograph † it was found that, as was to be expected, no

* It can be shown that when a series inductance L_3 is connected between the point J (fig. 1) and the condenser, the expression for the coupling becomes $\frac{L_{12}L_{21}}{(L_1+L_3)L_2}-\frac{(s-1)L}{(L_1+L_3)}$, and that the ratio $\frac{(L_1+L_3)C_1}{L_2C_2}$ takes the place of u in the equations. For the purposes of the present experiments, however, the effect of an auxiliary inductance in reducing the coupling was determined experimentally (see Section 5 below).

† Phil. Mag. Aug. 1907, p. 238.

curves showing the oscillations of the circuits could be obtained. The frequencies of the circuits are so high that probably no oscillograph with material moving parts would be suitable for the purpose. The difficulty cannot be overcome by merely connecting condensers of large capacity with the circuits in order to increase the periods, for since the effective resistances are very considerable the logarithmic decrements of the oscillations then become too great.

When, however, a suitable coreless coil was connected in series with the primary or the secondary coil, as well as a condenser, the damping factor was so far reduced that curves suitable for the measurement of frequency and effective resistance could be obtained. This method was accordingly adopted in the following determinations of the inductances and effective resistances of the circuits. Although the results were thus obtained for frequencies considerably below those which the circuits possess when unprovided with such additional inertia and capacity, they nevertheless correspond much more closely with actual working conditions than would results obtained by the use of slowly alternating currents or by other "slow" methods.

In the following experiments the various coefficients of the circuits, as well as those of the coils used as auxiliaries, were all determined for frequencies of about 600 oscillations per second. The quantities regarded as known and used as standards in the measurements were the frequency of a certain tuning-fork, the capacities of certain standard mica condensers, and the self-inductance of a certain air-core coil.

The self-inductance of the primary coil of the magneto was determined by connecting in the primary circuit, between the point J and the condenser (fig. 1), an air-core coil the self-inductance, L_3, of which was 0·0609 henry. Across the interrupter was connected a mica condenser of 0·6 microfarad. The oscillograph was connected to the H.T. terminal and to the frame of the machine. In these circumstances the circuits are loosely coupled and the oscillation-constant of the secondary is very small in comparison with that of the primary. Consequently, the frequency, n, of the oscillation—excited by interrupting a measured current in the primary circuit—gives the value of $(L_1+L_3)C_1$ subject to a small correction for the effect of the secondary. The expression is

$$(L_1+L_3)C_1 = \frac{1}{4\pi^2 n^2}\left\{1-\frac{(s-1+k^2)L_2C_2}{(L_1+L_3)C_1}\right\},$$

the second term in the brackets amounting in the present case to 0·005. Thus L_3 and C_1 being known, L_1 can be calculated. The result was $L_1=0\cdot0135$ henry at 0·4 ampere.

This value of L_1 is considerably smaller than that found by the galvanometer method, which was, moreover, determined at a much smaller current.

For the determination of the self-inductance of the secondary coil, the H.T. terminal of this coil was connected through a large air-core coil (70·15 henries) with one plate of an oil condenser (0·00088 microfarad), the other plate of this condenser being connected with the frame of the machine. The oscillograph was connected with the plates of the condenser. The primary circuit contained neither series inductance nor a condenser. In these circumstances the period of the oscillation is equal to $2\pi\sqrt{(sL_2+L_4)C_2}$, where C_2 is the capacity of the oil condenser with certain small additions for the capacities of the coils and the oscillograph, and L_4 represents the self-inductance of the air-core coil. The total value of C_2 in this experiment was 0·00091 microfarad. The value of L_4 being known, and s being taken as 1·05, L_2 was calculated from the observed frequency. The result was $L_2=19\cdot3$ henries, which value is again much smaller than that determined by the galvanometer method.

In this experiment the oscillation was started by interrupting a primary current, but it may instead be started by sparking with a small induction-coil to the terminals of the oil condenser. The value of L_2 found from the oscillation excited in this way was found to be 1 per cent. less than the value given above, and this difference is probably to be accounted for by a slight difference in the degree of magnetization of the core, the amplitude of the oscillation being smaller in the sparking method than in the other.

It should be observed that the value of L_2 given above holds for oscillating currents which are nearly uniformly distributed along the secondary wire ; when the secondary terminals are not connected with a condenser, the value of L_2 is still smaller.

The sparking method of excitation, still with the oil condenser and the large air-core coil, was also used for the determination of the coupling. By equation (16) the ratio of the squares of the frequencies of the system with the primary coil open and with it closed is $(1-k^2)/s'$, where s' now refers to the whole secondary inductance, including that of the air-core coil. The value of k^2 so found was 0·199. Hence without the air-core coil the coupling is $0{\cdot}199 \times \frac{L_2 + L_4}{L_2} = 0{\cdot}92$. This is the coupling when the secondary terminals are connected with the oil condenser. The mutual inductance L_{21} is therefore 0·49 henry.

The same experiment was repeated with the primary circuit closed through the small auxiliary coil (0·00099 henry) used in the determination of the curve of fig. 5. The result in this case was $k^2 = 0{\cdot}80$. Now from fig. 5 we concluded that the coupling in the experiment in which that curve was determined was 0·7. Hence the removal of the oil condenser from the secondary circuit, by rendering the secondary current less uniformly distributed, reduces the coupling from 0·8 to 0·7, and we may assume that the same proportional reduction will take place whether the auxiliary coil be present in the primary or not, since the presence of this coil cannot affect the coefficients L_{21}, L_{12}, or L_2.

Thus without series inductance in either circuit, and without a secondary condenser, the coupling of the magneto circuits is

$$k^2 = 0{\cdot}92 \times \frac{0{\cdot}7}{0{\cdot}8} = 0{\cdot}80.$$

The effective resistances of the primary and secondary circuits were determined from the logarithmic decrements and periods of the oscillograph curves, with due allowance in each case for the resistance of the auxiliary coil, which was determined independently for about the same frequency. It was found that the effective resistances of the magneto circuits for frequency 600 were very much greater than the steady-current values. Thus the resistance of the primary coil for steady currents was 0·85 ohm, and its effective resistance at frequency 600 was found to be 49 ohms. The secondary coil gave for steady currents 2115 ohms, for the oscillations 42,670 ohms. These very great differences arise mainly from core losses occurring during the oscillations, and are not found in air-core coils. For example, the large air-core coil had a steady-current resistance of 14,000 ohms, and an effective resistance at frequency 600 of 15,300 ohms.

Specimens of the photographic curves used in these measurements are shown in Plate VII. figs. 7 and 8 *. In these cases the oscillations were started by the sparking

* Full details as to the manner in which the frequencies are determined from the photographs have been given in a former paper (Phil. Mag. Aug. 1907, p. 242).

method, the secondary circuit including the large air-core coil and the oil condenser. The oscillograph being used idiostatically, the deflexion is proportional to the square of the difference of potential, and each elevation in the curve represents a half-oscillation. The first half-wave or two represent the period during which the exciting spark is passing, the remainder the free oscillation of the magneto circuit. In fig. 7 the primary circuit was open and unconnected with a condenser. In the case of fig. 8 this circuit was closed. The curves illustrate the large damping-effect of the core, which is much reduced when the primary is closed owing to the fact that the core is then partially shielded from the magnetic action of the secondary current.

6. *Calculation of the Capacity of the Secondary Circuit.*

We are now in a position to form an estimate of the value of the capacity of the secondary circuit of the magneto. Since

$$L_2C_2 = 1{\cdot}48 \,.\, 10^{-9} \text{ c.g.s.} \quad \text{and} \quad L_2 = 19{\cdot}3 \,.\, 10^{9} \text{ c.g.s.},$$

we have

$$C_2 = 0{\cdot}000077 \text{ microfarad.}$$

This estimate is, however, rather too low on account of the fact that we have assumed too great a value for L_2—the value for uniformly distributed currents. Probably the secondary capacity does not fall far short of 0·0001 microfarad, a value which is not exceeded by that of the secondary of a very large induction-coil. It is about equal to the capacity of a spherical condenser of radii 3 and 3·1 cm. Large secondary capacity must be a feature of all H T. magnetos, owing to the fact that the coils are closely surrounded by metallic surfaces at zero potential; and this fact must exercise a great influence, not only on the spark-length, but also on the character of the spark and the quantity of electricity discharged in it.

In the case of an induction-coil, when a condenser is connected with the secondary terminals in parallel with the gap, the discharge at moderate currents—currents which are considerably greater than the minimum required to produce the discharge—takes the form of a "multiple spark," a large number of sparks sometimes passing (all in the same direction) at each break of the primary current. When the primary current is increased to a certain value the discharge changes to the type usually found when there is no secondary condenser—a single spark followed by an arc.

In the magneto also the discharge at moderate currents may take the form of a multiple spark (see Plate VII. fig. 10), doubtless owing to the large capacity of its secondary coil.

Another effect of connecting a condenser with the secondary terminals of an induction-coil is to cause, for a given small or moderate current and a given spark-gap, a large diminution in the quantity of electricity discharged in the spark.

In some ways the large capacity of the magneto may act beneficially: for example, by increasing the periods of oscillation, and thus lengthening the duration of the high potential, it enables the spark to appear more readily; again, a large secondary capacity involves a large optimum primary capacity, which is an advantage, since the interrupter works better when associated with a condenser of large capacity. It should also be remembered that, other things being the same, an increase in the secondary capacity does not necessarily cause a diminution of the secondary potential for a given primary current; from the point of view of secondary *potential* there is an optimum secondary as well as primary capacity. It is also said that condenser-discharge sparks are specially favourable to the production of ignition.

7. *Oscillations during the Discharge.*

Some photographs were taken of the magneto spark by focussing the image on a sensitive plate with a rotating concave mirror. Two of these are shown in Plate VII. figs. 9 and 10.

The photograph in fig. 9 was taken at the interruption of a primary current of 1·8 amperes. It shows an initial spark followed by an arc on which are superposed a number of fine regularly-spaced bands representing small oscillations. Six or eight of these bands are visible on the negative, and though faint they could be measured with fair accuracy under a low-power microscope. Their frequency is about 15,800 per second. These are the oscillations of the system with the secondary closed by the arc, and their period is, by equation (15) with $C_2 = \infty$, given by the expression

$$2\pi\sqrt{\frac{L_1C_1(1-k^2)}{s}},$$

where k^2 is the coupling for uniformly distributed currents of the mean value used in this experiment.

In the experiment the primary condenser was of capacity $C_1=0{\cdot}2$ mfd.—rather greater than that of the condenser attached to the armature; and taking $L_1=0{\cdot}0135$ henry, $s=1{\cdot}05$, we find $k^2=0{\cdot}959$. This value of the coupling is greater than that found by the other methods; but when we remember the difference in the circumstances of the experiments, and especially in the value of the current, it is probably not inconsistent with them.

A similar photograph was obtained with a primary capacity of 1 microfarad, and the frequency in this case was about 7300 per second. The ratio of the two frequencies is about 2·16, which is not far from the inverse ratio of the square roots of the capacities, *i. e.* $\sqrt{5}$.

Fig. 10 was obtained at the interruption of a weaker primary current, in this case about 1·0 ampere. In this photograph the bands are much more clearly separated, reminding one rather of a "multiple spark" than of a "pulsating arc." As already remarked, this effect arises from the large capacity of the secondary coil.

8. *Calculation of the Maximum Secondary Potential from the Constants of the Circuits.*

We can now calculate the frequencies, amplitudes, and damping factors of the oscillations of the magneto circuits, and hence the maximum secondary potential.

The capacity of the condenser attached to the armature is $C_1=0{\cdot}175$ microfarad, and with $L_1=0{\cdot}0135$ henry we have

$$L_1C_1 = 2{\cdot}36\,.\,10^{-9} \text{ c.g.s.}$$

Also

$$L_2C_2 = 1{\cdot}48\,.\,10^{-9} \quad ,,$$

$$L_{21} = 0{\cdot}49 \text{ henry},$$

$$k^2 = 0{\cdot}8,$$

$$s = 1{\cdot}05,$$

$$\frac{R_1}{2L_1} = 1815,$$

$$\frac{R_2}{2L_2} = 1100.$$

Hence, by (15), the frequencies are

$$n_1 = 2607 \text{ per second},$$

$$n_2 = 11624 \quad ,,$$

and by (28) the amplitudes are (with $i_0=1$ ampere)

$$B_1 = \frac{2\pi(L_1+L_{21})i_0 n_1 n_2^2}{n_2^2-n_1^2} = 8680 \text{ volts},$$

$$B_2 = \frac{2\pi(L_1+L_{21})n_1^2 n_2}{n_2^2-n_1^2} = 1950 \quad ,,$$

Further,

$$\theta_1 = \frac{R_1}{2L_1} . L_1C_1 = 4{\cdot}29 . 10^{-6},$$

$$\theta_2 = \frac{R_2}{2L_2} . L_2C_2 = 1{\cdot}63 . 10^{-6}.$$

Hence by (33)

$$\beta = -2{\cdot}187 . 10^{-6},$$

and by (31)

$$k_1 = 1380,$$

$$k_2 = 4120.$$

The expression for the secondary potential is therefore (in volts)

$$V_2 = 8680\, e^{-1380t} \sin 938500t - 1950\, e^{-4120t} \sin 4185000t,$$

t being in seconds and the angles in degrees.

The phase angles δ_1, δ_2 are here neglected. They are small, and being approximately proportional to the frequencies their effect is merely to alter slightly the time at which the maximum potential occurs without altering its value to any appreciable extent.

The maximum value of V_2 is 8530 volts at $t=0{\cdot}000069$ sec. ($\phi=64^\circ\ 45'$).

If damping had been neglected the maximum would have been 9710 volts at $t=0{\cdot}00007$ sec. If, further, the circuits had been better adjusted, so that the positive maxima in the two oscillations occurred simultaneously, the maximum would have been (for the same amplitudes) 10,630 volts.

We may therefore say that there is a drop of 9 per cent. in the maximum secondary voltage due to difference of phase, and a drop of 12 per cent. due to damping.

Now the least current observed to produce a 0·2 mm. spark was 0·232 ampere. Taking the sparking potential at 0·2 mm. as 1550 volts, the smallest primary current required to give this spark is, according to the above expression for V_2, about 0·182 ampere. The actual potential generated by the magneto, as estimated from the spark-length, therefore falls short of the value calculated by the above expression by about 25 per cent.

This difference must be attributed to

(1) the "lag" of the spark,

(2) imperfect interruption of the primary current,

(3) the fact that the inductances were determined for frequencies considerably below those of the actual magneto circuits.

If the inductances had been measured for a frequency of several thousands per second the values obtained would presumably have been all considerably smaller, and the theoretical value of the secondary potential would have been correspondingly reduced.

If the steady-current values of the inductances had been used in the calculation of the secondary potential the result would, on the other hand, have been considerably greater—probably about 50 per cent. greater than the calculated value given above. It is clear, therefore, that only a small proportion of the initial magnetic energy $\frac{1}{2}L_1 i_0^2$ (L_1 being here the primary self-inductance for slowly varying currents) appears as electrostatic energy in the secondary circuit. Regarded as an arrangement for producing high potential, the magneto is therefore a machine of low efficiency. In view of the fact that high secondary potential is the chief condition for spark production, and that in the opinion of some authorities it is also one of the controlling factors in the process of ignition—it has been suggested that a sufficiently high potential will produce ignition even though no spark actually passes,—there appears to be need for improvement in this respect in the design of high-tension magnetos.

XII. *Forced Vibrations Experimentally Illustrated.* By

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[Plates VIII. & IX.]

Contents. Page

I. Introduction.

It is well known that forced vibrations play an important part in most branches of physics. We may mention in this connexion: resonance tubes, fluorescence, Lodge's

* Communicated by the Authors.

syntonic jars, Hertz's oscillator and resonator, wireless telegraphy. Possibly we might be justified in adding to this list the sensitive parts of the ear and eye.

It accordingly seems desirable to have some quite simple vivid mechanical illustration that exhibits qualitatively and quantitatively the chief phenomena concerned. Types of such experiments are here described.

The apparatus consists of a single heavy driving pendulum and a number of light driven ones, of graduated lengths all suspended from the same tightly stretched cord. Thus the various effects of tuning and mistuning may be observed simultaneously. By a steady view the variation of amplitude with tuning is seen in spite of the phase differences involved. By a stroboscopic view (or illumination) the variation of phase with tuning is exhibited to a small class (or a larger audience).

The above remarks refer to the experiment in various forms. For confirmation of exact quantitative relations the experiment needs arranging with special attention to certain details. It is then found to confirm the theory in every respect.

By changing to responding bobs of greater density the increased sharpness of resonance with smaller damping is shown.

Photographic reproductions are given showing four time exposures and eight instantaneous views of the responding pendulums. These exhibit all the features enumerated above.

II. General Theory.

The equation of a single particle of mass m with restoring force s times its displacement y and r times its velocity under the action of a sustained harmonic impressed force, may be written

$$m\frac{d^2y}{dt^2}+r\frac{dy}{dt}+sy=\mathrm{F}\sin nt, \quad . \quad . \quad . \quad . \quad (1)$$

or

$$\frac{d^2y}{dt^2}+2k\frac{dy}{dt}+p^2y=f\sin nt, \quad . \quad . \quad . \quad . \quad (2)$$

where

$$2k=\frac{r}{m}, \quad p^2=\frac{s}{m}, \quad \text{and } f=\frac{\mathrm{F}}{m}. \quad . \quad . \quad . \quad (3)$$

The solution of this may be written

$$y=\frac{f\sin(nt-\delta)}{\sqrt{\{(p^2-n^2)^2+(2kn)^2\}}}+\mathrm{E}e^{-kt}\sin(qt+\epsilon), \quad . \quad . \quad (4)$$

or,

$$\left\{\begin{matrix}\text{Resultant}\\ \text{Displacement}\end{matrix}\right\}=(\text{Forced Vibration})+(\text{Free Vibration}).$$

In equation (4)

$$\tan\delta=\frac{2kn}{p^2-n^2}, \quad q^2=p^2-k^2, \quad . \quad . \quad . \quad . \quad (5)$$

and E and ϵ are arbitrary constants to be chosen to fit the initial conditions.

As indicated, the first term on the right side of (4) represents the forced vibration with which we are here chiefly concerned. The second term denotes the free vibration of the system, and this must be present to complete the solution. If the responding particles were at rest in the zero position when the impressed force was started, then the values of E and ϵ would have to be such as to express a free vibration which would annul both displacement and velocity as given by the forced vibration, whose amplitude and phase have nothing arbitrary.

If the forced and free vibrations coexist of differing periods and comparable amplitudes, beats will occur between them. These are easily obtained but are usually best avoided.

When, in virtue of the damping factor involving k, the free vibration has practically disappeared, the forced vibration is left in possession of the field. No beats are then possible. While the free vibration is dying away, the resultant motion which is under observation grows from nothing to the fixed amplitude and phase of the forced vibration.

Considering now the forced vibration itself, we may note, from the first term on the right side of equation (4), the following points.

1. The period of the forced vibration is identical with that of the impressed forces whatever the period natural to the responding system.

2. The best response occurs for the best tuning. This is a brief statement which may convey the right idea with sufficient accuracy for our present purpose. To make the statement precise we must define best as applied both to response and to tuning. This has already been done by one of the present writers in "Range and Sharpness of Resonance, &c." (Phil. Mag. July 1913).

3. The phase of the forced vibration varies continuously between 0 and π with the tuning. Thus the phase angle δ is almost zero for p^2 much greater than n^2, *i. e.*, for a responding system whose natural frequency is much greater than that of the impressed force. On the other hand, δ is almost π for p^2 much less than n^2, *i. e.*, for a responding system of natural frequency much less than that of the

impressed force. Finally, for $p^2=n^2$, $\delta=\pi/2$, and this corresponds with the case of maximum amplitude of response.

4. The smaller the damping of the responding system the sharper is its resonance, the greater the damping the greater is its range of resonance. That is to say, the smaller the value of k the greater is the falling off of the response for a given mistuning, and *vice versa*. For it is seen from the first term on the right side of equation (4) that when $p^2=n^2$ the amplitude is a maximum, for n constant while p varies. Further, when p^2-n^2 is finite and of a given value it has a less effect on the amplitude if the other term in the denominator $(2kn)^2$ is large.

By reference to the second term on the right side of equation (4) we see that the ratio of successive amplitudes of the free vibrations is $e^{k\pi/q}=e^{k\pi/p}$ nearly. But the logarithmic increment λ (per half wave) for this system is the logarithm to the base e of this ratio. Hence we have

$$\lambda=\frac{k\pi}{p} \text{ or } k=\frac{p\lambda}{\pi}=\frac{n\lambda_0}{\pi}, \quad . \quad . \quad . \quad . \quad (6)$$

where $\lambda_0=$ the log. dec. for the responding pendulum of the same period as the forces. Thus by observations on the free vibrations of a responding pendulum the value of k may be found.

It might be urged that in the experimental arrangement specified we have strictly speaking an instance of coupled vibrations, and have not reached the ideal of forced vibrations. That this is not the case may be ascertained as follows.

On reference to "Coupled Vibrations, II." (Phil. Mag. Jan. 1918) we see that in coupled systems two superposed vibrations occur, the ratio of their frequencies being $p/q=\sqrt{(1+\beta)}$. Also by equation (24) p. 65 and (43*a*) p. 68 of the same paper, we see that the ratio of the amplitudes of these quick and slow vibrations for our responding systems is given by

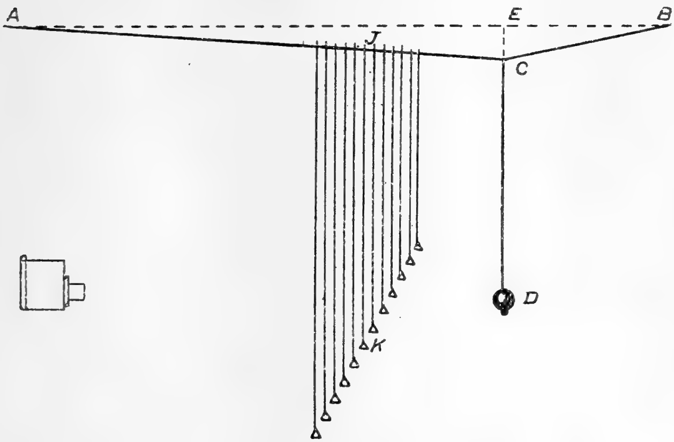
$$\frac{-\rho}{(1+\rho)\beta}e^{-(\rho-1)kt/(1+\rho)}=\frac{e^{-kt}}{\beta} \text{ nearly for } \rho \text{ large.} \quad . \quad (7)$$

In our experimental case ρ exceeds 2000 (being 700 gm. /0·3 gm.), k is of the order one fifth, and β about one third. Thus after 20 and 40 seconds, the ratio in question has fallen to 1/20 and 1/1000 respectively.

III. Illustrative Experiments.

In fig. 1 is shown an experimental arrangement that was found convenient when exact quantitative work was not the aim but rather a lecture demonstration of the general features involved was required.

Fig. 1.



Forced Vibration Apparatus with Differing Forces.

The tightly stretched cord AB is drawn down to a peak at C by the weight of the driving pendulum CD about 60 cm. long with bob of iron about 6 cm. diameter. The responding systems are pendulums of graduated lengths and with very light bobs so that their free vibrations are quickly damped. These pendulums should be placed fairly near to the driver and the point A kept far from them, so that their points of suspension all have approximately the same motion from the vibrations of the heavy bob D. The bobs of these responding pendulums may be—

(*a*) of solid cork about 2·3 cm. long, 1·2 cm. diameter, and 0·4 gm. mass.

(*b*) of hollow paper cones, semi-vertical angle 20°, of mass 0·2 gm.

(*c*) of paper cones, semi-vertical angle 45°, of mass 0·3 gm.

(*d*) of blown-glass spheres in imitation of pearls, diameter 6 mm.

The attachments to the cord AC may be made by passing the cotton suspension through it with a needle and leaving the end free. They are then sufficiently held by friction and may be adjusted, at will, by simple pulling. The paper cones may each have a little soft wax inside and the cotton suspension passed through by a needle and the end left free. The cone may then be slid up and down the cotton at pleasure to adjust in line with the others, and will stay where left.

The white paper cones are much easier seen (or photographed) stroboscopically than the corks, and are on the whole more satisfactory than the corks or the blown-glass spheres.

Since the phase of the forced vibration varies with the natural period and therefore with the length of the responding pendulum, the full displacements are not attained simultaneously. But on viewing the apparatus steadily from one end, A say, the full displacements are seen to be reached successively. Hence one may see a resonance curve in which the squares of the various periods or lengths of the pendulums are disposed vertically while the corresponding amplitudes exhibit themselves horizontally. Thus the limits to which the light bobs swing on each side form there a resonance curve in which the squares of the periods are the vertical abscissæ and the amplitudes are the horizontal ordinates. Thus a time exposure will give a photograph exhibiting this resonance curve in duplicate to right and left of the central line. The effect is shown for various types of responding pendulums in figs. 1, 2, and 3 of Plate VIII. It is seen that the blunt cones (fig. 1) give curves showing the sharpest resonance, the small blown-glass spheres (fig. 3) give the greatest range of resonance, and the sharp cones (fig. 2) show an intermediate type of resonance. This is in accordance with theory, since the values of k for these three kinds of bob (as found from their logarithmic decrements when vibrating alone) are 0·16, 0·265, and 0·2 respectively.

In order to appreciate the various phases of the vibrating systems of differing periods an instantaneous view of the bobs is needed. The motion is so slow that it seemed quite unnecessary to make any elaborate electric timing arrangement. At first the camera was instantaneously exposed 40 times at the desired instant as judged by sight, and this gave the result reproduced in fig. 4 (Pl. VIII.). Better results shown in figs. 5 and 6 were obtained by the ordinary flash-light process. One of these, fig. 5, corresponds to the central position of the driver, and exhibits what may be called an exaggerated resonance curve. This is because when the driver is at the centre, the driven bob of about the same length and having maximum response is then at one end of its swing and therefore shows its full amplitude. But as we pass to pendulums shorter or longer than this one, we gradually change to like phase with the driver or opposite phase respectively. Hence the horizontal ordinates of the curve rapidly diminish from their maximum both because the amplitude is less and the phase is not right to exhibit it fully. The comparison of fig. 5 with fig. 2 makes this point clearer.

Special interest attaches to the instantaneous view shown in fig. 6, and taken when the driving bob was at one end of its swing. The responding bob of about the same length as the driver is then at the centre, the much shorter responders are nearly in phase with the driver, the much longer ones in the opposite phase nearly. The resulting curve may be approximately represented by

$$y = \frac{\pm x}{a^2 + x^2}.$$

Other powers of x would be needed to represent more precisely the exact curve for any given arrangement of the experiment. This will be dealt with later.

To exhibit these instantaneous effects to a single observer stroboscopic vision is desirable. This was easily arranged by using a card with a vertical slit in its centre, each end of the card being carried by a pendulum. The period of this pendulum should bear a simple relation to that of the driver. In the actual experiments it was made of four times the length of the driver, as that suited the position of a purlin in the roof. The moving slit at the middle of its swing passes a slit of the same size in a fixed card. The period of coincidence of these slits can be shortened at will by increasing the amplitude of the pendulums carrying the moving card. For about six observers we may use a camera and focussing screen instead of a fixed slit. For a larger audience the same arrangement of fixed and moving cards may be used as for a single observer, but the light from an arc-lamp should be passed through the slits on to the bobs while the room is otherwise in darkness.

IV. Detailed Theory.

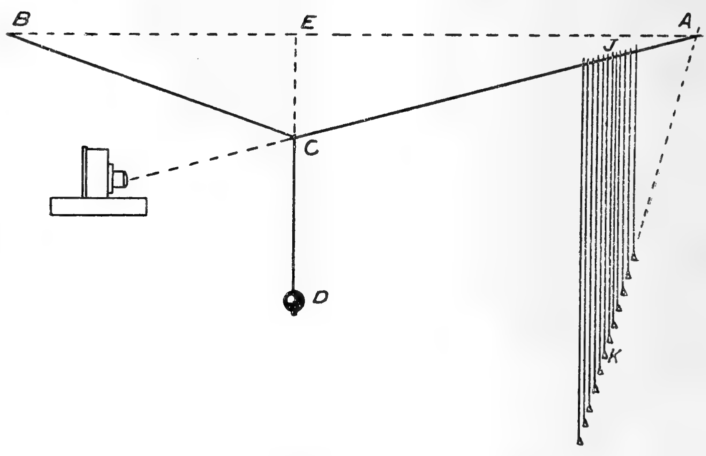
Let us now pass from general ideas as illustrated by the apparatus in fig. 1 to an experimental arrangement more suitable for a strict quantitative examination of the phenomena involved. Referring to equations (1) to (3) we see that in the set of responding pendulums we naturally keep m

and r constant throughout the series but vary the natural periods. For quantitative work it is also desirable to keep f constant throughout.

Now the period depends upon $s=mp^2=mg/x$ where x is the length JK of the pendulum in question. Further, $F=mg\times$ (inclination of JK) due to full amplitude of heavy driving bob D.

Now this inclination of JK (to the vertical) is the displacement of J divided by the length $JK=x$. Hence to keep F and f of same value for all the responding pendulums we must have their inclinations equal for a given displacement of D. And this is obviously obtained by arranging the bobs so that the straight line AK passes through them all as shown in fig. 2. For when the pendulum length is halved the displacement of the point of suspension is halved also, and thus the inclination retains the same value.

Fig. 2.



Forced Vibration Apparatus with Equal Forces.

Again, to have on the photographic plate coincidence of all the points J we must have the camera-lens in the line AJC when all is at rest. This coincidence is desirable so that the length x for each pendulum shall reckon from a definite invariable origin. Further, to have the displacements of the bobs K measured from the same vertical line on the plate, we must have the centre of the camera in the vertical plane through ABCD when at rest. But this has already been secured.

Finally, to avoid unequal treatment of the displacements of the various bobs K, their distances from the camera must be nearly equal. Hence they should be set well away from the camera but as close together as will avoid entanglement.

As regards the length JK for the best tuning with DE, it should be noticed that no equality will be apparent on the photographs. First, because E is not shown at all, and second, because the length DC which is shown is greatly magnified relatively to the lengths JK. To confirm the theory in this respect actual measurements of these lengths should be made on the apparatus itself.

Consider the time after the dying away of the free vibrations. Then equation (4) has reduced to

$$y=\frac{f\sin(nt-\delta)}{\sqrt{\{(p^2-n^2)^2+(2kn)^2\}}}, \quad . \quad . \quad . \quad . \quad (8)$$

which expresses the forced vibration only.

Case I. Take first the variation of amplitude y, of the forced vibration with frequency natural to the responding system. We have already from (6), $k=n\lambda_0/\pi$, let us now write

$$p^2=g/x, \quad n^2=g/l, \quad \text{then } kn=g\lambda_0/\pi l. \quad . \quad . \quad . \quad (9)$$

And (9) in (8) leads to

$$y_1{}^2=\frac{\pi^2f^2l^2x^2}{g^2\{\pi^2(l-x)^2+4\lambda_0{}^2x^2\}}. \quad . \quad . \quad . \quad . \quad (10)$$

Case II. For the second case take the instant when the heavy bob D is undisplaced but is moving in the positive direction. Then we may write $\sin nt=0$, and $\cos nt=1$. Inserting these in (8) we have

$$y_2=\frac{-f\sin\delta}{\sqrt{\{(p^2-n^2)^2+(2kn)^2\}}}=\frac{-f(2kn)}{(p^2-n^2)^2+(2kn)^2}. \quad . \quad (11)$$

Then using (9), (11) becomes

$$y^2=\frac{-2\pi f\lambda_0lx^2}{g\{\pi^2(l-x)^2+4\lambda_0{}^2x^2\}}, \quad . \quad . \quad . \quad . \quad (12)$$

Case III. Consider next the instant when the heavy bob D has its maximum displacement in the positive direction. Then we may write $\cos nt=0$, and $\sin nt=1$. Substituting these in (8) we have

$$y_3=\frac{f\cos\delta}{\sqrt{\{(p^2-n^2)^2+(2kn)^2}}=\frac{f(p^2-n^2)}{(p^2-n^2)^2+(2kn)^2}. \quad . \quad (13)$$

Inserting the values of p^2, n^2, and kn given in (9), equation (13) becomes

$$y_3 = \frac{\pi^2 f l (l-x) x}{g\{\pi^2 (l-x)^2 + 4\lambda_0^2 x^2\}}. \quad . \quad . \quad . \quad (14)$$

V. Experimental Results.

The photographs shown in figs. 7–12, Pl. IX., were taken with the apparatus arranged as in fig. 2 so as to keep the value of f due to the big bob the same for each responding pendulum. One kind of light bob only was used, viz. the sharp-angled paper cones. The curves obtained on the plates differed so little from those used in the first arrangement that it seemed unnecessary to repeat experiments with bobs having different dampings. Fig. 8 represents the resonance curve in duplicate to the right and left of the central line, and was obtained by a time exposure. The maximum swing of the lower cones is seen to be greater than that of the upper cones. This is because the vertical abscissæ are lengths as x in (10) and not the squares of the frequencies as p^2 in (8). These curves agree with equation (10).

Figs. 7, 9, 10, 11, 12, show instantaneous views taken by flash-powder. Fig. 7 shows the state when the heavy bob was passing the centre towards the right. The figure shows the bob slightly beyond the centre, but this is a small fraction of the amplitude and involves a still smaller fraction of the quarter period. Then it is well seen from the curve (*a*) that the upper responding bobs are in phase with the driving bob and therefore at the middle of their swing towards the right, (*b*) that the lower ones are also at the middle of their swing though they are in opposite phase and moving to the left, and (*c*) the middle bobs are at the end of their swing with a lag of about 90° phase angle behind the driver.

Fig. 9 shows the curve obtained with the large bob at the end of its swing to the right. It will be noticed that the upper bobs are less displaced from the centre than the lower ones. This asymmetry was to be expected from the form of equation (14), with which it is in entire accord.

Figs. 10–12 are intermediate stages with the large bob partly displaced. They show the gradual melting of the curve from the case of exaggerated resonance with the bobs all on one side, fig. 7, to the state of fig. 9 with half the bobs on each side. The set of figures 7, 10, 11, 12, 9 correspond to intervals of about the tenth of a second in the motions of the actual pendulums.

Nottingham,
May 28, 1918.

XIII. *Problems of Denudation.* By HAROLD JEFFREYS, *M.A., D.Sc., Fellow of St. John's College, Cambridge**.

THE major phenomena of physical geology may be divided into three main groups, namely crust-movements, denudation, and sedimentation. They are closely interrelated; the occurrence of sedimentary rocks in high mountains and the frequency of synclinal mountains show further that they are of the same order of magnitude. Their dynamical treatment has not been extensive in the past, and geologists have, as a rule, been content with qualitative explanations of the observed configurations of the surface rocks. Such treatment is, however, highly desirable; for a mathematical investigation enables us to specify accurately the causes we are taking into account, and the correspondence or divergence between the effects it predicts and the actual phenomena indicates the extent to which we have succeeded in tracing the most important causes. The differences revealed may then lead to the discovery of further causes, and thus observed facts may gradually become understood in greater completeness and detail.

The present paper deals with problems of the flow of surface water during rain. The ground is supposed completely covered with a thin layer of water, supplied at a known rate all over it. The movement of the water is found to be completely determinable in ordinary conditions. It must be carefully distinguished from the flow of a stream; in the present problem the surface of the water may be considerably inclined, for it closely follows that of the ground, whereas in a stream the section of the free surface by a plane across the lines of flow is always nearly horizontal however much the bed may be inclined. In English conditions the frictional resistance to the motion is usually mostly due to viscosity, turbulence being important only in mountainous regions. The form of an ideal peneplain that would sink at a uniform rate all over owing to the denudation caused by such flow is then determined, and its stability considered.

I. *The Flow of Surface-water during Rain.*

During rain water is supplied at a fairly uniform rate over wide areas, and as fast as it falls it runs away to lower ground under the action of gravity. The supply being practically continuous, the whole surface is always covered, the depth at

* Communicated by the Author.

any point depending on the shape of the neighbouring ground. The present problem is to find out how the water will flow.

Consider any particular small portion of the surface, of area dS, and let the depth of the water be ζ and its density ρ. Then the forces acting on the element of water of mass $\rho\zeta dS$ and affecting its movement over the ground are:—

(1) The tangential component of gravity, of amount $g\rho\zeta dS \sin\alpha$, along the line of greatest slope, where α is the angle between the normal to the surface and the vertical.

(2) The friction of the ground; when the velocity is considerable it has the value $f\rho V^2 dS$, where f is a constant of order 0·004 and V is the mean resultant velocity of the water within the element. The direction is against the resultant velocity.

(3) The pressure of the surrounding water. Now the pressure is zero at the upper surface of the water, and as there is practically no movement perpendicular to this surface, the normal component of gravity must be almost exactly balanced by the pressure. If ν be the distance from the surface of the ground, this makes the pressure equal to $g\rho(\zeta-\nu)\cos\alpha$; and therefore the difference between the pressures at two points ds apart and at the same distance from the bottom is of order $g\rho\frac{\partial}{\partial s}(\zeta\cos\alpha)ds$. Thus the whole thrust on the element dS in the direction of ds is of order $g\rho\frac{\partial}{\partial s}(\zeta\cos\alpha)\,dS$.

The resultant of these three forces is the rate of change of momentum of the element of water $\rho\zeta dS$.

Now the ratio of the third force to the first is $\frac{\partial}{\partial s}(\zeta\cos\alpha):\sin\alpha$, and provided that the depth of the water does not change rapidly in comparison with the height of the ground above sea-level, which is obviously a legitimate assumption, it appears that the pressure variation can be neglected.

Next, suppose for a moment that most of the force of gravity is used in producing acceleration, friction never exceeding a certain definite fraction of it. Then the velocity acquired in descending through a vertical height h is given by $V^2=2gh$, less a correction for friction. This makes the frictional force equal to $2fg\rho h dS$, whose ratio to the force due to gravity is $2fh:\zeta\sin\alpha$, which is of the order of the ratio of ·008 of the linear dimensions of the area to the depth of the water, and is obviously very large in all ordinary cases.

Thus on these hypotheses friction would considerably exceed gravity, which contradicts the initial assumption. Hence the hypothesis that friction is less than a fraction of gravity which never approaches near to unity is untenable, and therefore friction must be nearly equal to gravity, and the accelerations can be neglected in the equations of motion.

This holds for any depth of the liquid; when the depth is small enough to make viscous resistance (simply proportional to the velocity) exceed the type here considered, the resistance here assumed is still great in comparison with the accelerations, and *a fortiori* the viscous resistance is more important than the accelerations.

The forces acting on the element of fluid therefore reduce to two: gravity acting down the line of greatest slope, and friction acting opposite to the velocity, these two being practically equal and opposite. It follows at once first, that the motion of the liquid is always down the line of greatest slope, and second, that the velocity is given by

$$fV^2 = g\zeta \sin \alpha.$$

For a different law of resistance the velocity will have a different value.

The equation of continuity has not yet been considered. Consider the surface of the ground covered by two orthogonal systems of curves, specified by λ=constant and μ=constant, where λ and μ are functions of the position. Let the elements of length along these curves be ds_1 and ds_2, where

$$h_1 ds_1 = d\lambda \quad \text{and} \quad h_2 ds_2 = d\mu, \quad . \quad . \quad . \quad . \quad (1)$$

h_1 and h_2 being in general functions of λ and μ.

If the velocity at any point has components (u, v) in the directions of $d\lambda$ and $d\mu$ respectively, the amount of liquid crossing ds_1 in unit time is $v\zeta ds_1$; and accordingly it is seen that the equation of continuity is

$$\frac{\partial}{\partial\lambda}\left(\frac{u\zeta}{h_2}\right) + \frac{\partial}{\partial\mu}\left(\frac{v\zeta}{h_1}\right) = \frac{1}{h_1 h_2}\left(A - \frac{\partial\zeta}{\partial t}\right), \quad . \quad . \quad (2)$$

where A is the rate of supply of water per unit area. When the motion is steady $\partial\zeta/\partial t$ is zero. Now u and v are known in terms of ζ, so that this becomes a partial differential equation to find ζ. Again, ζ only enters through the combination ζV. If the law of resistance were different from that assumed here, the equation would still hold, but V would be a different function of ζ; nevertheless the equation would be satisfied by the same value of Vζ, so that the solution for one law of resistance can easily be deduced from that for any other law.

So far the systems of orthogonal curves are unspecified; the contour-lines, or lines of strike, will be taken to be λ=constant, and the dip-lines will be μ= constant. Thus $u=V$, $v=0$. Now if ds denote an element of length in any direction on the surface, and rectangular coordinate axes are chosen, the axis of z being upwards, the contour-lines are specified by the condition that z= constant along them. If the surface has the equation $z=f(x, y)$, then along a contour-line $\frac{dz}{ds}=0$, and therefore $\frac{\partial f}{\partial x}+\frac{\partial f}{\partial y}\frac{dy}{dx}=0$.

Thus λ can be taken equal to z, and as the contours are parallel to the plane of (x, y), it follows that their projections on the plane of (x, y) will also satisfy the differential equation

$$\frac{dy}{dx}=-\frac{\partial f}{\partial x}\Big/\frac{\partial f}{\partial y} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

Now the projections of the dip-lines must be perpendicular to those of the contour-lines, and therefore along them we must have

$$\frac{dy}{dx}=\frac{\partial f}{\partial y}\Big/\frac{\partial f}{\partial x} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (4)$$

The solution of this equation can be put in the form

Known function of x and y=arbitrary constant, . (5)

and this function can then be taken to be μ.

The element of length on a dip-line measured in the direction of the flow is cosec $\alpha\, dz$, and therefore $h_1=-\sin\alpha$. h_2 is as yet undetermined. The equation of continuity is

$$\sin\alpha\frac{\partial}{\partial z}\left(\frac{V\zeta}{h_2}\right)+\frac{A}{h_2}=0. \quad . \quad . \quad . \quad . \quad . \quad . \quad (6)$$

The above treatment is independent of the form of the surface considered, and shows that if the value of $V\zeta$ is known along any contour, it can be found along any other by an integration along the dip-lines. Thus a general solution can always be found.

At this stage the law followed by the friction may be investigated.

As long as V is a function of ζ, $V\zeta$ must be of order $\int A \operatorname{cosec}\alpha\, dz$, or Al, where l is of the order of the horizontal dimensions of the area. The condition that the friction may be proportional to the square of the velocity is the Osborne Reynolds criterion, that $V\zeta$ shall be greater than

$1000k$, where k is the kinematic coefficient of viscosity, practically $0{\cdot}02$ cm.^{2}/1 sec. Thus we must have $Al > 20$ cm.^{2}/sec. A moderate rainfall would be one centimetre in an hour, or about 3×10^{-4} cm./sec. For the law to be correct l must then be greater than $0{\cdot}7$ kilometre. Again, using the relation $fV^2 = g\zeta \sin\alpha$, we have

$$fV^3 = O^*(gAl \sin\alpha)$$
$$= O(gAh),$$

where h is the height of the highest point. Thus we have

$$\zeta = O\left\{ l \left(\frac{A^2 f}{gh} \right)^{\frac{1}{3}} \right\}$$
$$= O(7 \times 10^{-5}) l h^{-\frac{1}{3}} \quad . \quad . \quad . \quad . \quad . \quad (7)$$

with the above data. Taking $h = 10^5$ cm. and $l = 10^6$ cm., these figures corresponding to a mountainous region, we have $\zeta = O(0{\cdot}7)$ cm.; thus a steady depth under a moderate steady rainfall would be attained after a time of the order of an hour. In flat regions, on the other hand, we may have $h = 10^4$ cm. and $l = 10^7$ cm.; then $\zeta = O(30)$ cm. This is evidently incorrect, for rain does not ordinarily last long enough to flood the ground to this depth. It follows that $V\zeta$ must be less than $1000k$, and the friction is not due to turbulence, but to ordinary viscosity.

Now if u be the velocity at distance ν above the solid surface, the equation of viscous motion is

$$k \frac{\partial^2 u}{\partial \nu^2} = -g \sin\alpha, \quad . \quad . \quad . \quad . \quad . \quad . \quad (8)$$

while $u = 0$ when $\nu = 0$, and $\dfrac{\partial u}{\partial \nu} = 0$ when $\nu = \zeta$.

These give

$$u = \frac{g \sin\alpha}{2k} (2\nu\zeta - \nu^2). \quad . \quad . \quad . \quad . \quad . \quad . \quad (9)$$

In the equation of continuity we can take V to be the mean value of u over a normal section. Then

$$V = \frac{1}{\zeta} \int u \, d\nu = \frac{1}{3} \frac{g\zeta^2 \sin\alpha}{k}. \quad . \quad . \quad . \quad . \quad (10)$$

Thus $$V\zeta = g\zeta^3 \sin\alpha / 3k \quad . \quad . \quad . \quad . \quad . \quad . \quad (11)$$

* $O(x)$ denotes a "quantity of the order of x."

The equation of continuity therefore reduces to

$$\zeta^3 = -\frac{3kh_2}{g\sin\alpha}\int\frac{\text{A}\operatorname{cosec}\alpha}{h_2}\,dz, \quad . \quad . \quad . \quad (12)$$

the integration being along the lines of greatest slope. At the top of the slope $\text{V}\zeta$ must evidently be zero, so that the lower limit of the integral must correspond to the summit. The problem is thus reduced to one of quadratures.

II. *Denudation by Surface Water.*

The friction of surface water on the ground tends to remove the finer particles and carry them away. Solution also occurs in certain cases, but the purely mechanical effect is always one of the most important, and will be considered first. If M denote the mass of a particle of density ρ', it will be detached when the frictional force on it reaches a certain proper fraction of the normal force between it and the surface. The velocity in its neighbourhood being u, and the linear dimensions of order a, the frictional force is $O(k\rho ua)$ in the case of pure viscosity. Now u is practically $a\partial u/\partial\nu$ evaluated for $\nu=0$, and this is $ag\zeta\sin\alpha/k$. Thus the frictional force is $O(\rho a^2 g\zeta\sin\alpha)$. The normal force is $O\{(\rho'-\rho)a^3 g\cos\alpha\}$. The ratio of the two being a definite number, say λ, we see that the size of a particle that would just be moved is given by

$$a = O\left\{\frac{\rho}{\lambda(\rho'-\rho)}\zeta\tan\alpha\right\}. \quad . \quad . \quad . \quad . \quad (1)$$

Hence the mass of the largest particle that can be transported by viscosity without sticking in the first small hollow it comes to is proportional to $\zeta^3\tan^3\alpha$. The rate of denudation is therefore a function of $\zeta\tan\alpha$, its form depending on the distribution in the soil of particles of different sizes. It is to be noted that when the depth and slope are the same the limiting mass is independent of the kinematic viscosity.

An interesting case arises if $\zeta\tan\alpha$ and a are constants, α being small. The surface sinks at a uniform rate all over, retaining its size and shape, but progressively sinking. This represents one case of the "peneplain." When the surface is such that all the contour-lines are parallel to the axis of y, its form can be found. For I. (12) when ζ is proportional to $\cot\alpha$ and A to $\cos\alpha$, corresponding to rain falling, on the average, vertically, becomes

$$\int\cot\alpha\,dz = -\text{B}\sin\alpha\cot^3\alpha, \quad . \quad . \quad . \quad . \quad . \quad (2)$$

where B is a constant; then

$$\frac{dz}{d\alpha} = B(2 \cot \alpha \operatorname{cosec} \alpha + \cos \alpha), \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

$$\frac{dx}{d\alpha} = B(2 \cot^2 \alpha \operatorname{cosec} \alpha + \operatorname{cosec} \alpha - \sin \alpha), \quad . \quad . \quad (4)$$

$$z = z_0 - B(2 \operatorname{cosec} \alpha - \sin \alpha), \quad . \quad . \quad . \quad . \quad . \quad . \quad (5)$$

$$x = x_0 - B(\operatorname{cosec} \alpha \cot \alpha - \cos \alpha), \quad . \quad . \quad . \quad . \quad . \quad (6)$$

where x_0 and z_0 are arbitrary constants.

When $z = z_0 - B$, we have $\alpha = \frac{1}{2}\pi$ and $x = x_0$. For smaller values of α, x and z become steadily smaller; and when α is small z behaves like $z_0 - 2B/\alpha$ and x like $x_0 - B/\alpha^2$. Thus the surface is very steep near the top, the slope gradually decreasing as we recede from this. Again, if the surface is flatter at a distance from the ridge than the peneplain form and steeper near the ridge, we see that $\zeta \tan \alpha$ is greater near the ridge and less away from it; thus the inside will tend to sink more slowly than the outside, the peneplain conditions being thereby restored. For displacements of this type the peneplain is therefore stable.

Except close to the ridge, where the surface is nearly vertical, the form is practically parabolic, the axis being horizontal and the latus rectum being 4B. The depth and the mean velocity are both zero at the top, the water running away as fast as supplied; at other places both increase like $\cot \alpha$, or practically like the square root of the horizontal distance traversed. The amount of water crossing a contour in unit time is proportional to the product of the depth and the mean velocity, and therefore to the horizontal distance. Thus it steadily increases the further we go down the slope, the increase being supplied by the rain gathered on the way.

Evidently the theory cannot be expected to hold close to the ideal ridge; the removal of solid matter would cause the surface after a short time to be at a uniform *normal* distance from the original surface, and not vertically below it. As long as the slope is small this will not make much difference, but when it is great there will be a considerable cutting back in a horizontal direction; any sharp angle will therefore be exposed to denudation on two sides, and will tend to be rounded off. This effect will be accentuated by the fact that rain falling on a steep slope would retain for some time part of the velocity acquired in its fall through the air, so that the velocity would be greater than on the theory. As long as we are concerned with only moderate gradients, however, the theory will hold. In dealing with the summits of hills, other factors must be taken into account which can be safely neglected in the present problem.

All peneplains with parallel contours produced by surface water must be geometrically similar, for the only arbitrary quantities involved in the equations obtained are x_0 and z_0, determining the position, and B, determining the scale.

In the actual character of the motion produced in the soil-particles this viscous flow would be expected to differ considerably from the ordinary turbulent flow of a stream. The latter involves great vertical agitation, and particles are lifted up and down by the vertical movements, travelling considerable horizontal distances during each jump. In viscous motion there is little or no turbulence, and little chance therefore of particles being removed from contact with the surface of the soil. The effect of the tangential forces will be merely to roll the particles along rather than to carry them. The actual rate of denudation will depend on the frequency in the soil of particles of different sizes, and accordingly while it will be a function of the frictional force its form cannot be predicted without a knowledge of the soil-composition. So far it has only been assumed uniform, requiring the rate of shearing at the bottom to be constant, so that the question of the effect of variations in the soil-composition has not arisen. When changes of the topography are considered, on the other hand, precise information on the relation between the rate of denudation and the rate of shearing at the surface will be needed.

When the denudation has proceeded a considerable time the lowest parts may reach sea-level. They will then cease to be denuded, while the inland parts will continue to sink. This will proceed till the whole is reduced to so low a level that surface-water can no longer attain sufficient velocities to carry débris with it. This corresponds to the "base-level" of geologists. Its form will naturally depend on the size of the particles of the soil. The length $\zeta \tan \alpha$ is somewhat less than a on account of the factor λ: we have, in fact,

$$B = O\left(\frac{\lambda^3 a^3 g}{3kA}\right). \quad . \quad . \quad . \quad . \quad . \quad . \quad (7)$$

Thus B increases rapidly with a. In other words, since at a given horizontal distance from the ridge α is proportional to $B^{\frac{1}{2}}$, we see that the larger the particles the greater is the slope needed to transport them when the water supply is the same. The variations in height would therefore be expected to be greater on a coarse soil than on a fine one; at the same time we see that a heavy rainfall will act in the same direction as a fine soil. If as a preliminary estimate we assume $\lambda^* = \frac{1}{10}$, $a=0{\cdot}1$ cm. (corresponding to a sand), $k=0{\cdot}02$ cm.^{2}/sec., $A=3\times10^{-4}$ cm./sec., we find $B=O(200)$ cm. Thus at a distance of 10 km. from the high ground the depression of the surface would be of order 200 metres. With a finer soil it would be less, and the scales indicated appear to be of the same order of magnitude as those observed.

III. *The Stability of the Peneplain.*

It was shown in the last section that a peneplain with uniform soil could retain parallel contours indefinitely if there were no external disturbance. Suppose, however, that on account of some local irregularity in the rainfall or the soil or some other factor the perfect peneplain form were slightly altered, would subsequent denudation increase or decrease the alteration? If it decreased it, the peneplain would be stable, and would be expected to persist for long intervals without considerable change. If it increased it, on the other hand, the peneplain would be unstable; the particular type of variation that increased most rapidly would become the most important, and in time would dominate all others.

Let the equation of the peneplain be $z=z_1$, where z_1 is a function of x only, and suppose it to be slightly disturbed, so that its equation is changed to

$$z=z_1+\phi(x, y), \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

where ϕ and all its derivatives are small quantities of the first order. Then neglecting second-order terms, we see that the dip-lines satisfy the equation

$$\frac{dy}{dx} = \frac{\partial\phi}{\partial y}\bigg/\frac{\partial z_1}{\partial x}. \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

Hence to this order $y-\int\frac{1}{z_1'}\frac{\partial\phi}{\partial y}dx$ is constant along a dip-line, and can be put equal to μ. The integral is to be taken along the dip-line or (with a second-order error) along a section of the surface by a plane parallel to the axis of x. Now ds_2 is the element of length along a contour, and we have

$$ds_2^2=dx^2+dy^2.$$

* We should expect λ to be less than the ordinary coefficient of sliding friction, for the motion is partly rolling, and the water must have some lubricating action. Experiments on traction in channels concern turbulent friction, and give no information on the present problem

As $\frac{dy}{dx}$ is small when we move along a contour we can put $ds_2 = dy$, and as $h_2 ds_2 = d\mu$, we find

$$h_2 = 1 - \int \frac{1}{z_1'} \frac{\partial^2 \phi}{\partial y^2} dx. \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

Again, the direction cosines of the normal to the surface are proportional to $z_1' + \frac{\partial \phi}{\partial x}, \frac{\partial \phi}{\partial y}, -1$ respectively. Thus

$$\cos \alpha = \left\{ 1 + \left(z_1' + \frac{\partial \phi}{\partial x} \right)^2 \right\}^{-\frac{1}{2}}, \quad . \quad . \quad . \quad . \quad . \quad . \quad (4)$$

$$\sin \alpha = \left(z_1' + \frac{\partial \phi}{\partial x} \right) \left\{ 1 + \left(z_1' + \frac{\partial \phi}{\partial x} \right) \right\}^{-\frac{1}{2}}. \quad . \quad . \quad (5)$$

If $A = A_0 \cos \alpha$, the equation of continuity now becomes

$$\frac{g\zeta^3 \tan^3 \alpha}{3kA_0} = h_2 \tan^2 \alpha \sec \alpha \int \frac{dx}{h_2}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (6)$$

$$= \left[x \tan^2 \alpha \sec \alpha \right] - h_2 \tan^2 \alpha \sec \alpha \int x \frac{d}{dx} \left(\frac{1}{h_2} \right) dx. \quad . \quad . \quad . \quad (7)$$

It is fairly evident without mathematical treatment that the greatest instabilities, if any, will occur for displacements forming corrugations along or down the slope, and not for those running obliquely. Consider first those running along the slope, so that ϕ does not involve y. Then $h_2 = 1$ and

$$\frac{g\zeta^3 \tan^3 \alpha}{3kA_0} = x \tan^2 \alpha \sec \alpha, \quad . \quad . \quad . \quad . \quad . \quad . \quad (8)$$

where x is the horizontal distance from the top of the slope. Thus where the distortion increases $\tan^2 \alpha \sec \alpha$ it will increase $\zeta \tan \alpha$, and hence the erosion. So long as the slope is not very great the variation in $\sec \alpha$ will always be small compared with that in $\tan \alpha$, and thus the surface will sink fastest where the relative increase in $\tan \alpha$ is greatest. Now considering a series of elevations of the same height and horizontal extent down the slope, we see that $\tan \alpha$ is greatest on the lower side of each, and the relative amount by which it exceeds the undisturbed value is greater the greater $\frac{1}{\phi z_1'} \frac{\partial \phi}{\partial x}$ is. Thus if A be the top of one ridge, C that of the next in order downwards, B the bottom of the intermediate hollow (the depth being measured normally to the general slope), and D that of the hollow below, the denudation in AB is less than that in CD, for z_1' is less in CD. Thus the denudation in AC is less than that in BD, and therefore the ridges are more denuded than the hollows and the system is stable for corrugations along the slope.

Consider next the case of ridges running down the slope. The line of greatest slope starting at any point will ascend rapidly towards the nearest ridge and gradually turn round till at a great enough height it is almost parallel with it and near the top of it. Thus lines of greatest slope equally spaced at the top of the general slope will tend to rearrange themselves lower down, so as to be more densely packed in the hollows and less densely on the ridges. Now water cannot cross a line of greatest slope, and as it is supplied uniformly all over it must tend to congregate in the hollows. Thus denudation is greatest in the hollows, since the slope there does not differ appreciably from that on the ridges, and therefore the peneplain is essentially unstable for distortions consisting of corrugations running down the slope. Again, α is independent of y to the first order, and therefore the difference in ζ^3 between ridges and hollows can only arise through the term in $\partial^2\phi/\partial y^2$ in h_2. This, other things being equal, is evidently proportional to a/λ^2, where a is the average extent of the elevations above the peneplain and λ the distance between consecutive ridges. So long as this is small, the difference between the rates of denudation in the ridges and hollows is proportional to a/λ^2, and thus the relative rate of increase of any disturbance is proportional to $1/\lambda^2$. The shorter the distance between consecutive crests, then, the more rapidly the disturbance will increase. As any type of disturbance is initially possible, it follows that surface-water alone is capable of cutting up a uniform surface into an indefinitely complicated pattern if no other agency exists that can counteract the instability.

This result does not agree with the observed frequency of remarkably uniform peneplains, and some stabilising cause must therefore exist. One possible cause is the friability of soils, which would soon cause local irregularities of considerable steepness to break up and spread themselves out again under the action of gravity. Sand spreads itself out when wet in a similar way. On a tenacious clay soil such irregularities can persist for a considerable time, and then the result is well confirmed by the extremely rough and angular forms developed by exposed masses of bare clay *. Clay covered

* See, for instance, Pirsson and Schuchert, 'Textbook of Geology,' figs. 19 & 21.

with vegetation is not affected in the same way, and it seems likely that vegetation does have a stabilising influence. It reduces the rate of denudation as a whole, and when the soil under a grass or other plant with fibrous roots is removed it is possible that the exposed roots may act as a filter, thus increasing redeposition and counteracting denudational instability. The general result that the rate of denudation is a function of $\zeta \tan \alpha$ thus ceases to hold for these distortions of short wave-length, but remains true when areas large compared with the size of the plants are considered. The form of the peneplain deduced in Section II. will therefore still hold.

Summary.

In Section I. it is shown that the movement of surface water is controlled by gravity and friction; hydrostatic pressure and inertia are ordinarily negligible. In consequence of this the water always moves along the lines of greatest slope. In mountainous regions the friction may be due to turbulence, but in ordinary cases it is due to ordinary viscosity; in either case the motion is completely determinable when the form of the land and the distribution of rain are known.

In Section II. it is shown that in the case of viscous flow the rate of denudation with uniform soil is a function of $\zeta \tan \alpha$, where ζ is the depth of the water and α the slope. Thus, if $\zeta \tan \alpha$ is a constant, the whole surface will sink at a uniform rate: an example of this is a surface with straight contours and almost parabolic dip-lines with the concavity upwards and the axis horizontal, agreeing in general appearance with ordinary peneplains.

In Section III. the peneplain already described is shown to be stable for corrugations running along the slope; but corrugations running down the slope tend to increase in depth, and the shorter the distance between consecutive crests the more rapidly will this increase occur. This corresponds well with the complicated character of the surface of weathered clay; the smoother types of peneplain are probably able to persist because the instability is counteracted for these disturbances of short wave-length by friability and vegetation.

XIV. *A Diffraction Problem, and an Asymptotic Theorem in Bessel's Series.* *By* R. HARGREAVES, *M.A.**

THE first diffraction problem to which exact methods have been applied with success, is the problem in two dimensions solved by Sommerfeld. Its solution is here presented in a form which is, I think, in a sufficient degree simpler and more convenient than the original, to justify an independent statement of the arguments. I add also a solution of the problem in three dimensions, which arises when the plane of the incident wave is not parallel to the edge of the barrier. The solution appears first in the form of a definite integral, and a direct algebraical transformation is made to a series of Bessel's functions and Trigonometrical functions. When the latter form is got independently the crux of the problem lies in the asymptotic value of the series.

§ 1. The coordinates in the plane being (xy), the barrier occupies the half of the XZ plane for which x is positive. The condition at the barrier may be $\frac{\partial\psi}{\partial t}=0$ (*i. e.* $\psi=0$), or $\frac{\partial\psi}{\partial y}=0$; the first corresponding to zero pressure, the second to zero velocity in the acoustical problem. In constructing the functions it is convenient to take an incident wave $\cos k(\mathrm{V}t+y)$; the transition to oblique incidence is immediate and presents no difficulties. This form of incident wave involves two asymptotic conditions. For x infinite and negative, y finite, ψ must approach the limit $\cos k(\mathrm{V}t+y)$. For x infinite and positive, y finite, the asymptotic value must be zero for y negative, $\cos k(\mathrm{V}t+y)\mp\cos k(\mathrm{V}t-y)$ for y positive according as we are dealing with the barrier condition $\psi=0$ or $\frac{\partial\psi}{\partial y}=0$.

The solution is based on the function

$$\phi(r,y)=\frac{\sqrt{k}}{2\sqrt{\pi}}\cdot\int_0^{r+y}\cos\left\{\frac{\pi}{4}+k(\mathrm{V}t+y-u)\right\}\frac{du}{\sqrt{u}}. \quad . \quad (1)$$

which for $r+y$ very great approaches the value

$$\tfrac{1}{2}\cos k(\mathrm{V}t+y).$$

The physical conception suggesting the form of function is that a wave of type $\mathrm{V}t+y$ must be converted to one of type $\mathrm{V}t-r$, which will correspond to divergence from the

* Communicated by the Author.

edge. We therefore try a function of form $e^{\iota k(\mathrm{V}t+y)}\chi(r+y)$, and with ρ for $r+y$ the differential equation yields

$$\frac{\chi''}{\chi'}+\frac{1}{2\rho}+\iota k=0, \text{ or } \rho^{\frac{1}{2}}\chi'=\mathrm{C}e^{-\iota k\rho},$$

which corresponds to a function of the type in (1).

The function $\phi(r, y)$ vanishes for $r+y=0$, *i. e.* on the axis OY′. The use of $r+y$ involves a certain disability in respect to change of sign, which must be directly imposed if demanded by the continuity of $\frac{\partial\phi}{\partial t}$, $\frac{\partial\phi}{\partial y}$ or $\frac{\partial\phi}{\partial x}$. The first two vanish on OY′, but

$$\frac{\partial\phi}{\partial x}=\sqrt{\frac{k}{\pi}}\cdot\frac{x\cos\left\{\frac{\pi}{4}+k(\mathrm{V}t-r)\right\}}{2r\sqrt{r+y}},$$

while $x=\pm\sqrt{(r+y)(r-y)}$ according as x is positive or negative. Thus on the two sides of OY′ we have

$$\frac{\partial\phi}{\partial x}=\pm\sqrt{\frac{k}{2\pi r}}\cos\left\{\frac{\pi}{4}+k(\mathrm{V}t-r)\right\},$$

a finite quantity except at $r=0$. The continuity of $\frac{\partial\phi}{\partial x}$ therefore requires the change from $+\phi(r, y)$ to $-\phi(r, y)$ in crossing OY′.

Corresponding to the reflected wave we have a similar function with $-y$ for y, and here the change of sign occurs on the axis OY. We have now the material for constructing the solution, which, for the condition $\psi=0$ on the barrier, is

$$\left.\begin{aligned}\psi&=\tfrac{1}{2}\cos k(\mathrm{V}t+y)-\tfrac{1}{2}\cos k(\mathrm{V}t-y)+\phi(r, y)-\phi(r, -y)\\&\qquad\text{in 1st quadrant (region A)}\\&=\quad ,,\qquad ,,\qquad ,,\qquad +\phi(r, y)+\phi(r, -y)\\&\qquad\text{in 2nd \& 3rd quadrants (region C)}\\&=\quad ,,\qquad ,,\qquad ,,\qquad -\phi(r, y)+\phi(r, -y)\\&\qquad\text{in 4th quadrant (region B)}\end{aligned}\right\}\quad(2)$$

For the problem with zero velocity on the barrier, the signs of terms $\frac{1}{2}\cos k(\mathrm{V}t-y)$ and $\phi(r, -y)$ must be changed throughout. In the solution (2) it will be noted that for $y=0$ in C, $\frac{\partial\psi}{\partial y}$ has the value due to the incident wave only. In the other solution ψ has the value for incident wave only for $y=0$ in C.

To pass to the case of oblique incidence we write

$$\left.\begin{array}{l} x\sin\alpha+y\cos\alpha \text{ for } \;\; y \text{ in } \cos k(\mathrm{V}t+y) \text{ and in } \phi(r,\;\; y\;) \\ x\sin\alpha-y\cos\alpha \text{ for } -y \text{ in } \cos k(\mathrm{V}t-y) \text{ and in } \phi(r,-y) \end{array}\right\} \quad . \quad (3)$$

Thus for example $\phi(r, y, \alpha)$ being

$$\frac{1}{2}\sqrt{\frac{k}{\pi}}\int_0^{r+x\sin\alpha+y\cos\alpha} \cos\left\{\frac{\pi}{4}+k(\mathrm{V}t+x\sin\alpha+y\cos\alpha-u)\right\}\frac{du}{\sqrt{u}} \qquad (4)$$

$\phi(r, y, \alpha)$ replaces $\phi(r, y)$ in (2). The regions B and C are now separated by the line of the incident ray through O in place of OY′; the regions A and C by the line of the ray reflected from O instead of OY. These are the only changes needed.

Lastly for an incident wave $\cos k(\mathrm{V}t+lx+my+nz)$, let $\varpi=\sqrt{x^2+y^2}$, $n'=\sqrt{1-n^2}$, and $(l, m)=n'(\sin\alpha, \cos\alpha)$; then $\phi(\varpi, y, n')$ being

$$\frac{1}{2}\sqrt{\frac{kn'}{\pi}}\int_0^{\varpi+x\sin\alpha+y\cos\alpha} \cos\left\{\frac{\pi}{4}+k(\mathrm{V}t+lx+my+nz-n'u)\right\}\frac{du}{\sqrt{u}} \qquad (5)$$

$\phi(\varpi, y, n')$ replaces $\phi(r, y)$ in (2), while the opening terms are

$$\tfrac{1}{2}\cos k(\mathrm{V}t+lx+my+nz)-\tfrac{1}{2}\cos k(\mathrm{V}t+lx-my+nz).$$

These are the only changes needed, the separation of regions being as in the last case.

§ 2. The above constitutes a solution in terms of definite integrals the evaluation of which depends on well known series. To pass to the second solution in terms of Bessel's functions we set out from these series, which are therefore briefly quoted. If

$$\left.\begin{array}{l} \displaystyle\int_0^\rho \frac{\cos u\,du}{\sqrt{u}}=\mathrm{P}(\rho)\cos\rho+\mathrm{Q}(\rho)\sin\rho, \\ \displaystyle\int_0^\rho \frac{\sin u\,du}{\sqrt{u}}=\mathrm{P}(\rho)\sin\rho-\mathrm{Q}(\rho)\cos\rho \end{array}\right\} \quad . \quad . \quad (6\,a)$$

then

$$\mathrm{P}=2\rho^{\frac{1}{2}}-\frac{2.2.2}{1.3.5}\rho^{\frac{5}{2}}+\ldots, \quad \mathrm{Q}=\frac{2.2}{1.3}\rho^{\frac{3}{2}}-\frac{2.2.2.2}{1.3.5.7}\rho^{\frac{7}{2}}+\ldots \quad (7\,a)$$

series convergent for all values of ρ. For ρ great, asymptotic values are

$$\mathrm{P}=\sqrt{\frac{\pi}{2}}(\cos\rho+\sin\rho)+\mathrm{P}_a, \quad \mathrm{Q}=\sqrt{\frac{\pi}{2}}(\sin\rho-\cos\rho)+\mathrm{Q}_v,$$

where

$$P_a = -\frac{1}{2\rho^{\frac{3}{2}}} + \frac{1.3.5}{2.2.2}\frac{1}{\rho^{\frac{7}{2}}} + \dots, \quad Q_a = \frac{1}{\rho^{\frac{1}{2}}} - \frac{1.3}{2.2}\frac{1}{\rho^{\frac{5}{2}}} + \dots, \quad (7\,b)$$

series ultimately divergent for any value of ρ. For ρ great evidently

$$\left.\begin{aligned} \int_\rho^\infty \frac{\cos u\,du}{\sqrt{u}} &= -P_a(\rho)\cos\rho - Q_a(\rho)\sin\rho, \\ \int_\rho^\infty \frac{\sin u\,du}{\sqrt{u}} &= -P_a(\rho)\sin\rho + Q_a(\rho)\cos\rho \end{aligned}\right\}. \quad (6\,b)$$

The pairs (PQ) and (P_aQ_a) both satisfy equations :

$$\frac{dQ}{d\rho} - P = 0, \quad \frac{dP}{d\rho} + Q = \frac{1}{\sqrt{\rho}};$$

whence

$$\frac{d^2Q}{d\rho^2} + Q = \frac{1}{\sqrt{\rho}}, \quad \frac{d^2P}{d\rho^2} + P = -\frac{1}{2\rho^{\frac{3}{2}}} \dots \quad (7c)$$

The nature of the wave expressed by $\phi(r, y)$ is revealed more intimately by the use of $(6\,a)$, giving as connected forms

$$\left.\begin{aligned} \phi(r, y) &= \frac{1}{2\sqrt{\pi}}\Big[\{P(k\rho)\cos k\rho + Q(k\rho)\sin k\rho\}\cos\left\{\frac{\pi}{4} + k(Vt+y)\right\} \\ &\qquad + \{P(k\rho)\sin k\rho - Q(k\rho)\cos k\rho\}\sin\left\{\frac{\pi}{4} + k(Vt+y)\right\}\Big] \\ &= \frac{1}{2\sqrt{\pi}}\Big[P(k\rho)\cos\left\{\frac{\pi}{4} + k(Vt-r)\right\} - Q(k\rho) \\ &\qquad \times \sin\left\{\frac{\pi}{4} + k(Vt-r)\right\}\Big] \\ &= \frac{1}{2\sqrt{\pi}}\Big[\{P(k\rho)\cos kr + Q(k\rho)\sin kr\}\cos\left(\frac{\pi}{4} + kVt\right) \\ &\qquad + \{P(k\rho)\sin kr - Q(k\rho)\cos kr\}\sin\left(\frac{\pi}{4} + kVt\right)\Big]. \end{aligned}\right\} \quad (8)$$

We have here the change from plane to divergent wave, and in the last line the forms of the stationary solution.

By use of the series P and Q we obtain solutions in

Bessel's functions, viz. for the boundary condition $\psi=0$

$$\psi=\tfrac{1}{2}\cos k(\mathrm{V}t+y)-\tfrac{1}{2}\cos k(\mathrm{V}t-y)$$
$$+\sqrt{2}\cos\left(\frac{\pi}{4}+k\mathrm{V}t\right)\left[\mathrm{J}_{\frac{1}{2}}(kr)\sin\frac{\theta}{2}+\mathrm{J}_{\frac{5}{2}}(kr)\sin\frac{5\theta}{2}+\ldots\right]$$
$$-\sqrt{2}\sin\left(\frac{\pi}{4}+k\mathrm{V}t\right)\left[\mathrm{J}_{\frac{3}{2}}(kr)\sin\frac{3\theta}{2}+\mathrm{J}_{\frac{7}{2}}(kr)\sin\frac{7\theta}{2}+\ldots\right]; \quad . \; . \; . \quad (9\,a)$$

while for $\dfrac{\partial\psi}{\partial y}=0$ on the barrier

$$\psi=\tfrac{1}{2}\cos k(\mathrm{V}t+y)+\tfrac{1}{2}\cos k(\mathrm{V}t-y)$$
$$+\sqrt{2}\cos\left(\frac{\pi}{4}+k\mathrm{V}t\right)\left[\mathrm{J}_{\frac{1}{2}}(kr)\cos\frac{\theta}{2}+\mathrm{J}_{\frac{5}{2}}(kr)\cos\frac{5\theta}{2}+\ldots\right]$$
$$+\sqrt{2}\sin\left(\frac{\pi}{4}+k\mathrm{V}t\right)\left[\mathrm{J}_{\frac{3}{2}}(kr)\cos\frac{3\theta}{2}+\mathrm{J}_{\frac{7}{2}}(kr)\cos\frac{7\theta}{2}+\ldots\right]. \quad . \; . \; . \quad (9\,b)$$

§ 3. The first step is to obtain expressions for P, Q, and thereafter for the stationary forms, in which the variables r and θ are separated. We have

$$\rho=r+y=r(1+\sin\theta)=\frac{r}{2}\left(e^{\omega\iota}+e^{-\omega\iota}\right)^2 \text{ where } \omega=\frac{\theta}{2}-\frac{\pi}{4} \quad (10)$$

and then

$$\pm\rho^{m+\frac{1}{2}}=2\times\left(\frac{r}{2}\right)^{m+\frac{1}{2}}\sum_{p=0}^{m}\frac{\lfloor 2m+1}{\lfloor m-p\,\lfloor m+p+1}\cos(2p+1)\omega. \quad (11)$$

The series changes sign with $\cos\omega$, *i. e.* we have the positive sign on the left from $\theta=0$ to $\frac{3\pi}{2}$, and the negative sign from $\frac{3\pi}{2}$ to 2π. Linking (11) with (7 *a*) we get when the left-hand member of (11) has the positive sign

$$\left.\begin{aligned} &\mathrm{P}(\rho)=\sum_p \mathrm{P}_p(r)\cos(2p+1)\omega,\quad \mathrm{Q}(\rho)=\Sigma\,\mathrm{Q}_p(r)\cos(2p+1)\omega,\\ &\text{where}\\ &\mathrm{P}_p=\sum_n\frac{(-1)^n\lfloor 2n\; 2(2r)^{2n+\frac{1}{2}}}{\lfloor 2n-p\;\lfloor 2n+p+1},\; 2n\geq p;\\ &\mathrm{Q}_p=\sum_n\frac{(-1)^n\lfloor 2n+1\; 2(2r)^{2n+\frac{3}{2}}}{\lfloor 2n-p+1\;\lfloor 2n+p+2},\; 2n+1\geq p. \end{aligned}\right\} (12)$$

These last give

$$\left.\begin{aligned} P_{2p} &= (-1)^p 2\sqrt{\pi}\, J_{2p+\frac{1}{2}}(r)\cos r, \\ Q_{2p} &= (-1)^p 2\sqrt{\pi}\, J_{2p+\frac{1}{2}}(r)\sin r, \\ P_{2p+1} &= (-1)^{p+1} 2\sqrt{\pi}\, J_{2p+\frac{3}{2}}(r)\sin r, \\ Q_{2p+1} &= (-1)^p 2\sqrt{\pi}\, J_{2p+\frac{3}{2}}(r)\cos r. \end{aligned}\right\} \quad . \quad . \quad (13)$$

It will be sufficient to sketch the argument by which the results in (13) were reached, as I have since found they are particular cases of formulæ given in Nielsen *. Since

$$\sum_p (P_p \cos r + Q_p \sin r)\cos(2p+1)\omega,$$

$$\text{and} \quad \sum_p (P_p \sin r - Q_p \cos r)\cos(2p+1)\omega$$

are solutions of $(\nabla^2+1)f=0$, and $\omega=\frac{\theta}{2}-\frac{\pi}{4}$, we conjecture that one of the brackets with p as index will vanish and the other be proportional to $J_{p+\frac{1}{2}}$, since the function of type $J_{-p-\frac{1}{2}}$ is not admissible. If the expression for $J_{2p+\frac{1}{2}}$ in terms of $\sin r$ and $\cos r$ with polynomial coefficients is used, an expression for $J_{2p+\frac{1}{2}}(r)\cos r$ with polynomials multiplying $\sin 2r$ and $1+\cos 2r$ results, which gives for coefficient of the general term a finite series. This is identified with that required for (13) by means of

$${}_nC_p = \sum_{q=0}^{p} (-1)^q \underset{n+p+1\ n+q+1}{C}\ \underset{p+q\ q}{C} \quad . \quad . \quad . \quad (14)$$

which can be readily proved by a repeated use of

$${}_nC_r = \underset{n+1\ r+1}{C} - \underset{n\ r+1}{C} \text{ and } {}_nC_n = \underset{n+1\ n+1}{C},$$

so applied that each step raises by 1 the value of the prefix.

From (13) follow

$$\left.\begin{aligned} &P(\rho)\cos r + Q(\rho)\sin r = 2\sqrt{\pi}\sum_p (-1)^p J_{2p+\frac{1}{2}}(r)\cos(4p+1)\omega, \\ &\text{and} \\ &P(\rho)\sin r - Q(\rho)\cos r = 2\sqrt{\pi}\sum_p (-1)^{p+1} J_{2p+\frac{3}{2}}(r)\cos(4p+3)\omega, \end{aligned}\right\} (15)$$

and then (8) gives

$$\left.\begin{aligned} \phi(r,y) = {} & \cos\left(\frac{\pi}{4}+kVt\right)\left[J_{\frac{1}{2}}(kr)\cos\omega - J_{\frac{5}{2}}\cos 5\omega + \dots\right] \\ & - \sin\left(\frac{\pi}{4}+kVt\right)\left[J_{\frac{3}{2}}\cos 3\omega - J_{\frac{7}{2}}\cos 7\omega + \dots\right]. \end{aligned}\right\} (16)$$

* Nielsen, *Cylinder Functionen*, p. 20. Write $\nu=2p+\frac{1}{2}$ and $\nu=2p+\frac{3}{2}$ in (5) and (6) to obtain the four results.

Thus in view of $\omega=\frac{\theta}{2}-\frac{\pi}{4}$ we get

$$\left.\begin{aligned}&\phi(r,y)-\phi(r,-y)\\&=\sqrt{2}\cos\left(\frac{\pi}{4}+k\mathrm{V}t\right)\left[\mathrm{J}_{\frac{1}{2}}(kr)\sin\frac{\theta}{2}+\mathrm{J}_{\frac{5}{2}}\sin\frac{5\theta}{2}+\ldots\right]\\&\quad-\sqrt{2}\sin\left(\frac{\pi}{4}+k\mathrm{V}t\right)\left[\mathrm{J}_{\frac{3}{2}}\sin\frac{3\theta}{2}+\mathrm{J}_{\frac{7}{2}}\sin\frac{7\theta}{2}+\ldots\right],\end{aligned}\right\}(17a)$$

and

$$\left.\begin{aligned}&\phi(r,y)+\phi(r,-y)\\&=\sqrt{2}\cos\left(\frac{\pi}{4}+k\mathrm{V}t\right)\left(\mathrm{J}_{\frac{1}{2}}\cos\frac{\theta}{2}+\mathrm{J}_{\frac{5}{2}}\cos\frac{5\theta}{2}+\ldots\right)\\&\quad+\sqrt{2}\sin\left(\frac{\pi}{4}+k\mathrm{V}t\right)\left(\mathrm{J}_{\frac{3}{2}}\cos\frac{3\theta}{2}+\mathrm{J}_{\frac{7}{2}}\cos\frac{7\theta}{2}+\ldots\right).\end{aligned}\right\}(17b)$$

Recalling the statement in connexion with (11), the formulæ (15) represent the left-hand members from $\theta=0$ to $\frac{3\pi}{2}$ and from $\frac{3\pi}{2}$ to 2π the left-hand members with sign changed. Thus in (17a) the series represents the part of formula (2) which contains ϕ, with the signs attached for the different regions; and (17b) gives the corresponding forms for the 2nd solution. Thus (9a) and (9b) represent the original solution.

§4. For oblique incidence $r+x\sin\alpha+y\cos\alpha$ takes the place of $r(1+\sin\theta)$, and so in passing from $\phi(r,y)$ to $\phi(r,y,\alpha)$, ω is changed to $\frac{\theta+\alpha}{2}-\frac{\pi}{4}$; and in like manner ω' is changed to $\frac{\theta-\alpha}{2}+\frac{\pi}{4}$. Thus we get in the region A

$$\left.\begin{aligned}&\phi(r,y,\alpha)-\phi(r,-y,\alpha)\\&=\sqrt{2}\cos\left(\frac{\pi}{4}+k\mathrm{V}t\right)\left[\mathrm{J}_{\frac{1}{2}}(kr)\sin\frac{\theta}{2}\left(\cos\frac{\alpha}{2}-\sin\frac{\alpha}{2}\right)\right.\\&\qquad\left.+\mathrm{J}_{\frac{5}{2}}\sin\frac{5\theta}{2}\left(\cos\frac{5\alpha}{2}-\sin\frac{5\alpha}{2}\right)+\ldots\right]\\&\quad-\sqrt{2}\sin\left(\frac{\pi}{4}+k\mathrm{V}t\right)\left[\mathrm{J}_{\frac{3}{2}}\sin\frac{3\theta}{2}\left(\cos\frac{3\alpha}{2}+\sin\frac{3\alpha}{2}\right)\right.\\&\qquad\left.+\mathrm{J}_{\frac{7}{2}}\sin\frac{7\theta}{2}\left(\cos\frac{7\alpha}{2}+\sin\frac{7\alpha}{2}\right)\ldots\right]\end{aligned}\right\}(18a)$$

and

$$\left.\begin{aligned}\phi(r, y, \alpha)+\phi(r, -y, \alpha)&\\ =\sqrt{2}\cos\left(\frac{\pi}{4}+kVt\right)&\Big[J_{\frac{1}{2}}\cos\frac{\theta}{2}\left(\cos\frac{\alpha}{2}+\sin\frac{\alpha}{2}\right)\\ &+J_{\frac{5}{2}}\cos\frac{5\theta}{2}\left(\cos\frac{5\alpha}{2}+\sin\frac{5\alpha}{2}\right)+\dots\Big]\\ +\sqrt{2}\sin\left(\frac{\pi}{4}+kVt\right)&\Big[J_{\frac{3}{2}}\cos\frac{3\theta}{2}\left(\cos\frac{3\alpha}{2}-\sin\frac{3\alpha}{2}\right)\\ &+J_{\frac{7}{2}}\cos\frac{7\theta}{2}\left(\cos\frac{7\alpha}{2}-\sin\frac{7\alpha}{2}\right)+\dots\Big].\end{aligned}\right\}\quad(18\,b)$$

In dealing with the three-dimensional wave and $\phi(\varpi, y, n')$ the right-hand member has $kn'\varpi$ as argument for the Bessel's functions, and also $Vt+nz$ takes the place of Vt. The connected forms shown in (8) are for the 3-dimensional case

$$\left.\begin{aligned}\phi(\varpi, y, n')&=\frac{1}{2\sqrt{\pi}}\Big[\{P(kn'\rho)\cos kn'\rho+Q(kn'\rho)\sin kn'\rho\}\\ &\qquad\times\cos\left\{\frac{\pi}{4}+k(Vt+lx+my+nz)\right\}+\dots\Big]\\ &=\frac{1}{2\sqrt{\pi}}\Big[P(kn'\rho)\cos\left\{\frac{\pi}{4}+k(Vt+nz-n'\varpi)\right\}\\ &\qquad-Q(kn'\rho)\sin\left\{\frac{\pi}{4}+k(Vt+nz-n'\varpi)\right\}\Big]\\ &=\frac{1}{2\sqrt{\pi}}\Big[\{P(kn'\rho)\cos kn'\varpi+Q(kn'\rho)\sin kn'\varpi\}\\ &\qquad\times\cos\left\{\frac{\pi}{4}+k(Vt+nz)\right\}+\dots\Big],\end{aligned}\right\}\quad(8\,b)$$

where

$$\rho=\varpi+x\sin\alpha+y\cos\alpha\text{ and so }n'\rho=n'\varpi+lx+my.$$

In the plane oblique case write $n=0$, $n'=1$, $\varpi=r$ in $(8\,b)$.

§ 5. The asymptotic values of the series containing Bessel's functions are assigned by their equivalents in terms of P, Q, or ϕ. An asymptotic value attaches to positions for which ρ ($r+y$ in the simplest case) is sufficiently great; it is not essential that y should be finite, but a finite angular space must be excluded on both sides of the critical line, β say without further precision.

The position in respect to P, Q, or ϕ, is that for r very great, outside an angle β on each side of a critical line, a *single* asymptotic value exists. For the series we have *different* asymptotic values in the different ranges. Thus the series (17 a) has asymptotic values :—

in A, $\frac{1}{2}\cos k(\mathrm{V}t+y)-\frac{1}{2}\cos k(\mathrm{V}t-y)$;

in C, $\frac{1}{2}\cos k(\mathrm{V}t+y)+\frac{1}{2}\cos k(\mathrm{V}t-y)$;

and in B, $-\frac{1}{2}\cos k(\mathrm{V}t+y)+\frac{1}{2}\cos k(\mathrm{V}t-y)$.

These values hold in A from $\theta=0$ to $\theta=\frac{\pi}{2}-\beta$,

in C from $\frac{\pi}{2}+\beta$ to $\frac{3\pi}{2}-\beta$, and in B from $\frac{3\pi}{2}+\beta$ to 2π.

The modifications needed for (17 b) and (18 a, b) are of an obvious character. It will be noted that the range of validity of the asymptotic forms is wider than we were justified in demanding at the outset as a condition of solution. The simplicity of the changes needed to pass to the 3-dimensional plane wave is also noteworthy.

XV. *On the Influence of the Finite Volume of Molecules on the Equation of State.* By Megh Nad Shaha, *M.Sc., and* Satyendra Nath Basu, *M.Sc., Lecturers on Mathematical Physics, Calcutta University* *.

It is well known that the departure of the actual behaviour of gases from the ideal state. defined by the equation $p=\frac{\mathrm{NK}\theta}{v}$ is due to two causes :—(1) the finiteness of the volume of the molecules, (2) the influence of the forces of cohesion, *i. e.*, the attractive forces amongst the molecules. van der Waals was the first to deduce an equation of state in which all these factors are taken into account ; according to van der Waals, we have

$$p=\frac{\mathrm{NK}\theta}{v-b}-\frac{a}{v^2} \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

where $b=8\times$ volume of the molecules, a defines the forces of cohesion.

In all subsequent modifications of this equation (Clausius, Dieterici, or D. Berthelot), the changes which have been

* Communicated by the Authors.

proposed all relate to the influence of the cohesive forces; the part of the argument dealing with the finiteness of molecular volumes is generally left untouched.

But it has been found that the results of experiments do not agree with the predictions of theory if we regard a and b as absolute constants. Accordingly it has been proposed to regard both a and b as functions of volume and temperature *.

But before proceeding to these considerations, it is necessary to scrutinize whether the influence of finite molecular volumes is properly represented by the term b. From theoretical considerations, the conclusion has been reached that this is not the case. The argument is as follows: According to Boltzmann's theory,

$$\text{the entropy } S = K \log W + C,$$

where K = Boltzmann's gas-constant, W = probability of the state. Let us now calculate the probability that a number N of molecules originally confined within the volume V_0 and possessing finite volumes, shall be contained in a volume V. Neglecting the influence of internal forces, the probability for the first molecule is $\frac{V}{V_0}$, for the second molecule the probability is $\frac{V-\beta}{V_0-\beta}$, where $\beta = 8 \times$ volume of a single molecule, for when the first molecule is in position, the space enclosed by a concentric sphere of double the radius of the molecule will not be available for the second molecule. The available space is therefore $V-\beta$, whence the probability is $\frac{V-\beta}{V_0-\beta}$. Introducing similar considerations for the rest of the molecules, we have

$$W = \frac{V}{V_0} \cdot \frac{V-\beta}{V_0-\beta} \cdot \frac{V-2\beta}{V_0-2\beta} \quad \cdots \quad \frac{V-\overline{N-1}\beta}{V_0-\overline{N-1}\beta} . . \quad (2)$$

We are, of course, neglecting those cases in which partial overlapping of the regions occupied by two or more molecules occurs; for the number of such cases can at best be a small fraction of the total number. Even cases of actual association do not include these, for in that case, two discrete molecules become merged into one, without their outer surfaces being actually in contact.

* Compare van der Waals, *Proc. Amst.* 1916; Van Laar, *Proc. Amst.* vol. xvi. p. 44.

From the relations $S=K \log W + C$

and $$\left(\frac{\partial S}{\partial V}\right)_u = \frac{p}{\theta}$$

we can easily verify that

$$p = -\frac{K\theta}{\beta} \log \frac{V-n\beta}{V}$$

$$= -\frac{R\theta}{2b} \log \frac{V-2b}{V} \quad (R=NK) \quad . \quad . \quad . \quad (3)$$

As a first approximation, when b is small compared to v, we obtain $p=\frac{NK\theta}{v}$ (Boyle–Charles–Avogadro Law), and as a second approximation we obtain

$$p=\frac{NK\theta}{v-b} \text{ (van der Waals correction).}$$

We also note that

$$pV=NK\theta \cdot \frac{x}{1-e^{-x}}, \text{ where } x=\frac{\beta p}{K\theta}. \quad . \quad . \quad . \quad . \quad (4)$$

To account for the influence of internal forces, we multiply, following the lead of Dieterici, the above expression (3) by $e^{-\frac{a}{NK\theta v}}$, a having the same significance as before.

From this equation of state, we can easily verify the following results for the critical point :

Critical volume, $V_c=\frac{2e}{e-1}b=3{\cdot}166\,b$,

$$K=\frac{NK\theta_c}{p_c V_c}=3{\cdot}513.$$

The corresponding values of V_c from the van der Waals and the Dieterici equations are $(3b, 2b)$ respectively, and of

$$K \text{ are } \left(\frac{8}{3}=2{\cdot}66, \frac{e^2}{2}=3{\cdot}695\right) \text{ respectively.}$$

As a matter of fact, for the simpler gases, the value of 'K' obtained in this paper agrees better with the experimental results than the Dieterici value $\frac{e^2}{2}$; we have for oxygen * $K=3{\cdot}346$, for nitrogen † $K=3{\cdot}53$, for argon ‡

* Mathias and K. Onnes, *Proc. Amst.* Feb. 1911.

† Berthelot, *Bull. de la Soc. France de Phys.* 167 (1901).

‡ Mathias, Onnes, and Crommelin, *Proc. Amst.* 1913, p. 960, vol. xv

$K=3\cdot424$, for xenon * $K=3\cdot605$. We need not consider the van-der-Waals value $\frac{8}{3}$, for it fails entirely.

The most serious drawback to Dieterici's equation is, according to Prof. Lewis (*vide* Lewis's Physical Chemistry, vol. ii. p. 117) that it makes b or the limiting volume $=\frac{V_c}{2}$, while the limiting volume, obtained by the extrapolation of Cailletet-Mathias mean density line to the temperature $\theta=0^\circ$ K is about $\frac{V_c}{4}$. The value of b obtained in this paper, viz., $\frac{V_c}{3\cdot16}$ therefore agrees better with this value.

It is yet premature to predict what influence this investigation will have on the speculations concerning the variability of the volume of molecules with temperature. A more detailed investigation dwelling upon this point, and the application of the formula (4) to Amagat's (pv, p) curves, will be communicated shortly. Meanwhile we point out that the factor $e^{-\frac{a}{NK\theta v}}$ has been introduced into the expression for 'p' only as a provisional measure, though it is considered that this step, though not quite exact, is one in the right direction. In the next paper an attempt will be made to introduce energy into probability calculations.

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Calcutta.

Note added in proof.—On consulting the literature on the subject, we noticed that in several papers in the Amsterdam Proceedings (*vide* vol. xv. p. 240 *et seq.*), Dr. Keesom of Leyden had also made attempts to deduce the equation of state from Boltzmann's entropy principle. But, in the expression (2) for W, he introduces, before differentiation, an approximation in which terms up to second order in $\frac{b}{v}$ are retained only. In this way, he arrives at the van der Waals' form $v-b$ for the influence of finite molecular volumes. In obtaining our present equation of state (4), no such approximation has been made. (M. N. Shaha *and* S. N. Basu.)

* Paterson, Cripps, Whytlaw-Gray, Proc. Roy. Soc. Lond. A. lxxxvi. p. 579 (1912).

XVI. *The Secular Perturbations of the Inner Planets. Note by* HAROLD JEFFREYS, *M.A., D.Sc.**

ACCORDING to Dr. Silberstein's theory † the values of $e\frac{d\varpi}{dt}$ for the four inner planets are as follow, the unit being 1″ per century :—

Mercury.	Venus.	Earth.	Mars.
−1·48	−0·01	−0·01	−0·02

The observed values of the excesses above the perturbations calculated on the simple Newtonian law are

$$+8·48 \pm ·43, \quad -0·05 \pm ·25, \quad +0·10 \pm ·13, \quad +0·75 \pm ·34.$$

Thus if we adopt the new theory instead of the old, the excesses become

$$+9·96 \pm ·43, \quad -0·04 \pm ·25, \quad +0·11 \pm ·13, \quad +0·77 \pm ·34.$$

The eccentricities and the planes of the orbits are not affected by the alteration, and the other residuals are therefore the same as those found by Newcomb.

The question to be decided is whether they can, as a whole, be accounted for by the attraction of other matter. As I have pointed out in an earlier paper (M. N. Roy. Astr. Soc. Dec. 1916, p. 112), any such matter must be very near the sun ; so that the disturbing function can be expressed in the form

$$R = f\alpha r^{-5}\{r^2 - 3(lx + my + nz)^2\},$$

α, l, and m being unknown constants.

Taking $\beta = 1·296 \times 10^8 \alpha$, I showed that the perturbations produced by such a distribution of matter are given by the formulæ in the third column of the following table :—

	Element.	Calculated perturbation.	Observed perturbation.	Mean error.
Mercury	di/dt	$(-49·9l - 52·7m)\beta$	0·38	0·80
	$\sin i\, d\Omega/dt$ ...	$(52·7l - 49·9m - 8·79)\beta$.	0·61	0·51
	$e\,d\varpi/dt$	$(0·66l - 0·61m + 14·72)\beta$.	9·96	0·43
Venus ...	di/dt	$(-2·57l - 10·14m)\beta$	0·38	0·33
	$\sin i\, d\Omega/dt$...	$(10·14l - 2·57m - 0·62)\beta$.	0·60	0·17
	$e\,d\varpi/dt$	$0·07\beta$	−0·04	0·25
Mars ...	di/dt	$(-0·36l - 0·42m)\beta$	−0·01	0·20
	$\sin i\, d\Omega/dt$ ...	$(0·42l - 0·36m - 0·012)\beta$	0·03	0·22
	$e\,d\varpi/dt$	$0·05\beta$	0·77	0·34

Nine equations of condition are thus obtained to determine β, l, and m. They are then divided by their respective

* Communicated by the Author.

† "General Relativity without the Equivalence Hypothesis," Phil. Mag. for July 1918, pp. 94–128.

mean errors to make them all of equal weight, and solved by the method of least squares. The method adopted being the same as in my former paper, the best solution is found to be $\beta=0{\cdot}67$, $l=0{\cdot}102$, $m=-0{\cdot}097$. Thus the inclination to the ecliptic of the equatorial plane of the disturbing matter is $8^\circ\ 9'$ and the ascending node is in longitude $46^\circ\ 24'$. The calculated perturbations produced by this matter are compared with observation in the following table :—

	Element.	Calculated.	Observed.	Residual.	Mean error.	Residual / Mean error.
Mercury	di/dt	0·02	0·38	0·36	0·80	0·4
	$\sin i d\Omega/dt$ ..	0·94	0·61	−0·33	0·51	−0·6
	$ed\varpi/dt$	9·95	9·96	0·01	0·43	0·0
Venus ...	di/dt	0·49	0·38	−0·11	0·33	0·3
	$\sin i d\Omega/dt$...	0·45	0·60	0·15	0·17	0·9
	$ed\varpi/dt$	0·05	−0·04	−0·09	0·25	−0·4
Mars ...	di/dt	0·01	−0·01	−0·02	0·20	−0·1
	$\sin i d\Omega/dt$...	0·04	0·03	−0·01	0·22	−0·0
	$ed\varpi/dt$	0·03	0·77	0·74	0·34	2·2

It is seen that all the residuals except one are less than their mean errors. This result is of course better than we are entitled to expect; for of nine observed quantities, three would be expected to deviate by more than their mean errors. It shows, however, that the observed excesses corresponding to Dr. Silberstein's theory are very easily accounted for by an oblate distribution of gravitating matter around the sun, and that the actual distribution will not be very different from that found on this theory.

The progression of the perihelion of Mars has still a residual which is more than twice its mean error. It is impossible to state on such a small margin of safety whether any considerable part of this is real. It was shown by Newcomb that the earth's attraction contributed $21''{\cdot}4$ per century to $ed\varpi/dt$ for Mars. Thus a mass 1/25 of the earth's, and equally favourably placed, could account for the residual. Newcomb pointed out that the known asteroids could not attain anything like such a mass, but if the invisible matter very much exceeded the visible it might be possible to explain the motion of the perihelion of Mars. Even if it were not so possible, however, the residual is not large enough to invalidate the theory.

The nodes of the ecliptic on the equatorial plane of the disturbing matter would at the same time be expected to regress $0''{\cdot}28$ in a century, the inclination to this plane remaining unaltered. This would give rise to small changes in the inclination of the equator to the ecliptic and in the planetary precession, but these would be within the probable errors.

We can conclude therefore that Dr. Silberstein's theory, combined with gravitating matter near the sun, can give an excellent representation of the secular inequalities of the inner planets.

June 22, 1918.

XVII. *On Relativity and Electrodynamics.*

To the Editors of the Philosophical Magazine.

GENTLEMEN,—

IN your number for April 1918 Mr. G. W. Walker discusses the transverse inertia of an electron. After quoting "experiments by Kaufmann, Bestelmeyer, and others" which "have been offered as experimental proof that the formula for transverse inertia of a contracted electron on relativity doctrine is correct," he expresses himself as follows:—

"I doubt if many people in this country realize the very meagre character of the experimental results, and I therefore give a full-sized reproduction of the photographic plate from which Kaufmann made his measurements."

One is led to suppose that some important researches, achieved since the time Mr. Walker was working side by side with Mr. Kaufmann in the laboratory in Göttingen, must have escaped his notice. Moreover, the statements quoted above tend to raise a feeling of surprise, if one remembers that Bestelmeyer's experiments failed to decide either against or in favour of the Lorentz-Einstein formula, and that Kaufmann finally considered his ("meagre") results to plead against this formula.

It seems therefore to be worth while to draw attention to the experiments of G. Neumann *, who improved a method devised by Bucherer in order to meet the criticisms raised by Bestelmeyer against Bucherer. He obtained results which fully confirmed Bucherer's view and which appear to establish beyond doubt the correctness of the Lorentz-Einstein formula for electrons moving with speeds from 0·4 up to 0·7 of the velocity of light.

More recently Prof. Ch. E. Guye and Mr. Ch. Lavanchy †,

* G. Neumann, "Die träge Masse schnell bewegter Elektronen," *Ann. d. Phys.* xlv. p. 529 (1914). See also Cl. Schaefer, *Verh. d. D. Phys. Ges.* xv. p. 935 (1915) and *Phys. Zschr.* xiv. p. 1117 (1913).

† Ch. E. Guye et Ch. Lavanchy, "Vérification expérimentale de la formule de Lorentz-Einstein par les rayons cathodiques de grande vitesse," *Arch. d. Sc. phys. et nat.* xlii. (1916).

in Geneva, verified experimentally the Lorentz-Einstein formula for cathode-rays with speeds from 0·25 up to 0·48 of the velocity of light. They found it in excellent agreement with their measurements of a great number (over two thousand) of observed electric and magnetic deviations of the rays.

Of course one would wish a further verification for velocities below 0·25 or beyond 0·7 that of light, but the experimental evidence obtained so far in favour of the incriminated formula leaves nothing to be desired.

Yours sincerely,

A. D. Fokker.

University of Leiden,
26th June, 1918.

XVIII. *Proceedings of Learned Societies.*

GEOLOGICAL SOCIETY.

[Continued from vol. xxxv. p. 507.]

February 15th, 1918.—Dr. Alfred Harker, F.R.S., President, in the Chair.

The President delivered his Anniversary Address, giving first obituary notices of H. Émile Sauvage (elected Foreign Correspondent, 1879), W. Bullock Clark (For. Corr. 1904), T. McKenny Hughes (el. 1862), Edward Hull (1855), R. H. Tiddeman (1869), G. A. Lebour (1870), Arnold Hague (1880), Robert Bell (1865), G. F. Franks (1890), G. C. Crick (1881), H. P. Woodward (1883), Upfield Green (1889), C. O. Trechmann (1882), A. N. Leeds (1893), R. Boyle (1911), A. M. Finlayson (1909), and others.

The President went on to discuss the present position and outlook of the study of metamorphism. The rapid development of physical chemistry and the successful application of experimental methods to petrological questions have greatly changed the situation during recent years, and for the first time it seems possible to approach the subject of metamorphism systematically from the genetic standpoint. For the geologist this implies the critical study, not only of the great tracts of crystalline schists and gneisses, but equally of metamorphic aureoles, of pneumatolysis and other contact-effects, and of the phenomena, mechanical and mineralogical, related to faults and overthrusts. It implies, moreover, the recognition that these are all parts of one general problem, that of the reconstruction of rocks under varying conditions of temperature and stress. In practice, this problem is complicated by the fact that perfect adjustment of chemical equilibrium cannot be assumed, either in the rocks prior to metamorphism, or during the process of metamorphism itself.

Some consideration was devoted to the solvents which play an essential part in metamorphism and to the limits of migration of dissolved material within a rock-mass. The Address proceeded to the discussion of what is the most fundamental characteristic of metamorphism: namely, that recrystallization takes place in a solid environment, and so may be profoundly affected by the existence of shearing stress. Stress of this type, on the one hand, arises from the crystal growth itself, and on the other hand is called into play by external forces. The automatic adjustment of the internally created stress to neutralize that provoked from without affords the key to all structures of the nature of foliation. The mineralogical peculiarities characteristic of the crystalline schists must find their explanation in kindred considerations; for it can be shown that the chemistry of bodies under shearing stress differs in important respects from the chemistry of unstressed bodies. The result is seen in the appearance of a certain class of 'stress-minerals' where the dynamic element has figured largely in metamorphism, while in the same circumstances the formation of minerals of another class seems to have been inhibited. But, while some of the general principles governing the effects can be formulated, the explanation on these lines of the observed associations of minerals is a task for the future. It may be that many of the particular problems involved will in time be brought within the scope of laboratory experiment.

The conditions governing metamorphism are temperature and shearing stress, with uniform pressure as a factor of less general importance. If the orogenic forces are sufficient to maintain shearing stress everywhere at its maximum, the stress itself becomes a function of temperature, since this determines the elastic limit, and the principal conditions of metamorphism come to depend upon a single variable. This degree of simplification, however, is not to be expected universally. One disturbing factor is the local rise of temperature sometimes caused by the mechanical generation of heat in the crushing of rock-masses.

February 20th.—Mr. G, W. Lamplugh, F.R.S., President, in the Chair.

The following communication was read:—

'The Geological Aspects of the Coral-Reef Problem.' By Prof. William Morris Davis, For.Corr.G.S.

The communication is a critical review of the various theories that have been put forward up to the present time to explain the origin of coral-reefs. A voyage in the Pacific, made in the year 1914, enabled the author to collect new evidence bearing upon the question, and to make observations that have influenced him in his support of Darwin's theory.

After laying stress upon the embayment of shore-lines as a proof of subsidence, the author expresses the opinion that all theories that postulate a fixed relation between reef-formation and ocean-level are disproved, and are probably inapplicable to the case of atolls. It appears certain that reef-upgrowth is intimately associated with submergence wherever the matter can be tested. The solution of the coral-reef problem turns, at present, upon some means of discriminating between a submergence caused by subsidence, and a submergence caused either by a general rise of the ocean-level due to the uplift of the ocean-floor beyond the coral-reef region, or to the melting of the Pleistocene ice-sheets. Although no means of such discrimination are known, the author presents reasons that lead him to regard changes in ocean-level as of secondary importance, and that have caused him to attribute the submergence demanded by self-encircled islands to local subsidence, in accordance with the views of Darwin and Dana. He regards the theory that presupposes the raising of the ocean-level by uplift as extravagant in its demands, and he finds the theory of 'Glacial Control' inadequate when applied to barrier-reefs and encircled islands.

Stress is laid on the highly-significant unconformable relationship that exists between reef- and lagoon-limestones and their foundations —a feature that presents the strongest testimony for subsidence. In such a case the foundations must have suffered erosion for a considerable period before they were submerged, in preparation for the unconformable deposition of reef-limestones upon them. From a consideration of such unconformable relations it is concluded that fringing-reefs do not mark stationary or rising islands so generally as Darwin supposed.

With regard to elevated reefs, the author demonstrates the impossibility of explaining their features by regarding them as having been stationary while the ocean-surface was lowered, and holds that they must be due to local and diverse uplift affecting the islands themselves, following on epochs of subsidence which were the epochs of reef-formation. The theory that such reefs were formed during pauses in the elevation and emergence is considered to be seriously defective, and is contrary to Darwin's views.

The author discusses the studies of Semper on the reefs of the Pelew Islands, the origin of atolls as propounded by Rein, the views of Murray on barrier-reefs and atolls, and of Wharton on the truncation of atoll-foundations; but forms the opinion that the geological evidence for subsidence has been overlooked by these investigators, who paid no attention to the evidence afforded by unconformable contacts or embayed shore-lines.

The author feels that scientific opinion in regard to the origin of coral-reefs has been guided rather by subjective preference than by objective logic. He considers that Darwin's theory of intermittent subsidence is the most competent to explain the facts, and while he holds that other theories than Darwin's deserve cordial consideration, he feels that the burden of proof should be laid upon those who assume that reef-foundations have not subsided.

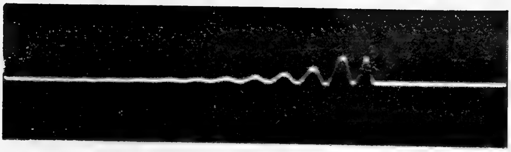


Fig. 7.—Oscillations with primary open.

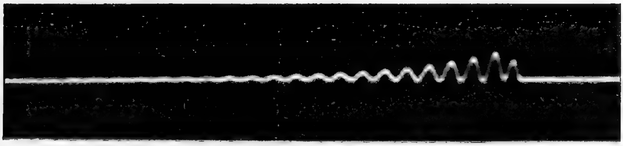


Fig. 8.—Oscillations with primary of magneto closed.

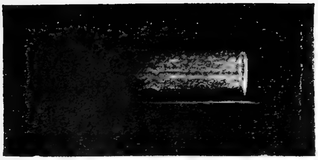
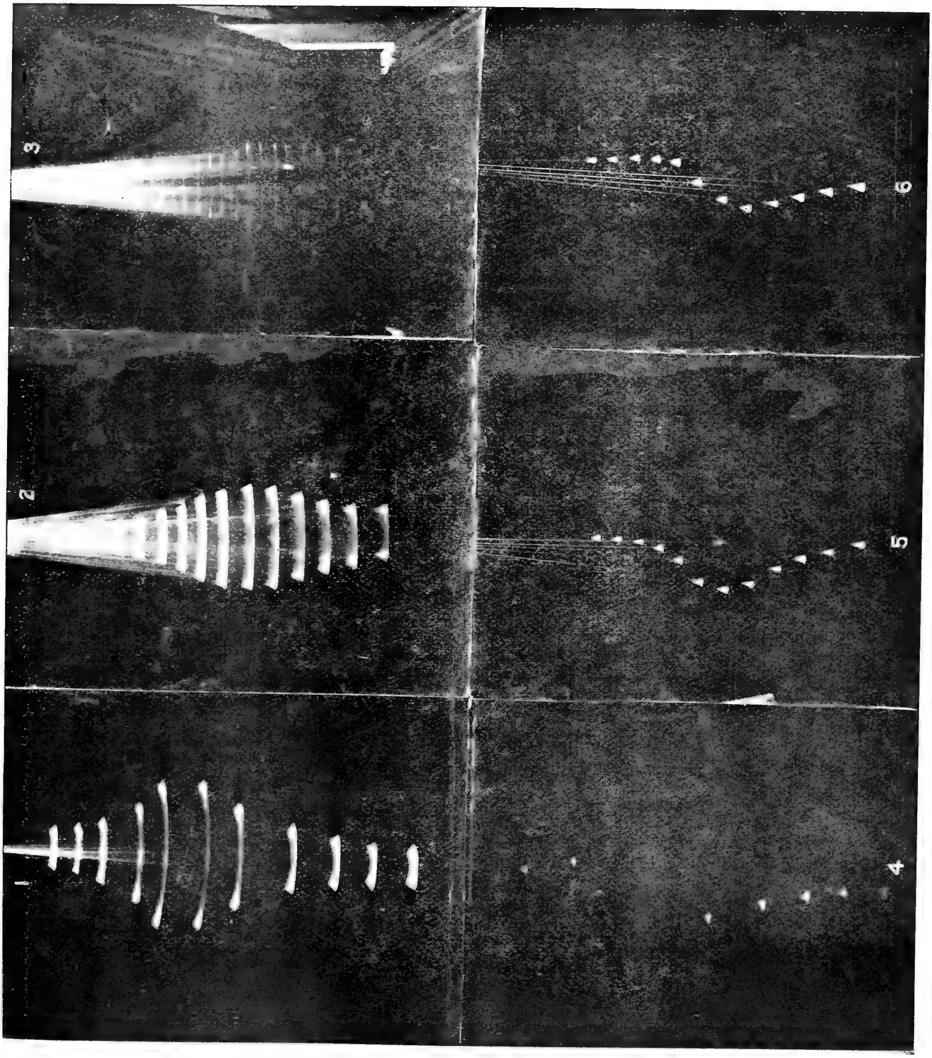


Fig. 9.—Magneto discharge showing pulsating arc.

Fig. 10.—Magneto discharge showing multiple spark.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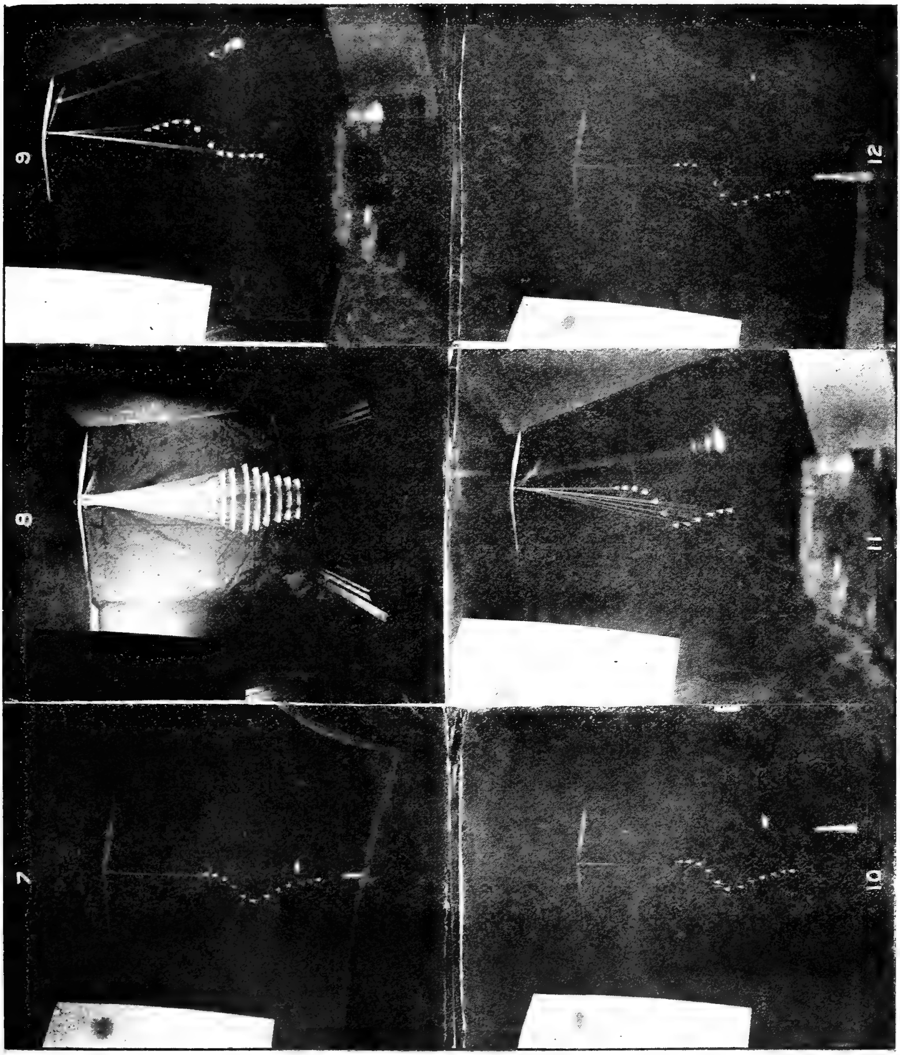
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THE
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AND
JOURNAL OF SCIENCE.

[SIXTH SERIES.]

SEPTEMBER 1918.

XIX. *A Comparative Study of the Flame and Furnace Spectra of Iron. By* G. A. HEMSALECH, *Honorary Research Fellow in the University of Manchester**.

[Plate X.]

§ 1. *Introduction.*

THE principal laboratory means of vaporizing substances at definite temperatures are provided by the flame and the electric-tube resistance furnace. If the determining factor in the emission of luminous radiations by a metal vapour in these two widely different sources of light were the same, namely heat, then the spectra observed in the various flames should be the same as those given by the furnace at corresponding temperatures. Thus at 1850° C. the furnace should emit a similar spectrum as the mantle of the air-coal gas flame, or at 2700° C. the furnace spectrum of a metal vapour should be the same as that observed in the oxy-acetylene flame. Now Dr. King, who has made a most exhaustive examination of the spectrum of iron as given by a tube furnace at various temperatures, has compared his results with those found for the flame-spectra of the same element by Dr. de Watteville and myself †, and, in conclusion, he has assigned certain values to the effective temperatures of our flames, which do not at all agree with those generally attributed to these flames. There is no doubt that a comparison of observational results obtained with different

* Communicated by Sir E. Rutherford, F.R.S.

† A. S. King, Astrophysical Journal, vol. xxxvii. p. 275 (1913).

spectroscopic appliances (Dr. King used a high-dispersion grating spectrograph, whereas our observations were made with an ordinary prism apparatus) presents some difficulty, inasmuch as a high dispersion will bring out a greater number of lines than a low dispersion, especially in the presence of a continuous spectrum. As a result of his comparison Dr. King has arranged the several flames used by Dr. de Watteville and myself in the following order with regard to their effective temperatures in producing radiations from iron vapour, as compared with the furnace :—

Flame.	Effective Temperature as derived by Dr. King.	Actual Flame Temperature (Dr. Bauer).
Air-coal gas (mantle).	below 1800° C.	1850° C.
Oxy-hydrogen.	about 1800 „	2550 „
Oxy-coal gas.	about 2000–2100 „	2450 „
Oxy-acetylene.	about 2000–2100 „	2700 „
Air-coal gas (cone).	about 2200 „	<1700 „

I have added to Dr. King's values those found by Dr. Bauer for the same flames by direct determinations*. Now it is of importance to mention here that Dr. Bauer used the same methods of colouring the flames as were devised and applied to their spectroscopic examination by Dr. de Watteville and myself. Moreover, in his final experiments on the high-temperature flames, namely the oxy-coal gas, oxy-hydrogen and oxy-acetylene flames, Dr. Bauer made use of our burners and, working for the time being in our laboratory, availed himself of our original equipment for the application of the spark method. Also I had the honour of assisting him in producing these high-temperature flames, and I am therefore in a position to state definitely that the flame conditions under which Dr. Bauer carried out his temperature determinations were identical with those used by Dr. de Watteville and myself in our spectroscopic researches on the same flames. But when we compare the results obtained by Dr. Bauer with those derived by Dr. King, we find discrepancies which appear to be very much in excess of any experimental errors that could reasonably be expected to affect Dr. Bauer's figures. With regard to the air-coal gas cone I have already shown† in two previous communications that the line emission is not due to temperature, but to chemical action. It is indeed inconceivable that the temperature of the cone film

* E. Bauer, *Thèses de Doctorat*, Paris, 1913.

† Hemsalech, Phil. Mag. vol. xxxiii. p. 1 (1917)—I.; ibid. vol. xxxiv. p. 221 (1917)—II.

could be so much higher than that of the mantle; for if it were so, the temperature in the zone of the mantle in immediate contact with the cone would naturally, by reason of convection, be at least of the same order of magnitude as that prevailing in the cone, and the luminous vibrations, if really they were controlled by temperature, would be observed to die out only gradually as the radiating centres passed from the cone into the surrounding mantle. But, as I have shown, this is not the case; for the characteristic cone emission stops abruptly at the boundary surface of the cone.

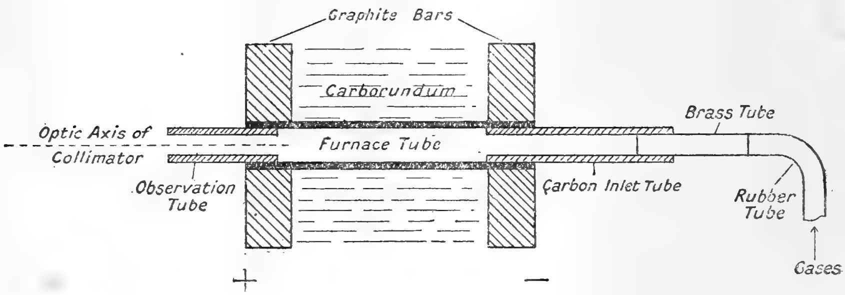
With regard to the mantle of the air-coal gas flame and those of the high-temperature flames, it is not possible to reconcile the figures given by Drs. King and Bauer on the basis of the experimental data so far available. It seemed to me that the only way in which the question could be settled was by a direct comparison between flame and furnace spectra made by the same experimenter. An opportunity for carrying out this test presented itself last autumn at the electro-chemical laboratory of this university, and with the kind permission of Sir Ernest Rutherford I was enabled to avail myself of the heavy current plant laid down specially for furnace work. I may say from the outset that the results of my investigation have established the existence of complete analogy between the characters of the flame and the corresponding furnace spectra of iron up to a temperature of about 2400° C.; and, further, they have shown that Dr. Bauer's results are in entire harmony with my own observations. Above the boiling-point of iron, namely at temperatures of over 2500° C., the furnace spectrum undergoes a very radical change. I had always suspected that when the furnace-tube is completely filled with metal vapour, the latter, if of adequate conductivity, would of necessity, in accordance with the fundamental electrical laws, carry part of the electric current which is supplied to the furnace. My experiments have brought most forcible evidence in favour of this view, and I have now no hesitation in ascribing the cause of the so-called high-temperature furnace emission of iron vapour to electric actions; it should indeed be classed as a low-tension arc spectrum.

The present and also the subsequent communication will contain an account of my own observations and experiments on the electric-tube resistance furnace. My observations, as will be noted, go to corroborate in many respects those made by Dr. King; but as a result of supplementary experiments and thanks to the experience gained in connexion with researches on flame and spark spectra, I have arrived at conclusions which differ materially from those advanced by Dr. King, and my interpretation of the phenomena will be found to present the whole question as to the origin of both flame and furnace spectra in a new light.

§ 2. *Description of Furnace.*

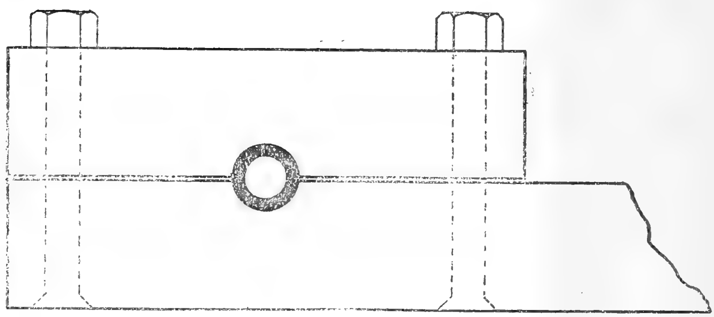
The furnace employed in this research was of a simple type and very similar in design to that used by Messrs. Duffield and Rossi*. Carbon tubes from 6 to 12 inches long with 20 mm. external and 14 mm. internal diameter, are tightly clamped between two pairs of graphite bars 1 inch thick by 2 inches high each. The lower bars are about two feet long,

Fig. 1.



Sectional View of Furnace.

Fig. 2.



Method of clamping Furnace Tube.

and are joined to the mains which communicate with the dynamos; the upper bars are one foot long. The clamping arrangement is easily understood from figs. 1 and 2. The

* W. G. Duffield and R. Rossi, Astrophysical Journal, vol. xxviii. p. 371 (1908).

space between the bars surrounding the exposed portion of the carbon tube is filled up with carborundum powder. An observation tube of carbon, about 2 inches long and having a bore of $\frac{3}{8}$ inch, is fitted into the furnace-tube end facing the projection-lens. The object of this tube is to prevent the brilliant light, radiated by the walls of the furnace, from reaching the lens. The other end of the furnace-tube is likewise provided with a carbon tube of $\frac{3}{8}$ inch bore and about 3 to 4 inches long. Into the free end of this tube is fitted a brass one, through which gases can be passed into the furnace. At first, furnace-tubes 12 inches in length were employed; but, after some trials, it was found that for the purpose of the present research tubes only 6 inches long did equally well. Thus with these shorter tubes the heated length of furnace was only 4 inches. The furnace was always set up in such a way that the axis of the tube was in a line with the optic axis of the spectrograph collimator, the middle part of the furnace and the spectrograph slit being at the foci of the projection objective. The spectrograph was the same one as that used in my work on flame spectra*. All experiments were conducted at atmospheric pressure. The furnace was heated by means of continuous current ranging from 160 to 600 amperes. The potential difference between the ends of the effective portion of the 6-inch tube varied from 6 to 13 volts. The temperatures were measured by means of a Wanner pyrometer, which could be directed upon the middle part of the interior furnace-wall after withdrawal of the carbon inlet tube from the end of the furnace. The range of temperatures employed was comprised between 1500° and 2700° C. Measurements were made in each case, and, during long exposures, readings were taken at regular intervals of time and, whenever necessary, the current was readjusted in order to keep the temperature constant. The pieces of metal to be vaporized were placed along the bottom of the furnace-tube. In the case of iron the arc method, formerly used for feeding flames, was successfully applied to the furnace. As will be remembered, this method consists in passing a steady current of air or oxygen through a glass bulb, which encloses an arc burning between iron poles. The issuing air, which carries in suspension the finely divided material from the arc (oxide of iron), is slowly passed through the furnace-tube.

* Hemsalech, *l. c.* I. p. 7.

§ 3. *The Luminous Phenomena exhibited by the Furnace as the Temperature gradually rises.*

Up to a temperature of about 2500° C. the aspect presented by the interior of the tube containing metallic iron is the same as when no iron is present. At low temperatures the space inside the furnace is filled with fumes or vapours giving out a strong continuous spectrum, which obliterates all but the strongest lines of the iron emission. It is not certain whether this continuous spectrum is actually emitted by these vapours, or whether it is merely reflected light from the inner surface of the white-hot carbon tube. It is, however, possible to greatly reduce the obnoxious effects of these vapours by passing a slow current of air or hydrogen through the tube. The velocity of the gas should be such as to produce only a very small flame at the opening of the carbon observation tube; this precaution is of special moment in the case of long exposures, because the too generous supply of fresh gas rapidly wears away the inner surface of the furnace, owing to chemical combination of the gases with the carbon (see § 10). As the temperature rises these luminous vapour clouds gradually dissipate and the tube appears fairly clear even without being constantly washed out by a current of gas. Up to a temperature of about from 2400° C. to 2500° C. the interior of the furnace, when free from clouds, glows in a beautiful purple tint. With cobalt metal in the tube, at about this temperature, a long luminous cloud of approximately cylindrical shape was observed to remain suspended in a position along the axis of the tube, as though held there in equilibrium by something expelled from all round the wall of the furnace-tube; its spectrum was continuous. The free space between this cloud cylinder and the furnace wall remained perfectly clear of mist and was of the same purple tint as before. When the temperature is raised to about 2500° C. the whole interior space gives out a white light showing strong continuous spectrum. I presume that this white light is caused by incandescent carbon particles shot off *en masse* in consequence of the more rapid disintegration, through the higher temperature, of the inner walls of the furnace. At 2700° C. the interior of the furnace is a blaze of dazzling white light, and the spectrum now shows the carbon bands in the green and blue; this band emission I take to indicate that an electric current now actually passes through conducting carbon vapour.

As already stated, the phenomena observed inside the furnace when the latter is charged with metallic iron, are the same as those described above when no metal is present, up to the temperature of about 2500° C., *i. e.* in the neighbourhood of the boiling-point of iron. Above this temperature, after the furnace space is completely filled with iron vapour from the boiling metal, the interior of the tube emits an extremely brilliant white light of greenish hue. An intensely bright line-spectrum, projected upon a most luminous continuous spectrum, is now visible, and among the lines those of classes II. and III. stand out more prominently. As will be shown in the next communication, at this stage, the metal vapour in the furnace carries part of the electric heating current, and the line-spectrum of iron as emitted under these conditions is of electric origin.

§ 4. *Origin of the Iron Spectrum emitted by the Furnace at Temperatures below* 2500° C.

The first traces of iron lines were obtained at a temperature of only 1500° C., or at about the melting-point of the metal, and the question naturally arises whether this emission is really caused by the action of heat on iron metal or on some compound of it. The number of lines and also the intensity of the spectrum increase rapidly as the temperature is raised, but the general character of the spectrum changes but slowly up to a temperature of about 2500° C., namely the boiling-point of iron, after which a great change occurs. Now it was found that iron spectra of precisely the same character were obtained when iron metal was in the tube or when no metal was present. Also the finely divided iron oxide blown through the tube gave an identical spectrum, only, if anything, a little more intense all round, as compared with the spectra observed in the first two cases. Hence when the furnace is run empty the iron spectrum emitted must be due to the existence of iron in the substance of the furnace-tube. There is little doubt that the iron, as well as most of the other impurities met with in the carbon, is chemically combined with the latter in the form of carbide. Thus, as the furnace gradually disintegrates in the interior, the iron carbide is set free, and under the action of the prevailing heat the compound molecule is dissociated or decomposed, which change is accompanied by the emission of luminous radiations. Since the spectrum emitted in this way is exactly the same in character as that given by the action of the furnace heat on iron oxide, the origin of the spectrum must be the same in the two cases, namely dissociation of an iron compound. It will be remembered that Dr. de Watteville and myself found a similarity of the like kind to exist between the spectra given by different compounds of iron when fed into flames. Now, as will be shown presently, a direct comparison between the spectra emitted by iron compounds in various flames and in the furnace at corresponding temperatures, has disclosed the interesting fact that these spectra are identical in character, from which we may conclude that the mode of excitation must be the same in these two very widely different sources of light. *It seems therefore evident that the furnace spectra of iron, up to a temperature of about* 2500° C., *are caused by the action of heat on a chemical compound, and not simply by vaporization of the metal which is placed in the furnace-tube.* The spectrum can therefore not be of purely thermal origin, because its emission necessarily involves also some process connected with the chemical change which the compound molecule undergoes as it is acted upon by heat. In order to better distinguish this mode of excitation from that which is supposed to represent the direct thermal action on a simple metal vapour, it will henceforth always be referred to as *thermo-chemical excitation.* In like manner the cone emission of iron in the air-coal gas flame will be considered as caused by *chemical excitation,* because in this case chemical actions evidently play the more important rôle.

§ 5. *General Character of the Furnace Spectrum of Iron.*

Most of the information regarding the character of the iron emission as excited by thermo-chemical actions in the furnace and in flames, was derived from an exhaustive examination of the many photographic records secured. All these photographs were taken on ordinary plates, and therefore they do not include the red end of the spectrum. This deficiency is, however, justified in the present circumstances because the low dispersion of my spectrograph did not allow of accurate observations in this part of the spectrum. The most objectionable factor in connexion with the low-dispersion spectrograph is the relative intensification, especially in the less refrangible region, of the continuous spectrum, which is always present at furnace temperatures above 1500° C. This continuous ground renders the observation of the line spectrum most difficult, especially in its denser parts.

As compared with the flame spectra of iron, the furnace spectra of this element are less well developed in the violet and ultra-violet parts of the spectrum; many of the lines in this region reverse at the higher temperatures, and it seems, as has already been remarked by Dr. King, that possibly the shorter wave-lengths suffer an appreciable absorption in passing out of the furnace through the cooler vapours near the end. It is well to remember in this connexion that the light radiations coming from the centre of the furnace, where we may rightly assume them to be of maximum intensity for a given temperature, have to pass in the present case through at least 2 inches of vapours, of which the last inch or so is probably well below the temperature prevailing at the centre. In flames the radiations do not traverse such a thick layer of cooler vapours, for practically the whole of the active volume of radiating vapour is confined within the limits of the flame envelope, which in the high-temperature flames rarely exceeds one centimetre in diameter. This fact no doubt accounts for the flame spectrum being so much better developed in the ultra-violet than the furnace spectrum, and it also explains the absence of reversals in the former. On the other hand, the furnace seems to be more efficient as regards line-intensity in the visible part of the spectrum, as though the longer wave-lengths were less absorbed than the short ones. This might indeed be the correct explanation, for the column of radiating vapour, even in my small furnace with an active length of 4 inches, must extend to at least 2 inches, as compared with an active depth of only from 5 to 10 mm. in flames. Hence, in judging the results of intensity estimations in flame and furnace spectra, account should be taken of the above considerations, and, further, it should always be remembered that it is not the real intensity of a line which counts, but its relative intensity and, particularly, the relative behaviour in each source of the various definite groups of related lines. The estimation of line intensities would, in the presence of a strong continuous background, be on a lower scale throughout than when this disturbing factor is absent. Thus it would be of no consequence if, for example, *both* group γ and the triplet at λ 4384 were observed to be intrinsically brighter in the flame spectrum than in the furnace spectrum at the same temperature, provided that the relative intensities of the triplet lines with regard to those of group γ be the same in the two cases. Hence in using the comparative table of flame and furnace spectra, which has been established as a result of my intensity estimations, these recommendations should be borne in mind, and attention be directed more specially to the relative intensities of the lines in any one particular spectrum, than to the real intensities of the lines in one spectrum as compared to those of the same lines in another spectrum.

The scale of intensities adopted is the same as that outlined in a former communication*. All wave-lengths are expressed in international units. A horizontal bar — means that no line has been observed. A *d* or *tr* attached to the wave-length

* Hemsalech, *l. c.* I. p. 9.

Thermo-chemical Excitation of Iron Radiations.

Wave-lengths.	Class.	Furnace.		Flame. Air-coal gas.	Furnace.			Flames. Oxy-coal gas.	Flames. Oxy-hydrogen.	Flames. Oxy-acetylene.
		1500°	1600°	1850°	1900°	2100°	2400°	2450°	2550°	2700°
3581·20	I.	—	—	—	—	—	—	—	00	00
3618·77	I.	—	—	—	—	—	—	—	000	00
3631·46	I.	—	—	—	—	—	—	—	00	0
3647·84	I.	—	—	—	—	—	—	000	0	0
3679·92	I.	—	—	000	—	—	00	0	2	2
3687·54*d*	I.	—	—	—	—	000	000	000	00	00
3705·56	I.	—	—	0	—	000	1	1	3	3
3707·91	I.	—	—	—	—	—	00	000	00	00
3709·24	I.	—	—	—	—	000	0	000	0	0
3719·93	I.	—	—	3	0	0 *r*	6 *r*	6	10	10
3722·57	I.	—	—	½	000	00	2	1	2	4
3727·63	I.	—	—	—	—	000	½	000	0	0
3733·32	I.	—	—	½	000	00	2	1	1	3
3734·86	II.	—	—	½	000	0	3	2	3	4
3737·13	I.	—	—	2	0	0 *r*	5 *r*	5	8	8
3743·37	II.	—	—	—	—	000	½	000	0	0
3745·73*d*	I.	—	—	2	½	0 *r*	4 *r*	4	10	10
3748·25	I.	—	—	1	00	0	3	2	4	4
3749·47	II.	—	—	00	00	0	2	1	3	3½
3758·23	II.	—	—	000	000	0	2	1	2	3
3763·80	II.	—	—	000	000	00	1½	½	1	2
3767·19	II.	—	—	000	000	00	1	0	1	1½
3787·88	II.	—	—	—	—	00	½	00	½	½
3795·00	II.	—	—	—	000	0	1	0	1	1
3798·50	II.	—	—	—	000	00	1	00	½	½
3799·55	II.	—	—	—	000	00	1	0	1	1
3812·88	II.	—	—	—	000	0	1	00	½	½
3815·84	II.	—	—	—	000	½	1	½	1	1
3820·44	II.	—	000	1	½	2	4	3	8	8
3824·44	I.	—	000	3	1	1½	3	3	8	8
3825·90	II.	—	000	½	½	1	2	2	6	6
3827·83	II.	—	—	—	—	½	1	0	1	1
3834·23	II.	—	—	0	½	1	2	1½	3	3
3840·44	II.	—	—	00	0	½	1	1	2	2
3849·99	II.	—	—	—	00	½	½	½	1	1
3856·38	I.	000	0	4	2	3	3	3	8	8
3859·90	I.	000	1	8	4	4 *r*	6 *r*	10	12	12
3865·53	II.	—	—	000	00	0	½	½	1	1
3872·51	II.	—	—	000	0	½	1	½	2	2
3878·02	II.	—	—	—	00	1	2	0	1	1
3878·70*d*	I.	—	½	4	2	2	2	3	8	10
3886·29	I.	000	¾	6	3	3	3	8	10	15
3887·05	II.	—	—	000	00	½	1	0	1	1
3888·52	II.	—	—	—	—	0	½	000	½	½
3895·65	I.	—	½	3	1½	2	2	3	6	6
3897·88	II.	—	—	—	—	—	—	—	00	00
3899·70	I.	—	½	3½	2	2	3	4	8	8
3902·95	II.	—	—	000	½	½	¾	0	1	1
3906·47	I.	—	0	1	1	1	1½	2	3	4
3917·17	II.	—	—	—	—	00	0	000	00	00

Wave-lengths.	Class.	Furnace.		Flame. Air-coal gas.	Furnace.			Flames. Oxy-coal gas.	Flames. Oxy-hydrogen.	Flames. Oxy-acetylene.
		1500°	1600°	1850°	1900°	2100°	2400°	2450°	2550°	2700°
3920·26	I.	000	¾	4	2	3	2	4	6	6
3922·92	I.	000	1	5	2½	4	3	6	8	8
3927·94	I.	000	1	5	2½	4	3	6	8	8
3930·30	I.	000	1	5	2½	4	3	6	8	10
3935·82	III.	—	—	—	—	—	—	—	—	000
3969·26	II.	—	—	000	00	1	1	1	2	2
4005·26	II.	—	—	—	—	½	½	½	2	2
4045·82	II.	—	000	1	½	1½	2	4	7	7
4063·61	II.	—	000	0	0	1	1½	2	4	4
4071·75	II.	—	—	00	00	1	1	1½	3	4
4132·08	II.	—	—	000	—	0	½	½	2	2
4143·66*d*	II.	—	—	000	000	½	1	1	3	3
4172·44*d*	II.	—	—	—	—	?	?	—	000	00
4177·60	II.	—	—	—	—	—	000	—	000	00
4187·43*d*	III.	—	—	—	—	—	—	—	—	00
4198·70*d*	III.	—	—	—	—	—	—	—	—	00
4202·04	II.	—	000	000	000	2	1	1	3	3
4206·91*d*	II.	—	—	00	000	0	½	0	½	½
4216·18	I.	—	0	2	1	1	2	½	2	2
4233·62	III.	—	—	—	—	—	—	—	—	000
4235·95	II.	—	—	—	—	—	—	—	00	0
4250·46*d*	II.	—	—	—	—	—	1	½	1	1
4260·48	II.	—	—	—	—	—	—	000	½	1
4271·46*d*	II.	—	?	1	½	½	2	3	5	5
4294·13	II.	—	—	—	—	—	½	00	½	½
4307·92	II.	—	?	½	½	1	2	3	5	5
4325·78	II.	—	0	½	½	1	2	3	5	5
4375·93	I.	—	2	4	2	2	2¼	2	4	4
4383·55	II.	—	½	2	1½	3	4	5	8	8
4404·75	II.	—	00	½	0	2	2½	2	4	4
4415·13	II.	—	—	00	000	½	1	½	2	2
4427·31	I.	—	1½	3	1½	1½	2	1	3	3
4447·72	III.	—	—	—	—	—	—	—	—	000
4461·65	I.	—	1	2	1	1	1½	½	2	3
4466·56	II.	—	—	000	—	000	00	000	00	00
4482·27	I.	—	0	1	0	¾	1	0	½	1
4489·91*d*	I.	—	000	0	000	0	½	00	0	0
4528·62	III.	—	—	—	—	—	—	—	—	0
4891·14*d*	III.	—	—	—	—	—	—	—	—	00
4919·76*d*	III.	—	—	—	—	—	—	—	—	0
4957·46*d*	III.	—	—	—	—	—	—	—	—	0
5012·07	I.	—	—	—	—	00	½	00	½	½
5110·42	I.	—	000	0	000	0	2	00	1	1
5167·49	I.	—	—	00	—	0	2	00	½	½
5268·82*tr*	I.	—	—	00	00	½	2	00	1½	2
5328·30*d*	I.	—	—	—	00	00	1	—	000	00
5371·50	I.	—	—	—	—	000	0	—	—	000
5397·12	I.	—	—	—	—	000	00	—	—	—
5404·96*d*	I.	—	—	—	—	000	00	—	—	—
5429·70	?.	—	—	—	—	—	000	—	—	—

number indicates that the line is double or treble, but has not been resolved by my spectrograph; in all these cases the mean value of the wave-lengths of the components has been given. An *r* attached to the intensity number signifies that the line is reversed. The class and therefore the character of each line is given in a separate column in accordance with the classification which I have proposed in the paper referred to. The numerical results are arranged in order of ascending temperatures, irrespective of source; this arrangement has the advantage of enabling the reader to follow in a convenient manner the gradual development of the spectrum of iron as the intensity of the thermal actions on the compounds involved increases.

§ 6. *Remarks on the tabulated Results.*

The emission of lines begins at the remarkably low temperature of 1500° C., and the spectrum, which consists of about seven lines, is practically identical with that given by iron in an air flame burning in an atmosphere of coal gas*. Thus already from the first signs of response to the thermal actions the luminous vibrations set up by the iron atom, both in flame and furnace, are of the same character. The next higher temperature, namely 1600° C., marks an interesting stage in the development of the iron spectrum, for at this point class I. quintets γ and ϵ (see § 8) form the most prominent feature in the visible part and the grouping of the lines is strikingly revealed. As the temperature rises the number of lines increases, as does also the brightness of the spectrum, but some lines gain more rapidly in intensity than others. The spectrum of the air-coal gas flame at 1850° C. and that of the furnace at 1900° C. are practically identical; so are also the spectra given by the furnace at 2400° C. and the oxy-coal gas flame at 2450° C. It will be noticed that in a number of cases feeble lines appear relatively more intense in the furnace spectrum than in the corresponding flame spectrum. This might be due to the fogging of the plate caused by the continuous spectrum always present in the furnace emission; for, as Professor R. W. Wood has shown, faint impressions on a photographic plate always show up in a remarkable manner when the background is slightly fogged.

The effect of temperature on lines of different character is well illustrated by the relative behaviour of class I. group γ

* Hemsalech, *l. c.* II. § 8, p. 233.

at 4376 and class II. triplet at 4384. At the low temperature of 1600° C. group γ stands out conspicuously, the line λ4376 being much more intense than its neighbour λ4384; at 2100° C. 4384 is brighter than 4376, and the relative intensities of the triplet lines with respect to those of group γ increase considerably at still higher furnace and flame temperatures.

The furnace spectra, as has already been explained, extend a little farther towards the red than the flame spectra. But it should also be remembered that I was unable for the reasons given elsewhere* to work the high-temperature flames to their full thermal advantage. Thus in the oxy-hydrogen flame spectrum of iron published by Dr. de Watteville and myself†, and in which we included for want of space only the lines down to intensity 2, the lines 5328 and 5371 are marked 5 and 2 respectively, so that most undoubtedly the other lines which are weaker, namely 5397, 5405, and 5430, should be expected to exist among those of intensities 1 to 000. As further evidence to the effect that my high temperature flames in the present experiments were not quite developed to their utmost perfection may be mentioned the fact, that no traces of class III. lines were observed with the oxy-acetylene flame. Their presence in this flame was, however, particularly noted by Dr. de Watteville and myself. Since the appearance of lines of this character in the oxy-acetylene flame is of the utmost importance for the true appreciation of furnace spectra, I have added them to the present list from the data previously published ‡. None of these lines are observed in the furnace spectrum below 2500° C., nor in any flame below the temperature of 2700° C. But they are easily emitted by chemical excitation in the air-coal gas cone, and some of them attain considerable prominence under the influence of electric actions, as for example the group at 4957. The fact that these lines appear only as feeble traces at the high temperature of the oxy-acetylene flame is another proof that the mode of excitation which is prevalent in the furnace up to 2400° C. and in the several flames examined is absolutely different from that which underlies their emission in the air-coal gas cone, arc, or spark. On the other hand, as is clearly demonstrated by my results, the mode of excitation in the furnace

* Hemsalech, *l. c.* I. p. 7.

† Hemsalech and de Watteville, *Comptes Rendus de l'Académie des Sciences*, vol. cxlvi. p. 962 (1908).

‡ Hemsalech and de Watteville, *Comptes Rendus de l'Académie des Sciences*, vol. cl. p. 330 (1910).

must be the same as in flames. Also the close agreement between the progressive development of the iron spectrum with the rise in temperature, both in furnace and in flames, testifies to the accuracy of Dr. Bauer's figures concerning the temperatures of the flames involved.

It should be possible with the help of the results here given for iron to estimate the temperature of a source of light in which thermo-chemical excitation prevails. Thus we may conclude that the temperature of an air flame burning in coal gas is about 1500° C.

§ 7. *Note on the Spectrum of Iron as excited by Chemical Actions.*

Whereas complete similarity has been found to exist between the spectra of iron as given by the mantles of various flames and those observed in the furnace up to a temperature of about 2400° C., no spectrum has been met with in the furnace corresponding to that given by chemical excitation in the air-coal gas cone. To judge by the development of this spectrum as regards mere number of lines, it seems to occupy a position intermediate between that of the oxy-acetylene flame and that of the self-induction spark, as is shown by the following figures derived from observations that were all made with the same spectrograph and therefore bear comparison:—

Mode of Excitation.	Number of Lines.
Thermo-chemical	100
Chemical.....................	220
Electrical	440

The origin of the cone emission has already been fully discussed in a previous communication.

§ 8. *Additions and corrections to the Line Groups of Class* I.

In the course of a previous research on flame spectra attention was directed to the existence, in the several classes of iron lines, of curious groupings of apparently related lines. In particular, among the lines of class I. several quartet groups were found in all of which the lines converge towards the red. The present experiments with the furnace, which as already explained in § 5 provided better opportunity for observing the less refrangible end of the spectrum, have disclosed the existence of a new outstanding group of five lines having its head at λ5270. It will hereafter be designated as group ε. This group is also observed in all the flames, but its less refrangible lines do not show on photographs taken on ordinary plates, and it had therefore escaped my attention. It was further found that group γ also consists of five lines and not of only four, as hitherto believed. Groups δ and ε overlap partly and some of their lines form close doublets, which it would have been impossible to separate with the low dispersion employed. But since I have now shown that the furnace spectra of iron up to 2400° C. are of identical character with the flame spectra of the same element, it has become possible to take advantage of Dr. King's most carefully prepared table of furnace lines based on observations made with a high-dispersion spectrograph. It was found that nearly all the lines of groups γ and ε belong to class I B of Dr. King's classification; those of group δ all with one exception to class II. I feel now, however, rather doubtful whether the lines of group δ are genuinely connected, but the decision must be left to further investigation. In any case group δ is not so prominent as the other two, which indeed seem to belong to the fundamental vibrations of the iron atom. The fifth line of group δ, namely 5341, has not been observed in any of the flames. A list of the revised groups γ and δ, and of the new group ε is given below.

	Wave-lengths.	Oscillation Frequency.	Δ_1.	Δ_2.	Dr. King's classification.
γ.	4375·93	22852·3	265·2		IB.
	4427·31	22587·1	173·9	91·3	IB.
	4461·65	22413·2	103·1	70·8	IB.
	4482·27	22310·1	37·1	66·0	IB.
	4489·74	22273·0			IA.
δ.	5012·07	19951·8	600·1		IB.
	5167·49	19351·7	377·6	222·5	II.
	5270·35	18974·1	207·2	170·4	II.
	5328·54	18766·9	43·9	163·3	II.
	5341·03 ?	18723·0			II.
ε.	5269·53	18977·0	208·4		IB
	5328·06	18768·6	151·8	56·6	IB.
	5371·50	18616·8	88·4	63·4	IB.
	5397·12	18528·4	29·7	58·7	IB.
	5405·78	18498·7			IB.

§ 9. *Observations on the Spectra of some other Elements contained as Impurities in the Carbon Tube of the Furnace.*

It is highly probable that all the foreign substances found in the carbon of the furnace tubes are in the form of carbon compounds, no doubt carbides, which are formed in the course of the heating to which the tubes are subjected during the process of their manufacture. The mode of excitation which gives rise to the spectra emitted by these substances will certainly be the same as that found for iron, namely dissociation of the carbide through the action of heat. In addition to sodium the presence of the following elements has been particularly noted :—

		Relative Intensity at	
Element.	Wave-length.	1500° C.	2400° C.
Al	3944·03	00	6*r*
	3961·54	$\frac{1}{2}$	8*r*
Ca	3933·67	—	00
	4226·72	5	15
	4302·53	—	1
	4318·64	—	00
	4435·32*d*	—	1
	4454·78	—	$\frac{3}{4}$
Mn	4030·80	1	2
	4033·06	$\frac{3}{4}$	$1\frac{1}{2}$
	4034·48	$\frac{1}{2}$	1
K	4044·15	2	4
	4047·21	1	3
Sr	4607·34	0	6
Pb	4057·84	000	1
Cr	4254·34	—	2
	4289·72	—	1

The appearance of the aluminium lines at so low a temperature as 1500° C. is most remarkable in view of the fact that all attempts to obtain them in the air-coal gas flame have so far proved unsuccessful. It may be that the carbide of aluminium is more readily dissociated than those compounds of this element which are generally employed in feeding flames.

§ 10. *Note on the Carbon Bands observed in the Furnace.*

No special attention was given to these bands, but the few observations made of them *en passant* are worth recording. The bands met with under various furnace conditions are those generally attributed to hydrocarbons, cyanogen, and carbon monoxide (Swan spectrum). Of these the last-named bands have been observed only at temperatures above 2500° C. in the presence of a strong ionization current. No bands were obtained at 1500° C. The relative behaviour of the hydrocarbon and cyanogen bands at various furnace temperatures is seen from the following table. At the lowest temperature, namely 1600° C., the observations were made with a current of hydrogen passing through the furnace. In all the other cases the furnace contained stagnant air.

		Relative Intensity at			
Edge.	Wave-length.	1600°	1900°	2100°	2400°
Red	3871	—	—	0	1
Red	3883	—	½	2	3
Violet	4241	½	—	1	1
Violet	4260	½	—	—	½

When a current of ammonia is passed through the furnace at 2400° C. the cyanogen bands become most intense and show a high degree of development. With a current of hydrogen at the same temperature they still show plainly, but their tails are imperfectly developed.

With regard to the origin of these bands, the following observations may possibly provide a clue. It was found that when air, hydrogen, &c. were passed through the carbon tube, the latter burnt through always near the end where the gases entered. This of course indicates a marked wear of the tube at the place upon which the gases impinge first. When no gases are passed through the furnace tube, the latter burns through almost invariably near the middle. There is no doubt that the wear noted in the former case is caused by the gas combining with the carbon at the lower temperature of the tube end. The newly formed compounds then enter the hot central region of the furnace, where they undergo dissociation. It may be the process involved in this dissociation which causes the emission of these bands, and the excitation would thus be due, as in the case of iron, to thermo-chemical actions. On the other hand, the Swan spectrum, which appears only at the highest furnace temperatures, seems to owe its emission to electric actions.

§ 11. *Relative Merits of Flame and Furnace as a means of obtaining the Spectrum of Iron by Thermo-chemical Excitation.*

Although the spectra given by the furnace are of the same character as those observed in flames, there are occasions when it is of advantage to give preference to either one or the other of these two light sources. Thus the furnace, for the reasons already suggested (§ 5), gives a better developed spectrum in the visible part. On the other hand, all the flames are superior to the furnace in the ultra-violet, where the latter absorbs a large percentage of the radiations. Also, with low-dispersion apparatus the flames give a purer spectrum of the metal owing to the relatively much smaller range of the continuous background. Moreover, the oxy-acetylene flame provides a means of investigating the effects of thermo-chemical excitation at a much higher temperature (2700°) than is possible with the tube furnace, since the latter exhibits a totally different phenomenon above 2500° C. Again, from the point of view of manipulating these sources of light, the flames possess this further advantage that their temperatures can be kept constant for any length of time without the least trouble, whereas the furnace has to be continuously watched and readjusted, because the gradual disintegration of the walls of the carbon tube entails changes in resistance and, consequently, in temperature. And, last not least, the spectra given by flames are free from the countless impurities which infect every furnace spectrum. On the other hand, the furnace permits to carry out the experiments at low or high pressures, as has been so effectively done by Dr. King, whereas flames can not be so conveniently subjected to such experimental variations.

With regard to the practical working of the furnace for spectroscopic purposes, it seems to me that improvements in several directions are possible. Thus the spark method, which has proved so successful with the high-temperature flames, might with advantage be applied to the furnace. Also the Gouy sprayer should give satisfactory results. Another method of obtaining the furnace spectra of metals would be to add these latter to the carbon or graphite from which the furnace tubes are made and turn them into carbides. With the disintegration of the tubes, these carbides would be set free and exposed to the thermal actions in the furnace.

§ 12. *Explanation of Plate X.*

The photographs here reproduced are enlarged copies (4 times) of the visible portion of the iron spectrum obtained under various conditions of excitation. Nos. 1–5 show the development of the iron spectrum at various stages of temperature both in furnace and flames. At the higher furnace temperatures, namely 2100° (No. 3) and 2400° (No. 4) a strong continuous spectrum impedes the distinctness of the iron lines. But if, as has been explained in § 5, attention be directed more particularly to the relative behaviour of class I. group γ and class II. triplet at 4384, it is easy to see that the latter gains steadily and continuously in relative intensity as the temperature rises. Thus at 1600° (No. 1) the line 4384 is much weaker than its neighbour 4376; at 2550° (No. 5), on the other hand, it is considerably brighter than 4376. No. 6 shows the spectrum of iron as emitted in the explosion region of the air-coal gas flame. This spectrum was obtained with burner No. 1 and the method of screening already described*. The most striking feature of this spectrum, although emitted at a temperature of less than 1700° C., is the presence of class III. group at 4957, which is entirely absent from the spectra given by flames, or the furnace up to 2500° C. As already mentioned in § 6, a mere trace of some of the lines in this group is observed in the oxy-acetylene flame at a temperature of 2700° C. In the explosion region, as will be seen from an inspection of No. 6, the lines in this group are quite as intense as those of class I. group γ. Class II. triplets at 4272 and 4384 are likewise relatively very bright as compared with group γ. The last spectrum, No. 7, was obtained with the self-induction spark, and represents an example of electrical excitation. It is characterized by the outstanding prominence of the groups of class II. and III. lines.

§ 13. *Summary.*

1. The spectra of iron as given by an electric-tube resistance furnace at atmospheric pressure and up to a temperature of about 2400° C., are caused by the action of heat on a chemical compound of the metal and not on the free metal itself. Hence these spectra are not of purely thermal origin. § 4.

* Hemsalech, *l. c.* I. pp. 9 & 10.

2. An iron spectrum has been observed at the low temperature of 1500° C., the spectrum being the same as that given by an air flame burning in coal gas. § 6.
3. The spectra emitted by iron compounds in flames are identical with those given by the furnace at corresponding temperatures up to about 2400° C. From this and other considerations it has been concluded that the mode of excitation must be the same in the two cases,—namely, chemical dissociation of an iron compound by the action of heat. §§ 4, 5, 6.
4. The character of the spectrum is independent of the nature of the iron compound, which is acted upon by the thermal forces in either flame or furnace; thus chlorides, oxides, &c. always give the same kind of spectrum in either of these sources at a given temperature. § 4.
5. The name *thermo-chemical excitation* has been adopted in order to designate the cause of emission of these spectra both in flame and furnace. They differ completely from the spectra given by the same compounds in the explosion region of the air-coal gas flame where the emission is due to *chemical excitation* at a comparatively low temperature. §§ 4 & 7.
6. A new group composed of class I. lines and possessing similar character as group γ, has been found with head line at λ5270. § 8.
7. The aluminium lines λλ3944 and 3962 have been observed at so low a temperature as 1500° C. § 9.

§ 14. *Concluding Remarks.*

Without entering into a fruitless discussion concerning the mechanism involved in the generation of the atomic vibrations, I hope, however, that a useful purpose will be served by briefly considering the possible changes in the state of the compound molecule when it is subjected to thermal actions. From all the evidence to hand it appears to me doubtful whether the iron atom in the compound is actually liberated either in the furnace or in flames. Thus, for example, if we heat iron oxide in a flame the product would hardly constitute a mere mixture of oxygen gas and iron vapour. I rather believe that in the particular case under consideration chemical affinity exerts its force even up to the temperature of 2700° C. and that the compound molecule, although undoubtedly changed in so far as the relative positions and the orbital motions of its component atoms are concerned, is not broken up, but retains its individuality throughout. The luminous vibrations set up by the iron atom forming part of a compound should, if my view were correct, show signs of the effects of the chemical forces which bind the atoms in the molecule. Now it has been shown in the course of this investigation that the line spectrum emitted by an iron compound at a temperature of 2700° C., although exceedingly intense, is nevertheless most appreciably restricted in range, it being composed almost entirely of class I. and II. lines only, as though some extraneous force were preventing the natural development of the luminous vibrations. This curtailment of the luminous vibrations could be satisfactorily accounted for if we accept the view expressed above, namely that the iron atom, even at the temperature of 2700°, is to some extent still associated with the other atoms in the compound. On the other hand, it is legitimate to assume that under the action of the powerful electric forces prevailing in the arc or condenser spark, the iron atoms are really set free and therefore enabled to execute their vibrations without restraint. This assumption seems to be amply borne out by the very high degree of development which characterizes the spectra emitted by iron vapour in these sources of light.

In this connexion it is interesting to compare the spectrum of the high temperature oxy-acetylene flame with that given by the low-temperature Bunsen cone. It will be remembered that I explained the origin of the cone emission by assuming the existence of a strong chemical affinity between the metal and the nitrogen, resulting in the formation of a nitride. Now if this hypothesis were well founded the iron atom would leave its partners in the original compound, which is fed into the flame (oxide, chloride, &c.), and join that of nitrogen. Hence, during the process of changing partner, the iron atom may be conceived to be quite free for a short moment and thus to be capable of executing its proper vibrations without hindrance. We should therefore expect, in accordance with the views put forth above, that its spectrum in this case would be better developed than in any of the high-temperature flames up to 2700° C., in all of which the iron atom is supposed never to be completely liberated from its partner in the compound. This conclusion is indeed substantiated by the facts observed, for the very lines which appear as mere traces with thermo-chemical excitation at 2700° C. stand out plainly in the spectrum of the cone emission, which as regards development approaches that of the self-induction spark.

Conversely, we may assume that at the lower flame and furnace temperatures the component atoms of a molecule become more firmly united, which, as has been shown, is accompanied by further restrictions in the line emission, and it is reasonable to conclude that the final extinction of the luminous vibrations would coincide with the completion of the chemical union of the atoms concerned. Thus the extinction of light which I observed* on passing a stream of oxygen or nitrogen through a coloured flame given by a weak gas mixture, receives a satisfactory explanation by supposing that the relaxed atoms of the heated salt molecule had completely recombined as a result of the cooling effected by the stream of gas. Moreover, the supposition that chemical union arrests the luminous vibrations of the atoms would at once enable us to account in a most plausible manner for the abrupt extinction of the cone emission as observed in the air-coal gas flame: namely, the emission of this spectrum would, in conformity with this view, stop instantly on the completion of the chemical union between the atoms of iron and nitrogen.

The hypothesis here developed is in short as follows:— The iron atom is never completely liberated by the action of heat either in flame or furnace, but remains always more or less chemically associated with the other atoms in the compound molecule; the light radiations which the atomic system of iron is capable of emitting under these conditions of restraint are always appreciably curtailed in development. On the other hand, in the explosion region of the air-coal gas flame the iron atom, thanks to its strong affinity for nitrogen, is severed from its partner in the original compound and the luminous vibrations, which are emitted whilst the atomic system is in the free state, show a high degree of development comparable to that observed in the arc and spark. Completion of chemical union is accompanied by the instant extinction of the line emission, as is shown by the abrupt cessation of the cone spectrum as the formation of the nitride is accomplished.

In conclusion I have great pleasure in placing on record my high appreciation of Dr. King's pioneering work on furnace spectra. It is mainly to the inspirations received through the medium of his important publications that the present research owes its origin.

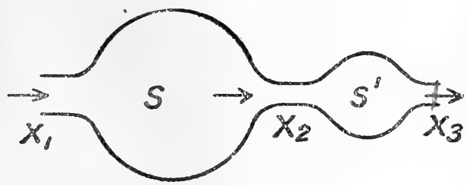
Manchester, May 16th, 1918.

* Hemsalech, Phil. Mag. vol. xxxv. p. 387 (1918).

XX. *Note on the Theory of the Double Resonator.* *By* Lord RAYLEIGH, *O.M., F.R.S.**

IN my book on the 'Theory of Sound'† I have considered the case of a *double* resonator (fig. 1), where

Fig. 1.



two reservoirs of volumes S, S' communicate with each other and with the external atmosphere by narrow passages or necks. If we were to treat SS' as a single reservoir and apply the usual formula, we should be led to an erroneous result; for that formula is founded on the assumption that within the reservoir the inertia of the air may be left out of account, whereas it is evident that the energy of the motion through the connecting passage may be as great as through the two others. However, an investigation on the same general plan meets the case perfectly. Denoting by X_1, X_2, X_3 the total transfers of fluid through the three passages, we have for the kinetic energy the expression

$$T = \tfrac{1}{2}\rho\left\{\frac{1}{c_1}\left(\frac{dX_1}{dt}\right)^2 + \frac{1}{c_2}\left(\frac{dX_2}{dt}\right)^2 + \frac{1}{c_3}\left(\frac{dX_3}{dt}\right)^2\right\}, \quad . \quad (1)$$

and for the potential energy

$$V = \tfrac{1}{2}\rho a^2\left\{\frac{(X_2-X_1)^2}{S} + \frac{(X_3-X_2)^2}{S'}\right\} \quad . \quad . \quad . \quad . \quad . \quad (2)$$

Here ρ denotes the density of the fluid, a the velocity of sound, while c_1, c_2, c_3 may be interpreted as the *electrical conductivities* of the passages. Thus for a long cylindrical neck of radius R and length L we should have $c=\pi R^2/L$.

* Communicated by the Author.

† § 310, first edition 1878, second edition 1896, Macmillan. Also Phil. Trans. 1870; Scientific Papers, vol. i. p. 41.

An application of Lagrange's method gives as the differential equations of motion,

$$\left.\begin{aligned} &\frac{1}{c_1}\frac{d^2X_1}{dt^2}+a^2\frac{X_1-X_2}{S}=0 \\ &\frac{1}{c_2}\frac{d^2X_2}{dt^2}+a^2\left\{\frac{X_2-X_1}{S}+\frac{X_2-X_3}{S'}\right\}=0 \\ &\frac{1}{c_3}\frac{d^2X_3}{dt^2}+a^2\frac{X_3-X_2}{S'}=0. \end{aligned}\right\} \quad . \quad . \quad . \quad (3)$$

By addition and integration

$$\frac{X_1}{c_1}+\frac{X_2}{c_2}+\frac{X_3}{c_3}=0, \quad . \quad . \quad . \quad . \quad . \quad . \quad (4)$$

since in the case of free vibrations all the quantities X may be supposed proportional to e^{pt}, so that d/dt may be replaced by p.

From (3) and (4) by elimination of X_3,

$$\left(\frac{p^2}{a^2c_1}+\frac{1}{S}\right)X_1-\frac{X_2}{S}=0, \quad . \quad . \quad . \quad . \quad . \quad (5)$$

$$\left(\frac{c_3}{c_1S'}-\frac{1}{S}\right)X_1+\left(\frac{p^2}{a^2c_2}+\frac{1}{S}+\frac{c_2+c_3}{c_2S'}\right)X_2=0,$$

whence as the equation for p^2

$$\frac{p^4}{a^4}+\frac{p^2}{a^2}\left\{\frac{c_1+c_2}{S}+\frac{c_2+c_3}{S'}\right\}+\frac{1}{SS'}\{c_1(c_2+c_3)+c_2c_3\}=0. \quad (6)$$

In the use of double resonance to secure an exalted effect, as in the experiments of Boys and of Callendar, we may suppress the direct communication between the second resonator S′ and the external air. Then $c_3=0$, and (6) becomes

$$\frac{p^4}{a^4}+\frac{p^2}{a^2}\left\{\frac{c_1+c_2}{S}+\frac{c_2}{S'}\right\}+\frac{c_1c_2}{SS'}=0. \quad . \quad . \quad . \quad . \quad (7)$$

To interpret the c's suppose first the passage between S and S′ abolished, so that $c_2=0$. The first resonator then acts as a simple resonator, and if p_1 be the corresponding p, we have $p_1^2/a^2=-c_1/S$, as usual. Again, if S be infinite, we have for the second resonator acting alone, $p_2^2/a^2=-c_2/S'$; and (7) may be written

$$p^4-p^2\left(p_1^2+p_2^2+\frac{S'}{S}p_2^2\right)+p_1^2p_2^2=0. \quad . \quad . \quad (8)$$

In (8) if S'/S be very small, p^2 approximates to p_1^2 or to p_2^2, and this is the case of greatest importance in experiment.

If p_1^2 and p_2^2 differ sufficiently, we may pursue an approximation from (8) founded on the smallness of S'/S. But it is of more interest to suppose that p_1^2 and p_2^2 are absolutely equal, which nothing precludes. Then

$$p^4-p^2\left(2p_1^2+\frac{S'}{S}\,p_1^2\right)+p_1^4=0, \quad . \quad . \quad . \quad . \quad (9)$$

whence

$$\frac{p^2}{p_1^2}=1+\frac{S'}{2S}\pm\sqrt{\left(\frac{S'}{S}+\frac{S'^2}{4S^2}\right)}; \quad . \quad . \quad . \quad . \quad (10)$$

or, if S'/S be small enough,

$$\frac{p^2}{p_1^2}=1\pm\sqrt{\left(\frac{S'}{S}\right)}, \quad . \quad . \quad . \quad . \quad . \quad . \quad (11)$$

p^2 differing but little from p_1^2 or p_2^2.

Referring back to (5), we have

$$\frac{X_1-X_2}{X_1}=-\frac{Sp^2}{a^2c_1}=\frac{p^2}{p_1^2}=1\pm\sqrt{\left(\frac{S'}{S}\right)},$$

when we introduce the value of p^2 from (11). Thus

$$\frac{X_2}{X_1}=\mp\sqrt{\left(\frac{S'}{S}\right)}. \quad . \quad . \quad . \quad . \quad . \quad . \quad (12)$$

We may now compare effects in the two component resonators, and here a certain choice presents itself. The *condensations* in the interiors are $(X_1-X_2)/S$ and X_2/S', and the *ratio* of condensations is

$$\frac{X_2/S'}{(X_1-X_2)/S}=\frac{\sqrt{(S/S')}}{1-\sqrt{(S'/S)}}=\sqrt{\left(\frac{S}{S'}\right)} \quad . \quad . \quad (13)$$

approximately. It appears that the condensation in the second resonator may be made to exceed to any extent that in the first by making the second resonator small enough, which sufficiently explains the advantage found in experiment to attend the combination.

In some forms of the experiment we may have to do rather with the flow through the passages than with the condensations in the interiors. In (12) we have the ratio of the total flows already expressed. But we may be more concerned with a comparison of flows reckoned per unit of area of the passages. In the case of passages which are mere circular apertures of radii R and R′ a simple result may be stated, for then $c_1 : c_2 = \mathrm{R} : \mathrm{R}'$; and, since $p_1^2 = p_2^2$, $c_1 : c_2 = \mathrm{S} : \mathrm{S}'$. Accordingly

$$\frac{\mathrm{X}_2/\mathrm{R}'^2}{\mathrm{X}_1/\mathrm{R}^2} = \sqrt{\left(\frac{\mathrm{S}'}{\mathrm{S}}\right)} \cdot \frac{\mathrm{S}^2}{\mathrm{S}'^2} = \left(\frac{\mathrm{S}}{\mathrm{S}'}\right)^{\frac{3}{2}}, \quad . \quad . \quad . \quad (14)$$

and the advantage of a small S′ is even more pronounced than in (13).

XXI. *The Addition Theorem of the Bessel Functions of Zero and Unit Orders.* By John R. Airey, *M.A., D.Sc.**

The Application of the Addition Theorem to the Calculation of Bessel Functions of Zero and Unit Orders.

The earliest form of the Addition Theorem of the $J_n(x)$ functions was found by Bessel. In his notation,

$$\mathrm{I}^i_{k+z} = \left(1+\frac{z}{k}\right)^i \left\{ \mathrm{I}^i_k - \mathrm{I}^{i+1}_k . z\left(1+\frac{z}{2k}\right) + \mathrm{I}^{i+2}_k . \frac{z^2}{2!}\left(1+\frac{z}{2k}\right)^2 - \dots \right\},$$

"welche Reihe zur Berechnung und Interpolation einer Tafel dieser Functionen angewendet werden kann" †.

The expressions given by Lommel and others,

$$\mathrm{J}_0(z+h) = \mathrm{J}_0(z)\mathrm{J}_0(h) - 2\mathrm{J}_1(z)\mathrm{J}_1(h) + 2\mathrm{J}_2(z)\mathrm{J}_2(h) - \dots,$$

$$\mathrm{J}_1(z+h) = \mathrm{J}_0(z)\mathrm{J}_1(h) - \mathrm{J}_1(z)\mathrm{J}_2(h) + \mathrm{J}_2(z)\mathrm{J}_3(h) - \dots$$
$$+ \mathrm{J}_1(z)\mathrm{J}_0(h) - \mathrm{J}_2(z)\mathrm{J}_1(h) + \mathrm{J}_3(z)\mathrm{J}_2(h) - \dots,$$

do not appear to have any useful application in the construction of tables.

A form of the Theorem, applicable over a wide range of values of the argument, can be found in which one of the terms in the argument is a root of a Bessel or Neumann function of zero or unit order.

The first differential coefficients of the functions satisfying Bessel's differential equation

$$\frac{d^2y}{dx^2} + \frac{1}{x} \cdot \frac{dy}{dx} + \left(1 - \frac{n^2}{x^2}\right) y = 0,$$

* Communicated by the Author.

† *E. g.*, Meissel's tables of $\mathrm{J}_n(x)$; $x=1, 2, 3, \dots 24$. Gray and Mathews, 'Bessel Functions,' pp. 266–279.

can be expressed in the form

$$\frac{dZ_n}{dx} = -\frac{n}{x}Z_n + Z_{n-1} = \frac{n}{x}Z_n - Z_{n+1},$$

where Z_n is written for J_n, G_n, Y_n, and other Bessel and Neumann functions. In the particular cases where $n=0$ and 1,

$$\frac{dZ_0}{dx} = -Z_1 \quad \text{and} \quad \frac{dZ_1}{dx} = Z_0 - \frac{Z_1}{x}.$$

From these results, addition theorems of Z_0 and Z_1 can be obtained. They are usually employed in connexion with tables of these functions where the intervals of the argument are small—say, 0·1 or 0·01; but as the formulæ can be expressed in a simple and concise form, the calculation can be carried out even when the increment or decrement is as great as $\frac{\pi}{2}$, which is approximately half the difference of two consecutive roots of the functions. Consequently, to evaluate any one of the Cylinder Functions, of the first or second kind, for values of the argument as far as 60, a short table of the first 19 or 20 roots of the Z_0 or Z_1 functions and the corresponding values of Z_1 or Z_0 for these arguments is required.

Bessel and Neumann Functions of Zero Order.

By Taylor's Theorem,

$$Z_0(x+h) = Z_0(x) + hZ_0'(x) + \frac{h^2}{2\,!}Z_0''(x) + \ldots,$$

and substituting for Z_0', Z_0'', etc. their values in terms of Z_0 and Z_1 we find

$$Z_0(x+h) = \left\{1 - \frac{h^2}{2} + \frac{h^3}{6x} + \frac{h^4}{24}\left(1 - \frac{3}{x^2}\right) - \ldots\right\}Z_0(x)$$
$$-\left\{h - \frac{h^2}{2x} - \frac{h^3}{6}\left(1 - \frac{2}{x^2}\right) + \frac{h^4}{12x}\left(1 - \frac{3}{x^2}\right) + \ldots\right\}Z_1(x).$$

If x is a root of $Z_0(x)$—say ρ,—the value of $Z_0(\rho+h)$ becomes

$$Z_0(\rho+h) = -\left[h - \frac{h^2}{2\,!\,\rho} - \frac{h^3}{3!}\left(1 - \frac{2}{\rho^2}\right) + \frac{h^4}{4\,!}\left(\frac{2}{\rho} - \frac{6}{\rho^3}\right)\right.$$
$$\left. + \frac{h^5}{5\,!}\left(1 - \frac{7}{\rho^2} + \frac{24}{\rho^4}\right) - \frac{h^6}{6\,!}\left(\frac{3}{\rho} - \frac{33}{\rho^3} + \frac{120}{\rho^5}\right) - \ldots\right]Z_1(\rho).$$

The expression in the bracket can be considerably simplified by comparing it with the series for $\sin h$. In fact, the coefficient of $Z_1(\rho)$

$$= -\Big[\Big(1-\frac{h}{2\rho}+\frac{h^2}{3\rho^2}-\frac{h^3}{4\rho^3}+\dots\Big)\sin h-\Big(\frac{h^5}{3\,.\,5!\,\rho^2}-\frac{h^6}{2\,.\,5!\,\rho^3}+\dots\Big)$$
$$+\Big(\frac{h^7}{7!\,\rho^2}-\frac{3h^8}{2\,.\,7!\,\rho^3}+\dots\Big)-\Big(\frac{2h^9}{9!\,\rho^2}-\dots\Big)+\dots\Big]$$

$$= -\Big[\frac{\rho}{h}\log_e\Big(1+\frac{h}{\rho}\Big)\sin h-\frac{h^5}{3\,.\,5!\,\rho^2}\Big(1-\frac{3h}{2\rho}+\frac{12h^2}{7\rho^2}-\frac{25h^3}{14\rho^3}+\dots\Big)$$
$$+\frac{h^7}{7!\,\rho^2}\Big(1-\frac{3h}{2\rho}+\frac{69h^2}{40\rho^2}-\dots\Big)-\frac{2h^9}{9!\,\rho^2}\Big(1-\frac{3h}{2\rho}+\dots\Big)+\dots\Big].$$

This is approximately equal to

$$-\Big[\frac{\rho}{h}\log_e\Big(1+\frac{h}{\rho}\Big)\sin h$$
$$-\frac{h^2}{360\rho^2}\Big(h^3-\frac{h^5}{14}+\frac{h^7}{504}-\dots\Big)\Big(1-\frac{3h}{2\rho}+\frac{12h^2}{7\rho^2}\dots\Big)\Big]$$

$$= -\Big[\frac{\rho}{h}\log_e\Big(1+\frac{h}{\rho}\Big)\sin h+\frac{h^2}{42\rho^2}\Big(\sin h-h+\frac{h^3}{20}\dots\Big)\Big(1+\frac{3h}{4\rho}\Big)^{-2}\Big].$$

Hence

$$Z_0(\rho+h) = -[\alpha_0\sin h-\beta_0\eta]\,Z_1(\rho),$$

where $\alpha_0 = \frac{\rho}{h}\log_e\Big(1+\frac{h}{\rho}\Big)$, $\beta_0 = \frac{h^2}{42\rho^2}\Big(1+\frac{3h}{4\rho}\Big)^{-2}$,

and $\eta = \Big(-\sin h+h-\frac{h^3}{20}+\frac{h^7}{30240}-\dots\Big)$.

When the value of ρ is comparatively large and h not greater than 1·6, the expression for $Z_0(\rho+h)$ reduces to one term,

$$Z_0(\rho+h) = -\frac{\rho}{h}\log_e\Big(1+\frac{h}{\rho}\Big)\,.\,\sin h\,.\,Z_1(\rho),$$

and $$\frac{Z_0(\rho-h)}{Z_0(\rho+h)} = \frac{\log(\rho-h)-\log\rho}{\log(\rho+h)-\log\rho}.$$

Bessel and Neumann Functions of Unit Order.

The Addition Theorem for the Cylinder functions $J_1(x)$, $Y_1(x)$, etc., can be found by expanding $Z_1(x+h)$ by Taylor's Theorem, and substituting the values of $Z_1'(x)$, $Z_1''(x)$, etc., in terms of Z_0 and Z_1; it is thus easily shown that

$$Z_1(x+h)=\left[1-\frac{h}{x}-\frac{h^2}{2}\left(1-\frac{2}{x^2}\right)+\frac{h^3}{3x}\left(1-\frac{3}{x^2}\right)+\ldots\right]Z_1(x)$$

$$+\left[h-\frac{h^2}{2x}-\frac{h^3}{6}\left(1-\frac{3}{x^2}\right)+\frac{h^4}{12x}\left(1-\frac{6}{x^2}\right)+\ldots\right]Z_0(x).$$

If x is a root of $Z_1(x)$, say r, then

$$Z_1(r+h)=\left[h-\frac{h^2}{2r}-\frac{h^3}{6}\left(1-\frac{3}{r^2}\right)+\frac{h^4}{12r}\left(1-\frac{6}{r^2}\right)+\ldots\right]Z_0(r)$$

$$=\left[\left(1-\frac{h}{2r}+\frac{h^2}{2r^2}-\ldots\right)\sin h+\frac{h^5}{5!\,r^2}\left(1-\frac{3h}{2r}+\frac{25h^2}{14r^2}-\ldots\right)\right.$$

$$\left.-\frac{3h^7}{7!\,r^2}\left(1-\frac{3h}{2r}+\frac{43h^2}{24r^2}-\ldots\right)+\frac{6h^9}{9!\,r^2}\left(1-\frac{3h}{2r}+\ldots\right)-\ldots\right]Z_0(r).$$

Therefore we have, approximately,

$$Z_1(r+h) = [\alpha_1 \sin h+\beta_1\eta]Z_0(r),$$

where

$$\alpha_1 = 1-\frac{h}{2(r+h)} \quad \text{and} \quad \beta_1 = \frac{h^2}{14r^2}\left(1+\frac{7h}{8r}\right)^{-\frac{12}{7}}.$$

As in the case of the Z_0 functions, for large values of the argument or for small values of h, the following simple formulæ may be derived :—

$$Z_1(r+h) = \left[1-\frac{h}{2(r+h)}\right].\sin h\,.\,Z_0(r),$$

and

$$\frac{Z_1(r-h)}{Z_1(r+h)} = -\frac{(2r-h)(r+h)}{(2r+h)(r-h)}.$$

$Z_0(x)$ and $Z_1(x)$ could be calculated by means of the above formulæ to seven places of decimals for values of x greater than twelve. The following tables of $\frac{\rho}{h}\log_e\left(1+\frac{h}{\rho}\right)$ etc. have been computed for values of $\frac{h}{\rho}$ and $\frac{h}{r}$ from 0·10 to $-0{\cdot}10$, so that $Z_0(x)$ and $Z_1(x)$ can be found to six places of decimals when x is greater than 15·5, the largest value of the argument in Meissel's tables *. Lommel gives $J_0(x)$ and $J_1(x)$ to six places of decimals from $x=0{\cdot}1$ to 20·0 by intervals of 0·1 ; $J_0'(x)$, $\frac{1}{2}J_0''(x)$ and $\frac{1}{6}J_0'''(x)$, etc. are also tabulated for purposes of interpolation.

In order to increase the accuracy of the tables, the last figure is given with or without a "point." This point means that the residue is greater than 0·25 and less than 0·75 units of the last place and is exactly equivalent to 5 in the first place of rejected decimals.

Values of α_0 and α_1, β_0 and β_1, with first and second differences Δ_1, Δ_2 in units of the sixth place of decimals.

$\frac{h}{\rho}$.	α_0.	Δ_1.	Δ_2.	β_0.	Δ_1.	Δ_2.
0,00	1,000000			0,000000		
		−4967			2 •	
0,01	0,995033		65 •	002 •		4
		−4901 •			6 •	
0,02	0,990131 •		63 •	009		5
		−4838			11 •	
0,03	0,985293 •		62 •	020 •		4
		−4775 •			15 •	
0,04	0,980518		61	036		4
		−4714 •			19 •	
0,05	0,975803 •		58 •	055 •		3 •
		−4655			23	
0,06	0,971148 •		58 •	078 •		4
		−4596 •			27	
0,07	0,966552		57 •	105 •		3
		−4539			30	
0,08	0,962013		56	135 •		3 •
		−4483			33 •	
0,09	0,957530		55	169		3 •
		−4428			37	
0,10	0,953102		54	206		3
−0,00	1,000000			0,000000		
		5033 •			2 •	
−0,01	1,005033 •		68 •	002 •		5
		5102			7 •	
−0,02	1,010135 •		69 •	010		5
		5171 •			12 •	
−0,03	1,015307		71 •	022 •		5 •
		5243			18	
−0,04	1,020550		73	040 •		6
		5316			24	
−0,05	1,025866		74 •	064 •		5 •
		5390 •			29 •	
−0,06	1,031256 •		77	094		6 •
		5467 •			36	
−0,07	1,036724		78 •	130		6 •
		5546			42 •	
−0,08	1,042270		80 •	172 •		7
		5626 •			49 •	
−0,09	1,047896 •		82	222		7
		5708 •			56 •	
−0,10	1,053605		84	278 •		7 •

* Gray and Mathews, 'Bessel Functions,' pp. 247–266.

$\frac{h}{r}$.	α_1.	Δ_1.	Δ_2.	β_1.	Δ_1.	Δ_2.
0,00	1,000000			0,000000		
		−4950 •			7	
0,01	0,995049 •		97	007		14
		−4853 •			21	
0,02	0,990196		94 •	028		12 •
		−4759			33 •	
0,03	0,985437		91	061 •		13
		−4668			46 •	
0,04	0,980769		89 •	108		11 •
		−4578 •			58	
0,05	0,976190 •		86	166		11 •
		−4492 •			69 •	
0,06	0,971698		84	235 •		11
		−4408 •			80 •	
0,07	0,967289 •		82	316		10 •
		−4326 •			91	
0,08	0,962963		79	407		10
		−4247 •			101	
0,09	0,958715 •		77 •	508		9 •
		−4170			110 •	
0,10	0,954545 •		76	618 •		9
−0,00	1,000000			0,000000		
		5050 •			7 •	
−0,01	1,005050 •		103	007 •		14 •
		5153 •			22	
−0,02	1,010204		106 •	029 •		16
		5260			38	
−0,03	1,015464		109 •	067 •		16
		5369 •			54	
−0,04	1,020833 •		113	121 •		17 •
		5482 •			71 •	
−0,05	1,026316		116 •	193		17 •
		5599			89	
−0,06	1,031915		120 •	282		19
		5719 •			108	
−0,07	1,037634 •		124 •	390		20
		5844			128	
−0,08	1,043478 •		128	518		20
		5972			148	
−0,09	1,049450 •		133	666		22
		6105			170	
−0,10	1,055555 •		138 •	836		23

The following short table gives the values of η when h varies from 0·1 to 1·6. For negative values of h, η is of course negative. It is not necessary to extend this table beyond the third place of decimals :—

h.	η.	*h.*	η.	*h.*	η.	*h.*	η.
0,05	0,000	0,45	0,010 •	0,85	0,068	1,25	0,203 •
0,10	0,000	0,50	0,014 •	0,90	0,080 •	1,30	0,227
0,15	0,000 •	0,55	0,019	0,95	0,093 •	1,35	0,251 •
0,20	0,001	0,60	0,024 •	1,00	0,108 •	1,40	0,278
0,25	0,002	0,65	0,031	1,05	0,124 •	1,45	0,305
0,30	0,003	0,70	0,038 •	1,10	0,142 •	1,50	0,334 •
0,35	0,005	0,75	0,047	1,15	0,161 •	1,55	0,364 •
0,40	0,007 •	0,80	0,057	1,20	0,182	1,60	0,396 •

The values of $\sin h$, the angle h being expressed in radians, are given in Burrau's tables * to six places of decimals

* Burrau, *Tafeln der Funktionen Cosinus und Sinus* (Reimer).

and in the Report of the Mathematical Tables Committee of the British Association (1916) to eleven places for $h=0{,}001$ to 1·600.

The first forty roots of $J_0(x)$ and the corresponding values of $J_1(x)$ were published by Willson and Peirce *, $J_1(x)$ being given to eight places of decimals (see p. 241). This table, in conjunction with those given above, can therefore be employed in calculating $J_0(x)$ for any value of x between 15·0 and 126·0 to six places of decimals.

The first fifty roots of $J_1(x)$ and the maximum and minimum values of $J_0(x)$ have been calculated by Meissel † to sixteen places of decimals (see p. 241). $J_1(x)$ can therefore be found for values of x from 15·0 to 159·0.

These tables, to four places only, are given in Jahnke u. Emde's *Funktiontafeln.*

The most complete tables of $J_0(x)$ and $J_1(x)$ are those calculated by Meissel ‡ from the ascending series to twelve places of decimals: $x=0·00$ to 15·50 by intervals of 0·01: for larger values of x, the asymptotic series can be employed where the calculation is not carried beyond the least term of $P_0(x)$, $Q_0(x)$, etc.

$$J_0(x) = \sqrt{\frac{2}{\pi x}}\left[P_0(x)\cos\left(x-\frac{\pi}{4}\right)-Q_0(x)\sin\left(x-\frac{\pi}{4}\right)\right]$$

and

$$J_1(x) = \sqrt{\frac{2}{\pi x}}\left[P_1(x)\sin\left(x-\frac{\pi}{4}\right)+Q_1(x)\cos\left(x-\frac{\pi}{4}\right)\right].$$

From a consideration of the divergent part of these series, it has been shown § that a greater degree of accuracy can be obtained by resolving these into series which can be evaluated by Euler's method of summation. In this way it is found that the divergent part of an asymptotic series of the first kind, where the signs of the terms alternate, is equivalent to the least term multiplied by a "converging factor." In these cases the term independent of x is $\frac{1}{2}$.

When x is an integer n, the "converging factors" for $P_0(x)$ and $Q_0(x)$ are:

$$\frac{1}{2}-\frac{1}{8n}+\frac{1}{8n^2}-\frac{15}{128n^3}+\frac{103}{1024n^4}\cdots$$

and

$$\frac{1}{2}+\frac{1}{8n}+\frac{0}{8n^2}-\frac{9}{128n^3}+\frac{159}{1024n^4}\cdots;$$

* Willson and Peirce, 'Bulletin of the American Mathematical Society,' vol. iii. 1896–97, pp. 153–5.

† Gray and Mathews, 'Bessel Functions,' p. 280.

‡ Gray and Mathews, 'Bessel Functions,' pp. 247–266.

§ *Archiv der Math. u. Phys.* 1914.

No. of root.	Roots of $J_0(x)$; ρ.	$J_1(\rho)$.	Roots of $J_1(x)$; r.	$J_0(r)$.
1	2·4048256	+0·5191475	3·8317060	−0·4027594
2	5·5200781	−0·3402648	7·0155867	+0·3001158
3	8·6537279	+0·2714523	10·1734681	−0·2497049
4	11·7915344	−0·2324598	13·3236919	+0·2183594
5	14·9309177	+0·2065464	16·4706301	−0·1964654
6	18·0710640	−0·1877288	19·6158585	+0·1800634
7	21·2116366	+0·1732659	22·7600844	−0·1671846
8	24·3524715	−0·1617016	25·9036721	+0·1567250
9	27·4934791	+0·1521812	29·0468285	−0·1480111
10	30·6346065	−0·1441660	32·1896799	+0·1406058
11	33·7758202	+0·1372969	35·3323076	−0·1342112
12	36·9170984	−0·1313246	38·4747662	+0·1286166
13	40·0584258	+0·1260695	41·6170942	−0·1236680
14	43·1997917	−0·1213986	44·7593190	+0·1192498
15	46·3411884	+0·1172112	47·9014609	−0·1152737
16	49·4826099	−0·1134292	51·0435352	+0·1116705
17	52·6240518	+0·1099911	54·1855536	−0·1083853
18	55·7655108	−0·1068479	57·3275254	+0·1053741
19	58·9069839	+0·1039596	60·4694578	−0·1026006
20	62·0484692	−0·1012935	63·6113567	+0·1000351
21	65·1899648	+0·0988226	66·7532267	−0·0976530
22	68·3314693	−0·0965240	69·8950718	+0·0954333
23	71·4729816	+0·0943788	73·0368952	−0·0933585
24	74·6145006	−0·0923705	76·1786996	+0·0914133
25	77·7560256	+0·0904852	79·3204872	−0·0895848
26	80·8975559	−0·0887108	82·4622599	+0·0878619
27	84·0390908	+0·0870369	85·6040194	−0·0862347
28	87·1806298	−0·0854542	88·7457671	+0·0846946
29	90·3221726	+0·0839549	91·8875043	−0·0832343
30	93·4637188	−0·0825319	95·0292318	+0·0818469
31	96·6052680	+0·0811788	98·1709507	−0·0805267
32	99·7468199	−0·0798902	101·3126618	+0·0792684
33	102·8883743	+0·0786610	104·4543658	−0·0780673
34	106·0299309	−0·0774869	107·5960633	+0·0769192
35	109·1714896	+0·0763591	110·7377548	−0·0758203
36	112·3130503	−0·0752882	113·8794408	+0·0747672
37	115·4546127	+0·0742568	117·0211219	−0·0737568
38	118·5961766	−0·0732667	120·1627983	+0·0727863
39	121·7377421	+0·0723152	123·3044705	−0·0718531
40	124·8793089	−0·0713997	126·4461387	+0·0709549

or in the general case, where $x=n+\alpha$:

$$\text{for } P_0(x), \text{ when } -\frac{3}{4}+\frac{1}{32n}\dots \overline{<} \alpha \overline{<} \frac{1}{4}+\frac{1}{32n}\dots,$$

$$\frac{1}{2}-\left(\frac{\alpha}{2}+\frac{1}{8}\right)\frac{1}{x}-\left(\frac{\alpha^2}{4}-\frac{\alpha}{8}-\frac{1}{8}\right)\frac{1}{x^2}+\left(\frac{5\alpha^3}{2}+\alpha^2-\frac{7\alpha}{32}-\frac{15}{128}\right)\frac{1}{x^3}-\dots;$$

and for $Q_0(x)$, when $-\frac{1}{4}+\frac{1}{32n}\cdots \overline{<} \alpha \overline{<} \frac{3}{4}+\frac{1}{32n}\cdots$,

$$\frac{1}{2}-\left(\frac{\alpha}{2}-\frac{1}{8}\right)\frac{1}{x}-\left(\frac{\alpha^2}{4}-\frac{3\alpha}{8}\right)\frac{1}{x^2}+\left(\frac{17\alpha^3}{6}-3\alpha^2+\frac{67\alpha}{96}-\frac{9}{128}\right)\frac{1}{x^3}-\cdots.$$

Similar expressions have been given for $P_1(x)$ and $Q_1(x)$.

When $x=9$ and the calculation is limited to the convergent parts of the asymptotic series, the values of $J_0(9)$ and $J_1(9)$ can be found to about eight places of decimals. The following table has been calculated by the above method, and gives results correct to fourteen places :—

	$x = 9.$	$x = 10.$
$J_0(x)$	−0·09033 36111 8287	−0·24593 57644 5134
$G_0(x)$	−0·39259 96475 9739	−0·08744 80650 7746
$Y_0(x)$	0·38212 71351 3807	0·05893 63591 5000
$J_1(x)$	0·24531 17865 7332	0·04347 27461 6886
$G_1(x)$	−0·16385 69515 5017	−0·39115 25136 5956
$Y_1(x)$ ...	0·19229 63187 7649	0·39619 23750 1275

Consequently the ascending series need only have been employed for values of x from 0·01 to 8·00 to give $J_0(x)$ and $J_1(x)$ to twelve places of decimals.

The Zonal Harmonic $P_n(\theta)$ can be expressed in terms of $J_0(z)$ and $J_1(z)$, where $z=\theta\sqrt{n(n+1)}$. Lord Rayleigh's formula,

$$P_n(\theta) = J_0(z) + \frac{1}{12n(n+1)}\{z^2J_0(z)-2zJ_1(z)\},$$

has been extended and employed in the calculation of $P_n(\theta)$ when n is large and θ is a small angle ; the extended formula, however, gives results correct to six places of decimals even for comparatively large angles, *e. g.* when $\theta=\frac{\pi}{2}$ and $n=20$.

XXII. *On Bohr's Hypothesis of Stationary States of Motion and the Radiation from an accelerated Electron.* By G. A. Schott, *B.A., D.Sc., Professor of Applied Mathematics, University College of Wales, Aberystwyth* *.

1. BOHR'S theory of the Balmer Series is based upon several novel hypotheses in greater or less contradiction with ordinary mechanics and electrodynamics, and amongst them the hypothesis of stationary states of motion occupies a prominent position. In his latest paper † on the subject Bohr states it in the following form :—

"A. An atomic system possesses a number of states in which no emission of energy radiation takes place, even if the particles are in motion, and such an emission is to be expected on ordinary electrodynamics. The states are denoted as the states of stationary motion of the system under consideration."

Although Nicholson's ‡ criticism of the theory indicates that it cannot be applied in its present form to elements other than hydrogen, and perhaps helium, yet the representation afforded by it of the line spectrum of hydrogen is so extraordinarily exact that a considerable substratum of truth can hardly be denied to it. Therefore it is a matter of great theoretical importance to examine how far really it is inconsistent with ordinary electrodynamics, and in what way it can be modified so as to remove the contradiction. The object of the present investigation is to consider Bohr's hypothesis A from this point of view.

2. In 1897 Liénard § published his well-known expression for the irreversible radiation from an accelerated electron. It is essentially positive and only vanishes when the acceleration vanishes, a possibility which is obviously excluded in the case of an electron moving in any way inside the atom. Thus it contradicts Bohr's hypothesis A unavoidably, and we must inquire how far Liénard's expression is a necessary consequence of ordinary electrodynamics.

An examination of Liénard's proof shows that it merely presupposes the usual expressions for the retarded scalar and vector potentials together with Poynting's expression

* Communicated by the Author.

† Bohr, Phil. Mag. ser. 6, vol. xxx. p. 394 (1915).

‡ Nicholson, Phil. Mag. ser. 6, vol. xxvii. p. 541, and vol. xxviii. p. 90 (1917).

§ Liénard, *L'Éclairage Électrique*, July 1898. Also Schott, 'Electromagnetic Radiation,' p. 251, § 231.

for the energy flux. It will doubtless be admitted generally that the retarded potentials represent those solutions of the equations of the electromagnetic field which are specially appropriate to the case of the accelerated electron. Hence in order to remove the contradiction with Bohr's hypothesis A, as it stands, only two alternatives appear to be possible: either we must reject the Poynting flux, or we must reject the retarded potentials together with the field equations of which they are the appropriate solutions.

3. Let us begin by considering the first alternative. It is well-known that neither the Poynting flux, nor the usual expressions for the energy densities, at any rate that for the magnetic energy density, corresponding to it, are at all unique. Livens * has recently examined this question in detail and gives several possible expressions for the energy flux, together with the corresponding ones for the magnetic energy density. At the same time he arrives at the conclusion that the Poynting energy flux and the classical energy densities corresponding to it are to be preferred to all others for physical reasons. It is, however, worthy of notice that there is one form of the energy flux which is consistent with Bohr's hypothesis A, viz. the expression $\phi\mathbf{C}$, where ϕ denotes the retarded scalar potential, and $\mathbf{C}$ the total electric current. It follows from what Livens calls the Macdonald theory generalized and corresponds to the magnetic energy density $\int(\mathbf{C}d\mathbf{A})/c$, where $\mathbf{A}$ denotes the retarded vector potential. This form for the energy flux, like the current $\mathbf{C}$ itself, is transverse to the radius vector at an infinite distance from the electron, and therefore merely gives rise to a flow of energy along the wave-front and no radiation across it, being fully consistent with Bohr's hypothesis A.

4. Unfortunately, quite apart from the physical reasons adduced by Livens, there are two strong reasons for preferring the Poynting energy flux to all other definitions of it.

In the first place, the Poynting energy flux and the corresponding expression for the density of electromagnetic momentum occupy a prominent place in the Theory of Relativity, and it is difficult to see how such an expression for the energy flux as that derived from the Macdonald theory generalized can be used effectively in this connexion.

Again, Liénard's expression for the radiation is perfectly consistent with the electron mechanics founded on the accepted equations of the Electron theory, and this fact constitutes a strong reason for preferring the Poynting flux, which is presupposed by Liénard's expression.

* Livens, Phil. Mag. ser. 6, vol. xxxiv. p. 386 (1917).

It may be objected that the electron mechanics itself presupposes the classical expressions for the energy densities and the Poynting flux corresponding to them, so that its agreement with Liénard's expression is to be expected *à priori* and cannot be used as an argument for the correctness of their common foundations.

5. It is indeed true that Abraham * in his classical deduction of electron mechanics uses the Lagrangian method with the classical expressions for the energy densities, the Poynting flux and the corresponding expression for the density of electromagnetic momentum, but none of them are really essential for the deduction and they are used merely for shortening the work. The true foundations of the theory are the equations of the electromagnetic field, the expression of Lorentz and Larmor for the mechanical force on a moving charge, and the definitions of force and mass afforded by Newton's first and second law of motion. Using these as a basis I have shown elsewhere † that the equations of motion of the electron can be obtained by direct integration over the space occupied by it in the form of series, which proceed according to ascending powers of a length determining the linear dimensions of the electron, with coefficients formed of multiple integrals extending over the electron and depending on its velocity, acceleration, and the other quantities determining its motion. The first approximation gives the electromagnetic momentum and mass; the second gives in addition the reaction due to radiation in the form first found by Abraham ‡ by an indirect method.

For our purpose it is important to observe that the equation of energy obtained from these equations of motion enables us to define the kinetic energy of the electron and the radiation from it without any assumptions as to the proper expressions for the energy densities of the surrounding field and the energy flux. In fact, the reaction due to radiation consumes work irreversibly as well as reversibly; the rate at which it consumes work irreversibly is given exactly by Liénard's expression for the radiation. Hence this expression is proved independently to be consistent with the ordinary electron mechanics, and the same thing follows for the Poynting flux.

6. Thus we are driven to consider the second alternative

* Abraham, *Ann. d. Phys.* ser. 4, vol. x. p. 105 (1903). French translation in Abraham et Langevin, 'Ions, Electrons, Corpuscules,' p. 1 (1905).

† Schott, 'Electromagnetic Radiation,' Ch. XI. and App. C, D, and F.

‡ Abraham, *Elektromagnetische Theorie der Strahlung*, p. 123 (1905).

and inquire whether it is possible to modify the equations of the electromagnetic field in such a way as to annul the radiation derived from the Poynting energy flux. We cannot change the Maxwell-Hertz equations for the distant field without risking the loss of the accepted theory of electromagnetic waves together with all that it implies; but perhaps it may prove possible to modify the electromagnetic equations for the interior of the electron so as to attain the desired object. The retarded potentials do depend on the form of the equations within this region, and there appears to be no reason *à priori* why a suitable modification should not enable us to annul the radiation from the electron. Only a calculation can decide, and we shall proceed to carry it out; it may, however, be stated at once that the result will prove to be negative.

We shall make use of a method of calculating the radiation on the basis of electromagnetic equations modified for the interior of the electron, together with the Poynting flux, a method developed for another purpose by Oseen *. For the outside space we write as usual

$$\operatorname{curl}\mathbf{h}-\frac{\partial\mathbf{d}}{c\partial t}=0,\quad \operatorname{curl}\mathbf{d}+\frac{\partial\mathbf{h}}{c\partial t}=0,\quad \operatorname{div}\mathbf{d}=0,\quad \operatorname{div}\mathbf{h}=0,$$

where **d** and **h** denote the electric force and magnetic force, and c the speed of light. For the interior of the electron we write

$$\operatorname{curl}\mathbf{h}-\frac{\partial\mathbf{d}}{c\partial t}=\mathbf{C},\quad \operatorname{curl}\mathbf{d}+\frac{\partial\mathbf{h}}{c\partial t}=\mathbf{K},\quad \operatorname{div}\mathbf{d}=\epsilon,\quad \operatorname{div}\mathbf{h}=\mu. \quad (1)$$

The equations (1) may be regarded as defining the scalar quantities ϵ and μ and the vector quantities **C** and **K** for the interior of the electron, and every part of space where they do not all vanish is to be regarded as part of an electron. They may be interpreted as densities of electric and magnetic charges and currents, but this must be regarded as a matter of terminology. In the usual electron theory we have

$$\mu=0,\quad \mathbf{C}=\epsilon\mathbf{v}/c,\quad \mathbf{K}=0,$$

where ϵ denotes the electric density of any element of the electron, and **v** its velocity. These relations are not supposed to hold in the present investigation, and in fact the only relations which will be assumed to subsist between the four quantities ϵ, μ, **C**, and **K** are the following two

* Oseen, *Ann. d. Phys.* ser. 4, vol. xliii. p. 639 (1914).

obtained by eliminating **d** and **h** between equations (1):

$$\frac{\partial \epsilon}{c\partial t} + \operatorname{div} \mathbf{C} = 0, \qquad \frac{\partial \mu}{c\partial t} - \operatorname{div} \mathbf{K} = 0. \quad . \quad . \quad . \quad (2)$$

In all other respects the four quantities are to be regarded as completely arbitrary, so that there is no limitation of the generality of the electromagnetic equations for the interior of the electron. For convenience of analysis we shall suppose that surface distributions are to be treated as limiting cases of surface layers in the usual way, so that the electric and magnetic forces, **d** and **h**, are continuous everywhere together with their first differential coefficients.

7. Using the Poynting energy flux Oseen derives the following expression for the radiation across any fixed surface S enclosing the radiating system:—

$$\left.\begin{aligned} &16\pi^2 c\mathrm{R} = \iint \mathrm{U}^2 d\Omega, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \\ &\text{where } \mathbf{U} = \iint \left\{ \left[\left[\frac{\partial \mathbf{h}}{\partial t} d\mathbf{S} \right] \mathbf{l} \right] + \left[\frac{\partial \mathbf{h}}{\partial t} d\mathbf{S} \right] - \left(\frac{\partial \mathbf{d}}{\partial t} d\mathbf{S} \,.\, \mathbf{l} \right) \mathbf{l} \right\} \\ &\text{and} \qquad t = t_1 + (\mathbf{rl})/c. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \end{aligned}\right\} (3)$$

Square brackets denote vector multiplication, round scalar multiplication as usual, $d\Omega$ denotes an element of solid angle in the direction of the unit vector **l**, **r** denotes the radius vector drawn from the origin to the vector element of surface $d\mathbf{S}$, whilst t_1 is a constant time, the same for all surface elements, and t a time varying from element to element as indicated. The units are Lorentz units.

By means of (1) we can express the vector **U** in the following more convenient form:

$$\mathbf{U} = \left[\mathbf{l} \iiint \left\{ \frac{\partial \mathbf{C}}{\partial t} + \left[\mathbf{l} \frac{\partial \mathbf{K}}{\partial t} \right] \right\}_t d\mathrm{V} \right], \quad . \quad . \quad . \quad (4)$$

where $d\mathrm{V}$ is an element of volume of the electron, that is, of the region throughout which **C**, or **K**, or both differ from zero.

8. In applying Oseen's formulæ (3) and (4) to our problem we shall find it convenient to use cylindrical coordinates (z, ϖ, χ) and to choose c, the speed of light, as the unit of velocity to save writing.

Let θ denote the colatitude and ϕ the longitude of the unit vector **l**; then the time t is given by the equation

$$t = t_1 + z\cos\theta + \varpi \sin\theta \cos(\phi - \chi). \quad . \quad . \quad . \quad . \quad (5)$$

Moreover, denoting vector components in the directions l, θ, and ϕ by subscripts, and similarly for other directions, we find

$$U_l=0, \quad U_\theta=-\int\left\{\frac{\partial C_\phi}{\partial t}+\frac{\partial K_\theta}{\partial t}\right\}dV, \quad U_\phi=\int\left\{\frac{\partial C_\theta}{\partial t}-\frac{\partial K_\phi}{\partial t}\right\}dV. \quad . \quad . \quad . \quad (6)$$

Lastly, as we shall have to deal with periodic motions involving one or more incommensurable periods, we shall calculate the average radiation by means of the equation

$$16\pi^2 TR=\int_0^T\int_0^\pi\int_0^{2\pi}\left\{U_\theta^2+U_\phi^2\right\}\sin\theta\, d\theta\, d\phi\, dt_1, \quad . \quad (7)$$

where T denotes the period, when the motion is mono-periodic, and an interval of time long compared with the longest period when it is polyperiodic. This equation is easily derived from the first equation (3).

9. Since the coordinate χ is the longitude, the quantities **C** and **K**, whatever their nature may be, must of necessity be periodic functions of χ with the period 2π. For the sake of generality we shall assume that they are also sums of periodic functions of the time, whose periods are not necessarily commensurable. Hence we may expand the components of **C** and **K** in series of exponentials of the longitude χ and the time t of the form

$$\left.\begin{aligned}\{C_z,\ C_\varpi,\ C_\chi\}&=\sum_{-\infty}^{\infty}\sum_{-\infty}^{\infty}\{C_1(j,k),\ C_2(j,k),\ C_3(j,k)\}\exp\iota\{jt+k\chi\},\\ \{K_z,\ K_\varpi,\ K_\chi\}&=\sum_{-\infty}^{\infty}\sum_{-\infty}^{\infty}\{K_1(j,k),\ K_2(j,k),\ K_3(j,k)\}\exp\iota\{jt+k\chi\},\end{aligned}\right\}\ (8)$$

where k is restricted to integral values, whilst j is not, but may take any values, whether they be commensurable with each other or not, whilst the coefficients C_1, &c., are explicit functions of z and ϖ, but not of χ or t.

10. For the sake of brevity we shall write

$$\psi=\chi-\phi-\pi/2, \qquad \tau=jt_1+k(\phi+\pi/2). \quad . \quad . \quad (9)$$

With these expressions we may write (5) in the form

$$jt+k\chi=\tau+jz\cos\theta+k\psi-j\varpi\sin\theta\sin\psi. \quad . \quad (10)$$

Substituting from (8) and (10) in (6), omitting the parameters j, k from the coefficients C_1, K_1, &c., as no longer necessary and putting for the volume element dV its value $\varpi dz d\varpi d\psi$, we obtain in succession

$$\frac{\partial C_\theta}{\partial t} = -\Sigma\Sigma\, \iota j\{C_1 \sin\theta + (C_2 \sin\psi + C_3 \cos\psi)\cos\theta\}$$
$$\exp \iota\{\tau + jz\cos\theta + k\psi - j\varpi \sin\theta \sin\psi\},$$

$$\frac{\partial C_\phi}{\partial t} = -\Sigma\Sigma\, \iota j\{-C_2 \cos\psi + C_3 \sin\psi\}$$
$$\exp \iota\{\tau + jz\cos\theta + k\psi - j\varpi \sin\theta \sin\psi\},$$

with similar expressions for $\partial K_\theta/\partial t$ and $\partial K_\phi/\partial t$; and also

$$\left.\begin{aligned} U_\theta = {} & \Sigma\Sigma \exp \iota\tau \,.\iiint \{-C_2\cos\psi + C_3 \sin\psi + K_1 \sin\theta \\ & + (K_2 \sin\psi + K_3\cos\psi)\cos\theta\} \,.\, \iota j\varpi \\ & .\exp\iota\{jz\cos\theta + k\psi - j\varpi\sin\theta\sin\psi\} dz\, d\varpi\, d\psi, \\ U_\phi = {} & -\Sigma\Sigma \exp\iota\tau \,.\iiint\{C_1 \sin\theta + (C_2\sin\psi \\ & + C_3\cos\psi)\cos\theta + K_2\cos\psi - K_3 \sin\psi\} \,.\, \iota j\varpi \\ & .\exp\iota\{jz\cos\theta + k\psi - j\varpi\sin\theta\sin\psi\} dz\, d\varpi\, d\psi \end{aligned}\right\} . \quad (11)$$

On the assumption which we have made the limits for ψ are 0 and 2π ; hence we find from (11) with the usual notation for Bessel Functions of order k,

$$\left.\begin{aligned} U_\theta = 2\pi\Sigma\Sigma \exp\iota\tau \,. & \iint\Big\{\iota\Big(K_1 j\varpi\sin\theta + \frac{K_3\cos\theta - C_2}{\sin\theta}k\Big)J_K(j\varpi\sin\theta) \\ & + (C_3 + K_2\cos\theta) j\varpi J_K'(j\varpi\sin\theta)\Big\}\exp \iota jz\cos\theta \,.\, d\varpi dz, \\ U_\phi = 2\pi\Sigma\Sigma \exp\iota\tau \,. & \iint\Big\{-\iota\Big(C_1 j\varpi\sin\theta + \frac{C_3\cos\theta + K_2}{\sin\theta}k\Big)J_K(j\varpi\sin\theta) \\ & + (K_3 - C_2\cos\theta) j\varpi J_K'(j\varpi\sin\theta)\Big\}\exp\iota jz\cos\theta \,.\, d\varpi dz, \end{aligned}\right\} \quad (12)$$

where the summations with respect to j and k are from $-\infty$ to ∞ as before, and we must remember that k is an integer, whilst j is not necessarily so. The limits for ϖ and z are determined by the form of the cross-section of the ring-shaped region swept out by the electron in its motion ; each element of charge has been implicitly supposed to describe a circle with its centre on the axis of z, otherwise we should have had to assume ϖ and z to depend on χ in an assigned manner. The motion is not, however, assumed to be uniform.

11. We now substitute from (12) in (7) and integrate with respect to t_1 from 0 to T and with respect to ϕ from 0 to 2π. All terms vanish except products of terms of U_θ and U_ϕ involving reciprocal time factors $\exp \iota\tau$ and $\exp(-\iota\tau)$, and these acquire the factor $2\pi T$, which cancels out from (7). From (9) we see that these pairs of terms correspond to equal and opposite pairs of values of j and k; clearly the reality of the motion requires that the coefficients C_1, &c., belonging to such pairs should be conjugate imaginaries. Moreover, the signs and magnitudes of the functions $J_K(j\varpi \sin\theta)$ and $J_K'(j\varpi \sin\theta)$ both remain unaltered when the signs of both j and k are changed. Under these circumstances the complex integrals in (12) change into their conjugate imaginaries, and each integral when multiplied by its conjugate gives a term of R.

In order to express these terms explicitly we write

$$\left.\begin{array}{ll} C_1(j,\,k)=A_1+\iota B, & C_1(-j,\,-k)=A_1-\iota B_1, \\ K_1(j,\,k)=L_1+\iota M_1, & K_1(-j,\,-k)=L_1-\iota M_1, \end{array}\right\} . \quad (13)$$

with similar equations for the remaining coefficients. Then the double integral in the first equation (12) becomes

$$\iint\Big[\Big\{(A_3+L_2\cos\theta)j\varpi J_K'(j\varpi\sin\theta) - \left(M_1 j\varpi\sin\theta+\frac{M_3\cos\theta-B_2}{\sin\theta}k\right)J_K(j\varpi\sin\theta)\Big\} + \iota\Big\{(B_3+M_2\cos\theta)j\varpi J_K'(j\varpi\sin\theta) + \left(L_1 j\varpi\sin\theta+\frac{L_3\cos\theta-A_2}{\sin\theta}k\right)J_K(j\varpi\sin\theta)\Big\}\Big] . \exp \iota jz\cos\theta \,.\, d\varpi\, dz. \quad (14)$$

For the conjugate we must of course change the sign of ι, but in addition it is convenient to replace the variables of integration, z, ϖ, by z', ϖ', the coordinates of a second element of the electron, and to write in place of the coefficients A_1, &c., the corresponding functions A_1', &c., of the new variables z', ϖ'.

On multiplying corresponding conjugate integrals together we obtain a fourfold integral with respect to the four variables z, ϖ, z', and ϖ', the integrations with respect to z and ϖ, as well as with respect to z' and ϖ', being extended over the area of the meridian plane swept over by the electron in its motion. This fourfold integral is itself complex, but the conjugate integral is obtained by changing the signs of the parameters j and k, and the two fourfold integrals together contribute their real part only to the radiation.

Treating the terms arising from U_ϕ in the same way we find

$$
\begin{aligned}
R = \tfrac{1}{2}\pi\Sigma\Sigma\int_0^\pi\iiiint \Bigg[& \Bigg\{ (A_3 + L_2\cos\theta) j\varpi J_K'(j\varpi\sin\theta) \\
& - \left(M_1 j\varpi\sin\theta + \frac{M_3\cos\theta - B_2}{\sin\theta}k\right) J_K(j\varpi\sin\theta) \Bigg\} \\
& . \Bigg\{ (A_3' + L_2'\cos\theta) j\varpi' J_K'(j\varpi'\sin\theta) \\
& - \left(M_1' j\varpi'\sin\theta + \frac{M_3'\cos\theta - B_2'}{\sin\theta}k\right) J_K(j\varpi'\sin\theta) \Bigg\} \\
& + \Bigg\{ (B_3 + M_2\cos\theta) j\varpi J_K'(j\varpi\sin\theta) \\
& + \left(L_1 j\varpi\sin\theta + \frac{L_3\cos\theta - A_2}{\sin\theta}k\right) J_K(j\varpi\sin\theta) \Bigg\} \\
& . \Bigg\{ (B_3' + M_2'\cos\theta) j\varpi' J_K'(j\varpi'\sin\theta) \\
& + \left(L_1' j\varpi'\sin\theta + \frac{L_3'\cos\theta - A_2'}{\sin\theta}k\right) J_K(j\varpi'\sin\theta) \Bigg\} \\
& + \Bigg\{ (L_3 - A_2\cos\theta) j\varpi J_K'(j\varpi\sin\theta) \\
& + \left(B_1 j\varpi\sin\theta + \frac{B_3\cos\theta + M_2}{\sin\theta}k\right) J_K(j\varpi\sin\theta) \Bigg\} \\
& . \Bigg\{ (L_3' - A_2'\cos\theta) j\varpi' J_K'(j\varpi'\sin\theta) \\
& + \left(B_1' j\varpi'\sin\theta + \frac{B_3'\cos\theta + M_2'}{\sin\theta}k\right) J_K(j\varpi'\sin\theta) \Bigg\} \\
& + \Bigg\{ (M_3 - B_2\cos\theta) j\varpi J_K'(j\varpi\sin\theta) \\
& - \left(A_1 j\varpi\sin\theta + \frac{A_3\cos\theta + L_2}{\sin\theta}k\right) J_K(j\varpi\sin\theta) \Bigg\} \\
& . \Bigg\{ (M_3' - B_2'\cos\theta) j\varpi' J_K'(j\varpi'\sin\theta) \\
& - \left(A_1' j\varpi'\sin\theta + \frac{A_3'\cos\theta + L_2'}{\sin\theta}k\right) J_K(j\varpi'\sin\theta) \Bigg\} \Bigg] \\
& . \cos\{j(z-z')\cos\theta\} \,.\, d\varpi\, dz\, d\varpi'\, dz' \sin\theta\, d\theta. \quad . \quad . \quad . \qquad (15)
\end{aligned}
$$

The summations must be taken for all positive and negative values of j and k, and both sets of integrations with respect to ϖ, z and ϖ', z' over the area swept out by the electron.

12. It is not difficult to reduce the Bessel Function integrals in (15) to simpler forms, but the results are complicated and not easy to interpret. Fortunately this reduction is not necessary for our purpose owing to the smallness of the electron. If ρ denote the radius of the circle described by the centre, and a a length comparable with the linear dimensions of the electron, for instance its radius if assumed spherical, which is, however, not necessary, then the coordinates ϖ and ϖ' differ from ρ by quantities of the order a, and $z-z'$ is of the same order of smallness. Thus, if we suppose the integrand of (15) expanded in a series of terms of increasing order of smallness, the first term will determine the sign of the radiation R, unless it should happen to vanish. To find this principal term we need only put ϖ and ϖ' each equal to ρ and $z-z'$ equal to zero in the Bessel Function and cosine terms respectively, but we shall refrain from doing this in the coefficients A_1, &c., because we know nothing as to their form. For the sake of brevity we shall write

$$a_1(j,\, k)=\iint A_1(j,\, k)\, d\varpi\, dz=\iint A_1'(j,\, k)\, d\varpi'\, dz' \quad . \ (16)$$

for all values of j and k, with similar expressions for the integrals of the remaining coefficients. Then we find to a first approximation

$$\begin{aligned} R=\tfrac{1}{2}\pi\Sigma\Sigma\int_0^{\pi}\Bigg[& \left\{ (a_3+l_2\cos\theta)j\rho J_K'(j\rho\sin\theta) - \left(m_1 j\rho\sin\theta+\frac{m_3\cos\theta-b_2}{\sin\theta}k\right)J_K(j\rho\sin\theta)\right\}^2 \\ & + \left\{ (b_3+m_2\cos\theta)j\rho J_K'(j\rho\sin\theta) + \left(l_1 j\rho\sin\theta+\frac{l_3\cos\theta-a_2}{\sin\theta}k\right)J_K(j\rho\sin\theta)\right\}^2 \\ & + \left\{ (l_3-a_2\cos\theta)j\rho J_K'(j\rho\sin\theta) + \left(b_1 j\rho\sin\theta+\frac{b_3\cos\theta+m_2}{\sin\theta}k\right)J_K(j\rho\sin\theta)\right\}^2 \\ & + \left\{ (m_3-b_2\cos\theta)j\rho J_K'(j\rho\sin\theta) - \left(a_1 j\rho\sin\theta+\frac{a_3 j\rho\cos\theta+l_2}{\sin\theta}k\right)J_K(j\rho\sin\theta)\right\}^2\Bigg] \\ & .\sin\theta\, d\theta. \quad . \quad . \quad . \quad . \quad . \quad (17) \end{aligned}$$

The new coefficients a_1, &c., are all real constants; hence each term of the integrand of (17) is essentially positive throughout the whole range of integration, the only exception being when j vanishes, in which case each term vanishes identically. Therefore the principal term (17) in the radiation R can only vanish if each of the four functions inside the curly brackets vanishes identically for every value of θ between 0 and π, and for every pair of values of j and k excepting only $j=0$.

13. There are two possible cases :—

(1) $\rho=0$.

The principal term in R vanishes identically whatever values the coefficients a_1, &c., may have, and the radiation becomes small of order a at least. This occurs for a uniform spherical electron rotating, or even oscillating about a diameter, a case already considered by Herglotz and Sommerfeld *, provided only that the period of oscillation be properly adjusted, and the electron have a surface-charge; also under similar conditions for a pair of spherical electrons oscillating about a common diameter (Oseen, *loc. cit.* p. 646); and lastly, as is well known, for an axially symmetrical system rotating uniformly about its axis. In all these examples, however, the centre of the electron remains at rest, and consequently not one of them has any bearing on Bohr's theory.

14. (2) $a_1=0$, &c., for all pairs of values of j, k except $j=0$.

We can express these conditions more conveniently by multiplying (8) by $d\varpi dz$, integrating over the area of the meridian plane swept through by the electron and using (13) and (16). The only terms left in the result are those for which $j=0$, and these are independent of the time t, so that we obtain

$$\left.\begin{aligned}&\iint\{C_z,\ C_\varpi,\ C_\chi\}\,d\varpi\,dz\\&\qquad=\sum_{-\infty}^{\infty}\{a_{10}+\iota b_{10},\ a_{20}+\iota b_{20},\ a_{30}+\iota b_{30}\}\exp \iota k\chi\\&\iint\{K_z,\ K_\varpi,\ K_\chi\}\,d\varpi\,dz\\&\qquad=\sum_{-\infty}^{\infty}\{l_{10}+\iota m_{10},\ l_{20}+\iota m_{20},\ l_{30}+\iota m_{30}\}\exp \iota k\chi\end{aligned}\right\}. \quad (18)$$

The zero suffix indicates that in each coefficient $j=0$, whilst k takes all integral values between $\pm\infty$.

* Sommerfeld, *Gött. Nach.* p. 431 (1904).

Corresponding conditions can be obtained for ϵ and μ by multiplying (2) by $\varpi d\varpi dz$, integrating over the area swept out by the electron, employing the usual expression for the operator div in cylindrical coordinates, and bearing in mind that **C** and **K** vanish at the surface of the electron by definition. Using (18) we find

$$\frac{\partial}{\partial t}\iint \epsilon\varpi\, d\varpi\, dz = -c\iint\left\{\frac{\partial\varpi C_z}{\partial z} + \frac{d\varpi C_\varpi}{\partial\varpi} + \frac{\partial C_\chi}{\partial\chi}\right\} d\varpi\, dz$$

$$= c\sum_{-\infty}^{\infty}(b_3 - \iota a_3)k \exp \iota k\chi,$$

with a similar equation for μ. In order that the integral involving ϵ, μ may not increase indefinitely with the time, which is clearly physically impossible, we must have

$$a_{30} = b_{30} = l_{30} = m_{30} = 0 \text{ for all integral values of } k. \quad (19)$$

Also performing the integrations and indicating initial values of ϵ, μ by a zero suffix we obtain

$$\iint \epsilon\varpi\, d\varpi\, dz = \iint \epsilon_0\varpi\, d\varpi\, dz, \quad \iint \mu\varpi\, d\varpi\, dz = \iint \mu_0\varpi\, d\varpi\, dz. \quad (20)$$

15. The conditions (18), (19), and (20) may be interpreted as follows:—

The radiation from an electron, which either moves uniformly or executes an oscillatory to and fro motion in a circular path, or from a system of electrons, which move in this manner in coaxal circular paths, can only vanish to a first approximation when the following conditions are satisfied:

(1) The mean values of the electric and magnetic currents for the whole area of a meridian plane swept out by the electron, or electrons, must be independent of the time, but may vary from one meridian plane to another. (2) The components of the currents perpendicular to the meridian plane must vanish on the average for each meridian plane and at each instant. (3) The mean values of the electric and magnetic densities for the whole area swept out by the electron, or electrons, on any meridian plane must be independent of the time, but may vary from one meridian plane to another.

Obviously these conditions cannot possibly be satisfied for any discontinuous distribution of charge in circular motion about an axis common to the whole distribution, such as a single electron moving in a circle of radius large compared with its own, or a stream of electrons following each other round such a circle in succession at distances apart large compared with their radii. If the radius of the path, or the distances between the electrons of a stream, were of the order of the electronic radius, the radiation would also be small; it would of course vanish if the stream of electrons coalesced into an anchor ring revolving about its axis, such as the Parson magneton, but not if the ring moved as a whole with acceleration.

Thus we see that for any discontinuous distribution of electrons, of the kind with which we are concerned in an atomic model such as is contemplated in Bohr's theory, radiation unavoidably results if we adopt the equations of Maxwell and Hertz for the field at a distance from an electron, together with Poynting's formula for the energy flux, at any rate for electrons moving in coaxal circles. When the paths are not circular, so that there is generally a tangential as well as a normal acceleration, we have every reason to suppose that the radiation is increased on that account, and there is little doubt that a formal proof could be given, although it would be much more complicated. There would be no alteration needed as far as § 11 and equation (12), but the approximation used thereafter would no longer apply, because the values of z, ϖ, z', ϖ' would no longer be restricted to a small area of the meridian plane for the single electron or stream, or several such areas for a system, but would be spread over a finite area bounded by the extreme values of these coordinates reached during the orbital motion.

16. Before proceeding to a consideration of the changes needed in the fundamental assumptions in order to remove the contradiction that we have found, we shall consider the case of a uniform spherical electron, which moves in a circle of any finite radius with uniform speed. We shall adopt the conventions of the accepted electron theory and write $\mathbf{C}=\epsilon\mathbf{v}$, $\mathbf{K}=0$. Hence we have $C_\chi=\epsilon v=3e\omega\varpi/4\pi a^3$ for all points which lie inside the electron at the time t, and $C_\chi=0$ for all outside points, whilst all the other components C_z, C_ϖ, K_z, K_ϖ, K_χ vanish.

In order to express this condition more precisely let us suppose the longitude of the centre of the electron at time t to be ωt; then C_χ is a function of the variables $\chi-\omega t$, z, and ϖ, and vanishes unless $\chi-\omega t$ lies between the limits $\pm\alpha$, where α is the least positive angle given by

$$\left.\begin{aligned} a^2&=z^2+\varpi^2+\rho^2-2\rho\varpi\cos\alpha, \\ \sin\tfrac{1}{2}\alpha&=\sqrt{\{a^2-z^2-(\varpi-\rho)^2\}}/2\sqrt{(\varpi\rho)}. \end{aligned}\right\} \quad . \quad (21)$$

We easily find by Fourier's method

$$C_\chi=\sum_{-\infty}^{\infty}\frac{3e\omega\varpi\sin k\alpha}{4\pi^2a^3k}\exp\iota k(\chi-\omega t). \quad . \quad . \quad (22)$$

Comparing (22) with (8) we see that all the coefficients vanish except $C_3(j, k)$, which takes the real value

$$3e\omega\varpi\sin k\alpha/4\pi^2a^3k \text{ when } j=-k\omega,$$

but vanishes for all other values of j. Using (13) and (16) we find

$$a_3=\frac{3e\omega}{4\pi^2a^3k}\iint\sin k\alpha\,.\,\varpi d\varpi dz \text{ for } j=-k\omega,$$

$$=0 \text{ for all other values of } j,$$

whilst all the other coefficients b_3, &c., vanish identically.

To evaluate the integral we put

$$f^2=(\varpi+\rho)^2+z^2,\quad g^2=(\varpi-\rho)^2+z^2;$$

neglecting higher powers of a/ρ we obtain from (21) and (22)

$$\left.\begin{aligned} a_3&=\frac{3e\omega\rho}{4\pi^2a^3k}\int_0^a\int_{2\rho-g}^{2\rho+g}\frac{\sin\{k\sqrt{(a^2-g^2)/\rho}\}gdfdg\}}{\sqrt{\{g^2-(f-2\rho)^2\}}}\\ &=\frac{e\omega}{4\pi}\left\{3\frac{\sin\gamma-\gamma\cos\gamma}{\gamma^3}\right\},\text{ where } \gamma=ka/\rho.\end{aligned}\right\} \quad (23)$$

Since $j=-k\omega$, and all coefficients vanish except a_3, we find from (17) on putting $\omega\rho=\beta$, the unit of speed being still that of light,

$$R=\pi\sum_1^{\infty}a_3^2k^2\int_0^{\pi}\left[\beta^2\{J_K'(k\beta\sin\theta)\}^2+\cot^2\theta\,.\,\{J_K(k\beta\sin\theta)\}^2\right]\sin\theta d\theta.$$

The coefficient of a_3^2 in the sum is equal to twice the integral I_1, introduced and evaluated by me elsewhere * with $l=m=k$; substituting its value and using (23) we obtain

$$R=\frac{e^2\omega^2}{8\pi\beta}\sum_1^{\infty}\left\{3\frac{\sin\gamma-\gamma\cos\gamma}{\gamma^3}\right\}^2\left[k\beta^2J_{2K}'(2k\beta)-k^2(1-\beta^2)\int_0^{\beta}J_{2K}(2kx)dx\right]. \quad (24)$$

In order to convert to electrostatic units we must replace

* 'Electromagnetic Radiation,' pp. 136, 137.

e by $e\sqrt{4\pi}$, ω by β/ρ and introduce c as a factor; then we obtain one-fourth of the value given elsewhere *, which is apparently due to an error in Oseen's equation for R, the first equation (3), where the factor 16 should be 4; this error, however, does not affect our argument, and the agreement in form, apart from the trigonometrical factor in (24), verifies the substantial correctness of the expression (17) for the radiation.

When γ, i.e. ka/ρ, is small, the trigonometrical factor is practically unity, but when k is comparable with ρ/a, i.e. of the order 50,000, this is no longer true. Its presence ensures the convergence of the series for all real values of β. For a surface charge the trigonometrical factor becomes

$$\{\sin(ka/\rho)/(ka/\rho)\}^2,$$

but this does not suffice to secure convergence when β exceeds unity †.

17. By the method of the last section we can also estimate the error committed in the present example by the approximation used in obtaining (17) from (15). To obtain an estimate we put $\varpi=\varpi'=\rho+a$ in the Bessel Function factors, and $z-z'=a$ in the cosine in (15). Expanding in powers of a/ρ and retaining only the first power in addition to the principal term represented by (17), we see that the cosine term contributes nothing to this order, whilst the Bessel Function factors contribute additional terms, which in the present example reduce to

$$\frac{2\pi a}{\rho}\sum_1^\infty a_{\frac{2}{3}}^2 k^3\beta\int_0^\pi\left[1-\beta^2\sin^2\theta+\cos^2\theta\right] \times J_K(k\beta\sin\theta)J_K'(k\beta\sin\theta)\,d\theta.$$

The coefficient of $a_{\frac{2}{3}}^2$ in the sum is $2k$ times the integral I_3 introduced and evaluated elsewhere, with $l=m=k$; using this value together with (23) we obtain for the additional term

$$\frac{e^2\omega^2 a}{8\pi\beta\rho}\sum_1^\infty\left\{3\frac{\sin\gamma-\gamma\cos\gamma}{\gamma^3}\right\}^2 k^2\Big[(1-\beta^2)J_{2K}(2k\beta) + (1+\beta^2)\int_0^\beta J_{2K}(2kx)\,dx\Big].$$

The form of this expression is quite similar to that of (24), and the same properties may presumably be predicated of it

* *Loc. cit.* p. 110.

† For these convergence results I am indebted to Prof. G. N. Watson.

as regards convergence, so that we conclude that the error committed in using (24) as a first approximation is relatively only of the order a/ρ.

18. Returning to the consideration of our main problem we must insist particularly on the fact that the contradiction we have found subsists between hypothesis A in the form adopted by Bohr on the one hand, and the four electrodynamic equations of Maxwell and Hertz for space at a distance from all electric charges, together with the Poynting energy flux, that is to say, not merely with the fundamental equations of the modern electron theory, but with those of the classical electromagnetic theory for the free æther, formulated by Maxwell and established by the experiments of Hertz and all the experience of wireless telegraphy. Doubtless no one will be willing to renounce so useful a theory as this is until much stronger reasons are forthcoming; hence there only remains the choice between the Poynting energy flux, together with the classical expressions for the electric and magnetic energies and the electromagnetic momentum which it implies, and Hypothesis A in its present form. Although the position as regards the Poynting flux is not so clear as that respecting the theory of Maxwell, yet, as we have seen above, there are very strong reasons for retaining the Poynting flux, so that it becomes necessary to consider the possibility of modifying Bohr's hypothesis A. After mature consideration the following wording has suggested itself to me as one which is sufficient for our purpose and at the same time satisfies the requirements of Bohr's theory in all essential respects :—

A. An atomic system possesses a number of states in which its electromagnetic energy continues unchanged, even if the particles are in motion and an emission of energy radiation is to be expected on ordinary electrodynamics. The states are denoted as the states of stationary motion of the system under consideration.

19. It will be seen that here the stress is laid on the constancy of the electromagnetic energy in spite of radiation, instead of on the total absence of emission of energy radiation. The emission of energy radiation in consequence of acceleration is supposed to take place continuously, not in quanta, and this may be objected to as contradicting the quantum hypothesis assumed by Bohr for the series spectrum emission in his hypothesis B. In reply, it may be urged that the original form of the quantum theory, in which all energy was assumed to occur only in quanta, has been abandoned

by most physicists; wherever emission in quanta occurs, it is attributed to some cause arising from the constitution of the atom rather than that of the radiant energy itself, a position taken up by Barkla * in his recent lecture on X-ray phenomena. We have sufficient reason for supposing that the emission of spectrum series is a process of a very special kind, to which the quantum hypothesis may perhaps be peculiarly applicable, whilst it may not hold for the ordinary emission of energy radiation which we have found to accompany all motions of electric charges involving acceleration.

20. It should be noticed that in our restatement of hypothesis A the constancy of the electromagnetic energy of the electron is expressly postulated, in spite of the fact that I have myself mentioned elsewhere two possible internal electromagnetic sources of energy from which the radiant energy might conceivably be derived. It is, however, easily shown that neither of these sources is available when we adopt Bohr's theory.

The first source of this kind is the acceleration energy, as I have called it elsewhere †, which is equal to

$$-2e^2\beta\dot{\beta}/3c(1-\beta^2)^2.$$

But in a stationary motion, such as is postulated in Bohr's theory, secular changes of β, $\dot{\beta}$ are clearly excluded, so that the acceleration energy cannot undergo any such change and therefore cannot supply the energy radiated.

The second source is the electrostatic energy of the electron, which can be tapped when the electron suffers a secular expansion, as I have shown elsewhere. But in this case the motion is only quasistationary and is subject to a secular variation, which however is much too fast to be reconcilable with the remaining hypotheses of Bohr's theory, in particular with Nicholson's hypothesis of constant angular momentum. In order to prove this we shall make use of the equations of motion of the electron ‡, adapted to the case of a fixed equal positive charge, but shall neglect the assumed small secular changes of the speed β and the radius of curvature of the path ρ wherever they occur in the small radiation

* Barkla, Proc. Roy. Soc. vol. xcii. A. p. 504 (1916).

† Schott, Phil. Mag. ser. 6, vol. xxix. p. 49 (1915); 'Electromagnetic Radiation,' p. 177.

‡ Schott, 'Electromagnetic Radiation,' pp. 188–192; *loc. cit.* p. 179, eq. (219).

terms. We find for a plane nearly circular orbit

$$\frac{dcm\beta}{dt}+\frac{2e^2\beta^3}{3\rho^2(1-\beta^2)^2}=-\frac{e^2\cos\phi}{r^2},\quad \frac{c^2m\beta^2}{\rho}=\frac{e^2\sin\phi}{r^2}=\frac{e^2p}{r^3}, \quad . \quad . \quad . \quad (25)$$

where p denotes the perpendicular from the positive nucleus on the tangent, and ϕ the angle between that tangent and the radius vector r. For the sake of brevity we shall write for the angular momentum of the electron

$$\mathrm{H}=cm\beta p. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (26)$$

Since $\rho=rdr/dp$, and $\dot{r}=c\beta\cos\phi$, we find from (25) and (26)

$$\frac{e^2\cos\phi}{r^2}=\frac{\mathrm{H}}{p^2}\frac{dp}{dt},\quad \frac{d\mathrm{H}}{dt}=-\frac{2e^2\beta^3p}{3\rho^2(1-\beta^2)^2}.$$

We have $r^2\dot{\theta}=c\beta p$; hence changing the variable from t to θ we find by means of (25) and (26)

$$3\mathrm{H}^2\frac{d\mathrm{H}}{d\theta}=-\frac{2e^6p^4}{c^3(1-\beta^2)^2r^4},\quad \mathrm{H}_0^3-\mathrm{H}^3=\frac{2e^6}{c^3}\int_0^\theta\frac{p^4d\theta}{(1-\beta^2)^2r^4}. \quad . \quad . \quad . \quad (27)$$

Thus the angular momentum of the electron, H, diminishes continually, instead of remaining constantly equal to $h/2\pi$, as it should do on the hypothesis adopted by Nicholson and Bohr. With the usual values of e and c we have

$$2e^6/c^3=8{\cdot}8\,.\,10^{-88},$$

whilst

$$h/2\pi=1{\cdot}05\,.\,10^{-27};$$

hence if H_0 be equal to $h/2\pi$, H will diminish to one-half of H_0 in about 180,000 revolutions, provided β^2 can be neglected, and the orbit is so nearly circular that we may put p/r equal to unity. So rapid a change of the angular momentum can hardly be regarded as consistent with stationary motion, and therefore the hypothesis of the expanding electron must for this reason alone be considered as incompatible with Bohr's theory.

The proof given here is extremely general; it assumes nothing whatever concerning the mass of the electron—its variation with the speed, or a possible secular change due to expansion or any other change of structure of the electron—or the force acting on it, beyond the fact that the latter must be central. Hence we cannot account for invariability of angular momentum by variation of mass, but must assume a force component in the direction of motion.

21. The considerations of § 20 make it clear that we cannot look to the electromagnetic energy of the electron itself as the source from which the energy lost by radiation is derived. There remain three possible sources to be discussed: (1) external electromagnetic energy, (2) internal nonelectromagnetic energy of the electron, and (3) external nonelectromagnetic energy; but the consideration of these sources must be reserved for a future communication.

In conclusion we may summarize the results of the present investigation as follows:—

(1) Bohr's hypothesis A is incompatible with the electromagnetic equations of Maxwell and Hertz, together with the Poynting energy flux for the free æther, at least for the case of uniform circular motion of the electron, and almost certainly for any other motion of translation.

(2) The hypothesis A can be rendered compatible by a restatement postulating no change in the electromagnetic energy of the electron in spite of the emission of radiant energy. For neither the acceleration energy, nor the electrostatic energy of an expanding electron, is available as a source of the radiant energy.

XXIII. *On Kirchhoff's Formulation of the Principle of Huygens.* *By* Prof. A. Anderson *.

THE usual method of establishing Kirchhoff's formula is to start with a function $V(x, y, z, t)$ of the co-ordinates of a point and the time, that satisfies the equation

$$\frac{d^2V}{dt^2} = a^2\nabla^2V,$$

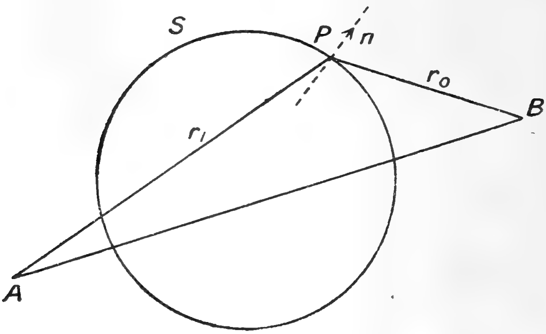
and to show that, if $t-\frac{r}{a}$ be written for t in V, we get a new function of x, y, z, r, and t, which satisfies a certain differential equation. A closed surface is then drawn bounding a space at every point of which the differential equation holds, and a point O is taken in this space. Both sides of the equation are then integrated throughout the space between the closed surface and the surface of a small sphere whose centre is O. This leads to an expression for the value of V at the point O in the

* Communicated by the Author.

form of an integral taken over the surface. The formula thus obtained can then be applied to the case of a single source or that of several sources of light emitting vibrations which travel through the æther with constant speed. It is, however, instructive and interesting to proceed differently, and begin with the case of a single source. It is assumed that the vibrational velocity or vibrational displacement at a distance r from a source of disturbance is equal to $\frac{M}{r}\phi\left(t-\frac{r}{a}\right)$, where M is a constant and a the velocity of propagation.

Let S, fig. 1, be a closed surface, and A, B two points outside it, and let r_1, r_0 be the distances of A and B from

Fig. 1.



an element dS of the surface at P, n being the outward drawn normal at that point. Many integrals whose subjects of integration depend on r_1 and r_0 and their rates of variation along the normal vanish when taken over the surface S. Thus, if $F_1(r_1, r_0)$ and $F_2(r_1, r_0)$ be two functions of r_1 and r_0 that are finite, continuous, and single-valued throughout the space S,

$$\iint_S \left(F_1\frac{\partial F_2}{\partial n}-F_2\frac{\partial F_1}{\partial n}\right)dS$$

will vanish if the integrand can be separated into a number of terms

$$U_1\frac{\partial V_1}{\partial n}-V_1\frac{\partial U_1}{\partial n},\quad U_2\frac{\partial V_2}{\partial n}-V_2\frac{\partial U_2}{\partial n},\quad U_3\frac{\partial V_3}{\partial n}-V_3\frac{\partial U_3}{\partial n},$$

&c., ... ,

where $U_1, U_2, U_3, \ldots, V_1, V_2, V_3, \ldots$ are such that the volume integral

$$\iiint[(U_1\nabla^2 V_1 - V_1\nabla^2 U_1) + (U_2\nabla^2 V_2 - V_2\nabla^2 U_2) + \ldots]\,dx\,dy\,dz$$

taken throughout the space S vanishes. This is merely a simple and obvious generalization of Green's theorem.

Consider now a function $\phi\left(t - \frac{r_1 + r_0}{a}\right)$, where t and a are for the present merely algebraic symbols, but which, subsequently, will be identified with the time and the velocity of propagation. We proceed to show that the surface integral

$$\iint\left[\left(\frac{1}{r_1^2 r_0}\frac{dr_1}{dn} - \frac{1}{r_0^2 r_1}\frac{dr_0}{dn}\right)\phi + \frac{1}{ar_1 r_0}\frac{d\phi}{dt}\left(\frac{dr_1}{dn} - \frac{dr_0}{dn}\right)\right]dS$$

vanishes when taken over the surface S.

Expanding the integrand by Taylor's theorem, we obtain for the surface integral the expression :—

$$\phi(t)\iint\left(\frac{1}{r_1^2 r_0}\frac{dr_1}{dn} - \frac{1}{r_0^2 r_1}\frac{dr_0}{dn}\right)dS$$

$$-\frac{1}{a}\frac{d\phi(t)}{dt}\iint\frac{1}{r_1 r_0}\left[\left(\frac{1}{r_1}\frac{dr_1}{dn} - \frac{1}{r_0}\frac{dr_0}{dn}\right)(r_1 + r_0) - \frac{dr_1}{dn} + \frac{dr_0}{dn}\right]dS$$

$$+\frac{1}{2a^2}\frac{d^2\phi(t)}{dt^2}\iint\frac{1}{r_1 r_0}\left[\left(\frac{1}{r_1}\frac{dr_1}{dn} - \frac{1}{r_0}\frac{dr_0}{dn}\right)(r_1 + r_0)^2 - 2\left(\frac{dr_1}{dn} - \frac{dr_0}{dn}\right)(r_1 + r_0)\right]dS$$

$$-\frac{1}{3!\,a^2}\frac{d^3\phi(t)}{dt^3}\iint\frac{1}{r_1 r_0}\left[\left(\frac{1}{r_1}\frac{dr_1}{dn} - \frac{1}{r_0}\frac{dr_0}{dn}\right)(r_1 + r_0)^3 - 3\left(\frac{dr_1}{dn} - \frac{dr_0}{dn}\right)(r_1 + r_0)^2\right]dS$$

$$+ \ldots\ldots +$$

$$\frac{(-1)^n}{n!\,a^n}\frac{d^n\phi(t)}{dt^n}\iint\frac{1}{r_1 r_0}\left[\left(\frac{1}{r_1}\frac{dr_1}{dn} - \frac{1}{r_0}\frac{dr_0}{dn}\right)(r_1 + r_0)^n - n\left(\frac{dr_1}{dn} - \frac{dr_0}{dn}\right)(r_1 + r_0)^{n-1}\right]dS$$

$$+ \ldots\ldots +$$

Each of the surface integrals in the above vanishes. The first is

$$\iint\left[\frac{1}{r_1}\frac{d}{dn}\left(\frac{1}{r_0}\right)-\frac{1}{r_0}\frac{d}{dn}\left(\frac{1}{r_1}\right)\right]d\mathrm{S},$$

which evidently vanishes, being equal to

$$\iiint\left[\frac{1}{r_1}\nabla^2\left(\frac{1}{r_0}\right)-\frac{1}{r_0}\nabla^2\left(\frac{1}{r_1}\right)\right]dx\,dy\,dz.$$

The vanishing of this surface integral leads at once to the expression for Green's Equivalent Stratum; and we shall see that the vanishing of *all* the integrals leads to Kirchhoff's formula. The second integral is equivalent to

$$\iint\left[\frac{d}{dn}\left(\frac{1}{r_0}\right)-\frac{d}{dn}\left(\frac{1}{r}\right)\right]d\mathrm{S},$$

which also clearly vanishes. The third reduces to

$$\iint\left[r_1\frac{d}{dn}\left(\frac{1}{r_0}\right)-\frac{1}{r_0}\frac{dr_1}{dn}-r_0\frac{d}{dn}\left(\frac{1}{r_1}\right)+\frac{1}{r_1}\frac{dr_0}{dn}\right]d\mathrm{S},$$

which is equal to

$$\iiint\left[r_1\nabla^2\left(\frac{1}{r_0}\right)-\frac{1}{r_0}\nabla^2 r_1-r_0\nabla^2\left(\frac{1}{r_1}\right)+\frac{1}{r_1}\nabla^2 r_0\right]dx\,dy\,dz$$

$$=\iiint\left(\frac{2}{r_1r_0}-\frac{2}{r_1r_0}\right)dx\,dy\,dz=0.$$

The fourth becomes, after a little reduction,

$$\iint\left[r_1^2\frac{d}{dn}\left(\frac{1}{r_0}\right)-\frac{1}{r_0}\frac{dr_1^2}{dn}-r_0^2\frac{d}{dn}\left(\frac{1}{r_1}\right)+\frac{1}{r_1}\frac{dr_0^2}{dn}-3\frac{dr_1}{dn}+3\frac{dr_0}{dn}\right]d\mathrm{S},$$

and the subject of integration of the equivalent volume integral is, consequently,

$$-\frac{1}{r_0}\nabla^2 r_1^2+\frac{1}{r_1}\nabla^2 r_0^2-3\nabla^2 r_1+3\nabla^2 r_0,$$

which is equal to

$$-\frac{6}{r_0}+\frac{6}{r_1}-\frac{6}{r_1}+\frac{6}{r_0}=0.$$

We must now show that the surface integral of the general term vanishes. The integrand is

$$\frac{dr_1}{dn}\left[\frac{(r_1+r_0)^n}{r_1^2 r_0}-\frac{n(r_1+r_0)^{n-1}}{r_1 r_0}\right]-\frac{dr_0}{dn}\left[\frac{(r_1+r_0)^n}{r_1 r_0^2}-\frac{n(r_1+r_0)^{n-1}}{r_1 r_0}\right]$$

$$=\frac{dr_1}{dn}\left[(1-n)\frac{r_1^{n-2}}{r_0}+n(2-n)r_1^{n-3}+\frac{n(n-1)}{1\,.\,2}(3-n)r_1^{n-4}r_0+\ldots\right.$$

$$\left.+\frac{n(n-1)(n-2)(-2)}{1\,.\,2\,.\,3}r_0^{n-4}r_1-\frac{n(n-1)}{1\,.\,2}r_0^{n-3}+\frac{r_0^{n-1}}{r_1^2}\right]$$

$$-\frac{dr_0}{dn}\left[(1-n)\frac{r_0^{n-2}}{r_1}+n(2-n)r_0^{n-3}+\frac{n(n-1)}{1\,.\,2}(3-n)r_0^{n-4}r_1+\ldots\right.$$

$$\left.+\frac{n(n-1)(n-2)(-2)}{1\,.\,2\,.\,3}r_1^{n-4}r_0-\frac{n(n-1)}{1\,.\,2}r_1^{n-3}+\frac{r_1^{n-1}}{r_0^2}\right],$$

which may be written

$$-\frac{1}{r_0}\frac{d}{dn}r_1^{n-1}+r_1^{n-1}\frac{d}{dn}\frac{1}{r_0}+\frac{1}{r_1}\frac{d}{dn}r_0^{n-1}-r_0^{n-1}\frac{d}{dn}\frac{1}{r_1}$$

$$-n\frac{d}{dn}r_1^{n-2}+n\frac{d}{dn}r_0^{n-2}$$

$$-\frac{n(n-1)}{1\,.\,2}\left[r_0\frac{d}{dn}r_1^{n-3}-r_1^{n-3}\frac{dr_0}{dn}\right]$$

$$+\frac{n(n-1)}{1\,.\,2}\left[r_1\frac{d}{dn}r_0^{n-3}-r_0^{n-3}\frac{dr_1}{dn}\right]$$

$$-\frac{n(n-1)(n-2)}{1\,.\,2\,.\,3}\left[r_0^2\frac{d}{dn}r_1^{n-4}-r_1^{n-4}\frac{dr_0^2}{dn}\right]$$

$$+\frac{n(n-1)(n-2)}{1\,.\,2\,.\,3}\left[r_1^2\frac{d}{dn}r_0^{n-4}-r_0^{n-4}\frac{dr_1^2}{dn}\right]$$

$$-\;\ldots\ldots\qquad\qquad+\;\ldots\ldots$$

$$-\;\ldots\ldots\qquad\qquad+\;\ldots\ldots$$

Remembering that $\nabla^2 r^n = n(n+1)r^{n-2}$, we see that the subject of integration of the equivalent volume integral is

$$n(n-1)\frac{r_0^{n-3}}{r_1}-n(n-1)\frac{r_1^{n-3}}{r_0}-n(n-1)(n-2)(r_1^{n-4}-r_0^{n-4})$$

$$-\frac{n(n-1)}{1\,.\,2}\left[(n-2)(n-3)r_0 r_1^{n-5}-2\frac{r_1^{n-3}}{r_0}\right]$$

$$+\frac{n(n-1)}{1\,.\,2}\left[(n-2)(n-3)r_1 r_0^{n-5}-2\frac{r_0^{n-3}}{r_1}\right]$$

$$-\frac{n(n-1)(n-2)}{1.2.3}\left[(n-3)(n-4)r_0^2 r_1^{n-6}-2\cdot 3r_1^{n-4}\right]$$

$$+\frac{n(n-1)(n-2)}{1.2.3}\left[(n-3)(n-4)r_1^2 r_0^{n-6}-2\cdot 3r_0^{n-4}\right]$$

$$-\frac{n(n-1)(n-2)(n-3)}{1.2.3.4}\left[(n-4)(n-5)r_0^3 r_1^{n-7}-3\cdot 4r_0 r_1^{n-5}\right]$$

$$+\frac{n(n-1)(n-2)(n-3)}{1.2.3.4}\left[(n-4)(n-5)r_1^3 r_0^{n-7}-3\cdot 4r_1 r_0^{n-5}\right]$$

$$- \ldots \qquad + \ldots$$

$$- \ldots \qquad + \ldots$$

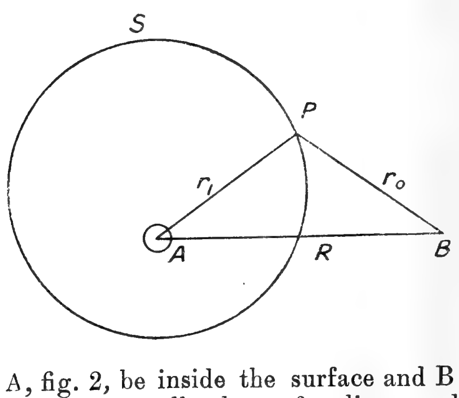
which vanishes identically.

Thus we have proved that the surface integral

$$\iint\left[\left(\frac{1}{r_1^2 r_0}\frac{dr_1}{dn}-\frac{1}{r_0^2 r_1}\frac{dr_0}{dn}\right)\phi\left(t-\frac{r_1+r_0}{a}\right)\right.$$
$$\left.+\frac{1}{ar_1 r_0}\left(\frac{dr_1}{dn}-\frac{dr_0}{dn}\right)\frac{\partial}{\partial t}\phi\left(t-\frac{r_1+r_0}{a}\right)\right]dS=0$$

for any closed surface S, r_1 and r_0 being the distances of a point P of the surface from two points A and B both lying outside it, $\phi(t)$ being such that, in the interval $\left(t-\frac{r_1+r_0}{a},\ t+\frac{r_1+r_0}{a}\right)$, the conditions for the validity of Taylor's series are satisfied.

Fig. 2.



Now let A, fig. 2, be inside the surface and B outside it, and surround A by a small sphere of radius ρ and centre A. A and B are both outside the space between the surface of the sphere and the surface S. The part of the surface integral pertaining to the sphere tends to the value $-\frac{4\pi}{R}\phi\left(t-\frac{R}{a}\right)$, as ρ approaches zero, and we have, therefore

$$\frac{M\phi\left(t-\frac{R}{a}\right)}{R}=\frac{M}{4\pi}\iint\left[\left(\frac{1}{r_1^2 r_0}\frac{dr_1}{dn}-\frac{1}{r_1 r_0^2}\frac{dr_0}{dn}\right)\phi\left(t-\frac{r_1+r_0}{a}\right)\right.$$
$$\left.+\frac{1}{ar_1 r_0}\left(\frac{dr_1}{dn}-\frac{dr_0}{dn}\right)\frac{\partial}{\partial t}\phi\left(t-\frac{r_1+r_0}{a}\right)\right]dS,$$

M being any constant. If we suppose A to be a source of disturbance, the vibrational velocity or displacement due to which at any point at a distance r can be expressed by $\frac{M}{r}\phi\left(t-\frac{r}{a}\right)$, t being the time and a the velocity of propagation, the above equation expresses the equivalence of the direct effect at B due to A at any instant to that due to a source distribution on the surface S, the secondary disturbance being sent out from each element of surface at a time $\frac{r_1}{a}$ after it was sent out from A and at a time $\frac{r_0}{a}$ previous to its arrival at B.

It is usual to write the surface integral in the form

$$\frac{M}{4\pi}\iint\left[\frac{\partial}{\partial n}\frac{\phi\left(t-\frac{r_1+r_0}{a}\right)}{r_1 r_0}-\frac{1}{r_0}\frac{\partial}{\partial n}\frac{\phi\left(t-\frac{r_1}{a}\right)}{r_1}_{\left(t-\frac{r_1+r_0}{a}\right)}\right]dS.$$

In the first term the differentiation with respect to the normal operates on r_0 only and in the second term $t-\frac{r_1+r_0}{a}$ is written for $t-\frac{r_1}{a}$ after differentiation with respect to the normal; but although there may be a gain in conciseness in writing the expression in this way, there is perhaps some loss in clearness.

Remembering now that at every point in space we have

$$\nabla^2 V=\frac{1}{a^2}\frac{\partial^2 V}{\partial t^2}-4\pi\phi,$$

where V denotes the vibrational displacement or velocity, and ϕ is a function of the co-ordinates of a point and the time which vanishes except at points where there are sources, and that the solution of the equation is

$$V = \iiint \frac{\phi\left(t-\frac{r}{a}\right)}{r} dx\, dy\, dz$$

throughout all space, we have Kirchhoff's formula for the most general case by writing V for ϕ. If, in addition to volume distributions, there are surface distributions of sources at which

$$\frac{\partial V}{\partial n_1}+\frac{\partial V}{\partial n_2}+4\pi\psi=0,$$

$$V=\iiint \frac{\phi\left(t-\frac{r}{a}\right)}{r} dx\, dy\, dz+\iint \frac{\psi\left(t-\frac{r}{a}\right)}{r} dS.$$

Thus V_0 the value of V at any point outside a surface enclosing all volume and surface distributions of sources is given by the formula

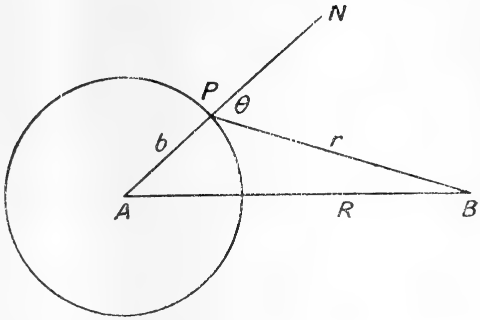
$$V_0 = \frac{1}{4\pi}\iint\left(\frac{\partial}{\partial n}\frac{V\left(t-\frac{r}{a}\right)}{r}-\frac{1}{r}\frac{\partial V}{\partial n}_{\left(t-\frac{r}{a}\right)}\right) dS,$$

r being the distance of the point from an element of the surface. In the first term the differentiation with respect to the normal is performed on r alone and, in the second term, $t-\frac{r}{a}$ is written for t after differentiation. Kirchhoff's formula has thus been shown to follow directly from a generalization of Green's theorem.

As remarked above, the development of $\phi\left(t-\frac{r_1+r_0}{a}\right)$ and of $\frac{d}{dt}\phi\left(t-\frac{r_1+r_0}{a}\right)$ by Taylor's theorem must conform to the conditions of validity, and these conditions must hold in the application of the series to any question considered. The time t is the time at which the actual disturbance reaches B, while $t-\frac{r_1+r_0}{a}$ is the time at which a secondary disturbance starts from A. The resultant at B of the secondary disturbances is made up of components which start from A at different times, and what is shown is that this resultant is the same as the actual disturbance at B at the time t.

As an example, let there be a single source at A the centre of a sphere of radius b, fig. 3, and let $\mathrm{PB}=r$, and denote the angle BPN by θ.

Fig. 3.



The formula gives

$$\frac{\mathrm{M}}{\mathrm{R}}\phi\left(t-\frac{\mathrm{R}}{a}\right)=\frac{\mathrm{M}}{4\pi}\iint\frac{1}{r}\left[\left(\frac{1}{b^2}+\frac{\cos\theta}{br}\right)\phi\left(t-\frac{b+r}{a}\right)\right.$$
$$\left.+\frac{1+\cos\theta}{ab}\frac{\partial}{\partial t}\phi\left(t-\frac{b+r}{a}\right)\right]d\mathrm{S}.$$

Let
$$\phi\left(t-\frac{r}{a}\right)=\sin\frac{2\pi a}{\lambda}\left(t-\frac{r}{a}\right),$$
then
$$\frac{\mathrm{M}}{\mathrm{R}}\sin\frac{2\pi a}{\lambda}\left(t-\frac{\mathrm{R}}{a}\right)=\frac{\mathrm{M}}{4\pi}\iint\frac{1}{r}\left[\left(\frac{1}{b^2}+\frac{\cos\theta}{br}\right)\sin\frac{2\pi a}{\lambda}\left(t-\frac{b+r}{a}\right)\right.$$
$$\left.+\frac{2\pi}{b\lambda}(1+\cos\theta)\cos\frac{2\pi a}{\lambda}\left(t-\frac{b+r}{a}\right)\right]d\mathrm{S}.$$

If b is very small in comparison with the other linear magnitudes involved, the right-hand side of the equation becomes

$$\frac{\mathrm{M}}{4\pi}\iint\frac{1}{rb^2}\sin\frac{2\pi a}{\lambda}\left(t-\frac{r+b}{a}\right)d\mathrm{S}$$
$$=\frac{\mathrm{M}}{\mathrm{R}}\sin\frac{2\pi a}{\lambda}\left(t-\frac{\mathrm{R}}{a}\right).$$

If all the linear magnitudes are large in comparison with λ, the right-hand side becomes

$$\frac{\mathrm{M}}{2b\lambda}\iint\frac{1+\cos\theta}{r}\cos\frac{2\pi a}{\lambda}\left(t-\frac{b+r}{a}\right)d\mathrm{S},$$

which is Stokes's expression for the secondary disturbance.

If we make $R=p+b$, write t for $t-\frac{b}{a}$, and make b indefinitely great, we get a plane wave-front, the distance of the point B from the plane being p. The right-hand side becomes

$$\frac{M}{4\pi}\iint\left[\frac{1}{r^2}\cos\theta\sin\frac{2\pi a}{\lambda}\left(t-\frac{r}{a}\right)+\frac{2\pi}{\lambda r}(1+\cos\theta)\cos\frac{2\pi a}{\lambda}\left(t-\frac{r}{a}\right)\right]dS$$

$$=\frac{M}{2}\int_p^\infty\left[\frac{\cos\theta}{r}\sin\frac{2\pi a}{\lambda}\left(t-\frac{r}{a}\right)+\frac{2\pi}{\lambda}(1+\cos\theta)\cos\frac{2\pi a}{\lambda}\left(t-\frac{r}{a}\right)\right]dr$$

$$=\frac{M}{2}\int_p^\infty\left[\frac{p}{r^2}\sin\frac{2\pi a}{\lambda}\left(t-\frac{r}{a}\right)+\frac{2\pi}{\lambda}\left(1+\frac{p}{r}\right)\cos\frac{2\pi a}{\lambda}\left(t-\frac{r}{a}\right)\right]dr$$

$$=M\sin\frac{2\pi a}{\lambda}\left(t-\frac{p}{a}\right).$$

XXIV. *On a New Secondary Radiation of Positive Rays.*

To the Editors of the Philosophical Magazine.

GENTLEMEN,

IN a recent publication (Phil. Mag. (6) xxxv. p. 59, 1918) on this subject I have expressed the belief that the penetrating radiation then observed was the characteristic radiation of tin and lead, and this conclusion was based on the marked differences observed in the intensity of the photographs according to different positions of the foils (see figure *l. c.*).

Further experiments have not, however, confirmed this supposition, but have led to the discovery of a source of error in my previous researches. When this source of error was eliminated, a uniform slight imprint only could be observed on the photograph.

Careful investigation of the nature of this radiation was carried out, and the effect of magnetic deflexion on the positive rays shows that the new secondary radiation is excited by the positive ions. From an approximate valuation the coefficient of absorption of the new radiation is estimated to be of the order of that of the characteristic K-radiation of aluminium.

Yours very truly,

M. WOLFKE.

The Physical Laboratory,
Technical High School of Zurich.
February 1918.

XXV. *On the Coefficient of Potential of Two Conducting Spheres.*

To the Editors of the Philosophical Magazine.

GENTLEMEN,

IN my paper "On the Coefficient of Potential of Two Conducting Spheres" (Phil. Mag. March 1918), there is an error in the determination of the values of one of the two series in terms of a, b, and c.

Denoting ab by p^2 and $c^2-a^2-b^2$ by k^2, the series G is

$$1+\frac{p^2}{k^2}+\frac{p^4}{k^4-p^4}+\frac{p^6}{k^6-2k^2p^4}+\frac{p^8}{k^8-3p^4k^4+p^8}+\ldots,$$

where each denominator is obtained from the two preceding ones by multiplying the immediately preceding one by k^2 and subtracting the other multiplied by p^4.

Similarly,

$$F_i=\frac{c}{c^2-b^2}\left[1+\frac{p^2}{h^2}+\frac{p^4}{h^2k^2-p^4}+\frac{p^6}{k^2(h^2k^2-p^4)-p^4h^2}+\ldots\right],$$

where h^2 denotes $\dfrac{(c^2-b^2+ac)(c^2-b^2-ac)}{c^2-b^2}$, and the same rule holds for determining the denominators inside the brackets. Any number of terms of q_{11} and q_{12} can be written down without difficulty.

Thus,

$$q_{11}=a+\frac{a^2b}{c^2-b^2}+\frac{a^3b^2}{(c^2-b^2+ac)(c^2-b^2-ac)}$$

$$+\frac{a^4b^3}{(c^2-b^2+ac)(c^2-b^2-ac)(c^2-a^2-b^2)-a^2b^2(c^2-b^2)}$$

$$+\frac{a^5b^4}{(c^2-a^2-b^2)[(c^2-b^2+ac)(c^2-b^2-ac)(c^2-a^2-b^2)-a^2b^2(c^2-b^2)]-a^2b^2(c^2-b^2+ac)(c^2-b^2-ac)}$$

$$+\ldots,$$

and

$$q_{12}=-\frac{ab}{c}\left[1+\frac{ab}{c^2-a^2-b^2}+\frac{a^2b^2}{(c^2-a^2-b^2)^2-a^2b^2}\right.$$

$$\left.+\frac{a^3b^3}{(c^2-a^2-b^2)^3-2a^2b^2(c^2-a^2-b^2)}+\ldots\right].$$

Yours faithfully,

ALEX. ANDERSON.

XXVI. *The Scattering of Light by Air Molecules.* *By* R. W. Wood, *Major U.S.R.**

A RECENT paper by Strutt (Proc. Roy. Soc. (last number) 1918) on the scattering of light by supposedly clean air makes it appear worth while to publish the results of some experiments which I made on the same subject in 1902, but did not publish at the time, as it was found that they were spurious. The apparatus, method, and results were identical with those of Strutt, in fact the diagram of his apparatus might have been a drawing made from the apparatus which I employed. This is merely a coincidence of course, resulting from the fact that the apparatus is the obvious one to use.

In my work I employed a spark instead of an arc, wishing to have available the shortest possible waves. Photographs of the cone of scattered light appeared in air which had been forced through long tubes filled with tightly packed cotton and dried over phosphorus pentoxide. The cone was also seen visually if the eyes were thoroughly rested in the dark.

This made me suspicious, and I varied the conditions under which the experiment was made employing eye observation.

It was soon found that if the spark was stopped and the tube thoroughly washed out with the purified air, absolutely no trace of cone of scattered light was visible on turning on the spark. In about ten seconds, however, a trace of the cone appeared, and after the spark had been in operation for a minute it was well developed. Interposition of a glass plate prevented the formation of the cone, if I remember correctly. This appeared to prove conclusively that the ultra-violet light caused a precipitation of something from the air, causing a slight cloud.

Substitution of sulphuric acid for the phosphorus pentoxide only made matters worse, a dense cloud forming in ten or fifteen seconds after starting the spark. I was unable to secure air in which the light of the spark failed to develop a visible fog, and consequently abandoned the experiments, which were designed to test experimentally Lord Rayleigh's theory of the blue sky. It would be well to try air vaporized from the liquid using no drying agents or cotton. Some six or seven years later some experiments were described by a French physicist, whose name I do not recall at the moment, which showed similar effects ascribed to the formation of nuclei by the ultra-violet light.

* Communicated by the Author.

In view of these facts it appears to me that Strutt's experiments should be repeated before we recognize the scattering of light by air molecules as demonstrated. It would be well to employ sunlight and a glass lens, and look for the cone with the eye. Its absence would prove that Strutt's results were due to a cloud resulting from the action of the ultra-violet rays on the air.

XXVII. *Some Two-Dimensional Potential Problems connected with the Circular Arc.* II. *By* W. G. Bickley, *B.Sc.**

§ 1. In a recent paper † the author has given the solution of some potential problems connected with the circular arc, and interpreted the results in terms of electricity and hydrodynamics. In particular, the velocity potential and stream functions for circulatory flow about an infinitely long lamina in the form of a circular arc, and for the disturbance of a stream due to such a lamina, were determined. It is now proposed to give drawings of the stream-lines in the latter case, to examine the case of rotation of the arc, and to give a brief discussion of the motion of the arc when free to move, and acted upon by the consequent fluid pressures.

§ 2. The stream-lines were obtained by first mapping out the z-plane by a system of orthogonal coordinates given by the relation

$$z = -\iota \frac{1+se^{\tau}}{1+se^{-\tau}}, \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

where $\tau = \rho + \iota\sigma$, and s is written, for brevity, for $\sin\frac{\alpha}{2}$. The results of the preceding paper show that for circulatory flow

$$w = \phi + \iota\psi = -\tau, \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

so that the figure of the last paper is the requisite map, for the particular case of the semicircle. Making the substitution (1) above, in equation (14) of the preceding paper, we obtain for the case of flow past the arc

$$w = 2\iota s \sinh(\tau - \iota\beta), \quad . \quad . \quad . \quad . \quad . \quad (3)$$

giving

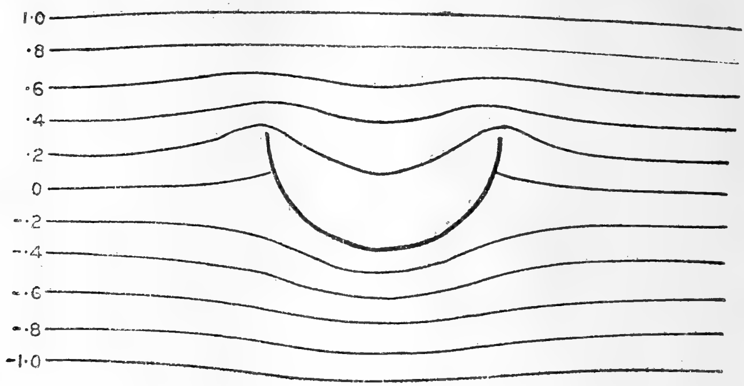
$$\psi = 2s \sinh\rho \cos(\sigma - \beta). \quad . \quad . \quad . \quad . \quad . \quad (4)$$

* Communicated by the Author.

† Phil. Mag. [6] vol. xxxv. p. 396 (May 1918).

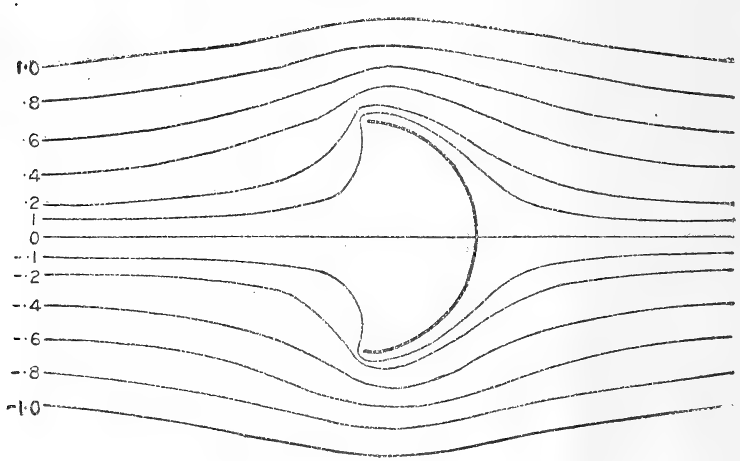
Corresponding values of ρ and σ are readily calculated and the stream-lines easily plotted on tracing-paper over the map above referred to. In this way figs. 1 to 4 have been

Fig. 1.



$\beta = 0$

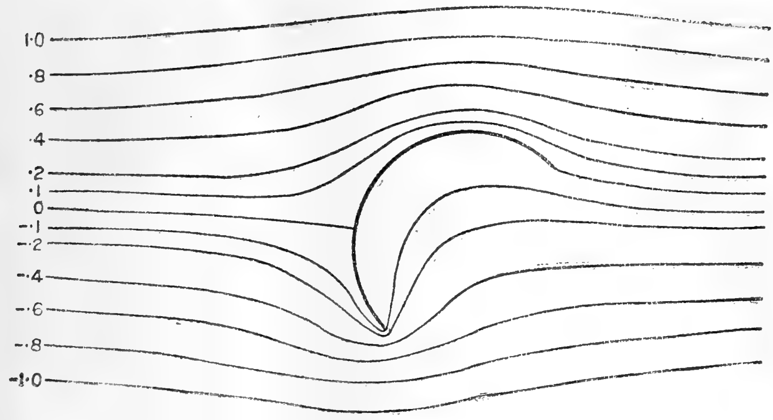
Fig. 2.



$\beta = 90°$

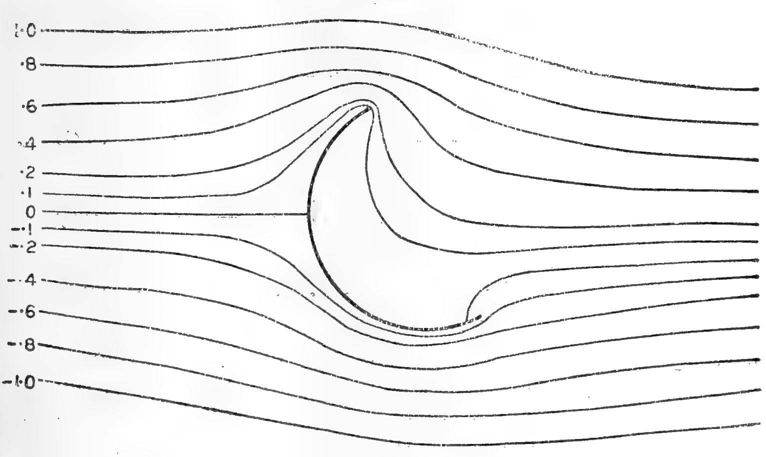
drawn, for the values of β, 0°, 90°, $-45°$, and 60° respectively, turned for convenience so that the undisturbed direction of the stream is horizontal. The third of these exemplifies the case of the stream dividing at the edge of the lamina, noted at the end of the preceding paper. (It may

Fig. 3.



$\beta = -45°$

Fig. 4.



$\beta = 60°$

be noted that equations (3) and (4) are in agreement with the results given in a recent paper by Dr. J. G. Leathem.)

§ 3. The method of sources used in the preceding paper is not effective when the motion due to rotation of the boundary is desired. For that purpose a method outlined in a recent note* by the present writer can be employed. The arc being, as before, that part of a circle of unit radius given by $z=-\iota e^{\iota\theta}$, for which $-\alpha \leqq \theta \leqq \alpha$, the doubly connected space outside is transformed into the upper half of the ζ-plane by the transformation

$$z=-\iota\frac{c\zeta^2+2\iota s\zeta+c}{c\zeta^2-2\iota s\zeta+c}, \quad . \quad . \quad . \quad . \quad . \quad (5)$$

where $c=\cos\frac{1}{2}\alpha$. On the arc, we have $\psi=\frac{1}{2}\omega\,|\,z-z_0\,|^2$, where ω is the angular velocity, and z_0 the axis of rotation. The choice of this axis is a matter of convenience, and is for simplicity chosen as $z_0=-\iota$. So that on the ξ-axis of the ζ-plane, on using (5), we get

$$\psi=\tfrac{1}{2}\omega\cdot\frac{16s^2\xi^2}{c^2(\xi^2+1)^2+4s^2\xi^2}. \quad . \quad . \quad . \quad . \quad (6)$$

The corresponding value of w, free from infinities in the upper half of the z-plane, is then, except as to an irrelevant constant,

$$w=\frac{1}{\pi}\int_{-\infty}^{\infty}\tfrac{1}{2}\omega\frac{16s^2\xi^2}{c^2(\xi^2+1)^2+4s^2\xi^2}\cdot\frac{d\xi}{\zeta-\xi}. \quad . \quad . \quad (7)$$

The integral is easily evaluated by the method of residues, and may be expressed in the three forms :—

$$w=\frac{4\omega s^2\zeta}{c\zeta^2+2\iota\zeta-c}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (8\,a)$$

$$=\frac{\omega(z+\iota)}{2z}\left\{(z+\iota)-\sqrt{z^2+2\iota z\cos\alpha-1}\right\}, \quad . \quad (8\,b)$$

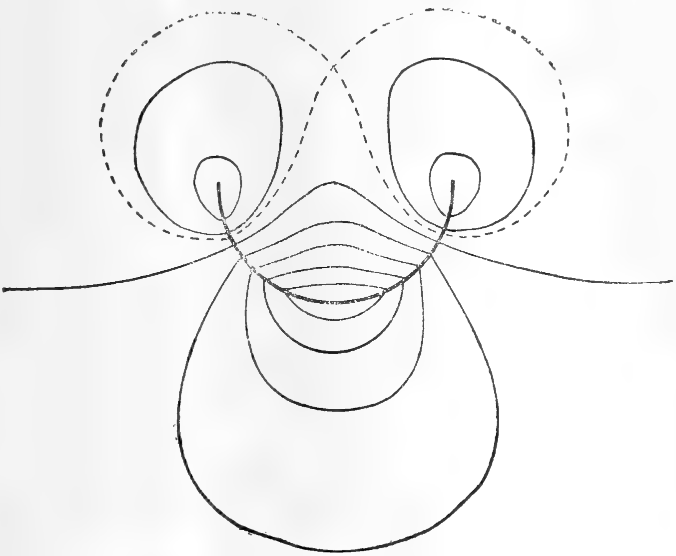
$$=\iota s^2-\iota s^2 e^{-2\tau}\frac{1+se^{\tau}}{1+se^{-\tau}}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (8\,c)$$

by the use of (5) and (1) above. Form (8 *c*) is convenient to enable the stream-lines to be drawn, and these are given for the case of a semicircle in fig. 5. An alternative, but special, method of obtaining the result is furnished by the fact that a rotation about the centre of the circle leaves the liquid undisturbed. This may be regarded as instantaneously

* Phil. Mag. June 1918.

compounded of a rotation about $z=-\iota$ and a translation parallel to the x-axis. The application of equation (14) of the preceding paper once more gives equation (8 *b*).

Fig. 5.



§ 4. On the boundary, $z=-\iota e^{\iota\theta}$, so that

$$\left.\begin{aligned} w &= 2\omega \sin\frac{\theta}{2}\left\{\sqrt{\sin^2\frac{\alpha}{2}-\sin^2\frac{\theta}{2}}+\iota\sin\frac{\theta}{2}\right\} \\ \phi &= \pm 2\omega\sin\frac{\theta}{2}\sqrt{\sin^2\frac{\alpha}{2}-\sin^2\frac{\theta}{2}},\ \psi=2\omega\sin^2\frac{\theta}{2}, \end{aligned}\right\} . \quad (9)$$

where the positive value of the root refers to the convex surface. For the energy of the motion, we have

$$2\mathrm{T}=\int\phi d\psi \text{ taken round the boundary}$$
$$=2\pi\omega^2\sin^4\frac{\alpha}{2}, \text{ upon evaluation.}$$

Introducing the radius a instead of unity, and the density ρ of the liquid,

$$\mathrm{T}=\pi\rho\omega^2 a^4\sin^4\frac{\alpha}{2}. \quad . \quad . \quad . \quad . \quad . \quad (10)$$

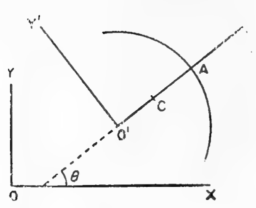
Proceeding to the limit of a plane lamina of breadth $2b$,

$$T = \tfrac{1}{16}\pi\rho\omega^2 b^4, \text{ a known result.}$$

The calculation of the fluid pressures on the lamina is not difficult, but the "end pressures" of the preceding paper must be taken into account. The final resultant is a force $2\pi\rho\omega^2 a^4 \sin^4\frac{\alpha}{2}$ acting along the y-axis. The result is also deducible from the general equations of motion now to be briefly discussed.

§ 5. Refer to moving axes O′X′, O′Y′, fixed with respect to the lamina as in fig. 6. Let the coordinates of O′ with respect to axes fixed in space be x, y, and the angle between O′X′ and OX be denoted by θ. A rotation θ does not disturb the liquid, and so contributes nothing to the kinetic energy. Denoting by U, V, the velocities of translation along O′X′, O′Y′ respectively, equation (18) of the preceding paper gives

Fig. 6.



$$2T = 2\pi\rho a^2 \left\{ U^2 \sin^2\frac{\alpha}{2}\left(1 + \cos^2\frac{\alpha}{2}\right) + V^2 \sin^4\frac{\alpha}{2} \right\}. \quad (11)$$

(This has also been deduced from $2T = \int \phi d\psi$.) The usual methods now give the forces acting on the lamina. For brevity, denote by A, $2\pi\rho a^2 \sin^4\frac{\alpha}{2}$, and by B, $\pi\rho a^2 \sin^2\alpha$. Then

$$2T = (A+B)U^2 + AV^2, \quad . \quad . \quad . \quad . \quad (11')$$

giving the forces and couple

$$\left.\begin{aligned} X &= -(A+B)\dot{U} + AV\dot{\theta}, \\ Y &= -A\dot{V} - (A+B)U\dot{\theta}, \\ L &= \quad BUV. \end{aligned}\right\} \quad . \quad . \quad . \quad (12)$$

These have also been deduced from the general pressure equation. On forming the general equations of motion from (11′), the consequences of the fact that T is independent of $\dot{\theta}$ are at once apparent, for these equations will be found incompatible unless $U=V=0$. However, this was to be expected, as there is, if the lamina be massless, nothing to enable it to disturb the fluid, so that the only possible motion is one which leaves the fluid undisturbed, *i.e.* for which $U=V=0$. Moreover, in consequence of its lack of inertia, any other motion, even if it could be started, would be instantaneously converted into the above type by the action of the finite pressures on the unsubstantial lamina.

If the lamina be supposed uniform, and of mass M, its kinetic energy T_1 is given by

$$2T_1 = M\left\{U^2+V^2+a^2\dot{\theta}^2+2a\frac{\sin\alpha}{\alpha}V\dot{\theta}\right\}. \quad . \quad . \quad (13)$$

By transference to a "centre of inertia" C, given by

$O'C=a\dfrac{M}{M+A}\cdot\dfrac{\sin\alpha}{\alpha}$, the total energy assumes the value given by

$$2T=(A+B+M)U^2+(A+M)V^2+\frac{A+M\left(1-\dfrac{\sin^2\alpha}{\alpha^2}\right)}{A+M}Ma^2\dot{\theta}^2, \quad (14)$$

and the motion of C is known (*cf.* for instance Lamb's 'Hydrodynamics,' 4th ed. p. 165), and so is that of the lamina. If ωa is sufficiently great compared with the velocity of translation, the path of A (in fig. 6) is looped, otherwise it is, in general, sinuous.

Loughborough,
June 1st, 1918.

XXVIII. *Proceedings of Learned Societies.*

GEOLOGICAL SOCIETY.

[Continued from p. 208.]

March 6th, 1918.—Mr. G. W. Lamplugh, F.R.S., President, in the Chair.

Mr. J. F. N. Green delivered a Lecture on the Igneous Rocks of the Lake District. He first drew attention to some of the manuscript 6-inch maps of the Lake District, prepared nearly fifty years ago, by the Geological Survey, and pointed out that, although undoubtedly most accurate, they differed greatly in the volcanic area from his own. He suggested that the reason was that there was a fundamental difference in the classification of tuffs and lavas. A large proportion of the Lake-District rocks were brecciated, and had been supposed to be altered tuffs. With the unbrecciated rocks into which they passed they had been mapped as ashes. A number of specimens and photographs were shown, indicating that the brecciation and apparent bedding were due to flow. Specimens were also shown of explosion-breccias, of the normal tuffs (which the Lecturer believed to be mainly the result of erosion between eruptions), and of rocks simulating true tuffs, but actually sandstones and conglomerates, composed of detrital igneous material. Attention was drawn to the criteria for distinguishing the various types. Recently, manuscripts had been found in the possession of the Geological Survey proving that Aveline, whose maps were extraordinarily accurate and detailed, had anticipated by thirty years the Lecturer's separation from the volcanic rocks of the basal beds of the Coniston Limestone Series.

When re-mapped on this basis, the Borrowdale Series appeared as a simple and regular sequence, strongly folded and cropping out in long bands. An interesting history of vulcanicity was revealed, beginning in many places with explosion-tuffs followed by a great series of pyroxene-andesites over the whole district. Then there was a pause during which fine-grained andesite-tuffs, with a tendency to produce true slates, accumulated. This was succeeded by a vast outpouring of andesites, of great thickness in the central mountain region, but dying out southwards and eastwards. Next a series of peculiar mixed tuffs, of special value in mapping, was covered by another mass of andesites dying out south-westwards. After this, soda-rhyolites covered the whole district, nothing later being preserved—with one possible known exception. These volcanic rocks were intersected by a varied series of intrusions.

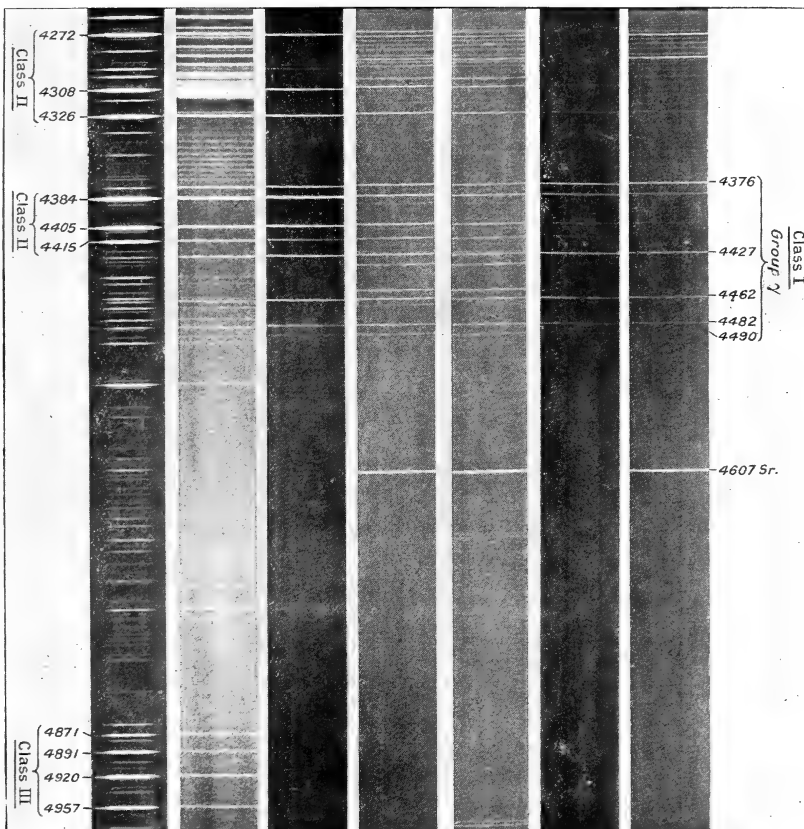
The solfataric phenomena were of interest, including the production of garnet and graphite, and a remarkable 'streaky' structure in the rhyolites.

An important question related to the age of the large acid intrusions associated with the volcanic rocks. Were they of the same age as, or later than, the Devonian folding? A sketch was given of the evidence on which the Lecturer assigned the Eskdale and Skiddaw granites to the Ordovician volcanic episode, and it was suggested that the great Skiddaw anticline was not due to regional folding, but a local structure connected with the vulcanicity.

Lantern-slides of Lake District country were shown, and the manner in which the volcanic rocks entered into the scenery was pointed out.

Spectra of Iron.

1. *Furnace* 1600° C.
2. *Flame* 1850° C.
3. *Furnace* 2100° C.
4. *Furnace* 2400° C.
5. *Flame* 2550° C.
6. *Explosion Region.*
7. *Self-induction Spark.*



Thermo-chemical Excitation. (1–5)

Chemical Excitation. (6)

Electrical Excitation. (7)

THE
LONDON, EDINBURGH, AND DUBLIN
PHILOSOPHICAL MAGAZINE
AND
JOURNAL OF SCIENCE.

[SIXTH SERIES.]

OCTOBER 1918.

XXIX. *On the Origin of the Line Spectrum emitted by Iron Vapour in an Electric Tube Resistance Furnace at Temperatures above* 2500° C. *By* G. A. Hemsalech, *Honorary Research Fellow in the University of Manchester**.

§ 1. *Introduction.*

It was shown in the preceding communication (this volume page 209) that the spectrum of iron as observed in the carbon tube resistance-furnace up to 2500° C. and in flames up to 2700° C. is caused by thermal actions on a compound of this metal and not by the direct action of heat on the pure metal. Now, as far as flames are concerned, the character of the spectrum changes only slowly, though progressively, as the temperature rises to 2700° C. But in the case of the tube furnace a great change is observed soon after the boiling-point of iron has been reached and the gases from the boiling metal have diffused into the interior space of the tube. These facts have led to the conclusion that the mode of excitation underlying the emission of the high-temperature furnace spectrum of iron is no longer the same as that which prevails in the furnace below 2500° and in flames up to 2700° C. Further, the appearance, at the high furnace temperature, of lines which are characteristic of the arc and spark, and their absence in flames of the same temperature, has suggested the idea that the spectrum of iron as emitted by the furnace under these conditions is of electric origin. The experiments described in this paper were accordingly based on this idea,

* Communicated by Sir E. Rutherford, F.R.S.

and, as will be seen from the various results obtained, there can be little doubt left, that the so-called high-temperature emission of iron vapour in an electric tube resistance-furnace is actually caused by the passage of an electric current through the vapour.

§ 2. *General Observations on the Furnace Spectrum of Iron Vapour at* 2700° C.

As has already been recorded, the interior of the tube furnace, working at atmospheric pressure, emits a purple light up to about 2400°; above this temperature and up to about 2500° the light emitted is of brilliant white, due no doubt to carbon particles, since it gives a continuous spectrum. Above the boiling-point of iron, when the gaseous metal has pervaded the whole interior of the furnace, the colour of the brilliant light emitted is of a decided greenish tint. Spectroscopic examination at this stage reveals, superposed on a bright continuous ground, a most brilliant iron spectrum, in which the group at 4957 is quite a prominent feature. Also the Swan bands at 4737 and 5165 are now visible. Owing to the low dispersion of my spectrograph, the finer details of the iron spectrum are unfortunately more or less destroyed by the continuous background which, even with short exposures of one second or less, is an annoying attribute of the photographic records secured. Nevertheless the general character of the spectrum is well brought out, as are also its distinguishing features as compared with the corresponding flame spectrum.

The spectrum of iron given by the furnace at 2700° differs entirely from that observed at the same temperature in the oxy-acetylene flame. Thus a large number of class III. lines and, further, lines so far only obtained by means of electric discharges, have been detected in this spectrum. Of class III. lines the four doublets λλ4872, 4891, 4920, and 4957 stand out prominently. It will be remembered that at the flame temperature of 2700° only traces have been observed of three of these doublets; they constitute, however, an important group in the spectrum of the explosion region of the air-coal gas-flame, and they are particularly marked in the self-induction spark where electric actions prevail. Now in electrical sources, such as arc and spark, both components of each doublet are well developed, whereas with chemical excitation in the explosion region only one component is brought out. The low dispersion employed has not enabled me to resolve these doublets, but according to Dr. King's observations of the high-temperature furnace spectrum of iron, made with a high dispersion, both components of each pair show in the furnace spectrum with approximately similar relative intensities as in the arc*. Hence the behaviour of these lines in the furnace is such as to suggest their emission being governed by electric rather than thermo-chemical or chemical actions.

§ 3. *Probable Mode of Excitation.*

The forcing of heavy electric currents through the carbon tube resistance-furnace entails the establishment between its extremities of a certain potential difference, the value of which depends upon the resistance of the tube and the temperature to which it is to be raised. Now, if with the rise in temperature the gases or vapours enclosed in the furnace became progressively ionized to a high degree, a stage should be reached at which part of the heating current will be carried by the ionized vapours according to the fundamental laws of electric conduction. The well-known experiments by Drs. Harker and Kaye † on the ionization in tube resistance-furnaces have furnished most important data on this point, and their results leave no doubt as to the relatively high conductivity of the ionized gases within the furnace-tube. In a first experiment these physicists showed that it was possible to send an electric current across a gap between two carbon rods held concentrically in the middle of the tube. With small potentials (up to 6 or 8 volts) measurable values of the current were obtained when the temperature rose above 1400° C., and at 2000° furnace temperature it reached the value of several amperes. In their own words: "The magnitude of the ionization currents indicated that, although the pressure was atmospheric, the atmosphere of the furnace was ionized to an unusual degree at high temperatures." In a further experiment, and one of still greater importance from our point of view, Drs. Harker and Kaye measured the current which leaks across the highly ionized space surrounding a heated carbon rod passing concentrically through a carbon tube. In this case the ionization current is part of the heating current supplied to the rod, and it flows across the ionized space when one end of the rod is joined to the

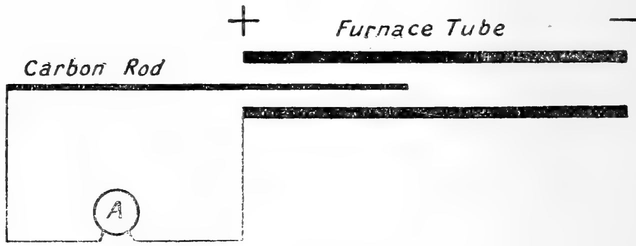
* A. S. King, Astrophysical Journal, vol. xxxvii. p. 259 (1913).

† J. A. Harker and G. W. C. Kaye, Proceedings Royal Society, Series A, vol. lxxxvi. p. 379 (1912); ibid. vol. lxxxviii. p. 522 (1913).

carbon tube. Under these conditions, and with a rod temperature not far from 3000° C., they obtained a steady ionization current of about $3\frac{1}{2}$ amperes. It seems to me that this experiment proves conclusively that at the higher furnace temperatures an electric current must pass through the ionized vapours contained between the extremities of the resistance-tube.

I have repeated the second experiment of Drs. Harker and Kaye with a slight modification, so as to approach more nearly the actual working conditions prevailing in the present investigation. The furnace was of the type already described in the preceding paper (§ 2), the carbon tube having an internal diameter of 14 mm. and an effective length of 4 inches between the graphite blocks. A carbon rod 4 mm. in diameter was mounted in such a way that it could be moved along the axis of the furnace whilst remaining in a concentric position with regard to the tube. Thus the radial distance between the carbon rod and the inner furnace-wall was always as nearly as possible 5 mm. The carbon rod was connected to one end of the furnace-tube with an ammeter in the circuit, as shown in fig. 1; and it was provided with a

Fig. 1.



Method of measuring Ionization Current.

division, so that the amount of its penetration into the tube could be ascertained. The experiments were made at a temperature of 2200° C.; at this temperature the carbon rod does not bend but remains perfectly straight throughout the time necessary for the various manipulations and readings involved. The difference of potential at the ends of the furnace-tube, at this temperature, was about 9·5 volts, or equal to a voltage drop of 1 volt per centimetre approximately. The following table gives the values of the ionization currents obtained under these conditions for various positions of the rod.

Amount of penetration of rod into heated portion of tube.	Ionization Current.
0 inch	0 ampere
0·5 ,,	0·1 ,,
1·0 ,,	0·6 ,,
1·5 ,,	1·0 ,,
2·0 ,,	1·5 ,,
2·5 ,,	2·0 ,,
3·0 ,,	2·3 ,,
3·5 ,,	2·6 ,,
4·0 ,,	2·8 ,,

From these results it will be seen that an appreciable current is obtained when the rod penetrates only ½ inch. The current then increases rapidly as the rod penetrates farther into the tube, partly on account of the higher temperature with consequent higher degree of ionization prevailing near the middle of the tube, and partly also on account of the increasing area of active rod surface and the rising potential difference between the extremity of the rod and the opposite furnace-wall. After the rod has passed 2·5 inches into the tube the rate of increase of the ionization current diminishes, no doubt because the rod now enters cooler regions of the furnace, in which the intensity of ionization declines again. It should of course always be borne in mind that the ionization current does not pass only between the extremity of the rod and the wall, but that the flow of electricity takes place, more or less, all along that portion of the rod which is within the heated zone of the furnace-tube.

A further set of experiments was carried out at higher temperatures, the extremity of the rod being at the middle of the furnace, namely 2·0 inches from the end of the heated portion of the furnace-tube. At the highest temperature observations were made both with and without iron vapour in the tube. The following values were obtained :—

Furnace Temperature.	Ionization Current.	
2400° C.	3·0 ampere	Without iron vapour.
2700 ,,	3·6 ,,	
2700 ,,	4·4 ,,	With iron vapour.

The last result shows that a heavy ionization current passes even when iron vapour is present, and I suppose that the higher value obtained is due to part of the current being now carried by iron vapour.

From the results of these experiments we may safely conclude that at temperatures above 2500° a column of iron vapour in the furnace will carry a small portion of the heating current supplied to the tube. Also it is highly probable that some connexion exists between the flow of electricity through the iron vapour and the brilliant line spectrum observed under these temperature conditions. The interior of the furnace-tube may in effect be regarded as a low-tension arc, in which the necessary degree of ionization and the gaseous state of the metal are maintained by the heat from the carbon tube. The spectrum of iron emitted under such conditions should therefore approach that given by an ordinary arc between iron poles, which indeed it does.

§ 4. *Persistence of the Iron Line Emission after the Electric Current through the Furnace is broken.*

As has been observed by Dr. King, the iron lines remain visible for some time after the current feeding the furnace is broken, and he has therefore concluded that the furnace radiation does not depend upon the existence of a potential difference. I quite agree that this conclusion holds as regards those lines which are caused by thermo-chemical excitation, and there is little doubt that a spectrum, composed of these lines, would be observed if the tube were heated by other than electrical means, as in fact it is observed in the mantles of the several flames examined. But with regard to the so-called high-temperature lines, which become prominent only after the metal has passed into the gaseous state and fills the interior of the furnace-tube, does their persistence, after the potential is taken off, really prove that they were not, in the first place, excited by electric actions?

I have shown in a series of experiments that in an electric spark discharge between metal electrodes the emission of luminous radiations by the metal vapour continues for an appreciable time after the discharge has passed*. In these experiments the spark employed was of the simplest type, consisting of only one single oscillation, so that the metal vapour, which was carried away from the spark-gap by means of a current of air, was no longer under the action

* Hemsalech, *Comptes Rendus de l'Académie des Sciences,* vol cl. p. 1743 (1910); *ibid.* vol. cli. p. 220 & p. 668 (1910).

of an electric field. Furthermore, I have furnished experimental evidence to the effect that the luminous vapour produced in these spark discharges is not the result of a vaporization of the electrodes by heat, but of some direct action of the discharge current upon the molecules on the surface of the electrodes*. Now in view of the fact that a small quantity of luminous metal vapour, although undergoing rapid cooling by a current of air, is capable of emitting light radiations for a measurable time after the exciting agent has ceased to act, should we not, by analogy, actually anticipate a continuation of the line emission, after the breaking of the electric current that caused it, in the case of a vapour which is completely shielded from the surrounding air by a slowly cooling furnace-tube? With his well-protected furnace Dr. King has been able to observe some of the iron lines for as long as 5 minutes after breaking the current. In my small furnace the luminous radiations die out much more rapidly, and the successive extinctions of the various groups of lines is most interesting to follow. The group of doublets at 4957 disappears first at from 5 to 10 seconds after the current is broken. The strong continuous spectrum, which until then masks many of the lines, begins now to clear, and at about 15 seconds after the breaking of the current class I. groups γ and ϵ stand out most conspicuously for a few moments on a dark background. These changes present quite a beautiful spectacle. Thanks to the relatively high luminosity of my spectrograph, it has been possible to secure photographic records, with exposures of only 1 or 2 seconds, at intervals of 10, 15, and 20 seconds after the interruption of the current. Further, temperature determinations of the inner surface of the furnace tube were made at corresponding intervals of time which furnish some interesting data with reference to the rate of cooling of the furnace. The values of these temperatures, which are the means of two readings, are as follows :—

Initial Furnace temperature.	Furnace temperature after an interval of		
	10 seconds.	15 seconds.	20 seconds.
2700° C.	2300°	2100°	2000°

In an additional series of experiments the field was left on partially, the current being dropped to about 180 amperes.

* Hemsalech, *Comptes Rendus de l'Académie des Sciences*, vol. cliv. p. 872 (1912).

In this case the mean values were:

Initial Furnace temperature.	Furnace temperature after an interval of	
	10 seconds.	15 seconds.
2700° C.	2200°	2100°

Thus the values obtained with part of the current on are, for the first 15 seconds and allowing for probable errors which amount to about $\pm 50°$ C., practically the same as with the current completely off.

As has already been stated above, according to visual observations the strong group at 4957 is the first to disappear after the current is broken, and this fact agrees with my former observations on the behaviour of these lines in the spark. Now the photographic records show this group after an interval of 10 seconds, but no longer after 15 seconds from the moment of breaking the current. But, as my observations indicate, the temperature of the furnace after an interval of 10 seconds is down at 2300°, that is to say well below the temperature at which these lines will appear in ordinary circumstances. Hence the spectrum in this case does not at all correspond to the temperature conditions of the furnace, and it seems therefore not to be controlled by temperature. Furthermore, when the current, instead of being broken, was only reduced to 180 amperes so that a slight potential gradient remained, the group at 4957 was photographed after an interval of 15 seconds, *i. e.* at a furnace temperature of only 2100°, and a trace is even visible on a photograph taken 20 seconds after the drop in current. These facts would seem to indicate that the potential, which subsisted after the current had been dropped, was still sufficient to appreciably prolong the life of this group in spite of the low temperature of the furnace-tube. The spectrum which remains visible after these lines have disappeared is caused by thermo-chemical excitation and is identical with that described in the preceding paper.

All the observations recorded in this paragraph are quite consistent with the view that the so-called high-temperature furnace spectrum of iron is of electric origin.

§ 5. *Observations on the Furnace Spectra of Zinc, Copper, Silver, Cobalt, and Nickel.*

Most of these spectra have already been investigated by Dr. King, and my observations go to corroborate in a general way his results. Thus, like Dr. King, I have been unable to obtain a spectroscopic reaction with zinc, even by subjecting it to furnace temperatures up to 2700° C. The open ends of the furnace were in this case provided with mica windows in order to exclude air, the presence of which caused a blue glow to appear near the opening, due no doubt to oxidation of the metal. This glow emitted only a strong continuous spectrum.

When copper was heated in the tube a band spectrum appeared in the blue and green at a temperature of about 2000° C. This band spectrum persisted after the boiling-point of the metal had been passed. But at no time was I able to observe or record photographically a line spectrum. The origin of the bands has not yet been investigated, but it may be connected with the formation and subsequent dissociation of a compound. Similarly, silver gave no line emission whatever, not even at the highest temperature. On the other hand, both nickel and cobalt emitted line spectra at 2700° C.

Thus there are, including iron, two groups of metals which, as regards their spectroscopic reaction in the furnace at high temperatures, behave very differently, namely zinc, copper, and silver which show no reaction, and iron, nickel, cobalt which give well-developed line spectra. It is interesting to inquire whether this difference in behaviour is consistent with the idea that the high-temperature furnace spectrum is caused by the passage of an electric current through the vapours of these metals. It will be remembered that in the course of my researches on the effect of self-induction on the lines emitted by metal vapours in the electric spark, I established the existence of two groups of metals which exhibited striking dissimilarity in so far as the appearance of nitrogen bands in their spectra was concerned *. One group, to which belong zinc, copper, and silver, showed the nitrogen bands very strongly in addition to the lines of the metal, and the other group, which includes iron, cobalt, and nickel, gave hardly a trace of them. Thus in the case of the former group the electric current in the discharge was partly carried by nitrogen ions, whereas in the second case almost entirely by metal vapour. These facts receive a plausible explanation by supposing that the vapours of iron, cobalt, and nickel are better conductors of electricity than those of the metals of the other group; and, if this were the right interpretation in the case of the spark spectra of these metals, it would equally well explain their relative behaviour in the

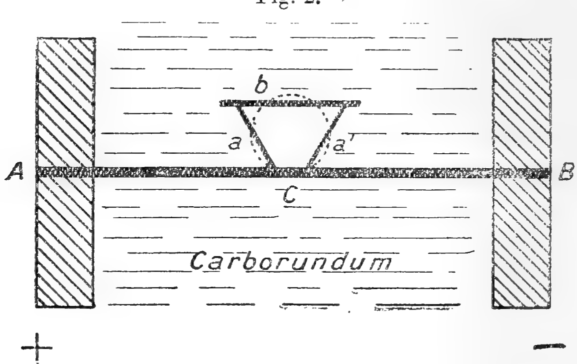
* Hemsalech, *Thèses de Doctorat*, p. 111, Paris, 1901.

furnace. For the better conducting vapours of iron, cobalt, and nickel could thus be conceived to convey an electric current under the pressure of the small potential gradient in the furnace-tube more easily than the less conducting vapours of zinc, copper, and silver; the first three metals would therefore be able to emit a line spectrum at lower potential gradients than the last three metals. Thus the inability of the vapours of zinc, copper, and silver to emit a spectrum at the highest furnace temperature can be satisfactorily accounted for by assuming them to possess a low degree of electric conductivity.

§ 6. *Observations on the Spectrum Emission of Metal Vapours in the absence of Electric Actions.*

All the facts observed so far point to electric actions as being the determining factor in the line emission of metal vapours in the high-temperature resistance-tube furnace. It was therefore felt desirable to devise a test experiment with a furnace in which electric actions were, if not altogether suppressed, at least reduced to a minimum. After several trials the following type of furnace was constructed which seemed to fulfil these conditions (fig. 2). A graphite

Fig. 2.



Sectional View of Plate Furnace.

plate AB, 245 mm. long, 20 mm. wide, and 1·9 mm. thick, is clamped in a horizontal position between two pairs of stout graphite bars which carry the current from the mains. On the middle part of AB, at C, a small furnace is built up of the small graphite plates a, a', and b; each of these plates is 20 mm. wide, and a, a' are about 20 mm. long each, b somewhat longer. The lower edges of a and a' are cut in such a way as to reduce the contact surfaces with the main plate to a few sharp points only, thus preventing any large currents from passing through the plates *a*, *b*, *a′*. Further, the plates *a* and *a′* are inclined at an angle of 60° or less, so that any potential gradient existing between *a* and *a′* would decrease rapidly in value on passing from the bottom of the furnace upwards. Two observation-tubes of carbon, each having an internal diameter of 14 mm. and a length of about 4 inches, are placed in a line, one in front and the other behind the furnace, as indicated by the dotted circle. Observations are made through these tubes, which afford an uninterrupted view of the metal vapours in the furnace. The whole space both beneath and above the main plate, furnace, and observation-tubes is filled up with carborundum-powder to a depth of at least two inches all round, so that this furnace is as efficiently protected as was the tube-furnace in a former experiment. The metal to be examined was laid on the bare portion of the plate AB comprised between the inclined plates *a* and *a′*. The plate was heated by means of direct current of over 300 amperes. As the temperature rose luminous vapours from the walls and carbon particles caused by the disintegration of the graphite gave out a strong continuous spectrum on which were visible the absorption-lines of Na, Ca, and Sr. When iron was boiled in this furnace the interior emitted a brilliant light, but at no time, even up to the burning through of the plate at C, was there observed any trace of an emission spectrum of this element, such as was beheld in the tube-furnace at 2700°. Nor were the Swan bands ever seen with this furnace.

Similar negative results were obtained with copper.

With thallium, however, the green line was observed first as an emission-line and then, at the higher temperatures, as an absorption-line. It is, however, doubtful whether in this case the emission was due to purely thermal excitation; it was indeed found that the metal, already at lower temperatures, rapidly formed a compound which adhered to the furnace walls in flaky masses; and it seems to me more probable that it was the action of heat on this compound which caused the emission of the green line; the emission would therefore have been due to thermo-chemical excitation.

§ 7. *Observations on the Spectrum Emission of Metal Vapours under the Influence of an Electric Field by Means of a New Type of Electric Furnace.*

Having failed to excite the line emission of the vapours of iron and copper by purely thermal actions, it was of course natural to try and obtain the desired effect by the simultaneous application of both thermal and electric forces, thus reproducing the particular conditions which are believed to exist in a resistance-tube furnace. To this end a special type of plate-furnace was built which permitted the establishment within the metal vapour of a potential gradient of any required strength. The principle of this furnace is illustrated in fig. 3. Two graphite plates AB and CD are placed

Fig. 3.

Principle of Two-plate Furnace.

with their flat sides parallel to each other at distances varying from 3 to 10 mm. or more. At one end, A and C, the plates communicate by means of a graphite block of the requisite thickness. At the other end the plates remain insulated from each other, and the extremities B and D are connected to the mains. Now it is evident that when an electric current is sent through the plates under these conditions a potential gradient will be established within the space between the plates, owing to the resistance of the latter. This gradient will have a maximum value between B and D and it will vanish at E. The magnitude of the gradient at the extremities B and D for a given value of the heating current and for graphite plates of given sectional area, will vary directly as the lengths of the plates, and inversely as the distance between them. If the current sent through the plates be of such strength as to raise the temperature of the plates sufficiently to ionize the space between them, an ionization current having a maximum value near the free ends B and D will pass across the space. Further, if the temperature attained be high enough to cause a piece of metal, placed on the lower plate near D, to boil, the whole or part of the ionization current will be carried by the metal vapour, provided that both the electric conductivity of the latter and the strength of the potential gradient be of the requisite magnitude. Hence if the so-called high-temperature line emission of iron vapour be really caused by the passage of an electric current through the vapour, we should with a furnace of this type observe its spectrum. Now, this is indeed what I have observed. Iron showed a most brilliant spectrum in which the group 4957 was as marked as in the tube-furnace at 2700°. Moreover, copper which gave no line spectrum in the tube-furnace, emitted one under the action of the stronger electric field which could be brought to bear upon it in the two-plate furnace.

A fuller discussion of these results, together with the description of a new and more practicable form of plate-furnace, based on the principle explained above, will be reserved for a subsequent communication. Suffice it to point out here that the mode of excitation under these conditions is similar to that underlying the emission of line spectra in the ordinary arc. But whereas in the present case the necessary ionization current is secured and maintained by special means at a relatively low potential gradient, in the arc it is produced and upheld automatically thanks to the existence of a high-potential gradient. The high-temperature furnace spectrum of iron as emitted either by a tube or a two-plate furnace should therefore be regarded as a low-tension arc spectrum. The line spectrum as obtained under these conditions is brought about by the simultaneous actions of heat and of electricity, and the process involved in its emission will be referred to as *thermo-electrical excitation* in distinction from the more purely electrical mode of excitation which occurs in the spark discharge as already mentioned in § 4.

§ 8. *Summary.*

I. All the results of the several observations and experiments carried out in the course of this investigation harmonize with the conclusion that the so-called high-temperature furnace spectrum of iron, which is emitted above the temperature of the boiling-point of this metal, is not caused by purely thermal actions, but requires for its emission the co-operation of electric forces. This conclusion is supported by the following observed facts:—

a. The furnace spectrum of iron at 2700° C. is entirely different from its flame spectrum at the same temperature. §§ 1 & 2.

b. The relative behaviour of class III. lines, especially the group of doublets at 4957, indicates that the high-temperature furnace spectrum of iron approaches in character that of the arc spectrum of this element. § 2.

c. Direct experimental evidence has been furnished to the effect that an ionization current will easily pass through iron vapour in a tube-furnace. § 3.

d. As in the electric spark, the line emission of iron vapour in the furnace does not stop abruptly on the electric field being removed, but continues for some time after, and the extinction of the luminous vibrations is accomplished gradually, class III. lines disappearing first. § 4.

e. The spectrum emitted after the current is broken is not controlled by the temperature of the furnace, as is evidenced by the observation of class III. group 4957 at 2300° C.; no trace of this group is seen in ordinary circumstances even when the temperature of the furnace has been raised to 2400°. § 4.

f. If, instead of completely breaking the current, the latter be only reduced to about 180 amperes, so that a feeble potential gradient is left on, class III. group 4957 remains visible much longer, and it has been photographed when the furnace temperature had fallen to 2100° C. § 4.

g. The absence of a line emission when the vapours of copper, silver, and zinc are subjected to thermo-electrical actions in the electric tube resistance-furnace at 2700°, receives a satisfactory explanation by supposing that they possess a low degree of electric conductivity as compared with the vapours of iron, cobalt, and nickel, which easily emit a line spectrum under the same furnace conditions. This supposition is supported by observations regarding the spark spectra of these metals. § 5.

h. The attempt to excite a line spectrum in iron vapour by purely thermal actions in a furnace of special construction has led to a negative result. § 6.

i. A brilliant line spectrum of iron, similar in character to that observed in the tube-furnace at 2700°, was obtained with a new type of electric furnace in which a potential gradient of any desired strength could be established. § 7.

II. As a result of my researches on flame and furnace spectra some light has been thrown on the various ways in which light radiations may be excited in iron vapour. For the sake of convenience, and also in order to facilitate the distinction between the several modes of excitation which prevail in the flames, furnace, arc and spark, the following denominations have been adopted:—

a. *Thermal excitation.* By this is understood the emission of luminous vibrations by the application of heat alone in the absence of chemical or electric actions. No line or band spectrum has been observed with iron vapour.

b. Thermo-chemical excitation. Here the emission of light radiations is caused by the action of heat on a chemical compound of iron. The component atoms in the compound remain chemically associated and therefore the vibrations emitted are observed to be restricted in development. This mode of excitation prevails in the mantles of all the low and high temperature flames so far examined, as also in the electric tube resistance-furnace up to a temperature of nearly 2500° C.

c. Chemical excitation. This involves the complete decomposition, at a relatively low temperature, of an iron compound and the formation of a new one, owing to the existence of a strong chemical affinity between iron and nitrogen. This mode of excitation has been met with for the first time in the explosion region of the air-coal gas flame. The spectrum to which it gives rise presents a high degree of development.

d. Thermo-electrical excitation. This accompanies the discharge of electricity through iron vapour, which has previously been strongly ionized through the action of heat. It occurs in the electric tube resistance-furnace at temperatures of over 2500° C. and also in the two-plate furnace. The ordinary electric arc between iron poles may be regarded as a special case in which the necessary degree of ionization is maintained automatically by the application of a high voltage.

e. Electrical excitation. Occurs in the capacity and self-induction sparks passing between iron electrodes at ordinary temperature. The radiating vapour is produced by a direct action of the electric discharge on the molecules in the surface-layer of the electrodes. The vapour is hurled into the spark-gap with definite velocity and its luminous vibrations, started in the first instance by the disruption of the molecules at the surface of the electrodes by the initial discharge (capacity spark) or the first oscillation (self-induction spark), are maintained or further developed by the subsequent oscillations.

§ 9. *Concluding Remarks.*

In considering all the various facts observed in connexion with the emission of the spectrum of iron we arrive at the general conclusion that temperature, although often playing an important rôle in bringing about conditions favourable to the effective actions of other agents, does not in itself suffice to excite characteristic line radiations. It would therefore appear premature to establish a temperature classification of the spectrum lines of iron which would embrace the lines observed in such sources as the high-temperature furnace and the arc, in both of which the prevalence of electric actions is so manifest. We know practically nothing about the state of temperature of the radiating atoms in these sources because all the measurements that have been made refer to the inner wall of the furnace or, as in the case of the arc, to the surface of the electrodes only. Further, all the experimental evidence seems to point against the view that the line emission in these sources is caused by direct thermal actions. As my experiments show, iron vapour may indeed be in a state of high temperature without emitting a line spectrum. Both the high degree of temperature in the vapour and the luminous vibrations of its atoms may, in the arc and furnace, represent concurrent manifestations of electric actions. On the other hand the results obtained with thermo-chemical excitation, both in flames and furnace, have revealed a gradual progression in the development of the iron spectrum as the temperature rises and, between the limits of 1500° and 2700° C., it should be possible, wherever this mode of excitation exists, to determine the state of temperature in the source from the degree of development of its spectrum. But whether we should be justified in deriving the state of temperature in an electrical source by extrapolating the values found for flames, is not at all certain. But supposing we did so, this would give us roughly for the high-temperature furnace spectrum of iron values of the order of at least 3000° if we judge by the development of class III. lines. This figure represents no doubt a possible result and, if such temperature determinations could be confirmed by more direct methods, it might perhaps lead to a closer coordination on the basis of equivalent temperature conditions, of the various modes of excitation discussed here.

Before concluding I desire to express my heartiest thanks to Sir Ernest Rutherford for the many acts of kindness with which he has favoured me and for the encouraging interest he has taken in the work.

My thanks are likewise due to Dr. Newbery for the cordial welcome extended to me in the electro-chemical laboratory.

Manchester, May 20th, 1918.

XXX. *The Buckling of Deep Beams.*

By J. PRESCOTT, *M.A., D.Sc. (Manc.)* *.

IT is a very well-known fact that a loaded beam may buckle sideways if the depth is much greater than the breadth, but so far no one seems to have given the mathematical theory of the subject. That theory is supplied in this paper. It is essentially a question of stability of the same type as Euler's problem of the strut but of rather more complexity. A sketch of a buckled beam is shown in fig. 1: it is drawn to suit Case 2.

Fig. 1.

It will be seen that the buckling load depends on the torsional rigidity of the beam as well as on the flexural rigidity for bending in a horizontal plane.

The first case to be considered, and one which leads to quite simple mathematics, is the case of a beam under a uniform bending moment. To make the problem quite clear, suppose a long strip of steel, such as a steel rule a yard long, is acted on at its ends by a pair of opposite couples the planes of which are parallel to the faces of the strip. If the couples are increased gradually there will be a certain limiting magnitude of the couples for which the strip is unstable, and at this stage it will bend sideways; that is, bend in a plane perpendicular to the one in which the couples are acting. The magnitude of this buckling couple will depend, of course, on the way in which the ends are held.

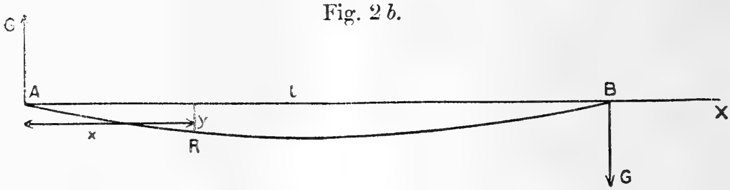
The method of attack is to assume that buckling has actually occurred and to find what couples at the end will maintain the buckled state of the beam.

* Communicated by the Author.

Case 1.—A pair of equal and opposite couples G act at the ends A and B in a vertical plane, as shown in fig. 2 *a*, causing the beam to buckle like a strut, to a greater extent on the thrust

Fig. 2 *a*.

Fig. 2 *b*.



Plan of Central Line of Section.

side than on the tension side, as fig. 2 *a* indicates. The deflexion of the beam in the vertical plane is assumed to be negligible, while the deflexion of the central line of the section in the horizontal plane is comparatively large. This means that the beam is twisted and bent, the curve of the central line being nearly a horizontal curve. Moreover, the amount of twist is so small everywhere that the displacement of a point on the central line may be considered to be perpendicular to the faces of the beam.

The couples G are represented as vectors in fig 2 *b* so that they can be resolved in the required directions.

Let E denote Young's modulus for the beam, C the least moment of inertia of the section, Kn the torsional rigidity, n being the modulus of rigidity.

τ is the angle of twist at any point R the coordinates of which are x, y, with A as origin.

Since the upper edge of the section through R is bent further out than the lower edge it follows that the couple G at the end A has a component about a line parallel to the twisted depth at R. This component has a magnitude Gτ and tends to increase the curvature of the central line. Consequently, by the usual equation for a loaded beam,

$$EC\frac{d^2y}{dx^2} = -G\tau. \qquad (1)$$

The couple G has also another component, of magnitude $G\frac{dy}{dx}$, about the tangent at R, and this component twists the beam just as it would twist a straight prism. Therefore

$$Kn\frac{d\tau}{dx}=N+G\frac{dy}{dx}. \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

The couple N is introduced as a twisting couple at the end to maintain the upright position of the end section if any couple is necessary for this.

Eliminating y from (1) and (2) by differentiating (2) and using the value of $\frac{d^2y}{dx^2}$ from (1) we get

$$Kn\frac{d^2\tau}{dx^2}=-\frac{G^2}{EC}\tau,$$

whence

$$\frac{d\tau^2}{dx^2}=-m^2\tau, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

where

$$m^2=\frac{G^2}{EnCK}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (4)$$

The solution of this equation is

$$\tau=A\sin mx+B\cos mx. \quad . \quad . \quad . \quad . \quad . \quad (5)$$

The conditions to be satisfied at the ends in the present case, since we have assumed that the end sections are held upright, are that

$$\tau=0 \text{ when } x=0$$

$$\text{and when } x=l,$$

l being the length of the beam.

The first of these conditions gives

$$B=0,$$

and the second

$$A\sin ml=0,$$

which means that either A is zero, in which case the beam is not buckled at all, or

$$\sin ml=0,$$

from which

$$ml=\pi,$$

that is,

$$Gl=\pi\sqrt{EnCK}. \quad . \quad . \quad . \quad . \quad . \quad (6)$$

This gives the couple G for which the beam is unstable when the ends are constrained in no way except that the end sections are maintained upright.

Case 2.—The problem just worked out is similar to the strut problem with pin joints at the end so that the direction of the elastic central line is not fixed at the ends. In the buckling question, just as in the strut question, we have also the case where $\frac{dy}{dx}$ is zero at the ends. In order to maintain these end conditions there must be applied at the ends of the beam another pair of couples, which we shall denote by M, these couples acting in a horizontal plane. Then the equations for the equilibrium of the portion AR become

$$\mathrm{EC}\frac{d^2y}{dx^2} = -\mathrm{G}\tau + \mathrm{M}, \quad \ldots \ldots \quad (7)$$

$$\mathrm{K}n\frac{d\tau}{dx} = \mathrm{N} + \mathrm{G}\frac{dy}{dx}. \quad \ldots \ldots \quad (8)$$

From these we get

$$\frac{d^2\tau}{dx^2} = -m^2\left(\tau - \frac{\mathrm{M}}{\mathrm{G}}\right), \quad \ldots \ldots \quad (9)$$

the solution of which is

$$\tau = \mathrm{A}\sin mx + \mathrm{B}\cos mx + \frac{\mathrm{M}}{\mathrm{G}}.$$

The condition $\tau = 0$ when $x = 0$ gives

$$\mathrm{B} = -\frac{\mathrm{M}}{\mathrm{G}}.$$

Therefore

$$\tau = \mathrm{A}\sin mx + \frac{\mathrm{M}}{\mathrm{G}}(1 - \cos mx). \quad \ldots \quad (10)$$

From (8) and (10)

$$\mathrm{G}\frac{dy}{dx} = \mathrm{K}nm\left(\mathrm{A}\cos mx + \frac{\mathrm{M}}{\mathrm{G}}\sin mx\right) - \mathrm{N}. \quad . \quad (11)$$

We have still to satisfy the three conditions

$$\tau = 0 \text{ and } \frac{dy}{dx} = 0 \text{ when } x = l$$

$$\text{and } \frac{dy}{dx} = 0 \text{ when } x = 0.$$

The last of these gives

$$\mathrm{N} = \mathrm{K}nm\mathrm{A}. \quad \ldots \ldots \quad (12)$$

Therefore

$$\mathrm{G}\frac{dy}{dx} = \mathrm{K}nm\left\{\frac{\mathrm{M}}{\mathrm{G}}\sin mx - \mathrm{A}(1 - \cos mx)\right\}.$$

The other two conditions give

$$0 = A \sin ml + \frac{M}{G}(1 - \cos ml),$$

$$0 = \frac{M}{G} \sin ml - A(1 - \cos ml).$$

From these last two equations we get

$$\frac{AM}{G} \sin^2 ml = -\frac{AM}{G}(1 - \cos ml)^2,$$

or

$$\sin^2 ml + (1 - \cos ml)^2 = 0,$$

which can only be satisfied when the two following equations are simultaneously true:

$$\sin ml = 0,$$

$$1 - \cos ml = 0.$$

Therefore

$$ml = 2\pi,$$

or

$$Gl = 2\pi\sqrt{EnCK}. \quad . \quad . \quad . \quad (13)$$

Thus the buckling couple is twice as great as when the ends were not constrained in the y direction.

There is one condition we have not used in arriving at the last solution, namely, that y has the same value at both ends. If we do make use of this condition it only tells us that the couple N is zero. The same is true for the first case we dealt with.

Case 3.—A beam is built into a wall at one end and is

Fig. 3.

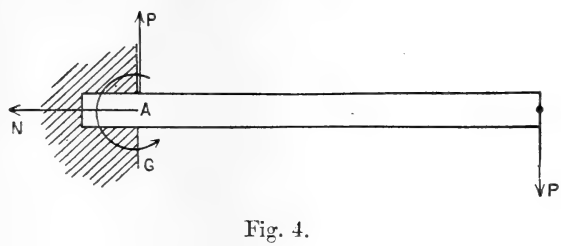
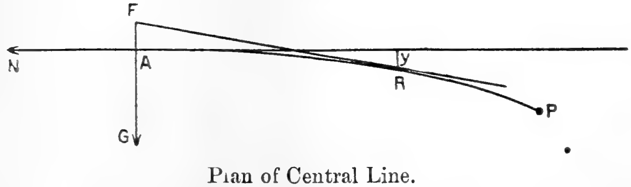


Fig. 4.



Plan of Central Line.

quite free at the other. A load P is applied to the middle of the section at the free end. For the equilibrium of the whole beam in the buckled state there must clearly be a couple N at the fixed end exerting a torsion on the beam, in addition to the couple G whose magnitude is Pl.

The bending moment equation at R is

$$EC\frac{d^2y}{dx^2}=G\tau-Px\tau, \quad . \quad . \quad . \quad . \quad . \quad (14)$$

x being measured from A.

The twisting moment on the section at R of the force P acting at A is

$$AF \times P=\left(x\frac{dy}{dx}-y\right)P.$$

The twisting effect at R of the couple G is the component of G about the tangent RF, that is, $-G\frac{dy}{dx}$ nearly. Therefore the equation for the torsion at R is

$$Kn\frac{d\tau}{dx}=N+\left(x\frac{dy}{dx}-y\right)P-G\frac{dy}{dx}. \quad . \quad . \quad (15)$$

Differentiating this last equation we get

$$Kn\frac{d^2\tau}{dx^2}=Px\frac{d^2y}{dx^2}-G\frac{d^2y}{dx^2}$$

$$=(Px-G)\frac{d^2y}{dx^2}. \quad . \quad . \quad . \quad . \quad . \quad (16)$$

Using the value of $\frac{d^2y}{dx^2}$ from (14) we get, on putting Pl for G,

$$Kn\frac{d^2\tau}{dx^2}=-\frac{P^2}{EC}(l-x)^2\tau$$

or

$$\frac{d^2\tau}{dx^2}=-m^4(l-x)^2\tau, \quad . \quad . \quad . \quad . \quad (17)$$

where

$$m^4=\frac{P^2}{EnCK}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (18)$$

It is clearly more convenient to measure x from the free end of the beam. This means using x for $(l-x)$. Then, since $\frac{d^2\tau}{dx^2}$ remains unaltered,

$$\frac{d^2\tau}{dx^2}=-m^4x^2\tau. \quad . \quad . \quad . \quad . \quad . \quad . \quad (19)$$

If we make the substitutions

$$s=\tfrac{1}{2}m^2x^2$$

and then

$$\tau=zs^{\frac{1}{4}},$$

the resulting differential equation is

$$\frac{d^2z}{ds^2}+\frac{1}{s}\frac{dz}{ds}+\left(1-\frac{1}{16s^2}\right)z=0, \quad . \quad . \quad . \quad (20)$$

which is the equation for Bessel Functions of order $\frac{1}{4}$. The solution is

$$z=a\mathrm{J}_{\frac{1}{4}}(s)+b\mathrm{J}_{-\frac{1}{4}}(s). \quad . \quad . \quad . \quad . \quad (21)$$

It is quite easy to solve (19) at once by a series of powers of x. The solution in series form, obtained either by using (21) or directly from (19) is

$$\tau=a\left\{x-\frac{m^4x^5}{4.5}+\frac{m^8x^9}{4.5.8.9}-\right\}$$

$$+b\left\{1-\frac{m^4x^4}{3.4}+\frac{m^8x^8}{3.4.7.8}-\right\}. \quad . \quad . \quad (22)$$

At the free end, where $x=0$, there is no twisting couple and therefore

$$\frac{d\tau}{dx}=0 \text{ when } x=0.$$

This makes $a=0$.

Another condition is that $\tau=0$ at the fixed end where $x=l$. This means that m is given by the equation

$$1-\frac{m^4l^4}{3.4}+\frac{m^8l^8}{3.4.7.8}-\&c.=0. \quad . \quad . \quad . \quad (23)$$

An approximate solution of this equation can be got by dropping all the terms except the first three. The result can then be improved by using the approximate result in the remaining terms. This method is not very laborious. Or we can get the solution from a formula for the zeros of Bessel functions, for this last equation amounts to the same as

$$\mathrm{J}_{-\frac{1}{4}}(\tfrac{1}{2}m^2l^2)=0. \quad . \quad . \quad . \quad . \quad . \quad (24)$$

The solution of this equation is approximately

$$\tfrac{1}{2}m^2l^2=2{\cdot}006,$$

whence

$$\mathrm{P}l^2=4{\cdot}012\sqrt{\mathrm{E}n\mathrm{CK}}. \quad . \quad . \quad . \quad . \quad (25)$$

Case 4.—The beam carries a load P at the middle and is supported at the ends, the only couples at the ends being such torsional couples as will keep the depth vertical there.

To make the conditions precise it should be stated that the load at the middle and the supporting forces at the ends are applied at the central points of the sections.

In this case a couple N acts at each end as shown at the end A in fig. 3. Also a force $\frac{1}{2}$P acts at each end to support the load. The couple G does not act in this case.

Measuring x from one end the equations for the equilibrium of a portion of the beam, obtained in the same way as equations (14) and (15), are

$$\mathrm{EC}\frac{d^2y}{dx^2} = -\tfrac{1}{2}\mathrm{P}x\tau, \qquad (26)$$

$$\mathrm{K}n\frac{d\tau}{dx} = \mathrm{N} + \tfrac{1}{2}\left(x\frac{dy}{dx} - y\right)\mathrm{P}. \qquad (27)$$

From these we get

$$\mathrm{K}n\frac{d^2\tau}{dx^2} = \tfrac{1}{2}\mathrm{P}x\frac{d^2y}{dx^2},$$

that is

$$\frac{d^2\tau}{dx^2} = -\frac{\mathrm{P}^2}{4\mathrm{E}n\mathrm{CK}}x^2\tau$$

$$= -m^4x^2\tau, \qquad (28)$$

where now

$$m^4 = \tfrac{1}{4}\frac{\mathrm{P}^2}{\mathrm{E}n\mathrm{CK}}. \qquad (29)$$

Equation (28) differs from equation (19) only in having $\frac{1}{2}$P instead of P. The conditions to be satisfied in this case are, however,

$$\tau = 0 \text{ when } x = 0$$

$$\frac{d\tau}{dx} = 0 \text{ when } x = \tfrac{1}{2}l.$$

This latter condition makes the twist a maximum at the middle, which is clearly the actual state of affairs in the most stable position of the beam.

The series for τ are exactly the same as in equation (22), and the condition that $\tau = 0$ when $x = 0$ makes the constant b

equal to zero. Then

$$\tau = ax\left\{1 - \frac{m^4x^4}{4\,.\,5} + \frac{m^8x^8}{4\,.\,5\,.\,8\,.\,9} - \frac{m^{12}x^{12}}{4\,.\,5\,.\,8\,.\,9\,.\,12\,.\,13} + \quad\right\}. \quad (30)$$

Therefore

$$\frac{d\tau}{dx} = a\left\{1 - \frac{m^4x^4}{4} + \frac{m^8x^8}{4\,.\,5\,.\,8} - \quad\right\}. \quad . \quad . \quad (31)$$

This has to be zero when $x = \frac{1}{2}l$.

Writing s for $\frac{1}{16}m^4l^4$ the equation for s is

$$1 - \frac{s}{4} + \frac{s^2}{4\,.\,5\,.\,8} - \frac{s^3}{4\,.\,5\,.\,8\,.\,9\,.\,12} + \quad = 0. \quad . \quad (32)$$

It is worth while to show how this can be solved.

The equation can be written, after multiplying up by 4 . 5 . 8,

$$s^2 - 40s = -160 + \frac{s^3}{9\,.\,12} - \frac{s^4}{9\,.\,12\,.\,13\,.\,16} + \quad ,$$

that is

$$(s-20)^2 = 400 - 160 + \frac{s^3}{9\,.\,12} -$$

$$= 240 + \frac{s^3}{9\,.\,12} - \quad .$$

Neglecting the cube and all higher powers of s we get

$$s - 20 = \pm\sqrt{240} = \pm 15{\cdot}5 \text{ approximately.}$$

Since we are seeking the smallest root of our equation, this smallest root being the one that corresponds to the most stable state, just as in the case of Euler's strut problems, we must take the negative sign on the right. Then

$$s = 4{\cdot}5 \text{ approximately.}$$

If we now use this approximate value of s in the terms containing s^3 and s^4, and add the correction for these terms on to the 240, we get

$$(s-20)^2 = 240 + \frac{4{\cdot}5^3}{9\,.\,12} - \frac{4{\cdot}5^4}{9\,.\,12\,.\,13\,.\,16}$$

$$= 240{\cdot}83,$$

$$s = 20 - \sqrt{240{\cdot}83}$$

$$= 4{\cdot}48.$$

Now using the last value of s in the s^3 and s^4 terms we get

$$(s-20)^2=240+\frac{4{\cdot}48^3}{9\,.\,12}-\frac{4{\cdot}48^4}{9\,.\,12\,.\,13\,.\,16}$$

$$=240{\cdot}814,$$

$$s=20-\sqrt{240{\cdot}814}$$

$$=20-15{\cdot}518$$

$$=4{\cdot}482. \quad \ldots \quad (33)$$

This is probably correct to the last figure. It follows that

$$P^2l^4=64\times 4{\cdot}482\,\mathrm{E}n\mathrm{CK},$$

$$Pl^2=16{\cdot}94\sqrt{\mathrm{E}n\mathrm{CK}}. \quad \ldots \quad (34)$$

Case 5.—The beam carries a concentrated load at the middle as in the last case, but the ends are constrained as in Case 2; that is, a pair of couples act on the ends in a horizontal plane preventing the ends from bending sideways.

The difference between this and the last case is that there is an unknown horizontal couple M at each end, and $\frac{dy}{dx}$ is zero at the ends.

Measuring x from one end the equations of equilibrium are

$$\mathrm{EC}\frac{d^2y}{dx^2}=-\tfrac{1}{2}\mathrm{P}x\tau+\mathrm{M}, \quad \ldots \quad (35)$$

$$\mathrm{K}n\frac{d\tau}{dx}=\mathrm{N}+\tfrac{1}{2}\left(x\frac{dy}{dx}-y\right)\mathrm{P}. \quad \ldots \quad (36)$$

From these we get

$$\mathrm{K}n\frac{d^2\tau}{dx^2}=\tfrac{1}{2}\mathrm{P}x\frac{d^2y}{dx^2}$$

$$=\frac{-\mathrm{P}^2}{4\mathrm{EC}}x^2\tau+\frac{\mathrm{PM}}{2\mathrm{EC}}x,$$

or

$$\frac{d^2\tau}{dx^2}=-m^4x^2\tau+6bx, \quad \ldots \quad (37)$$

where

$$m^4=\frac{\mathrm{P}^2}{4\mathrm{E}n\mathrm{CK}}, \quad \ldots \quad (38)$$

$$6b=\frac{\mathrm{PM}}{2\mathrm{E}n\mathrm{CK}}.$$

The solution of equation (37) in series is

$$\tau = a_0 \left\{1 - \frac{m^4x^4}{3\,.\,4} + \frac{m^8x^8}{3\,.\,4\,.\,7\,.\,8} - \ldots\right\}$$

$$+ a_1x\left\{1 - \frac{m^4x^4}{4\,.\,5} + \frac{m^8x^8}{4\,.\,5\,.\,8\,.\,9} - \ldots\right\}$$

$$+ bx^3\left\{1 - \frac{m^4x^4}{6\,.\,7} + \frac{m^8x^8}{6\,.\,7\,.\,10\,.\,11} - \ldots\right\}. \quad (39)$$

The conditions to be satisfied are

$$\left.\begin{aligned} \tau &= 0 \\ y &= 0 \\ \frac{dy}{dx} &= 0 \end{aligned}\right\} \text{when } x = 0,$$

$$\left.\begin{aligned} \frac{d\tau}{dx} &= 0 \\ \frac{dy}{dx} &= 0 \end{aligned}\right\} \text{when } x = \tfrac{1}{2}l.$$

To satisfy the first of these a_0 must be zero. In order to make use of all the other conditions we have to find y.

From (36)

$$\frac{1}{x}\frac{dy}{dx} - \frac{y}{x^2} = \frac{2\mathrm{K}n}{\mathrm{P}}\frac{1}{x^2}\frac{d\tau}{dx} - \frac{2\mathrm{N}}{\mathrm{P}x^2},$$

that is,

$$\frac{d}{dx}\left(\frac{y}{x}\right) = \frac{2\mathrm{K}n}{\mathrm{P}}\frac{1}{x^2}\frac{d\tau}{dx} - \frac{2\mathrm{N}}{\mathrm{P}x^2}$$

$$= \frac{2\mathrm{K}n}{\mathrm{P}}\left[\begin{aligned} &a_1\left\{\frac{1}{x^2} - \frac{m^4x^2}{4} + \frac{m^8x^6}{4\,.\,5\,.\,8} - \ldots\right\} \\ &+ b\ \left\{3 - \frac{m^4x^4}{6} + \frac{m^8x^8}{6.7.10} - \ldots\right\}\end{aligned}\right]$$

$$- \frac{2\mathrm{N}}{\mathrm{P}x^2}. \quad (40)$$

Therefore

$$\frac{y}{x}=\mathrm{H}+\frac{2\mathrm{N}}{\mathrm{P}x}$$
$$+\frac{2\mathrm{K}n}{\mathrm{P}}\left[\begin{array}{l} a_1\left\{-\frac{1}{x}-\frac{m^4x^3}{3.4}+\frac{m^8x^7}{4.5.7.8}-\ldots\right\} \\ +b\left\{3x-\frac{m^4x^5}{5.6}+\frac{m^8x^9}{6.7.9.10}-\ldots\right\}\end{array}\right]$$

and . . . (41)

$$\frac{dy}{dx}=\mathrm{H}+\frac{2\mathrm{K}n}{\mathrm{P}}\left[\begin{array}{l} a_1\left\{-\frac{m^4x^3}{3}+\frac{m^8x^7}{4.5.7}-\frac{m^{12}x^{11}}{4.5.8.9.11}+\right\} \\ +b\left\{6x-\frac{m^4x^5}{5}+\frac{m^8x^9}{6.7.9}-\ldots\right\}\end{array}\right]$$

. . . (42)

The condition

$$y=0 \text{ when } x=0$$

is satisfied by making

$$\frac{2\mathrm{N}}{\mathrm{P}}=\frac{2\mathrm{K}n}{\mathrm{P}}a_1.$$

The condition

$$\frac{dy}{dx}=0 \text{ when } x=0$$

is satisfied by making $\mathrm{H}=0$.

We have now satisfied all the conditions except those at the middle. If we write s for $\frac{ml^4}{16}$ the remaining conditions, namely,

$$\frac{d\tau}{dx}=0 \text{ and } \frac{dy}{dx}=0 \text{ when } x=\tfrac{1}{2}l,$$

are equivalent to

$$a_1\left\{1-\frac{s}{4}+\frac{s^2}{4.5.8}-\frac{s^3}{4.5.8.9.12}+\right\}$$
$$=-bx^2\left\{3-\frac{s}{6}+\frac{s^2}{6.7.10}-\frac{s^3}{6.7.10.11.14}+\ldots\right\} \quad (43)$$

and

$$a_1\left\{\frac{s}{3}-\frac{s^2}{4.5.7}+\frac{s^3}{4.5.8.9.11}-\ldots\right\}$$
$$=+bx^2\left\{6-\frac{s}{5}+\frac{s^2}{6.7.9}-\frac{s^3}{6.7.10.11.13}+\ldots\right\} \quad (44)$$

Writing, for the sake of shortness in the argument,

$$\left.\begin{aligned} X&=1-\frac{s}{4}+\frac{s^2}{4.5.8}-\frac{s^3}{4.5.8.9.12}+ \\ Y&=3-\frac{s}{6}+\frac{s^2}{6.7.10}-\frac{s^3}{6.7.10.11.14}+ \\ V&=\frac{s}{3}-\frac{s^2}{4.5.7}+\frac{s^3}{4.5.8.9.11}- \\ W&=6-\frac{s}{5}+\frac{s^2}{6.7.9}-\frac{s^3}{6.7.10.11.13}+ \end{aligned}\right\}, \ldots (45)$$

equations (43) and (44) become

$$a_1 X = -bx^2 Y, \quad . \quad . \quad (43)$$

$$a_1 V = \quad bx^2 W. \quad . \quad . \quad (44)$$

By division

$$\frac{X}{V} = -\frac{Y}{W}$$

or

$$XW + YV = 0. \quad . \quad . \quad . \quad . \quad . \quad (46)$$

Equation (46) has to be solved for s, and this will give the critical load.

The smallest value of s satisfying (46) is

$$s = 10{\cdot}47 \text{ approximately.}$$

Therefore

$$\frac{P^2 l^4}{64 EnCK} = 10{\cdot}47$$

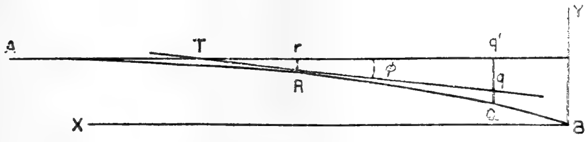
or

$$Pl^2 = 8 \times 3{\cdot}236 \sqrt{EnCK}$$

$$= 25{\cdot}89 \sqrt{EnCK}. \quad . \quad . \quad . \quad (47)$$

Case 6.—The beam carries a load W uniformly distributed

Fig. 5.



Plan of Central Line AB.

along its length, is quite free at one end, and held rigidly at the other.

Let w be the load per unit length, so that

$$W = wl.$$

The origin is taken at the end B, x being measured towards A, and y upwards in the figure. TRq is the tangent at R.

Consider the equilibrium of the portion RB of length x.

The moment about Rr of the weight RB is $\frac{1}{2}wx^2$, and this has a component about the depth of the twisted section at R of magnitude $\frac{1}{2}wx^2\tau$. This causes bending in the xy plane.

Therefore

$$EC\frac{d^2y}{dx^2} = -\tfrac{1}{2}wx^2\tau. \quad . \quad . \quad . \quad . \quad . \quad (48)$$

Let the coordinates of Q be (x', y'). Then the twisting couple on the section at R due to the weight wdx' near Q is

$$wdx' \times Qq.$$

Since τ decreases as x increases it follows that the total twisting couple at R is

$$-Kn\frac{d\tau}{dx} = \int_0^x wQq\,dx'. \quad . \quad . \quad . \quad . \quad (49)$$

Differentiating this with respect to the upper limit x,

$$-Kn\frac{d^2\tau}{dx^2} = \Big[w \cdot Qq\Big]^{x'=x} + w\int_0^x \frac{d(Qq)}{dx}dx'.$$

But
$$Qq = y - y' - rq' \times \tan\phi$$
$$= y - y' - (x - x')\frac{dy}{dx};$$

whence
$$\frac{d}{dx}(Qq) = \frac{dy}{dx} - \frac{dy}{dx} - (x-x')\frac{d^2y}{dx^2} = -(x-x')\frac{d^2y}{dx^2}$$

and
$$\Big[Qq\Big]^{x'=x} = 0.$$

Therefore

$$-Kn\frac{d^2\tau}{dx^2} = -w\int_0^x (x-x')\frac{d^2y}{dx^2}dx'$$
$$= -w\frac{d^2y}{dx^2}\Big[xx' - \tfrac{1}{2}x'^2\Big]_0^x$$
$$= -\tfrac{1}{2}wx^2\frac{d^2y}{dx^2}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (50)$$

From equations (48) and (50) we get

$$\mathrm{K}n\frac{d^2\tau}{dx^2}=-\frac{w^2}{4\mathrm{EC}}x^4\tau \quad . \quad . \quad . \quad . \quad (51)$$

or

$$\frac{d^2\tau}{dx^2}=-m^6x^4\tau, \quad . \quad . \quad . \quad . \quad . \quad (52)$$

where

$$m^6=\frac{w^2}{4\mathrm{E}n\mathrm{CK}}. \quad . \quad . \quad . \quad . \quad . \quad (53)$$

The solution of (52) in series is

$$\tau=a_0\left\{1-\frac{m^6x^6}{5\,.\,6}+\frac{m^{12}x^{12}}{5\,.\,6\,.\,11\,.\,12}-\ldots\right\}$$

$$+a_1x\left\{1-\frac{m^6x^6}{6\,.\,7}+\frac{m^{12}x^{12}}{6\,.\,7\,.\,12\,.\,13}-\ldots\right\} \quad . \quad (54)$$

At the free end, where x is zero, the twisting couple is zero; that is, $\frac{d\tau}{dx}$ is zero. This makes $a_1=0$.

At the other end, where $x=l$, the twist τ is zero. Therefore

$$0=1-\frac{m^6l^6}{5\,.\,6}+\frac{m^{12}l^{12}}{5\,.\,6\,.\,11\,.\,12}-\frac{m^{18}l^{18}}{5\,.\,6\,.\,11\,.\,12\,.\,17\,.\,18}+\ldots$$

The smallest root of this is

$$m^6l^6=41{\cdot}30:$$

that is,

$$\frac{w^2l^6}{4\mathrm{E}n\mathrm{CK}}=41{\cdot}30,$$

from which

$$\mathrm{W}l^2=2\sqrt{41{\cdot}30}\sqrt{\mathrm{E}n\mathrm{CK}}$$

$$=12{\cdot}86\sqrt{\mathrm{E}n\mathrm{CK}}. \quad . \quad . \quad . \quad . \quad (55)$$

Case 7.—The beam carries a total load W distributed as a

Fig. 6.

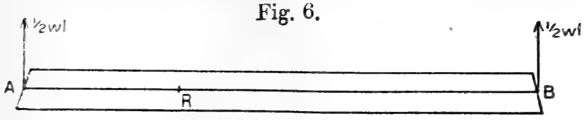
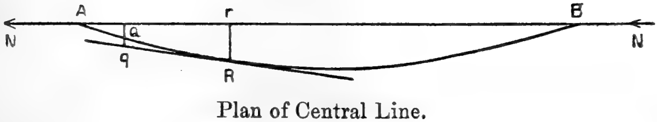


Fig. 7.



Plan of Central Line.

uniform load w per unit length and is supported at the ends as in Case 4.

Here the origin is taken at one end A, and y is measured towards the side to which the beam buckles. For the point R on the central line $Ar=x$, $Rr=y$.

A pair of couples N act at the ends in this case to keep the twist zero there.

The moment about Rr of the forces on the part AR is

$$\tfrac{1}{2}wlx-\tfrac{1}{2}wx^2.$$

The component of this about the depth of the twisted section at R is

$$\tfrac{1}{2}wx(l-x)\tau.$$

Therefore

$$EC\frac{d^2y}{dx^2}=-\tfrac{1}{2}wx(l-x)\tau. \quad . \quad . \quad . \quad . \quad (56)$$

The twisting couple at R due to the uniform load on AR is expressed by the same integral as in the last case. The twisting couple due to the force and couple at A is

$$N-\tfrac{1}{2}wl\left(y-x\frac{dy}{dx}\right).$$

In the present case τ increases as x increases so that $\frac{d\tau}{dx}$ is positive. The total twisting couple at R is

$$Kn\frac{d\tau}{dx}=N-\tfrac{1}{2}wl\left(y-x\frac{dy}{dx}\right)+\int_0^x wQq\,dx'. \quad . \quad (57)$$

Therefore

$$Kn\frac{d^2\tau}{dx^2}=\tfrac{1}{2}wlx\frac{d^2y}{dx^2}-\tfrac{1}{2}wx^2\frac{d^2y}{dx^2}$$

$$=\tfrac{1}{2}wx(l-x)\frac{d^2y}{dx^2}$$

$$=-\tfrac{1}{4}\frac{w^2}{EC}x^2(l-x)^2\tau, \quad . \quad . \quad . \quad . \quad . \quad . \quad (58)$$

and consequently

$$\frac{d^2\tau}{dx^2}=-m^2x^2(l-x)^2\tau, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (59)$$

where

$$m^2=\frac{w^2}{4EnCK}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (60)$$

Putting $X=x-\tfrac{1}{2}l$,

thus measuring X from the middle of the beam, our differential equation becomes

$$\frac{d^2\tau}{dX^2}=-m^2(X^2-\tfrac{1}{4}l^2)^2\tau. \quad . \quad . \quad . \quad . \quad . \quad (61)$$

Now putting

$$X = \tfrac{1}{2}ls$$

we get

$$\frac{d^2\tau}{ds^2} = -\frac{m^2l^6}{64}(s^2-1)^2\tau$$

$$= -c^2(s^2-1)^2\tau, \quad \ldots \ldots \quad (62)$$

where

$$c^2 = \frac{w^2l^6}{16^2 \mathrm{E}n\mathrm{CK}} \quad \ldots \ldots \ldots \quad (63)$$

We want a solution of this last equation which will make τ a maximum at the middle of the beam, where $s=0$, and τ zero at the ends, where $s=\pm 1$ or where $s^2=1$.

Assuming

$$\tau = a_0 + a_2s^2 + a_4s^4 + a_6s^6 + \quad \ldots \ldots \quad (64)$$

and substituting in the differential equation we get

$$2a_2 + 4\,.\,3a_4s^2 + 6\,.\,5a_6s^4 +$$
$$= -c^2(s^4 - 2s^2 + 1)\{a_0 + a_2s^2 + a_4s^4 + a_6s^6 + \ldots\}.$$

Equating coefficients of like powers of s we find

$$\left.\begin{aligned} 2a_2 &= -c^2a_0, \\ 4\,.\,3a_4 &= -c^2(a_2 - 2a_0), \\ 6\,.\,5a_6 &= -c^2(a_4 - 2a_2 + a_0), \\ 8\,.\,7a_8 &= -c^2(a_6 - 2a_4 + a_2), \\ &\text{etc.} \end{aligned}\right\} \quad \ldots \quad (65)$$

By means of these equations each of the coefficients can be expressed as the product of a_0 and a function of c. Thus

$$\tau = a_0\left\{1 - \frac{c^2s^2}{2} + \frac{c^2(c^2+4)}{\underline{|4}}s^4 - \ldots\right\} \quad \ldots \ldots \quad (66)$$

Since this involves only even powers of s it follows that τ must be either a maximum or a minimum when $s=0$, and if we choose the proper value of c then τ will be a maximum.

To make $\tau=0$ when $s=\pm 1$ we have to satisfy the equation

$$0 = 1 - \frac{c^2}{2} + \frac{c^2(c^2+4)}{\underline{|4}} - \quad \ldots \ldots \ldots \quad (67)$$

or

$$0 = 1 + \frac{a_2}{a_0} + \frac{a_4}{a_0} + \frac{a_6}{a_0} + \ldots \quad \ldots \ldots \quad (68)$$

The terms $\frac{a_2}{a_0}$, $\frac{a_4}{a_0}$, etc. are functions of c^2 only, the numerical values of which are quickly calculated for a given value of c^2 by means of equations (65). If we write

$$f(c^2) = 1 + \frac{a_2}{a_0} + \frac{a_4}{a_0} + \ldots .$$

then $f(2)$, $f(3)$, $f(4)$, can be calculated and the results plotted. The curve gives an approximate root of equation (68). The root is, in fact, very near 3. By this process and then by successive approximations it was found that

$$c^2 = 3{\cdot}131. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (69)$$

Therefore

$$wl^3 = 16\sqrt{3{\cdot}131}\sqrt{\text{E}n\text{CK}};$$

that is

$$\text{W}l^2 = 28{\cdot}31\sqrt{\text{E}n\text{CK}}. \quad . \quad . \quad . \quad . \quad . \quad (70)$$

The foregoing are the simplest cases. There are still many more cases to be worked out, as, for example, the beam with uniform load and clamped ends; the beam with a single load not equidistant from the ends; or again, the cases of a load applied at the top or bottom of a section instead of at the middle of a section. But most or all of these new cases will lead to troublesome equations for the critical loads, such equations as Case 5 led to, or worse. Time and assiduity are, however, all that are necessary for the solution of fresh cases.

It is worth while to make one comparison with Euler's strut formulæ.

The critical thrust R, applied at each end of a rod of length l, for which the rod just fails when the ends are not constrained in any way, is given by

$$\text{R}l^2 = \pi^2\text{EC}.$$

Thus R is proportional to the flexural rigidity of the beam and to the inverse square of the length.

The case of buckling that may be compared with the strut is Case 4. Here

$$\text{P}l^2 = 16{\cdot}94\sqrt{\text{ECK}n}.$$

Thus P is proportional to the geometric mean between the flexural rigidity EC and the torsional rigidity Kn, and to the inverse square of l. Also

$$\frac{\text{P}}{\text{R}} = \frac{16{\cdot}94}{\pi^2}\sqrt{\frac{\text{K}n}{\text{EC}}} = 1{\cdot}704\sqrt{\frac{\text{K}n}{\text{EC}}}.$$

That is, the ratio of the buckling load to the Euler thrust is proportional to the square root of the ratio of the torsional rigidity to the flexural rigidity.

XXXI. *A proposed Hydraulic Experiment.*

By Lord Rayleigh, *O.M., F.R.S.**

IN an early paper† Stokes showed "that in the case of a homogeneous incompressible fluid, whenever $udx + vdy + wdz$ is an exact differential, not only are the ordinary equations of fluid motion satisfied, but the equations obtained when friction is taken into account are satisfied likewise. It is only the equations of condition which belong to the boundaries of the fluid that are violated." In order to satisfy these also, it is only necessary to suppose that every part of the solid boundaries is made to move with the velocity which the fluid in irrotational motion would there assume. There is no difficulty in the supposition itself; but the only case in which it could readily be carried into effect with tolerable completeness is for the two-dimensional motion of fluid between coaxial cylinders, themselves made to rotate in the same direction with circumferential velocities which are inversely as the radii. Experiments upon these lines, but not I think quite satisfying the above conditions, have been made by Conette and Mallock. It would appear that, except at low velocities, the simple steady motion becomes unstable.

But the point of greatest interest is not touched in the above example. It arises when fluid passing along a uniform or contracting pipe, or channel, arrives at a place where the pipe expands. It is known that if the expansion be sufficiently gradual, the fluid generally speaking follows the walls, or, as it is often expressed, the pipe flows *full*; and the loss of velocity accompanying the increased section is represented by an augmentation of pressure, approximately according to Bernoulli's law. On the other hand, if in order to effect the conversion of velocity into pressure more rapidly, the expansion be made too violently, the fluid refuses to follow the walls, eddies result, and mechanical energy is lost by fluid friction. According to W. Froude's generally accepted view, the explanation is to be sought in the loss of velocity near the walls in consequence of fluid friction, which is such that the fluid in question is unable to penetrate into what should be the region of higher pressure beyond.

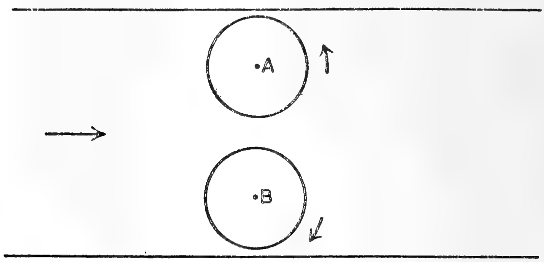
It would be a difficult matter to satisfy the necessary

* Communicated by the Author.

† Camb. Trans. vol. ix. p. [8], 1850; Math. and Phys. Papers, vol. iii. p. 73.

conditions for the walls of an expanding channel, even in two dimensions. The travelling bands of which the walls would be constituted should assume different velocities at different parts of their course. But it is quite possible that a very rough approximation to theoretical requirements would throw interesting light upon the subject, and I write in the hope of pursuading some one with the necessary facilities, such as are to be found in some hydraulic laboratories, to undertake a comparatively simple experiment.

What I propose is the observation of the flow of liquid between two cylinders A, B (probably brass tubes), revolving about their axes in opposite directions. The diagram will



sufficiently explain the idea. The circumferential velocity of the cylinders should not be less than that of irrotational fluid in contact with the walls at the narrowest place. The simple motion may be unstable; but, as I have had occasion to remark before*, the critical situation would be so quickly traversed that perhaps the instability may be of little consequence. If no marked difference in the character of the flow could be detected by colour streaks, whether the cylinders were turning or not, the inference would be that Froude's explanation is inadequate. In the contrary event the question would arise whether practical advantage could be taken by specially stimulating the motion of fluid near the walls of expanding channels, *e. g.* with the aid of steam jets.

* Phil. Mag. vol. xxvi. p. 776 (1913).

XXXII. *A Diffraction Problem. Supplementary Note.* *By* R. Hargreaves*.

A CORRESPONDENT writes with reference to my paper in the August number, "Your formula in terms of one potential applies I presume to electric waves as well as sound waves, when oblique." The formula does apply to electric waves; the function which for sound is a potential, is for electric waves a stream function. Also where the incident wave implies a third dimension, something more than the normal use of a stream function is involved.

As no reference to electric waves was made in the paper, it may be of advantage to supplement it by showing how the electromagnetic quantities are derived from the function, and what is the polarization in each case. The plane of polarization is, in general, subject to deflexion: a test could therefore be made by the use of short electric waves and a metallic screen†.

§1. If independence of z is assumed in Maxwell's equations of types

$$\frac{1}{V}\frac{\partial X}{\partial t}=\frac{\partial c}{\partial y}-\frac{\partial b}{\partial z} \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

and

$$\frac{1}{V}\frac{\partial a}{\partial t}=\frac{\partial Y}{\partial z}-\frac{\partial Z}{\partial y}, \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

the first two of (1) are equivalent to expressions for cXY in terms of ψ, viz.,

$$c=\frac{1}{V}\frac{\partial\psi}{\partial t},\ X=\frac{\partial\psi}{\partial y},\ Y=-\frac{\partial\psi}{\partial x};\ Z=0,\ a=0\quad b=0. \quad (3\,a)$$

The third equation of (2) then gives

$$\frac{1}{V^2}\frac{\partial^2\psi}{\partial t^2}=\frac{\partial^2\psi}{\partial x^2}+\frac{\partial^2\psi}{\partial y^2} \quad . \quad . \quad . \quad . \quad . \quad . \quad (4)$$

The components Zab are unconnected with cXY, and may

* Communicated by the Author.

† My correspondent is Sir Joseph Larmor, to whom I am indebted for the suggestion of this experimental test, as well as for the reminder that it may be of service to the physicist to make an explicit statement on the electrical problem.

be taken as zero to form the system written in (3 *a*). The form of (3 *a*) and the connexion with the divergence equation $\frac{\partial X}{\partial x}+\frac{\partial Y}{\partial y}=0$ for Z=0, show that ψ is a stream function in two dimensions. The condition $\frac{\partial \psi}{\partial y}$ on the barrier gives X=0 as well as Z=0, *i e.*, tangential electrical action vanishes. For sound waves where ψ is a potential, this is the condition of no velocity at right angles to the barrier.

Also the use of $\psi_0=\cos k(Vt+x\sin\alpha+y\cos\alpha)$ in (3*a*) for the incident wave, makes

$$c=-k\sin k(\quad),\ X=-k\cos\alpha\sin k(\quad),\ Y=k\sin\alpha\sin k(\quad).$$

The electric vector is therefore perpendicular to the axis of z and of amount

$$Y\sin\alpha-X\cos\alpha=k\sin k(Vt+x\sin\alpha+y\cos\alpha).$$

A second group has

$$Z=\frac{1}{V}\frac{\partial\psi}{\partial t},\ a=-\frac{\partial\psi}{\partial y},\ b=\frac{\partial\psi}{\partial x};\ c=0,\ X=0,\ Y=0. \quad (3\,b)$$

Here the condition $\frac{\partial\psi}{\partial t}=0$ gives a zero value for tangential electrical action on the barrier, while for sound it gives zero pressure. The electric vector in the incident wave is parallel to the axis of z.

To deal with an arbitrary plane of polarization, we may write $\psi_0=\cos\gamma\cos k(Vt+x\sin\alpha+y\cos\alpha)$ in (3*a*), $\sin\gamma\cos k(\quad)$ in (3*b*), and add the values of XYZ derived from the two solutions. For the incident wave then

$$X=-k\cos\alpha\cos\gamma\sin k(Vt+x\sin\alpha+y\cos\alpha),$$

$$Y=k\sin\alpha\cos\gamma\sin k(\quad),\qquad Z=-k\sin\gamma\sin k(\quad).$$

§ 2. Where the incident wave is

$$\psi_0=\cos k(Vt+lx+my+nz),$$

there cannot be complete independence of z as in the preceding work. It is a matter of intuition to perceive that the rôle of z is limited to its phase-effect. The mathematical expression of this limitation is that $\frac{\partial}{\partial z}=\frac{n}{V}\frac{\partial}{\partial t}$ in the equations (1) and (2). The first case is now

$$X=\frac{\partial\psi}{\partial y},\ Y=-\frac{\partial\psi}{\partial x},\ Z=0;$$

$$a=nY,\ b=-nX,\ c=\frac{1-n^2}{V}\frac{\partial\psi}{\partial t}, \quad . \quad . \quad (5\,a)$$

where

$$\frac{1-n^2}{V^2}\frac{\partial^2\psi}{\partial t^2}=\frac{\partial^2\psi}{\partial x^2}+\frac{\partial^2\psi}{\partial y^2},$$

or

$$\frac{1}{V^2}\frac{\partial^2\psi}{\partial t^2}=\frac{\partial^2\psi}{\partial x^2}+\frac{\partial^2\psi}{\partial y^2}+\frac{\partial^2\psi}{\partial z^2}. \quad . \quad . \quad . \quad . \quad (6)$$

Thus, for example, the first of (2) is

$$\frac{1}{V}\frac{\partial a}{\partial t}=\frac{\partial Y}{\partial z}=\frac{n}{V}\frac{\partial Y}{\partial t},\ \text{or}\ a=nY.$$

The polarization corresponds with that for (3 a) and the electric vector in the incident wave is

$$(lY-mX)/\sqrt{1-n^2},\ \text{or}\ k\sqrt{1-n^2}\sin k(Vt+lx+my+nz).$$

The second case is

$$a=-\frac{\partial\psi}{\partial y},\ b=\frac{\partial\psi}{\partial x},\ c=0;$$

$$X=-nb,\ Y=na,\ Z=\frac{1-n^2}{V}\frac{\partial\psi}{\partial t}. \quad . \quad (5\,b)$$

Here $X=-n\frac{\partial\psi}{\partial x}$, and when ψ or $\frac{\partial\psi}{\partial t}$ is made to vanish for all points of the barrier, X and Z also vanish. In the incident wave $X=kln\sin k(\quad)$, $Y=kmn\sin k(\quad)$, $Z=-k(1-n^2)\sin k(\quad)$; the magnetic vector is perpendicular to the axis of z, the electric vector has direction-cosines

$$\{ln,\ mn,\ -(1-n^2)\}/\sqrt{1-n^2},$$

and is of amount $k\sqrt{1-n^2}\sin k(Vt+lx+my+nz)$.

It is clear that (6) corresponds to the solution given in the paper, and that the same modification is applicable to other plane problems to meet the case where the incident wave has motion in a third dimension.

XXXIII. *The Scattering of Light by Air Molecules.*

By The Hon. R. J. STRUTT, *F.R.S.**

PROF. R. W. WOOD† has made some comments on my paper on this subject‡ to which it seems desirable to reply. I may say that at the time the paper was written I was fully aware of the pitfall which Wood refers to. I was working in the Cavendish Laboratory at the time that C. T. R. Wilson made his experiments there on the precipitation of clouds from moist air by ultra-violet light §, and have always borne in mind that this might occur, even with air that had been passed over phosphorus pentoxide: for complete drying is known to be a slow process.

On referring back to my paper, I see that I did not adequately explain the precautions taken on this point: thus Wood's demand for more evidence is quite justified, and it only remains to meet it.

Many of my experiments, including some of the earliest, have been made with a glass lens (a cheap plate-glass lantern condenser), and the visual intensity of the scattered light was not perceptibly less than with the quartz one, though I did not attempt any strictly quantitative comparison. I also found that a cell of quinine solution, which cuts out all rays more refrangible than λ4000, made no difference to the visual intensity, though naturally it reduced the photographic intensity considerably.

Another test often employed (and this *is* mentioned in my published paper) is to have a rapid current of filtered air going through the apparatus. This would prevent the accumulation of fog, and should at any rate greatly reduce the intensity of the scattered light, if due to fog; but in fact the intensity was the same as when the air was still.

In later experiments in course of publication by the Royal Society, I have worked with a variety of gases other than air, and in some cases the formation of fog has proved troublesome. There is not much uncertainty in practice as to whether a fog has been formed or not, in any given case; for in a series of exposures, the intensity varies with time, as Wood remarks. Frequently, too, a streakiness is observable in the photographed image of the scattered beam. I have gone into these questions more fully in the paper referred to, which was

* Communicated by the Author.

† Phil. Mag. vol. xxxvi. p. 272, Sept. 1918.

‡ Proc. Roy. Soc. A. vol. xciv. p. 453 (1918).

§ Phil. Trans. A. vol. cxcii. p. 412 (1899).

communicated to the Royal Society early in July. It is shown in that paper that the intensities from different gases are pretty closely proportional to the square of their refractivities as theory requires. Obviously this result could not be reconciled with the notion that the effects were spurious.

I do not think it is surprising that Wood could not observe any effect visually, in the absence of a fog. He does not tell us what was the photometric intensity of his spark; but unless with very special arrangements, it would probably not be more than a few candle-power. The arc I used was perhaps a thousand times as much: and even with that the scattered beam was not more than 20 times the minimum visible with well rested eyes. Thus the genuine effect would probably be considerably below the limit of visibility under the conditions of his experiment.

Finally, it may be asked, why did I not obtain a fog in air, when using a quartz lens, whereas Wood did obtain one? Probably because of the richness of his source in extreme ultra-violet rays.

XXXIV. *Elastic Solids under Body Forces.*

By D. N. Mallik, *Sc.D.*, *F.R.S.E.**

1. THE equations of equilibrium of an isotropic solid under body forces (X, Y, Z) are

$$(\lambda+\mu)\left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right)\Delta+\mu\nabla^2(u, v, w)+\rho(X, Y, Z)=0, \quad (1)$$

where

Δ is the cubical dilation,

u, v, w, displacements,

μ is rigidity,

and $\lambda+\frac{2}{3}\mu=$ modulus of compression.

Differentiating (1) with regard to x, y, z and adding, we get, since

$$\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}+\frac{\partial w}{\partial z}=0,$$

$$(\lambda+2\mu)\nabla^2\Delta+\rho\left(\frac{\partial X}{\partial x}+\frac{\partial Y}{\partial y}+\frac{\partial Z}{\partial z}\right)=0.$$

* Communicated by the Author.

Hence

$$\Delta = \frac{\rho}{4\pi(\lambda+2\mu)} \int \left(\frac{\partial X'}{\partial x'} + \frac{\partial Y'}{\partial y'} + \frac{\partial Z'}{\partial z'}\right) \frac{dx'\,dy'\,dz'}{r},$$

the integration extending over all space $\int dx'\,dy'\,dz'$.

Integrating by parts,

$$\Delta = \frac{\rho}{4\pi(\lambda+2\mu)} \int [X'\,dy'\,dz' + +] \frac{1}{r}$$

$$- \frac{\rho}{4\pi(\lambda+2\mu)} \int \left\{ X' \frac{\partial\left(\frac{1}{r}\right)}{\partial x'} + + \right\} dx'\,dy'\,dz',$$

where $$r^2 = (x-x')^2 + (y-y')^2 + (z-z')^2.$$

Now, since $$\frac{\partial}{\partial x} = -\frac{\partial}{\partial x'}, \text{ \&c.},$$

$$\Delta = \frac{\rho}{4\pi(\lambda+2\mu)} \int \left\{ X' \frac{\partial\left(\frac{1}{r}\right)}{\partial x} + + \right\} dx'\,dy'\,dz',$$

if the surface integrals vanish on the implied condition for the existence of body forces alone.

If X′, Y′, Z′ are constant over a sphere of radius a, and

$$\int \frac{\rho}{r}\,dx'\,dy'\,dz' = V,$$

then, since

$$V = \tfrac{4}{3}\pi\rho\frac{a^3}{R}, \text{ if } R > a \text{ where } R^2 = x^2 + y^2 + z^2,$$

and is $$= \tfrac{2}{3}\pi\rho(3a^2 - R^2), \text{ if } R < a,$$

$$\Delta = \frac{\rho}{3(\lambda+2\mu)} \left(X'\frac{\partial}{\partial x} + +\right)\left(\frac{a^3}{R}, \frac{3a^2-R^2}{2}\right), \ (R > a,\ R < a).$$

For an ellipsoidal distribution (uniform), we shall have

$$\Delta = \frac{\rho}{3(\lambda+2\mu)} \left(X'\frac{\partial}{\partial x} + +\right) \int_\lambda^\infty \left(1 - \frac{x^2}{a+\lambda} - \dots\right)$$

$$\times \frac{d\lambda}{\sqrt{\{(a^2+\lambda)(b^2+\lambda)(c^2+\lambda)\}}},$$

and $\lambda = 0$ or μ $\left(\text{where } \frac{x^2}{a^2+\mu} + \frac{y^2}{b^2+\mu} + \frac{z^2}{c^2+\mu} = 1\right)$

according as the point is inside or outside.

2. Returning now to the equations for displacements (1), let us put

$$u = u_1 + u_2, \text{ \&c.},$$

so that

$$\mu \nabla^2 u_1 + (\lambda + \mu) \frac{\partial \Delta}{\partial x} = 0$$

and

$$\mu \nabla^2 u_2 + \rho X = 0,$$

Let further

$$\Delta = \nabla^2 \phi,$$

where ϕ is a function of x, y, z to be determined.

Then

$$\mu u_1 + (\lambda + \mu) \frac{\partial \phi}{\partial x} = 0 \text{ \&c.},$$

as a particular solution, as also

$$u_2 = \frac{\rho}{4\pi\mu} \int \frac{X'}{r} dx'\, dy'\, dz'.$$

Again, since

$$\Delta = \frac{\rho}{4\pi(\lambda + 2\mu)} \int \left\{ X' \frac{\partial \left(\frac{1}{r}\right)}{\partial x'} + \dots \right\} dx'\, dy'\, dz'$$

and

$$\nabla^2 \left(\frac{dr}{dx}\right) = 2 \frac{\partial \left(\frac{1}{r}\right)}{\partial x}, \text{ \&c.}$$

we get

$$\phi = \frac{\rho}{8\pi(\lambda + 2\mu)} \int \left\{ X' \frac{\partial r}{\partial x} + \ + \right\} dx'\, dy'\, dz'$$

and

$$u_1 = -\frac{(\lambda + \mu)\rho}{8\pi\mu(\lambda + 2\mu)} \frac{\partial}{\partial x} \int \left(X' \frac{\partial}{\partial x} + \ + \right) r\, dx'\, dy'\, dz'.$$

Hence finally

$$u = \frac{\rho}{4\pi\mu} \int \frac{X'}{r} dx'\, dy'\, dz' - \frac{(\lambda + \mu)\rho}{8\pi\mu(\lambda + \mu)} \frac{\partial}{\partial x} \left[\int \left(X' \frac{x - x'}{r} + \ + \right) dx'\, dy'\, dz' \right].$$

The result was originally obtained by Lord Kelvin. His method was to find u, v, w for a distribution of body forces through the volume of a sphere as a potential problem and *then reduce the sphere to a point.* [See also Love's 'Elasticity.']

3. In the case of uniform distribution of body forces through

a sphere of radius a, we have to evaluate $\int X' r\,dx'\,dy'\,dz'$ over a sphere. For a point outside, this is

$$=2\pi X'\int_0^{\sin^{-1}\frac{a}{R}} \frac{r_1^4-r_2^4}{4}\sin\theta\,d\theta,$$

where r_1, r_2 are the distances of the point at which the displacements are to be found from the surface, measured in the direction θ, and R the distance of this point from the centre.

Now, since r_1, r_2 are the roots of $r^2+R^2-2rR\cos\theta=a^2$, the integral becomes, putting $a^2-R^2\sin^2\theta=\Theta$,

$$-\frac{2\pi X'}{R'}\int_{a^2}^{0}\Theta\{4\Theta+2(R^2-a^2)\}d\Theta$$

$$=\frac{4}{3}\frac{\pi a^3}{R}\cdot\left\{R^2+\frac{1}{5}a^2\right\}.$$

$\therefore$ if $R>a$, taking $\rho=1$,

$$u=\frac{1}{3\mu}\frac{a^3X'}{r}-\frac{(\lambda+\mu)a^3}{6\mu(\lambda+2\mu)}\frac{\partial}{\partial x}\left(X'\frac{\partial}{\partial x}+\ +\right)\left(R^2+\frac{1}{5}a^2\right).$$

if $R<a$,

$$\int r\,dx'\,dy'\,dz' \text{ over a sphere,}$$

$$=\pi\int_0^{\pi}\frac{r_1^4+r_2^4}{4}\sin\theta\,d\theta,$$

where r_1, r_2 are the roots of $r^2-Rr\cos\theta+\cos\theta=a^2$.

$$\text{This is }=-\pi\left(\frac{R^4}{15}-\frac{2}{3}a^2R^2-a^4\right).$$

4. If X′, Y′, Z′ are derived from gravitational potential, we may take

$$X'=\tfrac{4}{3}\pi x',\ \&c.,$$

since $X'\frac{\partial r}{\partial x}=\frac{\partial X'r}{\partial x}$, and we have to evaluate,

$$\int r\,dx'\,dy'\,dz'\left\{\frac{\partial r^2}{\partial x'}\right\}-2x\int r\,dx'\,dy'\,dz'.$$

The first is $= -\frac{\partial}{\partial x}\int r^3\,dx'\,dy'\,dz'$

$$= -2\pi\frac{\partial}{\partial x}\int_0^{\sin^{-1}\frac{R}{a}} \frac{r_1^6 - r_2^6}{6}\sin\theta\,d\theta$$

(with the same notation as before),

$$= \frac{2\pi}{3}\frac{\partial}{\partial x}\int\frac{d\Theta}{R}\{16\Theta^2 + 16\Theta\rho' + 3\rho'^2\}\sqrt{\Theta},$$

where $a^2 - R^2\sin^2\theta = \Theta$, as before,

and $R^2 - a^2 = \rho'$,

$$= -\tfrac{4}{3}\pi\,.\,a^3\frac{\partial}{\partial x}\left\{\frac{1}{R}\cdot\left(\frac{16}{7}a^4 + \frac{16}{5}\rho' a^2 + \rho'^2\right)\right\},$$

R being greater than a.

5. For an ellipsoidal distribution (uniform) we have to evaluate

$$\frac{\partial^2}{\partial x^2}\int r\,dx'\,dy'\,dz'$$

over an ellipsoid.

This is $= -\frac{\partial}{\partial x}\int\frac{\partial r}{\partial x'}dx'\,dy'\,dz'$

$$= \frac{\partial}{\partial x}\int\frac{x - x'}{r}dx'\,dy'\,dz'$$

$$= \frac{\partial}{\partial x}\{xV - V_x'\},$$

where

$V = \int\frac{dx'\,dy'\,dz'}{r}$ = potential due to a uniform distribution

$$= \pi abc\int\frac{d\lambda}{Q}\left(1 - \frac{x^2}{a^2+\lambda}\cdots\right),$$

and $V_x = \int\frac{x'}{r}dx'\,dy'\,dz'$

= potential due to a distribution of density varying as x

$$= \pi a^2bcx\int\frac{d\lambda}{Q(a^2+\lambda)}\left(1 - \frac{x^2}{a^2+\lambda}\cdots\right),$$

and $Q^2 = (a^2+\lambda)(b^2+\lambda)(c^2+\lambda)$.

6. If
$$X' = -\frac{\partial V}{\partial x'} \text{ \&c.} = Ax' \text{ \&c.},$$
we have to evaluate
$$\frac{\partial}{\partial x}\int r\,x'\,dx'\,dy'\,dz'.$$
This is
$$= \int \frac{x-x'}{r} x'\,dx'\,dy'\,dz',$$
$$= xV_x - V_{x^2},$$
where V_{x^2} = potential due to a distribution of density varying as x^2
$$= \pi a^3 bc \int \left\{ \tfrac{1}{4}\lambda \,.\, F^2 + \frac{a^2x^2F}{a^2+\lambda} \right\} \frac{d\lambda}{(a^2+\lambda)Q},$$
and
$$F \equiv \left\{ 1 - \frac{x^2}{a+\lambda} - \frac{y^2}{b^2+\lambda} - \frac{z^2}{c^2+\lambda} \right\}$$ [Routh, 'Statics,' vol. ii.]

A generalization of the above theory as well as its application to the case of the earth will be considered in a later paper.

XXXV. *Atomic Structure from the Physico-Chemical Standpoint.* By ALFRED W. STEWART, *D.Sc.**

THE theories put forward up to the present with regard to the structure of the atom have been based mainly upon physical data; but since the problem is a two-fold one, it appears possible that further light may be thrown upon it by a consideration of the chemical side of the question. Neither view alone will suffice to cover the whole ground; and the following is put forward with the idea of showing the essentials of the matter from the chemical standpoint, in the hope that it may prove suggestive to those who have hitherto regarded the problem chiefly from the physical aspect.

Any complete theory of atomic structure must account for the following facts concerning the elements:—

(1) That α- and β-ray changes are independent processes.
(2) That the electrons involved in valency changes occurring during ordinary chemical reactions originate in a region of the atom different from that occupied by the electrons which are ejected during β-ray changes.

* Communicated by the Author.

(3) That the "valency" electrons are easily removable in chemical reactions; whilst the β-ray electrons are ejected spontaneously and cannot be withdrawn from the atom by any process under our control.
(4) That the atomic number of an atom can be altered by either an α- or a β-ray change.
(5) That in an α-ray change the ejected material is always* a helium atom carrying two positive charges.
(6) That a change in the valency of an element produced by chemical means alters the chemical properties of that element in a manner similar to that which is observed when a β-ray is ejected; but that there is a difference in degree between the effects produced in the two cases.
(7) That certain atoms possessing different atomic weights show the same chemical properties, whilst other atoms having atomic weights identical with one another exhibit totally different chemical characteristics.

The model atom which will now be described covers these points; and it appears to possess certain features of novelty.

At the centre of the structure is a group of negative electrons travelling in closed orbits which, for the sake of clearness, may be assumed to be circular. Closely surrounding this negative group lies another series of orbits occupied by positive electrons † which, in some cases, are associated with negative electrons in a manner to be dealt with later. These orbits are assumed to be circular also; their extreme diameter may be taken, according to Rutherford's view‡, as not being greater than 10^{-12} cm.; and, as in the Rutherford atom, the mass of the system is assumed to be concentrated in this portion. Further still from the centre, other electrons move in orbits of an elliptical character, the ellipses being much elongated, so that the electrons travel in paths like those of comets in the solar system. The

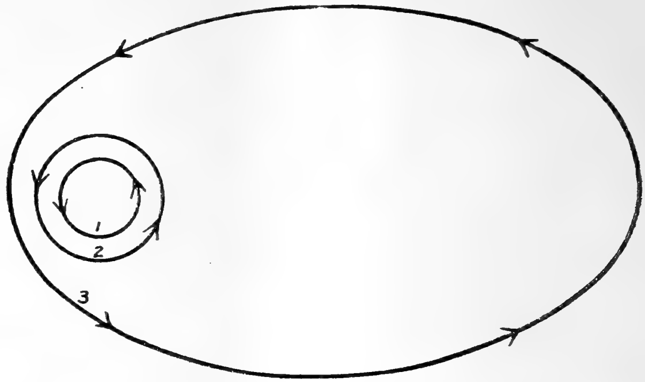
* The neon discovered in certain mineral springs is not yet proved to be of radioactive origin; and it is therefore left out of account here.

† This assumption as to the relative positions of the positive and negative zones is made purely for convenience. The general argument is not affected by an inversion of their positions, or even by assuming that they form a kind of double-star system.

‡ Rutherford, 'Radioactive Substances and their Radiations,' p. 621 (1913).

general appearance of the atomic mechanism is shown in fig. 1.

Fig. 1.



1. Orbits of negative electrons.
2. Orbits of positive electrons.
3. Cometary electronic orbits.

It is now necessary to consider each part of the system in detail. The central negative core is the point of origin of the β-rays; and since the electrons ejected by the atom during the β-ray changes travel at extremely high velocities, although they have passed through the positive zone during their flight, it is simplest to assume that under normal conditions they are moving at high speeds in their intra-atomic orbits. Charges moving with such high velocities would be difficult to deviate from their normal paths by external forces; and this accounts for the fact that chemical reactions fail to affect the intimate chemical structure of atoms. During phases of atomic instability, however, these electrons would leave the atom at high speeds.

The intermediate positive zone of the atom is occupied mainly—and in the non-radioactive elements exclusively—by positive electrons, the number of which is equal to the atomic number of the element. In the case of radioactive elements, a further complication most be postulated in order to account for the ejection of α-particles. In the case of these active elements it is assumed that in the positive zone some of the orbits are occupied by complex groups composed of two positive and one negative electron which together form a "planet and satellites" arrangement circulating as a whole about the central negative core. The number of these complexes depends upon the nature of the atom in question :

in the uranium atom, since it ejects eight α-particles in succession, there will be at least sixteen such systems. The ejection of the charged helium atom is supposed to take place when two of these complexes collide with one another either owing to a crossing of their orbits or by a disturbance of the stability conditions within the atom; and the collision produces a group of four positive and two negative charges, the arrangement of which will be clear when the next zone of the atomic structure has been considered.

The atomic number of the element and the general chemical character of the atom are governed by the nature of the two inner sections of the atomic system. A change in either the negative core or the intermediate positive zone alters the nature of the intra-atomic system and thus brings about a modification of the structure as a whole.

The external zone of the atom is the portion influenced by normal reactions resulting in chemical change or alteration in valency. The assumption that the orbits of the electrons in this zone are cometary in type has been made for the following reason. When the "cometary" electrons in their paths about the centre of the atom reach a position of aphelion to the nucleus, they will be travelling slowly in their orbits and hence will be less resistant to forces tending to remove them from the atom. Further, since they are far away from the centre of attraction under these conditions, the forces uniting them to it will be weakened; and it will be possible to abstract or insert electrons at this point much more readily than is the case with electrons in either of the other two zones. This serves to account for the ease with which the valency of certain elements can be altered by chemical or electrical means. In the case of elements which show no changes of valency, it may be assumed that the electronic orbits in the outer zone are more nearly circular in form than is the case with elements exhibiting variable valency. The inertness of the argon series is accounted for by assuming that in their case the attraction of the nucleus under normal conditions is insufficient to retain any electrons in an external zone.

At this point it may be well to indicate the conditions of attraction within the systems of ordinary elements; and the point may be illustrated by means of a metallic atom such as tin. In this case, the negative charges at the centre are assumed to be fewer in number than the charges in the positive zone. Owing to this preponderance of positive charges, the positive-negative nucleus as a whole will have a positive charge; and, acting as a unit, it will suffice to retain in their orbits the "cometary" negative electrons which circulate around it.

With regard to the expulsion of charged helium atoms from radioactive elements, it is assumed that the α-particle consists of four positive and two negative electrons: the pair of negative electrons being situated at the foci of an ellipse around the circumference of which two positive charges revolve. The extra pair of positive charges travel in longer, "cometary" orbits; so that they are easily detachable when in aphelion. It must be admitted that there is a difficulty in accounting for their attraction by the atomic nucleus, which in this case is electrically neutral; but as this attraction is a matter of practice and not of theory, it must be admitted as possible even if no theory can be adduced to account for it.

The formation of the α-particle is due, as has been said, to the collision of two systems each containing two positive and one negative electron. This does away with the necessity for postulating the presence of actual helium atoms within the structure of radioactive elements, an hypothesis which is fraught with difficulties owing to the fact that the helium atom has a volume of 26·6, whilst the uranium atom, which emits eight helium atoms, has a volume of only 12·8. The collision hypothesis also accounts for the presence of the two extra positive charges which invariably accompany the helium atom in its ejection.

In this model atom, as in most others, the valency of an element is taken as the difference between the total positive and the total negative charge of the atom; but the variation in valency caused by α- or β-ray changes is assumed to be brought about by alterations in the inner zones of the atomic structure, whilst chemical changes of valency are accounted for on the assumption that the number of the electrons in the cometary orbits is altered. No definite conclusions can be drawn with regard to the relative numbers of electrons in the various zones, beyond the suggestion put forward above that the number of electrons in the innermost negative core of metallic atoms is less than that of the electrons in the intermediate positive orbits; though probably, as Soddy has indicated, the surplus number of positive charges in the two inner zones combined is equal to the atomic number of the atom.

In order to test still further this conception of the atom, it is necessary to examine evidence in a different field. Among the radioactive elements, two classes can be distinguished. In the first place there are certain groups of elements which are chemically inseparable but which differ from one another in atomic weight. Since they are chemically indistinguishable from each other, they occupy the same place in the Periodic Table; and on this account Soddy named them isotopes (from *isos* equal, and *topos* a place). A second type of the radio-elements is exemplified by mesothorium-1, mesothorium-2, and radiothorium. These elements differ completely from one another in chemical character; but they all possess the same atomic weight. For this reason the name isobares* (from *isos* equal, and *baros* weight) is here suggested for them.

These isobaric elements result from the operations of β-ray changes in the radioactive series; and the generation of one element from another in this way is spontaneous and irreversible. On the other hand, a somewhat similar process occurs among the non-radioactive elements when an atom changes its valency; but in the latter case the process is controllable in the laboratory and is reversible under proper conditions. The two actions, then, are not identical†; but they appear to display a certain parallelism which is of considerable importance from the point of view of atomic structure. Unless a model atom is capable of throwing light upon this matter, it is evidently incomplete; and as the point forms a crucial test of the theory of atomic architecture, some details of it are given here, though the merest outline must suffice.

Ferrous iron and ferric iron will serve as a convenient example of the effects of changing the valency of an element by chemical reactions. Ferrous iron is divalent, whilst ferric iron is trivalent: the absorption spectra of the two materials are different from each other; and in chemical properties ferrous iron shows a close analogy with magnesium, whilst ferric iron is akin to aluminium in its reactions. A difference in chemical character such as this should, according to modern ideas of the atom, involve certain changes in the atomic nucleus; but at the same time it is hard to imagine that any changes in the nucleus can occur in ordinary inactive elements.

Turning to the case of the radioactive isobares, it is found that a very similar state of things prevails. Mesothorium-1 is divalent and resembles in its chemical relations the members of Group II. of the Periodic Table, which also contains magnesium. Mesothorium-2 is trivalent, and shows a close kinship with elements in the aluminium group.

* Isobars would be a better word, but unfortunately it is already in use in meteorology.

† Soddy, 'Nature,' xcii. p. 399 (1913); Fleck, Chem. Soc. Trans. cv. p. 247 (1914).

At first sight the main difference between the two phenomena appears to lie in the fact that the β-ray change is spontaneous, whilst the chemical change of valency is a controllable process; but even the spontaneity of the β-ray change finds its parallel among certain of the stable elements. Thus when the chloride of monovalent indium is dissolved in water, it is *spontaneously* converted into metallic indium and the chloride of trivalent indium. Reduced to its essentials, this change corresponds to the loss of two negative electrons from two of the monovalent indium atoms; and no external forces are required to bring about the phenomenon*. The case of indium is not an isolated one, as this type of reaction appears to be the most general which is exhibited by inorganic compounds.

Another parallelism between the β-ray change and the conversion of an ion into a new one of higher valency may be adduced. In several cases, elements are found which exist in monovalent and trivalent forms, or in the divalent and quadrivalent condition only, instead of yielding a complete series of mono-, di-, tri-, and quadrivalent varieties. Thus thallium forms the chlorides $TlCl$ and $TlCl_3$, but does not give rise to the intermediate $TlCl_2$. It may be asked why these intermediate forms are not isolated when electrical charges are removed step by step from substances of lower valency.

The state of affairs among the radio-elements throws some light upon this point. The conversion of $TlCl$ into $TlCl_3$ is paralleled by two consecutive β-ray changes in the radio-elements; and in the following table the results of such successive changes are given. These examples have been selected in which no disturbing factor in the form of an alternative α-ray change occurs. The figures† give the average life of the element.

Group N $\xrightarrow[\text{change}]{\beta\text{-ray}}$	Group (N+1) $\xrightarrow[\text{change}]{\beta\text{-ray}}$	Group (N+2).
Uranium-X_1 35·5 days	Uranium X_2 1·65 minutes	Uranium-2 3×10^6 years
Mesothorium-1 7·9 years	Mesothorium-2 8·9 hours	Radiothorium 2·01 years
Radium-D 24 years	Radium-E 7·20 days	Radium-F 196 days

* Even when solvent action is assumed, the spontaneity of the change retains its importance from the present point of view.

† Soddy, 'The Chemistry of the Radio-elements.'

Examination of the figures shows that the intermediate product in the double β-ray change has an average life very much shorter than those of the parent and the disintegration product. Applying the same reasoning to the case of the salts of thallium, it might be expected that when monovalent thallium loses an electrical charge and passes into divalent thallium, the latter substance readily loses an electrical charge and changes almost immediately into trivalent thallium, the intermediate stage $TlCl_2$ being too unstable for isolation.

Looking at the matter in its essentials, it is clear that both the β-ray change and the alteration of valency by chemical means produce a marked change in chemical character which is similar in both cases; and a true theory of the atomic structure must account for these phenomena.

The model atom described above furnishes a satisfactory explanation of the facts. In their paths, the "cometary" electrons periodically come into close proximity to the positive-negative system of the nucleus; and while they are in this position they will affect the centre of the atomic structure just as if they were travelling in the innermost negative orbit. In other words, at this stage in their career they behave as if they formed part of that portion of the atom in which the general chemical character is supposed to reside. At the same time, since their presence in this position is only periodic and temporary, they will not exert so much influence as is produced by the electrons of the innermost zone, which are always in touch with the positive electrons and which thus exert a permanent effect upon the atomic character.

This hypothesis, therefore, accounts for the fact that changes in the valency of an atom induced by chemical reactions do not completely and irreversibly alter its character as do modifications due to the expulsion of an α- or a β-ray; for in the last case the change takes place in the very core of the atomic structure, and its results are deep-seated and permanent.

Thus the model atom furnishes a solution of the questions arising from the chemical resemblances traceable between the uranous salts and the salts of thorium. The atomic number of thorium is 90, whilst that of uranium is 92; so the two elements are not isotopic. Uranium occurs in the hexavalent form and also in another modification which is quadrivalent like thorium. Quadrivalent uranium resembles thorium with a closeness approaching isotopy; but the similarity does not reach the point of identity, since the two substances are separable from one another by chemical means*. On the above view, the "pseudo-isotopy" is due to the influence of the "cometary" electrons upon the atomic nucleus of which they form a temporary part at certain points in their orbits. The removal of two positive charges from the intermediate zone of hexavalent uranium produces uranium-X_1, which is actually isotopic with thorium and has the atomic number 90; but the abstraction of two charges from the "cometary" orbits, although it has the same effect upon the total residual charge, is only sufficient to bring about a close resemblance between the product (quadrivalent uranium) and thorium; and is not enough to produce total identity and a change in atomic number†.

Taken in conjunction with the experimental evidence, the model suggests that those atoms which change their valency should really be regarded as "pseudo-elements," since they are capable of exhibiting two or more sets of distinct chemical characteristics according to the number of electrons which revolve in their "cometary" orbits. They are not "meta-elements" of the type suggested by Crookes‡; for they have definite atomic weights. They should rather be regarded as a new type of isobares similar to but not identical with the radioactive isobaric elements like mesothorium-1 and mesothorium-2.

The dynamic conception of the model atom set forth above suggests a possible solution of the problem of the atomic weights of the elements, though at the present stage the following suggestion must be treated with reserve.

The fact that negative electrons exist apart from matter as we know it, whilst positive electrons are never dissociated from masses of at least atomic magnitude, suggests that there is a close relation between mass and positive electricity. Further, the connexion between the two factors appears to be strengthened by the recognition that the atomic weight of an element is approximately twice its atomic

* Fleck, Trans. Chem. Soc. cv. p. 247 (1914).

† Similar reasoning may be applied to other cases. For example, Allen (Trans. Chem. Soc. cxiii. p. 389 (1918)) has pointed out that the "molecular number" of the ammonium group (NH_4) is 11, which is the same as the atomic number of sodium: and he has drawn the conclusion that this coincidence in value has some bearing upon the known resemblances between sodium and ammonium. In the model, the attachment of four hydrogen atoms to the nitrogen atom would entail the introduction of a corresponding number of electrical charges into the "cometary" orbits; and to the effect of these upon the central nucleus may be ascribed the change in character of the nitrogen atom.

‡ Crookes, Trans. Chem. Soc. liii. p. 487 (1888).

number. This correspondence in values is, however, only a rough one. The ratio of atomic number to atomic weight for calcium is found to be 1 : 2·00. For strontium it is 1 : 2·31; for barium it falls to 1 : 2·45 ; for radium it is still smaller, 1 : 2·57; and with uranium it reaches 1 : 2·59. In order to connect the number of positive charges in an atom with the atomic weight, therefore, it is necessary to provide some mechanism which will decrease the ratio of charge to mass from 1 : 2 to 1 : 2·59.

This can be done in the following way. The mass of an electrical charge depends upon the velocity with which the charge is moving, provided that this velocity be made approximate to that of light. In the model atom suggested above, it was assumed that the positive electrons were moving in their orbits; and if the further assumption be made that these charges revolve at speeds comparable with that of light, then the masses of the charges will vary according to the velocity with which they move*.

In the calcium atom, the positive electrons may be assumed to be travelling comparatively slowly; and as the series is ascended through strontium, barium, and radium, the intra-atomic velocities may be assumed to increase; with the result that each positive charge will gain in mass, and thus the ratio of charge to mass could be brought into accordance with the known data.

For example, the atomic number of radium-B is 82, whilst its atomic weight is 214. If the intra-atomic charges were moving within the radium-B atom with the same velocity as those of calcium, then the atomic weight of radium-B would be 164; so that it is necessary to calculate the velocity which will raise the mass of these charges from 164 to 214. This can be done by means of the Lorenz equation :—

$$\frac{m}{m_0} = \frac{1}{\sqrt{1 - \frac{v^2}{c^2}}},$$

in which m_0 is the mass of the charge moving at a low velocity; m is the mass of the same charge at the required velocity, v; and c is the velocity of light. Taking m_0 as 164 and m as 214, the equation yields $v = 0·64$, when the velocity of light is taken as unity.

This velocity may appear very high; but there is experimental proof that negative electrons† emerging from the

* The same question has been approached from a different standpoint by Comstock, Phil. Mag. [6] xv. p. 1 (1908).

† Owing to the assumptions made above, it is difficult to deduce intra-atomic velocities from the speed of the α-particles.

atom of radium-B do actually attain speeds of this order of magnitude; for it has been found that the β-rays from radium-B travel at velocities ranging between 0·36 and 0·70, when the speed of light is taken as unity. The value 0·64, calculated from purely theoretical considerations, certainly agrees closely with what is actually established with regard to those electronic velocities with which we are acquainted.

The velocity hypothesis furnishes an explanation of the case of the isotopic elements which, up to the present, have presented an unsolved problem. If it be assumed that in two isotopes the internal mechanism of the atoms is identical in every way, then the chemical inseparability of the two elements can be explained; and if it be further assumed that the intra-atomic velocities are different, then the masses of the two systems will also differ; all of which is exactly what is found in practice.

Further, a point of some interest arises when the Geiger-Nuttall relation is considered in this connexion. This empirical relationship establishes the fact that atoms throughout their various stages of disintegration still preserve a feature which is characteristic of their origin. This common characteristic pervades each of the three radioactive series and differentiates its members from those of the other series. It cannot be a chemical factor, for the elements belonging to the same series differ widely from each other in chemical character. It seems not unreasonable to suppose, however, that throughout the changes which the radio-elements undergo, one or more of the orbits within the atom remain unaffected by the process; and that the velocity of the electrons in these orbits may be the "distinctive feature" which survives the catastrophe of atomic disintegration.

The foregoing is sufficient to show that the suggested model atom meets the demands made upon it from the chemical side; and to this extent it justifies further consideration. An examination of it from the physical standpoint would be of interest. In the meantime, it may be pointed out once more that this view of atomic structure is to be regarded as suggestive rather than constructive.

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XXXVI. *Atomic Number and Frequency Differences in Spectral Series.* By HERBERT BELL *.

IT is a well-known law † that in the visible spectrum the wave-number differences ν between the components of a doublet series, and the differences ν_1, ν_2 between the components of a triplet series vary from element to element in such a way that, for the same family (column of the Mendelejeff table), the ν's are roughly proportional to the squares of the atomic weights. We have, for example, in Baly's 'Spectroscopy,' p. 624, the following table ‡ where the argument is 1000ν (or ν_1)/(atomic weight)^2 :—

Na 32·3	Mg 68·8	Al 152·8	O 14·5
K 37·8	Ca 66·1	Ga 168·6	S 17·7
Rb 32·3	Sr 51·5	In 172·1	Se 16·6
Cs 31·6	Ba 46·8	Tl 187·0	

If the law were accurate the arguments in the several columns would be constant. Perhaps the most thorough discussions from this point of view have been given by Rudorf § and Hicks ‖.

An attempt was made by Runge and Precht ¶ to show that the outstanding discrepancies in the above table, *e.g.* the case of thallium, are removed by assuming a different law. They stated that the logarithm of ν is linear in terms of the logarithms of the atomic weights. We shall have to refer to its correctness later.

Since Moseley's** discovery that the square root of the frequencies of the X-ray series is linear in terms of the atomic number, for all elements, more attention has naturally been turned to it than to atomic weight. In particular, Runge and Precht's method was modified in this direction by Ives and Stuhlman†† with a decided improvement especially in the case of potassium. Their paper contains no constants for the straight lines so that the agreement is only graphically demonstrated.

* Communicated by the Author.

† Due to Kayser and Runge, and Rydberg.

‡ Due to Rydberg, Intern. Reports, ii. p. 217 (Paris, 1900).

§ *Zt. Phys. Chemie*, l. p. 100 (1904).

‖ See *e.g.* Phil. Trans. A. ccx. (1910), ccxii. (1912), ccxiii. (1913), ccxvii. (1917).

¶ Phil. Mag. v. p. 476 (1903).

** Phil. Mag. xxvii. p. 703 (1914).

†† Phys. Rev. v. p. 703 (1915).

In connexion with the discovery of triplet differences in the Radium spectrum, the Misses Anslow and Howell * have reinvestigated the alkaline earth column, plotting, however, against log (atomic number), not log (doublet difference), as was done in the above paper, but the logarithm of $(\nu_1 + \nu_2)$. They obtain *graphically* good agreement with this law. They state also that the linearity is between alternate members of the same Mendelejeff column, a point of view which we shall have to modify.

There is reproduced below for reference a form of the atomic number table as given by Kossel †.

0.	I.	II.	III.	IV.	V.	VI.	VII	VIII.		
	H 1									
He 2	Li 3	Be 4	B 5	C 6	N 7	O 8	Fl 9			
Ne 10	Na 11	Mg 12	Al 13	Si 14	P 15	S 16	Cl 17			
Ar 18	K 19	Ca 20	Sc 21	Ti 22	V 23	Cr 24	Mn 25	Fe 26	Co 27	Ni 28
	Cu 29	Zn 30	Ga 31	Ge 32	As 33	Se 34	Br 35			
Kr 36	Rb 37	Sr 38	Y 39	Zr 40	Nb 41	Mo 42	43	Ru 44	Rh 45	Pd 46
	Ag 47	Cd 48	In 49	Sn 50	Sb 51	Te 52	I 53			
X 54	Cs 55	Ba 56	Earths	72	Ta 73	W 74	75	Os 76	Ir 77	Pt 78
	Au 79	Hg 80	Tl 81	Pb 82	Bi 83	84	85			
Eman. 86	87	Ra 88	89	Th 90	91	U 92				

In the two diagrams shown herewith the square root of wave-number difference per cm. is plotted against atomic number N as abscissa. Unless otherwise stated, the data have been taken from Dunz's *Bearbeitung unserer Kentnisse von den Serien* ‡, with the result that, in general, the wave-number differences refer, not to any particular member of a series, but to the calculated limiting frequencies.

Fig. 1 shows the result for the Lithium column, the constituents having only doublet series. $\sqrt{\nu}$ is seen to be linear.

* Proc. Nat. Acad. Sciences, iii. p. 409 (1917).
† *Ann. d. Phys.* lix. p. 247 (1916).
‡ Dissertation, Tübingen, 1911.

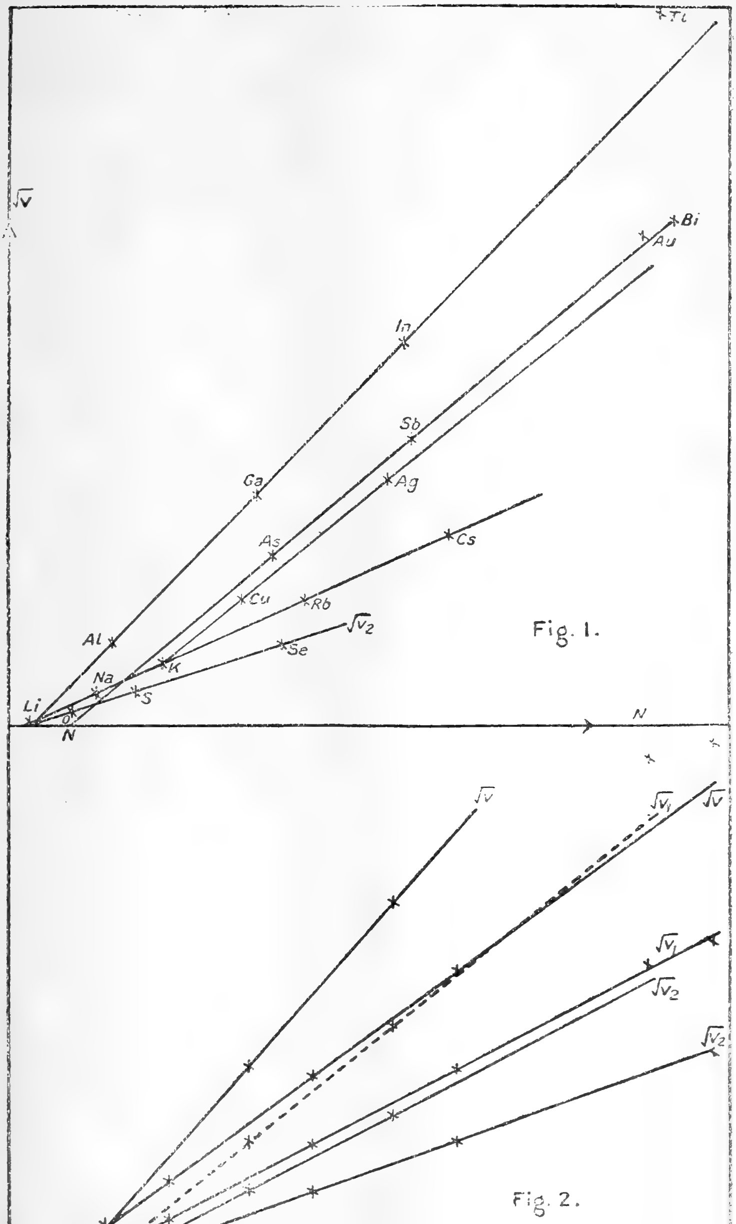
Tl
$\sqrt{v}$
Bi
Au
In
Sb
Ag
Ga
As
Cs
Cu
Rb
$\sqrt{v_2}$
Al
Se
K
Na
S
Li
O
N
N
$\sqrt{v}$
$\sqrt{v_1}$
$\sqrt{v}$
$\sqrt{v_1}$
$\sqrt{v_2}$
$\sqrt{v_2}$
Mg
Ca
Zn
Sr
Cd
Ba
Hg
Ra

Fig. 1.

Fig. 2.

in N for Li, Na, K, Rb, Cs, although the Cs value is slightly too great. Furthermore, two other elements in the column, Cu and Ag, fall on a branch line passing through K. There is thus a *two-fold collinearity in the same vertical column of the table, one line branching off from the other just where the column itself seems to divide.* A similar feature will be met with again in the next column. Au forms an apparent exception. It is sometimes regarded as having doublets (although no series is claimed) with $\sqrt{\nu} = \sqrt{3817}$ as shown by the cross, whereas the straight line indicates $\sqrt{\nu} = \sqrt{3167}$. This point will be discussed later.

In the alkaline earth column there are both doublets and triplets (ν_1, ν_2) and the resulting collinearities are shown in fig. 2. As regards the ν-line a lower value for Mg would fit in better with Ca, Sr, and Ba. It is noticeable, from the table below, that such a modified value falls well in line with the "branch" elements Zn and Cd. For the triplet series there is a pair of straight lines for each of ν_1 and ν_2, due to branching at Mg, as the table itself suggests. In the case of ν_2 Ba has too low a value, 370 cm.$^{-1}$ instead of 401 cm.$^{-1}$, and Mg is again anomalous. A dotted line has been drawn for ν_1 as being more hypothetical; it appears to branch off for Zn and Cd to the right of Mg, and the usually accepted series for Hg give differences that lie too high *†.

For the next column Al, Ga, In are collinear (fig. 1), but doubt has been expressed ‡ that the doublet intervals for Ga are known. The same remark applies still more strongly to Be. $\sqrt{\nu}$ for Tl is certainly not collinear with Al and In, being 7792 cm.$^{-1}$ instead of 6323 cm^{-1}. We might plead in extenuation that the rare earths intervene in this column, but the explanation given below seems more probable. Series for Scandium § and Yttrium are not known.

Differences in the Carbon column are so little known that it is not possible to discuss it in this manner.

In the Nitrogen column no series have been found, but in As, Sb, and Bi Kayser and Runge ‖ discovered a new type

* It is noteworthy that the interval 1545·45 cm.$^{-1}$, stated by Dunz, *loc. cit.*, as being preferred by Paschen, agrees fairly well with the value 1598 cm. required by the Zn, Cd line.

† The branching phenomena in these two columns is analogous to a result obtained by Rudorf, *loc. cit.* Plotting ν/A^2 against A, where A is atomic weight, he obtained *curves* for these families intersecting at these elements. There is no numerical test applied.

‡ Kayser, *Spektroscopie*, ii. p. 547.

§ The series in Sc suggested by Hicks, Phil. Trans. A. ccxiii. p. 408, give an interval of 320 cm.$^{-1}$; the line would give 350 cm.$^{-1}$.

‖ *Abh. Akad. Wiss. Berlin*, 1893.

of regularity. This consists in the recurrence of constant frequency differences between corresponding members of certain groups of lines, so that if A_r be the frequency for any line r in group A, then we have the frequency B_r in group B by adding a constant frequency β (say). Using this notation the groups for As are A_r+461, and A_r+8058. It is plain that besides the intervals 461 and 8058 there exists also $8058-461=7597$. For Sb, Kayser and Runge give $A'+2069$, $A'+8613$, $A'+9955$, $A'+12460$, and $A'+15023$, and for Bi, $A''+6225$, $A''+10245$, and $A''+21667$. By subtraction we have in Sb, 1342 and in Bi, 4020. In fig. 1 it is seen that these values, viz. 461, 1342, and 4020 for As, Sb, and Bi resp. lie on a line (very exactly as appears from the calculation below) passing nearly through a zero value for N and indicating a value 41·2 cm.$^{-1}$ for P. Examining Gautier's spectral measurements in P as quoted in Kayser's *Spektroscopie* we find the set

4649·23 (4)	41·83	4658·29 (6*r*)
4575·08 (3)	41·88	4583·86 (0)
4475·43 (3*r*)	41·86	4483·83 (3)
4102·3 (0)	41·8	4109·34 (5)

The interval 41·8 cm. is in good agreement, but the number of cases is too small to warrant any great confidence. There are about 150 differences within the range 40 cm.$^{-1}$ to 50 cm.$^{-1}$, so that the odds against four of these falling within a given region of ·05 cm.$^{-1}$ is *

$$1 \div \frac{(\cdot 75)^4}{4\,!} e^{-\cdot 75} = 160.$$

Hence the odds against this group being within ·60 cm.$^{-1}$ on either side of the given line is only about 13 to 1.

In the Oxygen column triplet series occur again and the values for $\sqrt{\nu_1}$ and $\sqrt{\nu_2}$ for O, S, Se are shown in fig. 1. Collinearity exists in ν_1 but not in ν_2. For the remainder of the column we have no data.

For the halogen column, again, the data are insufficient.

In the last column frequency differences have been found by Kayser † for Ru, Pd, and Pt, but none in the first row Fe, Co, and Ni, so that it is not possible to apply a test.

Turning now to the first column, He, Ne, &c., there is a further possibility of linearity. Rydberg ‡ has found in

* Using the formula $\frac{x}{r\,!}e^{-x}$, where x is the average density.

† *Abh. Berl. Akad.* 1897.

‡ Astroph. Journal, vi. p. 239 (1897).

Argon a relationship similar to that in As, Sb, Bi, the frequencies being $A_r + 846{\cdot}47$, $A_r + 1649{\cdot}68$, and $A_r + 2256{\cdot}71$. In Neon, again, according to Watson † there are triplets, one of the intervals being 417·45 cm.$^{-1}$. The line joining (1649·68, A) to (417·45, Ne) has for equation

$$\sqrt{n} = m(N - N_0) = 2{\cdot}523(N - 1{\cdot}90).$$

This indicates for He (N=2) the value $\nu = {\cdot}06$ cm.$^{-1}$ instead of about unity (Dunz gives 1·05). The line has not been drawn in the diagram; it requires for Krypton the interval 7402 cm.$^{-1}$.

The following tables contain the wave numbers whose square roots have been plotted in the diagrams. The first of the columns marked "ν calc." gives the values calculated from the equation $\sqrt{\nu} = m(N - N_0)$, m and N_0 having the values indicated. Least square methods were employed where more than two points were used to determine the line. The second "ν calc." column is calculated from $\log \nu = p \log N + q$. In each case only two pairs of values of ν and N were used to determine p and q; the results show clearly which values were chosen.

$$\sqrt{\nu} = m(N - N_0). \qquad \log \nu = p \log N + q.$$

Doublets.

	ν obs.	ν calc.	ν calc.
Li	·34*	·25	1·03
Na	17·21	16·48	17·21
K	57·90	58·0	56·17
Rb	237·71	244·0	237·7
Cs	564·10	558·3	560·8

$m = {\cdot}4447$, $N_0 = 1{\cdot}875$.
$p = 2{\cdot}1645$, $q = -1{\cdot}01832$.

	ν obs.	ν calc.	ν calc.
K	57·90	58·1	95·2
Cu	248·13	247·6	248·1
Ag	920·56	920·9	920·6
Au	3817·20	3172·0	3771·0

$m = {\cdot}8117$, $N_0 = 9{\cdot}619$.
$p = 2{\cdot}71512$, $q = -1{\cdot}57591$.

	ν obs.	ν calc.	ν calc.
Be		11·4	14·6
Mg	92·0*	84·5	81·6
Ca	222·9	225·5	222·9
Sr	801·3	794·7	788·2
Ba	1690·5	1699·0	1690·5
Ra	4858·0*	4162·0	4114·0

$m = {\cdot}7279$, $N_0 = -{\cdot}630$.
$p = 1{\cdot}96777$, $q = -{\cdot}21202$.

	ν obs.	ν calc.	ν calc.
Mg	92·0*	85·2	113·4
Zn	872·4	872·4	872·4
Cd	2484·1	2484·1	2484·1
Hg	(?)	7387·0	6904·0

$m = 1{\cdot}1280$, $N_0 = 3{\cdot}817$.
$p = 2{\cdot}22639$, $q = -{\cdot}34793$.

	ν obs.	ν calc.	ν calc.
B		6·14	
Al	112·07	112·07	112·1
Ga	823·6*	831·3	790·6
In	2212·63	2212·63	2212·6
Tl	7792·45*	6323·0	6859·0

$m = 1{\cdot}0136$, $N_0 = 2{\cdot}556$.
$p = 2{\cdot}24798$, $q = -{\cdot}45462$.

† Camb. Proc. xvi. p. 130 (1911); Astroph. Journal, xxxiii. p. 399 (1911).

Triplets, ν_1, ν_2.

	ν obs.	ν calc.	ν calc.		ν obs.	ν calc.	ν calc.
Be		2·50	2·35				
Mg	19·89	19·59	19·89	Mg	19·89	19·62	19·89
Ca	52·11	52·8	53·7	Zn	189·78	191·5	176·7
Sr	187·05	186·6	187·1	Cd	541·86	540·4	541·86
Ba	370·3*	402·2	397·6	Hg	1767·19*	1598·0	1831·0
Ra	1036·15*	987·0	957·3				

$m=\cdot 3552$, $N_0=-\cdot 459$. $p=1\cdot 94433$, $q=-\cdot 79965$.

$m=\cdot 5227$, $N_0=5\cdot 527$. $p=2\cdot 38390$, $q=-1\cdot 37403$.

Be		3·67	4·73				
Mg	40·95	38·06	40·95	Mg	40·95*	27·26	45·34
Ca	105·99	108·6	111·74	Zn	388·91	388·9	388·91
Sr	394·44	399·3	394·44	Cd	1171·05	1171·0	1171·05
Ba	878·4	872·9	845·03	Hg	4630·31*	3600·0	3880·0
Ra	2016·64*	2166·3	2054·0				

$m=\cdot 5312$, $N_0=\cdot 387$. $p=1\cdot 96502$, $q=-\cdot 50836$.

$m=\cdot 8055$, $N_0=5\cdot 517$. $p=2\cdot 34534$, $q=-\cdot 87450$.

O	3·38	3·12
S	17·90	18·89
Se	103·66	103·0
Te		254·0

$m=\cdot 8382$, $N_0=7\cdot 347$.

The ν_2 values for O, S, Se are resp. 2·76, 11·26, and 44·82 cm^{-1}.

N		
P	41·8*	41·2
As	461·36	462·5
Sb	1342·26	1339·1
Bi	4019·73	4021·7

$m=\cdot 8382$, $N_0=7\cdot 347$.

Values marked * are not used in determining m and N_0.

Au. The value given is criticised by Quincke, using later measurements. *Zt. f. Wiss. Phot.* xiv. p. 249 (1914).

Ba. Later measurements are by Schmitz, *Zt. Wiss. Ph.* xi. p. 209. Also by Lorenser, Dissertation, Tübingen, 1913, *Beiträge zur Kenntnis der Erdalkalien.* The latter contains a critical study of Mg, Ca, Sr, and Ba.

Hg. A detailed study of the Hg spectrum is given by Cardaun, *Zt. Wiss. Ph.* xiv. p. 89. The above values for ν_1, ν_2 are not much altered.

Sb. From later data by Schippers, *Zt. Wiss. Ph.* xi. p. 241, we have the interval 1341·17 cm. in better agreement with the straight line.

An examination of the above diagrams and tables shows, in general, an upward curvature of the observed points relative to the straight line, especially for large values of N. We notice further that, with the exception of two lines, N_0 is everywhere positive. This readily suggests that a curve passing through the origin and two of the given points might be an improvement. Runge and Precht's law mentioned above is in this direction, for, assuming $\nu=AN$, where s and A are arbitrary, we can choose s nearly equal to 0·5 and the necessity for passing through the origin gives the upward curvature. The tables contain values calculated on this assumption, and it will be seen that an improvement results, especially for large values of N.

This exception is to be expected, for, if

$$\sqrt{\nu}=m(\mathrm{N}-\mathrm{N}_0)=m\mathrm{N}(1-\mathrm{N}_0/\mathrm{N})$$

be the actual locus of the given points, *i. e.*

$$\tfrac{1}{2}\log\nu=\log m+\log\mathrm{N}-\mathrm{N}_0/\mathrm{N},$$

then, on a logarithmic diagram, the points will be above the logarithmic straight line (N_0/N being negative), *i. e.* the logarithms curve downwards relatively to the actual observations. This is seen to be the case from the tables, the logarithmic set of values being worse than the others. We are therefore compelled to regard Runge and Precht's law as being of the nature of an empiricism, especially since the improvement, where it exists, is not great. The logarithmic method fails to show graphically the branching relation of the columns, as is clearly shown by a reference to the papers already cited. A further empirical improvement would plainly be effected by assuming $\nu^s=\mathrm{A}(\mathrm{N}-\mathrm{N}_0)$.

The question of doublet and triplet differences has recently been gone into extensively by Sommerfeld *. If we write the equation to a series as

$$n=\mathrm{A}\left(\frac{1}{a^2}-\frac{1}{(m+\mu)^2}\right),$$

where m has integral values and μ, a, and A are curve-fitting constants, then his theory ascribes the constant doublet differences to the term $1/a^2$. The above expression for n is in fact proportional to the loss of energy for a revolving electron when falling from an outer to an inner Bohr ring. If the inner ring for the series be in reality double (of different eccentricities according to Sommerfeld) then a has two possible values, and we have the constant frequency difference

$$\nu=n_1-n_2=\mathrm{A}\left(\frac{1}{a_1^2}-\frac{1}{(m+\mu)^2}\right)-\mathrm{A}\left(\frac{1}{a_2^2}-\frac{1}{(m+\mu)^2}\right)$$
$$=\mathrm{A}\left(\frac{1}{a_1^2}-\frac{1}{a_2^2}\right),$$

Now Moseley's equation

$$n=\mathrm{R}(\mathrm{N}-1)^2(1/1^2-1/2^2),$$

where R is Rydberg's constant and in which Sommerfeld

* *Ann. d. Phys.* li. (1916).

substitutes N−3·5 for N−1, gives a good account of the X-ray spectra (K_α-line). This leads us to expect that the constant A contains $(N-N_0)^2$ as a factor, where N_0 may be interpreted as the shielding effect on the outer electrons by the inner ones, the nuclear positive charge being N*e*. Hence by the above equation, since *a* may be a constant for a given family, we might expect an equation of the form

$$\sqrt{\nu}=m(N-N_0),$$

as is found. Against this point of view stands the fact that the limiting frequencies for the series, $\nu_\infty = A/a^2$, decrease in a given column with increasing atomic number.

In discussing the X-ray series Sommerfeld finds that the electron's increase of inertia with speed has to be taken into account. Bohr * had shown that the divergence of the observed frequencies, in the case of hydrogen, from Balmer's formula

$$n=R(1/2^2-1/m^2)$$

could be explained in this manner and corrected the formula to

$$n=R(1/2^2-1/m^2)\{1+\alpha^2/8(1/2^2+1/m^2)\}.$$

Here α^2 is the small quantity $(2\pi e^2/hc)^2$, (where *e* is the electronic charge, *h* Planck's constant, and *c* the velocity of light) occurring again as a universal constant in the discussion of the fine structure of lines. Paschen † measures it more exactly and finds, in the case of helium, that it has the value $\alpha^2=5\cdot30\times10^{-10}$ in our units. Sommerfeld ‡ develops this idea in the case of X-rays and shows, *inter alia*, that the divergence of the observed frequencies for the K and L series can also be explained in this way. If the frequencies of the K series be written

$$n=A(1/a^2-1/n^2),$$

he finds that

$$A/a^2=\left(\frac{N-k}{p}\right)^2\left[1+\frac{a^2}{4}\left(\frac{N-k}{p}\right)^2+\frac{a^4}{8}\left(\frac{N-k}{p}\right)^4+\dots\right],$$

where $k=1\cdot6$ and $p=1$. We may therefore surmise a complete expression for $\sqrt{\nu}$ of the form

$$\sqrt{\nu}=n(N-N_0)[1+u(N-N_0)^2+v(N-N_0)^4+\dots],$$

where $u, v, \dots$ are decreasing small quantities and u is of the order of 10^{-5} or less.

* Phil. Mag. xxix. p. 332 (1915).
† *Ann. d. Phys.* l. p. 901 (1916).
‡ *Loc. cit.*

Now it is not possible with so few points to determine such an expansion with certainty, but we can obtain an approximation as follows:—Solving the equation

(1) $$N = N_0 + q\nu^{1/2} - r\nu^{3/2}$$

for $\sqrt{\nu}$ we obtain

(2)
$$\sqrt{\nu} = \left(\frac{N - N_0}{q}\right)\left[1 + \frac{r}{q}\left(\frac{N - N_0}{q}\right)^2 + 3\left(\frac{r}{q}\right)^2\left(\frac{N - N_0}{q}\right)^4 + \ldots\right].$$

The constants in (1) may be obtained from three points. If, for example, we take the values for Ga, In, Tl, we find that the series obtained from

$$N = \cdot 6843 + 1\cdot 07367\,\nu^{1/2} - 2\cdot 1024 \times 10^{-5}\,\nu^{3/2},$$

$$\textit{i. e. } \sqrt{\nu} = N'(1 + 1\cdot 9581 \times 10^{-5} N'^2 + 1\cdot 1503 \times 10^{-9} N'^4 + \ldots),$$

where N' has been written for $\left(\frac{N - \cdot 6843}{1\cdot 0737}\right)$, is satisfied by all three points. Substituting $N = 13$ for Al we obtain 138·5 instead of 112·07 cm. It is plainly possible to obtain an expansion of the form (2) passing through the Al point as well. What interests us is the fact that the first coefficient is of the same magnitude as Sommerfeld's calculation (10^{-5}) ascribes to the inertia effect. The exact values obtained have no particular interest at this stage, being so dependent on the particular function chosen, but a calculation shows that the same order of magnitude for r/q corrects the points for Au and Hg. It is noticeable, on the other hand, that Bi falls into line with Sb and As without this correction, but we are dealing here, perhaps, with a phenomenon of a different kind.

Summary.

The law of Rydberg, and Kayser and Runge that the square root of the doublet and triplet differences is proportional to the atomic weights, has been subjected to numerical tests, substituting, however, atomic number for atomic weight.

A similar relationship has been found among Kayser and Runge's frequency intervals in the nitrogen column, and a frequency difference of 41·8 cm.$^{-1}$ in the phosphorus

spectrum seems to fall into line. A further linearity is indicated for the helium column.

It is found that Runge and Precht's logarithmic law is not an essential improvement. The required correction is apparently necessitated on Relativity grounds.

It is shown that in two of the columns of the table there is a two-fold linearity, *the lines branching definitely at one of the elements.*

The Physical Laboratory,
The University of Michigan,
June 1918.

XXXVII. *The Genesis of the Law of Error.* *By* Prof. R. A. Sampson*.

IN the issue of this Journal for May of the present year, Prof. F. Y. Edgeworth does me the honour to criticise a paper on the law of distribution of errors which I contributed to the Fifth International Congress of Mathematicians in 1912 and have published in their Proceedings, vol. ii. p. 163. In the course of his remarks he points to an error in one of my formulæ, for which I desire to thank him. My excuse must be that the formula in question was thrown out collaterally and was unnecessary to support the point which I wished to make. Therefore it escaped, I suppose, sufficient examination. Apart from this,—and in itself it hardly seems sufficient reason,—after reading Prof. Edgeworth's paper somewhat carefully, I am a little at a loss to know why it was written; for while it certainly shows little agreement between us, the points of difference appear to me equally unsubstantial. The basis which he dubs my "peculiar notion of the nature of an error of observation" seems to me identically the same thing as he refers to earlier under the name of "some instructive remarks on the nature of errors in astronomical observations" by Morgan Crofton, in Phil. Trans. 1870; while for a text for the whole of my paper I might have taken, had I chosen, a sentence from his own article on "Probability" in *Enc. Brit.* 11th edition, p. 376—"the paths struck out by Laplace and Gauss have hardly yet been completed and made quite secure,"—and indeed would prefer this to his present statement that my "attack on the proof given by Poisson after Laplace strikes at all the

* Communicated by the Author.

applications of the law." I have, I feel, failed to convey my point to Prof. Edgeworth. Within ten minutes of the delivery of the paper at the Congress he had banned every detail of it, and now six years afterwards he puts into print his matured objections. I am not able to write more clearly than that paper is written; but in the hope that I may succeed better with those who view the subject with a less magisterial eye, I shall take the occasion to make a few supplementary remarks.

What is an Error, and why is an Error, of all things, subject to a Law? Has our notion of the nature of an error a definable character from which such a law may be deduced, or must we accept the existence of the law as a fact or as an axiom, without attempting to derive it as implicitly contained in an adopted definition of an error? There can hardly be a doubt as to the right answer to this question. The law is an approximation, and must follow as such, from some rough and ready, tacitly accepted, notion of what constitutes an error,—however difficult it may prove to assign the least restricting notions from which it can be shown to arise. The trouble is that proofs are in existence that seem to dispense, more or less completely, with *any* definition of an error, and which therefore derive without anterior conditions the conclusion that unconditioned errors occur according to a law of frequency of definite form. One must not hesitate to put such proofs aside, including any which begin by postulating the *existence* of an error function. Among these, to mention no more, are Gauss's original proof in the *Theoria Motus,* Herschel's proof from the distribution of shots on a target, and Morgan Crofton's proof by means of differential equations. The question is a logical one of the highest moment and well deserves a few sentences to make it clear.

If we postulate the existence of a law of frequency ruling the unknown and unknowable domain of errors, we so far limit Reality. We add to our view of the Nature of Things a new restriction. An exact analogue may be found in the domain of geometry. If we accept as an ascertained fact the twelfth of Euclid's "common notions," we make an affirmation as to the nature of real space, which in the same way has the character of a limitation of Reality, for we know that as a logical axiom it need not exist. This obstacle has been visible from the beginning. It did not escape the penetrating logical instinct of Gauss or of Laplace. Though Gauss wrote little to clear it up, it is quite evident through his terse expressions that he saw it exactly in its right position, and considered the postulated existence of a frequency function not as an axiom but as an hypothesis, while Laplace offered a proof that the known law of frequency would emerge from the mere superposition of indefinite numbers of small errors, which had *arbitrary laws of frequency of their own.* Poisson gave the same proof in a revised form. Such a theorem would relieve us of all difficulty, for though we may seem to have got something out of nothing, if the demonstration holds this paradox can only be apparent. It is a theorem of convergence, and must be judged so. It is either true or false. Such phrases as "*à trés-peu près,*" "*suivra sensiblement la loi de Gauss,*" or the charitable English equivalent "practically," with its power to cover a multitude of logical sins, are not in the first place admissible. If they are required to help the demonstration out, that means, the theorem is false; for Poisson in particular seems to have held that no conditions were necessary to impose upon the frequencies of the elementary contributing errors,—"*la fonction fx aura telle forme que l'on voudra.*"

I imagine that no one believes that the theorem is true in the form that Poisson gave it. Certainly not Prof. Edgeworth, who refers to instances given by himself in which it is falsified, and states conditions under which it may be true. Such conditions are an admission that the law does not exist unless the errors possess a defined character. They constitute implicitly a definition of errors as restricted to such a form as may be necessary to produce the law. That is to say the law is a consequence of limits tacitly imposed by accepted notions as to the nature of errors.

Where then does the Law of Error come from, and why does it apply, on the whole, so unerringly to the most diverse and unselected material? That it does not apply always and of necessity, may be taken as admitted. That it does apply very closely and very commonly is a matter of experience. Without questioning that Laplace's theorem, subject to restrictions the precise character of which it is at the moment immaterial to specify, gives with great generality an account of the origin of the law which is sufficient in the sense that on the whole it is analytically convincing, can we add anything from another point of view that will make its genesis and its progress as an approximation more visible? It is only here that a few remarks in my paper may claim some novelty. Yet so much has been written on the theory of errors that even for these I should not be surprised to find an anticipator.

If we take a distribution of errors subject to the regular law, that is to say occurring proportionately to $\exp(-h^2x^2)$, and replace each element of this by another distribution, also subject to the law, and collect the results, in the order of their magnitude, a third final distribution emerges which again is subject to the law. This is the reproductive property of the law of error, which has been proved apparently by a number of people. By itself such a property leads us nowhere, for its application is limited to domains already subject to the law. Therefore we must travel outside it to find the genesis of the law. But I make two further points. First, if we take a distribution not strictly under the law $\exp(-h^2x^2)$, but under one fluctuating about it, say $\exp(-h^2x^2).(1+a\cos kx)$ and go through the same operation of disturbing it by a second distribution of the same kind, say under the law $\exp(-h'^2x^2).(1+a'\cos k'x)$, we get a third resultant distribution in which the *fluctuating element tends to efface itself.* Hence if we go on piling error upon error, provided each has the fluctuating character indicated above, we shall as a limit converge to the pure law of Gauss. My other point is that to obtain an approximation to a set of numbers fluctuating about the law of distribution $\exp(-h^2x^2)$, where h is an adjustable constant, nothing more is requisite than to take as originating the error, say for precision, any holomorphic function and then to get the frequency curve register the number of times individual values occur, disregarding at the same time the order in which these values arise naturally. With a single-valued function possessing one maximum and one minimum, the resulting frequency graph will be two portions of the axis of x extending from plus and minus infinity respectively and a portion parallel to the axis between them. For example, a sine curve for the generating function gives such a distribution which may be considered the first rude approximation to an error curve modified by fluctuations. If Prof. Edgeworth's criticism of my treatment of this point implies that when observations are vitiated by the occurrence of a neglected term of the form $a\sin kt$, we should find among them infinitely more cases of occurrence of maxima than of zeroes, as it appears to do, then I beg to differ. I think all values within the limits $\pm a$ would be equally likely, and that is what the frequency graph described above would imply. If then we suppose that errors are not of mysterious character, *sui generis*, but are simply the mass of numberless neglected disturbances, each occurring according to regular law and order of its own, it is seen that we obtain the approximation to Gauss's law which is necessary to begin with, by the *operation of neglecting the circumstances and order of their origin, and scheduling merely in sequence of magnitude the number of times that each particular value occurs.* It is this operation that is the significant act which effaces the individuality of the contributing elements and permits us to obtain, apparently from nothing, the law of Gauss; for if we go on repeating it for more and more sources of error, we obtain the law with greater and greater purity. This is the view of the actual logical basis of the Error Law which I endeavoured to convey in my paper. It does not escape, of course, the difficulties of convergence which present themselves in Laplace's theorem, for these are inherent, and in a strict sense, fatal to absolute generality. Nor does it oust any other proof. But I offer it as a view by which we can *see the law coming into existence,* which I submit the other forms of proof one and all fail to supply.

XXXVIII. *On the Calculation of Magnetic and Electric Saturation Values. By* J. R. Ashworth, *D.Sc.**

THE principal object of this paper is to show how it is possible to calculate from two well-known constants the limiting value of the magnetic intensity of a magnetic substance for which Curie's law holds good, and by the same reasoning to estimate the limiting current density which a conductor can carry in the case of those metals in which the resistivity is directly proportional to the absolute temperature.

Magnetic Intensity.

In general, paramagnetic substances at all temperatures, and ferromagnetic bodies above their critical temperatures, obey a simple law which is analogous to the gas law.

* Communicated by the Author.

Let I = Intensity of Magnetization,
H = Field Strength,
T = Absolute Temperature,
and R' = a constant.

Then, if I is small compared with the saturation value (I_0),

$$\frac{\mathrm{H}}{\mathrm{I}} = \mathrm{R}'\mathrm{T}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

Here R′ is the reciprocal of Curie's constant (A)—that is to say, is the reciprocal of the product of the susceptibility into the absolute temperature.

If, however, I becomes appreciable compared with I_0 equation (1) must be extended to express the fact that I may reach a limiting value (I_0).

When the mutual control of the magnetic molecules is negligible compared with the external force the more general equation is

$$\mathrm{H}\left(\frac{1}{\mathrm{I}}-\frac{1}{\mathrm{I}_0}\right)=\mathrm{R}'\mathrm{T}. \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

To change the magnetic energy from HI_0 to HI when H is constant thermal energy must be supplied which may be expressed in terms of R, the gas constant, and T, the absolute temperature, RT being double the energy corresponding to each degree of freedom of the molecule.

Since there are two degrees of freedom which affect the magnetic moment, the mean kinetic energy under consideration will be RT, assuming that the vibrations and rotations take place with the same freedom as the translatory movements of the molecules of a gas.

Putting $\mathrm{I} = \frac{1}{n}\mathrm{I}_0$ at temperature T, then equation (2) becomes

$$\frac{\mathrm{H}}{\mathrm{I}_0}(n-1) = \mathrm{R}'\mathrm{T}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

and multiplying throughout by I_0^2 we have

$$\mathrm{HI}_0(n-1) = \mathrm{R}'\mathrm{T}\mathrm{I}_0^2. \quad . \quad . \quad . \quad . \quad . \quad . \quad (4)$$

The left side of this equation is the kinetic energy required to reduce the magnetic intensity from I_0 to

$\frac{1}{n} I_0$ at constant field-strength, and as it is proportional to the temperature, it may be put equal to RT. Hence

$$RT = R'TI_0^2, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (4a)$$

and therefore

$$I_0 = \sqrt{\frac{R}{R'}}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (5)$$

Thus the calculation of the limiting intensity of magnetization (I_0) only involves a knowledge of the well-known gas constant and the reciprocal of Curie's constant.

The truth of equation (5) may be tested by comparing the calculated values of the limiting magnetization with the experimentally determined maximum values of magnetization where they are known with approximate accuracy.

As examples the ferromagnetic metals Iron, Nickel, and Cobalt will be selected.

Iron.

The constant R must be taken for one cubic centimetre.

Putting the gas constant equal to $83{\cdot}15 \times 10^6$ ergs per degree centigrade for a gram molecule, and taking the atomic weight of iron to be 55 85 and the density to be 7·86, then

$$R = \frac{83{\cdot}15}{55{\cdot}85} \times 7{\cdot}86 \times 10^6 = 11{\cdot}7 \times 10^6,$$

assuming there is one atom in the molecule of iron in the solid state.

$$R' = 3{\cdot}56 \text{ when } A = 0{\cdot}281 \quad \text{(Curie, \textit{Œuvres}, p. 327)};$$

therefore

$$I_0 = \sqrt{\frac{11{\cdot}7}{3{\cdot}56}} \times 10^3 = 1817.$$

This number for the calculated limiting magnetization compares favourably with the following experimental saturation values :

I_0.		
1706	Weiss, *J. de Phys.* ix. p. 373.	1910.
1730	Ewing, Phil. Trans. clxxx. p. 221.	1889.
1798	Taylor Jones, Phil. Mag. xli. p. 161.	1896.
1798	Williams, Phys. Rev. vi. p. 404.	1915.

Nickel.

$$R = 12{\cdot}6 \times 10^6$$

if the atomic weight is 58·68, the density 8·9, and if the molecule contains one atom.

$$R' = 20{\cdot}8 \text{ when } A = 0{\cdot}048$$

(Weiss & Bloch, *Arch. des Sc.* t. xxxiii. p. 293);

therefore

$$I_0 = \sqrt{\frac{12{\cdot}6}{20{\cdot}8}} \times 10^3 = 781.$$

This value for I_0 is considerably higher than what has been observed. The calculation has been made on the supposition that the molecule of nickel contains one atom; if, however, the molecule contains two atoms, then, putting the molecular weight equal to $2 \times 58{\cdot}68$ instead of 58·68, the formula gives

$$I_0 = 552,$$

and this is in reasonable agreement with the facts. Experimental values are:

I_0.		
479	Weiss, *J. de Phys.* ix. p. 161.	1910.
540	Ewing, Phil. Trans. clxxx. A. p. 221.	1889.

Cobalt.

$$R = 12{\cdot}14 \times 10^6$$

if the atomic weight is 58·97, the density 8·6, and if there is one atom in the molecule.

$$R' = 6{\cdot}0 \text{ when } A = 0{\cdot}166$$

(Weiss, *Arch. des Sc.* 4 ser. t. xxxi. pp. 5 & 89).

Hence

$$I_0 = \sqrt{\frac{12{\cdot}14}{6{\cdot}0}} \times 10^3 = 1422,$$

which is very nearly the maximum magnetization which different observers have found, namely,

I_0.		
1310	Ewing, Phil. Trans. clxxx. A. p. 121.	1889.
1412	Weiss, *J. de Phys.* ix. p. 373.	1910.
1421	Stifler, Phys. Rev. xxxiii. p. 268.	1911.

Electric Current Density.

At ordinary temperatures and for pure metals in which the resistivity is proportional to the absolute temperature the law of conduction of electric currents is analogous to the gas law.

Let E = Fall of potential per cm. in c.g.s. units,
i = Current in c.g.s. units per sq. cm.,
T = Absolute temperature,
and S = a constant.

Then
$$\frac{E}{i} = ST. \quad \quad (6)$$

If σ is the resistivity in c.g.s. units, Ohm's law is

$$\frac{E}{i} = \sigma, \quad \quad (7)$$

and therefore
$$S = \frac{\sigma}{T}. \quad \quad (8)$$

This quantity S has the same importance in electrical theory as Curie's constant has in the theory of magnetism.

If, however, i can reach a limiting value i_0 then equation (6) must be written more generally as in magnetism, thus

$$E\left(\frac{1}{i} - \frac{1}{i_0}\right) = ST. \quad \quad (9)$$

Putting $i = \frac{1}{n} i_0$ the equation becomes

$$\frac{E}{i_0}(n-1) = ST, \quad \quad (10)$$

and multiplying throughout by i_0^2 we have

$$Ei_0(n-1) = STi_0^2. \quad \quad (11)$$

The left side of this equation when E is constant is the variation with temperature of electrical energy per unit of time, and according to a theory of metallic conduction to be referred to later on we may write this change of energy per unit of time in terms of its thermal equivalent, as in the magnetic problem, and put it equal to $\mathrm{RT}\frac{1}{t}$, where R is the gas constant and t is the time.

Hence

$$\mathrm{RT}\frac{1}{t} = \mathrm{ST}i_0^2, \quad . \quad . \quad . \quad . \quad . \quad . \quad (12)$$

and if v is written for $\frac{1}{t}$ we obtain

$$i_0 = \sqrt{\frac{\mathrm{R}}{\mathrm{S}}}\, v. \quad . \quad . \quad . \quad . \quad . \quad . \quad (13)$$

Thus the maximum current density can be calculated from a knowledge of the constants R and S, which can easily be obtained for most of the pure metals, and from a knowledge of v the velocity of the electron as it passes along a conductor.

For the sake of estimating a limiting value to the current density the velocity of the electron will be taken to be of the same order as that of the cathode ray, namely, 10^9 cm. per second.

The following, then, are some examples of the calculation of i_0, the maximum current density, according to equation (13), expressed as *amperes* per sq. cm., the other quantities in the table being in C.G.S. units.

Metal.	Atomic weight.	Density.	R. per cb. cm.	S.	i_0.
Silver	107·9	10·5	$8{\cdot}09\times10^6$	5·7	$3{\cdot}8\times10^8$
Copper	63·57	8·93	$11{\cdot}66\times10^6$	5·5	$4{\cdot}5\times10^8$
Aluminium	27·1	2·65	$6{\cdot}60\times10^6$	10·1	$2{\cdot}5\times10^8$
Tin	119·1	7·29	$5{\cdot}08\times10^6$	38·8	$1{\cdot}2\times10^8$
Lead	207·1	11·37	$4{\cdot}56\times10^6$	71·4	$0{\cdot}8\times10^8$

The calculation is made on the supposition that the molecule contains one atom; if it contains n atoms R must be divided by n and i_0 by $\sqrt{n}$.

Nernst ('Theory of the Solid State,' p. 81) states that silver, copper, aluminium, and lead are probably monatomic in the solid state, and, if so, i_0 for these metals must be of the order 10^8 amperes per sq. cm.

These examples include good and bad conductors of electricity, metals of high and low atomic weight, of high and low valency, and of high and low density. The temperature coefficient of resistivity is nearly the same for all, namely, about 0·0038, a number which shows that the resistivity is approximately proportional to the absolute temperature.

According to the calculation the saturation current density in these metals is of the order 10^8 amperes per sq. cm., and this is considerably in excess of any observations of high current densities which have been recorded. A recent experiment by Trauenberg (Trauenberg, *Phys. Zeits.* xviii. p. 75, 1917) shows that Ohm's law holds good up to 8×10^6 amperes per sq. cm., presumably for silver; but this enormous current density would have to be increased more than tenfold before Ohm's law would fail. There is nothing then in this experimental value to make the saturation current densities which have just been calculated at all improbable.

A direct proof that Ohm's law will fail for current densities of the order 10^8 amperes per sq. cm. seems at present beyond the reach of experimental demonstration.

Theory of Metallic Conduction.

In his Presidential Address to the Physical Society (Thomson, Phys. Soc. Proc. vol. xxvii. part 5, p. 527; Phil. Mag. xxix. pp. 192–202; also 'Corpuscular Theory of Matter,' p. 86) Sir J. J. Thomson has outlined a theory of metallic conduction based on the hypothesis that in a metal there are electric doublets which under an electric force can be orientated, and this is a principal function of an electromotive force. These doublets, like the magnetic molecules of a magnetic substance, have their alignment with the direction of the force opposed by thermal agitation, and according to the conditions of field-strength and temperature they may be free from each other's control or subject to each other's influence. So far the theory would apply to electric insulators as well as to conductors, but the distinguishing feature of a conductor is that the doublets very easily part with electrons, which pass from atom to atom of a polarized chain "like a company in single file passing over a series of stepping-stones."

If the intensity of the polarization and the charge determined by it can be calculated, the strength of the current will be given by multiplying this charge by the velocity of movement of the electron.

The problem is solved in the same way as for the determination of the intensity of magnetization of an assemblage of magnetic molecules.

In general symbols,

Let $Y =$ The component of intensity of magnetization, or, of electric polarization parallel to the directive force,

$X =$ The applied force,

$T =$ The absolute temperature,

$\alpha = \frac{\mu X}{R_\mu T}$, where μ is the magnetic or electric moment of the molecule and R_μ the gas constant for one molecule,

then

$$\frac{Y}{Y_0} = \coth \alpha - \frac{1}{\alpha}, \quad . \quad . \quad . \quad . \quad . \quad . \quad (14)$$

Y_0 being the maximum value of Y.

For small values of Y this becomes

$$Y = \frac{Y_0 \alpha}{3} = \frac{Y_0 \mu X}{3 R_\mu T}, \quad . \quad . \quad . \quad . \quad . \quad . \quad (15)$$

or

$$\frac{X}{Y} = CT, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (16)$$

C being the constant $\frac{3R_\mu}{Y_0 \mu}$.

This equation has the same form as the gas law.

In passing it may be noticed that if μ be multiplied by the number of molecules in unit volume and the product be put equal to Y_0, and if the appropriate value of R be used, then

$$Y_0 = \sqrt{\frac{3R}{C}},$$

a formula which differs from the one employed above in the calculation of maximum values by the insertion of the factor 3 in the numerator under the root sign; when it is applied, all the saturation values given above must be multiplied by $\sqrt{3}$. In magnetism the agreement between the theoretical and observed saturation values would remain the same as before if it be assumed that iron and cobalt have each *three* atoms in the molecule and that the molecule of nickel contains *six* atoms. (See Kunz, Phys. Rev. vol. xxx. p. 359.)

Super-Conductivity.

The fundamentally important experiments of Kamerlingh Onnes, which show that there is a critical temperature for electric conductivity in some metals and that below this temperature they pass into a state of super-conductivity, can be explained by an extension of the theory given above. The state of super-conductivity, in which it is possible for a current to continue after the removal of the applied electromotive force, is analogous to residual magnetization in a ferromagnetic body which persists after the applied magnetizing force is removed, and it may be explained, as for magnetism, by the hypothesis that there is an intrinsic field in action, which is a function of the polarization, in addition to the externally applied force. Thus in equation (16) X must be replaced by $X+f(Y)$, and then, although X may become zero, the intrinsic field $f(Y)$ may persist, under proper temperature conditions, giving rise to a persistent electric current.

The problem may be treated in the same way as when the gas law is made to include liquids by the introduction of an intrinsic pressure.

The extended gas law in general symbols will then be

$$(X+f(Y))\left(\frac{1}{Y}-\frac{1}{Y_0}\right)=KT, \quad . \quad . \quad . \quad (17)$$

K being a constant analogous to R,

and if van der Waals's expression for $f(Y)$ be adopted we have

$$(X+aY^2)\left(\frac{1}{Y}-\frac{1}{Y_0}\right)=KT. \quad . \quad . \quad . \quad . \quad (18)$$

This equation when applied to ferromagnetism yields numerical results which meet with the same success as those derived from the kinetic theory, and it represents in the main the chief experimental facts of magnetism, so that it may be applied with some confidence to electric polarizations and currents the theory of which is like the theory of magnetism. Equation (18) then becomes

$$(E+ai^2)\left(\frac{1}{i}-\frac{1}{i_0}\right)=ST, \quad . \quad . \quad . \quad . \quad (19)$$

and this equation implies that there are critical constants for electric polarizations and currents.

The critical temperature will be given by

$$T_c = \frac{8}{27}\frac{ai_0}{S}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (20)$$

from which it is possible to estimate a and therefore to estimate the magnitude of the intrinsic field. The calculation can only give an upper limit to a, since the critical temperatures for conductivity determined up to now are not far above the absolute zero, and at such low temperatures the atomic heat is a very small quantity and the kinetic energy in question is no longer equal to RT.

Taking Lead as an example in which the critical temperature is a little less than 4° absolute, and using the c.g.s. values of i_0 and S found above, then a must be less than $1{\cdot}2 \times 10^{-4}$ and consequently the maximum intrinsic field (ai_0^2) is less than $7{\cdot}9 \times 10^9$ c.g.s. units or 79 volts per cm. As the current densities commonly employed are only about a millionth of the limiting values calculated above, it follows that the intrinsic field in such a conductor as Lead, when it carries even a high current density at temperatures above the critical temperature, must be extremely small, indeed negligible compared with the applied electromotive force. This, however, is to be expected since Ohm's law is obeyed with very great accuracy at ordinary temperatures, which would not be the case if the intrinsic field made itself felt. Below the critical temperature current densities approaching the maximum should be attainable, and it is of interest to find that Kamerlingh Onnes has observed a current density in mercury at 2°·45 absolute, which is lower than the critical temperature, of more than 10^5 amperes per sq. cm. (K. Onnes, Elect. lxxi. p. 855, 1913).

From what has been said above, it is seen that the facts of electric conduction at very low temperatures as well as the like facts of ferromagnetic induction are in agreement with the ideas which underlie the fluid equation, and thus both magnetic and electric experiments give to the fluid law a generality wider than has commonly been accorded to it; and in the particular case in which it becomes the gas law it may be said that it governs not only the free translatory movements of molecules which determine the behaviour of a gas, but also the free vibrations and rotations of molecules which are manifested in the magnetic and electric behaviour of substances in general.

XXXIX. *Notices respecting New Books.*

Elements of the Electromagnetic Theory of Light. By LUDWIK SILBERSTEIN, Ph.D. Pp. vii+48. Longmans, Green & Co. Price 3*s.* 6*d.* net.

IN this elegant little volume of 48 pages are condensed the chief consequences which follow mathematically from the Hertz-Heaviside form of Maxwell's equations. The vectorial treatment effects a great economy of space as compared with the old Cartesian splitting of every vector into its three components, and, since the time has now been reached when, it is hoped, the elementary vectorial operations are familiar to every student, the method is the natural one to adopt in expounding the subject. Plane waves only are considered, and their reflexion and refraction, polarization, and double refraction in crystals handled briefly yet convincingly. This development of the simple theory is clear and satisfactory, and likely to be extremely useful to one whose acquaintance with the subject is not very deep: to the maturer student the chief interest of the book is the excellent historical account of the work preceding the electromagnetic theory, which occupies the first fifteen pages or so. In this the successive difficulties met by the elastic solid theory are succinctly exposed, and the many ingenious hypotheses put forward to solve the vexed question of the longitudinal waves are detailed. The striking advantages of the electromagnetic theory are thus thrown into relief. There are, we think, few who will not find something new to them in this well-planned critical sketch of one of the most interesting chapters of physics.

Stoichiometry. By SYDNEY YOUNG, D.Sc., F.R.S. Longmans, Green & Co. Second Edition. Pp. xii+363. Price 12*s.* 6*d.* net.

IT is eleven years since the first edition of Professor Young's book appeared, and during that time the discovery of isotopes, the other developments of the study of radioactivity, and the measurement of X-ray spectra have led to considerable modifications of and additions to previous ideas on the subject of atomic weight. In this second edition Professor Young has introduced a short discussion of the modern views of the nature of an element and the existence of isotopes, and gives an account of Soddy's theories, but it is remarkable that he contents himself with a passing reference, which is not even indexed, to X-ray spectra, and has no word of Moseley's atomic *numbers*, which confirm the impression given by the chemical properties of the elements concerned that there is something wrong in the position of Argon and Potassium, Tellurium and Iodine, Cobalt and Nickel in the periodic table, when the elements are arranged in order of the atomic *weight*. The physicist will also miss the lack of any reference in the sketch of the kinetic theory to recent experimental confirmation, or to Ramsay's determination of the atomic weight of radium emanation by the density method where that method is treated. Recent work on osmotic pressure is well discussed.

The book possesses all the valuable features of the first edition, and the larger format in which it is now printed shows a great improvement on the old in both appearance and convenience in handling.

X Rays and Crystal Structure. By W. H. Bragg, M.A., D.Sc., F.R.S., and W. L. Bragg, B.A. George Bell & Sons. Third Edition. Pp. vii+229. Price 8*s.* 6*d.* net.

It is pleasant to find that, in spite of the war, a third edition of this book has been called for. No alteration of any importance has been made since the first edition, which gave a wonderfully clear and concise account of the researches which led to the use of crystals as diffraction gratings for X rays, and afterwards revealed so much of the structure of crystals; researches to which Professor Bragg and his son contributed so much. It is to be hoped that the next edition will see the war ended, and the authors, at present employing their ingenuity in the fight against the common enemy, continuing their investigations in a field which they have cultivated to such purpose.

XL. *Proceedings of Learned Societies.*

GEOLOGICAL SOCIETY.

[Continued from p. 280.]

March 20th, 1918.—Mr. G. W. Lamplugh, F.R.S., President, in the Chair.

Dr. W. F. Smeeth delivered a Lecture on the Geology of Southern India, with particular reference to the Archæan Rocks of the Mysore State. With the aid of a map, prepared by the Geological Survey of India, the Lecturer pointed out the general character of the geological formations of Southern India, which consist, very largely, of a highly folded and foliated complex of Archæan gneisses and schists, followed by some considerable patches of pre-Cambrian slates, limestones, and quartzites; with these are associated basic lava-flows and ferruginous jaspers. The remaining formations consist of remnants of the Gondwana Beds (coal-measures of Permo-Carboniferous age), a few patches of Cretaceous rocks, some Tertiary and Pleistocene deposits, and recent sands and alluvium, all situated along the coastal margins of the Peninsula. He contrasted the scanty post-Archæan record of Southern India, the apparent non-submergence of the greater portion of the area and its freedom from great earth-movements since Archæan times, with the widely-extended formations of Northern India which recorded oft-repeated movements of depression and elevation, culminating in the rise of the Himalaya in Tertiary times and accompanied by igneous activity on a gigantic scale, as proved by the outpourings of the Deccan Trap.

In discussing the Archæan complex, the Lecturer traced the history of the various views which have been held. Newbold (1850) regarded the complex as formed of Protogene schists and gneisses intruded into by granites. Bruce Foote (1880) separated the schists (to which he gave the name 'Dhárwár System') from the gneisses, and regarded them as laid down unconformably upon the gneisses and granites which, for many years thereafter, were embraced in the term 'Fundamental Gneissic Complex.' He regarded the Dhárwár System as transition-rocks between the old gneisses and the older Palæozoic rocks (Cuddapa, etc.). Holland (1898) differentiated the Charnockites, showing that they formed a distinct petrographical province with intrusive relations to the main members of the gneissic complex, and in 1906 he proposed to regard the Cuddapa System as pre-Cambrian, and separated by a great Eparchæan Interval from the Dhárwár System which, together with the gneissic complex, he classed as Archæan. In 1913, Holland added a group of post-Dhárwár eruptive rocks, and produced a classification of the pre-Cambrian rocks of India which exhibits a remarkable parallelism with that given by Lawson (1913) for the pre-Cambrian of Canada.

The work of the Mysore Geological Survey from 1899 to 1914 had gradually eliminated the Fundamental Gneissic Complex, and shown that within the area of the Mysore State—representing some 29,000 square miles of the Archæan complex—the oldest rocks were the Dhárwar System, which had been intruded into by at least four successive granite-gneisses, namely: the Champion Gneiss, the Peninsular Gneiss (forming the greater part of the area), the Charnockites, and the Closepet Granite Series. If we compared this succession with Holland's 1913 classification, without assuming any real correlation with the Canadian rocks, but viewing the Dhárwár rocks as Huronian, as suggested by Holland, then his post-Dhárwár eruptive series (Algoman) included the whole of the gneisses of Mysore, while equivalents of the Laurentian and Ontarian formations were wanting. On the other hand, if the Dhárwár rocks were regarded as Keewatin, then the gneisses of Mysore might represent Laurentian and, possibly, Algoman formations, while representatives of the Huronian would be non-existent. Obviously, therefore, the Mysore Archæan succession was either very incomplete, or it did not fit in with the classifications of Holland and Lawson. It was to be remembered that Holland's classification dealt with a much wider area than Southern India, and the essential problem appeared to be whether his Bundelkhand gneiss (Laurentian) and the Bengal gneisses (Keewatin) were really older than, and unconformable to, the Dhárwár System—as represented by him—, or whether they were post-Dhárwár eruptives corresponding to portions of the Mysore gneissic complex. In favour of the latter view it was noted that observers acquainted with both have appeared to recognize the Bundelkhand and Bengal types of gneisses in and around Mysore, and that all of these gneisses have, until recently, been regarded as forming part of the great Fundamental Gneissic Complex of India.

The Lecturer then described the map of Mysore which, on a scale of 8 miles to the inch (1 : 506,880), presented a simplified summary of the work of the Mysore Geological Survey. On lithological grounds the Dhárwár System was divided into an Upper and a Lower Division. The former was composed largely of basic flows and sills with their schistose representatives. Whether some of the chloritic schists, slates, phyllites, and argillites were of sedimentary origin was still doubtful. In the series as a whole, chlorite predominated and hornblende was subordinate. The presence of carbonate of lime, magnesia, and iron was a strikingly prevalent feature. The Lower Division was composed of dark hornblendic epidiorites and schists, which were distinguishable from the greenstones of the Upper Division by their dark colour and practical absence of chlorite. Many of the greenstones and schists of the Upper Division appeared to resemble Keewatin rocks of Lake Superior, such as the Ely Greenstone series (save that augite is conspicuously absent in the Mysore rocks), and it had been suggested that the dark epidiorites, which naturally crop out between the rocks of the Upper Division and the intruding gneisses, might be merely metamorphosed portions of the greenstones and chlorite-schists. This might be true in some cases, but the independent existence of the dark hornblendic rocks of the Lower Division was supported by the fact that they do not exist in many places where the gneisses come into contact with the greenstones; that many of the former retain original igneous structures, which would be unlikely to survive the chloritization and the subsequent change to epidiorite; and, finally, that the amphibolitization of the rocks of the Lower Division appears to have been complete before the intrusion of the earliest of the gneisses which, with its associated pegmatites and quartz-veins, has developed secondary augite in the hornblendic rocks along intrusive contacts.

The Lecturer referred briefly to the autoclastic conglomerates which were usually associated with intrusions of the Champion Gneiss, to the intrusive character of some of the quartzites or quartz-schists, and to the evidence that the limestones were, partly if not wholly, due to metasomatic replacement of other rocks by carbonates of lime and magnesia.

The Dhárwár schists of Mysore contain a widely extended series of banded quartz iron-ore rocks, very similar to those of the Lake Superior district, the origin of which has been the subject of much discussion, and is still very perplexing. Some of the earlier American geologists considered them to be directly igneous in origin, but these views are now discredited, and replaced by an interesting and ingenious theory of chemical precipitation from liquids associated with subaqueous lavas. The Lecturer suggested that some of these rocks might be pegmatitic intrusions of quartz and magnetite, and that some might be the metamorphosed relics of igneous rocks composed, largely, of highly ferruginous amphiboles (such as cummingtonite) or other chemically allied minerals.

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XLI. *The Dispersal of Light by a Dielectric Cylinder.* *By* Lord RAYLEIGH, *O.M., F.R.S.**

THE problem of the incidence of plane electric waves on an insulating dielectric cylinder was treated by me as long ago as 1881 †. Further investigations upon the same subject have been published by Seitz ‡ and by Ignatowski § who corrects some of Seitz's results. Neither of these authors appears to have been acquainted with my much earlier work. The purpose of the present paper is little more than numerical calculations from the expressions formerly given, but in order to make them intelligible it will be well to quote what was then said. The notation is for the most part Maxwell's.

"We will now return to the two-dimension problem with the view of determining the disturbance resulting from the impact of plane waves upon a cylindrical obstacle whose axis is parallel to the plane of the waves. There are, as in the problem of reflection from plane surfaces, two principal cases—(1) when the electric displacements are parallel to the axis of the cylinder taken as axis of z, (2) when the electric displacements are perpendicular to this direction."

* Communicated by the Author.
† Phil. Mag. vol. xii. p. 81 (1881); Sci. Papers, vol. i. p. 533.
‡ *Ann. d. Physik*, xvi. p. 746 (1905); xix. p. 554 (1906).
§ *Ann. d. Physik*, xviii. p. 495 (1905).

" *Case* 1. [From the general equation with conductivity (C) zero and magnetic permeability (μ) constant],

$$\left(\frac{d^2}{dx^2}+\frac{d^2}{dy^2}\right)\frac{h}{\mathrm{K}}+n^2\mu\mathrm{K}\frac{h}{\mathrm{K}}=0\;;\quad . \quad . \quad . \quad (1)^*$$

or if, as before, $k=2\pi/\lambda$,

$$\left(\frac{d^2}{dx^2}+\frac{d^2}{dy^2}+k^2\right)\frac{h}{\mathrm{K}}=0,\quad . \quad . \quad . \quad . \quad . \quad (2)$$

in which k is constant in each medium, but changes as we pass from one medium to another. From (2) we see that the problem now before us is analytically identical with that treated in my book on Sound †, § 343, to which I must refer for more detailed explanations. The incident plane waves are represented by

$$e^{int}\,e^{ikx}=e^{int}\,e^{ikr\cos\theta}$$
$$=e^{int}\{\mathrm{J}_0(kr)+2i\,\mathrm{J}_1(kr)\cos\theta+\ldots$$
$$+2i^m\mathrm{J}_m(kr)\cos m\theta+\ldots\}\;;\quad . \quad . \quad . \quad (3)$$

and we have to find for each value of m an internal motion finite at the centre, and an external motion representing a divergent wave, which shall in conjunction with (3) satisfy at the surface of the cylinder ($r=c$) the condition that the function (h/K) and its differential coefficient with respect to r shall be continuous. The divergent wave is expressed by

$$\mathrm{B}_0\psi_0+\mathrm{B}_1\psi_1\cos\theta+\mathrm{B}_2\psi_2\cos 2\theta+\ldots,\quad . \quad . \quad (4)$$

where ψ_0, ψ_1, &c. are the functions of kr defined in § 341. The coefficients B are determined in accordance with

$$\mathrm{B}_m\left\{kc\frac{d\psi_m}{d\,.\,kc}\mathrm{J}_m(k'c)-k'c\,\psi_m\frac{d}{d\,.\,k'c}\mathrm{J}_m(k'c)\right\}$$
$$=2i^m\{k'c\,\mathrm{J}_m(kc)\,\mathrm{J}_m'(k'c)-kc\,\mathrm{J}_m(k'c)\,\mathrm{J}_m'(kc)\},\quad . \quad . \quad (5)$$

except in the case of $m=0$, when $2i^m$ on the right-hand side is to be replaced by i^m ‡. In working out the result we

* The numbering of the equations is changed. h is the component of electric displacement parallel to z, K the specific inductive capacity, and λ the wave-length.

† 'Theory of Sound,' vol. ii. Macmillan, 1st ed. 1878, 2nd ed. 1896.

‡ Here k' relates to the cylindrical obstacle and k to the external medium.

suppose kc and $k'c$ to be small; and we find approximately for the secondary disturbance corresponding to (3)

$$\psi = \left(\frac{\pi}{2ikr}\right)^{\frac{1}{2}} e^{i(nt-kr)} \left[\frac{k'^2c^2 - k^2c^2}{2} - \frac{k^2c^2(k'^2c^2 - k^2c^2)}{8} \cos\theta\right]; \quad (6)$$

showing, as was to be expected, that the leading term is independent of θ."

"For case 2, which is of greater interest, we have (from the general equations)

$$\left(\frac{d}{dx}\frac{1}{k^2}\frac{d}{dx} + \frac{d}{dy}\frac{1}{k^2}\frac{d}{dy} + 1\right)c = 0. \quad . \quad . \quad . \quad (7)^*$$

This is of the same form as (2) within a uniform medium, but gives a different boundary condition at a surface of transition. In both cases the function itself is to be continuous; but in that with which we are now concerned the second condition requires the continuity of the differential coefficient *after division by* k^2. The equation for B_m (or B_m' as we may write it for distinctiveness) is therefore

$$B_m' \left\{ k'c \frac{d\psi_m}{d.kc} J_m(k'c) - kc\psi_m \frac{dJ_m(k'c)}{d.k'c} \right\}$$

$$= 2i^m\{kc . J_m(kc) J_m'(k'c) - k'c J_m(k'c) J_m'(kc)\}, \quad . \quad . \quad (8)$$

with the understanding that the 2 is to be omitted when $m=0$. Corresponding to the primary wave $e^{i(nt+kx)}$, we find as the (approximate) expression of the secondary wave at a great distance from the cylinder,

$$\psi = \left(\frac{\pi}{2ikr}\right)^{\frac{1}{2}} e^{i(nt-kr)} \left[-\frac{k^2c^2}{16}(k^2c^2 - k'^2c^2)\right.$$

$$\left. - k^2c^2 \frac{k'^2 - k^2}{k'^2 + k^2} \cos\theta - \frac{1}{8} k^4c^4 \frac{k^2 - k'^2}{k^2 + k'^2} \cos 2\theta \right]. \quad . \quad . \quad (9)$$

The term in $\cos\theta$ is now the leading term; so that the secondary disturbance approximately vanishes in the direction of the primary electrical displacements, agreeably with what has been proved before. It should be stated here that (9) is not complete to the order k^4c^4 in the terms containing $\cos\theta$. The calculation of the part omitted is somewhat tedious in general; but if we introduce the supposition that the difference between k'^2 and k^2 is small, its effect is to bring in the factor $(1 - \frac{1}{4}k^2c^2)$."

* In (7) c is the magnetic component, and not the radius of the cylinder. So many letters are employed in the electromagnetic theory, that it is difficult to hit upon a satisfactory notation.

"Extracting the factor (k'^2-k^2), we may conveniently write (9)

$$\psi=-k^2c^2\frac{k'^2-k^2}{k'^2+k^2}\left(\frac{\pi}{2ikr}\right)^{\frac{1}{2}}e^{i(nt-kr)}\left[\cos\theta-\frac{k'^2c^2+k^2c^2}{16}\right.$$
$$\left.-\frac{k^2c^2}{8}\cos 2\theta\right], \quad . \quad . \quad . \quad (10)$$

in which

$$\cos\theta-\frac{k'^2c^2+k^2c^2}{16}-\frac{k^2c^2}{8}\cos 2\theta$$
$$=\cos\theta-\frac{k'^2c^2-kc^2}{16}-\frac{k^2c^2}{4}\cos^2\theta. \quad . \quad . \quad (11)$$

"In the direction $\cos\theta=0$, the secondary light is thus not only of high order in kc, but is also of the second order in $(k'-k)$. For the direction in which the secondary light vanishes to the next approximation, we have

$$\tfrac{1}{2}\pi-\theta=\tfrac{1}{16}(k'^2c^2-k^2c^2)=\frac{k^2c^2}{16}\frac{\text{K}'-\text{K}}{\text{K}}. \quad . \quad . \quad (12)$$

This . . . is true if kc, $k'c$ be small enough, whatever may be the relation of k' and k. For the cylinder, as for the sphere, the direction is such that the primary light would be bent through an angle *greater* than a right angle. . . ."

"If we suppose the cylinder to be extremely small, we may confine ourselves to the leading terms in (6) and (9). Let us compare the intensities of the secondary lights emitted in the two cases along $\theta=0$, *i. e.* directly backwards. From (6)

$$\psi\propto\tfrac{1}{2}(k'^2c^2-k^2c^2),$$

while from (9)

$$\psi\propto-k^2c^2(k'^2-k^2)/(k'^2+k^2).$$

The opposition of sign is apparent only, and relates to the different methods of measurement adopted in the two cases. In (6) the primary and secondary disturbances are represented by h/K, but in (9) by the magnetic function c. . . ."

It may be remarked that Ignatowski's equation agrees with (5) for this case, and that his corresponding equation (11) for the second case also agrees with (8) after correction of some misprints. His function Q corresponds with my ψ, at least when we observe that the introduction of a constant multiplier, even if a function of m, does not influence the final result.

In proceeding to numerical calculations we must choose a refractive index. I take for this index 1·5, as in similar work for a transparent *sphere* *, so that $k'/k=1\cdot5$. And before employing the more general formulæ, I commence with the approximations of (6) and (9), assuming $kc=\cdot10$, $k'c=\cdot15$. When we introduce these values into (6), we get

$$\psi=\frac{h}{K}=\left(\frac{\pi}{2ikr}\right)^{\frac{1}{2}}e^{i(nt-kr)}\left[\cdot00625-\cdot156\times10^{-4}\cos\theta\right], \quad . \quad (13)$$

in response to the incident wave $h/K=e^{i(nt+kx)}$. Again, from (9)

$$\psi=c=\left(\frac{\pi}{2ikr}\right)^{\frac{1}{2}}e^{i(nt-kr)}\left[10^{-4}(\cdot0781+\cdot0481\cos 2\theta)\right.$$

$$\left.-\cdot00385\cos\theta\right], \quad . \quad . \quad . \quad . \quad (14)$$

corresponding with $c=e^{i(nt+kx)}$ for the incident wave.

In using the general formulæ the next step is to express ψ_m, representing a divergent wave, by means of functions already tabulated. I am indebted to Prof. Nicholson for valuable information under this head. It appears that we may take

$$\psi_m(z)=G_m(z)-\tfrac{1}{2}i\pi J_m(z), \quad . \quad . \quad . \quad . \quad (15)$$

where z is written for kr, and the real and imaginary parts are separated. When z is very great

$$i^m\psi_m(z)=\left(\frac{\pi}{2iz}\right)^{\frac{1}{2}}e^{-iz}. \quad . \quad . \quad . \quad . \quad (16)$$

$J_m(z)$ is the usual Bessel's function; the G-functions are tabulated in Brit. Assoc. Reports †. The Bessel's functions satisfy the relations

$$J_{m+1}=\frac{2m}{z}J_m-J_{m-1}, \quad . \quad . \quad . \quad . \quad (17)$$

$$J_m'=J_{m-1}-\frac{m}{z}J_m; \quad . \quad . \quad . \quad . \quad (18)$$

and relations of the same form are satisfied by functions G. When $m=0$, $J_0'=-J_1$, $G_0'=-G_1$.

Writing z for kc and z' for $k'c$ and with use of (18), we have for the coefficient D_m of $2i^m$ on the right-hand side of (5)

$$D_m=z'J_m(z)J_{m-1}(z')-zJ_m(z')J_{m-1}(z); \quad . \quad . \quad (19)$$

and for the coefficient of B_m on the left

$$N_m+\tfrac{1}{2}i\pi D_m,$$

* Proc. Roy. Soc. A, vol. lxxxiv. p. 25 (1910); Sci. Papers, vol. v. p. 547.

† Reports for 1913, p. 30; 1914, p. 9.

where $$\mathrm{N}_m = z\mathrm{G}_m'(z)\mathrm{J}_m(z') - z'\mathrm{G}_m(z)\mathrm{J}_m'(z')$$

$$= z\mathrm{G}_{m-1}(z)\mathrm{J}_m(z') - z'\mathrm{G}_m(z)\mathrm{J}_{m-1}(z'). \quad . \quad (20)$$

Thus

$$\mathrm{B}_m = \frac{2i^m}{\mathrm{N}_m/\mathrm{D}_m + \frac{1}{2}i\pi}, \quad . \quad . \quad . \quad . \quad . \quad (21)$$

where, however, the 2 is to be omitted when $m=0$. Thus by (4) and (16) the divergent wave at a great distance r is expressed by

$$h/\mathrm{K} = \left(\frac{\pi}{2ikr}\right)^{\frac{1}{2}} e^{i(nt-kr)} \left[\frac{1}{\mathrm{N}_0/\mathrm{D}_0 + \frac{1}{2}i\pi} + \sum_1^\infty \frac{2(-1)^m \cos m\theta}{\mathrm{N}_m/\mathrm{D}_m + \frac{1}{2}i\pi}\right]. \quad (22)$$

Here N_m, D_m are given by (19), (20), and are real.

In like manner (8) may be put into the form

$$\mathrm{B}_m' = \frac{2i^m}{\mathrm{N}_m'/\mathrm{D}_m' + \frac{1}{2}i\pi}, \quad . \quad . \quad . \quad . \quad (23)$$

where

$$\mathrm{N}_m' = z'\mathrm{J}_m(z')\mathrm{G}_m'(z) - z\,\mathrm{J}_m'(z')\mathrm{G}_m(z)$$

$$= z'\mathrm{J}_m(z')\mathrm{G}_{m-1}(z) - z\,\mathrm{J}_{m-1}(z')\mathrm{G}_m(z)$$

$$- m\left(\frac{z'}{z} - \frac{z}{z'}\right)\mathrm{J}_m(z')\mathrm{G}_m(z), \quad . \quad . \quad . \quad . \quad . \quad (24)$$

$$\mathrm{D}_m' = z\mathrm{J}_m(z)\mathrm{J}_{m-1}(z') - z'\mathrm{J}_m(z')\mathrm{J}_{m-1}(z)$$

$$+ m\left(\frac{z'}{z} - \frac{z}{z'}\right)\mathrm{J}_m(z)\mathrm{J}_m(z'). \quad . \quad . \quad . \quad . \quad . \quad (25)$$

And, as in (22), the expression for the diverging wave at a distance is

$$c = \left(\frac{\pi}{2ikr}\right)^{\frac{1}{2}} e^{i(nt-kr)} \left[\frac{1}{\mathrm{N}_0'/\mathrm{D}_0' + \frac{1}{2}i\pi} + \sum_1^\infty \frac{2(-1)^m \cos m\theta}{\mathrm{N}_m'/\mathrm{D}_m' + \frac{1}{2}i\pi}\right]. \quad (26)$$

When we fix the refractive index at 1·5, the value of $z'/z - z/z'$ in (24), (25) is 5/6.

The values of N_1, N_1', D_1, D_1' may be deduced from the corresponding quantities with $m=0$ by means of the relations

$$\mathrm{N}_1 = \mathrm{N}_0', \quad \mathrm{N}_1' = \mathrm{N}_0 - \tfrac{5}{6}\mathrm{J}_1(z')\mathrm{G}_1(z), \quad . \quad . \quad (27)$$

$$\mathrm{D}_1 = \mathrm{D}_0', \quad \mathrm{D}_1' = \mathrm{D}_0 + \tfrac{5}{6}\mathrm{J}_1(z')\mathrm{J}_1(z) \quad . \quad . \quad (28)$$

For numerical calculation we have also to specify the values of z, or kc. For this purpose we take $z=$·4, ·8, 1·2, 1·6, 2·0, 2·4, where z denotes the ratio of the circumference of the cylinder to the wave-length in air; the corresponding values of $(\mathrm{N}/\mathrm{D} + \frac{1}{2}i\pi)^{-1}$ and of $(\mathrm{N}'/\mathrm{D}' + \frac{1}{2}i\pi)^{-1}$ may then be tabulated.

TABLE I.

z.	$[N_0/D_0+\frac{1}{2}i\pi]^{-1}$	$[N_0'/D_0'+\frac{1}{2}i\pi]^{-1}$	
·4	·10624 − i × ·01825	·00202 − i × ·00001	0
·8	·29104 − i × ·18940	·03397 − i × ·00182	
1·2	·31827 − i × ·32283	·17667 − i × ·05353	
1·6	·31745 − i × ·34157	·31764 − i × ·33892	
2·0	·31565 − i × ·35939	·23337 − i × ·53480	
2·4	·26905 − i × ·48842	·19953 − i × ·56634	
z.	$[N_1/D_1+\frac{1}{2}i\pi]^{-1}$	$[N_1'/D_1'+\frac{1}{2}i\pi]^{-1}$	
·4	·00202 − i × ·00001	·03066 − i × ·00148	1
·8	·03397 − i × ·00182	·10872 − i × ·01914	
1·2	·17667 − i × ·05353	·18711 − i × ·06080	
1·6	·31764 − i × ·33892	·24560 − i × ·11581	
2·0	·23337 − i × ·53480	·30426 − i × ·22477	
2·4	·19953 − i × ·56634	·27720 − i × ·47478	
z.	$[N_2/D_2+\frac{1}{2}i\pi]^{-1}$	$[N_2'/D_2'+\frac{1}{2}i\pi]^{-1}$	
·4	·00001 − i × 0	·00061 − i × 0	2
·8	·00084 − i × 0	·00931 − i × ·00014	
1·2	·00946 − i × ·00014	·04392 − i × ·00304	
1·6	·05510 − i × ·00481	·12114 − i × ·02395	
2·0	·21352 − i × ·08223	·22506 − i × ·09321	
2·4	·28583 − i × ·45838	·30204 − i × ·21784	
z.	$[N_3/D_3+\frac{1}{2}i\pi]^{-1}$	$[N_3'/D_3'+\frac{1}{2}i\pi]^{-1}$	
·4	....	....	3
·8	·00001 − i × 0	·00025 − i × 0	
1·2	·00027 − i × 0	·00262 − i × ·00001	
1·6	·00259 − i × ·00001	·01346 − i × ·00028	
2·0	·01514 − i × ·00036	·04636 − i × ·00339	
2·4	·06724 − i × ·00718	·12088 − i × ·02384	
z.	$[N_4/D_4+\frac{1}{2}i\pi]^{-1}$	$[N_4'/D_4'+\frac{1}{2}i\pi]^{-1}$	
1·2	....	·00008 − i × 0	4
1·6	·00008 − i × 0	·00072 − i × 0	
2·0	·00071 − i × 0	·00388 − i × ·00002	
2·4	·00415 − i × ·00003	·01482 − i × ·00035	
z.	$[N_5/D_5+\frac{1}{2}i\pi]^{-1}$	$[N_5'/D_5'+\frac{1}{2}i\pi]^{-1}$	
1·6	....	·00002 − i × 0	5
2·0	·00002 − i × 0	·00020 − i × 0	
2·4	·00019 − i × 0	·00109 − i × 0	
z.	$[N_6/D_6+\frac{1}{2}i\pi]^{-1}$	$[N_6'/D_6'+\frac{1}{2}i\pi]^{-1}$	
2·0	....	·00001 − i × 0	6
2·4	·00001 − i × 0	·00005 − i × 0	

The next step is the calculation of the series included in the square brackets of (22) and (26) for various values of θ from $\theta=0$ in the direction *backwards* along the primary ray to $\theta=180°$ in the direction of the primary ray produced. If we add the terms due to even and odd values of m separately, we may include in one calculation the results for θ and for $180-\theta$, since $(-1)^m \cos m(180-\theta)=\cos m\theta$ simply.

In illustration we may take the numerically simple case where $\theta=0$ and $\theta=180$, choosing as an example $z=2{\cdot}4$ in (22). Thus

m.		m.	
0	$\cdot 26905 - i \times \cdot 48842$	1	$\cdot 39906 - i \times 1{\cdot}13268$
2	$\cdot 57166 - i \times \cdot 91676$	3	$\cdot 13448 - i \times \cdot 01436$
4	$\cdot\ \ 830 - i \times \cdot\ \ \ 6$	5	$\cdot\ \ 38 - i\ \ \ \ 0$
6	$\cdot\ \ \ 2\ \ \ \ 0$		
$\Sigma(\text{even})=\cdot 84903 - i \times 1{\cdot}40524$		$\Sigma(\text{odd})=\cdot 53392 - i \times 1{\cdot}14704$	

Accordingly for $\theta=0$, we have

$$\Sigma_{\text{even}}-\Sigma_{\text{odd}}=\ \cdot 31511-i \times\ \ \cdot 25820,$$

and for $\theta=180°$

$$\Sigma_{\text{even}}+\Sigma_{\text{odd}}=1{\cdot}38295-i \times 2{\cdot}55228.$$

These are the multipliers of

$$\left(\frac{\pi}{2ikr}\right)^{\frac{1}{2}} e^{i(nt-kr)}$$

in (22). For most purposes we need only the modulus. We find

$$(\cdot 3151)^2+(\cdot 2582)^2=(\cdot 4074)^2,$$

and

$$(1{\cdot}383)^2+(2{\cdot}552)^2=(2{\cdot}903)^2.$$

As might have been expected, the modulus, representing the amplitude of vibration, is greater in the second case, that is in the direction of the primary ray produced.

For other angles, except 90°, the calculation is longer on account of the factor $\cos m\theta$. The angles chosen as about sufficient are 0, 30°, 60°, 90° and their supplements. For 2 or 3 of the larger z's the angles 45° and its supplement were added. The results are embodied in Table II., and a

TABLE II.

$z = \cdot 4$.

θ.	[ ] in (22).	Modulus.	[ ] in (26).	Modulus.
0	·10222 − i × ·01823	·1038	−·05808 + i × ·00295	·0582
30	·10275 − i × ·01824	·1044	−·05048 + i × ·00255	·0505
60	·10421 − i × ·01824	·1058	−·02925 + i × ·00147	·0293
90	·10622 − i × ·01825	·1078	+·00080 − i × ·00001	·0008
120	·10825 − i × ·01826	·1098	·03207 − i × ·00149	·0321
150	·10975 − i × ·01826	·1113	·05574 − i × ·00257	·0558
180	·11030 − i × ·01827	·1118	·06456 − i × ·00297	·0646

$z = \cdot 8$.

θ.	[ ] in (22).	Modulus.	[ ] in (26).	Modulus.
0	·22476 − i × ·18576	·2916	−·16535 + i × ·03618	·1693
30	·23303 − i × ·18625	·2983	−·14502 + i × ·03119	·1483
60	·25625 − i × ·18758	·3176	−·08356 + i × ·01746	·0854
90	·28936 − i × ·18940	·3458	+·01535 − i × ·00154	·0154
120	·32415 − i × ·19122	·3763	·13288 − i × ·02082	·1345
150	·35071 − i × ·19255	·4001	·23158 − i × ·03511	·2342
180	·36068 − i × ·19304	·4091	·27053 − i × ·04038	·2735

$z = 1 \cdot 2$.

θ.	[ ] in (22).	Modulus.	[ ] in (26).	Modulus.
0	−·01669 − i × ·21605	·2167	−·11469 + i × ·06201	·1305
30	+·02173 − i × ·23024	·2313	−·10357 + i × ·04872	·1145
60	·13268 − i × ·26916	·3001	−·04920 + i × ·01029	·0503
90	·29935 − i × ·32255	·4401	+·08899 − i × ·04745	·1009
120	·48494 − i × ·37622	·6138	·31454 − i × ·11127	·3337
150	·63373 − i × ·41570	·7579	·54459 − i × ·16188	·5681
180	·69107 − i × ·43017	·8141	·64413 − i × ·18123	·6691

$z = 1 \cdot 6$.

θ.	[ ] in (22).	Modulus.	[ ] in (26).	Modulus.
0	−·21265 + i × ·32667	·3898	·04320 − i × ·15464	·1606
30	−·17770 + i × ·24065	·2991	·01272 − i × ·16229	·1628
45	−·12826 + i × ·13772	·1882	−·01207 − i × ·17555	·1760
60	−·05019 + i × ·00214	·0502	−·02292 − i × ·19972	·2010
90	+·20741 − i × ·33195	·3914	+·07680 − i × ·29102	·3010
120	·57473 − i × ·67566	·8870	·41448 − i × ·43022	·5974
135	·76284 − i × ·82086	1·112	·64445 − i × ·50229	·8171
150	·92264 − i × ·93341	1·312	·86340 − i × ·56345	1·031
180	1·06827 − i × 1·02905	1·483	1·07952 − i × ·61900	1·244

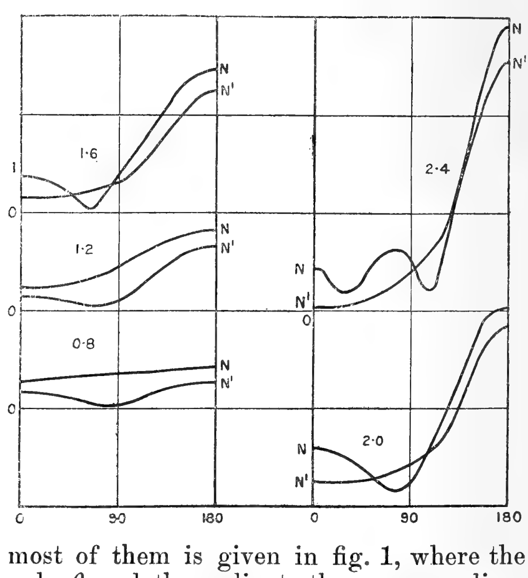
$z = 2\cdot0.$

θ.	[ ] in (22).	Modulus.	[ ] in (26).	Modulus.
0	·24705 + i × ·54647	·5997	− ·01037 − i × ·26494	·2651
30	·12429 + i × ·48468	·5004	− ·07211 − i × ·23868	·2493
60	− ·10169 + i × ·25692	·2763	− ·20729 − i × ·22358	·3049
90	− ·10997 − i × ·19493	·2238	− ·20901 − i × ·34842	·4063
120	+ ·30453 − i × ·81124	·8665	+ ·21619 − i × ·65956	·6941
150	·93263 − i × 1·36792	1·656	·98119 − i × 1·01730	1·413
180	1·24117 − i × 1·59417	2·020	1·39291 − i × 1·17758	1·824

$z = 2\cdot4.$

θ.	[ ] in (22).	Modulus.	[ ] in (26).	Modulus.
0	·31511 − i × ·25820	·4074	·03501 − i × ·00548	·0354
30	·20547 + i × ·03416	·2083	·00846 + i × ·03851	·0394
45	·07387 + i × ·30240	·3113	− ·04963 + i × ·07206	·0875
60	− ·08615 + i × ·52197	·5290	− ·15376 + i × ·07895	·1728
90	− ·29433 + i × ·42828	·5196	− ·37501 − i × ·13136	·3973
120	+ ·04433 − i × ·58199	·5837	− ·08070 − i × ·77525	·7794
135	·44763 − i × 1·27912	1·355	+ ·38943 − i × 1·20336	1·265
150	·89597 − i × 1·92770	2·126	·96494 − i × 1·60617	1·874
180	1·38295 − i × 2·55228	2·903	1·63169 − i × 1·99996	2·581

Fig. 1.

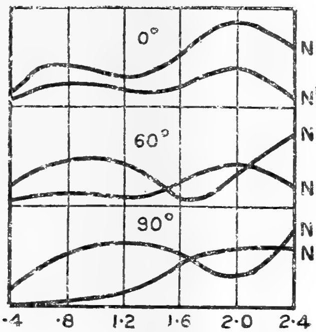


plot of most of them is given in fig. 1, where the abscissa is the angle θ and the ordinate the corresponding modulus from the table. The curve marked N corresponds to (22) and that marked N′ to (26). A few points have been derived from values not tabulated. From the nature of the functions represented both curves are horizontal at the limits 0° and 180°.

When $z = \cdot 8$, the curves show the characteristics of a very thin cylinder. At 90° N′ nearly vanishes, indicating that in this direction little light is scattered whose vibrations are perpendicular to the axis. When $z = 1\cdot 2$, the maximum polarization is still pretty complete, but the direction in which it occurs is at a smaller angle θ. For $z = 1\cdot 6$ the polarization is reversed over most of the range between 45° and 90°. By the time z has risen to 2·4 a good deal of complication enters, at any rate for the curve N.

In fig. 2 are plotted curves showing the variation with z at given angles of $\theta = 0°$, 60°, and 90°. At 0° the polarization is all in one direction over the whole range from 0 to 2·4.

Fig. 2.



At 60° there are reversals of polarization at $z = 1\cdot 5$ and $z = 2\cdot 05$. At 90° these reversals occur when $z = 1\cdot 7$ and $z = 2\cdot 3$.

The curves stop at $z = 2\cdot 4$. It would have been of interest to carry them further, but the calculations would soon become very laborious. As it is, they apply only to visible light dispersed by the very finest fibres, inasmuch as z is the ratio of the *circumference* of the cylinder to the wave-length of the light.

When z, or kc, is greater than 2·4, we may get an idea of the course of events by falling back upon the case where the refractivity $(\mu - 1)$ is very small, treated in my 1881 paper.

In our present notation the light dispersed in direction θ depends upon

$$\frac{\pi c^2}{z \cos \frac{1}{2}\theta} J_1(2z \cos \tfrac{1}{2}\theta). \quad . \quad . \quad . \quad . \quad (29)$$

When $\theta = 180°$, *i.e.* in the direction of primary propagation,

$$J_1(2z \cos \tfrac{1}{2}\theta) = z \cos \tfrac{1}{2}\theta,$$

and (29) reduces to πc^2. In this direction every element of the obstacle acts alike, and the dispersed light is a maximum. In leaving this direction the dispersed light first vanishes when

$$\cos \tfrac{1}{2}\theta = 3 \cdot 8317/2z,$$

and afterwards when

$$2z \cos \tfrac{1}{2}\theta = 7 \cdot 0156,\ 10 \cdot 173,\ 13 \cdot 324,\ \&c.$$

The factor (29) is applicable, whether the primary vibrations be parallel or perpendicular to the axis of the cylinder. The remaining factors may be deduced by comparison with the case of an infinitely small cylinder. Thus for vibrations parallel to the axis, we obtain from (6)

$$\psi = \left(\frac{\pi}{2ikr}\right)^{\frac{1}{2}} e^{i(nt-kr)} \times \frac{(k'c - kc) J_1(2kc \cos \frac{1}{2}\theta)}{\cos \frac{1}{2}\theta}, \quad . \quad . \quad (30)$$

applicable however large c may be, provided $(k' - k)$ be small enough.

In like manner for vibrations perpendicular to the axis we get from (9)

$$\psi = \left(\frac{\pi}{2ikr}\right)^{\frac{1}{2}} e^{i(nt-kr)} \times \frac{(kc - k'c) \cos \theta \,.\, J_1(2kc \cos \frac{1}{2}\theta)}{\cos \frac{1}{2}\theta}, \quad (31)$$

vanishing when $\theta = 90°$, whatever may be the value of kc. It will be seen that (30) and (31) differ only by the factor $-\cos \theta$, and that this is unity in the direction of the primary light.

XLII. *Interfacial Tension and Complex Molecules.*

By Prof. G. N. ANTONOFF *.

§ 1. *A Theory of Surface Tension.*

IN order to explain the phenomena of surface tension, it is usual to postulate the existence of forces, sensible only at very small distances, between the molecules of a liquid. The distance at which these forces are still effective is known as the radius of molecular action. They are assumed to be inversely proportional to a sufficiently high power of the distance apart of the molecules. Several writers, after Lord Kelvin, have regarded them as proportional to the inverse fifth power of distance, but Sutherland † has given strong evidence, based mainly on experimental results, that the inverse fourth power is more suitable.

Theories of surface tension based on the existence of such forces involve of necessity the conception of "Molecular pressure." But while surface tension is a real and tangible phenomenon, the same cannot be said of the molecular pressure. The absence of direct methods for its determination has even led some writers ‡ to regard it as a purely metaphysical magnitude, and no clear account has apparently been given which expresses precisely the mutual dependence of molecular pressure and surface tension.

One of the most widely known of such theories, which undoubtedly plays an important rôle, is that of Laplace, which is, however, of a very general type. But at the present time, our knowledge of the nature of the molecule and the forces which can be associated with it, is much more definite, and it is desirable to work out the consequences of a more definite hypothesis of molecular action which is in general agreement with the present conception of the molecule. For it is possible to explain the existence of attractive forces between molecules, diminishing rapidly with distance, without making any special hypothesis for the purpose §. We may suppose that the forces exerted by atoms and molecules are essentially of electromagnetic origin, for into the composition of atoms and molecules apparently enter only

* Communicated by Prof. J. W. Nicholson, F.R.S.
† Phil. Mag. [5] xxvii. p. 305; *ibid.* [5] xxxv. p. 112 (1893).
‡ *Kapillarchemie*, Leipzig, 1909, p. 9.
§ See also Crehore, Phil. Mag. xxvi. p. 25 (1913).

positive and negative charges of approximately equal magnitude on the whole. On the other hand, molecules must apparently be regarded as asymmetric, and therefore in many cases can be treated as mathematical bipoles or doublets in which the positive and negative charges are effectively concentrated in points at definite distances apart, such distances being characteristic of the molecules concerned. These distances may as usual be called the lengths of the molecular doublets. The use of molecular doublets, as a means of interpreting the phenomena of surface tension. was suggested by Sir Oliver Lodge *. In the following calculations, these doublets may be regarded as purely electrical in type, and the paper, in one of its aspects, indicates the extent to which a purely electrical theory of the forces operative in liquids between contiguous molecules can account for the observed phenomena. But magnetic polarity, if present, would also be subject to the same laws of action between neighbouring doublets. The investigation therefore does not preclude the existence of magnetic forces also. Their only effect would be to alter the absolute values of surface tension and molecular pressure, and not their ratio or the nature of the laws regulating their action. It is not without interest that this theory, or even the combined electrical and magnetic theory, at once necessitates that the molecular attractions must be proportional to the inverse fourth power of the distances in agreement with the conclusion reached by Sutherland on experimental grounds †.

If it be supposed that the molecules of liquids act as doublets,—in all the considerations advanced in this paper, only transparent liquids are under review,—they must be arranged in such a manner that the extremities of opposite sign are adjacent. Let the length of a doublet be l, and the charges on its poles $\pm e$. The component forces between two such doublets in any relative positions are known.

A single doublet at the origin O, pointing along the axis of x, produces an external field whose potential at a point P or (x, y, z) is, if $r = \text{OP}$,

$$V = elx/r^3.$$

If a second doublet is situated at P, in the plane xy, and if the projections of its length parallel to the axes are δx, δy,

* Proc. Inst. Elec. Eng. Part 159, vol. xxxii. (1903).
† *Loc. cit.*

the mutual potential energy of the two is

$$W = e\left(\frac{\partial V}{\partial x}\delta x + \frac{\partial V}{\partial y}\delta y\right).$$

But $\delta x = l\cos\theta$, $\delta y = l\sin\theta$, if θ is the angle between the axes of the doublets, and accordingly

$$W = el\left(\cos\theta\frac{\partial V}{\partial x} + \sin\theta\frac{\partial V}{\partial y}\right).$$

For our purposes in the present theory, it is only necessary to consider parallel doublets, for which $\theta = 0$, so that we may write

$$W = el\frac{\partial V}{\partial x} = e^2l^2\frac{\partial}{\partial x}\left(\frac{x}{r^3}\right)$$

$$= e^2l^2\left(\frac{1}{r^3} - \frac{3x^2}{r^5}\right).$$

The force acting on the second doublet, parallel to the axis of x, is X where

$$X = -\frac{\partial W}{\partial x} = e^2l^2\left(\frac{9x}{r^5} - \frac{15x^2}{r^7}\right),$$

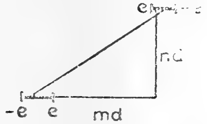
and that on the doublet at the origin is equal and opposite to this, or

$$e^2l^2\left(\frac{15x^2}{r^7} - \frac{9x}{r^5}\right) = \frac{3e^2l^2}{r^7}(5x^3 - 3xr^2) = \frac{3e^2l^2x}{r^7}(2x^2 - 3y^2).$$

The extremities of opposite sign being adjacent, the force is an attraction.

Fig. 1 shows two doublets whose distance apart has two

Fig. 1.



components md, nd, where m and n are integers. We can regard d as the average distance apart, in an assemblage arranged in regular order, of two adjacent doublets.

In this case, we have at once, writing $x=md$, $y=nd$, $r=d\sqrt{m^2+n^2}$, the expression for the force

$$\frac{3me^2l^2}{d^4} \cdot \frac{2m^2-3n^2}{(m^2+n^2)^{7/2}}.$$

It is understood that if the lengths of the doublets have different values (l_1, l_2), l^2 is replaced by l_1l_2 in this formula. The law is that of the inverse fourth power as already stated, and as could in fact be shown at once from a consideration of dimensions.

When $m=0$, we have practically a very simple case with the doublets facing one another, the force in the perpendicular direction being of order $\frac{l^3}{d^3}$. When $n=0$, the doublets are in line, and at distance md apart, the force becoming $\frac{6e^2l^2}{(md)^4}$.

For a given value of nd, a line of doublets is specified, and the whole attraction of this line on one side of the original doublet, is

$$\frac{3e^2l^2}{d^4}\sum_{n=0}^{\infty}\frac{m(2m^2-3n^2)}{(m^2+n^2)^{7/2}}$$

$$=\frac{3e^2l^2}{d^4}\left\{\frac{2-3n^2}{(1+n^2)^{7/2}}+\frac{2(8-3n^2)}{(4+n^2)^{7/2}}+\frac{3(18-3n^2)}{(9+n^2)^{7/2}}+\dots\right\}.$$

This series cannot be summed in a convenient manner, but a sufficient approximation may be obtained by noticing that each term decreases as n increases. When $n=0$, the terms are of order m^{-4}, and the second is only about 1/16 of the first. Thereafter the convergence is very rapid, and it is sufficient for our purpose to ignore all but the first two terms.

The bracket changes its sign when $n=1$. Thus its values are effectively, for $n=0, 1, 2$,

$$2+\frac{16}{4^{7/2}}=2\tfrac{1}{8}=2\cdot125,$$

$$-\frac{1}{2^{7/2}}+\frac{10}{5^{7/2}}=-\frac{1}{8\sqrt{2}}+\frac{2}{25\sqrt{5}},$$

$$-\frac{10}{5^{7/2}}-\frac{8}{4^7},$$

and rapidly decrease. Only the first two values of n need to be retained, and we may write, for the total attraction of a doublet towards one side, along its length,

$$\frac{3e^2l^2}{d^4}\left\{2+\frac{16}{4^{7/2}}-\frac{1}{2^{7/2}}\right\}=6{\cdot}13\frac{e^2l^2}{d^4}.$$

This does not differ appreciably from the force due to the next consecutive doublet in line. The force in the perpendicular direction similarly is effectively $\frac{3e^2l^2}{d^4}$, or half the above value. The problem so far has been two-dimensional, but it is evident that the three-dimensional problem gives the same approximate solution, and we may conclude that when such a doublet is one of a regularly disposed arrangement, all doublets being parallel, it is pulled in each direction in its own line by a force $\frac{6e^2l^2}{d^4}$, and in a perpendicular direction by half this force.

In a length nd parallel to the doublets or perpendicular to them, n doublets are situated. The number in unit length is $\frac{1}{d}$, and if p is the number of doublets or molecules in unit volume of the liquid,

$$\frac{1}{d^3}=p.$$

If the surface doublets were arranged parallel to the surface, the surface tension, or attraction along the surface per unit length, would be

$$\frac{6e^2l^2}{d^4}\Big/d=6e^2l^2p^{5/3}$$

in one direction, but only half this value in the perpendicular direction along the surface. We must therefore reject this case, and adopt, on the other hand, that with all the doublets arranged normally to the surface, the poles in any line being alternately positive and negative. The surface attraction per unit length is then $3e^2l^2p^{5/3}$ and the inward normal attraction is $6e^2l^2p^{4/3}$ on each doublet. It is not, of course, implied that the surface poles form a rectangular lattice arrangement at any instant. The magnitude d is the average distance apart of contiguous poles belonging to different molecular doublets, which are in fact continually in a state of vibration and of translatory motion with a definite free path. The surface force in either direction on a row of doublets is a mean value and actually must on the average be the same in every direction, thus producing the ordinary phenomenon of surface tension.

The number of doublets in unit area of the surface is $\frac{1}{d^2}$, so that the inward pull per unit area is

$$\frac{6e^2l^2p^{4/3}}{d^2}=6e^2l^2p^2.$$

The inward pull $6e^2l^2v^2$ on unit surface is the molecular pressure, which we denote by the symbol P. The surface tension is α. Thus

$$P=6e^2l^2p^2, \qquad \alpha=3e^2l^2p^{5/3}. \quad . \quad . \quad . \quad . \quad (1)$$

Thus

$$P=k\alpha p^{1/3},$$

where k is a numerical quantity, practically equal to 2 if the magnetic forces are negligible compared with those of electric origin. As an example we may calculate the value of P for benzene, assuming the following data:—

Weight of an atom of hydrogen $= 1{\cdot}64\times10^{-24}$ gr.

Molecular weight of benzene ... = 78.

Specific gravity of benzene at ordinary temperature ...... = 0·890.

Thus

$$p=\frac{0{\cdot}890}{78\times1{\cdot}64\times10^{-24}}=6{\cdot}8\times10^{21}.$$

At ordinary temperature, the surface tension of benzene is 32 dynes per cm. Therefore

$$P=32k(6{\cdot}8\times10^{21})^{1/3}=12\times10^6 \text{ dynes per sq. cm.},$$

with $k=2$. This is approximately 1200 atmospheres, and its order of magnitude is in accord with indirect evidence. The expression for the molecular pressure can be somewhat modified. Write $6e^2=J$. The length l of a doublet is a magnitude which cannot exceed the molecular dimension. Some evidence exists which tends to show that l is the same for various liquids at corresponding temperatures, and in particular therefore at absolute zero which is a corresponding temperature for all liquids.

When the temperature rises the effect on the liquid can be represented by an equivalent diminution in the effective value of l, which approaches zero in the neighbourhood of the critical point. In order to express the diminution of the attractive forces between the doublets when the temperature rises it is sufficient, for example, to suppose that the doublets, instead of being arranged vertically, commence to move in such a manner that their charges describe circles while their centres remain stationary, so that their areas in fact describe cones.

As the temperature rises, these cones tend to become flatter, until finally the two charges are describing the same circle, and the entire doublet moves in a horizontal plane. If we recollect that this type of movement must increase with the temperature, it is evident that the effective value of l must decrease, and reach the value zero when the doublet no longer exercises attractive forces on the average.

As for the magnitude p, it is merely the specific gravity of the liquid divided by its molecular weight. For certain reasons, however, it is necessary to replace this specific gravity by a smaller value, the difference between the density of the liquid d_1 and that of its saturated vapour d_2. In this case the formula for the normal pressure becomes

$$P = Jf(t)\left(\frac{d_1 - d_2}{M}\right)^2, \quad . \quad . \quad . \quad . \quad . \quad (2)$$

where $f(t)$ replaces l^2, and J is constant. The molecular weight is M. In a paper by Kleeman * a formula very similar to this is derived from considerations of a quite different character.

According to Kleeman, the surface tension is

$$\lambda = K'''\left(\frac{\rho_1 - \rho_2}{m}\right)^2 (\Sigma Ca)^2,$$

where $(\Sigma Ca)^2$ is a constant, K''' is a quantity which is the same for all liquids at corresponding temperatures, ρ_1 and ρ_2 are the densities of the liquid and of its saturated vapour, and m is the molecular weight of the liquid. This expression for the surface tension accords with the properties of

* Phil. Mag. xix. p. 784 (1910).

liquids in so far as it vanishes at the critical point. Moreover, d_1-d_2 becomes indefinitely small near the critical point and appears in the expression to the second power, while $f(t)$ also approaches zero at the critical point. The formula, in fact, indicates the same phenomenon which is found in practice, for the surface tension is effectively zero somewhat before the critical point. Laplace's theory, while embracing a whole series of phenomena, is not satisfactory in this respect, and the theory of van der Waals, which is based on the conception of a continual passage from the liquid to the gaseous state, appears to be more suitable; we must admit also that the density of the liquid is variable, and that near the surface it passes by degrees into the density of the vapour of the same liquid.

Let us consider how such phenomena can be represented from the point of view of the kinetic theory.

The particles of a liquid are in motion like those of a gas, but are characterized by a much smaller mean free path. Some particles, with a velocity greater than the mean, detach themselves from the liquid surface and enter the surrounding medium to form a saturated vapour. When equilibrium is reached, equal numbers of particles enter the surface and are detached from it. In this manner, the surface is in continual bombardment on two sides, and it is therefore quite natural to attribute special properties to it.

But in this case the properties of the surface must change radically, if it is in contact with another liquid instead of its own vapour. In fact, in the latter case, particles approach the surface which have, in the two media, very different free paths, whereas at the boundary of two liquids, the molecules in the two media have mean paths of the same order of magnitude and characteristic for the two liquids.

The available evidence appears to support the opinion, that the surface of a liquid, when in contact with another liquid, retains the same properties which it had while in contact with its own vapour (opinion of Planck) *. The opposite view does not, in fact, lead us to results in agreement with experiment (Kantor) †. Tamman ‡ considers that if the liquid passed to the gaseous state by jumps the law

$$\frac{P}{\alpha} = \text{const.}$$

should be true. But according to the theory developed in

* *Thermodynamik,* Leipzig, 1905, p. 175.
† Wied. *Ann.* lxvii. p. 687 (1899).
‡ *Ueber die Beziehungen,* p. 175.

this paper, even admitting the sharp passage from the liquid to the gaseous state, the constancy of this ratio cannot occur, for, as we have seen,

$$\frac{P}{\alpha} = kp^{1/3},$$

in which p is not constant, but in all cases is a function of the temperature.

We shall see later that systems consisting of two liquids in contact (liquids with limited mutual solubility) can throw some light on the nature of the liquids and we shall discuss their properties a little more completely.

§ 2. *Some Properties of Two Superposed Liquids.*

Considering liquid systems with limited solubility we have difficulties in explaining some laws governing that phenomenon and in understanding how the two layers can coexist without having the tendency to mix completely. It can be shown for example in certain cases (near the critical point of dissolution), that we can have two liquids of very different concentrations, but equal (within the limits of experimental error) as regards surface tension *. Two layers obtained when the liquid is disturbed correspond to this condition if the critical point is gently departed from. Two liquid layers have, according to Konovaloff †, an equal vapour tension as well as the same vapour composition, while their own composition is very different. If certain properties of a solution remain the same, even with change of concentration (for example, surface tension and vapour tension), we can suggest, as an explanation, the hypothesis that at the surface of a solution the concentration of the dissolved body is different from that in the interior. If, in general, it be admitted that such a change of concentration can occur in the surface layer, the formation of two solutions of different concentrations but identical surface tensions becomes admissible, and it is only necessary to suppose equality in the surface concentrations.

Such an interpretation is necessitated otherwise, in that there exists a theory according to which the surface concentration of solutions must differ from that of the deeper layers.

* G. N. Antonoff, *J. Ch. Phys.* v. p. 372 (1907).

† D. Konovaloff, Wied. *Ann.* xiv. (1881). See also Nernst, *Theor. Chemie*, p. 525 (1913).

The origin of this theory is due to Gibbs * and Thomson †, who have applied it to gaseous mixtures, but its detailed development in the application to solutions is due to Freundlich ‡, Milner §, Lewis ||, and others.

On the basis of the theory of Gibbs and Thomson, Freundlich and Lewis have given the formula

$$u = -\frac{c}{RT}\frac{d\alpha}{dc},$$

where u is the excess of the mass of dissolved body, expressed in grams per square cm. of the surface, c = concentration of the dissolved body in the depth of the solution, R = gas constant, T = absolute temperature, α = surface tension, $\frac{d\alpha}{dc}$ = change of surface tension with the concentration of the dissolved substance.

It is usual to apply the term "adsorption" to this change of concentration at the surface of a liquid. Let us consider some consequences of applying this formula to the critical points of solutions. We have seen that for a whole series of concentrations, $\frac{d\alpha}{dc} = 0$, and therefore we should also have

$$u = 0.$$

In other words, with the increase of concentration the new substance introduced distributes itself in the interior of the liquid and does not enrich the surface layer.

But Lewis ¶, in order to verify this formula, has made some researches on the subject and arrived at results which do not agree with the theory.

With the experimental results of Lewis as a basis, Arrhenius** has been led to conclude that the phenomena of absorption are not in simple dependence on those of capillarity, and all attempts to relate these phenomena directly are, in his opinion, doomed to failure. However,

* *Thermod. Studien*, p. 271.

† J. J. Thomson, 'Applications of Dynamics to Phys. & Chem.' p. 191.

‡ *Kapillarchemie*, p. 50.

§ Phil. Mag. [6] xiii. p. 96 (1907).

|| Lewis, Phil. Mag. [6] xvii. p. 466 (1909).

¶ *Ibid.*

** *Meddelanden f. K. Vetenskapakademiens Nobelinstitut*, Band 2, No. 7 (1911).

in order to explain the possibility of the existence of solutions of various concentrations, but the same vapour tensions, the equality of superficial concentration is not the only possible hypothesis or condition. For it is quite conceivable that near the critical point, the concentration of the surface layers plays a certain rôle in the phenomenon of equilibrium of two layers. Effectively if the two concentrations come to be equal for two different solutions, the equality of vapour tension and of surface tension becomes admissible. Nevertheless it is not possible by this hypothesis to explain a whole series of properties of these systems.

In a departure from the critical point, the surface tensions, as is known *, commence to differ sensibly for the layers, while the vapour tension remains in all cases the same for the two layers. It is also known that the two superposed liquids boil at an equal temperature and have the same freezing temperature. These properties obviously cannot be accounted for from the standpoint of the above theory. Evidently, in order to explain these phenomena, it is necessary to take into account the molecular state of the body dissolved in the solution.

A satisfactory hypothesis can be found, however, on the basis of the following considerations exposed in the next section.

§ 3. *The Tension at the Interface of Two Liquids with Limited Solubility in a State of Equilibrium.*

In the following we are going to call α_{12} the interfacial tension, α_1 and α_2 the surface tensions against the air of two superposed layers in a state of equilibrium; we will call them solutions 1 and 2. Thus α_1 and α_2 are not tensions of separate liquids but of the saturated solutions the two liquids form when in equilibrium.

The attempts to formulate a law connecting the surface tension at the limit of the two liquids (α_{12}) with the tension of the different phases (α_1 and α_2) have not up to the present given any very satisfactory result. The theory of Rayleigh has led to this result:

$$\sqrt{\alpha_{12}} = \sqrt{\alpha_1} - \sqrt{\alpha_2},$$

based on certain hypotheses regarding the layer of transition which do not agree with the experiments.

* See G. N. Antonoff, *loc. cit.*

We will take it for granted that when two liquids which do not completely mix are in equilibrium at the limit of separation, the following must hold good :

$$P_{12} = P_1 - P_2,$$

where P_{12} is the resulting normal pressure at the interface, P_1 and P_2 the normal pressures of the solutions 1 and 2.

According to the present theory there must be a definite relation between such quantities as P and α. For the solution 1 there must be

$$P_1 = k\alpha_1 p_1^{1/3}.$$

For the solution 2,

$$P_2 = k\alpha_2 p_2^{1/3};$$

and similarly at the interface of the two layers an expression must hold good of the following type :

$$P_{12} = k\alpha_{12} p^{1/3}.$$

Thus

$$P_{12} = P_1 - P_2 = k(\alpha_1 p_1^{1/3} - \alpha_2 p_2^{1/3}). \quad . \quad . \quad . \quad (3)$$

We will show in the subsequent paragraph that when two liquid layers are in equilibrium both superposed solutions are *equimolecular*, i. e. contain an equal number of molecules per unit volume, or we may put

$$p_1 = p_2.$$

From the expression (3) we shall then obtain

$$P_{12} = kp^{1/3}(\alpha_1 - \alpha_2),$$

but since $$P_{12} = k\alpha_{12} p^{1/3},$$

therefore $$\alpha_{12} = \alpha_1 - \alpha_2. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (4)$$

This is perhaps the most fundamental result required by the theory outlined above.

The equations (3) and (4) are identical provided that

$$p_1 = p_2 ;$$

which means that in *two layers there are an equal number of* *particles per unit volume* (AVOGADRO'S LAW). That is, however, only possible if there are phenomena of association in the solutions. This will be further discussed and a definite proof given in the subsequent paragraph.

We shall now show that the relation (4) is in agreement with the experimental evidence. This can be illustrated by the following table, where the figures are given for a state of complete equilibrium *.

TABLE I.

	a_1	a_2	a_1-a_2	a_{12}
Water-isobutyl alcohol	23·9	22·5	1·4	1·76
,, isoamylic alcohol	25·7	21·1	4·6	4·4
,, ether	26·7	17·3	9·4	9·1
,, aniline	44·4	40·0	4·4	5·1
,, chloroform	54·0	26·6	27·4	27·7
,, benzene	60·0	28·2	31·8	32·6

It is not easy to extend the above table for the following reasons. The pairs of liquids with considerable mutual solubility give saturated solutions with nearly equal surface tension, *e. g.* aniline-amylene (trimethylethylene), isobutyric acid-water, carbon disulphide-methyl alcohol, whose critical points of separation into two layers are not far from the ordinary temperature. In the proximity of that point the tensions of both layers are nearly equal, and α_{12} is nearly zero †. No accurate results can be obtained under such

* G. N. Antonoff, *J. Ch. Phys.* v. p. 372 (1907).

† In such a case the meniscus of the separation of two layers is nearly flat. As a rule the meniscus is curved according to the values of α_1 and α_2, the liquid with higher tension wets the glass at the interface and forms a concave surface. The explanation of these phenomena can be given by means of the above theory. The theory of the doublets can also permit of the explanation of the phenomena of humectation and of adsorption that both result in molecular attraction. We have seen that the attraction between two doublets of the same nature is expressed by

$$\frac{3e^2l^2}{d_1^4}.$$

Here d is the distance between the molecules. For a liquid of which the doublets l_1l_2 are of a different size, the attraction between the particles will be expressed by

$$\frac{3e^2l_1l_2}{d_2^4}.$$

circumstances. To obtain more or less reliable results only liquids with considerable differences α_{12} must be chosen to prove the above law. For this reason in the above table water exists as one of the constituents in all pairs. Water has an exceptionally high surface tension of the order of magnitude 70 $\frac{\text{dynes}}{\text{cent.}}$ and the majority of other liquids about 20–30 $\frac{\text{dynes}}{\text{cent.}}$. Therefore only in solutions having water as

---

It is apparent that the liquid will wet the glass in the case where the value of the attraction between the glass and the liquid is greater than between the particles of the liquid, *i. e.* if

$$\frac{l_2}{d_2^4} > \frac{l_1}{d_1^4}.$$

It must be admitted that d_1 and d_2 are little different from each other (and in all cases different less than l_1 and l_2), and in this case the condition of humectation will be

$$l_2 > l_1.$$

Let us try to explain the fact observed empirically that when two liquids are in contact with the glass (each of which wets the glass), the liquid possessing the highest surface tension will wet the glass replacing the other liquid.

Let us designate by l_1, l_2, and l_3 the reciprocal dimensions of the doublets of the first and second liquids and of the glass. We shall have that the attraction between the glass and the first liquid will be given by

$$\frac{3e^2 l_1 l_3}{d^4}.$$

The attraction between the glass and the second liquid will be

$$\frac{3e^2 l_2 l_3}{d^4}.$$

It is evident that the first liquid will wet if

$$l_1 > l_2.$$

However, experience shows that the condition of humectation is always

$$\alpha_1 > \alpha_2.$$

If we assume $p_1 = p_2$,

then the condition $l_1 > l_2$

is equivalent to $a_1 > a_2$.

one constituent can considerable differences be expected provided the mutual solubility is not very high. But in systems with low solubility there arises another difficulty which is not so easy to overcome. *E. g.*, for the pairs of liquids like water-benzene and water-chloroform, both α_2 and α_{12} can be determined pretty accurately, whereas α_1 cannot be estimated by ordinary methods, *e. g.* the capillary method cannot be used at all. To explain this, general properties of systems with limited mutual solubility have to be considered.

As a rule, the surface tension as a function of concentration varies in the following way, as can be seen on fig. 2 and fig. 3, where the concentrations are plotted along the abscissæ

Fig. 2.

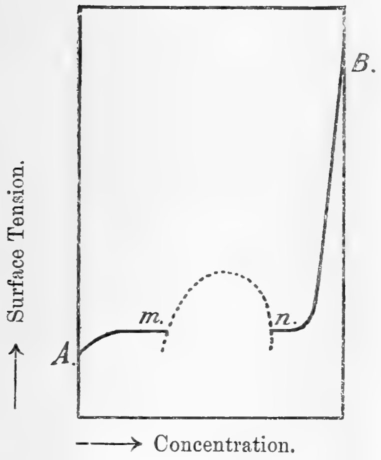
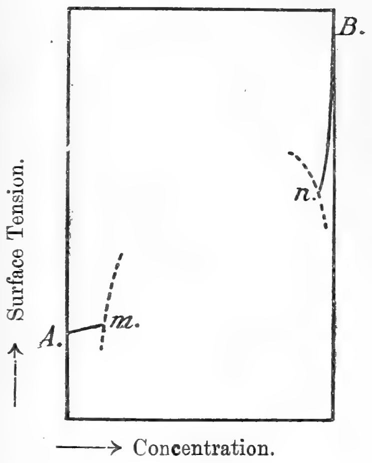


Fig. 3.



and the surface tensions along the ordinates. The dotted curves represent the solubility curves inside which there is a region of two layers. If the mutual solubility is considerable, the solubility curve intersects almost horizontal parts of the surface-tension curve as in fig. 2. Whereas when the mutual solubilities are small, the solubility curve intersects a nearly vertical part of the surface-tension curve on the side where the liquid with the higher surface tension is in excess, *i. e.* near B on fig. 3. In such a case $\frac{d\alpha}{dc}$ is large, where α is the surface tension and c the concentration, *i. e.* the slightest change in the concentration provokes a considerable change of the surface tension. If, moreover, one or both of the components are volatile at the temperature of the experiment, as in the case of water-benzene and water-chloroform, it is almost impossible to obtain the right value of surface tension for the aqueous layer. This can be demonstrated by the following table:—

TABLE II.

Liquids.	Layer with higher surface tension. Solubility.	a_1.	$\frac{da_1}{dc}$.	Layer with smaller surface tension. Solubility.	a_2.	$\frac{da_2}{dc}$.
Isobutyric alcohol-water ...	5·4%	23·9	9·4	15 %	22·7	nearly 0
Aniline-water	3·2	44·0	10	4·5	40·0	·4
Water-isoamylic alcohol ...	2·6	25·7	19	4·2	21·1	·55
Chloroform-water	·8	54·0	24	1·2	26·6	·3
Benzene-water	·1	60·0	100	·1	28·2	nearly 0

It is obvious from the above that in the case of almost immiscible liquids the experiment is unable to settle the question definitely.

For the same reason the so-called Gibbs-Konovaloff's law cannot be proved for immiscible liquids. Konovaloff *, who first succeeded in proving the law experimentally, could only show qualitatively that in the system water-carbon disulphide the aqueous layer gives off vapours containing very much more carbon disulphide than was to be expected as the result of small solubility in water. All the same, that law proved experimentally for miscible liquids is generally accepted and is believed to be true for all pairs of liquids forming two layers, however small their mutual solubility may be. This law is an essential condition of equilibrium, so that it is bound to be extended over nearly immiscible pairs.

For the same reasons I believe also the relation

$$\alpha_{12} = \alpha_1 - \alpha_2$$

to be a general law.

It can only be deduced theoretically if a definite assumption be made with regard to the molecular construction of the solutions. The equality of molecular concentration in

* Konovaloff's law:—Two equal layers in equilibrium have equal vapour pressure and equal composition of vapour.

both layers is in this case the essential condition of equilibrium. All the above considerations must be true for almost immiscible liquids if they are true in the case of the liquids with finite solubility.

The experiment may fail in proving the correctness of the above law for almost immiscible liquids, and yet it will not convince me that it does not hold true.

For that reason I do not attribute much importance to the remark by W. B. Hardy * that it is not a general law, not being applicable in the case of immiscible liquids. He does not give any examples nor figures. It would also be important to know how the figures were obtained, and whether all the necessary conditions to maintain equilibrium were satisfied.

§ 4. *The Existence of Complex Molecules in the Solutions.*

In the preceding paragraph we admitted that the equality of molecular concentration of two coexisting liquid phases must be an essential condition of equilibrium from the standpoint of the above theory, only under those conditions two layers of different composition may coexist permanently without having the tendency to mix with one another by diffusion.

However absurd this may appear at first sight, the assumption may be demonstrated to be quite necessary in the following way:—

It is known that the two superposed liquid layers in equilibrium boil at the same temperature and they have the same freezing-point. For the systems forming two layers the following types of freezing curves are known (see figs. 4 and 5)†, where the concentrations are plotted along the abscissæ and the freezing-points along the ordinates. (By the freezing-point is understood the temperature at which the solution can coexist with very small quantity of its ice.)

Consider first the curve of fig. 4.

Point A gives the freezing-point of substance A, and B, that of B. Between A and C the substance A freezes out, and between C and B the substance B, C being the so-called eutectic point where both A and B fall out simultaneously.

* Proc. Roy. Soc. lxxxviii. 1913, A, p. 325.

† The dotted curve shows the limits of solubility of one liquid in another, inside which there are two layers.

The substance which freezes out from the solution is generally called the dissolvant. The points *m* and *n* represent the freezing-point of the two saturated layers. At *m* the

Fig. 4.

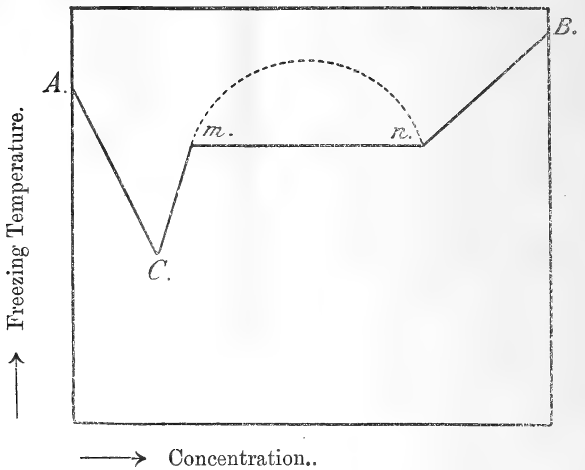
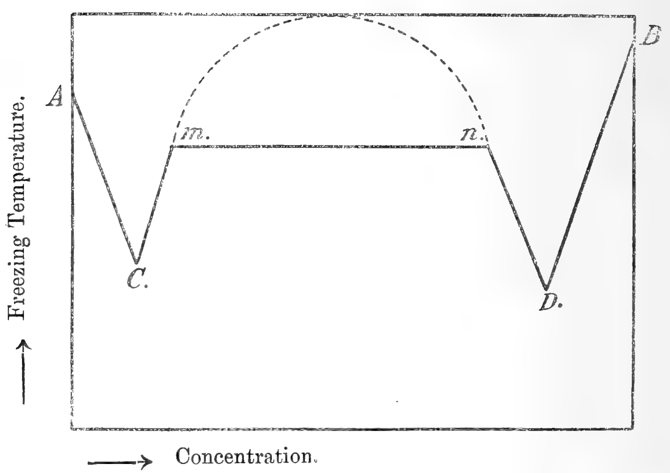


Fig. 5.



solution becomes saturated; further addition of substance B produces a second layer which increases in quantity until point B is reached where only the second layer is present. On further addition of B, the solution again becomes homogeneous and its freezing-point begins to rise.

The points *m* and *n*, which correspond to the concentrations of the two saturated layers, are situated on one side of the eutectic point and must be regarded as solutions in the same dissolvant. If two solutions in the same dissolvant have the same freezing-point (resp. boiling-point) they must contain an equal number of molecules per unit volume.

Such is only possible if the molecules of B form a compound with some of the molecules in the solutions, and further addition of B would not increase the number of molecules present in the solution and all properties depending on the number of molecules (and not their dimensions) would remain invariable. The Phase Rule specifies some conditions under which monovariant systems can be formed *, *e. g.* in a system water-salt, the addition of salt to its solution alters the properties thereof, until the solution is saturated. Adding more salt has only one effect, it only increases the quantity of solid salt with which the solution is in equilibrium. The properties of the solution are monovariant (*i. e.* they only depend upon the temperature) until the solid phase and liquid are coexisting. In this case the system remains monovariant while it is heterogeneous.

But if the molecules of the added substance, instead of forming a precipitate, adhere to the molecules in solution forming complex molecules, then the same monovariance may be attained in a quite homogeneous system.

Such cases of monovariance (*i. e.* when some properties remain invariable with a change of concentration at a given temperature) are actually known for some pairs of liquids not far from the critical point of the separation into two liquid layers †. Such properties may be the vapour pressure, the surface tension, freezing-point, &c.

All the above considerations are equally applicable to the case represented in fig. 5. In this case the points *m* and *n* are situated between the two eutectic points, and in this region the ice formed at the freezing-points is not one of the substances A or B, but a compound of $A_q B_r$.

The solutions *m* and *n* are therefore also solutions in the same dissolvant whatever it may be with regard to its chemical nature, and must also contain equal numbers of molecules per unit of volume.

The above must also be true for pairs with very small

* Systems for which some properties depend upon the temperature only being independent of the concentration.

† G. N. Antonoff, *loc. cit.*

mutual solubility. In this case the eutectic point corresponds to a very low concentration, *i. e.* approaches very closely to one of the ordinates. But such a point must exist if the liquids are at all soluble in each other. It is obvious that if the eutectic point exist

$$p_1 = p_2$$

must certainly hold true and the relation

$$\alpha_{12} = \alpha_1 - \alpha_2$$

must necessarily be satisfied.

Summary.

1. A theory of molecular attraction has been developed. The theory detailed above follows from the ordinary modern representation of the nature of atoms and molecules. Starting from this point of view, the phenomena of molecular attraction depend on the same forces as chemical affinity.

2. A relation between surface tension and molecular pressure has been deduced.

3. It was deduced theoretically that the interfacial tension α_{12} is equal to the difference of the surface tensions against the air of both superposed liquid layers $\alpha_1 - \alpha_2$ in equilibrium, which is in agreement with experiment.

4. Two superposed layers in equilibrium must be regarded as solutions in the same dissolvant.

5. They must contain an equal number of molecules per unit volume (Avogadro's Law).

6. Statement (5) is a result of formation of complex molecules in the solution.

7. The so-called monovariant systems may be obtained without fulfilment of the requirements of the Phase Rule, if the molecules of the component added combine with those in solution without increasing their number.

The author, in conclusion, wishes to express his sincere thanks to Prof. Svante Arrhenius and Prof. Arthur Schuster, both of whom have kindly read this paper and made valuable criticisms and suggestions.

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XLIII. *On some Properties of the Active Deposit of Radium.*

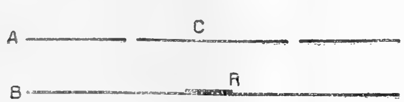
By S. Ratner (*Petrograd*), *Research Student, University of Manchester**.

1. *INTRODUCTION.*—The phenomenon of the spreading of the active deposit of Radium has been observed long ago, and recorded by a large number of experimenters. In some cases (for instance, in that described by Miss Brooks†) the phenomenon is now easily explained by the action of radio-active recoil; in other cases, however, it appears to be of a more complicated nature, and some further assumptions are required for its explanation. Fajans ‡ and others have assumed that the active deposit of radium is slightly volatile at ordinary temperatures, while Russ and Makower § suggested that the radio-active atoms are partly deposited on the surface in groups which may be set free by recoil when an α-particle is ejected from one of the atoms in the group. The phenomenon, however, has never been subjected to special investigation, and is recorded only as a source of error.

In the present paper the results are given of various experiments undertaken with the purpose of a detailed study of the phenomenon.

2. *Procedure of experiments.*—For the greater part of the experiments a simple apparatus was used, consisting of two insulated brass plates A and B (fig. 1), the distance between

Fig. 1.



which could be varied from zero to a few cm. The central part of the upper plate A consisted of a disk C which could easily be removed from the apparatus and replaced by a similar one. In the centre of the plate B a small plate R coated with the active deposit of radium could be fixed. The active matter was always found to expand from the plate R to the disk C; and the experiments mainly consisted in analysing the activity acquired under different conditions by the disks. When RaA was expected to be present on the plate R, a strong electric field (of the order of 10,000 v/cm.,

* Communicated by Prof. Sir E. Rutherford, F.R.S.

† Miss Brooks, 'Nature,' 1904.

‡ Fajans, *Phys. Zeit.* xii. p. 369 (1911).

§ Russ and Makower, Phil. Mag. xix. p. 100 (1910).

was established between A and B, A being positively charged) in order to prevent the recoil atoms of RaB from reaching the disk. The experiments were carried out in air at atmospheric pressure.

As far as the radioactive products constituting the active deposit can be separated from each other, the phenomenon has been investigated for each product separately. Thus sufficiently pure RaA was obtained by exposing the plate R to emanation for a short time and quickly removing it to the testing apparatus. RaB was obtained by recoil from RaA, and RaC by dipping a nickel plate into a solution of the active deposit. A great number of experiments, however, have been also carried out with radium (B+C) on the plate R.

As in the course of this work activities were dealt with varying in strength as 1 to 100,000, great care had to be taken to protect the disks from contamination with the radioactive matter from the plates. The amount of emanation used varied in different experiments from 10 to more than 200 millicuries. The activities of the disks were usually measured by a Wilson's tilted electroscope, that of the plates by a β- and γ-rays electroscope of the ordinary type.

As shown later, it was very important in the course of these experiments to be able to follow the rate of change with time of the quantity of active matter acquired by the disk C. This was realized by exposing to the action of the active plate R a number of disks at definite intervals one after another, and by measuring their activities. In this way the period T could be easily determined during which the quantity of active matter reaching the disks per unit time falls to a half value. If this quantity were proportional to the amount of active matter present on the plate R, T would be equal to the half-value period of the corresponding radioactive product. The experiments show, however, that these two periods as a rule differ from each other.

3. *Experimental results.*—In the first place it appeared necessary to ascertain whether the active matter acquired by the disk C belongs to the same product as that on the plate R. The activity of the disks was carefully analysed and identified beyond doubt with the active matter on the plate R. Thus, in the case of RaA on R the active matter on the disk was found to be an α-ray product with a half-value period of 3 min., giving up RaB by recoil. In general, the curves of decay or recovery drawn for the activities on the disks were found to coincide with the characteristic curves of the products covering the plate R.

Further, the question of the charge of the radioactive atoms expanding from the plate has been investigated. The amount of active matter deposited on the disk increased many times when an electric field was established between the plates A and B, but was found to be independent of the direction of the field. On the other hand, a stream of air maintained between A and B seemed to sweep away the active matter in spite of a strong electric field between the plates. In connexion with some experiments described in another place it must be supposed that the expanding atoms are uncharged, and are brought to the disk C by the electric wind*.

A great number of experiments have been carried out with the object of determining the relative quantity of active matter expanding from the plate. Since the plates R during the first experiments were usually slightly heated in a Bunsen flame or washed in alcohol before being introduced into the apparatus, it seemed necessary to find out to what extent the phenomenon is affected by the process of heating or washing. It was found that when the plate is introduced into the apparatus without being heated or washed, the amount of active matter expanding to the disk increases enormously, reaching in some cases 20 times its normal value. It may be shown, however, that this effect is not due to traces of emanation which could adhere to the plate after its removal from the exposure vessel and then diffuse towards the disk. Apart from the fact that this supposition is not justified by the analysis of the activity of the disk, the same effect can be observed when the plate used is coated with RaB by recoil from RaA, and has never been exposed to emanation. The experiments have shown, however, that the activity given *up* by a plate once slightly heated or washed decreases but slowly with further heating or washing of the plate.

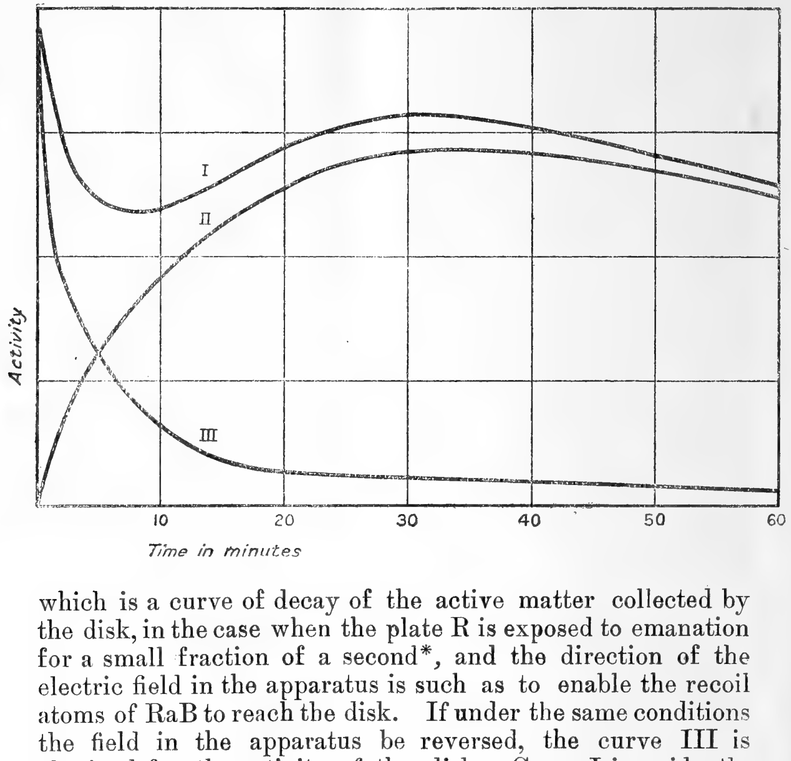
Further experiments have been greatly complicated by the lack of constancy in the relative quantity of active matter given off by the plate R. When the experiments are carried out under similar conditions, the ratio of activities on the disk and the plate is sufficiently constant and independent of the amount of emanation used ; but this ratio varies within large limits with the time of exposure of the plate R to the emanation, increasing considerably in case of small exposures. This is more marked in the case of RaA, when the total amount of active matter received by the disk is almost independent of the time of exposure, so that a plate exposed to emanation for a small fraction of a second gives *up* as much

* Phil. Mag. xxxiv. November 1917.

RaA as a plate exposed for several minutes. This effect is always observed whether the plate is heated or not. As stated above, however, the quantity of RaA received by the collecting disk is proportional to the amount of emanation used.

In the case of very short exposures the relative quantity of RaA expanding from the plate is so large that it becomes comparable with the amount of RaB projected from the plate by recoil. This is clearly shown in curve I (fig. 2),

Fig. 2.



which is a curve of decay of the active matter collected by the disk, in the case when the plate R is exposed to emanation for a small fraction of a second*, and the direction of the electric field in the apparatus is such as to enable the recoil atoms of RaB to reach the disk. If under the same conditions the field in the apparatus be reversed, the curve III is obtained for the activity of the disk. Curve I is evidently the sum of the two curves, II and III, corresponding to RaB and RaA, and it may be easily seen from the curves that the amounts of RaB and RaA deposited on the disk are as 10 to 1 respectively. Putting 0·6 for the efficiency of

* This may be easily realized when Wertenstein's exposure vessel is used. Wertenstein, *Thèse,* Paris 1912.

recoil of RaB from the surface of the plate, it appears that the relative quantity of RaA expanding from the plate under the described conditions is about 3 per cent. In the case when the plate is exposed to emanation for several minutes, the same quantity was found by direct measurements to be of the order of one ten-thousandth. To obtain distinctly the first part of curve I the disk must be exposed in the apparatus for a short time not exceeding one minute.

The order of magnitude of the amount of expanding active matter in the case of Ra(B+C) is well illustrated by the following experiments. If the plate R coated with Ra(B+C) be thoroughly washed in water and alcohol and then strongly heated for a considerable time in a Bunsen flame in order to reduce as much as possible the amount of expanding matter, the activity of the collecting disk, when measured by β-rays, shows a well-marked decrease during the first 3 or 4 minutes. This is undoubtedly due to the presence of RaC_2 on the disk, since the effect is observed only when the disk is negatively charged. When the plate is but slightly heated this fall in the activity of the disk cannot be detected. It appears that the amount of active matter expanding from the plate is usually large compared with the amount of RaC_2 given up by recoil from the active deposit, and that only under special conditions does it diminish to the same order of magnitude. Direct measurements show that the relative quantity of Ra(B+C) expanding from the plate varies from $\frac{1}{1000}$ to $\frac{1}{25,000}$.

Some experiments were also made with a plate coated with a strong layer of polonium (in equilibrium with RaD). No traces of activity could be detected on the collecting disk after an exposure of more than two weeks.

For a more complete study of the phenomenon, it appeared necessary to investigate the rate of change with time of the amount of active matter expanding from the plate; and for this purpose experiments have been undertaken in order to determine the period T (see Sec. 2). Most surprising results were obtained in case of RaA, when this period appeared to be of striking regularity and constancy under different experimental conditions. The same period T, viz. 1·4 min., for RaA was found in the preliminary, as well as in the final, experiments, although they were carried out in two different laboratories and after an interval of more than three years. In Table I. the results are shown of one series of these experiments. Four disks, I, II, III, and IV, were exposed in the apparatus to a plate coated with RaA for 1 min. each and

1 min. 25 sec., one after another. Column A_0 shows the activity of the disks measured immediately after their removal from the apparatus, columns A_3, A_6, and A_9 their

TABLE I.

	A_0.	A_3.	A_6.	A_9.
I.	95	48	25	13
II.	48	24	13	7
III.	25	14	7·5	4
IV.	15	8	5	3

activities measured 3, 6, and 9 min. after the first measurement. From column A_0 the period T for RaA can be deduced, while the rows I, II, III, and IV give the analysis of active matter on the corresponding disks. On the disk IV the presence of RaB and RaC is well marked, since at the time of its exposure the RaA on the plate had already disintegrated to a large degree.

In the case of RaB and RaC the period T is by far not so constant, and varies from one series of experiments to another within large limits, viz. 10–40 min. Some experiments carried out with pure RaB and RaC show that on the average the period T is smaller for RaB than for RaC. This is clearly seen in the case of Ra(B+C) on the plate R, when the analysis of the activities on the disks usually shows that in the active matter expanding from the plate the ratio of RaC to RaB increases with time. In some experiments, when the plate R was introduced into the apparatus 3 or 4 hours after its exposure to emanation, almost pure RaC could be obtained on the collecting disks. It must be pointed out, however, that this effect is not always observed.

4. *Discussion on the nature of the Phenomenon.*—The results given in the previous section seem to be very complicated, and throw but little light on the nature of the phenomenon. It appeared of interest therefore to test experimentally different assumptions which may be put forward for the interpretation of the phenomenon. First, the usual assumption that the active deposit of radium is slightly volatile at ordinary temperatures was investigated. In a series of experiments the disks, while exposed in the apparatus, were heated in a gas-flame to about 300°–400° C., and the amount of active matter deposited on them compared with that acquired by cold disks under the same conditions. It was found that the high temperature of the disks does not prevent the active matter from being deposited on them. It is obvious that the

assumption must be rejected, since the volatilization of the active deposit at lower, and its condensation at higher, temperatures would contradict the fundamental principles of Thermodynamics.

The suggestion made by Russ and Makower (*loc. cit.*), namely, that groups of radioactive atoms may be set free by recoil when an α-particle is ejected by one of the atoms in the group, was tested in the following way. A number of disks were exposed in the apparatus, 3 min. one after another, to a plate covered with RaB. It is evident that with the increase of RaC on the plate the chances for the groups to be set free also increase, and therefore the quantity of active matter expanding from the plate should be expected to increase with time, if the assumption were true. The experiments show, however, that this quantity decreases with a certain period T varying within the limits given above.

The striking regularity of the phenomenon in the case of RaA led to the assumption that the spreading of the active matter is not a secondary mechanical effect on the surface of the plate, but is due in some way to interatomic forces in the active deposit. With the knowledge now available one could easily imagine that a number of branch products are present in small quantities in the active deposit of radium, giving up by recoil the active matter found on the collecting disks. The period $T=1{\cdot}4$ min. for RaA, for instance, would be nothing else but the half-value period of the unknown branch product giving up RaA by recoil. This assumption seemed to be a very promising one, since it could serve as a guide for the investigation not only of the phenomenon itself, but also of the supposed branch products of radium. The results of numerous experiments carried out in this direction failed, however, to be in favour of this theory; and some of them, on the contrary, furnished sufficient evidence against it. Thus, as mentioned above, the amount of active matter given off by RaA does not depend on the time of exposure of the plate to emanation, though one can hardly imagine a radioactive product accumulating from the emanation to its full value during a small fraction of a second and decaying with a half-value period of 1·4 min. Further, the collecting disk, after being exposed to a plate coated with RaA, was put in place of the plate, and was found to give off RaA in its turn, though it is evident that this disk could not contain the branch product giving up RaA by recoil.

It could also be suggested that some of the radioactive atoms (or particles) are but slightly attached to the active surface of the plate, so that small air-disturbances could easily set them free. This is apparently supported by the fact that the washing or heating of the plate considerably reduces the effect. It must be remembered, however, that the phenomenon has been observed by Russ and Makower (*loc. cit.*) at high vacua where no air-disturbances could arise. Furthermore, the plate after being repeatedly heated and washed several times still continues to give off the active matter. In some experiments the active surface of the plate (not heated or washed) was exposed to a violent stream of gas coming from a high-pressure bottle at 80 atmospheres, and this did not reduce appreciably the amount of expanding active matter, although the stream of gas was certainly strong enough to remove the slightly attached particles from the active surface.

A number of experiments have been made in order to ascertain whether the phenomenon is affected by physical or chemical conditions on the surface of the plate. The results were entirely in the negative. A clean and well-polished platinum surface was found to give up the same amount of active matter, and with the same period T as a rough surface of brass oxidized in air or covered with grease.

5. *General Conclusions.*—If the experiments carried out in this work have failed to disclose the nature of the phenomenon, they give nevertheless a detailed description of some occurences taking place in the active deposit of radium, which for some time now have served as a grave source of error in many investigations in radioactivity. In the work of Fajans this source of error could be overcome owing to the fact that the amount of the branch product given up by RaC is not too small compared with the quantity of RaC expanding to the collecting disk. In other cases, however, this source of error renders the investigation impossible. That, for instance, is the case in some work carried out with the object of investigating the recoil phenomena due to β-rays. Various experiments described in this paper show clearly that, if the recoil of RaC from RaB does exist in reality, the amount of RaC given up by this process must be vanishingly small, compared with the activity expanding from the surface coated with the active deposit. A survey of the work dealing with the questions of recoil of RaC from RaB leads to the conclusion that this phenomenon has hardly ever been observed. Unless the source of error referred to above is completely eliminated, all attempts to detect

recoil phenomena due to β-rays must be considered as hopeless.

The results given in the present paper also throw some light on the question of the charge of the recoil atoms. Apart from the fact that the recoil atoms from the emanations are usually believed to be partly negatively charged in order to explain the origin of the anode activity, the atoms of RaB given up by RaA were also found to carry partly a negative charge. To account for the activity of a positively charged collecting disk placed over a plate coated with RaA, Wertenstein* and others have assumed that from 2 to 5 per cent. of the recoil atoms of RaB carry a negative charge. In all these cases the activity of the collecting disks (probably consisting of RaA) was not analysed. In some of the experiments described in this paper, when the disk has collected about $\frac{1}{10,000}$ of the total amount of RaA on the plate, no traces of RaB could be found on the disk. Since the presence of RaB in the proportion of $\frac{1}{20}$ of the amount of active matter could easily be detected on the disk, it follows that the proportion of recoil atoms of RaB carrying a negative charge is certainly less than 1 to 100,000.

My best thanks are due to Prof. Sir Ernest Rutherford for his kind interest in this work and for the supply of large quantities of radium emanation.

XLIV. *The Correction of Telescopic Objectives.* *By* T. Smith, *B.A., Optical Department, National Physical Laboratory* †.

IN the Philosophical Magazine for June Mr. A. O. Allen has pointed out the possibility of expressing in a small compass all the information contained in the N. P. L. tables of constructional data for small objectives, and much more besides, by means of a few formulæ and other methods. He gives expressions for this purpose and works out a number of numerical illustrations. The formulæ as he presents them are open to criticism, and the same may be said of several statements made in the course of the paper. I propose in this note to deal briefly with a few of the more important of these.

* Wertenstein, *Thèse*, Paris 1912.

† Communicated by the Author.

Formulæ for the purpose of such calculations, though not usually expressed in a form resembling that adopted by Mr. Allen, have been known for many years. Although a simple calculation by known formulæ would furnish equivalent information, it was considered desirable, in view of the special circumstances existing at the time, to publish tables in the form of those issued by the National Physical Laboratory. These appear to have served satisfactorily the limited function for which they were intended, and any value they still possess may be regarded as accidental.

It has been customary for a maker of telescopes or other optical instruments, when working out a new objective, to rely upon his previous experience to enable him to set down approximate curves on which to base his calculations. Under favourable conditions a very limited amount of trigonometrical ray tracing enables him to reach a satisfactory final solution. This method works satisfactorily in experienced hands, but such experience becomes quite unnecessary if other methods are adopted. Without any experience whatever it is possible, with the aid of a little algebra, to obtain in a few minutes an approximately correct form for an objective provided the conditions to be satisfied are stated in a suitable form. Mr. Allen's formulæ enable such calculations to be made, but they are cast in a form which involves an unnecessary amount of arithmetical work. Fourteen coefficients occur in his two expressions for the spherical aberration and the sine error. It is obvious that many of these do not involve separate computation—for instance, several identical relations exist between A, B, C, D, E, F, P, Q, and R. It seems preferable to express the fundamental quantities in a form which takes advantage of these relations.

The writer has pointed out elsewhere * that all the first order aberrations of any thin objective for light of a given wave-length are determined by three quantities which depend upon the refractive indices of the glasses and the curvatures of the surfaces, but not on the position of the object. If the three quantities be denoted by α, β, and ϖ † the factors which involve the constructional data of the objective in the expressions for the spherical aberration, the coma, and the

* Proc. Phys. Soc. vol. xxvii. p. 485.

† In the standard notation these quantities are denoted by $4C+2\varpi+1$, $B'-B$, and ϖ respectively.

departure from the sine condition are

$$\alpha-4\beta M+(2\varpi+3)M^2, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

$$\alpha-\beta(3M+S)+\varpi M(M+S)+M(M+2S), \quad . \quad (2)$$

and $$\alpha-\beta(3M-1)+\varpi M(M-1)+M(M-2) \quad . \quad . \quad (3)$$

respectively, where

$$M=\frac{1+m}{1-m}, \quad S=\frac{1+s}{1-s},$$

m and s being the magnifications for the object and for the aperture stop respectively. It will be noted that (3) may be derived from (2) by putting $S=-1$, *i. e.* $s=\infty$, showing that apart from the satisfaction of an aberrational condition the coma and the departure from the sine condition are only measured by the same expression when the centre of the aperture stop is situated at the first principal focus *.

If $\frac{1}{m}$ and $\frac{1}{s}$ are substituted for m and s, M and S are changed in sign but not in magnitude. This is sufficient to indicate that α, β, and ϖ are symmetrical † functions of the curvatures and refractive indices of the system. It is easy to show that if the system is reversed, thus changing the sign of the curvature of every surface, α and ϖ are unaltered and β is only changed in sign.

When the form of the lens is varied by making the same change in the curvature of each surface, ϖ, which is the Petzval sum, remains unchanged, but the other two quantities are altered. By choosing a suitable zero conformation to which such deformations may be referred, the change in α and β due to the impression of the additional curvature r on each surface of the system may be expressed in the form ‡

$$\alpha=\alpha_0+4r^2(2\varpi+1), \quad . \quad . \quad . \quad . \quad . \quad (4)$$

$$\beta=\beta_0+2r(\varpi+1), \quad . \quad . \quad . \quad . \quad . \quad . \quad (5)$$

which involve no new constants. Although it does not appear in these equations there is in effect one additional constant involved, inasmuch as a standard conformation for the system has been introduced by imposing the condition

* A detailed discussion of the relation between the spherical aberration, the coma, and the sine condition is given in Proc. Phys. Soc. vol. xxix. p. 293.

† In Mr. Allen's expressions nine unsymmetrical coefficients occur.

‡ Proc. Phys. Soc. vol. xxvii. p. 485.

that there should be no term in r on the right side of equation (4).

The substitution of the above values for α and β in (3) yields a value for the departure from the sine condition which should be comparable with Mr. Allen's second condition. The fact that their character is essentially distinct shows that one of them at any rate is not the sine error. As a matter of fact his second expression is comparable with the coefficient of S in (2). One of the factors multiplying (2) in the complete expression for the comatic displacement of a ray is $1-s$, and the coefficient of S therefore takes the place of (2) as the important factor in the value of the coma when the aperture stop is in contact with the objective. The statement that this expression only measures "the amount of coma provided there is no spherical aberration" is incorrect.

The expressions quoted above for spherical aberration and coma hold for any thin system, no matter how complex its structure may be, and whether the surfaces are cemented together or there are air-gaps. It is a simple matter, if desired, to introduce additional variables to show the effect of varying the curvature differences bounding these gaps. There will be no change in ϖ, but α and β will be respectively quadratic and linear functions of such curvature differences. This follows at once by noting that such gaps are created by bending part of the system relatively to the rest, thus causing alterations in the aberrational coefficients of the two parts of the kind indicated in equations (4) and (5). The additional coefficients in α and β are necessarily of a symmetrical form. When the objective is a doublet with one air-gap, one * additional coefficient will occur in the general expression for α, one in the expression for β, and one in the formulæ for the curvatures of the system in its zero conformation. Thus in the most general case considered by Mr. Allen only seven quantities are needed in place of the fourteen he tabulates. As a rule, however, there is not much point in taking the air-gap into consideration as a separate variable. It is usually possible to employ cemented objectives, and in most instruments this is very desirable on account of the better light transmission so obtained.

The reduction in the labour of calculation obtained by the arrangement described above does not exhaust the advantages of the system. If a triple lens is to be calculated in place of a doublet the α_0 and β_0 of the triple objective may be derived

* If g is the gap the coefficient of g^2 in α is one quarter the coefficient of r^2.

very simply from those of the doublet *. The ϖ is the same for both forms. Again it will be obvious from earlier remarks that the same coefficients apply to a doublet with the flint component leading as to one with the crown in front. But perhaps more important than either of these is the indication afforded by the magnitude of α_0 of the purpose for which a given combination of glasses will be useful. The value of ϖ always lies between very narrow limits ; β_0 is always small, being zero for a single lens or for a cemented combination of different glasses of the same refractive index, and a small positive quantity if, as is usual, the component made from glass with the greater dispersive power has the higher refractive index. On the other hand α_0 varies through a wide range of values. For a single lens of refractive index μ its value is $\mu^2/(\mu-1)^2$. In a doublet of the usual cemented type it falls from this value as the difference between the refractive indices of the two glasses increases. The rate of fall increases with the power of the components relative to that of the complete lens. Generally speaking the possibility of obtaining similar corrections with two different combinations of glasses depends upon their having approximately equal values of α_0. For example, the simultaneous correction of spherical aberration and coma for unit magnification ($m=-1$, $M=0$) requires, from equations (1) and (2), $\alpha=\beta=0$, and from equations (4) and (5) it follows that it will be necessary for α_0 to be approximately zero since β_0 is small.

This property of α_0 in determining the type of correction that is possible leads to a novel method of designing instruments which are built up of a number of separate lenses when each may be regarded as approximately thin. Each lens is assumed to have the same value of ϖ—a value about the middle of the possible range is chosen. The β_0 of each lens is assumed to be zero, and the conditions to be satisfied then lead to connected series of values of α_0 for the various component lenses. As the types of glass available do not form a continuous series it will not be possible to realize the majority of these series, but a few can usually be selected with very little difficulty in which these simplified conditions are very approximately satisfied. The most favourable case—often determined by the magnitude of the curvatures involved—may be adopted for more detailed investigation with corrected values of α_0, β_0, and ϖ based upon the glasses selected. This method of calculation is the inverse of that usually employed, the selection of the refractive indices of

* Proc. Phys. Soc. vol. xxviii. p. 232.

the glasses being the last step to be taken instead of being assumed initially.

The selection of glasses for this purpose is greatly facilitated by drawing on tracing-paper a chart containing curves corresponding to constant values of α_0. This is used in conjunction* with a diagram on which the available kinds of glass are plotted, the variables employed being the refractive indices for a definite wave-length—usually the D line—and the logarithm of the ν†. The chart on the tracing-paper is moved over the glass diagram in slide-rule fashion, and suitable pairs of glasses are selected by noticing that when the zero of the chart lies on the point representing one of the glasses the other glass lies on the line corresponding to the required value of α_0. This is one of the few directions in which I have found graphical methods of distinct value in lens calculations. I do not regard Mr. Allen's graphical suggestion as a useful one for the practical computer, because it is not only much easier to solve a quadratic equation directly than by graphical methods which involve the construction of a templet, but also, as will be explained later, there is a very good reason for solving the spherical aberration equation with greater accuracy than a graphical method will generally afford.

Before leaving the discussion of the detailed formulæ it may be pointed out that Problem (2) is not stated in a satisfactory form, for the result obtained will depend upon the interpretation given to "least aberration" as the object point is varied. The boundary condition may be that the lens aperture subtends a definite angle at the object, or at the theoretical image point. The most natural assumption in the absence of any statement would be that the linear aperture of the lens is kept constant. In all cases, however, it is to be remembered that expressions such as (1) are multiplied by other factors which involve m, or the position of the object, and these factors must be taken into account when the expression for the aberration is differentiated to find the stationary values. The result obtained will vary with the criterion adopted for the measurement of the aberration. For instance, the position of the object which gives minimum longitudinal aberration will differ from that for which the latitudinal aberration is least, and both will be distinct from the one for which the difference of path between axial and marginal rays is a minimum.

* Proc. Phys. Soc. vol. xxviii. p. 220.

† When the ordinary type of colour correction is not desired a modified quantity is substituted for ν.

I now turn to an entirely distinct question—the reliability of the results of the calculations and their subsequent treatment. Is the solution of the algebraic equations simply an equivalent of "experience" in affording a favourable basis on which subsequent trigonometrical work is grounded, or is it more? On this score Mr. Allen is very decided: "both the tables and the equivalent calculations lead to figures such as no manufacturer with a reputation to keep up would employ." I am bound to differ from Mr. Allen. So far as my experience goes a manufacturer's reputation will be quite safe if he solves the algebraic equations (not graphically, since the solution will not be sufficiently accurate) for a thin cemented objective, inserts the necessary thicknesses without altering the curvatures found for the surfaces, and leaves the objective as it is without troubling about trigonometrical calculations *. Naturally this only holds within limits, and may fail for abnormal combinations of glasses or for abnormal apertures. It applies, however, to the general run of objectives which are required in large numbers. In cases where this procedure does not yield the particular type of correction which the maker finds most pleasing, a slight alteration should be made in the conditions imposed on the thin objective.

The low esteem in which Mr. Allen holds the algebraic solution can hardly occasion surprise in view of a subsequent statement. In obtaining his algebraic expressions he says it is assumed "that all the angles in the calculation are so small that the excess of any angle above its sine is exactly equal to a sixth of the cube of the angle. In other words the rays could all travel within a capillary tube lying along the axis of the lens." In saying "exactly" a somewhat unhappy word has been chosen. To give a meaning to the statement we may consider that what is meant is that, when a definite number of figures are retained, the error resulting from the neglect of the next term would only involve an alteration of the last decimal place by unity, or alternatively would just fail to alter it, the error not exceeding five units in the next decimal place. Let four and five figures be taken as illustrations since these are the number of figures used in the majority of optical calculations by trigonometrical methods. The subjoined table gives the angles for which the errors due to the neglect of (*a*) the second term in the cosine expansion, (*b*) the second term in the sine expansion, (*c*) the third term in the cosine expansion, and (*d*) the

* For the theory underlying this use of the algebraic solution of the first order conditions see Proc. Phys. Soc. vol. xxx. p. 119.

third term in the sine expansion, amount to the values given at the head of the respective columns.

Error....	·000005	·00001	·00005	·0001
(*a*)	0° 11′	0° 15′	0° 34′	0° 48′
(*b*)	1° 46′	2° 15′	3° 50′	4° 50′
(*c*)	6° 0′	7° 8′	10° 40′	12° 41′
(*d*)	10° 49′	14° 56′	17° 9′	23° 40′

It is evident that the figures in the last row correspond to rays which are very far from travelling within a capillary tube lying along the axis of the lens. The figures of row (*b*), on the other hand, indicate that the number of figures used for trigonometrical calculations may frequently involve the neglect of aberrations altogether, though these would not be omitted in the corresponding algebraic operations. The table shows that the statement made above in discussing the reliability of results derived from algebraic calculation should occasion no surprise. It is, however, important that more should not be read into that statement than it contains. The field over which such calculations are reliable does not extend to the limits given in line (*d*) of the above table or indeed to line (*c*). The neglect of the third term in the cosine series (not the sine series) defines the theoretical limit of accuracy, but this limit is not applicable to the algebraic expansion for a series of surfaces owing to the neglect of product terms which are not necessarily of little account. Mr. Allen has simply taken the traditional view of algebraic calculations without investigating its accuracy. The true position I believe to be that the importance and reliability of algebraic calculations in the determination of aberrations has been underestimated, and that of trigonometrical work as it is usually carried out overestimated. In both methods of calculation it is desirable to employ about two more figures than can be said to correspond in the final rays with the mechanical accuracy attainable in the concrete instrument. When the calculations are completed the last two figures may be neglected. The reason for this is that aberrations are eliminated by opposing aberrations of different signs and necessarily large magnitude. A typical illustration is afforded by the values found by Mr. Allen for the coefficients L of a doublet and of one of its components. It is an instructive exercise to carry out calculations for corrected systems retaining in turn four, five, six, and even seven figures. The values of the outstanding aberrations given by the earlier calculations will occasionally be found to require appreciable modification.

XLV. *The Electron Theory of Metallic Conductors applied to Electrostatic Distribution Problems.* By L. SILBERSTEIN, *Ph.D.**

THE electron theory of metallic conductors, as propounded by Riecke and Drude, and developed by J. J. Thomson, Lorentz, and others, has almost exclusively been treated in connexion with problems of current-conduction and allied questions, this being undoubtedly the most vital and promising field of application of the theory, especially for the experimentalist. From a more theoretical standpoint, however, electrostatic applications may not be devoid of interest. As far as I could gather, investigations of this kind are limited to an incidental rough estimate of "the thickness" of the layer of electricity in a conductor, due to J. J. Thomson †.

It has seemed, therefore, worth while to represent the general problem of electrostatic distribution in terms of the electron theory. This, together with a full solution in the case of some of the most simple illustrative problems, is the object of the present paper.

1. Consider a metallic conductor or, more generally, any system of insulated conductors at uniform absolute temperature T. The latter will enter into our formulæ through a magnitude fundamental in every kinetic theory, viz. the average kinetic energy of a molecule or of a free electron, per degree of freedom,

$$\kappa = \frac{1}{3}\alpha T, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

where α is the "universal" constant, equal to $\frac{3}{2}$ of the gas constant divided by Avogadro's number, *i. e.* about 2.10^{-16} erg per degree centigrade ‡. The classical problem of distribution can be put as follows: Given the total charge of each conductor and the potential ϕ_e of the external field, due to charges fixed outside the conducting masses, find the electrostatic or equilibrium distribution of electricity over each of the conductors. The solution of the problem in its classical

* Communicated by the Author.

† 'The Corpuscular Theory of Matter' (1907), p. 82. The example treated by Thomson, which concerns an infinite plane as boundary of the conductor, has more recently been taken up again and dealt with on almost identical lines by Lorentz, who does not seem to have noticed Thomson's estimate; cf. *Vortraege ueber die kinet. Theorie d. Materie & Elektr.* Leipzig (1914), pp. 191–192.

‡ The symbol k used by some authors stands for $\frac{2}{3}\alpha$.

aspect is ultimately reduced to finding appropriate integrals of Laplace's equation and adapting them to the surfaces of the conductors, or, equivalently, to solving a linear integral equation in which the unknown function appears under an integral to be extended over *the surfaces* of the conductors. In its electronic form the problem relates essentially to *the interior* of the conducting bodies. No matter how rapidly the density of charge decreases with increasing depth below the surface, the problem is here, mathematically as well as physically, a volume problem.

2. By a fundamental theorem of the general kinetic theory *, and by the well-known assumptions of the current electron theory of metallic conductors, the number of free electrons whose velocities and positions fall within the element $d\varpi = du\,dv\,dw$ of the velocity-space and within the element $d\tau = dx\,dy\,dz$ of ordinary space occupied by metal will, in electrostatic statistical equilibrium, be proportional to

$$e^{-\frac{1}{2\kappa}\left(\frac{1}{2}mc^2+\psi\right)} \,.\, d\varpi\, d\tau, \quad \ldots \ldots \quad (2)$$

where κ is as in (1), m the mass, c the resultant velocity of an electron, and ψ, here an unknown function of x, y, z, the potential energy of the electron in the resultant field of force. Integrating (2) over the velocity-space, the number of free electrons per unit volume will be

$$C\,.\,e^{-\frac{1}{2\kappa}\psi} = C\,.\,e^{\frac{\epsilon}{2\kappa}\left(\phi_e+\phi_i\right)}, \quad \ldots \ldots \quad (3)$$

where ϵ is the absolute value of the charge of an electron, ϕ_e, as above, the given potential of the external field, and ϕ_i the potential of the (unknown) distribution of resultant charge within the conductors. The constant factor C will be determined presently.

Let n be the number of free electrons per unit volume of each conductor, in absence of the external field and in the (macroscopically) neutral or unelectrified state of the conductors. Then, by the assumption of the theory, n is also the number per unit volume of positively electrified atoms (which will be assumed to be rigorously fixed), each carrying the charge $+\epsilon$. Thus the resultant density ρ of electric charge at a point x, y, z, within any conductor of the system

* See, for instance, J. H. Jeans' 'Dynamical Theory of Gases' (1916), p. 89, and *passim*.

will be equal to the difference of $n\epsilon$ and the expression (3), *i. e.*

$$\rho = n\epsilon - C \,.\, e^{\frac{\epsilon}{2\kappa}\left(\phi_e+\phi_i\right)} = n\epsilon - C \,.\, e^{\frac{\epsilon\phi}{2\kappa}}, \quad . \quad . \quad (4')$$

where $\phi = \phi_e + \phi_i$ has been written for the resultant potential. If the conductors are neutral and if there is no external field, we have $\rho = 0$, $\phi = 0$, and therefore $C = n\epsilon$. Thus, ultimately, the equation for the unknown density of distribution $\rho = \rho(x, y, z)$ becomes

$$\log(1 - \rho/n\epsilon) = \frac{\epsilon}{2\kappa}\phi = \frac{\epsilon}{2\kappa}(\phi_e + \phi_i). \quad . \quad . \quad . \quad (4)$$

If ρ' be the density in any element $d\tau'$ and r' the distance of $d\tau'$ from the point x, y, z, then, taking the charges in rational units,

$$\phi_i = \frac{1}{4\pi}\int \frac{\rho' d\tau'}{r'},$$

integrated over the volume of all conductors of the system.

Thus the equation (4) becomes

$$\log(1 - \rho/n\epsilon) = \frac{\epsilon}{2\kappa}\phi_e + \frac{\epsilon}{8\pi\kappa}\int \frac{\rho' d\tau'}{r'}. \quad . \quad . \quad . \quad (5)$$

This is an *integral equation of the second kind*, with $\phi_e = \phi_e(x, y, z)$ as the given, and ρ as the unknown, function. Since, on the left hand, ρ appears through the log, our equation is a *non-linear* one, and thus differs from those hitherto studied by Fredholm, Hilbert, E. Schmidt, and other mathematicians.

Owing to its non-linearity, the solutions of this rigorous integral equation would obviously be deprived of the classical property of superponibility. On the other hand, we know from experience that this property does hold, at least—it would seem,—with a good approximation. If so, and if the assumptions of the electron theory of conductors are essentially sound, we can draw from the experimental facts the conclusion that, at least for such inducing fields, charges, etc., as are at our disposal, the left-hand member of (5) has to become sensibly linear—that is to say, that ρ is a small fraction of $n\epsilon$, *i. e.* that the defect or the excess of free electrons in a given volume is but a small fraction of the normal number of free electrons contained in that volume *.

* If the degree of accuracy with which superponibility holds were ascertained by experiments especially undertaken then one could form an idea of the upper limit of $\rho/n\epsilon$ (with the greatest attainable ρ, say) and therefore of the lower limit of n, the number of free electrons per cm.^{3}. I do not know whether such an (electrostatic) estimate of the lower limit of n has ever been contemplated.

Let us therefore make the assumption (which, as far as I know, has actually been made by the leading electronists) that $\rho/n\epsilon$ is a small fraction. Then $\log(1-\rho/n\epsilon) \doteq \frac{\rho}{n\epsilon}$, and eq. (5) becomes

$$\frac{2\kappa}{n\epsilon^2}\rho = -\phi_e - \frac{1}{4\pi}\int \frac{\rho' d\tau'}{r'} .$$

The coefficient on the left hand is a certain squared length, since such is the dimension of the ratio of ϕ and ρ. Calling the coefficient in question L^2, we have, by (1),

$$L^2 = \frac{2}{3}\frac{\alpha T}{n\epsilon^2} . \qquad \qquad (6)$$

The length L is identical, in fact, with J. J. Thomson's $1/p$ which he takes as the measure of the thickness of the layer in the example mentioned above. As concerns the value of the fundamental length L, we have, by (6), for, say, $T=300$ (or 27° C.),

$$L^2 = \frac{2 \cdot 10^{-14}}{4\pi \cdot 9{\cdot}5 \cdot 10^{-20}} \cdot \frac{1}{n} \doteq \frac{1{\cdot}7 \cdot 10^4}{n},$$

i. e. in round figures,

$$L = 130/\sqrt{n}.$$

Thus, for instance, if there is one free electron per each atom of the metal, say, of copper, or $n \doteq 10^{22}$, then L is of the order of $1{\cdot}3 \cdot 10^{-9}$. But as far as is known, there may be only one free electron for every 1000 or 10,000 atoms; in the latter case we should have $L \doteq 10^{-7}$ cm. As a matter of fact the number n is not even coarsely known, so that all such estimates, especially in the domain of electrostatics, are, for the present, pretty useless. (See also the preceding footnote.)

With the above notation the last approximate equation for ρ becomes

$$-L^2\rho = \phi_e + \frac{1}{4\pi}\int \frac{\rho' d\tau'}{r'}, \qquad . \; . \; . \qquad (\mathrm{I})$$

or written shortly,

$$-L^2\rho = \phi_e + \operatorname{pot}\rho,$$

a linear integral equation of the second kind, the integral on the right hand to be extended over the volume of all conductors of the system which, with ϕ_e and the total charges of the conductors given, suffices for the determination of ρ.

The circumstance that there are three independent variables instead of one or two does not, mathematically, make much difference, so that (I) could be investigated in concrete cases by the usual methods of the theory of integral equations. That task, however, will be left to the specialist in this new and promising branch of mathematics. Here it is enough to state that, the square and the higher powers of $\rho/n\epsilon$ being neglected, all cases of electrostatic distribution treated on the lines of the electron theory will obey the linear integral equation (I).

Introducing, as before, the resultant potential $\phi=\phi_e+\phi_i$, that equation amounts to

$$L^2\rho+\phi=0. \quad . \quad . \quad . \quad . \quad . \quad . \quad (7)$$

On the other hand, we have, by the very definition of ϕ,

$$\nabla^2\phi=-\rho,$$

so that $\nabla^2\phi=\frac{1}{L^2}\phi$, or to eliminate the auxiliary potential altogether, again by (7),

$$\nabla^2\rho=\frac{1}{L^2}\rho. \quad . \quad . \quad . \quad . \quad . \quad . \quad (II)$$

This is a common partial differential equation * for ρ of a form familiar from many chapters of mathematical physics. It is a consequence of the integral equation (I), but does not, of course, replace it completely. For in (II) every trace of the given external field and of everything that concerns the shape, the size, and the configuration of the conductors has disappeared. In short, (II) is more general than (I). However, although the equation (II) does not fully replace (I), it may help us in solving (I), if we are not in the position of solving it systematically by the methods of integral equations. In fact, it is enough to find a sufficiently general integral of (II) and to determine its more particular form or its coefficients by substituting it into (I) and by using the given total charges $\int\rho\, d\tau_i=q_i$ †.

In order to explain the latter method and to illustrate, at the same time, the meaning of the above general equations, let us work out a pair of examples of the most simple kind.

* The equation obtained by J. J. Thomson, *loc. cit.*, is a special (one-dimensional) case of the general equation (II).

† Another way would be to attempt to supplement the differential equation (II) by some plausible general surface-conditions. Such conditions, however, would seem to be artificial from the point of view of the electron theory. It is therefore that any conjectures about such supplementary conditions are here omitted.

4. *Full spherical conductor.*—Let the conductor have the total charge $\int \rho \, d\tau = q$, and let there be no "external" field. Then ρ is obviously a function of r alone, r being the distance from the centre O of the conducting sphere. Under these circumstances the differential equation (II) becomes

$$\frac{d^2}{dr^2}(r\rho) = \frac{1}{L^2}(r\rho),$$

and its most general integral

$$\rho = \frac{1}{r}(Ae^{r/L} + Be^{-r/L}). \quad . \quad . \quad . \quad . \quad . \quad (8)$$

Both of the arbitrary constants A, B could be determined from the integral equation and from the given q. In the present case, however, the procedure can be simplified by noting that to avoid $\rho = \infty$ we must have $B = -A$ *. Thus,

$$\rho = \frac{A}{r}(e^{r/L} - e^{-r/L}),$$

and the single constant A will be determined from the total charge. (Notice that the relation between B and A could, but the value of A could not, be determined from (I), since, in the present case, $\phi_e = 0$, it is a homogeneous equation.) If R be the radius of the sphere, the total charge is

$$q = 4\pi \int_0^R r^2 \rho \, dr,$$

which on substitution of the above ρ gives A without trouble. It is convenient to replace A by the density ρ_0 at the centre of the sphere, which is $\rho_0 = 2A/L$. Thus the final solution becomes

$$\rho = \rho_0 \frac{L}{r} \sinh \frac{r}{L}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (9)$$

where $$\rho_0 = q/4\pi L^3 \left(\frac{R}{L}\cosh\frac{R}{L} - \sinh\frac{R}{L}\right).$$

In order to bring into evidence the rapid decrease of density below the surface, compare ρ with the density ρ_R at the surface, thus:

$$\frac{\rho}{\rho_R} = \frac{R}{r} \frac{\sinh(r/L)}{\sinh(R/L)},$$

* We shall see, in fact, from the next example that the relation $B = -A$ would follow automatically from the integral equation.

or denoting by x the depth $R-r$, and neglecting the square of $\frac{x}{R}$,

$$\frac{\rho}{\rho_R} \doteqdot \left(1+\frac{x}{R}\right)\sinh\left(\frac{R-x}{L}\right) : \sinh\frac{R}{L},$$

i. e. ultimately, remembering that R/L is at any rate a very large number,

$$\rho/\rho_R \doteqdot \left(1+\frac{x}{R}\right)e^{-\frac{x}{L}},$$

which, for any x equal to several L and to a small fraction of R, reduces simply to the exponential decrease $e^{-x/L}$, agreeing with Thomson's result.

5. *Hollow sphere.*—Again, let there be no charges outside the metal (nor in the cavity), *i. e.* $\phi_e=0$ or const. Then we have again (8) as the general solution of the differential equation. In this case there is, of course, no reason for rejecting any of the constants A, B. Both have to be determined, from the given charge and from the integral equation (I), which in the present case becomes

$$L^2\rho + \frac{1}{4\pi}\int\frac{\rho'\,d\tau'}{r'} = \text{const.},$$

the value of the constant being irrelevant. Substituting ρ from (8), denoting the inner and the outer radii of the conductor by R_1 and R_2 respectively, and throwing the constant term of the space integral upon the "const." on the left hand, the reader will easily find

$$\frac{A}{B} = \frac{R_1+L}{R_1-L}e^{-\frac{2R_1}{L}},$$

the required ratio of the coefficients*. It is remarkable that this ratio contains only the inner radius of the conductor. The solution (8) becomes

$$\rho = \frac{B}{r}\left(e^{-\frac{r}{L}} + \frac{R_1+L}{R_1-L}e^{\frac{r-2R_1}{L}}\right). \quad . \quad . \quad . \quad (10)$$

The remaining factor B can easily be determined from the given total charge, but this need not detain us here. It is interesting to notice that the particular law of *distribution* depends only upon *the inner* and not on the outer radius (the latter entering only through the factor B and being thus

* For a full sphere, *i. e.* for $R_1=0$, this reduces to $A=-B$, as stated before.

related solely to the total charge). Practically, for any appreciable cavity, $\frac{R_1+L}{R_1-L} \doteq 1$ *, so that (10) becomes for any hollow sphere,

$$\rho = \frac{B}{r}\left(e^{-\frac{r}{L}} + e^{\frac{r-2R_1}{L}}\right). \quad . \quad . \quad . \quad . \quad (10\,a)$$

The ratio of the densities at the inner and outer surfaces is, with L as unit length,

$$\frac{\rho_1}{\rho_2} = \frac{R_2}{R_1} \cdot \frac{2e^{-R_1}}{e^{-R_2}+e^{R_2-2R_1}} = \frac{2R_2/R_1}{e^{+\Delta}+e^{-\Delta}},$$

where $\Delta = R_2 - R_1$ is the thickness of the spherical shell. If Δ contains many units (L), we have simply

$$\frac{\rho_1}{\rho_2} = 2\frac{R_2}{R_1}e^{-\Delta}, \quad . \quad . \quad . \quad . \quad . \quad (10\,b)$$

and if the shell is comparatively thin (Δ/R_2 small), the ratio becomes $2e^{-\Delta}$.

Some further points of the subject of the present paper, and especially those concerning a possible electrostatic estimate of n in connexion with the rigorous non-linearity of the integral equation (non-superponibility) may be reserved for a later opportunity.

4 Anson Road, N.W. 2.
October 8, 1918.

XLVI. *Diffraction of Plane Waves by a Screen bounded by a Straight Edge.* *By* F. J. W. Whipple†.

In an article by Mr. R. Hargreaves in a recent number of this Magazine ‡ the diffraction of plane waves by a half-plane is discussed. Mr. Hargreaves is concerned primarily with the case of wave motion according to the simple harmonic law. His method can be adapted, however, to the problem of the diffraction of waves of arbitrary type. The solution of this problem does not appear to have been derived hitherto in terms of such simple analysis §.

* Only for a full sphere this coefficient becomes −1, as before.

† Communicated by the Author.

‡ Phil. Mag. ser. 6, vol. xxxvi. p. 191.

§ *Cf* Lamb, Proc. London Math. Soc. ser. 2, vol. viii. p. 422, and Whipple, *idem,* vol. xvi. p. 106.

It will be assumed that the wave-fronts are parallel to the diffracting edge and that the disturbance has lasted only for a finite time. We take rectangular axes, the diffracting edge being the axis of z and the axis of y perpendicular to the wave-fronts, the waves approaching along the negative half of this axis. The distance from the diffracting edge will be denoted by ϖ, t will be written for the time, and c for the speed of propagation of the disturbance.

V, the measure of disturbance, must satisfy the differential equation

$$\left[\nabla^2-\frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right]\mathrm{V}=0. \quad . \quad . \quad . \quad . \quad . \quad (1)$$

The incident wave is determined by an equation such as

$$\mathrm{V}=\psi(ct-y), \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

where ψ is any continuous function satisfying the condition that, for all values of T greater than some constant K, $\psi(-c\mathrm{T})$ and its derivatives are zero.

It is proposed to construct an expression suitable for representing the disturbance diffracted into a shadow. For this purpose consider the integral

$$f=\int_1^\infty \psi(ct-y-u\xi)\,\mathrm{U}\,du, \quad . \quad . \quad . \quad . \quad (3)$$

in which ξ represents $\varpi-y$ and U is a function of u.

The parameters of ψ in this integral range from $ct-\varpi$ to $-\infty$ and suggest the passage of elementary waves by the direct line from the edge to x, y, z and by longer routes. If the integral f can be made to satisfy the fundamental differential equation (1) it may serve for constructing the solution of the diffraction problem.

It is easy to verify that

$$\left[\nabla^2-\frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right]f=\frac{1}{\varpi}\int_1^\infty \{2(u^2-u)\mathrm{U}\xi\psi''-u\mathrm{U}\psi'\}\,du, \quad (4)$$

differentiation of ψ being represented by dashes.

The integration can be effected if the expression in the larger brackets is a perfect differential. The condition for this, viz.,

$$\frac{\partial}{\partial u}\{2(u^2-u)\mathrm{U}\}=u\mathrm{U}, \quad . \quad . \quad . \quad . \quad . \quad . \quad (5)$$

is satisfied if

$$2(u^2-u)\mathrm{U}=\mathrm{C}(u-1)^{1/2} \quad . \quad . \quad . \quad . \quad . \quad . \quad (6)$$

or
$$U=\frac{C}{2u(u-1)^{1/2}}, \quad . \quad . \quad . \quad . \quad . \quad (7)$$

where C is a constant.

On making the substitution and integrating, it is found that

$$\left[\nabla^2-\frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right]f=-\frac{C}{\varpi}\Big[(u-1)^{1/2}\psi'(ct-y-u\xi)\Big]_{u=1}^{u\to\infty}. \quad (8)$$

Since by definition ψ' vanishes for large negative values of its parameter, its product by $(u-1)^{1/2}$ is also zero for such values and remains zero as u proceeds to infinity. Accordingly the right-hand side of equation (8) vanishes and therefore

$$\left(\nabla^2-\frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right)f=0,$$

where

$$f=\frac{C}{2}\int_1^\infty \psi(ct-y-u\xi)\frac{du}{u(u-1)^{1/2}}, \quad . \quad . \quad . \quad (9)$$

this expression being derived from (3) by substitution of its value $\frac{C}{2u(u-1)^{1/2}}$ for U.

The integral in (9) assumes a neater form if $\sec^2\alpha$ be written in place of u, when it becomes

$$f=C\int_0^{\frac{\pi}{2}} \psi(ct-y-\xi\sec^2\alpha)\,d\alpha. \quad . \quad . \quad . \quad (10)$$

It is convenient to take $1/\pi$ as the value of C so that

$$f=\frac{1}{\pi}\int_0^{\frac{\pi}{2}} \psi(ct-y-\xi\sec^2\alpha)\,d\alpha. \quad . \quad . \quad . \quad (11)$$

It has been shown that f satisfies the wave-equation; it reduces to $\frac{1}{2}\psi(ct-y)$ when $\xi=0$, *i.e.* on the + axis of y. It can be seen that $\frac{\partial f}{\partial x}=0$ at points on the same axis whilst $\frac{\partial f}{\partial y}$ reduces to $-\frac{1}{2}\psi'(ct-y)$.

It follows that if we make $V=f$ in the shadow, and

$V=\psi-f$ outside the shadow,

V will satisfy the wave-equation throughout the whole space. This solution provides for waves approaching the screen but not for any reflected waves, *i. e.* it is correct for a screen which absorbs all the energy of the waves falling on it.

In the cases more usually dealt with allowance must be made for the reflected waves.

To retain symmetry in the notation let the axis of y_1 be in the direction of propagation of the incident waves and the axis of y_2 in the direction of propagation of waves reflected according to the laws of geometrical optics. The reflecting plane is given by $y_1=y_2$.

Write

$$\left.\begin{aligned}\psi_1=\psi(ct-y_1),\quad \psi_2=\psi(ct-y_2),\\ f_1=\frac{1}{\pi}\int_0^{\frac{\pi}{2}}\psi(ct-y_1-\xi_1\sec^2\alpha)d\alpha,\end{aligned}\right\} \quad . \quad . \quad (12)$$

with a similar definition for f_2.

In the case in which the boundary condition to be satisfied on either face of the diffracting half-plane is $V=0$ the required solution is

$$\left.\begin{aligned}&V=\psi_1-f_1-\psi_2+f_2 \text{ in the region A, where the ordinary reflected waves occur,}\\ &V=f_1-f_2 \text{ in the region B, the geometrical shadow,}\\ &\text{and } V=\psi_1-f_1-f_2 \text{ in the remaining region C.}\end{aligned}\right\}(13)$$

The conditions of continuity of V and its differential coefficients are satisfied, the values of V on the boundaries between the regions A and C, B and C being $\psi_1-f_1-\frac{1}{2}\psi_2$ and $\frac{1}{2}\psi_1-f_2$ respectively.

The formula (11) is equivalent to one found as a special case in my paper * on "Diffraction by a Wedge and Kindred Problems." The weakness in the present demonstration lies in the vagueness of the argument which leads to the trial of the integral (3) proposed in the first instance. As a matter of fact, in the integrals which serve for the solution of the problem of diffraction by a reflecting wedge, the factor corresponding with the U of equation (3) is not merely a function of u, it depends on the azimuth.

The proof that f of equation (9) satisfied the fundamental differential equation was based on the condition that the waves had been passing for only a finite time. If this condition is removed then $(u-1)^{1/2}\psi'(ct-y-u\xi)$ does not vanish as $\xi\to\infty$. From the physical point of view the limitation can

* Proc. London Math. Soc. ser. 2, vol. xvi. at the foot of p. 106, $\tan\alpha\sin\frac{\phi}{2}$ being written for $\sinh\frac{u}{2}$.

not be fundamental, and it should be possible to modify the mathematics to allow for an infinite train of waves. The most important case is that of a simple harmonic disturbance, and it is found that in this case our solution reduces to Sommerfeld's and is therefore justified.

If the approaching waves are represented by

$$V=\cos\kappa(ct-y) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (14)$$

we have to investigate

$$f=\frac{1}{\pi}\int_0^{\frac{\pi}{2}}\cos\kappa(ct-y-\xi\sec^2\alpha)d\alpha. \quad . \quad . \quad (15)$$

We have

$$f=\text{Real Part of } \frac{1}{\pi}\int_0^{\frac{\pi}{2}} e^{\iota\kappa(ct-y-\xi\sec^2\alpha)}\,d\alpha$$

$$= \quad ,, \quad ,, \quad \frac{e^{\iota\kappa(ct-y)}}{\pi}\int_0^{\frac{\pi}{2}} e^{-\iota\kappa\xi\sec^2\alpha}\,d\alpha.$$

Now

$$e^{-\iota\kappa\xi\sec^2\alpha}=1-\iota\kappa\int_0^{\xi} e^{-\iota\kappa v\sec^2\alpha}\sec^2\alpha\, dv$$

$$=1-\iota\kappa\int_0^{\xi} e^{-\iota\kappa v}e^{-\iota\kappa v\tan^2\alpha}\sec^2\alpha\, dv.$$

Hence

$$\int_0^{\frac{\pi}{2}} e^{-\iota\kappa\xi\sec\alpha}\,d\alpha=\frac{\pi}{2}-\iota\kappa\int_0^{\xi} e^{-\iota\kappa v}\frac{\sqrt{\pi}}{2\sqrt{\kappa v}}e^{-\iota\pi/4}dv,$$

and, finally,

$$f=\frac{1}{2}\cos\kappa(ct-y)-\frac{1}{2}\sqrt{\frac{\kappa}{\pi}}\int_0^{\xi}\cos\left\{\frac{\pi}{4}+\kappa(ct-y-v)\right\}\frac{dv}{\sqrt{v}}.$$

$$. \quad . \quad (16)$$

As this formula can be identified with that of Sommerfeld as quoted by Mr. Hargreaves *, the demonstration is complete.

The result is of interest, not only as containing the first complete solution of a diffraction problem but also as showing that Fresnel's integrals, devised for an approximate solution of the problem, suffice for the complete one. This aspect of the subject is discussed at length in Drude's 'Optics.'

* The change in the direction of the axis of y should be noted.

XLVII. *Notices respecting New Books.*

An Introductory Treatise on Dynamical Astronomy. By H. C. PLUMMER, M.A. Pp. 343+xix. Price 18*s*. net. Camb. Univ. Press.

THE publication of an English work on dynamical astronomy is a rare event, and it is with more than usual anticipation that we examine Prof. Plummer's treatise; for there can be few branches of science which have suffered so much from the lack of a suitable textbook. The scope of the book is wide, the title "Dynamical Astronomy" being interpreted liberally. About half the pages are concerned with the subject of planetary and lunar perturbations; the remainder treat, amongst other matters, of the determination of orbits, including orbits of double stars and spectroscopic binaries, the libration of the moon, the phenomena of the earth's rotation, and the formulæ of numerical interpolation and quadrature. In each case the subject is developed far beyond an elementary stage, and it is surprising that the author has been able to keep the work within moderate compass. The general design will be especially welcomed by those who have at one time gained some acquaintance with the subject, and desire to revise and extend their knowledge. For the university student also, it will prove a valuable supplement to oral teaching, and assist in systematizing knowledge. Perhaps the reader who is trying to begin the subject unaided will at first be less appreciative. Some parts indeed are well adapted to his needs, and we would especially commend the two chapters on the lunar theory. But in general he may prefer to make his first approach to the subject through the leisurely expositions of Tisserand and Klinkerfues; he would certainly feel himself "hustled" by Prof. Plummer. For a subsequent reading—and the subject needs to be read again and again—the conciseness of the present work is an advantage.

The brevity is partly gained by the entire omission of worked examples; this is referred to in the Preface, and is a deliberate policy. Yet we wish that the author could have departed from his rule in some places at least. Worked examples of the calculation of orbits and of special perturbations are essential for a proper appreciation of the results obtained; and the inexperienced reader will find great difficulty in supplying these for himself. It is not that we lay stress on learning the best form of computation; the labour-saving devices of the computing-bureau are only necessary when large numbers of applications of the formulæ have to be made. But the unassisted reader will fail to provide himself with satisfactory examples, if only because he will inevitably make numerical mistakes in the lengthy computation. On much the same principle the author often leaves his analytical results to speak for themselves, where a few words of comment might have been helpful. Thus a novel theorem due to E. T. Whittaker is given in § 217, but we are left in doubt whether it is inserted solely for its theoretical elegance or is appropriate for practical application. In some parts of the subject we should have preferred a more geometrical treatment; but there may well be divergence of view as to this. The extent to which Prof. Plummer relies on analytical methods may be gauged by the fact that there are only eight diagrams in the book. We need not dwell further on these minor points in which we happen to take a different view from the author; but rather would hasten to express our gratitude to him for a book which we have placed on the shelf reserved for those in most constant use.

A subject on which so many of the great mathematicians of the last century have laboured tends to take a stereotyped form; but the author has succeeded in introducing much freshness of treatment, and he dispels any impression that the subject is played out. Many results are included that are not readily found elsewhere; and good use is made both of modern researches and half-forgotten results of the past. The account of the determination of spectroscopic orbits meets a need of recent growth. The collection of interpolation and numerical integration formulæ in Chapter XXIV. is the best we have seen, though we miss our own particular favourite (the quadrature formula of Darwin, 'Collected Papers,' vol. iv. p. 17), which has yet to find a place in any textbook.

The writing of this treatise must have cost a vast amount of labour, and we congratulate Prof. Plummer on a most successful result, which should aid and stimulate the study of dynamical astronomy in this country.

XLVIII. *Proceedings of Learned Societies.*

GEOLOGICAL SOCIETY.

[Continued from p. 364.]

May 1st, 1918.—Mr. G. W. Lamplugh, F.R.S., President, in the Chair.

Dr. A. Hubert Cox, M.Sc., F.G.S., delivered a Lecture on the Relationship between Geological Structure and Magnetic Disturbance, with especial reference to Leicestershire and the Concealed Coalfield of Nottinghamshire.

Before the Lecture, at the request of the President, Dr. A. Strahan, F.R.S., Director of the Geological Survey, briefly outlined the circumstances that had led to an investigation into a possible connexion between geological structure and magnetic disturbances. The magnetic surveys conducted by Rücker and Thorpe in 1886 and 1891 had proved the existence of certain lines and centres of disturbance, but those authors observed that 'the magnetic indications appear to be quite independent of the disposition of the newer strata,' and he (the speaker) had not been able to detect any obvious connexion with the form and structure of the Palæozoic rocks below. In 1914–15 a new magnetic survey was made by Mr. G. W. Walker, who confirmed the existence of certain areas of disturbance. It was suggested that the effects might be due to concealed masses of iron-ore, and the matter was referred to the Conjoint Board of Scientific Societies, who appointed an Iron-Ores Committee to consider what further steps should be taken. The Committee recommended that attention should be concentrated on certain areas of marked magnetic disturbance, and that a more detailed magnetic survey of these areas, accompanied by a petrological survey and an examination of the magnetic properties of the rocks of the neighbourhood, should be made. He (the speaker) had been approached with a view to the petrological work being undertaken by the Geological Survey, and it had been arranged by the Board of Education, with the consent of H.M. Treasury, that a geologist should be temporarily appointed as a member of the staff for the purposes of the investigation. Dr. Cox had received the appointment, and the lecture which he was about to deliver would show that results of great significance had been obtained by him. The new magnetic observations had been made by Mr. Walker, and the examination of the specimens collected, in regard to their magnetic susceptibility, had been conducted by Prof. Ernest Wilson.

Dr. Cox then described the selected areas, which lay on Lias and Keuper Marl between Melton Mowbray and Nottingham, and in the neighbourhood of Irthlingborough, where the Northampton Sands are being worked as iron-ores. The Middle Lias iron-ores, consisting essentially of limonite, which crop out near Melton Mowbray, have been proved incapable, by reason of their low magnetic susceptibility, of causing disturbances of the magnitudes observed, while the distribution of the disturbances showed no correspondence with the outcrop of the iron-ores. Nor was any other formation among the Secondary rocks found capable of exerting any appreciable influence. It appeared, therefore, that the origin of the magnetic disturbances must be deep-seated.

Investigation showed that the disturbances were arranged along the lines of a system of faults ranging in direction from north-west to nearly west. The faults near Melton Mowbray have not been proved in the Palæozoic rocks, and, so far as their effects on the Secondary rocks are concerned, they would appear to be only minor dislocations. But farther north, near Nottingham, faults which take a parallel course, and probably belong to the same system of faulting as those near Melton Mowbray, are known from evidence obtained in underground workings to have a much greater throw in the Coal Measures than in the Permian and Triassic rocks at the surface. It appears therefore that movement took place along the same lines at more than one period, the earlier and more powerful movement being of post-Carboniferous but pre-Permian age, the later movement being post-Triassic. Accordingly, it is probable that the small dislocations in the Mesozoic rocks indicate the presence of important faults in the underlying Palæozoic.

The faults can only give rise to magnetic disturbances if they are associated with rocks of high magnetic susceptibility. It is known from deep borings that the concealed coalfield of Nottinghamshire extends into Leicestershire, but how far is not known. Deep borings have proved that intrusions of dolerite occur in the Coal Measures at several localities in the south-eastern portion of the concealed coalfield and always, so far as observed, in the immediate vicinity of faults. It has been established that dolerites may exert a considerable magnetic effect; and the susceptibility of those that occur in the Coal Measures is above the general average. Further, no other rocks that are known to occur, or are likely to occur under the area, have susceptibilities as high as the dolerites found in the Coal Measures. These facts suggest the possibility of the occurrence of dolerites intrusive into Coal Measures beneath the Mesozoic rocks of the Melton Mowbray district.

The distribution of the dolerites actually proved, and of those the presence of which is suspected by reason of the magnetic disturbances, appears to be controlled by the faulting. Moreover, whereas the character of the magnetic disturbances is such that it would not be explained by a sill or laccolite faulted down to the north, in the manner demanded by the observed throw of the principal fault, it would be explained by an intrusion that had arisen along the fault-plane. The faulting itself is connected with a change of strike in the concealed Coal Measures, and the incoming of doleritic intrusions in the concealed coalfield, in contrast with their absence from the exposed coalfield, appears to depend upon the changed tectonic features. The change of strike is apparent, but to a less degree, in the Mesozoic rocks which, in the neighbourhood of Melton Mowbray, have suffered a local twist due to the development of an east-and-west anticlinal structure.

In view of the evidence that later movements have, in this district, followed the lines of earlier and more powerful movements, it appears possible and even probable that this post-Jurassic (probably post-Cretaceous) anticline is situated along the line of a more pronounced post-Carboniferous but pre-Permian anticline. In this connexion the isolated position of Charnwood Forest has a considerable significance. The Forest is situated on the prolongation of the east-and-west line of uplift, and just at the point where this uplift crosses the line of the more powerful north-westerly and south-easterly (Charnian) uplift. Where the two lines of uplift cross the elevation attains its maximum, and the oldest rocks appear.

The main line of faulting and of magnetic disturbance is parallel with and on the northern side of the east-and-west anticline, and the faulting is of such a nature that it serves to relieve the folding while accentuating the anticlinal structure. It is possible that this belt of magnetic and geological disturbance marks the southern limit of the concealed coalfield. The results obtained by joint magnetic and geological work have thus served to emphasize the real importance of a structure which, when judged merely from its effects on the surface-rocks, appears to be of only minor importance.

A further series of observations was carried out on the Jurassic iron-ores of the Irthlingborough district of Northamptonshire. The ores occur in the form of a nearly horizontal sheet of weakly susceptible ferrous carbonate partly oxidized to hydrated oxides. They give rise to small magnetic disturbances which are quite capable of detection, and these may be of use in determining the boundaries of the sheets in areas not affected by larger disturbances of deep-seated origin.

The results obtained by the joint magnetic and geological work in the two areas show that this method of investigation may be used to extend our knowledge of the underground structure. It appears also that an extension of the method to other parts of the country would yield information of considerable scientific and economic importance.

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XLIX. *On the Light emitted from a Random Distribution of Luminous Sources.* *By* Lord RAYLEIGH, *O.M., F.R.S.**

RECENT researches have emphasized the importance of a clear comprehension of the operation under various conditions of a group of similar unit sources, or centres, of iso-periodic vibrations, *e. g.* of sound or of light. The sources, supposed to be concentrated in points, may be independently excited (as probably in a soda flame), or they may be constituted of similar small obstacles in an otherwise uniform medium, dispersing plane waves incident upon them. We inquire into an effect, such as the intensity, at a great distance from the cloud, either in a particular direction, or in the average of all directions. For convenience of calculation and statement we shall consider especially sonorous vibrations; but most of the results are equally applicable to electric vibrations, as in light, the additional complication being merely such as arises from the vibrations being transverse to the direction of propagation.

If the centres, supposed to be distributed at random in a region whose three dimensions are all large, are spaced widely enough in relation to the wave-length (λ) to act independently, the question reduces itself to one formerly treated†, for it then becomes merely one of the composition of a large

* Communicated by the Author.

† Phil. Mag. vol. x. p. 73 (1880); Scientific Papers, vol. i. p. 491. For another method see 'Theory of Sound,' 2nd ed. § 42*a*, and for a more complete theory K. Pearson's Math. Contributions to the Theory of Evolution, XV, Dulau, London.

number (n) of unit vibrations of arbitrary phases. It is known that the "expectation" of intensity in any direction is n times that due to a single centre, or (as we may say) is equal to n. The word "expectation" is here used in the technical sense to represent the mean of a large number of independent trials, or combinations, in each of which the phases are redistributed at random. It is important to remember that it is infinitely improbable that the expectation will be confirmed in a single trial, however large n may be. Thus in a single combination of many vibrations of arbitrary phase there is about an even chance that the intensity will be less than $\cdot 7n$. The general formula is that the probability of an *amplitude* between r and $r+dr$ is

$$\frac{2}{n}e^{-r^2/n}\,r\,dr=\frac{1}{n}e^{-\mathrm{I}/n}\,d\mathrm{I}, \quad . \quad . \quad . \quad . \quad . \quad (1)$$

if I denote the *intensity* *.

As regards the "expectation" of intensity merely, the question is very simple. If θ, θ', θ'' .... be the n individual phases, the expectation is

$$\int_0^{2\pi}\int_0^{2\pi}\int_0^{2\pi}\cdots\frac{d\theta}{2\pi}\,\frac{d\theta'}{2\pi}\,\frac{d\theta''}{2\pi}\cdots\left[(\cos\theta+\cos\theta'+\ldots.)^2+(\sin\theta+\sin\theta'+\ldots.)^2\right].$$

Effecting the integration with respect to θ, we have

$$\int_0^{2\pi}\int_0^{2\pi}\cdots\frac{d\theta'}{2\pi}\,\frac{d\theta''}{2\pi}\cdots\left[1+(\cos\theta'+\cos\theta''+\ldots.)^2+(\sin\theta'+\sin\theta''+\ldots.)^2\right];$$

and when we continue the process over all the n phases we get finally

$$\text{Expectation of Intensity}=n.$$

The same result follows of course from (1). The "expectation" is

$$\int_0^{\infty}e^{-\mathrm{I}/n}\,\mathrm{I}.d\mathrm{I}/n=n. \quad . \quad . \quad . \quad . \quad . \quad (2)$$

But if we are not to expect any particular intensity when a large number of vibrations of unit amplitude and arbitrary

* An interesting example of variable intensity when phases are at random is afforded by the observations of De Haas (Amsterdam Proceedings, vol. xx. p. 1278 (1918)) on the granular structure of the field when a *corona* is formed from homogeneous light. The results of various combinations are exhibited to the eye simultaneously.

phase are combined, what precisely is the significance to be attached to this result? As has already been suggested, we must look to what is likely to happen when we have to do with a large number m of independent trials, in each of which the n phases are redistributed at random. By (1) the chance of the separate intensities $I_1, I_2, \dots I_m$, lying between $I_1 + dI_1$, $I_2 + dI_2$, &c. is

$$n^{-m}e^{-(I_1+I_2+\dots)/n}dI_1\,dI_2\dots dI_m\,;$$

and we may inquire what is altogether the chance of the sum of intensities, represented by J, lying between J and $J+dJ$. Over the range concerned the factor $e^{-J/n}$ may be treated as constant, and so the question is reduced to finding the value of

$$\iint\dots\, dI_1\,dI_2\dots dI_m$$

under the condition that $I_1+I_2+\dots$ lies between J and $J+dJ$. This is *

$$\frac{J^{m-1}}{(m-1)\,!}dJ\,;$$

so that the chance of $I_1+I_2+\dots$ lying between J and $J+dJ$ is

$$\frac{e^{-J/n}J^{m-1}\,dJ}{n^m\,.\,(m-1)\,!}\,; \qquad (3)$$

or, if we employ the mean value of the I's instead of the sum, the chance of the mean, viz. $(I_1+I_2+\dots)/m$, lying between K and $K+dK$ is

$$\frac{e^{-mK/n}\,.\,m^{m+1}K^{m-1}\,dK}{n^m\,.\,m\,!}. \qquad (4)$$

We may compare this with the corresponding expression when $m=1$, where we have to do with a single I, to which K then reduces. The ratio

$$R=(4):(1)=\frac{e^{-(m-1)K/n}m^{m+1}K^{m-1}}{n^{m-1}\,.\,m\,!}. \qquad (5)$$

When we treat m as very large, we may take

$$m\,!=m^m\sqrt{(2\pi m)}\,.\,e^{-m},$$

so that (5) becomes

$$\frac{e\sqrt{m}}{\sqrt{(2\pi)}}\left\{\frac{e^{-(K/n-1)}K}{n}\right\}^{m-1}. \qquad (6)$$

* See for example Todhunter's Int. Calc. § 272.

If in (6) $K=n$ absolutely, the second factor is unity, and since the first factor increases indefinitely with m, there is a concentration of probability upon the value n, as compared with what obtains for a single combination.

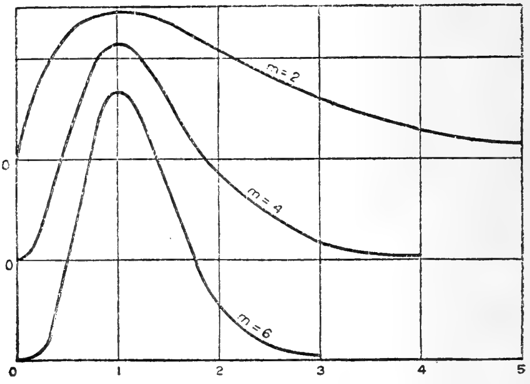
In general we have to consider what becomes of

$$\sqrt{m}\,.\{x\,e^{1-x}\}^{m-1}, \quad . \quad . \quad . \quad . \quad . \quad . \quad (7)$$

when $m=\infty$, and x, written for K/n, is positive. Here $x\,e^{1-x}$ vanishes when $x=0$ and when $x=\infty$, and it has but one maximum when $x=1$, $x\,e^{1-x}=1$. We conclude that $x\,e^{1-x}$ is a positive quantity, in general less than unity. The ratio of consecutive values when m in (7) increases to $m+1$ is $x\,e^{1-x}\sqrt{(1+1/m)}$, and thus when $m=\infty$, (7) diminishes without limit, unless $x=1$ absolutely. Ultimately there is no probability of any mean value K which is not infinitely near the value n.

Fig. 1 gives a plot of R in (5) as a function of x, or K/n, for $m=2$, 4, 6. It will be observed that for $m > 2$, $dR/dx=0$ when $x=0$, but that for $m=2$, $dR/dx=4$.

Fig. 1.



The corresponding question for J may be worth a moment's notice. We have

$$R'=(3):(1)=\frac{mJ^{m-1}}{n^{m-1}\,.\,m\,!}; \quad . \quad . \quad . \quad . \quad (8)$$

so that R' goes to zero as m increases, if J be comparable with n, as might have been expected.

It must not be overlooked that when the random distribution of phases is due to a random spatial distribution of centres, it fails to satisfy strictly the requirement that all the centres act independently, for some of them will lie at distances from nearest neighbours less than the number of wave-lengths necessary for approximate independence. The simple conditions just discussed are thus an ideal, approached only when the spacing is very open.

We have now to consider how the question is affected when we abandon the restriction that the spacing of the unit centres is very open. The work to be done at each centre then depends not only upon the pressure due to itself but also upon that due to not too distant neighbours. Beginning with a single source, we may take as the velocity-potential

$$\phi = -\frac{\cos k(at-r)}{4\pi r}, \quad . \quad . \quad . \quad . \quad . \quad . \quad (9)$$

where a is the velocity of propagation, $k=2\pi/\lambda$, and r is the distance from the centre. The rate of passage of fluid across the sphere of radius r is

$$4\pi r^2 d\phi/dr = \cos k(at-r) - kr\sin k(at-r). \quad . \quad (10)$$

If δp denote the variable part of the pressure at the same time and place, and ρ be the density,

$$\delta p = -\rho\frac{d\phi}{dt} = -\frac{\rho ka \sin k(at-r)}{4\pi r}. \quad . \quad . \quad (11)$$

The rate at which work (W) has to be done is given by

$$\frac{dW}{dt} = \delta p \, . \, 4\pi r^2 \frac{d\phi}{dr} = \frac{\rho ka \sin k(at-r)}{4\pi r} \times [kr\sin k(at-r) - \cos k(at-r)], \quad . \quad . \quad (12)$$

of which the mean value depends upon the first term only. In the long run

$$W/t = \rho k^2 a/8\pi. \quad , \quad . \quad . \quad . \quad . \quad . \quad (13)$$

It is to be observed that although the pressure is infinite at the source, the work done there is nevertheless finite on account of the pressure being in quadrature with the principal part of the rate of total flow expressed in (10).

When there are two unit sources distant D from one another and in the same initial phase, the potentials may be taken to be

$$\phi = -\frac{\cos k(at-r)}{4\pi r}, \qquad \psi = -\frac{\cos k(at-r')}{4\pi r'}. \quad . \quad . \quad (11)$$

At the first source where $r=0$

$$4\pi r^2\, d\phi/dr = \cos kat - kr \sin kat,$$

$$\frac{d\phi}{dt}+\frac{d\psi}{dt}=\frac{ka\sin kat}{4\pi r}+\frac{ka}{4\pi\mathrm{D}}\sin k(at-\mathrm{D}).$$

The work done by the source at $r=0$ is accordingly proportional to

$$1+\frac{\sin k\mathrm{D}}{k\mathrm{D}}, \quad \ldots \ldots \quad (12)$$

and an equal amount of work is done by the source at $r'=0$. If D be infinitely great, the sources act independently, and thus the scale of measurement in (12) is such that unity represents the work done by each source when isolated. If $\mathrm{D}=0$, the work done by each source is doubled, and the sources become equivalent to one of doubled magnitude.

If D be equal to $\frac{1}{2}\lambda$, or to any multiple thereof, $\sin k\mathrm{D}=0$, and we see from (12) that the work done by each source is unaffected by the presence of the other. This conclusion may be generalized. If any number (n) of equal sources in the same phase be arranged in (say a vertical) line so that the distance between immediate neighbours is $\frac{1}{2}\lambda$, the work done by each is the same as if the others did not exist. The whole work accordingly is n, whereas the work to be done by a single source of magnitude n would be n^2. Thus if sound be wanted only in the horizontal plane where there is agreement of phase, the distribution into n parts effects an economy in the proportion of $n:1$.

A similar calculation would apply when the initial phases differ, but we will now take up the problem in a more general form where there are any number (n) of unit sources, and by another method *. The various centres are situated at points finitely distant from the origin O. The velocity-potential of one of these at (x, y, z), estimated at any point Q, is

$$\phi=-\frac{\cos(pt+\epsilon-k\mathrm{R})}{4\pi\mathrm{R}}, \quad \ldots \ldots \quad (13)$$

where R is the distance between Q and (x, y, z). At a great distance from the origin we may identify R in the denominator with OQ, or R_0; while under the cosine we write

$$\mathrm{R}=\mathrm{R}_0-(lx+my+nz), \quad \ldots \ldots \quad (14)$$

* "On the Production and Distribution of Sound," Phil. Mag. vol. vi. p. 289 (1903); Scientific Papers, vol. v. p. 136.

l, m, n being the direction cosines of OQ. On the whole

$$-4\pi R_0 \phi = \Sigma \cos \{pt + \epsilon - kR_0 + k(lx + my + nz)\}, \quad . \quad (15)$$

in which R_0 is a constant for all the sources, but ϵ, x, y, z vary from one source to another. The *intensity* in the direction (l, m, n) is thus represented by

$$[\Sigma \cos \{\epsilon + k(lx + my + nz)\}]^2 + [\Sigma \sin \{\epsilon + k(lx + my + nz)\}]^2,$$

or by

$$n + 2\Sigma \cos [\epsilon_1 - \epsilon_2 + k\{l(x_1 - x_2) + m(y_1 - y_2) + n(z_1 - z_2)\}], \quad (16)$$

the second summation being for all the $\frac{1}{2}n(n-1)$ pairs of sources. In order to find the work done we have now to integrate (16) over angular space.

It will suffice if we effect the integration for the specimen term; and we shall do this most easily if we take the line through the points (x_1, y_1, z_1), (x_2, y_2, z_2) as axis of reference, the distance between them being denoted by D. If (l, m, n) make an angle with D whose cosine is μ,

$$D\mu = l(x_1 - x_2) + m(y_1 - y_2) + n(z_1 - z_2), \quad . \quad (17)^*$$

and the value of the specimen term is

$$\int_{-1}^{+1} \cos (\epsilon_1 - \epsilon_2 + kD\mu) d\mu,$$

that is

$$\frac{2 \sin kD \cos (\epsilon_1 - \epsilon_2)}{kD}. \quad . \quad . \quad . \quad . \quad . \quad (18)$$

The mean value of (16) over angular space is thus

$$n + 2\Sigma \frac{\sin kD \cos (\epsilon_1 - \epsilon_2)}{kD}, \quad . \quad . \quad . \quad . \quad (19)$$

where ϵ_1, ϵ_2 refer to any pair of sources and D denotes the distance between them. If all the sources are in the same initial phase, $\cos (\epsilon_1 - \epsilon_2) = 1$. If the distance between every pair of sources is a multiple of $\frac{1}{2}\lambda$, $\sin kD = 0$, and (19) reduces to its first term.

We fall back upon a former particular case if we suppose that there are only two sources and that they are in the same phase.

* In the paper referred to, equation (19), μ was inadvertently used in two senses.

If the question of the phases of the two sources be left open, (19) gives

$$2+2\cos(\epsilon_1-\epsilon_2)\frac{\sin k\mathrm{D}}{k\mathrm{D}} \quad . \quad . \quad . \quad . \quad . \quad . \qquad (20)$$

If D be small, this reduces to

$$2+2\cos(\epsilon_1-\epsilon_2),$$

which is zero if the sources be in opposite phases, and is equal to 4 if the phases be the same.

If in (20) the phases are 90° apart, the cosine vanishes. The work done is then simply the double of what would be done by either source acting alone, and this whatever the distance D may be. If this conclusion appear paradoxical, it may be illustrated by considering the case where D is very small. Then

$$-4\pi\mathrm{R}_0\phi=\cos(pt+\epsilon-k\mathrm{R}_0)+\cos(pt+\epsilon\pm\tfrac{1}{2}\pi-k\mathrm{R}_0)$$
$$=\sqrt{2}\,.\cos(pt+\epsilon\pm\tfrac{1}{4}\pi-\mathrm{R}_0),$$

representing a single source of strength $\sqrt{2}$, giving intensity 2 simply.

We have seen that the effect of a number n of unit sources depends upon the initial phases and the spatial distribution, and this not merely in a specified direction, but in the mean of all directions, representing the work done. We have now to consider what happens when the initial phases are at random, or when the spatial distribution is at random within a limited region. Obviously we cannot say what the effect will be in any particular case. But we may inquire what is the expectation of intensity, that is the mean intensity in a great number of separate trials, in each of which there is an independent random distribution.

The question is simplest when the individual initial phases are at random in separate trials, and the result is then the same whether the spatial distribution be at random or prescribed. For the mean value of every single term under the sign of summation in (19) is then zero, D meanwhile being constant for a given pair of sources, while

$$\int_0^{2\pi}\cos(\epsilon_1-\epsilon_2)\frac{d\epsilon_2}{2\pi}=0.$$

The mean intensity, whether reckoned in all directions, or even in a specified direction (16), reduces to n simply.

If the sources are all in the same phase, or even if each individual source retains its phase, $\cos(\epsilon_1-\epsilon_2)$ in (19) remains constant in the various trials for each pair, and we have to deal with the mean value of $\sin k\mathrm{D} \div k\mathrm{D}$ when the spatial distribution is at random. We may begin by supposing two sources constrained to lie upon a straight line of limited length l, where, however, l includes a very large number of wave-lengths (λ).

If the first source occupies a position sufficiently remote from the ends of the line, so that the two parts on either side (l_1 and l_2) are large multiples of λ, the mean required, represented by

$$\frac{l_1}{l}\int_0^{l_1} \frac{\sin k\mathrm{D}}{k\mathrm{D}}\frac{d\mathrm{D}}{l_1} + \frac{l_2}{l}\int_0^{l_2} \frac{\sin k\mathrm{D}}{k\mathrm{D}}\frac{d\mathrm{D}}{l_2}, \quad . \quad . \quad . \quad (21)$$

may be identified with π/kl, since both upper limits may be treated as infinite. Moreover, π/kl may be regarded as evanescent, kl being by supposition a large quantity.

So far positions of the first source near the ends of the line have been excluded. If the neglect of these positions can be justified, (20) reduces to 2 simply.

It is not difficult to see that the suggested simplification is admissible under the conditions contemplated. If x, x' be the distances of the two sources from one end of the line, the question is as to the value of

$$\int_0^l \frac{dx}{l}\int_0^l \frac{dx'}{l}\frac{\sin k(x'-x)}{k(x'-x)}, \quad . \quad . \quad . \quad . \quad . \quad (22)$$

where the integration with respect to x' may be taken first. Let X denote a length large in comparison with λ, but at the same time small in comparison with l. If x lie between X and $l-\mathrm{X}$, the integral with respect to x' may be identified with π/kl, and neglected, as we have seen. We have still to include the ranges from $x=0$ to $x=\mathrm{X}$, and from $x=l-\mathrm{X}$ to $x=l$, of which it suffices to consider the former. The range for x' may be divided into two parts, from 0 to x, and from x to l. For the latter we may take

$$\int_x^l \frac{dx'}{l}\frac{\sin k(x'-x)}{k(x'-x)} = \frac{\pi}{2kl},$$

so that this part yields finally after integration with respect to x,

$$\frac{\mathrm{X}}{l}\cdot\frac{\pi}{2kl}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (23)$$

As regards the former part, we observe that since $\theta^{-1}\sin\theta$ can never exceed unity,

$$\int_0^x \frac{dx'}{l}\,\frac{\sin k(x'-x)}{k(x'-x)} < \frac{x}{l}, \quad . \quad . \quad . \quad . \quad . \quad (24)$$

in which again $x<\mathrm{X}$. The result of the second integration leaves us with a quantity less than X^2/l^2. The anomalous part, both ends included, is less than

$$\frac{2\mathrm{X}}{l}\left(\frac{\mathrm{X}}{l}+\frac{\pi}{2kl}\right), \quad . \quad . \quad . \quad . \quad . \quad (25)$$

which is small in comparison with the principal part, of the order π/kl and itself negligible. We conclude that here again the mean intensity in a great number of trials is 2 simply. It may be remarked that this would not apply to the mean intensity in a specified direction, as we may see from the case where the initial phases are the same. In a direction perpendicular to the line on which the sources lie, the phases on arrival are always in agreement, and the intensity is 4, wherever upon the line the sources may be situated. The conclusion involves the mean in *all* directions, as well as the mean of a large number of trials.

Under a certain restriction this argument may be extended to a large number n of unit sources, since it applies to every term under the summation in (19). But inasmuch as the evanescence is but approximate, we have to consider what may happen when n is exceedingly great. The number of terms is of order n^2, so that the question arises whether $n^2\pi/kl$ can be neglected in comparison with n. The ratio is of the order $n\lambda/l$, and it cannot be neglected unless the mean distance of consecutive sources is much greater than λ. It is only under this restriction that we can assert the reduction of the mean intensity to the value n when the initial phases are not at random.

The next problem proposed is the application of (19) when the n sources are distributed at random over the volume of a *sphere* of radius R. In this case the distinction between the mean in one direction and in the mean of all directions disappears. If for the moment we limit our attention to a single pair of sources, the chance of the first source lying in the element of volume $d\mathrm{V}$ is $d\mathrm{V}/\mathrm{V}$, and similarly of the second source lying in $d\mathrm{V}'$ is $d\mathrm{V}'/\mathrm{V}$. As the individual sources may be interchanged, the chance of the pair occupying the elements dV, dV' is $2dV\,dV'/V^2$, so that from the second part of (19) we get for a single pair the expectation of intensity

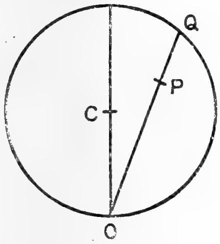
$$4\iint \frac{\sin kr}{kr}\frac{dV}{V}\frac{dV'}{V},$$

and for the $\frac{1}{2}n(n-1)$ pairs

$$\frac{2n(n-1)}{V^2}\iint \frac{\sin kr}{kr} dV\,dV'. \quad . \quad . \quad . \quad (26)$$

Here V is the whole volume of the sphere, viz. $\frac{4}{3}\pi R^3$, and r is written in place of D. The function of r may be regarded as a kind of potential, so that the integral in (26) represents the work required to separate thoroughly every pair of elements. As in 'Theory of Sound,' § 302, we may estimate this by successive removals to infinity of outer thin shells of thickness dR. The first step is the calculation of the potential at O, a point on the surface of the sphere.

Fig. 2.



The polar element of volume at P is $r^2 \sin\theta\, d\omega\, d\theta\, dr$, where $r=$OP, $\theta=$angle COP. The integration with respect to ω will merely introduce the factor 2π. For the integration with regard to r, we have

$$\int_0^r \frac{\sin kr}{kr} r^2 dr = \frac{\sin kr + kr\cos kr}{k^3},$$

r now standing for OQ. In terms of μ $(=\cos\theta)$, $r=2R\mu$, and we have next to integrate with respect to μ. We get

$$\int_0^1 \frac{\sin kr - kr\cos kr}{k^3} d\mu = \frac{1-\cos 2kR - kR\sin 2kR}{k^4 R},$$

which, multiplied by 2π, now expresses the potential at O.

This potential is next to be multiplied by $4\pi R^2 dR$ and integrated from 0 to R. We find

$$\iint \frac{\sin kr}{kr} dV\, dV' = \frac{8\pi^2}{k^6}(\sin kR - kR\cos kR)^2. \quad . \quad (27)$$

We have now to divide by V^2, or $16\pi^2 R^6/9$; and finally we get

$$(19) = n + \frac{9n(n-1)}{k^6 R^6}(\sin kR - kR\cos kR)^2, \quad . \quad (28)^*$$

where kR will now be regarded as very large. When n is moderate, or at any rate does not exceed k^3R^3, the second term is relatively negligible, that is reduction occurs to n simply, provided n be not higher than of order R^3/λ^3, corresponding to one source for each cubic wave-length †. But evidently n may be so great that this reduction fails, unless otherwise justified by a random distribution of initial phases.

At the other extreme of an altogether preponderant n, the second term in (19) dominates the first, and we get in the case of constant initial phases and a very large kR,

$$(19) = \frac{9n^2\cos^2 kR}{k^4 R^4}. \quad . \quad . \quad . \quad . \quad . \quad (29)$$

Under the suppositions hitherto made of a random spatial distribution within the sphere (R), and of uniformity of initial phases, there is no escape from the conclusion that the reduction to the simple value n fails when n is great enough. Nevertheless, there is a sense in which the reduction may take place, and the point is of importance, especially in the application to the dispersal of primary waves by a cloud of small obstacles. In order better to understand the significance of the term in n^2, let us calculate the intensity due to an absolutely uniform distribution of source of total amount n over the spherical volume. Since there is complete symmetry, it suffices to consider a a single specified direction which we take as axis of z. As in (15), we have

$$-4\pi R_0\,\phi = \frac{n e^{i(pt-kR_0)}}{V}\iiint e^{ikz}\,dx\,dy\,dz, \quad . \quad . \quad (30)$$

as the symbolical expression for the velocity-potential, from

* We may confirm (28) by supposing kR very *small*, when the right-hand member reduces to n^2.

† The number of molecules per cubic wave-length in a gas under standard conditions is of the order of a million.

which finally the imaginary part is to be rejected. The integral over the sphere is easily evaluated, either as it stands, or with introduction of polar coordinates (r, θ, ω) which will afterwards be required. Thus with μ written for $\cos\theta$,

$$\iiint e^{ikz}dx\,dy\,dz = 2\pi\int_0^{\mathrm{R}}\int_{-1}^{+1} e^{ikr\mu}r^2\,dr\,d\mu$$

$$= \frac{4\pi}{k}\int_0^{\mathrm{R}} \sin kr\,.\,r\,dr = \frac{4\pi}{k^3}(\sin k\mathrm{R} - k\mathrm{R}\cos k\mathrm{R}). \quad . \quad (31)$$

Accordingly

$$-4\pi\mathrm{R}_0\,\phi = \frac{3n}{k^3\mathrm{R}^3}(\sin k\mathrm{R} - k\mathrm{R}\cos k\mathrm{R}), \quad . \quad . \quad (32)$$

reducing to n simply when $k\mathrm{R}$ is very small. The intensity due to the uniform distribution is thus

$$\frac{9n^2}{k^6\mathrm{R}^6}(\sin k\mathrm{R} - k\mathrm{R}\cos k\mathrm{R})^2, \quad . \quad . \quad . \quad . \quad (33)$$

exactly the n^2 term of (28). The distinction between (28) and (32), at least when $k\mathrm{R}$ is very great, has its origin in the circumstance that in the first case the n separate centres, however numerous, are discrete and scattered at random, while in the second case the distribution of the same total is uniform and continuous.

When we examine more attentively the composition of the velocity-potential ϕ in (30), we recognize that it may be regarded as originating at the *surface* of the sphere R. Along any line parallel to z, the phase varies uniformly, so that every complete cycle occupying a length λ contributes nothing. Any contribution which the entire chord may make depends upon the immediate neighbourhood of the ends, where incomplete cycles may stand over. And, since this is true of every chord parallel to z, we may infer that the total depends upon the manner in which the volume terminates, viz. upon the surface. At this rate the n^2 term in (28) must be regarded as due to the surface of the sphere, and if we limit attention to what originates in the interior this term disappears, and ($k\mathrm{R}$ being sufficiently large) (19) reduces to n.

When we speak of an effect being due to the *surface*, we can only mean the discontinuity of distribution which occurs there, and the best test is the consideration of what happens when the discontinuity is eased off. Let us then in the integration with respect to r in (31) extend the range beyond

R to R′ with introduction of a factor decreasing from unity (the value from 0 to R), as we pass outwards from R to R′. The form of the factor is largely a matter of mathematical convenience.

As an example we may take $e^{-h'(r-R)}$, or $e^{-hk(r-R)}$, which is equal to unity when $r=R$ and diminishes from R to R′. The complete integral (31) is now

$$\frac{4\pi}{k}\int_0^R \sin kr\,.\,r\,dr+\frac{4\pi}{k}\int_R^{R'} e^{-hk(r-R)}\sin kr\,.\,r\,dr. \qquad (34)$$

From the second integral we may extract the constant factor e^{hkR}, and if we then treat $\sin kr$ as the imaginary part of e^{ikr}, we have to evaluate

$$\int_R^{R'} e^{(i-h)kr}r\,dr.$$

We thus obtain for (34)

$$\frac{4\pi}{k^3}(\sin kR-kR\cos kR)$$
$$-\frac{4\pi e^{-h'(R'-R)}}{k^3(1+h^2)^2}\left[\cos kR'\{(h^2+1)kR'+2h\}\right.$$
$$\left.+\sin kR'\{(h^2+1)hkR'+h^2-1\}\right]$$
$$+\frac{4\pi}{k^3(1+h^2)^2}\left[\cos kR\{(h^2+1)kR+2h\}\right.$$
$$\left.+\sin kR\{(h^2+1)hkR+h^2-1\}\right]. \qquad (35)$$

When we combine the first and third parts, in which R′ does not appear, we get

$$\frac{4\pi}{k^3(1+h^2)^2}\left[\cos kR\{2h-h^2(h^2+1)kR\}\right.$$
$$\left.+\sin kR\{h^4+3h^2+h(h^2+1)kR\}\right]. \qquad (36)$$

The first part of (35), representing the effect due to the sphere R suddenly terminated, is of order kR; and our object is to ascertain whether by suitable choice of h and R′ we can secure the relative annulment of (35). As regards (36), it suffices to suppose h small enough. In the second part of (35) the principal term is of relative order $(R'/R)e^{-hk(R'-R)}$ and can be annulled by sufficiently increasing R′, however small h may be.

Suppose, to take a numerical example, that $h=\frac{1}{1000}$, and that $e^{-hk(R'-R)}$ is also $\frac{1}{1000}$. Then

$$R'-R=\frac{3\lambda}{2\pi h\log_{10}e}=\frac{1\cdot 1\lambda}{h}=1100\lambda.$$

With such a value of R′−R the factor R′/R may be disregarded *.

It appears then that it is quite legitimate to regard the intensity due to the simple sphere, expressed in (33), as a surface effect; and this conclusion may be extended to the corresponding term involving n^2 in (28), relating to discrete centres scattered at random.

This extension being important, it may be well to illustrate it further. Returning to the consideration of n sources in the same initial phase distributed at random along a limited straight line, let us inquire what is to be expected at a distant point along the line produced. The first question which suggests itself is—Are the phases on arrival distributed at random? Not in all cases, but only when the limited line contains exactly an integral number of wave-lengths. Then the phases on arrival are absolutely at random over the whole period, and accordingly the expectation of intensity is n precisely. If, however, there be a fractional part of a wave-length outstanding, the arrival phases are no longer absolutely at random, and the conclusion that the expectation of intensity is n simply cannot be maintained. Suppose further that n is so great that the average distance between consecutive sources is a very small fraction of a wave-length. The conclusion that when an exact number of wave-lengths is included the expectation is n remains undisturbed, and this although the effect due to any small part, supposed to act alone, is proportional to n^2. But the influence of any outstanding fraction of a wave-length is now of increased importance. If we do not look too minutely, the distribution of sources is approximately uniform. If it were completely so, the whole intensity would be attributable to the fractions at the ends †, and would be proportional to n^2. In general we may expect a part proportional to n^2 due to the ends and another part proportional to n due to incomplete uniformity of distribution over the whole length. When n is small the latter part preponderates, but when n is great the situation is reversed, unless the number of wave-lengths included be very nearly integral. And it is apparent that the n^2 part has its origin in the discontinuity involved in the sharp limitation of the line, and may be got rid of by a tapering away of the terminal distribution.

Similar ideas are applicable to a random distribution in three dimensions over a volume, such as a sphere, which may be regarded as composed of chords parallel to the direction

* The application to light is here especially in view.

† It is indifferent how the fraction is divided between the two ends.

in which the effect is to be estimated. The n^2 term corresponds to what would be due to a continuous uniform distribution over the volume of the same total source, and it may be regarded as due to the discontinuity at the surface. In addition there is a term in n, due to the lack of complete uniformity of distribution and issuing from every part of the interior.

Thus far we have been considering the operation of given unit sources, by which in the case of sound is meant centres where a given periodic introduction (and abstraction) of fluid is imposed. We now pass to the problem of equal small obstacles distributed at random and under the influence of primary plane waves. It is easy to recognize that these obstacles act as secondary sources, but it is not so obvious that the strength of each source may be treated as given, without regard to the action of neighbours. I apprehend, however, that this assumption is legitimate; in the case of aerial waves it may be justified by a calculation upon the lines of 'Theory of Sound,' § 335. For this purpose we may suppose the density σ of the gas to be unchanged at the obstacles, while the compressibility is altered from m to m', so that the secondary disturbance issuing from each obstacle is symmetrical, of zero order in spherical harmonics. The expressions for the primary waves and of the disturbance inside the spherical obstacle under consideration remain as if the obstacle were isolated. But for the secondary disturbance external to the obstacle we must include also that due to neighbours. On forming the conditions to be satisfied at the surface of the sphere, expressing the equality on the two sides of pressure (or potential) and of radial velocity, we find that when the radii are small enough, the obstacle acts as a source whose strength is independent of neighbours.

The operation of a cloud of similar particles may now be deduced without much difficulty from what has already been proved. We suppose that the individual particles are so small that the cloud has no sensible effect upon the progress of the primary waves. Each particle then acts as a source of given strength. But the initial phase for the various particles is not constant, being dependent upon the situation along the primary rays. This is, in fact, the only new feature of which we have to take account.

Perhaps the most important difference thence arising is that there is no longer equality of radiation in various directions, even from a spherical cloud, and that, whatever may be the shape of the cloud, the radiation in the direction of the primary rays produced is specially favoured. In this direction any retardation along the primary ray is exactly compensated by a corresponding acceleration along the secondary ray, so that on arrival at a distant point the phases due to all parts are the same. But, except in this direction and in others approximating to it, the argument that the effect may be attributed to the *surface* still applies. If in a continuous uniform distribution we take chords in the direction, for example, of either the incident or the scattered rays, we see as before that the effect of any chord depends entirely on how it terminates *. In forming an integral analogous to that of (30), in addition to the factor e^{ikz} expressive of retardation along the secondary ray, we must include another in respect of the primary ray. If the direction cosines of the latter be α, β, γ, the factor in question is $e^{ik(\alpha x+\beta y+\gamma z)}$, γ being -1 when the directions of the primary and secondary rays are the same. The complete exponent in the phase-factor is thus

$$ik\{\alpha x+\beta y+(\gamma+1)z\} = ik\sqrt{(2+2\gamma)}\,.\,\frac{\alpha x+\beta y+(\gamma+1)z}{\sqrt{\{\alpha^2+\beta^2+(\gamma+1)^2\}}}\,.$$

The fraction on the right represents merely a new coordinate (ζ), measured in a direction bisecting the angle between the primary and secondary rays, so that the phase-factor may be written $e^{i\sqrt{(2+2\gamma)}\,.\,k\zeta}$, γ being the cosine of the angle (χ) between the rays. In integrating for the sphere the only change required in the integrand is the substitution of $2k\cos\frac{1}{2}\chi$ for k. With this alteration equations (31), (32), (33) are still applicable. When the secondary ray is perpendicular to the primary,

$$2k\cos\tfrac{1}{2}\chi=\sqrt{2}\,.\,k.$$

In order to find the mean intensity in all directions we have to integrate (33) over angular space and divide the result by 4π. It may be remarked that although $\cos^6\frac{1}{2}\chi$ appears in the denominator of (33), it is compensated when

* It may be remarked that the same argument applies to the particles of a crystal forming a regular space lattice. If the wave-length be large in comparison with the molecular distance, no light can be scattered from the interior of such a body. For X rays this condition is not satisfied, and regular reflexions from the interior are possible. Comparison may be made with the behaviour of a grating referred to below.

$\cos\frac{1}{2}\chi=0$ by a similar factor in the numerator. In the integration with respect to χ

$$\sin\chi\,d\chi=-4\cos\tfrac{1}{2}\chi\,.\,d(\cos\tfrac{1}{2}\chi).$$

If we write ψ for $2k\mathrm{R}\cos\frac{1}{2}\chi$, the mean sought may be written

$$\frac{9n^2}{2k^2\mathrm{R}^2}\int\frac{(\sin\psi-\psi\cos\psi)^2}{\psi^5}\,d\psi, \quad . \quad . \quad . \quad (37)$$

the range for ψ being from 0 to $2k\mathrm{R}$. The integration can be effected by "parts." We have

$$\int\frac{(\sin\psi-\psi\cos\psi)^2}{\psi^5}\,d\psi=-\frac{\sin^2\psi-2\psi\sin\psi\cos\psi+\psi^2}{4\psi^4}. \quad . \quad . \quad (38)$$

When ψ is small, the expression on the right becomes

$$-\frac{1}{4}+\frac{\psi^2}{18},$$

so that the integral between 0 and ψ is $\psi^2/18$ simply. In general, the mean intensity is

$$\frac{9n^2}{8k^2\mathrm{R}^2}\,\frac{2\psi\sin\psi\cos\psi-\sin^2\psi-\psi^2+\psi^4}{\psi^4}, \quad . \quad (39)$$

in which ψ stands for $2k\mathrm{R}$.

That the intensity, whether in one direction or in the mean of all directions, should be proportional to n^2 is, of course, what was to be expected. And, since the effect is here a surface effect, it may be identified with the ordinary surface reflexion which occurs at a sudden transition between two media of slightly differing refrangibilities, and is proportional to the square of that difference. If, as in a former problem, we suppose the discontinuity of the transition to be eased off, this reflexion may be attenuated to any extent until finally there is no dispersed wave at all *.

When we pass from the continuous uniform distribution to the random distribution of n discrete and very small obstacles, the term in n^2 representing reflexion from the surface remains, and is now supplemented by the term in n, due to irregular distribution in the interior. It is the latter part only with which we are concerned in a question such as that of the blue of the sky.

It must never be forgotten that it is the "expectation" of

* *Conf.* Proc. Lond. Math. Soc. vol. xi. p. 51 (1880); Scientific Papers, vol. i. p. 460.

intensity which is proved to be n. In any particular arrangement of particles the intensity may be anything from 0 to n^2. But in the application to a gas dispersing light, the motion of the particles ensures that a random redistribution of phases takes place any number of times during an interval of time less than any which the eye could appreciate, so that in ordinary observation we are concerned only with what is called the expectation.

It is hoped that the explanations and calculations here given may help to remove the difficulties which have been felt in connexion with this subject. The main point would seem to be the interpretation of the n^2 term as representing the surface reflexion when a cloud is supposed to be abruptly terminated. For myself, I have always regarded the light internally dispersed as proportional to n, even when n is very great, though it may have been rather by instinct than on sufficiently reasoned grounds. Any other view would appear to be inconsistent with the results of my son's recent laboratory experiments on dust-free air.

The reader interested in optics may be reminded of the application of similar ideas to a *grating* on which fall plane waves of homogeneous light. If the spacing be quite uniform, the light behind is limited to special directions. Seen from other directions the interior of the grating appears dark. But if the ruling be irregular, light is emitted in all directions and the interior of the grating, previously dark, becomes luminous.

In the problems considered above the space occupied by a source, whether primary or secondary, has been supposed infinitely small. Probably it would be premature to try to include sources of finite extension, but merely as an illustration of what is to be expected we may take the question of n phases distributed at random over a complete period (2π), but under the limitation that the distance between neighbours is never to be less than a fixed quantity δ. All other situations along the range are to be regarded as equally probable.

As we have seen, the expectation of intensity may be equated to

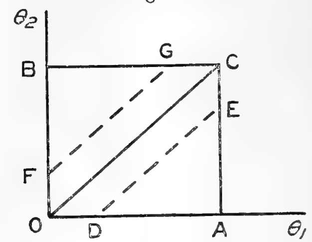
$$n+2\iiint \ldots . \Sigma \cos(\theta_\sigma-\theta_\tau)d\theta_1\, d\theta_2 \ldots . d\theta_n \div \iiint \ldots . d\theta_1 \ldots . d\theta_n, \quad (40)$$

and the question turns upon the limits of the integrals.

The case where there are only two phases ($n=2$) is simple.

Taking θ_1, θ_2 as coordinates of a representative point, fig. 3, the sides of the square OACB are 2π. Along the diagonal

Fig. 3.



OC, θ_1 and θ_2 are equal. If DE, FG be drawn parallel to OC, so that OD, OF are equal to δ, the prohibited region is that part of the square lying between these lines. Our integrations are to be extended over the remainder, viz. the triangles FBG, DAE, and every point, or rather every infinitely small region of given area, is to be regarded as equally probable. Evidently it suffices to consider one triangle, say the upper one, where $\theta_2 > \theta_1$.

For the denominator in (40) we have

$$\iint d\theta_1\, d\theta_2 = \text{area of triangle FBG} = \tfrac{1}{2}(2\pi - \delta)^2.$$

In the double integral containing the cosine, let us take first the integration with respect to θ_2, for which the limits are $\theta_1 + \delta$ and 2π. We have

$$\int_{\theta_1+\delta}^{2\pi} \cos(\theta_2 - \theta_1) d\theta_2 = \cos\delta - 1 - (2\pi - \delta)\sin\delta;$$

and since the limits for θ_1 are 0 and $2\pi - \delta$, we get as the expectation of intensity

$$2 - 4\frac{1 - \cos\delta + (2\pi - \delta)\sin\delta}{(2\pi - \delta)^2}. \quad . \quad . \quad . \quad (41)$$

If δ^2 be neglected, this reduces to

$$2(1 - \delta/\pi). \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (42)$$

If $\delta = \pi$, we have $2(1 - 4/\pi^2)$; and if $\delta = 2\pi$, we have 4, the only available situations being $\theta_1 = 0$, $\theta_2 = 2\pi$, equivalent to phase identity.

This treatment might perhaps be extended to a greater value, or even to the general (integral) value, of n; but I content myself with the simplifying supposition that δ is very small.

In (40) the integration with respect to θ_n supposes

$\theta_1, \theta_2 \ldots . \theta_{n-1}$ already fixed. If $\delta=0$, every term such as

$$\iiint \ldots . \cos(\theta_\tau-\theta_\sigma)d\theta_1 \ldots . d\theta_n \div \iiint \ldots . d\theta_1 \ldots . d\theta_n$$

$$=\iint \cos(\theta_\tau-\theta_\sigma)d\theta_\sigma\, d\theta_\tau \div \iint d\theta_\sigma\, d\theta_\tau$$

$$=\int_0^{2\pi} d\theta_\sigma\{\sin(2\pi-\sigma)+\sin\sigma\}\div 4\pi^2=0,$$

and the expectation is n simply, as we have already seen. In the next approximation the correction to n will be of order δ, and we neglect δ^2.

In evaluating (40) there are $\frac{1}{2}n(n-1)$ terms under the sign of summation, but these are all equal, since there is really nothing to distinguish one pair from another. If we put $\sigma=1$, $\tau=2$, we have to consider

$$\iiint \ldots . \cos(\theta_2-\theta_1)\, d\theta_1\, d\theta_2 \ldots . d\theta_n$$
$$\div \iiint \ldots . d\theta_1\, d\theta_2 \ldots . d\theta_n, \quad . \quad (43)$$

The integration with respect to θ_n extends over the range from 0 to 2π with avoidance of the neighbourhood of $\theta_1, \theta_2, \cdot \ldots \theta_{n-1}$. For each of these there is usually a range 2δ to be omitted, but this does not apply when any of them happen to be too near the ends of the range or too near one another. This complication, however, may be neglected in the present approximation. Then

$$\int \cos(\theta_2-\theta_1)d\theta_n=\cos(\theta_2-\theta_1)\,.\,\{2\pi-2\delta(n-1)\},$$

and in like manner

$$\int d\theta_n=2\pi-2\delta(n-1),$$

so that this factor disappears. Continuing the process, we get approximately

$$\iint \cos(\theta_2-\theta_1)d\theta_1\, d\theta_2 \div \iint d\theta_1\, d\theta_2,$$

as when there were only two phases to be regarded.

Accordingly, the expectation of intensity for n phases is

$$n\{1-(n-1)\delta/\pi\}, \quad . \quad . \quad . \quad . \quad . \quad (44)$$

less than when $\delta=0$, as was to be expected, since the cases excluded are specially favourable. But in order that this formula may be applicable, not merely δ, but also $n\delta$, must be small relatively to 2π.

A similar calculation is admissible when the whole range is $2m\pi$, instead of 2π, where m is an integer.

Terling Place, Witham.
Nov. 1.

L. *On the Ultraviolet Spectra of Magnesium and Selenium.*

By Professor J. C. McLennan, *F.R.S.*, *and* J. F. T. Young, *M.A.*, *University of Toronto* *.

[Plates XI. & XII.]

Introduction.

IN view of the fact that a number of the theoretical considerations being put forward at the present time regarding atomic structure are intimately related to spectral series which for many of the elements lie far down in the ultraviolet, it has become desirable to make the observations on the spectra in that region as extensive and as complete as possible. With the object of doing so a systematic study of the spectra of the elements in the ultraviolet and Schumann regions was recently begun by one of us. It was proposed to investigate the region between 3000 Å.U. and 2000 Å.U. with a Hilger quartz spectrograph, Type C, that between 2200 Å.U. and 1850 Å.U. with a Hilger quartz spectrograph, Type A, and that between 2000 Å.U. and 1400 Å.U. with a specially constructed fluorite spectrograph. The Schumann region to slightly below 600 Å.U. it was proposed to examine with a vacuum grating spectrograph recently designed and constructed for the Physical Laboratory at Toronto by the Adam Hilger Co.

The results obtained with silicon † and with cadmium ‡ have already been published elsewhere, and the present paper contains an account of the work done so far by the writers on the spectra of magnesium and selenium. With the former element the quartz spectrograph Type C was used, and with the latter the quartz spectrograph Type A. In all some fifty-eight new lines have been observed below 2600 Å.U. in the spectrum of magnesium, and some fourteen new ones in the spectrum of selenium.

Magnesium.

I. *Experiments.*

In studying the spectra of magnesium, the spark in air, the arc in air, and the arc *in vacuo*, were used in turn. The spectrum for the spark in air was obtained from the condensed discharge of a Clapp-Eastham half-kilowatt transformer rated to give 10,000 volts at the secondary terminals. With this arrangement the spark was quite thick and many of

* Communicated by the Authors.

† McLennan and Edwards, Phil. Mag. vol. xxx. p. 482 (1915).

‡ McLennan and Edwards, Proc. Roy. Soc. Canada, vol. ix. ser. iii. p. 167 (1915).

the lines below 2852 Å.U., especially the one at 2026 Å.U., could be observed visually with ease by means of a fluorescent eyepiece.

The arc in air was obtained by putting rods of magnesium metal in the carbon-holders of an ordinary hand-feed rectangular arc-lamp. The potential fall used was that of the mains, 110 volts, and the current varied from four to six amperes. For the arc *in vacuo* a quartz lamp of the type developed by McLennan and Henderson * was used. The side tubes were supplied with magnesium rod electrodes, and the arc was started by bombarding the magnesium vapour with electrons from the auxiliary incandescent tungsten cathode. The vapour condensed on the walls of the tube near the arc, but the lamp carried an additional side tube provided with a crystal quartz window and through it the light passed into the spectrograph.

With 220 volts across the magnesium terminals and a current of from 8 to 10 amperes, it was found that a brilliant arc could be maintained for an hour or two without the continued use of the Wehnelt cathode. The latter was therefore always cut out of the circuit as soon as the arc struck.

In taking photographs of the different spectra Schumann plates, made by the Adam Hilger Co., were used. When suitable precautions were taken to avoid fogging these gave spectrograms with sharp lines and clear definition over the whole range in the ultraviolet covered by the optical train.

Some of the results obtained are reproduced in fig. 1 (Pl. XI.) In the illustration, the upper spectrum is that of the spark between zinc terminals in air. The next is that of the magnesium spark in air; and the third and fourth spectrograms are respectively those of the arc in air and the arc *in vacuo*. In the spectrogram of the spark in air, the line $\lambda = 2852$ Å.U., of which the frequency is given by $\nu = (1{\cdot}5, S) - (2, P)$, sometimes appeared reversed. In the arc in air this line showed a broad though faint reversal, and the line $\lambda = 3838$ Å.U. was always strongly reversed. The spectrum of the arc *in vacuo* was readily obtained without any reversals showing on the plates. The line $\lambda = 4571$ Å.U. was never observed to exhibit reversal either in the arc or the spark spectra.

In the absorption spectrum of non-luminous magnesium vapour too, no absorption was ever observed at $\lambda = 4571$ Å.U. This result is rather interesting since it will be recalled that with the non-luminous vapours of mercury, zinc, and cadmium,

* McLennan and Henderson, Proc. Roy. Soc. A. vol. xci. p. 485 (1915).

when suitable densities were used, absorption was always obtained for the frequency $\nu=(1\cdot5, S)-(2, p_2)$, which in the magnesium spectrum is the series frequency of the line $\lambda=4571$ Å.U.

Some difficulty seems to have been experienced in observing the spectrum of magnesium in the region lying between $\lambda=2600$ Å.U. and $\lambda=2000$ Å.U., for while Handke *, Lyman †, and Saunders ‡ record lines between $\lambda=2000$ Å.U. and $\lambda=1700$ Å.U., the only one who appears to have recorded any line in the first-mentioned region is Saunders.

Lorenser § in his Inaugural Dissertation gives the second line of the series $\nu=(1\cdot5\ S)-(m, P)$ as $\lambda=2026$ Å.U., and this line Saunders ‖ reports that he observed.

From the illustrations given in fig. 1 it will be seen that in the region between $\lambda=2500$ Å.U. and $\lambda=2000$ Å.U. the spark-lines are rather faint. In the spectrum of the arc in air the line $\lambda=2026$ Å.U. is very strong, but the rest of the spectrum in this region is faint. In the spectrum of the arc *in vacuo* the line $\lambda=2026$ Å.U. is clearly marked, as also are some lines between $\lambda=2500$ Å.U. and $\lambda=2200$ Å.U. The rest of the spectrum below $\lambda=2600$ Å.U. is faint.

In working out the spectrum of magnesium in the present investigation a great many plates were taken with each source of light, and different samples of magnesium were used in order to eliminate any lines due to impurities which might be present as traces in the metal. In determining the wave-lengths four of the best plates obtained with each source were selected, and only lines common to all four were measured up.

The wave-lengths of the lines were determined from a calibration curve constructed for the spectrograph by using the following prominent zinc spark-lines :—

Wave-lengths of Zinc lines ¶.

$\lambda=6588\cdot65$ Å.U.	$\lambda=2658\cdot27$ Å.U.
6362·98 ,,	2502·20 ,,
6103·58 ,,	2418·95 ,,
5675·30 ,,	2346·80 ,,
4925·37 ,,	2265·08 ,,
3988·75 ,,	2138·66 ,,
3282·49 ,,	2100·06 ,,
3076·03 ,,	2062·08 ,,
2801·15 ,,	2025·56 ,,

* Handke, *Inaug. Diss.* Berlin, 1909, p. 18.

† Lyman, 'Spectroscopy of the Extreme Ultra-Violet' (Longmans, Green & Co.), p. 117.

‡ Saunders, The Astrophys. Jl. vol. xliii. no. 3, p. 234 (1916).

§ Lorenser, *Inaug. Diss.* Tübingen, 1913.

‖ Saunders, *loc. cit.*

¶ Eder and Valenta, *Atlas Typischer Spectren*, Wien.

The wave-lengths of the lines measured and their relative intensities are given in Table I. (p. 454). In the same table the lines recorded by other observers for the range of wave-lengths shorter than $\lambda=2852\cdot22$ Å.U. are included.

II. *Series Relations.*

The most recent work on the series spectra of magnesium is the Inaugural Dissertation recently published by E. Lorenser of Tübingen. In this paper he draws attention to a series calculated from the formula

$$A-\nu=(m, X)-\frac{N}{\left(m+X+\frac{x}{m}\right)^2},$$

where $A=56098\cdot8$, and for which the wave-lengths of the different members are :—

$m=$	2	3	4	5	6
$\lambda=$	2852·22	2219·8 (?)	2019·2 (?)	1930·9	1886·3

The first member of this series is well known, and the fifth and sixth it will be noted were observed by Handke. Lorenser, however, was not able to observe the second and third members, and concluded that the series as calculated was based on false assumptions. The results of the present investigation, as the table of wave-lengths shows, goes to support Lorenser's view, for no lines were observed with wave-lengths which could be supposed to represent $\lambda=2219\cdot8$ Å.U. and $\lambda=2019\cdot2$ Å.U.

A series of single lines and represented by $\nu=(1\cdot5, S)-(m, P)$ has been calculated by Lorenser as follows :—

$m=$	2	3	4	5	6	7	8
$\lambda=$	2852·22	2025·08	1828·1	1748·09	1707·3	1683·64	1668 04

For this series $\lambda=2852\cdot22$ Å.U. is well known. The line $\lambda=2025\cdot08$ Å.U. was observed by Saunders and is brought out clearly in the present investigation as a prominent line. Both Lyman and Saunders have observed the line $\lambda=1828\cdot1$ Å.U. No line has been observed as yet exactly at $\lambda=1748\cdot09$ Å.U., but Handke gives two which are close to it. The lines $\lambda=2852\cdot22$ Å.U. and $\lambda=2025\cdot08$ Å.U. have been shown by one of us to be strongly absorbed by non-luminous magnesium vapour, and the line $\lambda=2852\cdot22$ Å.U., it will be recalled, is easily reversed.

Moreover, it will be recalled that when magnesium vapour is bombarded by electrons*, the line $\lambda=2852\cdot22$ Å.U. is the

* McLennan, Proc. Roy. Soc. A. vol. xcii. p. 574.

TABLE I.

Kayser and Runge.		Exner and Haschek.				Eder.			
Arc.		Arc.		Spark.		Arc.		Spark.	
λ.	I.	λ.	I.	λ.	I.	λ.	I.	λ.	I.
2852·22	10 R	2852·25	500 R	2852·20	100 R	2852·22	10	2852·29	7 R
48·53	4 U	48·7	3	—		—		48·44	2
46·91	4 U	47·1	2	—		—		47·08	2
—		—		—		—		—	
—		17·2	1	17·0	2	—		17·29	2
—		15·8	1	15·8	2 u	—		15·67	2
—		12·0	1	—		—		—	
—		11·2	1	11·6	2 u	—		11·35	2
—		09·2	2	—		—		09·88	2
02·80	10 R	02·82	100 R	02·80	500 R	02·80	8	02·81	10 R
2798·07	4	2798·10	2	2798·17	100 R	2798·07	1	2798·12	5 R
95·63	10 R	95·64	200 R	95·62	500 R	95·63	8	95·63	10 R
—		95·01	2 R	—		—		—	
90·88	4	90·97	5	90·99	100 R	—		90·97	10 R
83·08	8 R	83·08	20 R	83·08	6	83·08	5	83·08	6
81·53	8 R	81·51	20 R	81·52	5	81·53	5	81·52	5
79·94	10 R	79·95	30 R	79·93	10 R	79·94	6	79·94	10
78·36	8 R	78·40	20 R	78·34	5	78·36	5	78·38	5
76·80	8 R	76·82	20 R	76·77	6	76·80	5	76·80	6
68·57	4 r	68·6	1 u	—		—		—	
65·47	4 r	65·5	1 u	—		—		—	
36·8	2 U	36·8	1 u	—		36·84	3	36·81	1 u
33·80	2 U	33·7	1 u	—		33·80	3	33·3	1 u
32·35	2 U	—		—		—		—	
—		—		—		—		—	
2698·44	2 u	2698·5	1 u	—		—		—	
95·53	2 u	—		—		—		—	
93·97	2 u	—		—		—		—	
72·90	1 u	73·0	2 U	—		2672·90	2	—	
69·84	1 U	—		—		69·84	2	—	
68·26	1 U	—		—		68·26	2	—	
—		—		2659·5	1 u	—		2660·0	1 U
49·30	1 U	—		—		—		—	
46·61	1 U	—		—		—		—	
45·22	1 U	—		—		—		—	
33·13	1 U	—		—		33·13	1	—	
30·52	1 U	—		—		30·52	1	—	
(05·4)		—		—		05·4	1	—	

THE AUTHORS.

Arc in air. λ.	I.	Arc in vacuo. λ.	I.	Spark in air. λ.	I.
2852	10	2852	10	2852	10
—		—		—	
—		—		—	
—		28	1	—	
—		—		—	
15	8	—		—	
—		—		—	
—		—		—	
—		—		—	
03	10	03	9	03	8
—		—		—	
—		2796	9	2796	10
2794	8	—		—	
—		91	9	91	8
—		—		—	
—		—		—	
79	10	80	10	—	
—		—		78	10
—		—		—	
—		68	1	—	
—		—		—	
35	7	36	4	35	1
—		32	4	—	
—		—		—	
19	2	—		—	
2698	2	2697	2	—	
—		—		—	
94	1	93	2	—	
71	5	—		—	
—		70	4	—	
—		67	4	—	
—		59	5	—	
—		—		—	
—		—		—	
—		—		—	
32	3	32	3	—	
—		28	3	—	
05	1	06	3	2605	1
—		2602	3	—	
2598	1	—		2598	2
93	1	—		93	2
—		87	2	—	
—		84	2	—	
75	1	—		75	2
—		73	2	—	
—		71	2	—	
28	1	—		29	4
23	2	—		22	4
16	2	—		—	
11	1	—		—	
07	1	—		—	
2491	3	—		—	
88	3	—		—	
83	3	—		—	
—		—		2478	2
—		2450	4	—	
—		—		45	4
—		—		33	3

Arc in air. λ.	I.	Arc in vacuo. λ.	I.	Spark in air. λ.	I.
2396	1	—		—	
82	2	—		81	2
48	1	—		—	
44	1	—		—	
—		2329	4	—	
—		—		2317	6
2299	1	—		2299	2
—		—		96	2
89	3	—		—	
—		2253	4	—	
—		—		47	4
30	2	—		—	
28	1	—		—	
26	1	—		—	
17	1	—		—	
14	1	—		—	
11	1	—		—	
08	1	—		—	
—		02	3	—	
2199	2	—		—	
96	1	—		—	
91	1	—		2191	4
88	1	—		88	4
2181	1	—		—	
78	2	—		—	
66	1	2165	2	—	
64	1	—		—	
54	1	53	1	—	
—		—		2148	1
38	4	38	1	—	
35	1	—		36	2
—		—		34	2
24	1	—		—	
—		18	1	—	
—		—		2097	2
—		—		91	1
—		—		64	4
—		—		62	4
2026	6	2026	3	26	6

HANDKE. Spark in air.		LYMAN. Spark in Hydrogen.		SAUNDERS. Vacuum arc.
1930·9	6			
1886·8	5			
64·1	4			
55·9	5			
39·6	3			
—		1828·1	1	1828·06
1753·0	3	1753·6	6	
50·7	2	50·9	5	
50·0	1			
46·7	1			
44·1	5			
41·4	5			
		37·8	7	
36·3	2			
		35·0	6	
34·0	1			

one most easily stimulated. It is also the only line which comes out in the spectrum of the light from a gently burning bunsen flame * fed with magnesium vapour. All these considerations point to the series $\nu=(1\cdot5, S)-(m, P)$ as given above by Lorenser as being correct. The evidence goes to show further that this series represents frequencies of fundamental importance in the spectrum of magnesium just as corresponding series have been shown to do for the spectra of mercury, zinc, and cadmium.

Assuming the wave-lengths given above for the series $\nu=(1\cdot5, S)-(m, P)$ as being correct, one may calculate the series given by $\nu=(1\cdot5, S)-(m, p_2)$.

This has been done by Lorenser, and the wave-lengths of the different members are as follows :—

$m=$	2	3	4	....	
$\lambda=$	4571·27	2090·08	1843·08		1621·00

Although series given by $\nu=(1\cdot5, S)-(m, p_2)$ have been identified in the spectra of mercury, zinc, and cadmium, no such series, with the exception of the first member of $\lambda=4571\cdot27$ Å.U., has as yet been observed with magnesium. The real existence of the series in the spectrum of magnesium, moreover, has been questioned. It will be noted, however, that both in the spectrum of the arc in air, and in that of the spark in air, a faint line was observed at $\lambda=2091$ Å.U. As the calculated value of the second member of the series $\lambda=2090\cdot08$ Å.U. is within the possible error of measurement of this line, it would appear therefore that it represents the second member. The series would then seem to have a real existence.

Summary.

1. The spectra of magnesium for (*a*) the spark in air, (*b*) the arc in air, and (*c*) the arc *in vacuo*, have been investigated in the region between $\lambda=2852\cdot22$ Å.U. and $\lambda=2000$ Å.U., and in all some fifty-eight new lines have been measured.

2. The existence of the line $\lambda=2026$ Å.U., first measured by Saunders, has been confirmed and considerations have been brought forward supporting the view that the series $\nu=(1\cdot5, S)-(m, p_2)$ has a real existence.

* McLennan and Thomson, Proc. Roy. Soc. A. vol. xcii. p. 584.

Selenium.

1. *Spark Spectrum Experiments.*—The most important work on the emission spectrum of selenium appears to have been done by Messerschmidt * and by Berndt †. The electrical conductivity of metallic selenium, as is well known, is exceedingly low, and as a consequence it is impossible to produce an arc or a spark in the usual manner. In his experiments Messerschmidt used a strong condensed discharge through a quartz Geissler tube containing a small bead of selenium.

Berndt investigated the spark spectrum by melting small globules of selenium on the tips of platinum wires about 1·2 mm. in diameter and then passing a condensed discharge across the terminals. The selenium metal was vaporized by the heat of the spark, and its spectrum was obtained as well as that of the spark spectrum of platinum. According to Kayser's *Handbuch der Spectroscopie*, the lowest limit reached was about $\lambda=2340$ Å.U.

In the present investigation the selenium spark spectrum was obtained superimposed upon that of carbon. Two commercial solid carbons were used. The lower one was cratered and filled with a bead of gray vitreous selenium metal, while the upper electrode was pointed and placed centrally over the lower one. When the condensed discharge from a Clapp-Eastham half-kilowatt transformer of 10,000 volts was passed across the gap, it gave a cone of light reaching from the tip of the upper electrode to the periphery of the crater of the lower one. Owing to the fact that carbon is a poor conductor of heat, the energy in the discharge was sufficient to boil the selenium in the crater and the vapour passed out through the cone discharge. The spectrum of the carbon spark alone was first photographed and then that of the carbon and selenium combined. As stated already the spectrograms were taken with a quartz Hilger spectrograph type A. In all the photographs of the spectrum taken in this way no lines due to selenium were obtained of wave-length longer than 2200 Å.U. Berndt's method was also tried with aluminium wires in place of platinum ones, but with the same result. This lack of lines in the longer wave-lengths was probably due to the way the spectrum

* J. Messerschmidt, *Dissertation*, Bonn, 1907; *Zs. Wiss. Photographie*, v. p. 249 (1907).

† G. Berndt, *Ann. der Phys.* xii. p. 1115 (1903).

was excited, for the voltage used was only a fraction of that employed by Berndt in his experiments.

In order to make certain that no chance impurities were the cause of lines coming out which were observed in these experiments below λ=2200 Å.U., different pieces of selenium were used as well as different carbon electrodes. The same spectrum, however, was invariably obtained. Many plates were taken, and four of the best of them were used in measuring up the wave-lengths of the lines. Twelve lines in all were obtained and ascribed to the selenium spark.

In determining the wave-length of the selenium lines the following prominent aluminium *, zinc †, and cadmium † lines were used:—

Aluminium lines.	Zinc lines.	Cadmium lines.
λ=1990·57 Å.U.	λ=2558·20 Å.U.	λ=2748·68 Å.U.
35·90 ,,	02·20 ,,	2573 15 ,,
1862·81 ,,	2138·66 ,,	2313·88 ,,
58·20 ,,	00·06 ,,	2288·12 ,,
54·80 ,,	2062·08 ,,	2265·04 ,,
	25·51 ,,	2194·71 ,,
		44·44 ,,

In fig. 2 (Pl. XII.) the first reproduction, "*a*," is the spectrum of the aluminium spark in air, the second, "*b*," that of the carbon spark in air, and the third, "*c*," that of the combined carbon and selenium spark in air.

In measuring up a plate the distances of the various aluminium, zinc, cadmium, and selenium lines from the aluminium line λ=1854·8 Å.U. were carefully measured with a Hilger comparator. The distances of the aluminium, zinc, and cadmium lines given above from the aluminium line λ=1854·8 Å.U. were used as the ordinates of a calibration curve and the wave-lengths of the lines as abscissæ. This calibration curve was then used to determine the wave-lengths of the selenium lines.

The results are probably accurate to one Ångström unit. The relative intensities of the lines were estimated by giving the strongest line the arbitrary value 10 and referring to this as the standard. The mean values of the measurements of all the selenium spark-lines observed are given in Table II.

* Handke, *Dissertation*, Berlin, 1909, p. 18.

† Eder and Valenta, *Atlas Typischer Spektren*, Wien.

TABLE II.—Selenium Spark Lines.

Wave-length.	Intensity.	Wave-length.	Intensity.
λ=2155 Å.U.	1	λ=1960 Å.U.	10
2073 „	3	15 „	3
63 „	8	1897 „	3
38 „	8	93 „	2
25 „	2	58 „	7
1993 „	5	54 „	7

2. *Arc Spectrum Experiments.*—An investigation was also made of the spectrum of selenium in the carbon arc. Solid carbons were used and a small bead of selenium metal was placed on the vertical carbon. The contact was then made and the selenium boiled away in the arc, which was fed with a current of 10 amperes from the 110 volt direct-current mains. The spectrum was photographed with the same instrument as in the experiments with the spark, and tests were made with different carbons and pieces of selenium.

The fourth reproduction "*d*," in fig. 2 is the spectrum of the carbon arc in air, and the fifth, "*e*," is that of the carbon and selenium arc in air.

Four of the best plates taken were used in making the measurements, and the observations showed that but five lines, all occurring in the ultraviolet, were due to the selenium arc. Their wave-lengths are given in Table III.

TABLE III.—Selenium Arc Lines.

Wave-length.	Intensity.
λ=2073 Å.U.	8
63 „	8
38 „	2
1988 „	1
60 ,	10
30 „	2

3. *Absorption Spectrum Experiments.*—An attempt was also made to see if an absorption spectrum for selenium vapour could be obtained.

The absorption spectrum of selenium vapour under various conditions has been exhaustively studied by Evans and Antonoff *, who found that for high vapour-pressure there is continuous absorption below λ=5800 Å.U. As the vapour-pressure decreased they found that absorption bands appeared in the green, blue, and violet, and for low pressure values in the ultraviolet. No bands were discovered below

* Evans and Antonoff, Astrophys. Jl. xxxiv. p. 277 (1911).

$\lambda=3200$ Å.U. The approximate wave-lengths of the bands found in the ultraviolet are given as $\lambda=3240$, 3255, 3280, 3295, 3317, 3338, 3363, 3387, 3412, 3435, 3460, 3483, 3510, 3537, 3592, 3614, 3640, 3663, 3684, 3700, 3715, 3730, 3742, 3755, 3763, 3774, and 3802 Å.U.

As none of these bands are in the region in which the emission lines, given above, were found, some experiments were performed to find if absorption did take place at any of these wave-lengths. Some selenium metal was enclosed in a fused quartz tube, highly exhausted and sealed off. But owing to the difficulty of obtaining a continuous spectrum in the region from $\lambda=2200$ Å.U. to $\lambda=1850$ Å.U., this method was abandoned. The photograph (*f*) on the plate showing a clear reversal of the selenium line $\lambda=1960$ Å.U. was obtained in the carbon arc. Two solid carbons with flat ends were used. A large bead of selenium metal was placed on the vertical carbon and then the arc was struck behind the selenium metal. As the carbons became hot the selenium boiled up in front of the arc and gave a sharp but narrow reversal-band at $\lambda=1960$ Å.U. This was the only absorption band observed, and it is possible that this frequency on account of its intensity and easy reversal is intimately connected with one of the series $\nu=(1\cdot5, S)-(m, P)$ or $\nu=(1\cdot5, S)-(m, p_2)$, the remainder of the series lying below $\lambda=1850$ Å.U. To investigate this further work with a fluorite or a vacuum grating spectrograph will be necessary.

Summary.

1. Twelve new lines have been recorded in the selenium spark spectrum between $\lambda=2200$ Å.U. and $\lambda=1850$ Å.U.

2. Five lines in the selenium arc have been found between the same limits.

3. In the sources used no part of the spectrum longer in wave-length than $\lambda=2200$ Å.U. was present.

4. The absorption spectrum of selenium metal in the carbon arc was investigated and a reversal was found at $\lambda=1960$ Å.U., which is the strongest line in both the arc and spark spectra.

5. If the absorption of selenium vapour should prove to be analogous to that of mercury, zinc, and cadmium, this would indicate that the two series $\nu=(1\cdot5, S)-(m, P)$ and $\nu=(1\cdot5, S)-(m, p_2)$ for selenium are in the extreme ultra-violet.

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LI. *On Fundamental Frequencies in the Spectra of Various Elements.* By Professor J. C. McLennan, *F.R.S., and* H. J. C. Ireton*.

[Plates XIII.-XV.]

Part I.

1. *Introduction.*

IN a series of papers by McLennan and Henderson† and others it has been shown that when the vapours of mercury, zinc, cadmium, and magnesium were subjected to bombardment by electrons whose velocity was gradually increased, these vapours were stimulated to the emission of a monochromatic radiation. With mercury the radiation emitted had the wave-length $\lambda=2536{\cdot}72$ Å.U., with zinc $\lambda=3075{\cdot}99$ Å.U., with cadmium $\lambda=3260{\cdot}17$ Å.U., and with magnesium $\lambda=2852{\cdot}22$ Å.U. In order to bring the vapours to the emission of these respective radiations, it was found that the bombarding electrons had to have kinetic energy corresponding to a fall of potential given by the quantum relation $Ve=h\nu$, where ν is the frequency of the monochromatic radiation emitted.

When the velocity of the electrons was increased beyond that given by the quantum relation for these frequencies, there did not appear to be any radiation emitted of shorter wave-length than those given above by any of the vapours mentioned until the electrons possessed the requisite energy to ionize the vapours. When this occurred arcs were struck and the many-lined spectra were obtained.

In order to obtain the many-lined spectra it was found that the electrons required to have kinetic energy corresponding to a potential fall given by the quantum relation $Ve=h\nu$ where ν was the frequency, $\nu=(1{\cdot}5, S)$, namely that of the shortest wave-length of the series $\nu=(1{\cdot}5, S)-(m, P)$.

It was thought that, in these experiments in which an incandescent limed platinum cathode was used, it might be possible, by giving the electrons kinetic energy intermediate between that which would bring on the monochromatic radiation and that which sufficed for striking the arc, to cause the vapours to emit radiations which became shorter

* Communicated by the Authors.

† McLennan and Henderson, Proc. Roy. Soc. A, vol. xci. p. 425 (1915).

and shorter in wave-length as the speed of the electrons was increased. In particular, experiments were directed to this end with mercury vapour, but it was found that when the electrons were given kinetic energy corresponding to 4·9 volts, the radiation of wave-length $\lambda=2536{\cdot}72$ flashed out; and then, as the velocity of the electrons was increased no photographic record was obtained of any wave-lengths shorter than $\lambda=2536{\cdot}72$ Å.U., until the velocity corresponding to 10·2 volts was reached, when the arc struck and the many-lined spectra came out.

The question has, however, been re-examined by Bergen Davis and Goucher *, and in a series of brilliantly designed experiments in which the photoelectric effect was used for detecting the existence of particular radiations, they have shown that when mercury vapour of very low density was bombarded by electrons, radiation of wave-length $\lambda=2536{\cdot}72$A.U. was emitted without ionization at an impact voltage of 4·9 volts, and that when the impact voltage was increased to 6·7 volts a radiation of wave-length $\lambda=1849$ Å.U. came out as well. With still higher impact voltages no additional types of radiation appeared before ionization of the vapour occurred, which took place with an impact of voltage of about 10·4 volts.

With a view to confirming this result by the photographic method the original experiments of one of us have been repeated by the writers and extended to include vapours other than mercury. The following paper contains an account of these experiments.

II. *Experiments.*

In carrying out the experiments, the form of vacuum arc-lamp used is shown in fig. 1, similar to the one described by McLennan and Henderson †. It consisted of a tube of fused quartz possessing three arms, R, S, and M N, and a receptacle L. Some of the metal to be used in producing the vapour was placed in the receptacle L. The arms were about 40 cm. long, so that when the receptacle was heated all wax joints remained quite cool. A short piece of tungsten was attached to two wires which constituted a heating circuit, these being passed through an ebonite plug and sealed in at A. A short iron tube, in which was sealed a crystal quartz plate,

* Bergen Davis and Goucher, Phys. Rev. vol. x. no. 2, p. 101.
† McLennan and Henderson, *loc. cit.*

was sealed on the end B. Connected to this iron tube, a small iron rod passed along S. At the farther end it had a ring from which a small chain dipped into the mercury.

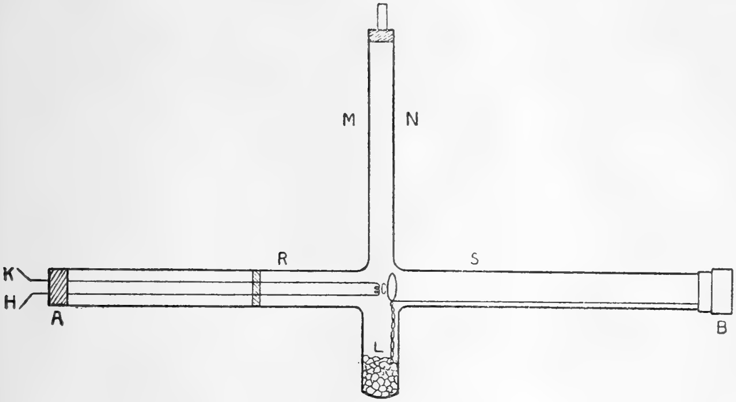


FIG. 1.

This was to keep the mercury in electrical contact with the iron electrode. The tube was then highly exhausted by a Gaede pump through a brass tube sealed on M N, and when a low vacuum was reached the metal was vaporized by heating the receptacle L with a Bunsen burner. When the tube was in operation, the terminals of an auxiliary heating circuit were attached at H and K. The impact voltage was applied between K and B, the latter being the positive terminal. In taking photographs, the tungsten was brought to incandescence by means of the auxiliary heating circuit, the metal in L was heated by the flame of a Bunsen burner to produce vapour of the metal, and the collimator of a small quartz Hilger spectrograph of type A was lined up with the arm S in front of the window at the end B.

(a) *Mercury Vapour.*

With mercury vapour, the results obtained were no better than those published originally by McLennan and Henderson. With an impact voltage of about 5 volts it was found an easy matter to obtain the monochromatic radiation of wave-length $\lambda = 2536 \cdot 72$ Å.U. With still higher impact voltages, no trace of shorter wave-lengths even with long exposure was obtained until the impact voltage was sufficiently great to cause the arc to strike. Reproduction No. 2, fig. II. (Pl. XIII.)

shows the single line $\lambda=2536{\cdot}72$ Å.U. obtained with an impact voltage of 6·1 volts. Although Schumann plates were used, exposures as long as 10 hours, with an impact voltage of from 8 to 8·5 volts, failed to give any trace of the line $\lambda=1849{\cdot}6$ Å.U. In view of the results obtained by Bergen Davis and Goucher, we can only conclude that in all our experiments the density of the vapour used was too great. It is known that radiation of wave-length $\lambda=1849{\cdot}6$ Å.U. is strongly absorbed and scattered by mercury vapour even of low density, and it is possible that this accounts for the non-appearance of any line at this wave-length.

(b) *Zinc.*

With zinc much better results were obtained. When the vapour of this metal was bombarded with electrons whose impact voltage was about 4 volts, monochromatic radiation of wave-length $\lambda=3075{\cdot}99$ Å.U. was recorded. As the impact voltage was increased no additional indication was observed until about 6 volts was reached, when the line corresponding to $\lambda=2139{\cdot}33$ Å.U came out on the plates. Reproduction No. 1, fig. III. shows the many-lined spectrum of the zinc spark. Reproduction No. 2, fig. III. shows the line $\lambda=3076{\cdot}00$ Å.U., which was brought out when the impact voltage was 5·6 volts. The plate for this spectrogram showed in addition the line corresponding to $\lambda=3260{\cdot}17$ Å.U., which would indicate, since the line was extremely faint, that a trace of cadmium was present as an impurity in the zinc. Reproduction No. 3 shows both the lines corresponding to $\lambda=3075{\cdot}99$ Å.U. and $\lambda=2139{\cdot}33$ Å.U., and was obtained with an impact voltage of 7·5 volts.

(c) *Cadmium.*

With cadmium vapour, results were obtained similar in character to those recorded with zinc. No photographic record was obtained of any radiations until an impact voltage of about 4 volts was reached. Under these circumstances, the line at wave-length $\lambda=3260{\cdot}17$ Å.U. came out on the plates. With still higher impact voltages no additional radiation appeared until an impact voltage slightly less than 6 volts was obtained. With this impact voltage the line at $\lambda=2288{\cdot}29$ came out on the plates in addition to the line at $\lambda=3260{\cdot}17$ Å.U.

Reproduction No. 1, fig. IV. is that of the many-lined spectrum of the cadmium spark. No. 2 shows the single line at $\lambda=3260{\cdot}17$ Å.U., and was obtained with an impact voltage of 5·2 volts. No. 3 shows the lines at $\lambda=3260{\cdot}17$ Å.U. and $\lambda=2288{\cdot}79$ Å.U. It was obtained with an impact voltage equal to 7·5 volts.

III. *Discussion of Results.*

The results described above combined with those of Bergen Davis and Goucher go to support the view that it is possible to stimulate the atoms of mercury, zinc, and cadmium to the emission of definite and distinct types of monochromatic radiation by choosing definite impact voltages which are given by the quantum relation. A view has been put forward that when the kinetic energy of the impinging electrons is sufficient to bring out the radiations $\lambda=2536{\cdot}72$ Å.U, $\lambda=3075{\cdot}99$ Å.U., and $\lambda=3260{\cdot}17$ Å.U., for mercury, zinc, and cadmium respectively, other radiations of still shorter wave-length are present, but their intensity is too weak to produce records of their presence on the photographic plates. The experiments we have carried out, however, do not support that view. Even with exposures as long as 10 hours no trace of lines at $\lambda=2139{\cdot}33$ Å.U. for zinc and at $\lambda=2288{\cdot}79$ Å.U. for cadmium came out when the impact voltage was less than that given by the quantum relation for their respective frequencies. When impact voltages corresponding to their frequencies were applied, these lines were at once obtained on the plates even with comparatively short exposures.

From Table I. it will be noted that the wave-lengths $\lambda=2536{\cdot}72$ Å.U., $\lambda=3075{\cdot}99$ Å.U., and $\lambda=3260{\cdot}17$ Å.U. are respectively the first members of the combination series $\nu=(1{\cdot}5, \mathrm{S})-(m, p_2)$, and the wave-lengths $\lambda=1849{\cdot}6$ Å.U., $\lambda=2139{\cdot}33$ Å.U., and $\lambda=2288{\cdot}79$ Å.U., the first members of the singlet principal series $\nu=(1{\cdot}5, \mathrm{S})-(m, \mathrm{P})$. The other members of both these series for the three metals are all beyond the range of wave-lengths which can be recorded by a spectroscope with an optical train of quartz. It would be interesting to extend the experiments described in this paper so as to see if the higher members of these two series came out on the plate one by one as the impact voltage of the electrons was increased to that given by the quantum relation for their frequencies. To do this it would be necessary to use a fluorite spectrograph or a vacuum grating spectrograph for the range intermediate between $\lambda=1900$ Å.U. and $\lambda=1400$ Å.U., and a vacuum grating spectrograph for the range below $\lambda=1400$ Å.U.

Series Spectra.

1·5, S—*m*, p_2 series.

	Mercury.	Zinc.	Cadmium.	*m*.
	2536·72	3076·99	3260·17	2
	1435·59	1632·08	1710·58	3
	1307·83	1468·90	1537·89	4
	1259·31	1408·86	1474·06	5
	1235·91	1379·38	1442·60	6
	1222·44	1362·59	1424·40	7
	1213·97	—	—	8
	—	—	—	—
	—	—	—	—
Lt	1188·0	1320·0	1378·7	α

1·5, S—*m*, P series.

	Mercury.	Zinc.	Cadmium.	*m*.
	1849·6	2139·33	2288·79	2
	1402·71	1589·64	1669·3	3
	1268·9	1457·64	1526·73	4
	1250·6	1376·97	1469·35	5
	—	—	—	—
Lt	1188·0	1320·0	1378·7	α

To work in this direction some experiments were set on foot by McLennan and Ainslie with a fluorite spectrograph, and others by McLennan and Lang with a vacuum grating spectrograph. It was found that with both instruments much time was consumed in working out technical details. The results obtained to date with them will be published shortly, and they will show that with the fluorite spectrograph it is now easy to obtain spectrograms down to $\lambda=1400$ Å.U. With the vacuum grating spectrograph spectrograms well below $\lambda = 600$ Å.U. can be readily obtained.

IV. *Fundamental Series.*

From the experiments described above it will be seen that it is possible with the vapours of mercury, zinc, and cadmium to stimulate at will the radiation—and this radiation only—given by the first member of the combination series $\nu=(1{\cdot}5,\ \mathrm{S})-(2,\ p_2)$. It is also possible to cause the atoms of the same vapours at will to emit the radiation given by the first member of the singlet principal series given by $\nu=(1\cdot5, S)—(m, P)$.

The question naturally arises then as to which of these two series is the more fundamental in character from the point of view of atomic structure.

It is impossible as yet to decide, but everything goes to show that, while the radiation given by $\nu=(1\cdot5, S)—(2, p_2)$ is the more easily stimulated by electronic bombardment of mercury, zinc, and cadmium atoms, the series of wave-lengths given by $\nu=(1\cdot5, S)—(m, P)$ is the one which corresponds to some very simple type of electronic vibrations within the atom.

It has been shown by McLennan and Edwards * that with the vapours of mercury, zinc, and cadmium absorption is more marked in the region corresponding to $\nu=(1\cdot5, S)—(2, P)$ than it is in the region about $\nu=(1\cdot5\ S)—(2, p_2)$. With magnesium vapour too, McLennan † has shown that absorption at $\nu=(1\cdot5, S)—(2, P)$ is very much more marked than it is at $\nu=(1\cdot5, S)—(2, p_2)$. More recently still McLennan and Young‡ have shown that with the vapours of calcium, strontium, and barium, absorption and reversal can be obtained much more readily at $\nu=(1\cdot5, S)—(2, P)$ than it can at $\nu=(1\cdot5, S)—(2, p_2)$.

It will also be recalled that McLennan § has shown that when magnesium was bombarded by electrons whose kinetic energy was gradually increased, no radiation of wave-length at $\lambda=4571$ Å.U. was obtained until the arc struck, while the line at $\lambda=2852\cdot22$ Å.U., $\nu=(1\cdot5, S)—(2, P)$ came out on the plates as soon as the electrons attained kinetic energy corresponding to the impact voltage given by the quantum relation for the frequency of this wave-length.

It would appear, then, that of the two series $\nu=(1\cdot5, S)—(m, p_2)$ and $\nu=(1\cdot5, S)—(m, P)$, the latter is the more fundamental in character, and that through it our attention is directed to vibrations within the atom which are of a principal or main type.

* McLennan and Edwards, Phil. Mag. vol. xxx. p. 695, Nov. 1915.
† McLennan, Proc. Roy. Soc. A, vol. xcii. p. 307.
‡ McLennan and Young. Communicated to the Royal Soc. Oct. 1918.
§ McLennan, *loc. cit.*

Part II.

1. *Introduction.*

In a paper by McLennan and Thomson on Bunsen Flame Spectra some experiments are described which were designed to throw light on the question of which of the two series $\nu=(1\cdot5, S)-(m, p_2)$ and $\nu=(1\cdot5, S)-(m, P)$ was the more fundamental from the point of view of vibrations within the atom. A Bunsen flame was chosen as being perhaps the most simple method of stimulating atoms to the emission of radiation, and the vapours of pure metals rather than those of their salts were used with a view to realizing the simplest possible conditions within the flame from a chemical point of view.

With mercury vapour the monochromatic radiation of wave-length $\lambda=2536\cdot72$ Å.U., $\nu=(1\cdot5, S)-(2, p_2)$ was obtained, but no trace of the line at $\lambda=1849\cdot6$ Å.U., $\nu=(1\cdot5, S)-(2, P)$ appeared, even when the Bunsen flame was strongly forced. With cadmium vapour the line at $\lambda=3260\cdot17$ Å.U., $\nu=(1\cdot5, S)-(2, p_2)$ came out when the flame was burning gently, and the line at $\lambda=2288\cdot79$ Å.U., $\nu=(1\cdot5, S)-(2, P)$ as well when the flame was forced. With magnesium vapour the line at $\lambda=2852\cdot22$ Å.U., $\nu=(1\cdot5, S)-(2, P)$ was obtained, but no trace of the line at $\lambda=4571$ Å.U., $\nu=(1\cdot5, S)-(2, p_2)$ appeared on the plates unless the many-lined spectrum appeared. With zinc no photographs belonging to the spectrum of the metal were obtained unless the metal was very strongly heated so that a copious supply of vapour was sent into the flame. Under these latter circumstances oxidation was intense and the vapour frequently took fire. The spectrum of zinc which was then obtained consisted of a large number of lines of greater or less intensity. From these results it will be seen that while the importance of the two series $\nu=(1\cdot5, S)-(2, p_2)$ and $\nu=(1\cdot5, S)-(m, P)$ was emphasized, there was little evidenee brought forward as to which of the two series was the more fundamental.

Ramage * in his admirable paper on "Relations of Spectra, etc., to Atomic Mass," was the first to identify the line $\lambda=3075\cdot99$ Å.U .in the flame spectrum of zinc. It appears, he pointed out, among the series of lines constituting the strongest water-vapour group. Our attention has also been

* Ramage, Proc. Roy. Soc. No. 459, vol. lxx. p. 1 (1901).

called by Hemsalech * to the fact that Charles de Watteville† in his exhaustive paper on Flame Spectra has recorded that the wave-length $\lambda = 3075 \cdot 99$ Å.U. was the only radiation belonging to the zinc spectrum which was emitted by the flame of a burner supplied with the spray from a solution of zinc chloride. In view of these results, it was thought that the line $\lambda = 3075 \cdot 99$ Å.U. should have come out on the plates of McLennan and Thomson when the Bunsen flame was fed with the vapour from heated metallic zinc.

The experiments with zinc were therefore repeated by us, with the result that the line was identified among those constituting the water-vapour group referred to by Ramage.

In these experiments the particular type of Bunsen burner, fig. V., used by McLennan and Thomson was adopted. To

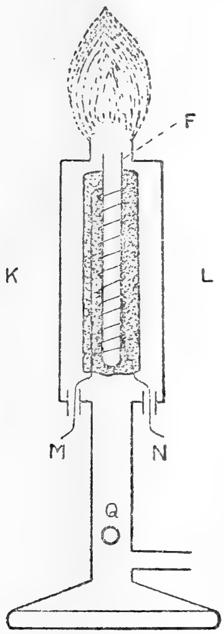


FIG. V.

the top of any ordinary Bunsen burner Q a brass cylinder KL, 3·8 cm. in diameter and 8 cm. high, was soldered. The top was closed by a lid containing an aperture about 2 cm. in

* Hemsalech, Phil. Mag. vol. xxxiv. p. 221 (1917).

† De Watteville, Phil. Trans. Roy. Soc. ser. A, vol. cciv. pp. 139–168

diameter. Another brass cylinder, 2·8 cm. in diameter and 7 cm. in length, was supported in the centre and coaxially with KL by means of three brass plugs placed between the cylinders. The inner cylinder contained a quartz tube F about 7 cm. in length. A coil of nichrome wire MN was wound around this tube, and the ends were led through openings fitted with small porcelain plugs in the bottom of KL. A layer of asbestos paper was placed around the wire, and the whole space between the quartz tube and brass cylinder packed with asbestos powder. When the gas was lighted a clear Bunsen flame was maintained above the mouth of the burner. The metals to be vaporized were placed within the quartz tube F, and the furnace was raised to whatever temperature was desired by applying a current of suitable strength to the circuit MN. The photographs were taken with a large Hilger quartz spectrograph, type C. Wratten and Wainwright Panchromatic plates manufactured by the Eastman Kodak Co. of New York were used.

The spectrograms taken are shown at the end of this paper. No. 1, fig. VI. (Pl. XIV.) is a reproduction of the spectrum of the zinc spark taken from a condensed discharge. No. 2 was obtained with zinc vapour in the Bunsen flame, and No. 3 shows the spectrum of the Bunsen flame free from the zinc vapour. In addition to the ordinary Bunsen flame spectrum, spectrogram No. 2 shows that the zinc line $\lambda = 3076{\cdot}03$ Å.U. came out strongly. This is well shown in No. 2 of the enlarged reproduction in fig. VII. (Pl. XV.). In no case did the flame spectrum show any trace of the line $\lambda = 2139{\cdot}33$ Å.U. Some experiments were also made with calcium, using the same type of Bunsen burner, and No. 1, fig. VIII. (Pl. XIV.) is the spectrogram of the calcium arc *in vacuo*, taken with an arc-lamp similar to the type described by McLennan and Henderson *. No. 2, fig. VIII. is a spectrogram of the Bunsen flame fed with calcium vapour. In addition to the ordinary flame spectrum, it shows the calcium line $\lambda = 4226{\cdot}91$ Å.U. of frequency $\nu = (1{\cdot}5, S)-(2, P)$. Another line is shown at about $\lambda = 4059$ Å.U., but this line must have been due to an impurity in the calcium metal, as no line is given at this wave-length by Eder and Valenta †. No. 3, fig. VIII. is a spectrogram of the ordinary Bunsen flame.

Since the line $\lambda = 2288{\cdot}79$ Å.U. came out in strong flames with cadmium vapour, it was thought the corresponding line

* McLennan and Henderson, *loc. cit.*

† Eder and Valenta, *Atlas Typischer Spektren*, Wien.

in the zinc flame spectrum $\lambda=2139 \cdot 66$ Å.U. might come out too. No trace of it, however, was found. With calcium no trace of the line $\lambda=2721$ Å.U. of frequency $\nu=(1 \cdot 5, S)-(3, P)$ was found in the flame spectrum. It is also of interest to note that with calcium no trace of the line $\lambda=6598$ Å.U. of frequency $\nu=(1 \cdot 5, S)-(2, p_2)$ was obtained. Since the line $\lambda=4226 \cdot 91$ Å.U. of frequency $\nu=(1 \cdot 5, S)-(2, P)$ came out so feebly, it was scarcely to be expected that the line $\lambda=2721$ Å.U., $\nu=(1 \cdot 5, S)-(3, P)$, or others of higher frequency in the series $\nu=(1 \cdot 5, S)-(m, P)$ would have been obtained. It should be remembered, however, that even in the spark or the arc spectrum of calcium the line $\lambda=2721$ Å.U. possesses relatively small intensity.

The results obtained with flame spectra in the present investigation as well as those obtained by McLennan and Thomson, it will be seen, do not afford much information as to the relative importance of the two series $\nu=(1 \cdot 5, S)-(m, p_2)$ and $\nu=(1 \cdot 5, S)-(m, P)$ from the point of view of fundamentality.

Summary of Results.

1. It has been shown that when zinc and cadmium vapours respectively are bombarded by electrons whose kinetic energy is gradually increased, monochromatic radiation is suddenly emitted by the vapour when the impact voltage is that given by the quantum relation for the frequency $\nu=(1 \cdot 5, S)-(2, p_2)$. When the impact voltage was increased beyond this amount no additional radiation was observed until that corresponding to the frequency $\nu=(1 \cdot 5, S)-(2, P)$ was applied. When these conditions were realized the wave-lengths whose frequencies are given by $\nu=(1 \cdot 5, S)-(2, p_2)$ and $\nu=(1 \cdot 5, S)-(2, P)$ were then recorded on the plates.

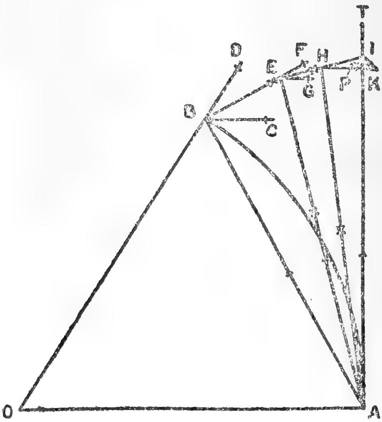
2. It has been shown that when a Bunsen flame is fed with the vapour of heated zinc, it is possible to obtain monochromatic radiation of wave-length $\lambda=3075 \cdot 99$ Å.U.

3. The evidence adduced goes to show that the series of wave-lengths given by $\nu=(1 \cdot 5, S)-(m, P)$ is probably fundamental from the point of view of electronic vibrations within the atoms of the elements mercury, zinc, cadmium, magnesium, calcium, and probably also strontium, and barium.

The Physical Laboratory,
University of Toronto.

LII. *Notes on a Geometrical Construction for rectifying any Arc of a Circle.* By F. A. LINDEMANN *.

NOTES have been published recently by M. de Pulligny and by R. E. Baynes giving geometrical constructions for the ratio π or some simple function of π. All of these are based upon some numerical coincidence which enables π or the function in question to be represented very closely by a ratio of fairly small whole numbers such as 355/113. The following construction may perhaps be of interest as it allows any arc of a circle to be rectified, and as it is based upon no such numerical coincidence but represents an extremely rapidly converging series. In principle, an extraordinary degree of accuracy is obtainable in a very short time; in practice, it need scarcely be said, it is of no more value for this purpose than any of the constructions whose accuracy can only be verified *a posteriori.*



Let AB be the arc whose length is to be determined.

Draw AT the tangent to AB at the point A.

Continue OB to D and draw BC parallel to OA.

Bisect ∢ DBC by line BF and ∢ BAT by line AE which cuts BF at E.

Draw EG parallel to OA.

Bisect ∢ FEG by line EH and ∢ EAT by line AH which cuts EH at H.

This process may be repeated as often as desired. In the present instance, for the sake of clearness in the diagram, no

* Communicated by the Author.

further bisection will be undertaken, and the point H will be used to determine the final result.

Draw HK parallel to OA meeting AT at K and continue EH until it cuts AT at I.

Divide KI in the ratio 1 : 2 at point P.

Then the straight line AP will be very nearly equal in length to the arc AB.

It is easy to demonstrate that this result is true. If $\measuredangle$AOB is called α and OA = 1, then $AI = 2^n \tan \frac{\alpha}{2^n}$, where n represents the number of times (in the present instance 2) that the process of bisecting the angles took place. Similarly

$$AK = 2 \sin \frac{\alpha}{2^n}.$$

Therefore

$$AP = 2^n \left\{ \sin \frac{\alpha}{2^n} + 1/3 \left(\tan \frac{\alpha}{2^n} - \sin \frac{\alpha}{2^n} \right) \right\}.$$

Expanding this one finds

$$\begin{aligned} AP = 2^n\{ &(\alpha/2^n) - 1/6(\alpha/2^n)^3 + 1/120(\alpha/2^n)^5 - \ldots \\ &+ 1/3((\alpha/2^n) + 1/3(\alpha/2^n)^3 + 2/15(\alpha/2^n)^5 + \ldots \\ &- (\alpha/2^n) + 1/6(\alpha/2^n)^3 - 1/120(\alpha/2^n)^5 + \ldots)\} \\ = 2^n\{ &(\alpha/2^n) + 1/20(x/2^n)^5 + \ldots\} = \alpha(1 + 1/20(\alpha/2^n)^4). \end{aligned}$$

The residual error $1/20(\alpha/2^n)^4$ is obviously reduced to 1/16 by each repetition of the bisecting process, and may therefore in theory be made very small indeed in a very short time. Even with but two bisections as in the above diagram, the error is only of the order of 1 part in 5000. A similar construction with a 90° arc would give π to 6 places of decimals if the bisecting process were repeated 5 times.

Farnborough.
June 8th, 1918.

[NOTE.

The method is interesting though hardly practical.

The details seem to be these :—

(1) The angles ABE, AEH are right angles.

For $A\hat{B}C = B\hat{A}O = 90° - \frac{\alpha}{2}$, as is seen by dropping a perpendicular from O on AB.

Whence also, incidentally, $B\hat{A}T=\frac{1}{2}\alpha$,

$E\hat{A}T=\frac{1}{4}\alpha$,

$H\hat{A}T=\frac{1}{8}\alpha$,

Now, $A\hat{B}C=90^\circ-\frac{\alpha}{2}$
and $C\hat{B}E=\frac{\alpha}{2}$ } therefore $A\hat{B}E=90^\circ$.

Similarly

$$A\hat{E}H=A\hat{E}G+G\hat{E}H=E\hat{A}O+G\hat{E}H$$

$$=90^\circ-\frac{\alpha}{4}+\frac{\alpha}{4}=90^\circ.$$

(2) $AB=2\sin\frac{\alpha}{2}$, seen by dropping same perpendicular from O on AB. Therefore

$$AE=2\sin\frac{\alpha}{2}\div\cos\frac{\alpha}{4}=2^2\sin\frac{\alpha}{2^2}=AK$$

$\left(AH=2^2\sin\frac{\alpha}{2^2}\div\cos\frac{\alpha}{2^3}=2^3\sin\frac{\alpha}{2^3}\right.$, not needed here though.)

$$AI=AE\div\cos\frac{\alpha}{2^2}=2^2\tan\frac{\alpha}{2^2}.$$

It is interesting to note that with A as origin and AT the initial line, the points B, E, H, ... all lie on the curve whose polar equation is $r=\frac{\alpha\sin\theta}{\theta}$. For, taking any one of the radii vectores, say AH, when $\theta=\frac{\alpha}{2^3}$ then $2^3=\frac{\alpha}{\theta}$, whence $AH=\frac{\alpha}{\theta}\sin\theta$.]

LIII. *On a Peculiarity of the Normal Component of the Attraction due to certain Surface Distributions.* By GANESH PRASAD, *M.A., D.Sc., Professor of Mathematics and Principal in the Hindu University of Benares* *.

THE object of this paper is to point out certain surface distributions for each of which the component N of the Newtonian attraction at a point P along the normal, which passes through P and meets the surface at a point O, tends to *no* limit as P approaches O along the normal. It is believed that such surface distributions have not been pointed out by any previous writer.

1. At P, let N be equal to N_1+N_2, where N_1 corresponds to a small area S round O, and N_2 to the remaining part of the surface. Then it is obvious that the limit of N_2 is existent; we have to consider the limit of N_1. For the sake of simplicity, the surface may be taken to be regular in the neighbourhood of O and, consequently, S may be taken to be a circle of centre O, radius a, and density σ.

Case 1. $\sigma = \cos\log\frac{1}{r}$.

2. First let $\sigma = \cos\log\frac{1}{r}$, where r is the distance between O and the point Q where the density is σ. Then it will be shown that the limit of N_1 is non-existent.

Divide the circle S into thin concentric rings. Then, taking the origin at O and the axis of z as the normal at O, we have

$$N_1 = -2\pi\int_0^a \frac{z\sigma r\,dr}{(z^2+r^2)^{3/2}}.$$

Thus we have to investigate

$$\operatorname*{Lim}_{z=0} N_1, \quad i.e., \quad -2\pi \operatorname*{Lim}_{z=0}\int_0^{a/z}\frac{t\sigma\,dt}{(1+t^2)^{3/2}},$$

where $zt=r$.

3. Now let C be a sufficiently large quantity independent of z. Then

$$\int_0^{a/z}\frac{t\sigma\,dt}{(1+t^2)^{3/2}} = \int_0^{C} + \int_C^{a/z}\frac{t\sigma\,dt}{(1+t^2)^{3/2}}.$$

* Communicated by the Author.

But

$$\sigma = \cos \log \frac{1}{r} = \cos \left\{ \log \frac{1}{z} + \log \frac{1}{t} \right\}.$$

Therefore

$$\int_0^C \frac{t\sigma dt}{(1+t^2)^{3/2}} = \mathrm{R} \cos \left\{ \log \frac{1}{z} + \gamma \right\}$$

approximately, where

$$\mathrm{R} \cos \gamma = \int_0^\infty \frac{t \cos \log \frac{1}{t}}{(1+t^2)^{3/2}} dt,$$

$$\mathrm{R} \sin \gamma = \int_0^\infty \frac{t \sin \log \frac{1}{t}}{(1+t^2)^{3/2}} dt.$$

Again,

$$\left| \int_C^{a/z} \frac{t\sigma\, dt}{(1+t^2)^{3/2}} \right| < \int_C^{a/z} \frac{t\, dt}{(1+t^2)^{3/2}}, \text{ i. e. } \left\{ \frac{1}{\sqrt{1+C^2}} - \frac{1}{\sqrt{1+\frac{a^2}{z^2}}} \right\}$$

which can be made as small as we please by choosing z to be sufficiently small and C to be sufficiently large. Thus it is proved that N_1 behaves as

$$-2\pi \mathrm{R} \cos \left\{ \log \frac{1}{z} + \gamma \right\},$$

as z tends to 0.

Therefore the limit of N_1, and, consequently, that of N are non-existent.

Case II. $\sigma = \cos \chi(r)$.

4. Take the general case in which $\sigma = \cos \chi(r)$, where $\operatorname*{Lim}_{r=0} \chi(r)$ is infinite. Then the same peculiarity is noticed as in Case I, if

$$\operatorname*{Lim}_{r=0} \frac{\chi(r)}{\log \frac{1}{r}}$$

is zero or a *finite* quantity different from zero. For the proof of this statement, see a paper of mine which will appear shortly in the 'Bulletin of the Calcutta Mathematical Society,' vol. ix.

LIV. *The Double Suspension Mirror.* *By* L. SOUTHERNS, *M.A., B.Sc., Assistant Lecturer in Physics in the University of Sheffield* *.

DURING the course of a series of experiments with a very delicate balance, in which a modification of the "double suspension mirror" method of observing deflexions was used, it became necessary to consider in detail the effect of the suspended system on the sensitiveness of the balance. As the method is capable of other applications, and especially as the attachment employed affords a convenient means of varying the sensitiveness through any required range, a description may be of interest to other users of sensitive instruments of the balance type.

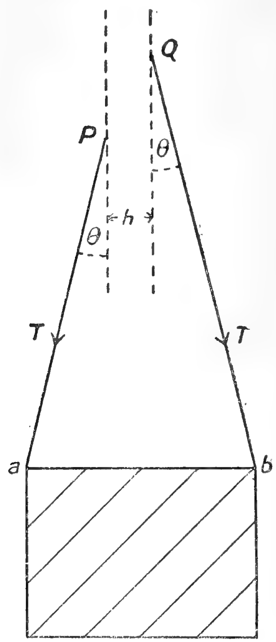
A small mirror $a\,b$ is suspended by two fibres as indicated in fig. 1—the plane of the diagram being the vertical plane in which the knife-edge of the balance lies—from Q, the end of the balance-pointer, and P, a support capable of adjustment, by means of a screw, in a horizontal line parallel to the knife-edge. A second screw for adjusting P perpendicularly to the plane of the figure is sometimes necessary. We shall first consider the deflexions of the balance-beam, and in this case the mirror will act merely as a weight causing tension in the fibres. Afterwards deflexions of the mirror itself will be considered, these of course being much greater than those of the beam. In this latter case, but hardly in the former, it is necessary to damp the vibrations of the mirror by means of a vertical wire projecting downward from it, carrying at its extremity a disk or set of vanes dipping into a small vessel containing oil. The viscosity of the oil should be as small as is consistent with effectiveness in stilling the vibrations. A good deal probably depends on the design of this damping arrangement, but the matter has not yet been investigated. It is desirable for practical reasons to arrange the points P Q at different levels; they can then be brought as nearly as may be required into the same vertical line without danger of actual contact of the fibres. The inclination of the fibres to the vertical has an important bearing on the theory of the method.

We suppose then for the present that $a\,b$ represents a weight attached to the fibres, and giving rise in them to

* Communicated by Dr. W. M. Hicks, F.R.S.

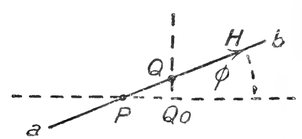
a tension T. In fig. 1 θ represents the inclination of either fibre to the vertical and h the horizontal distance between P and Q. The distance P Q is supposed to be small compared

Fig. 1.



with the length of the fibres, but great with respect to h. Small motions of P or Q will not appreciably affect either θ or T. In fig. 2, which represents a plan of the arrangement,

Fig. 2.



the point Q has moved perpendicularly to the plane of fig. 1 from its normal position Q_0 by reason of a small deflexion, say ψ, of the balance-pointer and beam. The deflexion of the mirror is represented by ϕ. H represents the horizontal component of the tension in the fibre Qb; this and the vertical component, say V, we shall regard as unaltered by small motions of P and Q. The component H cos ϕ of H is parallel to the knife-edge and does not tend to deflect the beam. The component H sin ϕ tends to increase this deflexion, while V tends to diminish it, the moment about the knife-edge being $Hl \sin \phi - Vl \sin \psi$, where l is the length of the pointer.

It will be convenient to compare the effect on the sensitiveness with that due to an imaginary alteration of the level of the centre of gravity of the beam, which may be effected by raising an imaginary weight w originally coincident with the knife-edge, through a distance r along a vertical wire attached to the beam. For a deflexion ψ the moment due to w will be $wr \sin \psi$. Thus the effect of the tension will be equal to that due to w provided that

$$Hl \sin \phi - Vl \sin \psi = wr \sin \psi,$$

or, since ψ and ϕ are small,

$$Hl\phi - Vl\psi = wr\psi. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

But $Q\,Q_0$ or $h \tan \phi$ is equal to $l \tan \psi$ or say

$$h\phi = l\psi, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

thus (1) becomes

$$Hl - Vh = \frac{wrh}{l}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

and we may say that for small values of ϕ the effect of the fibre on the balance is the same as that which would be due to a weight w placed at a distance r above the knife-edge, r being given by (3).

Since V is proportional to H and l is a constant, we may write $V = nlH$; then putting k^2 for $\frac{Hl^2}{w}$ the above condition becomes

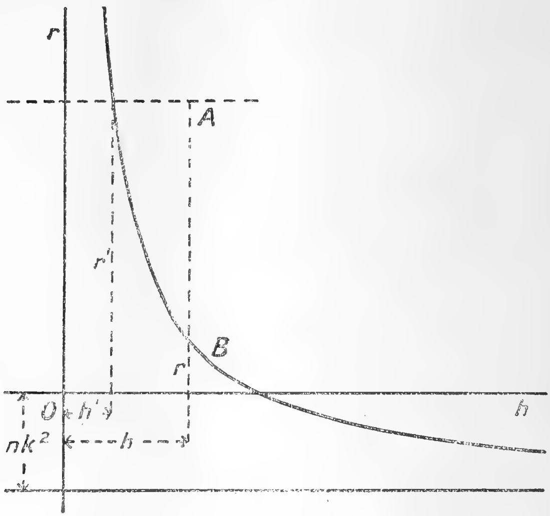
$$h(r + nk^2) = k^2, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (4)$$

where n and k^2 are constants. It is represented in fig. 3 by a rectangular hyperbola, whose centre is nk^2 below the axis of h. The part of the curve below Oh represents cases in which the fibres are separated so far as to diverge upwards, when they cause a reduction of sensitiveness instead of an increase. But our conditions do not apply to these extreme cases, since evidently H would not remain constant.

The variation of r with h is given by $\frac{dr}{dh} = -\frac{Hl^2}{wh^2}$. If

$h=0$, $r=\infty$; thus if P is brought into the same vertical as Q, the effect is equal to that of raising w to infinity. Thus whatever (finite) stability the beam may have when the fibres

Fig. 3.



are at an initial distance say h_0 apart, it will become unstable or infinitely sensitive for some smaller but still finite value of h. If the balance is originally very stable, this of course will only apply to extremely small values of ψ, since the above only holds for small values of ϕ, which is great compared with ψ when h is very small. In order then to make the stable balance as sensitive as we please, we need only move up P towards the vertical through Q by means of its adjusting screw. This may be done from the outside of the balance-case without touching or even arresting the beam.

Now suppose that in a given case, in order to reduce the balance to the point of instability, it would be sufficient to raise w to a height r' above the knife-edge. The same result will be produced by decreasing the value of h to h', as shown in fig. 3. Short of this, say for the value h, the sensitiveness as measured by the deflexion of the beam for a given small load in one scale-pan will be proportional to $\frac{1}{\text{AB}}$ on the diagram, for AB represents to some scale the vertical distance of the centre of gravity of the beam below the knife-edge, which is clearly zero when w is at the level A.

Let S_ψ denote the sensitiveness as defined above, then

$$S_\psi \propto \frac{1}{r'-r} \propto \frac{1}{r' - \frac{k^2}{h} + nk^2}$$

by (4), or say

$$S_\psi = \frac{q^2}{p - \frac{k^2}{h}}, \quad \ldots \ldots \ldots \quad (5)$$

where

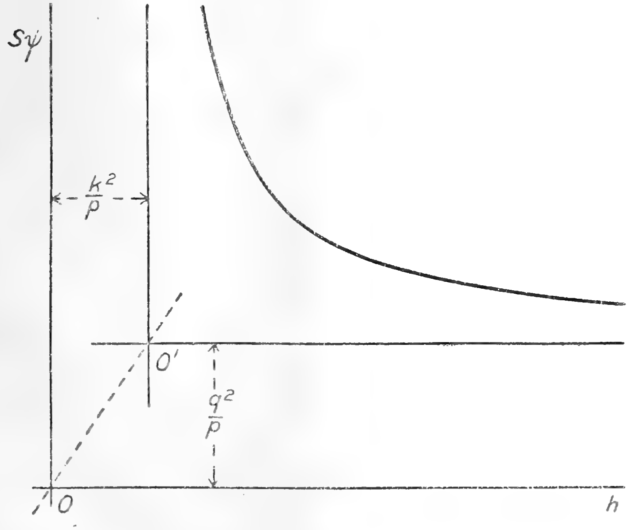
$$p = r' + nk^2, \quad \ldots \ldots \ldots \quad (6)$$

and q^2 is a constant. This equation may be written

$$\left(h - \frac{k^2}{p}\right)\left(S_\psi - \frac{q^2}{p}\right) = \frac{q^2k^2}{p^2},$$

and is represented by the rectangular hyperbola in fig. 4, its

Fig. 4.



centre O′ being at the point $\left(\frac{k^2}{p}, \frac{q^2}{p}\right)$ and its constant $\frac{k^2}{p} \cdot \frac{q^2}{p}$.
As the fibres are separated the sensitiveness tends to the

value $\frac{q^2}{p}$, but the law will alter for such wide separation. Actually this value will be reached when the fibres are parallel. When h is reduced to $\frac{k^2}{p}$ the sensitiveness becomes infinite.

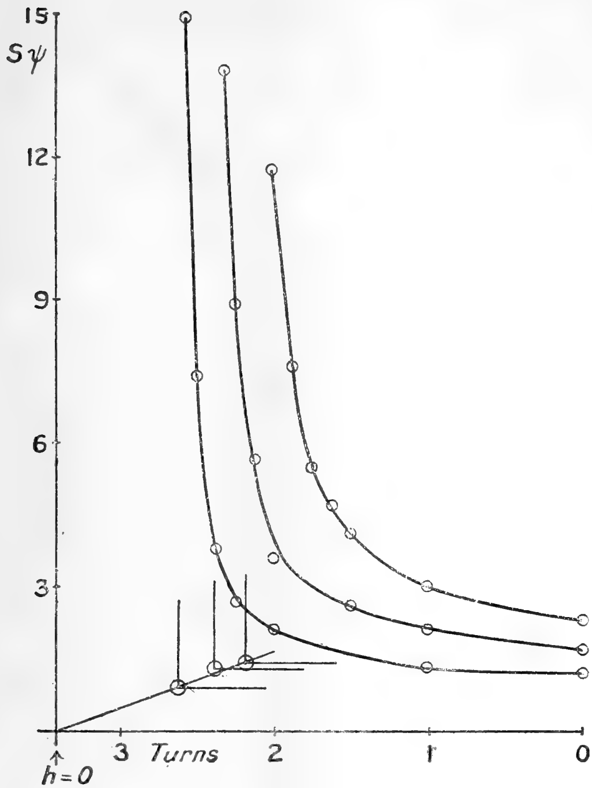
Such a curve as that given in fig. 4 may be plotted from the results of experiment. Distances Oh will not be known accurately, but if the ordinates be plotted against differences of Oh, which may be taken to be proportional to rotations of the adjusting screw, the position of O$'$ may be found by graphical means, thus giving the value of $\frac{q^2}{p}$ and also of the constant $\frac{q^2}{p} \cdot \frac{k^2}{p}$, from which $\frac{k^2}{p}$, and thus the position of O, can be determined.

Now suppose the balance to be made less sensitive by lowering the centre of gravity of the beam. This corresponds to an increase in r' the distance which w must be raised to give infinite sensitiveness. This means an alteration of the centre O$'$ in fig. 4, both its co-ordinates being reduced in the same ratio, as reference to (6) will show. Thus the locus of O$'$ is a straight line through O. This fact affords another means of obtaining the position of O, and the absolute horizontal distance between P and Q.

Fig. 5 gives a series of curves obtained by experiment on a Curie balance hastily fitted with roughly adjustable fibres carrying a small weight. Deflexions of the beam were read by means of the microscope which is permanently fitted to the balance. Ordinates of the curves represent deflexions for a small weight added to one scale-pan, and abscissæ rotations of the adjusting screw of the fibres. The curves are not perfect hyperbolas, but deflexions in some cases were by no means "small." The three curves correspond to three separate adjustments of the centre of gravity of the beam, the lower the centre of gravity the nearer the curve approaches to coincidence with the axes of h and Sψ. The centres lie approximately on a straight line which cuts the axis at a point corresponding to 3·4 turns from the initial position of the screw. Thus 3·4 turns would bring P, fig. 1, into the same vertical as Q.

Next consider the effect of increasing the weight of $a\,b$, fig. 1. It is clear from fig. 2 that the tendency will be to increase the sensitiveness, for a greater pull in the direction of increasing ψ will result. An increase of $k^2\left(=\frac{\mathrm{H}l^2}{w}\right)$ takes

Fig. 5



place due to increased tension of the fibres. From (5) and (6) we obtain

$$\left(\frac{\partial S_\psi}{\partial k^2}\right)_{h\ \text{const.}} = \frac{S_\psi\left(\frac{1}{h} - n\right)}{p - \frac{k^2}{h}}$$

$$= \frac{S_\psi^{\,2}\left(\frac{1}{h} - n\right)}{q^2};$$

or if W be the load *ab*,

$$\left(\frac{\partial S_\psi}{\partial W}\right)_{h\ \text{const.}} \propto S_\psi^{\,2}.$$

If $\frac{1}{h}-n$ is positive, S_ψ increases with k^2 or W. This condition roughly indicates that the fibres must be inclined sufficiently to the vertical to cause always a divergent force on the pointer.

A curve obtained by loading the fibres of the Curie balance is given in fig. 6. It should be part of a rectangular

Fig. 6

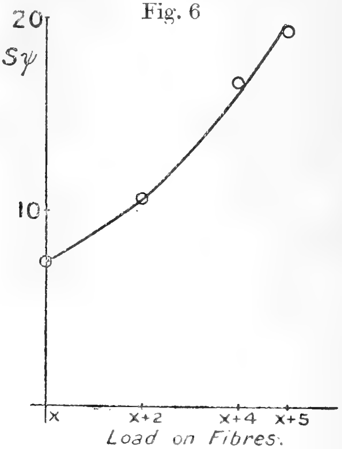
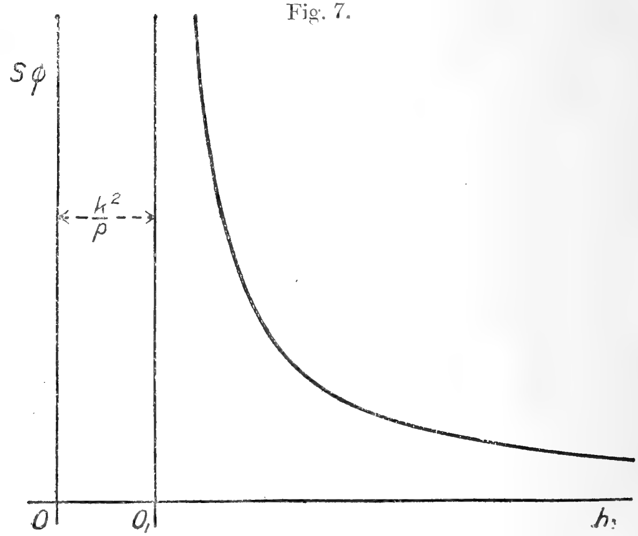


Fig. 7.



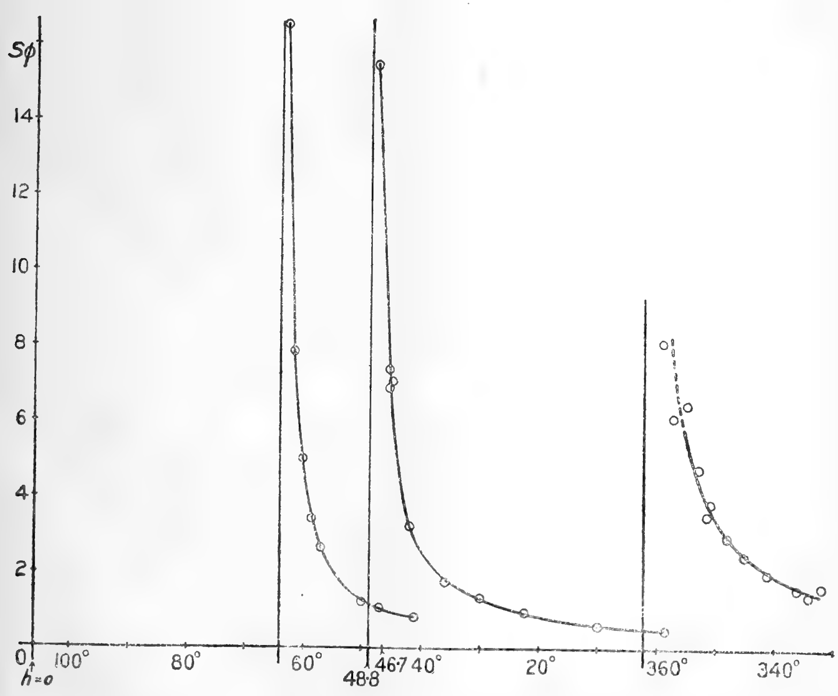
hyperbola in the left-hand upper quadrant. A finite load of this kind will produce infinite sensitiveness. From (5) and (6) we may easily show that this will occur when $k^2=\frac{hr'}{1-nh}$.

So far we have been concerned with sensitiveness as obtained by measuring deflexions of the beam itself. If now we consider $a\,b$ to represent a mirror whose deflexions are to be observed, a little further consideration is necessary. By (2) the deflexions ϕ of the mirror and ψ of the beam have the relation $\frac{\phi}{\psi}=\frac{l}{h}$, and our new sensitiveness S_ϕ will be similarly related to S_ψ or $\frac{S_\phi}{S_\psi}=\frac{l}{h}$. S_ϕ then represents the original sensitiveness multiplied by the magnification $\frac{l}{h}$ which becomes great when h is small. From this combined with (5) we obtain

$$S_\phi\left(h-\frac{k^2}{p}\right)=\frac{q^2 l}{p}, \quad . \quad . \quad . \quad . \quad . \quad (7)$$

which is a rectangular hyperbola whose centre O_1, fig. 7, is on Oh at distance $\frac{k^2}{p}$ from O. It lies vertically below O′ the

Fig. 8.



centre of the hyperbola giving deflexions of the beam under the same conditions of experiment. The two hyperbolas would intersect for the value $h=l$, but this is far beyond the range within which the above relations hold.

If we consider the values of S_ϕ for different positions of the centre of gravity of the beam, *i. e.* for different values of r' or of p, we shall obtain different hyperbolas of the type (7) whose centres lie on Oh. The constants of these hyperbolas vary directly as O O_1; and as they can be determined graphically for curves plotted from experimental values of the sensitiveness, the position of O can readily be obtained.

Fig. 8 gives a set of such hyperbolas obtained by means of the balance referred to at the beginning of this note. The mass of the beam is over 200 grms. The deflexions were observed by means of a telescope and scale. The vertical scale of the diagram represents deflexions in cms. caused by a load of $\frac{1}{100}$ mgm. at one end of the beam. The horizontal readings are taken from the graduated head of the adjusting screw. The position of O derived graphically from the 1st and 3rd curves corresponds to 106°. Using this value, and the constant of the 2nd curve obtained graphically, the centre of this curve is found to correspond to 46°·7 instead of 48°·8 as obtained from the curve alone. This gives an indication of the degree of approximation of the experimental values to the theory.

LV. *Notices respecting New Books.*

A Simplified Method of Tracing Rays through any Optical System. By LUDWIK SILBERSTEIN, Ph.D. Pp. vii+37. Longmans, Green & Co. Price 5*s.* net.

THIS little book deals with a subject which is of the utmost practical importance to all concerned with optical instruments. By using throughout the vectorial method the author has effected a considerable simplification of what is usually a laborious and complicated task, namely, the following of a ray through a system composed of any number of lenses, prisms, and mirrors. The knowledge of vectors required to enable the reader to make effective use of the book can be obtained in a few hours by anyone possessing the mathematical acquirements of the average worker in optics. The deduction of the vectorial form of the refraction (which really includes the reflexion) formula and the transfer formula for spherical surfaces occupies only a few pages, and the rest of the book is devoted to showing how the formulæ may be applied to numerical cases, and to expounding the author's dyadic operator for multiple reflexions. It is an original and suggestive little book, and we note with interest that it is written from the research department of Messrs. Hilger, whose name stands for so much in the realm of optical instruments.

Differential Equations. By H. Bateman. Pp. xi+306. Longmans, Green & Co. Price 16*s.* net.

In his preface the author states that he has endeavoured to supply some elementary material suitable for students studying the subject for the first time, and some more advanced work for mathematical physicists. We scarcely think that the book will appeal to young students, for such elementary matter as it contains is handled in a way too general and too little explicit to be grasped by a mind of small mathematical experience. To the more practised mathematician, however, the author offers an attractive treatment of the differential equations of types most commonly met with in physical problems, together with a chapter on mechanical integration which contains an account of Pascal's recent work. The book is completely in touch and sympathy with the most modern methods, and frequently introduces applications to recent work in mathematical physics, such as the author's treatment of the system of linear equations governing successive radioactive transformations, and Lorentz's electron theory equations.

There is an excellent selection of problems taken from a wide range of sources, and ample references for those who wish to pursue deeper the study of any particular subject. While the book is, perhaps, not well suited to form a first introduction to the subject, it is admirably adapted to be used in conjunction with one of the standard textbooks, and cannot fail to be useful to students of mathematical physics, especially those interested in research on this subject.

A History of Chemistry. By F. J. Moore, Ph.D. Pp. xiv+292. McGraw-Hill Book Company, New York, and Hill Publishing Company, 6 & 8 Bouverie Street, London, E.C. 4. Price 12*s.* 6*d.* net.

The value of a knowledge of the history of science in studying modern theories is fast being realized, and nowhere more so than in America. It is hard to find a subject of greater intrinsic interest, and on the history of chemistry Professor Moore writes with a knowledge and enthusiasm that makes his little book a fascinating one. A prominent feature are the illustrations, which not only include a wide range of portraits, from Basil Valentine to Sir Ernest Rutherford (this last a most excellent portrait), but also pictures of historical apparatus and laboratories, which give a clear idea of the conditions in which the older investigators carried out fundamental experiments and measurements of surprising accuracy. A praiseworthy effort is made to concentrate on work which has proved really basic in character. The book is notable in that the early developments, which receive much attention in most histories of chemistry, are comparatively briefly treated, attention being concentrated on the hard task of giving some idea of the great work of the nineteenth century and after. This has been very successfully carried out, with a good sense of proportion. The account is carried right down to the present day, and includes a description of the recent work on X-ray spectra and Moseley's atomic numbers. No student of science can fail to be interested by this book.

LVI. *Proceedings of Learned Societies.*

GEOLOGICAL SOCIETY.

[Continued from p. 428.]

June 5th, 1918.—Mr. G. W. Lamplugh, F.R.S., President, in the Chair.

THE following communications were read:—

1. 'The Kelestominæ, a Sub-Family of Cretaceous Cribrimorph Polyzoa.' By William Dickson Lang, M.A., F.G.S.

2. 'The Geology and Genesis of the Trefriw Pyrites Deposit.' By Robert Lionel Sherlock, D.Sc., A.R.C.Sc., F.G.S.

This pyrites deposit is worked at Cae Coch Mine, on the western side of the Conway Valley (North Wales), about 1 mile north of Trefriw.

A band of pyrites, about 6 feet thick, and of considerable purity, rests on the inclined top of a thick mass of diabase which is shown to be intruded into the Bala shales that cover the ore-body. The shales immediately above the pyrites are shown by the graptolites contained to belong to the zone of *Nemagraptus gracilis*, and are the equivalents of the Mydrim Limestone of South Wales and of part of the Lower Cadnant Shales of the Conway Mt. succession: that is, they are near the base of the Bala Series according to the Geological Survey classification (Carmarthen Memoir, 1909). Northwards the intrusive is bounded by an overthrust mass of volcanic ash, which itself is cut off by an east-and-west fault against rhyolite, well seen in a roadside quarry and in the crags of Clogwyn Mawr. Intrusions of dolerite of much later age, probably late Devonian, or Carboniferous, are found in the rhyolite, and form the plateau above the mine, passing over shales, diabase, ash, and rhyolite in turn.

Pyrites deposits are classified by Beyschlag, Vogt, and Krusch in four groups:—(1) Magmatic segregations, (2) formed by contact-metamorphism, (3) lodes, (4) of sedimentary origin. None of these modes of origin, however, will account for the Trefriw pyrites. The conclusion arrived at is that the diabase was intruded below a bed of pisolitic iron-ore. Hot water containing sulphuretted hydrogen given off from the intrusion, combined readily with the pisolites, which were in the form either of oxide or of silicate of iron, and formed pyrites. The graptolitic horizon at which the pisolitic ore occurs usually contains some pyrites, and this would be added to that derived from the above reaction. The pyrites was not formed by ordinary contact-metamorphism; because the intrusion is seen, at places where the pyrites is absent, to exert only a slight hardening effect on the shale. In North Wales pisolitic iron-ore is known to occur in several places at the horizon of *Nemagraptus gracilis*. From the mode of origin assigned above to the pyrites it follows that the mineral is of Bala age, since it was formed before the intrusion, itself of Bala age, had cooled. The pisolitic ironstone must have been in existence in Bala times, and this supports the idea that the ironstone is a bedded contemporaneous deposit.

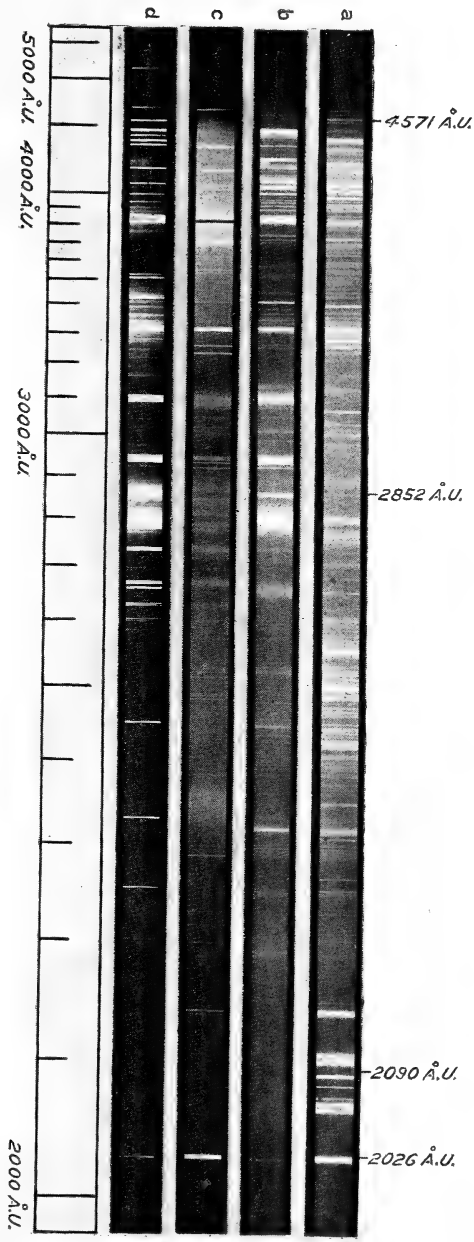

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Fig. I.

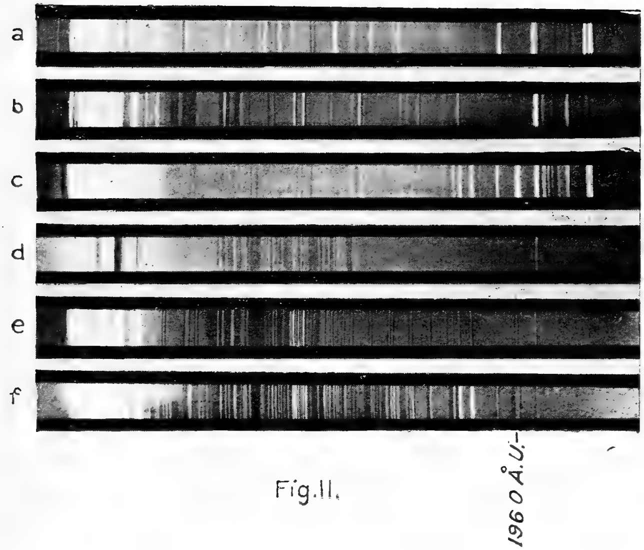

Fig. 11.

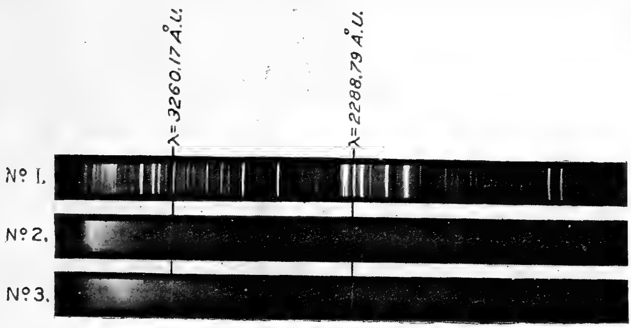

Fig. IV.

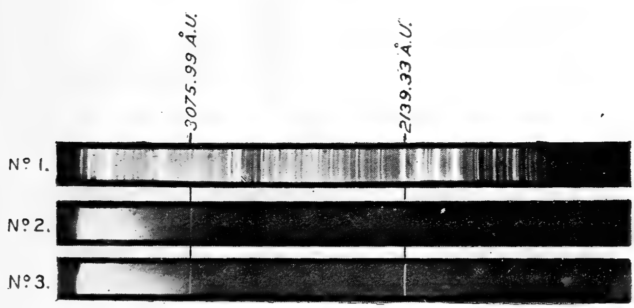

Fig. III.

Fig. II.

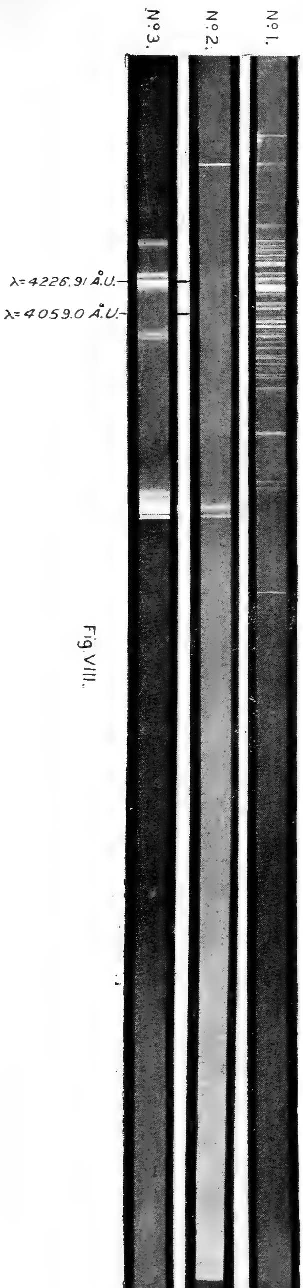

Fig. VIII.

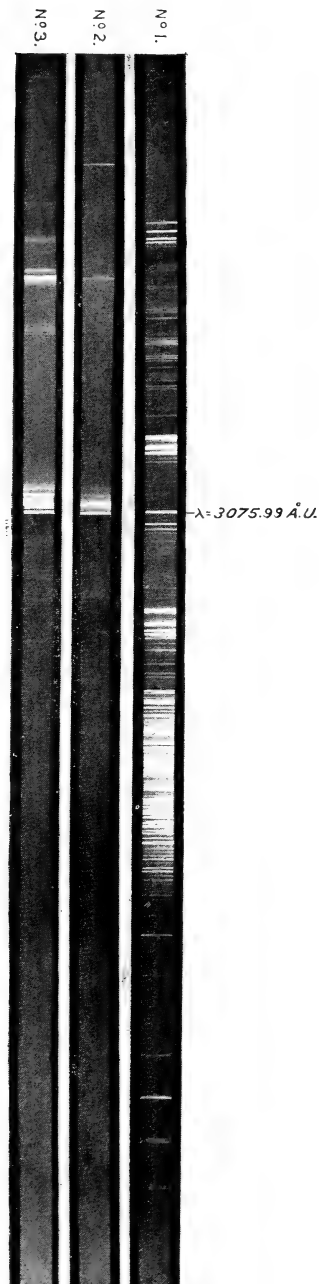

Fig. VI.

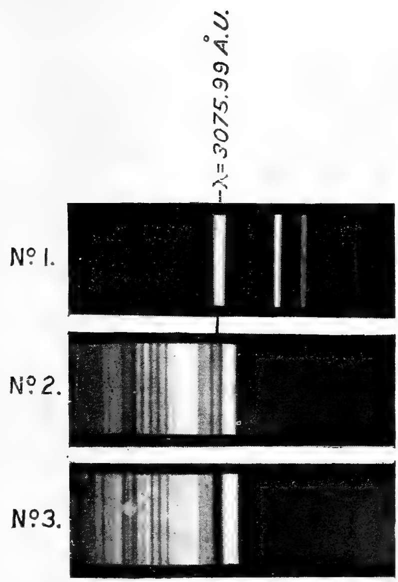

Fig. VII.

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