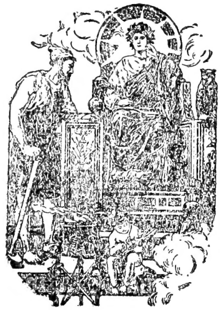


4

THE
LONDON, EDINBURGH, AND DUBLIN
PHILOSOPHICAL MAGAZINE
AND
JOURNAL OF SCIENCE.

CONDUCTED BY

SIR OLIVER JOSEPH LODGE, D.Sc., LL.D., F.R.S.
SIR JOSEPH JOHN THOMSON, O.M., M.A., Sc.D., LL.D., F.R.S.
JOHN JOLY, M.A., D.Sc., F.R.S., F.G.S.
GEORGE CAREY FOSTER, B.A., LL.D., F.R.S.
AND
WILLIAM FRANCIS, F.L.S.

"Nec aranearum sane textus ideo melior quia ex se fila gignunt, nec noster vilior quia ex alienis libamus ut apes." JUST. LIPS. *Polit.* lib. i. cap. 1. Not.

VOL. XXXVII.—SIXTH SERIES.

JANUARY—JUNE 1919.

LONDON:

TAYLOR AND FRANCIS, RED LION COURT, FLEET STREET.

SOLD BY SIMPKIN, MARSHALL, HAMILTON, KENT, AND CO., LD
SMITH AND SON, GLASGOW;—HODGES, FIGGIS, AND CO. DUBLIN —
AND VEUVE J. BOYVEAU, PARIS

"Meditationis est perscrutari occulta; contemplationis est admirari perspicua .... Admiratio generat quæstionem, quæstio investigationem, investigatio inventionem."—*Hugo de S. Victore.*

——"Cur spirent venti, cur terra dehiscat,
Cur mare turgescat, pelago cur tantus amaror,
Cur caput obscura Phœbus ferrugine condat,
Quid toties diros cogat flagrare cometas,
Quid pariat nubes, veniant cur fulmina cœlo,
Quo micet igne Iris, superos quis conciat orbes
Tam vario motu."

J. B. Pinelli ad Mazonium.

ALERE FLAMMAM.

CONTENTS OF VOL. XXXVII.

(SIXTH SERIES).

NUMBER CCXVII.—JANUARY 1919.

NUMBER CCXVIII.—FEBRUARY.

NUMBER CCXIX.—MARCH.

NUMBER CCXX.—APRIL.

NUMBER CCXXI.—MAY.

NUMBER CCXXII.—JUNE.

PLATES.

I. Illustrative of Mr. T. K. Chinmayam's Paper on the Flow of Energy in the Electromagnetic Field surrounding a Perfectly Reflecting Cylinder.

II. Illustrative of Mr. K. S. Mitra's Paper on Sommerfeld's Treatment of the Problem of Diffraction by a Semi-infinite Screen.

III. & IV. Illustrative of Prof. S. Banerji's Paper on the Radiation of Light from the Boundaries of Diffracting Apertures.

V. & VI. Illustrative of Prof. Barton and Miss Browning's Paper on Mechanical "Resonators" under Double Forcing.

THE
LONDON, EDINBURGH, AND DUBLIN
PHILOSOPHICAL MAGAZINE
AND
JOURNAL OF SCIENCE.

[SIXTH SERIES.]

JANUARY 1919.

I. *On Travelling Atmospheric Disturbances.*

By HAROLD JEFFREYS, *M.A.*, *D.Sc.*[*]

IT has been shown by Lamb [†] that an arbitrary disturbance of the uniform distribution of density in the atmosphere in horizontal layers would ordinarily give rise to a motion of the nature of a wave spreading out from the originally disturbed region, the velocity of propagation being of the same order of magnitude as that of sound. Thus a local variation in mass distribution would be rapidly dispersed over a wide area and the original uniform state restored. Any disturbance of even moderately permanent character must therefore be of a very special type. Now the ordinary cyclone is able to retain its size and pressure distribution about its centre for days, the velocity with which it moves being of the order of twenty feet per second, very much smaller than the speed of an atmospheric wave. Thus the first question with regard to the nature of a cyclone is, why does it not spread out like an ordinary wave and disperse in an hour or two? The answer seems to be that a cyclone is of the character of a *standing wave*, the pressure and velocity distribution being such that for this peculiar kind of disturbance the velocity of propagation is practically zero. The realization of this fact has led to the assumption of the well-known "Gradient relation," according to which the

---

[*] Communicated by the Author.

[†] "On Atmospheric Oscillations," Proc. Roy. Soc. lxxxiv. A. pp. 551–572 (1910).

wave velocity is very small, or, what is equivalent, that the acceleration terms in the equations of motion of the air can be neglected. If (u, v, w) be the components of velocity of the air at the point (x, y, z), at time t, the axis of z being vertical, the equations of motion take the form [*]

$$\left.\begin{aligned} \frac{du}{dt}-2\omega v &= -\frac{1}{\rho}\frac{\partial p}{\partial x} \\ \frac{dv}{dt}+2\omega u &= -\frac{1}{\rho}\frac{\partial p}{\partial y} \\ \frac{dw}{dt} &= -g-\frac{1}{\rho}\frac{dp}{dz} \end{aligned}\right\}, \quad \ldots\ldots \quad (1)$$

where

$$\frac{d}{dt}=\frac{\partial}{\partial t}+u\frac{\partial}{\partial x}+v\frac{\partial}{\partial y}+w\frac{\partial}{\partial z},$$

p is the pressure,
ρ is the density,
g is the acceleration due to gravity,
and ω is the component about the vertical of the earth's angular velocity of rotation.

Friction, which is only effective in the surface-layers, is ignored.

It is assumed first that the vertical velocity can be neglected. The writer has shown elsewhere that [†] this is justifiable in problems of winds caused by temperature variations of horizontal extent large compared with the height of the atmosphere, and as an average cyclone is of the order of 1000 km. across the same is probably true here. The assumption must not, however, be pushed too far; it would be in serious error for small islands, and probably also for land and sea breezes of the usual diurnal type. For the ordinary widespread depression, however, it is probably correct, and the third equation of motion becomes

$$p=\int_z^\infty g\rho dz. \quad \ldots\ldots\ldots \quad (2)$$

The assumption that the disturbance is permanent gives at once that $\frac{\partial u}{\partial t}$ and $\frac{\partial v}{\partial t}$ are zero; hence the two equations of horizontal motion become

$$\left.\begin{aligned} u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}-2\omega v &= -\frac{1}{\rho}\frac{\partial p}{\partial x} \\ u\frac{\partial v}{\partial x}+v\frac{\partial v}{\partial y}+2\omega u &= -\frac{1}{\rho}\frac{\partial p}{\partial y} \end{aligned}\right\} \quad \ldots\ldots \quad (3)$$

---

[*] *Cf.* Lamb, 'Hydrodynamics,' p. 302.

[†] Phil. Mag. vol. xxxiv. pp. 449–458 (1917).

Further simplification is not possible without some knowledge of the size of the quantities involved. If it be assumed, as is usually correct in these latitudes, that the pressure gradients and wind velocities are small enough for their squares to be neglected in a first approximation, we have nearly

$$\left.\begin{aligned}-2\omega v &= -\frac{1}{\rho}\frac{\partial p}{\partial x}\\ 2\omega u &= -\frac{1}{\rho}\frac{\partial p}{\partial y}\end{aligned}\right\}, \quad . \quad . \quad . \quad . \quad . \quad . \quad (4)$$

which gives the so-called "Geostrophic relation." If the isobars are concentric circles we have as a second approximation, the density being assumed constant, the equation

$$\frac{V^2}{r}+2\omega V=\frac{1}{\rho}\frac{\partial p}{\partial r}, \quad . \quad . \quad . \quad . \quad . \quad . \quad (5)$$

where V is the resultant velocity and r the distance from the common centre, which is the usual form of the equation giving the wind in terms of the pressure gradient. It is often assumed to be correct even when the pressure distribution is changing with the time ; but this involves the assumption that the air is not accelerated along its path, which means that $\frac{dV}{dt}$ is neglected while $\frac{\partial p}{\partial t}$ is retained. This is a very uncertain hypothesis, for in general these two quantities would be expected to be of the same order of magnitude.

When the pressure distribution is varying, the problem becomes much more complex. A cyclone in most cases moves fairly steadily in one direction, the isobars remaining approximately concentric. Sometimes the depression in the centre deepens as it moves, more often it becomes shallower and spreads out, but frequently it travels for thousands of miles practically unchanged. Now if it were merely a wave free to spread out, it would, as was said before, do so with a velocity comparable with that of sound, and would therefore disappear in a few hours. On the other hand, the motion of the depression is not itself of the character of the propagation of a wave, for it only takes place at the rate of some feet per second. Thus the moving depression, like the stationary one, requires peculiar conditions for its maintenance, and it is intended to indicate some of these in the present paper. The method adopted is that of successive approximation according to powers of the pressure gradient.

Suppose that the speed of translation of the cyclone is small, of the same order of magnitude at least as that of the winds themselves. Then $\partial/\partial t$ is of the order $u\partial/\partial x$ and the only first order terms in the equations of motion are those in the equations

$$\left.\begin{aligned} -2\omega v &= -\frac{1}{\rho}\frac{\partial p}{\partial x} \\ 2\omega u &= -\frac{1}{\rho}\frac{\partial p}{\partial y} \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad . \quad (6)$$

Thus the geostrophic relation still holds as a first approximation. When, however, the second powers of the velocities are negligible, no disturbance can travel. For the rate of increase of the surface pressure at any point is given by

$$\frac{\partial p_0}{\partial t} = \int_0^\infty g\frac{\partial \rho}{\partial t}dz = -g\int_0^\infty \left\{\frac{\partial}{\partial x}(\rho u) + \frac{\partial}{\partial y}(\rho v) + \frac{\partial}{\partial z}(\rho w)\right\}dz, \quad (7)$$

by the equation of continuity.

Now ρw is zero when $z=0$ and when z is infinite, since there is no vertical velocity on the ground and the density tends to zero at a great height. Hence

$$\frac{\partial p_0}{\partial t} = -g\int_0^\infty \left\{\frac{\partial}{\partial x}(\rho u) + \frac{\partial}{\partial y}(\rho v)\right\}dz. \quad . \quad . \quad (8)$$

Substituting in this from the equations (6) we have identically

$$\frac{\partial p_0}{\partial t} = 0. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (9)$$

Thus to this order the pressure distribution is not varying. Hence if it is changing at all it must depend on powers of the pressure gradient higher than the first. The assumption that this kind of approximation is possible is therefore so far justified.

A second approximation may now be obtained by substituting the values of u and v given by (6) into the terms in (1) depending on the squares of the velocities, and again determining u and v as far as the second powers of the pressure gradients. As $\partial p/\partial t$ is of higher order than $u\partial p/\partial x$, it follows that $\partial u/\partial t$ and $\partial v/\partial t$ are of higher order than $u\partial u/\partial x$; thus they contain no terms of the second order and therefore must be neglected. The vertical velocity also must be zero unless the distribution is changing; hence to this order it also may be neglected in the equations of motion. Then

$$\left.\begin{aligned} 2\omega\rho u &= -\frac{\partial p}{\partial y} + \frac{1}{2\omega}\left(\frac{\partial p}{\partial y}\frac{\partial v}{\partial x} - \frac{\partial p}{\partial x}\frac{\partial v}{\partial y}\right) \\ 2\omega\rho v &= \frac{\partial p}{\partial x} - \frac{1}{2\omega}\left(\frac{\partial p}{\partial y}\frac{\partial u}{\partial x} - \frac{\partial p}{\partial x}\frac{\partial u}{\partial y}\right) \end{aligned}\right\}, \quad . \quad (10)$$

and

$$\frac{\partial p_0}{dt} = -\frac{g}{4\omega^2}\int_0^\infty \left\{\frac{\partial}{\partial x}\left(\frac{\partial p}{\partial y}\frac{\partial v}{\partial x} - \frac{\partial p}{\partial x}\frac{\partial v}{\partial y}\right) - \frac{\partial}{\partial y}\left(\frac{\partial p}{\partial y}\frac{\partial u}{\partial x} - \frac{\partial p}{\partial x}\frac{\partial u}{\partial y}\right)\right\} dz, \quad (11).$$

where for u and v we must substitute their values from (4).

So far these results are general, subject to the validity of the approximations made, which seems satisfactory in ordinary cases. Further progress, however, requires some knowledge of the relations connecting pressure and density with position ; and when the actual laws that hold in the atmosphere are substituted the formulæ soon become unmanageable. An approximation can nevertheless be employed that enables the actual conditions to be imitated without making the algebra quite intractable. In the ordinary cyclone below the stratosphere the difference of pressure from normal does not vary greatly with the height, and it appears as if the disturbance arose from a change within the stratosphere : thus the conditions within the troposphere could be represented by a variation in the height of the free surface of an incompressible fluid. At the same time there is a well marked temperature gradient, the temperature usually increasing towards the south in the troposphere ; the opposite seems to hold in the stratosphere*. Thus the troposphere can be represented by an incompressible fluid of finite depth whose temperature and therefore whose density are monotonic functions of one of the horizontal coordinates. Let the mean height of the free surface be H, and the excess of the actual height of any point of the free surface above this be ζ. Also let the density be given by $\rho = \rho_0 + \rho_1$, where ρ_1 is small and a function of x and z only.

It is further supposed that ζ and ρ_1 are small enough for

---

* W. J. Humphreys, Bull. Mt. Weather Obs. vol. ii. pp. 292–297 (1910).

their third powers to be neglected in the present approximation. Then

$$u=-\frac{g}{2\omega}\frac{\partial\zeta}{\partial y};\quad v=\frac{g}{2\omega}\frac{\partial\zeta}{\partial x}+\frac{g}{2\omega\rho_0}\int_z^{\mathrm{H}}\frac{\partial\rho_1}{\partial x}dz\;\ldots\quad(12)$$

to the first order, and substituting from these into (11) we find

$$\begin{aligned}-\frac{8\omega^3}{g^3\rho_0}\frac{\partial\rho_0}{\partial t}&=\left(\frac{\partial\zeta}{\partial y}\frac{\partial^3\zeta}{\partial x^3}-\frac{\partial\zeta}{\partial x}\frac{\partial^3\zeta}{\partial x^2\partial y}-\frac{\partial\zeta}{\partial x}\frac{\partial^3\zeta}{\partial y^3}+\frac{\partial\zeta}{\partial y}\frac{\partial^3\zeta}{\partial x\,\partial y^2}\right)\mathrm{H}\\&\quad-\left(\frac{\partial^3\zeta}{\partial x^2\partial y}+\frac{\partial^3\zeta}{\partial y^3}\right)\frac{1}{\rho_0}\int_0^{\mathrm{H}}\int_z^{\mathrm{H}}\frac{\partial\rho_1}{\partial x}dz+\frac{1}{\rho_0}\frac{\partial\zeta}{\partial y}\int_0^{\mathrm{H}}\int_z^{\mathrm{H}}\frac{\partial^3\rho_1}{\partial x^3}dz\,dz\\&=\mathrm{H}\left(\frac{\partial\zeta}{\partial y}\frac{\partial}{\partial x}\nabla_1^2\zeta-\frac{\partial\zeta}{\partial x}\frac{\partial}{\partial y}\nabla_1^2\zeta\right)\\&\quad+\frac{1}{\rho_0}\frac{\partial\zeta}{\partial y}\int_0^{\mathrm{H}}\int_z^{\mathrm{H}}\frac{\partial^3\rho_1}{\partial x^3}dz\,dz-\frac{1}{\rho_0}\frac{\partial}{\partial y}\nabla_1^2\zeta\int_0^{\mathrm{H}}\int_z^{\mathrm{H}}\frac{\partial\rho_1}{\partial x}dz\,dz\\&=\frac{\mathrm{H}\,\partial(\nabla_1^2\zeta',\zeta')}{\partial(x,y)},\quad\ldots\ldots\ldots\quad(13)\end{aligned}$$

where $$\zeta'=\zeta+\frac{1}{\rho_0\mathrm{H}}\int_0^{\mathrm{H}}\int_z^{\mathrm{H}}\rho_1\,dz\,dz,\quad\ldots\quad(14)$$

and $$\nabla_1^2\equiv\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}.$$

Now $g\rho_0\zeta'$ is the mean of the excesses of the pressure above normal at all points of a vertical column. Also by hypothesis $\partial\rho/\partial t=0$, and therefore if $\mathrm{P}=g\rho_0\zeta'$, P satisfies the equation

$$-\frac{8\omega^3\rho_0}{g\mathrm{H}}\frac{\partial\mathrm{P}}{\partial t}=\frac{\partial(\nabla_1^2\mathrm{P},\mathrm{P})}{\partial(x,y)}.\quad\ldots\quad(15)$$

From this several interesting consequences can be deduced at once. If for instance this mean pressure anomaly is constant over concentric circles, it is a function of the distance from the common centre of these circles, and therefore $\nabla_1^2\mathrm{P}$ is a function of P and the Jacobian vanishes. Thus if a depression is perfectly symmetrical the pressure cannot vary with the time, and therefore all pressure changes must be caused by departures from circular symmetry. Again, if the curves P=constant are symmetrical with respect to two perpendicular axes, let us take these to be the axes of x and y; then P is an even function of x and y, and therefore so is $\nabla_1^2\mathrm{P}$; hence the Jacobian is an even function of x and y and the same must apply to the pressure changes.

Motion of the depression in one definite direction cannot therefore occur, though it may change its form subject to the condition that it must remain symmetrical with respect to these axes. Hence a cyclone cannot travel if it is symmetrical with respect to two axes; if it is to do so it must be definitely ovoid in form. A case of motion produced by asymmetry is afforded by the ordinary cyclone. This forms at the edge of a region of low pressure, so we shall take the general distribution of pressure to be decreasing in the direction of x increasing. In such a system a depression forms, the lowest pressure being near its centre. It seems, however, that the distribution does not consist of a symmetrical distribution on which a uniform increase in one direction is superposed. For if it were so, we should have

$$P = -Bx + R,$$

where R is a function of x^2+y^2 only, and B is a constant. Then

$$\frac{\partial(\nabla_1^2 P, P)}{\partial(x, y)} = B\frac{\partial \nabla_1^2 R}{\partial y},$$

and the rate of variation of P is given by

$$\frac{\partial P}{\partial t} = -\frac{gHB}{8\omega^3\rho_0}\frac{\partial \nabla_1^2 R}{\partial y}.$$

Now if the disturbance is travelling unaltered we must have

$$\frac{\partial}{\partial t} + U\frac{\partial}{\partial x} + V\frac{\partial}{\partial y} = 0,$$

where U and V are the components of the velocity of translation, and the object of the operation is any of the physical quantities of the system. But the density does not depend on t or y, and therefore this operation on it shows that U is zero. Next applying it to P we find

$$V\frac{\partial R}{\partial y} - \frac{gHB}{8\omega^3\rho_0}\frac{\partial \nabla_1^2 R}{\partial y} = 0.$$

First, consider the order of magnitude of the velocity of translation thus indicated. Let the horizontal dimensions of the cyclone be of order a; then $\nabla_1^2 R$ is comparable with R/a^2, and V is of order $-\dfrac{gHB}{8\omega^3\rho_0 a^2}$.

Now $B/2\omega\rho_0$ is the geostrophic wind corresponding to the general pressure gradient; and $gH/4\omega^2$ is the square of the distance sound would travel in an interval comparable with 12 hours, which is itself much greater than the linear dimensions of an ordinary cyclone, as has already been mentioned. Thus if the pressure were distributed in the simple way here supposed, the velocity of translation would be much greater than the general velocity of the winds around the cyclone. This does not appear to be the case; for it would imply that the depression moved at such a rate that no part of the surrounding air could keep pace with it, and probably none of the air it itself contains. This contradicts the fact of the existence of a tornado centre *, or portion of air within the cyclone moving with the cyclone as a whole. The explanation seems to be that the actual effect of superposing a general pressure gradient on a mass of rotating air is to cause a considerably smaller change in its symmetry than would be expected, the reason being that some internal compensation reduces the asymmetry. That this reduction is not complete is, on the other hand, seen from the fact that the cyclone does move, and usually does so in the direction of the general gradient wind.

Summary.

The geostrophic relation between the wind and the surface pressure gradient is incapable of accounting for any variation whatever with time in the pressure distribution. All changes in this arise from those terms in the equations of motion that are neglected when the geostrophic relation is assumed. When these terms, which depend on the squares and differential coefficients of the velocities, are taken into account, it is found that an asymmetrical cyclone can move. It seems, however, from the low speed of travel of these depressions, that a simple superposition of a general pressure gradient on a rotating system must be compensated internally in some way, so as to reduce the asymmetry introduced. Thus the remarkable circularity of the isobars in a cyclone is seen to be a condition of its slow movement. It is indicated that the cyclone itself is a very special type of disturbance, in which the pressure, temperature, and velocity are so distributed as to make the wave tending to readjust it travel with extreme slowness; other types of disturbance spread out much more rapidly (with velocities of the order of that of sound) and are dissipated, and this fact is probably the reason why of all the irregularities possible the cyclone is the most conspicuous, other forms dissipating before they can be observed.

---

* Sir Napier Shaw, Geophysical Memoirs of the Meteorological Office, No. 12.

II. *On the Flow of Energy in the Electromagnetic Field surrounding a Perfectly Reflecting Cylinder.* *By* T. K. Chinmayam, *B.A.* (*Hons.*), *Research Scholar in the Indian Association for the Cultivation of Science, Calcutta* *.

[Plate I.]

1. *Introduction.*

IN a paper recently contributed to this Journal, Mr. N. Basu † has discussed the general features of the phenomena observed in the immediate neighbourhood of a perfectly reflecting cylinder on which plane light waves are incident in a direction at right angles to its axis. Further investigation was, however, necessary in order to establish the formulæ for the distribution of light intensity in the various parts of the field. These formulæ have now been obtained and subjected to a detailed experimental test. Besides describing the results of a photometric study that has been carried out, the present paper also deals with the form of the lines of flow of energy through the field which, it is thought, may prove of interest with reference to the work of Profs. R. W. Wood ‡ and Max Mason § on the simpler case of the interference field due to two-point sources of light.

It may be remarked here that the phenomena which form the subject of this paper may be strikingly shown on a large scale without the aid of a microscope by using a cylindrical surface of very large radius as the diffracting "edge." A strip of thick plate glass, two inches wide and about a yard long, may be bent into a circle of some yards radius by resting it on supports near the two ends and loading the latter sufficiently. A slit illuminated by a Cooper-Hewitt lamp and placed at some distance from the surface in a line with it, may be used as the source of light. A very large number of fringes may then be seen with a low-power eyepiece if the plane of observation be within a few feet of the cylindrical "edge." At greater distances, the fringes widen out; their visibility and number decrease, and their spacing alters with increasing distances from the cylinder in such manner as to approximate more and more closely to that of

---

* Communicated by Prof. C. V. Raman, M.A.

† N. Basu: "On the Diffraction of Light by Cylinders of large radius." Phil. Mag. xxxv. p. 79 (1918).

‡ R. W. Wood: "On the Flow of Energy in a System of Interference Fringes." Phil. Mag. vol. xviii. p. 250 (1909).

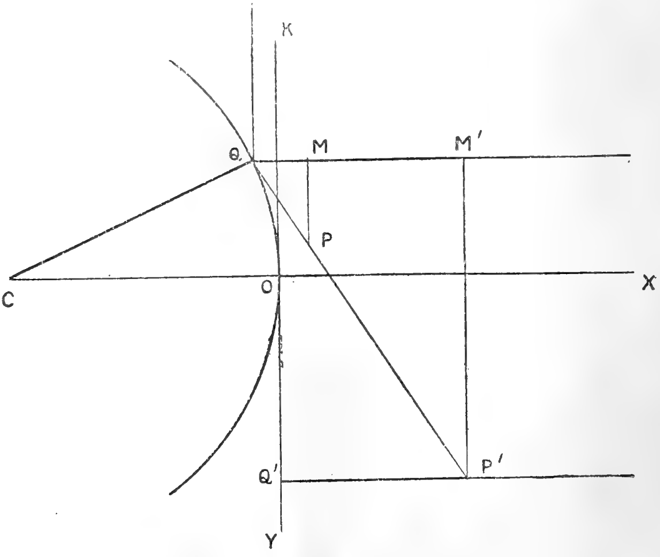
§ Max Mason: "The Flow of Energy in an Interference Field." Phil. Mag. vol. xx. p. 290 (1910).

the diffraction-fringes due to a straight edge. Some photographs taken with the arrangement described above are shown in Plate I., the figures (*a*), (*b*), (*c*) corresponding to the phenomena in planes at increasing distances from the "edge."

2. *The Form of the Illumination Curves.*

Debye [*] has shown from the electromagnetic theory that at a great distance from a cylinder (assumed to be of a large radius) on which plane waves are incident, the disturbance due to it is practically the same as that to be expected from the principles of geometrical optics, this statement, however, not being taken as correct in respect of points lying in a direction very nearly the same as that of the incident rays. Debye's results suggest a simple method of finding the distribution of intensity at points lying within the region of light in the immediate neighbourhood of the cylinder. Let QO (fig. 1) represent the section of the cylinder and KOY the direction of the incident rays. We may assume that the fringes observed in the plane OX containing the "edge" O

Fig. 1.



---

[*] Debye: *Phys. Zeitschr.* ix. pp. 775–778, Nov. 1908; also 'Science Abstracts,' 1909, p. 88.

of the cylinder grazed by the rays are due solely to the interference of these rays with those reflected from the surface of the cylinder at varying angles. This would also be the case as regards any plane such as QM′ in advance of the edge. But the phenomena in a plane such as Q′P′, which lies on the remote side of the edge, would not admit of such simple treatment, especially when we consider the effect at points lying not far from the boundary OY of the direct and reflected rays. In such a plane, the intensity at any point on the right of the boundary may be regarded as due to the superposition of three factors: (*a*) the effect due to the direct rays, (*b*) that due to the reflected rays, and (*c*) a diffraction effect mainly perceptible in the neighbourhood of the boundary. If the cylinder were replaced by a perfectly reflecting semi-infinite screen lying in the plane CO with its edge at O, the diffraction-effect would be found as in Sommerfeld's * well-known investigation, by superposing upon the direct rays a radiation emitted by the edge of the screen. It will be observed that in the present case the intensity of the rays regularly reflected from the surface of the cylinder, as given by the formulæ of Geometrical Optics, is zero along the boundary OY, and increases slowly as we move away from the boundary into the region of light, and thus presents no discontinuity. It thus seems justifiable to assume that so far as regards the phenomena in the region on the right-hand side of the boundary, the diffraction-effect (*c*) is practically the same as in the case of a semi-infinite screen with its edge at O.

The distribution of intensity in the field may be readily found on the foregoing assumptions. Take the "edge" O as origin of coordinates, and rectangular axes OX, OY, perpendicular and parallel respectively to the direction of the incident rays, and let the angle OCQ (assumed to be small) be denoted by θ. The path difference between the direct and the reflected rays reaching any point P is

$$\begin{aligned}\delta &= (\mathrm{QP}-\mathrm{MP})2\pi/\lambda+\pi\\ &= 2k\theta^2(y+a\theta)+\pi \text{ approximately,}\end{aligned}$$

where $k=2\pi/\lambda$.

Again, if the amplitude of the incident light be taken as unity, that of the reflected light may be written as

$$\{\rho/(\rho+\mathrm{QP})\}^{\frac{1}{2}},$$

---

* Sommerfeld: "On the Math. Theory of Diffraction," *Math. Annalen*, vol. xlvii. p. 317 (1895).

where ρ is the radius of curvature of the reflected wave on emergence at Q. Since $\rho = a\theta/2$ approximately,

$$\frac{\rho}{\rho + \mathrm{QP}} = \frac{a\theta}{2y + 3a\theta}.$$

In any plane in advance of that passing through the edge of the cylinder ($y = -d$, say), the expression for the intensity of illumination at any point is

$$\mathrm{I} = 1 + s - 2\sqrt{s}\cos\phi, \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

where $\phi = 2k\theta^2(y + a\theta)$ and $s = a\theta/(3a\theta + 2y)$.

The positions of the maxima and minima of illumination are given by

$$\frac{d\mathrm{I}}{d\theta} = \frac{ds}{d\theta}\left(1 - \frac{\cos\phi}{\sqrt{s}}\right) + 2\sqrt{s}\sin\phi\frac{d\phi}{d\theta} = 0. \quad . \quad . \quad (2)$$

Since $d\phi/d\theta = 4k\theta(y + a\theta) + 2k\theta^2 a$, and contains the factor $2\pi/\lambda$, it will be large, so that the first term in equation (2) is negligible. That equation hence becomes

$$\sin\phi = 0,$$

or

$$2\theta^2(y + a\theta) = m\lambda/2, \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

while the relation between θ and x is given by

$$x = (y + a\theta)2\theta - a\theta^2/2 = 2y\theta + 3a\theta^2/2. \quad . \quad . \quad (4)$$

From (1) and (3) it is seen that the intensities of the successive maxima and minima are respectively proportional to

$$\mathrm{I}_{\text{max.}} = \left\{1 + \left(\frac{a\theta}{3a\theta - 2d}\right)^{\frac{1}{2}}\right\}^2,$$

$$\mathrm{I}_{\text{min.}} = \left\{1 - \left(\frac{a\theta}{3a\theta - 2d}\right)^{\frac{1}{2}}\right\}^2.$$

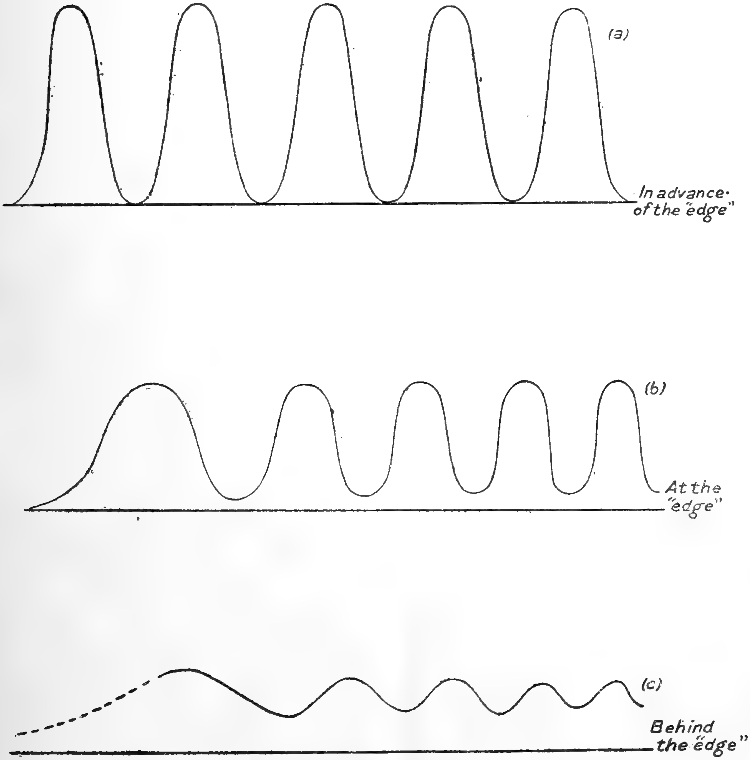
The intensity-curve has been plotted out in fig. 2 (*a*) for the case $a = 1{\cdot}5$ cm. and $d = 0{\cdot}2$ cm. It will be seen that in this plane $\mathrm{I}_{\text{max.}}$ begins nearly with a value $(1+1)^2 = 4$ and drops down gradually in successive fringes to a limiting value of $(1 + 1/\sqrt{3})^2 = 2{\cdot}49$ approx. $\mathrm{I}_{\text{min.}}$ increases from a value nearly zero to a limiting value $(1 - 1/\sqrt{3})^2 = 0{\cdot}18$ approx. The visibility of the successive fringes in the plane of observation therefore decreases slowly.

At the plane $y = 0$, the illumination is given by

$$\mathrm{I} = 1 + \tfrac{1}{3} - 2\cos\phi/\sqrt{3}. \quad . \quad . \quad . \quad . \quad . \quad . \quad (5)$$

The intensity-curve is shown in fig. 2 (*b*). $\mathrm{I}_{\text{max.}}$ has a constant value of $(1+1/\sqrt{3})^2$ and $I_{\text{min.}}$ a value of $(1-1/\sqrt{3})^2$. The visibility of the fringes is thus stationary along this plane for a considerable distance from the edge.

Fig. 2.



Intensity-curves at different parts of the Field.

Passing on to consider the distribution of intensity in any plane Q′P′ below the plane $y=0$, we have, as remarked above, to add to the effect of the direct and reflected rays a diffraction effect. We shall represent the latter effect by that due to a single source placed at the edge O, the amplitude of the disturbance due to it at a point in the region bounded by Ox and Oy being given by Sommerfeld's expression *

$$\frac{1}{4\pi}\sqrt{\frac{\lambda}{r}}\cos\left(kr-nt+\frac{\pi}{4}\right)\left\{\pm\frac{1}{\cos\frac{\phi+\phi'}{2}}-\frac{1}{\cos\frac{\phi-\phi'}{2}}\right\}, \qquad (6)$$

---

* Sommerfeld, *loc. cit.*

where r is the distance of the point from the edge O, ϕ, ϕ' are the angles which the diffracted and incident beams respectively make with the direction xO. If we denote the angle P′OY (fig. 1) by α, $\phi=\frac{3\pi}{2}-\alpha$; $\phi'=\frac{\pi}{2}$, so that the expression (6) becomes

$$\frac{1}{4\pi}\sqrt{\frac{\lambda}{r}}\cos\left(kr-nt+\frac{\pi}{4}\right)\left(\pm 1-\frac{1}{\sin\frac{\alpha}{2}}\right)$$

$$=-\frac{1}{4\pi}\cdot\frac{2}{\alpha}\sqrt{\frac{\lambda}{r}}\cos\left(kr-nt+\frac{\pi}{4}\right),$$

since α is small so far as our present investigation is concerned. Now $r=\mathrm{P'O}=y+x^2/2y$; $\alpha=x/y$ approximately. Hence the above expression may be written

$$-\frac{\sqrt{y\lambda}}{2\pi x}\cos\left(k\,\overline{y+\frac{x^2}{2y}}-nt+\frac{\pi}{4}\right).$$

The total disturbance at P′ is thus

$$\xi=\cos(ky-nt)-\sqrt{s}\cos\left(ky-nt+k\,\overline{2a\theta^3+2y\theta^2}\right)$$
$$-\frac{\sqrt{y\lambda}}{2\pi x}\cos\left(k\,\overline{y+\frac{x^2}{2y}}-nt+\frac{\pi}{4}\right).$$

Remembering that

$$x=3a\theta^2/2+2y\theta,\quad x^2/2y=2y\theta^2+3a\theta^3,$$

we get for the intensity of illumination at any point the expression

$$\mathrm{I}=1+\mathrm{S}-2\sqrt{\mathrm{S}}\cos\chi, \qquad (7)$$

where
$$\mathrm{S}=s+\frac{y\lambda}{4\pi^2x^2}+\frac{\sqrt{sy\lambda}}{\pi x}\cos\left(ka\theta^3+\frac{\pi}{4}\right), \qquad (8)$$

$$\chi=k(2a\theta^3+2y\theta^2)+\epsilon, \qquad (9)$$

and
$$\tan\epsilon=\frac{\frac{\sqrt{y\lambda}}{\pi x}\sin\left(ka\theta^3+\frac{\pi}{4}\right)}{2\sqrt{s}+\frac{\sqrt{y\lambda}}{\pi x}\cos\left(ka\theta^3+\frac{\pi}{4}\right)}. \qquad (10)$$

Equations (4) and (7) give for the positions of the maxima and minima of illumination

$$\left.\begin{aligned}x&=2y\theta+3a\theta^2/2,\\ 2a\theta^3+2y\theta^2&=m\lambda/2-\epsilon\lambda/2\pi\end{aligned}\right\} \qquad (11)$$

We see that the introduction of the diffraction term has slightly changed the positions of the maxima and minima. The magnitude of the change depends upon ϵ, which is zero when $y=0$ (equation 10), and steadily increases with y to a limiting value of $(ka\theta^3+\pi/4)$ or $\pi/4$, since $ka\theta^3$ is negligible over the first few bands when y is sufficiently large. By actual calculation it is found that this limit is practically reached when y is over three times the radius of the cylinder. Under these conditions equations (11) reduce to

$$x^2=y\lambda(4m-1)/4. \quad . \quad . \quad . \quad . \quad . \quad . \qquad (12)$$

Formula (12) is identical with Schuster's formula for the case of diffraction of plane waves by a straight edge.

Returning to equation (7) we see that the intensity of illumination of the successive maxima and minima is given by

$$\left.\begin{array}{l} I_{\text{max.}}=(1+\sqrt{S})^2 \\ I_{\text{min.}}=(1-\sqrt{S})^2 \end{array}\right\}, \quad . \quad . \quad . \quad . \quad . \quad . \qquad (13)$$

where S is given by equation (8). The intensity-curve for a plane 5 mm. behind the edge is shown in fig. 2 (*c*). It will be seen that the ratio of the minima to the maxima is considerably greater than in (*a*) and (*b*). Calculation also shows that the visibility of successive fringes in this plane of observation should decrease, though somewhat slowly. For still greater distances from the edge of the cylinder the illumination curves become practically identical with those of the Fresnel type due to a straight edge, the intensity of the reflected rays becoming negligible in comparison with that of the incident and the diffracted rays. The ordinates of the curves (*a*), (*b*), (*c*), (though not the abscissæ), have all been drawn to the same scale, and the curves illustrate the fact that the luminosity of the field as a whole decreases as we recede from the cylinder.

3. *Photometric Study of the Field.*

The formulæ obtained above have been tested by two independent methods, (1) by photometric comparison of the maxima and minima of illumination, and (2) by determination of their relative positions.

A small polished cylinder of glass, of about 1·5 cm. radius, was used. It was mounted on one of the stands of an optical bench, and a microscope objective mounted on another of those stands was brought up close to the cylinder. Light from a narrow slit was passed through a collimating lens and was allowed to fall grazingly on the cylinder. The field was viewed through a micrometer eyepiece placed at a distance behind the objective. By moving the cylinder towards or away from the objective, the phenomena at different planes $y=d$ could be observed.

The photometric arrangement used to study the relative intensity of the fringes was based upon a polarization method. The beam of light was plane-polarized by passage through a nicol before falling on the cylinder. The eyepiece (a low-power one) was moved off to a pretty large distance behind the objective, and two narrow slits cut out of aluminium-foil and pasted on two glass strips were mounted, one above the other, between the eyepiece and the objective, so as to allow a small relative motion which could be controlled by a micrometer-screw. A thin mica plate was also fixed up on the upper one, which was movable in the experiment, the thickness of the plate and its orientation with respect to the slit being adjusted by trial so that under the conditions of the experiment it circularly polarized the light falling on it. The field was viewed through another nicol fitted with a graduated circle and mounted just behind the slits. The lower slit was always set on the first bright band, while the upper one was set successively on the different maxima and minima. Equality of illumination was obtained in each case by rotating the analysing nicol. The reading for the crossed position of the analyser being also taken, the ratio of the intensities of illumination could be easily calculated. Thus if ψ be the orientation of the analyser in any case (ψ being reckoned from the crossed position) and I, I_0 the intensities of illumination of the upper and lower slits respectively,

$$I/I_0=\sin^2\psi.$$

Readings were taken for the first few bands on the planes $y=0$ and $y=0{\cdot}5$ cm., and are given in Table I. with the corresponding values calculated from theory.

TABLE I.

I_n/I_1.	$y=0$.				$y=0{\cdot}5$ cm.			
	Maxima.		Minima.		Maxima.		Minima.	
	Obsd.	Calcd.	Obsd.	Calcd.	Obsd.	Calcd.	Obsd.	Calcd.
1	1·00	1·00	0·09	0·07	1·00	1·00	0·46	0·42
2	0·96	1·00	0·08	0·07	0·91	0·93	0·45	0·44
3	1·00	1·00	0·12	0·07	0·91	0·91	0·52	0·46
4	0·94	1·00	0·15	0·07	0·83	0·90	0·52	0·46

The discrepancies are within the limits of experimental error, so that the theory developed above appears to be substantially correct.

As has been remarked already, the theory was also tested by measurements of the positions of the minima of illumination in different parts of the field. Some explanation is here necessary with regard to the measurements of fringes in a plane in advance of the "edge" (*y* negative). With the ordinary arrangement as described above, if the microscope is moved near so that its focal plane may be in advance of the "edge," we are unable to see the exact phenomena in that plane, since the light has to come past the edge before it can fall on the objective, and secondly, as has been fully described in Mr. Basu's paper, the field is complicated by the occurrence of the caustic and its accompanying fringes formed by reflexion from the surface of the cylinder. This difficulty was got over in the present work by turning the cylinder around its axis till the desired plane of observation coincided with the boundary of the polished surface. What is meant may be better understood by a reference to fig. 1, the process described being equivalent to cutting off the cylinder along CQ and removing the lower half. It is not easy, however, with this arrangement directly to determine the value of *y* when it is negative; and in Table II. below it has been calculated from a pair of readings. The source of light was a quartz mercury lamp with a green ray filter.

Table II.

Fringes between the cylindrical edge and the source of light

n.	$x_n - x_1$ in mms.	$y=0$.		$y=-0{\cdot}2$ cm.		$y=-0{\cdot}29$ cm.	
		Obsd.	Calcd.	Obsd.	Calcd.	Obsd.	Calcd.
3 ...		0·0169	0·0169	0·0082	0·0082	0·0062	0·0060
5 ...		0·0307	0·0302	0·0165	0·0163	0·0123	0·0120
7 ...		0·0420	0·0417	0·0240	0·0240	0·0183	0·0182
9 ...		0·0524	0·0522	0·0310	0·0317	0·0235	0·0235
11 ...		0·0620	0·0619	0·0378	0·0395	0·0290	0·0293

Table III.

Fringes behind the cylindrical "edge."

n.	x_n-x_1 in mms.	$y=0·1$ cm.		$y=0·3$ cm.		$y=0·5$ cm.		$y=0·7$ cm.	
		Obsd.	Calcd.	Obsd.	Calcd.	Obsd.	Calcd.	Obsd.	Calcd.
2 ...		0·0159	0·0158	0·0250	0·0251	0·0341	0·0326	0·0370	0·0370
3 ...		0·0284	0·0284	0·0439	0·0450	0·0580	0·0580	0·0660	0·0661
4 ...		0·0385	0·0390	0·0600	0·0609	0·0776	0·0782	0·0887	0·0897
5 ...		0·0483	0·0480	0·0740	0·0750	0·0928	0·0938		
6 ...		0·0568	0·0570	0·0867	0·0879	0·1116	0·1134		
7 ...		0·0651	0·0650	0·0990	0·0997				
8 ...		0·0715	0·0723	0·1108	0·1110				
9 ...		0·0794	0·0796						
10 ...		0·0862	0·0863						

The agreement between the calculated and the observed values is very close and confirms the theory.

4. *The Loci of Maxima and Minima of Illumination.*

These curves have an interesting property which may be briefly considered here. The equation to the loci is obviously

$$r(1-\cos 2\theta)=m\lambda/2 \quad \text{or} \quad r\theta^2=m\lambda/4 \text{ approx.}, \quad . \quad (14)$$

where r is the distance of the point from the surface of the cylinder, measured along the reflected ray which passes through the point. The shape of the curves is indicated by the thick lines in fig. 4, for the case $a=5$ inches, $\lambda=0·016$ inch, λ being taken so large for convenience of representation to scale; only the first, third, fifth, &c. loci are drawn.

The equation to the loci can also be got in terms of θ and x or y. Thus on eliminating y from equations (3) and (4) we get

$$x=\frac{m\lambda}{2\theta}-\frac{a\theta^2}{2}, \quad . \quad . \quad . \quad . \quad . \quad . \quad (15)$$

which gives the abscissæ of the points at which the loci cut the straight lines $\theta=$const. The ordinates of these points are given by

$$2y\theta = x - \frac{3}{2}a\theta^2 = \frac{m\lambda}{2\theta} - 2a\theta^2,$$

so that

$$y = \frac{m\lambda}{4\theta^2} - a\theta. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (16)$$

From (15) and (16) it is seen that

$$dx = -\left(\frac{m\lambda}{2\theta^2} + a\theta\right)\delta\theta$$

and

$$dy = -\left(\frac{m\lambda}{2\theta^3} + a\right)\delta\theta,$$

and hence $dy/dx = \theta$. At any point, therefore, the curves bisect the angle between the directions of the incident and the reflected rays which pass through that point.

As might be expected, the formulæ obtained above for the case of diffraction by a cylinder reduce to the ordinary formulæ for diffraction by a straight edge on writing $a=0$, provided of course that the light from the other side of the cylinder is cut off by a semi-infinite plane extending to the left of the origin O. For then equations (11) become

$$\left.\begin{aligned} x &= 2y\theta, \\ 2y\theta^2 &= m\lambda/2 - \epsilon\lambda/2\pi \end{aligned}\right\},$$

and $\epsilon = \frac{\pi}{4}$, so that $x^2 = y\lambda(4m-1)/4$. This is Schuster's formula for diffraction of plane waves at a straight edge. The results regarding the loci of maxima and minima of illumination will also apply for diffraction at a straight edge under the same conditions.

5. *The Flow of Energy in the Field.*

We will first take into account only the effects due to the interference of the direct and the reflected rays. The effect due to diffraction at the edge of the cylinder can be brought in later as a correction.

Let us assume, for simplicity, that the light is polarized in the plane of incidence, so that the electric intensity is perpendicular to that plane and the magnetic intensity lies in it. Then at any point P (fig. 1) the resultant electric intensity is

$$\mathrm{E} = \cos n\left(t - \frac{r_1}{c}\right) + \left(\frac{\rho}{\rho + r_2}\right)^{\frac{1}{2}} \cos\left\{n\left(t - \frac{r_2}{c}\right) + \pi\right\},$$

where $r_1 = \mathrm{PM}$, $r_2 = \mathrm{PQ}$, and ρ is the radius of curvature of the reflected wave at Q. The expression $(\rho/\rho+r_2)^{\frac{1}{2}}$ will in the rest of the paper be denoted by κ. The resultant magnetic intensity at P is

$$H=\overrightarrow{\cos n\left(t-\frac{r_1}{c}\right)}+\kappa\,\overrightarrow{\cos\left\{n\left(t-\frac{r_2}{c}\right)+\pi\right\}}.$$

Let now k_1, k_2 be unit vectors at P in the direction of the incident and the reflected rays respectively. The flow of energy is determined by the Poynting vector S where

$$S=\frac{c}{4\pi}[EH]=\frac{c}{4\pi}\left\{\cos n\left(t-\frac{r_1}{c}\right)+\kappa\cos\left(\overline{nt-\frac{r_2}{c}+\pi}\right)\right\}$$
$$\times\left\{k_1\left[\cos n\left(t-\frac{r_1}{c}\right)\right]+k_2\left[\kappa\cos\left(\overline{nt-\frac{r_2}{c}+\pi}\right)\right]\right\},$$

or

$$\frac{4\pi S}{c}=k_1[\cos^2\chi+\kappa\cos\chi\cos\chi']+k_2[\kappa\cos\chi\cos\chi'+\kappa^2\cos^2\chi'],$$

where $\chi=n\left(t-\frac{r_1}{c}\right)$ and $\chi'=n\left(t-\frac{r_2}{c}\right)+\pi.$

The time-mean $\overline{S}$ of the flow of energy is given by

$$\frac{4\pi\overline{S}}{c}=\tfrac{1}{2}k_1[1+\kappa\cos(\chi'-\chi)]+\tfrac{1}{2}k_2[\kappa^2+\kappa\cos(\chi'-\chi)],$$

$$\frac{8\pi\overline{S}}{c}=k_1[a_1]+k_2[a_2],$$

if $a_1=1+\kappa\cos(\chi'-\chi)$ and $a_2=\kappa^2+\kappa\cos(\chi'-\chi)$.

If ϕ be the angle which the direction of $\overline{S}$ makes with that of the incident rays, it can be easily shown that

$$\tan\phi=\frac{a_2\sin 2\theta}{a_1+a_2\cos 2\theta}=\frac{2a_2\theta}{a_1+a_2}\text{ approx.}$$

Thus $$\tan\phi=\frac{2\theta\{\kappa^2+\kappa\cos(\chi'-\chi)\}}{1+2\kappa\cos(\chi'-\chi)+\kappa^2}\quad\ldots\ldots\quad(17)$$

Now $\chi'-\chi=\frac{n(r_2-r_1)}{c}+\pi=\frac{n}{c}2r_2\theta^2+\pi$ approx.

Hence (17) becomes

$$\tan\phi=\frac{2\theta\left\{\kappa^2-\kappa\cos\frac{n}{c}2r_2\theta^2\right\}}{1+\kappa^2-2\kappa\cos\frac{n}{c}2r_2\theta^2}\quad\ldots\quad(18)$$

The current of energy at the point is

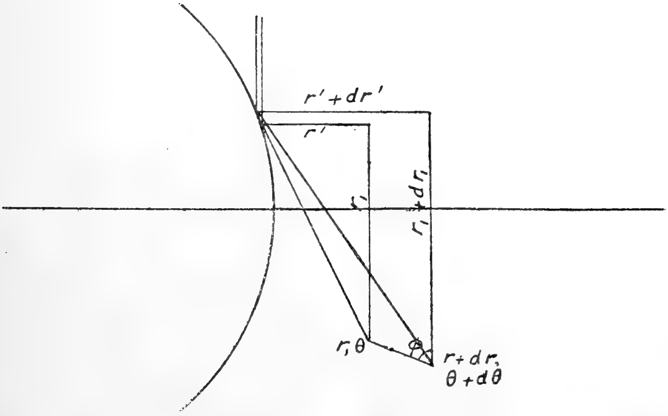
$$\bar{\mathrm{S}}=\frac{c}{8\pi}\{a_1{}^2+a_2{}^2+2a_1a_2\cos 2\theta\}^{\frac{1}{2}}$$

$$=\frac{c}{8\pi}\left\{1+\kappa^2-2\kappa\cos\frac{n}{c}2r_2\theta^2\right\}\quad . \quad . \quad . \quad (19)$$

The lines of flow of energy are determined by the condition that at any point (r_2, θ) the inclination to the direct rays is ϕ, where ϕ is given by equation (18). From this we can get the differential equation to the lines of flow in terms of r_2 and θ, or, dropping the suffix, r and θ. It is seen from fig. 3 that

$$\tan\phi=\frac{dr'-a\theta\,d\theta}{dr_1-a\,d\theta}.$$

Fig. 3.



But $r'=r\sin 2\theta=2r\theta$ and $r_1=r\cos 2\theta=r$ approximately.

Hence $$\tan\phi=\frac{2(r\,d\theta+\theta\,dr)-a\theta\,d\theta}{dr--a\,d\theta},\quad . \quad . \quad . \quad (20\,a)$$

or $$\tan\phi\left(1-a\frac{d\theta}{dr}\right)=2\theta+(2r-a\theta)\frac{d\theta}{dr}.\quad . \quad . \quad (20)$$

From (18) and (20), we get

$$\frac{d\theta}{dr}(2r-a\theta)+2\theta=\left(1-a\frac{d\theta}{dr}\right)\frac{2\kappa\theta\left(\kappa-\cos\frac{n}{c}2r\theta^2\right)}{1+\kappa^2-2\kappa\cos\frac{n}{c}2r\theta^2},$$

which on reduction becomes

$$\frac{d\theta}{dr}=-\frac{\theta}{r}\,\frac{1-\kappa\cos\frac{n}{c}\,2r\theta^2}{1-2\kappa\cos\frac{n}{c}\,2r\theta^2}. \quad . \quad . \quad . \quad (21)$$

Equation (21) may also be written

$$d(r\theta)-\frac{\kappa}{2\theta}\cdot\frac{c}{n}\cdot\cos\psi\,d\psi=0, \quad . \quad . \quad . \quad (22)$$

where $\psi=2r\theta^2\cdot\frac{n}{c}$. In the immediate neighbourhood of a point we can regard $\kappa/\theta=\{a/\theta(2r+a\theta)\}^{\frac{1}{2}}$ as constant, since its variation with r and θ will be small compared with the periodic part. On integrating (22) we get

$$r\theta-\left(\frac{\kappa}{2\theta}\right)\frac{c}{n}\cdot\sin\psi=\text{const.}, \quad . \quad . \quad . \quad (23)$$

which determines the shape of the lines of flow in the immediate neighbourhood of a point. The points of intersection of these curves with the loci of maximum and minimum of illumination ($\psi=m\pi$) are given by the equation

$$r\theta=\text{const.} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (24)$$

This gives the "mean lines of flow" about which energy crinkles down. They are shown in fig. 4 (thin lines). We find that so long as θ is not very large, these curves are inclined to the direct rays at angles smaller than those corresponding to the maxima and minima loci. If we imagine the latter set of curves as forming successive bright and dark tubes, energy will flow down across the tubes, its direction being periodically shifted such that it tends to flow along the bright tubes and to cut across the dark tubes. The shift in its direction goes through one complete cycle as the energy passes from one dark tube to the next or from one bright tube to the next, so that the "wave-length" of a crinkle in the neighbourhood of a point (r,θ) may be determined by finding the distance along the curve $r\theta=$const. between two successive points of intersection of that curve and the family of curves $r\theta^2=m\lambda/2$ (the loci of minima of illumination). Thus

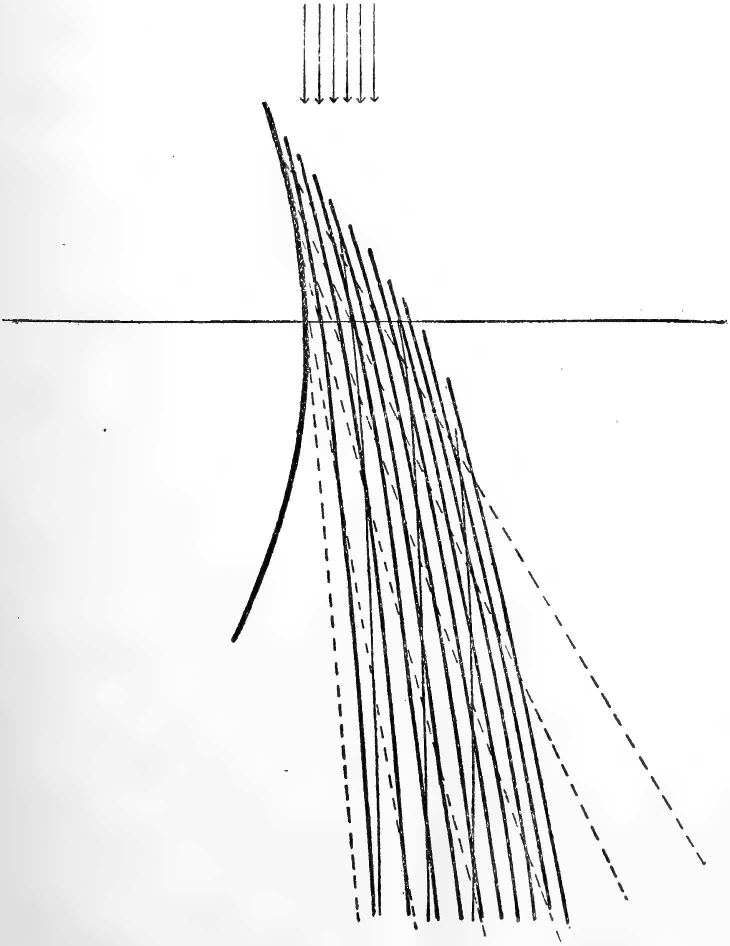
$$r\theta^2=m\lambda/2;\ r\theta=\text{C, giving } \theta=m\lambda/2\text{C}$$

or

$$\delta\theta=\lambda/2\text{C},$$

if $\delta\theta$ be the difference in the θ-coordinates between two successive points of intersection. Also if σ be the arc

Fig. 4.



Dotted lines represent the reflected rays, the heavy lines the loci of minimum illumination, and the thin lines the mean lines of flow of energy.

measured along the curve $r\theta = C$, we have from equation (20 *a*)

$$\left(\frac{d\sigma}{d\theta}\right)^2 = \left\{(2r - a\theta) + 2\theta\frac{dr}{d\theta}\right\}^2 + \left\{\frac{dr}{d\theta} - a\right\}^2. \quad (25)$$

Along the curve $r\theta = \mathrm{C}$, $dr/d\theta = -r/\theta$. Hence from (25)

$$\left(\frac{d\sigma}{d\theta}\right)^2 = a^2 + \frac{r^2}{\theta^2} + \frac{2ar}{\theta} \text{ approx.},$$

$$\frac{d\sigma}{d\theta} = a + \frac{r}{\theta} = a + \frac{\mathrm{C}}{\theta^2}.$$

If l be the "wave-length" of a crinkle,

$$l = \left(a + \frac{\mathrm{C}}{\theta^2}\right)\delta\theta = \left(a + \frac{\mathrm{C}}{\theta^2}\right)\frac{\lambda}{2\mathrm{C}}; \quad . \quad . \quad . \quad (26)$$

C being the same, l increases as θ decreases, the rate of increase getting larger as the absolute value of θ diminishes; $l = \infty$ when $\theta = 0$. Again, as we move away from the cylinder to the right (fig. 1), both C and θ increase so that l decreases. These points are brought out in fig. 5.

Turning back to equation (23), the lines of flow near the point (r_1, θ_1) are given by

$$r\theta = \mathrm{C} - \frac{\kappa_1}{\theta_1}\frac{\lambda}{4\pi}\sin\psi \ldots$$

The curves will obviously lie between the curves

$$r\theta = \mathrm{C} \pm \frac{\kappa_1}{\theta_1} \cdot \frac{\lambda}{4\pi}.$$

The deviation from the mean line $r\theta = \mathrm{C}$ is proportional to

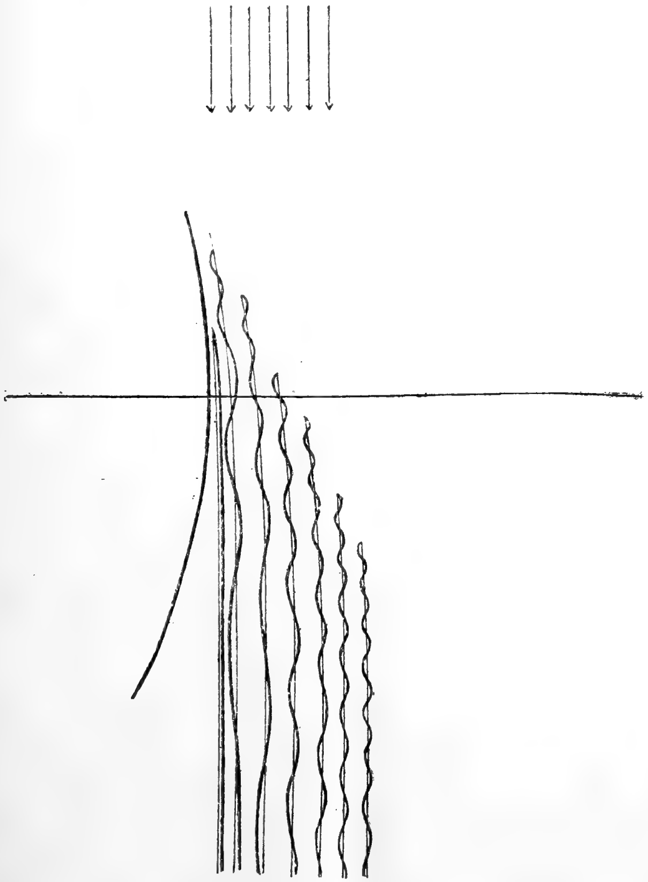
$$\frac{\kappa_1}{\theta_1} = \left\{\frac{a}{\theta(2r + a\theta)}\right\}^{\frac{1}{2}}.$$

As either r or θ increases, this will decrease. The "amplitude" of these crinkles therefore gets smaller and smaller as r and θ increase and the crinkles vanish at sufficiently large distances from the cylinder.

The shape of the lines of flow very near the surface of the cylinder is of special interest. The energy which comes crinkling down successive loci of maxima and minima of illumination, when it reaches the first maximum, flows down in a smooth curve which will meet the mean line of flow only at infinity. The energy does not crinkle along after it has crossed the first maximum of illumination.

It is thus seen that the introduction of a perfectly reflecting cylinder into a field through which plane waves are passing has the effect of (1) altering the general direction of flow of energy, and (2) giving a crinkled microscopic structure to the energy current at any point. But we have still to seek an explanation as to how a flow of energy in the manner described above leads to the actual distribution of maxima and minima of illumination in the field. The current of energy at any point is by (19)

Fig. 5.



Illustrating the form of the lines of flow of energy in different parts of the field. The thin lines are the "mean lines of flow" and the heavy lines the actual lines of flow.

$$\overline{S}=\frac{c}{8\pi}\left(1-2\kappa\cos\frac{n}{c}2r\theta^2+\kappa^2\right),$$

and this varies from point to point along each line of flow, being a maximum and a minimum respectively where it cuts successive curves $2r\theta^2 = m\lambda/2$. This variation of the current of energy along its own line of flow can be explained only as due to the change in cross-section of the tube of flow formed by two lines of flow close to each other. The conception of energy as flowing through tubes must therefore give us a better idea of what happens in the field.

The curves which are at every point normal to the lines of flow, *i. e.* the curves analogous to the equipotential curves, can easily be obtained. For these curves

$$\tan\phi = -\frac{1 - 2\kappa\cos\frac{n}{c}2r\theta^2 + \kappa^2}{2\theta\left(\kappa^2 - \kappa\cos\frac{n}{c}2r\theta^2\right)} \qquad \text{(comp. eq. 18).}$$

The differential equation in terms of θ and r is found to be

$$(dr - a\,d\theta) + \frac{2\kappa}{1+\kappa^2}\cos\frac{n}{c}2r\theta^2\left[(a - 2r\theta)\,d\theta - dr\right] = 0 \text{ (approx.).}$$

In the immediate neighbourhood of a point, we can as before leave out all variations other than the periodic one and integrate. Then we get

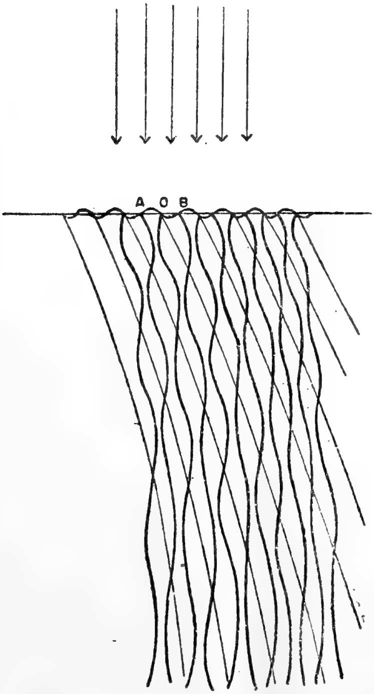
$$r - a\theta + \frac{2\kappa_1}{1+\kappa_1^2}\sin\frac{n}{c}2r\theta^2\left[\frac{a - 2r_1\theta_1}{4r\theta\frac{n}{c}} - \frac{1}{2\theta^2\frac{n}{c}}\right] = \text{const.,} \qquad (27)$$

which cuts the successive loci of maxima and minima of illumination at points lying on the curve

$$r - a\theta = \text{const.,} \qquad \dots\dots \qquad (28)$$

which is hence the mean curve about which the actual curve crinkles round. In the rectangular coordinates (x, y), equation (28) becomes approximately $y = \text{const.}$, *i. e.* the mean curves are nearly straight lines parallel to the x-axis. One of these curves is shown in fig. 6 in relation to the lines of flow and to the loci of minimum illumination. Suppose it cuts two successive minima loci at points A and B. Then by (27), there is one complete crinkle between A and B. Consider the tube of flow bounded by the lines of flow which pass through A and B. Since the flow of energy should be everywhere normal to the wavy curve AOB, energy is concentrated in the right half of the tube and "rarefied" in the left half. If we draw the line of flow passing through the point O midway between A and B, we see that we can conceive of the tube AB as made up of two tubes each of which widens and contracts periodically, and one of which is so shifted relatively to the other that the broadened part of one falls by the side of the narrow part of the other. We see also from fig. 6 that the contracted parts of successive tubes lie along the loci of maximum illumination.

Fig. 6.



Illustrating the "microscopic" structure of the Field.

We may now take into account the diffraction at the "edge" of the cylinder, and obtain the correction to our results necessary for points below the plane $y=0$. Using Sommerfeld's expression as before, the electric intensity at a point P′ (fig. 1) due to this alone is from (6)

$$\frac{1}{4\pi}\sqrt{\frac{\lambda}{r'}}\cos\left\{\frac{n}{c}(r'-ct)+\frac{\pi}{4}\right\}\left\{1-\frac{1}{\sin\frac{\alpha}{2}}\right\}$$

$$=-\frac{1}{2\pi\alpha}\sqrt{\frac{\lambda}{r'}}\cos\left\{\frac{n}{c}(r'-ct)+\frac{\pi}{4}\right\}$$

since α $(=\mathrm{P'\hat{O}Y})$ is small.

$$r'=\mathrm{P'O}=(r_1-a\theta)\sec\alpha=(r_1-a\theta)\sec 2\theta \text{ (approx.)}.$$

$$\therefore \qquad r'+a\theta=r_1(1+2\theta^2), \text{ where } r_1=\mathrm{P'M'}.$$

Also if $\mathrm{P'Q}=r$, $\alpha=2r\theta/(r-a\theta)$ approx.

If E, H be the resultant electric and magnetic intensities at P′,

$$\mathrm{E}=\cos n\left(t-\frac{r_1}{c}\right)-\kappa\cos n\left(t-\frac{r}{c}\right)$$
$$-\frac{1}{2\pi\alpha}\sqrt{\frac{\lambda}{r'}}\cos\left\{n\left(t-\frac{r'+a\theta}{c}\right)-\frac{\pi}{4}\right\},$$

and

$$\mathrm{H}=\overrightarrow{\cos n\left(t-\frac{r_1}{c}\right)}-\kappa\,\overrightarrow{\cos n\left(t-\frac{r}{c}\right)}$$
$$-\frac{1}{2\pi\alpha}\sqrt{\frac{\lambda}{r'}}\,\overrightarrow{\cos\left\{n\left(t-\frac{r'+a\theta}{c}\right)\right.}\left.-\frac{\pi}{4}\right\}.$$

If k_1, k_2, k_3 be unit vectors measured along r_1, r, and r' respectively, the time-mean of the flow of energy is found to be given by

$$\frac{8\pi\overline{\mathrm{S}}}{c}=k_1\left[1-\kappa\cos\frac{n}{c}2r\theta^2-\kappa'\cos\left(\frac{n}{c}2r\theta^2+\frac{\pi}{4}\right)\right]$$
$$+k_2\left[\kappa^2-\kappa\cos\frac{n}{c}2r\theta^2+\kappa\kappa'\cos\frac{\pi}{4}\right]$$
$$+k_3\left[\kappa'^2-\kappa'\cos\left(\frac{n}{c}2r\theta^2+\frac{\pi}{4}\right)+\kappa\,\kappa'\cos\frac{\pi}{4}\right],$$

κ' being written for $\frac{1}{2\pi\alpha}\sqrt{\frac{\lambda}{r'}}$.

Resolving the vectors along and perpendicular to the direction of the incident rays, we get

$$\frac{8\pi}{c}\overline{\mathrm{S}}_x=1+\kappa^2+\kappa'^2-2\kappa\cos\frac{n}{c}2r\theta^2$$
$$-2\kappa'\cos\left(\frac{n}{c}2r\theta^2+\frac{\pi}{4}\right)+2\kappa\kappa'\cos\frac{\pi}{4},$$

$$\frac{8\pi}{c}\overline{\mathrm{S}}_y=2\theta\left\{-\kappa\cos\frac{n}{c}2r\theta^2-\kappa'\cos\left(\frac{n}{c}2r\theta^2+\frac{\pi}{4}\right)\right.$$
$$\left.+2\kappa\kappa'\cos\frac{\pi}{4}+\kappa^2+\kappa'^2\right\}, \quad (29)$$

where θ^2 is neglected in comparison with unity and α and θ are considered to be the same in the small term $\overline{\mathrm{S}}_y$. Since κ, κ' are each small in the present problem, terms involving their squares and products may be neglected. The expression for $\tan\phi$ may then be written

$$\tan\phi=\frac{\overline{\mathrm{S}}_y}{\overline{\mathrm{S}}_x}=\frac{2\theta\left\{-\kappa\cos\frac{n}{c}2r\theta^2-\kappa'\cos\left(\frac{n}{c}2r\theta^2+\frac{\pi}{4}\right)\right\}}{1-2\kappa\cos\frac{n}{c}2r\theta^2-2\kappa'\cos\left(\frac{n}{c}2r\theta^2+\frac{\pi}{4}\right)}. \quad (30)$$

But by equation (20),

$$\frac{d\theta}{dr}(2r-a\theta)+2\theta=\left(1-a\frac{d\theta}{dr}\right)\tan\phi.$$

Hence (30) becomes, after some reduction,

$$\frac{d\theta}{dr}=-\frac{\theta}{r}\cdot\frac{1-\kappa\cos\frac{n}{c}2r\theta^2-\kappa'\cos\left(\frac{n}{c}2r\theta^2+\frac{\pi}{4}\right)}{1-2\kappa\cos\frac{n}{c}2r\theta^2-2\kappa'\cos\left(\frac{n}{c}2r\theta^2+\frac{\pi}{4}\right)}. \quad (31)$$

Now

$$\kappa\cos\frac{n}{c}2r\theta^2+\kappa'\cos\left(\frac{n}{c}2r\theta^2+\frac{\pi}{4}\right)=\mathrm{D}\cos\left(\frac{n}{c}2r\theta^2+\epsilon\right),$$

if $\mathrm{D}^2=(\kappa+\kappa'/\sqrt{2})^2+\kappa'^2/2=\kappa^2+\kappa'^2+\sqrt{2}\kappa\kappa'$

and $\tan\epsilon=\dfrac{\kappa'}{\kappa'+\kappa\sqrt{2}}$.

Then (31) may be written

$$\frac{d\theta}{dr}=-\frac{\theta}{r}\cdot\frac{1-\mathrm{D}\cos\left(\frac{n}{c}2r\theta^2+\epsilon\right)}{1-2\mathrm{D}\cos\left(\frac{n}{c}2r\theta^2+\epsilon\right)}. \quad . \quad . \quad (32)$$

In the neighbourhood of any point, D and ϵ can be regarded as constant and equation (32) integrated.

$$r\theta = C - \frac{D_1}{\theta_1}\sin\left(\frac{n}{c}2r\theta^2 + \epsilon\right).$$

The "mean lines of flow" are still given by $r\theta = C$; but the actual lines of flow which wind about them cut them along the curves

$$\frac{n}{c}2r\theta^2 + \epsilon = m\pi, \quad . \quad . \quad . \quad . \quad . \quad . \quad (33)$$

instead of the curves $\frac{n}{c}2r\theta^2 = m\pi$. It may be noted that it is equation (33) that determines the position of the maxima and minima of illumination when the effect of diffraction is also taken into account, so that the points of intersection of the lines of flow with the loci of maxima and minima of illumination still lie on the "mean lines of flow."

If the squares and products of κ, κ' are not neglected it can be shown that the mean lines are given by the equation

$$r\,d\theta(1+\kappa'^2+\kappa\kappa'\sqrt{2}) + \theta\,dr = 0$$

or

$$r\theta^{1+\alpha} = \text{const.},$$

where $\alpha = \kappa'^2 + \sqrt{2}\kappa\kappa'$, and is hence very small.

It will be interesting finally to deduce the results for the case of diffraction by a straight edge from the above investigation. Putting $a=0$, κ becomes zero also. We have then directly from equation (29),

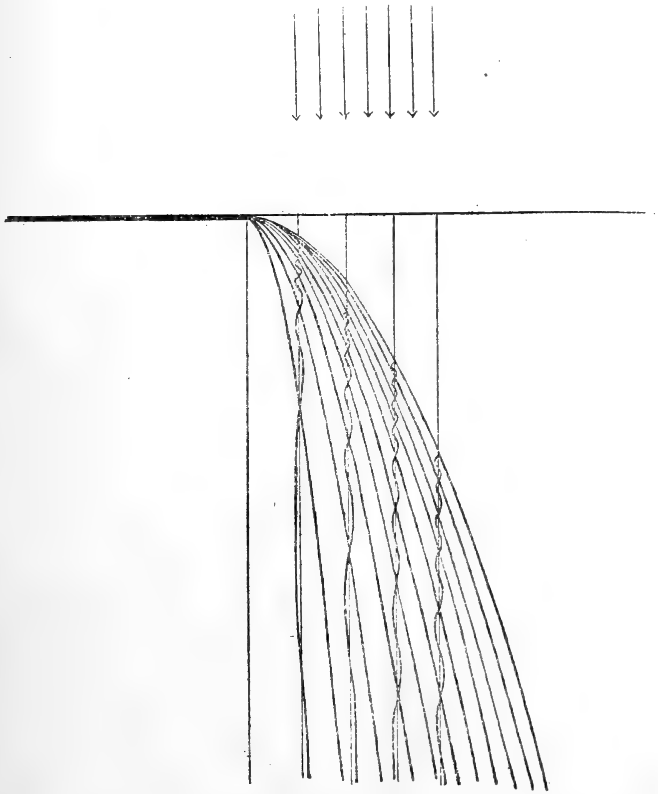
$$\tan\phi = \frac{2\theta\left\{\kappa'^2 - \kappa'\cos\left(\frac{n}{c}2r\theta^2 + \frac{\pi}{4}\right)\right\}}{1+\kappa'^2 - 2\kappa'\cos\left(\frac{n}{c}2r\theta^2 + \frac{\pi}{4}\right)},$$

and

$$\frac{d\theta}{dr} = -\frac{\theta}{r}\,\frac{1-\kappa'\cos\left(\frac{n}{c}2r\theta^2 + \frac{\pi}{4}\right)}{1-2\kappa'\cos\left(\frac{n}{c}2r\theta^2 + \frac{\pi}{4}\right)},$$

an expression very similar to (21). The "mean lines of flow" are given by $r\theta = \text{const.}$, or in the rectangular coordinates used, by $x = \text{const.}$ They are straight lines parallel to the y-axis; the mean direction of energy-flow is apparently not altered by the presence of the edge. The actual lines of flow cut the successive loci of maxima and minima of illumination (*i. e.* the curves $2r\theta^2 + \lambda/8 = m\lambda/2$) at points which lie along the mean lines of flow. The shape of the lines is indicated in fig. 7, which has been drawn for a large value (0·5 inch) of λ for convenience of representation.

Fig. 7.



The lines of flow of energy in relation to the loci of minimum illumination in the Diffraction Field due to a straight edge.

6. *Summary and Conclusion.*

The present paper deals with the distribution and flow of energy in the immediate neighbourhood of a perfectly reflecting cylinder on which plane light waves are grazingly incident in a direction at right angles to its axis. The following are the principal results which are indicated by theory and have been verified by photometric study of the field.

(*a*) The positions of maxima and minima of illumination are determined by eliminating θ from the pair of equations

$$2y\theta + 3a\theta^2/2 = x\ ;\ \ 2\theta^2(y + a\theta) = m\lambda/2\ ;$$

or from the pair of equations

$$2y\theta + 3a\theta^2/2 = x\ ;\ \ 2\theta^2(y + a\theta) = m\lambda/2 - \epsilon\lambda/2\pi,$$

according as the plane of observation is in front of the cylindrical "edge" or behind it. x, y are the coordinates of any point, the origin being the "edge" of the cylinder. ϵ is a small angle which for large values of y becomes equal to $\pi/4$.

(*b*) The visibility of the fringes varies in an interesting manner with the position of the part of the field under observation. It is practically constant over the entire plane of observation when this coincides with the plane passing through the "edge"; but it falls off when it is moved farther away from the source of light, the decrease being greatest for the regions farthest from the surface. When the part of the field under observation is between the "edge" and the source of light, the visibilility of the fringes reaches the maximum value at the surface of the cylinder, falling off slowly as we recede from it.

(*c*) The loci of maxima and minima of illumination are given by $r\theta^2 = m\lambda/4$, and the "mean lines of flow" of energy are given by $r\theta = \text{const.}$ and are for small values of θ less inclined to the direction of the incident rays than the former set of curves. The actual lines of flow crinkle about these mean lines; the "wave-length" of the crinkles increases as we move along the direction of the incident rays, and decreases as we move in a direction at right angles to it away from the cylinder. The "amplitude" of the crinkles decreases as r and θ increase, and vanishes at sufficiently large distances from the cylinder. A good conception of the actual phenomena is obtained, if we imagine energy as flowing through tubes which widen and contract periodically, the widened parts of successive tubes lying on the loci of minimum illumination, and the contracted parts on the loci of maximum illumination (see fig. 6). When the radius of the cylinder is very small, the results are practically identical with those obtained in the case of diffraction by a straight edge.

No reference has so far been made to the phenomena noticed within the region of the geometrical shadow of the cylinder. The writer has made some preliminary observations on this subject, and hopes to continue the work, which might prove of interest in relation to the general problem of the diffraction of electromagnetic waves by cylindrical or spherical surfaces of large radius.

In conclusion, the writer wishes to express his best thanks to Prof. C. V. Raman, who suggested the investigation and took much interest in its progress.

Calcutta,
November 15, 1917.

III. *Ionization and Resonance Potentials for Electrons in Vapours of Magnesium and Thallium.* By PAUL D. FOOTE, *Ph.D.*, and FRED. L. MOHLER, *Ph.D.*[*]

TWO types of inelastic collision between electrons and atoms of metallic vapours are known to exist. One type of collision results in the expulsion of an electron from the atom, and the other type of collision produces an agitation, or transfer from one orbit to another, of one or more electrons bound in the atom. Collision resulting in ionization is accompanied by the emission of the complicated series spectra of the metals, presumably due to recombination following ionization. The second type of collision is generally supposed to result in the emission of a single line spectrum[†].

The potential difference through which an electron must fall in order to acquire sufficient energy to produce ionization upon collision with an atom is known as the ionization potential, and the corresponding potential difference for the second type of collision is known as the resonance potential for the particular vapour in question. Dr. Tate and one of the writers[‡] have determined the ionization and resonance

---

[*] Communicated by Dr. S. W. Stratton, Director Bureau of Standards.

[†] Dr. F. A. Saunders suggests there is reason for believing that the term single line spectrum is a misnomer. It is possible that the entire series of which the so-called single line is the first member appears simultaneously with this line. In most cases where single line spectra have been studied, experimental difficulties are such that other members of the series could not be detected. This suggestion raises a very important question which might be decided experimentally by a spectroscopic study of low voltage discharge through sodium or potassium vapour. It may be pointed out, however, that conclusive evidence in this regard will be difficult to obtain; for if the other lines of the series should be found to be present, they would appear in such low intensity that their emission might be accounted for by the presence of a few electrons of velocity greater than that corresponding to the ionizing potential. These high velocity electrons are always present on account of the Maxwell distribution of velocities of the electrons emitted by the hot cathode. A second difficulty, even if a univelocity stream of electrons were employed, would be to assure that the collision of the electron was with the normal atom, and not with an atom having an electron already displaced from its normal orbit by a previous collision. This latter question has been considered by Van der Bijl, Phys. R. x. p. 546 (1917).

[‡] Tate and Foote, J. Wash. Acad. Sci. vii. p. 519 (1917); Bur. Stds. Sci. Paper No. 317; Phil. Mag. xxxvi. p. 64 (July 1918).

potentials for electrons in vapours of zinc, cadmium, sodium, and potassium. The present paper is a continuation of this work, using vapours of magnesium and thallium.

The experimental arrangements have been fully described in the above-referred to papers. One modification has been introduced for the work with thallium in that an equipotential surface was used for the hot cathode. This method of producing a more nearly uniform distribution of velocities of the electrons, first suggested by Goucher*, has proved most satisfactory. A helix of tungsten wire is threaded through a small porcelain cylinder, and the equipotential surface from which the electrons are accelerated consists of a platinum or preferably, for thallium, a nickel cylinder, coated with lime, fitting tightly over the porcelain. This surface is heated to about 1000° C. by the current through the tungsten helix, and a *thermionic* current of several milliamperes may be maintained with an accelerating potential of a few volts. The metal and ionization-chamber were heated in a glazed porcelain tube to about 600° C. for magnesium and 900° C. for thallium. Hard glass and iron tubes were also used for magnesium. Both of these metals attack porcelain very rapidly at these temperatures. At temperatures above a glow the thermionic emission from the cylinders of the ionization-chamber and from the walls of the porcelain tube become very troublesome, and for this reason the accuracy obtained in earlier work upon metals of lower boiling-point is not to be expected. The apparatus was evacuated by means of a Stimson and a Langmuir condensation-pump to a pressure of less than 0·002 mm. Hg as read by a McLeod gauge.

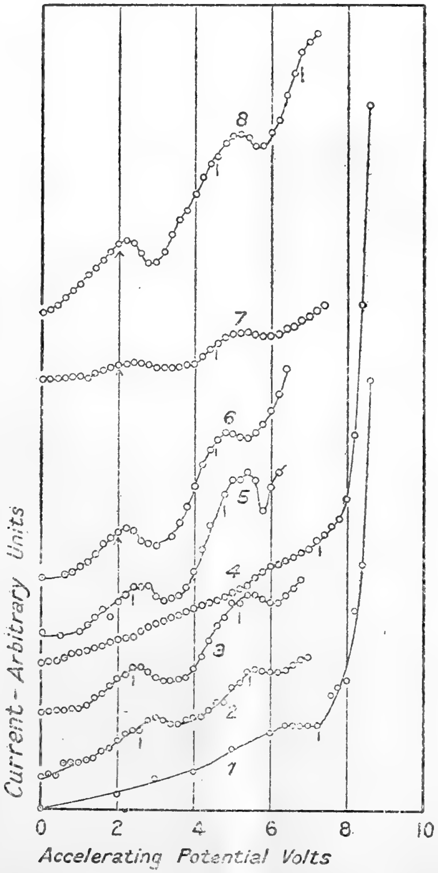
Fig. 1, curves 2, 3, 5, 6, 7, and 8, and fig. 2, curves 1, 3, 4, 5, 6, 7, and 8, for magnesium, represent the current between the inner net and outside cylinder, through a retarding field of about 0·7 to 1 volt, as a function of the accelerating field applied between the hot wire and net. The total current from the hot wire as a function of the accelerating field is represented by fig. 1, curves 1 and 4, and by fig. 2, curves 2, 2 *a*, 9, 10, and 11. The analysis of these curves is given in Tables I. and II. The partial current curves are characterized by successive maxima and minima due to inelastic collision of resonance. We have chosen as before the points at which such inelastic collisions begin to show an appreciable effect

---

* Goucher, Phys. R. viii. p. 561 (1916).

upon the current, rather than the position of maximum effect. These are determined by carefully plotting and extrapolating the lower portion of each curve. The point selected as

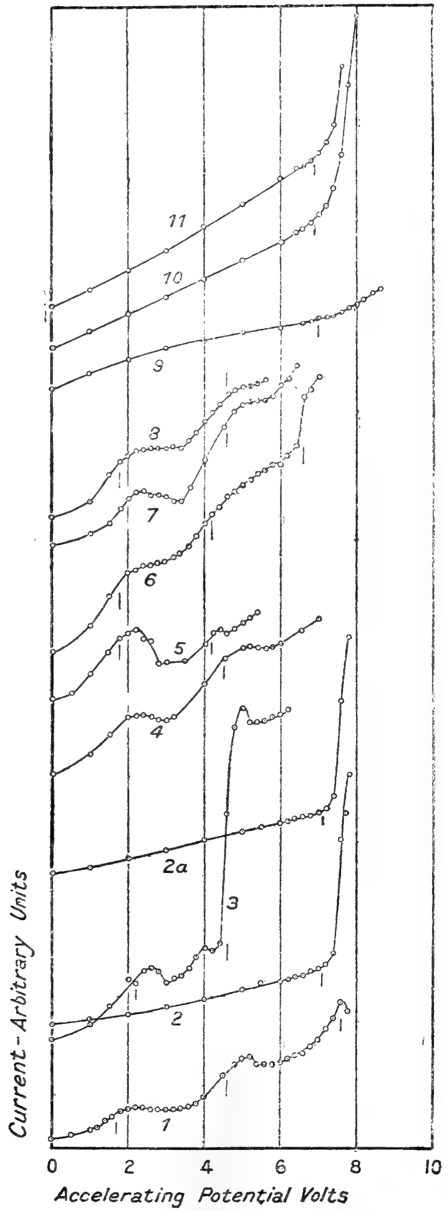
Fig. 1.



Variation with accelerating potential of current to outside cylinder and of total current from hot wire. Magnesium.

a, *b*, *c*, &c., is the voltage at which the observed curve deviates from the extrapolated curve by a small pre-assigned amount. The difference in the voltages corresponding to

Fig. 2.



Variation with accelerating potential of current to outside cylinder and of total current from hot wire. Magnesium.

these points of resonance gives the resonance potential directly. Since we are concerned only with differences of points similarly selected, the arbitrary method of selection can introduce no error. The total current curves, for example, fig. 1, curves 1 and 4, show a rapid increase in current at an applied potential of 7·3 volts, due to ionization. These points are selected in the same manner as the points of resonance, both curves being plotted on the same scale of coordinates. In general the total current is from 10 to 50 times greater than the partial current. The initial velocity of the electrons is determined from the partial current curves. Thus, referring to Table I., the first resonance occurs at 2·6 volts, for curve 2, whereas if the initial velocity were zero it would occur at 2·69 volts. The initial velocity of the electrons is accordingly 0·09 volt. The mean value of the initial potential for any one group of curves is added to the mean value of the applied potential at ionization to obtain the true ionization potential. The data represented by Tables I. and II. were obtained with entirely different construction of apparatus, and the agreement in the mean values of the resonance and ionization potentials illustrate the reproducibility of this work. The final mean values for magnesium are 7·75 volts for ionization and 2·65 volts for resonance.

TABLE I.

Ionization and Resonance Potentials for Magnesium
Referring to fig. 1.

Curve.	*a.*	*b.*	*c.*	Applied Potential for Ionization.	Resonance Potential.		Initial Potential.
					b–a.	*c–b.*	
1	...	...	...	7·3			
2	2·6	5·5	...	...	2·9	...	0·09
3	2·4	5·2	...	...	2·8	...	0·3
4	...	...	...	7·3			
5	2·4	4·8	...	...	2·4	...	0·3
6	2·0	4·6	...	...	2.6	...	0·7
7	2·0	4·6	...	...	2·6	...	0·7
8	2·0	4·6	6·8	...	2·6	2·2	0·7
			Means......	7·3	2·69 volts		0·5

Mean Initial Potential ... 0·5

Ionization Potential ...... 7·8 volts.

Table II.

Ionization and Resonance Potentials for Magnesium.
Referring to fig. 2.

Curve.	*a.*	*b.*	*c.*	Applied Potential for Ionization.	Resonance Potential.		Initial Potential.
					b–a.	*c–b.*	
1	1·7	4·6	7·6	...	2·9	...	0·9
2	...	...	...	7·1			
2*a*	...	...	...	7·1			
3	2·2	4·6	...	...	2·4	...	0·4
4	2·0	4·5	...	...	2·5	..	0·6
5	1·8	4·2	...	...	2·4	...	0·8
6	1·8	4·2	6·6	...	2·4	2·4	0·8
7	2·0	4·6	...	...	2·6	...	0·6
8	1·8	4·6	...	...	2·8	...	0·8
9	...	...	...	7·0			
10	...	...	...	6·9			
11	...	...	...	6·9			
			Means......	7·0	2·60 volts.		0·7

Mean Initial Potential... 0·7

Ionization Potential...... 7·7 volts.

Final Mean Ionization Potential=7·75 volts.
Final Mean Resonance Potential=2·65 volts.

The curves obtained for thallium vapour are shown by fig. 3. The total current curve 4 clearly indicates ionization at an applied potential of 6·6 volts. The other curves represent partial current between the net and cylinder against a retarding field of about 0·2 volt. Analysis of these curves is given by Table III. It will be noted that

Table III.

Ionization and Resonance Potentials for Thallium.
Referring to fig. 3.

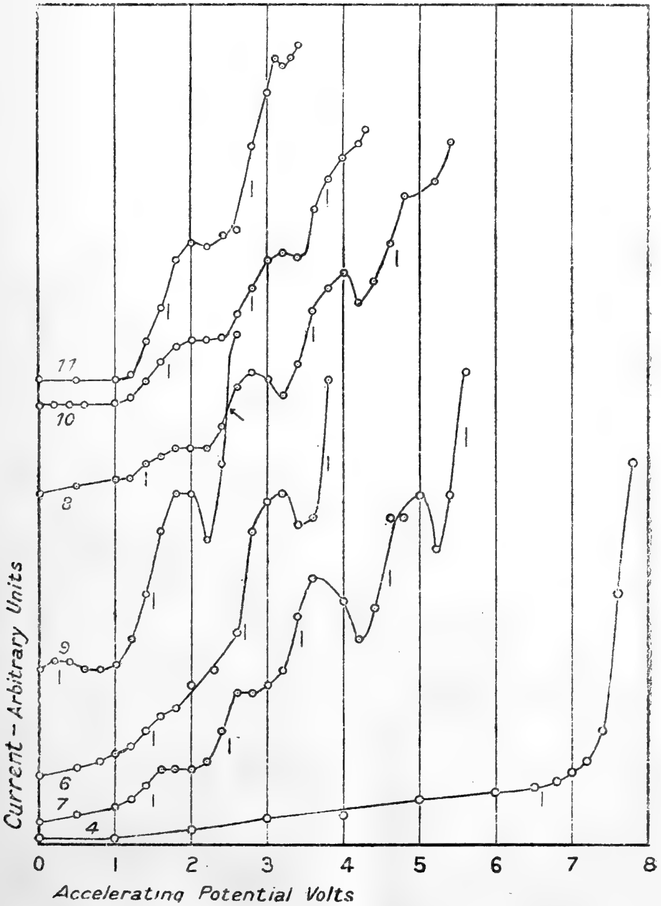
Curve.	*a.*	*b.*	*c.*	*d.*	*e.*	*f.*	Applied Potential for Ionization.	Resonance Potential.				Initial Potential.
								c–b.	*d–c.*	*e–d.*	*f–e.*	
6	...	1·5	2·7	3·8	...	...	...	1·2	1·1	...	...	0·6+
7	...	1·5	2·5	3·45	4·6	5·6	...	1·0	0·95	1·15	1·0	0·6+
8	...	1·4	2·5	3·6	4·7	...	...	1·1	1·1	1·1	...	0·7+
9	0·25?	1·5	2·5	...	...	...	...	1·0	...	...	...	0·6+
10	...	1·7	2·8	3·8	...	...	...	1·1	1·1			
11	...	1·7	2·8	...	...	...	...	1·1				
4	...			...	...	...	6·6					
						Means	6·6	1·07 volts.				0·7

Mean Initial Potential .... 0·7

Mean Ionization Potential ...... 7·3 volts.

the first point of resonance was very difficult to obtain. Only one curve, 9, shows this point clearly; and even in this case the value of the potential could not be observed with great enough precision to justify its use in determining

Fig. 3.



Variation with accelerating potential of current to outside cylinder and of total current from hot wire. Thallium.

the final value of the resonance potential. The fact that this point does not show up as distinctly as the other resonance points is due to the temperature distribution of initial velocities of the electrons, and to the thermionic emission of all parts of the apparatus on account of the high temperature, 900° C., required. The partial current curves 6, 7, 8, 9 correspond to the total current curve 4, and were accordingly used for determining the initial potential. The initial potentials for curves 10 and 11, taken some time later, are a trifle higher. The final values obtained for thallium were 7·3 volts for the ionization potential and 1·07 volts for the resonance potential.

Discussion of the Results.

If we apply the quantum relation $h\nu = eV$ to the experimental data obtained for magnesium, we find that the line $\lambda = 4571$ corresponds to resonance potential and the convergence or limiting wave-length, $\lambda = 1622$, corresponds to ionization potential. The limiting wave-length, $\nu = 1{\cdot}5\,S$, has the value 1622 Å., so that the ionization potential is determined by the same spectral series relation as in the case of other metals so far studied. It was found in the earlier work that the resonance potential is determined by the single line spectrum of the metal; but magnesium would appear to offer an exception to this rule. McLennan* has investigated the single line and absorption spectrum of magnesium, and has concluded that $\lambda = 4571$ is not the single line spectrum, but rather the line $\lambda = 2852$. Although special effort was made to detect the presence of $\lambda = 4571$ as a single line, the line 2852 always appeared with the line 4571, and only the line 2852 could be made to appear alone. If the single line spectrum is $\lambda = 2852$, we have the first exception to the rule that resonance potential is determined by the single line spectrum, for our curves show no indication of resonance at multiples of 4·3 volts as would be required by the line 2852.

The behaviour of the various metals as regards ionization and resonance potentials is apparently determined by their position in the periodic table of the elements. Thus in the case of sodium and potassium, group I., ionization is determined by the convergence of the principal series and resonance by the first line of this series. For magnesium, zinc, cadmium, and mercury, group II., ionization is determined by the convergence of a combination series $1{\cdot}5\,S - mp_2$, where mp_2 is the variable term of the second line of the principal series of triplets, and resonance is given by the first term of this series where $m = 2$. The $1{\cdot}5\,S$ term is the limit of the principal series of single lines, and may be derived from the variable term in the second

---

* Proc. Roy. Soc. Lond. xcii. p. 574 (1916); xcii. p. 305 (1916).

subordinate series of single lines. Thallium, group III., behaves in a still different manner, as will appear below.

The following table shows the critical wave-lengths determining resonance and ionization for the metals of group II.

TABLE IV.

Metal.	$1{\cdot}5\,S-2p_2$.	$1{\cdot}5\,S$.
Mg	4571·38 Å	1621·7 Å
Zn	3075·99	1319·95
Cd	3260·17	1378·69
Hg	2536·72	1187·96

The ionization potential is determined for all of these metals by the lines in the third column, and the resonance potential by the lines in the second column. If $\lambda=4571$ were the single line spectrum of magnesium, the second column would give correctly the single line spectrum of these elements of the same family. The experiments of Prof. McLennan do not permit the above classification of the elements of the same group as regards the single line spectrum, although the classification is correct as regards ionization and resonance potentials.

When the potential difference between the cathode and net was equal to or greater than the ionization potential, about 7·8 volts, a strong arc was formed, and the entire ionization-chamber was filled with a green glow. A faint blue-green glow, probably $\lambda=4571$, could be detected at potentials as low as 5·5 volts. This fact is a strong argument that the line 2852 is not the single line spectrum of magnesium; for if so, one would not expect the line 4571 to appear below 7·8 volts. On the other hand, if the line 4571 were the single line spectrum, one would not expect the line 2852 to appear below 7·8 volts. McLennan observed the latter line at 5·9 volts. A possible explanation of these contradictory results would be had if two resonance potentials existed, one corresponding to $\lambda=4571$ and the other to $\lambda=2852$. We have no experimental evidence of this fact. We would further expect that if two resonance potentials existed for magnesium, we would have two resonance potentials for cadmium, mercury, and zinc. Experimental evidence here again is to the contrary, although in the case of zinc peculiar maxima were found in the total current curves, the exact meaning of which has not been determined.

If, referring to the values for magnesium in Table IV., we apply the quantum relation $h\nu=eV$, using $h=6{\cdot}547\,.\,10^{-27}$ erg sec., we obtain the theoretical values of 2·70 volts for resonance potential, and 7·61 volts for ionization potential. The observed values of 2·65 volts and 7·75 volts respectively are in satisfactory agreement with the theoretical values.

In the case of thallium the observed value of the resonance potential, 1·07 volts, agrees exactly with that computed from the quantum relation $h\nu = eV$ if ν is taken as the frequency of the shorter wave-length member of the first term of the principal doublet series. The resonance potentials of sodium and of potassium are also determined by the first line of the principal series for these elements; but here the doublets are so close together that, as far as experimental confirmation is concerned, either member of the doublet may be used for the critical frequency ν. In thallium, however, the two lines of the doublet are widely separated, one lying at $\lambda = 11513$ Å. and the other at $\lambda = 13014$ Å. The former line corresponds to a resonance potential of 1·07 volts, and the latter line to 0·95 volt. The curves of fig. 3 show no indication of resonance collision at multiples of 0·95 volt. This fact raises the question as to whether the components of the thallium doublet can be separately excited by electronic impact. To determine this spectroscopically would be difficult, since the two lines lie in a region of the spectrum where temperature radiation from the surrounding walls is especially high.

A spectroscopic study of the radiation emitted when low velocity electrons collide with sodium atoms might prove fruitful. If the D-lines can be excited separately by electronic impact, we would expect one component alone to appear with two volt electrons, and would not expect the other component to appear until the ionizing potential was attained. Wood and Mohler* have shown that in the excitation of sodium vapour by incident radiation the D-lines may be excited separately, although the influence of the surrounding vapour may cause a transfer of energy from the excited line to the other component of the doublet. A similar transfer of energy might take place in thallium, sodium, &c., vapour excited by electronic impact. The fact that the higher frequency line of the thallium doublet determines the value of the energy quantum absorbed may be analogous to the emission of characteristic X-rays under electronic bombardment. D. L. Webster† concluded that the K group does not appear until the energy of the impacting electrons is greater than that corresponding to the highest frequency of the K group, namely K_γ. At this voltage all of the lines of the K group appear, and the ratio of the

---

* Wood and Mohler, Phys. Rev. xi. p. 70 (1918).

† D. L. Webster, Phys. Rev. vii. p. 599 (1916).

intensity of the lines remains the same when the voltage is further increased. However, if the K group is to be looked upon as a series of lines, the above comparison to the behaviour of thallium is not justified.

The single line spectrum of thallium has never been observed. McLennan has suggested that it lies far in the ultra-violet. Our work indicates that $\lambda = 11515$ Å., lying in the infra-red, is the true single line spectrum of thallium.

If we continue the analogy of the behaviour of thallium to that of sodium and potassium we shall expect to find the ionization potential determined by the convergence of the principal series, $\lambda = 4389$. This corresponds to an ionization potential of 2·81 volts, a value which is clearly disproven by the present work. Accordingly thallium does not behave similar to sodium and potassium, and actually in the light of the work with other metals we should expect a different behaviour, since it belongs to a new family in the periodic table as discussed above. The observed ionization occurred at 7·3 volts corresponding to a convergence frequency (wave number) of about 59000. The highest convergence frequency so far known in the thallium spectrum is that of the shorter wave-length member of the 2nd subordinate series, $\nu = 49263$. The other member of this series converges at 41470. McLennan * has attempted to predict 1·5 S for thallium by assuming, as in the case of zinc, mercury, etc., that the single line spectrum is given by the notation $\nu = 1{\cdot}5\ \mathrm{S} - 2p_2$ where ν corresponds to $\lambda = 5351$, the line appearing prominently in the flame-spectrum of thallium, and $2p_2 = 49263$ the convergence of one member of the first subordinate doublet series. In this manner he obtained the value $\nu = 1{\cdot}5\ \mathrm{S} = 67947$. If one employs the proper value for the single line spectrum, one obtains $\nu = 1{\cdot}5\ \mathrm{S} = 57946$. This corresponds to an ionization potential of 7·2 volts in good agreement with the experimentally determined value of 7·3 volts. However, the above method of deducing the convergence frequency 1·5 S is incorrect. The term 1·5 S belongs to the principal series of single lines, and $2p_2$ properly belongs to the principal series of triplets. Since neither single line series nor triplet series are known in thallium there is no means of determining 1·5 S or $2p_2$. If $2p_2$ is taken to denote the convergence of one member of the first subordinate series, $\nu = 49263$, then the only series deducible is $\nu = 1{\cdot}5\,s - mp_2$, which represents the principal series converging at $\nu = 1{\cdot}5\,s = 22786$. Our work accordingly predicts for thallium a new series of the form $1{\cdot}5\ \mathrm{S} - m\mathrm{P}$, a single line series

---

* McLennan, J. Frank. Inst. clxxxi. p. 201 (1916).

converging near $\nu=59000$, but a closer determination of this frequency must be made from spectroscopic data as yet not correlated in series. Prof. Saunders suggests that the lines ought to be of high intensity and that it is quite likely one or two of the lines are already known.

When the ionization potential of thallium was reached, a brilliant green glow filled the ionization-chamber No glow whatever could be detected below this potential in agreement with the assumption that the infra-red line $\lambda=11513$ is the single line spectrum. It is interesting to note that in all of this work on ionization and resonance we were unable to operate the apparatus when wireless signals were being transmitted in the wireless laboratories of the Bureau, as the ionization-chamber constitutes an efficient detector.

Dr. Tate and one of the writers have pointed out in earlier work that the long wave-length limit for the photoelectric effect in a metallic vapour is probably identical with the wave-length limit determining ionization potential. Prof. Kunz * has discussed this point in some detail, using our data on Na, Zn, Cd, and K. Thus, for example, Tate and Foote observed an ionization potential of 5·13 volts for sodium and a resonance potential of 2·12 volts. Metallic sodium is photoelectrically active to radiation of wave-length $\lambda=5893$, corresponding to 2·12 volts. Accordingly, if sodium vapour were photoelectrically active in the same manner as the solid metal we should have observed ionization when the D-line was emitted at resonance of 2·12 volts. Actually no ionization was detected until the voltage reached 5·13 volts. This fact proves that sodium vapour is not photoelectrically active to yellow light, and while not proving definitely that ultra-violet radiation is required to produce the photoelectric effect in the vapour, it is an indication that such is the case. For at 5·13 volts the ultra-violet lines of sodium are excited. Whether the photoelectric effect in sodium vapour can be produced by radiation of wave-length greater than that corresponding to 5·13 volts is a question which must be decided experimentally. It is hoped that work now in progress may offer conclusive evidence. Applying the above reasoning to the present work we may conclude that magnesium vapour is not photoelectrically active to radiation of wave-length 4571, and that very likely the limiting wave-lengths for photoelectric sensibility of magnesium and thallium vapours are $\lambda = 1622$ Å. and $\lambda =$ about 1700 Å. respectively.

Van der Bijl † has recently discussed the relation between

---

* Phys. R. xi. p. 246 (1918).

† Phys. R. x. p. 552 (1917).

the ionizing potential of a metallic vapour and the limiting wave-length for photoelectric sensibility of the solid. The data available at the time his paper was written were "too meagre to warrant any importance being attached to the fact that the difference between V_0, the voltage corresponding to the limiting line for photoelectric sensibility of the solid metal and V_c the ionization potential for the vapour, is nearly constant, and equal to 6 volts." If such a relation did exist as Van der Bijl points out, it would mean that a material having an ionization potential of less than 6 volts would be photoelectrically active in the dark. Both sodium and potassium have ionization potentials less than 6 volts. In the case of lithium, rubidium, and cæsium we should expect the ionization potentials to be 5·37, 4·16, and 3·88 volts respectively, although these values have not been determined experimentally*. All of these materials accordingly would be photoelectrically active in the dark. That the above empirical relation is not justified is seen from the following table of experimental data :—

TABLE V.

Element.	Photoelectric V_0.	Ionization V_c.	$V_c - V_0$.
Mg	3·08	7·75	4·67
Cd	3·49	8·92	5·43
Zn	3·77	9·5	5·73
Na	1·81	5·13	3·32
Hg	4·44	10·35	5·91

Thus, $V_c - V_0$ is definitely not equal to 6 volts and no simple accurate relation is apparent.

Hebb † has shown that ionization of mercury vapour is not produced by the radiation $\lambda = 2537$ acting photoelectrically. This fact is in accord with the above discussion. We should expect the long wave-length limit to be $\lambda = 1188$ Å. That radiation of wave-length $\lambda = 2537$ cannot ionize mercury vapour is evidenced by the failure of Tate ‡, Davis and Goucher §, and several other investigators to detect any sign of ionization of a *normal* mercury atom suffering resonance collision, as during such collisions the line $\lambda = 2537$ is emitted.

Gilbreath ‖ has concluded that potassium vapour is acted

---

* *Note added Nov.* 11.—We have since experimentally determined and confirmed these values for Rb and Cs. This paper including data for As will appear in Phys. R.

† Phys. R. xi. p. 179 (1918).

‡ Tate, Phys. R. vii. p. 686 (1916); idem. x. p. 81 (1917).

§ Davis and Goucher, x. p. 101 (1917).

‖ Phys. R. x. p. 166 (1917).

upon photoelectrically by ordinary light. The work of Tate and Foote* proves that light of wave-length $\lambda = 7685$ cannot produce ionization because this radiation is emitted during a resonance collision, and the total current-curves showed no trace of ionization at this point. We should expect the limiting long wave-length of photosensibility to be $\lambda = 2857$, and believe that the experimental evidence so far published is not sufficient to show that ionization of potassium vapour can be produced by ordinary light.

Determination of h.

Since the ionization and resonance potentials of electrons in metallic vapours obey the quantum relation $eV = h\nu$, this method is one which lends itself for a determination of h, Planck's constant of action. If we use the value of $e = 4{\cdot}774 \,.\, 10^{-10}$ determined by Millikan we obtain $h = 0{\cdot}5308\, \lambda V \,.\, 10^{-30}$ erg sec. where V is the value of the ionization or resonance potential in volts and λ is the wave-length in Ångström units corresponding to the frequency $\nu = 1{\cdot}5\ S$ or $\nu = 1{\cdot}5\ S - 2p_2$. Table VI. presents the data so far accumulated :—

TABLE VI.

Determination of h.

Element.	λ (1·5 S).	Observed Ionization.	Observer.	Observed $h \,.\, 10^{27}$.
Cd	1378·69	8·92 volts.	*b*	6·53
Na	2412·83 (*a*)	5·13	*b*	6·57
K	2856·65 (*a*)	4·1	*b*	6·22
Zn	1319·95	9·5	*b*	6·66
Hg	1187·96	10·35	*e*	6·53
Mg	1621·7	7·75	*c*	6·67
Tl	?	7·3	*c*	
	λ (1·5 S$-2p_2$).	Observed Resonance.		
Cd	3260·17	3·88	*b*	6·71
Na	5893· (*a*)	2·12	*b*	6·63
K	7685· (*a*)	1·55	*b*	6·32
Zn	3075·99	4·1	*b*	6·70
Hg	2536·72	4·9	*d*, *e*	6·60
Mg	4571·38	2·65	*c*	6·43
Tl	11513·22 (*a*)	1·07	*c*	6·54
			Mean	6·55

(*a*) Properly denoted by 1·5 *s* or 1·5 $s - 2p$, except for thallium, where $2p_2$ refers to the shorter wave-length of the doublet.

(*b*) Tate and Foote.

(*c*) Foote and Mohler.

(*d*) Franck and Hertz.

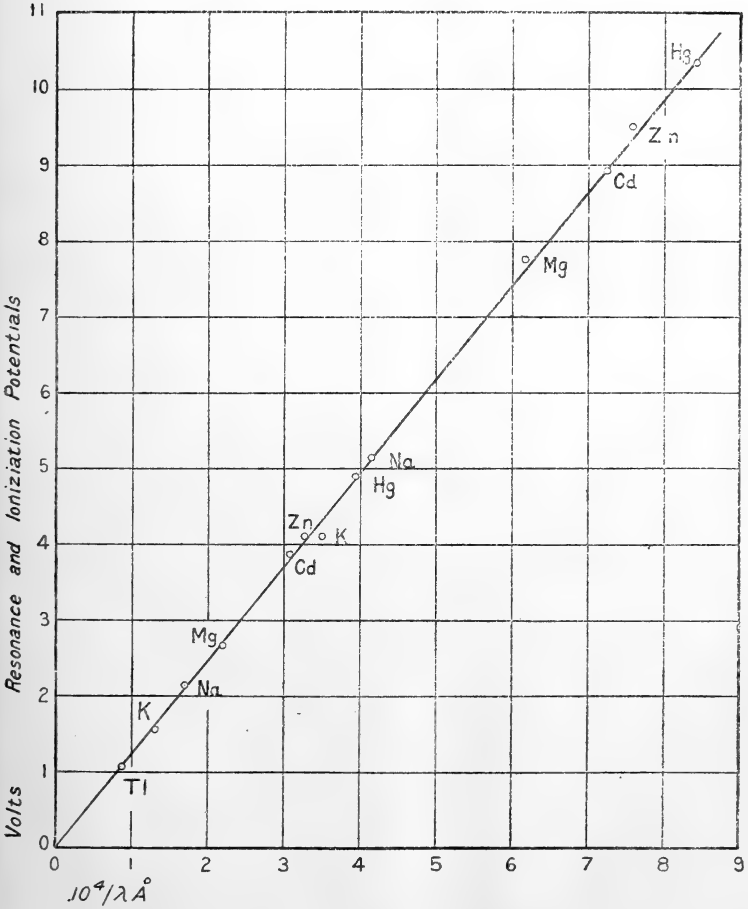
(*e*) Tate, Davis and Goucher, and others.

---

* Phil. Mag. *l. c.*

The above values are plotted in fig. 4, V versus $1/\lambda$. A straight line passing through the origin is obtained, confirming the quantum relation. The mean value of h obtained either from the curve or the above table is $h=6{\cdot}55 \,.\, 10^{-27}$ erg sec. In obtaining this mean value and in drawing the straight line of fig. 4 each determination was given equal weight. The final value of $h=6{\cdot}55 \,.\, 10^{-27}$ is in excellent agreement with the value of $6{\cdot}547 \,.\, 10^{-27}$ which Millikan * suggests as the most probable value.

Fig. 4.



Determination of h by ionization and resonance potentials.

---

* Phil. Mag. xxxiv. pp. 1–30 (1917).

The writers believe that the above method of determining h is capable of much higher accuracy than the photoelectric method if developed to the degree of refinement which the latter method has received. For very high precision a source of electrons possessing a uniform velocity is desired. This can be realized by magnetic separation similar to the method recently employed by Richardson and Bazzoni [*], or it can be accomplished by the use of a properly arranged electric field. Dr. Tate suggested a year ago the great value to be derived from such an experimental procedure. The writers contemplate using in the near future a univelocity stream of electrons in mercury vapour. Inasmuch as the present determination of h is by an entirely new method, it is of interest to compare determinations by other methods. The following table presents a summary statement of recent data :—

TABLE VII.

Determination of h.

Method.	Observer.	$h \,.\, 10^{27}$.
Black body radiation.	Coblentz.	6·56 erg sec.
X-rays.	Blake and Duane.	6·56
Photoelectric effect.	Millikan.	6·57
Ionization and Resonance Potentials.	Foote and Mohler.	6·55

In a recent paper Sanford [†] has developed a theory of the relation between ionizing potentials and atomic charges. The writers are unable to reconcile this theory with the Einstein equation or with experimental data. Sanford proposes the following relations :—

(1) Nuclear charge $=Q=2{\cdot}95\,.\,10^{-12}/\sqrt{\lambda}$ where λ is the wave-length of the radiation emitted.

(2) Ionization potential $=V=(Q-2{\cdot}42)\,1{\cdot}4$, whence

(3) $V=412/\sqrt{\lambda}-3{\cdot}39$ for λ expressed in Ångström units. On multiplying (3) by e, the equation takes the following form :—

(4) $eV=\alpha\sqrt{\nu}-\beta$, whereas the quantum relation which is known to be correct (see fig. 4 as one example of its verification) is given by eq. (5).

(5) $eV=h\nu$.

It is evident that Sanford's equation (4) is incompatible with the quantum relation (5).

---

[*] Phil. Mag. xxxiv. p. 285 (1917).

[†] Phys. R. ix. p. 575 (1917).

Summary.

The ionization and resonance potentials for electrons in vapours of magnesium and thallium have been measured, using the method of Franck and Hertz as modified by Tate. The experimentally determined values for magnesium vapour were 2·65 volts for resonance potential and 7·75 volts for ionization potential. The corresponding theoretical values on the basis of the quantum relation $h\nu=e\mathrm{V}$ are 2·70 volts and 7·61 volts respectively. The experimentally determined values for thallium vapour were 1·07 volts for resonance potential and 7·3 volts for ionization potential. The theoretical value for resonance potential is 1·07 volts. The theoretical value for ionization potential is not known. For magnesium the ionization potential is determined by the limit of the combination series $1{\cdot}5\,\mathrm{S}-mp_2$, and the resonance potential is determined by the first line in this series, $m=2$. For thallium the resonance potential is determined by the shorter wave-length member of the first term of the principal series of doublets. No known series in thallium has a convergence frequency greater than 49263. The observed ionization potential for thallium suggests the presence of an undiscovered series of single lines converging at $\nu=1{\cdot}5\,\mathrm{S}=57000$ to 60000. The present work offers evidence that the single line spectra of magnesium and thallium are $\lambda=4571$ and 11513 Å. respectively. The general behaviour of the metals as regards ionization and resonance potentials appears to be identical for metals in the same group of the periodic table. In the case of thallium the component of the doublet having the higher frequency determines the value of the energy quantum absorbed by the atom. This fact suggests either the possibility of separate excitation of the components of a doublet by electronic impact or a behaviour of thallium vapour, thus excited, analogous to the emission of characteristic X-rays for which the K group does not appear until the energy of the impacting electrons is greater than that corresponding to K_γ. From work carried on mainly at the Bureau of Standards the writers have obtained a value of Planck's constant h by the method of ionization and resonance potentials. Using 13 determinations on 7 different metals, each determination a mean of several experiments, the final value of $h=6{\cdot}55\,.\,10^{-27}$ erg sec. was obtained in excellent agreement with recent determinations by other methods. The question of photoelectric sensibility of metallic vapours has been briefly discussed, and several recent papers upon the subject of ionization and

resonance potentials have been considered from a critical standpoint. In conclusion, the writers desire to acknowledge their indebtedness to Prof. F. A. Saunders, of Vassar College, for suggestions in the preparation of the manuscript.

Bureau of Standards,
Washington, D.C.
April 18, 1918.

Note added Nov. 11, 1918.

Since the above paper was written the writers have studied electronic discharge in sodium vapour below the ionization potential and have reported upon this work in J. Wash. Acad. Sci. viii. p. 513 (1918). *Both* D-lines appear in the normal ratio of intensities at the resonance potential.

IV. *On Sommerfeld's Treatment of the Problem of Diffraction by a Semi-infinite Screen.* *By* SISIR KUMAR MITRA, *M.Sc., Lecturer in Physics at the Calcutta University*[*].

[Plate II.]

Introduction.

THE rigorous treatment of the effect of obstacles on the propagation of light, considered as a boundary-value problem in analysis, has received much attention from mathematical physicists during recent years[†]. The particular case of the diffraction of light by a semi-infinite perfectly reflecting screen, for which the complete solution was first given by Sommerfeld[‡] (and later by Carslaw[§]), has been more recently dealt with by Lamb[||] in a paper characterized by very simple and elegant mathematical analysis. As remarked by Lamb in his paper, the principal interest to the physicist of investigations such as these lies in the fact that they afford a check on the accuracy of the results obtained by less rigorous methods, and also enable a comparison of the theory with experiment to be carried out for cases in which the ordinary treatment can hardly be regarded as applicable. A comparison of the results of the approximate theory with those

---

[*] Communicated by Prof. C. V. Raman.

[†] An excellent summary of the literature of the subject with references to the original papers will be found in the article by Epstein in the *Encycl. Math. Wiss.* (section on Wave Optics, 1914).

[‡] Sommerfeld, *Math. Annal.* vol. xlvii. p. 317 (1895).

[§] Carslaw, Proc. Lond. Math. Soc. vol. xxx. p. 121 (1899).

[||] Lamb, Proc. Lond. Math. Soc. (2) vol. iv. p. 190 (1906).

deduced from the rigorous analysis for the case of the semi-infinite screen has been made by Sommerfeld himself, and also by Drude*, who has used the Cornu Spirals with good effect in his discussion of the value of Sommerfeld's integrals. According to these writers, Kirchhoff's formula should give the value of the intensity of illumination with sufficient accuracy when the angle of diffraction is small, that is at all points of the field (except very near the edge of the screen) which are not far removed from the boundary between light and shadow; but for large angles of diffraction, Kirchhoff's formula is inapplicable. It appears, however, from a careful examination of the formulæ given by Sommerfeld and Drude, that the statement made by them on this point requires to be qualified in one important respect. I propose in the present paper to show by a detailed discussion that *when the screen is held very obliquely in the path of the incident waves,* the rigorous treatment gives results differing from those of the approximate theory *even in regard to small angles of diffraction.* Experimental work recently carried out by me and described in the course of the paper confirms this, and shows that the approximate theory of diffraction fails to represent the facts correctly under these conditions. Incidentally it is found that in the case of light polarized in a plane perpendicular to that of incidence, the boundary condition at the screen assumed by Sommerfeld leads to results differing very widely from the observed optical behaviour of any actual screen at very oblique incidences, and a suggestion is made as to the manner in which the rigorous solution should be modified in order to secure an agreement with the result obtained experimentally in this case.

Theory.

It is convenient here to state Sommerfeld's results in the simplified form obtained by him from a semi-convergent expansion of the integrals representing the complete solution. This is

$$s=\cos\left[\frac{2\pi}{\lambda}r\cos(\phi-\phi')+nt\right]\mp\cos\left[\frac{2\pi}{\lambda}r\cos(\phi+\phi')+nt\right]$$

$$+\frac{1}{4\pi}\sqrt{\frac{\lambda}{r}}\left[\pm\frac{1}{\cos\frac{\phi+\phi'}{2}}-\frac{1}{\cos\frac{\phi-\phi'}{2}}\right]\cos\left[\frac{2\pi}{\lambda}r+\frac{\pi}{4}-nt\right],$$

where s is the light disturbance, ϕ' and ϕ are respectively

---

[*] Drude, 'Theory of Optics,' English translation by Mann and Millikan, p. 203.

the angles made by the incident rays and by the radius vector with the plane of the screen, and r is the distance of the point of observation from the edge of the screen. The alternative signs refer to the state of polarization of the incident light. The upper sign should be taken in the case when the incident light is polarized in a plane perpendicular to the edge of the screen, *i. e.* when the electric vector is parallel to the edge, and the lower sign in the case when the light is polarized in a plane parallel to the edge, *i. e.* when the magnetic vector is parallel to the edge. The first and second terms in the expression represent the incident and the reflected waves respectively, while the third term gives the wave of diffraction. In the region of shadow, only the third term should be taken into account; in the region of transmission we have to take the first and third terms only, while in the region of reflexion all the three terms in the expression for the light disturbance have to be retained. Thus, beyond the path of the rays determined by geometrical optics, there is a wave of diffraction whose phase is determined by the factor

$$\cos\left(\frac{2\pi}{\lambda}r+\frac{\pi}{4}-nt\right),$$

and whose amplitude by the factor

$$\frac{1}{4\pi}\sqrt{\frac{\lambda}{r}}\left[\pm\frac{1}{\cos\frac{\phi+\phi'}{2}}-\frac{1}{\cos\frac{\phi-\phi'}{2}}\right].$$

The lines of equal phase in this wave-train are circles round the point $r=0$, so that from this point rays start out on all sides in the direction of the radius vector, straight on as if the edge of the screen were a linear source of light. The intensity of these cylindrical waves is, however, not the same in all directions. It is greatest near the region of the two boundaries separating the different parts of the field, and gradually diminishes as we go away from these boundaries.

As is well known, the most remarkable result indicated by Sommerfeld's analysis, and which is in substantial agreement with the experimental observations of Gouy, Wien, and others, is that the amplitude of the diffracted waves is different for light polarized in and at right angles to the plane of incidence. This is sufficiently clear from the expression given above. For small angles of diffraction, however, that is in the neighbourhood of the planes defined by $\phi=\pi-\phi'$ and $\phi=\pi+\phi'$, the difference in the magnitude of the two components is generally quite negligible. For, in the neighbourhood of the first plane, $\dfrac{1}{\cos\dfrac{\phi+\phi'}{2}}$ is numerically very large compared with $\dfrac{1}{\cos\dfrac{\phi-\phi'}{2}}$, and similarly in the neighbourhood of the second plane $\dfrac{1}{\cos\dfrac{\phi-\phi'}{2}}$ is very large compared with $\dfrac{1}{\cos\dfrac{\phi+\phi'}{2}}$. Accordingly, in the neighbourhood of these two planes, it is ordinarily sufficient to retain one of the two terms and neglect the other in the expression for the amplitude of the diffracted waves, which is thus numerically the same for both states of polarization, and for the region considered is in substantial agreement with that found from the approximate theory. But the preceding argument fails entirely when ϕ' is nearly equal to either π or zero, that is when the incidence of the light at the screen is very oblique. For the two terms $\dfrac{1}{\cos\dfrac{\phi+\phi'}{2}}$ and $\dfrac{1}{\cos\dfrac{\phi-\phi'}{2}}$ are then always of comparable magnitude, and have both to be retained. It is thus clear that some special features are to be expected when the incidence of light on the screen is very oblique.

We may now consider separately the two cases in which ϕ' is nearly equal to π and zero respectively, as they present distinctive features. When ϕ' is nearly equal to π (see figs. 1*a* and 1*b*), by far the largest part of the field is occupied by the region of transmission (marked II. in the fig.), and the remaining part of the field is equally divided between regions of shadow and of reflexion (marked I. and III. respectively in the figure) which lie on opposite sides of the screen and are completely separated by it. If we wish to observe the phenomena in the neighbourhood of the boundaries separating the different parts of the field, we have two distinct choices open to us. We may either study the phenomena near the region of shadow which lies on one side of the screen, in which case ϕ is nearly equal to 2π (fig. 1*a*), or we may study the region which lies on the other side of the screen, in which case ϕ is nearly equal to zero (fig. 1*b*). The case in which ϕ is nearly equal to 2π is the simpler of the two, as we are then concerned only with the transmitted and diffracted wave-trains.

Fig. 1 *a*.

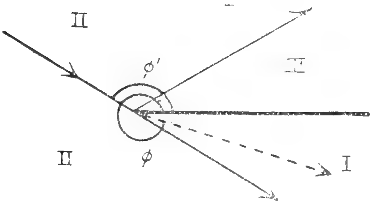
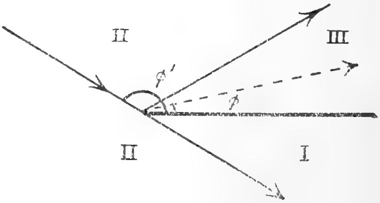


Fig. 1 *b*.



Putting $\phi' = \pi - \alpha$ and $\phi = 2\pi - \beta$ where α and β are small angles (fig. 1*a*), it is found that $\dfrac{1}{\cos\dfrac{\phi+\phi'}{2}}$, and $\dfrac{1}{\cos\dfrac{\phi-\phi'}{2}}$, are of comparable magnitude. Sommerfeld's formula thus leads to the striking result that in the case of a very obliquely-held screen, the intensity of the diffraction-fringes seen near the boundary between light and shadow should depend to a very considerable extent upon the plane of polarization of the incident light, and should be quite different from that given by Kirchhoff's formula. Similarly on the other side of the screen (fig. 1*b*), putting $\phi' = \pi - \alpha$ and $\phi = \beta$, where α and β are small angles, we may work out the expression for the light disturbance. If $\beta > \alpha$, we are only concerned with the interference of the incident and diffracted wave-trains; while if $\beta < \alpha$, we have to consider the stationary waves formed by the interference of incident and reflected wave-trains and modified by the superposition of the cylindrical waves radiated by the edge of the screen. If the light be polarized in the plane of incidence, we find that when ϕ is vanishingly small, the expression for the light disturbance is zero. This shows that the surface of the mirror is a nodal plane for the light vector in this case. On the other hand, if the light be polarized perpendicular to the plane of incidence, it is found that for vanishingly small values of ϕ, that is along the surface of the mirror, the intensity of the diffracted waves does *not* vanish and that, moreover, the incident and reflected wave-trains reinforce one another, the light vector being thus a maximum at the surface of the mirror. The last-mentioned result is a direct consequence of the boundary condition $\left(\frac{\partial s}{\partial z}=0\right)$ at the surface of the mirror assumed by Sommerfeld as the basis of his work, but is contrary to the observed optical behaviour of any actual screen at very oblique incidences. In practice any polished surface is, at such incidences, nearly a perfect reflector; but, as shown by Lloyd's experiment, both the electric and magnetic vectors in the incident and reflected waves are of opposite signs at the surface irrespective of the plane of polarization of the incident light, and their resultant is zero.

The preceding discussion makes it clear that the solution obtained with the boundary condition $\frac{\partial s}{\partial z}=0$ for the light vector at the surface of the mirror is entirely inapplicable under experimental conditions for oblique incidences. The solution obtained with the boundary condition ($s=0$) may, however, for our present purpose be regarded as practically valid at all incidences for any screen which is a sufficiently near approach to a perfect reflector, *provided the light be polarized in the plane of incidence.* The experimental investigation described in the present paper shows that *the same solution may also be regarded as applicable for light polarized at right angles to the plane of incidence* provided the incidence be very oblique, and attention is confined to the phenomena observed at small angles of diffraction.

Experimental Methods and Results.

The diffraction fringes of the Fresnel type bordering the shadow of an obliquely-held screen may be observed with the arrangement shown diagrammatically in fig. 1 *a*. A front-silvered glass plate bordered by parallel straight edges may be used as the screen. The first edge diffracts the incident light, and by observing the fringes near the surface of the mirror at the second edge, the necessity for using an infinitely extended screen is avoided. The contrast between the maxima and minima of illumination is found to be not entirely independent of the inclination of the surface of the screen to the direction of the incident rays. When the inclination is considerable, the fringes at the edge of the shadow are of course of the usual Fresnel type, few in number and very diffuse. But as the surface is gradually brought up to the position in which it just begins to graze the incident light, the contrast between the maxima and minima of illumination in the fringes gradually increases, their relative position remaining unaltered, and at the same time the falling off of the intensity to zero inside the geometrical shadow of the screen becomes more rapid than in the diffraction fringes of the ordinary type due to a normally held screen. On examination of the fringes through a nicol, it is found that the intensity of the fringes is independent of the plane of polarization of the incident light.

A detailed comparison has been carried out between the position and the intensities of the fringes as observed experimentally with those calculated from the theoretical expression

$$s=\cos\left[\frac{2\pi}{\lambda}r\cos(\phi-\phi')+nt\right]$$
$$+\frac{1}{4\pi}\sqrt{\frac{\lambda}{r}}\left[\frac{1}{\cos\frac{\phi+\phi'}{2}}-\frac{1}{\cos\frac{\phi-\phi'}{2}}\right]\cos\left[\frac{2\pi}{\lambda}r+\frac{\pi}{4}-nt\right],$$

the second term in which satisfies the boundary condition $s=0$ at the surface of the mirror. When ϕ' is much less than π, the intensity curve given by the expression is practically the same as that obtained from the usual Fresnel integrals, and is shown in the dotted line in fig. 2 (*c*)*. As ϕ' is gradually increased so as to approach the value π, the maxima and minima of the illumination remain unaltered in position, but the contrast between them gradually increases. The full line in fig. 2 (*c*) shows the calculated intensity curve in the limiting case in which ϕ' is equal to π and the screen just grazes the incident light. The illumination is seen to be zero on the surface of the mirror.

Table I. shows in the first column the calculated intensities of the maxima and minima in the diffraction fringes of the Fresnel type due to a normally held screen, and in the second column those due to a screen grazed by the incident rays, the intensity in the incident waves being taken as unity. The calculated positions of the maxima and minima are given in the third column, these being, of course, the same in both cases.

---

* The asymptotic expansion given by Sommerfeld is inapplicable over a very small part of the field on either side of the boundaries $\phi=\pi+\phi'$ and $\pi-\phi'$. A small part of each of the curves shown in fig. 3 has accordingly been filled in in free-hand so as to represent as closely as possible the general trend of the curve.

Fig. 2.

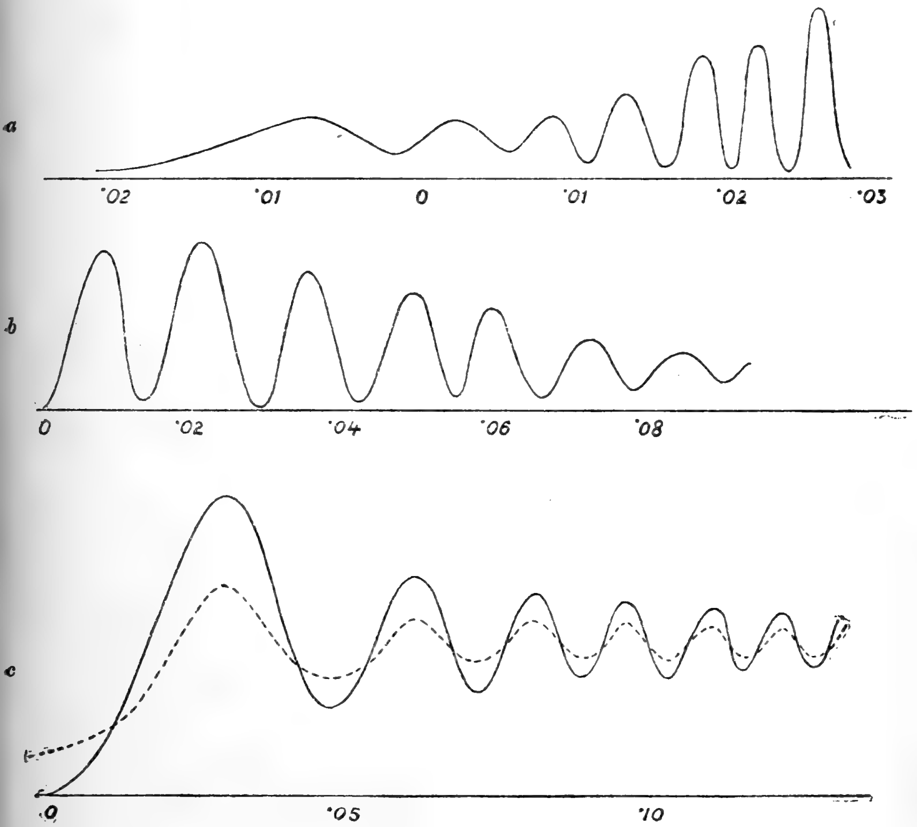


TABLE I.—$\lambda = 4377$ A.U. $r = 30{\cdot}75$ cm.

Calculated intensity at the maxima and minima. Case (1).	Calculated intensity at the maxima and minima. Case (2).	Calculated position of the maxima and minima $\times\sqrt{\frac{2}{r\lambda}}$.	Observed position of the maxima and minima $\times\sqrt{\frac{2}{r\lambda}}$.
1·374	1·920	1·225	1·192
·777	·590	1·871	1·887
1·199	1·238	2·345	2·345
·843	·722	2·739	2·746
1·151	1·300	3·082	3·077
·872	·757	3·391	3·409
1·126	1·243	3·674	3·692
·889	·786	3·937	3·960
1·110	1·219	4·183	4·155
·900	·820	4·416	4·422
1·099	1·199	4·637	4·615
·909	·820	4·848	4·850
1·091	1·153	5·050	5·000

The fourth column in the Table gives the positions of the maxima and minima of illumination measured on a photograph taken of the fringes due to a screen grazed by the incident rays. They agree closely with the theoretical values given in the third column.

The ratios of the intensity of illumination at the maxima and minima have been determined photometrically for comparison with the theoretical values shown in the second column of Table I. As mentioned above, the intensities were found to be independent of the plane of polarization of the incident light. The method adopted for the photometric work was as follows. The incident light was plane polarized by passage through a nicol. Two narrow slits were mounted one above the other in the plane of the diffraction fringes, and a thin plate of mica of proper thickness (·032 mm.) was fixed up on the upper one, so oriented that it circularly polarized the light falling on it. The field was observed through an eyepiece and an analysing nicol mounted in a graduated circle. The lower and the upper slit were then respectively set on the first maximum and the first minimum, or the second maximum and the second minimum, and so on. The illumination of the upper and the lower slit was equalized by rotating the analysing nicol, and the position of the analyser at which the light from the lower slit was extinguished was also noted. The angular difference of the two positions suffices to give the ratio of the illumination. Thus if θ_1 and θ_2 be the two positions and I_1 and I_2 the intensities of illumination of the upper and lower slit respectively,

$$I_1/I_2 = \sin^2(\theta_1 - \theta_2).$$

A correction was made for the loss of light in transmission through the mica sheet covering the upper slit. Table II.

TABLE II.

Ratio of intensity of Minima and Maxima.	Observed value.	Value calculated from Sommerfeld's expression.	Value calculated from Fresnel's integrals.
1st min. and 1st. max.	·36	·30	·58
2nd min. and 2nd max.	·53	·50	·67
3rd min. and 3rd max.	·60	·57	·79

shows the results of the photometric work which was carried out on the fringes obtained with monochromatic light. It was not found practicable to carry the measurements beyond the third fringe. The photograph reproduced in Plate II. fig. 4 (*c*) clearly shows that the contrasts between the maxima and minima of illumination in this case are greater than in the fringes of the Fresnel type.

Two other cases besides that described above have been investigated experimentally. In one of these shown diagrammatically in fig. 1 (*b*), ϕ' was nearly equal to π and ϕ was small and positive. The positions and magnitudes of the maxima and minima of illumination were calculated from Sommerfeld's solution (the upper signs only being taken), and the experimental data are shown in Table III. for comparison. The illumination curve for this case is shown in fig. 2 (*b*), and the photograph on which the measurements were made is reproduced in Plate II. fig. 4 (*b*). In this case on the side of the screen under observation, only the reflected wave-front is limited by a boundary passing through the edge of the screen, and suffers diffraction, and the general agreement between experiment and theory is only rendered possible by taking for the latter the rigorous solution obtained by Sommerfeld, the boundary condition $s=0$ being assumed to be satisfied at the surface of the screen.

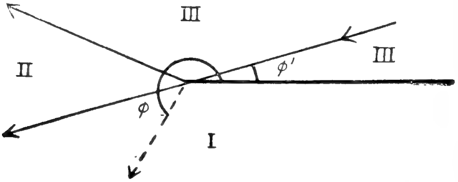
Table III.

$r=30{\cdot}75$ cm. $\lambda=4377$ A.U. $\phi'=179^\circ\ 54'{\cdot}56$.

Calculated Intensity of the maxima and minima.	Calculated distance of the maxima and minima from the edge of the screen in cm.	Observed distance of the minima from the edge of the screen.
Max. 3·83	·006	
Min. 0·18	·0140	·0140
Max. 4·52	·020	
Min. 0·02	·0278	·0275
Max. 3·38	·036	
Min. 0·20	·0423	·0421
Max. 3·20	·0490	
Min. 0·21	·0558	·0554
Max. 2·45	·060	
Min. 0·32	·0682	·0681
Max. 1·17	·074	
Min. 0·60	·0798	·0796
Max. 1·47	·082	
Min. 0·64	·0912	·0907
Max. 1·39	·092	
	·1014	·1008

The third case investigated is that shown in fig. 3. ϕ' is a small positive angle and ϕ is nearly equal to π.

Fig. 3.



In this case practically the whole of the field is divided equally between the region of shadow and reflexion (marked I. and III. in the figures) and only a comparatively small portion (marked II. in the figure) is that occupied by the region of transmission. Both of the dividing boundaries fall within the region of the field under observation; and so far as this part of the field is concerned, the approximate theory of diffraction gives the same result as that found from the complete analytical solution satisfying the boundary condition $s=0$. The theoretical form of the illumination curve from a typical case of this kind is shown in fig. 2 (*a*), and a photograph of the diffraction fringes is reproduced in Pl. II. fig. 4 (*a*). Table IV. shows the calculated positions of the minima of illumination and the experimental data for comparison. The agreement is satisfactory.

Table IV.

$r=3$ cm. $\lambda=4410$ A.U. $\phi'=19'\cdot97$.

Calculated widths of the successive bands between the minima of illumination.	Observed widths.
0·0071 cm.	0·0071
0·0052	0·0053
0·0046	0·0048
0·0044	0·0042
0·0039	0·0041
0·0037	0·0038

Summary and Conclusion.

The results arrived at from this investigation may be summarized as follows :—

When plane waves of light are diffracted by the edge of a very obliquely held screen, the fringe systems observed *at and near the surface of the screen on either side* show features which require for their explanation the complete analytical investigation of diffraction given by Sommerfeld. On one side of the screen we have the region of shadow, and adjoining it diffraction fringes, the maxima and minima of illumination in which show contrasts more marked than those in the diffraction fringes of the Fresnel type, their positions, however, being the same. This has been verified by photometric observation. On the other side of the screen, the fringes due to the interference of the direct and reflected wave-trains are observed, and these are modified by diffraction in a manner which can be fully explained only in terms of the complete analytical solution of the diffraction problem.

The solution obtained by Sommerfeld with the boundary condition $s=0$ at the surface of the mirror, agrees with the results observed at oblique incidences in the part of the field under discussion, irrespective of the plane of polarization of the incident light.

The investigation described in the paper was carried out in the Palit Laboratory of Physics.

Calcutta,
16th March, 1918.

V. *The Light from Mercury Vapour.*

By C. D. Child [*].

In a recent article by Strutt [†] a description is given of experiments on the luminous vapour coming from the discharge through gases at low pressure. The conclusion reached in that paper is that the light given off by the vapour very largely comes from the negative ions. The conclusion reached during an investigation of the luminous vapour distilled from the mercury arc performed by myself was that the light did not come from either the positive or the negative ions alone, but from the two at the moment of recombining [‡].

Two questions present themselves in connexion with these papers. First, is the same fundamental action being studied in the two sets of experiments? Secondly, do the experiments of Strutt prove his conclusions? Apparently neither of these

---

[*] Communicated by the Author.

[†] Proc. Roy. Soc. A, xciv. p. 88 (1917).

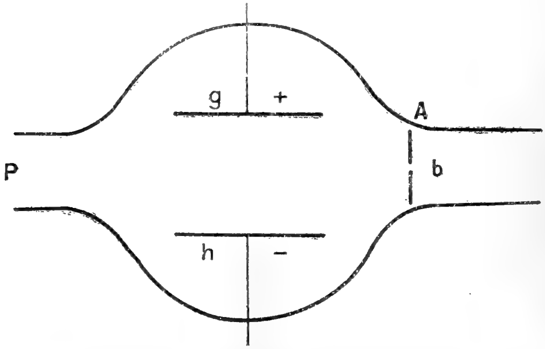
[‡] Phil. Mag. [6] xxvi. p. 906 (1913).

questions can be answered definitely until further experimental work has been done; and, since I do not have apparatus making it possible to repeat Strutt's experiments, I am taking the liberty of pointing out the desirability of further investigation of this subject.

In regard to the first question, the following statement may be made. In the experiments of Strutt, a current of a few milliamperes at a high voltage was passed through mercury vapour which had a slight admixture of some other gas. The presence of the additional gas was found to be necessary in order to produce the effects observed. In the experiments by myself there was a current of several amperes passing through a gas that was in all probability pure mercury vapour. The necessity of some impurity in the former experiments would indicate that two different phenomena were being studied. On the other hand, the fact that the luminosity lasts for practically the same length of time indicates that the cases are identical. We should be in a better position to decide this question if the spectrum of the luminosity observed by Strutt could be obtained. Unfortunately, the light was too faint to allow this to be done.

Again, we could better decide regarding the identity of the two phenomena, if it were certain that the light examined by Strutt was not caused by recombination—that is, if his experiments prove his conclusions. Certainly an explanation different from that given by him can be suggested, as is shown by the following discussion.

Fig. 1.



In Strutt's experiments a discharge is passed through a tube at the right of the part shown in fig. 1. A is the anode for this discharge. In this anode there is a small opening, and gas from the discharge is blown through this opening into the tube shown in the drawing. P is the connexion to the pump; *g* and *h* are electrodes to which an E.M.F. may be applied, so as to deflect any ions which come into this region. When these electrodes are uncharged, the luminosity spreads out from the opening *b* in a symmetrical manner, becoming fainter toward the left of the drawing. When an E.M.F. of 160 volts was applied between *g* and *h*, the position of the luminosity was shifted.

In all of the photographs shown by Strutt there was an increase of the luminosity in the neighbourhood of the positive electrode. In the majority of the photographs there was also an increase in the light near the negative electrode. According to Strutt, the positive and negative ions are separated by the field, and the luminosity near the positive electrode is caused by the negative ions alone, and that near the negative electrode by the positive ions alone.

As has often been shown, it is impossible to separate positive and negative ions by this method unless there is a very high vacuum on account of the formation of new ions. Thus in the stream of gas coming from A there are electrons, and since the potential difference between *g* and *h* is 160 volts, the electrons will be drawn toward the positive electrode with sufficient velocity to ionize by impact any molecules which may be present. The recombination of the ions thus formed will then give out light. That is, a second discharge is produced between *g* and *h*, giving out light as any other discharge through a rarefied gas gives out light. In fact, the potential difference is sufficiently large here to make it possible for the positive ions as well as the negative ones to ionize by impact, so that new ions may be formed both in the region of the positive and in that of the negative electrode.

The only way to avoid such action is to obtain a gas so rarefied that there are no molecules present. Unfortunately, no measurements were made of the pressure of the gas in Strutt's experiment; but it is evident that there was a sufficient number of molecules present to allow of ionization by impact, since an arc was apt to start between *g* and *h* when the voltage was raised above 160 volts (p. 911). It is impossible to have an arc in a region where there are no molecules, for no ions can be formed without molecules from which to form them. This is shown, for example, in the Coolidge X-ray tube, where the voltage may be several thousand volts without starting an arc.

Furthermore, it was found that there was the greatest tendency to start an arc when the gas was least dense. If there had been very few molecules present, the lower the density of the gas the less possibility there would have been of the formation of an arc.

Again, to start an arc there must be ionization by the positive ions as well as by the electrons. Consequently, with 160 volts a condition was being approached where both positive and negative ions were able to ionize by impact.

The appearance of the photographs favours the view that there was not a displacement of the light, but an actual production of new centres of luminosity. When no field is present, the luminosity gradually fades out as the gas passes from the opening. When the field is applied, it does not in any case gradually fade out. In some cases the whole region between the electrodes becomes luminous. In all cases the region near the positive terminal is luminous. In photograph No. 11, for example, which was a photograph of mercury vapour, the luminosity first decreases as the gas leaves the opening, and then increases as the positive terminal is approached. It is very difficult to see how the luminosity could increase, unless new centres of radiation were formed. The ions do not give out more light as they move from the opening, as is shown in the photograph where there is no field. The application of an electric field to this region causes the ions to move out of the region, so that there are fewer ions per unit volume. Moreover, the strength of the field will become greater in the neighbourhood of the electrodes, and the ions will move faster, so that the number of the ions per unit volume will be less, unless new ions are formed near the electrodes. If, then, the photograph represents correctly the appearance of the glow, there must be new centres of illumination formed near the positive electrode. It is difficult to see how this can occur, unless there is ionization by impact near the electrodes.

Again, the increase in luminosity in the neighbourhood of the negative electrode is most prominent when the density of the gas is least. This is also what would be expected if the luminosity is due to ions which are produced by positive ions. It would only be with the more rarefied gas and longer mean free path that the positive ions would be able to ionize. When both the positive ions and the electrons are able to ionize, it only requires a small increase in voltage to start an arc; and this, in fact, was what was found to occur.

It would, therefore, appear possible that the luminosity observed by Strutt was caused by recombination of positive and negative ions, and not by any action of the positive or negative ions alone; and that further experiments are necessary before a definite conclusion is reached concerning this phenomenon.

Colgate University.
Sept. 1918.

VI. *Molecular Attraction and Attraction of Mass, and some New Gas Equations.* *By* JAMES KAM *.

SINCE the publication of my article, "A Criticism on van der Waals' Equation, and some new equations derived therefrom" †, I notice I have omitted an argument without which the relation $2p_cv_c=$constant and its deduction cannot be understood.

In submitting the explanation here, some repetition for the sake of continuity of argument was unavoidable; but as the subject leads to entirely new expressions, and seems to point to a general law ruling the cohesive forces, I publish the results so far obtained under a new title.

Introduction.

The relation between pressure, volume, and temperature during isothermal compression of a "perfect" gas is given by the expression

$$\Pi=\frac{R.T}{v},$$

the law of Boyle-Mariotte.

For an actual gas according to van der Waals, the total pressure Π (equal to the sum of the pressure p from the exterior and the "Inward Pressure" p_1) is expressed by

$$\Pi=\frac{R.T}{v-\beta},$$

if β is the "co-volume" of the molecules of the gas contained in v.

The volume v is expressed in respect of $v_0=1$, 0° C. and 760 mm., and β is a function of v; and it follows that for the same pressure Π but a temperature T_1, both v and β change at the rate $\frac{T_1}{T}$.

The expression can therefore hardly be a correct version of the physical process of isothermal compression.

For suppose the same pressure Π to be exerted at a lower temperature T_1. Then the velocity of mean square c decreases at the rate $\frac{T_1}{T}$, *i. e.* proportional to the temperature. If the absolute value of v remains the same, the density increases at a rate smaller than $\frac{T_1}{T}$, say x times, and the co-volume

---

* Communicated by the Author.

† Kam, Phil. Mag. vol. xxxi. January 1916, p. 22.

becomes $\beta_1 = x \, . \, \beta$; so that the combined effect of these last two increases compensates the decrease of c.

This might be expressed as follows :—

$$\Pi = \frac{R \, . \, T}{v - \beta} = \frac{x \, . \, R \, . \, T_1}{v - x \, . \, \beta} = \frac{R \, . \, T_1}{\frac{v}{x} - \beta};$$

and it will be seen, since $x < \frac{T}{T_1}$, that a gas reaches a total pressure Π at a *lower* temperature sooner than at a *higher* temperature. The same changes of volume cause larger changes of pressure at lower than at higher temperatures.

Another objection lies in the solution of the equation of van der Waals for the critical state. If we convert it into the form

$$p + \frac{a}{(\phi + \beta)^2} = \frac{R \, . \, T}{\phi},$$

making $v = \phi + \beta$, and solve for the critical state, we find

$$\frac{dp}{d\phi} = 0, \quad \frac{2a}{(\phi_c + \beta)^3} = \frac{R \, . \, T_c}{\phi_c^2}, \quad . \quad . \quad . \quad . \quad (i.)$$

$$\frac{d^2 p}{d\phi^2} = 0, \quad \frac{6a}{(\phi_c + \beta)^4} = \frac{2R \, . \, T_c}{\phi_c^3}, \quad . \quad . \quad . \quad . \quad (ii.)$$

whence

$$\phi_c = 2\beta.$$

Since ϕ here is certainly a volume but *not* a density, we may not calculate the values of the critical data by substituting $\phi_c = 2\beta$ in (i.) or (ii.). Yet if we do so, we obtain those values as found by van der Waals. The operation is the same though in inverse order as the one of multiplying and arranging according to powers of ϕ_c, and the same error is made.

Substituting *e. g.* in (i.), we find

$$\frac{2a}{27\beta^3} = \frac{R \, . \, T_c}{4\beta^2}$$

and

$$a = \frac{27}{8} . \, \beta \, . \, R \, . \, T_c,$$

$$p_c = \frac{a}{27\beta^2} = \frac{1}{3} \frac{a}{v_c^2} = \frac{1}{3} p_{1_c},$$

$$T_c = \frac{8}{27} \cdot \frac{a}{R \, . \, \beta},$$

which for a, p_c, p_{1c}, and T_c are the values found by van der Waals.

It would have been more correct to substitute $\phi_c = 2\beta$ in the original equation, converting the latter into the quadratic form

$$p_c + \frac{a}{v_c^2} = \frac{3}{2} \frac{\mathrm{R}\,.\,\mathrm{T}_c}{v_c},$$

an equation in which v_c has the same meaning on either side, and which is a special form of a general quadratic expression into which we can reduce a cubic gas equation. In this manner we take into account that the two extreme values of v for the roots of every cubic equation are points of the quadratic "Border-curve."

I.

In order to meet the two objections mentioned, I have found it useful to introduce, apart from the factor β, another factor b.

β is the actual co-volume ;
b is the "theoretical" co-volume.

Imagine an actual gas with a co-volume β exerting at the volume v and the temperature T the same pressure $\Pi = p + p_1$ (pressure + the effect of the cohesive forces) as exerted by a "perfect" gas at the same volume and temperature. If the number of molecules in the latter is $\mathrm{N} = \frac{1}{v}$, then the number in the former is

$$\mathrm{N}_1 = \frac{v}{v+b} \mathrm{N},$$

with the co-volume

$$\beta = \frac{v}{v+b} \,.\, b.$$

Consequently b is the "theoretical" co-volume of N molecules of an actual gas at the "theoretical" volume $(v+b)$, which at the "measured" volume v would exert the pressure Π of N molecules of a "perfect" gas at the same temperature and volume v. The volume v contains N_1 molecules of the actual gas, and we have the relations :

$$\mathrm{N}_1 = \frac{v}{v+b} \mathrm{N} = \frac{v-\beta}{v} \mathrm{N} = \nu\,.\,\mathrm{N}, \quad . \quad . \quad . \quad . \quad (1)$$

$$\beta = \nu\,.\,b, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

$$b = \nu^{-1}\,.\,\beta, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

$$\nu = \frac{\mathrm{N}_1}{\mathrm{N}} = \frac{v}{v+b} = \frac{v-\beta}{v}. \quad . \quad . \quad . \quad . \quad . \quad . \quad (4)$$

We may call ν the "specific density" of a gas in respect of a perfect gas at the same volume, temperature, and pressure Π. It is always <1, and in respect of Π it compensates the effect of β, viz.

$$\Pi = \frac{\nu\,.\,\mathrm{R}\,.\,\mathrm{T}}{v-\beta} = \frac{\mathrm{R}\,.\,\mathrm{T}}{v}. \quad . \quad . \quad . \quad . \quad . \quad . \quad (5)$$

If, however, the volume of the actual gas is given in respect of $v_0=1$, 0° C. and 760 mm., and such is naturally the case, then the pressure is

$$\Pi_1 = \frac{\mathrm{R}\,.\,\mathrm{T}}{v-\beta} = \frac{v+b}{v}\;\frac{\mathrm{R}\,.\,\mathrm{T}}{v}, \quad . \quad . \quad . \quad . \quad . \quad (6)$$

and we consider either β or b as a constant, but the other as a variable.

We can, *e.g.*, consider b_{T_c} as the theoretical co-volume for the critical state. Then β_{T_c} and v_c have the values of the experiment. At any other state the relative values are the same.

At small densities the values of b and β are practically the same. For large pressures they differ considerably. Another characteristic of b is, by definition, that its value is *inversely proportional* to the temperature T,—which, as was pointed out previously, is *not* the case for β. This characteristic will presently prove most useful.

It will be seen that for a definite temperature T we can express v in b_T or β_T, viz.

$$v = n\,.\,b_\mathrm{T} = n\,.\,\nu^{-1}\,.\,\beta_\mathrm{T} = (n+1)\beta_\mathrm{T}\,; \quad . \quad . \quad . \quad . \quad (7)$$

hence (*vide* equation 6),

$$\Pi = \frac{\mathrm{R}\,.\,\mathrm{T}}{v-\beta_\mathrm{T}} = \frac{\mathrm{R}\,.\,\mathrm{T}}{n\,.\,\beta_\mathrm{T}} = \frac{(n+1)b_\mathrm{T}}{n\,.\,b_\mathrm{T}}\,.\,\frac{\mathrm{R}\,.\,\mathrm{T}}{n\,.\,b_\mathrm{T}} = \frac{v+b_\mathrm{T}}{v}\,.\,\frac{\mathrm{R}\,.\,\mathrm{T}}{v}. \quad (8)$$

The total pressure Π of equation (8) is equal to the pressure p from the exterior plus the effect of the cohesive forces p_1. We make

$$\Pi = p+2p_1, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (9)$$

for the following reason.

A molecule travelling towards the surface conquers the attraction p_1. The normal component of its impact is diminished by p_1, but its temperature is not affected, as the gas is in thermic equilibrium with its surroundings. After the impact it at once proceeds with the velocity corresponding to its temperature, and on impinging towards the interior its impact is supported by the molecular attraction. Hence the pressure p from the exterior is supported by p_1, but the pressure Π from the interior towards the surface is opposed by the same force. The condition of equilibrium is satisfied when

$$\Pi - p_1 = p + p_1$$

or

$$p + 2p_1 = \Pi,$$

and it is immaterial whether the pressure Π exists between the surface and the interior conforming to the principle of the equality of action and reaction, or in the interior. In the former case only, Π would be identical with the "Intrinsic Pressure" of Laplace in the case of liquids, and the pressure p in the interior would be transmitted undiminished towards the exterior and be equal to the pressure p from the exterior on the gas, no matter how great the intermediate "Intrinsic Pressure" Π.

Making the total pressure Π of an actual gas equal to Π of a perfect gas at the same volume (absolute) and temperature, we then find

$$p + \frac{2a}{(v+b)^2} = \frac{\mathrm{R.T}}{v}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (\mathrm{A})$$

in which a is the constant of attraction if the density of the actual gas were proportional to the pressure Π, and $\frac{a}{(v+b)^2} = p_1$. In whatever way the volume v is measured, the density of the gas is $\frac{1}{v+b}$ when it exerts a pressure $\frac{\mathrm{R.T}}{v}$.

At the critical state the three points of the curve of equation (A) denoting the three values of v satisfying the equation coincide with the two points for which the first and second derivatives are equal to 0.

We obtain

$$\frac{dp}{dv} = 0, \qquad \frac{4a}{(v_c+b)^3} = \frac{\mathrm{R.T}_c}{v_c^2}, \quad . \quad . \quad . \quad . \quad . \qquad (\mathrm{i.})$$

$$\frac{d^2p}{dv^2} = 0, \qquad \frac{12a}{(v_c+b)^4} = \frac{2\mathrm{R.T}_c}{v_c^3}, \quad . \quad . \quad . \quad . \qquad (\mathrm{ii.})$$

whence

$$v_c = 2b. \quad . \quad . \quad . \quad . \quad . \quad . \qquad (10)$$

Hence (*v.* equations (1) and (2)),

$$\left.\begin{aligned} \mathrm{N}_1 &= \tfrac{2}{3}\mathrm{N}, \\ \beta &= \tfrac{2}{3}b, \\ \nu &= \tfrac{2}{3}, \end{aligned}\right\} ; \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (11)$$

i. e., at the critical state the density of an actual gas is $\frac{2}{3}$ of the density of a perfect gas at the same temperature and pressure Π_c.

The experiment, however, gives us the volume of an actual gas reduced to $v_0=1$, 0° C. and 760 mm. If c is the constant of attraction at $v_0=1$, we can express equation (6) in two forms, (B) and (C), in a cubic and in a quadratic form, viz.

$$p+\frac{2c}{v^2}=\frac{\mathrm{R}\,.\,\mathrm{T}}{v-\beta}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad \text{(B)}$$

or

$$p+\frac{2c}{v^2}=\frac{v+b}{v}\,.\,\frac{\mathrm{R}\,.\,\mathrm{T}}{v}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad \text{(C)}$$

Treating equation (B) in the same manner as equation (A), we have for the critical state,

$$\frac{dp}{dv}=0, \qquad \frac{4c}{v_c^{\,3}}=\frac{\mathrm{R}\,.\,\mathrm{T}_c}{(v_c-\beta)^2}, \quad . \quad . \quad . \quad . \qquad \text{(i.)}$$

$$\frac{d^2p}{dv^2}=0, \qquad \frac{12c}{v_c^{\,4}}=\frac{2\mathrm{R}\,.\,\mathrm{T}_c}{(v_c-\beta)^3}; \quad . \quad . \quad . \quad . \qquad \text{(ii.)}$$

whence

$$v_c = 3\beta, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (12)$$

in accordance with equations (7) and (11).

The curve of equation (C) is a quadratic curve. For every value of p there are two values of v satisfying the equation, which coincide with the two extreme values of v on the curve of equation (B). At the critical point these two values coincide in the vertex of the curve for which $\frac{dp}{dv}=0$. We find

$$\frac{dp}{dv}=0, \qquad \frac{4c}{v_c^{\,3}}=\frac{2\mathrm{R}\,.\,\mathrm{T}_c(v_c+b)}{v_c^{\,3}}-\frac{\mathrm{R}\,.\,\mathrm{T}_c}{v_c^{\,2}}, \text{ and}$$

$$(v.\ \text{eq. } 10) \qquad c = \tfrac{1}{4}\mathrm{R}\,.\,\mathrm{T}_c(v_c+2b) = \tfrac{1}{2}\mathrm{R}\,.\,\mathrm{T}_c\,.\,v_c, \quad . \quad . \qquad (13)$$

and the critical state can only be attained if this equation (13) is realized.

On substituting this value of c in the original equation, we find at once

$$p_c = \frac{1}{2}\frac{\mathrm{R}\,.\,\mathrm{T}_c}{v_c} \text{ or } 2p_c\,.\,v_c = \mathrm{R}\,.\,\mathrm{T}_c,\ 2p_c\phi_c = 1, \text{ if } \phi_c \text{ is}$$

$$\text{expressed in } \phi=1, \text{ at } \mathrm{T}_c \text{ and } \Pi = 1. \quad . \quad . \quad . \qquad (14)$$

$$\left.\begin{aligned} p_c &= p_{1c} = \frac{c}{v_c^{\,2}} = \frac{1}{2}\,.\,\frac{\mathrm{R}\,.\,\mathrm{T}_c}{v_c}, \\ \Pi_c &= 3p_c = 3p_{1_c} = \frac{3}{2}\,.\,\frac{\mathrm{R}\,.\,\mathrm{T}_c}{v_c}. \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (15)$$

The critical point thus forms the point of intersection of three curves belonging to the equations (A) or (B), (C), and (14).

The equality of the critical pressure and critical "Inward Pressure" accounts for the disappearance of the phenomena of surface-tension and of the latent heat of vaporization at the critical state. For the attraction is compensated by the thermic pressure, and the transference of a molecule from the interior of the fluid towards the space over its surface does not require work done against a force p_{1_c}. This equality can be deduced entirely independently of the preceding, as will be shown presently. But it is as well to point out that it demands that the principle of the equality of action and reaction applies to the cohesive forces as to all other forces*.

According to the preceding, we should get in a p.v.t.-system a series of cubic and quadratic curves in vertical planes. Every equal-pressure line cuts the two curves of each plane in their points of intersection, denoting the volumes of the saturated vapour and its liquid. At the critical point the equal-pressure line cuts through the point of intersection of three curves.

Subtraction of $\frac{c}{v_c^2}=\frac{1}{2}\,\frac{\mathrm{R}\,.\,\mathrm{T}_c}{v_c}$ (equation 14) from equation (C) for the critical state leads at once to

$$p_c+\frac{c}{v_c^2}=\frac{\mathrm{R}\,.\,\mathrm{T}_c}{v_c}, \quad \dots\dots \quad (\mathrm{C}_1)$$

a similar equation to equation (C), and naturally affording the same values for the vertex of its curve for which $\frac{dp}{dv}=0$.

We have not so far considered the influence of the temperature on the relative values of b and β in respect of v. Still such an influence must exist, and is bound to affect the increase of Π with increasing density. If we compress the same gas at different temperatures from the volume 1 and the total pressure $\Pi=1$, it is clear that the same compression causes a greater pressure-increase at the lower than at the higher temperatures. For β at the lower temperature is relatively and absolutely greater than at a higher temperature. A change of density at lower temperatures causes greater changes of Π than at higher temperatures.

Our equations (A), (B), and (C) do not allow for this temperature influence. Their solution causes the temperature factor to divide out. But it is here that the introduction of the factor b, the "theoretical co-volume," proves most useful.

---

* *Vide* Kam, Phil. Mag. vol. xxxi. Jan. 1916, p. 35.

II.

A General Gas Equation which considers the effect of the temperature on the relation of b and β to v.

For the same volume v and total pressure Π, the theoretical co-volume b is by definition inversely proportional to the absolute temperature T.

If v is the volume of a gas at the pressure Π, one time at a temperature 0° C. (T=273) and another time at T, then

$$v=n\,.\,b_0=\frac{T}{273}n\,.\,b_t$$

if b_0 and b_t are the respective theoretical co-volumes.

If v is expressed in $v_0=1$, 0°, and $\Pi=1$ (760 mm.), then the total pressure of the gas is for both temperatures (*vide* equations (7) & (8))

$$\Pi=\frac{v+b_0}{v}.\frac{1}{v}=\frac{1}{v-\beta_0}=\frac{n+1}{n}.\frac{1}{n\,.\,b_0}. \quad . \quad . \quad (16a)$$

Giving b_0 the value $\frac{T}{273}\,.\,b_t$,

$$\Pi=\frac{n+1}{n}.\frac{1}{\frac{T}{273}\,.\,n\,.\,b_t}=\frac{n+1}{n}.\frac{273/T}{n\,.\,b_t}, \quad . \quad . \quad (16\,b)$$

in which Π is equal *either* to the pressure of the gas *at the temperature* 0° (T=273) and the volume $\frac{T}{273}\,.\,n\,.\,b_t$, or *at the temperature* T and the *volume* $n\,.\,b_t$. In the latter case we have compressed $\left(\frac{273}{T}\right)^2 N_1$ molecules of the gas at T from a pressure $\Pi=\frac{273}{T}$ to a pressure $\Pi_1=\Pi$ (16 b) if N_1 was the number of molecules of the gas at 0° exerting the pressure Π at the volume v.

If we make the pressure Π of a gas at T equal to the pressure Π of the same gas at 0° C., and express b_t in b_0, then, if the volumes are expressed in equal multiples n of b_0 and b_t, $\Pi=1$, we have the relation

$$\Pi=\frac{v+b_0}{v}.\frac{1}{v}=\frac{1}{v-\beta_0}=\frac{n+1}{n}.\frac{1}{n\,.\,b_0}=\frac{n+1}{n}.\frac{273/T}{n\,.\,b_t}$$

$$=\frac{\phi+b_t}{\phi}.\frac{273/T}{\phi}=\frac{273/T}{\phi-\beta_t}=\frac{1}{v-\frac{T}{273}\beta_t}, \quad (16\,c)$$

in which $\phi=\frac{273}{T}\cdot v$, β_0 and β_t are the co-volumes at the volumes v and ϕ and 0°. In this expression consequently v is the volume of ϕ reduced to 0° and $v_0=1$ at $\Pi_0=1$, *i. e.* the experimental value of the volume.

In other words, for the same *relative* change of the "free space" $(\phi-\beta_t)$ or $\left(v-\frac{T}{273}\beta_t\right)$ for a gas at 0°, we obtain the same change of pressure Π : we have reduced the actual gas as regards pressure-changes to a perfect gas for which they are only dependent on the changes of volume, whatever the temperature T. The volume at the temperature T is $\phi=nb_t$, and at 0° $v=n\cdot b_0$, but the pressure Π is the same ; and the expression given by (16 *c*) applies to any temperature between T_c and 273, if $T_c>273$, whatever the compression at the temperature T from $\Pi_0=1$ to Π.

If $T_c<273$, then for any temperature between T_c and 273 we have similarly :

$$\Pi = \frac{\phi+b_t}{\phi}\cdot\frac{1}{\phi} = \frac{(n+1)b_t}{n\cdot b_t}\cdot\frac{1}{n\cdot b_t} = \frac{(n+1)b_0}{n\cdot b_0}\cdot\frac{1}{\frac{273}{T}n\cdot b_0}$$

$$= \frac{v+b_0}{v}\cdot\frac{1}{\frac{273}{T}\cdot v} = \frac{1}{\frac{273}{T}(v-\beta_0)} = \frac{1}{\phi-\beta_t}, \qquad (16\,d)$$

in which ϕ is expressed in $\phi_t=1$, T, and $\Pi=1$, $b_0=\frac{T}{273}b_t$, $v=\frac{T}{273}\phi$, and v has the experimental value, reduced to $\phi_t=1$, T°, and $\Pi_0=1$, and we have the gas for temperatures below 0° in the same condition as above that temperature.

(We could of course just as well reverse the proceedings and express the volume of a gas for temperatures >273 in ϕ in respect of $\phi_t=1$, at T_c° and $\Pi=1$, etc.)

Making Π of equations (16 *a*) and (16 *b*) equal to $p+2p_1$, we obtain for $T>273$,

$$p+\frac{2c_0}{\left\{\frac{T}{273}\right\}^2\cdot\phi^2} = \frac{\frac{T}{273}(\phi+b_t)}{\frac{T}{273}\phi}\cdot\frac{1}{\frac{T}{273}\cdot\phi} = \frac{1}{\frac{T}{273}(\phi-\beta_t)} \qquad (D)$$

for any temperature T (in which c_0 is the constant of attraction at $v_0=1$, 0° C., and 760 mm.), and always equal to

$$p+\frac{2c_0}{v^2} = \frac{v+b}{v}\cdot\frac{1}{v} = \frac{1}{v-\beta}, \quad . \; . \; . \; . \; . \qquad (D_1)$$

in which v (the volume reduced to $v_0 = 1$, 0° and $\Pi = 1 = 760$ mm.) is always $= \frac{T}{273}\phi$.

Solving the cubic expression for the critical state, we find (equation D_1):

$$\frac{dp}{dv^2} = 0, \qquad \frac{4c}{v_c^3} = \frac{1}{(v_c - \beta)^2}, \quad . \quad . \quad . \quad . \quad . \quad . \qquad \text{(i.)}$$

$$\frac{d^2p}{dv^2} = 0, \qquad \frac{12}{v_c^4} = \frac{2}{(v_c - \beta)^3}; \quad . \quad . \quad . \quad . \quad . \quad . \qquad \text{(ii.)}$$

whence $\left(\frac{\text{(ii.)}}{\text{(i.)}}\right)$

$$v_c = 3\beta = 3\frac{T_c}{273}\beta_t.$$

Hence (*vide* equations 1–4)

$$b = \frac{1}{\nu}\,.\,\beta = \frac{3}{2}\beta,$$

$$v_c = 2b = 2\frac{T}{273}\,.\,b_t.$$

Solving the quadratic equation (D_1), we find for the critical state,

$$\frac{dp}{dv} = 0, \qquad \frac{4c_0}{v_c^3} = \frac{2(v_c + b)}{v_c^3} - \frac{1}{v_c^2};$$

whence

$$c_0 = \frac{1}{2}v_c = \frac{1}{2}\phi_c\,.\,R\,.\,T_c, \quad . \quad . \quad . \quad . \quad . \qquad (17)$$

but in respect of $\phi_0 = 1$,

T_c and $\Pi_0 = 1$, $\quad c_{t_c} = \frac{1}{2}v_c\left\{\frac{273}{T_c}\right\}^2 = \frac{1}{2}\phi_c\,.\,\frac{273}{T}. \quad . \quad . \qquad (18)$

Substitution of $c_0 = \frac{1}{2}v_c$ in the original equation leads at once to

$$p_c = p_{1_c} = \frac{c_0}{v_c^2} = \frac{1}{2}\,.\,\frac{1}{v_c}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (19)$$

$$2p_c\,.\,v_c = 1 \text{ (constant)}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (20)$$

$$3p_c = 3p_{1_c} = p + 2p_{1_c} = \frac{3}{2}\,.\,\frac{1}{v_c} = \Pi_c. \quad . \quad . \qquad (21)$$

For temperatures <273 we have the equations

$$p + \frac{2c_t}{\left\{\frac{273}{T}\,.\,v\right\}^2} = \frac{v+b}{v}\,.\,\frac{1}{\frac{273}{T}\,.\,v} = \frac{R\,.\,T}{v - \beta}, \quad . \quad . \quad . \quad . \qquad (E)$$

$$p + \frac{2c_t}{\phi^2} = \frac{\phi + b_t}{\phi}\,.\,\frac{1}{\phi} = \frac{1}{\phi - \frac{273}{T}\beta}, \quad . \quad . \quad . \quad . \quad . \quad . \qquad (E_1)$$

of which (E) again applies to any temperature, *i. e.* any reduced value of ϕ in equation (E_1), and always is equal to (E_1).

The same operations lead in the case of the cubic expression (E_1) to the similar expressions,

$$\frac{dp}{d\phi} = 0, \qquad \frac{4c_T}{\phi_c^3} = \frac{1}{\left(\phi_c - \frac{273}{T_c}\beta_{T_c}\right)^2}, \quad . \quad . \quad . \quad . \quad . \qquad (i.)$$

$$\frac{d^2p}{d\phi^2} = 0, \qquad \frac{12c_T}{\phi_c^4} = \frac{2}{\left(\phi_c - \frac{273}{T_c}\beta_{T_c}\right)^3}; \quad . \quad . \quad . \quad . \quad . \qquad (ii.)$$

whence $\frac{(ii.)}{(i.)}$,

$$\phi_c = 3\beta \, . \frac{273}{T_c} = 3\beta_{T_c},$$

$$\nu = \frac{2}{3},$$

$$b_{T_c} = \frac{3}{2}\beta_{T_c} \, . \frac{273}{T_c} = \frac{1}{2}\phi_c,$$

and in the case of the quadratic form of (E_1)

$$\frac{dp}{d\phi} = 0, \qquad \frac{4c_T}{\phi_c^3} = \frac{2(\phi_c + b_{T_c})}{\phi_c^3} - \frac{1}{\phi_c^2},$$

$c_T = \frac{1}{2}\phi_c$, in respect of $\phi = 1$ at T_c and $\Pi = 1$. . (22)

As $\phi_c = \frac{273}{T_c} \, . \, v_c$,

$c_T = \frac{1}{2} . \frac{273}{T_c} \, . \, v_c$ in respect of $\phi = 1$, the temperature T_c and $\Pi = 1$. . . . (22 *a*)

In respect of $v_0 = 1$,

$$c_T = \tfrac{1}{2}\, v_c, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (22\ b)$$

$$c_0 = \tfrac{1}{2}\, v_c \left\{\frac{T_c}{273}\right\}^2 = \tfrac{1}{2}\,\phi_c \left\{\frac{T_c}{273}\right\}^3 . \quad . \qquad (22\ c)$$

The values thus calculated for c_0 (equations 17 and 23 *c*) will be derived presently in a completely independent and different manner.

The value of c_T (equation 22 *a*) substituted in the original equation affords at once

$$p_c = p_{1_c} = \frac{c_T}{\phi_c^2} = \frac{1}{2\phi_c} = \frac{1}{2v_c} \cdot \frac{T_c}{273} = \tfrac{1}{3}\Pi_c, \quad . \quad . \qquad (23\ a)$$

$$2p_c \, . \, v_c \, . \frac{273}{T_c} = 1 \text{ (constant)}. \quad . \quad . \quad . \quad . \quad . \qquad (23\ b)$$

The accuracy of equation (20),

$$2p_c \cdot v_c = 1,$$

is sufficiently demonstrated by the figures of the larger Table * (I.), in which will be found the experimental and calculated values of

$$p_c = \frac{1}{2v_c}.$$

In the smaller Table † (II.) the values of (equation 23 *b*) $2 \cdot p_c \cdot v_c \cdot \frac{273}{T_c} =$ constant are calculated for gases whose critical temperatures are below 273 (0° C.)‡. Though the values thus obtained are smaller than 1, they are remarkably constant and equal to about ·75, with the exception of air, which shows a value of ·69. But the latter is a mixture.

The constancy is such that the deviation from the value 1 must have a general cause. It is conceivable that at the extremely low temperatures of the critical state the co-volume β adopts a relatively smaller value. In any case, the fact that the mono-atomic gases with extremely low critical temperatures fall into line with hydrogen, nitrogen, oxygen, CO, CH_4, and NO deserves attention.

Deduction of $c_0 = \frac{1}{2}v_c = b$, $c_{T_c} = \frac{1}{2}\phi_c = b_T$ from the corresponding equations (D_1) and (E) leads at once to

$$p + \frac{c}{v^2} = \frac{1}{v} \quad . \; . \; . \; . \; . \; . \; . \; . \quad (D_2)$$

and

$$p + \frac{c_T}{\phi^2} = \frac{1}{\phi}. \quad . \; . \; . \; . \; . \; . \; . \; . \quad (E_2)$$

It will also be of interest to compare the value of c_0 (eq. 17) and p_c (eq. 19) with a few experiments.

---

* Landolt-Börnstein, 'Physikalische Tabellen,' Berlin, 1905, pp. 181–186. Values marked * are borrowed from U. Winkelmann, 'Handbuch der Physik,' vol. iii. Leipzig, 1906, pp. 859–868.

† G. W. C. Kaye & F. H. Laby, 'Physical and Chemical Constants,' 1911, p. 34.

‡ $p_c \cdot v_c \cdot \frac{273}{T_c} = \frac{3}{8}$ as found by van der Waals is for such gases evidently more in correspondence with the experimental values.

TABLE I.

	V_c.	P_c (exp.).	P (calc.).
Acetic acid	·0066	57·11	75·7
Alcohol	·00713	62·76	70
Amyl formate	·01710	34·12	29·2
Benzene	·00981	47 9	50·9 *
Benzene chloride	·01175	44·62	42·5 *
Ether	·01334	37·5	37·8
Ethyl acetate	·01222	39·65	41
Ethyl butyrate	·01744	30·24	28·80
Ethyl dichloride	·00982	53	51
Ethyl isobutyrate	·01749	30·13	28·6 *
Ethyl propionate	·01482	33·86	33·75
Isobutyl acetate	·01717	31·4	29·41
Isobutyl formate	·01472	38·29	37·86
Methyl acetate	·0096	47·54	52
Methyl butyrate	·01455	36·02	34·4
Methyl ethyl ether	·00873	46·27	57
Methyl propionate	·01224	39·88	40·8
Methyl valerate	·01728	31·5	29
Propyl acetate	·01464	34·8	34·5
Propyl alcohol	·00968	50·16	51·3
Propyl chloride	·00982	49	50·9 *
Propyl formate	·01203	42·7	41·5
H_2O	·003864	194·61	130
N_2O	·00436	77·5	114
SO_2	·00587	78·9	85
CO_2	·0066	77	76
CS_2	·009011	72·868	60

TABLE II.

	t_c.	T_c.	v_c.	$2 \,.\, p_c \,.\, v_c \,.\, \frac{273}{T_c} = \text{const.}$
Hydrogen	−234	39	·00264	$2 \,.\, 20 \,.\, v_c \,.\, 7 = {\cdot}7392$
Oxygen	−118	155	·00426	$2 \,.\, 50 \,.\, v_c \,.\, 1{\cdot}76 = {\cdot}75$
Helium	−268	5	·00299	$2 \,.\, 2{\cdot}3 \,.\, v_c \,.\, 54{\cdot}6 = {\cdot}763$
Argon	−117·4	155·6	·00404	$2 \,.\, 52{\cdot}9 \,.\, v_c \,.\, 1{\cdot}76 = {\cdot}7523$
CO	−141·1	131·9	·00505	$2 \,.\, 35{\cdot}9 \,.\, v_c \,.\, 2{\cdot}07 = {\cdot}7506$
CH_4	− 95·5	177·5	·00488	$2 \,.\, 50 \,.\, v_c \,.\, 1{\cdot}54 = {\cdot}7515$
Nitrogen	−146	127	·00517	$2 \,.\, 33 \,.\, v_c \,.\, 2{\cdot}14 = {\cdot}7302$
Krypton	− 62·5	210·5	·00532	$2 \,.\, 54{\cdot}3 \,.\, v_c \,.\, 1{\cdot}3 = {\cdot}751$
Air	−140	133	·00468	$2 \,.\, 39 \,.\, v_c \,.\, 1{\cdot}9 = {\cdot}69$
NO	− 93·5	179·5	·00347	$2 \,.\, 71{\cdot}2 \,.\, v_c \,.\, 1{\cdot}52 = {\cdot}754$

For Carbonic Acid, Andrews found

$$v_c = \cdot 0066,$$

$$p_c = 77 \text{ atmospheres (experimental value).}$$

We have (equation 17)

$$c = \tfrac{1}{2} v_c = \cdot 0033\,;$$

hence, according to equation (19), we calculate

$$p_c = \frac{c}{v_c^2} = \frac{\cdot 0033}{\cdot 0066^2} = 76 \text{ atmospheres.}$$

For Methyl Acetate we have

$$v_c = \cdot 0096,$$

$$p_c = 47{\cdot}54 \text{ (exp. value)}\,;$$

hence

$$p_c = \frac{c}{v_c^2} = \frac{\cdot 0048}{\cdot 0096^2} = 52{\cdot}08 \text{ (calc. value).}$$

For Propyl Alcohol,

$$v_c = \cdot 0098,$$

$$p_c = 50{\cdot}16 \text{ (exp. value)}\,;$$

hence

$$p_c = \frac{\cdot 0049}{\cdot 0098^2} = 51{\cdot}65 \text{ (calc. value)}\,;$$

and so on. All substances following the rule $2 \,.\, p_c \,.\, v_c = 1$ must naturally comply. In case of dissociation or condensation during the experiment, large deviations may be expected. Thus acetic acid and compounds of [OH] generally show deviations beyond the somewhat large errors of observation.

Equations (D) and (E) hold good for any temperature between T_c and 273. They are, in fact, quadratic equations with the three variables p, v, and T. For every value of p and T we have two values of v satisfying the equation, the one being the volume of the saturated vapour, the other the volume of the liquid of the vapour the moment it has entirely disappeared. The curves of these equations consist of the points in which the equal-pressure lines cut the isothermals, and denote the two extreme values of v. Each point of these curves is a point of intersection of the quadratic curve, an equal-pressure line, and an isothermal. At the vertex of the quadratic curve the two points indicating the two values of v coincide with the three points of the critical isothermal denoting the three values of v, and the two points of the latter curve for which $\frac{dp}{dv}=0$, $\frac{d^2p}{dv^2}=0$. The curves described by (D) and (E) are therefore identical with the "Border-curve." Their vertices are the critical points, and at these points $\frac{\partial p}{\partial v}=0$, $\frac{\partial t}{\partial v}=0$.

We have already solved (D) and (E) for $\frac{\partial p}{\partial v}=0$. We get the same results for $\frac{\partial t}{\partial v}=0$.

Differentiating equation (D) we find :

$$-\frac{4c_0}{\phi_c^3}\left\{\frac{273}{T_c}\right\}^2\partial\phi-\frac{4c_0}{\phi_c^2}\cdot\frac{273^2}{T_c^3}\partial t = -\frac{2(\phi_c+b_{t_c})\cdot\frac{273}{T_c}}{\phi_c^3}\partial\phi$$

$$+\frac{273}{T_c}\cdot\frac{1}{\phi_c^2}\partial\phi-\frac{\phi_c+b_{t_c}}{\phi_c^2}\cdot\frac{273}{T_c^2}\partial t.$$

As $\frac{\partial t}{\partial\phi}=0$,

$$4c_0=\frac{2(\phi_c+b_{t_c})}{\frac{273}{T_c}}-\frac{T_c}{273}\phi_c\,;\quad \text{and as } \phi_c=2b_{t_c},$$

$$c_0=\tfrac{1}{2}\phi_c\frac{T_c}{273}=\tfrac{1}{2}v_c,$$

$$c_{T_c}=\tfrac{1}{2}\cdot\phi_c\left\{\frac{273}{T}\right\}=\tfrac{1}{2}v_c\cdot\left(\frac{273}{T}\right)^2,$$

which are the values obtained in equations (17) and (18).

These values substituted in the original equation lead to the expressions (19), (20), and (21).

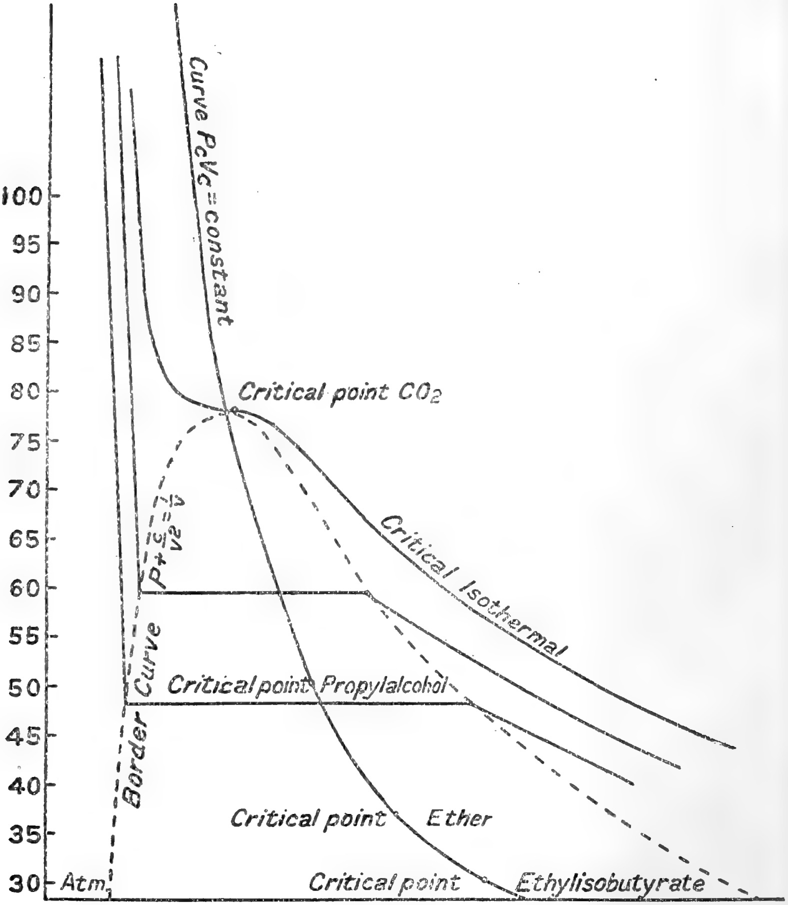
The critical point consequently is the point of intersection of the Critical Isothermal, the Border-curve, and the Curve $2p_c\cdot v_c=1$. For gases with critical temperatures below 273 the analogous curve of the latter is $2p_c\cdot\phi_c=1$. It is therefore defined by three equations. (*Vide* fig. 1.)

Differentiating equation (E), we find (for $T_c<273$)

$$-\frac{4c_{T_c}}{v_c^3}\left\{\frac{T_c}{273}\right\}^2\partial v+\frac{4T_c}{273^2}\cdot\frac{1}{v_c^2}\partial t=-\frac{2(v_c+b_c)}{v_c^3}\cdot\frac{T_c}{273}\partial v$$

$$+\frac{T_c}{273}\cdot\frac{1}{v_c^2}\partial v+\frac{v_c+b_c}{v_c^2}\cdot\frac{1}{273}\partial t,$$

Fig. 1.



or

$$-\frac{4c_{T_c}}{v_c^3}+\frac{2(v_c+b_c)}{v_c^3}\cdot\frac{273}{T_c}+\frac{1}{v_c^2}\cdot\frac{273}{T_c}$$

$$=\left(\frac{4}{T_c}\cdot\frac{1}{v_c^2}+\frac{273}{T_c^2}\cdot\frac{v_c+b_c}{v_c^2}\right)\frac{\partial t}{\partial v}=0,$$

$$4c_{T_c}=\frac{273}{T_c}\cdot 2(v_c+b_c)-v_c\frac{273}{T_c};$$

whence

$$c_{T_c} = \tfrac{1}{2} \cdot v_c \cdot \frac{273}{T_c} = \tfrac{1}{2}\phi_c$$

in respect of $\phi=1$, T_c and $\Pi=1$;

$$c_0 = c_{T_c}\left\{\frac{T_c}{273}\right\}^2 = \tfrac{1}{2}v_c \cdot \frac{T_c}{273}$$

or

$$c_0 = \tfrac{1}{2}v_c\left\{\frac{T_c}{273}\right\}^2$$

in respect of $v_0=1$, 0° and $\Pi=1$,

which are the values of equations 22 *a*, *b*, *c*.

For substances with $T_c \vee 273$ it is clear that equations (17) to (21) afford an opportunity to calculate T_c if the volume ϕ_c is given, *i.e.* the critical volume in respect of $\phi=1$, $\Pi=1$, and at the temperature T_c.

For we have

$$2p_c \cdot \phi_c \cdot \frac{T_c}{273} = 1\,;$$

and thus

$$T_c = \frac{273}{2 \cdot p_c \cdot \phi_c}.$$

The argument leading up to the equations (D) and (E) can be also expressed more shortly as follows:—

The "theoretical co-volume" b is inversely proportional to the absolute temperature T, *i. e.*

$$b_0 = T/273\, b_t, \quad \text{if } b_0 \text{ is given.}$$

Compressed at 0° from the volume $v_0=1$, $\Pi_0=1$, to the volume v the pressure is

$$\Pi = \frac{v+b_0}{v} \cdot \frac{1}{v} = \frac{n+1}{n} \cdot \frac{1}{n \cdot b_0}.$$

To attain the same pressure Π at the temperature T, the compression must be different; and

$$\Pi = \frac{v+b_T}{v} \cdot \frac{x}{v} = \frac{T/273\, n+1}{T/273\, n} \cdot \frac{x}{n \cdot b_0}$$

$$= \frac{T/273\, n+1}{T/273\, n} \cdot \frac{x}{T/273 \cdot n \cdot b_T}.$$

Equating the two expressions for Π,

$$x = \frac{T/273(n+1)}{T/273\, n+1}.$$

It will be seen that "reduction of $\phi = n \, . \, b_t$ to $v_0 = 1$" reduces the value of x to 1, whatever the temperature. If $T > 273$, $b_t = \frac{273}{T} \, . \, b_0$ and consequently $x > 1$,—*i. e.*, at the higher temperature the compression must be greater. If $T < 273$, $x < 1$: at the lower temperature less compression is required to attain the same Π.

Substituting x in the equation for the temperature T,

$$\Pi = \frac{v+b_t}{v} \, . \, \frac{T/273(n+1)}{(T/273n+1)v} = \frac{v+b_0}{v} \, . \, \frac{1}{v}$$

$$= \frac{n+1}{n} \, . \, \frac{273/T}{n \, . \, b_t} = \frac{\phi+b_t}{\phi} \, . \, \frac{273/T}{\phi}$$

$$= \frac{\phi+b_t}{\phi} \, . \, \frac{1}{T/273 \, . \, \phi}.$$

The last equation but one indicates that $(273/T)^2$ N molecules at T° compressed from the volume $\phi_0 = 273/T$ at T° and $\Pi_0 = 1$ to a volume ϕ exert the same pressure Π as at the temperature 0° compressed from the volume $v_0 = 1$ to a volume $v = T/273 \, . \, \phi$. This volume v is the experimental value, and is the reduced value of $\phi = n \, . \, b_t$ in respect of $v_0 = 1$. For b_t expressed in ϕ grows at the same rate as ϕ through the reduction.

Or again,

Imagine two quantities of gas at the two temperatures T and 273 exerting at the same volume 1 the infinitely small but still appreciable total pressure $\Pi_0 = 1$. Then

$$\beta_t = \frac{273}{T} \, . \, \beta_0.$$

The same pressure Π will be reached at the volumes $v_0 = (n+1)\beta_0$ and $v_t = (y+1)\beta_t$, if v_0 and v_t are expressed in the volume 1, at $\Pi_0 = 1$, *i. e.*,

$$\Pi = \frac{1}{v_0 - \beta_0} = \frac{1}{v_t - \beta_t},$$

or

$$\frac{1}{n \, . \, \beta_0} = \frac{1}{y \, . \, \beta_t} = \frac{1}{y \, . \, \frac{273}{T} \, . \, \beta_0};$$

$$\therefore \quad y = \frac{T}{273} \, . \, n.$$

"Reducing" the volume $y \, . \, \beta_t$ to $v_0 = 1$, 0°, $\Pi_0 = 1$,

$$\Pi = \frac{R \, . \, T}{y \, . \, \beta_0} = \frac{R \, . \, T}{T/273 \, . \, n \, . \, \beta_0} = \frac{1}{v_0 - \beta_0} = \frac{273/T_c}{n \, . \, \beta_T}.$$

If $T < 273$,

$$\Pi = \frac{1}{v_t - \beta_t} = \frac{1}{v_0 - \beta_0}, \quad \beta_0 = \frac{T}{273}\beta_t,$$

$$\frac{1}{n \,.\, \beta_t} = \frac{1}{y \,.\, \frac{T}{273}\beta_t}, \quad \therefore \quad n = \frac{T}{273} \,.\, y.$$

Reduced to v_0, 0°, and $\Pi = 1$,

$$\frac{1}{\frac{273}{T} \,.\, n \,.\, \beta_0} = \frac{R \,.\, T}{n \,.\, \beta_0} = \frac{R \,.\, T}{v_0 - \beta_0};$$

and in both cases v_0 is the experimental value.

Note.—It is evident that for gases with $T > 273$ the total pressure is inversely proportional to the reduced "free space."

If $$\Pi_1 \,.\, n \,.\, \beta_0 = 1 \text{ at } 0° \text{ C.},$$

then for the same pressure Π_1

$$\Pi_1 \,.\, n \,.\, \beta_T = \frac{273}{T}, \text{ at a temperature } T > 273,$$

$n \,.\, \beta_0$ and $n \,.\, \beta_T$ being "free spaces." β_T is a definite fraction of β_0 dependent on T, and β_0 has an absolute value expressed in $v_0 = 1$ at $\Pi_0 = 1$.

The number of molecules with the free space $n \,.\, \beta_T$ is $\left(\frac{273}{T}\right)^2$ times the number of molecules with the free space $n \,.\, \beta_0$, and their co-volume is $\left(\frac{273}{T}\right)^2 \,.\, \beta_0$ or $\left(\frac{273}{T}\right) \beta_T$.

Hence expressing its volume in respect of $v_0 = 1$ at which it would exert a pressure $\Pi_T = \frac{273}{T}$, the gas at the temperature T exerts a pressure *inversely proportional to the volume*, i. e.

$$\Pi = \frac{273/T}{n \,.\, \beta_T}.$$

But if we express that volume $n \,.\, \beta_T$ in respect of the volume $\frac{273}{T}$ at which the gas exerts the pressure $\Pi_0 = 1$, we find the total pressure inversely proportional to the *reduced free space*. Thus we found *e. g.* for the critical state:

$$\Pi_c \,.\, 2\beta_{T_c} = \frac{273}{T_c},$$

or reduced to 0° C. and 760 mm.

$$\Pi_c = \frac{1}{v_c - \beta_0} = \frac{3}{2}\frac{1}{v_c},$$

and

$$p_c = \frac{1}{2v_c} \text{ (eqs. 20 \& 21)}$$

for gases with $T_c > 273$.

For gases with $T_c < 273$ we should find for the same reason:

$$\Pi_c = \frac{273/T_c}{v_c - \beta_0}, \qquad p_c = \frac{273/T_c}{2v_c},$$

and v_c is in both cases the reduced volume.

This view is entirely in accordance with the formula

$$\Pi = \frac{R \, . \, T}{v}$$

for gases like H_2, O_2, N_2, &c. at temperatures T above 273.

For expressing β_T and β_0 in β_{T_c} as fraction of the volume 1 at $\Pi = 1$ and the critical temperature T_c, we have

$$\Pi = \frac{T_c/T}{n \, . \, \beta_T} = \frac{T_c/273}{n \, . \, \beta_0} = \frac{1}{n \, . \, \beta_{T_c}},$$

in which $n \, . \, \beta_T$, &c. are the "free spaces."

It is evident that

$$\frac{T_c/273}{n \, . \, \beta_0} = \frac{R \, . \, T \, . \, T_c/T}{n \, . \, \beta_0},$$

i. e. equal to the pressure of the gas at T and the "reduced volume" $(n+1)\beta_0$.

III.

Reduced Isothermals and Reduced Border-curve.

For every saturated vapour-pressure p_c we have one temperature but either two or three volumes. Where in our equations T^2 is found it only appears as a volume factor. In fact, as is shown by equations (D) and (E), the temperature divides out, and the volumes are determined by the value of p, the pressure of the saturated vapour.

If, following van der Waals, we make

$$p = e \, . \, p_c,$$

$$v = n \, . \, v_c,$$

and give p_c, c_0, and $\frac{1}{v_c}$ the values given by equations (17) and (19) for $T<273$, and p_c, c_{T_c}, and $\frac{1}{\phi}$ the values of equations (22) and (23 *a*), we obtain at once

$$e+\frac{1}{n^2}=\frac{2}{n}, \quad . \; . \; . \; . \; . \; . \; . \quad (G)$$

a quadratic equation in which all constants belonging to a particular gas have disappeared.

The corresponding cubic equation is

$$e+\frac{2}{n^2}=\frac{1}{n-q}, \quad . \; . \; . \; . \; . \; . \; . \quad (G_1)$$

in which

$$q=\frac{n}{2n+1},$$

but which for given values of n at once converts into the quadratic form.

For every value of e we have two values of n_1 and n_2, denoting the volume of the saturated vapour $n_1 v_c$, and of the liquid the moment all vapour has been liquefied. At the vertex of the curve for which $\frac{de}{dn}=0$, the two values of n coincide, and

$$n=1,$$
$$e=1;$$
$$i.\,e., \qquad p=p_c, \qquad v=v_c.$$

Giving e successive values for the *same* gas, then

$n_1 v_c$ is the volume of the saturated vapour,

$n_2 v_c$ is the volume of the liquid thereof,

at the pressure ep_c, the vapour-pressure.

Giving e the same value for various substances, then the volumes at the saturated-vapour pressures $e\,.\,p_c$ must be equal multiples of $n_1\,.\,v_c$ and $n_2\,.\,v_c$. I have not yet verified the latter deduction. The former I have examined for pressures of 1 atmosphere. Thus for $e=\frac{1}{p_c}$:

Solving equation (G) we find

$$n_1=\frac{1}{e}+\left\{\frac{1}{e^2}-\frac{1}{e}\right\}^{\frac{1}{2}}, \quad . \; . \; . \; . \; . \; . \quad (24)$$

$$n_2=\frac{1}{e}-\left\{\frac{1}{e^2}-\frac{1}{e}\right\}^{\frac{1}{2}}, \quad . \; . \; . \; . \; . \; . \quad (25)$$

and must obtain such values for n that for the vapour

$n_1 v_c = 1$, at the temperature T, the temperature of the saturated vapour at the pressure $p = 1$, and for the liquid at the same pressure p and temperature T, the volume is $n_2 . v_c$. In this manner we find for, *e. g.*:—

TABLE III.

	p_c.		$v_1 = n_1 . v_c$.	$v_2 = n_2 . v_c$.
CO_2 .... $v_c = {\cdot}0066$	77	$n_1 = 153{\cdot}4984$ $n_2 = {\cdot}5016$	1·0103	·5016 v_c
Ethyl propionate...... $v_c = {\cdot}01482$	33·86	$n_1 = 66{\cdot}716$ $n_2 = {\cdot}504$	·996	·504 v_c
Propyl acetate .... $v_c = {\cdot}01464$	34·80	$n_1 = 69{\cdot}10$ $n_2 = {\cdot}504$	1·0116	·5036 v_c
Ether .... v_c ... ·01334	37·50	$n_1 = 74{\cdot}50$ $n_2 = {\cdot}503$	·994	·503 v_c

A few examples taken from Table I. (p. 77) will suffice, as all others following the rule

$$2p_c . v_c = 1$$

must necessarily comply.

It is interesting to note that n_2 cannot be smaller than ·5, which value is approached as e becomes extremely small, *i. e.* at very low temperature.

For gases with $T_c < 273$ we have (*v.* eq. 23 *b*)

$$2p_c . \phi_c = {\cdot}75 \text{ (constant)},$$

in which $\phi_c = v_c . \frac{273}{T_c}$.

We may therefore expect a similar deviation from 1 for the values $n_1 \phi_c$ and $n_2 \phi_c$.

E.g., for Hydrogen ($T_c = 39$, $p_c = 20$, $v = {\cdot}00264$) we find

$$\phi_1 = n_1 . \phi_c = \frac{273}{39} . 39{\cdot}494 . {\cdot}00264 = {\cdot}727,$$

$$\phi_2 = n_2 . \phi_c = \frac{273}{39} . {\cdot}506 \phi_c \qquad = 7 . {\cdot}506 v_c = {\cdot}506 \phi_c ;$$

for Helium ($T_c = 5$, $p_c = 2{\cdot}3$, $v_c = {\cdot}00299$) we have

$$\phi_1 = n_1 . \phi_c = \frac{273}{5} . 4 . {\cdot}00299 \qquad = {\cdot}658,$$

$$\phi_2 = n_2 . \phi_c = \frac{273}{5} . {\cdot}6 . v_c \qquad = 54{\cdot}6 . {\cdot}6 v_c = {\cdot}6 . \phi_c.$$

We thus obtain a reduced "Border-curve" of the same kind as the "Border-curve" of any particular gas, but belonging to all substances that do not change their molecules during isothermal compression. The curve $2p_c \, . \, v_c = 1$ (*vide* fig. 1), consisting of vertices of various Border-curves, *i. e.* of critical points, has disappeared.

IV.

The equality of action and reaction and the "Intrinsic Pressure" it involves would explain the seemingly incongruous course of a continuous isothermal below T_c between the two extreme values of p. For the pressure in the interior must decrease as the intrinsic pressure increases, with the same force p_1. Once c has attained a value $>$ the value $\frac{1}{2}v_c$ or $\frac{1}{2}\phi_c$, $p_1 = \frac{c}{v_c^2}$ is bound to increase more rapidly than Π owing to the influence of β, as soon as the density $\frac{1}{v}$ exceeds $\frac{1}{v_c}$. This might also explain that this part of the curve cannot be realized by the experiment. For condensation would set in first where the pressure Π obtains, *i. e.* between the surface and the interior.

The densities corresponding with $\Pi_c = 3p_c$ in the layers between surface and interior and p_c in the interior should cause different refraction, and a falling of the heavier layers should be observed.

The equality of p_c and p_{1_c} can be proved also in the following and completely independent manner and substantiated by striking experimental evidence. As, according to van der Waals' equation, $p_{1_c} = 3p_c$, it was the following deduction and its experimental support which enforced the conviction that at least the solution of that equation must be inexact.

A second argument proving the Equality of p_c and p_{1_c}.

The "Inward Pressure" p_1 naturally opposes the thermic pressure p towards the exterior causing expansion, and the physical state of a substance must largely depend on the relative values of these two forces.

For a liquid (or solid) p_1 *must be larger* than p and the substance *must* remain a liquid (or solid) as long as $p_1 > p$; the minimum value of p_1 *cannot be less than* p and *must be equal to* p, in which case the substance is in an unstable condition.

For a gas (or vapour) p_1 *must be smaller* than p; its maximum value must be equal to p. For if p_1 becomes larger than p expansion is impossible, and the substance is bound to lose this characteristic of a gas (vapour).

Now if we heat a liquid in a closed vessel the density of its vapour must increase ; its own density must decrease as long as there is a difference between them and *until they are equal*, which point is reached at the critical state.

Consequently at the critical state the inward pressure of the liquid p_{1l} becomes equal to the inward pressure p_{1g} of the gas or vapour, for both converge in opposite sense towards the same value, the inward pressure of the critical fluid, and the same applies to the thermic pressures p_l and p_g.

Thus the critical state appears to be the limit which either of the two *must* reach before turning into the other, and we may consequently write the following equations :

$$\left.\begin{array}{l} p_{1l} \geqq p_l \left\{ \begin{array}{l} \textit{i. e.}\text{ The minimum value of the ``Inward} \\ \text{Pressure'' } p_{1l} \text{ is for a liquid equal to the} \\ \text{thermic pressure } p_l. \end{array}\right. \\ p_{1g} \leqq p_g \left\{ \begin{array}{l} \textit{i. e.}\text{ The maximum value of the ``Inward} \\ \text{Pressure'' } p_{1g} \text{ is for a gas equal to the} \\ \text{thermic pressure } p_g. \end{array}\right. \end{array}\right\} . \quad (26)$$

At the critical state, however,

$$p_{1l} = p_{1g},$$

$$p_l = p_g,$$

and it follows inevitably that at the critical state

$$p_{1l} = p_{1g} = p_l = p_g, \quad \ldots\ldots \quad (27)$$

or

> The "critical inward pressure" p_{1c} of a substance is equal to the critical thermic pressure p_c ; i. e.,

$$p_{1c} = p_c,$$

$$\Pi_c - p_{1c} = p_c + p_{1c},$$

$$\Pi_c = 3p_c = 3p_{1c}.$$

The "critical Intrinsic Pressure" is equal to three times the critical pressure or three times the critical inward pressure.

We thus arrive at the same results along an entirely different line of argument.

The exactitude of these rules, supported as they are by the substantial experimental proof already recorded and by that to follow presently, appears to be well established. The critical state is the only state at which we can directly measure the Inward Pressure.

A mathematical expression defining the value of the Inward Pressure p_{1_c} can be verified, if correct, at the critical state, inasmuch as it must give us the value of the critical pressure p_c.

In this manner we can test a general expression for the cohesive forces, deduced as follows.

V.

The cohesive forces causing the "Inward Pressure" appear to follow a law similar to the inverse square law of mass attraction.

If c is the "Inward Pressure" at the volume 1, then at the volume v,

$$p_1 = \frac{c}{v^2}$$

and

$$c = x \,.\, \mathrm{N}^2 \,.\, m^2 = x \,.\, \rho^2, \quad . \quad . \quad . \quad . \quad . \quad (28)$$

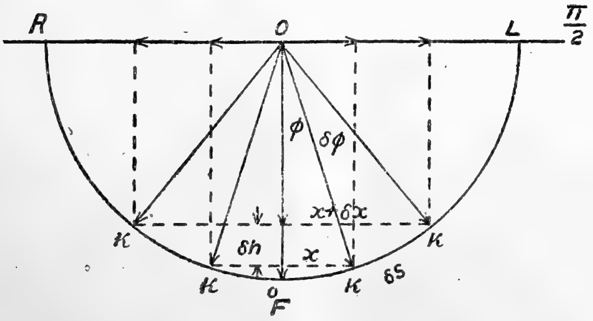
m being the mass of a molecule, N the number of molecules at the volume $v_0 = 1$, hence $\mathrm{N} \,.\, m$ the density ρ. The factor x is considered a constant peculiar to the nature of a gas. (Compare *e. g.* J. H. Jeans, 'Dynamic Theory of Gases,' Cambridge, 1904, pp. 118 & 114.)

In the following I intend to show that c is dependent on the mass or molecular weight M only and *x is a constant for all substances.*

Without discussing how the form $p_1 = \frac{c}{v^2}$ has been arrived at, I will here give my own deduction.

Suppose O (*vide* fig. 2) to be a molecule at the surface of a fluid attracted by the molecules below the surface in every direction at angles between $\frac{\Pi}{2}$ and O. Let us consider O

Fig. 2.



and every molecule in the "stationary motion" of Clausius and let OL be the radius of the "sphere of attraction" for O.

Every molecule on the spherical surface RFL will be just the last to attract O, and each radius through a molecule of that surface is the direction of a force attracting O.

Suppose κ to be such a force, and some function of distance, of the radius r,

$$\kappa = f(r).$$

The horizontal components compensate each other, but the vertical components of all forces κ form the normal resultant to which O is subjected.

If ϕ is the angle between such a force κ and OF, the normal component of κ is

$$k = \kappa \,.\, \cos\phi. \quad \ldots \ldots \quad (29)$$

All forces at the angle ϕ have this same component k and lie in the surface of the cone with the angle ϕ in O and go through the curve of intersection with the sphere which is the circle with the radius x.

If n is the number of molecules per unit of distance,

$$\mathrm{K} = \Sigma k = 2\pi \,.\, x \,.\, n \,.\, \kappa \,.\, \cos\phi, \quad \ldots \quad (30)$$

or, as $x = \sin\phi$,

$$\mathrm{K} = 2\pi \,.\, n \,.\, \kappa \,.\, \sin\phi \,.\, \cos\phi. \quad \ldots \quad (31)$$

A second cone with the angle $(\phi + d\phi)$ cuts the sphere a distance dh above the first. We obtain a second curve of intersection with the radius $(x + dx)$ parallel to the first one. The surface between the two planes is for the height dh the surface of a cone-frustrum, and equals

$$d\mathrm{O} = 2\Pi\{x + (x + dx)\}\frac{ds}{2} = 2 \,.\, \Pi \,.\, x ds. \quad \ldots \quad (32)$$

The number of molecules in this surface is

$$\mathrm{Z}_1 = 2\Pi \,.\, x \,.\, n^2 \,.\, ds. \quad \ldots \ldots \quad (33)$$

Through each of these molecules a force is directed, and the vertical component is $(ds = d\phi)$

$$d\mathrm{F} = 2\Pi \,.\, n^2 \,.\, \sin\phi \,.\, \cos\phi \,.\, d\phi. \quad \ldots \ldots \quad (34)$$

It is evident that

$$\mathrm{F} = \int d\mathrm{F} = 2\Pi \,.\, n^2 \,.\int d\,\frac{\sin^2\phi}{2},$$

and the sum of all normal components between $\phi=\frac{1}{2}\Pi$ and $\phi=0$ will be

$$\mathrm{F} = \Pi \,.\, n^2 \,.\, \kappa \int_{\phi=0}^{\phi=\frac{\Pi}{2}} d \sin^2 \phi = \Pi \,.\, n^2 \,.\, \kappa. \quad . \quad . \quad . \quad . \qquad (35)$$

Each molecule in the surface is attracted by this force F; so that the total attraction per unit of surface, *i. e.* the "Inward Pressure," becomes

$$p_{\mathrm{H}} = n^2 \,.\, \mathrm{F} = \Pi \,.\, n^4 \,.\, \kappa = \left\{v^{-\frac{2}{3}}\right\}^2 \kappa. \quad . \quad . \qquad (36)$$

Whatever the law of force, $\left\{\frac{1}{v^{\frac{2}{3}}}\right\}^2$ is a factor of the value for p_{H}, and p_{H} can only then adopt a form $\frac{c}{v^2}$, if κ follows a law similar to Newton's law of attraction of mass.

Since such an attraction between the molecules must certainly exist, I have in consideration of the preceding calculated κ according to the general law of attraction of mass.

If n is the number of molecules in unit distance, the average distance between two molecules is $r=\frac{1}{n}$.

If m is the mass of a molecule, the attraction in the direction OL of the first molecule on the molecule in O would be

$$f_1 = \mathrm{C} \,.\, \frac{m^2}{r^2};$$

of the second,

$$f_2 = \mathrm{C} \,.\, \frac{m^2}{4r^2};$$

of the nth,

$$f_n = \mathrm{C} \,.\, \frac{m^2}{n^2 \,.\, r^2},$$

C being a factor of proportionality similar to the astronomic unit of force $\frac{1}{15 \,.\, 10^6}$ dyne, if we express m in grams, r in cm.

The sum of the forces $f_1, f_2, \ldots f_n$ is the force κ, and we find

$$\kappa = \Sigma(f) = \mathrm{C} \,.\, \frac{m^2}{r^2}\left(1+\frac{1}{4}+\frac{1}{9}+\ldots\frac{1}{n^2}\right). \quad . \quad . \qquad (37)$$

For large values of n the series in this equation rapidly converges towards 1·7, and we obtain

$$\kappa = 1{\cdot}7 \,.\, \mathrm{C} \,.\, \frac{m^2}{r^2} = 1{\cdot}7 \,.\, \mathrm{C} \,.\, n^2 \,.\, m^2, \quad . \quad . \quad . \quad . \qquad (38)$$

i. e. for the same n only dependent on the square of the mass m.

The error we commit by assuming κ to be directed between O and *every* molecule of the surface RFL influences the result alike for any density, but is entirely eliminated by making it act between O and every three molecules of that surface; and we find, substituting eq. (38) in eq. (36),

$$p_{\mathrm{H}} = \frac{1{\cdot}7}{3} \,.\, \Pi \,.\, \mathrm{C} \,.\, n^6 \,.\, m^2 = \frac{1{\cdot}7}{3} \Pi \,.\, \mathrm{C} \,.\, \frac{m^2}{v_0^{\,2}}, \quad . \qquad (39)$$

if $v_0=1$ is the volume containing n^3 molecules.

If we consider m of equation (39) to be the mass of a molecule H_2, Hydrogen being the lightest of all gases, and then make $p_{\mathrm{H}}=x$ (*vide* equation 28), then the molecular attraction or Inward Pressure of a substance with the molecular weight M would be at $v_0=1$, 0° and 760 mm.,

$$c_0 = \frac{\mathrm{M}^2}{4} \,.\, x, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (40)$$

c_0 being the constant of attraction of the substance.

At the volume v in respect of $v_0=1$,

$$p_1 = \frac{\mathrm{M}^2}{4} \,.\, \frac{x}{v^2} = \frac{c_0}{v^2}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (41)$$

and for the critical state

$$p_{1_c} = \frac{c_0}{v_c^{\,2}} = \frac{\mathrm{M}^2}{4} \,.\, \frac{x}{v_c^{\,2}}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (42)$$

But as we found the critical Inward Pressure p_{1_c} equal to the critical pressure p_c, we must also find

$$p_c = \frac{\mathrm{M}^2}{4} \,.\, \frac{x}{v_c^{\,2}}; \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (43)$$

hence

$$x = 4\frac{p_c \,.\, v_c,}{\mathrm{M}^2} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (44)$$

and x is the "Inward Pressure" of Hydrogen at $v_0=1$, $t=0$, and 760 mm., expressed as a fraction of 1 atmosphere.

It will be seen (Table IV.) that x is indeed constant for substances widely different in chemical and physical properties.

The values of p_c and v_c have been borrowed from 'Physikalische Tabellen,' Landolt-Börnstein, Berlin, pp. 181–186; those marked (*) from 'Handbuch der Physik,' A. Winkelmann, vol. iii. Leipzig, 1906, pp. 859–868. As small changes of p_c cause large changes of v_c, care has been taken to select corresponding values of p_c and v_c, as will be noticed.

Adopting as the mean value of x,

$$x = 3 \cdot 10^{-6} P_0, \quad \text{or about 3 dynes}, \quad . \quad . \quad (45)$$

we find for the critical pressure p_c or the critical Inward Pressure p_{1_c},

$$p_c = p_{1_c} = \frac{1}{4}\frac{M^2}{v_c^2} \cdot 3 \cdot 10^{-6}, \text{ if } P_0 = 1; \quad . \quad . \quad (46)$$

and for the critical volume,

$$v_c = \left\{\frac{M^2}{4p_c} \cdot 3 \cdot 10^{-6}\right\}^{\frac{1}{2}},$$

and the values thus calculated for p_c and v_c appear to be well within the margin of observational error.

The equations deduced in Section II. afford some more relations that are of interest.

Substituting $2p_c \cdot v_c = 1$ (*v.* eq. 20), we find for equation (44)

$$x = 3 \cdot 10^{-6} = 2\frac{v_c}{M^2}. \quad . \quad . \quad . \quad . \quad . \quad (48)$$

The table shows also in this case great constancy.

We now possess two independent methods of fixing the value of c_0, the constant of molecular attraction at 0°, $v_0 = 1$, and 760 mm.

According to equation (17) we have for gases with $T_c > 273$,

$$c = \tfrac{1}{2} v_c;$$

whereas equation (40) gave

$$c = \frac{M^2}{4} \cdot 3 \cdot 10^{-6} \quad (x = 3 \cdot 10^{-6});$$

hence

$$c = \tfrac{1}{2} v_c = \frac{M^2}{4}\, 3 \cdot 10^{-6}. \quad . \quad . \quad . \quad . \quad . \quad . \quad (49)$$

	P_c. Expernt.	V_c. Expernt.	M.	$x=4\frac{P_c\,.\,V_c^2}{M^2}$.
Acetic acid .... $C_2H_4O_2$.	57·11	·0066	60	2·765 $.10^{-6}$
Amyl formate .... $C_6H_{12}O_2$.	34·12	·01710	116	2·966 . ,,
Benzene ....* C_6H_6.	47·90	·00981	78	3·031 . ,,
Ethyl acetate .... $C_4H_8O_2$.	39·65	·01222	88	3·058 . ,,
Ethyl butyrate .... $C_6H_{12}O_2$.	30·24	·01744	116	2·798 . ,,
Ethyl isobutyrate ...* $C_6H_{12}O_2$.	30·13	·01749	116	2·740 . ,,
Ethyl propionate ...... $C_5H_{10}O_2$.	33·86	·01482	102	2·870 . ,,
Isobutyl acetate......... $C_6H_{12}O_2$.	31·40	·01717	116	2·752 . ,,
Isobutyl formate ...... $C_5H_{10}O_2$.	38·29	·01472	102	3·186 . ,,
Methyl acetate .... $C_3H_6O_2$.	47·54	·00960	74	3·201 . ,,
Methyl butyrate ...... $C_5H_{10}O_2$.	36·02	·01455	102	2·638 . ,,
Methyl propionate ... $C_4H_8O_2$.	39·88	·01224	88	3·085 . ,,
Methyl valerate .... $C_6H_{12}O_2$.	31·50	·01728	116	2·796 . ,,
Propyl acetate .... $C_5H_{10}O_2$.	34·80	·01464	102	2·868 . ,,
Propyl chloride ......* C_3H_7Cl.	49·00	·00982	78·45	3·071 . ,,
Propyl formate .... $C_4H_8O_2$.	42·70	·01203	88	3·048 . ,,
N_2O	77·50	·00436	44	3·044 . ,,
SO_2	78·90	·00587	64	2·655 . ,,
N_2	33·00 35·33	·004603	28	3·568 . ,, 3·784 . ,,
O_2	50·00 50·8	·004042	32	2·8657. ,, 2·9997. ,,

The correspondence of the two values thus calculated in completely independent manner is striking, as the last two columns of Table IV. show. Gases with $T_c < 273$ vary; but for those we had equation (22 *c*),

$$c_0 = \tfrac{1}{2} v_c \,.\left\{\frac{T_c}{273}\right\}^2.$$

TABLE IV.

$x=2\frac{V_c}{M^2}$.	$P_c=\frac{M^2}{4V_c^2}.x.$ ($x=3.10^{-6}$.)	$V_c=\sqrt{\frac{M^2}{4P_c}.x}$. ($x=3.10^{-6}$.)	$c_0=\frac{V_c}{2}$.	$c_0=\frac{M^2}{4}.x.$ ($x=3.10^{-6}$.)
3·67 $.10^{-6}$	61·98	·00687	·0033	·0027
2·542 . „	38·51	·017189	·00855	·01
3·225 . „	47·425	·00976	·00495	·004563
3·15 . „	38·89	·01213	·00611	·005808
2·592 . „	33·18	·01827	·00872	·01
2·5996 . „	32·99	·01830	·008745	·01
2·8489 . „	38·53	·01518	·00741	·0078
2·552 . „	34·23	·01792	·00885	·01
2·8297 . „	36·01	·014274	·00736	·0078
3·506 . „	44·65	·0093	·0048	·00411
2·797 . „	36·86	·01472	·007275	·0078
3·1612 . „	38·77	·01207	·00612	·00581
2·5743 . „	38·80	·01790	·00864	·01
2·8143 . „	36·24	·014974	·00732	·0078
3·1986 . „	47·87	·009755	·00491	·00462
3·1069 . „	40·133	·011662	·006015	·00581
4·5041 . „	76·38	·0043284	·00218	·001452
2·8862 . „	89·20	·00624	·002935	·003072
1·2 $.10^{-5}$ 1·046 . „	27·752	·004212 ·0040795	·0023015	·000588
7·65 $.10^{-6}$ 7·60 . „	50·055	·0039192 ·0038882	·002021	·000768

Hence

$$c_0 = \tfrac{1}{2}v_c\left\{\frac{T_c}{273}\right\}^2 = \frac{M^2}{4}.3.10^{-6}. \quad . \quad (50)$$

The separate table for such gases shows this deduction to be approximately correct for O_2, N_2, CO, NO, but less for the mono-atomic gases, CH_4 and H_2.

TABLE V.

	$\frac{1}{2}$M.	T$_c$.	$\frac{1}{2}v_c$.	$\frac{1}{4} . M^2 . 3 . 10^{-6} = c = \frac{1}{2}v_c\{T_c/273\}^2$.	
O_2	16	155	·00213	7·68 . 10^{-4}	6·90 . 10^{-4}
N_2	14	127	·002585	5·88 . 10^{-4}	5·60 . 10^{-4}
CO	14	131·9	·0025	5·88 . 10^{-4}	5·84 . 10^{-4}
NO	15	179·5	·001735	6·75 . 10^{-4}	7·50 . 10^{-4}
Kr	41·45	210·5	·00266	5·15 . 10^{-3}	1·65 . 10^{-3}
A	20	155·6	·00202	1·20 . 10^{-3}	0·66 . 10^{-3}
He	2	5	·001495	1·20 . 10^{-6}	0·50 . 10^{-6}
H_2	1	39	·00132	3·00 . 10^{-6}	2·694 . 10^{-5}
CH_4	8	177·5	·00244	1·92 . 10^{-4}	10·32 . 10^{-4}

If the actual molecular weight is different from the one adopted, then large deviations may be expected from the expressions (46) and (48) and in some cases (42). Thus water only satisfies equation (42) if at the critical state it consists of equal numbers of molecules $[H_2O]_3$ and $[H_2O]_4$. (Vernon assumes for water the formula $[H_2O]_4$, Chem. News, lxvi. p. 54, 1891.)

A substance following the rule $2p_c . v_c = \text{const.}$ can hardly during the experiment dissociate or polymerize. But if it deviates from this rule, it need not do so from the rule of equation (42), as $1/v_c$ and M would be equally affected, and any change in the number of molecules would alter M in the opposite sense. This would of course not be so in the case of equation (48).

The material is, however, too large to allow of detailed discussion, and I only wish to point out how the compliance, or non-compliance with the preceding rules can assist us in forming an idea with regard to association or dissociation of molecules during the experiment and the molecular weight.

If the cohesive forces account partly for the deviations of the gaseous laws, by opposing the thermic pressure and expansion, they must also cause resistance against pull in the case of solids or liquids. In some solids the number of molecules (atoms) is so great that translatory velocity seems hardly possible, especially in crystalline structures. Association of molecules would in itself not necessarily increase the inward pressure $p_1 = \frac{c}{v^2} = c . N^2 . m^2$, as N decreases as m increases. But the number of molecules, the number of impacts, *i. e.* the thermic pressure decreases.

Substances like Fe and C (diamond) may belong to those which, but for association to complex-molecules, would contain the largest number of molecules per unit of volume. Compared with their specific weights, their atomic weights are relatively lowest. If for these substances all translatory motion ceases, Π becomes 0, and

$$p = -p_1.$$

Instead of exerting pressure, the molecules require a force to separate them equal to the cohesive force of iron, and p_1 would be the maximum pull iron can oppose before breaking.

As $\frac{1}{2}.\mathrm{N}.\mathrm{M}$ represents the relative density ρ in respect of Hydrogen, 1000 c.cm. of iron weigh 7·7 kilogram, of Hydrogen $8{\cdot}987\,.\,10^{-5}$ kilogram ; consequently,

$$\rho = \frac{7{\cdot}7}{8{\cdot}987\,.\,10^{-5}},$$

the cohesive force p_1 per cm.^{2} is, roughly,

$$p_1 = \rho^2\,.\,3\,.\,10^{-6}\text{ kilogram}\quad(\textit{vide}\text{ eq. 45}),$$

and per mm.^{2}

$$p_1 = \rho^2\,.\,3\cdot 10^{-8}\text{ kilogram.}$$

Thus

$$p_1 = \left\{\frac{7{\cdot}7}{8{\cdot}987\,.\,10^{-5}}\right\}^2.\,3\,.\,10^{-8}\text{ kilogram}$$
$$= 224{\cdot}1\text{ kilogram.}$$

This is considerably higher than the pull necessary to break iron. With decrease of diameter the experiment, however, affords rapidly increasing values. Thus a wire of Swedish iron of ·72 mm. diameter will stand a pull of 64 kilograms before breaking. But a wire of ·10 mm. diameter already requires a pull of 123 kilograms to break it (both, of course, per unit of surface)*. But p_1 is the maximum value if all translatory motion has ceased : the wire breaks where it is weakest, the temperature rises during the experiment, and there may remain a degree of translatory movement.

It is, however, interesting to note that we arrive at a value for the tensile strength of iron which is of the same order as the experimental value, out of the deviations from the gaseous laws. Whatever the ultimate nature of the cohesive forces may be, they seem proportional to the square of the molecular weight, and to obey the inverse square law, —a law consequently similar to Newton's law of attraction of mass.

Chemical Department,
University of Bristol.

---

* Poynting and Thomson, 'Properties of Matter,' London, 1915, p. 61.

VII. *On the Optical Character of some Brilliant Animal Colours.* *By* Lord Rayleigh, *O.M., F.R.S.*[*]

It is singular that the explanation of some of the most striking and beautiful of optical phenomena should be still matters of controversy. I allude to the brilliant colours displayed by many birds (*e. g.* humming-birds), butterflies, and beetles, colours which vary greatly with the incidence of the light, and so cannot well be referred to the ordinary operation of dyes. In an early paper[†], being occupied at the time with the remarkable coloured reflexions from certain crystals of chlorate of potash described by Stokes, and which I attributed to a periodic twinning[‡], I accepted, perhaps too hastily, the view generally current among naturalists that these colours were "structure-colours," more or less like those of thin plates, as in the soap-bubble. Among the supporters of this view[§] in more recent times may be especially mentioned Poulton and Hodgkinson. In Poulton's paper[||] the main purpose was to examine the history of the very remarkable connexion between the metallic colours of certain pupæ (especially *Vanessa urticæ*) and the character of the light to which the larvæ are exposed *before* pupation. In a passage describing the metallic colour itself he remarks:—

"*The Nature of the Effects Produced.*—The gilded appearance is one of the most metal-like appearances in any non-metallic substance. The optical explanation has never been understood. It has, however, been long known that it depends upon the cuticle, and needs the presence of moisture, and that it can be renewed when the dry cuticle is moistened. Hence it can be preserved for any time in spirit. If a piece of dry cuticle be moistened on its upper surface the colour is not renewed, but almost instantly follows the application of spirit to the lower surface. Sections of the cuticle resemble those of *Papilio machaon* described in a previous paper (Roy. Soc. Proc. vol. xxxviii. p. 279, 1885), and show an upper thin layer and a lower, much thicker, finely laminated layer which is also striated vertically to the surface. With Prof. Clifton's kind assistance I have been

---

[*] Communicated by the Author.

[†] Phil. Mag. vol. xxiv. p. 145 (1887); Scientific Papers, vol. iii. p. 13, see footnote.

[‡] Phil. Mag. vol. xxvi. p. 256 (1888); Scientific Papers, vol iii. p. 204.

[§] Distinctly suggested by Hooke in his 'Micrographia' (1665).

[||] Roy. Soc. Proc. vol. xlii. p. 94 (1887).

able to show that the appearances follow from interference of light, due to the presence of films of liquid between the lamellæ of the lower layer. The microscope shows brilliant red and green tints by reflected light, while in transmitted light the complementary colours are distinct, but without brilliancy. The latter colours are seen to change when pressure is applied to the surface of the cuticle, and when the process of drying is watched under the microscope, owing in both cases to the liquid films becoming thinner. In the dry cuticle the solid lamellæ probably come into contact, and prevent the admission of air, which, if present, would cause even greater brilliancy than liquid. The spectroscope shows broad interference-bands in the transmitted light, which change their position on altering the angle of incidence of the light which passes through the cuticle. Precisely similar colours, metallic on reflexion, non-metallic and with complementary tints on transmission, with the same spectroscopic appearances and changes induced by the same means, are seen in the surface films which are formed on bottle-glass after prolonged exposure to earth and moisture. In the alternating layers of the pupa the chitinous lamellæ are of higher, the liquid films of lower refractive index; hence water or alcohol produces brilliant appearances, while liquids of higher refractive indices produce less effect."

I owe to Prof. Poulton the opportunity of repeating some of these observations, such as the loss of metallic appearance on drying and of recovery under alcohol. On substitution of benzol with a little bisulphide of carbon for alcohol, the surface became very dark, but regained the golden glitter on going back to alcohol.

Of a specimen of another kind Prof. Poulton writes that the bug has been in the Oxford Museum Collection for 30 or 40 years judged by the pin. It is brown when dry, but when soaked in water becomes green like a leaf with bright iridescent green stripes on the under side. This observation also I have been able to repeat. All of which, it need hardly be said, is strongly suggestive of interference.

Dr. A. Hodgkinson also has described interesting observations. In his early papers* he distinctly refers the colours to Newton's scale, which in strictness would imply a limitation to a *single* thin plate. He emphasises the importance, for purposes of identification, of recording the colours of feathers &c. as seen by *perpendicular* reflexion, a condition best secured by illumination from a small perforated mirror,

---

* Manchester Memoirs, 1889; 1892, p. 149.

behind which the eye is placed. When daylight is used, it often suffices to examine the object with one's back to the window and at some distance from it. I shall have occasion later to refer again to Hodgkinson's work.

The first, so far as I know, to challenge the "structure" theory was Dr. B. Walter, whose tract* includes an elaborate discussion, accompanied by original observations, of the colours which may arise in the act of reflexion, and decides unequivocally that the colours now in question, with one or two possible exceptions, are due to surface, or quasi-metallic, reflexion as described by Haidinger, Brewster, and Stokes. The first of these writers formulated a law, named after him, which identifies the surface-colour with those rays which would be most intensely absorbed within the substance. The theory of "anomalous dispersion" since developed shows, however, that the matter does not stop there, and Walter emphasises that much of the surface-colour may be ascribed to rays which are not themselves intensely absorbed, but being situated *near* an absorption-band are abnormally *refracted*, and hence in accordance with Fresnel's laws are abnormally reflected. On the red side of the band the refractive index is increased and on the blue side diminished, so that when the substance is in air the surface reflexion is redder than according to Haidinger's law; but this conclusion may need to be modified when the substance is in contact with a strongly refractive solid, as when a dye spread upon a glass plate is examined from the glass side. In some cases it appears that the surface-colour is due as much, or even more, to these rays excessively refracted (and consequently reflected) as to those which would be intensely absorbed and are reflected in accordance with Haidinger's rule.

The departure from Haidinger's rule is specially important when we consider what happens at oblique incidences and with polarized light. The rays reflected in virtue of the extreme opacity of the substance to them are comparatively unaffected, and are indeed rendered more prominent by the appropriate use of a nicol. As Stokes says †: "In the case of the substances at present considered, the reflected light does not vanish, but at a considerable angle of incidence the pencil polarized perpendicularly to the plane of incidence becomes usually of a richer colour, in consequence of the removal, in great measure, of that portion of the reflected light which is independent of the metallic properties of the

---

* *Die Oberflächen oder Schillerfarben*, Braunschweig, 1895.

† Phil. Mag. vol. vi. Dec. 1853, p. 393; Math. and Phys. Papers, vol. iv. p. 42.

medium; it commonly becomes, also, more strictly related to that light which is absorbed with such great intensity." But, as Walter appears to have been the first to explain, there is a further important change of colour with the angle of incidence, when the light-vibrations are in the plane of incidence, in virtue of the abnormal refraction with its accompanying abnormal polarizing angle. In the usual case, where the dye is in contact with air, the polarizing angle for the rays lying on the red side of the absorption-band is unusually high, so that these rays, which at moderate angles of incidence contribute largely to the resultant colour, are extinguished at incidences of from 60° to 70°. In consequence, the colour of the reflected light moves towards the blue with increasing obliquity.

As an example, fuchsin may be referred to, a dye specially studied by Walter, who thus (p. 52) describes the surface-colour as seen from the air side:—

"(*a*) For light polarized in the plane of incidence:

"At small angles of incidence the reflexion is yellow-green, and at increasing angles becomes ever yellower and brighter.

"(*b*) For light polarized perpendicularly to the plane of incidence (that is, vibrating *in* this plane):

"At perpendicular incidence the reflexion is the same as under (*a*), and remains approximately so up to incidences of 50°. At about 60° it becomes rapidly blue-green and at 70° an almost pure blue, attaining its greatest purity at about 72°. At still greater angles the colour passes rapidly into a bright violet, and at 85° into white.

"When ordinary unpolarized light is employed, the colour of the reflexion is intermediate between (*a*) and (*b*), but always nearer to (*a*) than to (*b*) on account of the greater intensity of reflexion under (*a*)."

It is this movement of surface-reflexions towards the blue with increasing obliquity which is regarded by Walter and Michelson * as annulling the presumption in favour of the structure theory of the animal colours, which also move in this direction; and it must, of course, be admitted that the criterion is somewhat blurred thereby. Walter, indeed, maintains that thin plate colours change too much with angle to meet the requirements of the case. To this point I will return presently; but what I wish to remark at the moment is that with ordinary unpolarized light the surface-colours

---

* Phil. Mag. vol. xxi. p. 554 (1911). "On Metallic Colouring in Birds and Insects."

appear to change too little. Neither in the case of fuchsin nor of diamond green G—the second dye specially discussed by Walter,—or with any other dye hitherto examined*, have I seen an adequate change of colour without the use of the nicol to eliminate vibrations in the plane perpendicular to that of incidence. In the absence of a nicol there is little sign of the blue seen with it from fuchsin at 70° incidence. Much greater changes with more saturated colour are exhibited by the wing-cases of beetles when so examined.

As to the adequacy of the surface-colours Michelson himself remarks:—"indeed, it may perhaps be objected that the (animal) colours are far more vivid than any of the reflexion hues of the aniline dyes, or of any other case of 'surface-colour' hitherto observed." But perhaps this objection should not be very much insisted on in our ignorance of nature's operations and with regard to the known existence of powerful dyes, *e. g.* in feathers. It is rather the rapid loss of purity with obliquity in surface-colour which appears significant.

If a dye capable of surface-reflexion is present, there are still alternatives open. The pure or nearly pure dye may be on the outside so as to be in contact with air, or it may be overlaid by a colourless skin of horny material (chitin) in optical contact with it. The former case would be the more favourable for vivid and variable colour, but then one would expect to be able to remove the dye by solvents. So far as I am aware this has not been done, and my own trials with various solvents upon the wing-cases of beetles have not succeeded. The most satisfactory demonstration of the surface-colour theory would indeed be the extraction of the dye and its exhibition as a thin layer spread upon glass.

If, on the other hand, the dye is imprisoned within a layer of colourless chitin, the range of obliquities available in ordinary observation would be restricted and the difficulty of accounting for the variety of nearly saturated hues actually seen would be increased, more especially when we remember the dilution with white light reflected at the external surface.

---

* Through the kindness of Sir J. Dewar I have had the opportunity of experimenting with a good many dyes from the Badische Anilin-Fabrik. Following Walter, I have used warm alcoholic solutions spread upon previously warmed glass plates. Latterly I have examined some more dyes, for which I am indebted to Prof. Green. In no case have I seen any considerable change of well-developed colour unless the light was polarized.

There is still another view, which indeed is that actually maintained by Walter, whose argument and conclusion [*] it may be well to quote :—

" A further striking and at the same time more instructive proof of the equivalence of the lustre of butterfly-scales and the surface-colours of strongly absorbing dyes is to be found in the changes which the colours of these organs exhibit when immersed in fluids of varied refrangibility. These experiments are instructive because they disclose the manner in which the dye is contained in animal substances.

" The experiments show that, except when it is deep blue or violet, the lustre moves one or two colour-intervals in the direction from the blue towards the red end of the spectrum with increasing refrangibility of the surrounding medium, but at the same time becomes weaker. For example, the scales of *Morpho menelaus*, L., which glitter green-blue in air, become in ether ($n = 1{\cdot}36$) a pure green, shining less strongly, again in chloroform ($n = 1{\cdot}45$) a yellowish green and now decidedly weaker than in ether. In benzol ($n = 1{\cdot}52$) and in bisulphide of carbon ($n = 1{\cdot}64$) the weak yellow-green lustre is perceptible only with direct sunshine in a dark room. In a similar manner the scales of *Urania ripheus* shining green in air, in ether, alcohol or water become golden yellow, the yellow red and the red blue, while in benzol and bisulphide of carbon scarcely a trace of glitter remains.

" Where we know that the cause of the lustre is a dye, the latter facts admit of but one interpretation—that in the case of butterfly-scales we have to do with *solutions of the dyes in chitin*, solutions whose refractivity for most of the spectrum colours is nearly equal to those of benzol and bisulphide of carbon, so that these colours, unless they are very strongly absorbed by the solution, are practically not reflected in their passage from the colourless liquids. Accordingly, the dyes which give rise to lustre in the chitin-skin of insects, and, as we shall see presently, in the horny skin in birds, are dissolved in the same fashion as cobalt oxide in blue glass or organic dyes in a layer of solid gelatine, a conception suggested in the simple observation of the scales by transmitted light and confirmed by the facts above adduced."

If Walter's argument and conclusion are accepted, the difficulty, already considerable, of explaining the richness of the animal colours is enhanced by the supposed dilution of the dyes, and one can hardly fail to observe that a simpler

---

[*] *Loc. cit.* p. 96.

explanation is to reject the dye theory and refer the colours to interference. The facts recorded agree pretty closely with what happens in the case of films of old decomposed glass.

Indeed, Walter, in a later passage, very candidly admits a difficulty. He says (p. 98):—

"Finally, it must not be passed over in silence that there is a circumstance which makes a difficulty for the view here propounded of the lustre colours of butterflies. This is the fact that the lustre practically disappears in benzol and bisulphide of carbon, whereas in treating the theory of surface-colours we have several times insisted that a ray strongly absorbed must under all circumstances be vigorously reflected."

Before leaving the question of the colours it may be well to consider an objection strongly urged by Walter against the interference theory, viz., that the colours of thin plates change too much with obliquity. As regards a *single* thin plate, which alone Walter seems to have contemplated, it is true, I think, that the more pronounced colours of the 2nd and 3rd order in Newton's scale change more rapidly with the retardation* than could well be harmonized with what is observed of the animal colours. But the difficulty disappears when we admit a structure several times repeated with approximate periodicity. The changes in chlorate of potash crystals with obliquity seem to agree well enough with what is required, and this form of the interference theory has the advantage of greater elasticity, *e.g.* meeting Walter's objection that the colours of a single thin plate constitute a simple series with but one independent variable. Indeed, the purity of the *reds* often to be observed from beetles' wing-cases seems to exclude an interference theory limited to a single thin plate, inasmuch as the reds from such a plate are distinctly inferior, especially when diluted with white light reflected from an outer surface not forming part of the boundary of the thin plate †.

Michelson, who with his great authority supports the surface-colour theory, mentions several tests under four headings (p. 561). To my mind these tests are as well, if not better, borne by an interference theory. But reliance seems to be chiefly placed upon "the more rigorous optical test of the measurement of the phase-difference and

---

* See a diagram of the Colours of Thin Plates, Ed. Trans. vol. xxxiii. p. 157 (1886); Scientific Papers, vol .ii. p. 498.

† Ed. Trans. *loc. cit.*

amplitude-ratios" when polarized light is reflected. I agree that this is a cogent argument, and unless it can be met the balance of evidence derived from simple observation would perhaps incline to the surface-colour theory. It is, I think, the fact that many beetles exhibit a less well-marked polarizing-angle than could be reconciled with the usual theory of thin plates constituted of non-absorbent material. An escape from the difficulty might perhaps be found in imagining a stratification composed of more than two materials, so that, for instance, the polarizing-angle for the first and second might differ considerably from that corresponding to the second and third. But such a structure seems rather improbable, and any combination of thin plates composed of two transparent materials only should give a definite polarizing-angle, abstraction being made from the minor deviations observed by Airy and Jamin.

At this point it may be recalled that a well-marked polarizing-angle and a sudden change of relative phase through two right angles are more closely connected than is sometimes realized. The latter without the former would involve a physical discontinuity. Michelson considers that in practice the phase-change affords the more delicate criterion *, and that in most cases it is decisive in favour of surface-colour.

A circumstance which may perhaps be regarded as telling upon the other side is afforded by the variety of colouring at different parts, but at the same angle (*e. g.* at perpendicular incidence) seen in certain beetles—Dr. Hodgkinson mentions *Chrysochroa fulminans*. The "colours vary in an indescribable manner when attentively examined at different angles of incident light with the eye alone; with the mirror (viz., at perpendicular incidence) the wing-cases are seen to be coloured successively from base to tip iridescent green, yellow, orange, and red, and these tints remain unaltered by change of position of the object." I have confirmed generally this observation, and other beetles show something similar. The explanation makes large demands upon the surface-colour theory; but a moderate change of structure is all that would be required by interference.

A caution is perhaps required against regarding the two theories as mutually exclusive. Both Walter and Michelson

---

* Some of Michelson's diagrams are rather confusing in that they suggest a phase-difference of 180° between the two polarized components reflected *perpendicularly*, when evidently the distinction between the two components disappears.

admit exceptions, and certainly there is no improbability in surface reflexion playing a part. It may be that both causes are operative in a single specimen and even at the same part of it.

The next contribution to the discussion is an important one by Mallock *, who brings to bear the instinct and experience of a naturalist as well as of a physicist. His observations were mainly on the feathers of birds and the scales of insects, and they lead him to regard interference rather than selective reflexion as the origin of the iridescent colours. "The transparency or, at any rate, the vanishing of the characteristic transmitted colour in the case of all animal tissues when immersed and permeated by a fluid of the same refractive index is strongly in favour of interference being the source of the colour, but even stronger evidence is given by the behaviour of the structure under mechanical pressure.

"If the grain or peculiarities which favour the reflexion or transmission of particular colours is of molecular size, there is no reason to suppose that pressure insufficient to cause molecular disruption would alter the action of the material on light. On the other hand, if the colours are due to interference, that is, to cavities or strata of different optical properties, compression would alter the spacing of these, and thus give rise either to different colours or, with more than a very slight compression, to the transmission and reflexion of white light."

"In every experiment of this kind that I have made either on feathers or insect scales the effect of pressure has been to destroy the colour altogether.... With many feathers the colour returns when the pressure is taken off, but with insect scales the structure seems to be permanently injured by compression, and though when allowed to expand again the material is not colourless, the brilliancy which belonged to the uninjured scale is gone, and the colour in general changed."

"The facts above mentioned seem to offer stronger reasons in favour of interference than the polarization phenomena referred to by Michelson and Walter do against it."

I have already commented on the importance of the evidence afforded by observations with polarized light; and if we have to choose between selective reflexion and thin plates of the type usually considered in theoretical writings, we may find ourselves in a position of much difficulty. The

---

* Proc. Roy. Soc. vol. lxxxv. A, p. 598 (1911).

question then arises, Is there any loophole for escape? I think there may be. The polarizing angle, as given by Brewster's law, depends much upon what we may call the smoothness of the reflecting surface. A moderate curvature is of no significance in this connexion, but when the radius of curvature becomes comparable with the wave-length of the light it is another matter. Thus in the case of smooth glass the polarizing angle is about 57°—that is, light incident at this angle with the normal and vibrating in the plane of incidence is not reflected. In this observation the reflected light (if there were any) would be deviated from its original direction through an angle of $2\,(90° - 57°) = 66°$, and this is the direction in which light initially unpolarized would appear completely polarized. Now replace the flat glass by a sphere of the same material, whose diameter is small in comparison with the wave-length. Light is now scattered in various directions, but the direction in which light originally unpolarized becomes completely polarized is at 90° with the original direction, instead of 66°. As the sphere grows, the polarization ceases to be complete, and the direction of best polarization moves oppositely to what would be expected—that is, still further *away* from 66°. When the circumference of the sphere is equal to twice the wave-length, the polarization, still pretty good, occurs at an angle of 135° with the original direction of the light[*]. In order to carry out the suggestion, we must abandon the supposition of uniform plane strata, inapplicable anyhow in its integrity to the case where one of the alternate plates is of air, and substitute a structure in which one of the alternatives takes a form such as the spherical. A layer of equal spheres, with centres disposed upon a plane, would give a specular reflexion and a polarizing angle dependent upon the diameter of the spheres and upon the intervals between them. In certain cases, *e. g.* when the circumference of the sphere (of glass) is equal to $1{\cdot}75 \times$ wave-length, the polarization is very imperfect. To explain a brilliant and highly-coloured reflexion there would need to be several layers of spheres, and it might be supposed that the diameter varied in different layers. In this way it would seem possible to combine a specular and highly-coloured reflexion with a very imperfectly developed polarization, and thus to evade the difficulty which meets us when we confine ourselves to "thin plates." Spheres have been spoken of for simplicity and because some of

---

[*] Phil. Mag. vol. xii. p. 81 (1881); Proc. Roy. Soc. A. vol. lxxxiv. p. 25 (1910); Scientific Papers, vol. i. p. 518, vol. v. p. 564.

the effects have been calculated in this case, but it is evident that similar phenomena would be produced by obstacles of other and perhaps more probable forms. The obstacles must have a different index from that of the medium in which they are embedded, and there is no need for absorption.

It may perhaps be objected that though a layer of spheres may give a specular reflexion there would be an accompaniment of light dispersed at other angles, forming in the case of a regular pattern "diffraction spectra." It is uncertain whether or not this occurs. If it does not, the explanation may be that the pattern is too fine.

The above remarks are intended merely to attenuate the difficulty arising from the absence of a well-marked polarizing angle, and the details need not be insisted on. No surprise is felt at the deficiency of polarization in the light reflected from unpressed and unglazed paper, of which the fibres are quite large enough to be the seat of interference effects. In illustration the transverse reflexion from glass rods and fibres may be mentioned. When we examine with a nicol the reflexion from a rod ¼ inch (6 mm.) in diameter, we can verify the extinction at a suitable angle of the light reflected from the first surface, although abundance of other light still reaches the eye. When we replace the rod by a fine fibre, this discrimination is lost, and the rotation of the nicol may make no difference, or even a difference in the wrong direction.

The greater part of the preceding discussion was written about a year and a half ago. I am now able to supplement it with further observations of my own and of others who have been kind enough to help me. Most of my experiments have been made on wing-cases of beetles found in my garden (June and July 1917). Usually attention is first attracted by the display of a vivid green coloration, but on indoor examination the variation with angle is found to be about the same as is observed with brilliant specimens from abroad. At perpendicular incidence the colour is an orange with approach to red, passing with increasing obliquity through yellow and green to a blue-green. Ordinary solvents such as water even at the boiling-point, ether, alcohol, benzol, bisulphide of carbon, acetic acid, &c., seem to be without effect, even when the precaution is taken to separate a wing-case into two parts so as to allow access to the interior of the cuticle. A treatment with hot caustic potash has more effect, in one experiment shifting the colour at perpendicular incidence from orange to a brilliant scarlet. By the action of hot somewhat diluted nitric acid* the black underlying pigment may be removed without much affecting the dye, or the structure, which is the seat of the coloration.

Several experiments were made to test whether air-cavities existed. For this purpose the wing-case was exposed for some time to the action of a vacuum, into which afterwards water or benzol was admitted. But no distinct evidence of the penetration of liquid could be recorded. I understand that Prof. Poulton has had a similar experience.

Again, it has been noticed by Mr. H. Onslow and myself that considerable pressure fails to alter the colour of beetles and of the wings of some iridescent dragon-flies, though (Poulton, Mallock) effective in some other cases. It would seem that the hypothesis of air-cavities must be abandoned.

In the absence of air-cavities the alternating structure demanded by the interference theory would require two kinds of matter capable of resisting pressure and of sensibly different refractive indices. Probably both would be solids; and since the range of relative index is then much restricted, the brightness of the reflected light could hardly be explained without supposing more than the two or three alternations which might suffice were air in question. Mr. Onslow thinks that there may then be a difficulty in finding room for the alternating structure and the protective covering.

An important question is whether the change of colour with angle is such as can plausibly be attributed to a periodic structure. As Walter points out, a good deal depends upon whether, or not, there is a limitation upon the obliquity of the rays *within* the thin plate, or plates. In the ordinary arrangement for Newton's rings there is no limitation, the direction in the air-film being parallel to that of the rays before incidence upon the first plate. The optical retardation may then vary from its maximum at perpendicular incidence to zero at 90° obliquity. According to this, it should always be possible to push the colour out of the spectrum at the blue end by sufficiently increasing the obliquity, but it must be remembered that unless special provision is made the colour effects would be overlaid by the white light reflected at these angles from the first glass surface encountered.

From what we have seen in the case of the beetle colours where we must suppose that the refractive index does not differ greatly from that (1·6) of the chitin, there is a limit to the obliquity within the thin films even when externally the incidence is grazing. If θ be the angle in the thin film and μ the refractive index, the retardation is proportional to $\cos\theta$, and in the limiting case

$$\cos\theta = \sqrt{(\mu^2-1)}/\mu.$$

If we take $\mu=1\cdot5$, the minimum retardation is represented by ·746, the maximum retardation at perpendicularity being taken as unity. It may be remarked that the minimum retardation may practically be secured without pushing very far the obliquity outside. If we suppose the maximum retardation to give a coloration corresponding to the Fraunhofer line C ($\lambda=6563$), the minimum will correspond to $\lambda=4896$, pretty close to the line F. According to the interference theory, then, the range of coloration should be from the full red of C to the blue-green of F, and this is just about what is observed. The agreement must be admitted to be a strong argument in favour of the theory. So far as I have seen, so great a range cannot be found in the surface colour of any dye, even with the aid of polarized light.

I have already mentioned that the opaque backing behind the seat of coloration can be attacked, and for the most part removed, with nitric acid, so as to allow the transmission colour to be observed. But a much superior effect has been obtained by Dr. Eltringham, using *eau de javelle* (hypochlorite) *. After removal of the backing, the wing-case was mounted with balsam in a slide, which Dr. Eltringham has kindly left in my possession. Close observation of this specimen has yielded results which I think interesting and telling. Seen by transmitted light with the aid of a Coddington lens, the slide shows a pale green over the larger part of the area, which by perpendicular reflexion is a full red. The green is fairly uniform except where it appears perforated with small circular spots, which look reddish, but perhaps only by contrast. Especially to be noted is the fact that there is no colour seen by transmission at all comparable in saturation with those exhibited by reflexion. For observation of the reflected light it is advantageous, though not necessary, to renew the opaque backing, which was done by coating the under surface of the *glass* with gelatine darkened with ink. A good deal depends upon the source of light. In the first detailed examination, the source (a gas-mantle) happened to be highly localized, and I was puzzled to reconcile the highly spotty character of the reflexion, varying from red to green or green-blue according to the incidence, with the uniformity of the transmission tint. Similar appearances could of course be observed in direct sunlight. But when the slide was held

---

* Dr. Eltringham's label runs:—

Mimela leei. Elytron after prolonged eau-de-javelle. Only surface-film left. Transmits complementary colours to those it reflects, and reflects same colours from both sides (1917).

very close to a large window facing a nearly uniform sky, the intervals between the spots filled up with colour, for the most part of approximately the same hue, and the reflexion was nearly uniform except for the small round holes already mentioned. Evidently the reflexion of the gas-mantle had failed to reach the eye, except from a relatively small area presenting the proper angle, thus explaining the spotty appearance observed with this illumination.

The colours reflected at moderate angles seem highly saturated. At perpendicular incidence the prism shows next to nothing beyond the uninterrupted red and red-orange, and on inclination the green region appears well isolated. The impression left upon my mind is that the phenomena cannot plausibly be explained as due to surface-colour, which in my experience is always *less* saturated than the transmission colour, and that, on the other hand, the interference theory presents no particular difficulty, unless it be that of finding sufficient room within the thickness of the cuticle. But the alternations cannot be those of plane strata, extending without interruption over the whole area of the colour.

As regards the difficulty of finding room sufficient for an optical structure of the kind contemplated, Mr. Onslow estimates the available thickness at from 0·75 μ * to 2 μ in the case of many butterfly scales, and this is little enough. Even the larger estimate would amount to only about 9 or 10 half wave-lengths, even when allowance is made for the wave-length being less than in air, and the lower limit would apparently not suffice. But these measurements are not easy, and may perhaps be disturbed by refraction effects. Mr. Onslow has shown me many drawings of sections in planes perpendicular to the surface from many butterfly scales and from two or three beetle wing-cases. Most of these exhibit structures approximately periodic along the surface, but in no case a structure periodic in going inwards along the normal. But a structure of the latter kind adequate to the purpose may probably lie close upon the microscopic limit, unless, indeed, it could be made evident in a section cut very obliquely.

It must be confessed that much still remains to be effected towards a complete demonstration of the origin of these colours. Even if we admit an interference character, questions arise as to the particular manner, and there are perhaps possibilities not hitherto contemplated.

November 20, 1918.

---

* $\mu = \frac{1}{1000}$ mm.

VIII. *On the Radiation of Light from the Boundaries of Diffracting Apertures.* *By* SUDHANSUKUMAR BANERJI, *M.Sc., Assistant Professor of Applied Mathematics, Calcutta University* *.

[Plates III. & IV.]

I. *Introduction.*

IN his famous memoir on the mathematical theory of diffraction, Sommerfeld † has given a rigorous treatment of the effect of a semi-infinite perfectly reflecting screen on the propagation of plane waves of light through the medium. One of the most important results indicated by his investigation is that the diffraction effect due to the screen may be regarded as due to cylindrical waves emitted by its edge, the intensity of which is different in different directions, these waves alone being operative in the region of shadow, but in the other regions appearing superposed upon and interfering with the reflected and transmitted waves. Among the more recent writers who have observed and studied the phenomena of the luminosity of a diffracting edge experimentally may be mentioned Gouy ‡, Wien §, E. Maey ‖, and Kalaschnikow ¶.

The present paper deals experimentally and theoretically with the problem of the emission of light by the boundary of a diffracting *aperture* of limited area and of specified form. I consider the actual form of the luminous fringes seen at the boundary of a diffracting aperture when it is viewed solely by the diffracted light. To give the problem definiteness we have to assume that the aperture is viewed through a telescope focussed on its plane, the object-glass of the telescope being itself covered by a screen containing one or more apertures through which the diffracted light enters the field of view. In these circumstances the luminosity appears practically confined to certain more or less well-defined regions lying near the boundary of the aperture. When the wave entering the first aperture is convergent and the object-glass of the observing telescope is placed in the focal plane, the mathematical treatment becomes analogous to that given in a recent paper on the theory

---

* Communicated by Prof. C. V. Raman.

† *Math. Ann.* Bd. xlvii. (1896).

‡ Gouy, *Ann. d. Phys. et de Chim.* (6), (8), p. 145 (1886).

§ Wien, Inaug. Diss., Berlin, 1886.

‖ E. Maey, Wiedemann's *Annalen*, xlix. (1893).

¶ Kalaschnikow, *Journ. Russ. Phys. Ges.* xliv. (1912).

of Foucault's test by Lord Rayleigh *. The present investigation was, in fact, suggested by a perusal of Lord Rayleigh's paper, and the photographs reproduced in Plate IV. were taken with an arrangement analogous to that used in Foucault's test, though I have also made visual observations in the cases in which the incident waves do not converge to a focus in the plane of the second screen.

The most remarkable result found in the course of the experimental work (and which, so far as I know, has not been noticed by any previous writer) is that in all cases in which the apertures in the focal plane through which the diffracted rays pass (whatever be their actual form) are *symmetrically* disposed about the centre of the field, the latter itself being excluded, the image of the boundary of the diffracting surface appears as *a perfectly black line* surrounded on either side by luminous bands. This is irrespective of the actual form of the boundary itself, that is, whether it is circular or of any other shape whatsoever. A similar result is also found when the screen with the apertures is placed symmetrically in any plane either in advance of or behind the focus. When the apertures on the screen are wide enough to admit a large portion of the diffraction pattern formed at the screen into the field of view of the observing telescope, this black line is characterized by extreme fineness and is surrounded on either side by broad luminous bands. Narrowing the apertures is, however, attended by an increase in the width of this black line and the appearance of a large number of well-defined fringes on either side. This remarkable feature is explained in a general way if we regard each element of the edge of a diffracting aperture as sending out two streams of light in directions more or less normal to itself (one on each side of the wave-normal), and that these streams are in opposite phases. This is distinctly suggested by Sommerfeld's well-known investigation on the diffraction of plane waves of light by a semi-infinite screen. The expression given by him for the radiation emitted by the edge is

$$\frac{1}{4\pi}\cos\left\{2\pi\left(\frac{r}{\lambda}-\frac{t}{\mathrm{T}}\right)+\frac{\pi}{4}\right\}\sqrt{\frac{\lambda}{r}}\left[\frac{\pm 1}{\cos\frac{\phi+\phi'}{2}}-\frac{1}{\cos\frac{\phi-\phi'}{2}}\right],$$

where (r, ϕ) are the coordinates of a point in the medium

---

* Lord Rayleigh, "On Methods for detecting small Optical Retardations, and on the Theory of Foucault's Test," Phil. Mag. Feb. 1917.

and ϕ' is the angle which the incident beam makes with the screen. This is valid everywhere except in two extremely limited regions lying in the neighbourhood of the two directions $\phi=\pi-\phi'$ and $\phi=\pi+\phi'$, which indicate the planes of transition between the regions of shadow and of transmission and between the regions of transmission and reflexion respectively. It will be noticed that in the neighbourhood of these two planes of transition, one of the two terms within the square bracket is very small in comparison with the other, and may therefore be neglected. The term which is retained changes sign when we pass from one side of the plane of transition to the other. It is thus seen that the phase of the radiation emitted by the edge changes by π when we move from the region of shadow into the region of light. If similarly we assume that each element of the boundary of a diffracting *aperture* emits radiations, the phase of which differs by π on the two sides of the wave-normal passing through it, then the phenomena described by me would be qualitatively explained. The detailed mathematical treatment will, however, be given in the course of the paper for the case of the circular and rectangular boundaries.

2. *Case of the Circular Boundary.*

The optical surfaces examined by the well-known "knife-edge" test due to Foucault are most frequently limited by a circular aperture[*], the illumination being that due to a point source. Fig. 9 (Pl. III.) reproduces a photograph of the luminosity observed at the boundary when the knife-edge is put in horizontally into the focal plane from below so as to cut off most of the light. It will be noticed that the luminosity is a maximum on the upper and the lower boundaries, and diminishes to zero at the ends of a horizontal diameter. In order to give definiteness to a discussion of this effect, it is necessary to postulate some specified forms for the boundaries of the apertures in the focal plane which admit the diffracted rays into the field of view of the observing telescope. For instance, we may assume that a horizontal slit is placed in the focal plane below the centre of the field. Fig. 12, Plate III., reproduces the beautiful lunette-shaped diffraction-fringes that

---

[*] See the memoirs by Draper and Ritchey "On the Construction of a Silvered Glass Telescope," Smithsonian Contributions to Knowledge, vol. xxxiv. (1904).

appear on either side of the boundary of the circular aperture with this arrangement. The luminosity, as in the case of the simple knife-edge test, tends to zero at the ends of a horizontal diameter. The explanation of this fact and of the peculiar form of the fringes will appear later.

More striking still are the interference phenomena obtained when the boundary is observed through a pair of apertures of the same form placed in the focal plane. With two horizontal slits placed *on the same side* of the centre of the field, lunette-shaped *interference* fringes are observed, the central fringe which coincides with the boundary being white (Pl. III. fig. 15). But when two horizontal slits are placed on opposite sides of the centre of the field, *the central fringe is black*—in other words, the boundary of the aperture is itself non-luminous but appears surrounded on either side by luminous bands (Pl. III. fig. 6). As has been remarked in the introduction, this remarkable fact is one of great generality. Fig. 16 in Pl. III. represents the appearance of the circular boundary when a horizontal wire is placed across the centre of the focal plane. A fine black line may be seen running through the luminous arcs and dividing them into two. A case that admits of detailed mathematical treatment is that in which the arrangement is completely symmetrical about the axis. Figs. 7 & 10 in Pl. III. show the results obtained when the central part of the field at the focal plane is blocked out by a circular disk and only the diffracted rays passing through an annulus of greater or less width surrounding it enter the observing telescope. It will be seen that in both photographs the boundary appears as a perfectly black circle, with luminous rings on either side of it.

Let R be the radius of the circular aperture of the lens, and assume that in the focal plane there is a screen containing an annular aperture, R_1, R_2 being the radii of the circles defining the annulus. Let ϕ be the angle of diffraction of parallel rays which meet at any point Q in the focal plane. Since the path-difference between the rays leaving a point (r, θ) and the centre of the diffracting aperture is evidently $r \cos \theta \sin \phi$, the diffracted disturbance at a point in the focal plane due to an area $r\,d\theta\,dr$ can be written in the form

$$r \sin 2\pi \left(\frac{t}{T} - \frac{r \cos \theta \sin \phi}{\lambda}\right) d\theta\, dr.$$

The total disturbance at Q in the focal plane is therefore

$$\int_0^{2\pi}\int_0^{R} r\sin 2\pi\left(\frac{t}{T}-\frac{r\cos\theta\sin\phi}{\lambda}\right)d\theta\,dr. \quad . \quad . \quad . \qquad (1)$$

The rays from the various elements $r'\,d\theta'\,dr'$ of the second aperture may be regarded as meeting in the field of the observing telescope proceeding at an angle ϕ' with the axis and producing the observed effect. The total disturbance in this direction due to the second aperture is therefore;

$$\int_0^{2\pi}\int_{R_1}^{R_2}\int_0^{2\pi}\int_0^{R} rr'\sin 2\pi\left(\frac{t}{T}-\frac{r\cos\theta\sin\phi}{\lambda}\right.$$
$$\left.-\frac{r'\cos\theta'\sin\phi'}{\lambda}\right)dr\,d\theta\,dr'\,d\theta'. \qquad (2)$$

Since ϕ and ϕ' are small quantities, the above expression can be written as

$$\int_0^{2\pi}\int_{R_1}^{R_2}\int_0^{2\pi}\int_0^{R} rr'\sin 2\pi\left(\frac{t}{T}-\frac{\phi\,.\,r\cos\theta}{\lambda}\right.$$
$$\left.-\frac{\phi'\,.\,r'\cos\theta'}{\lambda}\right)dr\,d\theta\,dr'\,d\theta'. \qquad (3)$$

This expression can be reduced to the form

$$\int_0^{2\pi}\int_{R_1}^{R_2}\left[r'\sin 2\pi\left(\frac{t}{T}-\frac{\phi'\,.\,r'\cos\theta'}{\lambda}\right)dr'\,d\theta'\right.$$
$$\left.\times\int^{2\pi}\int_0^{R} r\cos\left(2\pi\frac{\phi\,.\,r\cos\theta}{\lambda}\right)dr\,d\theta\right], \qquad (4)$$

the other integral being zero on account of symmetry of the diffracting aperture.

But the integral

$$\int_0^{2\pi}\int_0^{R} r\cos\left(2\pi\,.\,\frac{\phi\,.\,r\cos\theta}{\lambda}\right)dr\,d\theta=\frac{2}{\pi}\cdot\frac{R\lambda}{2\pi\phi}J_1\left(\frac{2\pi}{\lambda}R\phi\right).$$

Therefore the expression (4) becomes

$$\int_0^{2\pi}\int_{R_1}^{R_2} r'\,d\theta'\,dr'\sin 2\pi\left(\frac{t}{T}-\frac{\phi'\,.\,r'\cos\theta'}{\lambda}\right).\frac{1}{\phi}J_1\left(\frac{2\pi}{\lambda}R\phi\right), \qquad (5)$$

neglecting a constant factor.

If f is the focal length of the lens, then $\phi=\frac{r'}{f}$. The expression (5) can be written in the form

$$\sin 2\pi\frac{t}{T}\int_0^{2\pi}\int_{R_1}^{R_2} r'\,d\theta'\,dr'\cos\left(2\pi\frac{\phi'.r'\cos\theta'}{\lambda}\right)\frac{f}{r'}J_1\left(\frac{2\pi R}{\lambda f}.r'\right)$$

$$+\cos 2\pi\frac{t}{T}\int_0^{2\pi}\int_{R_1}^{R_2} r'\,d\theta'\,dr'\sin\left(2\pi\frac{\phi'.r'\cos\theta'}{\lambda}\right)\frac{f}{r'}J_1\left(\frac{2\pi R}{\lambda f}.r'\right).$$

Since the second aperture is also symmetrical about the axis, the second integral is zero, for the elements of it arising from two points situated at equal distances on opposite sides of a diameter are equal and of opposite signs. Therefore the intensity as viewed in the direction ϕ' is

$$I=\left[\int_0^{2\pi}\int_{R_1}^{R_2} d\theta'\,dr'\cos\left(2\pi\frac{\phi'.r'\cos\theta'}{\lambda}\right)J_1\left(\frac{2\pi R}{\lambda f}.r'\right)\right]^2.$$

Integrating with respect to ϕ', we get

$$I=\left[\int_{R_1}^{R_2} J_0\left(\frac{2\pi\phi'}{\lambda}.r'\right)J_1\left(\frac{2\pi R}{\lambda f}.r'\right)dr'\right]^2$$

(neglecting a constant factor).

If the angular semi-diameter of the lens be denoted by ψ, then $R=f\psi$. The expansion for the intensity can therefore be written in the form

$$I=\left[\int_{R_1}^{R_2} J_0\left(\frac{2\pi\phi'}{\lambda}.r'\right)J_1\left(\frac{2\pi\psi}{\lambda}.r'\right)dr'\right]^2.$$

Since $\frac{1}{\lambda}$ is a very large quantity, it is convenient to use semi-convergent expansions for J_0 and J_1. We have

$$J_0(x)=\sqrt{\frac{2}{\pi x}}\left[\cos\left(x-\frac{\pi}{4}\right)\left\{1-\frac{1^2.3^2}{2!\,(8x)^2}+\frac{1^2.3^2.5^2.7^2}{4!\,(8x)^4}-\dots\right\}\right.$$
$$\left.+\sin\left(x-\frac{\pi}{4}\right)\left\{\frac{1}{8x}-\frac{1^2.3^2.5^2}{3!\,(8x)^3}+\frac{1^2.3^2.5^2.7^2.9^2}{5!\,(8x)^5}-\dots\right\}\right],$$

$$J_1(x)=\sqrt{\frac{2}{\pi x}}\left[\sin\left(x-\frac{\pi}{4}\right)\left\{1+\frac{3.5.1}{8.16}\left(\frac{1}{x}\right)^2\right.\right.$$
$$\left.-\frac{3.5.7.9.1.3.5}{8.16.24.32}\left(\frac{1}{x}\right)^4+\dots\right\}$$
$$+\cos\left(x-\frac{\pi}{4}\right)\left\{\frac{3}{8}.\frac{1}{x}-\frac{3.5.7.1.3}{8.16.24}\left(\frac{1}{x}\right)^3\right.$$
$$\left.\left.+\frac{3.5.7.9.11.1.3.5.7}{8.16.24.32.40}\left(\frac{1}{x}\right)^5-\dots\right\}\right].$$

Thus, when x is large,

$$J_0(x)=\sqrt{\frac{2}{\pi x}}\left[\cos\left(x-\frac{\pi}{4}\right)+\frac{1}{8}\,\frac{\sin\left(x-\frac{\pi}{4}\right)}{x}\right],$$

$$J_1(x)=\sqrt{\frac{2}{\pi x}}\left[\sin\left(x-\frac{\pi}{4}\right)+\frac{3}{8}\,\frac{\cos\left(x-\frac{\pi}{4}\right)}{x}\right].$$

Therefore

$$I=\left[\int_{R_1}^{R_2}J_0\left(\frac{2\pi\phi'}{\lambda}x\right)J_1\left(\frac{2\pi\psi}{\lambda}x\right)dx\right]^2$$

$$=\frac{1}{\psi\phi'}\left[\frac{\lambda}{2\pi^2}\int_{R_1}^{R_2}\frac{\sin\frac{2\pi}{\lambda}(\psi-\phi')x}{x}dx-\frac{\lambda}{2\pi^2}\int_{R_1}^{R_2}\frac{\cos\frac{2\pi}{\lambda}(\psi+\phi')}{x}dx\right.$$

$$+\frac{\lambda^2}{32\pi^3}\left(\frac{1}{\phi'}+\frac{3}{\psi}\right)\int_{R_1}^{R_2}\frac{\cos\frac{2\pi}{\lambda}(\psi-\phi')x}{x}dx$$

$$-\frac{\lambda^2}{32\pi^3}\left(\frac{1}{\phi'}-\frac{3}{\psi}\right)\int_{R_1}^{R_2}\frac{\sin\frac{2\pi}{\lambda}(\psi+\phi')x}{x}dx$$

$$\left.+\text{ terms involving higher powers of }\lambda\right]^2.$$

Taking $R_1=\frac{3\lambda}{2\pi\psi}$, $R_2=\frac{50\lambda}{2\pi\psi}$, we obtain (neglecting a constant factor),

$$I=\frac{1}{\phi'\psi}\left[\left\{\mathrm{Si}\,50\left(1-\frac{\phi'}{\psi}\right)-\mathrm{Si}\,3\left(1-\frac{\phi'}{\psi}\right)\right\}\right.$$

$$-\left\{\mathrm{Ci}\,50\left(1+\frac{\phi'}{\psi}\right)-\mathrm{Ci}\,3\left(1+\frac{\phi'}{\psi}\right)\right\}$$

$$\left.+\lambda\left(\dots\quad\dots\right)+\dots\right]^2.$$

Calculating the values of this expression for different values of $\frac{\phi'}{\psi}$, we construct the following table (Table I.).

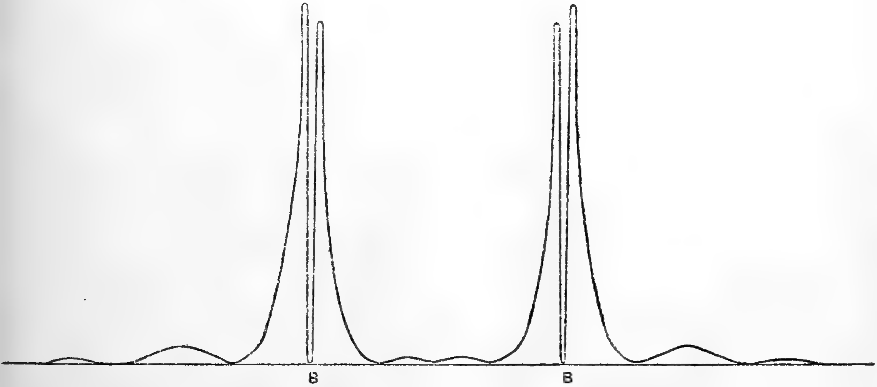
Plotting the values, we obtain a curve (fig. 1) representing the distribution of intensity along any given diameter. The fringes that appear on either side of the boundary are clearly shown, the most remarkable feature being the extreme rapidity with which the intensity falls practically to zero on

TABLE I.

ϕ'/ψ.	$\sqrt{\text{I}}$.	I (×const. factor).	ϕ'/ψ.	$\sqrt{\text{I}}$.	I (×const. factor).
0·00	− ·162	262	1·02	− ·889	7905
0·20	− ·219	478	1·04	−1·527	23307
0·40	− ·119	142	1·06	−1·701	28911
0·60	+ ·255	650	1·08	−1·549	23909
0·80	+ ·819	6711	1·10	−1·269	16096
0·90	+1·139	12902	1·20	−1·047	10964
0·92	+1·417	20076	1·40	− ·359	1294
0·94	+1·662	27613	1·60	+ ·058	34
0·96	+1·405	19754	1·80	+ ·254	645
0·98	+ ·815	6649	2·00	+ ·358	1281
0·99	+ ·396	1571	2·40	+ ·101	102
1·00	+ ·073	53	2·60	− ·054	29
1·01	− ·524	2746			

the boundary itself (BB in the figure) from a large value on either side of it. This feature depends on the inner radius R_1 of the annulus being small and the outer radius R_2 being very large, and is entirely confirmed by observations under these conditions.

Fig. 1.



If the radii R_1 and R_2 of the annulus in the focal plane do not differ much, or if they are both large, the brightness falls off to zero on the boundary, but not very suddenly. A large number of well-defined fringes also appear on either side of the boundary in the former case (see, for instance, Pl. III. fig. 13). This will be shown from the following calculations based on the data obtained from an actual experiment. The data were :

$$R = {\cdot}94 \text{ mm.}, \quad f = 272 \text{ mm.}, \quad R_1 = 1{\cdot}72 \text{ mm.},$$
$$R_2 = 2{\cdot}61 \text{ mm.}, \quad \lambda = {\cdot}00045 \text{ mm.}$$

We thus obtain for this case

$$I = \frac{1}{\phi'\psi}\Bigg[\left\{\text{Si } 122{\cdot}67\left(1-\frac{\phi'}{\psi}\right) - \text{Si } 80{\cdot}84\left(1-\frac{\phi'}{\psi}\right)\right\}$$
$$-\left\{\text{Ci } 122{\cdot}67\left(1+\frac{\phi'}{\psi}\right) - \text{Ci } 80{\cdot}84\left(1+\frac{\phi'}{\psi}\right)\right\}$$
$$+\lambda\left(\dots \quad \dots\right) + \&\text{c.}\Bigg]^2.$$

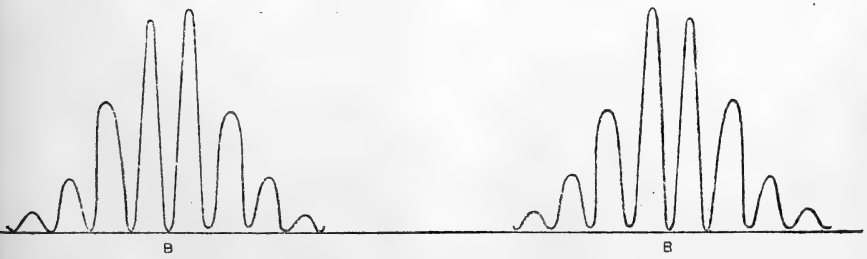
The values of this expression for different values of ϕ'/ψ are shown in Table II., in which the calculated and the observed values of the ratio ϕ'/ψ at which the illumination is a maximum or a minimum are given for comparison. It will be seen from the table that $\sqrt{I}$ changes sign at the minima, and these are therefore absolute zeros. The values for I shown in Table II. have been plotted in fig. 2. It will be seen from Tables I. and II. that $\sqrt{I}$ changes sign as it passes through its values at the boundary $\phi'/\psi=1$, showing that the radiations from the edge on the two sides of the boundary differ in phase by π.

The investigation given above may be modified to suit the case in which we have one or more slits (instead of an annulus) in the focal plane by suitably altering the limits of integration in the expression (5) given above. The writer hopes to give the detailed numerical calculations in a later paper. It is possible, however, to understand in a general way the reason for the peculiar configuration of the fringes shown in Pl. III. figs. 6, 12, & 15. If instead of a slit we have in the focal plane a small aperture at (r', θ') through which the boundary is viewed, the luminosity of the latter appears confined to certain regions lying in the neighbourhood of the two points $r=R$, $\theta=\theta'$, and $\theta=\pi+\theta'$, and which are more or less well-defined according to the size and the position of the aperture. The further the aperture is from the centre of the focal plane, the feebler is the luminosity observed through it. Accordingly, if we regard the horizontal slit placed in the focal plane as consisting of a number of elements along its length, the vanishing of the luminosity of the boundary at the ends of a horizontal diameter is seen to follow as a consequence. At

TABLE II.

ϕ'/ψ.	$\sqrt{I}$.	I (× const. factor).	Calculated values of ϕ'/ψ for maximum or minimum.	Observed values of ϕ'/ψ for maximum or minimum.
0·880	−·0536	2,873	0·885 min.	
0·890	−·1257	15,801	0·892 max.	0·894
0·900	−·0804	6,464	0·904 min.	0·901
0·910	+·0908	8,245		
0·920	+·2179	47,481	0·922 max.	0·930
0·930	+·1543	23,808		
0·940	−·0779	6,068	0·935 min.	0·940
0·950	−·3149	99,168	0·953 max.	0·957
0·960	−·2761	76,231		
0·970	+·0599	3,588	0·971 min.	0·971
0·975	+·2423	58,709		
0·980	+·3779	142,809		
0·985	+·4268	182,158	0·983 max.	0·985
0·990	+·3548	125,883		
0·995	+·2244	50,355		
1·000	+·0053	28	1·000 min.	1·000
1·005	−·2134	45,539		
1·010	−·3448	118,889		
1·015	−·4158	172,889	1·017 max.	1·015
1·020	−·3673	134,908		
1·025	−·2209	48,797		
1·030	−·0491	2,411	1·028 min.	1·030
1·040	+·2866	82,139		
1·050	+·3251	105,690	1·047 max.	1·046
1·060	+·0875	7,656	1·063 min.	1·061
1·070	−·1478	21,844		
1·080	−·2141	45,839	1·077 max.	1·078
1·090	−·0859	7,379		
1·100	+·0854	7,293	1·095 min.	1·096
1·110	+·1354	18,333	1·107 max.	1·109
1·120	+·0596	3,552	1·125 min.	1·132

Fig. 2.



and near these points also the radial width of the luminosity should obviously be the least, as the latter is, roughly speaking, in inverse proportion to the corresponding radial width of the aperture in the focal plane. This gives us a qualitative explanation of the lunette-shaped form of the fringes in these cases. Figs. 5, 8, 11, & 14 in Pl. III. illustrate the remarks made above regarding the localization of the luminosity of the boundary observed in certain cases. Fig. 5 represents the effect observed when there were two small circular apertures in the focal plane not lying on the same radius vector. Accordingly we have on the boundary four separate regions of luminosity. Fig. 8 represents a photograph obtained when a ring of six circular holes was placed symmetrically in the focal plane. Each of the six spots seen along the boundary is crossed by very fine fringes, due to the interference of the effects produced by the pair of apertures at the end of each diameter. Fig. 14 was obtained when the ring of holes was slightly displaced in the focal plane. Twelve spots appear on the boundary. Fig. 11 represents the effect observed when the screen in the focal plane was so placed that two out of the three pairs of apertures fell on lines passing through the centre of the field. Accordingly only eight spots are seen, the four larger ones being crossed by fine interference fringes.

3. *Case of the Rectangular Boundary.*

It has been shown by Lord Rayleigh* that when an optical surface bounded by parallel straight edges and illuminated by a linear source of light is examined by the "knife-edge" test, the intensity of the field as viewed in the direction ϕ is given by

$$\begin{aligned} \mathrm{I} = \Big[&\mathrm{Si}\Big\{\frac{2\pi}{\lambda}(\theta+\phi)\xi_2\Big\} - \mathrm{Si}\Big\{\frac{2\pi}{\lambda}(\theta+\phi)\xi_1\Big\} \\ &+ \mathrm{Si}\Big\{\frac{2\pi}{\lambda}(\theta-\phi)\xi_2\Big\} - \mathrm{Si}\Big\{\frac{2\pi}{\lambda}(\theta-\phi)\xi_1\Big\}\Big]^2 \\ + \Big[&\mathrm{Ci}\Big\{\frac{2\pi}{\lambda}(\theta-\phi)\xi_2\Big\} - \mathrm{Ci}\Big\{\frac{2\pi}{\lambda}(\theta-\phi)\xi_1\Big\} \\ &- \mathrm{Ci}\Big\{\frac{2\pi}{\lambda}(\theta+\phi)\xi_2\Big\} + \mathrm{Ci}\Big\{\frac{2\pi}{\lambda}(\theta+\phi)\xi_1\Big\}\Big]^2, \end{aligned}$$

where θ is the angular semi-aperture of the lens, ξ_1 denotes

---

* *Loc. cit.*

the extent to which the knife-edge has been advanced in the focal plane beyond the centre of the field, and ξ_2 defines an upper limit for the aperture in the focal plane. If $\xi_1=0$ and ξ_2 is very large, this expression has the value

$$\mathrm{I} = \tfrac{1}{4}\pi + \left(\log\frac{2\pi\theta}{\lambda}\xi_2\right)^2$$

at the boundaries $\phi/\theta=\pm1$, and becomes logarithmically infinite with ξ_2. If ξ_1 and ξ_2 are both finite, the intensity at the boundaries is given by

$$\mathrm{I} = \left[\log\frac{\xi_2}{\xi_1} - \mathrm{Ci}\frac{4\pi\theta}{\lambda}\xi_2 + \mathrm{Ci}\frac{4\pi\theta}{\lambda}\xi_1\right]^2 + \left[\mathrm{Si}\frac{4\pi\theta}{\lambda}\xi_2 - \mathrm{Si}\frac{4\pi\theta}{\lambda}\xi_1\right]^2,$$

which is also very large compared with the intensity of the other parts of the field.

Fig. 23 (Pl. IV.) represents the luminosity observed at the edges of a rectangular diffracting aperture in Foucault's test. In taking this photograph, ξ_1 was small and ξ_2 large. The luminosity accordingly appears highly condensed at the edges. Fig. 20 reproduces a photograph obtained when ξ_1, ξ_2 did not differ very considerably. Diffraction fringes are clearly seen on either side of the boundary in this case. Fig. 17 reproduces a photograph of the aperture obtained with two parallel slits in the focal plane on the same side. It will be noticed that the central fringe which coincides with each boundary is *white*. Figs. 21 and 24 represent photographs obtained when the central band and a few fringes on either side of the diffraction-pattern at the focal plane were cut off by a wire parallel to the edges of the aperture. It will be observed that the positions of the boundaries in these two photographs appear as *fine black lines* with luminous bands on either side. The same feature, but with the dark lines at the boundaries much broader, is shown in figs. 18 & 25, which were secured by placing two parallel slits symmetrically in the focal plane—that is, one on either side of the centre of the field.

We proceed to consider the explanation of the black lines marking the positions of the boundaries in the four photographs mentioned in the preceding paragraph. In the focal plane we have two apertures extending from ξ_1 to ξ_2 and from $-\xi_1$ to $-\xi_2$ respectively. On account of the symmetry, the Ci-functions disappear from the expression for the intensity of the field as viewed in the direction ϕ, which may be written in the form

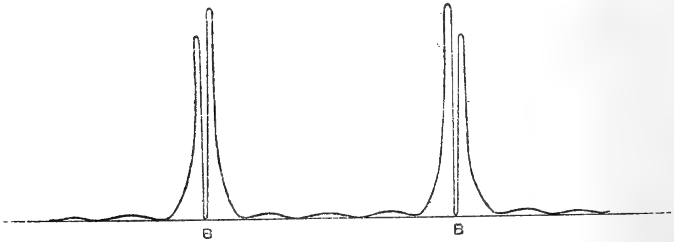
$$I = 4\left[\mathrm{Si}\left\{\frac{2\pi\theta}{\lambda}\left(1+\frac{\phi}{\theta}\right)\xi_2\right\} - \mathrm{Si}\left\{\frac{2\pi\theta}{\lambda}\left(1+\frac{\phi}{\theta}\right)\xi_1\right\} + \mathrm{Si}\left\{\frac{2\pi\theta}{\lambda}\left(1-\frac{\phi}{\theta}\right)\xi_2\right\} - \mathrm{Si}\left\{\frac{2\pi\theta}{\lambda}\left(1-\frac{\phi}{\theta}\right)\xi_1\right\}\right]^2,$$

where θ is the angular semi-aperture of the lens. When $\phi/\theta = \pm 1$—that is, at the boundaries—this expression becomes very small, but the suddenness with which the illumination falls to zero at these points depends very much on the magnitudes of ξ_1 and ξ_2. To illustrate this statement, I have calculated the distribution of intensity for a hypothetical case in which $\xi_1 = \frac{\lambda}{\theta}$ and $\xi_2 = \frac{50\lambda}{\theta}$. The values are shown in Table III. (in which the factor 4 in the expression for the intensity has been neglected), and these have been plotted in fig. 3.

TABLE III.

ϕ/θ.	I.	ϕ/θ.	I.
1·000	0·0045	1·000	0·0045
1·002	1·7956	0·998	2·2201
1·005	1·5129	0·995	1·8496
1·010	2·9584	0·990	3·4225
1·020	1·5376	0·980	2·1609
1·030	2·0164	0·970	2·4025
1·100	0·8281	0·900	1·0050
1·300	0·0020	0·700	0·0000
1·500	0·0605	0·500	0·0357
1·800	0·0000	0·200	0·0057
2·000	0·0182	0·000	0·0853

Fig. 3.



Another case, in which the disparity between ξ_1 and ξ_2 was much smaller, was chosen for experimental verification of the position of the diffraction maxima and minima given by theory. It is found that the illumination falls off to

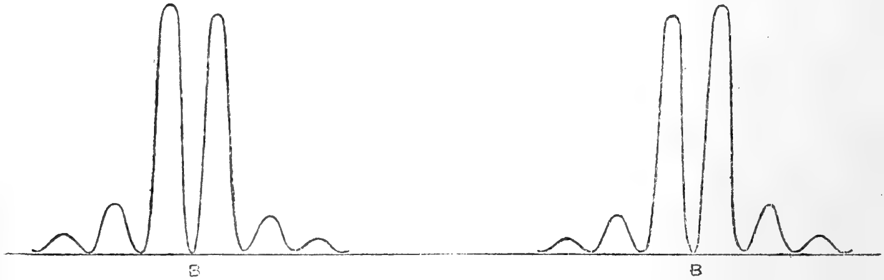
TABLE IV.

ξ_1=·654 mm., ξ_2=·963 mm., f=13·4 cm.,
λ=·00045 mm.,
width of the aperture =0·955 mm.

ϕ/θ.	$\sqrt{I}$.	I (×const. factor).	Calculated values of ϕ/θ for maximum or minimum.	Observed values of ϕ/θ for maximum or minimum.
·760	−·0002	0	·760 min.	·756
·780	+·1468	204		
·800	+·2123	449	·800 max.	·802
·820	+·1463	201		
·840	−·0018	0	·840 min.	·843
·860	−·1601	256		
·880	−·3084	957	·888 max.	·882
·900	−·2369	559		
·920	−·0065	0	·923 min.	·927
·930	+·1337	179		
·940	+·2386	567		
·950	+·3416	1167		
·960	+·6707	4498	·960 max.	·961
·970	+·3431	1177		
·980	+·2398	575		
·990	+·1350	182		
·995	+·0620	38		
1·000	−·0145	2	1·000 min.	1·000
1·005	−·0918	84		
1·010	−·1528	233		
1·020	−·2587	669		
1·030	−·3701	1370		
1·040	−·6898	4758	1·04 max.	1·038
1·050	−·3648	1331		
1·060	−·2517	634		
1·070	−·1513	229		
1·080	−·0059	0	1·080 min.	1·076
1·100	+·2431	589		
1·120	+·3196	1021	1·114 max.	1·115
1·140	+·1713	293		
1·160	−·0024	0	1·16 min.	1·157
1·180	−·1642	270		
1·200	−·2128	453	1·198 max.	1·205
1·220	−·1478	218		
1·240	−·0002	0	1·24 min.	1·238

practically zero-value on the boundaries, but much less suddenly than in the case previously discussed. The experimental data, the calculated intensities of the illumination, and the theoretical and the experimentally observed positions of maxima and minima are shown in Table IV. The agreement is fairly satisfactory. For comparison with the preceding case, the illumination curve has been plotted in fig. 4. As in the case of the circular boundary, the minima of illumination are absolute zeros. It will be noticed also that $\sqrt{I}$ changes sign as it passes through its value at the boundary ($\phi/\theta = \pm 1$), showing that the radiations emitted by the edge on the two sides of the boundary differ in phase by π.

Fig. 4.



4. *Other Forms of Boundary.*

The cases in which the surface is bounded by forms of apertures other than those considered previously are of interest from the point of view of the general theory of diffraction. Figs. 19 and 22 (Pl. IV.) represent photographs of the effect observed when the surface is bounded by quadrilateral and triangular apertures respectively. These photographs were obtained using a point source of light and an annular aperture placed symmetrically in the focal plane. It will be noticed that the boundaries appear as black lines with luminous fringes on either side, thus showing a complete analogy with the case of the circular and rectangular boundaries previously considered.

In order more fully to study the luminosity at the boundaries of the triangular, quadrilateral, and other forms of aperture, an arrangement was devised in which a screen containing a small circular hole could be placed excentrically in the focal plane and rotated in this plane. This hole comes successively over different parts of the diffraction-pattern formed at the focus, and the luminosity at the boundaries observed through it undergoes a series of changes. For instance, with a triangular aperture, it is known that the diffraction-pattern at the focus consists of a six-rayed "star," the "rays" being perpendicular to the three sides of the triangle respectively. When the hole comes over any one of the rays the corresponding boundary appears luminous, but in other cases it becomes practically invisible. Similarly, with a quadrilateral aperture, the diffraction-pattern is a "star" with eight rays perpendicular to its four sides, and each of these appears luminous when the excentrically-placed hole in the focal plane comes over the corresponding ray of the pattern.

Whether any particular part of the boundary appears luminous or not seems in general to depend on the normal to the boundary at that point being parallel to the radius vector from the centre of the focal plane to the aperture in the screen through which it is viewed. This is stated here as an experimental fact, the detailed mathematical explanation of which is deferred till a future occasion. An interesting illustration of its generality is furnished by the observation that minute irregularities on the boundary often appear luminous when the adjoining parts which are straight are invisible from any given point in the focal plane. A discussion of the cases in which the boundary is a complicated figure such as a grating or a series of parallel apertures is also reserved for a future occasion.

5. *On the Flow of Energy in a Diffraction Field.*

The phenomena described in the preceding sections suggest two important problems for study. In the ordinary Fresnel-Kirchhoff treatment of diffraction problems, the disturbance at any point of the field is expressed as an integral taken over a surface bounded by the diffracting aperture. It is a subject for investigation whether, in any circumstances, the surface integral can be resolved either wholly or partially to a line integral taken over the boundary, and whether the disturbance in the region of shadow could be expressed practically in terms of the line-integral alone. Another interesting problem which also suggests itself for investigation is the determination of the forms of the lines of flow of energy in the optical field due to rectangular or circular boundaries in convergent light. The shapes of these lines of flow of energy would be specially interesting in the neighbourhood of the focus. As a first step towards a detailed study of these problems, a few experimental observations have been made by placing a narrow screen (such as a needle or a plate of very small dimensions) or else a wide screen containing a narrow aperture in some selected part of the field and tracing the phenomena observed in its rear. Some very striking results have been obtained, especially with a circular aperture illuminated by a point source of light. If a small screen is placed in the focal plane so as to cut off the entire geometrical cone of rays, a bright image of the source may be traced along the axis behind the screen and for a considerable distance beyond. This effect is entirely due to the light diffracted by the boundary of the circular aperture*. With a narrow screen placed in the diffraction field due to a rectangular aperture illuminated by a linear source of light, two independent shadows each bordered by diffraction fringes and differing in intensity have been observed behind the screen, these being formed respectively by the two luminous edges of the aperture. If the narrow screen be placed in the region of shadow close to the geometrical pencil of rays, the two shadows differ considerably in their intensity, and the difference becomes less and less as the screen is moved more and more into the region of shadow. The writer hopes to take up the fuller study of these problems at an early opportunity.

The investigation described in this paper was carried out in the Laboratory of the Indian Association for the Cultivation of Science. The writer has much pleasure in gratefully acknowledging the helpful interest taken by Prof. C. V. Raman during the progress of the work.

Calcutta,
February 8th, 1918.

---

* Porter and Hufford have about the same time observed that the rays diffracted by a circular disk can form an optical image of the source along the axis of symmetry (see Phil. Mag. and Phys. Rev. April 1913). The phenomenon observed by me is somewhat analogous, but differs from that observed by these writers, as the image in this case is formed by the rays diffracted by the boundary of a circular *aperture*.

IX. *Note on the Effects of Grid Currents in Three-Electrode Ionic Tubes.* *By* E. V. Appleton, *M.A.*, *B.Sc.*, *St. John's College, Cambridge* *.

In the interpretation of the functioning of three-electrode ionic tubes, which are now used so extensively in the reception of wireless signals, we have to consider simultaneous effects in two electrical circuits, which may be defined as follows:—

(*a*) The grid or input circuit, which is completed inside the tube by the space between the incandescent filament and the perforated grid, and
(*b*) the plate or output circuit, which is completed inside the tube by the space between the filament and the metal plate.

The variables to be considered are thus as follows:—Plate current, plate voltage, grid current, grid voltage, and filament temperature, the values of which may be represented by the letters I, V, i, v, and θ respectively. The complete working of the tube can only be interpreted when the static characteristic surfaces representing the relations between any three of the above quantities are known. In the case of a tube acting as a relay or amplifier, the main surface to be considered is that represented by the equation $f(\mathrm{I}, \mathrm{V}, v)=0$. Vallauri † has shown how this may be expressed approximately as $\mathrm{I}=av+b\mathrm{V}+c$, where a, b, and c are constants, and has applied it to the elucidation of many wireless circuits. Hazeltine ‡ has recently made substantial additions to that part of Vallauri's work dealing with oscillating audion circuits. In both cases the effects of the currents flowing in the grid circuit were neglected.

The magnitude of the grid currents involved is known if we know the contour of the (i, v, V) surface. In general, this surface cannot be represented by a simple expression, but for small changes of the quantities concerned we may write

$$\Delta i=\kappa_1\Delta v+\kappa_2\Delta \mathrm{V}, \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

where κ_1 is the slope of the $(v, i)_{\mathrm{V}=\mathrm{V}_0}$ curve and κ_2 is the slope of the $(\mathrm{V}, i)_{v=v_0}$ curve.

The quantity κ_1 may be either positive or zero in a tube of

---

* Communicated by the Author.

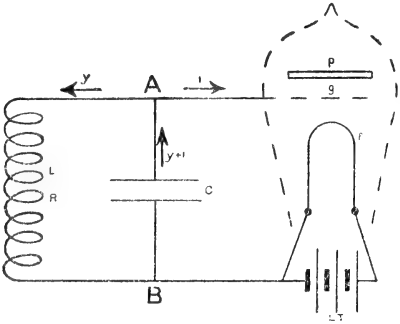
† G. Vallauri, *L'Elettrotecnica,* iv. 3, p. 43 (1917).

‡ L. A. Hazeltine, Proc. Inst. Rad. Eng., April 1918.

extreme exhaustion, but may be positive or negative in a tube possessing an appreciable amount of gas. The quantity κ_2 is usually negative and small numerically compared with κ_1. As an approximation we may therefore regard the grid circuit inside the tube as possessing a conductance κ_1, which may be either positive or negative.

In the use of the tube as an amplifier or rectifier of high frequency oscillations, the closed oscillatory circuit of the receiver is connected between the grid and the filament (see fig. 1), and the conductance κ_1 thus acts as a leak to the condenser.

Fig. 1.



p, plate; *g*, grid; *f*, filament; L.T., filament heating battery.

If y represents the current through the inductance and E the potential difference between the points A and B, we have

$$y=\frac{E}{R}-\frac{L}{R}\frac{dy}{dt},$$

$$y+i=-C\frac{dE}{dt},$$

$$i=\kappa_1 E.$$

Eliminating y and i we get

$$\frac{d^2E}{dt^2}+\left(\frac{R}{L}+\frac{\kappa_1}{C}\right)\frac{dE}{dt}+\frac{E}{LC}(1+R\kappa_1)=0,$$

the solution of which is oscillatory when

$$\left(\frac{R}{L}-\frac{\kappa_1}{C}\right)^2-\frac{4}{LC}<0,$$

and is given by

$$E = Ae^{-\frac{1}{2}\left(\frac{R}{L}-\frac{\kappa_1}{C}\right)t} \cos\left(\sqrt{\frac{1}{LC}-\left(\frac{R}{2L}-\frac{\kappa_1}{2C}\right)^2}\, t\right), \quad . \quad (2)$$

which represents a damped oscillation of frequency n given by

$$n = \frac{1}{2\pi}\sqrt{\frac{1}{LC}-\left(\frac{R}{2L}-\frac{\kappa_1}{2C}\right)^2}, \quad . \quad . \quad . \quad . \quad (3)$$

and where the damping factor α is given by

$$\alpha = \frac{1}{2}\left(\frac{R}{L}+\frac{\kappa_1}{C}\right). \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (4)$$

These equations show that the damping factor of a free oscillation is altered by the presence of the conductance, being increased or decreased according as the conductance is positive or negative.

Let us consider the case where the conductance is positive. For any given frequency there is a particular division of inductance and capacity which produces a minimum damping factor. This can obviously be obtained by finding when α in (4) is a minimum. Now, let $\frac{1}{LC}=\omega^2$, where $\omega=2\pi n$ and is therefore approximately constant. After substituting for L, the value of capacity which gives minimum damping is found to be $\frac{1}{\omega}\sqrt{\frac{\kappa_1}{R}}$.

Thus for a wave-length of λ cm. the particular value of capacity producing minimum damping, and therefore maximum electrostatic effect on the grid, is given by

$$C = \frac{\lambda}{6\pi}\sqrt{\frac{\kappa_1}{R}} \times 10^{-10} \text{ farad.} \quad . \quad . \quad . \quad . \quad (5)$$

This result is best illustrated by the consideration of a typical case where the conductance is positive for the particular operating point considered.

Suppose $R=4$ ohms,

$L=10^{-4}$ henry,

$C=2\times10^{-9}$ farad,

* $\kappa_1=5\times10^{-6}$ mho.

---

* See Bown, Phys. Rev. 10, pp. 253–265.

For various arrangements of inductance and capacity consistent with the relation LC=const., the values of frequency as calculated from (3) are practically identical. Thus we may tabulate values of L, C, and α as follows :—

L microhenries.	C microfarads.	α ohms per henry.
800	·00025	$12{\cdot}5\times10^3$
600	·00033	$10{\cdot}83\times10^3$
400	·00050	10×10^3
300	·000666	$10{\cdot}415\times10^3$
200	·00100	$12{\cdot}5\times10^3$
100	·00200	$21{\cdot}25\times10^3$
50	·00400	$40{\cdot}625\times10^3$
25	·00800	$80{\cdot}31\times10^3$

From these figures it is obvious that there is a minimum value of damping in the neighbourhood of ·0005 mfd. Substituting in the formula (5) gives this value exactly as ·0005 mfd.

Thus in a case with no conductance damping (*e. g.* with a hard tube and applied negative grid potentials) minimum damping is obtained when C is as small as possible, but where grid currents are appreciable maximum control effects are obtained by using capacity values given by equation (5).

When the conductance is negative (*e. g.* in a soft tube with applied negative grid potentials) the natural damping of the oscillatory circuit is in general reduced, and for capacity values below a certain amount is negative, bringing about the possibility of free electrical oscillations. For zero damping we must have κ_1 negative and $\frac{R}{L}$ numerically equal to $\frac{\kappa_1}{C}$. For a definite wave-length λ this is obtained when C is numerically equal to $\frac{\lambda}{6\pi}\sqrt{\frac{\kappa_1}{R}}\times10^{-10}$ farad. To illustrate this we may again consider the numerical example, taking κ_1 as $-\,002\times10^{-3}$ mho.* In this case for capacities below $3{\cdot}16\times10^{-4}$ mfd. persistent oscillations are obtained in the control oscillatory circuit, the usual retroactive action from the plate circuit being unnecessary.

The magnitude of the effects of the grid conductance can

---

* *E. g.* Bown, *loc. cit.*

be further illustrated by considering the amplitude reached by an impressed oscillation. Assume a sinusoidal E.M.F. impressed on the inductance L. We then have

$$L\frac{dy}{dt}+Ry-v=-E_0\sin\omega t,$$

$$y+i=-C\frac{dE}{dt},$$

$$i=\kappa_1 E.$$

These equations yield a solution

$$v=V\sin(\omega t+\phi), \quad \ldots \ldots \quad (6)$$

where

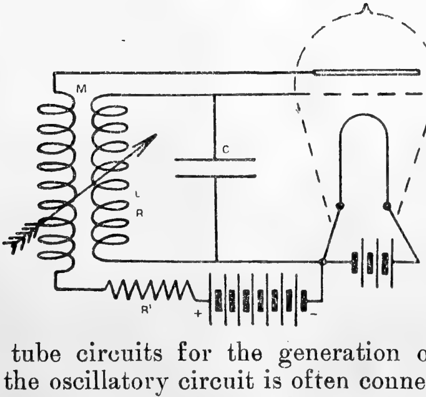
$$V=\frac{E_0}{\omega C\sqrt{\left(R+\frac{L}{C}\kappa_1\right)^2+\left(\omega L-\frac{1+\kappa_1 R}{\omega C}\right)^2}}. \quad . \quad (7)$$

From this it is clear that a positive grid conductance tends to lower the effective electrostatic action of the grid at resonance frequencies, but that with a negative conductance amplification may occur. In the case of positive conductance the maximum control action is again obtained when

$$C=\frac{\lambda}{2\pi v}\sqrt{\frac{\kappa_1}{R}}.$$

In the case of an extremely hard valve the conductance is positive for positive grid potentials and zero for all negative values. Thus for an impressed oscillation the amplitude reached for positive half oscillations is less than that reached during negative halves. The difference can be got definitely from (7), and gives a measure of the rectification obtained.

Fig. 2.



In ionic tube circuits for the generation of continuous oscillations the oscillatory circuit is often connected between the grid and the filament. A typical case is depicted in fig. 2.

It can be shown that, neglecting grid conductance, persistent oscillations are produced when M, the mutual inductance, is given by the equation

$$aM+\frac{bM^2}{L}+RC(1+bR')=0. \quad . \quad . \quad . \quad . \quad (8)$$

If we take into account the grid conductance we must substitute for R the effective resistance value αL (see (4)). Thus (8) becomes

$$aM+\frac{bM^2}{L}+(CR+L\kappa_1)(1+bR')=0. \quad . \quad . \quad (9)$$

Summary.

The effects of grid currents in a three-electrode vacuum tube are discussed, treating the conductance of the grid circuit inside the tube as a high-resistance leak across the condenser of the oscillatory circuit connected to the grid and filament. The effect of this leak in amplifying and oscillation circuits is investigated quantitatively.

X. *On the Lags of Thermometers with Spherical and Cylindrical Bulbs in a Medium whose Temperature is changing at a Constant Rate.*

By A. R. McLeod, *M.A.*, 1851 *Exhibition Science Research Scholar of the University of Toronto, and Wollaston Scholar of Gonville and Caius College, Cambridge*[*].

Summary.

THE lag of a thermometer, situated in a medium whose temperature is changing, is of practical interest in the case of ascending aeroplanes and balloons. The medium in this case is the atmosphere; but application might also be made to the case where the variable temperature of any other medium, say a liquid, is to be determined by the thermometer as a function of the time. In the following paper we suppose the temperature of the medium to change uniformly. Expressions are given for the steady values of the lags for spherical and cylindrical bulbs, when the surface conductivity is finite; and for the lags at any instant when the surface conductivity is infinite.

The lag of the mean temperature of the bulb behind that

---

[*] Communicated by Dr. G. F. C. Searle, F.R.S.

of the medium is, when steady, of the form $MG\rho\sigma/K$, where G is the constant rate of change of temperature of the medium, ρ is the density, σ the specific heat, and K the thermal conductivity of the substance in the thermometer bulb. M is a numerical constant whose value depends on the shape and dimensions of the bulb.

It should be noted that for a given dilatation of the liquid filling the bulb, the lag in a uniform gradient varies almost inversely as the conductivity, since the product $\rho\sigma$ is approximately constant for most substances used in thermometers.

Infinite surface conductivity implies, in the case of a cooling bulb, that the medium can carry away heat from the surface as fast as or faster than it can arrive there. If the fluid medium in which the bulb is situated has a low conductivity, as is the case with air, this can only be attained by a sufficiently rapid movement of the medium past the bulb. For air, this implies a much greater velocity relative to the surface of the bulb than for liquids (of the order of 3000 times for considerable temperature gradients in the bulb). When the surface conductivity is finite, there is a definite, discontinuous change in the temperature at the surface of the bulb. The expression for this temperature difference is of the form $NG\rho\sigma/h$, where h is the surface conductivity, and N is a constant of the same nature as M.

Radiation, and convection in the liquid in the bulb of the thermometer, are neglected; and no attempt has been made to correct for the exposed stem or for the glass wall of the bulb.

Numerical results are given for mercury and alcohol, in spherical and cylindrical bulbs, on a descending aeroplane. The cylindrical bulbs may, in practice, be wound in the form of a flat spiral. These calculated results show that for spherical bulbs of mercury and alcohol giving the same volume expansion, the lags are about the same.

With the same thermometric substance, the lag is less for cylinders 10 cm. long than for spheres giving the same volume expansion, and it is less for a cylinder of alcohol than for a cylinder of mercury.

1. *Lag for a Sphere with Variable Surface Temperature and Infinite Surface Conductivity.*

Let u be the temperature at a distance r from the centre of the sphere, and let t be the time. The equation to be satisfied is :

$$\frac{\partial u}{\partial t} = a^2\left(\frac{\partial^2 u}{\partial r^2} + \frac{2}{r}\frac{\partial u}{\partial r}\right), \quad \ldots \ldots \quad (1)$$

the temperature being supposed to depend only on r. The value of a^2 is $K/\rho\sigma$, where K is the thermal conductivity, ρ the density, and σ the specific heat of the substance in the sphere.

Writing $v = ru$, we get :

$$\frac{\partial v}{\partial t} = a^2\frac{\partial^2 v}{\partial r^2}. \quad \ldots \ldots \ldots \quad (2)$$

Since $u \neq 0$ when $r=0$, we must have $v=0$ for $r=0$.

To get the temperature as a function of r and t, we require a solution of (2) which satisfies the conditions

$$\begin{aligned} v &= 0 \quad \text{for } t = 0, \\ v &= 0 \quad \text{for } r = 0, \\ v &= \phi(t) \text{ for } r = c = \text{radius of sphere.} \end{aligned}$$

Such a solution is (Weber, *Partielle Differential-Gleichungen*, vol. ii. § 48) :

$$v = -\frac{2\pi a^2}{c^2}\sum_{n=1}^{\infty}(-1)^n n \sin\frac{n\pi r}{c}\int_0^t e^{-n^2\lambda^2(t-\tau)}\phi(\tau)\,d\tau,$$

where

$$\lambda = \frac{\pi^2 a^2}{c^2} = \frac{\pi^2 K}{c^2\rho\sigma}.$$

Suppose $\phi(t) = Gct$, where G is a constant. Then, on integration,

$$u = \frac{2Gc}{\pi r}\sum_{n=1}^{\infty}\frac{(-1)^{n+1}\sin(n\pi r/c)}{n}\left\{t + \frac{1}{n^2\lambda}\left(e^{-n^2\lambda t} - 1\right)\right\}.$$

The mean temperature is given by

$$\bar{u} = \frac{3}{c^3}\int_0^c r^2 u\,dr = \frac{6G}{\pi^2}\sum_{n=1}^{\infty}\frac{1}{n^2}\left\{t + \frac{1}{n^2\lambda}\left(e^{-n^2\lambda t} - 1\right)\right\}.$$

Since $\Sigma\, 1/n^2 = \pi^2/6$ and $\Sigma\, 1/n^4 = \pi^4/90$, this becomes

$$\bar{u} = Gt - \frac{\pi^2 G}{15\lambda} + \frac{6G}{\pi^2\lambda}\sum_{n=1}^{\infty} n^{-4}e^{-n^2\lambda t}. \quad \ldots \quad (3)$$

The lag of a thermometer, whose spherical bulb is subject to the conditions just considered, is therefore

$$\frac{\pi^2 G}{15\lambda}-\frac{6G}{\pi^2\lambda}\sum_{n=1}^{\infty} n^{-4}e^{-n^2\lambda t}. \quad . \quad . \quad . \quad . \quad (4)$$

This approaches a steady value, L, as the time t increases, which is

$$L=\frac{\pi^2 G}{15\lambda}=\frac{Gc^2}{15a^2}=\frac{Gc^2\rho\sigma}{15K}. \quad . \quad . \quad . \quad . \quad . \quad (5)$$

The steady lag is thus determinate, for a given gradient G, when λ is known. We can measure λ experimentally by observing the rate of cooling of the spherical bulb. For suppose the temperature of the medium is zero, and the initial distribution of temperature in the sphere is given by $v=\phi(r)$. We neglect the effect of the glass wall, if we are dealing with a liquid-in-glass thermometer, and suppose the surface conductivity to be infinite as before. Then, to get the temperature in the cooling bulb at any instant, we require a solution of (2) which satisfies the conditions

$$\begin{aligned} v &= \phi(r) && \text{for } t = 0, \\ v &= 0 && \text{for } r = 0, \\ v &= 0 && \text{for } r = c = \text{radius of sphere.} \end{aligned}$$

Such a solution is (Weber, vol. ii. § 46):

$$v=\frac{2}{c}\sum_{n=1}^{\infty} e^{-n^2\lambda t}\sin\frac{n\pi r}{c}\int_0^c \phi(\alpha)\sin(n\pi\alpha/c)\,d\alpha.$$

If the initial temperature is constant and equal to u_0, we have $\phi(r)=ru_0$. On substituting, and taking the mean temperature, we get, since $v=ru$,

$$\bar{u}=\frac{6u_0}{\pi^2}\sum_{n=1}^{\infty} n^{-2}e^{-n^2\lambda t}.$$

If t_1 is the time required for the mean temperature to drop from u_0 to fu_0, where f is some positive fraction, we have

$$f\pi^2/6=e^{-\lambda t_1}+\tfrac{1}{4}e^{-4\lambda t_1}+\tfrac{1}{9}e^{-9\lambda t_1}+\ldots \quad . \quad . \quad . \quad (6)$$

And λ may be found, by successive approximations, from this equation, one or two terms on the right sufficing for most cases.

2. *Lag for a Cylinder whose Surface Temperature is Variable, the Surface Conductivity being Infinite.*

Taking cylindrical coordinates, and neglecting the effects of the ends of the cylinder, whose length is supposed great in comparison with its diameter, the temperature u is a function only of r, the distance from the axis of the cylinder. The equation for u is then

$$\frac{1}{a^2}\frac{\partial u}{\partial t}=\frac{1}{r}\frac{\partial u}{\partial r}+\frac{\partial^2 u}{\partial r^2}. \quad \ldots \ldots \quad (7)$$

Writing $u=\mathrm{R}e^{-a^2\alpha_n^2 t}$, the equation reduces to

$$\frac{d^2\mathrm{R}}{dr^2}+\frac{1}{r}\frac{d\mathrm{R}}{dr}+\alpha_n^2\mathrm{R}=0.$$

Thus R need be a function of r only, and we have the following particular solution of (7) in terms of the Bessel Function of order zero:

$$u=e^{-a^2\alpha_n^2 t}\,\mathrm{J}_0(\alpha_n r).$$

The general solution of this type is thus

$$u=\sum_{n=1}^{\infty}\mathrm{A}_n e^{-a^2\alpha_n^2 t}\mathrm{J}_0(\alpha_n r). \quad \ldots \quad (8)$$

We must determine the constants A so that (8) shall satisfy the conditions

$$u=0 \quad \text{for } t=0,$$
$$u=\phi(t) \text{ for } r=c.$$

We know that

$$1=\sum_{n=1}^{\infty}\frac{2\mathrm{J}_0(\alpha_n r/c)}{\{\mathrm{J}_1(\alpha_n)\}^2}\int_0^1\xi\,\mathrm{J}_0(\alpha_n\xi)\partial\xi,$$

where α_n is the nth root of $\mathrm{J}_0(\alpha)=0$.

Hence, since the value of the integral is $\alpha_n^{-1}\mathrm{J}_1(\alpha_n)$,

$$1-\sum_{n=1}^{\infty}\frac{2\mathrm{J}_0(\alpha_n r/c)}{\alpha_n\mathrm{J}_1(\alpha_n)}=0\,;$$

and so

$$1-\sum_{n=1}^{\infty}\frac{2\mathrm{J}_0(\alpha_n r/c)}{\alpha_n\mathrm{J}_1(\alpha_n)}e^{-\frac{a^2\alpha_n^2}{c^2}t}$$

vanishes for $t=0$, and satisfies the differential equation (7).

Following a method of Riemann's, let us consider the function $\chi(r, t)$ defined by

$$\chi(r, t) = 0, \qquad t \leq 0,$$

$$\chi(r, t) = 1 - \sum_{n=1}^{\infty} \frac{2J_0(\alpha_n r/c) e^{-\frac{a^2\alpha_n^2}{c^2\alpha_n^2}t}}{\alpha_n J_1(\alpha_n)}, \qquad t > 0.$$

This function satisfies the differential equation (7) and vanishes for $t=0$. Further,

$$\chi(c, t) = 0, \qquad t \leq 0;$$

$$\chi(c, t) = 1, \qquad t > 0.$$

Take a series of small intervals $\xi_p = t_{p+1} - t_p$, defined by the fixed values of t,

$$0, \quad t_1, \quad t_2, \quad \dots, \quad t_p, \quad \dots .$$

Now consider

$$u_1 = \sum_{p=0}^{n-1} \phi(t_p)[\chi(r, t-t_p) - \chi(r, t-t_{p+1})].$$

We see that when $r=c$,

$$u_1 = \phi(t_p) \text{ for } t_p < t \leq t_{p+1}.$$

Hence in the limit, as $\xi \to 0$, we get, since u_1 satisfies (7), and vanishes for $t=0$, and in addition is equal to $\phi(t)$ at time t, when $r=c$,

$$u_1 = u = \int_0^t \phi(\tau) \frac{\partial \chi(r, t-\tau)}{\partial t} d\tau.$$

Performing the differentiation, we have the following solution for our problem:

$$u = \int_0^t \phi(\tau) \sum_{n=1}^{\infty} \frac{2J_0(\alpha_n r/c)}{\alpha_n J_1(\alpha_n)} \cdot \frac{a^2\alpha_n^2}{c^2} e^{-\frac{a^2\alpha_n^2}{c^2}(t-\tau)} d\tau$$

$$= \frac{2a^2}{c^2} \sum_{n=1}^{\infty} \frac{\alpha_n J_0(\alpha_n r/c)}{J_1(\alpha_n)} \int_0^t \phi(\tau) e^{-\frac{a^2\alpha_n^2}{c^2}(t-\tau)} d\tau.$$

Take $\varphi(t) = Gt$, where G is constant. Then we find, on integrating,

$$u = 2G \sum_{n=1}^{\infty} \frac{J_0(\alpha_n r/c)}{\alpha_n J_1(\alpha_n)} \left\{ t + \frac{c^2}{a^2\alpha_n^2} \left(e^{-\frac{a^2\alpha_n^2}{c^2}t} - 1 \right) \right\}. \quad . \quad (9)$$

The mean value of u is

$$\bar{u} = \frac{2}{c^2}\int_0^c ru\,dr = 4\mathrm{G}\sum_{n=1}^{\infty}\alpha_n^{-2}\left\{t+\frac{c^2}{a^2\alpha_n^2}\left(e^{-\frac{a^2\alpha_n^2}{c^2}t}-1\right)\right\}$$

$$= \mathrm{G}t - \frac{4\mathrm{G}c^2}{a^2}\sum_{n=1}^{\infty}\alpha_n^{-4} + \frac{4\mathrm{G}c^2}{a^2}\sum_{n=1}^{\infty}\alpha_n^{-4}e^{-\frac{a^2\alpha_n^2}{c^2}t} \quad . \; . \; . \qquad (10)$$

The lag is thus

$$\frac{4\mathrm{G}c^2}{a^2}\left[\sum_{n=1}^{\infty}\alpha_n^{-4} - \sum_{n=1}^{\infty}\alpha_n^{-4}e^{-\frac{a^2\alpha_n^2}{c^2}t}\right], \quad . \; . \; . \; . \; . \qquad (11)$$

and its steady value is :

$$\frac{4\mathrm{G}c^2}{a^2}\sum_{n=1}^{\infty}\alpha_n^{-4}.$$

Computing the series, we find for the sum of the first 10 terms the value ·031246 which is practically 1/32. Hence

$$\mathrm{L} = \frac{\mathrm{G}c^2}{8a^2} = \frac{\mathrm{G}c^2\rho\sigma}{8\mathrm{K}}. \quad . \; . \; . \; . \; . \; . \qquad (12)$$

We shall see in § 5, on writing $h=\infty$ in (20), that this value is rigorously correct ; and so

$$\sum_{n=1}^{\infty}\frac{1}{\alpha_n^4} = \frac{1}{32}.$$

The temperature of a cylinder with infinite surface conductivity, cooling from a uniform initial temperature u_0, the temperature of the surrounding medium being zero, is given by

$$u = 2u_0\sum_{n=1}^{\infty}e^{-\frac{a^2\alpha_n^2}{c^2}t}\,\frac{\mathrm{J}_0(\alpha_n r/c)}{\alpha_n\mathrm{J}_1(\alpha_n)}\,;$$

and the mean temperature is

$$\bar{u} = 4u_0\sum_{n=1}^{\infty}\alpha_n^{-2}e^{-\frac{a^2\alpha_n^2}{c^2}}.$$

This equation could be used for calculating the value of a^2 from the curve for the cooling of the cylinder.

From this last formula, putting $t=0$, we find

$$\sum_{n=1}^{\infty} \alpha_n^{-2} = 1/4.$$

3. *Numerical Conclusions for Mercury and Alcohol.*

For Mercury	$a^2 = \cdot 0437$	c.g.s. units.
For Alcohol (C_2H_5OH 100 °/ₒ) . .	$a^2 = \cdot 0009026$	,, ,,

Taking the same expansion (same thermometric scale) in each example, and reckoning the coefficient of expansion of alcohol as six times that of mercury, we have for the dimensions of equivalent bulbs :

Mercury Sphere	$c = 1$ cm.
Alcohol Sphere	$c = \cdot 550$ cm.
Mercury Cylinder, 10 cm. long . .	$c = \cdot 36$ cm.
Alcohol Cylinder, 10 cm. long . .	$c = \cdot 145$ cm.

Taking $G = \cdot 032°$ C. per second, which corresponds to a gradient of 1°·9 C. per thousand feet of height in the air, and a rate of descent of one thousand feet per minute, we have, taking the first exponential terms only, the following lags for the above bulbs :—

Lag of Mercury Sphere	$= \cdot 049 - \cdot 045\, e^{-\cdot 431 t}$.
Lag of Alcohol Sphere	$= \cdot 713 - \cdot 657\, e^{-\cdot 0285 t}$
Lag of Mercury Cylinder . .	$= \cdot 012 - \cdot 012\, e^{-1 \cdot 94 t}$.
Lag of Alcohol Cylinder . .	$= \cdot 095 - \cdot 089\, e^{-\cdot 248 t}$.

Here t is the time in seconds. These lags are not all zero when $t=0$, because first exponential terms alone have been taken. The other exponential terms, however, become very rapidly less and less important as t becomes appreciable. The above lags reach 95 per cent. of their steady values in the following times :—

Mercury Sphere	7 seconds.
Alcohol Sphere	103 ,,
Mercury Cylinder	1·5 ,,
Alcohol Cylinder	12 ,,

4. *Steady Lag for a Sphere in a Medium whose Temperature changes at a Uniform Rate, the Surface Conductivity being Finite.*

We require a solution of (1) which satisfies the conditions

$$u = 0, \text{ for } t = 0,$$

$$K\frac{\partial u}{\partial r} = h(Gt-u), \text{ for } r = c. \quad . \quad . \quad . \quad . \quad (13)$$

The temperature of the medium is Gt, and h is the (constant) surface conductivity. Let the lag at any point in the sphere distant r from the centre be θ, and write $u=Gt-\theta$ in (1). Then we get

$$G-\frac{\partial\theta}{\partial t} = a^2\left(-\frac{\partial^2\theta}{\partial r^2}-\frac{2}{r}\frac{\partial\theta}{\partial r}\right);$$

and for the steady lag $\partial\theta/\partial t=0$.

Hence the equation for the lag, when it has reached its steady value, is

$$\frac{d^2\theta}{dr^2}+\frac{2}{r}\frac{d\theta}{dr}+\frac{G}{a^2} = 0. \quad . \quad . \quad . \quad . \quad . \quad . \quad (14)$$

The solution of (14) is

$$\theta = A+\frac{B}{r}-\frac{Gr^2}{6a^2}. \quad . \quad . \quad . \quad . \quad . \quad . \quad (15)$$

Since θ is finite for $r=0$, we must have $B=0$.

The mean lag for the sphere is

$$\overline{\theta} = \frac{3}{c^3}\int_0^c\left(A-\frac{Gr^2}{6a^2}\right)r^2\,dr = A-\frac{Gc^2}{10a^2}. \quad . \quad . \quad (16)$$

If there is a temperature lag, θ_c, at the surface, it is given by

$$K\frac{\partial u}{\partial r} = h(Gt-u), \text{ where } r = c.$$

This becomes

$$-K\left(\frac{\partial\theta}{\partial r}\right)_{r=c} = h\theta_c. \quad . \quad . \quad . \quad . \quad . \quad . \quad (17)$$

Hence by (15), since $B=0$, we get

$$\frac{KGc}{3a^2} = h\theta_c,$$

or

$$\theta_c = \frac{KGc}{3ha^2} = A - \frac{Gc^2}{6a^2};$$

and so

$$A = \frac{Gc^2}{a^2}\left(\frac{K}{3ch}+\frac{1}{6}\right).$$

Thus from (16) we get the mean lag as follows:

$$\bar{\theta} = \frac{Gc^2}{15a^2}+\theta_c = \frac{Gc^2}{a^2}\left(\frac{1}{15}+\frac{K}{3ch}\right). \quad . \quad . \quad . \quad . \quad (18)$$

5. *Steady Lag for a Cylinder in a Medium whose Temperature changes at a Uniform Rate, the Surface Conductivity being Finite.*

Substituting $u=Gt-\theta$ in (7), we get the following equation for the steady lag, $\frac{\partial\theta}{\partial t}$ vanishing:

$$\frac{d^2\theta}{dr^2}+\frac{1}{r}\frac{d\theta}{dr}+\frac{G}{a^2} = 0,$$

which has the solution

$$\theta = A + B\log r - \frac{Gr}{4a^2}, \quad . \quad . \quad . \quad . \quad . \quad . \quad (19)$$

and, as before, $B=0$. The mean lag for the cylinder is found to be

$$\bar{\theta} = A - \frac{Gc^2}{8a^2}.$$

From (17) and (19) the surface lag is obtained:

$$\theta_c = \frac{KGc}{2ha^2} = A - \frac{Gc^2}{4a^2}.$$

Hence

$$A = \frac{Gc^2}{a^2}\left(\frac{1}{4}+\frac{K}{2hc}\right),$$

and

$$\bar{\theta} = \frac{Gc^2}{8a^2}+\theta_c = \frac{Gc^2}{a^2}\left(\frac{1}{8}+\frac{K}{2hc}\right). \quad . \quad (20)$$

6. *Numerical Results for Thermometers moving through the Air at Aeroplane Speeds.*

The following approximate expression for the surface conductivity, h, has been derived from experimental data, referring to the case of air moving with a uniform velocity of V miles per hour past the thermometer:

$$h = \cdot0000515\,\mathrm{V}.$$

This expression holds only for ordinary aeroplane speeds of, say, 60–100 miles per hour. Taking $\mathrm{G}=\cdot032^\circ$ C. per sec. and $\mathrm{V}=60$ m.h p., we have the following steady lags for the bulbs already considered (which have each the same volume expansion).

Table I.—Lags.

Shape and Dimension.	Internal lag.	Surface lag.	Total lag.
Mercury.... $\mathrm{K}=\cdot0197$, $a^2=\cdot0437$.			
Sphere, $c=1$ cm.......	$\frac{\mathrm{G}c^2}{5a^2}=\cdot05^\circ$	$\theta_c=1^\circ\cdot56$	1°·61
Cylinder, $c=\cdot36$ cm. Length $=10$ cm.	$\frac{\mathrm{G}c^2}{8a^2}=\cdot01^\circ$	$\theta_c=\cdot84^\circ$	·85°
Alcohol .... $\mathrm{K}=\cdot00043$, $a=\cdot0009026$.			
Sphere, $c=\cdot55$ cm. ...	$\frac{\mathrm{G}c^2}{15a^2}=\cdot71^\circ$	$\theta_c=\cdot91^\circ$	1°·62
Cylinder, $c=\cdot145$ cm. Length $=10$ cm.	$\frac{\mathrm{G}c^2}{8a^2}=\cdot09^\circ$	$\theta_c=\cdot36^\circ$	·45°

The thanks of the author are due to Dr. G. F. C. Searle for his kind assistance in revising this paper for publication, and in checking the formulæ.

XI. *Experimental Demonstration of the Constancy of Velocity of the Light emitted by a Moving Source.* *By* Q. MAJORANA, *Professor of Physics at the Polytechnic School of Turin* *.

IN a preceding paper† on the second postulate of the theory of relativity I described an experimental arrangement of mine by means of which I was able to demonstrate that light propagates itself with constant velocity, independently of the conditions of movement or rest of the mirror by which the light is reflected. At the end of the above-mentioned paper I hinted at my intention of studying experimentally the eventual influence of the movement of the source on the velocity of the propagation of the light; the object of the present note is to communicate the result of these researches.

As is known, the only studies made with luminous sources in motion are astronomical ones, and those with the canal rays. Particularly with the former it has been possible to deduce the measure of the Doppler effect (and therefore the value of the velocity of displacement) for different sources, such as the fixed stars or planets and the limb of the sun. I am not aware of any attempt to prove the Doppler effect with the artificial movement of a common luminous source; the difficulty of this research consists principally in the necessity for giving a specially high rate of velocity of displacement to the source.

But even if an arrangement of this kind could be realized, its interest does not lie in the verification of the Doppler effect (change of frequency), upon which no doubt any longer exists, so much as in the control of the value of the velocity of the propagation of the light, also in the case of a moving source. This is the reason why the examination of the latter must not be made either with prisms, as in the arrangement of Belopolski, or with diffraction-gratings, as I have before explained. In making my preparations to set up an apparatus with moving source, I resolved, from the first, to examine the latter with the interference method already described by me, which is founded on the use of the

---

* Communicated by the Author.

† Phil. Mag. Feb. 1918, p. 163; on the same argument see also the papers of Michelson, Astrophysical Journal, xlii. p. 19 (1913), and of Fabry and Buisson, *C. R.* clviii. p. 1438 (1914). These works, of which I heard only lately, arrive in different ways at the same conclusions.

Michelson interferometer, with a great difference in path. Admitting the second postulate of the theory of relativity, or if, in any way, the velocity of propagation of light by earthly sources seems to us unchangeable, supposing such an apparatus were realized, a certain number f of fringes would be seen to cross the wire of the eyepiece, when the source passes from rest to velocity v,

$$f = \frac{lv}{\lambda c},$$

where l is the difference of path of the interfering rays, λ the length of the wave, and c the velocity of light. This is analogous to my affirmation previously made with regard to moving mirrors. Now, the values which it is hoped may be attained in a laboratory for v are rather modest, even with regard to those of astronomical luminous sources; it is therefore necessary to give to l the highest possible value, which is only to be obtained by using a source of light with a very long visibility curve.

This can only be the green line of mercury, which, however, being more complex in its structure than the lines of cadmium, allows the observation of the fringes up to the value of $l = 32$ cm.*, using excitation of vapours by means of a voltaic arc in vacuum. Besides, this source is particularly suited for the present researches because of its exceptional intensity. I believe, therefore, that they would hardly be repeated, using a different source.

I established a new plan of experiments, intending to endow with swift rotatory movement some mercurial arcs held by airless glass tubes, and to examine by means of the Michelson interferometer the light emitted by them tangentially to the trajectory line. Now in the attaining of a peripheral velocity of nearly 100 m. per second, this being the necessary velocity for a sure appreciation of a displacement in the fringes, two principal mechanical difficulties are found: the enormous centrifugal force, and the very great resistance of air. To diminish the first, it is convenient to enlarge as much as possible the diameter of the trajectory and lessen the number

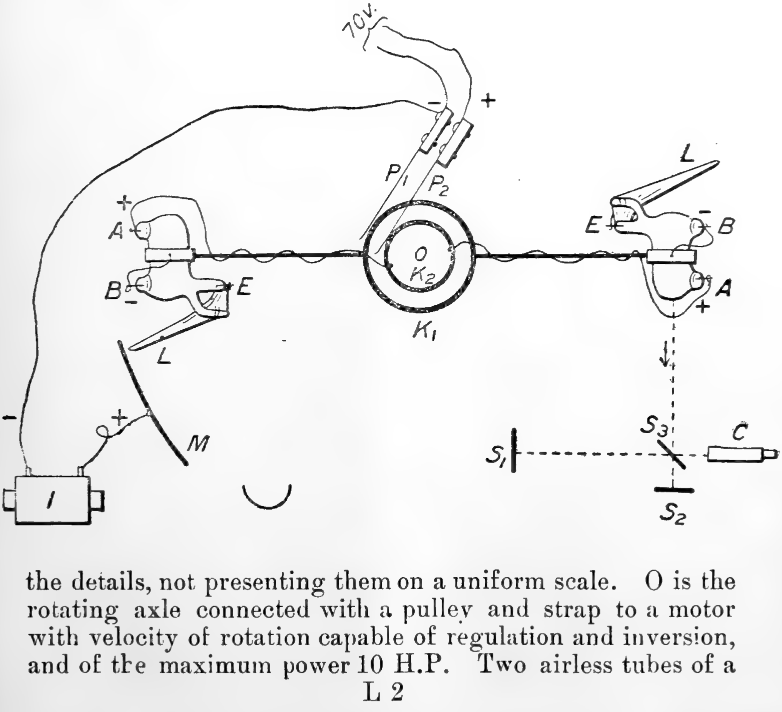
---

* It must be noted that Michelson observed fringes up to $l = 40$ cm. But that scientist used Geissler tubes with mercury vapour; it seems that the excitation with the voltaic arc in vacuum, used by me, changes the visibility curve.

per second of the revolutions. It is known that the centrifugal velocity increases as the square of that number, and the velocity of displacement grows as the first power. After different attempts I fixed the diameter at 2 m.: because since the small tubes with the mercury weigh 35 gr., the centrifugal force that excites them at a velocity of only 14 revolutions per second (corresponding to nearly 90 m. of peripheral velocity) amounts to 30 kg.

That is nearly the highest point of capacity of charge to which the glass material can be brought. In my experiments the glass tubes still broke very often, although allowing a sufficient interval of time for the observation and measures.

In respect to the resistance of air, it has been reduced to the minimum by using fine steel wires of high mechanical resistance as connexions between the tubes and the rotating axle. Notwithstanding the aforesaid conditions of velocity, the apparatus being provided with only two tubes in diametral position, a power of about 5 kw. was necessary. I give now a short description of it. The figure illustrates schematically



the details, not presenting them on a uniform scale. O is the rotating axle connected with a pulley and strap to a motor with velocity of rotation capable of regulation and inversion, and of the maximum power 10 H.P. Two airless tubes of a particular form (maximum diameter 4 cm.) bear each three electrodes A, B, E, and are joined by means of strong steel wires with the axle O. The latter is encircled by two collector-rings K_1, K_2 communicating with the electrodes A, B. Two brushes P_1, P_2 bring a continuous current at 70 v. A certain quantity of mercury is contained in each tube (28 gr.) and is distributed on A and B when the apparatus is rotating. The electrodes E are provided with light segments of aluminium L, which, when the tubes revolve, pass near a metallic arc M, which has the centrum O. An induction-coil I with its terminals connected as in the figure, placed in action for few seconds, gives discharges, which excite the tubes passing near M. In these the current can be regulated, by interposing the necessary resistance, to between 2 or 3 amperes. I must observe that it is often useless to employ the coil I after having started the rotation. Although in fact the quantity of mercury contained in each tube is absolutely insufficient to establish a continued metallic connexion between A and B, the excitement frequently happens spontaneously by reason of ionization of the remaining gas, caused by the mechanical shaking during rotation. Besides K_1 and K_2, the axle O is surrounded by a series of small metallic blocks (not visible in the figure) across which wipes a third brush. This apparatus, like an electric siren, allows a sound to be heard by means of the necessary connexion with a battery and a telephone, by which it is possible to deduce the value of the velocity of the rotation. The light emitted by the tubes is the highest, for constructive reasons, in the tangential direction of the movement. The Michelson interferometer is disposed as in the figure, and on it the light arrives parallel on the mirror S_2 by means of a lens not shown. With the telescope C it is possible to receive a luminous sensation, sufficiently intense in spite of its discontinuity. (20–30 spots per second.)

Thanks to a sufficient intensity of light, I can use (instead of the above-mentioned experiments with mirrors) a value of $l=232$ mm., by which I have observed a maximum of visibility of fringes. In these conditions, and giving to the apparatus a velocity of from 10 to 14 rotations a second, one perceives a displacement of the fringes when the velocity *passes from one part to the other*. This displacement observed with an eyepiece with micrometer has really the direction that is demanded by the principle of constancy of velocity of propagation of light. Let us foresee its value on this basis. During a long series of observations there is an average:

$v=79{\cdot}77$ m./sec.; $l=232$ mm., $\lambda=0{\cdot}546\ \mu$; so that a displacement is foreseen of

$$f=\frac{232\,.\,79{\cdot}77}{546{\cdot}3}10^{-2}=0{\cdot}113,\text{ and } 2f=0{\cdot}226.$$

Experimentally, I have been able to augment considerably the exactitude of observation of the fringes; and this because of the higher luminosity of the phenomena. The circular fringes follow each other as is known, with increasing diameter according to the law:

$$n=\frac{l}{\lambda}\left(1-\frac{\alpha^2}{2}\right),$$

in which l and λ have the usual meaning, n is the increasing ordinal number of the fringes, beginning at the centre, and α is their radius measured in visual angle from the eye of the observer. So that when a displacement of a fringe is observed, for obtaining a higher precision it is necessary to keep in mind that parabolical law referring to the value of the length of the wave. This is what I have done, studying previously the distribution of the above-mentioned fringes in the field of the telescope. Want of space does not allow me to explain this more at length. I may only say that in the series of observations quoted, I obtained an average of displacement

$$2f=0{\cdot}238.$$

As we see, this value is somewhat superior to that expected, about 5 per cent. Until now, although I took the greatest care to keep in mind the precision of the different measures which are necessary to arrive at this result, I do not know if any systematic error is the reason for this small difference. Certainly it appears superior to the probable error of the result; and that is why I have mentioned it. But in consideration of the delicacy of the measures I do not register the value of the displacement of the fringes before admitting the above-mentioned discordancy (however slight). For now we may conclude that, under the conditions of the experiment, and within the limits of exactitude of the observations made, *the velocity of light does not change by the movement of the source along the direction of propagation.*

From the researches made by Michelson, Fabry and Buisson, and by myself, it results that the velocity of light is not influenced by reflexion on mirrors, or reflecting surfaces; from those now described by me, it results that the said velocity does not change by the movement of the source. These facts are surely in harmony with the theory of relativity; but really, in spite of their evident interest, they cannot logically be cited as sure experimental proof of this theory. In fact, two experimental circumstances must not be forgotten: first, the presence of materials which are traversed by the interfering rays (air, glass, metals); and second, the gravitation field of our earth. While it is possible to imagine experiments entirely apart from the former, it cannot be foreseen if later experimental results will bring into evidence the eventual influence of the second.

XII. *An Attempt to explain the Michelson Interference-Experiment. By* RICHARD BIRKELAND, *Dr. phil., Professor of Mathematics at the Technical High School, Trondhjem*[*].

1. THE problem to determine the influence, if any, exercised by the earth's motion on optic phenomena on the earth's surface, is one of great theoretical interest and importance, and a vast amount of speculation and research has been devoted to the subject.

The earth's mean velocity in its orbit is $v=30$ km./sec. Even compared to the velocity of light $c=300,000$ km./sec., v is not a negligible quantity in all circumstances. All attempts to register effects of the $\frac{v}{c}=10^{-4}$ order of magnitude have been in vain. In 1881 Professor A. A. Michelson devised his now famous experiment[†], by which it would be possible to discover effects of the $\frac{v^2}{c^2}=10^{-8}$ order of magnitude. The expected effect was not registered. The experiment has afterwards been repeated with still greater accuracy, and at present most physicists feel sure that the effect, which was to be expected, does really not occur.

It had been possible to explain all previous experiments

---

[*] Communicated by the Author.

[†] American Journal of Science, (3) xxii. p. 128 (1881).

by the assumption of a stationary æther, a property which has been attributed as a necessary one to the æther by the majority of physicists. But to explain the experiment of Michelson it was necessary to introduce new hypotheses, and one of these we will here take into consideration. We shall first show the general arrangements of the apparatus in Michelson's experiment.

From the source L (fig. 1) light-rays are emitted to the glass S slightly silvered on one side, so as to reflect a portion of the light to the mirror S_1 and to allow the rest to go through to the mirror S_2. From S_1 as well as from S_2 the light is once more reflected and the rays meet at last in the telescope K, and produce by their interference a system of bright and dark fringes. The fundamental idea of the experiment is that, if *the æther remains at rest*, a translation given to the apparatus must of necessity produce a change in the differences of phase. The whole arrangement was mounted upon a slab of stone floating on mercury. The initial situation was: SS_2 coinciding with the direction of the earth's translation, SA and SS_1 perpendicular to it and to a vertical axis. In addition was $SS_1=SS_2=l$. Neglecting terms of higher order than $\frac{v^2}{c^2}$, one can show that the light would take the time

$$T=2\frac{l}{c}\left(1+\frac{v^2}{c^2}\right)$$

to go to and fro between S and S_1, and the time

$$t=2\frac{l}{c}\left(1+\frac{1}{2}\frac{v^2}{c^2}\right)$$

to and fro between S and S_1. The motion produces a difference of phase between the two beams to the extent of

$$l\frac{v^2}{c^3}.$$

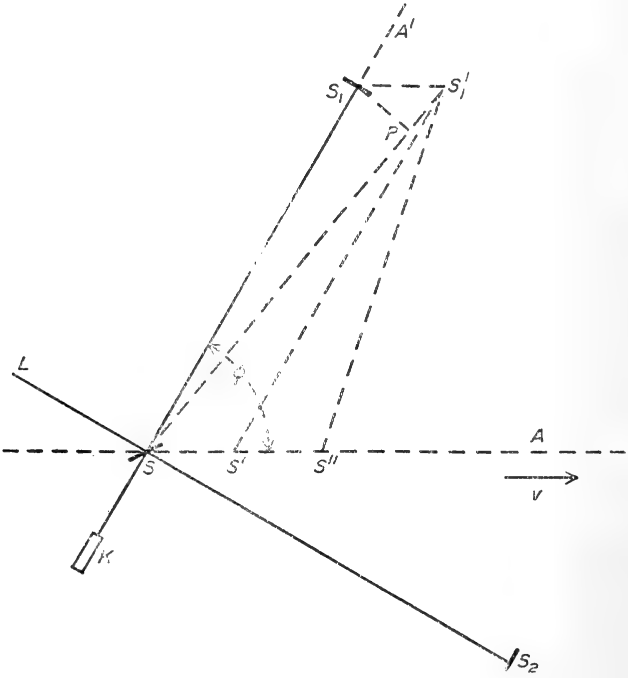
The apparatus was turned 90° about the vertical axis. A measurable displacement of the interference-bands should then have been observed. No such displacement was, however, discovered however the apparatus was orientated with respect to the direction of the earth's translation.

2. The attempt at an explanation of Michelson's experi-

ment here offered will appear to some to lie somewhat outside the domain where physicists would at present prefer to seek it. I have for some years delayed the publishing of it, but I now find, by the advice of some physicists, that I ought no longer to hold it back, as every new idea in this field of keen discussion should be produced that it may be carefully examined.

Imagine a point S on the earth's surface emitting light-rays in all directions. The earth, and the light-source with it, move in the direction SA (fig. 1) with a constant rectilinear velocity $=v\frac{\text{km.}}{\text{sec.}}$. We assume that the æther all around the earth has become anisotropic * (for instance, as a

Fig. 1.



---

* This anisotropy must be assumed to diminish with increasing distance from the earth. Otherwise the result of the Michelson experiment should be dependent upon the motion of every globe in the universe.

consequence of the motion of the earth) in the following way:—

1°. The velocity of propagation of light being c km./sec. in the direction of translation SA; along SA′ (fig. 1) forming an angle ϕ with SA, it is then

$$c_\phi = c\left(1 - \tfrac{1}{2}\frac{v^2}{c^2}\sin^2\phi\right); \quad c_\phi{}^2 + (v\sin\phi)^2 = c^2.$$

2°. The light has the same frequency in all directions from S.

We note that $v\sin\phi$ is the component of the velocity of translation perpendicular to the direction SA′. The velocity of light is according to this hypothesis not very different in different directions, the deviation being of the $\frac{v^2}{c^2} = 10^{-8}$ order of magnitude. It is therefore not in the least strange that this difference of the light's velocity of propagation has not been detected by direct measuring, the accuracy here obtained being no greater than that of $\frac{v}{c} = 10^{-4}$ order of magnitude. The æther becomes symmetrical about an axis SA parallel to the direction of translation. The velocity of light is least perpendicular to this axis and equal to $c_{\frac{\pi}{2}} = c\left(1 - \tfrac{1}{2}\frac{v^2}{c^2}\right)$.

It can easily be shown that the expected effect of the Michelson experiment *must fail to appear* according to this hypothesis.

Imagine the arrangement of the experiment swung so as to make the line SS_1 (fig. 1) form the angle ϕ with the direction of translation SA. During the time t_1, which the light consumes in going from S to the mirror S_1, the latter will have moved to S_1' and the mirror S to S′ and

$$SS' = S_1 S_1' = vt_1.$$

The light covers the distance $SS_1' = l_1$, where

$$l_1^2 = l^2 + v^2 t_1^2 + 2vt_1 l\cos\phi.$$

According to our hypothesis the light's velocity along SS_1 is $c_\phi = c\left(1 - \tfrac{1}{2}\frac{v^2}{c^2}\sin^2\phi\right)$ and consequently* $l_1 = c_\phi t_1$.

---

* It will be shown later that taking the velocity along SS_1 instead of that along SS_1' is without influence if we want an accuracy of the second order with respect to $\frac{v}{c}$.

Introducing this value of l_1, we obtain the following equation, determining t_1:—

$$t_1^2(c_\phi^2 - v^2) - 2vl\cos\phi t_1 - l^2 = 0.$$

Hence

$$t_1 = \frac{vl\cos\phi + \sqrt{v^2l^2\cos^2\phi + l^2(c_\phi^2 - v^2)}}{c_\phi^2 - v^2}.$$

The other solution is negative and must be rejected. On the way back to S a time t_2 is consumed and the light has to cover the distance $S_1' S'' = l_2$, the mirror S being at S'' after the elapse of the time $t_1 + t_2$. We find, the assumption as to the light's velocity being the same,

$$c_\phi^2 t_2^2 = l_2^2 = l^2 + v^2t_2^2 - 2vt_2l\cos\phi;$$

hence

$$t_2 = \frac{-vl\cos\phi + \sqrt{v^2l^2\cos^2\phi + l^2(c_\phi^2 - v^2)}}{c_\phi^2 - v^2}.$$

The total time, consumed in going from S to S_1 and back, is

$$T = t_1 + t_2 = \frac{2l}{\sqrt{c_\phi^2 - v^2}}\sqrt{1 + \frac{v^2\cos^2\phi.}{c_\phi^2 - v^2}}.$$

Introducing the value of c_ϕ, we obtain

$$c_\phi^2 - v^2 = c^2\left[1 - \frac{v^2}{c^2}(1 + \sin^2\phi)\right].$$

With an accuracy of the second order with respect to $\frac{v}{c}$ we obtain

$$\frac{1}{\sqrt{c_\phi^2 - v^2}} = \frac{1}{c}\left[1 + \frac{1}{2}\frac{v^2}{c^2}(1 + \sin^2\phi)\right]$$

$$\sqrt{1 + \frac{v^2\cos^2\phi}{c_\phi^2 - v^2}} = \sqrt{1 + \frac{v^2}{c^2}\cos^2\phi} = 1 + \frac{1}{2}\frac{v^2}{c^2}\cos^2\phi;$$

hence

$$T = t_1 + t_2 = \frac{2l}{c}\left(1 + \frac{v^2}{c^2}\right). \quad . \quad . \quad . \quad . \quad . \quad (1)$$

The angle ϕ does not enter into this formula. The light consumes the same time to go from S to S_1 and back to S, however the apparatus of Michelson is set relative to the direction of the earth's translation. Especially the same time T is consumed from S to S_1 and back to S as from S to S_2 and back to S. As we have further assumed the frequency to be the same for all light-rays emitted from S, the number of wave-lengths will be the same to and fro between S and S_1 as between S and S_2, however the arrangement is set. No displacement of the interference-bands can therefore appear by the Michelson experiment, however the apparatus is swung with respect to the direction of translation.

It is still to be proved that no error has been committed by substituting the velocity c_ϕ along SS_1 for the velocity of light $c_{\phi'}$ along SS_1', forming an angle ϕ' with the direction of translation. We find

$$c_{\phi'}^2 - v^2 = c^2\left[1 - \frac{v^2}{c^2}(1 + \sin^2\phi')\right]$$

$$= c^2\left[1 - \frac{v^2}{c^2}\left(1 + (\sin\phi\cos\Delta\phi - \cos\phi\sin\Delta\phi)^2\right)\right],$$

$\Delta\phi$ being $=\phi-\phi' = S_1SS_1'$ (fig. 1). Projecting S_1 on SS_1' to the point P (fig. 1), we obtain

$$S_1P = vt_1\cos\left(\frac{\pi}{2} - \phi'\right) = vt_1\sin\phi' = l\sin\Delta\phi.$$

Hence

$$\sin\Delta\phi = \frac{vt_1\sin\phi'}{l}$$

t_1 is to be sure $< 2\frac{l}{c}$, and consequently $|\sin\Delta\phi| < 2\frac{v}{c}$. The difference between the two angles ϕ and ϕ' is therefore of the order $\frac{v}{c} = 10^{-4}$. c_ϕ and $c_{\phi'}$ will consequently differ only in terms of the third order with respect to $\frac{v}{c}$. The time t_2 will, with the same accuracy, be the same replacing the velocity along $S_1'S''$ by c_ϕ.

3. Adopting this hypothesis, we meet with no contradiction if we determine the time by means of light-signals

emitted in different directions from a point on the earth's surface.

Suppose that an observer B at the time $t=0$ emits a flash of light from S (fig. 1) to a point P on the earth at a distance of l km. from S. The time T consumed by the light in going from S to P and back to P is according to (1),

$$T=\frac{2l}{c}\left(1+\frac{v^2}{c^2}\right).$$

B's watch can consequently be adjusted by means of light-signals without contradiction.

Trondhjem, Norway,
June, 1918.

XIII. *On Fermat's Law.*

India Meteorological Department,
Simla, 16th April, 1918.

To the Editors of the Philosophical Magazine.

GENTLEMEN,—

IN a paper on Fermat's law by Professor D. N. Mallik, in your issue of July 1913, he deduces (p. 152) that "optical energy is entirely kinetic," and hence that the phenomena of elasticity and electrostatics are also kinetic. These results, if valid, are of very great generality and importance, and they have been restated on pp. 12 and 13 of Professor Mallik's recent volume on 'Optical Theories'*. The idea that all potential energy is capable of interpretation as kinetic was worked out by Helmholtz, and J. J. Thomson has examined many of its consequences; but the conclusion that optical energy must be kinetic is, I believe, entirely new, and as I do not follow the argument employed, I desire to append the following criticism in the hope that a decision may be reached in a matter so far-reaching in its consequences.

2. Professor Mallik says on page 149:—

"the configuration of equilibrium and motion of a dynamical system is defined by $\delta\int(T-V)dt=0$, where

$T=$kinetic energy,
$V=$potential energy.

---

* Cambridge University Press, 1917.

If this is to be consistent with Fermat's law we must have, for light propagation,

$$T-V=C \text{ (constant)}. \quad . \quad . \quad . \quad . \quad . \qquad (1)\text{''}$$

Later on, in §17, he says :—

"Again, from the principle of energy

$$T+V=C' \text{ (constant)}. \quad . \quad . \quad . \quad . \quad . \quad . \qquad (3)$$

Therefore from (1) and (3) we get

$$2T=C'+C,$$

$$2V=C'-C.$$

But this is meaningless, since the mean potential energy and the mean kinetic energy are alone constant, as these quantities are understood to mean in the above equations. Accordingly, the only conclusion that seems to be consistent with all the equations is that the optical energy is entirely kinetic."

"Again, if the potential energy of deformation of the æthereal medium involved in light propagation is to be regarded as essentially kinetic, we are led to conclude that all energy is kinetic."

3. For a statement of the principle of Hamilton to which appeal is first made we may refer to Lamb's article on Dynamics in the 'Encyclopedia Britannica'[*]. It is

$$\delta\int_t^{t'}(T-V)\,dt=0,$$

the time of transit being the same for the hypothetical as for the actual motion, and the initial and final configurations prescribed. But in Fermat's law we have

$$\delta\int dt=0, \quad \text{or} \quad \delta\int\frac{ds}{v}=0,$$

where the co-ordinates are no longer those of particles of matter as in Hamilton's principle, but successive points on the ray as the light-wave travels along, and the velocity of the ray is entirely different from the velocity of the individual material particles whose motion constitutes the light. Also, while the initial and final points are prescribed in Fermat's equation the time of transit is not.

---

[*] Eleventh Edition, p. 762.

Thus the expressions to be integrated in the equations expressing Hamilton's principle and Fermat's law are quite distinct, and cannot be treated as identical. Dr. Mallik's conclusion appears to me therefore not proved.

4. That this must be the case may be seen in the following manner. In the various elastic solid theories it has been shown that certain assumptions as to the elastic constants will lead to results consistent with the laws of refraction, and so of propagation through an isotropic medium in which the refractive index varies from point to point. This is sufficient to ensure that Fermat's law shall be completely obeyed; and the analytical condition that Fermat's law shall be obeyed will therefore give no additional result regarding the potential energy.

In the corresponding case of sound also the velocity of the particles of air and of the velocity of sound are obviously quite distinct. So that though Fermat's law of least time also holds as a consequence of the ordinary sound equations* there is no fresh information to be deduced from Fermat's law.

5. All this argument applies merely to dynamical theories of light, to which alone is there a satisfactory justification for applying Hamilton's principle. It cannot therefore be convincing to those who regard the electromagnetic theory, with certain modifications, as holding the field. It is quite true, as Professor Mallik says in section 14 of his paper, that we *can* "interpret the intimate nature of the electric field" as possessing the same qualities as MacCullagh's medium; but this is different from being "led to conclude that all energy is kinetic."

GILBERT T. WALKER.

XIV. *On Fermat's Law.*

Presidency College, Calcutta,
27th April, 1918.

To the Editors of the Philosophical Magazine.

GENTLEMEN,—

THROUGH the courtesy of Dr. Gilbert Walker, I have been permitted to see in advance the letter which he has written to the Philosophical Magazine criticising a

---

* See Rayleigh's 'Sound,' vol. ii. §289 in the Second Edition.

certain interpretation of Fermat's law which I have given. He has very kindly suggested to me that I might offer a reply, if I wished, in the same number of the Magazine. I trust, therefore, that you will kindly permit me to offer not a reply, but some explanations on the subject.

1. It will be useful in the first place to state my exact point of view.

We have the dynamical equation $\delta\int(\mathrm{T}-\mathrm{V})dt=0$ which gives a complete account of the motion of a dynamical system—in this case, the disturbed optical medium.

We have also the equation $\delta\int dt=0$ (the dynamical significance of which requires investigation).

Are these independent of each other? If so, the second equation can only be the equation of constraint. No such constraint can, so far as we can see at present, be well associated with the medium considered.

If no such constraint can be postulated, we can only regard the second equation as identical with the first, in this particular case.

2. Now in order that we should be justified in doing so, it is necessary to admit that t has the same meaning in both Hamilton's principle and in Fermat's law.

3. Dr. Walker maintains that this is not permissible as "in Fermat's law we have $\delta\int dt=0$ or $\delta\int\frac{ds}{v}=0$, where the co-ordinates are no longer those of particles of matter as in Hamilton's principle, but successive points on the ray as the light-wave travels along, and the velocity of the ray is entirely different from the velocity of the individual material particles whose motion constitutes the light." In other words, Dr. Walker bases his argument on (2), which is derived from (1) by a mere analytical transformation which apparently imposes limitations (from the point of view of the present line of argument) on equation (1), not necessarily involved in it. In fact, the second equation may well be taken as $\delta\int\frac{d\phi}{\phi'}$ where $\phi'=\frac{d\phi}{dt}$ and may, as such, be held to give information regarding ϕ and ϕ', *whatever these may be* (not merely s and v), so long as these quantities are related in any manner to the phenomenon of light propagation. But the particular co-ordinates involved in ϕ and ϕ' or their nature cannot well be regarded as alone implied in (1). I conceive, therefore, that the present line of argument is not crucial against my theory.

4. Moreover, it stands to reason that Hamilton's principle, although it directly deals with a certain volume distribution of energy, will naturally lead to equations giving propagation of energy or disturbance, if T and V are appropriate to such a propagation. The latter equations may, in this event, involve a set of co-ordinates distinct from that involved in Hamilton's principle and yet we may not argue that these equations are independent of that principle.

5. It is now necessary to consider another difficulty, which Dr. Walker raises and which has always appeared to me of great importance. We know that Hamilton's principle postulates that the initial and final configuration of a dynamical system are prescribed and the time of transit of the system from the initial to the final configuration must remain unchanged. Now the first condition applies also to Fermat's law and the second condition *may* be imposed on it, if we take the time of transit to be that from one wave-front to the next. This, however, may deprive my conclusions of a part of their generality, but only in a manner which it is not possible to decipher at present.

6. As to the contention that "in Hamilton's principle the co-ordinates are those of particles of matter," I am doubtful whether this limitation will be universally acceptable. I rather think, given the forms of T and V, in a medium which is the seat of energy, Hamilton's principle will be applicable, though we may be unable to determine the intimate nature of the constitution of the medium which determines T and V. In order to arrive at these forms, various hypotheses have to be framed, and we thus get various forms of T and V and corresponding optical theories. From this point of view, the electro-magnetic theory with or without modifications may well be regarded as a dynamical theory. This may meet the difficulty to which Dr. Walker refers in para. 5.

7. I must admit, however, that even if the identity between Fermat's law and Hamilton's principle can be established, I have not been able as yet to *prove* that $T-V=$ constant is the *only* solution. Therefore, although I cannot think of any other solution ($T-V=f(t)$ being inadmissible, on the principle of energy), I have modified my conclusion and now content myself with saying (in my book) that we "*may take* $T-V=$constant."

8. This seems to be all the more desirable in view of what I have stated in paras. 1 and 5. I do not therefore

claim to have proved that all energy is kinetic. I only suggest that Fermat's law is capable of an interpretation which will yield this conclusion. I trust the proof will be forthcoming in due course. At present, however, I have no illusion on that point.

Yours truly,
D. N. MALLIK.

XV. *The Principle of Molecular Scattering of Radiation.*
By Sir JOSEPH LARMOR, *F.R.S.*[*]

A FUNDAMENTAL element in the theory of radiation is the principle first elucidated by Lord Rayleigh [†], that when light is scattered by the particles of a fog or haze, or even by the molecules of the air, they act independently, without sensible mutual interference as regards the distribution of the energy.

The condition necessary for this independence is that the disturbances (such as strain, velocity) must arrive from the scattering particles in phases which are entirely uncorrelated: so that on an average taken over a short interval of time, the square of the sum of the disturbances is equal to the sum of their squares, and thus the total energy would come from addition of energies of independent scattered disturbances.

This condition will be secured if the scattering particles are distributed at random, provided the intervals between adjacent ones are substantial fractions of the wave-length of the radiation that is being scattered: for then the phases of the scattered disturbances coming from adjacent particles will be uncorrelated. It holds good usually for particles of dust in the atmosphere. But in the case of a gas there are 10^6 molecules in a cubic wave-length, and in the case of a liquid or solid 10^9, giving differences of adjacent phases of the order of only 10^{-2} of the period in the former case and 10^{-3} in the latter.

Even for a gaseous medium the question thus arises, whether it is wrong to consider the distribution of the scattering molecules as based upon uniform spacing, but subject to uncorrelated deviations from this regularity which

---

[*] Communicated by the Author.

[†] Proc. London Math. Soc. 1870; Phil. Mag. x. 1880; and later papers, including the one under special reference in Phil. Mag., Dec. 1918.

obey the law of statistics of gas-theory and amount at most to a few hundredths of the wave-length. This representation would seem to be permissible, at any rate for each group of say about 10^6 molecules occupying the cubic wave-length; and such groups will be practically independent.

Now if the molecules were spaced with exact uniformity at distances of smaller order than the wave-length, as they are in a crystal, the disturbances scattered from an incident beam, instead of being additive as regards their energy, would interfere completely; so that there ought to be no radiation scattered in traversing a crystalline medium. This has in fact been remarked by Lord Rayleigh in his recent paper*, and I think previously by Prof. Lorentz. A beautiful experiment by Prof. R. J. Strutt, which I had the advantage of seeing some time ago, showed that the actual scattering in a column of quartz crystal was small compared with what occurs in optical glass or even in a liquid such as ether. Quantitative comparison would be of interest on various grounds.

This principle that a crystal should scatter no radiation seems to be unimpeachable, provided the molecules are fixed and do not partake of thermal agitation. And it seems difficult to see why it should not also apply to the molecules of a gas, if they could be regarded as fixed while the radiation is passing, subject to correction for the statistical deviations aforesaid from their mean positions.

If this were so the individual molecules of a gas, and *a fortiori* of a liquid or a solid on account of their closer packing, ought in conjunction to scatter radiation far less than they would do separately, the reason being the vast number contained in a cubic wave-length and the statistical regularity of their distribution. And accordingly Lord Rayleigh's announcement † that the blue sky could be due to scattering by the molecules of the air itself came as a surprise, which subsequent quantitative verifications did not wholly resolve.

The suggestion now to be advanced for consideration is that the principle is to be maintained, even to some extent for crystals, but its logical basis is to be shifted.

The molecules of the atmosphere are in thermal motion, with velocities in uncorrelated directions which are at ordinary temperatures of the order of 10^{-6} of that of radiation. The wave-length of the radiation scattered from

---

* Phil. Mag., Dec. 1918, p. 445, footnote.

† Phil. Mag. 1899; Scientific Papers, iv. p. 397.

them will thus vary within a range of 10^{-6} of itself. If the phases of the scattered radiations are correlated at first, after traversing 10^6 wave-lengths or 50 cm. they will have become fortuitous, and the energy-effects thus additive.

This consideration, if justified, would find the source of Lord Rayleigh's principle in the uncoordinated thermal motions of the molecules. And as in a crystal thermal vibrations are contemplated about the mean positions of the molecules in the space-lattice, there ought to be some degree of scattering in traversing a crystal.

This way of envisaging the matter contemplates a scattered beam of radiation of wave-length slightly indefinite, on Doppler principles, in which therefore the phases of the constituent elements become fortuitous after travelling a substantial though not very great distance. Other influences of radiation on the gas which are also adjusted over a considerable range in distance would be implied: the molecular effects connected with pressure of radiation would be expected to belong to this class.

Some confirmation from another point of view seems desirable. The scattered radiation, forming a spectral band of some slight breadth, may be considered as resolved into more homogeneous constituents. The radiation in one of them has been scattered at each instant by molecules whose thermal velocities agree within close limits as regards both direction and magnitude: these molecules constitute a sparsely distributed group whose distances apart can be of the order of the wave-length, so that the condition necessary for fortuitous phases in the scattered disturbances is satisfied.

The interesting remark is made by Lord Rayleigh (p. 445) that the radiation scattered nearly in the direction of the primary rays is specially favoured, in that all its components nearly agree in phase. The question arises whether this would not make a clear sky very much brighter within a few degrees of the sun than at some distance away from it. Here also the slight variety of wave-lengths arising from the thermal motions of the scattering molecules would seem to reduce or perhaps nearly remove such disparity.

Cambridge, Dec. 16, 1918.

XVI. *The Energy in the Electromagnetic Field.*

To the Editors of the Philosophical Magazine.

The University,
Sheffield.
Nov. 6th, 1918.

DEAR SIRS,—

IN your issue of August, 1917*, Mr. A. E. Biedermann raises certain objections against my criticism of his note on the "Energy of the Electromagnetic Field," on the score that this criticism does not in fact dispose of the fundamental difficulty which necessitated his modification of the usual expression for the magnetic energy density.

The main point at issue seems to be whether the extra term in the magnetic energy, viz.

$$\frac{1}{8\pi}(\mathrm{div}\ \mathrm{A})^2,$$

A being the vector potential, is zero or not in the cases with which Mr. Biedermann deals. I asserted in fact that it was zero, without offering a proof of the statement, but he still contends that this assertion cannot be justified. However, he agrees that if it is possible to prove that a certain double line integral vanishes, then his contention falls to the ground and there is then nothing in his modification. This integral is

$$\iint \frac{\cos\epsilon - \cos\alpha_1 \cos\alpha_2}{r} ds_1\, ds_2$$

taken round any two closed curves in space, α_1, α_2, ϵ being the angles the elements ds_1, ds_2 make respectively with the radius r joining them and with one another.

Now it is easy to prove† that

$$\cos\epsilon - \cos\alpha_1 \cos\alpha_2 = r\frac{d^2 r}{ds_1\, ds_2},$$

and thus the integral is

$$\iint \frac{d^2 r}{ds_1\, ds_2} ds_1\, ds_2$$

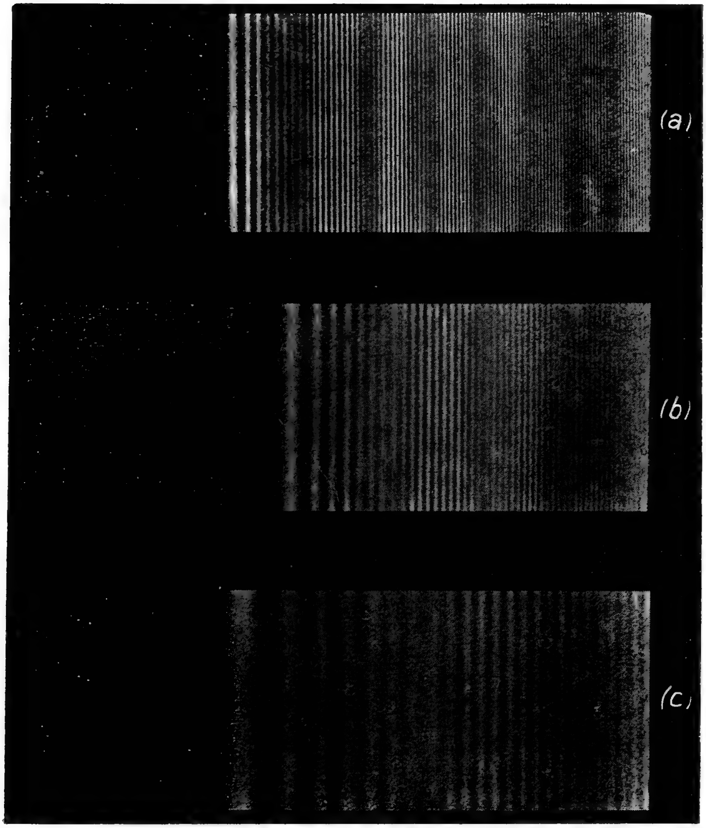
and obviously vanishes.

I am,
Yours very truly,
G. H. LIVENS.

---

* Circumstances over which I have no control have prevented my answering this letter before.

† The details of the calculation are given in Poincaré, *Electricité et Optique* (2nd Ed. Paris 1901), p. 233.



Illustrating the Interferences in the Field surrounding a reflecting cylinder and the decrease in the visibility of the fringes with increasing distance from the cylinder.

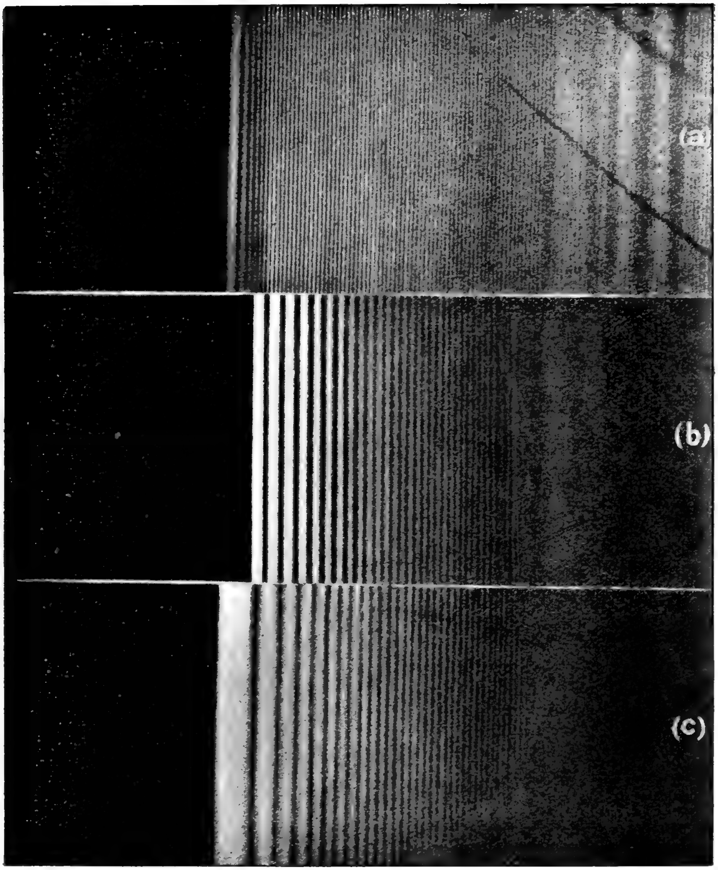
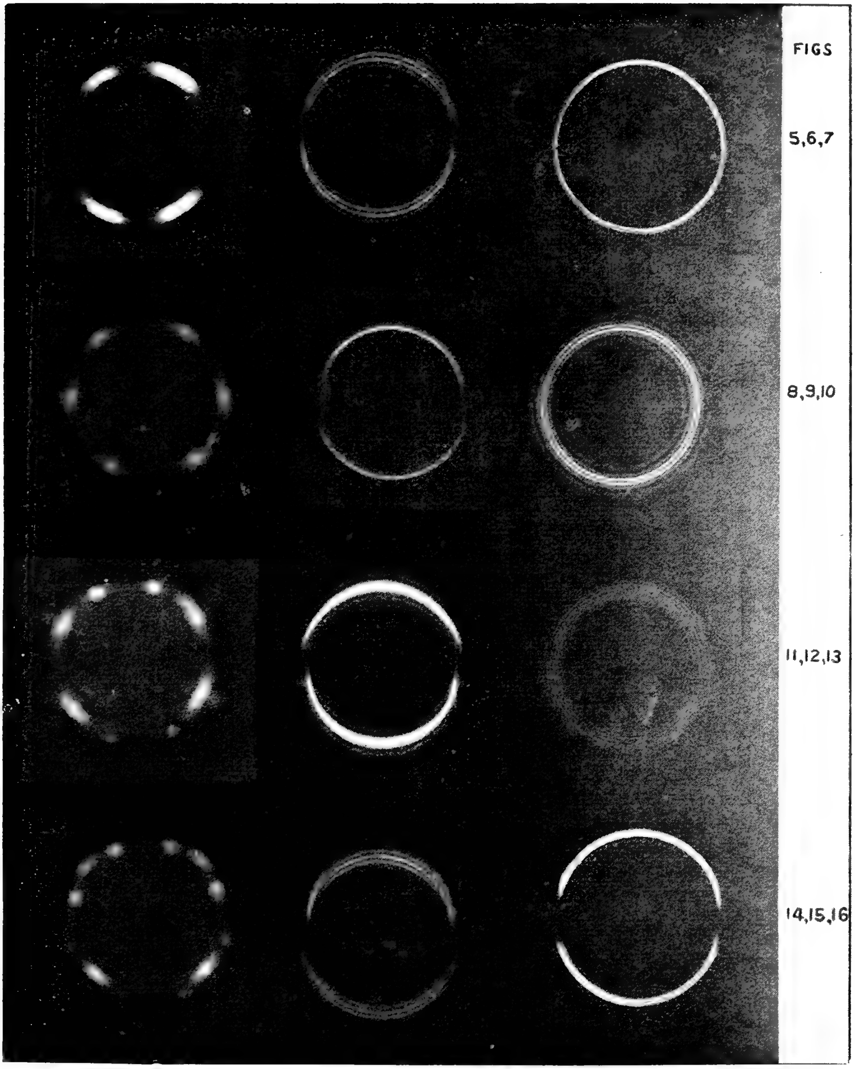
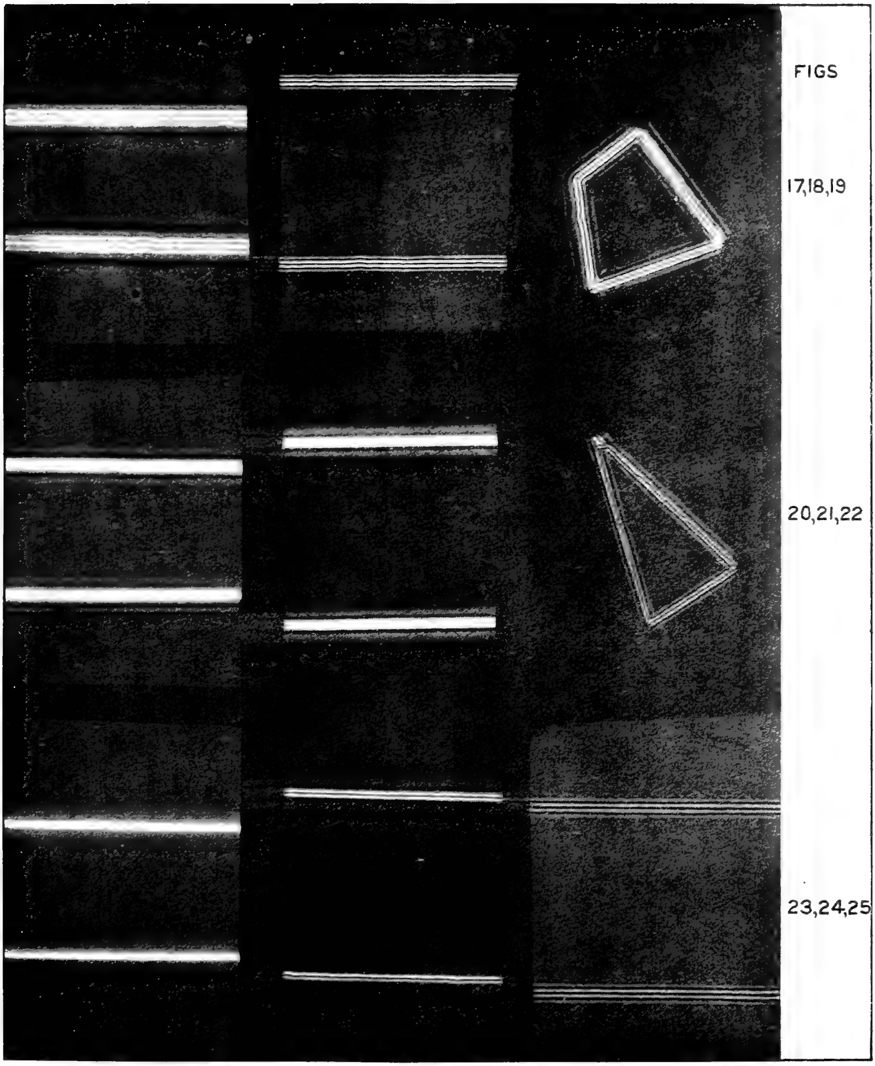


FIG. 4.—Illustrating Oblique Diffraction by a Semi-Infinite Screen.



Illustrating the Emission of Light by the Boundary of a Circular Diffracting Aperture.



Illustrating the Emission of Light by the Edges of Apertures with Rectilinear Boundaries.

THE
LONDON, EDINBURGH, AND DUBLIN
PHILOSOPHICAL MAGAZINE
AND
JOURNAL OF SCIENCE.

[SIXTH SERIES.]

FEBRUARY 1919.

XVII. *The Absorption of X-Rays.*

By TYCHO E:SON AURÉN, *Dr. phil.**

I. *Method of Investigation.*

SINCE Moseley has ascertained that there is a simple relation between the frequency of X-ray spectra and atomic number, we may readily assume that there will be a similar relation when examining X-ray absorption in different elements. By way of ascertaining this kind of relation, I have made determinations of absorption in a number of elements. In so doing I have introduced a method of investigation which I have more particularly described in a preliminary paper †. This method has in view, by means of a procedure of compensation and by using a non-homogeneous radiation, the endeavour to determine the relation between the absorption coefficient of a certain element and the same coefficient of water. As materials for investigation I have preferably made use of chemical solutions, because a great number of elements are not obtainable in solid form, whereas they are easily obtainable in soluble compounds. As the method is adapted for determination of relative absorption coefficients, and these, only excepting the lightest elements, at least within a not too wide range,

---

* Communicated by the Author.

† Phil. Mag. xxxiii. p. 471 (1917).

are independent of the wave-length, we have reason to believe that the fact that the radiation has not been quite homogeneous must be of less importance. Corrections can also easily be made for the absorption produced by scattering in the lighter elements, as will be shown below. In determining the absolute absorption coefficients by means of the method generally adopted, the irregularities in the action of the generator and the variations of the gas pressure in the bulb create appreciable difficulties by altering the composition of the radiation. Using homogeneous radiation obtained by reflexion on a crystal and using the ordinary method, Bragg and Peirce * have made a series of experiments, but the authors represented them as being merely preliminary and announce that they intend to renew the experiments by the aid of a compensation method. These experiments recently have, however, been repeated by Owen †. In the following pages I shall revert to an experiment performed by Hull and Rice ‡ with the aid of very hard homogeneous rays, which research, however, only includes the elements Al, Cu, Pb. I shall also review a research recently published by Barkla and White §, which has also been worked out by using a homogeneous radiation and which likewise includes a few elements only.

My preliminary experiments, published in my paper above mentioned, have been continued, and I am now going to give the results obtained. In these experiments I have particularly endeavoured to make determinations with more accuracy and to perform them so as to make them comparable as far as possible. In order to increase the sensibility of the method I have in the later experiments used a more sensitive electrometer—a string-electrometer from A. B. Vetenskapliga Instrument, Lund, Sweden, which gave an indication of about 12 divisions on the scale for 1 volt, and the current in the bulb had amounted to 2 milliamperes and in this way the intensity of radiation appreciably increased. By way of obtaining fully comparable results I have tried, in all my experiments, to keep the composition of the radiation as constant as possible. The generator used, a high-tension transformer of 4 kilowatts, has been working very evenly. The generator was charged so as to keep a constant potential difference between the electrodes in

---

* Bragg and Peirce, Phil. Mag. xxviii. p. 626 (1914).

† Owen, Proc. Roy. Soc. xciv. no. A 664, p. 510 (1918).

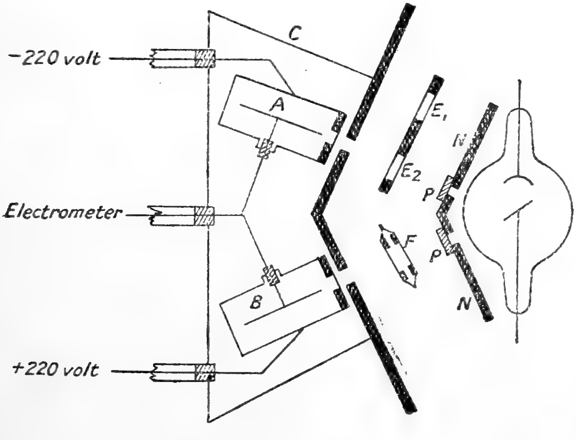
‡ Hull and Rice, Phys. Rev. ix. p. 326 (1916).

§ Barkla and White, Phil. Mag. xxxiv. p. 270 (1917).

the bulb. The same bulb, a Siemens' tungsten bulb of Grundelach type, with fin radiator for cooling the anticathode, has been employed in all experiments. But as the composition of radiation, in the case of an intermittent current, depends not only on the potential difference at the bulb but also on the gas-pressure in it, the greatest precaution has been taken to keep the gas-pressure as constant as possible. For this purpose the current in the bulb has been steadily controlled by means of a milliamperemeter, and by a careful and continual regeneration of the bulb it has been possible to keep the current very nearly constant. That the composition of the radiation has not undergone any alterations such as to have appreciably influenced the results is evidenced by the fact that different experiments repeated on different occasions with one and the same element have given very nearly the same results.

Both absorption vessels (E and F in fig. 1) having a diameter of 3 cm. are placed at a distance of 12 cm. from their respective ionization chambers, whose front is made of lead plates, in which there are round openings of a radius of 2·5 cm. covered with mica. The position of the bulb has been adjusted so that with the same thickness of the layer and the same liquid in the two absorption vessels the intensity of both the ray-pencils penetrating into the ionization chambers should be as nearly alike as possible. The distance from the focus to the ionization chambers is 42 cm. Between the absorption vessels and the bulb there

Fig. 1.



is a leaden screen (N) at a distance of 20 cm. from the focus. In the screen there are also made two round openings of a radius of 2 cm., through which are admitted the ray-pencils that are to pass through the absorption vessels. In front of the openings are placed aluminium plates (P), which in the experiments have been used as filters for the rays. These filters were of four thicknesses—*i. e.* 1·25, 2·5, 5·0, and 10·0 mm.—and are marked, in the following, by I., II., III., IV. The mean wave-length of the four different compositions of radiation when using these filters is indicated in Table I.

Concerning the γ-rays Keetman [*] has shown that, in order to obtain reliable values of the absorption coefficients, two conditions must be fulfilled. First, the absorbing layer should be exposed as little as possible to oblique radiation; secondly, the rays coming out of the absorbing layer in such a direction should, as much as possible, be excluded from the ionization chamber. There can be no doubt but that experiments for the purpose of determining the absorption coefficient in the case of hard X-rays give more or less inaccurate results if the said conditions are not approximately fulfilled. As in the experimental arrangements above mentioned the ray-pencil entering the ionization chamber seen from the focus and the absorbing layer has not had greater opening angles than $3^{\circ}\cdot 5$ and 12° respectively, and as the diameter of the absorbing layer has not been more than 3 cm., the conditions mentioned may safely be considered as very nearly fulfilled. To be sure that no error might arise from the possible admission of scattered rays from the absorbing layer into the ionization chamber, special experiments were undertaken, in which the absorption vessel (E_1 and E_2) when being irradiated was displaced so that the distance to the ionization chamber was diminished from 12 to 1·5 cm. without the direction of the incident rays being changed in relation to the vessel. The experiments have shown that when the distance is greater than 3 or 4 cm. the position of the absorption vessel has scarcely any measurable influence on the results. In regard to this fact, O. Klein [†] has undertaken a mathematical explanation, to which I here beg to refer. The same problem has also recently been discussed by Glocher [‡], and his results agree very well with those found by Klein.

The relation between the intensity of both the ray-pencils

---

[*] Keetmann, *Ann. der Ph.* lii. p. 720 (1917).

[†] O. Klein, *infrà*, p. 207.

[‡] Glocher, *Phys. Z.* xix. p. 251 (1918).

sometimes happened to undergo small changes, which have had a disturbing effect on the measurements. These changes seem likely to depend on the position of the focus being changed. To be independent, in the highest degree possible, of this source of errors I have used two cuvettes beside each other (E_1 and E_2), which have been so arranged as to be pushed alternately into the path of the rays. One of the cuvettes has always been filled with water, while the other has been filled with the liquid whose absorption was to be determined. When solid elements were examined the latter cuvette was emptied and in front of it a plate of the element in question was placed.

The procedure of measuring has been as follows:—When the cuvette filled with water had at first been placed in the path of the rays and the thickness of the water layer in comparator (cuvette F, fig 1) had been adjusted so that the electrometer gave no indication, the other cuvette, containing the liquid whose absorption was to be determined, was immediately pushed into the same position; whereupon a new adjustment was effected. The difference between the two adjustments then indicates the difference of absorption in the two cuvettes. As a rule several such adjustments have been made, which, however, have seldom differed by more than *in maximo* 5 per cent. The absorption coefficients given in the following pages are calculated from the mean values of the experimental observations. The thickness of the liquid layers contained in both the cuvettes was about 1 cm. By special observations the thicknesses of the layers have been accurately determined. From these determinations I could measure the thickness of a water layer that absorbed just as much as a layer of the thickness of 1 cm. of a liquid or a layer of known thickness of the solid substance in question.

If the thickness of the water layer whose absorption is equal to a layer of the thickness of 1 cm. of a certain substance be supposed to be d cm., and the respective absorption coefficients of the water and of the liquid in question to be μ_w and μ_s, then by means of the absorption law $I=I_0e^{-\mu d}$, where I stands for the intensity of transmitted radiation and I_0 for the intensity of incident radiation, we obtain the equation:—

$$I_0e^{-\mu_w d} = I_0e^{-\mu_s},$$

from which follows that

$$\frac{\mu_s}{\mu_w} = d.$$

If κ_{A/H_2O} designates the molecular absorption coefficient of a certain substance (A) in relation to the molecular coefficient of water, then when the said substance is supposed to contain m and water 55·5 mols per litre, we obtain the following formula for the calculation of this coefficient :

$$\kappa_{A/H_2O} = \frac{55{\cdot}5d}{m} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

As shown by researches by Winaver and Sachs*, Glocker†, Barkla and White ‡, etc., the absorption coefficient of water increases with increasing wave-length much more slowly than is the case with absorption coefficients of substances containing elements of higher atomic weight. In section IV. I shall give a detailed account of the matter and its cause. For that reason it is less suitable, in values obtained when the composition of radiation has been varied, to use the molecular absorption coefficient of water as a unit. The coefficients calculated after formula (1) have therefore been recalculated so as to give values corresponding to the atomic absorption coefficient of copper. If κ_{Cu/H_2O} designates the atomic absorption coefficient of copper in relation to the molecular absorption coefficient of water, we thus obtain

$$\kappa_{A/Cu} = \frac{\kappa_{A/H_2O}}{\kappa_{Cu/H_2O}} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

Benoist was the first to advance the hypothesis that in chemical compounds, as well as what has been ascertained in regard to mixtures, the total absorption can be calculated from the absorption of the respective components by simple addition. This assumption has later on been regarded as satisfactory, a fact which is also in accordance with my former experiments. If in a molecule of a certain substance (A) there are n_1 atoms of the element a_1, n_2 atoms of the element a_2, &c., we consequently suppose that

$$\kappa_{A/Cu} = n_1\kappa_{a_1/Cu} + n_2\kappa_{a_2/Cu} + n_3\kappa_{a_3/Cu} + \ldots \quad . \quad . \quad . \quad (3)$$

When chemical compounds are used it is evidently most profitable for attaining a greater accuracy to choose

---

* Winaver and Sachs, *Phys. Z.* xvi. p. 258 (1915).
† Glocker, *Phys. Z.* xviii. p. 332 (1917).
‡ Barkla and White, Phil. Mag. xxxiv. p. 270 (1917).

compounds in which absorption is dependent as much as possible on the element examined. When we calculate absorption of substances in solutions we have to correct the observed values in respect to the influence of the solvent. All the values below are corrected in this manner.

In calculating $\kappa_{A/Cu}$ I have made use of the mean values of κ_{Cu/H_2O} obtained when examining Cu in metal form and in water solutions of $CuCl_2$, $CuSO_4$, $CuNO_3$ (Table VII.). By the aid of the values exhibited in Barkla and White's paper on the mass absorption coefficient of copper and water in case of different wave-lengths κ_{Cu/H_2O} can be calculated, and by interpolation one can readily find the wave-lengths corresponding to the values found by the writer for κ_{Cu/H_2O}. In the Table below I have given the values obtained for the mean wave-lengths (λ_m) in the four cases with different compositions of radiation. Likewise I have indicated the values of the mass-absorption coefficients of Cu $\left(\frac{\mu}{\rho}\text{Cu}\right)$ corresponding to these wave-lengths, which values have been obtained by interpolation between the values given by Barkla and White.

TABLE I.

Notation.	Filter. Thickness in mm.	κ_{Cu/H_2O}.	λ_m.	$\frac{\mu}{\rho}$ Cu.
I..........	1·12	78·9	0·38	7·10
II..........	2·50	67·8	0·36	5·92
III..........	5·00	57·6	0·34	4·89
IV..........	10·00	48·4	0·30	3·73

In the table, as well as in the following demonstration, the wave-lengths are denoted by the Ångström units (1 A.E. $=10^{-8}$ cm.).

By filtering, most of the waves of greater wave-lengths in proportion are done away with, by which means radiation for an increasing thickness of filter is more and more apt to be homogeneous. Of course the range in which the wave-lengths fall has had an appreciable extension on both sides of the above-mentioned mean wave-lengths. Among the substances examined, certainly the characteristic wave-lengths of K-radiation for Sn, I, and Ba have fallen within the range mentioned. This is evident from the strong increase of κ_{a/H_2O} (see Table VII.) for these elements; while there is decrease in intensity of rays of a greater wave-length on increasing the thickness of the filter. Values for elements that fall within what is called selective absorption range are not comparable with the other values. No doubt this range has also extended to substances whose atomic numbers lie next below the numbers of these elements, but probably the intensity of radiation within this section of the wave-length range must have been much less. In order, as much as possible, to do away with errors that would arise from this cause, it does not seem to be advisable to include the values obtained when using filter I. for the substances Sr ($\lambda_{\kappa\beta_1} = 0{\cdot}767$), Mo ($\lambda_{\kappa\beta_2} = 0{\cdot}633$), Rh ($\lambda_{\kappa\beta_1} = 0{\cdot}537$), and for the substances Pd ($\lambda_{\kappa\beta_2} = 0{\cdot}521$) and Ag ($\lambda_{\kappa\beta_1} = 0{\cdot}491$) when filtering with I. and II.

II. *Experimental Results.*

By comparing the absorption coefficients in compounds whose molecules only differ from one another by one or more atoms of oxygen the relative atomic absorption coefficient of this element can be determined. In my first experiments water-solutions of $NaClO_3$ and NaCl were employed for this purpose; but in the experiments described below I have preferred to use certain pure organic compounds which, at ordinary temperature, are obtained in the form of liquids. This is done on the ground of absorption for these substances being determined to an appreciably greater extent by the absorption of oxygen than is the case with $NaClO_3$ and similar combinations, which, in addition to oxygen, contain heavier and at the same time more strongly absorbing substances. The substances used have been free from water and, moreover, in chemical purity the best obtainable. In the following table are noted down the substances used as well as the values found for κ_{A/H_2O} in the compositions of radiation which have been got by means of the above-mentioned four different aluminium filters (I., II., III., IV.), and also the differences of these values for two compounds which in regard to molecular composition differ from each other only by one atom of oxygen. These differences (κ_{O/H_2O}) thus indicate the atomic absorption coefficient of oxygen in relation to the molecular absorption coefficient of water.

Table II.

Atomic absorption coefficient of oxygen in relation to the molecular absorption coefficient of water.

Chemical combination.	I.		II.		III.		IV.	
	κ_{A/H_2O}.	κ_{O/H_2O}.	κ_{A/H_2O}.	κ_{O/H_2O}.	κ_{A/H_2O}.	κ_{O/H_2O}.	κ_{A/H_2O}.	κ_{O/H_2O}.
Methyl formate, $C_2H_4O_2$	2·90	0·87	2·95	0·87	2·98	0·83	3·05	0·81
Paraldehyde, $\frac{1}{3}(C_6H_{12}O_3)$	2·03		2·08		2·15		2·24	
Propionic acid, $C_3H_6O_2$	3·47	0·88	3·58	0·83	3·62	0·74	3·76	0·75
Allyl alcohol, C_3H_6O	2·59		2·75		2·88		3·01	
Propionic acid, $C_3H_6O_2$	3·47	0·88	3·58	0·87	3·62	0·84	3·76	0·88
Acetone, C_3H_6O....................................	2·59		2·71		2·78		2·88	
Carbonic acid dimethylester, $C_3H_6O_3$	4·35	0·89	4·46	0·90	4·59	0·94	4·61	0·80
Ethyl formate, $C_3H_6O_2$	3·46		3·56		3·66		3·81	
Methylal, $C_3H_8O_2$	3·57	0·92	3·68	0·83	3·85	0·78	4·02	0·84
Isopropyl alcohol, $C_3H_8O_2$	2·65		2·85		3·07		3·16	
Ethyl acetate, $C_4H_8O_2$	4·15	0·96	4·28	0·90	4·43	0·82	4·63	0·73
Methyl ethyl ketone, C_4H_8O	3·19		3·38		3·61		3·90	
Benzyl alcohol, $C_6H_5CH_3O$......................	4·58	0·92	4·77	0·92	5·02	0·94	5·28	0·86
Toluol, $C_6H_5CH_3$	3·66		3·85		4·08		4·42	
Octyl alcohol, $C_8H_{18}O$	5·54	0·84	5·92	0·90	6·28	0·86	6·64	0·82
Octane, C_5H_{18}	4·70		5·02		5·42		5·82	
Mean		0·895		0·878		0·844		0·811

The values found for κ_{O/H_2O}, as shown from the table, agree very fairly. On the ground of the additive law and by the aid of the mean values above, we find corresponding values for hydrogen, κ_{H/H_2O}, if the molecular absorption coefficient of water be denoted $=1$:

		I.	II.	III.	IV.
κ_{H/H_2O}	=	0·052	0·061	0·078	0·094

In the same manner, by means of the values found for κ_{H/H_2O} and κ_{O/H_2O}, the corresponding coefficients for C may be computed. In the following table we again find the values obtained from the observations for κ_{A/H_2O} indicated in Table II., and also those obtained from observations relatively to methyl alcohol and benzene (Table IV.).

TABLE III.

Atomic absorption coefficients for carbon in relation to the molecular absorption coefficient of water.

Substance.	I. κ_{C/H_2O}.	II. κ_{C/H_2O}.	III. κ_{C/H_2O}.	IV. κ_{C/H_2O}.
Methyl formate, $C_2H_4O_2$	0·450	0·475	0·490	0·525
Paraldehyde, $C_6H_{12}O_3$	0·463	0·469	0·497	0·526
Propionic acid, $C_3H_6O_2$	0·457	0·487	0·488	0·524
Allyl alcohol, C_3H_6O	0·460	0·502	0·522	0·544
Acetone, C_3H_6O	0·460	0·489	0·489	0·503
Carbonic acid dimethylester, $C_3H_6O_3$	0·450	0·487	0·530	0·536
Ethyl formate, $C_3H_6O_2$	0·452	0·479	0·505	0·540
Methylal, $C_3H_8O_2$	0·453	0·479	0·512	0·547
Isopropyl alcohol, C_3H_8O	0·445	0·495	0·534	0·531
Ethyl acetate, $C_4H_8O_2$	0·485	0·509	0·529	0·566
Methyl ethyl ketone, C_4H_8O	0·469	0·504	0·518	0·572
Benzyl alcohol, $C_6H_5CH_3O$	0·467	0·485	0·507	0·547
Toluol, $C_6H_5CH_3$	0·463	0·478	0·492	0·505
Octyl alcohol, $C_8H_{18}O$	0·463	0·493	0·504	0·516
Octane, C_8H_{18}	0·470	0·490	0·502	0·515
Methyl alcohol, CH_4O	0·485	0·508	0·494	0·521
Benzene, C_6H_6	0·488	0·506	0·514	0·543
Mean	0·462	0·490	0·508	0·533

There is a very fair agreement between the values found from different compounds, as appears from the table. There does not appear to be any difference between carbon from alifatic and aromatic compounds. If the mean values are computed from the alitatic substances (14 in number) in the first case and from the aromatic compounds (3 in number) in the second, in which latter case an atom of carbon in toluol and benzyl alcohol is supposed to have the same absorption as in alifatic compounds, then we get:

	I.	II.	III.	IV.
Alifatic compounds ...	0·46	0·49	0·50	0·53
Aromatic ,, ...	0·47	0·49	0·50	0·53

Thus we get almost identical values from the two kinds of compounds. A difference which I formerly have found in this respect is entirely elucidated by the fact that, as I pointed out at the time, the values for hydrogen and oxygen were not determined with sufficient accuracy.

It has seemed to me to be of particular interest to examine how far the absorption of carbon in a solid form has the same magnitude as that of carbon in chemical compounds in liquid state. Therefore I have made several experiments on graphite plates. Here, however, we have the difficulty that completely pure graphite is hardly obtainable. Interspersed extraneous matters, most frequently consisting of heavier and hence more strongly absorbing elements, may, even in minute quantities, play an appreciable part in absorption. Professor C. Benedicks has kindly placed Acheson graphite plates at my disposal, which on chemical analysis have proved to contain solely 0·2 per cent. of ash substances, chiefly consisting of Fe. On examining this sort of graphite, the following values have been established:—

	I.	II.	III.	IV.
κ_{C/H_2O} (Acheson graphite) ...	0·59	0·61	0·62	0·62

The values thus obtained are markedly higher than those found by examining organic compounds. Owing to the included metal particles these values are certainly a little too high, but the stated proportion of ashes does not seem to be so great that the difference can be fully explained by it. It may therefore be possible that carbon in the form of graphite is more strongly absorbing than in the form of organic compounds. Possibly the crystallic structure may have some influence. As, on the other hand, the influence of the included particles cannot with certainty be established, the importance of the here observed difference can hardly be discussed more in detail. Yet I should wish to add that on examining other kinds of graphite I have found about the same difference. Since the chemical compounds examined must be regarded as particularly pure, and in any case not to contain heavier elements, we may surely assume that the values found for κ_{C/H_2O} are thoroughly reliable.

In the following table are given, by means of the values found for κ_{H/H_2O}, κ_{C/H_2O}, and κ_{O/H_2O}, the calculated values of κ_{A/H_2O} for the organic compounds mentioned above, compared with corresponding observed values.

Table IV.

Comparison between calculated and observed values for κ_{A/H_2O} in organic compounds.

Substances.	I.		II.		III.		IV.	
	Obs.	Calc.	Obs.	Calc.	Obs.	Calc.	Obs.	Calc.
Methyl alcohol, CH_4O	1·59	1·57	1·62	1·61	1·65	1·66	1·71	1·72
Methyl formate, $C_2H_4O_2$	2·90	2·92	2·95	2·98	2·98	3·02	3·05	3·07
Allyl alcohol, C_3H_6O	2·59	2·60	2·75	2·71	2·88	2·84	3·01	2·98
Acetone, C_3H_6O	2·59	2·60	2·71	2·71	2·78	2·84	2·88	2·98
Propionic acid, $C_3H_6O_2$	3·47	3·49	3·58	3·59	3·62	3·68	3·76	3·79
Ethyl formate, $C_3H_6O_2$	3·46	3·49	3·56	3·59	3·66	3·68	3·81	3·79
Carbonic acid dimetylester, $C_3H_6O_3$	4·35	4·39	4·46	4·47	4·59	4·52	4·61	4·60
Isopropyl alcohol, C_3H_8O	2·65	2·70	2·85	2·84	3·07	2·99	3·16	3·17
Methylal, $C_3H_8O_2$	3·57	3·60	3·68	3·71	3·85	3·84	4·02	3·98
Methyl ethyl ketone, C_4H_8O	3·19	3·16	3·38	3·33	3·61	3·50	3·90	3·70
Ethyl acetate, $C_4H_8O_2$	4·15	4·06	4·28	4·20	4·43	4·34	4·63	4·51
Benzene, C_6H_6	3·25	3·09	3·40	3·31	3·55	3·52	3·83	3·77
Toluol, $C_6H_5CH_3$	3·66	3·65	3·85	3·92	4·08	4·18	4·42	4·49
Benzyl alcohol, $C_6H_5CH_3O$	4·58	4·55	4·77	4·80	5·02	5·02	5·28	5·30
Paraldehyde, $C_6H_{12}O_3$	6·09	6·09	6·24	6·31	6·46	6·52	6·71	6·77
Octane, C_8H_{18}	4·70	4·64	5·02	5·02	5·42	5·47	5·82	5·96
Octyl alcohol, $C_8H_{18}O$	5·54	5·54	5·92	5·90	6·28	6·31	6·64	6·78

For nitrogen I have calculated κ_{N/H_2O} from the determinations of $\kappa_{A_2H/O}$ for the compounds given in Table V., where besides N only H, O, and C are component parts.

In determining κ_{a/H_2O} for other elements than those now dealt with, chemical solutions of different kinds have been used. By determining κ_{A/H_2O} for H_2SO_4, $Na_2S_2O_3$, HCl, NaCl, and NaOH (Table VII.), and also by the aid of

Table V.

Atomic absorption coefficient for nitrogen in relation to the molecular absorption coefficient of water.

Chemical combination.	I.		II.		III.		IV.	
	κ_{A/H_2O}.	κ_{N/H_2O}.	κ_{A/H_2O}.	κ_{N/H_2O}.	κ_{A/H_2O}.	κ_{N/H_2O}	κ_{A/H_2O}.	κ_{N/H_2O}.
Nitric acid, HNO_3	3·17	0·43	3·33	0·64	3·38	0·77	3·39	0·86
Ammonium nitrate, $H_4N_2O_3$	4·11	0·61	4·14	0·63	4·50	0·83	4·71	0·95
Pyridine, C_5H_5N	3·26	0·69	3·49	0·74	3·63	0·70	3·85	0·71
Aniline, $C_6H_5NH_2$	3·78	0·64	4·11	0·74	4·39	0·80	4·70	0·84
Ethyl nitrate, $C_2H_5NO_3$	4·40	0·53	4·50	0·58	4·63	0·69	4·73	0·76
Nitro-benzene, $C_6H_5NO_2$	5·43	0·61	5·70	0·70	5·92	0·79	6·03	0·74
Amyl nitrate, $C_5H_{11}NO_3$	6·17	0·60	6·41	0·66	6·67	0·74	6·91	0·77
Mean		0·59		0·67		0·76		0·80

values found before of κ_{O/H_2O} and κ_{H/H_2O}, I have calculated κ_{A/H_2O} for the radicles given in the table below.

TABLE VI.

Absorption coefficients of some frequently-used radicles in relation to the molecular absorption coefficient of water.

Radicle.	I.	II.	III.	IV.
OH …………	0·95	0·94	0·92	0·91
NH_4 ………	0·80	0·91	1·07	1·18
NO_3 ………	3·27	3·30	3·29	3·24
SO_4 …………	11·57	10·57	9·46	8·26

From the observed values for κ_{A/H_2O} in the compounds examined have been deduced the above values, when with the respective elements the mentioned radicles have been allied, or when other radicles have been allied, the values in the following table :—

TABLE VII.

Atomic absorption coefficients in relation to the molecular absorption coefficient of water.

Atomic number. Z.	Element.	Substance examined.	I.	II.	III.	IV.
1	H	(Table II.) ………	0·052	0·061	0·078	0·094
3	Li	LiOH ……………	0·27	0·29	0·30	0·30
		$LiNO_3$ ……………	0·16	0·24	0·30	0·37
		Mean ……	0·21	0·26	0·30	0·33
4	Be	$Be(NO_3)_2$ …………	0·31	0·32	0·42	0·56
5	B	B_2O_3 (solution in meth. alcohol)	0·50	0·49	0·56	0·60
6	C	(Table III.) ………	0·46	0·49	0·51	0·53
7	N	(Table V.) ………	0·59	0·67	0·76	0·80
8	O	(Table II.) ………	0·90	0·88	0·84	0·81
9	F	C_6H_5F ……………	1·43	1·35	1·29	1·14
11	Na	NaOH ……………	2·16	2·01	1·87	1·66
12	Mg	$MgSO_4$ ……………	2·73	2·26	2·26	1·77
		$Mg(NO_3)_2$ ………	2·48	2·36	1·86	1·58
		Mean ……	2·60	2·31	2·06	1·67

Atomic number. Z.	Element.	Substance examined.	I.	II.	III.	IV.
13	Al	Metal	4·08	3·66	3·23	2·79
14	Si	$C_6H_5Si(CH_3)_3$	5·88	5·05	4·39	3·70
15	P	H_3PO_4	6·16	5·47	4·82	4·18
16	S	H_2SO_4	7·99	6·81	6·04	5·21
		$Na_2S_2O_3$	7·99	7·30	6·12	4·83
		Mean	7·99	7·06	6·08	5·02
17	Cl	HCl	10·25	8·94	7·65	6·39
		NaCl	10·01	8·94	7·53	7·14
		Mean	10·13	8·94	7·59	6·76
19	K	KCl	14·7	12·6	11·6	9·2
		K_2S	15·1	13·3	11·5	9·7
		Mean	14·9	12·9	11·5	9·4
20	Ca	$CaCl_2$	21·3	18·9	15·6	11·2
		$Ca(NO_3)_2$	19·8	18·0	15·4	12·9
		Mean	20·5	18·4	15·5	12·0
23	V	$Na_4V_2O_7$	34·8	30·6	27·2	22·5
24	Cr	K_2CrO_4	40·1	35·7	30·5	25·3
		$K_2Cr_2O_7$	39·6	36·5	30·1	25·6
		$CrCl_3$	39·7	36·6	31·9	24·8
		Mean	39·8	36·3	30·8	25·2
25	Mn	$MnCl_2$	46·2	40·9	35·6	28·9
		$MnSO_4$	48·6	42·0	36·4	28·8
		Mean	47·4	41·4	36·0	28·9
26	Fe	Metal	55·1	48·7	40·8	33·3
		$FeCl_3$	51·5	44·3	39·7	32·1
		$Fe(NH_4)_2(SO_4)_2$	57·2	48·1	41·4	33·5
		Mean	54·6	47·0	40·6	33·0
27	Co	$CoCl_2$	67·6	57·0	51·8	40·6
		$Co(NO_3)_2$	64·0	55·5	47·2	36·9
		Mean	65·8	56·2	49·5	38·8
28	Ni	Metal	69·7	63·8	54·5	46·8
		$NiCl_2$	70·1	63·0	55·0	43·2
		Mean	69·9	63·4	54·7	45·0
29	Cu	Metal	81·0	66·9	54·8	47·4
		$CuCl_2$	78·0	69·5	59·1	48·5
		$CuSO_4$	78·4	70·0	61·0	49·4
		$Cu(NO_3)_2$	78·4	64·8	55·5	48·3
		Mean	78·9	67·8	57·6	48·4

Atomic number. Z.	Element.	Substance examined.	I.	II.	III.	IV.
30	Zn	$ZnCl_2$	88·2	75·6	65·4	52·8
		$ZnSO_4$	88·2	79·6	68·7	56·5
		Mean	88·2	77·6	67·0	54·6
33	As	Na_2HAsO_4	125·6	113·7	98·4	80·1
34	Se	SeO_2	143·0	124·0	109·5	92·2
35	Br	NaBr	152·5	137·7	119·9	100·0
37	Rb	RbCl	204·3	182·1	153·0	128·4
38	Sr	$SrCl_2$	207·2	183·8	164·6	140·4
		$Sr(NO_3)_2$	207·8	188·0	168·8	142·6
		Mean	207·5	185·9	166·7	141·5
42	Mo	$Mo_7O_{34}(NH_4)_6$	267	239	219	185
45	Rh	$RhCl_3$	310	293	266	213
46	Pd	$Pd(NO_3)_2$	333	321	288	235
47	Ag	$AgNO_3$	327	322	313	247
50	Sn	Metal	331	364	372	347
53	I	NaI	326	352	356	366
56	Ba	$BaCl_2$	340	357	387	425
82	Pb	$Pb(NO_3)_2$	675	608	544	460

Generally the values of one and the same element which have been obtained from different solutions agree very closely. Only when using the hardest radiation (IV. 10 mm. Al-filter) the difference seems to be somewhat greater on certain occasions. The agreement in this case being less good may in some part be due to decreased intensity of radiation, and, therefore, the adjustment of the comparator is more difficult. The principal cause is surely that a small variation of the wave-length, in the almost homogeneous radiation, must have a proportionally greater influence than when radiation has been less homogeneous. The agreement of the values found in solutions of chemical compounds with those found in the scrutiny of metals in solid form (Fe, Ni, Cu) is particularly good. So there does not appear to be any occasion for doubting the correctness of the idea that absorption is purely an atomic quality, and consequently that it is independent as well of the aggregation form—except perhaps for C, as was said above—in which the substance is taken as of the manner in which it enters into compounds. More especially the strong rule of the additative law is corroborated by the experiments shown in Table IV., for which the absorption coefficients of the constituent parts are more elaborately determined than in respect to other elements and for which the agreement of observed and calculated coefficients is also the best possible. The deviations from the said law that certain authors think they have found may probably be due either to the error of neglecting the absorption of the lightest elements or else to experimental errors.

For reasons previously mentioned (p. 170) the values of κ_{a/H_2O} found for the different elements have been recalculated for the purpose of obtaining corresponding coefficients in relation to the atomic absorption coefficient of copper; κ_{Cu} is arbitrarily put equal to 100. For the elements Li–V, besides the values found directly from the observations, the corrected values are also stated, which have been obtained in a way that is stated more in full below (p. 187). The values given in parentheses are not included in the calculated mean values.

Table VIII.

Atomic absorption coefficients for the elements below in relation to the atomic absorption coefficient of copper.

Z.	Element.		I.	II.	III.	IV.	Mean.
1	H	obs.	0·067	0·090	0·136	0·195	
		corr.	0·00	0·00	0·00	0·00	0·00
3	Li	obs.	0·27	0·38	0·52	0·68	
		corr.	0·07	0·11	0·11	0·10	0·10
4	Be	obs.	0·39	0·47	0·73	1·16	
		corr.	0·12	0·11	0·19	0·38	0·20
5	B	obs.	0·63	0·72	0·97	1·24	
		corr.	0·30	0·27	0·19	0·26	0·25
6	C	obs.	0·58	0·72	0·89	1·10	
		corr.	0·31	0·36	0·34	0·32	0·33
7	N	obs.	0·75	0·99	1·32	1·66	
		corr.	0·28	0·36	0·37	0·29	0·32
8	O	obs.	1·13	1·30	1·46	1·67	
		corr.	0·86	0·94	0·92	0·89	0·91
9	F	obs.	1·81	1·99	2·24	2·36	
		corr.	1·48	1·54	1·56	1·37	1·50
11	Na	obs.	2·74	2·96	3·25	3·43	
		corr.	2·27	2·33	2·29	2·06	2·24
12	Mg	obs.	3·30	3·41	3·58	3·45	
		corr.	3·03	3·05	3·04	2·67	2·95
13	Al	obs.	5·17	5·40	5·61	5·76	
		corr.	4·84	4·95	4·93	4·78	4·87

Z.	Element.		I.	II.	III.	IV.	Mean.
14	Si	obs.	7·45	7·45	7·62	7·65	
		corr.	7·18	7·09	7·08	6·77	7·03
15	P	obs.	7·81	8·07	8·37	8·64	
		corr.	7·34	7·44	7·42	7·27	7·37
16	S	obs.	10·13	10·40	10·55	10·36	
		corr.	9·86	10·04	10·01	9·58	9·87
17	Cl	obs.	12·84	13·19	13·17	13·96	
		corr.	12·51	12·74	12·49	12·98	12·68
19	K	obs.	18·9	19·0	20·0	19·4	
		corr.	18·4	18·4	19·0	18·0	18·5
20	Ca	obs.	26·0	27·1	26·9	24·8	
		corr.	25·7	26·7	26·4	24·0	25·7
23	V	obs.	44·1	45·1	47·2	46·5	
		corr.	43·6	44·5	46·2	45·1	44·9
24	Cr		50·4	53·6	53·5	52·1	52·4
25	Mn		60·1	61·1	62·5	59·7	60·9
26	Fe		69·2	69·4	70·5	68·2	69·3
27	Co		83·4	82·9	85·9	80·2	83·1
28	Ni		88·6	93·5	95·0	93·0	92·5
29	Cu		100	100	100	100	100
30	Zn		112	114	116	113	114
33	As		159	168	171	166	166
34	Se		181	183	190	191	186
35	Br		193	203	208	206	203
37	Rb		259	269	266	265	265
38	Sr		(263)	274	289	292	285
42	Mo		(338)	353	380	382	371
45	Rh		(393)	432	462	440	445
46	Pd		(422)	(473)	500	486	493
47	Ag		(414)	(475)	544	510	527
50	Sn		420	536	646	717	—
53	I		413	519	618	756	—
56	Ba		431	527	672	878	—
82	Pb		855	898	945	950	—

It appears from the table that $\kappa_{a/\mathrm{Cu}}$ for the lightest elements increases with a decreasing wave-length, but starting from Mg this progress lessens more and more when the atomic number increases. As for the elements Sn, I, and Ba, selective absorption plays a considerable part: that is why the values found for these elements are not comparable with the other values. At Pb, $\kappa_{\mathrm{Pb/Cu}}$ increases markedly as the wave-length is diminishing. By means of very hard rays ($\lambda = 0{\cdot}360$–$0{\cdot}145$) Hull and Rice recently have determined $\frac{\mu}{\rho}$ for Pb and Cu, in which case the following formulæ have been found to hold:—

$$\frac{\mu}{\rho}_{\text{Pb}} = 0{\cdot}12 + 430\lambda^3,$$

$$\frac{\mu}{\rho}_{\text{Cu}} = 0{\cdot}12 + 150\lambda^3.$$

If $\kappa_{\text{Pb/Cu}}$ is calculated from these formulæ for the mean wave-lengths used by the author (Table I.), we obtain

$\kappa_{\text{Pb/Cu}}$:—	I.	II.	III.	IV.	Mean.
according to Hull and Rice.	925	925	922	918	922
,, ,, Table VIII. .	855	898	945	950	912

The mean values found according to the said formulæ and the mean values from my experiments agree very closely; but by comparing the original values, it is seen that whereas in my experiments $\kappa_{\text{Pb/Cu}}$ increases with a decreasing wave-length, these values according to the Hull and Rice formulæ are fairly constant. After Barkla's opinion, however, the absorption coefficient cannot be exactly expressed by the formula $\frac{\mu}{\rho} = \text{C} + k\lambda^b$, because neither C nor b is perfectly constant within a large wave-length range. In my experiments, it is true, the wave-length range has extended markedly on both sides of the mean wave-lengths stated above; but this can hardly account for the difference between the values found according to the Hull and Rice formulæ and those in my experiments. On the other hand, the deviations can possibly be explained by the circumstance that C has not, as these physicists have suggested, the same value in all elements, but is greater for Pb than for Cu. In section IV. I shall return to this question.

If by the Hull and Rice formula for aluminium,

$$\frac{\mu}{\rho}_{\text{Al}} = 0{\cdot}12 + 14{\cdot}9\,\lambda^3,$$

$\kappa_{\text{Al/Cu}}$ is calculated, we obtain the values stated in the table below. In Barkla and White's paper values are given of $\frac{\mu}{\rho}$ for a very wide wave-length range in regard to Al and Cu. By interpolation the necessary values may be obtained for calculating $\kappa_{\mathrm{Al/Cu}}$ in the mean wave-lengths used by the author. The calculated values are given below:

κ Al/Cu :—	I.	II.	III.	IV.
according to Hull and Rice	4·77	4·88	4·99	5·34
,, ,, Barkla and White	4·89	5·52	6·05	5·85
,, ,, Table VIII.	5·17	5·40	5·61	5·76

There is, as appears from the table, a fairly good agreement between my values and those of Barkla and White, whereas those of Hull and Rice are more deviating. The reason why, especially for the lighter elements, there are so great differences between the values of the absorption coefficients recorded by different observers may probably first of all be ascribed to the experimental method. As for Al, the observers may have disregarded that a source of errors lies in the fact that the metal plates existing on the market are seldom perfectly pure. The impurities are always Fe and Si, often Cu and also other elements. From a chemical-technical point of view, common aluminium metal, according to Haber, is regarded as satisfactorily pure if it does not contain more than 1 per cent. Fe and Cu and no more than 0·8 per cent. Si. Supposing that in the metal there is 1 per cent. Fe, this would enhance absorption by 6 or 7 per cent., and if we also admit that there are other impurities the error seems not unlikely to amount to as much as 10 per cent. or more. When analysing the metal used by me, it has been found to contain 0·21 per cent. Fe, 0·29 per cent. Si, and only traces of Cu. Hence the metal is comparatively pure. The error caused by the impurities cannot exceed 2 or 3 per cent., and as this is within the limits of experimental errors no correction has been made in this respect. The circumstance pointed out here as to the quality of the common aluminium metal will suffice to show that it is not quite appropriate to be a standard metal for absorption measures. Besides, Barkla has pointed out that Al in regard to absorption with a diminishing wave-length shows a slight, certainly, but very unmistakable increase at $\lambda = 0{\cdot}37$, which Barkla thinks must depend on the metal sending out a characteristic so-called J-radiation of that wave-length. Barkla, therefore, prefers copper as a standard metal, and I have made use of it also and found it to be quite appropriate for this purpose. The difference observed by different authors in regard to Al absorption may at least partly be explained by difference in quality of the experimental material. In a still higher degree may the impurities disadvantageously influence the X-ray absorption of the lightest elements, a case which was already pointed out in respect to carbon.

III. *Distribution of Electrons between the inner and outer region.*

In accordance with what has been shown in previous experiments, the atomic absorption coefficient, at least within a not too wide wave-length range, may be expressed by the formula

$$\kappa_a = C_a + k_a \lambda^b, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (4)$$

where C_a is a constant depending on the absorption due to the diffusedly reflected radiation, "scattering," k_a is a constant exclusively depending on the chemical nature of the substance, and b has very nearly the value 3.

If we proceed from the theory that absorption is exclusively determined by the number of electrons in the atom and by the manner in which these are attached to the nucleus, it seems highly probable that the electrons arranged in the periphery of the atom, which in the following are designated as *outer electrons* (p), must play quite another part than the inner, more closely-attached electrons. When, in the lines below, the expression *inner electrons* (c) is used it does not designate those hypothetic electrons being a part of the very nucleus, but only those arranged nearer to the nucleus and the number of which corresponds to the atomic number on deducting the number of the outer electrons. These outer electrons [*], on which the optical and chemical qualities of the atom are generally supposed to depend, and which, therefore, are usually designated as "dispersions" or respectively "valence-electrons," must be considered much more elastically attached to the nucleus than the others. Working with X-ray crystal analysis of different substances, also, Hull [†] makes the conclusion that certain electrons constituting the atom are more closely attached to the nucleus than others which show a certain degree of freedom.

The theory brought forward by J. J. Thomson on the scattering phenomenon is founded on the assumption that the electrons are quite independent of each other when subjected to the primary radiation. This assumption may be thought to be fulfilled in respect to the electrons of the surface part of the atom, but not in respect to the electrons

---

[*] Comp. Sommerfeld, *Ann. d. Phys.* B. 53, p. 513 (1917).

[†] Hull, Phys. Rev. x. ser. 2, p. 663 (1917).

arranged in the proximity of the nucleus, more especially if they are arranged in some sort of planetary orbits. If the scattering phenomenon is produced solely by the outer electrons, the absorption due to this process according to the said theory must be proportional to the number of such electrons. Of the two terms of the formula for κ_a, the term C_a seems to be determined by the number of outer electrons.

In the elements, except in the lightest, absorption is chiefly due to the inner electrons and the manner in which they are grouped in relation to the nucleus. On increasing the atomic number, presumably rearrangements of electrons must take place in certain elements, whether according to Bohr one assumes that new electron rings are formed or simply assumes a discontinuous change of concentration by sudden leaps. When such a rearrangement takes place it must make itself perceivable by a sudden increase in absorption. In previous experiments I found that a perceivably sudden increase in absorption takes place in the elements O, S, Mn, and Cu, which in accordance with the same view might be assumed to be due to rearrangements of the inner electrons of these elements. In the experiments I have given an account of above, a similar sudden increase in absorption has been found again in the same elements except Mn. Owing to an error in the analysis of the first-examined solution of $CrCl_3$, somewhat too low a value was obtained for Cr. The point where rearrangements seem to be occurs at this element. In addition, similar points seem to be at the elements Mg and Ca.

At the elements following from Cu there seems not to be any rearrangement of the inner electrons to Pd; but as the experimental errors may possibly be a little greater here than in the preceding elements, this point may not be quite certain.

To obtain comparable values of the absorption coefficients for the lighter elements, it is necessary to correct the values found in relation to the absorption due to scattering. For Cu, with the mean wave-lengths used by me, the term C_a in the formula (4) compared with $k_a\lambda^b$ is certainly very insignificant and may, in maxima, be estimated to 1 per cent. of the whole. If this term is neglected for Cu, we obtain the following formula for calculating $\kappa_{a/\text{Cu}}$:—

$$\kappa_{a/\text{Cu}} = \frac{C_a}{k_{\text{Cu}}\lambda^b} + \frac{k_a}{k_{\text{Cu}}}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (5)$$

or

$$\bar{\kappa}_{a/\text{Cu}} = \kappa_{a/\text{Cu}} - \frac{C_a}{k_{\text{Cu}}\lambda^b} = \frac{k_a}{k_{\text{Cu}}}. \quad . \quad . \quad . \quad (6)$$

The designation $\bar{\kappa}_{a/\mathrm{Cu}}$ below stands for the relative values of the atomic absorption coefficients, which have been corrected for scattering [*]. The correction, however, is of no significance in heavier elements, and in elements of higher atomic number than 23 $\bar{\kappa}_{a/\mathrm{Cu}}$ is therefore assumed to be equal to $\kappa_{a/\mathrm{Cu}}$.

As pointed out above, according to the theory on scattering of J. J. Thomson, every one of the outer electrons must have an equal influence. If we approve of the common view that the hydrogen atom consists of a nucleus combined only with one electron, we can readily find the value of the scattering effect (q) excited by one single electron. Without any noteworthy error we may assume that in the formula (5) $\frac{k_a}{k_{\mathrm{Cu}}}$ may be put equal to zero for H. Thus the absorption of the hydrogen atom would entirely be determined by the scattering effect produced by the electron combined with the nucleus. In the case of the four different compositions of radiation used in my experiments, the absorption coefficient depending on an outer electron in relation to the atomic coefficient of copper will thence give the following values :—

	I.	II.	III.	IV.
$q =$	0·067	0·090	0·136	0·195

Supposing the number of the outer electrons (p) to be known, the correction for scattering, after what has been said above, will simply be pq.

If Li is regarded as being built up of 3, Be of 4, and B of 5 electrons, then the corrected values ($\bar{\kappa}_{a/\mathrm{Cu}}$) for these elements will be obtained by deducting $3q$, $4q$, $5q$ respectively from the observed values. These values are, as seen from Table VIII., fairly constant, except for $\mathrm{Be}_{\mathrm{IV}}$. An experimental error might be responsible for this difference—a circumstance which is easily accounted for, partly by the difficulty in accurately determining absorption in case of the slightly absorbing lightest elements, partly by the above-mentioned difficulty in keeping radiation unchanged when using the hardest rays.

As for C, the simplest thing would evidently be to regard all of the 6 electrons to be arranged as outer electrons ; but if the values found of $\kappa_{\mathrm{C/Cu}}$ are corrected by $6q$, we get:

	I.	II.	III.	IV.
$\kappa_{\mathrm{C/Cu}}-6q =$	0·186	0·183	0·080	−0·069,

---

[*] Comp. Glocker, *Phys. Z.* xix. p. 68 (1918).

which shows that all of the electrons presumably are not active as outer electrons. On the other hand, if we assume that 2 of the electrons are more intimately bound up to the nucleus, and consequently that only 4 electrons are electrons of the remotest orbit, then we obtain the values seen in Table VIII., which, indeed, agree very well. Now, in regard to N, all of the electrons seem to be outer electrons, and the values corrected by $7q$ accordingly agree rather closely. Similarly, we find that O is probably built up of 4 outer electrons; thus the remaining 4 electrons must be regarded as arranged in an inner region. This rearrangement to an inner region may be regarded as causing the above change by a sudden leap in the absorption of the said element. For F ($Z=9$) we are probably right in regarding 5 and for Na ($Z=11$) 7 electrons to be outer electrons. In the elements O–Na the inner region, with 4 electrons as constituent parts, would then remain unchanged. How the four electrons, which so would form the inner region of the elements O–Na, are arranged in relation to each other, is a question to which, of course, my experiments cannot give an answer. Starting from the Rutherford-Bohr atom model, the simplest view seems to be that all of the four electrons are arranged in one ring. It may be well to remember that Vegard* has shown that from quantum-theory this atom model would lead to the result that the inner ring may be built up of either three or four electrons. With four electrons we arrive, according to Vegard, at a particularly simple formula to calculate the frequency in spectra of X-rays; but the author, however, for certain reasons, thinks it more plausible that the ring is built up of only three electrons.

Another rearrangement, due to the passing of 4 other electrons to the inner region, seems to take place at Mg ($Z=12$). The outer region of Mg-atom would then contain 4 electrons, and after correction by $4q$ we get almost constant values of $\kappa_{\mathrm{Mg/Cu}}$. The found values of $\kappa_{a/\mathrm{Cu}}$ give no occasion to think of any change at Al and P. The outer region of Al would consist of 5 and that of P of 7 electrons, which, as shown by the table, is in good agreement with the observed values. Concerning Si ($Z=14$), we obtain the following values after correcting by $6q$:—

	I.	II.	III.	IV.
$\kappa_{\mathrm{Si/Cu}}-6q =$	7·05	6·91	6·81	6·48

To estimate with greater degree of certainty the number of outer electrons by means of the said correction is

---

* Vegard, Phil. Mag. xxxv. p. 302 (1918).

not possible for elements which show greater absorption. Already at Si this method must be regarded as less reliable. The values for Si just mentioned plausibly authorize the assumption that the number of the outer electrons must be less than 6. There is not a long way to the suggestion that the atomic structure of Si is analogous to that of C, which, indeed, in the periodic system stands next above Si in the same vertical row. Consequently, if we suppose 4 outer electrons, the values given in the table will be obtained, which may be regarded as almost constant. In the Si-atom, just as was the case in C, 2 electrons would be more closely attached to the nucleus than the 4 outer electrons. Thus we should be able to account for the value of $\bar{\kappa}_{\mathrm{Si/Cu}}$ being somewhat higher than it ought to be if this element, like the next preceding and following elements, only contained an inner region with 8 electrons. From the sudden increase of absorption at S (Z=16) apparently a new rearrangement takes place at this element, and the above reasons speak in favour of the assumption that here 4 electrons pass from the inner region, which thus should contain 12 electrons. The capacity of absorption indicating that the arrangement of electrons in the inner region of the elements Cl (Z=17) and K (Z=19) remains unchanged, these atoms would contain 5 and 7 electrons respectively in the outer region. At Ca (Z=20) there would occur another rearrangement of the electrons, by which change four other electrons would pass from the outer to the inner region. Thus the inner region of the elements Ca–V would contain 16 electrons and the outer region, as in the preceding group, 4–7 electrons respectively.

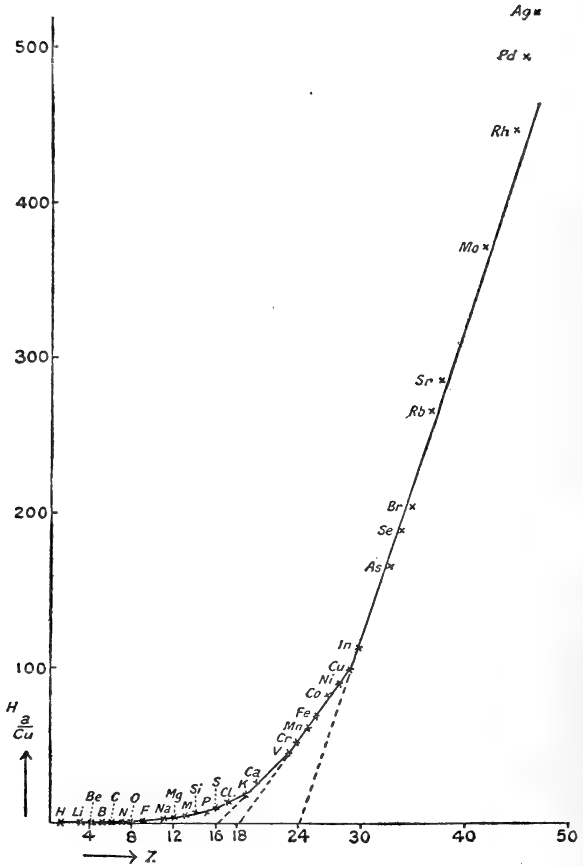
With increasing atomic weight the term $\frac{C_a}{k_{\mathrm{Cu}}\lambda^b}$ becomes, in comparison with the term $\frac{k_a}{k_{\mathrm{Cu}}}$ (5), of still less importance and, certainly in the latter of the just-mentioned elements, lies within the limits of experimental errors. If, however, the relation between $\bar{\kappa}_{a/\mathrm{Cu}}$ and Z is pictured graphically (fig. 2), it is seen that $\kappa_{a/\mathrm{Cu}}$ within each group appears to be very nearly a linear function of the atomic number. Thus the relation between the two quantities may be expressed:

$$\bar{\kappa}_{a/\mathrm{Cu}} = \mathrm{K}(\mathrm{Z}-c), \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (7)$$

where c stands for the number of inner electrons of the elements constituting the group, and K is a constant characteristic of the group. If the straight line designating

$\kappa_{a/\mathrm{Cu}}$ for a certain group is drawn so as to cut the X-axis, the section thus cut off corresponding to the value $Z=c$ will give the number of electrons of the inner region in the group mentioned. By means of a graphical construction we accordingly find that the elements Cr ($Z=24$)—Ni ($Z=28$) form a group having an inner region containing 18 electrons,

Fig. 2.



and that the elements Cu ($Z=29$)—Rh ($Z=45$) form another group whose inner region is characterized by 24 electrons. It must be mentioned that for the last elements in the latter of the just-mentioned groups there may be some uncertainty for the above-given reasons, the more so as the values exhibited in the figure fall a little above the straight line that represents the group. In the group Cr—Ni the number of outer electrons is 6–10 respectively; but presumably there are in these elements two or more electrons more closely attached to the nucleus, as we found it likely to be the case for C and Si.

At Cu the outer electrons would be 5, which will then successively be augmented by one electron at each element of the following group. Now it does not seem very likely that in so great a number of outer electrons as would be present in the outer region for the latter elements in this group all the electrons would occupy an equivalent position. Possibly here, too, some kind of ring formation may take place by means of which certain among the outer electrons may concentrate towards the inner region and so more closely adhere to the nucleus. Thus the circumstance pointed out, that $\kappa_{a/\mathrm{Cu}}$ in the latter elements of this group is somewhat greater than what ought to have been the case if the formula (7) were quite correct, might be explained.

We may well anticipate that the just mentioned rearrangements may be perceivable in the X-ray spectra of the elements. As Sommerfeld * has made probable, the N-ring ought to commence by Mg and the M-ring by Ca. This is in good agreement with my results. That there are no new lines in the spectra of S and Cr—as up to the present had been detected—may be explained easily, if we assume that by rearrangements of the electrons no new ring has been formed, but only the number of electrons in the already existing inner rings has been increased. The sudden increase in absorption by $Z=29$ (or perhaps $Z=30$) seems to stand in some relation to the at-this-point-beginning L-series, which, after our present knowledge, just begins at $Z=30$. This question is, however, not so simple, as it is not evidently proved that elements with lower atomic number than 30 do not show L-lines. Perhaps these lines have escaped detection owing to their considerably great wave-length.

In Table IX. there is an arrangement indicating, in accordance with what is said above, the most probable distribution of electrons between the inner and outer region in the first 23 elements. The first figure given for the respective elements indicates the number of electrons in the inner region and the second figure gives the number of electrons in the outer region. The elements are arranged in accordance with the periodic system.

---

* Sommerfeld, *Phys. Z.* xix. p. 298 (1918).

TABLE IX.—Distribution of Electrons between the outer and inner regions.

	0.	I.	II.	III.	IV.	V.	VI.	VII.
H, 0, 1	He, 0, 2	Li, 0, 3	Be, 0, 4	B, 0, 5	C, 0, 2+4	N, 0, 7	O, 4, 4	F, 4, 5
	Ne, 4, 6	Na, 4, 7	Mg, 8, 4	Al, 8, 5	Si, 8, 2+4	P, 8, 7	S, 12, 4	Cl, 12, 5
	Ar, 12, 6	K, 12, 7	Ca, 16, 4	Sc, 16, 5	Ti, 16, 2+4	V, 16, 7		

As appears from the table, elements with similar chemical qualities will in this distribution of electrons have the same number of outer electrons—only the elements He and Li form exceptions to this rule,—and the periodicity exhibited by the Mendelejeff scheme comes out plainly. The hypothesis of Kossel * regarding the arrangement of electrons in the atom also may be applied here. Instead of 8 electrons, 2 or 6 should, however, be the number of electrons which form a stabile outer ring. I will further discuss this question in a following paper. I only want to mention that the cause of the two periods of 8 elements at the beginning of the scheme seems to be that in the first four rearrangements taking place when the number of electrons is successively augmented, 4 electrons each time pass from the outer to the inner region.

To draw conclusions from the experiments related here regarding the arrangement of electrons within either of the two regions can hardly be possible. That the electrons of the inner region should be distributed to different rings (K-, L-, M-, etc. rings) is in no way in contradiction to the suppositions made here. Neither seems it unlikely that a similar formation of rings can take place also among the outer electrons. It is evident that the values given for the number of electrons of the different elements in the inner regions do not claim to be absolute, as they apparently only give the difference between the respective atomic number and the number of outer electrons.

Bragg and Peirce have, for the relation of the atomic number, established the following formula:

$$\kappa = \mathrm{K} \, . \, Z^4 \, . \, \lambda^{5/2}, \quad . \; . \; . \; . \; . \; . \; . \qquad (8)$$

where K is a constant. Owen † shows that this formula gives better results if the values are corrected for scattering.

---

* Kossel, "Über Molekülbildung als Frage des Atomhaus," *Ann. d. Phys.* xlix. p. 229 (1916).

† Owen, Proc. Roy. Soc. xciv. no. A 664, p. 552 (1918).

Thus we obtain :

$$\bar{\kappa}_{a/\mathrm{Cu}}=\left(\frac{Z_a}{Z_{\mathrm{Cu}}}\right)^4 . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (9)$$

Moore * confirms this equation, but instead of the atomic number he puts the atomic weight (M) ; consequently,

$$\bar{\kappa}_{a/\mathrm{Cu}}=\left(\frac{M_a}{M_{\mathrm{Cu}}}\right)^4 . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (10)$$

Yet Moore admits that the determinations hitherto performed are not sufficiently exact to allow of asserting whether atomic weight or atomic number should properly appear in the formula.

Glocker † recently has derived the following formulæ for the mass absorption coefficient :

$$\frac{\bar{\mu}}{\rho}=0{\cdot}0004\,Z^{3{\cdot}14}\lambda^{2{\cdot}8} \text{ for } \lambda>\lambda_{\mathrm{A}}, \quad . \quad . \quad . \quad (11)$$

$$\frac{\bar{\mu}}{\rho}=0{\cdot}0195\,Z^{2{\cdot}58}\lambda^{2{\cdot}8} \text{ for } \lambda\leqq\lambda_{\mathrm{A}}, \quad . \quad . \quad . \quad (12)$$

where λ_{A} indicates the wave-length of the absorption-zone limit for K-radiation. Now, as in all examined elements with $Z<47\lambda_m<\lambda_{\mathrm{A}}$, it follows that $\kappa_{a/\mathrm{Cu}}$ for these elements might be calculated after the formula

$$\bar{\kappa}_{a/\mathrm{Cu}}=\left(\frac{Z_a}{Z_{\mathrm{Cu}}}\right)^{2{\cdot}58} \cdot \frac{M}{M_{\mathrm{Cu}}} . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (13)$$

In Table XI. are arranged the values of $\kappa_{a/\mathrm{Cu}}$ which have been obtained from observations, mean amounts from Table VIII., with values of the same constant that have been calculated by the aid of the formulæ (7), (9), (10), and (13). In the formula (7) the values of K seen in the following table have been used.

Table X.—Values of K in Formula (7).

Group.	Z.	Electrons of the inner regions.	K.
I.	1– 7	0	0·05
II.	8–11	4	0·26
III.	12–15	8	0·98
IV.	16–19	12	2·55
V.	20–23	16	6·42
VI.	24–28	18	9·00
VII.	29–46 ?	24	19·8

---

* Moore, Phys. Soc. of London, xxvii. p. 432 (1918).

† Glocker, *Phys. Z.* xix. p. 71 (1918).

Table XI.

Comparison between observed and calculated values of $\bar{\kappa}_{a/\mathrm{Cu}}$.

Element.	Atomic number. Z.	Atomic weight. M.	$\bar{\kappa}_{a/\mathrm{Cu}}$, after Table VIII.	K(Z−*c*).	$100\left(\frac{Z_a}{Z_{\mathrm{Cu}}}\right)^4$.	$100\left(\frac{M_a}{M_{\mathrm{Cu}}}\right)^4$.	$100\left(\frac{Z_a}{Z_{\mathrm{Cu}}}\right)^{2·58}\frac{M_a}{M_{\mathrm{Cu}}}$
H	1	1·0	—	—	—	—	—
Li	3	6·94	*0·10*	0·15	0·00	0·01	0·03
Be	4	9·1	*0·20*	0·20	0·04	0·04	0·09
B	5	11·0	*0·25*	0·25	0·09	0·09	0·19
C	6	12·0	*0·33*	0·30	0·18	0·13	0·32
N	7	14·0	*0·32*	0·35	0·34	0·24	0·56
O	8	16·0	*0·91*	1·04	0·58	0·40	0·91
F	9	19·0	*1·50*	1·30	0·93	0·80	1·46
Na	11	23·0	*2·24*	1·82	2·07	1·71	2·96
Mg	12	24·3	*2·95*	3·92	2·93	2·13	3·92
Al	13	27·1	*4·87*	4·90	4·04	3·30	5·37
Si	14	28·3	*7·03*	5·88	5·44	3·92	6·80
P	15	31·0	*7·37*	6·86	7·17	5·64	8·90
S	16	32·1	*9·87*	10·20	9·27	6·49	10·90
Cl	17	35·5	*12·68*	12·75	11·81	9·70	14·10

K	19	39·1	*18·5*	17·85	18·4	14·3	20·7
Ca	20	40·1	*25·7*	25·6	22·6	15·8	24·2
V	23	51·0	*44·9*	44·9	39·6	41·3	44·1
Cr	24	52·0	*52·4*	54·0	46·9	44·7	50·1
Mn	25	54·9	*60·9*	63·0	55·2	55·5	58·8
Fe	26	55·8	*69·3*	72·0	64·6	59·2	66·0
Co	27	59·0	*83·1*	81·0	75·1	74·1	77·1
Ni	28	58·7	*92·5*	90·0	87·0	72·5	84·3
Cu	29	63·6	*100·0*	99·0	100·0	100·0	100·0
Zn	30	65·4	*114*	119	114	112	112
As	33	75·0	*166*	178	168	193	165
Se	34	79·2	*186*	198	189	240	188
Br	35	79·9	*203*	218	212	249	201
Rb	37	85·4	*265*	257	265	325	252
Sr	38	87·6	*285*	277	295	360	277
Mo	42	96·0	*371*	356	441	519	392
Rh	45	102·9	*445*	416	580	685	503
Pd	46	106·7	*493*	—	633	793	550
Ag	47	107·9	*527*	—	690	828	590

From the table it appears that the values calculated by means of the formula (9) undoubtedly agree much better with the observed values than those obtained by means of the formula (10). The formula (9), as Owen using the values in my first paper also shows, approximately gives the relation between the relative atomic absorption coefficient and the atomic number of a great number of elements rather satisfactorily, but we cannot pretend that it has proved to be the proper expression of the real facts over the whole range of the section examined. For the elements Li–F they will thus turn out to be too low, for Ca–Ni all of the calculated values are lower than the observed, and for Sr–Ag all the calculated values are too high. On the whole, the Glocker's formula (13) describes the relation of $\overline{\kappa}_{a/\mathrm{Cu}}$ and Z in a better way than that just mentioned; but here, too, marked deviations occur, for instance at N, P, Cl, K, and for Ca–Ni as well as for Mo–Ag the same view will be true as for the formula (9), though the deviations here are considerably less.

Thus, although the atomic absorption coefficient in a great number of elements may be pronounced to be proportional to a power of the atomic number, such a formula, however, cannot render the real relation. From the above experiments it is clear that in certain elements there are some very marked discontinuities in the values of $\overline{\kappa}_{a/\mathrm{Cu}}$, which in the lighter elements are still more marked by the change by sudden leaps of the absorption effected by scattering (fig. 4). As shown by Table XI. and fig. 2, the different elements may be arranged into groups within which $\overline{\kappa}_{a/\mathrm{Cu}}$ according to formula (7) is expressed as a linear function of the atomic number. Great deviations do not occur except at Mg and Si. In respect to the latter element, the probable cause of the deviation has already been discussed. As to the former element, the deviation presumably depends on the fact that in the examined salts absorption for about 80 per cent. is determined by the negative radicle, and that therefore the errors in determining the absorption of these radicles will proportionally too much influence the calculation of $\kappa_{\mathrm{Mg}/\mathrm{Cu}}$.

IV. *Outer Electrons and Scattered Radiation.*

It was shown previously that the part of the relative absorption coefficient which was thought to be due to the outer electrons is in a high degree dependent on the wave-length. Now, as the mean wave-lengths employed in my experiments have only comprised a relatively limited wave-length range ($\lambda = 0{\cdot}38-0{\cdot}30$), the said experiments do not afford sufficiently safe starting-points for judging the connexion between absorption due to the outer electrons and the wave-length. Barkla and White have in their above-mentioned work given the results of a set of observations regarding $\frac{\mu}{\rho}$ for Cu, Al, and H_2O, comprising a very large wave-length range. In the table below we find again both the values taken from the said work for $\frac{\mu}{\rho}$ of Cu and H_2O and the values of κ_{Cu/H_2O} and q calculated by that means. The last-mentioned values have been obtained by aid of the value of $\bar{\kappa}_{a/Cu}$ for Cu and O,

Table XII.

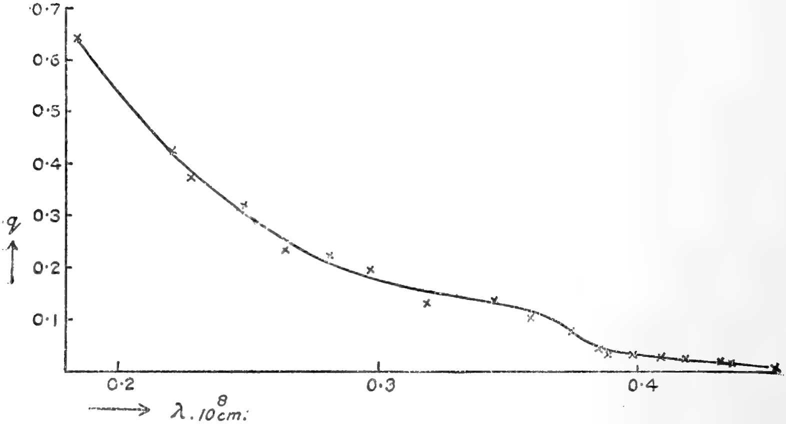
κ_{Cu/H_2O} and q calculated from the table of Barkla and White.

λ.	$\frac{\mu}{\rho}$ Cu.	$\frac{\mu}{\rho}$ H_2O.	κ_{Cu/H_2O}.	q.	$q\frac{\mu}{\rho}$ Cu.	Mean.
0·504	18·10	0·539	118·7	−0·011		
0·499	17·7	0·524	119·4	−0·012		
0·477	15·4	0·479	113·6	−0·005		
0·468	14·55	0·473	108·7	+0·002		
0·460	13·75	0·436	111·5	−0·002		
0·454	13·10	0·411	112·7	−0·004		
0·437	11·6	0·400	102·5	+0·011		
0·433	11·2	0·388	102·0	0·012		
0·419	9·98	0·371	95·1	0·024		
0·409	9·28	0·349	94·0	0·024		
0·400	8·55	0·329	92·1	0·030		
0·390	7·82	0·302	91·5	0·031		
0·387	7·61	0·312	86·2	0·042		
0·376	6·90	0·335	72·8	0·078	0·538	0·678
0·359	5·87	0·311	66·7	0·100	0·587	
0·346	5·17	0·314	58·2	0·137	0·709	
0·320	4·54	0·270	59·4	0·131	0·594	
0·299	3·71	0·271	48·4	0·196	0·727	
0·283	3·20	0·253	44·7	0·226	0·723	
0·266	2·77	0·226	43·3	0·238	0·660	
0·250	2·33	0·229	35·9	0·320	0·722	
0·229	1·86	0·203	32·4	0·376	0·699	
0·222	1·71	0·204	29·6	0·423	0·723	
0·187	1·20	0·196	21·6	0·645	0·774	

assuming that there are 5 outer electrons in Cu and 6 in H_2O, consequently after the formula

$$\frac{100+5q}{0{\cdot}91+6q}=\kappa_{\mathrm{Cu/H_2O}}. \qquad (14)$$

Fig. 3.



It appears from the table that q practically is 0 for radiation of greater wave-length than $\lambda=0{\cdot}42$, which roughly indicates the limit of the absorption-zone in the J-radiation for carbon found by Barkla. With diminishing wave-length q increases, as shown in fig. 3, where the values of q found in Table XII. are graphically indicated, very rapidly to a wave-length of about 0·37, whereupon q continually increases with a still more diminishing wave-length. In regard to the fact that characteristic radiation of the wave-length indicated for J-radiation according to what has been mentioned by Siegbahn[†] has not been possible to state directly, there is good evidence for believing that this same increase of scattering within the range of $\lambda=0{\cdot}42$–$0{\cdot}37$ is the cause of the increase of absorption that has been the occasion of supposing an especial J-radiation.

It is seen from the table that $q\frac{\mu}{\rho}_{\mathrm{Cu}}$, beginning at a wave-length of $\lambda=0{\cdot}37$, is nearly constant with a diminishing wave-length. Starting from the medium value (0·678) of the values found for $q\frac{\mu}{\rho}_{\mathrm{Cu}}$, one readily finds the value of q when $\frac{\mu}{\rho}_{\mathrm{Cu}}$ for the wave-length in question

---

[*] Siegbahn, *Die Naturwissenschaften*, Heft xxxiii. (1917).

is known. In the table below are arranged the values of q found in this way, from the table of Barkla and White, corresponding to values of κ_{Al/H_2O} (Obs.) calculated from the table referred to, and (Calc.) calculated by aid of my values of $\bar{\kappa}_{Al/Cu}$ and $\bar{\kappa}_{O/Cu}$ previously found thus, after the formula

$$\kappa_{Al/H_2O} = \frac{4\cdot87+5q}{0\cdot91+6q}. \quad . \quad . \quad . \quad . \quad . \quad . \qquad (15)$$

Table XIII.

Comparison between observed and calculated values of κ_{Al/H_2O}.

λ.	$\frac{\mu}{\rho}$ Cu.	$\frac{\mu}{\rho}$ Al.	$\frac{\mu}{\rho}$ H_2O.	$q=\frac{0\cdot678}{\frac{\mu}{\rho}\text{ Cu}}$.	κ_{Al/H_2O}.		$\frac{4\cdot87+6q}{0\cdot91+6q}$.
					Obs.	Calc.	
0·376	6·90	0·802 *	0·335	0·098	3·60	3·58	3·64
0·368	6·38	0·707	0·327	0·106	3·25	3·51	3·55
0·345	5·00	0·713	0·305 *	0·136	3·52	3·21	3·29
0·320	4·54	0·622 *	0·270	0·149	3·47	3·12	3·20
0·303	3·77	0·519	0·265 *	0·180	2·95	2·90	2·99
0·283	3·20	0·450 *	0·253	0·212	2·69	2·72	2·82
0·266	2·77	0·391	0·226	0·245	2·60	2·56	2·66
0·257	2·58	0·382	0·224	0·263	2·57	2·49	2·59
0·229	1·94	0·300	0·203	0·349	2·22	2·20	2·32
0·212	1·54	0·263	0·199	0·440	1·99	1·99	2·12
0·192	1·29	0·228	0·196 *	0·526	1·75	1·84	1·97
0·175	1·03	0·210	0·194 *	0·658	1·63	1·68	1·82
0·166	0·95	0·192	0·194 *	0·714	1·49	1·63	1·76

* Inter- or extra-polated values.

There is a very good agreement between the observed and calculated values of κ_{Al/H_2O}, as the table shows. If we compare the values of q that have been calculated in the above way for the medium wave-length used in my experiments with the values resulting from the experiments, we obtain :

	I.	II.	III.	IV.
$q=\frac{0\cdot678}{\frac{\mu}{\rho}\text{ Cu}}$. . .	0·095	0·115	0·139	0·182
$q_{obs.}$	0·067	0·090	0·136	0·195

The values observed for q corresponding to the radiations

I. and II. being somewhat lower than those calculated may be due to the fact that, with a thinner filter, besides waves of shorter wave-lengths there have occurred rays of an appreciable intensity of greater wave-length than $\lambda=0{\cdot}37$. As the found values of $\overline{\kappa}_{a/\mathrm{Cu}}$ for the lighter elements have been calculated by aid of the observed values of q for the radiation referred to, no error can have originated in the fact that the wave-length range has partly extended beyond the limit just mentioned. From what is said it appears, however, that the formula $\kappa=\mathrm{C}+\mathrm{A}\lambda^b$ is not applicable on different sides of the limit unless C gets the value corresponding to respective wave-length, and more especially this holds regarding the lighter elements, in which C plays a proportionally great part.

In case we assume that in Al 6, instead of 5, electrons are arranged as outer electrons, and accordingly $\kappa_{\mathrm{Al/H_2O}}$ is calculated by the formula $\frac{4{\cdot}87+6q}{0{\cdot}91+6q}$, the values stated in the last column of Table XIII. will be obtained. The agreement of the thus calculated and the observed values is for the greater wave-lengths about as good as in the case where the calculation is made after the formula (14); but, on the other hand, the agreement for the shorter wave-lengths is less good and shows a clear divergence.

If we believe that the number of electrons is the same as the atomic number, then with decreasing wave-length the limiting value for $\kappa_{\mathrm{Al/H_2O}}$ must be $\frac{13}{10}$. But Barkla and White have shown that $\frac{\mu}{\rho}$ for Al, in case of shorter waves, tends towards being less than for H_2O, and, in case of sufficiently short waves $\kappa_{\mathrm{Al/H_2O}}$ must be apt to be less than 1. If, according to the formula (15), we merely suppose 5 outer electrons in Al and 6 in H_2O, then $\kappa_{\mathrm{Al/H_2O}}$ will get the value of 1 when q has reached the value of $3{\cdot}96$; and as this value still increases with decreasing wave-length, $\kappa_{\mathrm{Al/H_2O}}$ will be <1. Thus we have reasons for believing that the number of outer electrons must be less in Al than in H_2O, and the observations (Table XIII.) show that the relation of these electron numbers may very closely be regarded as 5 : 6.

From the known values of q the mass scattering coefficient $\left(\frac{s}{\rho}\right)$ may be calculated. According to what was said above, it is admitted that

$$\kappa_{a/\mathrm{Cu}} = \frac{\frac{\mu}{\rho} a \, \mathrm{M}_a}{\frac{\mu}{\rho}\mathrm{Cu} \;\; \mathrm{M}_{\mathrm{Cu}}} = \frac{\mathrm{K}_a + p_a q}{100 + p_{\mathrm{Cu}}\, q},$$

or

$$\frac{\mu}{\rho} a = \frac{\mathrm{M}_{\mathrm{Cu}} \,.\, \mathrm{K}_a}{\mathrm{M}_a(100 + p_{\mathrm{Cu}} q)} \frac{\mu}{\rho}\,\mathrm{Cu} + \frac{\mathrm{M}_{\mathrm{Cu}}\, p_a q}{\mathrm{M}_a(100 + p_{\mathrm{Cu}} q)} \frac{\mu}{\rho}\,\mathrm{Cu}\,; \quad . \qquad (16)$$

and as $\frac{\mu}{\rho} = \mathrm{A}\lambda^b + \frac{s}{\rho}$, then will

$$\frac{s}{\rho} = \frac{\mathrm{M}_{\mathrm{Cu}}\, p_a q}{\mathrm{M}_a(100 + p_{\mathrm{Cu}}\, q)} \cdot \frac{\mu}{\rho}\,\mathrm{Cu}, \quad . \quad . \quad . \quad . \qquad (17)$$

or as $q \frac{\mu}{\rho}\,\mathrm{Cu} = 0{\cdot}678$ and $p_{\mathrm{Cu}} q$ is supposed to be negligible,

$$\frac{s}{\rho} = \frac{\mathrm{M}_{\mathrm{Cu}}}{\mathrm{M}_a} \cdot p_a \,.\, 0{\cdot}00678. \quad . \quad . \quad . \quad . \quad . \quad . \qquad (18)$$

In the following table we again find the mass scattering coefficients for the elements 1–20, calculated by the formula (18).

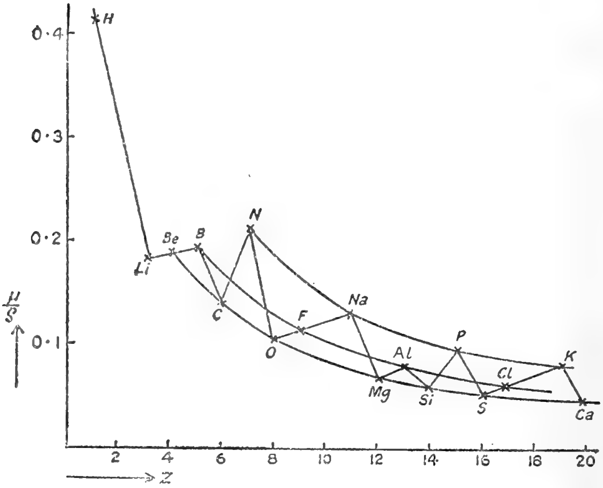
Table XIV.

Mass scattering coefficients for the elements 1–20.

Atomic number.	Element.	Number of outer electrons.	$\frac{s}{\rho}$, calculated according to	
Z.		p.	formula (18).	formula (19).
1	H	1	0·432	0·205
3	Li	3	0·187	0·205
4	Be	4	0·190	0·205
5	B	5	0·196	0·205
6	C	4	0·144	0·137
7	N	7	0·216	0·205
8	O	4	0·108	0·103
9	F	5	0·114	0·114
11	Na	7	0·132	0·130
12	Mg	4	0·071	0·068
13	Al	5	0·080	0·079
14	Si	4	0·061	0·059
15	P	7	0·097	0·096
16	S	4	0·052	0·051
17	Cl	5	0·061	0·060
19	K	7	0·077	0·076
20	Ca	4	0·043	0·041

In fig. 4 the connexion between $\frac{\mu}{\rho}$, calculated by the formula (18), and Z is given graphically. As we find from the figure, the values, except for H and Li, are distributed to three hyperbolas corresponding to $p=7$, $p=5$, $p=4$ respectively.

Fig. 4.



But in regard to H, the Table XIV. shows that the mass scattering coefficient also may be expressed by the formula

$$\frac{s}{\rho}=\frac{0{\cdot}205p}{Z}. \quad \ldots \ldots \quad (19)$$

For elements, in case of which $p=Z$, that is identical with the known value given by Barkla for the light elements. In case of H, the formula (19) is not applicable and by means of it we get a value that is merely about half the real. This, however, is in good agreement with experience; for after what is shown by the experiments of Barkla and Crowther, $\frac{s}{\rho}$ is twice as great for H as for other lighter elements. The exceptional position seemingly occupied by H is in appearance only, and may be fully explained if one takes as granted that not all electrons which constitute the atom, but merely the outer ones, give off scattered rays. On the other hand, as has been clearly made out, not only by above-mentioned experiments on C, N, and O, but also by the Barkla and White determinations of $\frac{\mu}{\rho}$ for Al and H_2O, the assumption that $\frac{s}{\rho}$ for all light elements has the same value cannot be considered to be in harmony with reality; and still less, I think, can this be the case for the heavier elements, as was assumed by Hull and Rice. So it follows that all evidence of the assumption that the number of electrons in the lightest elements should be the same as the atomic number, which is based on the estimate of mass scattering coefficients, cannot be thought conclusive except perhaps for the elements H, Li, Be, B, and N.

Glocker has pointed out that $\frac{s}{\rho}$ in case of Al must be less than 0·2, because already at the wave-length $\lambda=0{\cdot}16$ the uncorrected mass absorption coefficient $\left(\frac{\mu}{\rho}\right)$ is 0·18, and $\frac{s}{\rho}$, as we know, can only be a part of the whole absorption coefficient. Now, for the lightest elements (Li, Be, B, N) the value 0·2 must undoubtedly be nearly correct; thus $\frac{s}{\rho}$ cannot have the same value for all the lighter elements. As I have shown before (Table XIII.), the value 0·080 may well agree with the determinations of $\frac{\mu}{\rho}$ by Barkla and White for Al.

As was previously pointed out (p. 183), $\kappa_{\mathrm{Pb/Cu}}$ rapidly increases with decreasing wave-length. This would, according to what has been said above, mean that absorption dependent on scattering is much greater in case of Pb than of Cu, *i. e.* that in an atom of Pb there would be present a considerably greater number of outer electrons than in an atom of Cu. Yet whether the simple law that defines the scattering power in case of the lighter elements be in force in case of the heaviest must for the present be left undecided. Presumably the electrons in case of the heavier elements are not active individually but in groups, as is the opinion of Barkla and White. By direct determinations of $\frac{s}{\rho}$, Barkla and Dunlop* have shown that $\frac{\left(\frac{s}{\rho}\right)_{\mathrm{Pb}}}{\left(\frac{s}{\rho}\right)_{\mathrm{Al}}}$ is about 2 or 4 times greater than $\frac{\left(\frac{s}{\rho}\right)_{\mathrm{Cu}}}{\left(\frac{s}{\rho}\right)_{\mathrm{Al}}}$.

---

* Barkla and Dunlop, Phil. Mag. xxxi. p. 222 (1916).

V. *Absorption and Scattering of γ-rays.*

Formula (18) is valid merely on condition that the term $p_{\mathrm{Cu}}q$ is so small relatively to 100 that it is negligible. Already, at a wave-length of $\lambda=0{\cdot}166$, on the assumption that in Cu there are 5 outer electrons, absorption produced by scattering amounts to 3·4 per cent. of the whole absorption; and in case of a still shorter wave-length this part is very rapidly augmented. According to what was pointed out by Barkla and Glocker, absorption for the heavier elements will therefore in γ-radiation in a great measure, and for the lighter elements necessarily mostly, depend on scattering. When calculating $\frac{s}{\rho}$ of γ-radiation, the term $p_{\mathrm{Cu}}q$ is therefore not negligible, and in this case formula (17) must be used.

Without too great a mistake we may assume that the ratio $q\frac{\mu}{\rho}_{\mathrm{Cu}}=0{\cdot}678$ is correct also for shorter wave-lengths than those given in Table XII.; then, in consequence, the value of q can be found when $\frac{\mu}{\rho}_{\mathrm{Cu}}$ is known. In Table XV. we again find the last-mentioned values for γ-rays emitted from some radioactive elements in accordance with determinations performed by Soddy and Russell. Indeed, λ_m may not be thought the same for the different substances, but the values of $\frac{\mu}{\rho}$ showing that the difference cannot be very great, mean values for $\frac{\mu}{\rho}_{\mathrm{Cu}}$ may be used for approximate calculation. By aid of this mean value, 0·0407, we find q to be 16·7, which, put in formula (17), for Al gives the value 0·043. By direct observations, Ishino [*] has for the same metal found the value 0·045, which very nearly agrees with the value calculated here.

When inserting the just-named value of q, also $\kappa_{a/\mathrm{Cu}}$ for Fe, S and Al may be calculated if the values found for $\bar{\kappa}_{a/\mathrm{Cu}}$ are assumed to be valid for γ-rays. Using the values given in Table VIII., and assuming that p-values for Al, S, Cu according to the above are 5, 4, 5 respectively, and for Fe 5, we in the same way as in the case of formula (14) obtain the values given in the table below. The assumption that there are 5 outer electrons in Fe is quite arbitrary, but

---

[*] Ishino, Phil. Mag. xxxiii. p. 129 (1917).

in accordance with the circumstance that, as was previously stated (p. 191), the number may probably be less than 8.

Table XV.

$\kappa_{a/\text{Cu}}$ in γ-rays, according to Soddy and Russell.

Element.	Thorium D.		Radium C.		Mesothorium 2.		Uranium X.		Mean.	
	$\frac{\mu}{\rho}$.	$\kappa_{a/\text{Cu}}$.	$\frac{\mu}{\rho}$.	$\kappa_{a/\text{Cu}}$.	$\frac{\mu}{\rho}$.	$\kappa_{a/\text{Cu}}$.	$\frac{\mu}{\rho}$	$\kappa_{a/\text{Cu}}$.	Obs.	Calc.
Cu …………	0·0334	100	0·0398	100	0·0423	100	0·0472	100	100	
Fe …………	0·0328	86·2	0·0399	87·9	0·0415	86·1	0·0472	87·8	87·1	83·7
S ……………	0·0369	55·8	0·0438	55·6	0·0465	55·5	0·0516	55·2	55·5	41·7
Al……………	0·0324	41·3	0·0406	43·5	0·0421	42·4	0·0469	42·3	42·3	48·1
Paraffin-wax.	0·0361		0·0464		0·0580		0·0502			

Merely approximate calculations being here at issue, the agreement of the medium amounts of the observed and calculated values of $\kappa_{a/\text{Cu}}$ seems to be as good as can reasonably be demanded, more especially when considering that determinations of $\frac{\mu}{\rho}$ for γ-rays are in a high degree dependent on the experimental arrangements *.

In the case of the lighter elements absorption of γ-rays will chiefly depend on the number of outer electrons, and this being the same or nearly the same the values of $\frac{\mu}{\rho}$ for these elements must tend towards being the same. As for S, however, the observed absorption coefficients seem to be somewhat greater than what would have been expected on account of the above-assumed number of outer electrons; but this question, I think, ought to remain undecided till a wider range of observations on absorption of γ-rays is at hand.

The apparent anomaly that has been observed in respect to absorption in paraffin-wax, and that consists in $\frac{\mu}{\rho}$ for this substance being equally great or even on some occasions greater than for the heavy metals, may be explained in the same way as was done for absorption of X-rays of very short wave-length in case of Al and H_2O. In paraffin-wax there are, as we know, a great number of hydrogen atoms—a fact that makes the number of outer electrons greater, counted by mass unit, than in metals.

---

* Comp. Kohlrausch, *Jahrb. d. Rad. u. Elektronik*, xv. p. 93 (1918).

Taking as granted that scattering merely depends on outer electrons, we thus can easily account for the observations performed. It may therefore be superfluous to assume that the electrons should behave otherwise for different wave-lengths. On the contrary, we have every reason for believing, a fact advocated by Glocker, the simple law that was found valid for X-rays also to be in force for γ-rays.

Summary.

(1) By aid of the compensation method described by the author for the purpose of determining the relative absorption coefficients of X-radiation, a number of different elements have been examined.

(2) In the chemical compounds examined the additive law has been found thoroughly valid. Perhaps with the exception of carbon, the state of aggregation appears to have had no influence on the quantity of absorption. In examining chemical compounds of which the same element but in different valences is a constituent part, I have not been able to ascertain any difference as to the quantity of absorption of the element at issue.

(3) The relation between the atomic absorption coefficients for most elements, starting from H and inclusive of Ag and for Pb, and the same coefficient for Cu, have been determined at the medium wave-lengths $\lambda=0{\cdot}38$, $0{\cdot}36$, $0{\cdot}34$, and $0{\cdot}30 \,.\, 10^{-8}$ cm.

(4) On assuming that absorption for H is exclusively due to scattering produced by the electron combined with the atomic nucleus, scattering for other elements has been found likely to be solely due to the electrons constituting the outer layer of the respective atoms, *i. e.* to the "outer electrons," and by aid of the relative atomic absorption coefficient for H, the number of the outer electrons has been estimated for the lighter elements.

(5) The atomic absorption coefficient increases for different elements by groups nearly proportionally to the atomic number. If we accept that the atomic number gives the number of electrons combined with the atomic nucleus, we may by the increase of absorption with increasing atomic number determine the distribution of electrons between the outer and inner region.

(6) From the determinations of $\frac{\mu}{\rho}$ for Cu and H_2O of Barkla and White a formula of the mass scattering coefficient for different elements has been calculated.

(7) The values calculated for the relative absorption coefficients for γ-rays agree fairly with the observed absorption coefficients. The value of mass scattering coefficient of Al for γ-radiation, calculated by means of the said formula, agrees very nearly with the value directly observed by Ishino.

(8) The number of the outer electrons in the lighter elements seem to be the same for the element placed in the same vertical row of the periodic system; and the distribution of electrons thus appears to be in close connexion with the periodicity of the chemical qualities of the elements as expressed by this system.

Nobel Institute for Physical Chemistry,
Stockholm, June 1918.

XVIII. *Calculation of Scattered Radiation from a Plate exposed to a beam of X-rays.* *By* Oskar Klein, *Lic. phil.*[*]

In a paper treating of the absorption and scattering of γ-rays, Ishino [†] has undertaken a few theoretical calculations of scattered radiation issuing from the layer of a substance. Such calculations combined with measurements of the radiation scattered within known solid angles would lead to a determination of the scattering coefficient.

Some time ago T. E. Aurén, in connexion with his researches on the absorption of X-rays, performed some experiments in this direction. He measured the absorption coefficient in different positions of the plate exposed to the primary radiation. If the scattered radiation had been of any considerable amount, he would have found different values for the apparent absorption coefficient in the different positions. Now the differences were very small. Therefore it was of interest to Aurén to know how far this was in agreement with the theory. For this purpose I have undertaken the calculation below. As mathematical formulæ are obtained which may be used in numerical calculation, the results may perhaps have a certain interest for those who treat these questions experimentally. In this connexion it must be mentioned that Ishino in his paper just cited has made an error in reckoning, which has caused his final value to take the

---

[*] Communicated by the Author.
Since this paper was finished and sent to the Phil. Mag. a quite similar calculation has been published by Glocher, *Phys. Zeitschr.* xix. p. 251 (1918).

[†] Ishino, Phil. Mag. vol. xxxiii. p. 129 (1917).

form of a divergent integral. Moreover, he never tried to evaluate this integral, a fact which accounts for his not observing the mistake. At the outset of the calculation I have been guided completely by Ishino. There is nothing new in what follows as regards the mathematics.

Suppose a plate of thickness l is struck by a radiation, whose intensity is I. The absorption coefficient may be κ, the scattering coefficient σ. When the radiation has penetrated the plate to the depth x, its intensity has been diminished to $\mathrm{I}e^{-(\kappa+\sigma)x}$. The quantity of scattered radiation issuing from the distance between x and $x+dx$ is evidently $\sigma\mathrm{I}e^{-(\kappa+\sigma)x}dx$. But it is not distributed alike at all angles. J. J. Thomson has theoretically arrived at the following expression for that part which is radiated from the solid angle dw, whose direction makes the angle θ with the incident radiation :

$$d\sigma = k(1+\cos^2\theta)dw.$$

If this expression be integrated over all directions we ought to have σ.

Consequently

$$\sigma = 2\pi k\int_0^\pi (1+\cos^2\theta)\sin\theta\,d\theta = \frac{16\pi}{3}k.$$

$$\therefore \qquad k = \frac{3\sigma}{16\pi},$$

$$d\sigma = \tfrac{3}{8}\sigma(1+\cos^2\theta)\sin\theta\,d\theta.$$

This expression, according to experimental investigations, does not seem to be quite correct. For what we have in view, however, the error will scarcely be of any importance. I therefore adopt the following expression for the radiation that is scattered from the path dx between two cones, whose generatrices form the angles θ and $\theta+d\theta$ with the incident radiation when leaving the plate :

$$\tfrac{3}{8}\mathrm{I}\sigma(1+\cos^2\theta)e^{-(\kappa+\sigma)x}\sin\theta\,d\theta\,dx.$$

Owing to absorption and scattering, the intensity of this radiation when leaving the plate will be

$$\tfrac{3}{8}\mathrm{I}\sigma(1+\cos^2\theta)\sin\theta\,e^{-(\kappa+\sigma)x}e^{-(\kappa+\sigma)\frac{(l-x)}{\cos\theta}}d\theta\,dx,$$

for the radiation must travel the distance $\dfrac{l-x}{\cos\theta}$. The radiation issuing from the whole plate and falling within a cone whose generatrices form the angle θ with the incident radiation will then be

$$S=\int_0^{\theta_1}\int_0^{l}\tfrac{3}{8}I\sigma(1+\cos^2\theta)\sin\theta\, e^{-\frac{(\kappa+\sigma)}{\cos\theta}l} \times e^{-(\kappa+\sigma)\left(1-\frac{1}{\cos\theta}\right)x}\,dx\,d\theta. \quad . \quad (1)$$

In the first place, I effect the integration in regard to x and obtain

$$S=\int_0^{\theta_1}\tfrac{3}{8}I\frac{\sigma}{\kappa+\sigma}\frac{(1+\cos^2\theta)\sin\theta}{1-\dfrac{1}{\cos\theta}}e^{-\frac{(\kappa+\sigma)}{\cos\theta}l} \times\left(1-e^{-(\kappa+\sigma)\left(1-\frac{1}{\cos\theta}\right)l}\right)d\theta. \quad . \quad (2)$$

The next thing is to find an expression for this integral adapted to numerical calculation. For this purpose I put

$$(\kappa+\sigma)l=k,$$

$$\frac{1}{\cos\theta}-1=x,$$

$$\frac{1}{\cos\theta_1}-1=z,$$

and obtain

$$S=\frac{3\sigma}{8(\kappa+\sigma)}\int_0^{z}\frac{1-e^{-kx}}{x}\left(\frac{1}{(1+x)^2}+\frac{1}{(1+x)^4}\right)dx. \quad . \quad (3)$$

For the sake of brevity, I write

$$K=\int_0^{z}\frac{1-e^{-kx}}{x}\left(\frac{1}{(1+x)^2}+\frac{1}{(1+x)^4}\right)dx. \quad . \quad . \quad (4)$$

I now pass to an examination of the integral K. First of all we may write

$$\frac{1}{x(1+x)^m}=\frac{1}{x}-\frac{1}{1+x}\cdot\cdot\cdot\cdot\cdot-\frac{1}{(1+x)^m}.$$

By using this relation we obtain, after some reductions:

$$K=2\int_0^{z}\left(\frac{1-e^{-kx}}{x}-\frac{1}{1+x}\right)dx+2\int_0^{z}\frac{e^{-kx}}{1+x}dx$$
$$+2\int_0^{z}\frac{e^{-kx}}{(1+x)^2}dx+\int_0^{z}\frac{e^{-kx}}{(1+x)^3}dx+\int_0^{z}\frac{e^{-kx}}{(1+x)^4}dx$$
$$+2\left(\frac{1}{1+x}-1\right)+\frac{1}{2}\left(\frac{1}{(1+x)^2}-1\right)+\frac{1}{3}\left(\frac{1}{(1+x)^3}-1\right). \quad (5)$$

I commence by examining the integral

$$\mathrm{U}=\int_0^z \left(\frac{1-e^{-kx}}{x}-\frac{1}{1+x}\right)dx.$$

We may write

$$\mathrm{U}=\int_0^\infty \left(\frac{1-e^{-kx}}{x}-\frac{1}{1+x}\right)dx-\int_2^\infty \left(\frac{1-e^{-kx}}{x}-\frac{1}{1+x}\right)dx$$

and

$$\int_0^\infty \left(\frac{1-e^{-kx}}{x}-\frac{1}{1+x}\right)dx=\int_0^\infty \left(\frac{1-e^{-t}}{t}-\frac{1}{k+t}\right)dt.$$

But

$$\int_0^\infty \left(\frac{1-e^{-t}}{t}-\frac{1}{k+t}\right)dt-\int_0^\infty \left(\frac{1-e^{-t}}{t}-\frac{1}{1+t}\right)dt=\log k.$$

$$\mathrm{C}=\int_0^\infty \left(\frac{1-e^{-t}}{t}-\frac{1}{1+t}\right)dt$$

is equal to the constant of Euler :

$$\lim_{m=\infty}\left(1+\frac{1}{2}+++\ldots+\frac{1}{m}-\log m\right).$$

Besides, this may be proved without difficulty by starting from the ordinary integral form of C, which in the manner known may be directly derived from the above-mentioned definition, *i. e.*

$$\mathrm{C}=\int_0^\infty \left(\frac{e^{-t}}{1-e^{-t}}-\frac{e^{-t}}{t}\right)dt.$$

If we take the difference between the two integrals and exchange the limits 0 and ∞ for t and T, we obtain by effecting the integration :

$$\log(1+\mathrm{T})+\log(1-e^{-\mathrm{T}})-\log\mathrm{T}$$
$$-\log(1+t)-\log(1-e^{-t})+\log t.$$

The limit value of this expression when t goes to 0 and T to ∞ will evidently be 0. The other part of U may be written as follows :

$$\int_z^\infty \left(\frac{1-e^{-kx}}{x}-\frac{1}{1+x}\right)dx=\int_z^\infty \left(\frac{1}{x}-\frac{1}{1+x}\right)dx-\int_z^\infty \frac{e^{-kx}}{x}dx$$
$$=\log\frac{1+z}{z}-\int_z^\infty \frac{e^{-kx}}{x}dx.$$

Consequently

$$\mathrm{U}=\mathrm{C}+\log k+\log\frac{z}{1+z}+\int_z^\infty \frac{e^{-kx}}{x}dx. \quad . \quad . \quad (6)$$

To reduce the remaining integrals I consider the following functions :

$$\phi(\lambda)=\int_0^\infty \frac{e^{-\lambda t}}{1+t}dt, \quad . \quad . \quad . \quad . \quad . \quad . \qquad (7)$$

$$\phi_m(\lambda)=\int_0^\infty \frac{e^{-\lambda t}}{(1+t)^m}, \quad . \quad . \quad . \quad . \quad . \quad . \qquad (8)$$

(m an entire number).

It is readily shown that the subsequent expressions hold :

$$\phi(\lambda)=\int_0^\infty \frac{e^{-t}}{\lambda+t}dt, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (9)$$

$$\phi(\lambda)=e^\lambda\int_\lambda^\infty \frac{e^{-t}}{t}dt, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (10)$$

$$\phi_m(\lambda)=\lambda^{m-1}\int_0^\infty \frac{e^{-t}}{(t+\lambda)^m}dt, \quad . \quad . \quad . \qquad (11)$$

$$\phi_m(q\lambda)=\lambda^{m-1}\int_0^\infty \frac{e^{-qt}}{(\lambda+t)^m}dt. \quad . \quad . \quad . \qquad (12)$$

These will be employed below. $\phi_m(\lambda)$ may by partial integration be reduced to $\phi(\lambda)$. By simple calculation we get

$$\phi_m(\lambda)=\frac{(-1)^{m-1}\lambda^{m-1}}{(m-1)\,!}\phi(\lambda)+\sum_{\nu=0}^{\infty}\frac{(-1)^\nu\lambda^\nu}{(m-1)\ldots.(m-\nu-1)}. \qquad (13)$$

I am now going to show that the integrals found in K of the form

$$\int_0^z \frac{e^{-kx}}{(1+x)^m}dx$$

may be reduced to expressions which only contain the function ϕ together with rational functions of z and k. I therefore examine the integral

$$\int_0^z \frac{e^{-kx}}{(1+x)^m}=\phi_m(k)-\int_z^\infty \frac{e^{-kx}}{z(1+x)^m}dx.$$

I put $t+z=x$ and get

$$\int_0^z \frac{e^{-kx}}{(1+x)^m}dx=\phi_m(k)-e^{-kx}\int_0^\infty \frac{e^{-kt}}{(1+z+t)^m}dt.$$

But according to (12)

$$\int_0^\infty \frac{e^{-kt}}{(1+z+t)^m}dt=\frac{1}{(1+z)^{m-1}}\phi_m[k(1+z)].$$

Consequently

$$\int_0^z \frac{e^{-kx}}{(1+x)^m}\,dx = \phi_m(k) - \frac{e^{-kz}}{(1+z)^{m-1}}\,\phi_m[k(1+z)]. \quad (14)$$

For $m=1$ we get

$$\int_0^z \frac{e^{-kx}}{1+x}\,dx = \phi(k) - e^{-kz}\phi[k(1+z)]. \quad . \quad . \quad (15)$$

So we have reduced all the integrals of this form. There remains only

$$\int_z^\infty \frac{e^{-kx}}{x}\,dx = \int_{kz}^\infty \frac{e^{-t}}{t}\,dt.$$

This one is immediately reduced by the relation (10), which gives

$$\int_z^\infty \frac{e^{-kx}}{x}\,dx = e^{-kz}\phi(kz).$$

By effecting all these reductions of K, we finally get:

$$\begin{aligned} \mathrm{K} = {} & 2\log\frac{kz}{1+z} + 2e^{-kz}\phi(kz) \\ & + \left(2 - 2k + \frac{k^2}{2} - \frac{k^3}{6}\right)\Big(\phi(k) - e^{-kz}\phi[k(1+z)]\Big) \\ & - \frac{e^{-kz}}{1+z}\left(2 - \frac{k}{2} + \frac{k^2}{6} + \frac{1}{2}\,\frac{1}{1+z}\left(1 - \frac{k}{3}\right) + \frac{1}{3}\,\frac{1}{(1+z)^2}\right) \\ & + \frac{2}{1+z} + \frac{1}{2}\,\frac{1}{(1+z)^2} + \frac{1}{3}\,\frac{1}{(1+z)^3} + 2\mathrm{C} - \frac{1}{3}k\left(2 - \frac{k}{2}\right). \quad (16) \end{aligned}$$

For z equal to ∞, *i. e.* $\theta = \frac{\pi}{2}$, this expression is transformed to

$$\mathrm{K}_{\pi/2} = 2\log k + \left(2 - 2k + \frac{k^2}{2} - \frac{k^3}{6}\right)\phi(k) + 2\mathrm{C} - \frac{1}{3}k\left(2 - \frac{k}{2}\right). \quad (17)$$

This value must be used in order to calculate the scattered radiation entering the ionization-chamber, when the plate is quite close to its opening. It now remains to show how $\phi(\lambda)$ is connected with known tabulated functions. This is extremely simple and follows from (10). Then the function

$$\int_\infty^{-\lambda} \frac{e^{-t}}{t}\,dt$$

is found in the tables of Jancke and Emde under the designation $\mathrm{E}i(\lambda)$. (10) gives

$$\phi(\lambda) = -e^{\lambda}\mathrm{E}i(-\lambda). \quad . \quad . \quad . \quad . \quad . \quad . \quad (18)$$

Besides ϕ is in a close connexion with the well-known function

$$l(z)=\int_0^z \frac{dx}{\log x},$$

which under the denomination of integral logarithm has been the object of special study. Indeed, we get by exchange of variables

$$\phi(\lambda)=-e^{\lambda}l(e^{-\lambda}). \quad . \quad . \quad . \quad . \quad . \quad . \quad (19)$$

The expression obtained allows of calculating, without any difficulty, the scattered radiation, which under a given visual angle penetrates into the ionization-chamber, as soon as we know the intensity, absorption and scattering coefficient of the incident radiation. It is evident that we may by the aid of this expression be able to calculate the said coefficients from the experimental determinations. I leave to the experimental physicists the consideration of how this may be best executed in practice.

In conclusion I wish to make a numerical application of the above results to some data taken from the experiments of Aurén. He placed the plate in two different positions, and measured the radiation entering the ionization-chamber in both cases. The distance to the centre of the plate was in the former position 2·1 cm., in the latter 12·3 cm. The distance between the ionization-chamber and the focus of the bulb was 42 cm. In one case Aurén examined a graphite plate 1 cm. thick. The mass-scattering coefficient for C, according to Aurén, is 0·142. Thus the scattering coefficient $\sigma=1{\cdot}7\,.\,0{\cdot}142$, because the density is about 1·7. For one of the wave-lengths used by him $(0{\cdot}34\,.\,10^{-8}$ cm.) $\kappa=0{\cdot}094\,.\,1{\cdot}7$. Thus

$$k=l(\kappa+\sigma)=0{\cdot}4.$$

By the aid of these facts, I get the following values of K in both positions:

$$K_1=0{\cdot}0997,$$

$$K_2=0{\cdot}00408.$$

For the sake of comparison, I calculate K_∞, and get

$$K_\infty=1{\cdot}328.$$

The scattering leaving the plate in the different positions is moreover inversely proportional to the square of the distance of the plate from the focus. With this in view, I calculate the ratio of scattered radiation entering the chamber in the respective positions to that which would enter it if the plate were in the immediate proximity of the opening, *i. e.* if r_1, r_2, and r_∞ are the distances in the respective cases, I calculate $\frac{K_1}{r_1^2}\times\frac{r_\infty^2}{K_\infty}$, and $\frac{K_2}{r_2^2}\times\frac{r_\infty^2}{K_\infty}$. For the two expressions we obtain the values 0·0827 and 0·00602 respectively. The difference between the two expressions is 0·077. If I assume that about equal scattered radiation is propagated backwards, this difference, which comes out as a difference in the apparent absorption, will amount to ca. 3·9 per cent. of the absorption caused by scattering. According to the figures given above it only amounts to $\frac{0{\cdot}094}{0{\cdot}094+0{\cdot}142}$ of the total absorption. The difference will then be 1·5 per cent. of the total absorption. This is in fair agreement with the Aurén experiments.

In addition to the above calculation I wish to mention that, for small values of z, K is more readily denoted by developing in the formula (4), the integrand of a power series and integrating term by term. So the following expression is arrived at:

$$K=\sum_{s=0}^{\infty}\frac{z^{s+1}}{s+1}(-1)^s\sum_{\mu=0}^{s}\frac{s-\mu+1}{(\mu+1)\,!}\left(1+\frac{(s-\mu+2)(s-\mu+3)}{6}\right)k^{\mu+1}.$$

Nobel Institution of Physical Chemistry,
Stockholm, June 1918.

XIX. *Fizeau's Experiment and the Æther.* *By* Dr. R. A. Houstoun, *Lecturer on Physical Optics in the University of Glasgow**.

§ 1. In a celebrated experiment performed in 1859 Fizeau showed that the velocity of light in a tube containing running water could be explained on the assumption that the æther in the tube was dragged with the water with a velocity $v(1-1/\mu^2)$, where v is the velocity of the water. The experiment was repeated with greater accuracy by Michelson and Morley in 1886, and Fizeau's result verified. According to the theory of H. A. Lorentz published in 1895, however, the velocity of the æther-drift in the tube is given by

$$v\left(1-\frac{1}{\mu^2}-\frac{\lambda}{\mu}\frac{d\mu}{d\lambda}\right),$$

---

* Communicated by the Author.

and not by $v(1-1/\mu^2)$, the numerical value of the new term being much smaller than the numerical value of either expression for the drift. The experiment was repeated with extreme care and accuracy by Zeeman in 1915, and not only was the existence of the new term verified, but also its variation with the wave-length.

It has hitherto been assumed, that the Lorentz expression for the æther-drift can be derived only on the basis of the theory of electrons, or from the equations of the theory of Relativity. I desire here to call attention to the fact, that it follows much more simply on the basis of the older elastic theories of light.

§ 2. Let us suppose that

$$\rho\frac{\partial^2 y}{\partial t^2}=\mathrm{E}\frac{\partial^2 y}{\partial x^2} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

represents the propagation of a light-wave in the æther, y denoting the displacement of the æther, ρ its density and E its elasticity; ρ is constant throughout all space. Let us suppose that inside matter there are particles attached to points in the æther, the mass of the distribution per unit volume being σ. These particles may be regarded as constituting a kind of "virtual" æther. They execute vibrations about the points to which they are attached. If η denotes their average displacement, their motion is given by

$$\sigma\frac{d^2\eta}{dt^2}+k(\eta-y)=0. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

The force attaching them to the æther reacts on the latter, so that instead of (1) we obtain

$$\rho\frac{\partial^2 y}{\partial t^2}=\mathrm{E}\frac{\partial^2 y}{\partial x^2}+k(\eta-y). \quad . \quad . \quad . \quad . \quad . \quad (3)$$

Add (2) and (3), and we obtain

$$\rho\frac{\partial^2 y}{\partial t^2}+\sigma\frac{d^2\eta}{dt^2}=\mathrm{E}\frac{\partial^2 y}{\partial x^2} \quad . \quad . \quad . \quad . \quad . \quad . \quad (4)$$

Assume now that y and η vary as $\cos nt$. Then (2) gives

$$\eta(k-\sigma n^2)=ky,$$

and (4) becomes

$$\left(\rho+\frac{k\sigma}{k-\sigma n^2}\right)\frac{\partial^2 y}{\partial t^2}=\mathrm{E}\frac{\partial^2 y}{\partial x^2}.$$

The index of refraction is given by

$$\mu^2=\frac{c^2}{\mathrm{E}}\left(\rho+\frac{k\sigma}{k-\sigma n^2}\right)=1+\frac{k\sigma c^2}{\mathrm{E}(k-\sigma n^2)}, \quad . \quad . \quad (5)$$

where c is the velocity of light *in vacuo*.

Now $n=2\pi c/\lambda$, where λ is the wave-length *in vacuo*. It will be found, by differentiating (5) with respect to λ and making the necessary substitutions, that

$$\mu^2-1-\mu\lambda\frac{d\mu}{d\lambda}=\frac{k^2\sigma c^2}{\mathrm{E}(k-\sigma n^2)^2}. \quad . \quad . \quad . \quad . \quad (6)$$

Suppose now that the refracting medium is moving through the æther with velocity v, in the direction in which the light-wave is travelling. Our co-ordinate axes are fixed in the æther. We assume that the light-wave is given by

$$y=\cos n(t-x/\mathrm{V}).$$

Since there is no damping, η is in phase with y, and is proportional to the same cosine. Equations (2) and (3) still hold, but in equation (2) now

$$\frac{d}{dt}=\frac{\partial}{\partial t}+v\frac{\partial}{\partial x}.$$

Previously the $\partial/\partial x$ term was zero. This is the only difference between moving and stationary media. For one definite particle of the "virtual" æther

$$\frac{\partial}{\partial x}=-\frac{1}{\mathrm{V}}\frac{\partial}{\partial t}. \quad \text{Hence} \quad \frac{d}{dt}=\left(1-\frac{v}{\mathrm{V}}\right)\frac{\partial}{\partial t}.$$

Substitute now in (2) and (4), and we obtain

$$\left[\rho+\frac{k\sigma(1-v/\mathrm{V})^2}{k-\sigma(1-v/\mathrm{V})^2n^2}\right]\frac{\partial^2 y}{\partial t^2}=\mathrm{E}\frac{\partial^2 y}{\partial x^2}.$$

Hence

$$\frac{1}{\mathrm{V}^2}=\frac{1}{\mathrm{E}}\left[\rho+\frac{k\sigma(1-v/\mathrm{V})^2}{k-\sigma(1-v/\mathrm{V})^2n^2}\right]. \quad . \quad . \quad . \quad (7)$$

This is absolutely true, no matter what the size of v/V is. Assume that v/V is small, and the equation reduces to

$$\frac{1}{\mathrm{V}^2}=\frac{1}{\mathrm{E}}\left[\rho+\frac{k\sigma}{k-\sigma n^2}-\frac{2v}{\mathrm{V}}\,\frac{k^2\sigma}{(k-\sigma n^2)^2}\right]$$
$$=\frac{\mu^2}{c^2}\left[1-\frac{2v}{\mathrm{V}}\left(1-\frac{1}{\mu}-\frac{\lambda}{\mu}\frac{d\mu}{d\lambda}\right]\right..$$

Invert and take the root of both sides, remembering that v/V

is small. Then

$$V=\frac{c}{\mu}\left[1+\frac{v}{V}\left(1-\frac{1}{\mu^2}-\frac{\lambda}{\mu}\frac{d\mu}{d\lambda}\right)\right]$$
$$=\frac{c}{\mu}+v\left(1-\frac{1}{\mu^2}-\frac{\lambda}{\mu}\frac{d\mu}{d\lambda}\right),$$

on making the permissible assumption that $V=c/\mu$ in the coefficient of v.

Thus the formula is proved. The proof can easily be extended to take in the case of several "virtual" æthers each with its own free period. If the medium is not moving in the direction of the light-wave, v is to be understood as the component of its velocity in that direction.

§ 3. Any theory which is to explain the phenomena of light must involve two vectors vibrating at right angles to one another in the same way as the electric and magnetic intensities do in the electromagnetic theory, *i. e.* it must have equations formally the same as the equations of the electromagnetic theory, and it must also have boundary conditions formally the same as the boundary conditions of the electromagnetic theory. The two vectors may, for example, represent displacement and rotation of an æther. There are various theories of this type, *e. g.* the elastic theory of W. Voigt. The above discussion shows, that if such a system of equations is coupled with a theory of dispersion of the type given by Maxwell or Sellmeier, it will apparently do all that the original theory of Lorentz does for moving media, and more simply than the latter, for it is simpler to suppose particles reacting on an elastic æther than to connect electrons with the light-wave by means of the displacement current. It is a case of phenomena being represented well from two different standpoints, which shows that each has only one aspect of the truth.

Now Lorentz's original theory explained astronomical aberration and the null effect of the earth's motion through space on all optical experiments to the order v/c, but did not explain the null effect of the earth's motion through space on the Michelson-Morley experiment in the order $(v/c)^2$. It is exactly the same with this theory. To explain the Michelson-Morley experiment on the Lorentz theory it is necessary to make the FitzGerald-Lorentz assumption, that the earth contracts in the direction of its motion. But if we assume an elastic æther we have another loophole of escape; we may suppose that the æther is moving with the earth in space, that it is participating in the earth's motion of translation, but not in the earth's motion of rotation. This assumption is not permissible to the Lorentz theory, for according to the latter the æther is merely empty space.

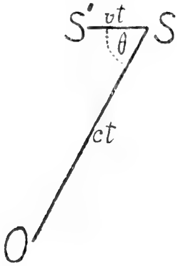
§ 4. Let us shortly examine the consequences involved in the hypothesis, that the æther has tbe same velocity of translation as the earth.

Seventy years ago this hypothesis would have been objected to on *a priori* grounds. The belief then was, that the æther was created first and then myriads of hard unchangeable material particles; these came together under the influence of certain "forces" of attraction, cohesion, heat, &c., and formed the planets. The object of science was to discover and interpret the Divine Plan, and, of course, it was unreasonable to suppose that the æther had any closer connexion with any one planet than any other.

Now the standpoint is rapidly changing; "laws of nature" have truth only with reference to our present state of knowledge, and to the degree of skill with which we have succeeded in giving expression to it. The æther is reached only by our intellectual processes; it is a conception to satisfy our phenomena, Hence we have a right to attach it to our own planet, if necessary, and the hypothesis must be judged solely on its merits.

If the æther participates in the earth's motion of translation, the only motion which the Michelson-Morley experiment might reveal, is that due to the earth's diurnal rotation. This amounts at the equator to ·291 miles per second, and hence gives a value of $v/c=1{\cdot}56\times10^{-6}$ instead of the 10^{-4} given by the orbital velocity. The effect to be observed would consequently diminish to $\frac{24}{100,000}$ of its value, and hence be 100 times too small for the sensitiveness of the apparatus. So the Michelson-Morley experiment is accounted for.

In addition to the annual astronomical aberration with the maximum value of $20''{\cdot}47$ there is a diurnal aberration ranging from zero to $0''{\cdot}31$. The peculiarity of this hypothesis is, that it explains them in different ways. The diurnal aberration is explained in the same way as the annual aberration is explained on the Lorentz theory. To explain the annual aberration it is necessary to proceed as is shown in the fig. O is the observer on the earth supposed at rest in the æther, S is the star and SO the path of a ray to the observer. When the ray arrives at O, the star has moved through the æther to S′, the earth and æther being regarded as fixed. The star is thus displaced behind its true position S′ by the angle $SS'\sin\theta/OS=v\sin\theta/c$, where v is the relative velocity of star and observer. The case is exactly the same as locating an aeroplane by pointing a resonator in the direction in which the noise of the engine is loudest; the aeroplane will appear displaced behind the true position. If



θ is the apparent latitude and θ' the true latitude, this explanation makes $v/c = \sin(\theta'-\theta)/\sin\theta'$, whereas the usual explanation makes $v/c = \sin(\theta'-\theta)/\sin\theta$. But the difference is too small to observe.

Thus the hypothesis accounts for aberration, and seems to give a means of escaping from the FitzGerald-Lorentz assumption and its consequence, the Principle of Relativity, at the expense, of course, of a strictly electromagnetic explanation of matter.

XX. *On General Relativity.*

To the Editors of the Philosophical Magazine.

GENTLEMEN,—

THE appearance of Dr. Silberstein's recent article * on "General Relativity without the Equivalence Hypothesis" encourages me to restate my own views on the subject. I am perhaps entitled to do this as my work on the subject of General Relativity was published before that of Einstein and Kottler, and appears to have been overlooked by recent writers. In 1909 I proposed a scheme of electromagnetic equations † which are covariant for all transformations of co-ordinates which are biuniform in the domain we are interested in. These equations were similar to Maxwell's equations, except that the familiar relations $B = \mu H$, $D = kE$ of Maxwell's theory were replaced by more

---

* Phil. Mag. July 1918.

† Proc. Lond. Math. Soc. ser. 2, vol. viii. p. 223 (1910).

general equations, which implied that two fundamental integral forms were reciprocals with regard to a quadratic differential form

$$\Sigma \Sigma g_{m,n} dx_m dx_n,$$

which was assumed to be invariant for all transformations of co-ordinates. The coefficients of the quadratic form were regarded as characteristics of the medium supporting the electromagnetic field and of the motion of the medium and its parts. The vanishing of the quadratic form was regarded as the condition that two neighbouring particles should be in positions such that a disturbance starting from one at the associated time should arrive at the other at its associated time *.

The idea that the coefficients of the quadratic form might be considered as characteristics of the mind interpreting the phenomena was also entertained †, and it was suggested that a correspondence or transformation of co-ordinates might be considered as a crude mathematical symbol for a mind.

The phenomena here considered were those occurring in the brain and body; and although the correspondence by which the universe is reconstructed, so to speak, may be totally different ‡ from the type contemplated here, yet it was thought that some of the general conclusions might still be valid if a transformation of co-ordinates was adopted as a working model of the correspondence. It was thought, for instance, that there was an analogy between the relativity principle that the earth's motion in space cannot be detected from experiments with terrestrial objects; and the interesting fact that we are unaware of the flow of blood and other processes taking place in our own bodies so long as they take place in the normal way. It was thought that the

---

* *Loc. cit.* p. 225. See also Amer. Journ. of Math. vol. xxxiv. p. 340 (1912).

† Memoirs of the Manchester Literary and Philosophical Society (1910). A full account of my ideas has not yet been published, owing to the difficulty of eliminating vagueness.

‡ The term correspondence is used here in a very general sense, and is by no means restricted to the familiar one to one correspondence of entities of the same type, such as points. We should say, for instance, that there is a correspondence between the disturbance running along a telephone-wire and the sound-waves which produce it, because we can pass from one to the other by mathematical equations of a definite type, or rather by solving the equations and the boundary conditions. A correspondence is, moreover, regarded as an entity which may have real existence and be capable of growth and variation.

correspondence by which the elementary processes in the brain are interpreted may be adjusted in such a way that some of the changes are obscured.

Again, if we assume that the nature of an electromagnetic field depends on the type of fundamental quadratic form which determines the constitutive relations, and thus depends indirectly on a transformation which alters the coefficients of this quadratic form, this dependence may be a symbol for the relation between physical and mental phenomena instead of giving the influence of gravitation on light as in Einstein's theory.

Einstein and others have attempted to formulate a set of equations of motion which will cover all physical phenomena; but the present writer does not feel inclined to accept them as final, because in his opinion the true equations of motion should be capable of accounting for the phenomena of life, which after all are the most important physical phenomena.

To make my position more definite, let us consider one of the methods by which the equations of motion of an electron are obtained in the usual electromagnetic theory. The principle is adopted that at each instant the integral over the electron of the total force on each element must be zero. Now before this principle can be used to write down the equations of motion we must know *the design of the electron*, and we must know *the way in which the motions of the different elements are co-ordinated.* This co-ordination or organization of the motions of the elements may be represented mathematically by a sequence of infinitesimal transformations, by which some of the features of the design are preserved. The design of the electron and the co-ordinated motion of its parts may, perhaps, be specified by a quadratic differential form in four variables, which determines a mapping of the interior of the electron on the interior of a stationary sphere; but I doubt if this is sufficiently general. A knowledge of this quadratic differential form is necessary then before we can write down the equations of motion of the electron as a whole. What we usually regard as the equations of motion of matter need then to be supplemented by geometrical conditions which specify the design and organization of each elementary portion of matter. Furthermore, when this design and organization is assumed to be known, the ordinary equations of motion may be regarded as a consequence of the electromagnetic laws and the abovementioned principle.

It must be confessed, however, that this principle does not seem satisfactory for a fundamental principle, and is probably a consequence of some deep underlying principles which are the true equations of motion. These new principles should indicate the reason for a similarity of design of the different electrons. One of the fundamental facts of life is that a good design is copied, and that there is a certain characteristic of the design of an object and its surrounding medium, depending perhaps on the closeness of fit of an imperfect correspondence, which determines the extent to which the design of the object is copied and preserved in the surrounding medium. This may be called the *value of the design in relation to the medium*, and it is a quantity which I feel must be taken into account in the true equations of motion, and a number assigned to it at each instant. As an example of standardization, the Ford motor-car is not in it with the electron; and, according to the above view, we must regard the design of the electron as one of very great value in relation to the surrounding medium.

Returning to our generalized scheme of electromagnetic equations, and looking at matters from the point of view of physical optics, it may be remarked that the scheme of constitutive relations mentioned above is not sufficiently general to cover the case of a doubly-refracting crystalline medium [*]. To remedy this defect we may use a biquadratic integral form instead of a quadratic differential form to specify the constitutive relations. The vanishing of the biquadratic integral form may perhaps be regarded as the condition for action of a moving curve on a particle, a type of condition that seems natural if we regard moving Faraday tubes as fundamental. With this generalized theory it is possible for the elementary wave surface in a medium to be a general Kummer surface, a surface of which Fresnel's wave surface is a particular case. It is doubtful whether this generalized theory is sufficiently general for all purposes, and the above example is given just to emphasize that the absolute calculus of Ricci and Levi Civita can be used to develop a theory of generalized relativity on many lines in addition to that adopted by Einstein.

Going back to the case in which a quadratic form is sufficient to determine the optical properties of a medium, we may remark that if Einstein's idea of the gravitational equations is accepted, it is still by no means certain that his

---

[*] Proc. London Math. Soc. ser. 2, vol. viii. p. 375. See also p. 251 of my first paper.

quadratic form from which the gravitational equations are derived is the same as the quadratic form which determines the optical properties of the medium. Indeed, the example which I considered on p. 262 of my first paper would seem to indicate that this was not the case. It should be mentioned that in the first seven equations in this example there is a misprint, $\epsilon\mu$ should be replaced by $(\epsilon\mu)^{-1}$. On the above view Einstein's idea of an influence of gravitation on light is simply an hypothesis, but a very interesting and reasonable one. It may be remarked, however, that in the theory of surfaces there are two fundamental quadratic forms, and we may perhaps expect something similar in general relativity.

With regard to possible extensions of the idea of relativity it may be worth while to consider transformations analogous to the contact transformations of dynamics in which the co-ordinates x, y, z, t and the component velocities u, v, w correspond to a new set $(x_1 y_1 z_1 t_1 u_1 v_1 w_1)$ in such a way that the differential equations

$$\frac{dx_1}{u_1} = \frac{dy_1}{v_1} = \frac{dz_1}{w_1} = dt_1$$

are a consequence of the equations

$$\frac{dx}{u} = \frac{dy}{v} = \frac{dz}{w} = dt.$$

This may be secured by making a single quadratic form, such as

$$(dx^2 + dy^2 + dz^2 - c^2dt^2)(c^2 - u^2 - v^2 - w^2)$$
$$+ (c^2dt - udx - vdy - wdz)^2, \qquad (c^2 > u^2 + v^2 + w^2),$$

an invariant*. Various other quadratic forms consisting of sums of squares may, of course, be adopted instead.

H. Bateman.

Throop College.
Pasandena, Cal.
Aug. 10th, 1918.

---

* This is a positive definite quadratic form in the variables $dx - udt$, $dy - vdt$, $dz - wdt$, and so can only vanish when all these quantities are zero.

XXI. *The Influence of Temperature on Homogeneous Gas Reactions.* *By* George W. Todd, *D.Sc.* (*Birm.*), *B.A.* (*Cantab.*), *and* S. P. Owen, *B.Sc.* (*Wales*) *.

It is well known that a small rise in temperature generally causes a big increase in the velocity of a chemical reaction—a rise of 10° C. often doubles or even trebles the velocity. On the kinetic theory of gases the frequency with which the molecules collide is proportional to the absolute temperature, so that a rise of 10° C. in normal regions of temperature would only result in increasing the frequency of collision by two per cent. For this reason the kinetic theory as a means of explaining the effect of temperature on reaction velocity and on chemical equilibrium has been left severely alone and artificial assumptions such as the "active" and "passive" molecules of Arrhenius have been brought forward.

Many years ago Boltzmann (Wied. *Ann.* xxii. p. 64) pointed out that a kinetic treatment of gaseous equilibrium ought to give better results than a thermodynamic treatment. As yet no results of practical importance have been obtained in this way.

The majority of the molecules in a gas are moving with a velocity not far removed from the average, but whatever the temperature there are at every instant a very few molecules practically motionless and a very few with extremely high velocities. Now it is quite certain that in a chemical reaction only a small proportion of the molecules are in the condition for combining, at any instant of time. We shall therefore assume that a gas molecule can never react with another gas molecule unless its energy (we shall take kinetic energy in what follows) surpasses a definite minimum.

According to Maxwell's theorem on the distribution of velocities the number of molecules in 1 c.c. with velocities between c and $c+dc$ is

$$dn=\frac{4}{\sqrt{\pi}}\,.\,\mathrm{N}\,.\left(\frac{m}{2\mathrm{R}\theta}\right)^{\frac{3}{2}}.\,e^{-\frac{mc^2}{2\mathrm{R}\theta}}\,.\,c^2\,.\,dc,$$

where N = number of molecules per cc.,
R = gas constant,
m = mass of a gas molecule,
θ = absolute temperature.

---

* Communicated by the Authors.

If n is the number of molecules having velocities greater than a critical value c, then

$$n=\frac{4}{\sqrt{\pi}}\mathrm{N}\left(\frac{m}{2\mathrm{R}\theta}\right)^{\frac{3}{2}}\int_{c}^{\infty}e^{-\frac{mc^2}{2\mathrm{R}\theta}}\,.\,c^2\,.\,dc.$$

We shall assume that chemical reaction can only take place in consequence of the collisions of these n molecules.

Before proceeding further we must obtain a suitable expression for the above integral. We shall show that

$$n=\frac{2}{\sqrt{\pi}}\mathrm{N}e^{-x^2}\left[x+\frac{1}{2x}+\text{higher powers of }\left(\frac{1}{x}\right)\right],$$

where

$$x=c\sqrt{\frac{m}{2\mathrm{R}\theta}}.$$

The asymptotic expansion rapidly decreases for large values of x, and generally we find it justifiable to assume

$$n=\frac{2}{\sqrt{\pi}}\mathrm{N}e^{-x^2}\left[x+\frac{1}{2x}\right].$$

The proof is as follows:—

We have

$$n=\frac{4}{\sqrt{\pi}}\mathrm{N}\int_{x_1}^{\infty}e^{-x^2}\,.\,x^2\,.\,dx, \quad \ldots \ldots \quad \text{(i.)}$$

where

$$x_1=c_1\sqrt{\frac{m}{2\mathrm{R}\theta}}.$$

Now

$$\int_{x_1}^{\infty}e^{-x^2}\,.\,x^2\,.\,dx=\tfrac{1}{2}\Big[-e^{-x^2}\,.\,x\Big]_{x_1}^{\infty}+\tfrac{1}{2}\int_{x_1}^{\infty}e^{-x^2}\,.\,dx$$

$$=\tfrac{1}{2}e^{-x_1^2}\,.\,x_1+\tfrac{1}{2}\int_{x_1}^{\infty}e^{-x^2}\,.\,dx. \quad \ldots \quad \text{(ii.)}$$

Putting $w=\dfrac{x}{x_1}$ we have

$$\int_{x_1}^{\infty}e^{-x^2}\,.\,dx=\int_{1}^{\infty}e^{-x_1^2w^2}\,.\,x_1dw$$

$$=e^{-x_1^2}\,.\,x_1\int_{1}^{\infty}e^{x_1^2(1-w^2)}\,.\,dw$$

$$=e^{-x_1^2}\,.\,x_1\int_{0}^{\infty}e^{-t}\,.\left(1+\frac{t}{x^2}\right)^{-\frac{1}{2}}\cdot\frac{1}{2x_1^2}\cdot dt$$

where $t=x_1^2(w^2-1)$

$$=\tfrac{1}{2}e^{-x_1^2}\,.\,x_1^{-1}\int_{0}^{\infty}e^{-t}\,.\left(1+\frac{t}{x^2}\right)^{-\frac{1}{2}}.\,dt. \quad \text{(iii.)}$$

Now

$$\left(1+\frac{t}{x^2}\right)^{-\frac{1}{2}} = 1 + \frac{\frac{1}{2}}{1}\left(-\frac{t}{x^2}\right) + \frac{\frac{1}{2}\cdot\frac{3}{2}}{\underline{|2}}\left(-\frac{t}{x^2}\right)^2 + \ldots\ldots$$

$$+ \frac{\frac{1}{2}\cdot\frac{3}{2}\cdot\ldots.(\frac{1}{2}+n-1)}{\underline{|n}}\left(-\frac{t}{x^2}\right)^n + \ldots\ldots$$

$$+ \frac{\frac{1}{2}\cdot\frac{3}{2}\cdot\ldots.(\frac{1}{2}+n)}{\underline{|n}}\left(1+\frac{t}{x^2}\right)^{-\frac{1}{2}}\int_0^{\frac{t}{x^2}} S^n.(1+S)^{-\frac{1}{2}}.dS.$$

Substituting in (iii.) and remembering that

$$\int_0^\infty e^{-t}.t^{n-1}.dt=\Gamma(n)$$

we get

$$\int_{x_1}^\infty e^{-x^2}.dx$$

$$= \tfrac{1}{2}e^{-x_1^2}.x_1^{-1}\left\{1-\frac{1}{2.x_1^2}+\frac{1.3}{2^2.x_1^4}-\frac{1.3.5}{2^3.x_1^6}+\ldots\ldots\right.$$

$$\left.+\frac{\frac{1}{2}\cdot\frac{3}{2}\cdot\ldots(\frac{1}{2}+n)}{\underline{|n}}\int_0^\infty e^{-t}\left(1+\frac{t}{x^2}\right)^{-\frac{1}{2}}\int_0^{\frac{t}{x_1^2}} S^n.(1+S)^{-\frac{1}{2}}dS.dt\right\}$$

If we take

$$S_n=\tfrac{1}{2}e^{-x_1^2}.x_1^{-1}\left\{1-\frac{1}{2x_1^2}+\frac{1.3}{2^2x_1^4}-\ldots.\right\}=\Sigma a_n,$$

then
$$\frac{a_{n+1}}{a_n}=\frac{2n+1}{2x_1^2},$$

which approaches ∞ as n approaches ∞.

Hence S_n is divergent for all values of x.

But we can make

$$\int_{x_1}^\infty e^{-x^2}.dx-S_n$$

as small as we please by taking large values of x_1, for

$$\int_{x_1}^\infty e^{-x^2}.dx-S_n=\tfrac{1}{2}e^{-x_1^2}.x_1^{-1}\frac{\frac{1}{2}\cdot\frac{3}{2}\cdot\ldots.(\frac{1}{2}+n)}{\underline{|n}}$$

$$\times\int_0^\infty e^{-t}\left(1+\frac{t}{x^2}\right)^{-\frac{1}{2}}\int_0^{\frac{t}{x_1^2}} S^n.(1+S)^{-\frac{1}{2}}.dS.dt$$

$$=R_n \text{ say.}$$

Since $1+\frac{t}{x^2}<1+t$ and $1+s<1+t$,

$$R_n<\tfrac{1}{2}e^{-x_1{}^2}.\,x_1{}^{-1}\frac{\frac{1}{2}\cdot\frac{3}{2}\cdot\ldots\cdot(\frac{1}{2}+n)}{|\underline{n}}$$

$$\times\int_0^\infty e^{-t}(1+t)^{\frac{1}{2}}(1+t)^{\frac{1}{2}}\cdot\frac{t^{n+1}}{x_1^{2n+2}}\cdot\frac{1}{n+1}\cdot dt$$

$$<\tfrac{1}{2}e^{-x_1{}^2}.\,x_1{}^{-1}\frac{\frac{1}{2}\cdot\frac{3}{2}\cdot\ldots\cdot(\frac{1}{2}+n)}{|\underline{n}}\cdot\frac{\Gamma(n+2)+\Gamma(n+3)}{(n+1)\,.\,x_1^{2n+2}}\,.$$

If we take $x=2$ (a low value) and neglect all terms after the first, $R_n<0{\cdot}002$. Thus we are justified in taking

$$\int_{x_1}^\infty e^{-x^2}.\,dx=\tfrac{1}{2}e^{-x_1{}^2}x_1^{-1}.$$

Substituting in equations (i.) and (ii.) we obtain

$$n=\frac{2}{\sqrt{\pi}}\mathrm{N}e^{-x_1{}^2}\left[x_1+\frac{1}{2x_1}\right]. \quad \ldots \quad \text{(iv.)}$$

Reaction Velocity and Temperature.

Let us apply our result to the homogeneous gas reaction represented by

$$A+B\rightarrow.$$

Let n_a be the number of A molecules per c.c. which may react with B molecules on collision, *i. e.*, the number of A molecules which have velocities greater than a certain critical value characteristic of the gas A. Let n_b have the same significance for the B gas. The molecules with less than the critical kinetic energy will be merely diluents, so that the velocity of the reaction will be

$$\begin{aligned}V &\propto n_a n_b\\ &\propto \mathrm{N}_a\mathrm{N}_b e^{-(x_a{}^2+x_b{}^2)}\left[x_a+\frac{1}{2x_a}\right]\left[x_b+\frac{1}{2x_b}\right]\end{aligned}$$

from equation (iv.).

The velocity of reaction is not only proportional to the product of the molecules having the requisite energy for combination but also to the frequency of collision. Maxwell showed that the frequency of collision is given by

$$\left(1+\frac{h}{\theta}\right)\sqrt{2}\,\pi s^2\frac{\Omega}{\lambda^3},$$

where $\left(1+\frac{h}{\theta}\right)$ is a term correcting for molecular attraction,

πs^2 is the cross-section of the sphere of action, λ^3 is the volume occupied by one molecule, and Ω is the average molecular velocity. The value of h/θ being small compared with unity we may take the frequency of collision proportional to Ω, *i. e.* proportional to θ. Hence

$$\mathrm{V} \propto \theta \mathrm{N}_a \mathrm{N}_b e^{-(x_a^2+x_b^2)} \left[x_a + \frac{1}{2x_a}\right]\left[x_b + \frac{1}{2x_b}\right].$$

But $\mathrm{V} = k\mathrm{N}_a\mathrm{N}_b$, where k is the "velocity constant." Therefore

$$k \propto \theta e^{-(x_a^2+x_b^2)} \left[x_a + \frac{1}{2x_a}\right]\left[x_b + \frac{1}{2x_b}\right].$$

If k_{θ_2} and k_{θ_1} are the velocity constants at the temperatures θ_2 and θ_1 we obtain

$$\frac{k_{\theta_2}}{k_{\theta_1}} = \frac{\theta_2\left(\frac{\mathrm{P}}{\theta_2} + \frac{\theta_2}{4\mathrm{P}} + \mathrm{Q}\right)}{\theta_1\left(\frac{\mathrm{P}}{\theta_1} + \frac{\theta_1}{4\mathrm{P}} + \mathrm{Q}\right)} \cdot e^{\mathrm{W}\left(\frac{1}{\theta_1} - \frac{1}{\theta_2}\right)},$$

P, Q, and W being constants having the values

$$\mathrm{P} = \frac{1}{\mathrm{R}}\sqrt{\mathrm{E}_a . \mathrm{E}_b}, \qquad \mathrm{Q} = \frac{1}{2}\frac{\mathrm{E}_a + \mathrm{E}_b}{\sqrt{\mathrm{E}_a\mathrm{E}_b}}, \qquad \mathrm{W} = \frac{1}{\mathrm{R}}(\mathrm{E}_a + \mathrm{E}_b),$$

E being the critical kinetic energy of the molecule.

In the general case $\mathrm{A} + \mathrm{B} + \mathrm{C} + \ldots . \rightarrow$ it is obvious that we shall obtain the expression

$$\frac{k_{\theta_2}}{k_{\theta_1}} = \frac{\phi(\theta_2)}{\phi(\theta_1)} \cdot e^{\mathrm{W}\left(\frac{1}{\theta_1} - \frac{1}{\theta_2}\right)},$$

where the form of ϕ is determined by the number of reactants.

The Effect of Temperature on the Equilibrium Constant.

Let us take for example the homogeneous gas reaction

$$\mathrm{A} + \mathrm{B} \rightleftharpoons \mathrm{C}.$$

For the forward reaction

$$k_{ab} = v_{ab} . \theta . e^{-(x_a^2+x_b^2)} \left[x_a + \frac{1}{2x_a}\right]\left[x_b + \frac{1}{2x_b}\right],$$

and for the back reaction

$$k_c = v_c . \theta . e^{-x_c^2} \left[x_c + \frac{1}{2x_c}\right],$$

giving the equilibrium constant

$$\mathrm{K}_\theta = \frac{k_{ab}}{k_c} = \frac{v_{ab} \cdot e^{-(x_a^2+x_b^2)}\left[x_a + \frac{1}{2x_a}\right]\left[x_b + \frac{1}{2x_b}\right]}{v_c e^{-x_c^2}\left[x_c + \frac{1}{2x_c}\right]}.$$

This reduces to the well-known expression

$$\log_e \mathrm{K}_\theta = \alpha + \frac{\beta}{\theta} + \gamma \, . \log \theta + \delta \, . \, \theta,$$

where $\alpha = \frac{1}{2}\left[\log_e \frac{\mathrm{E}_a}{\mathrm{R}} + \log_e \frac{\mathrm{E}_b}{\mathrm{R}} - \log_e \frac{\mathrm{E}_c}{\mathrm{R}}\right] + \log_e \frac{v_{ab}}{v_c},$

$$\beta = \frac{1}{\mathrm{R}}\left[\mathrm{E}_c - \mathrm{E}_a - \mathrm{E}_b\right],$$

$$\gamma = -\tfrac{1}{2},$$

$$\delta = \frac{\mathrm{R}}{2}\left[\frac{1}{\mathrm{E}_a} + \frac{1}{\mathrm{E}_b} - \frac{1}{\mathrm{E}_c}\right].$$

Take the reaction

$$n_a\mathrm{A} + n_b\mathrm{B} \rightleftharpoons n_c\mathrm{C} + n_d\mathrm{D}.$$

The equilibrium constant is

$$\mathrm{K} = \frac{v_{ab} \cdot e^{-n_a x_a^2 - n_b x_b^2}\left[x_a + \frac{1}{2x_a}\right]^{n_a}\left[x_b + \frac{1}{2x_b}\right]^{n_b}}{v_{cd} \cdot e^{-n_c^2 x_c^2 - n_d^2 x_d^2}\left[x_c + \frac{1}{2x_c}\right]^{n_c}\left[x_d + \frac{1}{2x_d}\right]^{n_d}}.$$

Again this reduces to the form,

$$\log_e \mathrm{K}_\theta = \alpha + \frac{\beta}{\theta} + \gamma \, . \log_e \theta + \delta \, . \, \theta,$$

where $\alpha = \frac{1}{2}\left[n_a \log_e \frac{\mathrm{E}_a}{\mathrm{R}} + n_b \log_e \frac{\mathrm{E}_b}{\mathrm{R}}\right.$

$$\left. - n_c \log_e \frac{\mathrm{E}_c}{\mathrm{R}} - n_d \log_e \frac{\mathrm{E}_d}{\mathrm{R}}\right] + \log_e \frac{v_{ab}}{v_{cd}},$$

$$\beta = \frac{1}{\mathrm{R}}\left[n_d\mathrm{E}_d + n_c\mathrm{E}_c - n_b\mathrm{E}_b - n_a\mathrm{E}_a\right],$$

$$\gamma = \tfrac{1}{2}\left[n_c + n_d - n_a - n_b\right],$$

$$\delta = \frac{\mathrm{R}}{2}\left[\frac{n_a}{\mathrm{E}_a} + \frac{n_b}{\mathrm{E}_b} - \frac{n_c}{\mathrm{E}_c} - \frac{n_d}{\mathrm{E}_d}\right].$$

In the general case it is obvious that we get the same form

$$\log_e K_\theta = \alpha + \frac{\beta}{\theta} + \gamma \log_e \theta + \delta . \theta,$$

where

$$\alpha = \tfrac{1}{2}\Sigma n \log_e \frac{E}{R} + \Sigma \log_e v,$$

$$\beta = -\frac{1}{R}\Sigma n E,$$

$$\gamma = -\tfrac{1}{2}\Sigma n,$$

$$\delta = \frac{R}{2}\Sigma \frac{n}{E}.$$

When the reaction takes place without change in the number of molecules, γ is zero.

Summary.

1. From Maxwell's distribution theorem we have deduced a simple expression for the number of gas molecules per c.c. having velocities greater than a particular value.

2. Assuming that only molecules with kinetic energies greater than a definite minimum have the power to react with other molecules, we have shown how the "velocity constant" of a gaseous reaction varies with the temperature.

3. Further, we obtain the usual form of expression for the variation of the "equilibrium constant" with the temperature.

London, November 1918.

XXII. *Boundary Difficulties of Einstein's Gravitation Theory.*

By L. SILBERSTEIN, *Ph.D.*[*]

IN a letter to the Editors of 'The Observatory' (vol. xli. Oct. 1918, p. 380), supplementing the deductions of my first note on this subject[†], I have derived from Einstein's field-equations[‡] the following cubic for the principal curva-

---

[*] Communicated by the Author.

[†] "Bizarre Conclusion, &c.", Monthly Notices of the Roy. Astronomical Society, vol. lxxviii. p. 465.

[‡] These equations are, in usual notation,

$$G_{ij} = \kappa\left[\tfrac{1}{2}g_{ij}T - T_{ij}\right] - \lambda g_{ij} (i, j = 1, 2, 3, 4),$$

where λ is a universal constant.

tures K_1, K_2, K_3 of the three-space (x_1, x_2, x_3) within a lump of matter

$$\det \left| (K-L)g_{ij} - \kappa T_{ij} \right| = 0, \quad . \quad . \quad . \quad . \quad (A)$$

where $L = \kappa(g^{44}T_{44} - \frac{1}{2}T) + \lambda$, and det stands for the determinant of the nine elements $(i, j = 1, 2, 3)$. All T_{ij} (stresses, &c.) being negligible with the exception of T_{44}, so that also $T = g^{44}T_{44} = \rho$ (density), the above cubic became

$$(K-L)^3 \cdot \det \left| g_{ij} \right| = 0, \quad . \quad . \quad . \quad . \quad (B)$$

and gave at once

$$K_1 = K_2 = K_3 = \tfrac{1}{2}\kappa\rho + \lambda,$$

i. e. an isotropic space, thus corroborating the original "bizarre" conclusion according to which "a homogeneous body could have only the shape of a sphere."

In deducing equation (A) I have based myself, among other things, upon certain differentially geometrical relations, due to Ricci, given in a paper by Levi-Civita (*Atti Lincei*, vol. xxvi. p. 641). Now, Prof. Levi-Civita, to whom I have communicated the above and some previous results, has recently (in a private letter) called my attention to the fact that the G_{ij} of his paper just quoted are not identical with my G_{ij}, but differ from them by

$$-\frac{1}{v} v_{ij} (i, j = 1, 2, 3),$$

where $v = \sqrt{g_{44}}$ and v_{ij} is the covariant derivative of v with respect to x_i, x_j, corresponding to the three-space element in question.

Thus, neglecting the stresses, as before, the so-called coefficients of Ricci become

$$\alpha_{ij} = -\tfrac{1}{2}\kappa T \cdot g_{ij} - \frac{1}{v} v_{ij},$$

and we have, instead of the above (B), the following cubic

$$\det \left| (K-L)g_{ij} - \frac{1}{v} v_{ij} \right| = 0, \quad . \quad . \quad . \quad . \quad (1)$$

with $L = \frac{1}{2}\kappa\rho + \lambda$, which gives in general *three different curvatures* K_1, K_2, K_3, depending essentially upon g_{44} or v^2. Thus the space is no more isotropic, and the original conclusion about the shape of a homogeneous body ceases to be necessary. But the geometrical boundary difficulties do not vanish therefore; on the contrary, unless some boundary conditions *ad hoc* are not made, the difficulties or incompatibilities at the surface may possibly become more acute, extending, that is, even to non-homogeneous lumps of matter.

Manifestly, the coefficient g_{44} appearing through v_{ij}/v, and therefore also the K's, can only be obtained after an integration of the differential field-equations, so that in general the whole problem becomes considerably more intricate. To shed some light upon it let us consider the well-known approximate solution embodied in the line-element

$$ds^2=\left(1-\frac{2\Omega}{c^2}\right)c^2dt^2-dl^2, \quad \ldots\ldots \quad (2)$$

$$dl^2=\left(1+\frac{2\Omega}{c^2}\right)(dx_1{}^2+dx_2{}^2+dx_3{}^2), \quad \ldots \quad (2\,a)$$

that is to say,

$$v^2=g_{44}=1-\frac{2}{c^2}\Omega, \quad \ldots\ldots\ldots \quad (2\,b)$$

where Ω is the Newtonian potential of the mass distribution ρ, *i. e.* satisfying the common Laplace-Poisson equation

$$\nabla^2\Omega=-\frac{c^2}{2}\kappa\rho.$$

The developed form of (1) is easily seen to be

$$|g_{ij}|\cdot(\mathrm{K}-\mathrm{L})^3-\frac{p_2}{v}(\mathrm{K}-\mathrm{L})^2+\frac{p_1}{v^2}(\mathrm{K}-\mathrm{L})-\frac{1}{v^3}|v_{ij}|=0,$$

where

$$p_2=\begin{vmatrix} v_{11} & v_{12} & v_{13} \\ g_{21} & g_{22} & g_{23} \\ g_{31} & g_{32} & g_{33} \end{vmatrix}+\begin{vmatrix} g_{11} & g_{12} & g_{13} \\ v_{21} & v_{22} & v_{23} \\ g_{31} & g_{32} & g_{33} \end{vmatrix}+\begin{vmatrix} g_{11} & g_{12} & g_{13} \\ g_{21} & g_{22} & g_{23} \\ v_{31} & v_{32} & v_{33} \end{vmatrix}.$$

$$p_1=\begin{vmatrix} g_{11} & g_{12} & g_{13} \\ v_{21} & v_{22} & v_{23} \\ v_{31} & v_{32} & v_{33} \end{vmatrix}+\text{two similar terms.}$$

Now, with the approximate solution (2), and taking c as unit velocity,

$$g_{44}=1-2\Omega,\quad g_{11}=g_{22}=g_{33}=-(1+2\Omega)\,;$$

all other $g_{\iota\kappa}=0$. Rejecting higher order terms we can write

$$v=1-\Omega,\quad |g_{ij}|=g_{11}g_{22}g_{33}=-(1+6\Omega),$$

$$p_2=(1+4\Omega)(v_{11}+v_{22}+v_{33}),$$

$$p_1=-(1+2\Omega)(v_{22}v_{33}-v_{23}^2+\ldots.).$$

Thus, writing for the sake of shortness

$$\sigma=\mathrm{K}-\mathrm{L}=\mathrm{K}-\frac{\kappa\rho}{2}-\lambda,$$

or, since (in the case of water, for instance) λ is more than 10^{10} smaller than $\frac{\kappa\rho}{2}$,

$$\sigma=\mathrm{K}-\tfrac{1}{2}\kappa\rho, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

the cubic (1) becomes, after the rejection of terms such as 2Ω, 6Ω in the presence of unity,

$$\sigma^3-\sigma^2(\Omega_{11}+\Omega_{22}+\Omega_{33})+\sigma(\Omega_{11}\Omega_{22}-\Omega_{12}^2+\&c.)-|\Omega_{ij}|=0, \quad (4)$$

where Ω_{ij} are covariant derivatives of the Newtonian potential Ω (with respect to x_i, x_j) corresponding to the *three-space* element (2 *a*). In this equation, generally speaking, no term can be neglected, and we have, therefore, three different principal curvatures K_1, K_2, K_3.

The covariant derivatives of the potential appearing in (4) are

$$\Omega_{ij}=\frac{\partial^2\Omega}{\partial x_i \partial x_j}-\sum_{k=1}^{3}\begin{Bmatrix} ij \\ k \end{Bmatrix}\frac{\partial\Omega}{\partial x_k},$$

where the three-index symbols are to be taken with respect to the line-element (2 *a*), *i. e.* for the tensor

$$\begin{matrix} 1+2\Omega, & 0, & 0, \\ 0, & 1+2\Omega, & 0, \\ 0, & 0, & 1+2\Omega. \end{matrix}$$

This gives, with the above degree of approximation,

$$\left.\begin{aligned} \Omega_{11}&=\frac{\partial^2\Omega}{\partial x_1^2}-\left(\frac{\partial\Omega}{\partial x_1}\right)^2+\left(\frac{\partial\Omega}{\partial x_2}\right)^2+\left(\frac{\partial\Omega}{\partial x_3}\right)^2 \\ \Omega_{12}&=\frac{\partial^2\Omega}{\partial x_1\partial x_2}-2\frac{\partial\Omega}{\partial x_1}\cdot\frac{\partial\Omega}{\partial x_2} \end{aligned}\right\}, \quad . \quad (5)$$

with similar expressions for Ω_{22}, Ω_{33}, and Ω_{23}, Ω_{31}. Thus the coefficient of σ^2, for instance, assumes the elegant form

$$\Omega_{11}+\Omega_{22}+\Omega_{33}=\nabla^2\Omega+(\nabla\Omega)^2, \quad . \quad . \quad (5\,a)$$

where ∇ is the Hamiltonian or, in our present case, the gradient. Similarly the remaining two coefficients of the cubic (4) will be determined by (5) in terms of the first and the second derivatives of the Newtonian potential Ω of the given body (mass distribution). Thus, if Ω is found with, say, the usual conditions of continuity and those "at infinity," the curvatures inside and outside the body can be calculated as the roots of (4) and then tested for their compatibility or incompatibility at the prescribed surface of the body. Let K_n stand for the Riemannian curvature near the boundary S of the body corresponding to a geodesic surface element oriented tangentially to S. Then if K_n is continuous across the boundary S, there will be no incompatibility. But if K_n is discontinuous, then, no matter whether the body is homogeneous or not, certain forms of S as boundary will obviously be excluded, and will—if forced upon the body—lead to contradictions. Now it seems very doubtful that, with the above determination of Ω, the curvature K_n should be generally continuous across the boundary, especially as K_n contains also the second derivatives of the potential. And, for the same reason, it seems doubtful whether the usual conditions of the integral of $\nabla^2\Omega=-\frac{1}{2}\kappa\rho$ can be successfully replaced once and for ever by some other boundary conditions. The circumstance that with the actually existing densities the amount of discrepancy may be only small * and even negligible for the physicist, does not change the position. For the question is one of principle, and would certainly call for a thorough mathematical investigation by the strict adherents of Einstein's theory.

No solution of this difficult general problem will be attempted here simply because it is beyond the powers of the present writer (and quite apart from his disinclination to Einstein's theory based on entirely different grounds). In order, however, to explain the above general remarks and formulæ, it may be well to give here the solution of our cubic for the simplest case of a spherical body, for which (as was to be expected) there is no geometrical incompatibility.

Let r be the distance from the centre of the sphere and R its radius. We might with equal ease treat the case of ρ equal to any function of r. But to fix the ideas, let the sphere be homogeneous. Then, α being a constant proportional to its total mass, we have

$$\Omega=\frac{\alpha}{r} \text{ outside, and } \Omega=\frac{\alpha}{\mathrm{R}^3}r^2 \text{ inside.}$$

Thus, writing **F** for the gradient $\nabla\Omega$, and

$$\mathrm{F}=\frac{d\Omega}{dr}$$

for its absolute value, and using the abbreviation

$$\Phi=\frac{d\mathrm{F}}{dr}-\frac{\mathrm{F}}{r}-2\mathrm{F}^2,$$

---

* The radius of curvature corresponding to $\frac{1}{2}\kappa\rho$ is for water of the order of 25 astronomical units.

we find, by (5),

$$\left.\begin{aligned}\Omega_{11}&=\mathrm{F}^2+\frac{1}{r}\mathrm{F}+\Phi\frac{x_1^{\,2}}{r^2},\ \&\text{c.}\\ \Omega_{12}&=\Phi\frac{x_1x_2}{r^2},\ \&\text{c.}\end{aligned}\right\}\quad .\ .\ .\ .\ (6)$$

These values of Ω_{ij} are to be introduced into the cubic (4), giving the three roots σ_1, σ_2, σ_3, of which two will, no doubt, be equal. These, however, can be found more simply by remembering that the cubic is precisely of the form of the equation for the principal axes of the linear vector operator ϖ whose constituents are Ω_{11}, $\Omega_{12}=\Omega_{21}$, &c. Call **P** the operand, denote $\varpi\mathbf{P}$ by $\mathbf{P}'$, and use the suffixes 1, 2, 3 for the components along the axes of x_1, x_2, x_3. Then, by (6), and putting for the moment $F^2+F/r=\Psi$,

$$P_1'=\Psi P_1+\frac{x_1}{r}\Phi\left(\frac{x_1}{r}\mathrm{P}_1+\frac{x_2}{r}\mathrm{P}_2+\frac{x_3}{r}\mathrm{P}_3\right),$$

or, calling **u** the unit of **r**,

$$\mathrm{P}_1'=\Psi P_1+\frac{x_1}{r}\Phi(\mathbf{Pu}),$$

with two similar expressions for P_2', P_3. Thus, for any vector operand **P**,

$$\varpi\mathbf{P}=\Psi\,.\,\mathbf{P}+\Phi\,.\,\mathbf{u}(\mathbf{uP}).$$

Whence we see at once that the principal axes are radial and all transversals. In other words, the operator is *radially symmetric*. To find the principal values, *i. e.* σ, take first $\mathbf{P}\perp\mathbf{u}$, and then $\mathbf{P}\parallel\mathbf{u}$, that is, radial, obtaining in the first case

$$\varpi\mathbf{P}=\Psi\mathbf{P},\ i.\ e.\ \sigma_1=\sigma_2=\Psi,$$

and in the second case,

$$\varpi\mathbf{P}=(\Psi+\Phi)\mathbf{P},\ i.\ e.\ \sigma_3=\Psi+\Phi.$$

Thus, remembering the meanings of Ψ and Φ, the required roots of the cubic (4) will be

$$\left.\begin{aligned}\sigma_1=\sigma_2&=\mathrm{F}^2+\frac{1}{r}\mathrm{F}\ \text{(transversal)}\\ \text{and}\qquad \sigma_3&=\frac{d\mathrm{F}}{dr}-\mathrm{F}^2\ \text{(radial)}\end{aligned}\right\}\quad .\ .\ .\ .\ (7)$$

The corresponding principal curvatures will be, by (3),

$$\mathrm{K}_i=\tfrac{1}{2}\kappa\rho+\sigma_i,\ \ i=1,\,2,\,3.$$

Now, let the body be a homogeneous sphere of radius R, and of density ρ; or let it consist of concentric strata, and let ρ be the density just below the surface. Then we shall have, at the *inner* side of the surface $r=R$,

$$\left.\begin{aligned} K_1=K_2&=\frac{\kappa\rho}{2}+F\left(F+\frac{1}{R}\right) \\ K_3&=-F\left(F+\frac{2}{R}\right) \end{aligned}\right\}, \quad . \quad . \quad . \quad (8\ in)$$

giving $K_1+K_2+K_3=F^2$, and at its *exterior* side ($\rho=0$),

$$\left.\begin{aligned} K_1=K_2&=F\left(F+\frac{1}{R}\right) \\ K_3&=-F\left(F+\frac{2}{R}\right) \end{aligned}\right\}, \quad . \quad . \quad . \quad (8\ out)$$

making $K_1+K_2+K_3=F^2+\frac{1}{2}\kappa\rho$. Thus, if $F=d\Omega/dr$ is continuous across the surface, so is $K_n=K_3$, the principal curvature corresponding to the radial or *normal* axis, and there is thus, for the sphere under consideration, no geometrical incompatibility.

But (to repeat it) whether such will also be the case for differently shaped bodies seems doubtful. Levi-Civita, in a private letter to the writer, expresses the opinion that the continuity of the normal principal curvature K_n (as the above K_3) will also in general be ensured automatically in virtue of a certain equation given in treatises on differential geometry. (Levi-Civita quotes Bianchi's "*Lezioni di geometria differenziale*," vol. i. p. 373, unfortunately not accessible for the present). But I do not see how such an equation can hold independently of the details of mass distribution and therefore of the distribution of the values of Ω and of its derivatives. The reader may find it worth his while to investigate on similar lines as our above sphere the case of, let us say, a homogeneous ellipsoid of revolution.

As I have attempted to point out in another paper (Phil. Mag. vol. xxxvi. p. 94) there are grave objections against Einstein's gravitation theory of an entirely different nature. Until these are removed, it would seem useless to insist any further upon the purely geometrical difficulties.

London,
December 28, 1918.

THE
LONDON, EDINBURGH, AND DUBLIN
PHILOSOPHICAL MAGAZINE
AND
JOURNAL OF SCIENCE.

[SIXTH SERIES.]

MARCH 1919.

XXIII. *On the X-Ray Spectra and the Constitution of the Atom.* II. *By* L. VEGARD, *Dr.Phil., Professor of Physics at the University of Christiania*[*].

Introduction.

§ 1. IN a previous paper[†] published in the Phil. Mag. for April this year, I have tried to explain the X-ray spectra from definite conceptions with regard to the system of electrons which surround the positive nucleus.

My theory of X-ray spectra was based on the assumption that the quant-number of the electrons surrounding the nucleus is not constant and equal to one (Bohr's hypothesis), but increases as we pass outwards from the nucleus—from one ring-system to another.

With regard to the way in which the X-ray spectra are produced, the following two possibilities were considered:—

(1) An X-ray line is produced by the recombination of an electron from a "secondary" stability circle of a higher quant-number to the "primary" circle which has lost one electron.

(2) The X-ray line is produced by the recombination of an electron belonging to one of the outer "primary" stability circles to the broken primary system (the one which has lost one electron).

---

[*] Communicated by the Author. Communicated as manuscript to the *Kristiania Vid. Selsk.* on the 18th of September, 1918.

[†] L. Vegard, "The X-Ray Spectra and the Constitution of the Atom," Phil. Mag. xxxv. p. 293 (1918). This paper will be referred to as Paper I.

General formulæ for the calculation of the frequency were given for both the possibilities mentioned (Equations 15 and 16, Paper I.). It should, however, be remembered that either of these two assumptions needs some additional hypothesis, if the frequency formula is to be derived without ambiguity.

It is my intention in this paper to give a more complete discussion of the various possibilities with regard to the process of recombination, and to draw some of the consequences to which they lead.

I.

On the Recombination between "Primaries."

§ 2. In order to deduce the frequency formula on the assumption of recombination between primary systems we must introduce some additional hypothesis with regard to the change of the electronic systems which accompanies the expulsion of an electron from one of the rings.

When an electron by some agency (X- or β-rays) is driven out from one of the rings the electrons left behind in the atom will change their motion, and not only those which belong to the broken ring-system, but also those belonging to the other rings and especially those outside the broken system, will have their motion changed.

The change taking place must be a very definite one if we are to get homogeneous radiation.

We have mainly two possibilities to consider :—

(1) The motion is changed in such a way that the *angular momentum* of each electron preserves its value unaltered, or it remains the same as in the normal state of the atom.

(2) The motion is changed in such a way that the *energy* of the unbroken electronic system remains unchanged.

A priori we cannot tell which of these hypotheses is the right one.

As the maintenance of a definite angular momentum of the electron seems to be a fundamental property of the atom, the first assumption might seem the more probable; still, in my previous paper, I gave up the assumption of unchangeable momentum because I found it to disagree by far too much with Kossel's empirical relation :

$$\nu_{K_\beta} - \nu_{K_\alpha} = \nu_{L_\alpha} \text{ (approximately)}. \quad . \quad . \quad . \quad (1)$$

The frequencies were therefore calculated on the assumption of conservation of energy, which is in accordance with the relation (1).

To fix the idea, let us recall to memory the way in which the frequency formulæ are deduced.

Let the element considered have an atomic number N. Let us consider a certain ring-system with quant-number n, consisting of q electrons. Let the total number of electrons between the nucleus and the ring be p, when the atom is in its normal state.

The effective nucleus charge of the ring considered will be $(\mathrm{N}-p)e$, and the energy of the ring

$$\mathrm{E}=\mathrm{C}-\frac{2qh\mathrm{R}}{\rho}\left\{1-\sqrt{1-\frac{\rho}{n^2}(\mathrm{N}-p-\mathrm{S}_q)^2}\right\}=\mathrm{C}-\mathrm{W}(n,p,q). \quad . \quad . \quad . \quad (2)$$

In this formula the variation of mass with velocity is taken into account.

R is Rydberg's constant, h Planck's constant, ρ a constant equal to $5\cdot30 \, . \, 10^5$, and

$$\mathrm{S}_q=\frac{1}{4}\sum_{i=1}^{i=q-1}\frac{1}{\sin i\frac{\pi}{q}}.$$

For a given value of N, W is a function of n, p, q, and for the sake of convenience we introduce

$$\frac{\mathrm{W}(n,p,q)}{h\mathrm{R}}=\mathrm{V}(n,p,q)=\frac{2q}{\rho}\left\{1-\sqrt{1-\frac{\rho}{n^2}(\mathrm{N}-p-\mathrm{S}_q)^2}\right\}. \quad . \quad . \quad . \quad (3\,a)$$

If we do not take into account the variation of the mass, we get

$$\mathrm{V}_0(n,p,q)=\frac{q}{n^2}(\mathrm{N}-p-\mathrm{S}_q)^2. \quad . \quad . \quad . \quad (3\,b)$$

As ρ is a small quantity, the expression to the right of (3 a) can be expanded into series, and to the first approximation:

$$\mathrm{V}(n,p,q)=\mathrm{V}_0+\frac{\rho}{4q}\mathrm{V}_0^2. \quad . \quad . \quad . \quad . \quad (3\,c)$$

The hypothesis of maintenance ot energy means that even after an electron is removed from a system inside the ring considered the energy remains unaltered and equal to $\mathrm{C}-\mathrm{W}(n,p,q)$, where p is the number of electrons inside the q ring before the removal of the electron.

If we suppose the angular momentum to be kept the energy of the ring after the removal of an internal electron will be $C-W(n, p-1, q)$.

The correctness of this statement is evident from the fact that the formula (2) gives the energy of the ring when each electron has an angular momentum of $\frac{h}{2\pi}$ and an effective nucleus charge $(N-p)e$. The expulsion of an electron from an interior system will change the effective nucleus charge of the ring to $(N-(p-1))e$.

When the momentum is maintained the expulsion of an electron is accompanied by a sudden change of energy of all the ring-systems outside the broken one. As a matter of fact, the energy of such a ring is diminished, and by the amount

$$\Delta E = W(n, p-1, q) - W(n, p, q).$$

This energy might either disappear in the form of radiation or it might be spent on the escaping electron on its way out of the atom.

The Frequency Formula.

§ 3. The general expression for the frequency on the assumption of conservation of energy was given in my previous paper, and we easily see that Kossel's relations are fulfilled.

On the second assumption of conservation of momentum the matter is not quite so simple.

Let the ring-systems—beginning from the nucleus—be indicated by the indices 1, 2, 3, ... Let the broken ring have an index i and the ring from which recombination takes place be k. The effective nucleus charge of any ring i is $(N-p_i)e$, where

$$p_i = q_1 + q_2 + \ldots q_{i-1}. \quad . \quad . \quad . \quad . \quad (4)$$

Now originally I supposed that it was only the energies of the rings q_i and q_k, between which recombination took place, which were engaged in the production of an X-ray line. On this assumption we get for the frequency:

$$\frac{\nu}{R} = V(n_i, p_i, q_i) - V(n_i, p_i, q_i - 1) - V(n_k, p_k - 1, q_k) \\ + V(n_k, p_k, q_k - 1).$$

Applying this formula to the lines K_β, K_α, L_α, we find :

$$\left(\frac{\nu}{R}\right)_{K_\beta} - \left(\frac{\nu}{R}\right)_{K_\alpha} - \left(\frac{\nu}{R}\right)_{L_\alpha} = V(2,\, q_1 - 1,\, q_2) - V(2,\, q_1,\, q_2).$$

If Kossel's relation (1) should be fulfilled the expression on the right should be approximately equal to zero ; but as a matter of fact it is equal to a positive quantity which may assume quite considerable values. This disagreement between theory and observation made me prefer the assumption of conservation of energy.

Later on I found that if we gave up the assumption that it is only the energy-changes of q_i and q_k which are engaged in the production of the X-ray line, the assumption of conservation of angular momentum can be made to agree with Kossel's relation.

During recombination the rings between q_i and q_k will undergo a change of energy, because the effective nucleus charge is diminished by one elementary unit, and we might assume *that also the change of energy of the intermediate rings enters into the energy quantum of radiation which is emitted as the result of recombination.*

On this assumption the expression for the frequency takes the form :

$$\left.\begin{aligned}\left(\frac{\nu}{R}\right)_i^k &= V(n_i,\, p_i,\, q_i) - V(n_i,\, p_i,\, q_i - 1)\\ &\quad + \sum_{l=i+1}^{l=k-1} \left[V(n_l,\, p_l,\, q_l) - V(n_l,\, p_l - 1,\, q_l)\right]\\ &\quad + V(n^k,\, p_k,\, q_k - 1) - V(n_k,\, p_k - 1,\, q_k).\end{aligned}\right\} \quad (5a)$$

Let us now suppose that another time the electron recombines from a ring q_j, where $i < j < k$, to the same ring q_i. Then we get a frequency $\left(\frac{\nu}{R}\right)_i^j$.

Finally we imagine the electron to be removed from the j-ring, and that the recombining electron comes from the k-ring ; then we get a frequency $\left(\frac{\nu}{R}\right)_j^k$.

Applying equation (5 *a*) in all cases, we easily deduce the identity:

$$\left(\frac{\nu}{R}\right)_i^k - \left(\frac{\nu}{R}\right)_i^j = \left(\frac{\nu}{R}\right)_j^k. \quad \ldots\ldots \quad (6)$$

Applied to the K_β, K_α, and L_α lines, this equation is identical with (1), and Kossel's relation is fulfilled.

The general equation for the frequency we get by inserting into equation (5 *a*) the expression for V (equations 3).

Instead of using the exact expression (3 *a*) we may take (3 *c*), which with the first approximation takes into account the variation of the mass with velocity. The expression for the frequency then takes the form:

$$\frac{\nu}{\mathrm{R}} = \frac{\nu_0}{\mathrm{R}} + \epsilon. \quad \dots\dots\dots \quad (5\,b)$$

ν_0 is the frequency we should get when the variation of the mass is disregarded, and ϵ is the correction for variation of mass. $\frac{\nu_0}{\mathrm{R}}$ and ϵ are given by the following expressions:

$$\left.\begin{aligned}
&\frac{\nu_0}{\mathrm{R}} = \left(\frac{1}{n_i^2} - \frac{1}{n_k^2}\right)\mathrm{N}^2 - \mathrm{BN} + \mathrm{C}, \\
&\text{where} \\
&\tfrac{1}{2}\mathrm{B} = \frac{\phi_1(p_i q_i)}{n_i^2} - \frac{\phi_1(p_k q_k) - q_k}{n_k^2} + \sum_{l=i+1}^{l=k-1} \frac{q_l}{n_l^2}, \\
&\mathrm{C} = \frac{\phi_2(p_i q_i)}{n_i^2} - \frac{\phi_2(p_k q_k) - 2q_k(p_k + \mathrm{S}_{q_k} - 1/2)}{n_k^2} \\
&\qquad + \sum_{l=i+1}^{l=k-1} \frac{2q_l}{n_l^2}(p_l + \mathrm{S}_{q_l} - 1/2).
\end{aligned}\right\} \quad (5\,c)$$

The functions ϕ_1 and ϕ_2, which were introduced in my previous paper, are given by the following expressions:

$$\left.\begin{aligned}
\phi_1(pq) &= q(\mathrm{S}_q - \mathrm{S}_{q-1}) + \mathrm{S}_{q-1} + p, \\
\phi_2(pq) &= q(2p + \mathrm{S}_q + \mathrm{S}_{q-1})(\mathrm{S}_q - \mathrm{S}_{q-1}) + (p + \mathrm{S}_{q-1})^2.
\end{aligned}\right\} \quad (7)$$

Finally:

$$\left.\begin{aligned}
\epsilon = \frac{\rho}{4}\Bigg\{ & q_i\left(\frac{\mathrm{N} - a_i}{n_i}\right)^4 - (q_i - 1)\left(\frac{\mathrm{N} - b_i}{n_i}\right)^4 \\
& - \left[q_k\left(\frac{\mathrm{N} - a_k + 1}{n_k}\right)^4 - (q_k - 1)\left(\frac{\mathrm{N} - b_k}{n_k}\right)^4\right] \\
& + \sum_{l=i+1}^{l=k-1} \frac{q_l}{n_l^4}\left[(\mathrm{N} - a_l)^4 - (\mathrm{N} - a_l + 1)^4\right].\Bigg\}
\end{aligned}\right\} \quad (5\,d)$$

where

$$a_l = p_l + \mathrm{S}_{q_l},$$

$$b_l = p_l + \mathrm{S}_{q\,-1}.$$

On the Determination of the Quant-number and Number of Electrons of the Ring-systems from the observed Frequencies.

§ 4. The problem before us is to find such whole numbers n, p, and q, that the calculated and observed frequencies of the various X-ray spectra are in agreement with each other. The procedure in the present case is the same as the one followed in my previous paper, only the equations and some of the results will be different.

From equations (5) we see that also the assumption of unchangeable momentum leads to an equation of the right type; for, as we know, the frequency of an X-ray line varies with the atomic number in such a way as to approximately satisfy an equation of the form

$$\frac{\nu}{\mathrm{R}}=\left(\frac{1}{n_i^2}-\frac{1}{n_k^2}\right)\mathrm{N}^2-\mathrm{BN}+\mathrm{C}. \quad . \quad . \quad . \quad (8)$$

At any rate this equation will hold for small atomic numbers.

Now the quant-numbers n_i and n_k will be determined from the coefficient of N^2.

The number of electrons in the rings must be determined from the coefficients B and C. To make the final test we can calculate ϵ by means of the known (or possible) values of n, p, q, and thus find the value of $\frac{\nu}{\mathrm{R}}$ corrected for variation of mass.

As mentioned in Paper I., we have first of all to find such values of q that we get the right value of the coefficient B.

Putting as before:

$$\phi(q)=\phi_1(pq)-p=q(\mathrm{S}_q-\mathrm{S}_{q-1})+\mathrm{S}_{q-1}, \quad . \quad . \quad (9)$$

our principal equation of condition now takes the form

$$1/2\,\mathrm{B}=\frac{\phi(q_i)+p_i}{n_i^2}-\frac{\phi(q_k)+p_k-q_k}{n_k^2}+\sum_{l=i+1}^{l=k-1}\frac{q_l}{n_l^2}. \quad . \quad (10)$$

In the case of conservation of energy the corresponding equation of condition is (see Paper I., equation 16)

$$1/2\,\mathrm{B}=\frac{\phi(q_i)+p_i}{n_i^2}-\frac{\phi(q_k)+p_k}{n_k^2}.$$

Taking first the important case when recombination takes place between successive rings, we have to put $k=i+1$, and $\sum\limits_{l=i+1}^{l=k-1} q_l/n_l^2=0$. Then the equation (10) takes the simpler form

$$\phi(q_{i+1})-q_{i+1}=\left(\frac{n_{i+1}}{n_i}\right)^2[\phi(q_i)+p_i]-p_{i+1}-1/2\,\mathrm{B}n_{i+1}^2. \quad (11\,a)$$

From equation (4) we see that p_{i+1} is independent of q_{i+1}. But p_{i+1} depends on q_i and the number of electrons between the q_i ring and the nucleus. In order to get an equation of condition between q_{i+1} and q_i we must know the rings inside the i ring, or p_i. Thus, the natural procedure will be to begin with the ring next to the nucleus and proceed outwards, as done in my previous paper.

Putting $p_{i+1}=q_i+p_i$ equation (11 a) takes the form

$$\phi(q_{i+1})-q_{i+1}=\left(\frac{n_{i+1}}{n_i}\right)^2\phi(q_i)-q_i+\mathrm{A}=f(q_i) \quad . \quad . \quad (11\,b)$$

where A is a constant independent of q_i and q_{i+1}.

$$\mathrm{A}=p_i\frac{n_{i+1}{}^2-n_i^2}{n_i^2}-1/2\,\mathrm{B}n_{i+1}^2. \quad . \quad . \quad . \quad (11\,c)$$

If we have calculated once for all the values of the function $\phi(q)-q$ for various values (whole numbers) of q, we have then in any particular case to determine the term to the right for various values of q_i, and we have to see whether any of the values of the right term equals any of the values found for $\phi(q)-q$.

The values of $\phi(q)$ and $\phi(q)-q$ corresponding to $q=1, 2, \ldots\ldots\ldots 16$ are given in Table I.

Table I.

q.	$\phi(q)$.	$\phi(q)-q$.	q.	$\phi(q)$.	$\phi(q)-q$.
1	0·000	−1·000	9	7·512	−1·488
2	0·500	−1·500	10	8·678	−1·322
3	1·231	−1·769	11	9·946	−1·054
4	2·097	−1·903	12	11·232	−0·768
5	3·052	−1·948	13	12·537	−0·463
6	4·082	−1·918	14	13·881	−0·119
7	5·173	−1·827	15	15·234	+0·234
8	6·305	−1·695	16	16·604	+0·604

From the variation shown by $\phi(q)-q$ we can draw some general conclusions.

In my previous paper it was shown that in a number of cases the X-ray spectra might be explained by recombination from secondary circles. If in our general equations (5) we put $k=i+1$ and $q_{i+1}=1$ we get the equation 15 of Paper I. corresponding to recombination from a secondary circle.

Thus we know that in a number of cases $q_{i+1}=1$ gives an approximate solution of (11); but then we see from Table I. that in any such case there will be another value of q_{i+1} which ought to give an equally good solution.

We also see that the values of $\phi(q)-q$ inside the interval of possible values of q show comparatively small variations, and, if $q=1$ is an approximate solution, any value of q between 1 and 12, say, would give a fairly good agreement with observations.

As we cannot, at any rate from the present scheme, claim or obtain a perfect agreement between observed and calculated values we cannot be sure that the value of q_{i+1} which gives the best numerical agreement is the true one. In other words, even if our present hypothesis should prove to be right, we cannot with any claim of accuracy determine the number of electrons of a ring only from a radiation process in which this ring is the one *from which* the electron starts recombination, thus *e. g.* we cannot by means of the K_α line determine exactly the number of electrons in the L-ring.

As it appears from my previous paper, this is no longer the case when we assume the energy to be maintained. In that case the K_α-line gave quite definite values for the number of electrons of both the K- and the L-rings.

On the assumption of conservation of momentum, however, the number of electrons of each ring-system must be determined from a line which is produced, when the ring in question is the one to which recombination takes place, or the number of electrons must be determined by the value of q_i which gives a possible value of the right term of equation (11 *b*).

Now $\left(\frac{n_{i+1}}{n_i}\right)^2\phi(q_i)-q_i$ is the variable part of the term to the right, and for small values of n_i the ratio will differ considerably from unity, and the expression to the right of 11 *b* will vary fairly rapidly with q. Thus, if there is a value of q_i which gives an approximate solution of (11 *b*) there will be no other value which satisfies it. When n_i becomes fairly large, however, $\frac{n_{i+1}}{n_i}$ approaches unity and also the right term will show a similar variation as $\phi(q)-q$.

Hence, we conclude that it is only for the rings nearest to the nucleus that we may expect to find definite values from the Röntgen-ray spectra for the number of electrons in the ring-systems—provided we stick to the hypothesis of recombination between primaries and conservation of momentum.

APPLICATIONS TO SPECIAL CASES.

a. *The* K_α-*line.*

§ 5. In the case of K_α $i=1$, $n_1=1$, $n_2=2$, $p_1=0$. Further, the empirical relation of Moseley shows that B is approximately equal to 1·5. Inserting these values into (11 *b*) the equation of condition takes the form

$$\phi(q_2)-q_2=4\phi(q_1)-q_1-3=f(q_1). \quad . \quad . \quad . \quad (12)$$

This equation is analogous to the one given in Paper I. corresponding to maintenance of energy. The only difference is that q_2 is subtracted from the left term.

In Table II. is given the variation of $f(q_1)$:—

TABLE II.

q_1.	$f(q_1)$.
1	−4·000
2	−3·000
3	−1·076
4	+1·388
5	+4·208
6	+7·328

Comparing the values of $f(q_1)$ with those of $\phi(q_2)-q_2$ (Table I.), we see that independent of q_2 the only possible value of q is 3.

With regard to the number of electrons in the second ring (q_2) we notice that $q_2=1$ gives the solution of Debye; but besides there should be good agreement with observations for values of q_2 between 7 and 11. The constants B and C corresponding to some values of q_2 are given in Table III.

TABLE III.

q_2.	B.	C.
1	1·464	−0·125
7	1·876	+3·38
8	1·810	+3·98
9	1·706	+4·42
10	1·623	+5·05
11	1·489	+5·43

For small atomic numbers we can put $\epsilon=0$ and $\nu/\mathrm{R}=3/4\,\mathrm{N}^2-\mathrm{BN}+\mathrm{C}$. Calculating ν/R for various values of q_2 we should find that $q_2=9$ gives the best agreement with observations for small atomic numbers; but also the other values of q_2 give a fairly good agreement, and, as already mentioned in the general discussion of the problem, q_2 must be determined from the L-radiation.

In Paper I. we found that $q_2=7$ and $q_3=1$ would give a good agreement in the case of L_α. As we shall see later on, the explanation of L_α on the assumption of recombination between primaries and conservation of momentum gives $q_2=7$.

Thus we put:

$$q_1=3, \quad q_2=7.$$

In accordance with Debye we introduce for the sake of convenience the quantity Δ given by the expression

$$\Delta=\frac{\nu}{\mathrm{R}}-3/4\mathrm{N}^2=-\mathrm{BN}+\mathrm{C}+\epsilon.$$

In the present case we have

$$\epsilon=1{\cdot}325\,10^{-5}\Big\{3(\mathrm{N}-0{\cdot}577)^4-2(\mathrm{N}-0{\cdot}25)^4-\frac{7}{16}(\mathrm{N}-4{\cdot}305)^4 \\ +\frac{6}{16}(\mathrm{N}-4{\cdot}827)^4\Big\}.$$

In Table IV. are given values of Δ corresponding to various values of N.

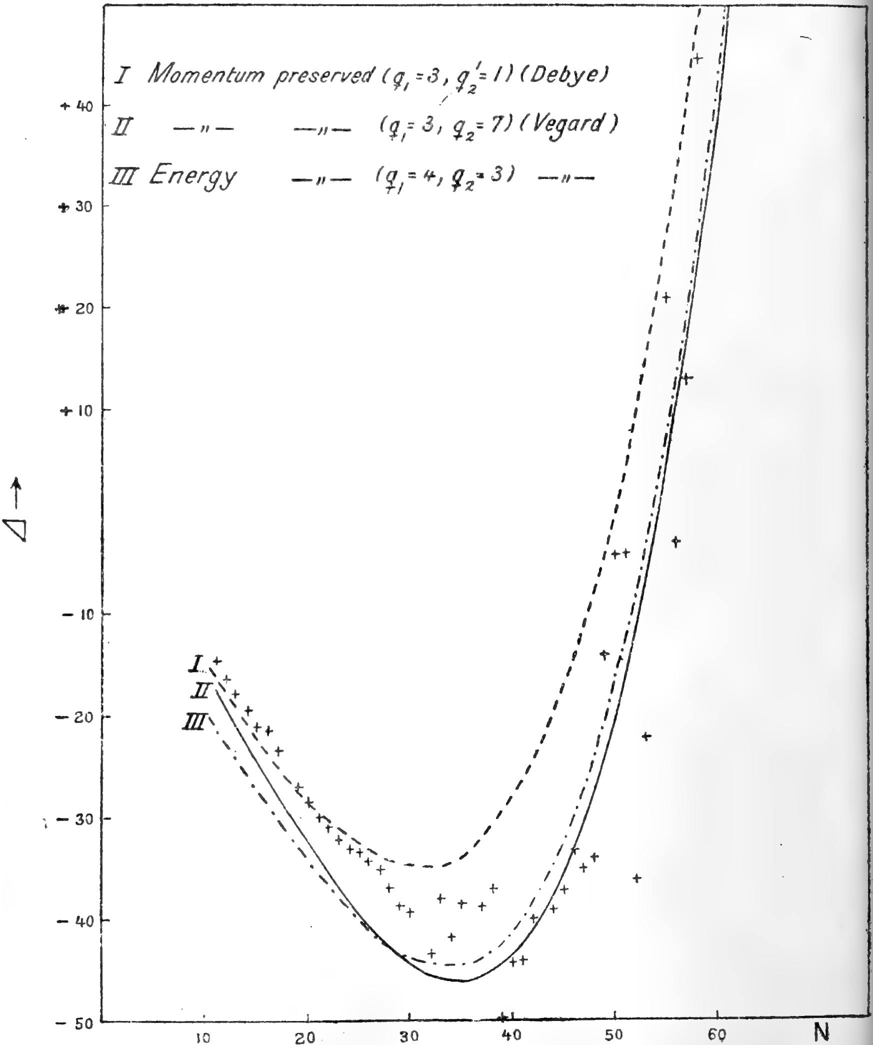
TABLE IV.

N ...	11	15	20	25	30	35	40	45	50	55	60
Δ ...	−17·2	−24·4	−32·6	−39·5	−44·4	−46·2	−43·6	−35·5	−20·4	+5·1	+38·7

In fig. 1 is drawn the curve showing the variation of Δ with the atomic number. The observed points are also marked off. For the sake of comparison two other curves are drawn, that of Debye ($q_1=3$, $q_2=1$) and the one corresponding to maintenance of energy ($q_1=4$, $q_1=3$). (See Paper I., p. 308.)

We see that the curve of Debye is the best for small atomic numbers; but the two other curves give a better agreement for high atomic numbers. From a mere numerical comparison it would hardly be possible to decide which curve and which assumption is the right one.

Fig. 1.



b. *The L_α-line.*

§ 6. The L_α line is produced by recombination between a system with quant-number 3 to one with quant-number 2. We shall first suppose that there is only one ring with quant-number 2 and that recombination takes place between successive systems. On this assumption we have

$$i=2,\quad n_2=2,\quad n_3=3,\quad p_2=3,\quad p_3=q_2+3,\quad \mathrm{B}=2{\cdot}06 \text{ (approx.)},$$

and the equation (11 *b*) takes the form

$$\phi(q_3)-q_3=9/4\,\phi(q_2)-q_2-5{\cdot}52=f(q_2). \quad . \quad . \quad (13)$$

Values of $f(q_2)$ corresponding to various values of q_2 are given in Table V.

TABLE V.

q_2 ...	1	2	3	4	5	6	7	8
$f(q_2)$...	−6·50	−6·40	−5·98	−4·80	−3·65	−2·34	−0·88	+0·67

Comparing the values of $f(q_2)$ with those of $\phi(q_3)-q_3$ (Table I.) we see that the equation of condition (13) is approximately fulfilled by the following two combinations:

$$q_2=7,\quad q_3=1\rightarrow 12,$$

$$q_2=8,\quad q_3=16.$$

The first solution corresponds to the one found in Paper I. and given by

$$q_2=7 \text{ and } q_3=1.$$

The second solution $q_2=8$, $q_3=16$ must be considered as improbable on account of the large number of electrons in the third system. Further, the application of the theory to K_β and M_α makes the assumption of $q_3=16$ impossible. The only possibility is to assume 7 electrons in the second ring, and the coefficients B and C are determined by the equations

$$\mathrm{B}=1{\cdot}865-\frac{2}{9}\left(\phi(q_3)-q_3\right),$$

$$\mathrm{C}=14{\cdot}30-\frac{1}{9}\Big[q_3(20+\mathrm{S}_{q_3}+\mathrm{S}_{q_3-1})(\mathrm{S}_{q_3}-\mathrm{S}_{q_3-1}) \\ +(10+\mathrm{S}_{q_3-1})^2-2q_3(\mathrm{S}_{q_3}+9{\cdot}5)\Big].$$

Values of B and C corresponding to various values of q_3 are given in Table VI.

The relative change of B and C with variation of q_3 in the case of L_α is even smaller than for K_α; and, although the values $q_3=1$, 11 or 12 give the best agreement, at any rate for small atomic numbers, we cannot be certain that the particular value which gives the best agreement is also the right one.

TABLE VI.

	B.	C.	q_3.	B.	C.
Empirical values.	+2·0556	+7·6	7	+2·271	+8·1
q_3 = 1	2·087	5·3	8	2·242	8·2
2	2·198	6·4	9	2·196	8·2
3	2·258	7·1	10	2·159	8·1
4	2·288	7·5	11	2·099	7·9
5	2·298	7·8	12	2·036	7·6
6	2·291	8·0	13	1·968	7·2

In order to make a choice between the q_3 values we have recourse to the M-series. As we shall see later on the explanation of the M_α-line gives the two possibilities $q_3=8$, $q_3=12$. The first value corresponds to the assumption of two—the second to the assumption of one system with quant-number 3.

In Table VII. are given observed and calculated values for various atomic numbers. The calculated values are corrected for variation of mass with velocity.

TABLE VII.

N.	$\nu/R_{obs.}$	$\nu/R_{cal.}$ $q_2=7$, $q_3=12$.	D.	P.	$\nu/R_{cal.}$ $q_2=7$, $q_3=8$.	D.	P.
30	73·8	71·6	− 2·2	−3·0	66·0	−7·8	−10·6
40	149·8	149·1	− 0·7	−0·5	141·4	−8·4	− 5·6
50	253·5	255·1	+ 1·6	+0·6	245·3	−8·2	− 3·2
60	384·6	390·3	+ 5·7	+1·5	378·4	−6·2	− 1·6
70	445·6	555·1	+ 9·5	+2·1	541·1	−4·5	− 1·0
80	734·9	751·4	+16·5	+2·2	735·3	+0·4	0·0
90	952·2	979·6	+27·4	+2·9	961·3	+9·1	+ 1·0

$$D=\nu/R_{cal.}-\nu/R_{obs.},\quad P=\frac{100\,D}{(\nu/R)_{obs.}}.$$

The assumption $q_3=12$ gives the best agreement for small, while $q_3=8$ is the better for high atomic numbers. As we shall see later on the frequencies of K_β are in favour of the assumption $q_3=8$.

C. *On the Possibility of two L-systems.*

§ 7. In Paper I. I was led to assume a second L-ring of 8 electrons. Building on the assumption of recombination from secondaries I was able by means of this assumption to indicate an explanation of the l-series. Now it would be of interest to see how far the assumption of two L-systems is consistent with our present hypothesis with regard to the process of recombination, and how far this hypothesis is able to yield any satisfactory explanation of the l-series.

In this case we must apply equation (10) because the recombination does not take place between successive rings.

Putting in equation (10)

$$i=2,\ k=4,\ n_2=2,\ n_3=2,\ n_4=3,\ p_2=3,$$
$$p_3=3+q_2,\ p_4=3+q_2+q_3,$$

the equation of condition takes the form

$$\phi(q_4)-q_4=f(q_2)+\frac{5}{4}q_3, \quad . \quad . \quad . \quad . \quad (14)$$

where $f(q_2)$ is the same function as in equation (13).

From the Tables I. and V. we find that we get a fairly satisfactory solution of (14) for a number of values of q_2 and q_3.

The values of $f(q_2)+5/4\,q_3$ for the possible combinations (q_2, q_3) are given in Table VIII.

TABLE VIII.

q_2	6	5	4	3	2	1
q_3	1	2	3	4	4	4
$f(q_2)+5/4q_3$.	−1·09	−1·15	−1·05	−0·98	−1·40	−1·50

With the exception of the last two combinations the total number of electrons in the L-systems (q_2+q_3) is equal to 7.

From a mere numerical point of view the above combinations would give a fairly satisfactory agreement, and we may say that our hypothesis of recombination between primaries and conservation of momentum is not against the assumption of two L-systems; but in this case—as far as I can see at present—such an assumption only leads to complications; for it does not lead to any satisfactory explanation of the l-series.

Putting into the equation (5 c)

$$i=3,\ k=4,\ n_3=2,\ n_4=3,\ p_3=3+q_2,\ p_4=3+q_2+q_3,$$

we shall find that for all the combinations (q_2, q_3) given in

Table VIII. the frequency would assume too high values as compared with l_α, and would be very nearly equal to that of L_α.

When we are going to proceed further we assume that the L-ring only consists of one system composed of 7 electrons.

d. *The M_α-line.*

§ 8. At present the M-radiation is only known for a few elements. As shown in Paper I. the frequencies of the M_α-lines can be expressed by the empirical formula

$$\frac{\nu}{R}=\left(\frac{1}{3^2}-\frac{1}{4^2}\right)N^2-2\cdot 37\,N+40. \quad . \quad . \quad . \quad (15)$$

According to this formula the M_α-line is produced by recombination from a circle with quant-number 4 to one with quant-number 3.

We shall treat the following two possibilities :—

(1) There is only one system with quant-number 3.
(2) There are two ring-systems.

First Possibility. One M-ring.

In this case we put into equation (11)

$$i=3, \quad n_3=3, \quad n_4=4, \quad p_3=10,$$

and we get :

$$\phi(q_4)-q_4=\frac{16}{9}\phi(q_3)-q_3+7\cdot 78-8B. \quad . \quad . \quad (16)$$

Taking B to be in the interval $2\cdot 1<B<2\cdot 5$, q_3 must be in the interval $12<q_3<14$.

Now, in order to determine the frequency, we should know the value of q_4 exactly. But this number cannot be exactly found by means of M_α.

In Table IX. will be found the values of B and C corresponding to some of the best combinations of q_3 and q_4.

Table IX.

q_3.	q_4.	B.	C.
13	10	+2·30	+22·4
12	10	+2·14	+19·8
12	8	+2·184	+20·5

In Table X. are given the values of ν_0/R corresponding to the above values of B and C. In the case of the combination ($q_3=12$, $q_4=8$) the corrected values of ν/R have been calculated and set out in the fifth column. Finally, ν/R has been calculated from the observed wave-length.

Table X.

N.	ν_0/R.			$\nu/\mathrm{R}_{\mathrm{cal.}}$	$\nu/\mathrm{R}_{\mathrm{obs.}}$	D.	P.
	$q_3=13$, $q_4=10$.	$q_3=12$, $q_4=10$.	$q_3=12$, $q_4=8$.	$q_2=12$· $q_4=8$.			
79	144	154	151·4	152·0	156·1	−4·1	−2·6
81	155	165	162·7	163·6	166·3	−2·7	−1·6
82	161	171	168·3	169·4	171·8	−2·4	−1·4
83	166	177	174·2	175·4	178·1	−2·7	−1·5
90	209	221	217·8	219·9	220·2	−0·3	−0·1
92	222	234	231·1	233·6	233·4	+0·2	+0·1

The agreement is quite good; but as we are now treating rings with a somewhat high index-number, the determination of the number of electrons becomes more uncertain, especially so long as the value of q_4 is left undetermined.

The Second Possibility. Two M-rings.

In this case we put into (10)

$$i = 3, \quad k = 5, \quad n_3 = 3, \quad n_4 = 3, \quad n_5 = 4,$$
$$p_3 = 10, \quad p_4 = 10+q_3, \quad p_5 = 10+q_3+q_4,$$

and we get the equation of condition:

$$\phi(q_5) - q_5 = f(q_3) + \frac{7}{9}q_4. \quad . \quad . \quad . \quad . \quad (17)$$

$f(q_3)$ is the right term of equation (16).

There are several combinations (q_3, q_4) which might approximately satisfy equation (17). Thus, *e. g.*,

$$(q_3=7,\ q_4=8), \quad (q_3=8,\ q_4=7), \quad (q_3=9,\ q_4=5)$$

give quite good agreement.

The combination $q_3=8$, $q_4=7$, and $q_5=10$ gives the following equation for the frequency:

$$\frac{\nu_0}{\mathrm{R}} = \left(\frac{1}{3^2} - \frac{1}{4^2}\right)\mathrm{N}^2 - 2{\cdot}22\mathrm{N} + 25. \quad . \quad . \quad . \quad (18)$$

The formula (18) will give an equally good agreement as the combination $q_3=12$, $q_4=8$ in the case of one M-ring.

It might also be of interest to try the values:

$$q_3 = 8, \quad q_4 = 10, \quad q_5 = 8,$$

which correspond to the number of electrons which were proposed in Paper I. for the third, fourth, and fifth rings.

The latter combination would lead to the formula

$$\frac{\nu_0}{R} = \left(\frac{1}{3^2} - \frac{1}{4^2}\right)N^2 - 2{\cdot}56N + 33. \qquad . \quad . \quad (19)$$

This formula is almost identical with the one developed in Paper I. on the assumption of recombination from secondaries; but in that case the L-system was supposed to consist of two rings.

It appears from the treatment of the M_α-line that quite a number of assumptions with regard to the composition of the ring-systems lead to a fairly good agreement with observations.

If we assume only one ring with quant-number 3 the number of electrons in this ring comes out fairly definite and equal to 12. On the assumption of two M-rings there are a number of possibilities, of which the one making $q_3=8$ is probably the best one.

e. *The K_β-line.*

§ 9. From the point of view here adopted, K_β should be produced by recombination from the M-ring to the K-ring.

In the case of the M-ring we have treated two possibilities, and for K_β we shall also consider the two combinations:

$$q_3 = 12 \text{ (only one M-ring)},$$
$$q_3 = 8 \text{ (two M-rings)}.$$

The frequency is found from equations (5 *c*) and (5 *d*) by putting $i=1$, $k=3$, $n_1=1$, $n_2=2$, $n_3=3$, $q_1=3$, $q_2=7$, and $q_3=12$ or 8, and

$$\left(\frac{\nu}{R}\right)_{K_\beta} = \frac{8}{9}N^2 - 3{\cdot}911N + 11{\cdot}0 + \epsilon'_{K_\beta} \quad (q_3=12), \qquad (20\,a)$$

$$\left(\frac{\nu}{R}\right)_{K_\beta} = \frac{8}{9}N^2 - 4{\cdot}117N + 11{\cdot}6 + \epsilon''_{K_\beta} \quad (q_3=8). \qquad (20\,b)$$

We need not, however, calculate the frequencies from these formulæ, for, as we have seen previously, the following relation is identically fulfilled:

$$\left(\frac{\nu}{\mathrm{R}}\right)_{\mathrm{K}_\beta} = \left(\frac{\nu}{\mathrm{R}}\right)_{\mathrm{K}_\alpha} + \left(\frac{\nu}{\mathrm{R}}\right)_{\mathrm{L}_\alpha} = \left(\frac{\nu_0}{\mathrm{R}}\right)_{\mathrm{K}_\alpha} + \left(\frac{\nu_0}{\mathrm{R}}\right)_{\mathrm{L}_\alpha} + \epsilon_{\mathrm{K}_\alpha} + \epsilon_{\mathrm{L}_\alpha}. \quad (21)$$

Here only $\left(\frac{\nu}{\mathrm{R}}\right)_{\mathrm{L}_\alpha}$ will change with q_3, and $\left(\frac{\nu}{\mathrm{R}}\right)_{\mathrm{K}_\beta}$ is easily found from the results for K_α and L_α.

Values of $\left(\frac{\nu}{\mathrm{R}}\right)_{\mathrm{K}_\beta}$ found in this way are given in Table XI.

TABLE XI.

N.	$(\nu/\mathrm{R})_{\mathrm{obs.}}$	$q_3=12.$			$q_3=8.$		
		$(\nu/\mathrm{R})_{\mathrm{cal.}}$	D.	P.	$(\nu/\mathrm{R})_{\mathrm{cal.}}$	D.	P.
20	295·8	287·9	− 7·9	−2·7	286·3	−9·5	−3·2
30	704·2	702·2	− 2·2	−0·31	696·6	−7·6	−1·1
40	1293	1305·5	+12·5	+1·0	1297·8	+4·8	+0·37
50	2109	2109·7	+ 0·7	+0·03	2099·9	−9·1	−0·43
60	3121	3129	+ 8·0	+0·26	3117·1	−3·9	−0·13

It appears from the table that both values of q_3 give a very close agreement with observations, and from the mere numerical comparison here given it would be difficult to decide in favour of either of the two possibilities.

There is, however, one circumstance in connexion with the production of K_β which might be of great value as a test of the correctness of the various hypotheses, which might come into consideration with respect to the emission process and the number of electrons in the third ring.

Observations show that K_β always accompanies K_α (perhaps with the exception of $\mathrm{N}=11$, for which no K_β-line is given in the table of Siegbahn).

If, now, the homogeneous X-radiation is produced in the way here considered, by recombination between primary rings, it follows that for lower atomic numbers the recombining electron must come from the surface ring; but the number of electrons in this ring will diminish as we pass towards lower values of N. For $\mathrm{N}=12$ there will be 2 electrons in the third ring, for $\mathrm{N}=13$ $q_3=3$, a. s. o. If we assume 8 electrons in the third ring, q_3 will keep on increasing until the element Ar ($\mathrm{N}=18$); if we put $q_3=12$, for the final M-ring, q_3 will keep on increasing till the element Ti ($\mathrm{N}=22$).

From the periodic variation of the properties of the elements we come (as we shall see later on) to the conclusion that we shall most likely in addition have an increase of the quant-number of the surface ring when we pass the element Argon. If, however, for a moment we look away from this possibility, we still have to consider the changes of frequency which are due to the variation of the number of electrons in the system of departure (system from which recombination starts). As we have seen, these changes are not large; but still they may be great enough to affect the curve $\nu=f(N)$ and give to it a characteristic curvature, which also should be found for the experimental curve.

In order to carry out this test, we shall calculate the frequencies for all atomic numbers between 11 and 30. We shall consider the quantity:

$$\Delta=\frac{\nu}{R}-\frac{8}{9}N^2=-BN+C+\epsilon. \quad . \quad . \quad . \quad (22)$$

In Table XII. are given the values of B and C for values of q_3 varying from 1 to 12.

TABLE XII.

q_3.	B.	C.	q_3.	B.	C.
1	3·962	8·7	7	4·146	11·5
2	4·073	9·8	8	4·117	11·6
3	4·133	10·5	9	4·071	11·6
4	4·163	10·9	10	4·034	11·5
5	4·173	11·2	11	3·974	11·3
6	4·166	11·4	12	3·911	11·0

From these values of B and C, and from the values of ϵ for K_α and L_α, we can determine Δ for various atomic numbers. The results are given in Table XIII.

The variation of Δ with the atomic number is represented in fig. 2.

Curve I. is drawn from Debye's values, which are calculated on the assumption of recombination from a secondary circle.

The curves II. and III. represent the above values of Δ for $q_3=12$ and $q_3=8$ respectively. Curve IV. is drawn so as to represent the observed values of Δ. It should be remembered that errors may be attached to the observed frequencies, and that the absolute error of frequency will greatly increase with increase of atomic number. Thus the curve Δ_{obs} can be drawn with a considerable accuracy up to about N=30, but the position of the true experimental curve is somewhat uncertain for higher values of N.

TABLE XIII.

N.	8/9 N^2.	$\Delta_{\text{obs.}}$	$\Delta_{\text{cal.}}$ q_3 (max.)=12.	$\Delta_{\text{cal.}}$ q_3 (max.)=8.
11	107·5	...	— 34·8	— 34·8
12	128·0	— 31·8	— 39·0	— 39·0
13	150·2	— 36·1	— 43·0	— 43·0
14	174·2	— 39·4	— 47·1	— 47·1
15	200·0	— 43·1	— 51·0	— 51·0
16	227·6	— 46·0	— 54·8	— 54·8
17	256·9	— 49·5	— 58·3	— 58·3
18	288·0	...	— 61·5	— 61·5
19	230·9	— 56·7	— 64·6	— 65·4
20	255·6	— 59·8	— 67·7	— 69·2
21	392·0	— 64·0	— 70·3	— 73·0
22	430·2	— 67·0	— 72·7	— 76·7
23	470·2	— 70·7	— 76·3	— 80·4
24	512·0	— 73·7	— 79·6	— 83·9
25	555·6	— 76·5	— 82·9	— 87·4
26	600·9	— 79·6	— 86·0	— 90·7
27	648·0	— 83·1	— 89·0	— 94·0
28	696·9	— 88·2	— 92·0	— 97·2
29	747·6	— 92·5	— 94·8	—100·2
30	800·0	— 95·8	— 97·7	—103·4
40	1422·2	—129·3	—116·7	—124·4
50	2222·2	—113·2	—112·5	—122·3
60	3200·0	— 79·0	— 71·0	— 82·9

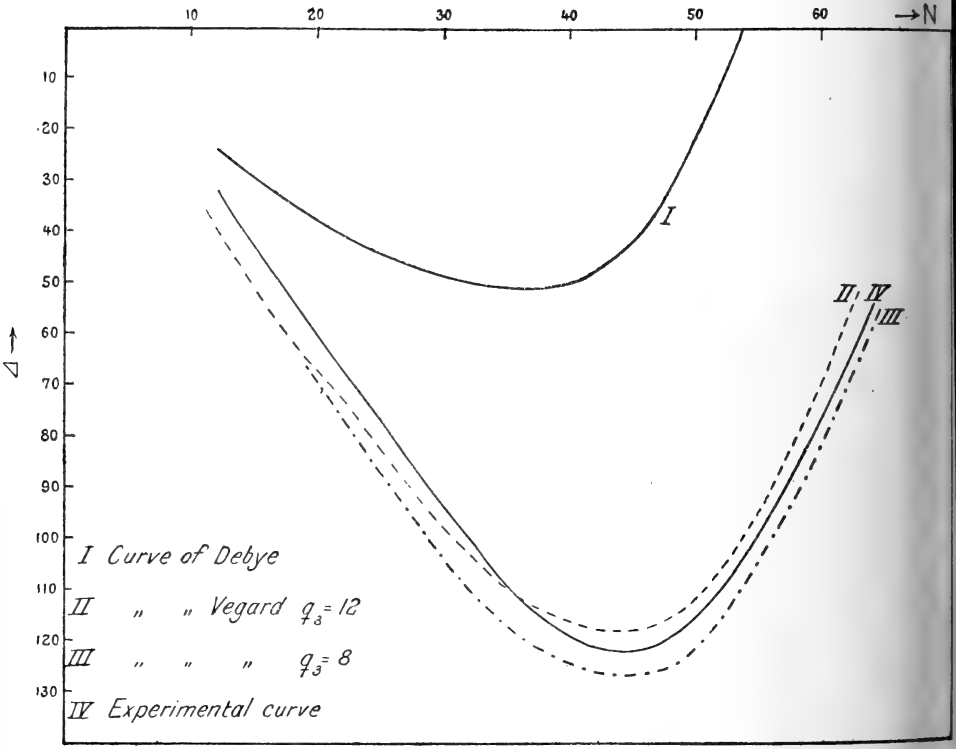
We thus see that both curves (II. and III.) run very near to the experimental curve IV., and they give a much better agreement than the curve of Debye.

The curve II. runs nearest to the experimental curve; but still—if we stick to the hypothesis of recombination between primaries with conservation of momentum—the assumption of 12 electrons in a single M-ring must be given up, because for small atomic numbers, owing to the change of q_3, the curve II. has a peculiar bend which is not at all indicated in the experimental curve.

Passing from low atomic numbers, the distance between

II. and IV. first increases to a maximum for N=18 and then decreases quite considerably till N=22. Then the curve II. has a point of inflexion and runs more nearly parallel to the experimental curve.

Fig. 2.



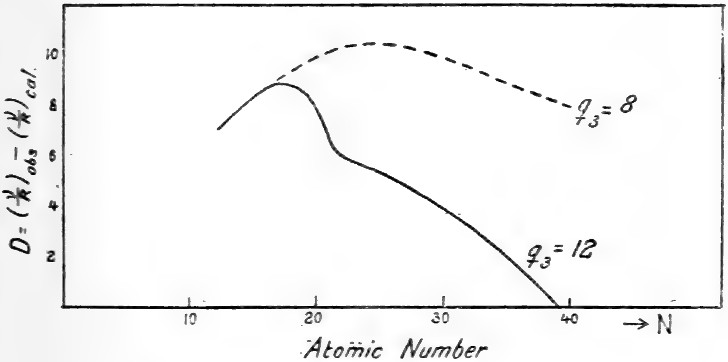
The peculiar variation of the distance between the two curves II. and IV. is shown in fig. 3.

The curves III. and IV. run more nearly parallel; but also in this case they show at the beginning a somewhat different curvature. Their distance first increases to a maximum and then diminishes slightly, as indicated by the dotted curve in fig. 3.

I think that from the preceding test we are justified in concluding that the assumption of 12 electrons in the third ring is inconsistent with the hypothesis of recombination between primaries and conservation of momentum.

The assumption of 8 electrons in the third ring, however, holds in this respect a somewhat stronger position.

Fig. 3.



f. *On the Origin of the other Principal L-lines.*

§ 10. In Paper I. I gave an explanation of the other L-lines, which—with the exception of the *l*-spectrum—can be followed on the present assumption of recombination between primaries.

In accordance with Sommerfeld, we assumed that L_α and L_β were the two components of a doublet; and from the fact that the difference of frequency between the two doublets was the same as that between the frequencies of the absorption edges, we concluded that in the normal atom the L-system could exist in two states—one circular giving, by the recombination process, L_α, and one elliptic giving L_β.

We were thus led to the assumption of *one fixed elliptic state for systems of electrons.* This, as we stated, involved a kind of mutual connexion between the electrons belonging to the same ring-system, because we had to assume that all of them were thrown simultaneously into the elliptic state where each electron followed an elliptic orbit of definite shape, magnitude, and position. We may, *e. g.*, imagine that the elliptic axes are arranged radially and with equal angular intervals, and that all electrons are in the same phase. Thus at any moment the electrons will be evenly distributed on the circumference of a circle, the radius of which undergoes periodic changes as time passes.

Now we may expect the next doublet, $L_\gamma L_\delta$ (Siegbahn $\beta_2 \gamma_1$), to be produced by recombination from the ring with quant-number 4 to the circular and elliptic state of the L-system. Just as in the case of the K_β-line, we have:

$$\left(\frac{\nu}{R}\right)_{L_\gamma} = \left(\frac{\nu}{R}\right)_{L_\alpha} + \left(\frac{\nu}{R}\right)_{M_\alpha}, \quad . \quad . \quad . \quad . \quad . \qquad (23\,a)$$

and for recombination to the elliptic state:

$$\left(\frac{\nu}{R}\right)_{L_\delta} = \left(\frac{\nu}{R}\right)_{L_\beta} + \left(\frac{\nu}{R}\right)_{M_\alpha}. \quad . \quad . \quad . \quad . \quad . \qquad (23\,b)$$

In Table XIV. are given the calculated values of $\left(\frac{\nu}{R}\right)_{L_\alpha}$, $\left(\frac{\nu}{R}\right)_{M_\alpha}$, $\left(\frac{\nu}{R}\right)_{L_\nu}$, and the observed values of $\left(\frac{\nu}{R}\right)_{L_\gamma}$. The frequency of M_α is calculated from the combination $q_3=8$, $q_4=7$, both with quant-number 3, and $q_5=10$ (quant-number 4).

Table XIV.

	Calculated.			Observed.		
N.	$\left(\frac{\nu}{R}\right)_{L_\alpha}$.	$\left(\frac{\nu}{R}\right)_{M_\alpha}$.	$\left(\frac{\nu}{R}\right)_{L_\gamma}$.	$\left(\frac{\nu}{R}\right)_{L_\gamma}$.	D.	P.
50	245·3	35·5	280·8	287·3	− 6·5	−2·3
60	378·4	66·8	445·2	448·0	− 2·8	−0·6
70	541·5	107·8	649·3	644·0	+ 5·3	+0·8
80	735·3	158·7	894·0	874·5	+19·5	+2·2
90	961·3	221·2	1182·5	1143	+39·5	+3·5

The agreement is seen to be remarkably good, and thus we see by comparison with Paper I. that also in the case of the L-series the assumption of recombination between primaries gives a better agreement for the second line of the series than the hypothesis of recombination from secondaries in the form in which this hypothesis was taken in my previous paper.

In this way we have succeeded in giving a fairly satisfactory explanation of the two doublets $(\alpha\,\beta)$ and $(\gamma\,\delta)$. If our interpretation is right, they should—in analogy with the K-series—rather be called $(\alpha_0\,\alpha_1)$ $(\beta_0\,\beta_1)$.

With regard to the *l*-series, we have seen that the assumption of recombination between primaries is not consistent with the explanation previously given to the *l*-series; but this fact alone must not be considered as fatal to the assumption of recombination between primaries. Further investigations may show us new possibilities for the production of lines; so, *e. g.*, it is not excluded that a ring-system, besides the circular and elliptic state, may take up some third state.

II.

On the Recombination from Secondaries.

§ 11. In Paper I. most of our calculations were carried out on the assumption that the recombining electron did not come from an atomic ring-system, but from independent stability circles (secondary circles).

It was further assumed, in accordance with Debye, that the recombination towards a broken system took place as if the exterior atomic rings had been removed.

This latter hypothesis is a very legitimate one so long as there is no electron-system between the secondary circle and the broken ring. This is, however, not always the case, thus *e.g.* the production of K_β would require a secondary with quant-number 3 recombining to the first ring-system. Now the radius of a circular electronic system is given by the expression (Paper I. eq. 25)

$$a_p = a_H \frac{n_k^2}{N - p_k - S_{q_k}} ; \quad . \quad . \quad . \quad . \quad . \quad (24\,a)$$

a_H is the radius of the normal stability circle of hydrogen. As a_p is proportional to the square of the quant-number, it follows that inside a secondary with quant-number τ will be situated all those primary rings for which

$$n < \tau.$$

Outside the secondary those rings must be situated for which

$$n > \tau.$$

With regard to the primary rings for which $n = \tau$, we may assume that they are situated outside the secondary. This assumption is in agreement with equation (24 a).

If n_k is the quant-number which is equal to τ, the radius of the primary is found from (24 a), and that of the secondary a_s will be

$$a_s = a_H \frac{\tau^2}{N + 1 - p_k} . \quad . \quad . \quad . \quad . \quad . \quad (24\,b)$$

p_k being the same in (24 a) and (24 b) when $\tau = n$, we see that under these conditions

$$a_s < a_p.$$

If there are systems between the secondary and the broken ring, it would be difficult on the basis of the hypothesis mentioned to convey any clear conception as to the process of light-emission ; the secondary stability circles could hardly have any real existence.

It is, however, possible to modify the hypothesis of recombination from secondaries in such a way that these difficulties are avoided.

Just as in the case of recombination between primaries, we may assume *that the energy changes of the intermediate rings which accompany the recombination process enter into the energy-quantum which is emitted as the result of the recombination.*

Proceeding in the same way as before, and using the same designations, we easily deduce the following general frequency formula :

$$\left.\begin{aligned} \frac{\nu}{R} &= V(n_i, p_i, q_i) - V(n_i, p_i, q_i - 1) \\ &\quad \sum_{l=i+1}^{l=k-1} [V(n_l, p_l, q_l) - V(n_l, p_l - 1, q_l) \\ &\quad - V(\tau, p_k - 1, 1)]. \end{aligned}\right\} \quad . \quad (25)$$

Comparing the equation (25) with (5 a), we see that the first can be derived from the latter simply by putting

$$n_k = \tau \quad \text{and} \quad q_k = 1.$$

In order to see the significance of this statement, we shall consider the recombination to the i-ring. In one case we suppose the electron to recombine from the *primary* k-ring with quant-number n_k. (If there is more than one ring with the same quant-number we suppose recombination to occur from the ring nearest to the nucleus.) The resulting frequency we call ν_p. In the second case we consider the recombination from a secondary with quant-number $\tau = n_k$, which results in a frequency ν_s. From what has been said about the position of the secondary circle relative to the primary ones we conclude that in both cases the recombining electron passes the same intermediate primary systems, and we get :

$$\frac{\nu_p}{R} - \frac{\nu_s}{R} = V(\tau, p_k, q_k - 1) - V(\tau, p_k - 1, q_k) + V(\tau, p_k - 1, 1).$$

Thus the difference is only due to the change of number of electrons of the system from which recombination takes place from q_k to 1, and, as we saw in the first part of this paper, the change of q_k between 1 and 12 has only a comparatively small influence on the frequency. Further we saw that $q_k=1$ was always among the values which gave the best agreement with observations.

Hence we conclude *that the determinations of the ring-systems from the Röntgen-ray spectra which were undertaken in the first part of the present paper also apply to this modified form of recombination from secondaries.* There is, however, the difference that some of those combinations of q values which were possible when q_k was variable will be much less good when q_k has the fixed value 1. Thus the last two combinations of Table VIII. ($q_2=2$, $q_3=4$) and ($q_2=1$, $q_3=4$) do not any longer give a satisfactory agreement.

Comparing the equation (25) with equation (15) of my previous paper, we notice that they are identical if

$$k=i+1.$$

Thus the modified and the original hypothesis of recombination from secondaries will be identical in all those cases in which there is no electronic system between the secondary circle and the broken ring. Consequently, the equation for the frequency of K_α will be the same in both cases and equal to that of Debye. As regards L_α we also get the same equation for the frequency in the two cases, provided we give up the assumption of two L-rings with 7 and 8 electrons. From the treatment of the L_α-line on the hypothesis of recombination between primaries, we conclude that also the modified hypothesis of recombination from secondaries may be consistent with two L-systems, for which $q_2+q_3=7$; but such an assumption was found to be of no use for the explanation of the l-line, and we shall also for the recombination from secondaries assume that there is only *one* ring with quant-number 2. If so, the equation (25) gives the same frequency for L_α as the equation (15) of Paper I., and the calculations of $\left(\frac{\nu}{R}\right)$ of my previous paper also apply to the modified hypothesis.

With regard to M_α there will be principally the same possibilities as in the case of recombination between primaries. *There may be two M-systems or only one.*

In the latter case the M-ring should consist of 12 or 13 electrons.

Now we saw from the treatment of K_β in the first part of this paper that if we would stick to the hypothesis of recombination between primaries the assumption of one M-ring with 12 or 13 electrons has to be abandoned.

This is no longer necessary when we assume recombination from secondaries; *because in that case there is always one and only one electron in the system of departure.*

As the simplest hypothesis we shall assume one, M-ring, and calculate the frequency for M_α on the assumption of $q_3=12$.

The values of B and C are found from (5 *c*) by putting:

$$i=3, \quad k=4, \quad n_3=3, \quad n_4=\tau=4, \quad p_3=10, \quad p_4=10+q_3$$

$$B=2\left[\frac{\phi(q_3)+10}{9}-\frac{10+q_3-1}{16}\right]=2\cdot093\ (q_3=12);$$

$$C=\frac{\phi_2(10,q_3)}{9}-\frac{\phi_2(10+q_3,1)-2(10+q_3-0\cdot5)}{16}$$
$$=17\cdot8\ (q_3=12).$$

Further:

$$\epsilon=1\cdot325\ 10^{-5}\left\{12\left(\frac{N-14\cdot98}{3}\right)^4-11\left(\frac{N-14\cdot42}{3}\right)^4-\left(\frac{N-21}{4}\right)^4\right\}.$$

In Table XV. are given calculated and observed values of $\frac{\nu}{R}$ for some atomic numbers.

TABLE XV. (M_α).

N.	$(\nu/R)_{\text{cal.}}$	$(\nu/R)_{\text{obs.}}$	P.
79	156·4	156·1	+0·02
82	174·2	171·8	+1·2
90	225·3	220·2	+2·3
92	239·2	233·4	+2·5

The agreement is seen to be very satisfactory. It should be remarked *that if we give up the assumptions of two L-rings the results given in the table would also have followed as a consequence of the assumption of recombination from secondaries in the original form of this hypothesis.*

Quite generally we might say that if we stick to the hypothesis that that there is only one system with a given quant-number, the modified form of the hypothesis of recombination from secondaries will give, for the α-line of each series, the same frequency as it did in its original form, because there is no electronic system between the secondary circle and the broken ring.

If we would assume two M-rings we can, just as in the case of recombination between primaries, explain M_α by means of combinations such as $q_3=8$, $q_4=7$.

The second (β) *Lines of each Series.*

§ 12. In the case of the second lines (β) the modified hypothesis of recombination from secondaries will give frequencies which are not very different from those we deduced on the assumption of recombination between primary systems: but they are essentially different from the frequencies we might deduce for these lines from the hypothesis of recombination from secondaries in its original form (Debye's hypothesis).

K_β.

In the case of K_β we put in equation (25)

$$i=1, \quad k=3, \quad n_1=1, \quad n_2=2, \quad n_3=3, \quad q_1=3, \quad q_2=7.$$

In fact the quantities B and C are already given in Table XII. corresponding to $q_3=1$, and we get the following equation for the frequency:

$$\frac{\nu}{R}=\frac{8}{9}N^2-3\cdot962\,N+8\cdot7+\epsilon.$$

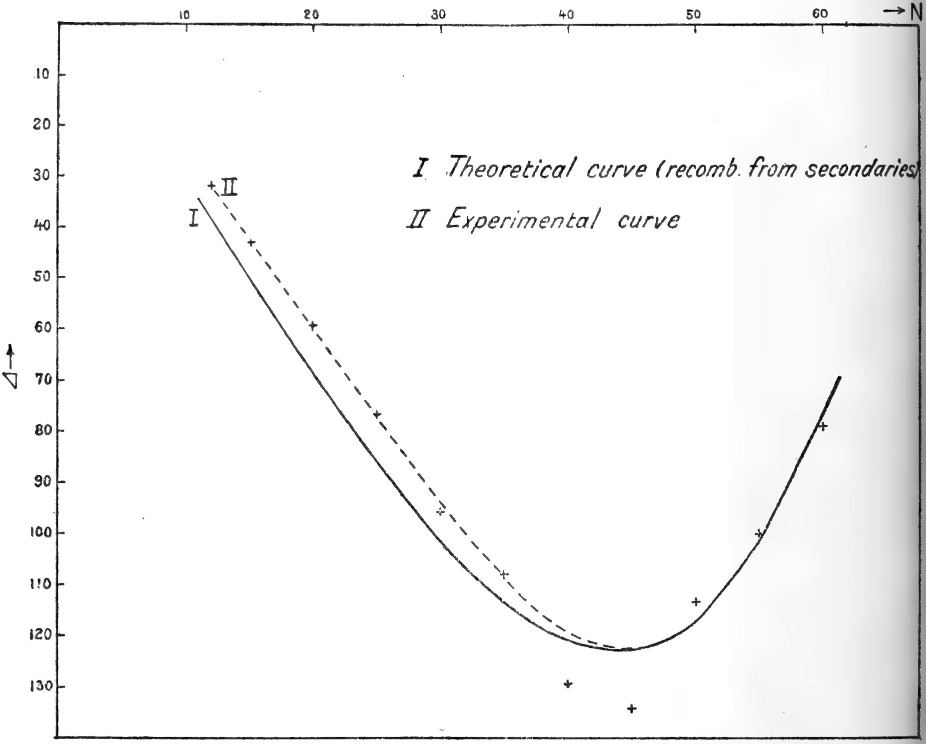
In Table XVI. are given the observed and calculated values of $\left(\frac{\nu}{R}\right)$ and Δ for some atomic numbers.

Table XVI.

N.	$(\nu/R)_{obs.}$	$(\nu/R)_{cal.}$	D.	P.	$\Delta_{obs.}$	cal.
11	...	72·7	...	...	...	− 34·8
12	96·2	89·4	− 6·8	−7·7	− 31·8	− 38·6
15	156·9	149·7	− 7·3	−4·9	− 43·1	− 50·3
20	295·8	286·6	− 9·2	−3·2	− 59·8	− 69·0
25	479·1	469·2	− 9·9	−2·1	− 76·5	− 86·4
30	704·2	698·4	− 5·8	−0·83	− 95·8	−101·6
35	980·9	975·3	− 5·6	−0·57	−108·0	−113·6
40	1293·0	1301·1	+ 8·1	+0·62	−129·3	−121·1
45	1666·0	1677·1	+11·1	+0·66	−134·0	−122·9
50	2109·0	2105·0	− 4·0	−0·19	−113·2	−117·2
55	2589·0	2588·0	− 1·0	−0·04	−100·0	−101·0
60	3121·0	3124·0	+ 3·0	+0·10	− 79·0	− 76·2

It appears from the table that the agreement is remarkably good, and in fact the best hitherto obtained. Fig. 4 represents the curves $\Delta = f(N)$ corresponding to calculated and observed values. A comparison with fig. 2 shows that the theoretical curve of fig. 4 gives a better agreement than any of those in fig. 2.

Fig. 4.



L_β.

In the case of the second line (L_β) of the L-radiation we are on less certain ground.

First of all there may be some doubt as to which of the β-lines of Siegbahn we might take as the one which is produced by recombination from $\tau = 4$ to the circular L-ring. As before, we take the β_1 line of Siegbahn to be produced by recombination from $\tau = 3$ to an elliptic state of the L-ring. But still we shall have the choice between β_2 and β_3 (Sommerfeld, $\gamma\ \phi$). As before, we shall consider β_2 as the second L-line and indicate it by L_{β_0}.

Further, there may also be some uncertainty with regard to the constitution of the M-system. In our calculations we suppose one M-ring with 12 electrons.

The equation of frequency is obtained from the general equation (5) by putting:

$$i=2, \quad k=4,$$
$$n_2=2, \quad n_3=3, \quad n_4=\tau=4,$$
$$q_2=7, \quad q_3=12, \quad q_4=1,$$

and we get:

$$\frac{\nu}{\mathrm{R}}=\frac{3}{16}\mathrm{N}^2-4{\cdot}128\,\mathrm{N}+25{\cdot}35+\epsilon.$$

$\frac{\nu}{\mathrm{R}}$ has been calculated for various atomic numbers, and the results are given in Table XVII.

TABLE XVII.

N.	$(\nu/\mathrm{R})_{\mathrm{cal.}}$	$(\nu/\mathrm{R})_{\mathrm{obs.}}$	P.
50	289·8	287·3	+0·87
60	457·8	448·0	2·2
70	665·0	644·0	3·3
80	914·1	874·5	4·5
90	1205·0	1143·0	5·4

The increase of P with increasing N indicates that perhaps β_3 and not β_2 is to be considered as the normal β-line of the L-series.

The considerations in § 10 with regard to the formation of doublets $(\alpha_0\,\alpha_1)$ $(\beta_0\,\beta_1)$ also apply to the present hypothesis.

Recombination from Secondaries and Kossel's Relations.

§ 13. On the assumption of recombination between primaries Kossel's relations were identically fulfilled. This is, however, not the case when we assume recombination from secondaries. From the general equation of frequency (25) we easily deduce the formula:

$$d=\left(\frac{\nu}{\mathrm{R}}\right)_i^j+\left(\frac{\nu}{\mathrm{R}}\right)_j^k-\left(\frac{\nu}{\mathrm{R}}\right)_i^k$$
$$=\mathrm{V}(n_j, p_j-1, q_j)-\mathrm{V}(n_j, p_j, q_j-1)-\mathrm{V}(n_j, p_j-1, 1)\,. \quad (26\,a)$$

or (approximately)

$$d=1/n_j^2[q_j(\mathrm{N}-p_j+1-\mathrm{S}_{q_j})^2$$
$$-(q_j-1)(\mathrm{N}-p_j-\mathrm{S}_{q_j-1})^2-(\mathrm{N}-p_j+1)^2].$$

Putting:

$$\begin{aligned} p_j + S_{q_j} - 1 &= a, \\ p_j + S_{q_j - 1} &= b, \\ p_j - 1 &= c. \end{aligned}$$

$$d = [N(c + (q_j - 1)b - q_j a) + q_j a^2 - (q_j - 1)b^2 - c^2]\, 1/n_j^2 . \quad (26\,b)$$

In the case of K_β, K_α, and L_α we get:

$$d = \left(\frac{\nu}{R}\right)_{K_\alpha} + \left(\frac{\nu}{R}\right)_{L_\alpha} - \left(\frac{\nu}{R}\right)_{K_\beta} = 0{\cdot}413\,N - 3{\cdot}53. \quad (26\,c)$$

This formula shows that the quantity d is not constant but increases with atomic number. The formula is not exact, because the variation of mass is disregarded.

Having previously calculated the frequencies for the three lines, we can from these values find d. Now, on the other hand, the frequencies derived from observations do not fulfil Kossel's relation identically. They show, in fact, a deviation which is too great to be due to errors of experiment.

In the Table XVIII. are given calculated and observed values of the deviation d for the lines K_β, K_α, and L_α.

Table XVIII.

N.	$d_{\text{obs.}}$	$d_{\text{cal.}}$
30	+ 5·5	+11·0
40	+12·8	+14·8
50	+23·6	+22·5
60	+44·6	+37·9

It appears from the table that the calculated and observed values of d, both as regards sign and absolute magnitude, agree in a very striking manner. The observed values of d increase with atomic number in the way required by theory.

This fact, that our theory is able to explain the deviation from Kossel's relation, is a matter of the greatest importance and weighs heavily in favour of the assumption of recombination from secondaries in its modified form.

A third Modification of the Hypothesis of Recombination from Secondaries.

§ 14. In our previous treatment of recombination from secondaries we assumed the electron before starting recombination to be moving in a stability circuit of its own with quant-number τ. We further assumed the secondary circuit to be inside the atomic rings with quant-number $n_k = \tau$.

There is, however, the possibility to consider, that the recombining electron is taken up in the atomic ring k, which has the quant-number $n_k = \tau$. If q_k is the number of electrons in the normal ring $q_k + 1$ would be the number just before recombination.

This hypothesis may also be considered as a higher union of the hypothesis of recombination between primaries and that of recombination from secondaries; because we may say that *recombination takes place from a deformed primary circle.*

The analogy between this hypothesis and the two previously treated will be apparent from the fact that the frequency formula is simply deduced from equation (5) by replacing q_k with $q_k + 1$.

From the fact that the change of number of electrons in the ring of departure has a very small influence on the frequency, we conclude that also the present hypothesis leads to the same number of electrons in the various rings, and that it will give a very good numerical agreement with observations.

In the case of K_α the agreement will be even better than that obtained on the assumption of recombination between primaries (Table IV.), at any rate for small atomic numbers. And for higher atomic numbers it will give a better agreement than the formula of Debye based on the hypothesis of recombination from secondaries.

Just as in the case of recombination between primaries, considerations with regard to K_β will show that we cannot assume one M-system with 12 or 13 electrons. Two M-systems, one with 8 another with 7, however, may still be possible.

With regard to Kossel's relation it is not identically, but merely approximately, fulfilled—in fact, the quantity d is given by the expression :

$$\left.\begin{aligned} d &= \left(\frac{\nu}{\mathrm{R}}\right)_i^j + \left(\frac{\nu}{\mathrm{R}}\right)_j^k - \left(\frac{\nu}{\mathrm{R}}\right)_i^k \\ &= \mathrm{V}(n_j, p_j, q_j) - \mathrm{V}(n_j, p_j, q_j - 1) \\ &\quad - \mathrm{V}(n_j, p_j - 1, q_j + 1) - \mathrm{V}(n_j, p_j - 1, q_j). \end{aligned}\right\} \quad (27)$$

To get approximate values of d we may put $\mathrm{V} = \mathrm{V}_0$; then in the case of K_β, K_α, L_α we get :

$$d = 0{\cdot}066\,\mathrm{N} + 0{\cdot}53.$$

Although the deviation d has the right sign and the right type of variation with N, its absolute magnitude is by far too small to account for the observed deviations from Kossel's relation.

The Frequency of the Absorption Edges.

§ 15. We assume that the necessary condition for absorption to take place is that the energy quantum $h\nu$ of the incident ray is equal to or greater than the energy which is required to remove an electron from its place in the atom and bring it to an infinite distance with a velocity equal to zero. Let this energy be ΔE, then :

$$h\nu \geqq \Delta E,$$

or if ν_A is the frequency of the absorption edge :

$$h\nu_A = \Delta E.$$

Assuming that during the removal the angular momentum of the remaining electrons is kept unaltered, we found that the expulsion of an electron from a ring-system would be accompanied with a change of energy of all electronic systems outside the broken one.

We mentioned the possibility that this change of energy escaped in the form of radiation ; but it is perhaps more likely that it is spent on the escaping electron.

This latter assumption would mean that each electron left behind in the rings at any moment during the time of escape keeps its angular momentum unaltered, and this tendency to keep a constant momentum is equivalent to a force which is exerted on the escaping electron, and which on an average is directed away from the nucleus.

If, however, no energy is wasted by radiation during the escape of the electron, *the energy necessary to remove the electron to infinity must be equal to the energy which must radiate out when the electron recombines from infinity to its original position.*

Let us suppose the recombination to take place in one step, or, that only one energy quantum of frequency ν_∞ is emitted ; then :

$$h\nu_\infty = \Delta E = h\nu_A \quad \text{or}$$

$$\nu_A = \nu_\infty.$$

Consequently ν_∞ should be equal to the frequency of the absorption edge.

At an infinite distance the quant-number τ is infinitely great, and $V(\tau p q)$ vanishes. Putting in equation (25) $\tau=\infty$, we get:

$$\left.\begin{aligned}\frac{\nu_A}{R}=\frac{\nu_\infty}{R}&=V(n_i, p_i, q_i)-V(n_i, p_i, q_i-1)\\&\quad+\sum_{l=i+1}^{l=m}\left[V(n_l, p_l, q_l)-V(n_l, p_l-1, q_l)\right],\end{aligned}\right\} \quad (28)$$

where m is the total number of electronic systems of the atom considered.

In order to calculate ν_A we must know the constitution of all the electronic systems surrounding the nucleus, but we only know with certainty the K- and L-system.

The best way of testing the correctness of the equation (28) would be to calculate ν_A for the K-absorption edge and for very low atomic numbers. In that case we know those systems which contribute most to the frequency ν_A, and differences with regard to the outer systems will not have any great effect on the frequency.

The lowest atomic numbers for which I have found determinations of the absorption edges are $N=26$ (Fe) and 28 (Ni).

Wagner * gives the following values:—

Fe............	$\lambda_A=1{\cdot}759 \ 10^8$ cm.
Ni............	$\lambda_A=1{\cdot}502$,, ,,

Now we assume a constitution of the electronic system similar to the one indicated in fig. 2 of my previous paper, but with a change of the quant-number of the third and fourth rings.

The following constitution is adopted:—

Ring	1.	2.	3.	4.
n	1	2	3	4
q (Fe)	3	7	8	8
q (Ni)	3	7	8	10
p	0	3	10	18

These values of (n, p, q) give us all we want for the determination of ν_A from (28).

Values of $\frac{\nu_A}{R}$ calculated from (28) and corrected for variation of mass with velocity, as also the values calculated from observations, are given in Table XIX.

---

* E. Wagner, *Phys. Zeitschr.* xviii. p. 436 (1917).

TABLE XIX.

N.	$\frac{\nu_A}{R}$ cal.	$\frac{\nu_A}{R}$ obs.	D.	P.
Fe 26	513·4	518·0	4·6	0·88
Ni 28	605·1	606·7	1·6	0·26

We see from the table that the agreement between observed and calculated values is remarkably good, the difference between observed and calculated values being less than 1 per cent.

I think this fact strongly supports the correctness of our formula and the assumption on which our theory of X-ray spectra is founded.

I also think that the absorption edges may give us some valuable information with regard to the constitution of the surface ring and those rings nearest to it.

Discussion of the Results.

§ 16. In the preceding part of this paper we have discussed the various possibilities with regard to the process of recombination. The various possibilities treated will be seen from the following scheme:—

I. Recombination between primaries:
  a. Conservation of momentum.
  b. ,, energy.
II. Recombination from a disturbed primary:
  a. Conservation of momentum.
  b. ,, energy.
III. Recombination from secondaries:
  a. Conservation of momentum;
  b. ,, energy.
  c. Debye's hypothesis.

The assumption I. *b* does not lead to any satisfactory result. Although we were able to get a fairly good formula for the K line, the assumption does not seem to give room for further extensions to the other X-ray series. Thus we cannot explain L_α by means of the electronic rings which were given through the calculation of K_α. For the same reason assumption II. *b* must be abandoned. In the case of recombination from secondaries the difference between III. *a*, III. *b*, and III. *c* would not come in so long as we only treat the principal (α) line of each series. But the difference is very marked indeed when we are considering the (β) lines and we find that the hypothesis III. *a* gives by far the better agreement for these lines.

Thus we come to the following conclusion :—

Whether *we assume recombination from a normal or deformed primary system or recombination from a secondary one, we have always to assume that the angular momentum of the electrons left behind in the atom remains unchanged.* And, further, we have to assume that *the changes of energy of the systems situated between the broken ring and the ring of departure enters into the energy quantum of radiation which is emitted as the result of the recombination.*

When we assume conservation of momentum the three assumptions I., II., III. will all give a very close agreement between calculated and observed values, and all three assumptions lead to the same values for the number of electrons in the various ring-systems. This is due to the fact that the energy changes which accompany the recombination are very little affected by a change of the number of electrons in the system of departure.

If we were merely regarding the numerical agreement for each line separately, it would be very difficult indeed to decide in favour of any of the three possibilities (I., II., III.) *a*. All of them explain the principal (α) lines as well as the secondary (β) lines almost equally well when each line is seen separately ; but still we found that the assumptions I. and II. meet with considerable difficulties.

First of all we found that the assumptions I. *a* and II. *a* in the case of K_β led to a characteristic form of the curve $\nu = f(N)$ which was not indicated by the experimental values. At any rate the assumptions I. *a* and II. *a* are not consistent with the assumption of only one M-ring. The assumption III *a*., however, gave the right curvature.

The strongest argument in favour of the assumption III *a*. we got from considerations with regard to the deviation from Kossel's relation. The assumption of recombination from secondaries gave just the right values of the deviation

$$d = \left(\frac{\nu}{R}\right)_{K_\alpha} + \left(\frac{\nu}{R}\right)_{L_\alpha} - \left(\frac{\nu}{R}\right)_{K_\beta}$$

both as regards sign and absolute magnitude. On the other hand, the assumption I. *a* gave $d \equiv 0$, and II. *a* led to values of d which were only a small fraction of those given by experiments.

Now we must remember that our theory—so far it has been carried—does not give an exact agreement between observed and calculated values. Thus our calculations will require some small corrections, probably due to the fact that our expression for the energy is not quite correct. But if these corrections are due to errors which enter into the energy, the correction terms of the frequency cannot essentially alter the values which the various hypotheses give for the deviation from Kossel's relation. Thus the assumption I.*a* makes *d* identically equal to zero independent of the special form of the energy function. And the assumption II., which is merely a slight modification of I., will make *d* approximately equal to zero.

Thus, so long as we build on the scheme here proposed for the production of X-rays, *I can see no escape from assumption III.a, that recombination takes place from secondary circles (systems) and that always the angular momentum of the electrons left behind in the atom is kept constant during expulsion and recombination.*

If the change of energy of the electronic systems outside the broken ring which accompanies the ionization process does not escape in the form of radiation, but is utilized for the motion of the escaping electron, we found the frequency of the absorption edge by putting $\tau=\infty$ in the frequency formula corresponding to III.*a*. And it was found that the agreement between the calculated and observed values was very good indeed.

The form here given to the hypothesis of recombination from secondaries will not only explain the principal (α) lines, but will also give a very close agreement for the second (β) lines of the K- and L-series. If we further assume that the L-ring has one circular and one elliptic state we may say that the following lines have been explained: K_α, K_β, the doublets of the L-series (α, β), (γ, δ) (Sommerfeld's denotation), and, finally, M_α.

An explanation of the *l*-series, such as the one given in Paper I. by means of two L-rings, is not consistent with the hypothesis of recombination from secondaries in its modified form; but we shall have to find some other explanation of this series of lines.

The line K_{β_2} of Siegbahn is no doubt produced by recombination from a secondary for which $\tau=4$ and ought to be denoted by K_γ. Also in the case of the L- and M-radiation some of the lines here left out are merely to be considered as members of the series corresponding to higher values of τ.

The modified form of the hypothesis of recombination from secondaries leads to the following constitution of the electronic systems nearest to the nucleus:—

1 K-system with quant-number 1 consisting of 3 electrons.
1 „ „ „ „ „ 2 „ „ 7 „
1 M- „ „ „ „ 3 „ „ 12 „

Our theory leaves no doubt with regard to the number of electrons of the K- and L-systems; but as regards the constitution of the M-system we are on less certain ground, and we might quite as well suppose two M-systems with quant-number 3.

The uncertainty in the case of the M-system is partly due to the limited experimental material and partly due to the fact that the determination of the number of electrons of a ring becomes less definite as we pass outwards from the centre, and, finally, matters may be more complicated and the identification of lines more difficult as we pass outwards.

General Remarks regarding Atomic Constitution.

§ 17. The constitution here given for the K-, L-, and M-ring will involve some change with regard to the general scheme given in my previous paper (fig. 2) with regard to the constitution of the elements.

The first two rings will be the same as before, and the surface systems for elements with atomic numbers lower than 10 (Ne) should be those given in Paper I.

The difference comes in at the third ring, which in Paper I. was given a quant-number 2, while our present results have given to it a quant-number 3. If, however, we assume two M-systems, we may still maintain the same *number of electrons* (8) of the third ring, and it is even possible that the fourth ring may have 10 electrons just as indicated in Paper I.

If, on the other hand, we assume only one M-ring, the third ring should be composed of 12 electrons, and provided this value is the right one, we should have to give up the principle that a ring once formed inside a surface ring should be maintained with the same number of electrons when we pass to higher atomic numbers; for on account of the periodic properties of the element we cannot assume that a third ring of 12 electrons has been formed as a surface ring for, if so, the element Ti $(N=22)$ should possess a surface ring of 12 electrons and Va $(N=23)$ a surface ring of one electron, which cannot be assumed. Although our present conception leads to some changes with regard to the constitution of the *internal rings*, still the periodic variations shown by the properties of the elements indicate that the number of electrons of the surface systems are those given in Paper I.

In my previous paper I found that the electric conductive capacity ought to vary in about the same way as the quantity:

$$1/\sigma=\left(\frac{n}{q-s_q}\right)^2, \quad . \quad . \quad . \quad . \quad . \quad . \quad (29)$$

where n is the quant-number and q the number of electrons of the surface ring.

Now we found that in order to get the observed typical variation of the conductive capacity with atomic number (Benedick's curve) we had to assume the *quant-number n to be constant for all elements belonging to the same period*, or *for all elements falling between two successive* inert gases.

On this assumption we got the right type of variation of the conductive capacity for both the short periods (see Paper I. fig. 4). But also for the two long periods (Ar–Kr) and Kr–Xe) we get the right type of variation, when we assume n constant inside each interval. Let us, *e. g.*, consider the period between Ar and Kr. At first we get a sudden fall of $1/\sigma$ with increasing q as we pass from K to Ni. Then a new ring is formed which should produce a sudden increase of $1/\sigma$ for the element Cu. In fact we find a sudden increase of the conductive capacity for Cu followed by a fall which is continued till the next period sets in with Kr.

There is, however, an apparent disagreement with regard to the magnitude of the conductive capacity.

The formula (29) would give both for K and Cu the same value of $1/\sigma$, while the observed conductive capacity is very much smaller for Cu. I think, however, that the smaller value in the case of Cu is just what we may expect from theory. The expression for $1/\sigma$ given in equation (29) is deduced on the assumption that the effect of the internal electrons is the same as if they were placed at the centres.

Now the radius of the surface ring is determined by the equation (Paper I., equation 25):

$$a=a_{\mathrm{H}}\frac{n^2}{q-\mathrm{S}_q}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (30)$$

where a_{H} is the radius of the surface ring of hydrogen. When we pass an inert gas—Ar, say—we shall have an increase of a from the two causes: increase of n and diminution of q. When, however, we pass from Ni to Cu we have a relatively smaller increase of a because now the quant-number is unaltered. Now this comparatively much smaller jump with regard to the radius which is found for Cu may account for its smaller conductive capacity as compared with that of K.

For chemically related elements q is the same, and the electric conductive capacity as mentioned in Paper I. should vary in a similar way as n^2.

Let us, *e.g.*, consider the elements for which $q=1$, and which follow immediately after an inert gas. These elements are Li, Na, K, Rb, and Cs. The curve of Benedicks (Paper I., fig. 3) shows a gradual increase of the conductive capacity indicating an increase of the quant-number. From the X-ray analysis we know that when we come to elements which give the L-radiation there is a second ring with quant-number 2, which has 7 electrons.

This result might indicate that the surface ring of the elements of the first short period had the quant-number 2, but we must also be aware of the possibility that some change of the quant-number of the second ring may have taken place, or we cannot be sure that also the elements before Krypton have a second ring of 7 electrons with quant-number 2. The very large increase of electric conductive capacity from Li to Na would be best accounted for by assuming that the surface electron of Li had quant-number 1. In that case the conductive capacity of the elements considered should vary as the numbers: 1, 4, 9, 16, 25.

These numbers give, at any rate, variations of the right order of magnitude. In fact, the conductive capacity of Cs is about exactly 25 times as large as that of Li.

But we ought not to lay too much stress on this coincidence, for the conductivity may be influenced by many other causes.

Putting the quant-number for the surface electron of Li equal to 2, the conductive capacities should vary as the numbers 4, 9, 16, 25, 36. It should be remembered that the constitution given to Fe and Ni, in deducing the values of ν_A for the K-absorption edge, was in accordance with the latter assumption.

A special interest is attached to the elements Cu, Ag, Au, which have a large conductive capacity and only one electron in the surface ring. We notice a very marked increase from Cu to Ag corresponding to an increase of quant-number; but for Au there is again a drop of conductive capacity, which might indicate a drop of quant-number. It seems as if the formation of the rare earth elements is accompanied with some kind of condensation process.

The atomic property which first of all suggests itself for examination is the atomic volume, because we might expect it to vary in a similar way as the radius of the surface ring in such a way that a greater radius would give a greater atomic volume.

Now the radius of the surface ring is at any rate approximately determined by the equation (30), and by means of the values previously given to q and n of the surface ring we can easily calculate the radius a and compare its variation with the atomic volume curve.

Such a comparison would immediately show some striking similarities.

For chemically related elements (q constant) we find, as a rule, an increase of atomic volume with increase of N in accordance with the variation of a. This increase is in particular very marked for the elements Li, Na, K, Rb, Cs, and Mg, Ca, Sr, Ba.

If, on the other hand, we consider the elements belonging to the same period we find a marked difference between the variation of the atomic volume and that of the radius.

Let us, *e. g.*, consider the short period from Ne to Ar. These elements should have the same quant-number of the surface ring; but the number of electrons should increase from 1 (Na) to 7 (Cl), and the radius should decrease. Also the atomic volume curve shows at first a rapid decrease; but it reaches a minimum at the element Al, after which it increases quite rapidly to meet the most conspicuous maximum at K. Taking one of the long periods we find a similar variation of the atomic volume and, curiously enough, we find no secondary maximum for Cu and Ag as was to be expected.

When we consider the result of this comparison on the basis of our scheme with regard to the constitution of the surface systems we arrive at the following conclusions:—

The atomic volume is no doubt greatly influenced by the magnitude of the radius of the surface ring; but besides it is greatly influenced from other causes, which in certain cases may even more than counterbalance the effect due to change of radius.

The particular manner in which the atoms are grouped together in the condensed aggregate forms is likely to play an important part in this respect.

Also the variation of the atomic volume is in favour of of our two fundamental assumptions with regard to the constitution of the surface systems:—

1. All elements belonging to the same period have the same quant-number.
2. The quant-number increases by one unit when we pass from one period to the next one. Also, when we consider the atomic volumes, the best agreement is got when the quant-number of the first period is put equal to 1.

The constitution of the surface systems previously assumed is represented in Table XX.

TABLE XX.

Quant-number of surface ring.			Number of electrons in surface ring.									
Either	or		1.	2.	3.	4.	5.	6.	7.	8.	9.	10.
1	2	....	Li	Ba	B	C	N	O	Fl	Ne		
2	3	....	Na	Mg	Al	Si	P	S	Cl	Ar		
3	4	....	K	Ca	Sc	Ti	Va	Cr	Mn	Fe	Co	Ni
		....	Cu	Zn	Ga	Ge	As	Se	Br	Kr		
4	5	....	Rb	Sr	Y	Zr	Nb	Mo	—	Ru	Rh	Pd
		....	Ag	Cd	In	Sn	Sb	Te	I	Xe		
5	6	....	Cs	Ba	La	Ce (rare earths)	Ta	W	—	Os	Ir	Pt
		....	Au	Hg	Tl	Pb	Bi	Po	—	Em		
?	?	....	—	Ra	—	Th	—	U				

This scheme accounts for a number of properties of the elements; but there are a few instances of disagreement which are very remarkable. Thus from the point of view of the atomic constitution here proposed the element Mn ought to show similar properties to the halogens, and, further, there is nothing in the present scheme which can explain the singular position of the inert gases.

I think the reason for this incompleteness of the scheme is that the whole mechanism of atomic constitution is not given only by the number of electrons of the rings and their quant-numbers.

The electrons as part of the atomic systems are not to be regarded as independent centres of force only acting on each other with ordinary repulsing forces; but they are attached to each other in some other way.

Thus the electrons of the same ring are forced to take up quite definite orbits, either circular or elliptic, and we may also imagine a mutual connexion between the motions of the various ring systems. Now it is quite possible that these mutual relations may modify the properties of the atoms both as regards spectra, chemical, and physical properties.

Further, the atomic properties may greatly depend on the constitution of the ring next to the surface system.

Thus the scheme here given for the surface systems may be essentially right so far as it goes; but it does not give the whole mechanism of atomic constitution.

Physical Institute, Christiania.
September 15, 1918.

XXIV. *Note on the Motion of a simple Pendulum after the String has become slack.* By W. B. Morton, *M.A.*, *Queen's University, Belfast**.

THE writers of text-books appear to lose interest in the motion of the bob of a pendulum once the string has become slack. It may be worth while to point out, for the benefit of teachers of dynamics, some features of interest presented by the subsequent motion, which lend themselves to graphical treatment. It is very unlikely that these are new; they have probably been hit upon by others engaged in the work of manufacturing examination questions, but I have not found them referred to anywhere.

When the tension of the thread vanishes the "level of no velocity" or, to use the older expression, the level of

---

* Communicated by the Author.

"potential ascent" lies above the point on the circle at a height which is half the vertical height of this point above the centre. The particle then describes a free parabola having its directrix at this same level until the string becomes taut at the intersection of parabola and circle. This takes place at a point whose angular distance from the highest point of the circle is three times the angular distance, measured in the opposite direction, of the point at which the thread became slack. It is easy to prove this analytically, but it is most quickly obtained as a consequence of the well-known geometrical theorem that the chords which join, in pairs, the four intersections of a circle and a conic are equally inclined to an axis of the conic. In the present case the parabola and the circle have the same curvature at their point of contact, and so three of the intersections coincide at this point. The common tangent is one of the two chords, and the other is the line joining the point of slackening to the point of tightening of the thread. These are equally inclined to the vertical, and the given result follows immediately.

If the thread be supposed inextensible the component velocity in its direction is destroyed by the jerk, and the bob resumes its movement in the circle with the tangential component of its previous velocity.

If α is the angular distance from the highest point at which the thread slackens, the tangential velocity when it tightens again comes out to be

$$\sqrt{ga\cos\alpha}\,(8\cos^4\alpha-12\cos^2\alpha+3),$$

where a is the length of the thread. This vanishes when $\cos^2\alpha=\frac{1}{4}(3-\sqrt{3})$ or $\alpha=55^\circ\ 44'$. Accordingly if the thread slackens at this point the bob will be brought instantaneously to rest when the thread makes with the downward vertical the angle $\pi-3\alpha=12^\circ\ 48'$, and will then oscillate with this amplitude.

In general, the level of no velocity after the jerk is at a height above the centre of the circle given by

$$\tfrac{1}{2}a\cos\alpha\,(3-64\sin^6\alpha\cos^2\alpha).$$

This is shown plotted against α in the middle curve on the diagram. The values of α run backward, so that the origin corresponds to the end of the horizontal diameter of the circle. It is convenient to start from this position in discussing the motion.

The upper curve gives the level of no velocity before the jerk, the lowest curve the level at which the thread tightens. These are the sine-curves $y=\frac{3}{2}a\cos\alpha$ and $y=a\cos 3\alpha$ respectively. The vertical distance between the top curve and the middle one represents the energy destroyed by the jerk. The distance of the middle curve above the bottom of the diagram is the total energy after the jerk, and it is divided by the lowest curve into two parts: the upper part, between the two curves, is the kinetic energy, and the lower part the potential energy measured from the lowest position of the bob. It will be seen that the curves touch for $\alpha=55° 44'$, *i. e.* the bob is then stopped dead as already explained.

By use of these graphs it is easy to follow the complete course of the motion of the bob when the thread slackens for the first time at a given position on the circle. If the level of no velocity after the jerk is below the centre the pendulum will oscillate below this level. If it is above the centre the thread will slacken again. To find where this takes place run a horizontal line from the point on the middle curve to meet the upper curve. Underneath the point of intersection is found the value of α at which the thread becomes slack for the second time.

When this process is repeated it is found that the ultimate motion alternates between two types:—

(1) Asymptotic approach to oscillation between the ends of the horizontal diameter. This will occur if the thread slackens anywhere on an arc of 13° 39′ above an end of this diameter (from $\alpha=90°$ to $\alpha=76° 21'$). The curve giving the level of zero velocity after the tightening of the thread is then above the axis, and if we apply the construction to find successive points of slackening we are led by a series of zig-zags into the corner, at $\alpha=90°$, between the upper and middle curves. The same terminal state of motion is reached if the 2nd, 3rd, 4th . . . . slackening falls within this region, which will be the case if the first slackening falls in the regions shown on the diagram by a thickening of the axis of α. The boundaries of these segments are found by construction as indicated by the lines drawn vertically and horizontally, beginning with the two points $\alpha=76° 21'$ and 41° 21′, where the middle curve crosses the axis. They can be calculated by solving by trial the series of equations

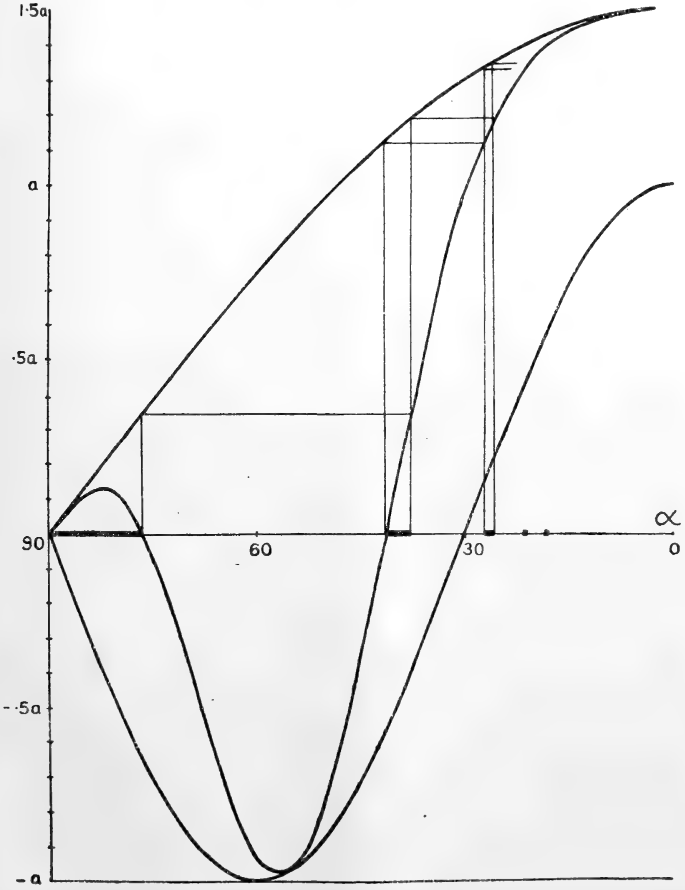
$$f(\alpha_1)=0,\ f(\alpha_2)=\tfrac{3}{2}\alpha_1,\ f(\alpha_3)=\tfrac{3}{2}\alpha_2,\ \text{and so on,}$$

where
$$f(\alpha)=\tfrac{1}{2}\cos\alpha(3-64\sin^6\alpha\cos^2\alpha).$$

It is thus found that the motion runs into the asymptotic type when the thread becomes slack between the values of α

90° and 76° 21′,
41° 21′ and 37° 45′,
27° 15′ and 25° 56′,
21° 29′ and 20° 50′,
18° 26′ and 18° 3′,
&c., &c.

These regions become very narrow compared with the spaces separating them and crowd indefinitely close together as $\alpha=0$ is approached.



(2) From other points on the circle the motion reaches, after a *finite* number of slackenings and tightenings, the

state of oscillation through a limited range. This corresponds to the region between 76° 21′ and 41° 21′, where the curve is below the axis and to the other regions from which this part of the axis is reached after one or more repetitions.

It will be seen that the highest point of the circle $\alpha=0$ resembles an "essential singularity" of a mathematical function. Within any distance from it, however small, the type of ultimate motion alternates an infinite number of times.

XXV. *Experiments on the High-Tension Magneto.*—I.

By NORMAN CAMPBELL, *Sc.D.*[*]

NOTE.—The work described in this paper was carried out at the National Physical Laboratory under the direction of the Advisory Committee for Aeronautics. The results have been communicated in a series of confidential reports to the Internal Combustion Sub-Committee of that Committee, who have now given their consent to the publication of any portions which appear of pure scientific interest.

I desire to express my obligations to Sir Richard Glazebrook, Director of the National Physical Laboratory, and more especially to Mr. C. C. Paterson, in whose department the experiments were performed, for invaluable advice and encouragement at all stages of the research. Prof. Taylor Jones was also good enough to discuss very fully the questions which arose in connexion with his theory on which the work is based.

Object of the Experiments.

1. IN the Philosophical Magazine, xxxvi. p. 145 (Aug. 1918), Prof. E. Taylor Jones has given a theory of the magneto; this theory is essentially the same as that which he had given previously for the induction-coil, and had shown to predict results in close accordance with observation[†]. It is not, however, immediately certain that so close an agreement between theory and experiment would be found with the magneto. The theory neglects altogether effects of hysteresis and eddy-currents in the iron core on which the armature is wound, or rather regards these effects as merely adding to the effective resistance of the circuits by which the damping of the oscillations is determined. Since the flux density in the iron is much greater in the magneto

[*] Communicated by the Author.

[†] See Phil. Mag. xxvii. p. 580 (April 1914).

than in the induction-coil, and the lamination of the iron is in general much less complete, it is possible that a source of error which is inappreciable in one instrument might be very serious in the other. Prof. Jones's experiments on the magneto are hardly sufficient to indicate how far the theory may be employed in the design of the magneto; for the methods of determining the periods of the oscillations set up in the circuits which he had developed in his work on the induction-coil were much less well suited to the more rapid and more highly damped oscillations of the magneto. It was thought that experimental methods which had already been developed for other purposes would enable some of the difficulties which Prof. Jones had encountered to be overcome, and accordingly an attempt was made by the use of these methods to investigate more nearly the applicability of the theory to the magneto.

Theory of the Experiments.

2. The general principles of the investigation were the same as those of Prof. Jones's work on the induction-coil, but in order to indicate clearly the bearing of the experiments which were made, it will be convenient to transform slightly some of the equations given by Prof. Jones. Adopting his notation with slight modifications, we shall write:—

L_1, C_1, R_1 the self-inductance, capacity, and resistance of the primary circuit; L_2, C_2, R_2 the corresponding quantities for the secondary circuit.

L_{21}, L_{12} the two mutual inductances, which become equal when the current is uniform throughout the secondary.

k^2 the coupling; $s=(L_2+L_{21}+L_{12}+L_1)/L_2$, $c=(1-k^2)/s$. (Prof. Jones puts $c=1-k^2$).

The values which L_2, k, c, s assume when a condenser of large capacity is inserted in the secondary, so that the current in it becomes uniform, will be denoted by L_2', k', c', s'.

$T_1^2=4\pi L_1C_1$, $T_2^2=4\pi sL_2C_2$, $N_1=1/T_1$, $N_2=1/T_2$;

τ, τ' the periods of the two components of the oscillation, τ being greater than τ'; $n=1/\tau$, $n'=1/\tau'$.

$r=n'/n_1$; $u=T_1^2/T_2^2$.

λ, λ' the damping coefficients of the two components of the oscillation; and $\mu=\lambda\tau$, $\mu'=\lambda'\tau'$, their logarithmic decrements.

Then from Prof. Jones's paper it is easy to show that

$$\tau^2+\tau'^2=T_1^2+T_2^2. \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

$$n^2+n'^2=\frac{1}{c}(N_1^2+N_2^2). \quad . \quad . \quad . \quad . \quad (2)$$

And from (31) of Prof. Jones's paper, so long as $cu(1+u^2)$ is small compared with unity,

$$2\lambda = \left(\frac{R_1}{L_1}T_1^4 + \frac{R_2}{L_2}T_2^4\right)\Big/(T_1^2+T_2^2)^2. \quad . \quad . \quad . \quad . \quad (3)$$

$$2\lambda' = \frac{1}{c}\left(\frac{R_1}{L_1} + \frac{R_2}{L_2}\right). \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (4)$$

It is possible to measure accurately the periods of the components of the oscillation only when these periods are very different and r is not nearly equal to 1. This condition will always be fulfilled so long as c is small; thus so long as c is less than 0·1, r is always greater than 6. Accordingly, in these conditions, we may neglect terms involving $1/r^2$, and write in place of (1) and (2)

$$\tau^2 = T_1^2 + T_2^2 \quad . \; . \; . \; . \quad (1\,.\,1)\,; \quad n'^2 = \frac{1}{c}(N_1^2+N_2^2). \quad (2\,.\,1)$$

We may now determine the constants of the circuits as follows:—

1. To determine L_1 and T_2.

Measure τ^2 for different values of C_1 and plot τ^2 against C_1. A straight line should be obtained, of which the intercept on the axis gives T_2, the inclination L_1.

2. To determine L_2'.

Measure τ^2 for different values of C_2', where C_2' is the capacity added to the secondary circuit. Again a straight line should be obtained of which the inclination gives L_2'. It is assumed that all values of C_2' are so great that the current in the secondary is uniform.

3. To determine c.

Measure τ' when C_1 is very large. Then 2 . 1 reduces to $\tau'^2 = cT_2^2$; and since T_2 is already known, c can be determined.

4. To determine c' and L_{21}.

Measure τ' when C_2 is very large. Then $\tau'^2 = c'T_1^2$. C_1 is always known and L_1 has been determined; hence T_1 is known and c' can be deduced. Since c' is a function of L_1, L_{21}, and L_2' only, we can now deduce L_{21}.

To determine the damping coefficients.

u or μ' can be determined directly by measuring the amplitudes of successive oscillations. If τ or τ' is determined at the same time, λ or λ' is known immediately.

Experimental Methods.

3. For the determination of the periods of the oscillations a Helmholtz pendulum was employed. The pendulum broke in succession two electrical contacts, the distance between which could be varied by moving one of them relatively to the other by means of a micrometer-screw. The first contact carried the primary current, the second made connexion between the circuit in which the oscillation was to be measured and an instrument for measuring the potential in that circuit. Consequently, when both contacts had been broken that instrument registered the potential in the circuit at a time after the break of the primary current determined by the distance between the two contacts.

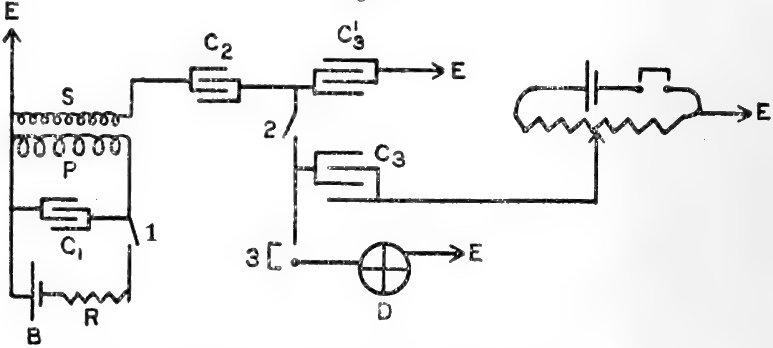
The speed of the pendulum over the distance between the two contacts was usually 250 cm./sec. One division of the scale of the micrometer-screw corresponded to a movement of 0·001 cm. Consequently a time interval of 4×10^{-6} sec. could be measured and reproduced.

The contacts consisted of pivoted steel rods resting against the ends of permanent magnets. The attraction of the magnet ensured that the rod always took up a definite position, and that the electrical resistance of the contact was approximately constant. For the successful working of the method it is necessary that at successive operations of the break of the primary current the potentials developed in the circuits should be the same; this condition will be fulfilled only if there is no sparking at the break. In a subsequent note the circumstances which are necessary and sufficient for the obtaining of a sparkless break will be described; for the present it need only be said that so long as the current broken is below a certain limit (determined by the nature of the contact and of the circuits), it is quite easy to obtain a sparkless break. In most of the measurements to be described the current broken was 0·5 amp. Preliminary observations had shown that the maximum potential developed in the primary or secondary circuit was very nearly proportional (as it should be according to the theory) to the current broken in the primary. Accordingly it was assumed that the periods of the oscillations did not depend on the current broken, and that a change in that current changed nothing but the amplitude of the oscillations.

4. The instrument for registering the potential attained needs some attention. Since the potential to be registered often amounted to several thousand volts, it was clearly impossible to obtain the desired result by simply breaking

the connexion between the source of potential and a voltmeter reading such potentials directly. For if the break occurred when the potential was rising, the subsequent increase of potential would be sufficient to cause a spark to pass across the gap as the terminals separated. Even if this difficulty could be surmounted, others obviously attending the breaking of a circuit carrying such high voltages would be encountered. Accordingly the device was adopted of reducing the potentials to be measured in a known ratio by means of condensers connected in series. The arrangement is shown in fig. 1, where it is supposed that the potential developed in the secondary circuit is being measured.

Fig. 1.



P is the primary circuit, across the terminals of which is connected the condenser C_1, supplied with current from the battery B controlled by the rheostat R. The primary current is broken by the opening of the contact 1 by the pendulum. (It should be observed that the primary condenser is connected across the primary coil and *not* across the contact; the battery and rheostat do not form part of the oscillating circuit.) The high-potential terminal of the secondary is connected to one plate of the air-condenser C_2 of about 10 mmf. capacity*; the other plate is connected to one side of condenser C_3', the other side of which is earthed, and, through the contact 2, to the parallel condenser C_3, which again is joined through the key 3 to the Dolezalek electrometer D. The other plate of C_3 is connected to earth through the potentiometer arrangement shown, by means of which any desired potential can be established between the second plate of C_3 and earth. The contact 2 is the second contact broken by the pendulum.

---

* 1 mmf. $=10^{-12}$ farad $=0·9$ cm.

If V_2 is the potential difference between the high-potential terminal of the secondary and earth at any moment after the opening of 1, then v, the potential difference across C_3 or C_3' at the same moment is given by

$$C_2(V_2-v)=(C_3+C_3')v. \quad . \quad . \quad . \quad . \quad . \quad (5)$$

If contact 2 is opened at this moment, contact 3 being closed, the electrometer will finally register the potential v. Since it is convenient to use the electrometer as a null instrument, the needle of the electrometer is brought back to zero by applying to the second plate of C_3 a potential difference $v'=-v$ by means of the potentiometer; V_2 is determined by the value of v' necessary to bring the needle back to zero. If the capacity of the electrometer itself and of the various leads is taken into account, the relation between V_2 and v' is rather more complicated than that stated; but it is easy to see that the ratio V_2/v' will be constant, so that it may readily be determined by calibration with a known V_2.

The object of the condenser C_3 should be explained. If it were absent, then after contact 2 is opened, the second plate of C_2 would have a very small capacity to earth, and its potential would vary nearly as rapidly as that of the first plate. A potential of several thousand volts might be established across contact 2 and a spark would pass. C_3', composed of leyden-jars, was so chosen that the potential across 2 could never rise above about 30 volts. C_3, composed chiefly of good mica condensers, was chosen so that v' was of the order of 2 volts, which could be measured with an accuracy of 1 in 500*.

In order to minimize insulation losses when the greatest accuracy was desired, other contacts were worked automatically by the pendulum. The sequence was then as follows. The pendulum first opened a contact disconnecting the electrometer from earth; it then operated contacts 1 and 2; it next closed the battery circuit of the potentiometer, thus applying to C_3 a potential which would very nearly bring the needle back to zero, and lastly broke

---

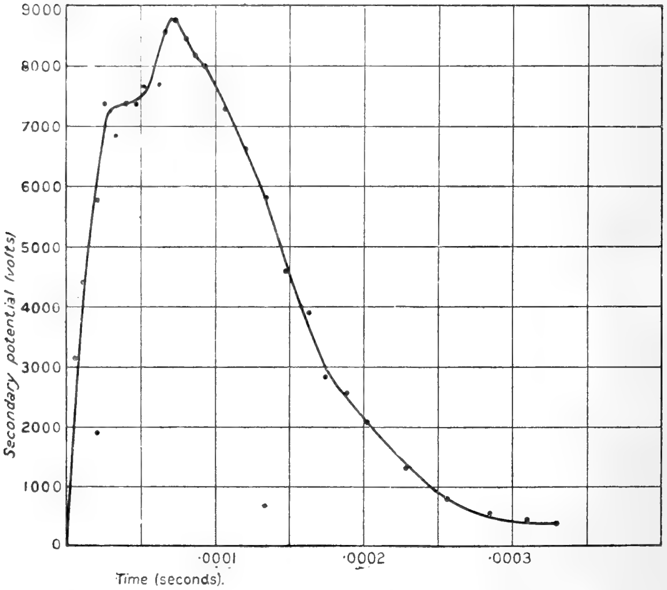
* C_3 included also the leads from the pendulum to the electrometer, which were some yards in length. For these leads rubber cable, such as is used for connecting a magneto to the sparking-plug, was employed, the outside of the cable being coated with an earthed conductor. Such cable has an insulation resistance of more than 10^{15} ohms per yard, and is extremely convenient where very high insulation is required; it may be used for all but the most delicate electrostatic experiments to replace the usual arrangement of bare wires supported on sulphur or sealing-wax supports. It has the drawback of considerable dielectric hysteresis, but it is often possible to arrange matters so that this is of no consequence.

contact 3 insulating the electrometer from the rest of the apparatus. The residual deflexion was then read and applied as a correction to that given by the potentiometer.

It will be observed that this method has the advantage that the potential in the secondary can be observed without adding to it any capacity greater than that of a small air-condenser C_2. The capacity *to earth* actually added was 11·8 mmf., and could have been made even smaller if it had been desired.

In order to indicate the accuracy obtainable with this method of measurement, there is given in fig. 2 the wave-form of the secondary of a magneto, all the experimental points being shown. It will be seen that two of them are clearly in error, but that the rest lie very closely on a smooth curve and enable the relation between the secondary potential and the time to be determined with an accuracy amply sufficient for all the purposes of this investigation. In general, since only the period of the oscillations was required, the whole wave-form was not traced, but only the part in the neighbourhood of the maxima and minima; such observations permit the period and the logarithmic decrement to be deduced.

Fig. 2.



5. The arrangement just described, in which the oscillations are excited by breaking a current in the primary while the secondary potential is measured, is suitable only if the period of the slower component of the oscillations is required. But if, as in the determinations of c and c', the period of the more rapid component is required, the method is not available; for if, as is necessary in those determinations, either C_1 or C_2 is made very large, the amplitude of the rapid component in the secondary potential becomes very small compared with that of the slow component. When C_2 is made large, and c' measured, the difficulty can be overcome by measuring the primary, instead of the secondary, potential; in such circumstances the amplitude of the more rapid component of the primary potential is the greater. Accordingly all that is necessary is to connect the measuring apparatus to the primary circuit instead of to the secondary potential*.

But when C_1 is large the slower component has the greater amplitude in both circuits, if the oscillations are excited by breaking the primary current. If, however, they are excited by starting the primary current, the amplitudes of the two components in the secondary circuit are the same. Accordingly, when c was to be measured, contact 1 was placed as a shunt across the primary terminals, so that the current started when it was broken†. The oscillations now take place with the primary circuit closed, *i. e.* with C_1 infinite. But since it is necessary for a measurement of c that C_1 should be very large, no limitation is thereby placed on the application of the method.

Experimental Results.

5. The first experiments were made on a Thomson-Bennett magneto, Type AD 4C; this machine has a laminated core but unlaminated pole-pieces. The armature was fixed in the position of maximum inductance, and the terminals of the primary circuit disconnected from the condenser and brought outside the machine. The air-condenser used for measuring the secondary potential was always connected to the secondary,

---

* Since the primary potentials are much less than the secondary, it is convenient also to increase the sensitivity. This is easily done by increasing the condenser C_2 by the substitution of a leyden-jar of about 2000 mmf. for the air-condenser. So long as this condenser has a capacity small compared with that of the rest of the circuit, its exact value is, of course, immaterial, and affects only the calibration constant.

† In order that the resistance of the shunt might be small compared with that of the primary, a few ohms extra resistance had to be inserted in the primary circuit.

and its capacity will be regarded throughout as part of the secondary capacity C_2. Any other capacity added to the secondary circuit will be denoted by C_2'.

Fig. 3.

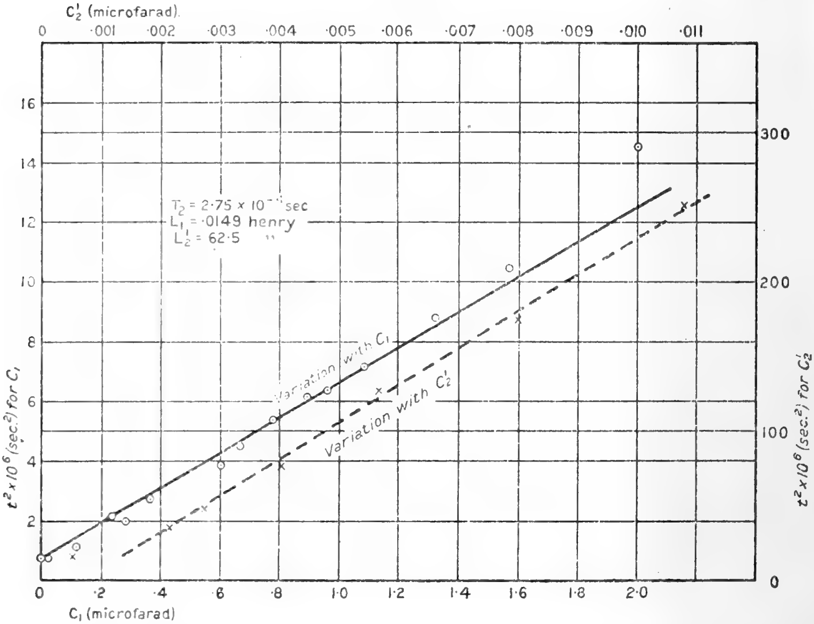


Fig. 3 gives the results of the measurements for determining L_1, T_2, L_2'. It will be seen that the values of τ^2 for different values of C_1 do not lie exactly on a straight line; for large values of C_1 the points lie above the straight line; for values between 0 and 0·7 mf. below it. The first deviation merely represents the fact that the self-inductance increases as the period of the oscillations decreases, and is to be expected; for the eddy-currents in the core will reduce the apparent permeability of the iron. Prof. Jones also found that the inductances were greater for the slower oscillations. The second deviation may be due to the fact that (1.1) has been substituted for (1), but is more probably due to errors in determining the period from the wave-form. The wave-form is not a sine curve, but composed of at least two harmonic components with different damping coefficients (see fig. 2), and it is not certain that the distance between successive maxima is truly equal to the period. Errors are especially likely to arise from this source when, as in the region between 0 and 0·7 mf., the periods of the two components are most nearly equal.

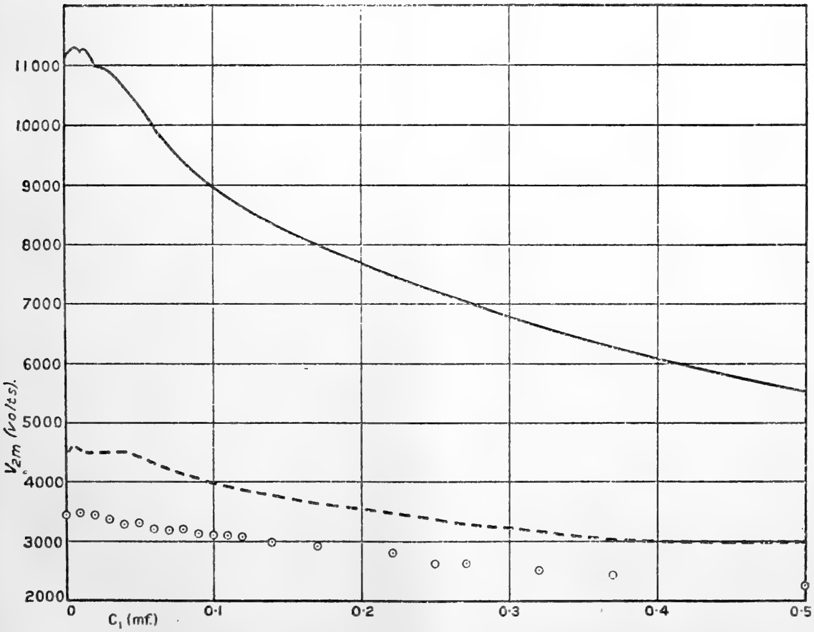
On the other hand, the points relating τ^2 and C_2' do lie on a straight line, but this line cuts the vertical axis below the origin, instead of above it as would be expected from (2 . 1). This discrepancy (which has been found in similar experiments on other machines) is not easily explained, but it would seem to suggest that the value of L_2' obtained is likely to be too large. If, however, we take the values given by the lines we obtain

$L_1=0{\cdot}0149$ henry; $L_2'=62{\cdot}5$ henry; $T_2=0{\cdot}000275$ sec.

Only a single measurement of c is possible, for it must be determined for the normal value of C_2. c was found to be 0·050.

c' was measured with $C_1=0{\cdot}5$, 2·0, 8·3 mf., and the values 0·0225, 0·0200, 0·0192 obtained. It is to be expected that c' should be rather less than c. L_{21} is now found to be 0·954 henry, and $s'=1{\cdot}031$.

Fig. 4.



We are now in a position to calculate the peak potential from Prof. Jones's theory on the assumption that there is no damping. These calculated values are plotted against C_1 in the full line curve of fig. 4; i_0, the primary current broken, is taken as 0·5 amp. The points marked ⊙ give the values determined experimentally by the method described in the note on p. 301 of this Journal. The experimental values are very much less than those calculated; at the maximum of the curve they are only about one third of those predicted by the theory. Moreover, in the experimental results there is no certain indication of the "arches" of the theoretical curve ; but since, when the coupling is so large, the arches are not very prominent, it is possible (though by no means probable) that their absence is due to insufficient accuracy of the measurements.

The Damping.

7. We must now inquire whether the discrepancy between theory and experiment can be removed by taking the damping of the oscillations into account.

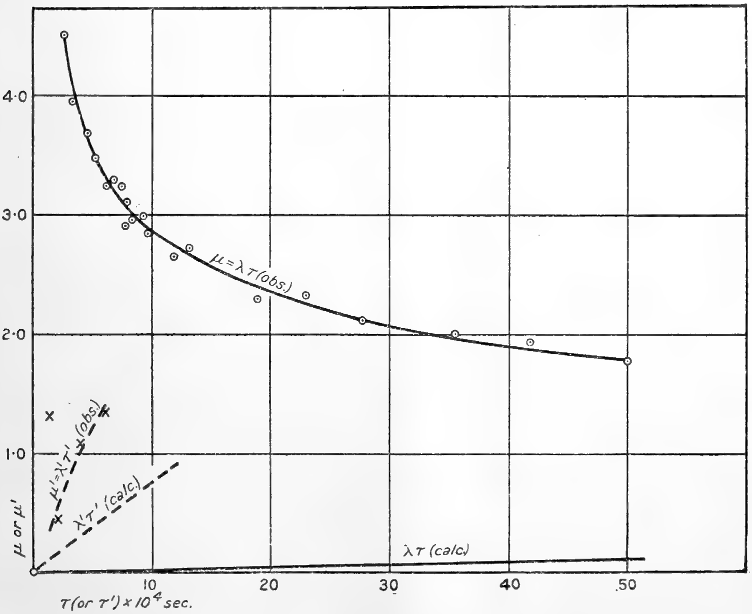
In fig. 5 the lines marked $\lambda\tau$ (calc.) and $\lambda'\tau'$ (calc.) show the relations between the logarithmic decrement of the oscillations and their period, for the slower and more rapid components respectively, calculated from (3) and (4); in this calculation the values of R_1 and R_2 are those determined with steady currents, namely $R_1=0{\cdot}623$ ohm, $R_2=2160$ ohms. The lines marked μ (obs.) and μ' (obs.) give the values of the decrements observed experimentally.

It is evident at once that, at any rate for the slower component of the oscillation, there is no agreement whatever between the observed and the calculated values. The wide difference shows that by far the greater part of the energy loss represented by the damping occurs in the iron core, and is not due to the cause contemplated by the theory, namely the ohmic resistance of the windings. Moreover, since the observed decrements decrease as the period increases, while the calculated decrements increase, it is not even legitimate to regard the loss in the core as adding to the effective resistance of the circuits. It should perhaps be pointed out that the excess of the damping observed over that calculated cannot be due to an increase of resistance of the windings at high frequencies owing to "skin effect," for the terms in R_1 and R_2 in (3) are of the same order ; R_1 may increase somewhat with the frequency, but R_2, being the resistance of wire of very small cross-section, can hardly change appreciably.

Even more noteworthy than the great difference between observed and calculated values for the slower component is the difference between the values for the slower and the more rapid component. According to (3) and (4) the damping of the more rapid component should be greater than that of the slower component (for the same values of the period) in a ratio of the order of $1/c$. But experiment shows that the damping of the more rapid component is very much less than that of the slower, and that both in its magnitude and in the sense of its change with the period, it agrees much more nearly with that calculated.

It should be remembered, of course, that the damping coefficients of the two components are determined in somewhat different conditions. The rapid component is only investigated when one of the two windings is short-circuited and the period of the slower component is infinite. It cannot be concluded immediately that, when the two periods are of the same order of magnitude, the damping of the more rapid component is as much less than that of the slower as fig. 5 would indicate. Evidence on this point is not available. It could only be obtained by the careful analysis of many complete wave-forms in which the two periods were of the same order, and the experiments have been brought to an end before a sufficient number of such wave-forms have been traced with the necessary accuracy. But such evidence as there is certainly indicates that, even when the periods of the two components are of the same order, the damping of

Fig. 5.



the slower is certainly not so much less than that of the more rapid as the theory would indicate. Moreover, in all the experiments on other machines which will be mentioned later all the characteristic features of fig. 5 were reproduced. It appears that whenever the damping is mainly due to losses in the core, for a component of a given frequency the damping is always much less if that component is the more rapid component of an oscillation than if it is the slower component. The result appears suggestive for any theory which would take into account the effect of the losses due to hysteresis and eddy-currents.

Some experiments were made to discover how far the damping of the oscillations was the same in the primary and secondary circuits. More definite information on this point will be given later, but it was clear that in general, and especially for the longer periods, the damping in the primary circuit was somewhat less than that in the secondary. It was also suspected that the periods of the oscillations in the two circuits were not quite equal, but the uncertainties in the measurement of the period when the damping is so large are too great (see p. 292) for any definite pronouncement on this point to be made.

Correction of the Peak Potential for Damping.

8. It is clear from these results that the correction of the calculated peak potentials for damping is not likely to improve greatly their agreement with observation, if the calculated values of the damping are used for the correction. In fact these calculated values are so small that they would change the calculated peak potentials by a few per cent. at most. But it is worth while to inquire whether a better agreement between the calculated and observed peak potentials can be obtained by correcting the former for the observed damping.

In order to apply the correction strictly the values for both the more rapid and the slower component ought to be taken into account; but the numerical work involved in applying the correction in this manner would be extremely heavy and certainly not justified in view of the uncertainties of the value of the damping of the rapid component when its period is of the same order as that of the slower. A simpler method has therefore been adopted. According to the theory the amplitude of the more rapid component should not be as much as one-tenth of that of the slower for any value of C_1 shown in fig. 4; it has been simply assumed, therefore, that the peak potential is determined wholly by the amplitude of the slower component, and to each point on the curve in that figure has been applied the correction appropriate to the slower component only. That is to say, it has been assumed that the wave-form is $V_2 = A \sin 2\pi nt \,.\, e^{-\lambda t}$, so that the reduction in the peak voltage due to damping is given by the factor $\sin\theta \,.\, e^{\frac{\mu\theta}{2\pi}}$, where $\tan\theta = \frac{2\pi}{\mu}$. This procedure is strictly accurate only when C_1 is very small or very large.

The resulting curve is shown by the dotted line in fig. 4. The ratio of observed to calculated peak potential is now 0·78 at $C_1 = 0$ and increases regularly to 0·88 at $C_1 = 4$ (which is not shown on the curve). Accordingly most of the discrepancy can be accounted for by the damping, but not all. It is, of course, hardly to be expected that any simple correction of this kind could bring theory and experiment into complete accord; for, as has been emphasized before, the actual damping which produces so great a reduction of the peak potential is wholly different in its nature from that contemplated by the theory.

Some Further Experiments.

9. Similar experiments were made on the armature of the same machine when it was completely removed from its housing, so that the only iron in the neighbourhood of the circuits was the core on which they were wound; the unlaminated pole-pieces could no longer exert any influence. It was thought that the iron losses would be greatly reduced, and that better agreement between theory and experiment might be found.

By exactly the same method as before the following values were found for the constants of the armature:—

> $L_1 = 0{\cdot}00045$ henry; $L_2' = 15{\cdot}3$ henry; $T_2 = 0{\cdot}000174$ sec.; $c = 0{\cdot}101$; $c' = 0{\cdot}043$; $L_{21} = 0{\cdot}282$ henry; $s' = 1{\cdot}037$. i_0 is again taken $= 0{\cdot}5$ amp.

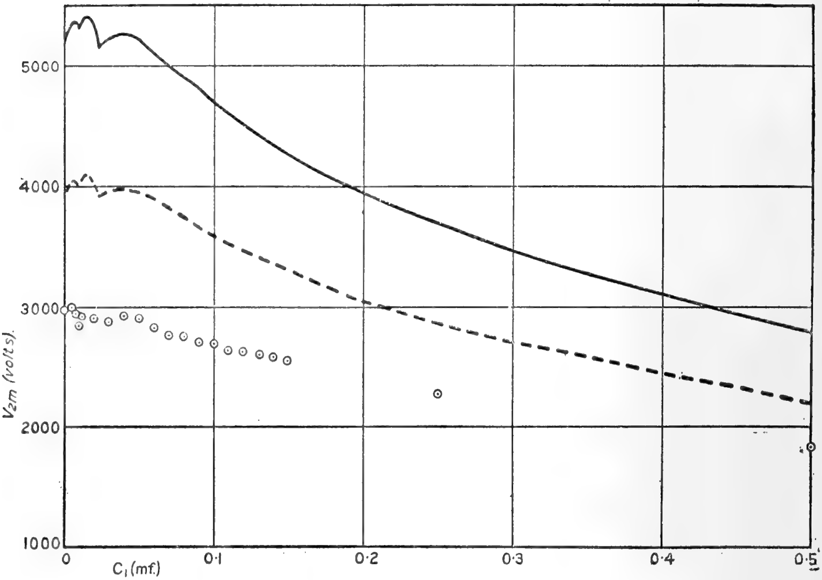
In fig. 6 the peak potential calculated from these values is plotted against C_1 in the full curve, the experimentally determined values being again given by the points ⊙.

The damping coefficients were not determined so completely as before, but the following values were obtained:—

$\tau = 0{\cdot}000175$	μ (obs.) $= 1{\cdot}20$	μ (calc.) $= 0{\cdot}0124$
0·00114	1·10	0·0650
0·00161	0·84	0·0922
$\tau' = 0{\cdot}000232$	μ' (obs.) $= 0{\cdot}76$	μ' (calc.) $= 0{\cdot}296$

As is to be expected, the damping is considerably less at the same frequency than before, but the same general features are shown. μ increases with the frequency and, for the same frequency, μ' is much less than μ.

Fig. 6.



The peak potentials were then corrected in the same manner as before for the observed damping, and the dotted curve in fig. 5 obtained. The ratio of observed to calculated peak potential is 0·76 at $C_1=0$ and 0·88 at $C_1=4$—almost exactly the same ratios as were found in the previous experiments; it will appear later that the same ratios reappear in experiments on an armature of quite different construction. It would seem, therefore, that the coincidence of the value of this ratio is not wholly accidental; but it cannot be determined at present what theoretical significance is to be attributed to the agreement.

General Conclusions.

10. The main conclusion to be derived from these experiments is that a theory which introduces only constants characteristic of the windings of the armature will not lead to values for the peak potential which are, even approximately, in accordance with experiment. By far the most important factor in determining what peak potential will be obtained is

the damping which is due to hysteresis and eddy-currents, and no theory which cannot predict this damping is of any use in predicting the peak potentials.

On the other hand, in some matters the theory gives results which are in general agreement with observation. Thus the relations between τ and C_1 or C_2' shown in fig. 2 are not very different from those predicted; the values for c' are approximately constant and not very different from the value of c determined experimentally or the value to be expected from the form and arrangement of the circuits. Again, the changes in the inductances which result from removing the armature from its housing are in the expected direction, and the primary and secondary inductances are reduced in approximately the same ratio (0·30 and 0·25 respectively); while removal from the housing is found to reduce the coupling, as it should. Indeed there is only one considerable discrepancy. When the armature is removed from its housing, the secondary capacity C_2 should be reduced; but if we put $L_2=L_2'$ and calculate C_2 from T_2, we find $C_2=30·8$ mmf. in the housing and $C_2=51·5$ mmf. out of the housing, a difference which can hardly be accounted for by an error introduced by identifying L_2 and L_2'. It is probable that the second value is the more nearly correct.

It is not surprising that the success of the theory should be so very different according as it is applied to the deduction of the constants of the circuits, or to the prediction of the peak potential. For in the former application we are only concerned with the periods of the oscillations, and the general theory of oscillations indicates that these periods will not be very greatly affected by damping, so long as it not so great as to reduce the oscillations to subsidences. On the other hand, the prediction of the peak potentials involves the amplitudes and the phase differences of the components of the oscillations, and these will be greatly affected by the presence of a source of damping which the theory does not contemplate.

Accordingly, in spite of the general failure of the theory to predict absolute values for the peak potential, it is worth while to inquire whether it can predict how those values vary with alterations in the constants of the circuits; for if it is successful in this respect, its value in the design of machines (the purpose to which the investigation was primarily directed) will scarcely be diminished.

The most remarkable and interesting feature of the theory is the relation which it predicts between the peak potential, the primary capacity C_1, and the coupling or the associated quantity c. For it is the peak potential which alone is of much practical importance, while C_1 and c are the quantities which are most completely at the disposal of the designer. According to the theory there is for each value of c an optimum value of C_1, which gives the greatest peak potential. The value of this peak potential with optimum C_1 does not vary very greatly with c; so long as c lies between 0 and 0·5, the greatest peak potential obtainable does not vary by more than 5 per cent.; but the optimum value of C_1 varies greatly. If $c=0$, the value is 0; but it can be increased considerably by increasing c. Now in practice a lower limit is set to C_1 by the necessity of avoiding sparking at the break in the primary circuit; indeed, until an adequate theory of the machine was produced, it was thought that the only purpose of inserting a condenser in parallel with the primary terminals was to suppress such sparking. It is therefore clear that, since this lower limit is set to C_1, the optimum value of c will not be 0, but will be some higher value such that the optimum value of C_1 is not less than the lower limit set by the necessity of avoiding sparking; and the theory indicates that the attempt, which seems to have been made in the design of most actual instruments, to make the coupling as close as possible and c as small as possible is mistaken, and that better results would be obtained by decreasing the coupling and increasing c. The second part of this paper will be devoted to examining how far this prediction is correct. But it should be noted that evidence is already available that the theory may fail in this respect also. For the relation predicted by it between the peak potential, C_1, and c is intimately connected with the occurrence of "arches" (such as are shown in the full-line curves of figs. 4 and 6) in the curve relating the peak potential and C_1. Now the experimental curves do not show these arches; there are indications of very shallow maxima and minima which scarcely exceed the possibilities of experimental error, but there is nothing whatever corresponding to the sharp cusps which are so notable a feature of the theoretical curves. It must, therefore, be doubtful from the outset whether the theoretical relation is fulfilled. [*To be continued.*]

Summary.

1. The object of the experiments is to investigate more nearly the applicability of the theory of the high-tension magneto which has been given by Prof. E. Taylor Jones in a recent paper.

2. The application of the theory to the experimental measurements is described.

3, 4, 5. The experimental methods for tracing the wave-form of the magneto are described.

6. The values of the constants of a certain machine obtained by these methods are given, and the values of the peak potential calculated from them by the theory. The observed values are very much less than those calculated.

7. The damping of the oscillations is investigated. It is shown that the main source of damping is one not contemplated by the theory, namely, the losses in the iron core due to hysteresis and eddy-currents.

8. The calculated peak potentials are corrected for the observed damping and values obtained which, though in much better agreement than before with the observed values, are still considerably greater than the observed values.

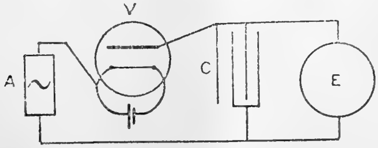
9. Some further experiments in which the iron losses are greatly reduced are described, but the same discrepancies as before are noted.

10. It is pointed out exactly in what points theory and experiment agree and in what points they differ. It is suggested that it is still possible that the theory may provide a useful guide to design of magnetos, and the considerations underlying the experiments to be described in a second part of the paper are stated.

XXVI. *Note on the Measurement of the Peak Potential of an Alternating Source.* *By* Clifford C. Paterson, *M.I.E.E.*, *and* Norman Campbell, *Sc.D.**

THE method of measuring the peak potential of an alternating source by means of a thermionic valve is now familiar, but the conditions necessary for accuracy do not seem always to be appreciated.

Fig. 1.



The connexions are shown in fig. 1. A is the source of alternating potential, V the thermionic valve with its heating

---

* Communicated by the Authors.

battery well insulated from earth, E an electrostatic voltmeter, and C a condenser connected across the terminals of the voltmeter. A separate condenser C is not always inserted; if it is not, the capacity of C represents simply the capacity of the voltmeter and its connexions. A is shown connected to the valve so that the negative peak potential is measured; if the positive peak potential is required, the leads from the valve to A and E must be interchanged.

The following are the conditions necessary for accuracy:—

(1) The valve must pass no reverse current.

(2) The insulation between the terminals of E must be so great that the potential between them does not fall appreciably in the interval between successive occurrences of the peak potential.

The necessity for these two conditions is obvious.

(3) Condition (2) could always be fulfilled by making C sufficiently large and sufficiently well insulated. But the insulation must fulfil a further condition; it must be so great that the amount of electricity which the source of potential has to supply at successive occurrences of the peak, in order to make up the loss occurring between them, is not so large as to alter the peak potential obtained. Suppose, for instance, that the source of potential can be shunted by a resistance of 1 megohm without affecting appreciably its peak potential; then, if the interval between successive occurrences of the peak potential is n times the duration of the peak potential, the least resistance which can be placed as a shunt across E will be of the order n megohms and not of 1 megohm. Thus, when the source of potential was a magneto, it was found that a resistance of 10,000 megohms across E reduced the peak potential recorded by nearly 10 per cent., although a resistance of 1 megohm placed across the terminals of the machine did not affect it more than 2 per cent., and although the potential recorded by E did not fall by as much as $\frac{1}{2}$ per cent. between successive occurrences of the peak potential.

It may be noted that it is a sufficient criterion that conditions (2) and (3) are fulfilled that the peak potential recorded is independent of the temperature of the filament of the valve. It will always be found that there is a lower limit to the temperature below which the peak potential recorded decreases: but unless (3) is fulfilled, there will be no well-defined upper limit above which the temperature of the filament is without effect on the recorded potential.

(4) The capacity C must be a sufficiently large multiple of the capacity of the valve. The necessity for this condition appears to be often overlooked.

Let C_1 be the capacity coefficient of the anode and cathode circuits of the valve; and let at any time E_1 be the potential difference between anode and cathode; E_2 the potential difference across the voltmeter E; E the potential given by the source A. Then

$$E = E_1 + E_2.$$

Since the anode of the valve is insulated, the charge on it is constant, that is,

$$C_1E_1 - CE_2 = \text{const.}$$

The constant is determined by the condition that, when $E = E_m$, where E_m is the peak voltage, $E_1 = 0$ owing to the action of the valve. Hence

$$(C_1 + C)E_2 = C_1E + CE_m.$$

The voltmeter reads the R.M.S. value of E_2. Denoting average values by a bar, we have

$$(C_1 + C)^2\overline{E_2^2} = C_1^2\overline{E^2} + 2C_1C\overline{E} \,.\, E_m + C^2E_m^2.$$

So that E_m is not equal to $\sqrt{\overline{E_2^2}}$, even if $\overline{E} = 0$ (as is usual), unless C_1 is very small compared to C; if an accuracy of 1 per cent. is required, C must be 200 times C_1 unless a correction involving a knowledge of the R.M.S. voltage of the source is applied. C_1 will not usually be less than 2 or 3 cm.; hence C_2 must be several hundred cm., which is much more than the capacity of most electrostatic voltmeters, so that the addition of a condenser in the position C is required. A well-insulated leyden-jar will usually meet the purpose, but if still higher insulation is found to be necessary, condensers for the purpose may be made readily by wrapping an external conductor round rubber cable, such as is used for connecting magnetos to their sparking-plugs. Such cable usually has a resistance of at least 10^{15} ohms per metre; the 7 mm. variety will stand up to 20,000 volts (direct).

If these conditions are fulfilled—and they are easily fulfilled so long as the peak potential is not greater than 15,000 volts—the only limit to the accuracy of the method is the accuracy of the voltmeter. And here a warning may be inserted. There are types of electrostatic voltmeter which do not give the same reading with alternating and direct current; that is to say, an alternating current of R.M.S. voltage E will not give the same reading as a direct current of steady voltage E. It is not safe therefore to calibrate the voltmeter by alternating current; it is best calibrated by means of the valve with an alternating source of which the wave-form is known.

Dec. 1918.

XXVII. *The Lateral Vibration of Loaded Shafts in the Neighbourhood of a Whirling Speed.—The Effect of Want of Balance.* By H. H. Jeffcott *.

1. WHEN a shaft whose centre of mass does not lie on its axis of figure (the shaft then being said to be "out of balance"), is rotated, its geometrical axis ceases to remain straight and in coincidence with the axis of the bearings, but becomes bent and rotates round the latter axis. The amount of this bending depends on, among other things, the relation of the speed of rotation to that of any of the free lateral vibrations of the shaft. So long as the speed of rotation is not too near that of any of these vibrations the shaft remains nearly straight unless it is badly "out of balance." If, however, the speed of rotation increases so as to approach that of any transverse vibration the bending becomes noticeable,—sooner or later according to the perfection of balance of the shaft—and increases rapidly; and the shaft bends in one or more loops according to the particular transverse vibration whose speed is being approached.

If the speed is maintained constant at such a value the bending may become excessive and is ultimately restrained by the action of the bearings. And if the speed changes from one slightly below to one slightly above that of the transverse vibration the character of the motion changes.

In particular, for the slowest transverse vibration of a shaft put out of balance by a single mass, the motion changes from one in which the centre of mass is nearer to the axis of the bearings than is the elastic centre to one in which the reverse is true. In this change the shaft appears to shiver.

Well above the speed of the transverse vibration the shaft settles down again to more or less steady running with the axis nearly straight until the speed approaches that of another transverse vibration.

When the shaft is considerably bent it is said to "whirl." The speeds of transverse vibration are spoken of as the "whirling speeds."

In the foregoing when the shaft is horizontal, the straight form, or the axis of the bearings, is to be understood to refer to the gravitationally deflected form.

It is proposed in this note to discuss to some extent how want of balance causes these phenomena, and to what extent in good practice balancing should be carried. It should be

---

* Communicated by the Author.

pointed out, however, that an adequate mathematical treatment of the general case of a loaded shaft is a matter of considerable difficulty, and that the discussion which follows does not pretend to be more than illustrative.

2. For the sake of simplicity in illustrating the character of these vibrations we will consider the case of a light uniform shaft supported freely in bearings at its ends and carrying a mass m at the centre of its span, the mass centre, however, being slightly eccentric by a distance a from the elastic centre of the shaft.

We will suppose the load to be of the nature of a thin pulley or disk of negligible moment of inertia. Accordingly we are dealing with simple lateral vibration, and are not concerned with oscillatory vibration about a diameter.

The conditions given permit a statement of the problem in a simple form. The motion of the cross-section at the centre of the span in its own plane need alone be considered, subject to (*a*) a restoring force varying as the distance of a point known as the elastic centre from the axis of the bearings, (*b*) a damping force, and (*c*) the disturbing effect produced by an impressed rotation in its own plane combined with the fact that the centre of mass is placed eccentrically with respect to the elastic centre. Thus the problem may be viewed as the motion in the plane XY of a disk on which is impressed a constant angular velocity ω.

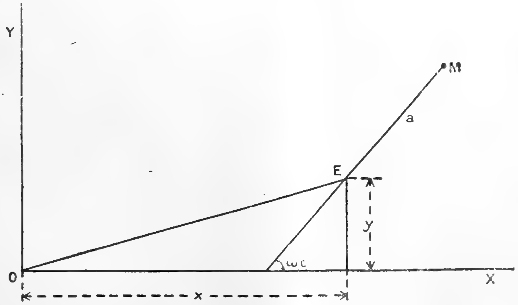


Fig. 1.

In the cross-section in fig. 1, let O be the intersection of the axis of bearings with the plane XY. Let x and y be the co-ordinates of E the elastic centre (which is also supposed to be the centre of figure) of the shaft at any instant; and let M be the position of the centre of mass at the same instant.

It is given that the shaft rotates with angular velocity ω. This angular velocity determines the angle (ωt) which any line in the disk such as EM makes with a line fixed in space such as OX. It would not be correct in the statement of the problem to assume that the inclination of the radii vectores OE or OM changes as ωt. It will be shown later that the path of E, when steady motion is established, is a circle described with angular velocity ω. Hence O is the instantaneous centre, and the inclinations of the lines OE and OM change as ωt.

The equation of motion of m parallel to OX is

$$m\frac{d^2}{dt^2}(x+a\cos\omega t)+b\frac{dx}{dt}+cx=0,$$

where b is the coefficient of damping due to viscous resistances, and c is the elastic force of restitution at unit displacement.

Hence

$$m\ddot{x}+b\dot{x}+cx=ma\omega^2\cos\omega t.$$

The solution of this equation is well known to be

$$x=\mathrm{A}e^{-\frac{bt}{2m}}\sin(qt+\alpha)+\frac{ma\omega^2}{\sqrt{(c-m\omega^2)^2+b^2\omega^2}}\cos(\omega t-\beta),$$

where $$\tan\beta=\frac{b\omega}{c-m\omega^2};\quad q=\frac{\sqrt{4mc-b^2}}{2m};$$

and A and α are arbitrary constants.

In like manner the equation of motion parallel to OY is

$$m\frac{d^2}{dt^2}(y+a\sin\omega t)+b\frac{dy}{dt}+cy=0,$$

and the solution is

$$y=\mathrm{A}'e^{-\frac{bt}{2m}}\sin(qt+\alpha')+\frac{ma\omega^2}{\sqrt{(c-m\omega^2)^2+b^2\omega^2}}\sin(\omega t-\beta),$$

where A′ and α' are new arbitrary constants.

The first term in the solution represents an oscillatory motion of amplitude

$$\mathrm{A}e^{-\frac{bt}{2m}}\text{ or }\mathrm{A}'e^{-\frac{bt}{2m}}.$$

This amplitude diminishes with increase of t, so that the term becomes negligible.

The second term persists, and is a vibration of amplitude

$$\frac{ma\omega^2}{\sqrt{(c-m\omega^2)^2+b^2\omega^2}}.$$

This forced vibration is caused by the disturbing action of the eccentric mass during rotation.

It is seen from the foregoing values of x and y that the terms representing the forced motion satisfy the relation

$$x^2+y^2=\text{constant}.$$

Accordingly the path of the elastic centre, when the oscillatory motion subsides, is a circle round the axis of the bearings of radius

$$\frac{ma\omega^2}{\sqrt{(c-m\omega^2)^2+b^2\omega^2}}.$$

The co-ordinates of M being $x+a\cos\omega t$ and $y+a\sin\omega t$, and x and y having the values corresponding to the forced motion as already given, it follows in the same way that the path of M is also a circle round the axis of the bearings when the steady motion is established. Thus

$$x'=\frac{ma\omega^2}{\sqrt{(c-m\omega^2)^2+b^2\omega^2}}\cos(\omega t-\beta)+a\cos\omega t,$$

$$y'=\frac{ma\omega^2}{\sqrt{(c-m\omega^2)^2+b^2\omega^2}}\sin(\omega t-\beta)+a\sin\omega t,$$

$$\text{OM}^2=x'^2+y'^2=x^2+y^2+a^2+2a(x\cos\omega t+y\sin\omega t).$$

Now

$$x\cos\omega t+y\sin\omega t=\frac{ma\omega^2}{\sqrt{(c-m\omega^2)^2+b^2\omega^2}}\cos\beta$$

and

$$\tan\beta=\frac{b\omega}{c-m\omega^2}.$$

Hence

$$\text{OM}=a\sqrt{\frac{c^2+b^2\omega^2}{(c-m\omega^2)^2+b^2\omega^2}}.$$

If all the quantities save ω are given the amplitude is maximum, or

$$\frac{ma\omega^2}{\sqrt{(c-m\omega^2)^2+b^2\omega^2}}$$

is maximum, when

$$\omega=\frac{2c}{\sqrt{4mc-2b^2}}.$$

This value of ω approximates to the speed

$$\frac{\sqrt{4mc-b^2}}{2m}$$

corresponding to the free period, when b is small.

At speeds in the neighbourhood of this value of ω the vibration, caused by a given eccentricity of mass centre, may be excessively great. This is the whirling speed of the shaft.

3. Looking next at the value of β which determines the phase of the displacement of the mass centre relatively to that of the elastic centre, we see that since

$$\tan\beta = \frac{b\omega}{c - m\omega^2},$$

the value of β is 0 when ω is zero. The value of β changes with increase of ω, until when $m\omega^2 = c$, $\beta = \frac{\pi}{2}$, or the phase of the mass displacement has now shifted by $\frac{\pi}{2}$ from that corresponding to very slow rotation.

Above this value of ω, the phase changes further, and at very high values of ω, $\beta \doteqdot \pi$.

Now the value of b is usually small, and on closer examination it will be seen that the whole phase-change through an angle π takes place practically entirely between a speed very slightly below and another very slightly above the critical speed. In other words, this change of phase takes place mainly within a comparatively small number of revolutions per minute on either side of the critical speed.

Thus at speeds appreciably below the whirling speed the shaft rotates with the mass centre farther from the axis of rotation than the elastic centre, while at speeds appreciably above the critical speed the mass centre is closer to the axis.

4. This result can easily be obtained directly by considering the steady motion in these extreme cases and omitting the damping altogether, as is well known.

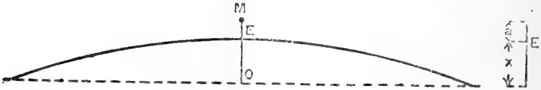


Fig. 2.

Thus in fig. 2, let $OE = x$, $EM = a$, where O, E, M represent the axis of rotation, the elastic centre, and the centre of mass. Equating the centrifugal force to the restoring elastic force acting on m in the assumed steady motion, we obtain

$$m\omega^2(x + a) = cx, \quad \text{or} \quad x = \frac{m\omega^2 a}{c - m\omega^2},$$

so that while $m\omega^2$ is less than c, we have the condition illustrated in fig. 2.

When $m\omega^2 = c$ we have x indefinitely great, and the value of ω is a critical speed.

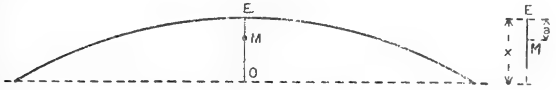


Fig. 3.

When $m\omega^2$ is greater than c, we have the condition illustrated in fig. 3. Then

$$m\omega^2(x-a) = cx \quad \text{and} \quad x = \frac{m\omega^2 a}{m\omega^2 - c}.$$

The transition from one condition to the other in passing through the critical speed is clearly brought out above when the damping is included.

5. The phenomena observed at balancing, as to the relation of the (marked) point of maximum deflexion on the shaft's periphery to the position of the mass centre, at speeds below, in the neighbourhood of, or above the whirling speed are well known to concord with the above phenomena due to damping, and a reversal of the direction of rotation gives results confirmatory of this.

Thus in machines designed for testing the dynamic balance of rotors, the part to be balanced is revolved in bearings which are free to move to and fro between controlling spring or rubber buffers which limit the amount of their motion in a horizontal plane when the shaft vibrates.

A scriber or pencil is held near the shaft at the section being examined, and thus it is marked at that part of its circumference that is most deflected.

It is observed that at all speeds below the critical speed an unbalanced body will be marked on the heavy side ; and conversely above the critical speed it will be marked on the light side.

The mark, however, is not at the exact heavy or light spot, but it is displaced angularly round the shaft, more or less according to the proximity to the critical speed. The direction of this angular displacement depends on the direction of revolution of the body, and if displaced in one direction by revolving clockwise, it will be equally displaced in the opposite direction by revolving counter clockwise at the same speed. Thus the true heavy or light spot lies midway between the centres of the markings in these two cases.

In most cases, when balancing, the machine is operated by running above the critical speed, thereby marking on the light side.

It is to be noted, however, that the cases of balancing met with in practice are usually complicated by having several masses along the shaft instead of only one as considered above, so that the resultant displacement at any section then depends on the joint action of all the loads, according to their several effects at that section.

6. It should be pointed out that, at speeds appreciably below the critical value, the phase-change due to damping is small. The introduction of the coefficient b in the equation of motion is mainly of interest in showing how the mass centre changes from being outside to being inside the elastic centre, relatively to the axis of rotation, as the speed rises from a little below to a little above the whirling speed; and also that the amplitude of vibration at the critical speed in the ideal case under consideration, in which no restraint to excessive vibration is assumed to be imposed by the bearings, would not become indefinitely great.

Returning now to the amplitude of the vibration, which is

$$\frac{ma\omega^2}{\sqrt{(c-m\omega^2)^2+b^2\omega^2}},$$

its value is not seriously altered by the omission of b except in the close neighbourhood of the value $\omega=\sqrt{\frac{c}{m}}$.

At this critical value of ω, the effect of b is to diminish the amplitude and prevent it increasing to an indefinitely great value, as has just been stated.

7. If $b=0$, and also $m\omega^2=c$, the equation of motion is

$$\ddot{x}+\omega^2x=a\omega^2\cos\omega t,$$

of which the solution is

$$x=\mathrm{A}\sin(\omega t+\alpha)+\tfrac{1}{2}a\omega t\sin\omega t,$$

where A and α are arbitrary constants.

The amplitude of the forced vibration would in this case increase continually with t. But the damping b is not zero, and the amplitude therefore does not increase indefinitely.

It is worthy of note, however, that if a shaft be run quickly through its whirling speed to a higher working speed, there may not be time for serious vibration to take place; in fact, the value of t may be so small that the amplitude of this forced vibration $\frac{1}{2}a\omega t$ may never be great.

8. Apart from a knowledge of the numerical value of b in any particular calculation, we cannot obtain the exact value of the amplitude of vibration; but, as we have seen, it is only in the close proximity of the whirling speed that the damping seriously modifies the amplitude, and we may omit it at all except such speeds.

Also when not too close to the whirling speed we may regard the displacements of the mass centre and elastic centre as taking place along the same radius-vector from the axis of rotation, which likewise may be obtained from the general solution by putting $b=0$.

Let u be the ultimate amplitude of the vibration at the elastic centre, then

$$u=\frac{ma\omega^2}{\sqrt{(c-m\omega^2)^2+b^2\omega^2}} \doteqdot \frac{ma\omega^2}{\pm(c-m\omega^2)}.$$

Put $k=\sqrt{\frac{c}{m}}$ = speed of free vibration.

If $\omega<k$, we have

$$u=\frac{\frac{a}{k^2}}{\frac{1}{\omega^2}-\frac{1}{k^2}}.$$

The centrifugal force is

$$\mathrm{F}=m\omega^2(u+a)=\frac{ma}{\frac{1}{\omega^2}-\frac{1}{k^2}}=mk^2u.$$

If $\omega>k$, we have

$$u=\frac{\frac{a}{k^2}}{\frac{1}{k^2}-\frac{1}{\omega^2}}$$

and

$$\mathrm{F}=m\omega^2(u-a)=\frac{ma}{\frac{1}{k^2}-\frac{1}{\omega^2}}=mk^2u.$$

Plotting these results against the speed ω, we see that, with a suitable alteration in scale, the same curves represent both u and F.

The curves (fig. 4) have not been drawn close to the position $\omega=k$, as in that region the damping prevents the values becoming indefinitely great, as would appear to be the case if the formulæ used for u and F were fully plotted.

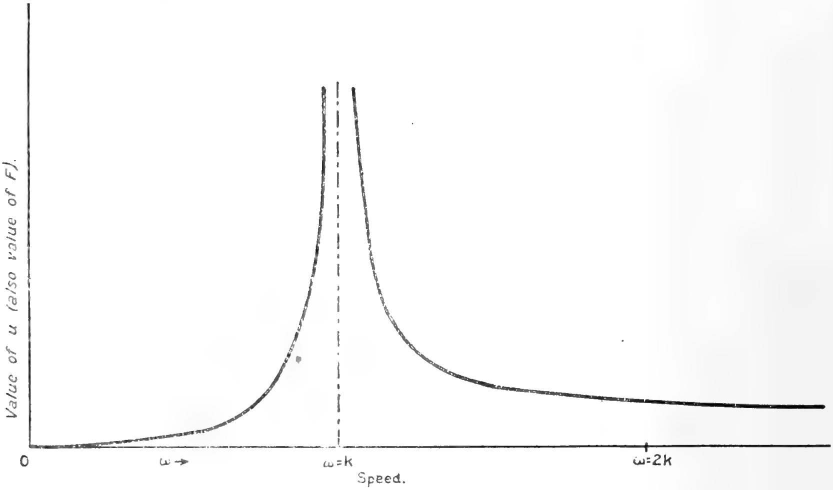


Fig. 4.

The curves exemplify clearly how that the critical speed is one that is particularly favourable to vibration, or one at which the shaft is very sensitive to lateral disturbing forces.

If in the choice of the working speed the critical speed is avoided by 10 per cent. on either side, *i. e.* if ω does not lie between $\cdot 9k$ and $1\cdot 1k$, then the amplitude of vibration will not be more than about five times the eccentricity of the mass centre.

If the shaft is in good balance this choice of working speed will therefore give good results and sufficiently steady running.

The better the balance the closer to the critical speed may the shaft be worked. For while keeping the amplitude of vibration u within a definite limit, we may approach k the more closely the smaller the value of a; and further, the damping will keep the amplitude rather lower than that plotted.

On the other hand, if the balance is bad the critical speed must be avoided by a large margin, and it may even be dangerous to run through the critical speed to a higher working speed.

9. If we include damping we have the displacement of the shaft $u = \mathrm{OE}$, and $\mathrm{F} = m\omega^2 . \mathrm{OM}$; see fig. 1.

But
$$\mathrm{OE} = \frac{ma\omega^2}{\sqrt{(c - m\omega^2)^2 + b^2\omega^2}},$$

and
$$\mathrm{OM} = a \times \sqrt{\frac{c^2 + b^2\omega^2}{(c - m\omega^2)^2 + b^2\omega^2}}.$$

So that we now have after putting $k = \sqrt{\frac{c}{m}}$ as before,

$$u = \frac{ma\omega^2}{\sqrt{m^2(k^2 - \omega^2)^2 + b^2\omega^2}},$$

and
$$\mathrm{F} = ma\omega^2 \times \sqrt{\frac{m^2k^4 + b^2\omega^2}{m^2(k^2 - \omega^2)^2 + b^2\omega^2}}.$$

It will be noticed that the previous results are a particular case of this.

On assuming a particular value for $\frac{b}{m}$ these results may be plotted similarly to those in fig. 4.

10. Reverting now to the formulæ for centrifugal force written above (§ 8) we may compare the forces and vibrations due to a given eccentricity of mass at speeds above and below the whirling speed.

Consider the design of a shaft for a given duty and operating at a given working speed.

Then the preceding formulæ indicate that it is better from the vibration point of view to design the shaft with its critical speed below the working speed rather than to have a critical speed the same proportion above the working speed. In the latter case the shaft will of course be thicker and somewhat heavier. This result, indeed, is well illustrated in the behaviour of the De Laval steam turbines.

Thus comparing the centrifugal forces in the two cases in the particular example with which we have been dealing, let F be the centrifugal force when the critical speed k is greater than the working speed ω; and let F′ be the centrifugal force when the critical speed k' is less than the working speed; the eccentricity of mass centre a is supposed to be the same in two cases, but the masses m and m' are not quite the same, the former being somewhat greater owing to the shaft being heavier.

Then we have $$\mathrm{F}=\frac{ma\omega^2k^2}{k^2-\omega^2}$$

and $$\mathrm{F}'=\frac{m'a\omega^2k'^2}{\omega^2-k'^2},$$

so that $$\frac{\mathrm{F}'}{\mathrm{F}}=\frac{m'}{m}\times\frac{k'^2(k^2-\omega^2)}{k^2(\omega^2-k'^2)}.$$

If now $k=(1+p)\omega$, $k'=(1-p)\omega$, we find

$$\frac{\mathrm{F}'}{\mathrm{F}}=\frac{m'}{m}\times\frac{2-p(3-p^2)}{2+p(3-p^2)},$$

which is less than 1.

Hence F′ is less than F, or the centrifugal force is less when the whirling speed is below the working speed rather than the same amount above.

11. To get a numerical idea of the degree of balance desirable in any machine, we will consider the case of a rotor consisting of a single heavy mass of 5 tons concentrated at the span centre.

Further, suppose the working speed to be 1000 r.p.m. and the critical speed 1250 r.p.m.

Then $$\frac{u}{a}=\frac{1}{1{\cdot}25^2-1}=1{\cdot}8,$$

thus $u=1{\cdot}8a$. If, therefore, the amplitude is to be limited to say ·0001 inch, a must be not more than about half that figure.

Also $$\mathrm{F}=mk^2u=\frac{5\times 2240}{32}\times\left(\frac{1250}{9{\cdot}55}\right)^2\times\frac{1{\cdot}8}{12}a$$
$$=900{,}000\,a,$$

where a is inches, and F is lb. Putting $a=\cdot 0001$ inch, we find F=90 lb.

If the rotor be out of balance to the extent of $\frac{1}{2}$ oz. at a radius of 10 inches, this corresponds to an eccentricity of the mass centre of ·00003 inch. Good balancing would realize this figure.

XXVIII. *Note on Æther and Motion.*

By Sir OLIVER LODGE[*].

REFERRING to Dr. Houstoun's paper in the February issue of the Phil. Mag., the supposition that the Æther of Space can be in any sense attached to the body of the Earth involves so many fundamental difficulties that it cannot be considered an attractive hypothesis, and to my mind it has been definitely negatived by a series of rather elaborate experiments which I made at Liverpool in the years 1892–7. See Phil. Trans. 1893, vol. 184, p. 727, and 1897, vol. 189, p. 149.

Possibly these papers have escaped Dr. Houstoun's attention. Taken in conjunction with the great experiment of Messrs. Michelson and Morley, the combined result definitely drives us to the view that the FitzGerald-Lorentz contraction—so probable on an electric theory of matter—is a reality.

Dr. Houstoun seems inclined to think that to admit this contraction is equivalent to admitting in full the negations of the Principle of Relativity—not merely as a practically convenient summary for dealing with phenomena which otherwise would need detailed consideration, but as a law of nature. I do not myself feel in the least compelled to admit such a limitation, and hope that before very long some method of detecting and measuring the drift of our solar and stellar system through the stationary æther of space will be forthcoming. At any rate the door ought not to be prematurely shut on attempts in that direction, though the problem is admittedly a curious and unexpectedly difficult one: the compensations being so numerous and apparently so complete.

Incidentally I am glad to see, from his method of tackling the matter, that Dr. Houstoun agrees with me in holding that Fizeau's experiment does not establish any motion of the æther of space inside a moving transparent medium: Fizeau's result, anticipated by Fresnel, definitely proves that the main body of Æther does not so move. All that moves with a stream of illuminated water is the ætherial modification or loading which is responsible for lessening the velocity of light when travelling through a region occupied by matter. In other words, the thing that travels is the extra $K\mu$ which belongs to the matter, over and above the value of the product of these two fundamental ætherial constants in free space.

Would that it were possible to determine the ratio or some other function of these two constants, as well as their product! Then our knowledge of the Æther would indeed begin to forge ahead.

---

[*] Communicated by the Author.

XXIX. *Notices respecting New Books.*

The Theory of the Relativity of Motion. By Professor R. C. Tolman. University of California Press. Pp. ix+225.

THE writer of a book on relativity has a double task, firstly to explain clearly the nature of the assumptions on which the theory rests and their experimental justification, and secondly to develop mathematically the characteristic transformations and their consequences. Of these the first is the harder, since, while the mathematics of the simple theory are straightforward, and present no difficulty to the student furnished with the necessary technical equipment, the fundamental concepts are rather in seeming contradiction to, than a natural development of, what he has already learnt in mechanics. That the mass of a moving body depends on its velocity and the length of it on its orientation, that the result of compounding any two velocities (even collinear) less than that of light gives a velocity still less than that of light—to make these things even reasonable demands a very careful and logical exposition of the bases of the theory. We do not think that Professor Tolman has devoted enough care to the preliminary explanations. The ideas of Fresnel and his successors receive a bare mention, yet it is by explaining these, and describing the difficulties solved and the fresh ones encountered that the mind can be gradually led to the standpoint of the relativists. Considering that the book is intended as an introduction to the subject, the difficulties presented to the mind trained in the older electrodynamics have been handled somewhat perfunctorily.

The order in which the subject is presented is the reverse of the historical. The electrodynamic equations, in which the theory originated, are not presented until Chapter XII., the previous chapters being devoted to the relativistic dynamics of particles and elastic bodies. The historical order is the more natural one, and the more convincing for the physicist, since, after all, the only experimental confirmations of the theory are to be found in the realm of electromagnetic phenomena (including, of course, light). The scope and nature of the experimental results in question might have been made clearer.

The chapter on chaotic motion of particles contains no reference to the work of Jüttner, and that on the dynamics of elastic bodies no reference to Herglotz, Born or Lamla. The book, being intended as an introductory treatise, does not handle the generalized theory of relativity. The last chapter contains an interesting exposition of four-dimensional analysis, based upon the work of Wilson and Lewis, and elsewhere the work of the American relativists is well represented.

We regret to announce the death on February 9th of Professor Carey Foster, one of the conductors of this Magazine. For many years previous to 1911, when he was persuaded to allow his name to appear on the wrapper, the Philosophical Magazine was indebted to him for invaluable advice and counsel. For this help, continued until within a few weeks of his death, we wish to record our deep sense of gratitude.

W. & R. T. Francis.

GEORGE CAREY FOSTER.

George Carey Foster was one of the most lovable of men. He was also a man of remarkably sound judgement, so that his opinion, when he would give it, was always welcome ; he had an instinct for discriminating between wise and foolish schemes or policies, and always lent the weight of his influence in what experience showed to be the right direction.

Originally a student of Chemistry, and of the more chemical side of Physics—his first paper was a purely chemical one—he trained himself to deal accurately with a wide range of physical knowledge, and to apply mathematics to it rather after the German model.

He was never a fluent lecturer : he thought too carefully over his sentences, and delivered them with a hesitancy that made him difficult to follow ; yet, when taken down, his lectures were a model of cautious exact exposition,—accurate and well-formed, like his hand-writing.

By senior and painstaking students he was much appreciated; and, though he made no claim to encyclopædic knowledge, his acquaintance with German treatises enabled him to direct enquirers to original sources, and to stimulate a certain type of research; while his natural bent led him to encourage exact measurement. He gave constant and conscientious attendance in the physical laboratory where the students worked; indeed in early days he allowed the more advanced ones to work in what should have been his private room, which he had fitted up with benches and glass cases, since no other place was available; and here he practically initiated a students' physical laboratory which must have been a pioneer institution in this Country—at any rate south of the Tweed.

He was very modest about his own work, and sometimes seemed rather surprised at the importance attached by physicists to his Paper before what was then the Society of Telegraph Engineers (now the Institution of Electrical Engineers) on the Wheatstone Bridge. But, at the time it was written, it was a most ingenious and instructive piece of work, and displayed a number of special features which anyone less critically thorough than himself would have overlooked. Undoubtedly he therein devised the method—not at all an obvious method—for extremely accurate comparison of standards of resistance, turning the bridge into an instrument for the measurement of differences instead as previously only of ratios, and thus importantly supplementing Wheatstone's or Christie's original design.

The principle of the Carey Foster Bridge was also used by him for a very accurate and convenient method of plotting the equipotential lines of a current flowing between electrodes situated anywhere in a plane conducting sheet. [See Phil. Mag. December 1875, a sequel to papers in May and June of the same year. A further paper in the

1876 volume was also the outcome of the work, though in that portion Foster did not join.]

His treatment of Thermodynamics, as in the Article 'Heat' of Watts's Dictionary of Chemistry, contained several interesting and suggestive features—at a time when that subject was comparatively in its infancy.

His interest and useful work in connexion with the Electrical Standards Committee of the British Association must be well known.

Many other things he did: but after all his desire to help senior students, and those members of the Staff who came under his personal influence, was perhaps the most prominent feature of his life and the one I have most pleasure in remembering: his house in the Hilldrop Road, Camden Town, was a centre of simple and kindly hospitality. Not at all likely to be deceived by mere plausibility, and of a very critical and shy disposition, he formed his judgement of men, and when it was favourable nothing that he could do to help them was too great a call upon his time.

Further details of his life and work are contained in an Article—presumably by his colleague and collaborator A. W. Porter, F.R.S.—which appeared in 'Nature' of Feb. 20, page 489.

Long and well did Carey Foster serve The Philosophical Magazine, and he must have acted as referee for an immense number of papers. Of Dr. William Francis senior he was an intimate friend, and gave him the benefit of counsel and support through difficult negotiations at a time when the policy of the Conductors of the Magazine was passing through what might have been a revolutionary period. He regarded the Magazine as one of the bulwarks of serious Physics in this Country, and exerted himself to preserve it practically in its ancient form.

He also took a leading part in the formation of the Physical Society of London: he and Professor Guthrie together were in my opinion its real founders. I was present at some of the preliminary discussions, and heard a good deal of what was going on. Both had apparently been co-students in Germany, and some of their reminiscences were very humorous.

Through the period of the seventies of last century it is not too much to say that Carey Foster was the leader of Physics in London; and a wiser, more cautious, and yet earnest and indefatigable man could not readily have been found.

He has lived to a good old age, and seen generations of his pupils take their place in the world of science. I for one gladly pay my tribute to the work and character of my old Chief. OLIVER J. LODGE.

22nd February, 1919.

THE
LONDON, EDINBURGH, AND DUBLIN
PHILOSOPHICAL MAGAZINE
AND
JOURNAL OF SCIENCE.

[SIXTH SERIES.]

APRIL 1919.

XXXI. *On the Problem of Random Vibrations, and of Random Flights in one, two, or three Dimensions.* By Lord RAYLEIGH, *O.M., F.R.S.*[*]

WHEN a number (n) of isoperiodic vibrations of unit amplitude are combined, the resultant depends upon the values assigned to the individual phases. When the phases are at random, the resultant amplitude is indeterminate, and all that can be said relates to the *probability* of various amplitudes (r), or more strictly to the probability that the amplitude lies within the limits r and $r+dr$. The important case where n is very great I considered a long time ago[†] with the conclusion that the probability in question is simply

$$\frac{2}{n}e^{-r^2/n}\,r\,dr. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (1)$$

The phase (θ) of the resultant is of course indeterminate, and all values are equally probable.

The method then followed began with the supposition that the phases of the unit components were limited to 0° and 180°, taken at random, so that the points (r, θ), representative of the vibrations, lie on the axis $\theta=0$, and indifferently on both sides of the origin. The resultant x, being the difference

---

[*] Communicated by the Author.

[†] Phil. Mag. vol. x. p. 73 (1880); Scientific Papers, vol. i. p. 491.

between the number of positive and negative components, is found from Bernoulli's theorem to have the probability

$$\frac{1}{\sqrt{(2\pi n)}} e^{-x^2/2n} dx. \quad . \quad . \quad . \quad . \quad . \quad . \qquad (2)^*$$

The next step was to admit also phases of 90° and 270°, the choice between these two being again at random. If we suppose $\frac{1}{2}n$ components at random along $\pm x$, and $\frac{1}{2}n$ also at random along $\pm y$, the chance of the representative point of the resultant lying within the area $dx\, dy$ is evidently

$$\frac{1}{\pi n} e^{-(x^2+y^2)/n} dx\, dy, \quad . \quad . \quad . \quad . \quad . \qquad (3)$$

or in terms of r, θ,

$$\frac{1}{\pi n} e^{-r^2/n} r\, dr\, d\theta. \quad . \quad . \quad . \quad . \quad . \quad . \qquad (4)$$

Thus all phases are equally probable, and the chance that the resultant amplitude lies between r and $r+dr$ is

$$\frac{2}{n} e^{-r^2/n} r\, dr. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (1)$$

This is the same as was before stated, but at present the conditions are limited to a distribution of precisely $\frac{1}{2}n$ components along x and a like number along y. It concerns us to remove this restriction, and to show that the result is the same when the distribution is perfectly arbitrary in respect to all four directions.

For this purpose let us suppose that $\frac{1}{2}n+m$ are distributed along $\pm x$ and $\frac{1}{2}n-m$ along $\pm y$, and inquire how far the result is influenced by the value of m. The chance of the representative point lying in $r\, dr\, d\theta$ is now expressed by

$$\frac{1}{\pi\sqrt{(n^2-4m^2)}} e^{-nr^2/(n^2-4m^2)} e^{-2mr^2 \cos 2\theta/(n^2-4m^2)} r\, dr\, d\theta.$$

Since r is of order $\sqrt{n}$, and m/n is small, the exponential containing θ may be expanded. Retaining the first four terms, we have on integration with respect to θ,

$$\frac{2r\, dr}{\sqrt{(n^2-4m^2)}} e^{-nr^2/(n^2-4m^2)} \left\{1+\frac{m^2r^4}{(n^2-4m^2)^2}+\ldots\right\},$$

as the chance of the amplitude lying between r and $r+dr$. Now if the distribution be entirely at random along the four

---

* See below.

directions, all the values of m of which there is a finite probability are of order not higher than $\sqrt{n}$, n being treated as infinite. But if m is of this order, the above expression becomes the same as if m were zero; and thus it makes no difference whether the number of components along $\pm x$ and along $\pm y$ are limited to be equal, or not. The previous result is accordingly applicable to a thoroughly arbitrary distribution along the four rectangular directions.

The next point to notice is that the result is symmetrical and independent of the directions of the rectangular axes, from which we may conclude that it has a still higher generality. If a total of n components, to be distributed along one set of rectangular axes, be divided into any number of large groups, it makes no difference whether we first obtain the probabilities of various resultants of the groups separately and afterwards of the final resultants, or whether we regard the whole n as one group. But the probability in each group is the same, notwithstanding a change in the system of rectangular axes; so that the probabilities of various resultants are unaltered, whether we suppose the whole number of components restricted to one set of rectangular axes or divided in any manner between any number of sets of axes. This last state of things is equivalent to no restriction at all; and we conclude that if n unit vibrations of equal pitch and of thoroughly arbitrary phases be compounded, then when n is very great the probability of various resultant amplitudes is given by (1).

If the amplitude of each component be l, instead of unity, as we have hitherto supposed for brevity, the probability of a resultant amplitude between r and $r+dr$ is

$$\frac{2}{\pi l^2} e^{-r^2/nl^2} r\,dr. \quad . \quad . \quad . \quad . \quad . \quad . \qquad (5)$$

In 'Theory of Sound,' 2nd edition, § 42*a* (1894), I indicated another method depending upon a transition from an equation in finite differences to a partial differential equation and the use of a Fourier solution. This method has the advantage of bringing out an important analogy between the present problems and those of gaseous diffusion, but the demonstration, though somewhat improved later*, was incomplete, especially in respect to the determination of a constant multiplier. At the present time it is hardly worth while to pursue it further, in view of the important improvements effected by Kluyver and Pearson. The latter

---

* Phil. Mag. vol. xlvii. p. 246 (1899); Scientific Papers, vol. v. p. 370.

was interested in the "Problem of the Random Walk," which he thus formulated:—"A man starts from a point O and walks l yards in a straight line; he then turns through any angle whatever and walks another l yards in a second straight line. He repeats this process n times. I require the probability that after these n stretches he is at a distance between r and $r+dr$ from his starting point O."

"The problem is one of considerable interest, but I have only succeeded in obtaining an integrated solution for *two* stretches. I think, however, that a solution ought to be found, if only in the form of a series in powers of $1/n$, when n is large"*. In response, I pointed out that this question is mathematically identical with that of the unit vibrations with phases at random, of which I had already given the solution for the case of n infinite†, the identity depending of course upon the vector character of the components.

In the present paper I propose to consider the question further with extension to *three* dimensions, and with a comparison of results for one, two, and three dimensions‡. The last case has no application to random vibrations but only to random *flights*.

One Dimension.

In this case the required information for any finite n is afforded by Bernoulli's theorem. There are $n+1$ possible resultants, and if we suppose the component amplitudes, or stretches, to be unity, they proceed by intervals of *two* from $+n$ to $-n$, values which are the largest possible. The probabilities of the various resultants are expressed by the corresponding terms in the expansion of $(\frac{1}{2}+\frac{1}{2})^n$. For instance the probabilities of the extreme values $\pm n$ are $(1/2)^n$. And the probability of a combination of a positive and b negative components is

$$(\tfrac{1}{2})^n \frac{n!}{a!\,b!}, \quad \ldots \ldots \quad (6)$$

in which $a+b=n$, making the resultant $a-b$. The largest values of (6) occur in the middle of the series, and here a distinction arises according as n is even or odd. In the

---

* 'Nature,' vol. lxxii. p. 294 (1905).

† 'Nature,' vol. lxxii. p. 318 (1905); Scientific Papers, vol. v. p. 256.

‡ It will be understood that we have nothing here to do with the direction in which the vibrations take place, or are supposed to take place. If that is variable, there must first be a resolution in fixed directions, and it is only after this operation that our present problems arise.

former alternative there is a unique middle term when $a=b=\frac{1}{2}n$; but in the latter a and b cannot be equated, and there are two equal middle terms corresponding to $a=\frac{1}{2}n+\frac{1}{2}$, $b=\frac{1}{2}n-\frac{1}{2}$, and to $a=\frac{1}{2}n-\frac{1}{2}$, $b=\frac{1}{2}n+\frac{1}{2}$. The values of the second fraction in (6) are the series of integers in what is known as the "arithmetical triangle."

We have now to consider the values of

$$\frac{n\,!}{a\,!\,b\,!} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (7)$$

to be found in the neighbourhood of the middle of the series. If n be even, the value of the term counted s onwards from the unique maximum is

$$\frac{n\,!}{(\frac{1}{2}n-s)\,!\,(\frac{1}{2}n+s)\,!}. \quad . \quad . \quad . \quad . \quad . \quad . \qquad (8)$$

If n be odd, we have to choose between the two middle terms. Taking for instance, $a=\frac{1}{2}n+\frac{1}{2}$, $b=\frac{1}{2}n-\frac{1}{2}$, the sth term onwards is

$$\frac{n\,!}{\{\frac{1}{2}n-(s-\frac{1}{2})\}\,!\,\{\frac{1}{2}n+(s-\frac{1}{2})\}\,!}. \quad . \quad . \quad . \qquad (9)$$

The expressions (8) and (9) are brought into the same form when we replace s by the resultant amplitude x. When n is even, $x=-2s$; when s is odd, x is $-2(s-\frac{1}{2})$, so that in both cases we have on restoration of the factor $(\frac{1}{2})^n$

$$\frac{n\,!}{2^n.(\frac{1}{2}n-\frac{1}{2}x)\,!\,(\frac{1}{2}n+\frac{1}{2}x)\,!}. \quad . \quad . \quad . \qquad (10)$$

The difference is that when n is even, x has the $(n+1)$ values

$$0, \qquad \pm 2, \qquad \pm 4, \qquad \pm 6, \ldots . \pm n;$$

and when n is odd, the $(n+1)$ values

$$\pm 1, \qquad \pm 3, \qquad \pm 5, \ldots . \pm n.$$

The expression (10) may be regarded as affording the complete solution of the problem proposed; it expresses the probability of any one of the possible resultants, but for practical purposes it requires transformation when we contemplate a very great n.

The necessary transformation can be obtained after Laplace with the aid of Stirling's theorem. The process is detailed in Todhunter's 'History of the Theory of Probability,' p. 548, but the corrections to the principal term there exhibited (of the first order in x) do not appear here where the probabilities of the *plus* and *minus* alternatives are equal. On account of the symmetry, no odd powers of x can occur. I have calculated the resulting expression with retention of the terms which are of the order $1/n^2$ in comparison with the principal term. The resultant x itself may be considered to be of order not higher than $\sqrt{n}$.

By Stirling's theorem

$$n\,! = \sqrt{(2\pi)}\,n^{n+\frac{1}{2}}e^{-n}C_n, \quad \ldots\ldots \quad (11)$$

where

$$C_n = 1 + \frac{1}{12n} + \frac{1}{288n^2} + \ldots, \quad \ldots\ldots \quad (12)$$

with similar expressions for $(\tfrac{1}{2}n - \tfrac{1}{2}x)\,!$ and $(\tfrac{1}{2}n + \tfrac{1}{2}x)\,!$ For the moment we omit the correcting factors C. Thus

$$\frac{1}{(\frac{1}{2}n-\frac{1}{2}x)!\,(\frac{1}{2}n+\frac{1}{2}x)!} = \frac{e^n}{2\pi}\left(\frac{n}{2}\right)^{-n-1}\left(1-\frac{x^2}{n^2}\right)^{-\frac{1}{2}n-\frac{1}{2}}\left(\frac{1-x/n}{1+x/n}\right)^{\frac{1}{2}x}.$$

For the logarithm of the product of the two last factors, we have

$$\frac{n+1}{2}\left\{\frac{x^2}{n^2}+\frac{x^4}{2n^4}+\frac{x^6}{3n^6}+\ldots\right\}-\frac{x^2}{n}-\frac{x^4}{3n^3}-\frac{x^6}{5n^5}-\ldots.$$

$$= -\frac{x^2}{2n}+\frac{x^2}{2n^2}-\frac{x^4}{4n^3}\left(\frac{1}{3}-\frac{1}{n}\right)-\frac{x^6}{6n^5}\left(\frac{1}{5}-\frac{1}{n}\right)-\ldots,$$

and for the product itself

$$e^{-x^2/2n}\left\{1+\frac{1}{2n}\left(\frac{x^2}{n}-\frac{x^4}{6n^2}\right)+\frac{1}{8n^2}\left(\frac{3x^4}{n^2}-\frac{3x^6}{5n^3}+\frac{x^8}{36n^4}\right)\right\}. \quad (13)$$

The principal term in (10) is

$$\frac{\sqrt{(2\pi)}\,.\,n^{n+\frac{1}{2}}e^{-n}}{2^n}\cdot\frac{e^n}{2\pi}\left(\frac{n}{2}\right)^{-n-1}e^{-x^2/2n} = \sqrt{\left(\frac{2}{n\pi}\right)}\,e^{-x^2/2n}.$$

There are still the factors C to be considered. We have

$$\frac{C_n}{C_{\frac{1}{2}(n-x)}C_{\frac{1}{2}(n+x)}} = \left\{1+\frac{1}{12n}+\frac{1}{288n^2}\right\}$$

$$\left\{1+\frac{1}{6(n-x)}+\frac{1}{72(n-x)^2}\right\}^{-1}\left\{1+\frac{1}{6(n+x)}+\frac{1}{72(n+x)^2}\right\}^{-1}$$

$$= \left\{1+\frac{1}{12n}+\frac{1}{288n^2}\right\}\left\{1-\frac{1}{3n}+\frac{1}{3n^2}\left(\frac{1}{6}-\frac{x^2}{n}\right)\right\}$$

$$= 1-\frac{1}{4n}+\frac{1}{32n^2}\left(1-\frac{32x^2}{3n}\right). \quad \ldots\ldots\ldots \quad (14)$$

Finally we obtain

$$\sqrt{\left(\frac{2}{n\pi}\right)}\, e^{-x^2/2n} \left\{1 - \frac{1}{4n}\left(1 - \frac{2x^2}{n} + \frac{x^4}{3n^2}\right) + \frac{1}{32n^2}\left(1 - \frac{44x^2}{3n} + \frac{38x^4}{3n^2} - \frac{12x^6}{5n^3} + \frac{x^8}{9n^4}\right)\right\}, \quad . \quad (15)$$

as the probability when n is large of the resultant amplitude x. It is to be remembered that x is limited to a series of discrete values with a common difference equal to 2, and that our approximation has proceeded upon the supposition that x is not of higher order than $\sqrt{n}$.

If the component amplitudes or stretches be l, in place of unity, we have merely to write x/l in place of x.

The special value of the series (15) is realized only when n is very great. But it affords a closer approximation to the true value than might be expected when n is only moderate. I have calculated the case of $n=10$, both directly from the exact expression (10) and from the series (15) for all the admissible values of x.

TABLE I.

$n=10$.

x.	From (10).	From (15).
0	·24609	·24608
2	·20508	·20509
4	·11719	·11722
6	·04394	·04392
8	·00977	·00975
10	·00098	·00102

The values for $x=0$ and twice those belonging to higher values of x should total unity. Those above from (10) give 1·00001 and those from (15) give 1·00008. It will be seen that except in the extreme case of $x=10$, the agreement between the two formulæ is very close. But, even for much higher values of n, the actual calculation is simpler from the exact formula (10).

When l is very small, while n is very great, we may be able for some purposes to disregard the discontinuous character of the probability as a function of x, replacing the isolated points by a continuous representative curve. The difference between the abscissæ of consecutive isolated points is $2l$; so that if dx be a large multiple of l, we may take

$$\sqrt{\left(\frac{1}{2n\pi}\right)}e^{-x^2/2nl^2}dx/l, \quad . \quad . \quad . \quad . \quad (16)$$

as the approximate expression of the probability that the resultant amplitude lies between x and $x+dx$.

Two Dimensions.

If there is but one stretch of length l, the only possible value of r is of course l.

When there are two stretches of lengths l_1 and l_2, r may vary from l_2-l_1 to l_2+l_1, and then if θ be the angle between them

$$r^2=l_1{}^2+l_2{}^2-2l_1l_2\cos\theta, \quad . \quad . \quad . \quad . \quad (17)$$

and

$$\sin\theta\, d\theta=r\,dr/l_1l_2. \quad . \quad . \quad . \quad . \quad . \quad . \quad (18)$$

Since all angles θ between 0 and π are deemed equally probable, the chance of an angle between θ and $\theta+d\theta$ is $d\theta/\pi$. Accordingly the chance that the resultant r lies between r and $r+dr$ is

$$\frac{r\,dr}{\pi l_1l_2\sin\theta}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (19)$$

or if with Prof. Pearson * we refer the probability to unit of area in the plane of representation,

$$\begin{aligned}\phi_2(r^2)&=\frac{1}{2\pi^2l_1l_2\sin\theta}\\ &=\frac{1}{\pi^2\sqrt{\{2r^2(l_1{}^2+l_2{}^2)-r^4-(l_1{}^2-l_2{}^2)^2\}}}, \quad . \quad . \quad (20)\end{aligned}$$

$\phi_2(r^2)dA$ denoting the chance of the representative point lying in a small area dA at distance r from the origin.

If the stretches l_1 and l_2 are equal, (20) reduces to

$$\phi_2(r^2)=\frac{1}{\pi^2r\sqrt{\{4l^2-r^2\}}}, \quad . \quad . \quad . \quad . \quad . \quad (21)$$

Prof. Pearson's expression, applicable when $r<2l$. When $r>2l$, $\phi_2(r^2)=0$.

When there are three equal stretches ($n=3$), $\phi_3(r^2)$ is

---

* Drapers' Company Research Memoirs, Biometric Series III., London, 1906.

expressible by elliptic functions[*] with a discontinuity in form as r passes through l.

For values of n from 4 to 7 inclusive, Pearson's work is founded upon the general functional relation[†]

$$\phi_{n+1}(r^2)=\frac{1}{\pi}\int_0^\pi \phi_n(r^2+l^2-2rl\cos\theta)d\theta. \quad . \quad . \quad (22)$$

Putting $r=0$, he deduces the special conclusion that

$$\phi_{n+1}(0)=\phi_n(l^2), \quad . \quad . \quad . \quad . \quad . \quad . \quad (23)$$

as is indeed evident *a priori*.

From (22) the successive forms are determined graphically. For values of n higher than 7 an analytical expression proceeding by powers of $1/n$ is available, and will be further referred to later.

A remarkable advance in the theory of random vibrations and of flights in two dimensions, when the number (n) is finite, is due to J. C. Kluyver[‡], who has discovered an expression for the probability of various resultants in the form of a definite integral involving Bessel's functions. His exposition is rather concise, and I think I shall be doing a service in reproducing it with some developments and slight changes of notation. It depends upon the use of a discontinuous integral evaluated by Weber, viz.

$$\int_0^\infty \mathrm{J}_1(bx)\mathrm{J}_0(ax)dx=u \text{ (say)}.$$

To examine this we substitute from

$$\pi\,.\,\mathrm{J}_1(bx)=2\int_0^{\frac{1}{2}\pi}\cos\theta\sin(bx\cos\theta)d\theta \text{ §},$$

and take first the integration with respect to x. We have ||

$$\int_0^\infty dx\sin(bx\cos\theta)\mathrm{J}_0(ax)=0, \qquad \text{if } a^2>b^2\cos^2\theta,$$

or

$$=(b^2\cos^2\theta-a^2)^{-\frac{1}{2}}, \qquad \text{if } b^2\cos^2\theta>a^2,$$

Thus, if $a^2>b^2$, $u=0$. If $b^2>a^2$,

$$u=\frac{2}{\pi}\int\frac{d\theta\cos\theta}{\sqrt{(b^2\cos^2\theta-a^2)}}=\frac{2}{\pi b}\sin^{-1}\frac{b\sin\theta}{\sqrt{(b^2-a^2)}}.$$

---

[*] Pearson (*l. c.*) attributes this evaluation to G. T. Bennett.

[†] Compare 'Theory of Sound,' § 42 *a*.

[‡] Amsterdam Proceedings, vol. viii. p. 341 (1905).

[§] Gray and Matthews, 'Bessel's Functions,' p. 18, equation (46).

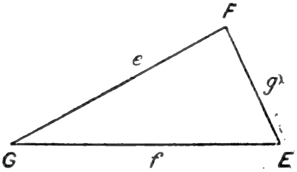
[||] G. and M. p. 73.

The lower limit for θ is 0, and the upper limit is given by $\cos^2\theta = a^2/b^2$. Hence $u=1/b$, and thus

$$\left.\begin{aligned} b\int_0^\infty J_1(bx)J_0(ax)dx &= 1, \quad (b^2>a^2) \\ \text{or} \qquad\qquad &= 0, \quad (a^2>b^2) \end{aligned}\right\} \quad . \quad . \quad (24)$$

A second lemma required is included in Neumann's theorem, and may be very simply arrived at. In fig. 1,

Fig. 1.



G and E being fixed points, the function at F denoted by

$$J_0(g), \quad \text{or} \quad J_0\sqrt{(e^2+f^2-2ef\cos G)},$$

is a potential satisfying everywhere the equation $\nabla^2+1=0$, and accordingly may be expanded round G in the Fourier series

$$A_0J_0(e) + A_1J_1(e)\cos G + A_2J_2(e)\cos 2G + \ldots,$$

the coefficients A being independent of e and G. Thus

$$\frac{1}{2\pi}\int_0^{2\pi} J_0\sqrt{(e^2+f^2-2ef\cos G)}dG = A_0J_0(e).$$

By parity of reasoning when E and F are interchanged, the same integral is proportional to $J_0(f)$, and may therefore be equated to $A_0'J_0(e)J_0(f)$, where A_0' is now an absolute constant, whose value is at once determined to be unity by making e, or f, vanish. The lemma

$$\int_0^{2\pi} J_0\sqrt{(e^2+f^2-2ef\cos G)}dG = 2\pi J_0(e)J_0(f), \quad (25)$$

is thus established *.

---

* Similar reasoning shows that if $D_0(g)$ represent a symmetrical purely divergent wave,

$$\int_0^{2\pi} D_0\sqrt{(e^2+f^2-2ef\cos G)}dG = 2\pi J_0(e)D_0(f);$$

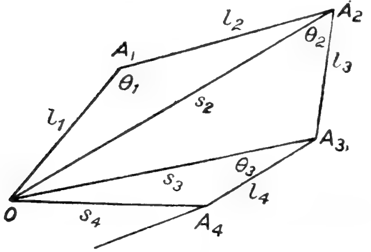
provided that $f>e$.

We are now prepared to investigate the probability

$$P_n(r\,;\ l_1,\ l_2,\ \dots\ l_n)$$

that after n stretches $l_1, l_2, \dots l_n$ taken in directions at random the distance from the starting-point O (fig. 2), shall be less

Fig. 2.



than an assigned magnitude r. The direction of the first stretch l_1 is plainly a matter of indifference. On the other hand the probability that the angles θ lie within the limits θ_1 and $\theta_1+d\theta_1$, θ_2 and $\theta_2+d\theta_2$, … θ_{n-1} and $\theta_{n-1}+d\theta_{n-1}$ is

$$\frac{1}{(2\pi)^{n-1}}\,d\theta_1\,d\theta_2\,\dots\dots d\theta_{n-1}, \quad . \quad . \quad . \quad (26)$$

which is now to be integrated under the condition that the nth radius vector s_n shall be less than r.

Let us commence with the case of two stretches l_1 and l_2. Then

$$P_2(r\,;\ l_1,\ l_2)=\frac{1}{2\pi}\int d\theta_1,$$

the integration being taken within such limits that $s_2<r$, where

$$s_2{}^2=l_1{}^2+l_2{}^2-2l_1l_2\cos\theta_1.$$

The required condition as to the limits can be secured by the introduction of the discontinuous function afforded by Weber's integral. For

$$r\int_0^\infty J_1(rx)\,J_0(s_2x)\,dx$$

vanishes when $s_2>r$, and is equal to unity when $s_2<r$. After the introduction of this factor, the integration with respect to θ_1 may be taken over the complete range from 0 to 2π. Thus

$$P_2(r\,;\ l_1,\ l_2)=\frac{r}{2\pi}\int_0^{2\pi}d\theta_1\int_0^\infty dx\,J_1(rx)\,J_0(s_2x).$$

Taking first the integration with respect to θ_1, we have by (25)

$$\frac{1}{2\pi}\int_0^{2\pi} d\theta_1 \mathrm{J}_0(s_2 x) = \mathrm{J}_0(l_1 x)\mathrm{J}_0(l_2 x),$$

and thus

$$\mathrm{P}_2(r\,;\,l_1,\,l_2) = r\int_0^{\infty} dx\,\mathrm{J}_1(rx)\mathrm{J}_0(l_1 x)\mathrm{J}_0(l_2 x). \quad . \quad (27)$$

The method can be extended to any number (n) of stretches. Beginning with the integration with respect to θ_{n-1} in (26), we have as before

$$\frac{1}{2\pi}\int d\theta_{n-1} = \frac{r}{2\pi}\int_0^{2\pi} d\theta_{n-1}\int_0^{\infty} dx\,\mathrm{J}_1(rx)\mathrm{J}_0(s_n x)$$

$$= r\int_0^{\infty} dx\,\mathrm{J}_1(rx)\mathrm{J}_0(l_n x)\mathrm{J}_0(s_{n-1}x).$$

The next integration gives

$$\frac{1}{(2\pi)^2}\iint d\theta_{n-2}d\theta_{n-1} = r\int_0^{\infty} \mathrm{J}_1(rx)\mathrm{J}_0(l_n x)\,\mathrm{J}_0(l_{n-1}x)\mathrm{J}_0(s_{n-2}x)dx,$$

and so on. Finally

$$\mathrm{P}_n(r\,;\,l_1,\,l_2,\,\ldots\,l_n) = \frac{1}{(2\pi)^{n-1}}\iint\ldots d\theta_1 d\theta_2\ldots d\theta_{n-1}$$

$$= r\int_0^{\infty} \mathrm{J}_1(rx)\mathrm{J}_0(l_1 x)\mathrm{J}_0(l_2 x)\,\ldots\,\mathrm{J}_0(l_n x)\,dx, \quad . \quad (28)$$

—the expression for P_n discovered by Kluyver.

It will be observed that (28) is symmetrical with respect to the l's ; the order in which they are taken is immaterial.

When all the l's are equal,

$$\mathrm{P}_n(r\,;\,l) = r\int_0^{\infty} \mathrm{J}_1(rx)\{\mathrm{J}_0(lx)\}^n dx. \quad . \quad . \quad (29)$$

If in (29) we suppose $r = l$,

$$\mathrm{P}_n(l\,;\,l) = -\int_0^{\infty} \{\mathrm{J}_0(lx)\}^n d\,\mathrm{J}_0(lx)$$

$$= -\left.\frac{\{\mathrm{J}_0(lx)\}^{n+1}}{n+1}\right|_0^{\infty} = \frac{1}{n+1}; \quad . \quad . \quad . \quad (30)$$

so that after n equal components have been combined the chance that the resultant shall be less than one of the components is $1/(n+1)$, an interesting result due to Kluyver. The same author notices some of the discontinuities which present themselves, but it will be more convenient to consider this in a modified form of the problem.

The modification consists in dealing, not with the chance of a resultant less than r, but with the chance that it lies between r and $r+dr$. It may seem easy to pass from the one to the other, as it involves merely a differentiation with respect to r. We have

$$\frac{d}{dr}\{r\mathrm{J}_1(rx)\} = -\frac{d}{dr}\{r\mathrm{J}_0'(rx)\}$$
$$= -\mathrm{J}_0'(rx) - rx\,\mathrm{J}_0''(rx) = rx\,\mathrm{J}_0(rx),$$

in virtue of the differential equation satisfied by J_0. Thus, if the differentiation under the integral sign is legitimate,

$$\frac{d\mathrm{P}_n}{dr} = 2\pi r\,\phi_n(r^2) = r\int_0^\infty x\,dx\,\mathrm{J}_0(rx)\mathrm{J}_0(l_1x)\mathrm{J}_0(l_2x)\ldots\mathrm{J}_0(l_nx)\ldots, \quad (31)$$

and, if all the l's are equal,

$$\phi_n(r^2) = \frac{1}{2\pi}\int_0^\infty x\,dx\,\mathrm{J}_0(rx)\{\mathrm{J}_0(lx)\}^n, \quad . \quad . \quad (32)$$

the form employed by Pearson, whose investigation is by a different method. If we put $n=1$ in (32),

$$\phi_1(r^2) = \frac{1}{2\pi}\int_0^\infty x\,dx\,\mathrm{J}_0(rx)\mathrm{J}_0(lx), \quad . \quad . \quad . \quad (33)$$

and this is in fact the equation from which Pearson starts. But it should be remarked that the integral (33), as it stands, is *not convergent*. For when z is very great,

$$\mathrm{J}_0(z) = \sqrt{\left(\frac{2}{\pi z}\right)}\cos\left(\frac{1}{4}\pi - z\right), \quad . \quad . \quad . \quad (34)$$

so that $(r \neq 0)$

$$\frac{1}{2\pi}\int^x x\,dx\,\mathrm{J}_0(rx)\mathrm{J}_0(lx) = \frac{1}{2\pi^2\sqrt{(rl)}}\int^x dx$$
$$\{\sin(r+l)x + \cos(r-l)x\},$$

and this is not convergent when $x=\infty$.

The criticism does not apply to (29) itself when $n=1$, but it leads back to the question of differentiation under the sign of integration. It appears at any rate that any number of such operations can be justified, provided that the integrals, resulting from these and the next following operation, are finite for the values of r in question. But this condition is not satisfied in the differentiation under the integral sign of (29) when $n=1$. For the next operation upon (32) then yields

$$\int_0 x^2\,dx\,\mathrm{J}_1(rx)\mathrm{J}_0(lx).$$

When we substitute for $J_0(lx)$ from (34) and for $J_1(rx)$ from

$$J_1(z)=\sqrt{\left(\frac{2}{\pi z}\right)}\cos\left(\frac{3\pi}{4}-z\right),$$

we get

$$\int^x x\,dx\cos\left(\frac{3\pi}{4}-rx\right)\cos\left(\frac{\pi}{4}-lx\right),$$

which becomes infinite with x, even for general values of r and l.

So much by way of explanation ; but of course we do not really need to discuss the cases $n=1$, $n=2$, or even $n=3$, for which exact solutions can be expressed in terms of functions which may be regarded as known.

For higher values of n it would be of interest to know how many differentiations with respect to r may be made under the sign of integration. It may be remarked that since all J's and their derivatives to any order are less than unity, the integral can become infinite only in virtue of that part of the range where x is very great, and that there we may introduce the asymptotic values.

We have thus to consider

$$\frac{d^p}{dr^p}\phi_n(r^2)=\frac{1}{2\pi}\int_0^\infty dx\,x^{p+1}J_0{}^p(rx)\{J_0(lx)\}^n. \quad . \quad (35)$$

For the leading term when z is very great, we have

$$J_0{}^p(z)=\frac{d^p}{dz^p}\left\{\sqrt{\left(\frac{2}{\pi z}\right)}\cos\left(\frac{1}{4}\pi-z\right)\right\}$$

$$=\sqrt{\left(\frac{2}{\pi z}\right)}\cos\left(\frac{1}{4}\pi-z-\frac{1}{2}p\pi\right), \quad . \quad . \quad (36)$$

$$\{J_0(z)\}^n=\left(\frac{2}{\pi z}\right)^{\frac{1}{2}n}\cos^n\left(\frac{1}{4}\pi-z\right), \quad . \quad . \quad . \quad . \quad . \quad (37)$$

so that with omission of constant factors our integral becomes

$$\int^\infty dx\,x^{p+\frac{1}{2}-\frac{1}{2}n}\cos\left(\frac{1}{4}\pi-rx-\frac{1}{2}p\pi\right)\cos^n\left(\frac{1}{4}\pi-lx\right). \quad (38)$$

In this $\cos^n(\frac{1}{4}\pi-lx)$ can be expanded in a series of cosines of multiples of $(\frac{1}{4}\pi-lx)$, commencing with $\cos n(\frac{1}{4}\pi-lx)$ and ending when n is odd with $\cos(\frac{1}{4}\pi-lx)$, and when n is even with a constant term. The various products of cosines are then to be replaced by cosines of sums and differences. The most unfavourable case occurs when this operation leaves a constant term, which can happen only for values of r which are multiples of l. We are then left with

$$\int^{\infty} dx\, x^{p+\frac{1}{2}-\frac{1}{2}n} = \left.\frac{x^{p+\frac{3}{2}-\frac{1}{2}n}}{p+\frac{3}{2}-\frac{1}{2}n}\right|^{\infty}.$$

The integral is thus finite or infinite according as

$$p < \text{ or } > \tfrac{1}{2}(n-3).$$

If, however, there arise no constant term, we have to consider

$$\int^{\infty} dx\, x^{s} \cos mx = \left.\frac{x^{s}}{m}\sin mx\right|^{\infty} - \frac{s}{m^2}\int^{\infty} dx\, x^{s-1}\sin mx,$$

where m is finite; and this is finite if s, that is $p+\frac{1}{2}-\frac{1}{2}n$, be negative. The differentiations are then valid, if

$$p < \tfrac{1}{2}(n-1).$$

We may now consider more especially the cases $n=4$, &c. When $n=4$, $s=p+\frac{1}{2}-\frac{1}{2}n=p-\frac{3}{2}$.

If $p=1$, $s=-\frac{1}{2}$, and the cosine factors in (38) become

$$\cos\left(\tfrac{1}{4}\pi+rx\right)\cos^4\left(\tfrac{1}{4}\pi-lx\right),$$

yielding finally

$$\cos\left(\frac{5\pi}{4}+rx-4lx\right),\ \cos\left(\frac{3\pi}{4}-rx-4lx\right),$$

$$\cos\left(\frac{3\pi}{4}+rx-2lx\right),\ \cos\left(\frac{\pi}{4}-rx-2lx\right),\ \cos\left(\frac{\pi}{4}+rx\right),$$

so that there is no constant term unless $r=4l$, or $2l$. With these exceptions, the original differentiation under the integral sign is justified.

We fall back upon ϕ_4 itself by putting $p=0$, making $s=-\frac{3}{2}$. The integral is then finite in all cases ($r\neq 0$), in agreement with Pearson's curve.

Next for $n=5$, $s=p-2$.

When $p=1$, $s=-1$, and we find that the cosine factors yield a constant term only when $r=3l$. Pearson's curve does not suggest anything special at $r=3l$; it may be remarked that the integral with $p=1$ is there only logarithmically infinite.

If $n=5$, $p=0$, $s=-2$; and the integral for ϕ_5 is finite for all values of r.

When $n=6$, $s=p-2\frac{1}{2}$. In this case, whether $p=1$, or 0, no question can arise. The integrals are finite for all values of r.

A fortiori is this so, when $n>6$.

If we suppose $p=2$, $s=\frac{1}{2}(5-n)$. Thus $n=7$ makes $s=-1$, and infinities might occur for special values of r. But if $n>7$, $s<\frac{3}{2}$, and infinities are excluded whatever may be the value of r.

Similarly if $p=3$, infinities are excluded if $n>9$, and so on.

Our discussion has not yielded all that could be wished; the subject may be commended to those better versed in pure mathematics. Probably what is required is a better criterion as to the differentiation under the integral sign.

We may now pass on to consider what becomes of Kluyver's integral when n is made infinite. As already remarked, Pearson has developed for it a series proceeding by powers of $1/n$, and it may be convenient to give a version of his derivation, without, however, carrying the process so far.

The evaluation of the principal term depends upon a formula due, I think, to Weber *, viz.

$$u=\int_0^\infty \mathrm{J}_0(rx)e^{-p^2x^2}x\,dx=\frac{1}{2p^2}e^{-r^2/4p^2}, \quad . \quad . \quad . \quad (39)$$

making †

$$\frac{du}{dr}=\int_0^\infty \mathrm{J}_0'(rx)e^{-p^2x^2}x^2\,dx=-\frac{1}{2p^2}\int_0^\infty \mathrm{J}_0'(rx)x\,de^{-p^2x^2}$$

$$=\frac{1}{2p^2}\int_0^\infty e^{-p^2x^2}\{\mathrm{J}_0'(rx)+rx\mathrm{J}_0''(rx)\}dx$$

$$=-\frac{r}{2p^2}\int_0^\infty \mathrm{J}_0(rx)e^{-p^2x^2}x\,dx=-\frac{r}{2p^2}u.$$

Hence $$u=\mathrm{C}e^{-r^2/4p^2}.$$

To determine C we have merely to make $r=0$. Thus

$$\mathrm{C}=u_{r=0}=\int_0^\infty e^{-p^2x^2}x\,dx=\frac{1}{2p^2},$$

by which (39) is established.

Unless lx is small, the factor $\{\mathrm{J}_0(lx)\}^n$ in (32) diminishes rapidly as n increases, inasmuch as $\mathrm{J}_0(lx)$ is less than unity for any finite lx. Thus when n is very great, the important part of the range of integration corresponds to a small lx.

---

[*] Gray and Matthews, *loc. cit.* p. 77.

[†] I apprehend that there can be no difficulty here as to the differentiation, the situation being dominated by the exponential factor.

Writing s for $\frac{1}{2}nl^2$, we have

$$\log \mathrm{J}_0(lx) = \log\left(1 - \frac{sx^2}{2n} + \frac{s^2x^4}{16n^2} - \frac{s^3x^6}{288n^3} + \dots\right)$$

$$= -\frac{sx^2}{2n} - \frac{s^2x^4}{16n^2} - \frac{s^3x^6}{72n^3} + \dots;$$

so that

$$\{\mathrm{J}_0(lx)\}^n = e^{-\frac{1}{2}sx^2}\left(1 - \frac{s^2x^4}{16n} - \frac{s^3x^6}{72n^2} + \frac{s^4x^8}{512n^2}\right),$$

making

$$2\pi\phi_n(r^2) = \int_0^\infty x\,dx\,\mathrm{J}_0(rx)e^{-\frac{1}{2}sx^2}\left(1 - \frac{s^2x^4}{16n} - \frac{s^3x^6}{72n^2} + \frac{s^4x^8}{512n^2}\right). \quad (40)$$

Calling the four integrals on the right I_1, I_2, I_3, and I_4, we have by (39)

$$\mathrm{I}_1 = \int_0^\infty x\,dx\,\mathrm{J}_0(rx)e^{-\frac{1}{2}sx^2} = \frac{1}{s}e^{-r^2/2s}, \quad . \quad . \quad . \quad (41)$$

$$-\mathrm{I}_2 = \frac{s^2}{4n}\frac{d^2\mathrm{I}_1}{ds^2} = \frac{s^2}{4n}\frac{d^2}{ds^2}\left(\frac{1}{s}e^{-r^2/2s}\right), \quad . \quad . \quad . \quad (42)$$

$$\mathrm{I}_3 = \frac{s^3}{9n^2}\frac{d^3\mathrm{I}_1}{ds^3} = \frac{s^3}{9n^2}\frac{d^3}{ds^3}\left(\frac{1}{s}e^{-r^2/2s}\right), \quad . \quad . \quad (43)$$

$$\mathrm{I}_4 = \frac{s^4}{32n^2}\frac{d^4\mathrm{I}_1}{ds^4} = \frac{s^4}{32n^2}\frac{d^4}{ds^4}\left(\frac{1}{s}e^{-r^2/2s}\right). \quad . \quad (44)$$

Thus

$$2\pi\phi_n(r^2) = \frac{e^{-r^2/2s}}{s}\left\{1 - \frac{1}{4n}\left(2 - \frac{2r^2}{s} + \frac{r^4}{4s^2}\right)\right.$$

$$- \frac{1}{9n^2}\left(6 - \frac{9r^2}{s} + \frac{9r^4}{4s^2} - \frac{r^6}{8s^3}\right)$$

$$\left. + \frac{1}{32n^2}\left(24 - \frac{48r^2}{s} + \frac{18r^4}{s^2} - \frac{2r^6}{s^3} + \frac{r^8}{16s^4}\right)\right\}$$

$$= \frac{e^{-r^2/2s}}{s}\left\{1 - \frac{1}{4n}\left(2 - \frac{2r^2}{s} + \frac{r^4}{4s^2}\right) + \frac{1}{12n^2}\left(1 - \frac{6r^2}{s}\right.\right.$$

$$\left.\left. + \frac{15r^4}{4s^2} - \frac{7r^6}{12s^3} + \frac{3r^8}{128s^4}\right)\right\}, \quad . \quad . \quad . \quad . \quad . \quad . \quad (45)$$

in agreement (so far as it goes) with Pearson, whose σ^2 is equal to our s. The leading term is that given in 1880.

Three Dimensions.

We may now pass on to the corresponding problem when flights take place in three dimensions, where we shall find, as might have been expected, that the mathematics are simpler. And first for *two* flights of length l_1 and l_2. If μ be the cosine of the angle between l_1 and l_2 and r the resultant,

$$r^2=l_1{}^2+l_2{}^2-2l_1l_2\mu,$$

giving

$$rdr=-l_1l_2d\mu. \quad . \quad . \quad . \quad . \quad . \quad . \quad (46)$$

The chance of r lying between r and $r+dr$ is the same as the chance of μ lying between μ and $\mu+d\mu$, that is $-\frac{1}{2}d\mu$, since all directions in space are to be treated as equally probable. Accordingly the chance of a resultant between r and $r+dr$ is

$$\frac{r\,dr}{2l_1l_2}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (47)$$

The corresponding volume is $4\pi r^2dr$, so that in the former notation

$$\phi_2(r\,;\,l)=\frac{1}{8\pi l^2r}, \quad . \quad . \quad . \quad . \quad . \quad . \quad (48)$$

l_1 and l_2 being supposed equal. It will be seen that this is simpler than (21). It applies, of course, only when $r<2l$. When $r>2l$, $\phi_2=0$.

In like manner when l_1 and l_2 differ, the chance of a resultant less than r is zero, when r falls short of the difference between l_2 and l_1, say l_2-l_1. Between l_2-l_1 and l_2+l_1 the chance is

$$\int_{l_2-l_1}^{r}\frac{r\,dr}{2l_1l_2}=\frac{r^2-(l_2-l_1)^2}{4l_1l_2}. \quad . \quad . \quad . \quad . \quad . \quad (49)$$

When r has its greatest value (l_2+l_1), (49) becomes

$$\frac{(l_2+l_1)^2-(l_2-l_1)^2}{4l_1l_2}=1. \quad . \quad . \quad . \quad . \quad . \quad (50)$$

The "chance" is then a certainty, as also when $r>l_1+l_2$.

In proceeding to the general value of n, we may conveniently follow the analogy of the two-dimensional investigation of Kluyver, for which purpose we require a function that shall be unity when $s<r$, and zero when $s>r$. Such a function is

$$\frac{2}{\pi}\int_0^\infty dx\frac{\sin sx}{sx}\,\frac{\sin rx-rx\cos rx}{x}; \quad . \quad . \quad (51)$$

for it may be written

$$-\frac{2r}{\pi s}\int_0^\infty \sin sx\, d\left(\frac{\sin rx}{rx}\right)=\frac{2}{\pi}\int_0^\infty \frac{\sin rx}{x}\cos sx\, dx$$

$$=\frac{1}{\pi}\int_0^\infty \frac{\sin(s+r)x-\sin(s-r)x}{x}\,dx = 1 \text{ or } 0,$$

according as s is less or greater than r.

In like manner for a second lemma, corresponding with (25), we may reason again from the triangle GFE (fig. 1). $J_0(g)$ is replaced by $\sin g/g$, a potential function symmetrical in three dimensions about E and satisfying *everywhere* $\nabla^2+1=0$. It may be expanded about G in Legendre's series *

$$A_0\frac{\sin e}{e}+A_1\mu\left(\frac{\sin e}{e^2}-\frac{\cos e}{e}\right)+\ldots,$$

μ being written for cos G, and accordingly

$$\tfrac{1}{2}\int_{-1}^{+1} d\mu\,\frac{\sin\sqrt{(e^2+f^2-2ef\mu)}}{\sqrt{(e^2+f^2-2ef\mu)}}=A_0\frac{\sin e}{e}.$$

When E and F are interchanged, the same integral is seen to be proportional to $\sin f/f$, and may therefore be equated to

$$A_0'\frac{\sin e}{e}\frac{\sin f}{f},$$

where A_0' is now an absolute constant, whose value is determined to be unity by putting e, or f, equal to zero. We may therefore write

$$\tfrac{1}{2}\int_{-1}^{+1} d\mu\,\frac{\sin\sqrt{(e^2+f^2-2ef\mu)}}{\sqrt{(e^2+f^2-2ef\mu)}}=\frac{\sin e}{e}\,\frac{\sin f}{f}. \quad . \quad (52)$$

As in the case of two dimensions, similar reasoning shows that

$$\tfrac{1}{2}\int_{-1}^{+1} d\mu\,\frac{\cos\sqrt{(e^2+f^2-2ef\mu)}}{\sqrt{(e^2+f^2-2ef\mu)}}=\frac{\sin e}{e}\,\frac{\cos f}{f}, \quad . \quad (53)$$

provided $e<f$.

With appropriate changes, we may now follow Kluyver's argument for two dimensions. The same diagram (fig. 2) will serve, only the successive triangles are no longer limited to lie in one plane. Instead of the angles θ, we have now to deal with their cosines, of which all values are to be

---

* 'Theory of Sound,' § 330.

regarded as equally probable. The probability that these cosines shall lie within the interval μ_1 and $\mu_1+d\mu_1$, μ_2 and $\mu_2+d\mu_2$, ... μ_{n-1} and $\mu_{n-1}+d\mu_{n-1}$ is

$$\frac{1}{2^{n-1}}d\mu_1\,d\mu_2\ldots.d\mu_{n-1}, \quad . \quad . \quad . \quad . \quad . \qquad (54)$$

which is now to be integrated under the condition that the nth radius s_n shall be less than r.

We begin with two stretches l_1 and l_2. Then, in the same notation as before, we have

$$\mathrm{P}_2(r\,;\,l_1,\,l_2)=\tfrac{1}{2}\int d\mu,$$

the integration being within such limits as make $s_2>r$, where

$$s_2{}^2=l_1{}^2+l_2{}^2-2l_1l_2\mu.$$

Hence, by introduction of the discontinuous function (51),

$$\mathrm{P}_2(r\,;\,l_1,\,l_2)=\frac{1}{\pi}\int_{-1}^{+1}d\mu\int_0^\infty dx\frac{\sin s_2x}{s_2x}\,\frac{\sin rx-rx\cos rx}{x}.$$

But by (52)

$$\tfrac{1}{2}\int_{-1}^{+1}d\mu\frac{\sin s_2x}{s_2x}=\frac{\sin l_1x}{l_1x}\,\frac{\sin l_2x}{l_2x},$$

and thus

$$\mathrm{P}_2(r\,;\,l_1,\,l_2)=\frac{2}{\pi}\int_0^\infty dx\frac{\sin rx-rx\cos rx}{x}\,\frac{\sin l_1x}{l_1x}\,\frac{\sin l_2x}{l_2x}. \qquad (55)$$

A simpler form is available for $d\mathrm{P}_2/dr$, since

$$\frac{d}{dr}(\sin rx-rx\cos rx)=rx^2\sin rx.$$

Thus

$$\frac{d\mathrm{P}_2}{dr}=\frac{2r}{\pi l_1l_2}\int_0^\infty\frac{dx}{x}\sin rx\sin l_1x\sin l_2x, \quad . \quad . \quad . \qquad (56)$$

in which we replace the product of sines by means of

$$4\sin rx\sin l_1x\sin l_2x=\sin(r+l_2-l_1)x$$
$$+\sin(r-l_2+l_1)x-\sin(r+l_2+l_1)x-\sin(r-l_2-l_1)x.$$

If r, l_2, l_1 are sides of a real triangle, any two of them together are in general greater than the third, and thus when the integration is effected by the formula

$$\int_0^\infty\frac{\sin u}{u}du=\tfrac{1}{2}\pi,$$

we obtain three positive and one negative term. Finally

$$\frac{dP_2}{dr}=\frac{r}{2l_1l_2},$$

in agreement with (47). The expression is applicable only when the triangle is possible. In the contrary case we find dP/dr equal to zero when r is less than the difference and greater than the sum of l_1 and l_2.

This argument must appear very roundabout, if the object were merely to obtain the result for $n=2$. The advantage is that it admits of easy extension to the general value of n. To this end we take the last stretch l_n and the immediately preceding radius s_{n-1} in place of l_2 and l_1 respectively, and then repeat the operation with l_{n-1}, s_{n-2}, and so on, until we reach l_2 and s_1 $(=l_1)$. The result is evidently

$$P_n(r\,;\ l_1, l_2, \ldots l_n)=\frac{2}{\pi}\int_0^\infty dx\frac{\sin rx-rx\cos rx}{x}$$
$$\frac{\sin l_1x}{l_1x}\frac{\sin l_2x}{l_2x}\cdots\frac{\sin l_nx}{l_nx}, \quad . \quad . \quad . \quad . \quad . \quad (57)$$

or if we suppose, as for the future we shall do, that the l's are all equal,

$$P_n(r\,;\ l)=\frac{2}{\pi}\int_0^\infty dx\frac{\sin rx-rx\cos rx}{x}\left(\frac{\sin lx}{lx}\right)^n. \quad (58)$$

This is the chance that the resultant is less than r. For the chance that the resultant lies between r and $r+dr$, we have, as the coefficient of dr,

$$\frac{dP_n}{dr}=\frac{2r}{\pi l^n}\int_0^\infty\frac{dx}{x^{n-1}}\sin rx\sin^n lx \quad . \quad . \quad . \quad (59)$$

Let us now consider the particular case of $n=3$, when

$$\frac{dP_3}{dr}=\frac{2r}{\pi l^3}\int_0^\infty\frac{dx}{x^2}\sin rx\sin^3 lx. \quad . \quad . \quad . \quad . \quad (60)$$

In this we have

$$\sin rx\sin^3 lx=\tfrac{1}{8}\{3\cos(r-l)x-3\cos(r+l)x$$
$$-\cos(r-3l)x+\cos(r+3l)x\}.$$

And
$$\int_0^\infty\frac{dx}{x^2}\{\cos(r-l)x-\cos(r+l)x\}$$
$$=2\int_0^\infty\frac{dx}{x^2}\left\{\sin^2\frac{(r+l)x}{2}-\sin^2\frac{(r-l)x}{2}\right\}$$
$$=\tfrac{1}{2}\pi\{r+l-|r-l|\}\,;$$

and in like manner for the second pair of cosines.

Thus

$$\frac{d\mathrm{P}_3}{dr}=\frac{r}{8l^3}\{2r-3\,|r-l|+|r-3l|\}. \quad . \quad . \quad (61)$$

expresses the complete solution. When

$$r<l, \quad d\mathrm{P}_3/dr=r^2/2l^3,$$

$$3l>r>l, \quad d\mathrm{P}_3/dr=(3lr-r^2)/4l^3,$$

$$r>3l, \quad d\mathrm{P}_3/dr=0.$$

It will be observed that $d\mathrm{P}_3/dr$ is itself continuous ; but the next derivative changes suddenly at $r=l$ and $r=3l$ from one finite value to another.

Next take $n=4$. From (59)

$$\frac{d\mathrm{P}_4}{dr}=\frac{2r}{\pi l^4}\int_0^\infty \frac{dx}{x^3}\sin rx \sin^4 lx,$$

and

$$-\frac{d^2}{dr^2}\left(\frac{1}{r}\frac{d\mathrm{P}_4}{dr}\right)=\frac{2}{\pi l^4}\int_0^\infty \frac{dx}{x}\sin rx \sin^4 lx$$

$$=\frac{1}{8\pi l^4}\int_0^\infty \frac{dx}{x}\{\sin(r+4l)x+\sin(r-4l)x$$

$$-4\sin(r+2l)x-4\sin(r-2l)x+6\sin rx\}$$

$$=\frac{1}{16l^4}\{1\pm1-4\mp4+6\}=\frac{1}{16l^4}\{3\pm1\mp4\},$$

the alternatives depending upon the signs of $r-4l$ and $r-2l$.

When $\quad r<2l, \quad -16l^4\frac{d^2}{dr^2}\left(\frac{1}{r}\frac{d\mathrm{P}_4}{dr}\right)=6,$

$$4l>r>2l, \quad -16l^4\frac{d^2}{dr^2}\left(\frac{1}{r}\frac{d\mathrm{P}_4}{dr}\right)=-2,$$

and when $r>4l$, the value is zero. In no case can the value be infinite, from which we may infer that

$$\frac{d}{dr}\left(\frac{1}{r}\frac{d\mathrm{P}_4}{dr}\right) \quad \text{and} \quad \frac{1}{r}\frac{d\mathrm{P}_4}{dr}$$

must be continuous throughout.

From these data we can determine the form of $d\mathrm{P}_4/dr$, working backwards from the large value of r, where all derivatives vanish.

$$(4l>r>2l) \quad -16l^4\frac{d}{dr}\left(\frac{1}{r}\frac{d\mathrm{P}_4}{dr}\right)=-2(r-4l),$$

$$(2l>r) \quad -16l^4\frac{d}{dr}\left(\frac{1}{r}\frac{d\mathrm{P}_4}{dr}\right)=6(r-2l)+4l=6r-8l,$$

giving continuity at $r=4l$ and $r=2l$. Again

$$(4l>r>2l) \quad -16l^4\frac{1}{r}\frac{dP_4}{dr}=-(r^2-16l^2)+8l(r-4l)$$
$$=-(r-4l)^2,$$

$$(2l>r) \quad -16l^4\frac{1}{r}\frac{dP_4}{dr}=3(r^2-4l^2)-8l(r-2l)-4l^2$$
$$=3r^2-8rl.$$

Finally

$$\left.\begin{aligned} \frac{dP_4}{dr}=4\pi r^2\phi_4(r\,;\,l)&=\frac{r^2(8l-3r)}{16l^4} \quad (r<2l) \\ \text{or} \qquad &=\frac{r(4l-r)^2}{16l^4} \quad (4l>r>2l) \end{aligned}\right\}, \quad (62)$$

and vanishes, of course, when $r>4l$.

From (61), (64) we may verify Pearson's relation, $\phi_4(0)=\phi_3(l)$.

From these examples the procedure will be understood. When n is even, we differentiate (59) $(n-2)$ times, thus obtaining

$$\frac{d^{n-2}}{dr^{n-2}}\left(\frac{1}{r}\frac{dP_n}{dr}\right)=\frac{2}{\pi l^n}\int_0^\infty\frac{dx}{x}\sin rx\sin^n lx, \quad . \quad (63)$$

in which $\sin^n lx$ is replaced by the series containing $\cos nlx$, $\cos(n-2)lx, \ldots$ and ending with a constant term. When this is multiplied by $\sin rx$, we get sines of $(r\pm nl)x$, $\{r\pm(n-2)l\}x, \ldots \sin rx$, and the integration can be effected. Over the various ranges of $2l$ the values are constant, but they change discontinuously when r is an *even* multiple of l. The actual forms for dP_n/dr can then be found, as already exemplified, by working backwards from $r>nl$, where all derivatives vanish, and so determining the constants of integration as to maintain continuity throughout. These forms are in all cases algebraic.

When n is odd, we differentiate $(n-3)$ times, thus obtaining a form similar to (60) where $n=3$. A similar procedure then shows that the result assumes constant values over finite ranges with discontinuities when r is an *odd* multiple of l. On integration the forms for dP_n/dr are again algebraic.

I have carried out the detailed calculation for $n=6$. It will suffice to record the principal results. For the values of

$$-2^6l^6\frac{d^4}{dr^4}\left(\frac{1}{r}\frac{dP_6}{dr}\right)$$

we find for the various ranges :

$$(r<2l),\ -20\,;\quad (2l<r<4l),\ +10\,;$$
$$(4l<r<6l),\ -2\,;\quad (6l<r),\ 0.$$

And on integration for

$$-2^6l^6\left(\frac{1}{r}\frac{dP_6}{dr}\right), \quad . \quad . \quad . \quad . \quad . \quad . \quad (64)$$

$$(0-2l) \qquad -\frac{5r^4}{6}+4lr^3-16l^3r,$$

$$(2l-4l) \qquad +\frac{5r^4}{12}-6lr^3+30l^2r^2-56l^3r+20l^4,$$

$$(4l-6l) \qquad -\frac{r^4}{12}+2lr^3-18l^2r^2+72l^3r-108l^4,$$

$$(r>6l) \qquad 0.$$

We may now seek the form approximated to when n is very great. Setting for brevity $l=1$ in (59), we have

$$\log\left(\frac{\sin x}{x}\right)^n=n\left\{-\frac{x^2}{6}+h_4x^4+h_6x^6+\ldots\right\},$$

where

$$h_4=-\frac{1}{180}, \quad h_6=-\frac{1}{35\,.\,81}, \quad . \quad . \quad . \quad (65)$$

and

$$\left(\frac{\sin x}{x}\right)^n=e^{-nx^2/6}\{1+nh_4x^4+nh_6x^6+\tfrac{1}{2}n^2h_4{}^2x^8+\ldots\},$$

so that

$$\frac{1}{r}\frac{dP_n}{dr}=\frac{2}{\pi}\int_0^\infty x\,dx\,\sin rx\,e^{-nx^2/6}\{1+nh_4x^4+nh_6x^6$$
$$+\tfrac{1}{2}n^2h_4{}^2x^8+\ldots\}. \quad (66)$$

The expression for the principal term is a known definite integral, and we obtain for it

$$\frac{dP_n}{dr}=\frac{3\sqrt{6}\,.\,r^2}{\sqrt{\pi}\,.\,n^{3/2}}\,e^{-3r^2/2n}, \quad . \quad . \quad . \quad . \quad . \quad (67)$$

which may be regarded as the approximate value when n is very large. To restore l, we have merely to write r/l for r throughout.

In pursuing the approximation we have to consider the relative order of the various terms. Taking nx^2 as standard, so that x^2 is regarded as of the order $1/n$, nx^8 is of order n^{-3} and is omitted. But n^2x^8 is of order n^{-2} and is retained. The terms written down in (66) thus suffice for an approximation to the order n^{-2} inclusive.

The evaluation of the auxiliary terms in (66) can be effected by differentiating the principal term with respect to n. Each such differentiation brings in $-x^2/6$ as a factor, and thus four operations suffice for the inclusion of the term containing x^8. We get

$$\frac{dP_n}{dr}=\frac{3\sqrt{6}\,.\,r^2}{\sqrt{\pi}\,.\,l^3}\Big[\mathrm{N}+nh_4\,.\,6^2\frac{d^2\mathrm{N}}{dn^2}-nh_6\,.\,6^3\frac{d^3\mathrm{N}}{dn^3}+\tfrac{1}{2}n^2h^4\,.\,6^4\frac{d^4\mathrm{N}}{dn^4}\Big], \quad (68)$$

where

$$\mathrm{N}=n^{-3/2}e^{-3r^2/2nl^2}, \quad . \quad . \quad . \quad . \quad . \quad (69)$$

Finally

$$\frac{dP_n}{dr}=\frac{3\sqrt{6}\,.\,r^2e^{-3r^2/2nl^2}}{\sqrt{\pi}\,.\,l^3\,.\,n^{3/2}}\Big\{1-\frac{3}{20n}\Big(5-\frac{10r^2}{nl^2}+\frac{3r^4}{n^2l^4}\Big)+\frac{1}{40n^2}\Big(\frac{29}{4}-\frac{69r^2}{nl^2}+\frac{981r^4}{10n^2l^4}-\frac{1341r^6}{35n^3l^6}+\frac{81r^8}{20n^4l^8}\Big)\Big\}. \quad (70)$$

Here $dP_n/dr\,.\,dr$ is the chance that the resultant of a large number n of flights shall lie between r and $r+dr$. In Pearson's notation,

$$4\pi r^2\phi_n=dP_n/dr.$$

The maximum value of the principal term (67) occurs when $r/l=\sqrt{(2n/3)}$.

It is some check upon the formulæ to compare the exact results for $n=6$ in (64) with those derived for the case of n great in (70), although with such a moderate value of n no precise agreement could be expected. The following Table gives the numerical results for $l\,dP_6/dr$ in the two cases:—

r/l.	From (64).	From (70).
0	·2500 r^2/l^2	·2483 r^2/l^2
·5	·05900	·05886
1	·2005	·2007
2	·4167	·4169
3	·2930	·2922
4	·0833	·1055
5	·00652	·00716
6	·00000	......

So far as the principal term in (70) is concerned, the maximum value occurs when $r/l=2$.

It will be seen that the agreement of the two formulæ is in fact very good, so long as r/l does not much exceed $\sqrt{n}$. As the maximum value of r/l for which the true result differs from zero, is approached, the agreement necessarily falls off. Beyond $r/l=n$, when the true value is zero, (70) yields finite, though small, values.

Terling Place, Witham,
January 24th, 1919.

P.S. *March 3rd.*

In (45) we have the expression for the probability of a resultant (r) when a large number (n) of isoperiodic vibrations are combined, whose representative points are distributed at random along the circumference of a circle of radius l, so that the component amplitudes are all equal. It is of interest to extend the investigation to cover the case of a number of groups in which the amplitudes are different, say a group of p_1 components of amplitude l_1, a group containing p_2 of amplitude l_2, and so on to any number of groups, but always under the restriction that every p is very large. The total number (Σp) may still be denoted by n. The result will be applied to a case where the number of groups is infinite, the representative points of the components being distributed at random over the *area* of a circle of radius L. We start from (31), now taking the form

$$2\pi\phi_n(r^2)=\int_0^\infty x\,dx\,\mathrm{J}_0(rx)\{\mathrm{J}_0(l_1x)\}^{p_1}\{\mathrm{J}_0(l_2x)\}^{p_2}\ldots, \quad . \quad (71)$$

The derivation of the limiting form proceeds as before, where only one l was considered. Writing $s_1=\frac{1}{2}p_1l_1^2$, $s_2=\frac{1}{2}p_2l_2^2$, &c., we have

$$\log\left[\{\mathrm{J}_0(l_1x)\}^{p_1}\{\mathrm{J}_0(l_2x)\}^{p_2}\ldots\ldots\right]$$

$$=-\frac{x^2}{2}\Sigma(s)-\frac{x^4}{16}\Sigma\left(\frac{s^2}{p}\right)-\frac{x^6}{72}\Sigma\left(\frac{s^3}{p^2}\right),$$

and thus

$$2\pi\,\phi_n(r^2)=\int_0^\infty x\,dx\,\mathrm{J}_0(rx)e^{-\frac{1}{2}x^2\Sigma(s)}\left[1-\frac{x^4}{16}\Sigma\left(\frac{s^2}{p}\right)\right.$$

$$\left.-\frac{x^6}{72}\Sigma\left(\frac{s^3}{p^2}\right)+\frac{x^8}{512}\left\{\Sigma\left(\frac{s^2}{p}\right)\right\}^2\right] \quad . \quad . \quad . \quad (72)$$

As before, the leading term on the right is

$$I_1 = \frac{1}{\Sigma(s)} e^{-\frac{1}{2}r^2/\Sigma(s)}, \quad . \quad . \quad . \quad . \quad . \quad (73)$$

and the other integrals can be derived from it by differentiations with respect to $\Sigma(s)$. So far as the first two terms inclusive, we find

$$\begin{aligned} 2\pi\,\phi_n(r^2) = \frac{e^{-\frac{1}{2}r^2/\Sigma(s)}}{\Sigma(s)} \Bigg\{ 1 - \frac{\Sigma(s^2/p)}{4} \Bigg(& \frac{2}{\{\Sigma(s)\}^2} \\ & - \frac{2r^2}{\{\Sigma(s)\}^3} + \frac{r^4}{4\{\Sigma(s)\}^4} \Bigg) \Bigg\}, \quad . \quad . \quad . \quad (74) \end{aligned}$$

from which we may fall back upon (45) by dropping the Σ and making $p=n$. In general $\Sigma(p)=n$. The approximation could be pursued.

Let us now suppose that the representative points are distributed over the area of a circle of radius L, all infinitesimal equal areas being equally probable. Of the total n the number (p) which fall between l and $l+dl$ should be $n\,.\,(2l\,dl/L^2)$, and thus

$$\Sigma(s) = \tfrac{1}{2}\Sigma(pl^2) = \frac{n}{L^2}\int_0^L l^3\,dl = \frac{nL^2}{4}, \quad . \quad . \quad . \quad (75)$$

$$\Sigma(s^2/p) = \tfrac{1}{4}\Sigma(pl^4) = \frac{n}{2L^2}\int_0^L l^5\,dl = \frac{nL^4}{12}. \quad . \quad . \quad . \quad (76)$$

Introducing these values in (74), we get

$$2\pi\,\phi_n(r^2) = \frac{4e^{-2r^2/nL^2}}{nL^2}\left\{1 - \frac{2}{3n}\left(1 - \frac{4r^2}{L^2} + \frac{2r^4}{n^2L^4}\right)\right\}. \quad (77)^*$$

A similar extension may be made in the problem where the component vectors are drawn in three dimensions.

XXXII. *On the Fundamental Law of Electrical Action.* By MEGH NAD SAHA, *M.Sc., Research Scholar in Mathematical Physics, Sir T. N. Palit College of Science, Calcutta* †.

IN the present paper an attempt has been made to determine the law of attraction between two moving electrons, with the aid of the New Electrodynamics as modified by the Principle of Relativity. The problem is a

---

* The applicability of the second term (in $1/n$) to the case of an entirely random distribution over the area of the circle L is not over secure.

† Communicated by Prof. D. N. Mallik.

rather old one, and seems to have first occurred in 1835 to Gauss *, from whom the title of the paper has been borrowed. Before explaining my methods, I shall give a short history of the problem.

About the year 1826 Ampère published his celebrated laws of electrodynamic action, which enable us to calculate, with strict mathematical exactness, the action between two closed electric currents. If we assume that a current of electricity consists of streams of positive and negative charges moving in opposite directions, this action between two closed currents is seen to be composed of the elementary actions between the moving charges, taken two and two. The moving charges, therefore, cannot attract or repel in the same manner as two stationary charges $\left(\text{viz. force} = \frac{ee'}{r^2}\right)$, for in that case the total action would be zero. The natural assumption is that the law of attraction in this case is quite different, and it depends not only upon mutual distance between the two electrons, but also upon their velocities. This is the problem which Gauss set himself to answer; he does not of course speak of electrons, but of charged particles, which mathematically amounts to the same thing.

Gauss and his followers adopted a deductive method for solving this problem. Ampère had given the law which should subsist between two elements of current, *i. e.* the currents flowing through an element of length of a circuit in order to account for the action between two closed currents. This law was derived partly from the Geometry of lines, partly from experiments, and besides, involved a number of assumptions. The solution was therefore not quite convincing, and, indeed, as Grassmann † and Stefan ‡ subsequently proved, was not a unique one. Three other expressions were found to be as good as Ampère's expression for the action between two elements of current. Still, Ampère's solution seemed to be most likely, because the assumptions were simpler in this than in other cases.

Starting with Ampère's expression for the action between two elements of current, and introducing the further assumption that the current consists of discrete charged

---

* Much of the Introduction is taken from Maxwell's 'Electricity and Magnetism,' Chaps. II. and XXIII., see especially pp. 483 *et seq.*

† *Loc. cit.* p. 174.

‡ *Populäre Schriften-Boltzmann,* pp. 95 & 96.

particles in motion, Gauss deduced the following expression for the mutual attraction between two charges:

$$F=\frac{ee'}{r^2}\left[1+\frac{1}{c^2}\left(u^2-\frac{3}{2}\left(\frac{dv}{dt}\right)^2\right)\right],$$

where e, e' are the charges, r=mutual distance, u=relative velocity.

But the law was found to be inconsistent with the principle of conservation of energy, and naturally fell through.

Other physicists in turn took up the problem. The most celebrated formula is that of Weber [*], according to whom the mutual potential of two moving charges is given by the expression

$$\psi=\frac{ee'}{r}\left[1-\frac{1}{2c^2}\left(\frac{\partial r}{\partial t}\right)^2\right].$$

This formula is consistent with the principle of conservation of energy, but was nevertheless found by Helmholtz [†] to lead to improbable results.

These laws were all based on the idea of action at a distance. But in 1845, Gauss [‡] again returned to the problem (which he now calls the real keystone of electrodynamics), with the idea that the action, instead of being propagated instantaneously, may be propagated with a finite velocity in a manner similar to that of light. But he did not succeed, as he himself tells us, in forming any consistent mental picture of the manner in which the action is propagated, and seems to have given up the attempt.

Three other mathematicians, Riemann, Neumann, and Betti [§], followed in the wake of Gauss, and suggested solutions, but these also have been no more successful than their predecessors. According to Riemann [||], the force components between two charges are given by the Lagrangian derivatives of the function

$$\psi=\frac{ee'}{R}\left[1-\frac{(u-u')^2+(v-v')^2+(w-w')^2}{c^2}\right],$$

where (u, v, w) are the velocities of the one particle, (u', v', w') are the velocities of the other.

---

[*] Maxwell, *loc. cit.* pp. 484 & 485.
[†] Phil. Mag. December 1872.
[‡] Maxwell, *loc. cit.* p. 490.
[§] Maxwell, *loc. cit.* p. 490.
[||] Clausius, Phil. Mag. 1880.

According to all of these theories, the action depends on the relative velocity of the two particles. This can be at once perceived by a reference to the formulæ of Gauss, Weber, and Riemann. If both particles move with the same velocity, the action would be the same as that between two stationary ones, and there would not be any electrodynamical action. This is a very objectionable feature of these theories, and attention to this fact was first drawn, I believe, by Clausius *. Clausius is also the author of a series of elaborate investigations on this point. According to his theory, the components of the force between two electrified particles are the Lagrangian derivatives of the function

$$\phi=\frac{ee'}{\mathrm{R}}\left[1-\frac{uu'\cos\theta}{c^2}\right],$$

u and u' being the velocities of the two particles, θ being the angle between their directions of motion. The force components are given by the expressions

$$\mathrm{X}=\frac{\partial\phi}{\partial x}-\frac{d}{dt}\left(\frac{\partial\phi}{\partial\frac{dx}{dt}}\right),\quad \mathrm{Y}=\frac{\partial\phi}{\partial y}-\frac{d}{dt}\left(\frac{\partial\phi}{\partial\frac{dy}{dt}}\right),$$

$$\mathrm{Z}=\frac{\partial\phi}{\partial z}-\frac{d}{dt}\left(\frac{\partial\phi}{\partial\frac{dz}{dt}}\right).$$

It will be observed that the action depends not upon the relative velocity, but upon the absolute velocities of the two particles. Clausius indeed proceeds to show that his formula, besides leading to Ampère's laws of Electrodynamic action, is remarkably free from the objections which were raised against the other formulæ.

Clausius's formula may be said, in a way, to have been confirmed by the investigations of J. J. Thomson †. Thomson investigated, from Maxwell's theory of moving tubes of force, the action between two spheres of radii a and a', moving with the velocities u and u' and carrying the charges e and e'. The kinetic energy was found to be

$$\left(\frac{1}{2}m+\frac{2}{15}\frac{\mu e^2}{a}\right)u^2+\left(\frac{1}{2}m'+\frac{2}{15}\frac{\mu e'^2}{a'}\right)u'^2+\frac{\mu ee'\cos\theta\, uu'}{3\mathrm{R}}.$$

---

* *Journal für Mathematik* (Crelle's Journal), vols. lxxxii. & lxxxiii.; Phil. Mag. 1880.

† Phil. Mag. 1881. 'Application of Dynamics to Problems of Physics and Chemistry,' Chap. IV.

Neglecting the terms due to the Mass-motion, the Lagrangian Function T–U, for the two charged particles, is easily seen to be equivalent to

$$\frac{\mu}{3}\left(\frac{uu'\cos\theta}{\mathrm{R}}\right)-\frac{1}{k\mathrm{R}}.$$

The similarity of this form with the Clausius form is apparent. There is of course discrepancy in the $\left(\frac{\mu}{3}\right)$ term.

These formulæ are all limited to the case where the velocities of the moving charges are small compared with the velocity of light.

From what has been said before, it will be seen that the problem is still an open one. The investigations hitherto given are largely empirical, and not based on sufficient theoretical basis. In view of the recent extraordinary development of electronic physics, it cannot be said that the importance of the problem has been in any way diminished. On the contrary, a knowledge of the laws of electronic attraction and a clear formulation of the dynamics of the electron are necessary before we can satisfactorily handle any problem on electronic physics,—such as the atomic model, or radiation from atoms and electrons.

In the present investigation I have throughout used the New Electrodynamics (*i. e.*, as modified by Lorentz, Einstein, and Minkowski according to the Principle of Relativity). I have particularly used the method of four-dimensional analysis which was first initiated by Minkowski *. A large amount of work in this line has been done by Born † and Sommerfeld ‡, though not always with the same specific purpose which I have in this paper. Sommerfeld in particular, in connexion with his development of four-dimensional analysis, has investigated the law of attraction between two moving electrons; but the result obtained is so cumbrous as to make further progress almost impossible. This is due to the fact that for the scalar and vector potential of the field produced by a moving electron, they arrived at an expression which is only a partial statement of the complete result (see remarks at the end of § 8). When this complete result is introduced, the electric and magnetic forces as well as the ponderomotive force acting on an

---

* H. Minkowski, *Mathematische Annalen*, vol. lxviii. p. 472 *et seq.*

† Born, *Ann. d. Physik*, vol. xxviii. p. 571.

‡ Sommerfeld, *Ann. d. Physik,* vol. xxxiii. pp. 649 *et seq.*; vol. xxxii. pp. 749 *et seq.*

electron come out in very elegant forms, enabling us ultimately to write out the equations of motion of two electrons round each other in a Lagrangian form. When one electron is at rest, the equations lead to Darwin's results (Phil. Mag. 1915).

3. *Notation.*

The notation used in this paper is identical with that used by Minkowski and Sommerfeld in the memoirs just mentioned, and is to be found in any one of the general treatises on Relativity (Cunningham or Silberstein). However, for the convenience of the reader, it is explained below.

The unit of time used in this paper is $\frac{1}{c}$ times the ordinary unit (c, velocity of light measured in ordinary C.G.S. units), so that, with this notation, the velocity of light becomes unity.

We shall, in most cases, use $l=\sqrt{-1}t$, so that (x, y, z, l) denotes the space-time coordinates of a world-point (Weltpunkt).

The quantities

$$(w_1, w_2, w_3, w_y) = \frac{1}{\sqrt{1-\mathrm{u}^2}}(\mathrm{u}_1, \mathrm{u}_2, \mathrm{u}_3, \sqrt{-1}),$$

where $(\mathrm{u}_1, \mathrm{u}_2, \mathrm{u}_3)$ are the ordinary space components of the velocity of a material point, will denote the space-time components of the Velocity-four-vector. It should be noticed that $(\mathrm{n}_1, \mathrm{u}_2, \mathrm{u}_3) = \frac{d}{dt}(x, y, z)$, and if by τ we denote the proper-time (Eigenzeit) of motion of the material point, we shall have $d\tau = dt\sqrt{(1-\mathrm{u}^2)}$, and

$$(w_1, w_2, w_3, w_4) = \frac{d}{d\tau}(x, y, z, l).$$

$\mathfrak{a}$ will denote a four-vector of which the space components are equivalent to the vector-potentials used in Electrodynamics, the time-component $=\sqrt{-1}\phi$, where ϕ is the ordinary scalar-potential. This is known as the Potential-four-vector.

The operator

$$\left(i\frac{\partial}{\partial x} + j\frac{\partial}{\partial y} + k\frac{\partial}{\partial z} + \frac{\partial}{\partial l}\right),$$

which plays the same role in four-dimensional analysis as the familiar operator ∇ in three-dimensions

$$\left(\nabla = i\frac{\partial}{\partial x} + j\frac{\partial}{\partial y} + k\frac{\partial}{\partial z}\right),$$

was called by Minkowski "Lor," in honour of H. A. Lorentz, the discoverer of the Principle of Relativity.

It is denoted by $\square$.

The operator

$$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} + \frac{\partial^2}{\partial l^2}\right),$$

which corresponds to the three-dimensional operator

$$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}\right),$$

is generally denoted by $\square^2$.

The set of four quantities $\rho(u_1, u_2, u_3, i)$, where ρ = density of electricity at a point, is a four-vector according to Lorentz and Einstein. It is known as the Stream-four-vector and will be denoted by s.

4.

The potential-four-vector $\mathfrak{a}$ satisfies the equations *

$$\square^2\mathfrak{a} = -4\pi\mathsf{s}, \quad \text{or} \quad \square^2\mathfrak{a} = 0, \quad . \quad . \quad . \quad (1)$$

according as the world-point at which $\square^2\mathfrak{a}$ is taken is occupied by a stream-four-vector or is empty,

$\mathfrak{a}$ satisfies also the equation

$$\text{Div. } \mathfrak{a}, \quad \text{or} \quad (\square\, \mathfrak{a}) = 0 \; . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

at all points of the world-space.

Now the fundamental solution of equations (1), due to a single stream-four-vector s, occupying the world-point (x', y', z', l') is

$$\frac{A\mathsf{s}}{r^2}, \quad \text{or} \quad A\frac{\mathsf{s}}{(x-x')^2+(y-y')^2+(z-z')^2+(l-l')^2}, \quad (3)$$

where (x, y, z, l) is the world-point at which $\mathfrak{a}$ is to be estimated.

A can be proved to be equivalent to $\frac{1}{\pi}$.

---

* Born, *Ann. d. Physik*, vol. xxviii. p. 571.

Therefore the potential-four-vector at a world-point (x, y, z, l) due to a distribution in the world-space of the stream-four-vector s is

$$\mathsf{a} = \frac{1}{\pi}\iiiint \frac{s\,dx'dy'dz'dl'}{(x-x')^2+(y-y')^2+(z-z')^2+(l-l')^2}, \quad (3')$$

N.B. In modern methods of treating problems on Electrodynamics, the usual practice is to choose a unit of current which is $\sqrt{4\pi}$ times smaller than the ordinary unit, thereby instead of having $\Box^2\mathsf{a} = -4\pi s$, we have $\Box^2\mathsf{a} = -\mathsf{s}$. I have stuck to the older method, because this is more convenient for our purpose.

The fundamental solution $\frac{1}{r^2}$ seems to have been first obtained by Poincaré [*]. It corresponds to the solution $\frac{1}{r}$ in three-dimensional problems on Potential, and is a particular case of the following general result first obtained by Poincaré [†].

If $(x_1, x_2, \ldots x_n)$ be the coordinates of a point in space of n-dimensions, the fundamental solution of the generalized Laplacian

$$\left(\frac{\partial^2}{\partial x_1{}^2} + \frac{\partial^2}{\partial x_2{}^2} + \cdots \frac{\partial^2}{\partial x_n{}^2}\right) \mathrm{V} = 0,$$

is $\frac{\mathrm{A}}{r^{n-2}}$, where $r^2 = (x_1 - x_1')^2 + (x_2 - x_2')^2 + \ldots (x_n - x_n')^2$. (4)

5. *Potential-four-vector at an external point due to the motion of a point-charge.*

By a point-charge is meant a charge having no extension in ordinary space. In four-dimensions, however, it has extension in one direction, viz. in the direction of the time-axis if the electron be stationary, or along an axis making an angle of $(\tan^{-1}u)$ with the time-axis, if u be its velocity of motion.

Let (x, y, z, l) be the coordinates of the point-charge, which we suppose to have started from the origin at time $t=0$. Then we have $(x, y, z) = -\sqrt{-1}(\mathrm{u}_1, \mathrm{u}_2, \mathrm{u}_3)$. Let (a, b, c, λ) be the coordinates of the external point at which the potential a is sought. According to the general theorem

---

[*] Sommerfeld, *Ann. d. Physik*, vol. xxxiii. p. 663.

[†] *Théorie du Potential Newtonien.*

in the previous section, the potential-four-vector **a** is given by the integral

$$\int_{-\infty}^{\infty}\frac{\rho_0(w)dl'}{(x-a)^2+(y-b)^2+(z-c)^2+(l-\lambda)^2},$$

where $\rho_0(w)=\rho(\mathrm{u}_1,\ \mathrm{u}_2,\ \mathrm{u}_3,\ \sqrt{-1})$,

and therefore $\rho_0=\rho\sqrt{(1-\mathrm{u}^2)}$, the rest-density, which is an invariant according to Lorentz and Einstein,

dl' = an element of length along the axis of motion; dl' is easily seen to be equivalent to $dl\sqrt{(1-\mathrm{u}^2)}$.

Now $(x-a)^2+(y-b)^2+(z-c)^2+(l-\lambda)^2$

$$=l^2(1-\mathrm{u}^2)+2il(\mathrm{u}_1a+\mathrm{u}_2b+\mathrm{u}_3c+i\lambda)+a^2+b^2+c^2+\lambda^2$$

$$=l'^2+2il'(w_1a+w_2b+w_3c+w_4\lambda)+a^2+b^2+c^2+\lambda^2.$$

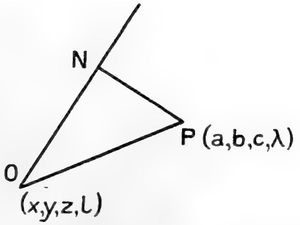
Putting $l'=l\sqrt{(1-\mathrm{u}^2)}$,

this integral is easily seen to be equivalent to

$$\frac{\rho_0(w)}{[a^2+b^2+c^2+\lambda^2+(aw_1+bw_2+cw_3+\lambda w_4)^2]^{\frac{1}{2}}}. \quad (5)$$

With the aid of four-dimensional geometry, we can give an interesting interpretation to this expression. The direction of motion of the charge (ρ) is given by the line

$$\frac{x}{w_1}=\frac{y}{w_2}=\frac{z}{w_3}=\frac{l}{w_4}.$$



Let P be the point $(a,\ b,\ c,\ \lambda)$. Then we have

$$PN^2=OP^2-ON^2$$

$$=(a^2+b^2+c^2+\lambda^2)+(aw_1+bw_2+cw_3+\lambda w_4)^2,$$

for ON = projection of OP on OA $=i(w_1a+w_2b+w_3c+w_4\lambda)$.

Thus the denominator in the expression (5) is seen to be equivalent to R, where R is the perpendicular distance from the external point on the axis of motion.

The result can also be easily proved if we introduce a Lorentz-transformation, by which the axis of motion becomes the new-time-axis. Then in the expression (4), the four-vector $\rho_0(w)$ becomes

$$\rho_0(0,\ 0,\ 0,\ \sqrt{-1}),$$

and the problem is reduced to one at rest. The denominator becomes equivalent to $R^2+l'^2$, where R is the perpendicular from P on the axis of motion.

We have therefore

$$\mathfrak{a}' = \frac{1}{\pi}\int_{-\infty}^{\infty}\frac{\rho_0(i)dl'}{R^2+l'^2} = \frac{\rho_0(0,\ 0,\ 0,\ i)}{R}.$$

Now $(0, 0, 0, \mathfrak{a}_4')$ are the components, in the transformed system, of potential-four-vector $\mathfrak{a}'$, whose components in the original system are $(\mathfrak{a}_1, \mathfrak{a}_2, \mathfrak{a}_3, \mathfrak{a}_4)$. Re-transforming to the original system, we have

$$[\mathfrak{a}_1,\ \mathfrak{a}_2,\ \mathfrak{a}_3,\ \mathfrak{a}_4] = \frac{\rho_0(w_1,\ w_2,\ w_3,\ w_4)}{R}. \quad . \quad . \quad . \quad (6)$$

Otherwise—When by means of an orthogonal Lorentz-transformation, we transform from the system (x, y, z, l) to the system (x', y', z', l'), the generalized Laplacian $\square^2\mathfrak{a}$ is transformed to

$$\left(\frac{\partial^2}{\partial x'^2}+\frac{\partial^2}{\partial y'^2}+\frac{\partial^2}{\partial z'^2}+\frac{\partial^2}{\partial l'^2}\right)\mathfrak{a}=0, \text{ or } -4\pi\mathfrak{s}'.$$

In the present case, the distribution on an infinite line is along the l'-axis. Therefore $\mathfrak{a}$ must be independent of l', from which

$$\mathfrak{a}=\frac{\mathfrak{s}'}{\sqrt{x'^2+y'^2+z'^2}}.$$

$\sqrt{x'^2+y'^2+z'^2}$ is easily seen to be equivalent to what we have called R previously.

Thus according to this method of investigation also, the potential-four-vector

$$\mathfrak{a}=\frac{\rho_0(w_1,\ w_2,\ w_3,\ w_4)}{R}, \quad . \quad . \quad . \quad . \quad (6)$$

where R=perpendicular distance from the external point (a, b, c, λ) on the axis of motion of the point-charge:—direction cosines $\sqrt{-1}(w_1, w_2, w_3, w_4)$.

6. *The Ponderomotive Force* *.

If a be the potential-four-vector in an electric field, and ρ be the electric space-density at a point, the force acting on this point is given by the matrix

$$\rho \left| \begin{array}{cccc} u_1, & u_2, & u_3, & i \\ \frac{\partial}{\partial x}, & \frac{\partial}{\partial y}, & \frac{\partial}{\partial z}, & \frac{\partial}{\partial l} \\ a_1, & a_2, & a_3, & a_4 \end{array} \right| \quad . \quad . \quad . \quad . \quad . \quad (7)$$

It should be noticed that the word "Force" is used in a generalized sense. The components of this four-vector are (X, Y, Z) the ordinary space-components, and

$$L = i(Xu_1 + Yu_2 + Zu_3),$$

i. e., $\sqrt{-1}$ times the rate of doing work. The four components are connected by the equation

$$Xw_1 + Yw_2 + Zw_3 + Lw_4 = 0,$$

i. e., the force-four-vector is always normal to the velocity-four-vector.

Writing (ϕ, F, G, H) for $(\sqrt{-1}a_4, a_1, a_2, a_3)$ and introducing the ordinary C.G.S. units, it can be easily verified that this expression is identical with Lorentz's expressions for Ponderomotive Force.

We shall now write (w_1, w_2, w_3, w_4) instead of $(u_1, u_2, u_3, \sqrt{-1})$. Then

$$[X, Y, Z, L] = \rho_0 \left| \begin{array}{cccc} w_1 & w_2 & w_3 & w_4 \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} & \frac{\partial}{\partial l} \\ a_1 & a_2 & a_3 & a_4 \end{array} \right| .$$

$\rho_0 = \rho\sqrt{(1-u^2)}$ is an invariant, and is generally known as the rest-density,

and $$(a_1, a_2, a_3, a_4) = \frac{\rho_0' w'}{R'}.$$

R′ = perpendicular distance from the external point (x, y, z, l) on the axis of motion of the charge ρ', which produces the field.

---

* Minkowski, *loc. cit.* § 11.

The coordinates here refer to the coordinates of the point-charges*.

7. *Law of Attraction between two Point-charges.*

We have now

$$X=\rho_0\Big[\Big\{w_1\frac{\partial a_1}{\partial x}+w_2\frac{\partial a_2}{\partial x}+w_3\frac{\partial a_3}{\partial x}+w_4\frac{\partial a_4}{\partial x}\Big\}$$
$$-\Big\{w_1\frac{\partial}{\partial x}+w_2\frac{\partial}{\partial y}+w_3\frac{\partial}{\partial z}+w_4\frac{\partial}{\partial l}\Big\}a_1,$$

i. e.,

$$X=\rho_0\rho_0'\Big[(w_1w_1'+w_2w_2'+w_3w_3'+w_4w_4')\frac{\partial}{\partial x}\Big(\frac{1}{R'}\Big)$$
$$-\Big(w_1\frac{\partial}{\partial x}+w_2\frac{\partial}{\partial y}+w_3\frac{\partial}{\partial z}-w_4\frac{\partial}{\partial l}\Big)\Big(\frac{w_1'}{R'}\Big)\Big];$$

for in the expression for a, R is the only term explicitly involving the coordinates (x, y, z, l), (ρ_0', w') being independent of them.

Now let $d\tau$ = proper time (Eigenzeit) of motion of A.

Then $$d\tau=dt\sqrt{(1-u^2)},$$

and $$(w_1, w_2, w_3, w_4)=\frac{d}{d\tau}(x, y, z, l).$$

We have therefore

$$\frac{d}{d\tau}=\frac{\partial}{\partial x}\cdot\frac{dx}{d\tau}+\frac{\partial}{\partial y}\cdot\frac{dy}{d\tau}+\frac{\partial}{\partial z}\cdot\frac{dz}{d\tau}+\frac{\partial}{\partial l}\cdot\frac{dl}{d\tau}$$
$$=w_1\frac{\partial}{\partial x}+w_2\frac{\partial}{\partial y}+w_3\frac{\partial}{\partial z}+w_4\frac{\partial}{\partial l}.$$

If we now put

$$\Phi=\frac{w_1w_1'+w_2w_2'+w_3w_3'+w_4w_4'}{R'}\rho_0\rho_0', \quad . \quad . \quad (8)$$

then, since

$$\frac{\rho_0\rho_0'w_1'}{R'}=\frac{\partial\Phi}{\partial w_1},$$

we have

$$X=\frac{\partial\Phi}{\partial x}-\frac{d}{d\tau}\Big(\frac{\partial\Phi}{\partial w_1}\Big).$$

Similarly for the other components (Y, Z, L).

---

* The matrix used for expressing the Ponderomotive Force (X, Y, Z, L) has not been used in the conventional sense (Sommerfeld, *Ann. der Physik*, vols. xxxii. & xxxiii.), as can be easily observed.

The form is Lagrangian, and the expression for (X, Y, Z, L) comes out in the form originally pointed out by Clausius.

We therefore prove that the force-four-vector on (A) can be put into the Lagrangian forms

$$\left.\begin{aligned} X&=\frac{\partial\Phi}{\partial x}-\frac{d}{d\tau}\left(\frac{\partial\Phi}{\partial w_1}\right)\\ Y&=\frac{\partial\Phi}{\partial y}-\frac{d}{d\tau}\left(\frac{\partial\Phi}{\partial w_2}\right)\\ Z&=\frac{\partial\Phi}{\partial z}-\frac{d}{d\tau}\left(\frac{\partial\Phi}{\partial w_3}\right)\\ L&=\frac{\partial\Phi}{\partial l}-\frac{d}{d\tau}\left(\frac{\partial\Phi}{\partial w_4}\right)\end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad (9)$$

Similarly, if R=perpendicular distance of the point B(a, b, c, λ) from the axis of motion of (A) (x, y, z, l),

i. e., $$R^2=(x-a)^2+(y-b)^2+(z-c)^2+(l-\lambda)^2$$
$$+[(x-a)w_1+(y-b)w_2+(z-c)w_3+(l-\lambda)w_4]^2,$$

and Φ' denotes the expression

$$\frac{\rho_0\rho_0'}{R'}(w_1w_1'+w_2w_2'+w_3w_3'+w_4w_4'),$$

the forces exerted by A on B are given by the equations

$$\left.\begin{aligned} X'&=\frac{\partial\Phi'}{\partial a}-\frac{d}{d\tau'}\left(\frac{\partial\Phi'}{\partial w_1'}\right) & Y'&=\frac{\partial\Phi'}{\partial b}-\frac{d}{d\tau}\left(\frac{\partial\Phi'}{\partial w_2'}\right)\\ Z'&=\frac{\partial\Phi'}{\partial c}-\frac{d}{d\tau'}\left(\frac{\partial\Phi'}{\partial w_3'}\right) & L'&=\frac{\partial\Phi'}{\partial \lambda}-\frac{d}{d\tau}\left(\frac{\partial\Phi'}{\partial w_4'}\right)\end{aligned}\right\} . \quad (10)$$

8. *Two Electrons in Motion.*

In the foregoing sections we treated the case of two point-charges. We shall now take the case of two electrons when these are in a state of motion. It will be shown that the same equations would hold if instead of the rest densities ρ_0, ρ_0', we substitute the invariant charges (e, e'), and suppose the whole charge to be concentrated at the centre of each.

The electron occupies the space

$$(x-x_0)^2+(y-y_0)^2+(z-z_0)^2 \leqq r^2;$$

where (x, y, z) are the space-components of any point within the electron, (x_0, y_0, z_0) the corresponding quantities for the centre, and r is the radius.

In three-dimensions this equation represents a sphere, but in four-dimensions this represents a spherical cylinder having infinite extension along the time-axis. The equation shows that the electron is at rest.

We shall now write down the equation of a spherical electron moving with a uniform velocity (u_1, u_2, u_3).

In three-dimensions, the equation of a circular cylinder having the line

$$\frac{x-x_0}{l}=\frac{y-y_0}{m}=\frac{z-z_0}{n}$$

as the axis is given by the equation

$$\begin{aligned}&(x-x_0)^2+(y-y_0)^2+(z-z_0)^2\\&\qquad-[l(x-x_0)+m(y-y_0)+n(z-z_0)]^2=r^2.\end{aligned}$$

Similarly, in four-dimensions, since the axis of motion is given by

$$\frac{x-x_0}{iw_1}=\frac{y-y_0}{iw_2}=\frac{z-z_0}{iw_3}=\frac{l-l_0}{iw_4},$$

therefore the equation of the cylinder having this as axis is

$$\begin{aligned}&(x-x_0)^2+(y-y_0)^2+(z-z_0)^2+(l-l_0)^2\\&\qquad+[w_1(x-x_0)+w_2(y-y_0)+w_3(z-z_0)+w_4(l-l_0)]^2=r^2.\end{aligned}$$

That this is so can easily be observed by introducing a Lorentz-transformation in which the line of motion is the new time-axis, and the velocity is equivalent to the moment of transformation. Then if (ξ, η, ζ, ν) be the new coordinates, we have

$$\begin{aligned}&(\xi-\xi_0)^2+(\eta-\eta_0)^2+(\zeta-\zeta_0)^2+(\nu-\nu_0)^2\\&\qquad=(x-x_0)^2+(y-y_0)^2+(z-z_0)^2+(l-l_0)^2\end{aligned}$$

and

$$i[w_1(x-x_0)+w_2(y-y_0)+w_3(z-z_0)+w_4(l-l_0)]=\nu-\nu_0.$$

$\therefore$ the equation of the electron becomes

$$(\xi-\xi_0)^2+(\eta-\eta_0)^2+(\zeta-\zeta_0)^2=r^2.$$

We shall now calculate the potential-four-vector due to the motion of electron at an external point (a, b, c, λ).

We have

$$\mathrm{a}=\frac{1}{\pi}\iiiint\frac{\rho_0(w)\,dx\,dy\,dz\,dl}{[(x-a)^2+(y-b)^2+(z-c)^2+(l-\lambda)^2]}, \quad (10a)$$

the integration being extended over the whole world-space enclosed by the electron.

We shall now introduce again the above-mentioned Lorentz-transformation. Then we can write

$$d\xi\, d\eta\, d\zeta\, d\nu \quad \text{for} \quad dx\, dy\, dz\, dl,$$

$$\rho_0(0,\, 0,\, 0,\, i) \quad \text{for} \quad \rho_0(w_1,\, w_2,\, w_3,\, w_4),$$

and $$(\xi-\xi')^2+(\eta-\eta')^2+(\zeta-\zeta')^2+(\nu-\nu')^2$$

for $$(x-a)^2+(y-b)^2+(z-c)^2+(l-\lambda)^2.$$

Now $\mathbf{a}'$, the transformed of $\mathbf{a}$ becomes

$$=\frac{1}{\pi}\iiiint\frac{\rho_0(0,\, 0,\, 0,\, i)\, d\xi\, d\eta\, d\zeta\, d\nu}{(\xi-\xi')^2+(\eta-\eta')^2+(\zeta-\zeta')^2+(\nu-\nu')^2},$$

integrated over the world-space

$$(\xi-\xi_0)^2+(\eta-\eta_0)^2+(\zeta-\zeta_0)^2 \leqq r^2 \quad \text{(A)}.$$

We shall first integrate over the new time-axis. The limits are then from $-\infty$ to ∞.

$$\therefore \qquad \mathbf{a}'=\iiint\frac{\rho_0(0,\, 0,\, 0,\, i)\, d\xi\, d\eta\, d\zeta}{\sqrt{(\xi-\xi')^2+(\eta-\eta')^2+(\zeta-\zeta')^2}},$$

over the Spherical Volume (A). This is a three-dimensional potential problem, and is easily seen to be

$$=\frac{e(0,\, 0,\, 0,\, i)}{\sqrt{(\xi_0-\xi')^2+(\eta_0-\eta')^2+(\zeta_0-\zeta')^2}},$$

where $$e=\iiint\rho_0\, d\xi\, d\eta\, d\zeta,$$

integrated over the spherical volume (A).

Now $$\sqrt{(\xi_0-\xi')^2+(\eta_0-\eta')^2+(\zeta-\zeta_0)^2}$$

is the perpendicular distance from the external point (ξ',η',ζ',ν') on the axis of motion; we can denote this by R.

Then $$\mathbf{a}'=\frac{(0,\, 0,\, 0,\, i)e}{\mathrm{R}}.$$

Now $\mathbf{a}'$ is what the potential-four-vector $\mathbf{a}$ with the components $(\mathbf{a}_1,\, \mathbf{a}_2,\, \mathbf{a}_3,\, \mathbf{a}_4)$ becomes when the transformation is introduced. Retransforming to the original coordinates, we have

$$[\mathbf{a}_1,\, \mathbf{a}_2,\, \mathbf{a}_3,\, \mathbf{a}_4]=\frac{(w_1,\, w_2,\, w_3,\, w_4)e}{\mathrm{R}}. \quad . \quad . \quad (11)$$

We can now express R in terms of the original system of coordinates.

$$R^2=(x_0-a)^2+(y_0-b)^2+(z_0-c)^2+(l_0-\lambda)^2$$
$$+[(x_0-a)w_1+(y_0-b)w_2+(z_0-c)w_3+(l_0-\lambda)w_4]^2,$$

where (x_0, y_0, z_0, l_0) are the coordinates of the centre of the electron, (a, b, c, λ) those of the external point.

N.B. The Scalar and Vector potentials due to the motion of an electron were first obtained by Lienard and Wiechert * about 1898. They were expressed in the forms

$$\phi=\frac{e}{r(1-u_r/c)}, \quad [F, G, H]=\frac{e(u_1/c, u_2/c, u_3/c)}{r(1-u_r/c)}, \tag{12}$$

where r is the distance of the external point from the point occupied by the electron at a time $(t-u_r/c)$, etc. (u_1, u_2, u_3) are the velocity components at the time $(t-r/c)$, u_r is the component of this velocity along the line of r.

The expression (11) is in fact equivalent to the expression (12), as the following reasoning will show. Suppose the time-coordinates are so chosen that

$$(x_0-a)^2+(y_0-b)^2+(z_0-c)^2+(l_0-\lambda)^2=0,$$

i. e. $$c(t_0-t')=-r,$$

or $$t=t_0+\frac{r}{c}.$$

We are in fact estimating the effect at the external point r/c seconds after the electron had been in the position (x_0, y_0, z_0).

Then, since

$$R^2=(x-a)^2+(y-b)^2+(z-c)^2+(l-\lambda)^2$$
$$+[(x-a)w_1+(y-b)w_2+(z-c)w_3+(l-\lambda)w_4]^2,$$

we can, denoting by R′ the four-vector with the components $\{(x-a), (y-b), (z-c), (l-\lambda)\}$,

write $$R^2=R'^2+(R'w)^2,$$

where $(R'w)$ denotes the scalar product of the four-vectors R′ and w.

---

* *L'Eclairage Electrique*, vol. xvi. (1898); Wiechert, *Ann. d. Physik*, vol. iv.

With the above assumption, we have $R'=0$,

$$\therefore \quad |R| = |(R'(w)| = \frac{u_1(x-a) + u_2(y-b) + u_3(z-c) - r}{\sqrt{(1-u^2)}},$$

∴ we can write

$$[a_1, a_2, a_3, a_4] = \frac{ew}{(R'w)} = \frac{r}{r(1-u_r)}[u_1, u_2, u_3, i].$$

Using the ordinary time-coordinate, we have

$$\phi = \frac{e}{r\left(1-\frac{u_r}{c}\right)}, \quad F, G, H = \frac{e(u_1, u_2, u_3)/c}{r\left(1-\frac{u_r}{c}\right)}. \quad . \quad (12)$$

This result has been obtained in various ways by Herglotz*, Sommerfeld†, and other workers. Sommerfeld effects the integration of equation (10 *a*), with the aid of Cauchy's law of residues, and confirms the result (previously obtained by Herglotz),

$$a = \frac{ew}{(R'w)}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (11')$$

But a comparison of the methods of arriving at the two formulæ will show that the expression (11′) is but a partial statement of the result, it being assumed from the very beginning that the time-coordinates are separated by the interval r/c, where r=three-dimensional distance between the points. The result $a=\frac{ew}{R}$ is perfectly general, and in full agreement with the requirements and the spirit of the principle of relativity. This reduces to the expression (11′), when for the purpose of forming an idea of the result in three-dimensions, we make the particular assumption just mentioned about the time-coordinates. Hence it is apparent that when we apply the result to the determination of the magnetic and electric forces, and the ponderomotive force, we must use the expression (11), and not (11′).

9. *The Ponderomotive Force on an Electron due to the field produced by the motion of another electron.*

In § 6 we investigated the action of a point-charge on another charge; in the present section we shall investigate the action of an electron (B) [coordinates of centre (a, b, c, λ),

---

* Herglotz, *Gött. Nach.* Heft 6 (1904).

† Sommerfeld, *Ann. d. Physik,* vol. xxxiii. p. 666.

velocity components (v_1, v_2, v_3)] upon another electron A [charge e, coordinates of centre (x, y, z, l), velocities (u_1, u_2, u_3)].

The components of the ponderomotive force upon a point (x', y', z', l') of the electron A are given by the four components of the matrix

$$\rho_0 \begin{vmatrix} w_1 & w_2 & w_3 & w_4 \\ \dfrac{\partial}{\partial x'} & \dfrac{\partial}{\partial y'} & \dfrac{\partial}{\partial z'} & \dfrac{\partial}{\partial l} \\ \mathrm{a}_1' & \mathrm{a}_2' & \mathrm{a}_3' & \mathrm{a}_4' \end{vmatrix},$$

where $\mathrm{a}' = \dfrac{e'w'}{\mathrm{R}'}$,

$$\mathrm{R}'^2 = (x-a)^2 + (y-b)^2 + (z-c)^2 + (l-\lambda)^2$$
$$+ [w_1'(x-a) + w_2'(y-b) + w_3'(z-c) + w_4'(l-\lambda)]^2$$

and $$[w_1', w_2', w_3', w_4'] = \frac{1}{\sqrt{(1-v^2)}}[v_1, v_2, v_3, i].$$

The total force is obtained by integrating each of these four expressions over the whole volume of the electron A. The X-component of force

$$= \rho_0 \left[\frac{\partial \phi'}{\partial \mathrm{X}'} - \frac{d}{d\tau'}\left(\frac{\partial \phi'}{\partial w'}\right)\right],$$

where $$d\tau' = \sqrt{(1-\mathrm{u}^2)}\,.\,dt,$$

and $$w_1 = \frac{\partial \mathrm{X}'}{\partial \tau} = \frac{\partial \mathrm{X}}{\partial \tau},$$

for all points of the electron move with the same velocity,

and $$\phi' = \frac{e'}{\mathrm{R}'}[w_1w_1' + w_2w_2' + w_3w_3' + w_4w_4'].$$

The total force

$$\mathrm{X} = \iiint \rho_0 \left[\frac{\partial \phi'}{\partial x'} - \frac{d}{d\tau}\left(\frac{\partial \phi}{\partial w_1}\right)\right] d\Omega;$$

$d\Omega$ being the contents of the normal section of the cylinder

$$(x-x_0)^2 + (y-y_0)^2 + (z-z_0)^2 + (l-l_0)^2$$
$$+ [(x'-x_0)w_1 + (y'-y_0)w_2 + (z'-z_0)w_3 + (l'-l_0)w_4]^2 = r^2. \quad . \quad . \quad . \quad (13)$$

It can be easily proved that

$$\frac{\partial}{\partial x'}\left(\frac{1}{R'}\right)=\frac{\partial}{\partial x_0}\left(\frac{1}{R}\right),$$

for the points x_0 and x are rigidly connected. Accordingly

$$X=\iiint\left\{\rho_0\left[\frac{\partial\phi'}{\partial x_0}-\frac{d}{d\tau}\left(\frac{\partial\phi'}{\partial w_1}\right)\right]\right\}d\Omega.$$

Let $$\int\rho_0\,\phi'\,d\Omega=\Phi.$$

Then $$X=\frac{\partial\Phi}{\partial x_0}-\frac{d}{d\tau}\left(\frac{\partial\Phi}{\partial w_1}\right).$$

To evaluate Φ we need only find out the value of the integral

$$I=\iiint\frac{d\Omega}{R'}.$$

Introducing the Lorentz-transformation, in which the axis of motion of the cylinder (13) becomes the new time-axis, we have now

$$I=\iiint\frac{d\xi\,d\eta\,d\zeta}{R'},$$

over the volume

$$(\xi-\xi_0)^2+(\eta-\eta_0)^2+(\zeta-\zeta_0)^2\overline{\overline{<}}\,r^2;$$

and R' is expressed in terms of the new-coordinate system.

Let $(\xi', \eta', \zeta', \nu')$ be the new coordinates of B.

Then

$$\begin{aligned}R'^2=&(\xi-\xi')^2+(\eta-\eta')^2+(\zeta-\zeta')^2+(\nu-\nu')^2\\&+[(\xi-\xi')w_1''+(\eta-\eta')w_2''+(\zeta-\zeta')w_3''+(\nu-\nu')w_4'']^2,\end{aligned}$$

where $(w_1'', w_2'', w_3'', w_4'')$ are the direction cosines of the axis of B in the new system. R' is therefore of the form

$$\begin{aligned}R'^2=&A\xi^2+B\eta^2+C\zeta^2+2H\xi'\eta'+2G\eta'\zeta'\\&+2F\eta'\zeta'+2U\xi'+2V\eta'+2W\zeta'+D.\end{aligned}$$

Let (R) be the same function of (ξ_0, η_0, ζ_0), *i. e.* R is now the perpendicular distance of the centre of A from the axis of B.

Then it can be proved that approximately

$$I=\frac{\Omega}{R}\left[1-\frac{(2-w_4'')}{R}r^2+\ldots\ldots\right].$$

Neglecting terms of higher order than the first, we have

$$I=\frac{\Omega}{R}.$$

(In view of the fact that the radius of the electron is extremely small, the second term must be infinitesimal of a higher order compared with the first.)

Therefore as a first approximation,

$$\Phi=\frac{ee'}{R}(w_1w_1'+w_2w_2'+w_3w_3'+w_4w_4').$$

$$\therefore\quad \left.\begin{aligned} X&=\frac{\partial\Phi}{\partial x}-\frac{d}{d\tau}\left(\frac{\partial\Phi}{\partial w_1}\right)\\ Y&=\frac{\partial\Phi}{\partial y}-\frac{d}{d\tau}\left(\frac{\partial\Phi}{\partial w_2}\right)\\ Z&=\frac{\partial\Phi}{\partial z}-\frac{d}{d\tau}\left(\frac{\partial\Phi}{\partial w_3}\right)\\ L&=\frac{\partial\Phi}{\partial l}-\frac{d}{d\tau}\left(\frac{\partial\Phi}{\partial w_4}\right)\end{aligned}\right\}, \quad \ldots\ldots \quad (14)$$

dropping the subscripts 0 (x, y, z, l), now denoting the coordinates of the centre.

10. *Laws of Electrodynamical Action.*

We shall now reduce the Lagrangian function to three-dimensions. We have

$$\Phi=\frac{ee'(w_1w_1'+w_2w_2'+w_3w_3'+w_4w_4')}{\{(x-a)^2+(y-b)^2+(z-c)^2+(l-\lambda)^2+[(x-a)w_1'+\ldots]^2\}^{\frac{1}{2}}}.$$

Putting $(x-a)^2+(y-b)^2+(z-c)^2+(l-\lambda)^2=0$ just as we did in the interpretation of the potential-four-vector, we have

$$\Phi=\frac{ee'(u_1v_1+u_2v_2+u_3v_3-1)}{r(1-v_r\sqrt{(1-u^2)}}, \quad \ldots\ldots \quad (15)$$

with the same interpretation for r and v_r as before.

Excepting for the factor $[(1-v_r)\sqrt{(1-u^2)}]$ in the denominator, the form for P is identical with that assumed by Clausius for explaining the laws of electrodynamic action. The occurrence of these terms need not cause us any confusion; following in the wake of Clausius, we can easily prove that this formula leads to the laws of electrodynamical action just as well as any one of the formulæ mentioned in the introduction. We have to take terms up to the second order, and instead of using r, we shall have to introduce the instantaneous distance r', which differs from $r(1-v_r)$ by terms of second order only. The second-order terms arising out of $r(1-v_r)$ and $\sqrt{(1-\mathrm{u}^2)}$ affect only one electron; while the term $(\mathrm{u}_1v_1+\mathrm{u}_2v_2+\mathrm{u}_3v_3)$ affects both of them. Remembering that current consists of equal quantities of positive and negative charges moving in opposite directions, there will be no difficulty in realizing that in the final process of summation, terms affecting only one electron would cancel out, and only terms involving both of the electrons would remain in the final result. For further particulars, I would refer the reader to the above-mentioned memoir of Clausius's, where the whole thing is worked out in a most elaborate and convincing manner.

11.

While the main object which I had in view when the work was undertaken has been achieved, viz. the deduction of the laws of electrodynamical action between two closed currents from the theory of electrons, I wish to point out certain other consequences to which this investigation may lead. With the help of Minkowski's four-dimensional analysis, I have succeeded in recasting the important result of Lienard and Wiechert (on the field produced by a moving electron) in an entirely novel form, and as I believe, the only form consistent with the principle of relativity. The potential-four-vector has been proved to be equivalent to $\left(\frac{ew}{\mathrm{R}}\right)$, where e=total charge, w=velocity-four-vector of the electron, and R is the four-dimensional perpendicular distance of the external point from the axis of motion of the electron. By applying the theorem in this simple form to Lorentz's equations for the ponderomotive force acting on an electron, it has been found possible to deduce a Lagrangian function controlling the motion of two electrons round each other. It has been shown that for small velocities, the result is practically identical with that tentatively assumed by Clausius in 1880 for explaining the laws of electrodynamical action on the atomistic hypothesis. There is one important distinction to which attention should be drawn.

In the usual form of Lagrangian equations of motion, we express the force X in the form

$$\mathrm{X}=\frac{\partial\phi}{\partial x}-\frac{d}{d\tau}\left(\frac{\partial\phi}{\partial\frac{dx}{dt}}\right).$$

But here we have

$$X=\frac{\partial\phi}{\partial x}-\frac{d}{d\tau}\left(\frac{\partial\phi}{\partial\frac{dx}{d\tau}}\right),$$

i. e. in place of the time t, we have to use the proper-time τ, where $d\tau=\sqrt{(1-u^2)}\,dt$.

The name proper-time for the function τ, suggests that it has some special relation with the time-coordinate, whereas in fact it is perfectly symmetrical, and similarly related to each of the four coordinates. To dispel any such false notion, it is now usual to designate $d\tau$ as an element of length of the world-line of motion. Thus

$$d\tau=ds=\sqrt{dt^2-dx^2-dy^2-dz^2},$$

and (w_1, w_2, w_3, w_4) becomes $\sqrt{-1}$ times the direction cosines of the element ds.

In a system consisting of two electrons only, the forces controlling the motion are due to electronic attraction only; the gravitational field, being 10^{-42} times smaller than the electronic field, can be entirely neglected. Following Minkowski *, the equations of motion can be written in the forms:

$$\left.\begin{array}{l|l}
c^2m_0\frac{d^2x}{ds^2}=\frac{\partial\phi}{\partial x}-\frac{d}{ds}\left(\frac{\partial\phi}{\partial\frac{dx}{ds}}\right) & m'\frac{d^2a}{ds'^2}=\frac{\partial\phi'}{\partial a}-\frac{d}{ds'}\left(\frac{\partial\phi'}{\partial\frac{da}{ds'}}\right)\\
c^2m_0\frac{d^2y}{ds^2}=\frac{\partial\phi}{\partial y}-\frac{d}{ds}\left(\frac{\partial\phi}{\partial\frac{dy}{ds}}\right) & m'\frac{d^2b}{ds^2}=\frac{\partial\phi'}{\partial b}-\frac{d}{ds'}\left(\frac{\partial\phi'}{\partial\frac{db}{ds'}}\right)\\
c^2m_0\frac{d^2z}{ds^2}=\frac{\partial\phi}{\partial z}-\frac{d}{ds}\left(\frac{\partial\phi}{\partial\frac{dz}{ds}}\right) & m'\frac{d^2c}{ds^2}=\frac{\partial\phi'}{\partial c}-\frac{d}{ds'}\left(\frac{\partial\phi'}{\partial\frac{dc}{ds'}}\right)\\
c^2m_0\frac{d^2l}{ds^2}=\frac{\partial\phi}{\partial l}-\frac{d}{ds}\left(\frac{\partial\phi}{\partial\frac{dl}{ds}}\right) & m'\frac{d^2}{ds^2}=\frac{\partial\phi'}{\partial\lambda}-\frac{d}{ds'}\left(\frac{\partial\phi'}{\partial\frac{d\lambda}{ds'}}\right)
\end{array}\right\}\quad(16)$$

These equations are a particular case of the general

---

* Minkowski, *loc. cit.* Appendix.

equations of motion of an electron

$$\left.\begin{aligned}
\frac{m_0c^2}{e}\frac{d^2x}{d\tau^2} &= \phantom{w_1f_{11}+{}} w_2f_{12}+w_3f_{13}+w_4f_{14} \\
\frac{m_0c^2}{e}\frac{d^2y}{d\tau^2} &= w_1f_{21} \phantom{{}+w_2f_{22}} +w_3f_{23}+w_4f_{24} \\
\frac{m_0c^2}{e}\frac{d^2z}{d\tau^2} &= w_1f_{31}+w_2f_{32} \phantom{{}+w_3f_{33}} +w_4f_{34} \\
\frac{m_0c^2}{e}\frac{d^2l}{d\tau^2} &= w_1f_{41}+w_2f_{42}+w_3f_{43}
\end{aligned}\right\} \quad . \quad . \quad (17)$$

These equations can be deduced from the Principle of Least Action in the following manner. The ordinary form of the Principle of Least Action is

$$\delta\int(\mathrm{T}-\mathrm{V})dt=0. \quad . \quad . \quad . \quad . \quad . \quad (18)$$

Instead of dt we write $d\tau=\sqrt{dt^2-dx^2-dy^2-dz^2}$, and for T we write m_0c^2, where m_0=rest-mass of the electron.

We have then

$$\delta\mathrm{V}=\mathrm{X}\delta x+\mathrm{Y}\delta y+\mathrm{Z}\delta z+\mathrm{L}\delta l,$$

where (X, Y, Z, L) are the components of the Ponderomotive Force-four-vector, (δx, δy, δz, δl) are the variational displacements.

Instead of the form, we have now

$$\delta\int m_0c^2d\tau-\int\delta\mathrm{V}\,.\,d\tau=0. \quad . \quad . \quad . \quad (18')$$

Now $$d\tau=-(w_1dx+w_2dy+w_3dz+w_4dl),$$

and $$\delta\mathrm{V}\,d\tau=-[\mathrm{X}\,\delta x+\mathrm{Y}\,\delta y+\mathrm{Z}\,\delta z+\mathrm{L}\,\delta l]ds$$

$$=-e[f_{12}(\delta x\,dy-dx\,\delta y)+f_{23}(\delta y\,dz-dy\,\delta z)+f_{31}(\delta z\,dx-dz\,\delta x)$$
$$+f_{14}(\delta x\,dl-dx\,\delta l)+f_{24}(\delta y\,dl-dy\,\delta l)+f_{34}(\delta z\,dl-dz\,\delta l)].$$

Now we shall prove an auxiliary theorem*; the ("X, Y, Z, L") used in this proof have no connexion with the force-components.

We have $$\delta\int\mathrm{X}\,dx+\mathrm{Y}\,dy+\mathrm{Z}\,dz+\mathrm{L}\,dl$$

$$=\Sigma\int\delta\mathrm{X}\,dx+\int\mathrm{X}\,\delta\,dx$$

$$=\Sigma\int\left(\frac{\partial\mathrm{X}}{\partial x}\delta x+\frac{\partial\mathrm{X}}{\partial y}\delta y+\frac{\partial\mathrm{X}}{\partial z}\delta z+\frac{\partial\mathrm{X}}{\partial l}\delta l\right)dx$$

$$+\Sigma\int\left[\mathrm{X}\left(\frac{\partial\,\delta x}{\partial x}dx+\frac{\partial\,\delta x}{\partial y}dy+\frac{\partial\,\delta x}{\partial z}dz+\frac{\partial\,\delta x}{\partial l}\right)dl\right].$$

---

* *Vide* Cunningham, 'Principle of Relativity,' Chap. viii.

After partial integration, the second term equals

$$\Sigma X \delta x \Big|_{\text{final}}^{\text{initial}} - \Sigma \int \left(\frac{\partial X}{\partial x} dx + \frac{\partial X}{\partial y} dy + \frac{\partial X}{\partial z} dz + \frac{\partial X}{\partial l} dl \right) \delta x.$$

Hence $\delta \int X dx + Y dy + Z dz + L dl$

$$= X\delta x + Y\delta y + Z\delta z + L\delta l \Big|_{\text{final}}^{\text{initial}} + \int \left(\frac{\partial Y}{\partial x} - \frac{\partial X}{\partial y} \right)$$

$$(\delta x\, dy - dx\, \delta y) + 5 \text{ other similar terms.}$$

Now for (X, Y, Z, L) substitute $m_0c^2 (w_1, w_2, w_3, w_4)$ and let us denote

$$\left(\frac{\partial w_k}{\partial x_h} - \frac{\partial w_h}{\partial x_k} \right) \text{ by } \Omega_{hk}.$$

Then we have

$$\delta \int m_0c^2 ds = \delta \int m_0c^2 (w_1 dx + w_2 dy + w_3 dz + w_4 dl)$$

$$= m_0c^2 (w_1 \delta x + w_2 \delta y + w_3 \delta z + w_4 \delta l) \Big|_{\text{final}}^{\text{initial}}$$

$$+ \int [\Omega_{12}(\delta x\, dy - dx\, \delta y) + \Omega_{23}(\delta y\, dz - dy\, \delta z) + \Omega_{31}(\delta z\, dx - dz \delta x)$$
$$+ \Omega_{14}(\delta x\, dl - dx\, \delta l) + \Omega_{24}(\delta y\, dl - dy \delta l) + \Omega_{34}(\delta z\, dl - dz\, \delta l)].$$

Putting the first term $=0$ as usual, we have from equation (18′)

$$\int [(m_0c^2\Omega_{12} + ef_{12}) (\delta x\, dy - dx\, \delta y) + 5 \text{ other similar terms}] = 0.$$

The six-components of the six-vector $(\delta s \times ds)$ are not independent, hence we cannot put their coefficients individually $=0$. If this were possible we would have obtained the system of equations

$$-\frac{m_0c^2}{e} = \frac{f_{12}}{\Omega_{12}} = \frac{f_{23}}{\Omega_{23}} = \frac{f_{31}}{\Omega_{31}} = \frac{f_{14}}{\Omega_{14}} = \frac{f_{24}}{\Omega_{24}} = \frac{f_{34}}{\Omega_{34}},$$

since (dx, dy, dz, dl) represent the actual displacement, $(\delta x, \delta y, \delta z, \delta l)$ the variational displacements.

We shall have to collect the coefficients of $(\delta x, \delta y, \delta z, \delta l)$ separately and put them individually equal to zero. In this way we obtain the four equations

$$\left.\begin{aligned} -\frac{m_0c^2}{e} &= \frac{f_{12}w_2 + f_{13}w_3 + f_{14}w_4}{\Omega_{12}w_2 + \Omega_{13}w_3 + \Omega_{14}w_4} = \frac{f_{21}w_1 + f_{23}w_3 + f_{24}w_4}{\Omega_{21}w_1 + \Omega_{23}w_3 + \Omega_{24}w_4} \\ &= \frac{f_{31}w_1 + f_{32}w_2 + f_{34}w_4}{\Omega_{31}w_1 + \Omega_{32}w_2 + \Omega_{34}w_4} = \frac{f_{41}w_1 + f_{41}w_2 + f_{43}w_3}{\Omega_{41}w_1 + \Omega_{42}w_2 + \Omega_{43}w_3} \end{aligned}\right\} \quad (17')$$

Of these, only three are independent.

It is easy to see that

$$m_0c^2[w_2\Omega_{12}+w_3\Omega_{13}+w_4\Omega_{14}]=m_0c^2\frac{d^2x}{ds^2},$$

for $$w_2\Omega_{12}+w_3\Omega_{13}+w_4\Omega_{14}=w_2\left(\frac{\partial w_2}{\partial x}-\frac{\partial w_1}{\partial y}\right)$$

$$+w_3\left(\frac{\partial w_3}{\partial x}-\frac{\partial w_1}{\partial x}\right)+w_4\left(\frac{\partial w_4}{\partial x}-\frac{\partial w_1}{\partial l}\right)$$

$$=\frac{1}{2}\frac{\partial}{\partial x}[w_1^2+w_2^2+w_3^2+w_4^2]$$

$$-\left(w_1\frac{\partial}{\partial x}+w_2\frac{\partial}{\partial y}+w_3\frac{\partial}{\partial z}+w_4\frac{\partial}{\partial l}\right)w_1$$

$$=-\frac{dw_1}{ds}=-\frac{d^2x}{ds^2}.$$

for $w_1^2+w_2^2+w_3^2+w_4^2=-1$, ∴ the first term $=0$.

The system of equations (17′) are thus practically identical with the equations (17), but for practical purposes this form may be more convenient than the Minkowskian form.

The six-vector Ω may be styled as the "acceleration" six-vector, (Ω_{23}, Ω_{31}, Ω_{12}) being connected with the three components of rotation, and (Ω_{14}, Ω_{24}, Ω_{34}) with the three components of acceleration

$$\left(\frac{d^2x}{dt^2},\ \frac{d^2y}{dt^2},\ \frac{d^2z}{dt^2}\right).$$

In conclusion, I wish to express my thanks to Prof. D. N. Mallik, and my friend Mr. Satyendra Nath Basu for their kind help and encouragement *.

---

* The paper was communicated about two years ago, but owing to irregularities of the mail service caused by the war, the publication has been rather delayed. Meanwhile much work has been published on the subject, especially several important papers by Crehore in the 'Physical Review.' The author takes this opportunity of expressing his regret that he has not been able to compare his results with those obtained by Crehore and other workers.

XXXIII. *Experiments on the High-Tension Magneto.*—II. *By* NORMAN CAMPBELL, *Sc.D.*[*]

(Continued from p. 301.)

11. AS was explained in the last paragraph of the first part of this paper, the object of the experiments about to be described was to discover whether the relation between the peak potential, the primary capacity, and the coupling is in accordance with that predicted by the theory of Prof. Taylor Jones.

Plan of the Experiments.

12. In most commercial magnetos the circuits are so arranged that the coupling is as great as is consistent with the mechanical limitations of the design; any modifications introduced must be in the direction of a reduction of the coupling. Further, a little consideration will show that the only method of reduction which would be possible in practice, is that of placing in series with the secondary windings, which are coupled with the primary windings, other secondary windings which are not so coupled; if it is desired during the change to keep the total secondary self-inductance constant, the addition of uncoupled secondary windings must be accompanied by a decrease of the windings coupled with the primary. A reduction of the coupling by adding uncoupled primary windings is impracticable, because such a change would certainly decrease the primary current; while separation of the primary and secondary windings (such as is easily possible in the induction-coil) is precluded, unless the general design of the machine is completely altered.

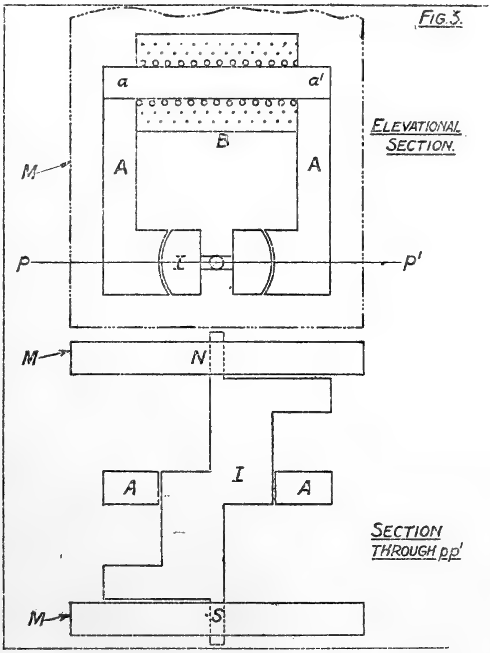
The machine selected for investigation was of the "Inductor" type made by the British Thomson-Houston Co. It has the advantage that the circuits are easily altered; since, moreover, it differs in important features from all machines of the rotating armature type, experiments made on it would give evidence of the generality of the conclusions already attained. The arrangement of the machine is shown diagrammatically in fig. 7. The primary and secondary circuits are wound on the portion $a a'$ of the laminated iron yoke A A. This yoke is fixed relatively to the permanent magnet M and the frame of the machine.

---

[*] Communicated by the Author.

The excitation of the primary current is effected by the rotation in the gap of the yoke of the laminated iron inductor I, which connects magnetically the ends of the yoke to the poles of the magnet and reverses the connexion twice in each revolution. The portion *a a′*, carrying the circuits, is easily detachable, so that one set of windings may be easily replaced by another.

Fig. 7.



Through the kindness of Mr. A. P. Young, of the British Thomson-Houston Co., special experimental armatures were wound to fit the standard machine. No. 1 had 160 turns in the primary and 10,000 in the secondary; No. 2 160 primary, 8000 secondary; No. 3 160 primary, 6000 secondary; No. 4 no primary, 4000 secondary; No. 5 no primary, 2000 secondary. A second housing was also provided. One of the armatures with primary windings was fitted into one housing (A), and one of the other armatures into the other housing (B); the secondaries of the two armatures were then connected in series. Thus only the part of the secondary circuit wound on the first armature was coupled with the primary, and by different arrangements of the armatures different couplings could be obtained. The various combinations will be represented by symbols such as 10,000/0 or 8000/4000 ; the first number gives the number of secondary turns coupled with 160 primary turns in housing A, the second the number of secondary turns uncoupled with the primary in housing B.

Preliminary Measurements.

13. The inductances of the circuits were first measured at a frequency of 100 cycles per sec. by determining the impedance and the ohmic resistance; the primary inductance was determined with a current of 0·5 amp. (which was used in all subsequent measurements), the secondary with 100 volts between the terminals. The inductances were measured with the armatures both in housing A and in housing B, for most of them were sometimes used in one housing and sometimes in the other. The following results were obtained :—

TABLE I.

Armature.	L_1.	L_2.	
		In A.	In B.
No. 1 ………	0·00664 henry.	28·6 henry.	—
No. 2 ………	0·00691	22·7	23·9 henry.
No. 3 ………	0·00688	11·9	13·8
No. 4 ………	—	—	6·32
No. 5 ………	—	—	1·68

An attempt was made to measure the mutual inductance L_{21} in Nos. 1, 2, 3, by determining the voltage across the secondary when an alternating current of known magnitude and frequency was passed through the primary. But from the values so obtained the calculated value of the coupling was greater than 1. This discrepancy is only an expression of the well-known fact that this method of determining a mutual induction is only applicable strictly when the circuit is non-magnetic, but the failure of the method makes it difficult to find a simple method for determining the mutual induction in the conditions of current and frequency which obtain in the measurement of the self-inductances. Accordingly, no further attempt was made to determine the mutual inductance, but it was clear that the coupling was not very different from unity.

Oscillation Measurements.

14. The next step was to measure the constants of the circuits by the methods, depending on the determination of the periods of the oscillations, described in the previous part of the paper. Such measurements were made on the following combinations:—10,000/0, 8000/2000, 8000/6000, 6000/4000, 6000/8000. The results are given in Table II., the figures derived from Table I. being given in brackets where they are available. The self-inductance of the primary is assumed to be independent of the uncoupled secondary turns in series, while the secondary self-inductance is taken to be the sum of those for the coupled and uncoupled turns which are in series. The values of c in brackets are calculated in this manner. It is assumed that the coupling and the value of s for armatures 1, 2, 3, when there are no uncoupled turns in series, are the same and equal to that determined by the oscillation method for No. 1. Since both s and k^2 are nearly 1, the precise choice of their values will make little difference. The coupling of any other combination will then bear to the coupling of 10,000/0 the ratio of the self-inductance of the coupled turns to that of the coupled and uncoupled turns in series.

Table II.

Combination.	L_1 (henry).	L_2' (henry).	C_2 (mmf.).*	c.
10000/0	·00623 (·00664)	28·2 (28·6)	43·5	0·040
8000/2000	·00599 (·00691)	23·3 (24·4)	52	0·062 (·106)
8000/6000	·00591 (·00691)	— (36·5)	—	0·360 (·378)
6000/4000	·00573 (·00688)	19·0 (18·25)	45	0·271 (·373)
6000/8000	·00552 (·00688)	— (35·8)	—	0·760 (·663)

* Calculated from T_2, assuming $L_2 = L_2'$.

The agreement between the secondary self-inductances determined by the two methods is within the limits of experimental error; it is certainly surprising that this quantity does not vary more rapidly with the frequency. On the other hand, the primary self-inductances are smaller (as would be anticipated) when determined by the oscillation method at the higher frequency. In the previous part it was noted that L_1 appeared to increase with the period of the oscillations, but the increase found here is considerably less than that found by Prof. Jones *.

The agreement in the value of c is not very close in most cases, but the variation of c found by the oscillation method is always in the right direction; as the uncoupled secondary inductance increases, c increases also. The measurement of C_2 is not capable of much accuracy, and the differences between the various figures can hardly be said to exceed experimental error. The values shown include the 11·8 mmf. added to the circuit for measuring purposes. It is perhaps rather surprising that the difference between C_2 for these armatures and that for the armature previously used is not greater; for in the inductor machine the windings are not surrounded on all sides by earthed conductors. On the other hand, it is to be expected that the addition of uncoupled turns should not increase C_2 materially, for the housing in which those turns were contained was insulated from earth, and the capacity to earth of the windings in it should be less than that of the housing, which was estimated to be 4 or 5 mmf. In what follows it will be assumed that C_2 may be taken the same for all the combinations.

In general, then, it may be said again that the oscillation method gives results which are consistent among themselves, and agree, as well as can be expected, with the values obtained by other methods. The theory which is involved in the measurements is reliable so far as the determination of the constants is concerned.

The Damping Coefficients.

15. Fig. 8, corresponding in all particulars to fig. 5 of the first part, shows μ plotted against τ. The full line refers to the oscillations in the secondary circuit; the dotted line to oscillations in the primary circuit. The suspicion previously noted that the damping in the two circuits is not strictly the same is thus confirmed. The lines refer equally to all the combinations examined; it could not be established that, at the same frequency, μ was different for different combinations of the armatures.

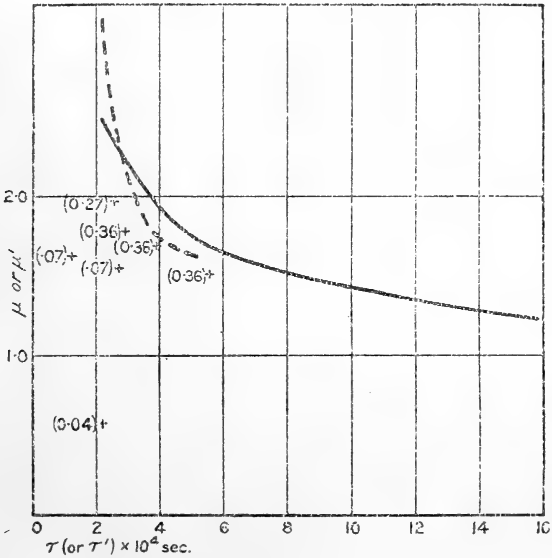
It was found, again, that, for the same frequency, μ' is

---

* For the larger values of c the estimate of L_1 by the oscillation method may be too low. For that method involves the substitution of (1·1) for (1), which is only legitimate when r is very large—a condition which is not fulfilled when c is as great as 0·4. An error of 10 per cent. might arise in this way.

much less than μ. Points corresponding to μ' are denoted by +, and against each is marked the value of c for the combination to which it refers. It will be seen that the difference between μ and μ' decreases with the coupling, and at the smallest coupling investigated the two are nearly equal. These relations, again, are suggestive for a theory which would take into account the effect of the losses in the iron cores.

Fig. 8.



As is to be expected, the damping coefficients are smaller for the inductor machine than for the Thomson-Bennett armature in its housing, although they are greater than those for the latter out of its housing. It cannot be immediately concluded that the difference is due to the better lamination of the magnetic circuit of the inductor machine, for if any part of the loss is due to hysteresis it will decrease rapidly with the flux density in the iron. This flux density is doubtless greatest in the Thomson-Bennett armature in its housing, and probably least in that armature out of its housing. The superiority of the inductor machine in this respect is of considerable practical importance, whatever may be its cause; but it should be noted that the rotating armature machine had unlaminated pole-pieces; in most modern machines the pole-pieces are laminated.

The Peak Potential.

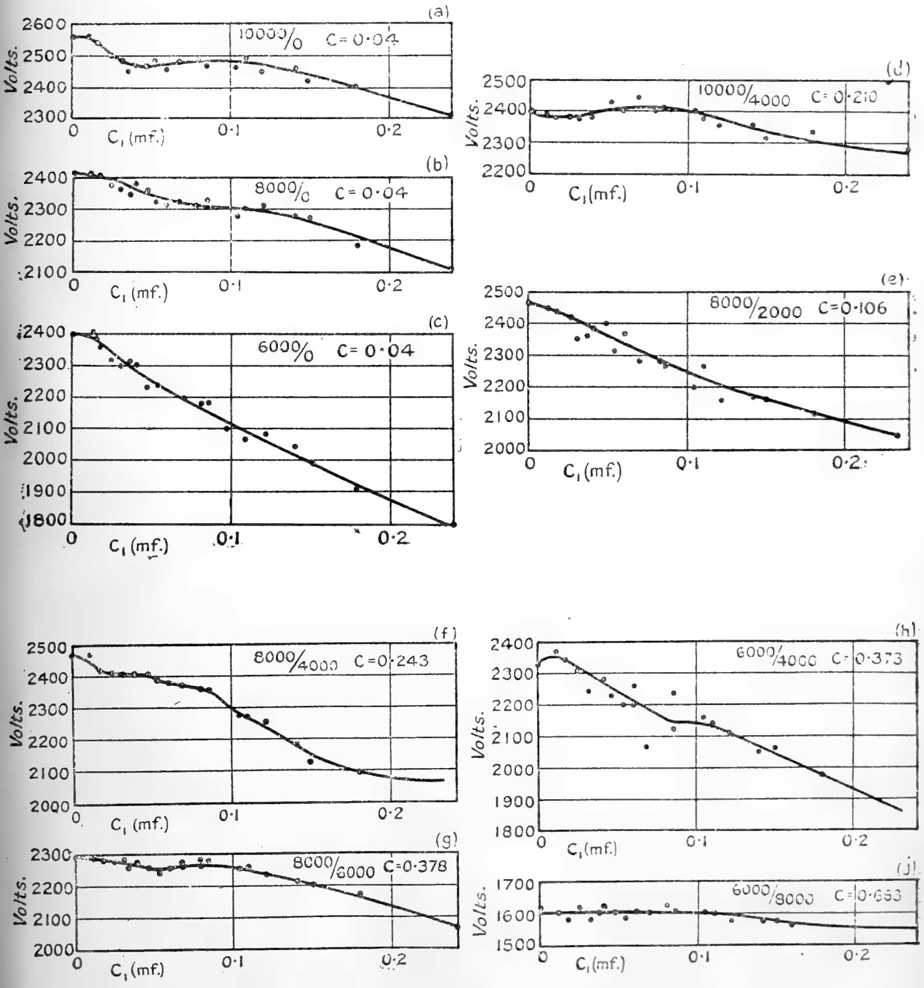
16. So far the experiments have been preliminary. The main object of the investigation is to determine the peak potentials given by the various combinations of armatures. These were determined as in the previous part, a primary current of 0·5 amp. being broken; the capacity added to the secondary circuit for purposes of measurement was 37·4 mmf., in place of 11·8 mmf.

In fig. 9 the resulting peak potentials are plotted against C_1 for the following combinations:—10,000/0, 8000/0, 6000/0, 8000/2000, 8000/4000, 8000/6000, 6000/4000, 6000/8000. On each curve is given the corresponding value of c, calculated as explained in § 14.

It should be noted at the outset that the divergence of the experimental points from the smooth curves drawn through them exceeds greatly the possible experimental error of a measurement; it is due to a real variability of the peak potential being measured, and not to a deficiency of the method of measurement. The best proof of this statement is to be found in figs. 12, 13 (which will be discussed presently), in which the method of measurement was exactly the same but the divergence of the points from a smooth curve very much less. Again, while the general shape of the curves, and even such minor features as could be detected with certainty, could be repeated at successive trials, the absolute values of the peak potential were found to vary considerably. Thus experiments were made six times on 10,000/0; the extreme values differed 5 per cent. from the mean, although on each occasion the variation of the peak potential with C_1 was almost exactly the same. The series of measurements selected for fig. 9 are believed to be closely comparable, but the ratio between the absolute values for any two curves cannot be relied upon to less than 3 per cent.

The reason for this variability of the peak potential has not been discovered, but it is clearly connected with the losses in the iron core. It has always been noticed in all experiments on magnetos, and was not mentioned in the previous part because the method of measurement had been less highly developed when those results were obtained, and it was less certain that the apparent variability was real.

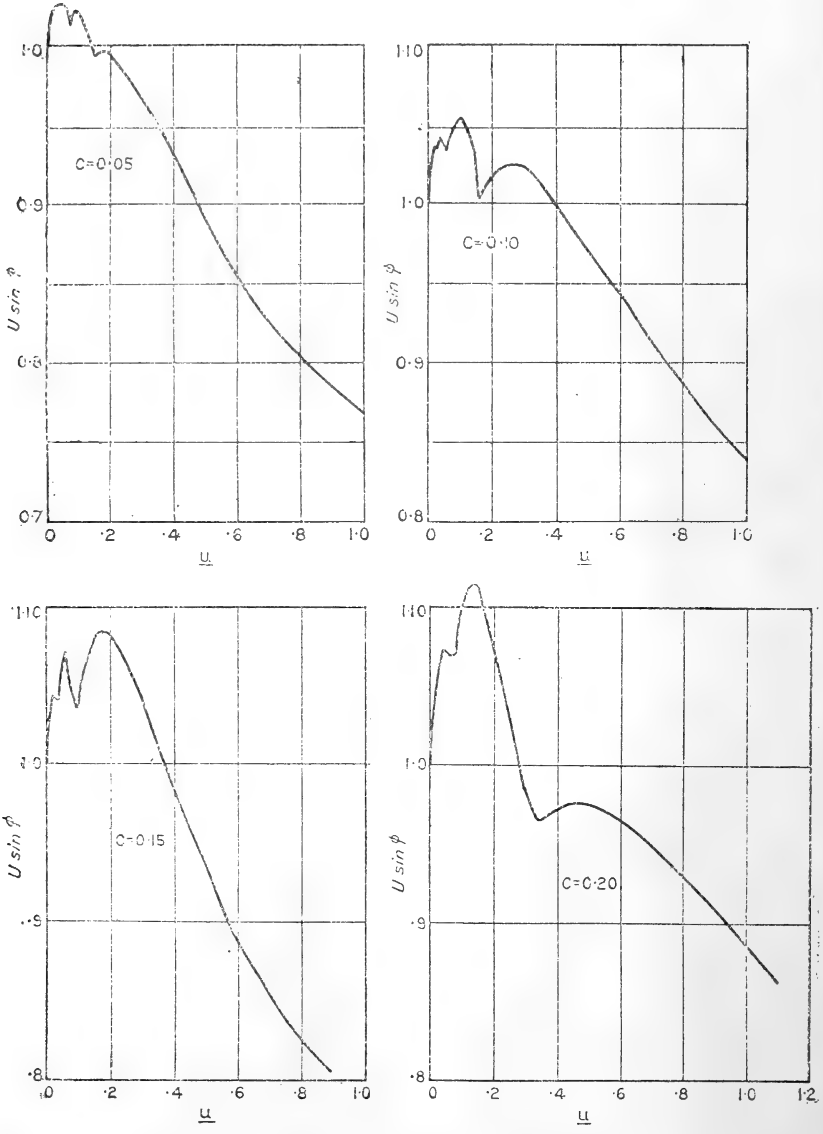
Fig. 9.

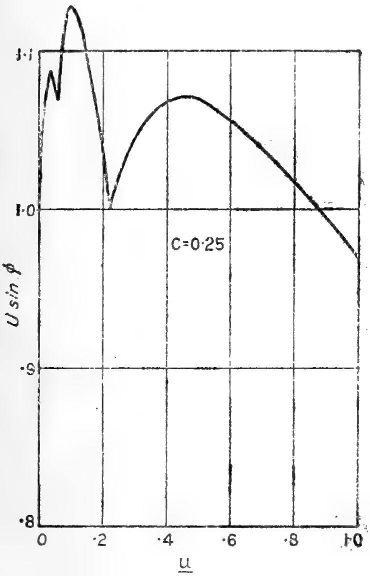


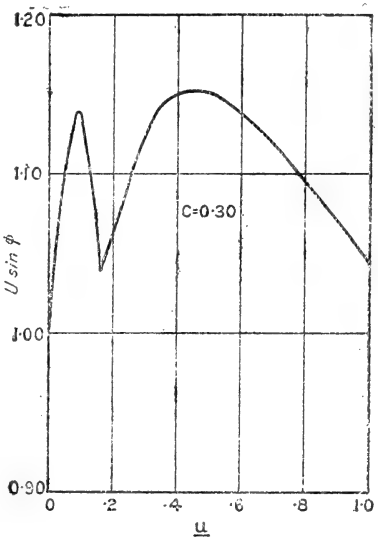
Comparison of Theory and Experiment.

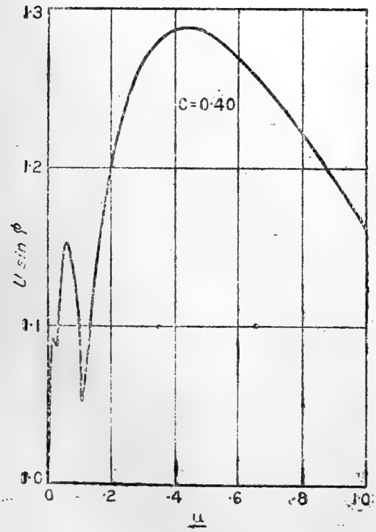
17. In fig. 10 the relation between the peak potential and u which is predicted by the theory is shown for various

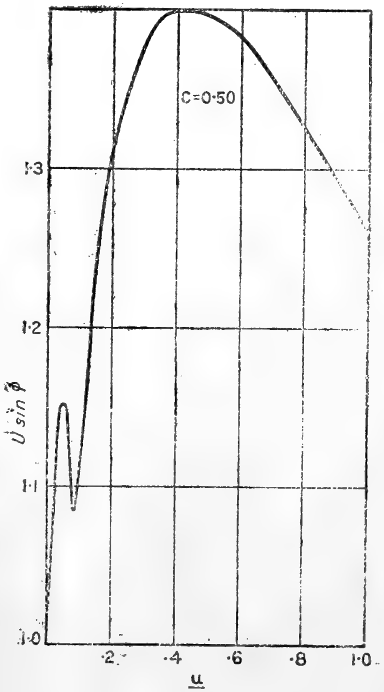
Fig. 10.




1·1
1·0
·9
·8
U sin φ
C=0·25
0
·2
·4
·6
·8
1·0
u


1·20
1·10
1·00
0·90
U sin φ
C=0·30
0
·2
·4
·6
·8
1·0
u


1·3
1·2
1·1
1·0
U sin φ
C=0·40
0
·2
·4
·6
·8
1·0
u


1·3
1·2
1·1
1·0
U sin φ
C=0·50
·2
·4
·6
·8
1·0
u

values of c. In the Appendix tables are given showing the numerical values of V_{2m} and also of r from which the curves are plotted; these tables may possibly be of use to others pursuing similar investigations. When all constants but C_1 are unchanged, u is proportional to C_1; and accordingly the experimental curves of fig. 9 should each resemble one of the theoretical curves of fig. 10, or a curve intermediate between those drawn for some intermediate value of c *.

It is evident at once that the theory is not applicable. In the experimental curves there is no trace of the sharply-defined minima which are so characteristic a feature of the theoretical curves. Minima can be traced in the experimental curves, and they appeared always to be accompanied by a variability in their neighbourhood greater than the average; but the minima are always shallow depressions and not sharp cusps. Further, if we neglect the subsidiary minima and consider only the general trend of the curves no better agreement is found. All the theoretical curves have main maxima at values of u which are markedly different from 0, and the value of u at which the maximum occurs varies notably with c. But all the experimental curves have maxima at or very near 0, and there is no sign of a variation of the position of the maximum with c.

In particular, the expectation in which the experiments were undertaken is completely falsified. There is no sign that by increasing c and decreasing the coupling the optimum value of C_1 can be increased, or even that the decrease of the peak potential with increase of C_1 can be diminished. Indeed, if the curves are compared for two combinations which have the same number of coupled turns but different numbers of uncoupled turns (*e. g. b* and *c* or *c* and *h*), it will be seen that the addition of uncoupled turns, which decreases the coupling, always makes the curve fall off more steeply as C_1 is increased.

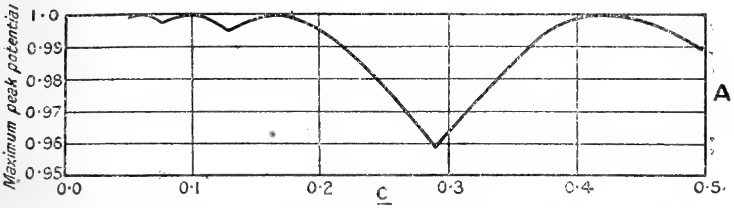
And when we turn our attention from the general shape of the curves, and compare merely the maximum peak potentials given by different combinations with the optimum value of C_1 in each case, no better agreement is found. If L_1 and C_2 are constant (as they are very approximately in these experiments) then the theory predicts that the relation between the peak potential with optimum C_1

---

* The abscissæ are so chosen that the scale of u in fig. 10 is nearly the same as the scale of C_1 in fig. 9. Differences between the two figures are not simply due to a wide difference in the scale of the abscissæ.

should vary with c in the manner shown in fig 11. The experimental curves show no such relation; the maximum peak potential decreases with the number of turns coupled with the primary, even if the coupling is unaltered, and

Fig. 11.



it decreases steadily as the coupling is decreased by the addition of uncoupled secondary turns. The best result is obtained by using as many secondary turns as possible and coupling them as closely as possible with the primary.

In all particulars, therefore, the theory proves false, and it provides no guidance whatever to the design of the armature. The methods which have been adopted in practice as a result of mere empiricism seem the most satisfactory, and it seems unlikely that any considerable improvement in the efficiency of the magneto can be produced except by reducing the iron losses in the core. Until these losses are made as small as they are in the induction-coil, an application of the theory which neglects those losses is useless.

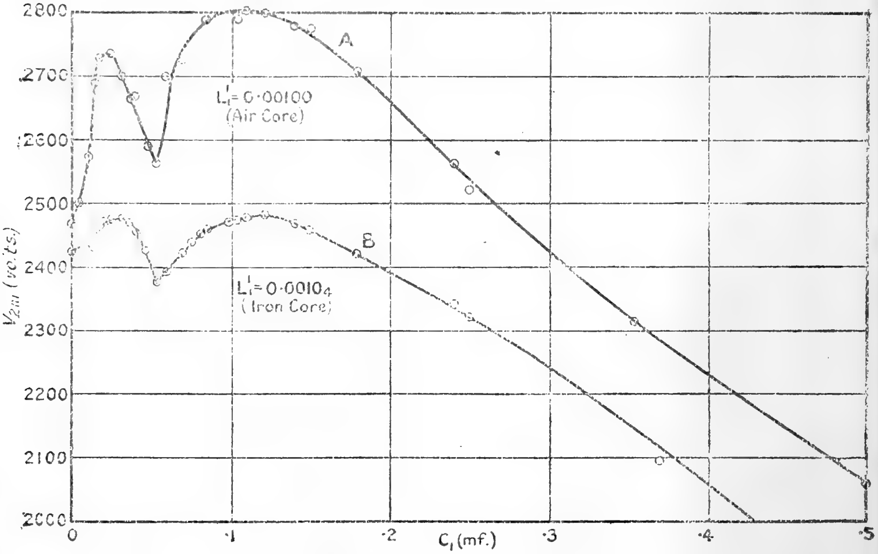
But before passing on it is interesting to notice one feature in which the experiments just described agree with those of the previous part. The maximum peak potential calculated by the theory (neglecting damping) for the combination 10,000/0 is 4750 volts; if this is corrected in the manner explained in § 8 for the observed damping, it becomes 3090 volts. The observed maximum peak potential is 2560 volts, and the ratio of observed to calculated is 0·83. It is worthy of remark that this ratio is not very different from that found for the totally different armatures examined previously, namely 0·76 and 0·78. It is possible that the constancy of this ratio has some theoretical significance.

Effect of an Air-core Coil in the Primary.

18. In his paper, on which the present work is based, Prof. Jones described measurements of the peak potential of a magneto in which results much more concordant with his theory were obtained. The curve which he gave showing the relation between the peak potential and C_1 was marked by well-defined minima such as are characteristic of the theoretical curves. The discrepancy between his results and those just described has to be explained.

At first it was thought possible that the discrepancy might arise from the methods adopted for measuring the peak potential; Prof. Jones employed a method depending on a spark-gap. But a few experiments were sufficient to show that the two methods gave precisely the same results within the limits of their respective errors. It was then noted that, in order to bring the frequency of the oscillations within the limits of the instrument which he had employed for measuring the wave-form, Prof. Jones had inserted in series with the primary of the magneto a choking coil with an air-core. Accordingly, the measurements which had been made were repeated with a choking coil with an air-core and a self-inductance of 0·001 henry inserted in series with the primary. Curve A of fig. 12 shows the relation that was then obtained between the peak potential and the primary capacity. The sharp minimum characteristic of the theoretical curve is now clearly shown; the

Fig. 12.



points lie much more closely to a smooth curve, and they could be repeated on successive trials very nearly within the limits of observational error. The nature of the result was thus entirely changed and observation brought much more closely into concordance with theory.

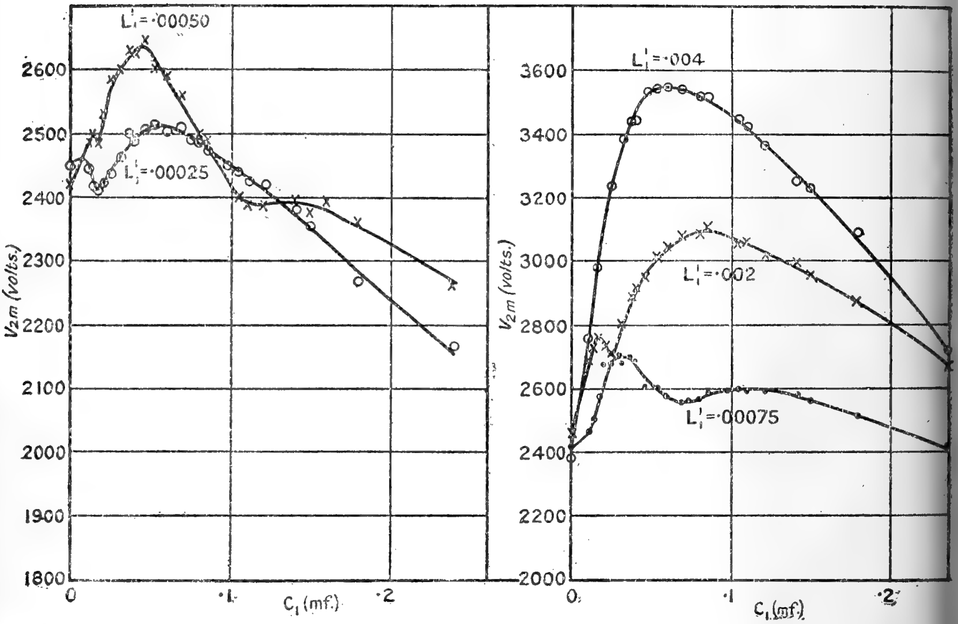
It was, of course, suspected at once that the great difference was due to the fact that the external choking coil was free from hysteresis and eddy-currents, and not merely to the presence of the external coil; but it was thought desirable to establish the fact by substituting for the air-core coil a coil of the same self-inductance wound on an iron core. Fig. 12 B shows the results which were then obtained; the sharp minimum has not altogether disappeared, but it is much reduced, while the whole curve is depressed some 12 per cent. Now the iron losses in the external coil must have been very much less than they would be in a coil of similar self-inductance constructed like the armature of a magneto, for the core on which the coil was wound consisted of a straight laminated bar and not of a nearly closed magnetic circuit. Accordingly, the difference in the curve introduced by the substitution of such a coil for one strictly free from iron losses is sufficient to indicate that the main reason for the complete absence of sharp minima in the curve taken without the external coil is to be found, as has always been suspected, in the iron losses. On the other hand, it is by no means clear immediately why the mere addition of an external coil, even if it is free from iron losses, should change so completely the character of the results; some further investigation into the matter seemed desirable.

The first step was to discover whether the damping of the oscillations had been greatly diminished by the air-core choking coil. The logarithmic decrement was found to be reduced, but only by about 10 per cent., an amount much too small to account for the great difference in the peak potential. But at the same time a remarkable feature appeared. The primary self-inductance without the choking coil in series was ·00623 henry, and it was accordingly suspected that with the choking coil in series it would be about ·0072; it was actually found that the self-inductance was only ·00578, less than that of the armature without any external coil. It is clear that the presence of the external coil modifies profoundly the oscillations in the rest of the primary circuit.

Deduction of the Constants from the (V_{2m}, C_1) *Curves.*

19. Measurements of the relation between the peak potential and the primary capacity were also made with other air-core coils in series with the primary. The results of these measurements are shown in fig. 13, where the self-inductance of the external air-core coil is denoted by L_1'.

Fig. 13.



It will be seen that, even when L_1' is as small as ·00025 and only 5 per cent. of L_1, the self-inductance of the primary of the armature itself, the presence of the external coil makes marked minima appear in the (V_{2m}, C_1) curve, though they are absent when it is not there. Since the appearance of these minima indicates that, in one respect at least, the predictions of the theory are fulfilled more accurately than before, it is worth while to inquire how far the agreement of theory and observation extends in the new circumstances.

Perhaps the best way to investigate the matter is to attempt to calculate from the experimental curves the constants of the circuits; the general consistency of the values obtained will show how far the theory used in their deduction is applicable. The attempt will also have some intrinsic interest, for, if it is successful, it will show a new method for deriving the constants of the circuits and one that has some advantages over methods which depend on the determination of wave-forms. For the tracing of wave-forms of such high frequency as is characteristic of magnetos and induction-coils must always involve rather elaborate apparatus, whereas in order to obtain (V_{2m}, C_1) curves only the simple apparatus for measuring the peak potential is required (see page 301 of the March number) together with variable inductances and capacities.

In the deduction of the constants of the circuits from the experimental curves, it is probably best to use only the positions of the various maxima and minima, and to base no conclusions on the relative height of these points; for the latter are likely to be more affected by slight imperfections of the theory than the former. But even when the inquiry is so limited, the best method of procedure requires a little consideration. For it must be remembered that the experimental curves give V_{2m} as a function of C_1, while the theory gives it as a function of u; u is proportional to C_1, but the factor of proportionality is not known at the outset; to discover this factor must be the first step in the process. We may proceed thus:

Each of the curves in fig. 13 applies to a different value of L_1' and therefore to a different value of the ratio u/C_1 and of c. The first step, then, will be to calculate from the theory the values of u at which maxima and minima occur for different values of c. According to Prof. Jones's theory (see (40), (41) of his paper), maxima occur when u and c are (very nearly) such that $r=4m-1$, and minima when they are such that $r=4m+1$, m being any integer. In fig. 14 the values of u at which minima occur for various values of m are plotted against c; fig. 15 gives the corresponding curves for the maxima. In the latter figure the portions of the curves which are drawn full instead of dotted are those for which the corresponding maximum is the highest maximum of all; the full lines, joined by vertical chain-dotted portions, give the relation between c and that value of u which corresponds to the optimum value of C_1 (see § 10).

From fig. 15 we see that the value of u which gives the maximum $m=1$ is very nearly independent of c within the limits $c=0{\cdot}2$ to $c=0{\cdot}4$ and equal to $0{\cdot}46$. Accordingly, if we can find the value of C_1, which gives the maximum $m=1$, we shall know that it corresponds to $u=0{\cdot}46$ and deduce at once the ratio u/C_1 for the corresponding curve. The maximum $m=1$ will, if it occurs at all, be always that furthest to the right, but if c is too small it may not occur at all. A very little consideration will show that in figs. 12,

Fig. 14.

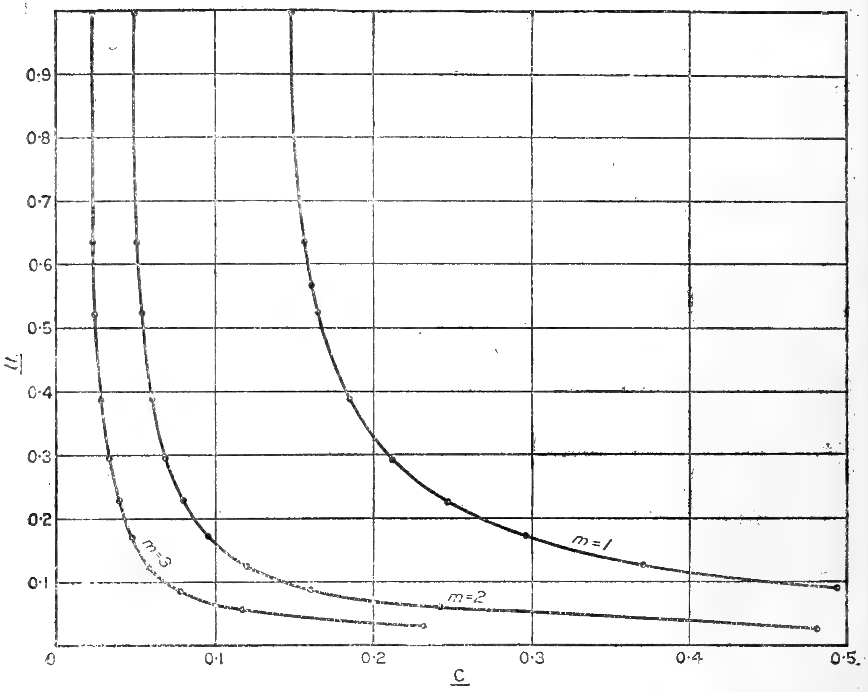
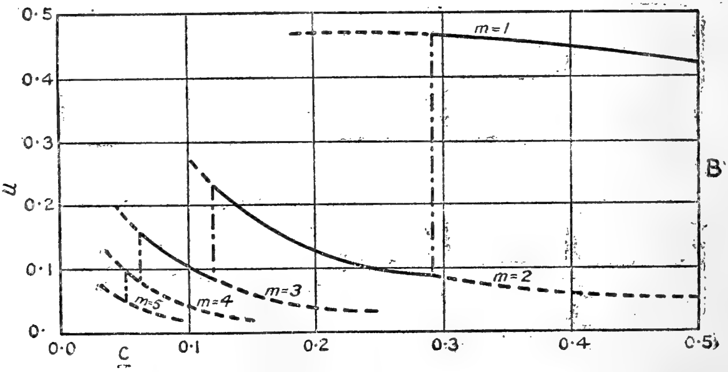


Fig. 15.



13 it occurs in all the curves except that for $L_1' = \cdot 00025$. Accordingly, we get the following table :—

TABLE III.

L_1'.	C_1 for max. at $m=1$.	u/C_1. Obs.	u/C_1. Calc.
·00025	——	——	3·07
·00050	0·136	3·38	3·45
·00075	0·108	4·25	3·81
·00100	0·113	4·06	4·17
·00200	0·082	5·60	5·63
·00400	0·057	8·07	8·08

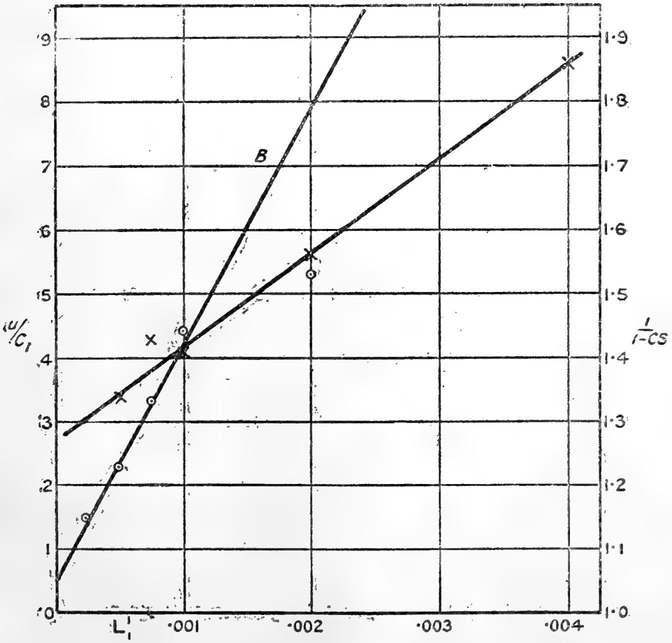
Here u/C_1 (obs.) is deduced from C_1 by putting $u=0·46$.

If the theory were correct we should have

$$u/C_1=4\pi^2(L_1+L_1')/T_2^2.$$

Hence if we plot u/C_1 against L_1' we should get a straight line of which the inclination is $T_2^2/4\pi^2$, the intercept on the u/C_1 axis $4\pi^2L_1/T_2^2$. Such a line is shown by A, fig. 16;

Fig. 16.



all the points except that for L_1' =·00075 lie well on it; the resulting values are L_1=·00178 henry, T_2=0·000172 sec. Both these values are much smaller than those estimated by the oscillation method, and so far the theory does not seem to work well; but since the probable error of the determination is large it is worth while to proceed. We will use the result only to give new "smoothed" values for u/C_1 and to extrapolate a value for $L_1'=\cdot 00025$; these values are given under u/C_1 (calc.).

The next step is to use the minima and the other maxima of the curves. The following table gives for each value of L_1', first C_1 corresponding to the various maxima and minima, second the corresponding value of m (easily determined by inspection or corrected later if an initial mistake is made), third the corresponding value of u taken from Table III., fourth the corresponding value of c taken from figs. 14 or 15.

Table IV.

L_1'.	C_1.	*m*.	*u*.	*c* (obs.).	*c* (calc.).
·00025	·060 (max.)	2	·185	·135	·119
	·017 (min.)	2	·052	·29*	
		3?		·13*	
·00050	·045 (max.)	2	·155	·175	·185
	·017 (min.)	2	·058	·24*	
	·119 (min.)	1	·410	·180	
·00075	·035 (max.)	2	·134	·210	·240
	·071 (min.)	1	·273	·240	
·00100	·024 (max.)	2	·100	·240*	·288
	·053 (min.)	1	·221	·297	
·00200	·017 (max.)	2	·096	·245*	·429
	·025 (min.)	1	·141	·337	

There are considerable discrepancies between the values of c determined for the same L_1'; but all those marked by asterisks are very uncertain owing to the shape of the curves from which they are determined; if these are neglected some of the greatest differences disappear. Further, for all the curves except $L_1'=\cdot 00025$ the value for the minimum is much more certain than that for the maximum and will alone be used in what follows. The case of $L_1'=\cdot 00025$ is peculiar; c determined from the minimum agrees well with that determined from the maximum if $m=3$, but the form of the curve shows that m should $=2$; but it must be remembered that the theory is already known to fail if L_1' is very small.

We have now obtained c as a function of L_1'. According to the theory

$$1-cs=\frac{L_{12}\,.\,L_{21}}{(L_1+L_1')L_2'}\,;$$

so that if we plot $1/(1-cs)$ against L_1', we should get a straight line of which the intercept on the vertical axis is $-L_1$, the slope $L_2'/L_{12}\,.\,L_{21}$. The points are denoted by $\odot$ in fig. 16, s being taken as 1·025. The point for

$L_1'=\cdot00200$ lies far from any straight line that can be drawn through the remainder; but if we neglect this point, we get the line B, giving $L_1=\cdot00283$ and $L_{12}L_{21}/L_2'=\cdot00270$. L_1 is again smaller than the value determined by the oscillation measurements, namely $\cdot00578-\cdot00100=\cdot00478$. Using the line to recalculate c we obtain the figures shown under c (calc.); it should be noted that for $L_1'=0$, c (calc.) is 0·046, agreeing well with the observed value by the oscillation method 0·040.

We may now return to Table III., and, using this value of L_1, calculate T_2 from u/C_1. We obtain for the various values of L_1', $T_2=\cdot000192$, ·000182, ·000194, ·000188, ·000182. T_2 determined by the oscillation method is ·000258 sec.

It is clear, then, that the values obtained from the (V_{2m}, C_1) curves are not perfectly consistent amongst themselves, and that different methods of calculation give somewhat different results; further, that these values do not agree with those found by the oscillation measurements. On the other hand, we obtain values of the right order of magnitude, and it is possible that the method, if applied to an induction-coil, for which the theory appears much more nearly correct, might give the constants with as great accuracy and much less trouble than by measurement of the periods. To complete the determination L_2' would be required. If the variation of the peak potential with C_2, instead of C_1, were studied, a simple extension of the method would give the required information; but it has not been thought worth while to pursue the matter further.

One final point may be investigated. Owing to the uncertainty in T_2 it is not worth while to attempt to compare measured and calculated values for the peak potentials, but we may investigate how the measured values vary with L_1'. According to the theory, the peak potential ought to be nearly proportional to $\sqrt{L_1+L_1'}$; applying this criterion we obtain the following table according as we take for L_1 the value found from the determination of the periods or the method just given:—

TABLE V.

L_1'.	$L_1=\cdot00478$.	$L_1=\cdot00283$.
0	$3\cdot7\times10^4$	$4\cdot82\times10^4$
·00025	3·54	4·53
·00050	3·83	4·57
·00075	3·70	4·61
·00100	3·69	4·54
·00200	3·76	4·45
·00400	3·78	4·27

It appears that the ratio is more nearly constant for the value of L_1 determined from the period of the oscillations, but an intermediate value would be better than either. The primary self-inductance appears to vary with every method used for measuring it.

General Conclusion.

20. It seems, then, that the theory of the magneto which neglects the effects of interaction between the circuits and the iron core is not applicable in any important respect to machines of ordinary construction. It will predict neither the absolute values of the peak potential nor the variation of that potential with the primary capacity and the coupling.

An important improvement in the agreement between theory and experiment can be produced by inserting in series with the primary circuit a choking coil with an air-core; even with this coil in place the quantitative predictions of the theory are not fulfilled, but there is a qualitative agreement between theory and experiment concerning the relation between the peak potential and the constants of the circuits.

These results were very disappointing in view of the purpose for which the work was undertaken, namely the practical improvement of the machine. It seems that the only considerable improvement which could be obtained would be produced by decreasing greatly the losses in the iron core; no alteration of the coupling, the inductances, and the capacities which is practically possible seems likely to lead to any useful result. The inclusion in the primary of the air-core choking coil increases the peak potential obtainable with a given primary current, but in practice the decrease which it would produce in the primary current would more than counterbalance the increase in the ratio of peak potential to primary current. In order to avoid the decrease in the primary current it would be necessary that the choking coil should have a resistance small compared with that of the primary in the armature; to attain this condition and at the same time make its inductance comparable with that of the armature would involve the use of a mass and volume of copper which is quite outside the bounds of practical possibilities.

How far a reduction of iron losses is possible is a matter to be determined by further investigation. If hysteresis is the main source of loss, little improvement can be hoped for; but this alternative is not very probable. If hysteresis were the main agent, the loss and the damping should vary rapidly with the total field, and therefore with the primary current; whereas observation shows that the peak potential developed is very closely proportional to the current broken. If eddy-currents are the main source of loss, finer laminations might produce the desired result; but it is impossible to say at present whether any lamination which is mechanically possible would produce a marked effect.

In the present state of the investigation hardly more light is thrown on matters of theoretical interest than on those of practical importance. But it may be urged that several of the relations found are extremely suggestive, and may help to indicate the path to a more complete theory. It is perfectly clear that the effect of the interaction between the circuits and the iron is to change profoundly the relation between the two components of the oscillation which are predicted by the simple theory; it is possible that other components are introduced. If the effect of the iron core is merely to alter the relation between the amplitudes, the phase angles, and the damping coefficients of the two components, it seems not wholly impossible that a theory might be produced which would be as readily applicable to numerical computation and practical design, as is the simpler theory which Prof. Jones has applied with such success to the induction-coil.

Further progress might doubtless be made on the experimental side by determining more carefully the exact wave-forms of the oscillations, by analysing them into their harmonic components, and inquiring in what manner the relations between their amplitudes, phase angles, and damping differ from those predicted by the theory. Experiments in this direction had been begun when the time came to put an end to the investigation. They were by no means complete, and few conclusions could be drawn. So far as could be discovered, there was no sign of more than two different periods concerned in the oscillations, but whether the two periods which could always be detected were those of two *harmonic* components had not been fully ascertained.

Summary.

11. The paper is a continuation of the one in the March number. The object of the experiments is to discover how far the relation between the peak potential, the primary capacity, and the coupling of the circuits which is predicted by the theory is found in experiment.

12. The machine investigated is described.

13, 14. The constants of the circuits are measured by a standard method and compared with those determined by methods involving the use of the theory.

15. The damping of the oscillations is investigated as in § 7.

16, 17. The peak potentials are measured and compared with those predicted by theory. No agreement in any important respect is found between theory and experiment.

18. In accordance with a suggestion arising out of Prof. Jones's work, the effect of the presence of a choking coil with an air-core in the primary circuit is investigated. It is found that such a choking coil improves in some respects the agreement between calculation and experiment.

19. An attempt is made to use the measurements of the relation between the peak potential and the primary capacity, when the choking coil is in series with the primary, to determine the inductances and coupling of the circuits. A method for such a determination is given, but the results show that the observations, though qualitatively in agreement with those predicted by the theory, are not so completely in agreement that consistent values for these constants can be obtained.

20. General conclusions arising out of the research.

APPENDIX.

Values of U, U sin ϕ, and r for different values of c and u.

$c=0·05$.

u.	U.	r.	U sin ϕ.
·01	1·017	45	1·017
·02	1·022	32	1·022
·03	1·025	26·2	1·025
·04	1·026	22·8	1·026
·05	1·026	20·5	1·026
·06	1·026	18·8	1·025
·07	1·025	17·5	1·017
·08	1·024	16·4	1·016
·09	1·023	15·5	1·022
·10	1·021	14·8	1·021
·12	1·018	14·2	1·013
·14	1·011	13·7	1·001
·16	1·008	12·8	·988
·18	1·005	12·1	·997
·20	1·000	11·8	·995
·30	·974	10·4	·968
·40	·947	9·77	·932
·50	·919	9·34	·890
·60	·893	9·11	·854
·70	·869	8·94	·828
·80	·845	8·89	·806
·90	·823	8·83	·788
1·00	·803	8·80	·768

$c=0·10$.

u.	U.	r.	U sin ϕ.
·02	1·035	22·6	1·035
·04	1·046	16·1	1·041
·06	1·052	13·3	1·034
·08	1·054	12·0	1·047
·10	1·054	11·0	1·054
·12	1·054	10·1	1·048
·14	1·053	9·5	1·038
·16	1·050	9·0	1·002
·18	1·047	8·64	1·009
·20	1·044	8·35	1·017
·25	1·035	7·84	1·024
·30	1·025	7·37	1·023
·35	1·013	7·13	1·013
·40	1·000	6·80	·999
·45	·987	6·71	·998
·50	·975	6·56	·995
·6	·954	6·37	·990
·7	·924	6·28	·987
·8	·900	6·21	·985
·9	·877	6·18	·983
1·0	·855	6·18	·983
1·5	·762	6·29	·988
2·0	·689	6·53	·994
2·5	·632	6·86	·999
3·0	·587	7·14	·999
3·5	·549	7·45	·996

$c=0{\cdot}15$.

u.	U.	r.	U $\sin\phi$.
·01	1·035	22·4	1·035
·02	1·048	18·4	1·045
·04	1·063	13·1	1·040
·06	1·072	11·1	1·072
·08	1·077	9·63	1·044
·10	1·081	8·81	1·036
·12	1·084	8·31	1·057
·14	1·085	7·68	1·076
·16	1·085	7·37	1·083
·18	1·084	7·02	1·084
·20	1·082	6·79	1·082
·25	1·077	6·32	1·065
·30	1·069	5·97	1·040
·35	1·059	5·72	1·011
·40	1·048	5·52	·982
·45	1·036	5·41	·959
·50	1·025	5·31	·935
·60	1·000	5·13	·888
·70	·975	5·06	·855
·80	·951	4·99	·825
·90	·926	4·96	·806
1·00	·904	4·96	·786

$c=0{\cdot}20$.

u.	U.	r.	U $\sin\phi$.
·01	1·042	22·4	1·042
·02	1·058	15·9	1·053
·03	1·070	13·1	1·058
·04	1·078	11·7	1·074
·06	1·091	9·55	1·068
·08	1·099	8·43	1·070
·10	1·106	7·65	1·099
·12	1·111	7·06	1·111
·14	1·114	6·68	1·114
·16	1·116	6·36	1·105
·18	1·118	6·03	1·091
·20	1·118	5·81	1·075
·22	1·118	5·62	1·059
·24	1·116	5·48	1·041
·26	1·116	5·35	1·021
·28	1·114	5·21	1·003
·30	1·111	5·11	·983
·35	1·104	4·96	·965
·40	1·095	4·75	·973
·45	1·085	4·60	·977
·50	1·074	4·52	·976
·55	1·062	4·45	·970
·60	1·050	4·39	·966
·70	1·025	4·31	·950
·80	1·000	4·26	·930
·90	·976	4·24	·909
1·0	·951	4·24	·885
1·2	·906	4·27	·841
1·4	·864	4·31	·800
1·6	·825	4·37	·759
1·8	·790	4·44	·723
2·0	·760	4·52	·690
2·5	·692	4·75	·614
3·0	·638	4·96	·555
4·0	·538	5·40	·515
5·0	·500	5·80	·483

$c=0{\cdot}25$.

u.	U.	r.	U $\sin\phi$.
·01	1·048	20·1	1·048
·02	1·067	14·2	1·063
·03	1·080	11·7	1·076
·04	1·091	10·4	1·088
·06	1·108	8·64	1·068
·08	1·120	7·48	1·116
·10	1·129	6·84	1·128
·12	1·136	6·32	1·124
·14	1·143	5·93	1·110
·16	1·148	5·63	1·084
·18	1·150	5·38	1·059
·20	1·152	5·19	1·033
·22	1·154	5·00	1·000
·25	1·155	4·80	1·021
·30	1·152	4·52	1·046
·35	1·148	4·34	1·059
·40	1·142	4·19	1·068
·45	1·133	4·08	1·069
·50	1·123	3·97	1·069
·60	1·100	3·87	1·056
·70	1·076	3·81	1·037
·80	1·051	3·77	1·016
·90	1·026	3·74	·994
1·00	1·000	3·73	·970
1·20	·952	3·75	·921
1·40	·908	3·78	·877
2·00	·794	3·99	·755

$c=0{\cdot}30$.

u.	U.	r.	U $\sin\phi$.
·02	1·075	13·0	1·049
·04	1·110	9·46	1·080
·06	1·124	7·84	1·112
·08	1·139	6·78	1·137
·10	1·152	6·19	1·134
·15	1·174	5·23	1·059
·20	1·187	4·70	1·060
·25	1·193	4·38	1·096
·30	1·195	4·09	1·127
·35	1·194	3·90	1·144
·40	1·188	3·77	1·149
·45	1·182	3·67	1·151
·47	1·179	3·64	1·151
·50	1·174	3·60	1·148
·60	1·154	3·48	1·138
·70	1·130	3·42	1·118
·80	1·104	3·38	1·094
·90	1·078	3·36	1·070
1·00	1·052	3·36	1·044

$c=0{\cdot}40.$

u.	U.	*r.*	U sin ϕ.
·02	1·091	11·2	1·091
·04	1·127	8·14	1·106
·06	1·155	6·72	1·153
·08	1·177	5·87	1·138
·10	1·195	5·32	1·086
·15	1·229	4·46	1·122
·20	1·255	3·95	1·199
·25	1·273	3·68	1·240
·30	1·284	3·46	1·267
·35	1·289	3·31	1·281
·40	1·291	3·20	1·285
·45	1·289	3·10	1·289
·50	1·285	3·02	1·285
·55	1·279	2·97	1·279
·60	1·269	2·93	1·268
·70	1·248	2·86	1·246
·80	1·222	2·83	1·220
·90	1·195	2·81	1·192
1·00	1·167	2·81	1·164

$c=0{\cdot}50.$

u.	U.	*r.*	U sin ϕ.
·02	1·104	10·0	1·093
·04	1·149	7·25	1·148
·06	1·184	5·97	1·151
·08	1·213	5·20	1·089
·10	1·238	4·71	1·102
·15	1·289	3·96	1·230
·20	1·328	3·49	1·308
·25	1·357	3·26	1·352
·30	1·380	3·03	1·380
·35	1·398	2·88	1·396
·40	1·406	2·77	1·399
·45	1·412	2·69	1·393
·50	1·414	2·62	1·396
·55	1·414	2·57	1·390
·60	1·413	2·53	1·384
·70	1·392	2·47	1·356
·80	1·368	2·44	1·324
·90	1·337	2·42	1·295
1·00	1·306	2·415	1·260

XXXIV. *On the Dispersion of Diamond.* *By* L. SILBERSTEIN, *Ph.D., Lecturer in Mathematical Physics at the University of Rome* *.

THE object of the present paper is to apply the concept of electrical interaction of atoms, developed in previous papers †, to the refractive properties of diamond considered as a known assemblage of fixed "atomic centres," each containing a single dispersive electron and becoming a doublet in presence of an external electric field. The assumption of the mutual immobility of the "centres" themselves, *i. e.* of the whole atoms, need not be given up until one comes to contemplate the infra-red free frequencies; the ultra-violet ones, which chiefly interest us for the present, can all be thrown upon the electrons.

Consider a space lattice of points, indefinitely extended in all directions. Let all these points be occupied by the centres of *equal* atoms, each containing a single electron whose charge and mass are e, m. Write $B=e^2/m$, and denote by the vector $\mathbf{p}_i$ the electric moment of the doublet produced at the i-th centre by the electric force $\mathbf{E}$ of the incident light-wave (and by the co-operation of all the remaining doublets). Then, for monochromatic light of

---

* Communicated by the Author.

† For what follows it will suffice to consult the *second* paper on "Refractivity and Atomic Interaction," Phil. Mag. vol. xxxiii. p. 521 (1917), more especially the general formulæ (2), (3) on p. 522.

squared frequency γ, the equation of motion of the i-th electron gives, for any arrangement of equal centres,

$$(\gamma_0-\gamma)\mathbf{p}_i-\frac{B}{4\pi}\sum\frac{1}{r^3}[3\mathbf{u}(\mathbf{u}\mathbf{p}_j)-\mathbf{p}_j]=B\mathbf{G},$$

where γ_0 is the squared free frequency, common to all the electrons, $\mathbf{u}$ the unit vector drawn from the i-th towards the j-th centre (or *vice versa*), r the mutual distance of these centres, and $\mathbf{G}$ the force on the i-th electron, per unit charge, due to the external field $\mathbf{E}$ and due to all those doublets not already taken account of under the sign of summation.

If the sum, in our case a triply infinite series, embraces *all* the centres, that is to say, all "neighbours" of i, no matter how distant, then the force $\mathbf{G}$ reduces simply to $\mathbf{E}$. For, according to the opinion generally held, electromagnetic phenomena are influenced by particles of matter only so far as these contain electric charges. Now, in the case of our present problem it will be easy to exhaust all centres.

Thus,

$$\frac{\gamma_0-\gamma}{B}\mathbf{p}_i-\frac{1}{4\pi}\sum\frac{1}{r^3}[3\mathbf{u}(\mathbf{u}\mathbf{p}_j)-\mathbf{p}_j]=\mathbf{E}. \quad . \quad . \quad (1)$$

According to Bragg, diamond consists of the superposition of two face-centred cubic lattices, one of which is obtained from the other by translating it rigidly along a cube diagonal one-quarter of the length of the diagonal *. Each point of both lattices is occupied by a carbon atom. The result of this superposition is that each carbon atom occupies the centre of a regular tetrahedron whose four corners are occupied by the four nearest neighbours of that atom. The relation of this tetrahedron to the original face-centred lattice is well-represented in fig. 1, drawn in perspective (for which I am indebted to Mr. E. Hatschek). The lattice points are the centres of the five spheres drawn so as to be in contact with one another. The rough sketch, fig. 2, corresponding to Bragg's photograph on Plate III. of his work, represents a number of centres linked up by straight lines to their four nearest neighbours. (The actual distance of nearest neighbours, *i. e.* centres, or in usual notation $\frac{3}{4}d_{111}$, is $1{\cdot}54 \,.\, 10^{-8}$ cm. This, however, is irrelevant for our present

---

* W. H. and W. L. Bragg, 'X-Rays and Crystal Structure,' London, Bell (1915), pp. 105 *et seq.* A beautiful construction of the "fundamental domains" of Bragg's diamond lattice was given by L. Föppl in *Phys. Zeitschrift*, vol. xv. p. 191 (1914).

purpose, owing to the ultimate cancelling of all interactions.) Details showing the very strong experimental evidence in support of the view that such is precisely the structure of diamond will be found in Bragg's work.

Fig. 1.

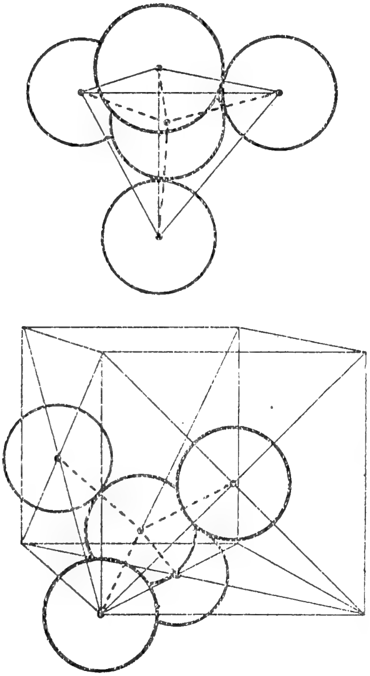
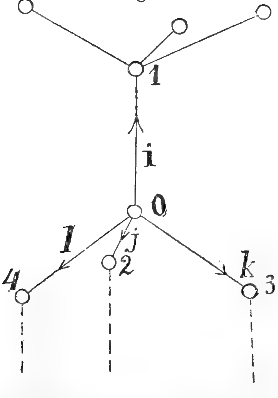


Fig. 2.



In order to evaluate the vector sum appearing in formula (1), fix as the i-th centre any of the lattice points, say O in fig. 2. Call 1, 2, 3, 4 its four nearest neighbours. The assemblage being unlimited and homogeneous, all the atomic centres are in equal conditions. Disregarding the free oscillations of the system, which would soon die away, let us take account only of the vibrations of $\mathbf{p}_i$ forced by the external field. Then, in a homogeneous* field $\mathbf{E}$, the moments $\mathbf{p}_i$ will either be all equal in size and direction, or, at the utmost, split into two classes, $\mathbf{p}'$ and $\mathbf{p}''$. In order to see this we can proceed as follows. Let, for instance, $\mathbf{E}$ be parallel to and concurrent with the vector $O1$. Then every atomic centre will be either such as O, in which $\mathbf{E}$ points *away* from it towards its next neighbour (viz. 1), or such as 1, or 2, etc., where $\mathbf{E}$ is directed *towards* them, away from their nearest neighbours. Let $\mathbf{p}'$ correspond to the first, and $\mathbf{p}''$ to the second class of centres. Then, denoting by Ω the linear vector operator in formula (1), *i. e.* writing it, as a dyadic,

$$\Omega=\frac{1}{4\pi}\sum\frac{1}{r^3}[3\mathbf{u}\,.\,\mathbf{u}-1], \quad . \quad . \quad . \quad . \quad (2)$$

the vector sum, say $\mathbf{P}$, representing the total action of all doublets upon the electron in O will be

$$\mathbf{P}=\Omega'\mathbf{p}'+\Omega''\mathbf{p}'',$$

where Ω' is as in (2) with $\mathbf{u}=\mathbf{u}'$, $r=r'$, and with Σ extended to the first class of points, and similarly for Ω''.

Now, $\mathbf{E}$ being along $O1$, the vectors $\mathbf{p}'$, $\mathbf{p}''$ can, by reasons of symmetry, have only the same or opposite directions. Thus, denoting the unit vector from O to 1 by $\mathbf{i}$,

$$\mathbf{p}'=p'\mathbf{i}, \quad \mathbf{p}''=sp'\mathbf{i},$$

where s is a scalar (undetermined thus far), and therefore,

$$\mathbf{P}=p'(\Omega'+s\Omega'')\mathbf{i}.$$

But if the field $\mathbf{E}$ is reversed, the rôles of the two classes of centres will be exchanged: the moment for each member of the first class will be $-sp'\mathbf{i}$, and for the second class $-p'\mathbf{i}$; at the same time the total $\mathbf{P}$ will be reversed, so that (the

---

* This will also hold with sufficient accuracy in the case of luminous oscillations, since their wave-length is very great as compared with the mutual distance of neighbour centres. The latter being of the order of 10^{-8} cm., a wave-length cube of visible, or of not too remote ultra-violet light, will contain 10^{11} or 10^{10} atomic doublets.

geometrical relations of all the centres to O remaining the same)

$$\mathbf{P}=p'(s\Omega'+\Omega'')\mathbf{i},$$

and by adding to the last equation,

$$\mathbf{P}=\frac{(1+s)p'}{2}(\Omega'+\Omega'')\mathbf{i},$$

that is,

$$\mathbf{P}=\tfrac{1}{2}(1+s)p'\Omega\mathbf{i}, \quad \ldots \ldots \quad (3)$$

where Ω is the total operator, taking account of all centres, as in (2). Thus, in order to show that the resultant interaction is nil *, it is enough to prove that $\Omega\mathbf{i}=0$.

Let us introduce as a reference system the four unit vectors $\mathbf{i}$, $\mathbf{j}$, $\mathbf{k}$, $\mathbf{l}$, drawn from a centre, say O, to its four nearest neighbours, 1, 2, 3, 4 (fig. 2). Then, first of all,

$$\mathbf{i}+\mathbf{j}+\mathbf{k}+\mathbf{l}=0, \quad \ldots \ldots \quad (4)$$

and $\mathbf{ij}=\mathbf{ik}=\mathbf{il}=\mathbf{jk}$, etc. Thus, multiplying (4) scalarly by $\mathbf{i}$, for instance, we have

$$\mathbf{ij}=\mathbf{ik}=\ldots.=-\tfrac{1}{3}, \quad \ldots \ldots \quad (5)$$

which, by the way, is a very simple deduction of the cosine of the equal angles $1O2$, $1O3$, etc. Using this, any vector $\mathbf{R}$ can at once be represented by means of its four projections upon $\mathbf{i}$, etc.,

$$R_1=\mathbf{Ri}, \quad R_2=\mathbf{Rj}, \quad R_3=\mathbf{Rk}, \quad R_4=\mathbf{Rl}.$$

In fact, write

$$\mathbf{R}=a_1\mathbf{i}+a_2\mathbf{j}+a_3\mathbf{k}+a_4\mathbf{l}\;;$$

then

$$R_1=a_1-\tfrac{1}{3}(a_2+a_3+a_4),$$

or

$$\mathrm{R}_1=\tfrac{4}{3}a_1-\tfrac{1}{3}\Sigma a_i, \text{ etc.}$$

Of these four equations three only are independent, since we have identically

$$R_1+R_2+R_3+R_4=\mathbf{R}(\mathbf{i}+\mathbf{j}+\mathbf{k}+\mathbf{l})=0.$$

Thus one of the four coefficients a_i remains arbitrary. We can put $\Sigma a_i=0$, so that $R_i=\frac{4}{3}a_i$, and therefore, the required

---

* When $\mathbf{E}$ falls into $\mathbf{i}$ or $O1$, and therefore also when $\mathbf{E}$ has the directions $O2$ or $O3$ or $O4$. This being the case, and $\mathbf{p}$ being at any rate a symmetrical or self-conjugate linear vector function of $\mathbf{E}$, the proof that $\Omega\mathbf{i}=0$ will imply also that the interaction is nil for any direction of $\mathbf{E}$.

representation of any vector,

$$\mathbf{R}=\tfrac{3}{4}\{R_1\mathbf{i}+R_2\mathbf{j}+R_3\mathbf{k}+R_4\mathbf{l}\},$$

or written as a dyadic,

$$\mathbf{i}\,.\,\mathbf{i}+\mathbf{j}\,.\,\mathbf{j}+\mathbf{k}\,.\,\mathbf{k}+\mathbf{l}\,.\,\mathbf{l}=\tfrac{4}{3}. \quad . \quad . \quad . \quad . \quad (6')$$

This may be a useful formula for various purposes *.

Returning to our problem, let us substitute in Ω

$$\mathbf{u}=\tfrac{3}{4}\{u_1\mathbf{i}+\ldots+u_4\mathbf{l}\}. \quad . \quad . \quad . \quad . \quad . \quad (6)$$

In the first place, it may be interesting to notice that if only the four nearest neighbours are taken into account, then $\Omega=0$ identically; in fact, in this case we have, by (2),

$$\Omega_{(1234)}=\frac{1}{4\pi r^3}\Sigma(3\mathbf{u}\,.\,\mathbf{u}-1)=\frac{1}{4\pi r^3}\{3(\mathbf{i}\,.\,\mathbf{i}+\ldots+\mathbf{l}\,.\,\mathbf{l})-4\},$$

which vanishes identically by (6′). Next, passing to the full operator, with the sum in (2) extended to the whole unlimited assemblage, notice that Ω is a symmetrical or self-conjugate linear vector operator and will thus have three mutually perpendicular principal axes. If the three corresponding principal values of such an operator are equal, the operator degenerates into an ordinary scalar or numerical factor. In that case, in fact, every direction is a principal one. Now, turning to our operator Ω, take $\mathbf{i}$, that is, a unit vector along $O1$, as operand. Then the result will obviously be a vector along $+\mathbf{i}$ or $-\mathbf{i}$, *i.e.*

$$\Omega\mathbf{i}=\Omega_{11}\,.\,\mathbf{i},$$

where Ω_{11} is a scalar, viz. $\mathbf{i}\Omega\mathbf{i}$. Again, remembering that $\mathbf{ij}=-\frac{1}{3}$, etc.,

$$\Omega_{12}=\mathbf{j}\,\Omega\mathbf{i}=\Omega_{11}\mathbf{ji}=-\tfrac{1}{3}\Omega_{11},$$

and similarly $\Omega_{13}=-\frac{1}{3}\Omega_{11}$, etc. Also, by the equivalence of $\mathbf{i}, \mathbf{j}, \mathbf{k}, \mathbf{l}$,

$$\Omega_{11}=\Omega_{22}=\text{etc.}$$

Thus, summarily,

$$\Omega_{11}=\Omega_{22}=\ldots=\Omega_{\alpha\alpha}=-3\Omega_{\alpha\beta}, \quad . \quad . \quad . \quad (7)$$

where $\alpha\neq\beta=1, 2, 3, 4$. Writing, for the moment, $\rho=\frac{1}{4\pi r^3}$, we have, by (2),

$$\Omega_{11}=\Sigma\rho(3u_1^2-1)\,;\ \Omega_{12}=\Sigma\rho(3u_1u_2-\mathbf{ij})=\Sigma\rho(3u_1u_2+\tfrac{1}{3}),$$

and since $\Omega_{12}=-\frac{1}{3}\Omega_{11}$,

$$\Sigma\rho(3u_1u_2+\tfrac{1}{3})=\Sigma\rho(\tfrac{1}{3}-u_1^2),\ \text{etc.}$$

---

* Similarly, if $\mathbf{i}, \mathbf{j}, \mathbf{k}$ are three unit vectors drawn from the centre of an equilateral triangle towards its corners, so that again $\mathbf{i}+\mathbf{j}+\mathbf{k}=0$, $\mathbf{ij}=\mathbf{jk}=\ldots=-\frac{1}{2}$ (=cos 120°), we have, for any operand,

$$\mathbf{i}\,.\,\mathbf{i}+\mathbf{j}\,.\,\mathbf{j}+\mathbf{k}\,.\,\mathbf{k}=3/2.$$

Hence, making use of (7),

$$\Sigma\rho(\tfrac{4}{3}-u_1^2-u_2^2-u_3^2-u_4^2)=4\Sigma\rho(3u_1u_2+\tfrac{1}{3})$$

or

$$\Sigma\rho\{u_1^2+u_2^2+u_3^2+u_4^2+12u_1u_2\}=0.$$

Now, by (6),

$$u_1^2+\ldots+u_4^2=\tfrac{2}{3}u_1u_2+\ldots+\tfrac{16}{9}\,; \quad \therefore\ \Sigma 16\rho u_1u_2+\tfrac{16}{9}\Sigma\rho=0,$$

i. e.

$$\Sigma\rho u_1u_2=-\tfrac{1}{9}\Sigma\rho,$$

and therefore,

$$\Omega_{12}=\Sigma\rho(-\tfrac{1}{3}+\tfrac{1}{3})=0,$$

so that, by (7), $\Omega_{\alpha\beta}=\Omega_{\alpha\alpha}=0$ for all values of the suffixes, *i. e.*

$$\Omega=0,$$

for any direction of the operand.

In plain language, *the resultant action* of all doublets in an unlimited * diamond lattice is *nil*.

Equation (1) thus becomes, for any i,

$$\frac{\gamma_0-\gamma}{B}\,\mathbf{p}_i=\mathbf{E},$$

as if there were no interaction between the carbon atoms.

If μ is the refractive index and $\mathfrak{N}$ the number of carbon atoms per unit volume of the crystal, then

$$(\mu^2-1)\mathbf{E}=\mathfrak{N}\overline{\mathbf{p}_i}=\mathfrak{N}\mathbf{p},$$

and therefore

$$\mu^2=1+\frac{\mathfrak{N}B}{\gamma_0-\gamma},$$

as in the elementary theory of dispersion.

If λ be the incident wave-length (*in vacuo*) and, similarly, λ_0 the free wave-length belonging to the electron of the carbon atom, then

$$\gamma=4\pi^2c^2/\lambda^2, \quad \gamma_0=4\pi^2c^2/\lambda_0^2,$$

c being the velocity of light *in vacuo*. Thus, with the abbreviations

$$u=\frac{1}{\lambda^2}, \quad a=\frac{\mathfrak{N}B}{4\pi^2c^2}=\mathfrak{N}\frac{e^2/m}{4\pi^2c^2}, \quad \ldots \quad (8)$$

---

* Such an assumption is physically equivalent to the limitation of the investigation to portions of the crystal distant enough from its surface. Deviations from the said simple behaviour will occur only in surface-layers whose thickness is comparable with the elementary spacings. It would be interesting to investigate them.

the dispersion formula of diamond becomes

$$\mu^2=1+\frac{a}{u_0-u}, \quad . \quad . \quad . \quad . \quad . \quad . \quad (9)$$

that is, the simplest, two-constant formula of the common type.

If, therefore, our assumptions are correct, this simple formula should represent the observed refractive index of diamond at least for not very long infra-red waves *, and also for not too short ones. For, in the latter case, the assumption of approximate, comparative homogeneity of the external **E** would break down.

Now, turning to facts, the best observations on diamond are, as far as I can make out, those due to Martens †, who measured the refractive index of this crystal from $\lambda=0{\cdot}643$ down to 0·313 microns. Martens himself represented his observations by means of a *three*-constant formula of the type $\mu^2=A+a/(u_0-u)$ (with A as much different from unity as 1·8755). His wave-lengths, however, attributed by him to the cadmium lines, deviate slightly but not insensibly from the values obtained more recently. Adopting the new λ-values, chiefly as they are given in Landolt-Börnstein's Tables (latest edition, 1912), I find that Martens' observations of μ are almost perfectly represented by our formula (9), to wit with

$$\left.\begin{array}{l} a=357{\cdot}40 \text{ micr.}^{-2} \\ u_0=76{\cdot}691 \quad ,, \end{array}\right\}, \quad . \quad . \quad . \quad . \quad . \quad (9a)$$

i. e. $\lambda_0=0{\cdot}1142$ micr. The values of the refractive index μ calculated by means of (9) and (9*a*) are given in the third column of the following Table. The first column contains Martens' short denominations of the spectrum lines used for his measurements, the second column gives their wave-lengths as here adopted, the fourth column contains Martens' observed μ (at 14° C.), and the last, the differences $\Delta\mu=\mu_{\text{calc.}}-\mu_{\text{obs.}}$ ‡. Below the horizontal bar three more observations are added, due to Walter, Schrauf, and Wülfing) as quoted in Landolt-Börnstein's Tables.

---

* According to Coblentz ("Infra-red," pt. vi. fig. 30, p. 46) the nearest absorption-band of diamond lies at about $\lambda=5$ microns. Coblentz has also observed one at 6·5 microns. At any rate these long free frequencies should not appreciably influence the dispersion in the domain here contemplated.

† F. F. Martens, *Ann. der Physik*, vol. viii. p. 459 (1902).

‡ Besides the 10 observations given in this Table there is one more, $\mu=2{\cdot}4253$ observed by Martens; this, however, being for a λ uncertain within the limits ·5338–·5379, is omitted here.

	λ.	$\mu_{\text{calc.}}$	$\mu_{\text{obs.}}$	$\Delta\mu$.
Cd 313	$0{\cdot}3133_1$	2·5247	2·5254	+·0003
,, 325	$0{\cdot}3251_4$	2·5132	2·5130	+·0002
,, 340	$0{\cdot}3403_7$	2·5003	2·5008	−·0005
,, 346	$0{\cdot}3446_3$	2·4955	2·4951	+·0004
,, 361	$0{\cdot}3610_7$	2·4856	2·4853	+·0003
,, 467	$0{\cdot}4678_4$	2·4401	2·4410	−·0009
,, 480	$0{\cdot}4799_9$	2·4371	2·4370	+·0001
,, 508	$0{\cdot}5085_8$	2·4306	2·4308	−·0002
Na 589	$0{\cdot}5890_2$	2·4171	2·4172	−·0001
Cd 643	$0{\cdot}6438_5$	2·4107	2·4109	−·0002
H_α 656	$0{\cdot}6563_0$	2·4095	2·4100 (Walter, 16° C.) 2·4103 (Wülfing)	−·0006
Li 670	$0{\cdot}6708_2$	2·4081	$2{\cdot}4084_5$ (Schrauf)	$-{\cdot}0003_5$
A 760	0·7604	2·4016	$2{\cdot}4024_5$ (Walter, 16° C.) 2·4024 (Wülfing)	−·0008

With the only exception of Cd 467 ($\Delta\mu = -{\cdot}0009$), the coincidence is almost perfect, and the deviations are, from ·313 to ·643 micr., pretty irregular. Even the three values obtained by other observers and, of course, with another sample of the crystal, although showing a tendency to increasing negative $\Delta\mu$, are too small to point to a systematic deviation of experiment from theory.

Future, more precise, measurements of the refractive index of diamond may reveal some discrepancies. But for the present we are justified in asserting that the one-term formula

$$\mu^2 - 1 = \frac{a}{u_0 - u} \quad \ldots\ldots \quad (9)$$

represents the observations with all desirable accuracy.

By (8) we have for the atomic constant of carbon denoted previously (*loc. cit.* p. 527) by k_0, and whose immediate meaning is

$$k_0 = \frac{1{\cdot}008}{3\pi} \cdot \frac{e'/c}{m} \cdot \frac{e'/c}{m_{\text{H}}}, \quad e' = e/\sqrt{4\pi}, \quad \ldots \quad (10)$$

the value

$$k_0 = \frac{1}{3}\frac{\text{M}}{d}a. \quad \ldots\ldots \quad (11)$$

It will be remembered that if e' be the charge of an electron proper, the numerical value of k_0 is

$$\epsilon = 1{\cdot}83 \,.\, 10^{10} \text{ cm. gr.}^{-1}.$$

Now, substituting in (11) the value (9 a) of a, and putting $M=12$, and the density of diamond, $d=3{\cdot}515$ (the mean of the best measurements [*]), we find as *the atomic coefficient of carbon*, entering into its "atomic refractivity" $C=\frac{k_0}{u_0-u}$,

$$k_0=4{\cdot}07\,.\,10^{10}\text{ cm. gr.}^{-1}. \quad . \quad . \quad . \quad (12\,a)$$

This exceeds the nearest multiple (2ϵ) of the proper electronic value by as much as $0{\cdot}22\,\epsilon$, a feature certainly not agreeable to the pan-electronists. I do not propose here to attempt to bring it down to an exact multiple of ϵ by some artificial assumptions.

The other atomic attribute of carbon, *its free wave-length*, would be, by (9 a),

$$\lambda_0=1142\text{ Å.U.}, \quad . \quad . \quad . \quad . \quad . \quad . \quad (12\,b)$$

as already quoted. There is nothing unlikely about this latter wave-length. Diamond has been known since 1862 (Miller, Phil. Trans. clii. pp. 861–887, quoted by Martens) to be transparent for ultraviolet rays as far down as 2240 Å.U. I do not know whether this crystal has been investigated for its absorption below this limit. It would be very interesting to undertake experiments in the region 2240→1000 Å.U. with the considerably improved modern means.

It will be kept in mind that the free wave-length (12 b) belongs to carbon inasmuch as we accept the absence of resultant interaction between the carbon atoms, which after what was said above it seems to me we have all reasons to accept. On the other hand, if (as we may abstractedly assume, for the sake of shedding some light upon the problem) there were some interaction, we should ultimately obtain a formula which would again be of the type of (9), viz.

$$\mu^2-1=\frac{a}{u_0'-u}, \quad . \quad . \quad . \quad . \quad . \quad . \quad (13)$$

the only difference consisting in a modified value of the "free wave-length" ($\lambda_0'=1:\sqrt{u_0'}$). This statement can easily be proved.

In fact if there is some resultant interaction, *i. e.* if $\Omega\neq0$, we have, by (1), with Ω as explained in (2), with equal $\mathbf{p}_i=\mathbf{p}$,

$$\frac{\gamma_0-\gamma}{B}\mathbf{p}=\mathbf{E}+\Omega\mathbf{p}.$$

---

[*] Martens does not quote the density of his piece of diamond.

Now, if Ω is a non-vanishing operator (that is to say, not an annihilator), it is, at any rate, self-conjugate and has *equal* principal values, at least along our previous **i**, **j**, **k**, and **l**. But such being the case, Ω can only be an ordinary scalar factor, say ϖ. Thus

$$\mathrm{p}\left[\frac{\gamma_0-\gamma}{B}-\varpi\right]=\mathbf{E},$$

and since, as on p. 402, $(\mu^2-1)\mathbf{E}=\mathfrak{N}\mathrm{p}$, we have, instead of $\mu^2-1=\frac{\mathfrak{N}B}{\gamma_0-\gamma}$, the dispersion formula

$$\mu^2-1=\frac{\mathfrak{N}B}{\gamma_0-B\varpi-\gamma}=\frac{a}{u_0-u-a\varpi/\mathfrak{N}}, \quad . \quad . \quad (14)$$

which is precisely of the form (13), with the same value of a as in absence of interaction, and with u_0 replaced by

$$u_0'=u_0-\frac{a\varpi}{\mathfrak{N}}. \quad . \quad . \quad . \quad . \quad . \quad . \quad (15)$$

Notice in passing that μ^2-1 being of the form $\frac{\mathrm{A}}{\mathrm{B}-u}$, where A, B are constants, the same is automatically true of such refractivity expressions as the much renowned $\frac{\mu^2-1}{\mu^2+2}$ of Lorenz-Lorentz. More generally, if κ be any constant number, and if

$$\mu^2-1=\frac{\mathrm{A}}{\mathrm{B}-u},$$

we have also, identically,

$$\frac{\mu^2-1}{\mu^2+(\kappa-1)}=\frac{\mathrm{A}}{(\mathrm{A}+\kappa\mathrm{B})-\kappa u},$$

that is, again of the form const. : (const. $-u$). This simple remark may be useful in discussing critically the Lorentz expression ($\kappa=3$) and similar ones appearing in more recent empirical formulæ.

Returning to formula (15) notice that (ϖ being at any rate positive) the effect of interaction in such crystals as the diamond would consist in *lowering the free frequency* (the latter being proportional to the square root of u) or in shifting it towards the red domain of the spectrum. A more detailed discussion of these theoretical relations may be left to the reader.

February 18th, 1919.
Research Dept., Adam Hilger, Ltd.

XXXV. *Examples of Operational Methods in Mathematical Physics.* *By* T. J. I'A. Bromwich, *Sc.D., F.R.S.*[*]

THE use of operational methods has been repeatedly urged, in electrical and other physical problems, by Heaviside[†]. It does not, however, appear that the method has been as widely used as it deserves to be, both on account of its simplicity and its elegance; and even when the method has been used, some hesitation has been shown in accepting the conclusions based upon this process.

Attacking certain problems by a different method, I was led to some general rules which have confirmed the accuracy of Heaviside's method[‡]; and I have recently developed a more elementary treatment of the method (given in § 3 below).

In order to illustrate the merits of the method a number of examples have been worked out in Conduction of Heat and in Electricity. Some of these will be found in §§ 1, 2 below: those in § 1 are suggested by the problem of thermometers carried by aeroplanes. It will be found, I think, that these solutions are more direct, as well as being shorter than those which have been given previously; it is easy, moreover, to take account of the effect of surface-conductivity without adding substantially to the difficulties of the discussion—see, for instance, examples (β), (γ) in § 1.

§ 1. *Some Operational Solutions in Heat-problems.*

In connexion with recent observations on thermometers carried on aeroplanes, it has been suggested that the surface-conditions may be represented by supposing the temperature in the surrounding atmosphere to be proportional to the time t. This represents the assumption of a uniform rate of descent and a uniform temperature-gradient in the atmosphere; and solutions of certain problems under this hypothesis have been worked out by Mr. A. R. McLeod[§].

It is easy to apply operational methods to solve such problems, using an interpretation of the formulæ which is an extension of a known formula due to Heaviside: fairly simple direct proofs will be given below (§ 3) both for the new formula and for Heaviside's original result. But in

---

[*] Communicated by the Author.

[†] Electrical Papers and Electromagnetic Theory, *passim*.

[‡] Proc. Lond. Math. Soc. vol. xv. ser. 2, p. 401 (1916). See in particular, §§ 3, 4.

[§] Phil. Mag. vol. xxxvii. ser. 6, p. 134 (January 1919).

the present section the formula will be used to obtain the results given by Mr. McLeod and some extensions of them—found by allowing for surface-conductivity.

The new formula may be stated as follows :—

Suppose that we have found, by operational methods, the symbolic equation

$$x=\frac{\mathrm{F}(p)}{\Delta(p)}(\mathrm{G}t), \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (1)$$

where p stands for the operator d/dt and G is a constant ; then the solution (1) is to be interpreted as

$$x=\mathrm{G}\left\{\mathrm{N}_0 t+\mathrm{N}_1+\sum_\alpha \frac{\mathrm{F}(\alpha)}{\alpha^2\Delta'(\alpha)}e^{\alpha t}\right\}, \quad . \quad . \quad . \quad . \qquad (2)$$

where $p=\alpha$ is any root of $\Delta(p)=0$, the summation extends to all such roots, and N_0, N_1 are defined by the algebraic expansion

$$\frac{\mathrm{F}(p)}{\Delta(p)}=\mathrm{N}_0+\mathrm{N}_1 p+\mathrm{N}_2 p^2+\ldots, \quad . \quad . \quad . \qquad (3)$$

The solution (2) has the property of reducing to zero at $t=0$.

Heaviside's equation * is given similarly in the form

$$\frac{\mathrm{F}(p)}{\Delta(p)}\mathrm{P}=\mathrm{P}\left\{\mathrm{N}_0+\sum_\alpha \frac{\mathrm{F}(\alpha)}{\alpha\Delta'(\alpha)}e^{\alpha t}\right\}, \quad . \quad . \quad . \quad . \qquad (4)$$

where P is a constant. It will be noticed that equation (4) can be derived from (2) by differentiation with respect to t ; but the reverse step of integrating (4) does not give the constant N_1 immediately. We proceed now to apply equation (2) to the heat-problems mentioned.

(α) *The surface of a sphere is maintained at temperature* $u=\mathrm{G}t$.

The differential equation for u in the sphere can be written

$$\left.\begin{aligned} a^2\frac{\partial^2}{\partial r^2}(ru)&=\frac{\partial}{\partial t}(ru), \\ \text{where} \qquad a^2&=\mathrm{K}/\rho\sigma, \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad . \qquad (5)$$

K being the thermal conductivity, ρ the density, and σ the specific heat of the substance of the sphere.

Now writing symbolically p for $\partial/\partial t$ and

$$q^2=pc^2/a^2, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (6)$$

---

* 'Electrical Papers,' vol. ii. pp. 226 and 373.

we obtain the solution of (5) in the form

$$\frac{ru}{c}=\frac{\sinh(qr/c)}{\sinh q}\,\mathrm{G}t, \quad . \quad . \quad . \quad . \quad . \quad . \quad (7)$$

because ru must vanish with r, and $u=\mathrm{G}t$ at the surface $r=c$.

It is quite easy to interpret (7) by the aid of equation (2)* ; but it seems unnecessary to write the result out at length, as the observations refer to the mean temperature $\bar{u}$, which we shall proceed to calculate in symbolic form.

Thus we have the mean temperature

$$\bar{u}=\frac{3}{c^3}\int_0^c r^2u\,dr=3\left(\frac{1}{q}\coth q-\frac{1}{q^2}\right)\mathrm{G}t, \quad . \quad . \quad (8)$$

and this has now to be interpreted by means of equation (2).

In the first place we know that

$$q\coth q=1+\frac{1}{3}q^2-\frac{1}{45}q^4+\dots$$

and so the expansion corresponding to (3) is

$$3\left(\frac{1}{q}\coth q-\frac{1}{q^2}\right)=1-\frac{1}{15}q^2+\dots=1-\frac{1}{15}\frac{pc^2}{a^2}+\dots$$

Thus here

$$\mathrm{N}_0=1, \quad \mathrm{N}_1=-\frac{1}{15}c^2/a^2. \quad . \quad . \quad . \quad . \quad . \quad (9)$$

Again in (8) we can write

$$\mathrm{F}(p)=3(q\cosh q-\sinh q)/q^3,$$
$$\Delta(p)=\sinh q/q,$$

because each of these functions can be expressed in positive integral powers of p. The roots α of $\Delta(p)=0$ are given by

$$q=n\pi\iota, \quad p=-n^2\pi^2a^2/c^2=\alpha, \quad . \quad . \quad . \quad . \quad (10)$$

where n is any positive integer †.

Then we find that ‡

$$\Delta'(\alpha)=\frac{1}{q}\frac{dq}{dp}\cosh q=\frac{1}{2\alpha}\cosh q$$

and by (6)

$$\mathrm{F}(\alpha)=\frac{3}{q^2}\cosh q=\frac{3a^2}{\alpha c^2}\cosh q.$$

---

* It will be found that the operator in equation (7) really involves only q^2 and so can be expressed as the quotient of two series in p; but the form used in (7) is more compact and is easier to work with.

† Negative integers do not give any fresh values for p: and so should not be included because we require only the complete set of values of α.

‡ Since $\Delta(\alpha)=0$ it is only necessary to differentiate $\sinh q$, in order to evaluate $\Delta'(\alpha)$.

Thus
$$\frac{F(\alpha)}{\Delta'(\alpha)}=\frac{6a^2}{c^2}. \quad . \quad . \quad . \quad . \quad . \quad . \quad (11)$$

Hence we have the interpretation of (8):
$$\bar{u}=G\left(t-\frac{c^2}{15a^2}+\frac{6a^2}{c^2}\sum_{\alpha}\frac{1}{\alpha^2}e^{\alpha t}\right)$$
$$=G\left\{t-\frac{c^2}{15a^2}+\frac{6c^2}{\pi^4a^2}\sum_{n=1}^{\infty}\frac{1}{n^4}e^{-n^2\pi^2a^2t/c^2}\right\}, \quad . \quad (12)$$

and this result agrees with that calculated by A. R. McLeod (see p. 136 of the paper quoted).

(β) *The same problem as (α), allowing for surface-conductivity.*

The surface-condition is then
$$K\frac{\partial u}{\partial r}=h(Gt-u), \quad \text{at } r=c,$$

where h is the surface-conductivity, supposed to be the same at all points on the surface of the spherical boundary.

Substituting in this condition the symbolic solution
$$ru=\sinh(qr/c)A,$$

where A is independent of r, we deduce the formula
$$\frac{ru}{c}=\frac{ch\sinh(qr/c)}{K(q\cosh q-\sinh q)+ch\sinh q}(Gt). \quad . \quad (13)$$

Then the mean temperature is given by the symbolic formula
$$\bar{u}=\frac{3ch(q\cosh q-\sinh q)}{q^2\{K(q\cosh q-\sinh q)+ch\sinh q\}}(Gt), \quad . \quad (14)$$

and it will be noted that the formula (14) reduces to (8) if $ch/K\to\infty$.

To obtain the interpretation of (14), we note that
$$q\cosh q-\sinh q=(\tfrac{1}{2}-\tfrac{1}{6})q^3+(\tfrac{1}{24}-\tfrac{1}{120})q^5+\dots$$
$$=\tfrac{1}{3}q^3+\tfrac{1}{30}q^5+\dots$$

Thus we find, on substituting in (14), and reducing
$$\frac{F(p)}{\Delta(p)}=\frac{ch(1+\frac{1}{10}q^2+\dots)}{ch(1+\frac{1}{6}q^2+\dots)+K(\frac{1}{3}q^2+\dots)}$$
$$=1+\left(\frac{1}{10}-\frac{1}{6}-\frac{K}{3ch}\right)q^2+\dots$$
$$=1-\left(\frac{1}{15}+\frac{K}{3ch}\right)\frac{pc^2}{a^2}+\dots$$

Hence, on comparison with (3), we have

$$\mathrm{N}_0=1, \quad \mathrm{N}_1=-\left(\frac{1}{15}+\frac{\mathrm{K}}{3ch}\right)\frac{c^2}{a^2}. \quad . \quad . \quad . \quad (15)$$

We can write, as in (8),

$$\mathrm{F}(p)=3(q\cosh q-\sinh q)/q^3$$

and then (14) leads to the form

$$\Delta(p)=\{\lambda(q\cosh q-\sinh q)+\sinh q\}/q,$$

where $\lambda=\mathrm{K}/(ch)$.

The roots α of $\Delta(p)=0$ are known to correspond to purely imaginary values of q *; and they are expressed by the formulæ

$$\left.\begin{aligned} q=\iota\omega, \quad p=-\omega^2a^2/c^2=\alpha \\ \omega\cot\omega=1-1/\lambda=1-ch/\mathrm{K}, \end{aligned}\right\} \quad . \quad . \quad . \quad (16)$$

where

and as in (10), only the positive values of ω are to be retained.

It will now be seen that, since $\Delta(\alpha)=0$,

$$\Delta'(\alpha)=\frac{1}{q}\frac{dq}{dp}(\lambda q\sinh q+\cosh q)=\frac{1}{2\alpha}(\cos\omega-\lambda\omega\sin\omega),$$

while

$$\mathrm{F}(\alpha)=\frac{3}{\omega^3}(\sin\omega-\omega\cos\omega).$$

Thus

$$\frac{\mathrm{F}(\alpha)}{\Delta'(\alpha)}=\frac{6\alpha}{\omega^3}\cdot\frac{\sin\omega-\omega\cos\omega}{\cos\omega-\lambda\omega\sin\omega}=\frac{6a^2}{c^2}\cdot\frac{\sin\omega-\omega\cos\omega}{\omega(\lambda\omega\sin\omega-\cos\omega)},$$

which can be rearranged in various forms, by making use of equation (16). The most compact formula appears to be

$$\frac{\mathrm{F}(\alpha)}{\Delta'(\alpha)}=\frac{6a^2}{c^2}\cdot\frac{1}{1-\lambda+\lambda^2\omega^2}=\frac{6a^2}{c^2}\cdot\frac{(ch)^2}{ch(ch-\mathrm{K})+\mathrm{K}^2\omega^2}. \quad (17)$$

Substituting in (2) from equations (15), (17) we now deduce the mean temperature

$$\bar{u}=\mathrm{G}\left\{t-\left(\frac{1}{15}+\frac{\mathrm{K}}{3ch}\right)\frac{c^2}{a^2}+\frac{6c^2}{a^2}\Sigma\frac{1}{\omega^4}\cdot\frac{(ch)^2e^{-\omega^2a^2t/c^2}}{ch(ch-\mathrm{K})+\mathrm{K}^2\omega^2}\right\}. (18)$$

The formula (18) gives the extension of (12); and agrees, as far as concerns the first and second terms, with A. R. McLeod's result (*l. c.* p. 143).

---

* This is a well-known result and has been proved in various ways in connexion with problems of conduction of heat in spheres. It is deducible from the general discussion given on p. 444 of my paper previously quoted (Proc. Lond. Math. Soc. vol. xv.); in the notation of that paper, we have to write $\rho=0$ in the general formulæ there established.

(γ) *Problems for a cylinder corresponding to those of* (α), (β).

It is now necessary to use the differential equation

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u}{\partial r}\right)=\frac{1}{a^2}\frac{\partial u}{\partial t},$$

which leads to the use of the function $I_0(qr/c)$ in place of $\{\sinh(qr/c)/(qr/c)\}$, where I_0 is the modified Bessel function *.

We then obtain for the mean temperature

$$\bar{u}=\frac{2}{c^2}\int_0^c ru\,dr=\frac{2}{q}\frac{I_0'(q)}{I_0(q)}Gt, \quad . \quad . \quad . \quad (19)$$

if the surface-conductivity is treated as infinite ; or

$$\bar{u}=\frac{2}{q}\frac{ch I_0'(q)}{ch I_0(q)+KqI_0'(q)}Gt. \quad . \quad . \quad . \quad (20)$$

in general.

On expanding in powers of q we find that

$$I_0(q)=1+\tfrac{1}{4}q^2+\tfrac{1}{64}q^4+\ldots$$
$$I_0'(q)=\tfrac{1}{2}q(1+\tfrac{1}{8}q^2+\ldots)$$

Thus the expansion of the operator in (20) is

$$\frac{ch(1+\frac{1}{8}q^2+\ldots)}{ch(1+\frac{1}{4}q^2+\ldots)+\frac{1}{2}Kq^2+\ldots}=1+\left(\frac{1}{8}-\frac{1}{4}-\frac{K}{2ch}\right)q^2+\ldots$$
$$=1-\left(\frac{1}{8}+\frac{K}{2ch}\right)p\frac{c^2}{a^2}+\ldots$$

Hence on comparison with (3) we find

$$N_0=1,\quad N_1=-\left(\frac{1}{8}+\frac{K}{2ch}\right)\frac{c^2}{a^2}. \quad . \quad . \quad . \quad . \quad (21)$$

In equations (19) and (20) we can take

$$F(p)=\frac{2}{q}I_0'(q),$$
$$\Delta(p)=I_0(q) \quad \text{or} \quad I_0(q)+\lambda qI_0'(q),$$

(where $\lambda=K/ch$, as before), because each of these functions can be expressed in positive integral powers of p.

---

* Thus $$I_0(z)=1+\frac{z^2}{2^2}+\frac{z^4}{2^2.4^2}+\frac{z^6}{2^2.4^2.6^2}+\ldots$$

The roots α of $\Delta(p)=0$ for equation (19) are given by

where
$$\left.\begin{array}{r} q=\iota\omega, \quad p=-\omega^2a^2/c^2=\alpha \\ J_0(\omega)=0, \end{array}\right\} \quad . \quad . \quad . \quad . \quad . \quad (22)$$

and as before only positive values of ω are to be included. We have then

$$\Delta'(\alpha)=\frac{dq}{dp}I_0'(q)=\frac{c^2}{2qa^2}I_0'(q)$$

and so

$$\frac{F(\alpha)}{\Delta'(\alpha)}=\frac{4a^2}{c^2} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (23)$$

Similarly the roots α of $\Delta(p)=0$ for equation (20) are given by

where
or
$$\left.\begin{array}{r} q=\iota\omega, \quad p=-\omega^2a^2/c^2=\alpha \\ J_0(\omega)+\lambda\omega J_0'(\omega)=0 \\ chJ_0(\omega)+K\omega J_0'(\omega)=0 \end{array}\right\} \quad . \quad . \quad . \quad (24)$$

and again positive values only are to be used for ω *.

We have then

$$\Delta'(\alpha)=\frac{dq}{dp}\{I_0'(q)+\lambda q I_0(q)\}$$

and so

$$\frac{F(\alpha)}{\Delta'(\alpha)}=\frac{2}{q}\frac{dp}{dq}\frac{I_0'(q)}{I_0'(q)+\lambda q I_0(q)}$$
$$=\frac{4a^2}{c^2}\frac{J_0'(\omega)}{J_0'(\omega)-\lambda\omega J_0(\omega)}=\frac{4a^2}{c^2}\frac{1}{1+\lambda^2\omega^2}, \quad . \quad (25)$$

where the final reduction follows from (24).

Substituting in (2) from (21), (23), and (25), we find the mean temperature

$$\bar{u}=G\left(t-\frac{1}{8}\frac{c^2}{a^2}+\frac{4c^2}{a^2}\Sigma\frac{1}{\omega^4}e^{-a^2\omega^2t/c^2}\right). \quad . \quad . \quad (26)$$

or

$$G\left\{t-\left(\frac{1}{8}+\frac{K}{2ch}\right)\frac{c^2}{a^2}+\frac{4c^2}{a^2}\Sigma\frac{1}{\omega^4}\cdot\frac{(ch)^2}{(ch)^2+(K\omega)^2}e^{-a^2\omega^2t/c^2}\right\}, (27)$$

where the values of ω are given by (22) or (24) respectively.

Of these formulæ the former agrees with a result given by A. R. McLeod (*l. c.* p. 140); and (27) agrees, so far as the first and second terms are concerned (*l. c.* p. 143).

---

* That the values of ω are real in (22) and (24) is well-known: and proofs can be obtained from the general discussion given on p. 444 of my paper already quoted on p. 411 above.

§ 2. *Some Applications of Heaviside's Formula.*

A simple case which can be dealt with at once by the aid of the results of § 1 is the following :—

(δ) *A sphere is heated to uniform temperature* $u = u_0$, *and allowed to cool, the surrounding atmosphere being at zero.*

Here $u_0 - u$ is initially zero ; and, proceeding as before, we find that

$$\frac{r}{c}(u_0 - u) = \frac{\sinh(qr/c)}{\sinh q} u_0, \quad . \quad . \quad . \quad . \quad (28)$$

taking for brevity the case in which the surface-conductivity h is treated as infinite.

Then the mean temperature is given by

$$u_0 - \bar{u} = 3\left(\frac{1}{q}\coth q - \frac{1}{q^2}\right)u_0. \quad . \quad . \quad . \quad . \quad (29)$$

To interpret (29), we observe that the operator is the same as in (8) ; and accordingly we need only substitute in (4) from equations (9), (10), and (11). The result is easily found to be

$$u_0 - \bar{u} = u_0\left\{1 - \frac{6}{\pi^2}\sum_{n=1}^{\infty}\frac{1}{n^2}e^{-n^2\pi^2a^2t/c^2}\right\}$$

or

$$\bar{u} = \frac{6u_0}{\pi^2}\sum_{n=1}^{\infty}\frac{1}{n^2}e^{-n^2\pi^2a^2t/c^2}. \quad . \quad . \quad . \quad . \quad (30)$$

It will be noticed that, as might be anticipated, the series (30) can be derived from (12) by differentiation with respect to t.

Further examples of Heaviside's formula can be derived similarly from the other problems of § 1.

(ε) *Problems of induction-balances.*

Electrical problems, in which the arms of a balance contain inductances and condensers (in addition to resistances) have been considered repeatedly by Heaviside* ; it is hardly too much to say that the formula (4) was constructed with a view to such problems—and Heaviside's own discussion of the formula is based upon electrical considerations.

---

* For example, see 'Electrical Papers,' vol. i. p. 412, vol. ii. pp. 256, 280, etc., and many other papers.

Nevertheless, it seems to be doubtful if the ease and simplicity of the operational method have been fully appreciated; and it is hoped that the following examples may do something to call fresh attention to the merits of the operational method.

Supposing an E.M.F. applied by a battery of strength P, it is usually easy to express the galvanometer current in the symbolic form

$$x=\frac{\mathrm{F}(p)}{\Delta(p)}\mathrm{P},$$

which is to be interpreted as

$$x=\left\{\frac{\mathrm{F}(0)}{\Delta(0)}+\sum_{\alpha}\frac{\mathrm{F}(\alpha)}{\alpha\Delta'(\alpha)}e^{\alpha t}\right\}\mathrm{P},$$

repeating the formula (4) quoted above.

Thus, to produce a complete balance we must have

$$\mathrm{F}(0)=0,\quad \mathrm{F}(\alpha)=0 \quad . \quad . \quad . \quad . \quad . \quad . \qquad (31)$$

for all roots $p=\alpha$ of $\Delta(p)=0$.

But, since $\mathrm{F}(p)$ is of lower degree * than $\Delta(p)$, it follows from (31) that $\mathrm{F}(p)$ must be identically zero. Hence we have the working rule † :—

To obtain the conditions for a complete balance, put zero for the galvanometer current; this will lead to a certain algebraic condition in p which must be satisfied identically, when p is treated as an algebraic variable.

It may, however, prove to be impossible to obtain a complete balance: then we should make the time-integral of the current zero, so that the galvanometer will show no ballistic effect.

Thus we wish to make

$$\int_0^\infty x\,dt=0$$

or

$$\int_0^\infty\left\{\frac{\mathrm{F}(0)}{\Delta(0)}+\sum_{\alpha}\frac{\mathrm{F}(\alpha)}{\alpha\Delta'(\alpha)}e^{\alpha t}\right\}dt=0.$$

Now, since all the roots α are negative in actual problems, this leads to the two conditions

$$\frac{\mathrm{F}(0)}{\Delta(0)}=0,\quad\text{and}\ -\sum_{\alpha}\frac{1}{\alpha^2}\frac{\mathrm{F}(\alpha)}{\Delta'(\alpha)}=0.$$

---

* See § 3 below.

† Heaviside, 'Electrical Papers,' vol. ii. p. 259.

But on reference to § 3 below, equations (38)–(40), it will be seen that

$$\frac{F(0)}{\Delta(0)} = N_0, \quad -\sum_{\alpha} \frac{1}{\alpha^2} \frac{F(\alpha)}{\Delta'(\alpha)} = N_1,$$

where N_0, N_1 are the two coefficients defined by the expansion

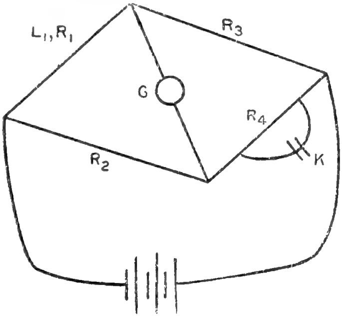
$$\frac{F(p)}{\Delta(p)} = N_0 + N_1 p + N_2 p^2 + \dots$$

used in equation (3) of § 1 above.

Thus we find the supplementary rule* :

To obtain the conditions for the time-integral to be zero, the algebraic condition (found as in the last statement) need vanish only so far as the constant term and the coefficient of p.

To illustrate, let us consider the simple example (due to Rimington †) in which one arm of the balance contains an inductance and the opposite arm is shunted with a condenser.



The resistance operators are easily seen to be

$$Z_1 = L_1 p + R_1, \quad Z_2 = R_2, \quad Z_3 = R_3,$$

$$Z_4 = \frac{1}{\frac{1}{r_4} + Kp} + R_4 - r_4 = \frac{R_4 + Kpr_4(R_4 - r_4)}{1 + Kpr_4},$$

where r_4 denotes the resistance of the part of R_4 which is shunted by the condenser K. Then the algebraic condition for a complete balance is

$$Z_1 Z_4 - Z_2 Z_3 = 0$$

or

$$(L_1 p + R_1)\{R_4 + Kpr_4(R_4 - r_4)\} = R_2 R_3 (1 + Kpr_4). \quad (32)$$

---

* Heaviside, 'Electrical Papers,' vol. ii. p. 260.

† Phil. Mag. vol. xxiv, 5th ser. July 1887.

Equation (32) can be satisfied identically only if $r_4=R_4$, so that the condenser is shunted against the *whole* resistance R_4: and then the balance reduces to an arrangement given by Maxwell [*]. Thus we find *the conditions for a complete balance*,

$$r_4=R_4, \quad R_1R_4=R_2R_3, \quad L_1=KR_2R_3=KR_1R_4. \qquad (33)$$

But, when r_4 is less than R_4, we can still arrange for zero time-integral by the conditions:

$$R_1R_4=R_2R_3, \quad L_1R_4+R_1Kr_4(R_4-r_4)=R_2R_3Kr_4,$$

found from the constant term and the coefficient of p in (32); these readily reduce to the forms

$$R_1R_4 = R_2R_3, \quad L_1 = KR_1r_4^2/R_4. \quad . \quad . \quad . \quad . \quad . \qquad (34)$$

The advantages of the operational method will be seen very clearly if the above simple calculations are compared with those given, for instance, by A. O. Allen [†].

Further examples of induction-balances are described in Allen's paper just quoted; and any of these will be found to give simple exercises in the practical application of operational methods. General discussions will be found in vol. ii. of Heaviside's 'Electrical Papers' [‡].

§ 3. *Direct Proofs of the Equations* (2), (4) *given in* § 1 *above.*

It is somewhat more natural to take Heaviside's equation (4) first, in which we suppose $F(p)$, $\Delta(p)$ to be simple polynomials, with $\Delta(p)$ of higher degree than $F(p)$.

Then, as a matter of elementary algebra, we can write

$$\frac{F(p)}{\Delta(p)} = \sum_{\alpha} \frac{A}{p-\alpha}, \text{ where } A=\frac{F(\alpha)}{\Delta'(\alpha)}. \quad . \quad . \quad . \qquad (35)$$

---

[*] 'Electricity and Magnetism,' vol. ii. Art. 778.

[†] Phil. Mag. vol. xxv. 6th series, 1913, p. 520; the conditions (33), (34) are considered in §§ 2, 3 of his paper. Reference may also be made (for a different method) to a paper by J. P. Dalton on p. 56 of the same volume.

[‡] See, for instance, pp. 33, 102, 260–297, etc.

To interpret the symbolic equation

$$z=\frac{1}{p-\alpha}\,\mathrm{P}$$

is now easy; for z must be that solution of the differential equation

$$(p-\alpha)z=\mathrm{P}$$

which reduces to zero at $t=0$. And on evaluating this solution, we obtain *

$$z=\frac{1}{a}(e^{at}-1)\,\mathrm{P}. \quad \ldots\ldots \quad (36)$$

Substituting in (35) from (36), we now see that

$$\frac{\mathrm{F}(p)}{\Delta(p)}\,\mathrm{P}=\sum_{\alpha}\frac{\mathrm{A}}{\alpha}(e^{\alpha t}-1)\,\mathrm{P}. \quad \ldots\ldots \quad (37)$$

Further, since (35) is an algebraic identity, we may write $p=0$, which gives

$$-\sum_{\alpha}\frac{\mathrm{A}}{\alpha}=\frac{\mathrm{F}(0)}{\Delta(0)}=\mathrm{N}_0. \quad \ldots\ldots \quad (38)$$

On combining (37) and (38), we obtain the formula

$$\frac{\mathrm{F}(p)}{\Delta(p)}\,\mathrm{P}=\mathrm{P}\left\{\mathrm{N}_0+\sum_{\alpha}\frac{1}{\alpha}\frac{\mathrm{F}(\alpha)}{\Delta'(\alpha)}\,e^{at}\right\},$$

which is Heaviside's equation as quoted in equation (4) above.

To deduce equation (2) we integrate the last result with respect to t and obtain

$$\frac{\mathrm{F}(p)}{\Delta(p)}\,\mathrm{G}t=\mathrm{G}\left\{\mathrm{N}_0t+\sum_{\alpha}\frac{\mathrm{A}}{\alpha^2}(e^{at}-1)\right\}, \quad \ldots \quad (39)$$

where the arbitrary constant of integration has been adjusted so as to make the solution zero at $t=0$.

Now returning to the identity (35), we see on expanding powers of p, that

$$\mathrm{N}_0+\mathrm{N}_1p+\mathrm{N}_2p^2+\ldots=-\sum_{\alpha}\frac{\mathrm{A}}{\alpha}\left(1+\frac{p}{\alpha}+\frac{p^2}{\alpha^2}+\ldots\right);$$

so that

$$\mathrm{N}_1=-\sum_{\alpha}\frac{\mathrm{A}}{\alpha^2}. \quad \ldots\ldots \quad (40)$$

---

* It should be noted that we may also write

$$\frac{1}{p-a}\,\mathrm{P}=\left(\frac{1}{p}+\frac{a}{p^2}+\frac{a^2}{p^3}+\ldots\right)\mathrm{P}=\left(t+\frac{at^2}{2!}+\frac{a^2t^3}{3!}+\ldots\right)\mathrm{P},$$

leading to the same result. But this process is less convincing than that given in the text.

Combining (39) and (40), we deduce that

$$\frac{F(p)}{\Delta(p)}Gt = G(N_0 t + N_1 + \sum_a \frac{A}{\alpha^2} e^{at}),$$

which is the same as equation (4) of § 1.

It will be noticed that in § 1 the equation (2) has been used with more general types of function taking the place of the polynomials $F(p)$, $\Delta(p)$. It is natural to suppose that the equation (2) still remains valid; but formal proofs are more troublesome, as might be expected *.

XXXVI. *On the Origin of Spectra and Planck's Law.*

By Sir J. J. Thomson, *O.M., P.R.S.*†

THE results of investigations on the number of electrons in the atoms of the various elements show that the structure of these atoms, if expressed in terms of electrical charges, is in some cases of a very simple character. The atom of hydrogen, for example, is believed to contain only one electron and one unit positive charge. The question arises whether, if we regard the electrons and the positive charge as centres of forces varying inversely as the square of the distance, we have the potentiality of explaining by mechanical principles the properties of the atom. The explanation of some of these properties such as, for example, the specific inductive capacity of the gas, the formation of molecules by union with other atoms, whether of hydrogen or of some other element, seems to be within the scope of this very simple system; there are, however, other properties of which this cannot be said. Prominent among these is the spectrum emitted by the gas. Hydrogen, as is well known, can emit several spectra. We need, however, for our purpose only refer to the best known ones: the second spectrum, which is a spectrum containing an exceedingly large number of lines and extending far into the ultra-violet, and the so-called four-line spectrum, which contains, we have reason to believe, an infinite number of lines, the frequencies of which are connected by a simple numerical relation discovered by Balmer. The vibrations which would

---

* This problem has been considered with the aid of complex integrals in my paper on "Normal Coordinates in Dynamical Systems" already quoted (Proc. Lond. Math. Soc. vol. xv.); the sections §§ 4, 5, 8 have special bearing on this question. Short summaries are also given in the "Abstracts" of the Proceedings (vol. xiii. ser. 2, 1914, p. xxvii, and vol. xviii. January 1919).

† Communicated by the Author.

be emitted by a single electron placed near a unit positive charge would not be of this character. Assuming the usual law of force for two charges, if the atom is to be in a steady condition the electron must describe an orbit round the positive charge. The time of rotation of the electron will depend on its distance from this charge, and if this varies continuously the times of rotation will do so also, and the spectrum whose frequencies are determined by these times would be a continuous one. If the charges are regarded merely as centres of inverse square forces there are no reasons for retaining some of these orbits as possible and neglecting the others. Mr. Bohr, in his theory of spectra, supposes that the only orbits which are possible are those where the ratio of the energy of the electron to its angular velocity is an integral multiple of a definite unit. This, however, is not the consequence of dynamical considerations; it is arithmetical rather than dynamical, and if it is true it must be the result of the action of forces whose existence has not been demonstrated. The investigation of such forces would be a problem of the highest interest and importance.

By the use of this principle and a further one, that when an electron passes from one orbit to another it gives out radiation whose frequency is proportional to the difference of the energy of the electron in the two orbits, Mr. Bohr obtains an expression which gives with quite remarkable accuracy the frequencies of the lines in the four-line spectrum of hydrogen. It is, I think, however, not unfair to say that to many minds the arithmetical basis of the theory seems much more satisfactory than the physical.

The vibrations which give rise to the spectrum do not on this theory correspond in frequency with any rotation or vibration in the atom when in the steady and normal state. That in the normal atom there is something which can vibrate with the frequency of the lines, or at any rate with that of some of the lines in the spectrum, seems to be proved almost irresistibly by the experiments of Professor Wood and Mr. Bevan on the absorption spectra of the vapours of the alkali metals. These vapours give as absorption spectra fine well-defined black lines coinciding in position with the lines in the principal series of the spectrum of the metal. Thus, for example, Professor Wood obtained 48 of these lines in the absorption spectrum of sodium vapour, and Mr. Bevan 24 for potassium, 30 for rubidium, 24 for cæsium. The sharpness and intensity of these absorption lines produced by comparatively cold vapour are so great that it is very difficult to believe that they are not due to a resonance effect caused by systems in the normal atom having periods of vibration identical with those of the lines absorbed, and that there are in the normal atom vibrators with the same period as some of those in the luminous atom. There are, however, lines in the subordinate series in the emission spectrum which do not appear in the absorption one: thus it would appear that some of the atoms in the luminous gas differ from those in the normal gas inasmuch as the electrons vibrate with different periods in the two cases. The absorption spectrum is, however, sufficiently complicated to prove that it cannot be represented by the vibrations of a few electrons under the influence of their own repulsions and the attraction exerted upon them by a positive charge, if these attractions and repulsions follow the simple law of the inverse square.

Though it is natural in any investigation of the equilibrium of electrons and positive charges to begin by assuming that the attraction between the positive and negative charges varies inversely as the square of the distance, we must bear in mind that we have no direct evidence from experiment that this law holds at distances comparable with those which separate the electrons and the positive charges in a molecule. The measurements and the phenomena which furnish the evidence for the inverse square law relate to distances which are enormous compared with atomic distances. If c_1, c_2, c_3 are of the order of atomic distances, there are no experiments yet made which would distinguish between a force of $1/r^2$ and one of

$$\frac{1}{r^2}\left(1-\frac{c_1}{r}\right)\left(1-\frac{c_2}{r}\right)\left(1-\frac{c_3}{r}\right),$$

and yet at atomic distances the two forces are quite dissimilar, the second changing from attraction to repulsion and back to attraction again as r passes through the values c_1, c_2, c_3, &c.

If the positive part of the atom is built up of distinct units the repulsion between them cannot continue down to distances such as those which occur in this part of the atom, otherwise an atom containing several of these units would explode.

I shall, therefore, consider the consequences of supposing that the field of force round the positive charge, although varying inversely as the square of the distance at large distances from the atom, yet in the atom itself changes backwards and forwards between attraction and repulsion. To fix our ideas let us suppose that the expression for the force contains the factor $\frac{\sin cu}{cu}$ where $u=1/r$, and r is the distance from the centre. When r is great compared with c, $\sin cu/cu$ is unity, so that this factor does not affect the force at great distances. Inside the atom, if atomic dimensions are comparable with c, there will be a series of positions of equilibrium determined by $cu=n\pi$ or $r=c/n\pi$ where n is an integer. Thus even if there is only one positive charge and one electron there may be a single infinite series of atoms with the electron at distances from the centre represented by $r=\frac{c}{n\pi}$; the times of vibrations of the electrons about these positions would be different, so that a collection of such atoms could give rise to an infinite number of lines both in the absorption and emission spectra. Each line would arise from a different kind of atom ; the brightness of the line would depend, along with other things, on the number of atoms of the kind giving out the particular line. The theory of dispersion enables us, if we know the connexion between the refractive index and the wave-length, to calculate the number of systems which vibrate in any particular period. Bevan applied this method to find the number of atoms in sodium vapour which could vibrate in unison with the different lines in the principal series of this metal. He found that the number of atoms which can give out or absorb the D line is about one twelfth of the total number of atoms, while only about one in 1000 can give out the pair 3303, and about one in 5500 the pair 2852.

Thus the number of atoms corresponding to a line in the principal series diminishes very rapidly as the number which represents the position of the line in the series increases, and though different kinds of atoms may exist, the great majority of them are of one kind.

It is remarkable that only about 9 per cent. of the Sodium atoms are of the type giving lines in the visible part of the principal series. This suggests that perhaps the strongest line in the series may be a line in the infra red. An atom of one kind may, by physical or chemical processes, be transformed to one of another kind. Thus, if an atom with the electron at P were ionized, it would be positively electrified and would attract an electron from outside ; this electron might, however, settle at another point of equilibrium Q, giving rise to a different type of atom.

It would seem that these different types of atoms would differ in other respects than the period of vibration of the electrons. They would, for example, differ in their specific inductive capacities so that the proportion of the different kinds would be different at different places in a rapidly varying electric field. Again, the atoms of one kind might more readily combine with an atom of a different element than those of another. So that if the gas in a spectroscopic tube were gradually absorbed by chemical means, as, for example, when oxygen is absorbed by sodium, the relative intensity of the lines at the end of the process might not be the same as at the beginning; or, again, when the spectrum of an element is obtained by decomposing a compound, as, for example, when the spectra of the alkali metals are obtained by putting one or other of their salts in a Bunsen flame, the relative intensity of the lines might depend upon the character of the salt. Bevan's experiment, already alluded to, indicated that the number of atoms giving the lines of shorter wave-length in the principal series increased with the temperature of the sodium vapour. I pointed out many years ago, that the magnitude of the refractive index of helium showed that only a small fraction of the helium atoms could vibrate with the frequency of any particular line in the helium spectrum. Many instances of the variability of the relative intensity of different lines in the same spectrum are given in Kayser's 'Spectroscopie.' Since the processes which make the gas luminous also ionize the gas and thus enable an atom of one kind to be converted into one of another, it does not seem probable that by any process of fractionation we should be able to obtain a gas containing nothing but atoms of one kind, and therefore giving out a spectrum consisting of a single line under all conditions of excitation.

The electric field inside an atom consists, according to the view we have just taken, of alternate shells of attractive and repulsive forces, the places of transition from attraction to repulsion being places where the force vanishes and where an electron could be in equilibrium; the distances of these places of equilibrium from the centre being in harmonic progression.

We now pass on to consider the nature of the forces which act upon the electron and cause it to vibrate in the frequencies of the spectral lines. It is possible, as we shall see, to postulate an infinite number of laws of electric force which could give rise to a sequence of vibrations represented by Rydberg's law.

We may also imagine that in addition to the electric field inside the atom there is also a magnetic one, and that the frequencies of the vibrations of the electrons are determined by the magnetic and not by the electric forces. An electron, when in a magnetic field, describes a spiral round a line of magnetic force completing a revolution in a time $2\pi \Big/ \frac{He}{m}$, where H is the magnetic force, e the charge, and m the mass of the electron; it thus gives out radiant energy whose frequency is $\frac{H}{2\pi}\frac{e}{m}$, which is independent of the energy of the electron. As this energy diminishes, the radius of the spiral described by the electron diminishes, but the frequency of the vibration is unchanged. Thus, if the magnetic force were predominant in determining the vibrations of electrons, the frequencies of the vibrations given out by the different kinds of atom would be

$$\frac{H_1}{2\pi}\frac{e}{m}, \quad \frac{H_2}{2\pi}\frac{e}{m}, \quad \frac{H_3}{2\pi}\frac{e}{m},$$

where H_1, H_2, H_3 are the values of the magnetic induction at the various places of equilibrium. Suppose, now, that the value of H at a point of equilibrium at a distance r from the centre were equal to $\mu(a^2-r^2)$, a distribution of magnetic force which *a priori* is not improbable, as it is that inside a sphere uniformly charged with electricity and rotating like a rigid body.

Since the positions of equilibrium are given by $\sin c/r=0$, *i. e.*, by $\frac{c}{r}=n\pi$ or $r=c/n\pi$, where n is an integer, the value of the magnetic force at the positions of equilibrium, and therefore the frequencies of vibration in these positions, would be proportional to

$$a^2-\frac{c^2}{n^2\pi^2} \quad \text{to} \quad \frac{c^2}{\pi^2}\left(\frac{a^2\pi}{c^2}-\frac{1}{n^2}\right),$$

and would thus form a series of the Balmer type. If, in addition, the place $r=a$, where the magnetic force vanishes, is also a place where the electric force vanishes, $\frac{c}{a}=m\pi$ where m is an integer, and the expression for the frequency becomes $C\left(\frac{1}{m^2}-\frac{1}{n^2}\right)$, where C is a constant and m and n integers.

Before proceeding to discuss this expression in detail, we shall consider the type of atom which is required to satisfy the assumption we have made in the preceding investigation. This atom consists of a field of electric force which may be regarded as made up of a series of shells of attractive and repulsive force following one another alternately, the radii of the boundary of these shells, which are places where an electron would be in equilibrium, being in harmonical progression. Superposed on the field of electric force is a field of magnetic force, also arranged in shells, the outer boundary of the magnetic field coinciding with a place where the electric force vanishes. I contemplate that when the atom is exposed to such conditions as may arise in strong electric discharges or other methods of producing spectra, the outer layers of this magnetic field may get detached and the boundary of the magnetic field come close up to the centre, that, in fact, there is what might be called a magnetic ionization of the atom, and that the atom resumes its normal magnetic state when the electric discharge, &c., ceases. The atoms in a luminous gas thus possess a double manifoldness, one arising from the different positions of the electrons in the atom, the other from the variations in the magnetic boundary of the atom. The first manifoldness would give rise to different lines in the same series, the second to a number of different series most of which would only be emitted by the special type of atoms produced when an electric discharge passes through the gas.

So far we have only considered the case when only one electron was in the atom, so that a position of equilibrium was a place where the force due to the positive charge vanishes. If there are more electrons than one, the position of equilibrium will not be where the force due to the positive charge vanishes, but where this force at any electron balances the repulsion due to the other electrons. This will displace the position of equilibrium, and instead of these being given by $\sin cu = 0$, or $cu = n\pi$, they will be given by $cu = \pi(n+\delta)$, where δ is a quantity depending on the repulsion of the electrons and perhaps also on n.

As the frequencies of the vibrations are proportional to $a^2 - r^2$ they will now be proportional to

$$\frac{1}{(m+\delta')^2} - \frac{1}{(n+\delta)^2}.$$

Let us now consider the various types of series that could arise on this view.

Principal Series.

If the magnetic boundary is constant and fixed by the parameter m, and the different lines are due to the vibration of electrons in the positions of equilibrium inside the magnetic boundary, the series will be expressed by

$$C\left(\frac{1}{(m+\delta')^2}-\frac{1}{(n+\delta)^2}\right), \quad . \quad . \quad . \quad . \quad (1)$$

where m is constant and n has successive integral values. When the boundary is that corresponding to the majority of the normal atoms and the number of electrons is the full number proper to the atom, this will be the principal series; as the atoms are in the normal state, the atoms of the cold vapour will contain electrons able to vibrate in these periods, so that the vapour will be able to give this series of lines as an absorption spectrum.

If the atoms were ionized so that the number of electrons inside were diminished, this would alter the positions of equilibrium and therefore δ, so that the series of lines given out by the atoms would be represented by

$$C\left(\frac{1}{(m+\delta')^2}-\frac{1}{(n+\delta'')^2}\right), \quad . \quad . \quad . \quad . \quad (2)$$

a series with the same limiting frequency as the preceding.

Inasmuch as these lines proceed from ionized atoms, their frequencies do not correspond to the vibrations of electrons in a normal atom, and so this series would not appear as an absorption spectrum of the cold vapour; it would not then be a principal series, as the reversibility of the lines is the characteristic of this type of series.

Let us now take the case when the magnetic boundary is not the same as in the last case, but now corresponds to the parameter m' instead of m. The series of lines will now be given by

$$C\left(\frac{1}{(m_1+\delta_1)^2}-\frac{1}{(n+\delta_2)^2}\right) \quad . \quad . \quad . \quad . \quad (3)$$

where n has the successive integral values and m_1 is a constant integer. This series has a different limit from the preceding. If the new magnetic boundary, which by hypothesis is a position of equilibrium for electrons, were to be the position of equilibrium next nearer the centre than the one for the atoms giving the series (1), then the electron in the gravest mode of vibration of the atoms in (1) would be on the magnetic boundary of the atoms in (2). In other words,

$$\frac{1}{(m_1+\delta_1)^2}=\frac{1}{(1+\delta)^2},$$

so that the series (3) could be written in the form

$$C\left(\frac{1}{(1+\delta)^2}-\frac{1}{(n+\delta_2)^2}\right). \quad . \quad . \quad . \quad . \quad . \qquad (4)$$

The difference between the limiting frequencies of the series (1) and (4) is thus

$$C\left(\frac{1}{(m+\delta')^2}-\frac{1}{(1+\delta)^2}\right)$$

and this, as we see from (1), is the gravest frequency of the principal series.

Comparing this result with the Rydberg-Schuster law that the gravest frequency of the principal series is equal to the difference in the limits of the principal and first subordinate series, we infer that the series (4) represents the first subordinate series. Just as in the previous case we may have atoms which are neutral, *i. e.* which contain the normal number of electrons, and also atoms which have lost one electron and thus have unit positive charge. The positions of equilibrium for these are not the same as for the neutral atom, and the frequencies will therefore be represented by

$$C\left(\frac{1}{(1+\delta)^2}-\frac{1}{(n+\delta_3)^2}\right). \quad . \quad . \quad . \quad . \quad . \qquad (5)$$

This has the same limit as (4), and therefore corresponds to the second subordinate series. If there are atoms which have two positive charges there would be another series with the same limits as (5), but with a different step between the various lines; or, again, if instead of being positively charged the atom were negatively charged, *i. e.,* had got one more electron than the normal, and in some gases (such as, for example, hydrogen and the electronegative gases oxygen, chlorine, and iodine) these, as Positive Ray Analysis shows, are plentiful, there might yet be another series again with the same limit but with a different step from any of the preceding series. Thus, on this view, one of the subordinate series, the one connected with the principal series, would be emitted by uncharged atoms, while other subordinate series would be emitted by charged ones.

Pairs and Triplets.

A common feature in spectra is that the series consists not of single lines, but of a series of pairs, or triplets.

These are not confined to any particular type of series but occur in the principal as well as the subordinate series, though the law of difference of frequencies is different; in the principal series the frequency difference between the constituents of the pair diminishes as the wave-length diminishes, while in the subordinate series the frequency difference is constant.

The two constituents of a pair arise from different electrons; for Wood has shown that one of the D lines of sodium can be excited without the other and, again, the Zeeman effect for one line of a pair is quite different from that of the other.

We should expect to get pairs or triplets when instead of a single electron we had a ring of electrons or a number arranged at the corners of a polyhedron, if the distribution were not perfectly symmetrical about the centre. Thus, for example, if we had a ring of four electrons, arranged not at the corners of a square, but at those of a rectangle whose diagonals were of slightly different lengths; two of the electrons would be at a greater distance from the centre than the other two, so that the magnetic force, and therefore the frequency, would be smaller for the first than for the second pair of electrons, and since the variable part of the magnetic field, being proportional to r^2, would vanish at the centre, the limiting frequency which corresponds to the vibrations of electrons close to the centre would be the same for each pair of electrons. We should thus get a series of pairs behaving like those in the principal series.

We have seen reason for ascribing the first subordinate series to the vibrations of the electron in an atom in which the magnetic boundary has crept up to the position occupied by the outer ring of electrons in an atom of the type of that emitting the principal series. If the configuration of the ring is slightly unsymmetrical, that of the magnetic boundary for the atoms giving out the first subordinate series may be expected to be so also.

If this is so, the term in the magnetic force which does not depend on r would be different in different directions, and therefore the value of the first term in the bracket in the expression for the frequency

$$C\left(\frac{1}{(m+\delta')^2}-\frac{1}{(n+\delta)^2}\right)$$

would be different for electrons lying on different radii drawn from the centre of the atoms, the limiting frequency of electrons lying along these radii would be different, and the lines of the pair would not close up as the wave-length diminished.

Again, the directions in which the electrons giving the principal series bulged out, and along which the vibrations had the smaller frequency, would be the directions where the magnetic boundary bulged out in the atoms giving the first subordinate series. The directions where the magnetic boundary bulges out correspond to places where the constant term inside the bracket is a maximum and therefore to directions along which the frequency is a maximum.

The line of longer wave-length in a pair in the principal series is thus analogous to that of shorter wave-length in one in the first subordinate series. This is in accordance with the behaviour of the pairs in the spectra of the alkali metals, for when the more refrangible line of a pair is the stronger in the principal series, in the first subordinate series the more refrangible line is the weaker, while the Zeeman effect for the more refrangible lines in the principal series is analogous to that of the less refrangible one in the first subordinate series.

In the spectra of the alkali metals the terms $1/(n+\mu)^2$, $1/(m+\mu)^2$, which occur in the expression for the series approximate to $1/(n+\frac{1}{2})^2$ in some cases and to $1/n^2$ in others where n is an integer. This would occur if the positions where the electrons are in equilibrium were given by the equation $\sin 2x=0$ rather than by $\sin x=0$, for then the solution of the equation would be $2x=p\pi$, where p is an integer. If p is an even integer this may be written $x=n\pi$, and if p is an odd one $x=(n+\frac{1}{2})\pi$ where n is an integer. Thus, if we call the former the "S" solutions and the latter the "C" ones and suppose that the magnetic boundary of the atom may coincide with either a C or an S solution, we can have the following types of series. I call the constant term inside the bracket in Rydberg's expression the limit of the series, and denote it by L_s or L_c according as it corresponds to an S or C solution; the steps, the variable part, will be denoted by S_s or S_c according as the electrons giving out the vibrations are in the position of equilibrium corresponding to the S or C solution respectively.

The series may be represented as

$$(1)\quad L_s - S_s = C\left(\frac{1}{m^2} - \frac{1}{n^2}\right),$$

$$(2)\quad L_s - S_c = C\left(\frac{1}{m^2} - \frac{1}{(n+\cdot 5)^2}\right),$$

$$(3)\quad L_c - S_c = C\left(\frac{1}{(m+\cdot 5)^2} - \frac{1}{(n+\cdot 5)^2}\right),$$

$$(4)\quad L_c - S_s = C\left(\frac{1}{(m+\cdot 5)^2} - \frac{1}{n^2}\right),$$

where L_s and L_c may have different values corresponding to different positions of the magnetic boundary.

We see from this that the difference in the frequencies of two lines in a series of types (1) or (3) will be the frequency of a line in another series of the same type.

The differences between the frequencies of any two lines in a series of type (2) will be the frequency of a line in another series of type (3).

The difference between the frequencies of any two lines in a series of type (4) will be the frequency of a line in another series of type (1).

Thus, if we take the lines belonging to any one series, the complete spectrum of the gas will contain a line whose frequency is the difference of the frequencies of the two lines in the series.

It must be remembered that the two lines must be selected from the same series, if the two lines are selected at random from the spectrum there need not be a line whose frequency is the difference in frequency of the two lines.

Planck's Law.

The view that the vibrations which give rise to radiation are determined by the magnetic forces, leads, if we assume a simple relation between the electric and magnetic forces, to Planck's law, and gives a physical concept for the Quantum Theory.

For suppose that an electric field is accompanied by a magnetic one, the magnetic induction B being in the direction of the electric force R and connected with it by the relation

$$B = \frac{2\pi m}{h}\int p\frac{d}{ds}(\omega R)\,ds,$$

where ds is an element of a line of electric force, ω the cross-section of a tube of force, m the mass of an electron, h Planck's constant, and p determined by the equation $\frac{dp}{ds}=-\frac{1}{\omega}$.

When the force is due to a single positive charge we may put $p=1/r$, $\omega=r^2$, where r is the distance from the charge.

We notice that when the law of force reduces to the inverse square law, ωR is constant and B vanishes.

Integrating by parts we have

$$\mathrm{B}=\frac{2\pi m}{h}\left[p\omega\mathrm{R}+\int\mathrm{R}\,ds\right]. \quad . \quad . \quad . \quad . \quad (1)$$

Hence, if B_1, B_2 be the values of B at two places of equilibrium where R vanishes,

$$\mathrm{B}_1-\mathrm{B}_2=\frac{2\pi m}{h}\int_{s_1}^{s_2}\mathrm{R}\,ds,$$

or if w be the work done on an electron in moving from one place to another,

$$\mathrm{B}_1-\mathrm{B}_2=\frac{2\pi m}{eh}w.$$

If n_1, n_2 are the frequencies of the vibrations of an electron at the two places

$$n_1=\frac{1}{2\pi}\frac{e}{m}\mathrm{B}_1\,; \qquad n_2=\frac{1}{2\pi}\frac{e}{m}\mathrm{B}_2.$$

Hence

$$n_1-n_2=\frac{w}{h}.$$

Thus, if an electron falls from a place where the magnetic force vanishes to another position of equilibrium, the frequency of the vibration is equal to w/h, where w is the energy converted into radiation. Thus the transference from potential energy to energy of radiation is in accordance with Planck's law.

Assuming the relation between the electric and magnetic force given in the preceding investigations, it is easy to find laws of electric force such that the frequencies of the vibrations of electrons would be connected by a relation similar to that expressed by a Rydberg series.

Thus, for example, suppose that the force R exerted by the positive charge on an electron is given by the equation

$$R=\frac{Qx^2}{c^2}\frac{d}{dx}\left\{\frac{\sin^2 x}{x^2}+\sin x\cos^2 x\right\},$$

where $x=c/r$, when x is small, *i. e.* when r is large, this expression reduces to

$$R=\frac{Q}{r^2},$$

so that it gives the right value of the force at a great distance from the atom. Again, when x is considerable, the positions of equilibrium are given by

$$\cos x=0, \quad \text{or} \quad x=(2p+1)\frac{\pi}{2},$$

where p is an integer.

Since

$$\int R\,dr=-c\int\frac{R\,dx}{x^2}=\frac{2Q}{c}\left\{\frac{\sin^2 x}{x^2}+\sin x\cos^2 x\right\}. \qquad (2)$$

At a position of equilibrium we see from equation (1), since $\sin^2 x=1$,

$$B=\frac{4\pi Q}{ch}m\left\{\beta-\frac{4}{\pi^2(2p+1)^2}\right\},$$

where β is a constant, if the magnetic induction vanishes at the point of equilibrium where $p=p_0$,

$$\beta=\frac{4}{\{\pi(2p_0+1)\}^2}.$$

Hence n, the frequency of the vibrations, is given by the equation

$$n=\frac{2Qe}{ch}\left\{\beta-\frac{1}{\pi^2(p+\frac{1}{2})^2}\right\},$$

which may be written in the form

$$n=\frac{2Qe}{\pi^2 ch}\left\{\frac{1}{(p_0+\frac{1}{2})^2}-\frac{1}{(p+\frac{1}{2})^2}\right\},$$

a series of the Rydberg type.

If
$$\frac{cQ}{Q}=\frac{4}{\pi^4}\frac{h^2}{em},$$

the coefficient of the variable term is the same as in Bohr's theory.

The lines in the spectrum only give information about the value of the magnetic force at a number of isolated points;

they cannot, even when combined with the condition that at great distances the electric force must be expressed by Qe/r^2, afford sufficient data to determine uniquely the law of force, and, in fact, we can find, without difficulty, different expression for R which would yet give the same series of spectral lines.

The quantity c, which occurs in the expressions for R and B inside the atom, is of the dimensions of a length, and if it were a universal constant, *i. e.*, the same for all atoms, there would be some standard length occurring either in all the atoms or in the medium surrounding them. In this case, since c would not vary from atom to atom, the frequency of the vibration would be proportional to Q, the positive charge in the atom. Moseley's experiments on the wave-length of the characteristic Röntgen radiation given out by the different elements, if taken in conjunction with the assumption that the positive charge is proportional to the atomic number, show that the frequencies are proportional to the square, and not to the first power of the positive charge. For this reason we must suppose that Q/c is proportional to Q^2, so that $c=c'/Q$ where c' and not c is a universal constant.

Since c' is of the dimensions of a length multiplied by a charge of electricity, it represents the moment of an electric doublet, and if it is taken as a universal constant we must suppose that electrical doublets with constant moments form a part either of the atoms of all the elements or else of the medium which surrounds them.

We can calculate the value of c or c' if we know the amount of work required to move an electron from one of its positions of equilibrium to an infinite distance from the atom. For we see by equation (2) that if the position of equilibrium of the electron is that corresponding to $x=\pi/2$, this work is equal to

$$\left(1-\frac{4}{\pi^2}\right)\frac{Qe}{c},$$

or if the work is expressed as Ve, where V is the potential through which the charge e must fall to acquire this amount of energy,

$$V=\left(1-\frac{4}{\pi^2}\right)\frac{Q}{c}.$$

If we take the ionizing potential as the measure of V, then for the atom of hydrogen $V=11$ volts, and $Q=e$, we find, putting $e=4{\cdot}7\times10^{-10}$, $V=11/300$,

$$c=7{\cdot}7\times10^{-9},$$

and $$c'=c\times Q=3{\cdot}6\times10^{-18}.$$

If we calculate c' from the relation

$$c'=\frac{4}{\pi^4}\frac{h^2}{em},$$

we find $$c'=3{\cdot}97\times10^{-18};$$

this differs by about 10 per cent. from that calculated from the ionizing potential. The determinations of this potential for the hydrogen *atom* are hardly definite enough to exclude the possibility of an error of 10 per cent., so that the value $3{\cdot}97\times10^{-18}$ for c' is admissible. This value would make the numerical constants in the series relation agree with those given by Bohr's theory which give values for the frequencies agreeing with observation.

The ionizing potential is usually measured by the potential difference through which cathode rays must fall to enable them to ionize the gas.

When, however, the atom is, as we have supposed, the seat of intense magnetic forces, it is possible that the energy which the cathode rays must possess to be able to ionize the gas may not measure the energy which the cathode rays have to give up to an electron to enable it to escape from the atom, but rather the energy which the cathode rays must possess if they are to penetrate far enough into the atom to reach that electron which has to be liberated.

Again, in many cases of ionization by cathode rays, the electrons are ejected with a finite amount of kinetic energy which is independent of that of the cathode rays; when this is so, the ionizing potential is greater than the line integral of the electric force from the initial to the final position of the electron.

When the spectrum of a gas is excited by the impact of cathode rays, the first process which goes on is the ejection of electrons from the atom; this does not, on our view, give rise to radiation corresponding to the series lines in the spectrum. This is due to a second process, the return of electrons to the ionized gas to replace those which have been ejected; these electrons are not the high speed cathode rays but slow speed electrons produced by the ionization of the atoms, and the energy which is converted into radiation is that which these electrons acquire when falling into the positions of equilibrium. From some experiments I have made on the radiation produced by the bombardment of substances by cathode rays, I conclude that by far the larger part of the radiant energy is concentrated in radiation of definite wavelength, for I found that when the energy of the cathode rays was gradually measured the character of the radiation did not at some stages alter appreciably, while at other stages an increase in the energy of the cathode rays produced considerable increase in the hardness of the radiation. This is what would happen if this radiation were a mixture of definite types (a, b, c, d ......) of characteristic radiations from the target struck by the cathode rays. The type a not being excited unless the energy of the cathode rays exceeded e_1, b not being excited unless the energy exceeded e_2, and so on. Thus, when the energy of the cathode rays was between e_1 and e_2, the radiation would be confined to the a type, it would be a mixture of the a and b types as soon as the energy of the cathode rays exceeded e_2, when it exceeded e_3 the c type of radiation would be added, and so on.

Number of Waves in a Train of Waves.

When an electron falls into a position of equilibrium and rotates round the lines of magnetic force, the energy it acquires by the fall is gradually converted into radiant energy, and the electron gradually comes to rest.

If f is the acceleration of an electron, the rate at which it emits energy is

$$\frac{2}{3}\frac{e^2 f^2}{V_0},$$

where e is the charge on the electron and V_0 the velocity of light.

If H is the magnetic force, v the velocity of the electron, then when the orbit is at right angles to the magnetic force

$$f=\frac{Hev}{m}=2\pi nv,$$

where n is the frequency of the vibration of the electrons, when this is governed by the magnetic force. Substituting this value for f, we find that the rate at which energy is emitted from the electron is equal to

$$\frac{8\pi^2}{3V_0}e^2n^2v^2=\frac{16\pi^2}{3V_0}\frac{e^2n^2}{m}E,$$

where E is the kinetic energy of the electron; hence when the loss of energy by the electron is entirely due to the radiation, we have

$$\frac{dE}{dt}=\frac{-16\pi^2}{3V_0}\frac{e^2}{m}n^2E=-kE, \text{ say.}$$

Thus $$E=c\epsilon^{-kt}.$$

Now
$$\frac{e}{m}=1{\cdot}78\times10^{7},$$
$$e=1{\cdot}6\times10^{-20},$$
hence
$$k=5\times10^{-22}\,n^{2},$$

and the time taken for the energy to fall to $1/\epsilon$ of its initial value is $10^{22}/5n^{2}$.

Thus, if the vibrations have the frequency $5{\cdot}1\times10^{14}$ of the D line in the sodium spectrum, this time is equal to $7{\cdot}7\times10^{-9}$ seconds, and the length of the train emitted in this time would be about 4 million wave-lengths. If the wave-lengths were 10^{-8} cm., about that usually given for the L characteristic Röntgen radiation from platinum, the time would be $\frac{1}{4{\cdot}5\times10^{15}}$, and the train emitted would contain about 660 wave-lengths. Thus, if the spectral lines arise from the effect of magnetic forces, they may be expected to consist of trains containing a very large number of wave-lengths, the number of wave-lengths being inversely proportional to the frequency of the vibrations.

Transference of Radiant Energy into Potential or Kinetic Energy.

Let us consider a stream of radiant energy passing through a piece of metal. In consequence of the magnetic fields due to the atoms, there will be in the metal magnetic forces reaching very high values in certain places, and falling to quite low ones in others; and between very wide limits of the magnetic force it will be possible to find places where the magnetic force has any assigned value, if at all such places an electron could be found, a piece of metal would be a system containing vibrators of every possible period within very wide limits.

In a metal it seems very probable that the electrons are not all contained within the atoms themselves, but that some of them are detached, helping by the forces they exert to make the atoms cohere and form a solid body. Unless, then, the position of these detached electrons were exactly co-ordinated with the intensity of the magnetic field, the effect would be much the same as if these electrons were distributed at random through the magnetic field; and we might expect to find some electrons in places where the magnetic force had, within wide limits, any specified value. A piece of metal would then be a system of the kind we are considering, possessing vibrators able to respond to any vibrations within a wide range of frequency. Thus, if light of any frequency between these limits were to fall on the metal it would find some electrons whose frequency under the magnetic force at the place where they were situated was the same as its own ; these would acquire a high velocity. If an electron is to get free, it must get to a place where the magnetic force vanishes, because, as we can easily prove, the effect of the magnetic field on an electron displaced from a position of equilibrium, is the same as if there were an attraction on the electron to the point equal to Be/m times the displacement of the electron. Suppose that P is a place where the magnetic induction vanishes, and where consequently an electron can get free. Suppose that Q is the place from which the electron to be liberated at P starts. To enable it to get from Q to P it must acquire an amount of energy equal to w, where w is the work required to move an electron against the electric field from Q to P. If it is to acquire this by radiation, it must be by resonance, so that the radiant energy from which it gets it must have the frequency of the free vibrations of the electron at Q. Since the magnetic force vanishes at P this frequency is, by equation (2), equal to w/h. Thus the electron at Q, by absorbing energy of frequency n, can be liberated at P with an amount of Potential Energy equal to hn.

This would produce the well-known photoelectric effects. If Q were to absorb a smaller amount than w of energy from the radiation, it would not get liberated. After the radiation had passed over it this energy would be again radiated by the electron at Q vibrating under the magnetic force at Q with the frequency n. Thus the absorbed radiation would be again radiated as radiation of the same frequency, and there would not be any transformation of energy. For the energy to be transformed, energy equal to w must be given to Q ; and since $w = hn$, we get the result given by the quantum theory that the transference from radiant to potential or kinetic energy takes place by definite quanta each equal to hn.

The explanation of the photoelectric effect assumes the possibility of finding in the metal an electron with a natural frequency n, where n may be any assigned number within a wide range of values. In a complex system like a metal this seems probable ; and it is noteworthy that the photoelectric effects are on quite a different scale in metals from what they are in gases ; thus Hughes found no trace of ionization when ultra-violet light was totally absorbed by the vapour of zinc ethyl, though when the same light fell on a zinc plate it excited a stream of electrons a million times greater than that which could have been detected if any had emerged from the vapour. Ultra-violet light of very small wave-length does undoubtedly produce ionization in gases, while one of the most characteristic properties of the Röntgen rays is their power of ionizing gases. We must remember, however, that as the absorption-bands in the ultra-violet of gases like CO_2 and O_2 show, gases may in many regions of the ultra-violet light spectrum be able to give out a spectrum which is continuous over many frequencies, and presumably, therefore, possesses electrons able to vibrate freely over a wide range of frequencies. Again, it is important to remember that the number of atoms ionized is an exceedingly small fraction of the number of atoms in the gas. The number varies, of course, with the intensity of the ionizing radiation; but under ordinary conditions of laboratory work the number of atoms ionized per second is much less than $1/10^{12}$ of the number of atoms exposed to the radiation; or, to put it another way, an individual atom would not be ionized nearly so often as once in one hundred thousand years.

Now, in a gas the atoms come into collision and may form temporary associations, and while in this state the natural frequencies are changed by an amount which will vary from one pair of atoms or molecules to another; hence, if we take these pairs into account we shall find in the gas, electrons whose natural frequencies are not merely those of the free atom, but extend continuously over the range of frequencies to be found in the pairs. Thus, as far as range of frequencies of the electron goes, the gas will be on an equality with a piece of metal. It is true that in the gas the number of electrons which can vibrate in what we may call these abnormal frequencies is an exceedingly small fraction of the whole number of electrons; but we have seen that if the electron gets into this state once in a hundred thousand years it will be able to do more than its full share of the ionization in the gas. The effect of proximity of the atoms in increasing the range of frequencies is shown by the enormous broadening of some lines when the pressure of the gas is increased. If we accept the view that in a gas we may expect to find electrons with frequencies varying continuously over a wide range, though the number which have frequencies differing from the normal natural frequencies may be exceedingly small, we may apply to gases the reasoning that we use for the photoelectric effect in metals, and assume, as the result, that the energy in the ejected electron will be proportional to the frequency of the radiation which supplied it with energy. In other words, that the energy in the corpuscular radiation will be proportional to the frequency of the primary Röntgen radiation; the experiments hitherto made on the expulsion of electrons by Röntgen rays are consistent with this law.

An atom from which an electron has been expelled will be an electron short when it gets free from the atom with which it was temporarily associated, and will be in a condition to give out radiation when an electron falls into it to replace the one that was lost; the atom is now free, so that the vibration it will emit will have the normal frequency, whatever may have been the frequency of the radiation which supplied the energy to eject the electron. Thus the character of the Röntgen radiation excited by the primary radiation will not depend on the nature of the radiation, but solely on the nature of the atom; on the other hand, the nature of the corpuscular radiation will depend solely on the nature of the primary radiation, and not upon the nature of the atom. These conclusions are in accordance with known properties of Röntgen radiation.

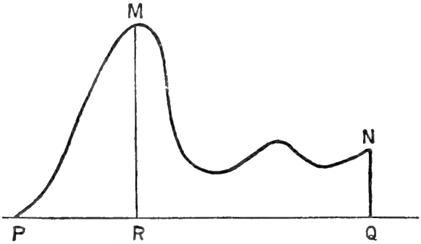
Again, we see that each electron ejected will correspond to one unit of energy of the characteristic radiation of the atom; this will be given out when an electron falls into the atom to replace the one that has been ejected. Thus whatever the energy in the ejected electron, the energy in the corresponding characteristic radiation will be constant.

Now, the energy in the ejected electron increases with the frequency of the rays; hence the ratio of the energy in the corpuscular radiation to that in the characteristic radiation will increase with the hardness of the primary Röntgen radiation. We see, too, that when the frequency of the primary Röntgen radiation is the same as that of the characteristic radiation, the energy in the corpuscular radiation will be equal to that in the characteristic.

We have, in the preceding discussion, neglected the possibility of the high-speed electrons ejected by the Röntgen rays exciting by their impact with the atoms the characteristic radiation of the substance. The justification for this is that in some experiments which I made on the excitation of Röntgen rays by cathode rays I found that the energy of the Röntgen radiation was concentrated in a very soft type of radiation, and that only an exceedingly small fraction of the energy of the cathode rays was transformed into a type of radiation comparable in hardness with that which would excite corpuscular radiation comparable in velocity with that of the cathode rays which produced it.

When the electric force changes from plus to minus, as we have supposed it to do inside the atom, the amount of energy which must be communicated to an electron to enable it to get from one place P to another Q may be greatly in excess of $\int_P^Q R\,ds$, where R is the force on an electron and *ds* an element of its path. For the potential distribution may then be represented by a curve such as that in fig. 1, when an electron could not get from P to Q

Fig. 1.



unless it surmounted the peak R. To do this it must receive energy represented by RM, which may be much greater than $\int_P^Q R\,ds$, which is represented by QN. In such a case as this the electron ejected from P will arrive at Q with a finite amount of kinetic energy, measured by the difference between RM and QN.

In a periodically varying electric field of the type we are considering we might expect the maximum value of RM to be approximately twice QN. For the work done between two places of equilibrium is equal to $B_1 - B_2$ when B_1 and B_2 are the values of the magnetic induction at the two places.

Now, if the numerical value of the magnetic force at P is not less than that at any other point nearer the surface of the atom, the greatest possible negative value of B_2 will be $-B_1$, and since B_1, to a rough approximation, varies harmonically, B_2 may be expected to approach this value. Thus $B_1 - B_2$ will have as its maximum value $2B_1$, *i. e.* the energy which must be communicated to the electron to enable it to surmount an obstacle like the peak at R is twice the increase in potential energy it gains by going to Q, where B_2 is assumed to vanish. Thus the electron will possess, when it emerges from the atom at Q, kinetic energy equal to half the energy communicated to it at P.

Let us suppose that the atom is being ionized by radiation whose frequency is n, where n is also the frequency of the characteristic radiation of the atom. The ejected electron begins by absorbing an amount of energy equal to $2hn$, of this, when it is liberated, it retains as kinetic energy hn—this energy appears as the energy of corpuscular radiation; the other half of the energy appears subsequently as characteristic radiation when an electron drops into the atom to take the place of the one displaced. In this case the energy in the corpuscular radiation is equal to that in the characteristic.

Next suppose that the frequency n_1 of the incident radiation is not the same as n, that of the characteristic radiation. The absorption of energy will take place when the atom has formed a temporary alliance with another atom, of such a kind that the forces between the atoms have altered the frequency of the electron from n to n_1. The absorption of energy by the electron is now $2hn_1$, of this hn_1 appears as the energy of corpuscular radiation, the atom recovering an electron after it has again become free will give out hn units of characteristic radiation; the difference

$$2hn_1 - h(n + n_1) = h(n_1 - n)$$

will be given up to the atoms, making them separate with more energy than they had before they came together, a fraction β of this energy might be converted into slow corpuscular or soft Röntgen radiation.

Thus the ejection of a high-speed electron will be accompanied by the following energy effects:

Energy in corpuscular radiation $= hn_1$.
Energy in characteristic radiation $= hn$.
Energy in soft Röntgen radiation $= \beta h(n_1 - n)$.

The corpuscular radiation will bear to the characteristic radiation the proportion of n_1 to n and will thus increase indefinitely as n_1 increases. If we include the soft Röntgen radiation with the corpuscular, and in many experiments they would not be distinguished, the ratio of the corpuscular energy to the characteristic energy would be

$$\frac{n_1 + \beta(n_1 - n)}{n}.$$

The view that when an electron absorbs radiant energy it is a constituent of a system in which it vibrates in unison with the radiation, is not without difficulties. It requires, for example, concentration of the energy of the light-wave in certain places instead of uniform distribution over the wave front. It has, however, the great advantage that it accounts for the electrical effects produced by light without introducing changes in our conception of light which give rise to difficulties in connexion with ordinary optical effects such as interference or scattering. These would not be affected by the resonance hypothesis.

In favour of this hypothesis may be urged :—

1. That it is an exceedingly minute fraction of the number of atoms or molecules of a substance which plays any part in the electrical effect. Even what is regarded as very intense ionization by Röntgen rays would be produced if on an average each atom got ionized once in some hundred thousand years. So that if each atom were to form a part of a suitable system once in this period it would be all that is required.
2. The number of lines in the spectrum of a substance which are theoretically possible is very large ; the electrons have a large number of possible frequencies, each of which would be disturbed by the electric and magnetic forces exerted by a neighbouring atom, so that the frequencies continuously covered by the system would extend over a wide range.
3. The difficulty of supposing that the influence of two neighbouring atoms of a light element could be so great that the frequency of an electron in one of the atoms could rise to that of the K radiation of a heavy element has been very greatly lessened by some recent researches. Barkla (Phil. Trans. vol. 217. p. 315) has shown that the lighter elements give out a type of radiation much harder than the types previously detected, and that this new type is comparable in hardness with the K type given out by the heavier elements. Again, Shearer (Phil. Mag. xxx. p. 644), who has investigated the ionization of hydrogen by hard Röntgen rays, has shown that the amount of it is, at most, surprisingly small, even when the density of hydrogen is taken into account, and that it is doubtful whether even this small amount is a genuine ionization of hydrogen ; the experiments seem to suggest that it was due to a trace of some heavier gas. This is an exceedingly important result, because it shows that the impact of Röntgen rays against an electron does not necessarily communicate very much energy to it, even though there may be a large amount of energy in the rays ; it differentiates the action of Röntgen rays from that of either cathode or α-rays.

We have supposed, hitherto, that resonance is obtained by the free periods of the electrons in one atom getting modified by the proximity of another. It is conceivable, however, that the same result might be obtained by processes localized in the atom itself and independent of neighbouring atoms. For the atoms of every substance except hydrogen contain more than one unit positive charge, and if the mass of each positive charge is equal to that of the atom of hydrogen, the number of positive charges will be equal to the atomic weight. If the atom is electrically neutral there must be, somewhere or other inside it, an equal number of electrons ; experiment shows, however, that only about half of these are free to exercise any independent effects, the other half must be bound up with the positive charges. If, at long intervals, changes occurred among these positive charges analogous, though on a very much reduced scale, to those which occur in radioactive substances, electrons might be temporarily detached from these charges, though not as in radioactive substances, with such energy as to drive them out of the atom in the form of high-speed β-rays. The displacement of the electrons in the non-radioactive substances would, however, modify the frequency of the vibrations of the electrons in the atom, and would furnish a small supply of atoms whose frequencies differ irregularly from those in the normal atom and which would resonate to vibrations to which the ordinary atom would not respond.

We have supposed that the periods of the oscillations of the electrons are determined solely by the magnetic forces, but even when those forces are predominant the electric forces which accompany them will modify, to some extent, the periods, and may make what, if these are neglected, would correspond to a single line become several lines separated by very small intervals.

It may, perhaps, be worth pointing out that one objection, often raised against the oscillations being due to electrostatic forces alone, viz. that under such forces, what is primarily given by the equations of motion is the square of the frequency, while the relations given by the series laws relate to the frequency itself and not to its square, does not apply to all types of electrostatic force. If, for example, the force of the positive charge on an electron were represented by $Ce\frac{d\phi^2}{dr}$, where C is a constant and ϕ a function of r, then one set of positions of equilibrium would be determined by $\phi=0$, and if ξ is the displacement from one of these positions the equation of motion is

$$m\frac{d^2\xi}{dt^2} = -2Ce\left\{\frac{d\phi}{dr}\right\}^2\xi,$$

where $\left\{\frac{d\phi}{dr}\right\}$ is the value of $\frac{d\phi}{dr}$ at the equilibrium position.

The frequency of the oscillation represented by this equation is equal to

$$\left\{\frac{2Ce}{m}\right\}^{\frac{1}{2}}\frac{d\phi}{dr},$$

so that the case resembles that where the oscillations are determined by the magnetic force, inasmuch as the first power of the frequency is given by a simple expression without square roots in the variable part. By taking appropriate values of ϕ we can get series for the frequencies of the Rydberg type.

When an electron in a magnetic field is acted upon by an electric force, the magnetic forces will deflect it as soon as it gets set in motion, and it will not move along the direction of the electric force. Thus if a plane polarized beam of light in which the electric force is parallel to the axis of x falls upon the electron, it will originate accelerations parallel to y and z as well as to x. The accelerations parallel to y and z will give rise to scattered waves which will not vanish along the axis of x; now we know that with either visible light or Röntgen rays the light scattered in this direction is in normal cases exceedingly small, so that it is necessary to see if this result is consistent with the existence of strong magnetic forces inside the atom.

Let the electric force in the wave be parallel to x and equal to $\mathrm{E}\cos pt$. Let ξ, n, ζ be the displacements of an electron parallel to x, y, z and a, b, c the components of the magnetic induction inside the atom; then the equations of motion, if we include an electric restoring force proportional to the displacement, are

$$\frac{md^2\xi}{dt^2} = \mathrm{E}e\cos pt - \mu\xi + \frac{celn}{dt} - \frac{bed\zeta}{dt},$$

$$\frac{md^2\eta}{dt^2} = -\mu\eta + \frac{aed\zeta}{dt} - \frac{ced\xi}{dt},$$

$$\frac{md^2\zeta}{dt^2} = -\mu\zeta + \frac{bed\xi}{dt} - \frac{acd\eta}{dt};$$

the solution of these equations when ξ, η, ζ vary as ϵ^{pt} is

$$\xi = \frac{\mathrm{E}e\cos pt}{\Delta}\{(\mu - mp^2)^2 - a^2e^2p^2\},$$

$$\eta = \frac{\mathrm{E}e\cos pt}{\Delta}\{-abe^2p^2 - \iota cp(\mu - mp^2)\},$$

$$\zeta = \frac{\mathrm{E}e\cos pt}{\Delta}\{-ace^2p^2 + ibp(\mu - mp^2)\},$$

where $$\Delta = (\mu - mp^2)^3 + (\mu - mp^2)e^2(a^2 + b^2 + c^2).$$

We see from this that the displacements of the electron are not wholly in the direction of the electric force; but from the form of the equations we see that if mp^2 is either very small or very large compared with μ, the displacement across the direction of the force is infinitesimal in comparison with that along it. In the theory of scattering of visible light mp^2 is assumed to be small compared with μ, while for Röntgen rays it is assumed to be large, so that the magnetic forces would not appreciably affect these types of radiation.

If there are several electrons with their centre of figure at the centre of the atoms, and if the axis of x is a principal axis, then Σab, Σac, Σc, and Σb vanish, and the scattered radiation will be in the same direction as that given by the ordinary theory for the non-magnetic atom. The effect of the transverse displacements, if the electrons in one atom were arranged unsymmetrically, so that Σab, etc. did not all vanish, would, if all the different atoms were orientated in exactly the same way, produce a rotation of the plane of polarization when plane polarized light passed through the collection of atoms. This effect would, however, disappear if the atoms were orientated at random.

There does not, therefore, appear to be anything in the scattering of light by the atoms of a gas or in the rotation of the plane of polarization of light inconsistent with the existence of a strong magnetic field inside the atom.

Scattering of Cathode particles by the Atom.

If the dominant forces inside the atom were magnetic, a charged particle moving through the atom would be deflected, but inasmuch as the force on a moving particle due to its magnetic field is always at right angles to the direction of motion of the particle, the particle would neither lose nor gain energy. The collisions would be what are often called elastic to distinguish them from those in which there is a loss of energy, which are called inelastic.

The experiments of Franck and Hertz show that in some cases, for example, argon and nitrogen, an electron may collide with a very large number of atoms without losing energy, and in cases where the electrons cannot make many collisions before being captured and attached to an atom, the effects seem to be reconcilable with the view that such collisions as occur are elastic, and that the alternatives are capture or escape with undiminished energy.

Since the frequency of vibrations of an electron is proportional to the magnetic force it is evident that it would be affected by an external magnetic field ; it would also be affected by an external electric field, since this would displace the position of equilibrium. The consideration of these effects must, however, be left for another occasion.

XXXVII. *A General Formula for the Moments of the Normal Correlation Function of any Number of Variates.* *By* S. D. Wicksell, *Lund, Sweden*[*].

If $x_1, x_2, \ldots . x_n$ be n variates reckoned from their respective means and expressed in their respective dispersions as units, and r_{pq} is the coefficient of correlation of x_p and x_q, the normal correlation function is given by

$$\phi(x_1, x_2, \ldots . x_n) = \frac{1}{(2\pi)^{n/2}\sqrt{S}}\, e^{-\frac{1}{2S}\Sigma\Sigma S_{pq}x_px_q}, \quad . \quad (1)$$

where

$$S = \begin{vmatrix} r_{11}, & r_{12}, & \ldots & r_{1n} \\ r_{21}, & r_{22}, & \ldots & r_{2n} \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ r_{n1}, & r_{n2}, & \ldots & r_{nn} \end{vmatrix} = \begin{vmatrix} r_{pq} \end{vmatrix}$$

and

$$S_{pq} = \frac{\partial S}{\partial r_{pq}}.$$

The function $\phi(x_1, x_2, \ldots . x_n)$ may also be written in the form of a certain multiple integral which will be

---

[*] Communicated by the Author.

found to be very useful in discussing the mathematical properties of the function. Indeed we have

$$\phi(x_1, x_2, \ldots . x_n) = \frac{1}{(2\pi)^n} \int_{-\infty}^{\infty} dw_1 \int_{-\infty}^{\infty} dw_2 \ldots . \int_{-\infty}^{\infty} dw_n \, e^{-\frac{1}{2}\Sigma\Sigma r_{pq} w_p w_q} \, e^{-i\Sigma x_p w_p}. \qquad (2)$$

This integral, which for $n=1$ reduces to the well-known integral of Laplace, has been given by Charlier for the special case of $n=2$. A proof of the theorem for any value of n will be given in the Appendix.

By partial integration it is further easily verified that

$$i^{k_1+k_2+\ldots .+k_n} x_1^{k_1} x_2^{k_2} \ldots . x_n^{k_n} \phi(x_1, x_2, \ldots . x_n) =$$

$$= \frac{1}{(2\pi)^n} \int_{-\infty}^{\infty} dw_1 \int_{-\infty}^{\infty} dw_2 \ldots . \int_{-\infty}^{\infty} dw_n \frac{\partial^{k_1+k_2+\ldots . k_n} e^{-\frac{1}{2}\Sigma\Sigma r_{pq} w_p w_q}}{\partial w_1^{k_1} \partial w_2^{k_2} \ldots . \partial w_n^{k_n}} e^{-i\Sigma x_p w_p}. \quad (3)$$

Hence by the theorem of Fourier we have

$$i^{k_1+k_2+\ldots .+k_n} \int_{-\infty}^{\infty} dx_1 \int_{-\infty}^{\infty} dx_2 \ldots . \int_{-\infty}^{\infty} dx_n \, x_1^{k_1} x_2^{k_2} \ldots . x_n^{k_n} \phi(x_1, x_2, \ldots . x_n) =$$

$$= \left[\frac{\partial^{k_1+k_2+\ldots .+k_n} e^{-\frac{1}{2}\Sigma\Sigma r_{pq} w_p w_q}}{\partial w_1^{k_1} \partial w_2^{k_2} \ldots . \partial w_n^{k_n}} \right]_{w_1 = w_2 = \ldots . = w_n = 0}.$$

Putting

$$\frac{\partial^{k_1+k_2+\ldots . k_n} e^{-\frac{1}{2}\Sigma\Sigma r_{pq} w_p w_q}}{\partial w_1^{k_1} \partial w_2^{k_2} \ldots . \partial w_n^{k_n}} = \underset{k_1 k_2 \ldots k_n}{\mathrm{H}} (w_1, w_2, \ldots . w_n) \, e^{-\frac{1}{2}\Sigma\Sigma r_{pq} w_p w_q},$$

we find, as

$$\underset{k_1 k_2 \ldots . k_n}{\mathrm{H}} (0, 0, \ldots . 0) = 0$$

when $k_1+k_2+\ldots .+k_n$ is an odd number, for the moments of any order of $\phi(x_1, x_2, \ldots . x_n)$ the formula

$$\int_{-\infty}^{\infty} dx_1 \int_{-\infty}^{\infty} dx_2 \ldots . \int_{-\infty}^{\infty} dx_n \, x_1^{k_1} x_2^{k_2} \ldots . x_n^{k_n} \phi(x_1, x_2, \ldots . x_n)$$

$$= (-1)^{\frac{k_1+k_2+\ldots .+k_n}{2}} \underset{k_1 k_2 \ldots . k_n}{\mathrm{H}} (0, 0, \ldots . 0). \quad (4)$$

We have now, remembering that in the case under consideration $r_{pq}=r_{qp}$ and $r_{pp}=1$,

$$\left.\begin{aligned}
H_{200\,\ldots.\,0} &= [\Sigma r_{1q}w_q]^2-1,\\
H_{110\,\ldots.\,0} &= \Sigma r_{1q}w_q\Sigma r_{2q}w_q-r_{12},\\
H_{400\,\ldots.\,0} &= [\Sigma r_{1q}w_q]^4-6[\Sigma r_{1q}w_q]^2+3,\\
H_{310\,\ldots.\,0} &= [\Sigma r_{1q}w_q]^3\,.\,\Sigma r_{2q}w_q-3\Sigma r_{1q}w_q\,.\,\Sigma r_{2q}w_q\\
&\qquad\qquad -3[\Sigma r_{1q}w_q]^2r_{12}+3r_{12},\\
H_{220\,\ldots.\,0} &= [\Sigma r_{1q}w_q]^2[\Sigma r_{2q}w_q]^2-[\Sigma r_{2q}w_q]^2-[\Sigma r_{1q}w_q]^2\\
&\qquad\qquad -4r_{12}\Sigma r_{1q}w_q\,.\,\Sigma r_{2q}w_q+1+2r_{12}^2,\\
H_{2110\,\ldots.\,0} &= [\Sigma r_{1q}w_q]^2\Sigma r_{2q}w_q\,.\,\Sigma r_{3q}w_q-\Sigma r_{2q}w_q\,\Sigma r_{3q}w_q\\
&\quad -2r_{12}\Sigma r_{1q}w_q\,\Sigma r_{3q}w_q-2r_{13}\Sigma r_{1q}w_q\,\Sigma r_{2q}w_q\\
&\quad -r_{23}[\Sigma r_{1q}w_q]^2+r_{23}+2r_{21}r_{13},\\
H_{11110\,\ldots.\,0} &= \Sigma r_{1q}w_q\,.\,\Sigma r_{2q}w_q\,.\,\Sigma r_{3q}w_q\,\Sigma r_{4q}w_q\\
&\quad -r_{12}\Sigma r_{3q}w_q\,\Sigma r_{4q}w_q-r_{13}\Sigma r_{2q}w_q\,\Sigma r_{4q}w_q\\
&\quad -r_{23}\Sigma r_{1q}w_q\,\Sigma r_{4q}w_q-r_{14}\Sigma r_{2q}w_q\,\Sigma r_{3q}w_q\\
&\quad -r_{24}\Sigma r_{1q}w_q\,\Sigma r_{3q}w_q-r_{34}\Sigma r_{1q}w_q\,\Sigma r_{2q}w_q\\
&\quad +r_{12}r_{34}+r_{13}r_{24}+r_{23}r_{14}.
\end{aligned}\right\}\,.\quad (5)$$

Putting

$$\int_{-\infty}^{\infty}dx_1\int_{-\infty}^{\infty}dx_2\,\ldots.\int_{-\infty}^{\infty}dx_n\,x_1^{k_1}x_2^{k_2}\ldots.\,x_n^{k_n}\,\phi(x_1,\,x_2,\,\ldots.\,x_n)=m_{k_1k_2\ldots.\,k_n},$$

we find consequently

$$\left.\begin{aligned}
m_{20\,\ldots.\,0} &= 1,\\
m_{110\,\ldots.\,0} &= r_{12},\\
m_{40\,\ldots.\,0} &= 3,\\
m_{310\,\ldots.\,0} &= 3r_{12},\\
m_{220\,\ldots.\,0} &= 1+2r_{12}^2,\\
m_{2110\,\ldots.\,0} &= r_{23}-2r_{12}r_{13},\\
m_{11110\,\ldots.\,0} &= r_{12}r_{34}+r_{13}r_{24}+r_{23}r_{14}
\end{aligned}\right\}\quad .\;.\quad (6)$$

and any other moment of the second and fourth orders may be obtained by permutation of indices.

Without deducing the polynoms H of the sixth order, the moments of the sixth order may be determined in the following way :—

Evidently we have

$$H_{k_1k_2\ldots.\,k_s+1\,\ldots.\,k_n}=\frac{\partial H_{k_1k_2\,\ldots.\,k_n}}{\partial w_s}-\Sigma r_{sq}w_qH_{k_1k_2\,\ldots.\,k_n},$$

Using this formula twice, we find, putting

$$w_1 = w_2 = \ldots\ldots = w_n = 0,$$

$$\begin{aligned}
&\mathrm{H}_{k_1k_2\ldots\ldots k_s+1\ldots\ldots k_t+1\ldots\ldots k_n}(0,0,\ldots\ldots 0)\\
&= r_{st}\,\mathrm{H}_{k_1k_2\ldots\ldots k_n}(0,0,\ldots\ldots 0)-(-1)^{\frac{k_1+k_2+\ldots\ldots k_n}{2}}\left[\frac{\partial^2\mathrm{H}_{k_1k_2\ldots\ldots k_n}}{\partial w_s\partial w_t}\right]_{w_1=w_2=\ldots\ldots=w_n=0};
\end{aligned}$$

or

$$\left.\begin{aligned}
&m_{k_1k_2\ldots\ldots k_s+2\ldots\ldots k_n}\\
&= m_{k_1+k_2\ldots\ldots k_n}-(-1)^{\frac{k_1+k_2+\ldots\ldots k}{2}}\left[\frac{\partial^2\mathrm{H}_{k_1k_2\ldots\ldots k_n}}{\partial w_s^2}\right]_{w_1=w_2=\ldots\ldots=w_n=0},\\
&m_{k_1k_2\ldots\ldots k_s+1\ldots\ldots k_t+1\ldots\ldots k_n}\\
&= r_{st}\,m_{k_1k_2\ldots\ldots k_n}-(-1)^{\frac{k_1+k_2+\ldots\ldots+k_n}{2}}\left[\frac{\partial^2\mathrm{H}_{k_1k_2\ldots\ldots k_n}}{\partial w_s\partial w_t}\right]_{w_1=w_2=\ldots\ldots w_n=0}.
\end{aligned}\right\}\quad(7)$$

Hence we find, with the aid of formulæ (5) and (6),

$$\left.\begin{aligned}
m_{60\ \ldots\ldots 0} &= 15,\\
m_{510\ \ldots\ldots 0} &= 15r_{12},\\
m_{420\ \ldots\ldots 0} &= 3+12r_{12}^2,\\
m_{4110\ \ldots\ldots 0} &= 3r_{23}+12r_{12}r_{13},\\
m_{330\ \ldots\ldots 0} &= 9r_{12}+6r_{12}^3,\\
m_{3210\ \ldots\ldots 0} &= 3r_{13}+6r_{12}r_{23}+6r_{12}^2r_{13},\\
m_{31110\ \ldots\ldots 0} &= 3r_{13}r_{24}+3r_{12}r_{34}+3r_{23}r_{14}+6r_{12}r_{14}r_{13},\\
m_{2220\ \ldots\ldots 0} &= 1+2r_{12}^2+2r_{13}^2+2r_{23}^2+8r_{12}r_{13}r_{23},\\
m_{22110\ \ldots\ldots 0} &= r_{34}+2r_{13}r_{14}+2r_{23}r_{24}+2r_{12}^2r_{34}+4r_{12}r_{13}r_{24}\\
&\qquad +4r_{12}r_{14}r_{23},\\
m_{211110\ \ldots\ldots 0} &= r_{23}r_{45}+r_{24}r_{35}+r_{25}r_{34}+2r_{12}r_{13}r_{45}+2r_{23}r_{14}r_{15}\\
&\quad +2r_{13}r_{14}r_{25}+2r_{13}r_{15}r_{24}+2r_{12}r_{14}r_{35}+2r_{12}r_{15}r_{34},\\
m_{1111110\ \ldots\ldots 0} &= r_{23}r_{14}r_{56}+r_{13}r_{24}r_{56}+r_{12}r_{34}r_{56}+r_{34}r_{15}r_{26}+r_{34}r_{16}r_{25}\\
&\quad +r_{24}r_{15}r_{36}+r_{24}r_{16}r_{35}+r_{14}r_{25}r_{36}+r_{14}r_{26}r_{35}\\
&\quad +r_{23}r_{15}r_{46}+r_{23}r_{16}r_{45}+r_{13}r_{25}r_{46}+r_{13}r_{26}r_{45}\\
&\quad +r_{12}r_{35}r_{46}+r_{12}r_{36}r_{45},
\end{aligned}\right\}\quad(8)$$

and any other moment of the sixth order is obtained by permutation of indices.

The moments of the variates expressed in any units being denoted by $\nu_{k_1 k_2 \ldots . k_n}$, we have further

$$\nu_{k_1 k_2 \ldots . k_n} = \sigma_1^{k_1} \sigma_2^{k_2} \ldots . \sigma_n^{k_n} m_{k_1 k_2 \ldots . k_n},$$

where $\sigma_1, \sigma_2, \ldots . \sigma_n$ are the respective dispersions of the variates.

On account of the reciprocity of the determinants S and Δ we have finally, putting

$$\frac{S_{pq}}{S} = a_{pq} \; ; \quad \Delta = \begin{vmatrix} a_{11}, & a_{12}, & \ldots . & a_{1n} \\ a_{21}, & a_{22}, & \ldots . & a_{2n} \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ a_{n1}, & a_{n2}, & \ldots . & a_{nn} \end{vmatrix} ; \quad \Delta_{pq} = \frac{\partial \Delta}{\partial a_{pq}}$$

and

$$\frac{\partial^{k_1+k_2+\ldots . +k_n} e^{-\frac{1}{2\Delta}\Sigma\Sigma\Delta_{pq}w_p w_q}}{\partial w_1^{k_1} \partial w_2^{k_2} \ldots . \partial w_n^{k_n}} = \mathrm{R}_{k_1 k_2 \ldots . k_n}(w_1, w_2, \ldots . w_n)\, e^{-\frac{1}{2\Delta}\Sigma\Sigma\Delta_{pq}w_p w_q},$$

the general formula

$$\frac{\sqrt{\Delta}}{(2\pi)^{n/2}} \int_{-\infty}^{\infty} dx_1 \int_{-\infty}^{\infty} dx_2 \ldots . \int_{-\infty}^{\infty} dx_n\, x_1^{k_1} x_2^{k_2} \ldots . x_n^{k_n}\, e^{-\frac{1}{2}\Sigma\Sigma a_{pq} x_p x_q}$$

$$= (-1)^{\frac{k_1+k_2+\ldots . +k_n}{2}} \mathrm{R}_{k_1 k_2 \ldots . k_n}(0, 0, \ldots . 0).$$

Here a_{pq} may be any real quantities, and up to the sixth order the value of the integral is obtained if in the respective equations (6) and (8) r_{pq} is exchanged for $\frac{\Delta_{pq}}{\Delta}$.

APPENDIX.

In order to prove equation (2) we put, using the theorem of Fourier,

$$\phi(x_1, x_2, \ldots . x_n) = \frac{1}{(2\pi)^n} \int_{-\infty}^{\infty} dw_1 \int_{-\infty}^{\infty} dw_2 \ldots . \int_{-\infty}^{\infty} dw_n\, e^{-i\Sigma x_p w_p}$$

$$\times \int_{-\infty}^{\infty} d\lambda_1 \int_{-\infty}^{\infty} d\lambda_2 \ldots . \int_{-\infty}^{\infty} d\lambda_n \phi(\lambda_1, \lambda_2 \ldots . \lambda_n)\, e^{i\Sigma w_p \lambda_p}.$$

It will thus be required to show that

$$\frac{1}{(2\pi)^n\sqrt{\mathrm{S}}}\int_{-\infty}^{\infty}d\lambda_1\int_{-\infty}^{\infty}d\lambda_2\ldots\ldots\int_{-\infty}^{\infty}d\lambda_n\,e^{-\frac{1}{2\mathrm{S}}\Sigma\Sigma\mathrm{S}_{pq}\lambda_p\lambda_q+i\Sigma w_p\lambda_p}$$

$$=e^{-\frac{1}{2}\Sigma\Sigma r_{pq}w_pw_q}. \quad (1)$$

The evaluation of this integral will be possible if the exponent $-\frac{1}{2\mathrm{S}}\Sigma\Sigma\mathrm{S}_{pq}\lambda_p\lambda_q+i\Sigma w_p\lambda_p$ is reduced to the form $-\frac{1}{2\mathrm{S}}\Sigma\Sigma\mathrm{S}_{pq}(\lambda_p+\alpha_p)(\lambda_q+\alpha_q)+\frac{1}{2\mathrm{S}}\Sigma\Sigma\mathrm{S}_{pq}\alpha_p\alpha_q$ and the integral (1) is then equal to

$$e^{\frac{1}{2\mathrm{S}}\Sigma\mathrm{S}_{pq}\alpha_p\alpha_q}.$$

Thus it remains only to show that

$$\frac{1}{2\mathrm{S}}\Sigma\mathrm{S}_{pq}\alpha_p\alpha_q=-\frac{1}{2}\Sigma\Sigma r_{pq}w_pw_q.$$

Putting $i\lambda_p=\lambda_p'$ and $i\alpha_p=\alpha_p'$, we have

$$\frac{1}{2\mathrm{S}}\Sigma\Sigma\mathrm{S}_{pq}\lambda_p'\lambda_q'+\Sigma w_p\lambda_p'$$

$$=\frac{1}{2\mathrm{S}}\Sigma\Sigma\mathrm{S}_{pq}(\lambda_p'+\alpha_p')(\lambda_q'+\alpha_q')-\frac{1}{2\mathrm{S}}\Sigma\Sigma\mathrm{S}_{pq}\alpha_p'\alpha_q'$$

$$=\frac{1}{2\mathrm{S}}\Sigma\Sigma\mathrm{S}_{pq}\lambda_p'\lambda_q'+\frac{1}{\mathrm{S}}\Sigma\Sigma\mathrm{S}_{pq}\lambda_p'\alpha_q'.$$

Hence we have

$$w_s=\frac{1}{\mathrm{S}}\sum_t\mathrm{S}_{st}\alpha_t'.$$

This is a system of linear equations to determine the different α_t'.

Putting

$$\Delta=\left|\frac{\mathrm{S}_{st}}{\mathrm{S}}\right|;\quad\Delta_{st}=\frac{\partial\Delta}{\partial\frac{\mathrm{S}_{st}}{\mathrm{S}}},$$

the solution is

$$\alpha_s'=\frac{1}{\Delta}\sum_t\Delta_{st}w_t.$$

But according to well-known theorems on determinants, we have

$$\frac{\Delta_{st}}{\Delta} = r_{st} ;$$

and hence

$$\alpha_s' = \sum_t r_{st} w_t .$$

For the sum

$$\frac{1}{2\mathrm{S}} \sum_p \sum_q \mathrm{S}_{pq} \alpha_p' \alpha_q'$$

we may now write

$$\frac{1}{2\mathrm{S}} \sum_p \sum_q \mathrm{S}_{pq} \sum_t r_{pt} w_t \sum_s r_{sq} w_s .$$

Changing the order of summation, we have this equal to

$$\frac{1}{2\mathrm{S}} \sum_s \sum_t w_s w_t \sum_p r_{pt} \sum_q \mathrm{S}_{pq} r_{sq} .$$

But

$$\sum_q \mathrm{S}_{pq} r_{sq} = \begin{cases} 0 & \text{when } p \neq s, \\ \mathrm{S} & \quad ,, \quad p = s. \end{cases}$$

Thus the quadruple sum is equal to

$$\frac{1}{2\mathrm{S}} \sum_s \sum_t w_s w_t r_{st} \mathrm{S},$$

or

$$\frac{1}{2\mathrm{S}} \sum_p \sum_q \mathrm{S}_{pq} \alpha_p' \alpha_q' = \frac{1}{2} \sum_p \sum_q r_{pq} w_p w_q .$$

As $\alpha_p' = i\alpha_p$, we have finally

$$\frac{1}{2\mathrm{S}} \sum_p \sum_q \mathrm{S}_{pq} \alpha_p \alpha_q = -\frac{1}{2} \sum_p \sum_q r_{pq} w_p w_q .$$

Q. E. D.

XXXVIII. *Mechanical "Resonators" under Double Forcing.*

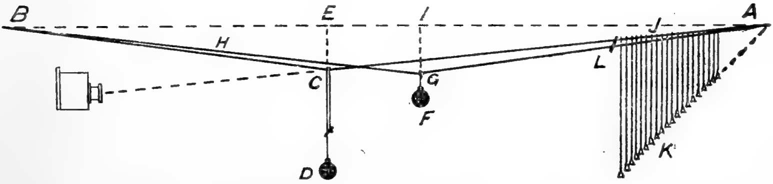
By Prof. E. H. BARTON, *F.R.S., and* H. M. BROWNING, *M.Sc.*[*]

[Plates V. & VI.]

FOLLOWING up the idea of a recent paper on Forced Vibrations[†], it seemed desirable to use a similar set of pendulum "resonators" to exhibit experimentally the effects upon them of two simultaneous harmonic forcings of different periods. Four sets of photographs of six each were taken. The first set had resonators highly damped. The second set had denser bobs and so less damping. In each case the two drivers differed greatly in their periods. The third set of six photographs had the less damped resonators and drivers whose periods differed slightly and were made to approach and finally coalesce. The final set of six had but one driver and was taken to demonstrate that the resonators could discriminate several periods of driver between those of adjacent resonators.

Experimental Arrangements.—Fig. 1 shows the experimental arrangement used throughout. Twenty "resonators" were made as pendulums with small paper cones as bobs and black threads as suspensions. These cones were used alone for the first set of six photographs, and afterwards with rings of copper wire on them to lessen their damping in the subsequent photographs. Their logarithmic decrements in the two states were respectively of the order 0·1 and 0·025 per half wave, but varied slightly from end to end of the system.

Fig. 1.



The pendulums J K were hung at equal distances and their lengths arranged so that a line drawn through the bobs pointed to A, one of the fixed ends of the cord. The extreme lengths of these responding pendulums were about as three to one. The direction of this cord AC pointed to the lens of the camera. At the point C on this cord ACB was hung a

---

[*] Communicated by the Authors.

[†] Phil. Mag. [6] vol. xxxvi. pp. 169–178 (Aug. 1918).

pendulum with heavy bob D. On an adjacent cord AGH was hung at G a similar pendulum, with bob F about equal in mass to D. This pendulum was made to affect the resonators by means of a wooden connector at L. This connector is far from both the large driving pendulums, and so the effects of the coupling between them were always small and never obtruded themselves.

Thus the two forcing pendulums or drivers could be of various lengths at will, and so subject the resonators to the corresponding double harmonic impressed forces. For each one was sufficiently connected to the resonators while the two were almost free from action and reaction on each other. The experiments consisted in adjusting the lengths of the drivers, starting their oscillations, and then taking either flash photographs or time exposures of the resonators.

It should be noted that the effective lengths of the driving pendulums are DE and FI respectively, those of the resonators being typified by JK.

Results.—Plate V. figs. 1–6 shows the effects obtained on the responding pendulums with plane paper cones by two drivers of distinctly different periods and kept of the same lengths throughout. The responders were highly damped, and soon settled to their steady state corresponding to their forced vibrations only. It is highly instructive to watch the resonators in these cases. They show two distinct humps or places of maximum resonance as exhibited in fig. 1, which is a time exposure. But also, when watching them, the quick vibrations of those pendulums in tune with the short driver are in striking contrast to the slower vibrations of those in tune with the longer driver. Thus the vibrators exhibiting the two humps are almost always out of phase. But near each hump or maximum there is the usual state of ordered phase relation corresponding to the resonance in question. These effects are partly indicated by the flash photographs of figs. 2–6. These five reproductions illustrate one of the effects shown in the paper already cited. Namely, that we have sometimes a hump all on one side of the central line, and sometimes the quasi-cubic with smaller humps one on each side. It should be noticed that in figures 1–6 of this Plate the resonance is not sharp but spreads over a considerable distance up and down owing to the very strong damping of the resonators in use.

Plate V. figs. 7–12 shows the effects obtained when the paper cones each carried a ring of copper wire and so were much less damped. Consequently the resonances are much sharper instead of being widely spread as before. This is best seen in the two time exposures, figs. 9 & 10. The flash photographs again show the random phases of the resonators responding to the two different periods of forcing in use. The cases of a hump at one side and the cubic are again illustrated. The increased sharpness of resonance suggested pushing the periods of the two drivers nearer to equality with the view of noting how closely two maxima of resonance could occur, and yet be detected without coalescence of behaviour on the part of the responders. This detection of separate maxima in spite of some spreading of the resonance is somewhat analogous to the case of resolution of points by lenses and lines by prisms in spite of diffraction.

The idea of such resolution was accordingly followed up in the experiments whose results are given in Plate VI. figs. 13–18, consisting of four time exposures and two flash photographs. Of these four time exposures, the driving pendulums had the greatest difference of lengths for fig. 16, they are closer for fig. 13, closer still for fig. 17, and are in exact agreement of lengths and periods in fig. 18. Indeed the result shown in fig. 18 is indistinguishable in character from that of fig. 19 (also on Plate VI.), in which only a single driver was employed. The two flash photographs, figs. 14 & 15, are for the same drivers as used in the time exposure fig. 13, and show the change of phase introduced by the slightly different periods of forcing.

In Plate VI. figs. 19–24 are given the results obtained by using a single driver whose length was very slightly changed from each figure to the next. Thus the discrimination of frequency of a driving vibrator which can be effected by a set of graduated responding vibrators is seen to be carried to *a fineness much closer than that of the difference of frequencies of the successive responders themselves.* For in passing through this series of six figures (19--24) we do not cover the interval between the frequency of one vibrator and its neighbour.

Conclusion.—Some find difficulties in accepting a resonance theory of audition on the ground that there may not be in the internal ear sufficient responding vibrators provided with separate nerves to give one nerve for each perceptible pitch throughout the range of hearing. To such the above simple experiments with pendulums may afford some help by analogy. These experiments, at any rate, show that the resonance theory of audition, whether it stands or falls, does not fall on this account.

It is hoped to deal more particularly with this and other points as to the resonance theory of hearing in another paper for which the experiments are already completed, and to which this paper forms a natural introduction.

Nottingham,
February 14th, 1919.

XXXIX. *Resonance Radiation of Sodium Vapour excited by One of the D Lines.* *By* R. W. Wood *and* F. L. Mohler[*].

IN a paper on "Resonance Radiation of Sodium Vapour"[†], published by one of us in 1905, it was shown that a glass tube containing the vapour of metallic sodium at a low temperature (about 200°), when illuminated by the light of a sodium flame, re-emitted a yellow light, which spectroscopic analysis showed was identical with the exciting light, in other words the two D lines.

In 1914 Wood and Dunoyer[‡] showed that the vapour when excited by the light of one D line only, emitted only this same wave-length, in other words, that it was possible to have sodium vapour emit only one of the D lines. This showed that the mechanisms producing the D lines could be separately excited.

Owing to the very long exposures which were necessary in these experiments, and the extreme faintness of the images obtained, it was impossible to tell whether the other D line was in reality completely absent, or merely greatly reduced in intensity. Moreover, in the few cases in which a trace of the other D line appeared, it was never quite certain that this wave-length was completely eliminated from the exciting beam, as the temperature of the room always varied considerably, and this altered the optical properties of the quartz block used in the polarization method of separating the two lines.

We have accordingly repeated the work, using a spectroscope of higher resolving power, and controlling the temperature of the optical system used for separating the D lines, with a thermostat.

We have, moreover, taken extreme precautions to remove completely the occluded hydrogen, and have examined the effect of the presence of small traces of hydrogen on the nature of the radiation.

This seemed very important, as it has been shown by one of us that when iodine vapour in an atmosphere of helium at low pressure is excited to resonance by the light of the mercury arc, there is a transfer of energy from one vibrating system to others, and the emitted spectrum is much more complicated than is the case when the iodine is in a high vacuum.

---

[*] Communicated by the Authors. This communication is an abstract of a paper which appears in full in the Physical Review.

[†] R. W. Wood, Phil. Mag. [6] x. p. 513 (1905).

[‡] R. W. Wood & L. Dunoyer, Phil. Mag. [6] xxvii. p. 1018 (1914).

Other resonance phenomena show that the centres of radiation in the sodium molecule are not entirely independent. One of us * showed in 1905 that the vapour, when excited by blue-green light, in the region of its band-absorption spectrum emitted a narrow yellow band coincident with the D lines.

More recently Strutt † has found that excitation by light of wave-length equal to that of the ultra-violet doublet (second member of the principal series, of which the D lines form the first member) caused the emission of the D lines.

In view of results which will be mentioned presently it seems possible that the effect observed by Strutt may have been due to traces of hydrogen in his bulbs.

The polarization method of separating close spectrum lines, which was described by one of us ‡ in 1914, has been employed in the present work. The apparatus has been improved, however, in a number of respects. Larger polarizing prisms have been employed, and the entire polarizing system was enclosed in a box and kept at a constant temperature by means of an electrical heater controlled by a thermostat. The thick quartz plate was re-figured and polished, as some slight errors had been found which affected the purity of the light delivered by the apparatus.

The spectroscope was built up with two large portrait objectives of 3 inches aperture and 24 inches focus, and two 5-inch prisms of flint glass.

The plates were clamped against the tube which held the eyepiece of the filar micrometer, and by moving this slightly with the screw between exposures, two or more images could be recorded on the same plate side by side.

A very great advance resulted from the discovery that the new "pyrex" glass made by the Corning Co. (Corning, N.Y.) resists the action of sodium vapour far better than any other glass in the market. Bulbs blown from tubes of this glass showed scarcely any discoloration after ten hours' heating, which is sufficient to colour bulbs of soft-glass a dark brown.

As a source of light we employed a Meker burner fed with sodium by a large disk of asbestos board charged with sodium chloride and rotated once in twelve hours by a clock. It is important that the disk touches only the edge of the flame on the side opposite to that which faces the illuminating system, otherwise there is a reversal of the D lines, which is very detrimental to the excitation of the resonance. This

---

* R. W. Wood, Phil. Mag. [6] x. p. 408 (1905).

† Strutt, Proc. Roy. Soc. Series A, xci. p. 511.

‡ Wood, Phil. Mag. [6] xxvii. p. 524 (1914).

arrangement worked quite as well as the more complicated atomizing apparatus employed in the earlier work.

The sodium bulbs were prepared in the same way as before, except that the sodium was distilled from one side of the bulb to the other a great many times by waving the flame of a burner around the bulb, the pump working continuously. We found that even after the sodium deposit had been driven across the bulb 170 times, the McLeod gauge still showed a small trace of hydrogen.

We have photographed the spectrum of the resonance radiation excited by both D_2 and D_1. In the earlier work we experimented only with the light of the D_2 line on account of its greater brilliancy.

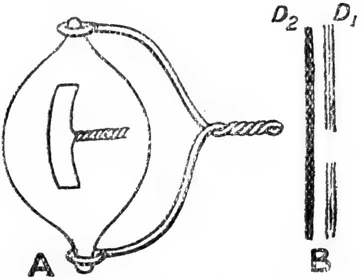
The efficiency of the polarizing separators was tested by photography, and it was estimated that when the quartz block was set for the extinction of D_1, D_2 was about 50 times brighter than D_1 in the transmitted light. Adjustments of the apparatus were made by coating the bulb with a layer of magnesium oxide, by holding it over burning magnesium wire. The coating was then wiped off with the exception of a narrow vertical strip. The bulb was mounted in a wire frame as before over a chimney heated by a quadruple burner, in such a manner that it could be rotated on a vertical axis. The image of the small window in the sheet-iron chimney surrounding the sodium flame, formed by the polarizing separator, was focussed on the patch of MgO, and an image of this thrown on the slit of the spectroscope. The quartz block was then turned until one only of the D lines appeared in the spectroscope. The bulb was now rotated until the image fell on the clear glass to one side of the oxide patch, and the quadruple burner lighted.

With the improved apparatus we could give exposures of ten or twelve hours, which was sufficient to yield a fully exposed spectrogram of the resonance.

In the case of the first photographs which we made we found that there was always a trace of the D_1 line when the illumination was by D_2.

All possible precautions were taken to exclude scattered radiation, and to still further eliminate every chance of error, we added a narrow horizontal strip of dark grey Mg oxide which cut across the patch of resonance, and matched it in intensity. This grey patch we made by heavily smoking the bulb in a gas-flame and then depositing a very light coat of MgO. The bulb with the large white patch and the narrow grey strip is shown in fig. 1 A. Fig. 1 B shows the appearance of the resulting spectrum line as it appeared when this method was used. Both D lines appear except at the place where the exciting light is diffusively reflected from the horizontal grey strip, and there only one line is recorded. If the appearance of both lines resulted from reflexion of stray light from the glass, there would be no break in the D_1 line.

Fig. 1.



In the early part of the work we obtained very conflicting results, for on some plates we found no trace of the line which was absent in the exciting light, and in others both lines were present. We finally found that bulbs very carefully freed from hydrogen, and operated at a comparatively low temperature, gave always a pure resonance radiation, *i. e.* one line only appeared. If hydrogen was present, or if the temperature (and consequently the density) was raised, both lines were recorded.

The effect of a change in the density of the pure sodium vapour was first investigated.

A bulb carefully freed from hydrogen was used for three exposures to resonance excited by D_2 at temperatures of 210°, 270°, and 340°, keeping all other conditions constant. D_1 was absent in the case of the exposure made at 210°, it was present though faint at 270°, while at 340° it was nearly as strong in respect to D_2 as in the case of a feeble sodium flame for which the intensity ratio $\frac{D_2}{D_1}=2$.

In making our exposures we always made a preliminary exposure of ten minutes on the MgO patch, following this with the resonance exposure, and ending with a third exposure on the MgO. This furnished a check on the purity of the exciting light, and showed that no change had occurred in the adjustments of the polarizing separator. In our photographs the two outside records were obtained by the light reflected from the MgO, and the middle record from the resonance.

The change resulting from the presence of hydrogen in the bulb was also very marked. In this case both lines were present even when the bulb was at a low temperature.

Similar investigations were made using D_1 as the exciting radiation. The difficulties are greater in this case, as D_1 has an intensity only half as great as that of D_2, and it is more difficult to completely exclude the light of the brighter line: moreover, the resonance is only half as bright as with D_2 excitation.

With pure sodium at 210° we obtained a faint trace of D_2, while with 0·1 mm. of hydrogen D_2 is about half as bright as D_1.

On some plates recording the effects with more hydrogen in the bulb, the intensity of the two lines was nearly equal. All of the results were verified by repetitions, about fifty plates being taken in all.

The observations may be summarized as follows:—

D_2 *Excitation.*

Bulb as free from hydrogen as possible.
    At 210° (no trace of D_1) intensity ratio of D_2 to D_1 at least 20 : 1.

    At 300° ratio of D_2 to D_1 about 5.

Bulb containing hydrogen at 0·25 mm.
    At 220° ratio of D_2 to D_1 about 4.
    At 300° ,, ,, 3.

D_1 *Excitation.*

Bulb free from hydrogen.
    At 220° a trace of D_2.
    At 300° ratio of D_1 to D_2 about 3.

Bulb containing 0·1 mm. hydrogen.
    At 220° ratio of D_1 to D_2 about 2.

Bulb containing 0·25 mm. hydrogen.
    At 250° ratio about 3/2.

In conclusion it seems safe to say that the transfer of energy from the excited line to its companion results from molecular collision, either of sodium with hydrogen or sodium with sodium, the effect in the latter case being evident only at high temperature and increased density. The effect is probably analogous to the similar one which occurs in the case of the iodine vapour in a tube containing helium, excited by the green line of the mercury arc. If the iodine is in a high vacuum the emission spectrum consists of a series of 28 members of close doublets. In helium the doublets are weakened and the band spectrum is strongly developed.

XL. *Notices respecting New Books.*

Matrices and Determinoids. By Dr. C. E. Cullis. Vol. II. Pp. xxiv+555. Cambridge: University Press, 1918.

THE first volume of these Readership Lectures at the University of Calcutta was published in 1912, and a third volume, "completing the theory of matrices and clearing the way for the applications" (p. xi), is in prospect of publication. This volume is full of elaborate detail, and what may seem to the unsympathetic the monotony of systematic deduction is not relieved for the benefit of lesser mortals by interesting geometrical and physical applications or references to the work of others. Indeed, other persons are not mentioned in this volume with the exception of those—such as Sylvester and Rodrigues—whose names are attached to theorems, and in a short acknowledgment on pp. vii–viii of certain books on higher algebra and geometry by Bôcher, Heffter and Koehler, Muth, Netto, Veronese, and Whitehead, and papers by Schläfli (1866) and Haripada Datta.

The absence of applications is at first sight somewhat surprising: the author was in the habit of using matrices freely in the solution of problems in algebra, geometry, and applied mathematics, and the chief aim of his book is to give a systematic account of certain applications of matrices (p. v). The first volume contained the foundations of a calculus of matrices; an account of the properties of the "determinoid" of a matrix, which becomes the determinant of the matrix when the matrix is square; an account of the solution of matrix equations of the first degree; and some other features. This second volume contains those parts of the theory which naturally precede any investigation of the special properties of those ("functional") matrices whose elements are rational integral functions of a finite number of variables (p. vi). Thus we see that, with regard to that part of Dr. Cullis's work which deals with the applications, the constant increase of the theoretical portion has resulted in the applications being driven further back, "though they still remain the ultimate object of the book" (p. v).

This volume contains Chapters XII–XIX of the whole work, and the chapters deal successively with definitions of compound and compartite matrices and other introductory matter (p. 1); relations between the elements and minor determinants of a matrix (p. 37); some properties of square matrices (p. 107); ranks of matrix products and matrix factors (p. 165); equigradent transformations of a matrix whose elements are constants (p. 228); some matrix equations of the second degree (p. 309); the extravagances of matrices and of spacelets in homogeneous space (p. 378); and the paratomy and orthotomy of two matrices and of two spacelets of homogeneous space (p. 463). There are also Appendixes and an Index.

The ideas and language are frequently geometrical: for example, we have "spacelets" (pp. vi, 78), but there seem to be—quite naturally, after what has been said above—no geometrical applications in this volume, and the only physical application seems to be the interesting treatment (pp. 330–332) of the rotation of a rigid body.

XLI. *Proceedings of Learned Societies.*

GEOLOGICAL SOCIETY.

[Continued from vol. xxxvi. p. 488.]

November 6th, 1918.—Mr. G. W. Lamplugh, F.R.S., President, in the Chair.

The President read a communication that he had received from Prof. Charles Barrois, D.Sc., F.M.G.S., in reply to congratulations sent on the occasion of the evacuation of Lille by the enemy forces.

A Discussion on the Antarctic Ice-Cap and its Borders was introduced by Sir Douglas Mawson, D.Sc., B.E., F.G.S.

Sir Douglas Mawson said that at the last Meeting of the Society the subject of the Antarctic Ice-Cap was reviewed in its broader aspects, chiefly with the view of promoting a discussion among those specially interested in Glaciology. The present occasion had been reserved for the Discussion, and he proposed to show certain lantern-slides in order to bring the salient features freshly to mind.

Though much of the foundation of the Antarctic Ice-Cap is certainly elevated land, it is quite possible that elsewhere the dome rests upon a floor actually below sea-level. In any case it is most probable that the smooth ice-surface masks a very irregular rock-basement. The thickness of the ice may, therefore, be expected to be extremely variable, no doubt reaching a maximum of several thousands of feet.

An ice-formation of such magnitude introduces questions relating to the flow of its substance and the abrasion of its foundations which do not enter into the physics of ice-masses of smaller dimensions.

Here the static pressure on the lower zones of the ice may reach 1 ton per square inch. At the same time, the temperature may be so increased by ground heat as to be much higher than that prevailing above. As a consequence, when the ice-formation is very thick, a more plastic base must be admitted.

The outflow of the inland ice is principally deflected at the coastal margin into depressed areas outlining the heads of gulfs and bays. In such localities the rate of movement and the volume of ice entering the sea are both great. So great indeed, that extensive floating 'glacier tongues' are a feature of such situations, often extending 40 to 50 miles from the shore.

Along other stretches of the coast less well placed for receiving contributions from the interior of the Continent, the outflow is so much less that the destructive influences at work on reaching the sea easily maintain its boundaries at approximately the true coast-line.

As exceptions to this latter prevailing condition, however, there are known already two notable localities where the general overflow from the land maintains itself as an immensely thick floating structure extending far out over the sea—a veritable oceanic ice-cap. To this type of formation we apply Prof. Nordenskjöld's term 'shelf-ice.' The formations referred to are the Great Ross Barrier at the head of the Ross Sea, and the Shackleton Shelf off the coast of Queen Mary Land.

The former occupies what is really the head of the Ross Sea—a somewhat triangular area. From apex to base it measures 500 miles, with a base-length of about 400 miles. This great raft of ice presses forward to the open sea at the rate of a few hundred yards per annum. The available figures, quoted by David and Priestly, show that, at the present rate of advance, the ice now appearing at the sea-face must have left the inner extremity of the floating sheet at some time during the 7th century. A survey of the ice-cliff forming the sea-face indicates by its changing height that the Ross Barrier is of varying thickness. This has been explained by the presence, in localities where it is thickest, of the remnants of the massive-ice contribution received during its course from certain of the large tributary glaciers. The ice from these glaciers, in fact, constitutes a strong framework which stiffens and contains the more crumbling structure derived from the consolidation of the annual snowfall.

To a great extent this must certainly be so; but the influence of a varying snowfall, and the effect of violent periodic winds—a feature of the region—in sweeping the loose snow from certain areas and depositing it in other favoured localities, must be reckoned with. The snowfall is lighter on the eastern side than on the western. Furthermore, the snow tends to accumulate on the western side owing to the fact that the winds regularly blow from the quarter south to east, and not from the west.

In the case of the Shackleton Shelf, this is the more remarkable because it maintains itself as a pontoon stretching into the open sea, even across the drift of the prevailing ocean-current.

The deluge of ice, after descending to the sea, presses northwards as an integral whole, at first touching bottom at intervals, then forcing its way past several islands, eventually reaches an extreme distance of 180 miles from the land before it is mastered by the swell and currents of the Southern Ocean. It is somewhat triangular in form, with the apex out to sea. The base against the land, though not completely charted, extends in all probability for a distance of about 200 miles.

The main body of the shelf-ice advances rather slowly, but the Denman Glacier, which contributes to it, has a much more rapid movement, very well illustrated by the fact of its ploughing through the other shelf-ice with such force that a shatter-zone some miles wide is developed.

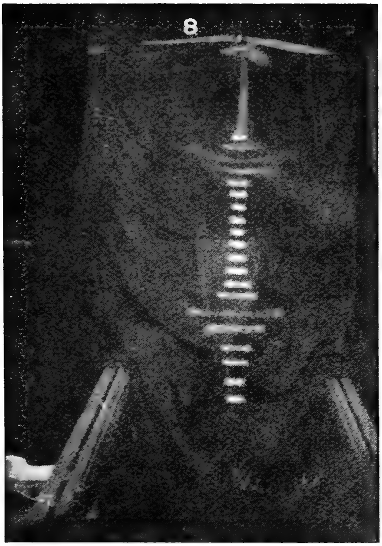
The wall of the shelf-ice on the west side offers an excellent example for study, as it is a section from the point of its departure from the land to its crumbling apex. In the case of the Ross Barrier, the cliff-face is a section across the direction of movement.

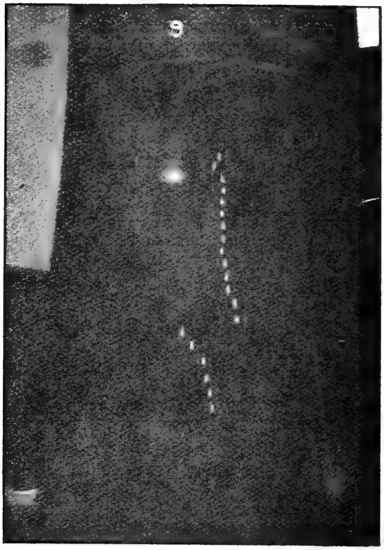
At the land end, the Shackleton Shelf, from the surface down, is hard glacier-ice breaking with a characteristic fracture. A few miles farther out, away from the influence of the winds descending from the land slopes, a nevé mantle commences to make its appearance over the original ice-formation. As one steams along the face away from the land, this capping is observed to increase steadily in thickness. The overburden of nevé is arranged in regular bands, each of which corresponds to a single year's addition. This being so, it is possible to make some sort of an estimate of the age of the formation.

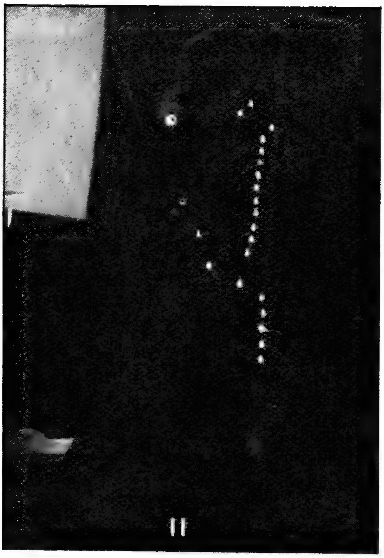
The weight of these additions depresses the top of the original ice below the surface of the water. Though there is a regular annual addition above, it must not be imagined that the total thickness of the pontoon is correspondingly increased; for the solution of the lower surface by the sea has also to be reckoned with. Very often, however, in the nevé sections of glacier-tongues the cliff-face above the water is observed to stand higher than in the wholly ice zone at the land end. This is to be expected on account of the lighter nature of the nevé ice added, there being a larger proportion of air sealed up in it.

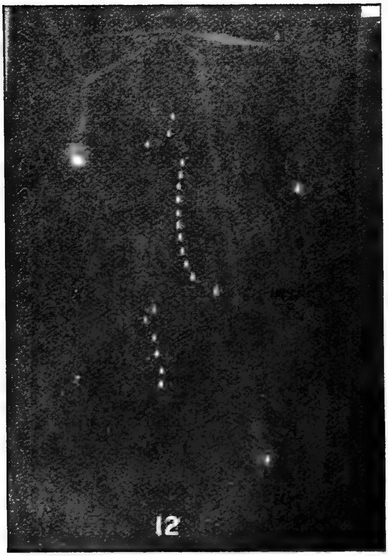
The observed height above sea-level of Antarctic shelf-ice so far recorded ranges from about 20 to over 200 feet. A common figure is from 90 to 120 feet, suggesitng a total thickness of 600 to 1000 feet.

Although the height of the cliff-face presented by shelf-ice gives some idea of its total thickness, a really accurate method of determination is badly needed. The Australasian Expedition hit upon a method which gives positive results in some cases at least. This consists in taking serial temperatures of the sea-water in depth near the face of the shelf-ice. As there is always a current flowing beneath the ice, the bottom of it is likely to be marked by a sudden slight change in the water temperature, easily observed when the observations are plotted as a graph.


8

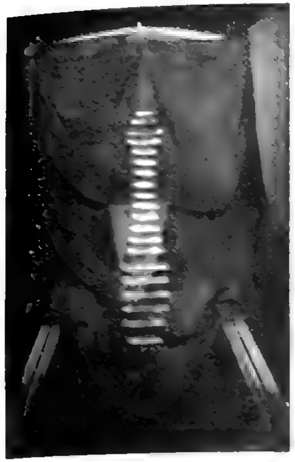

9

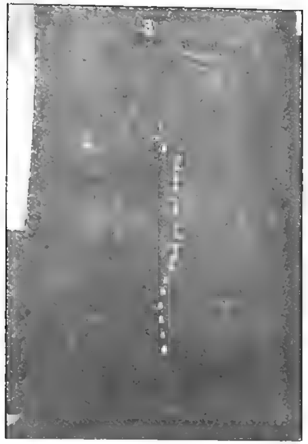

11

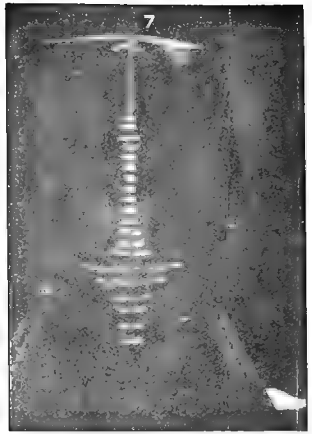

12

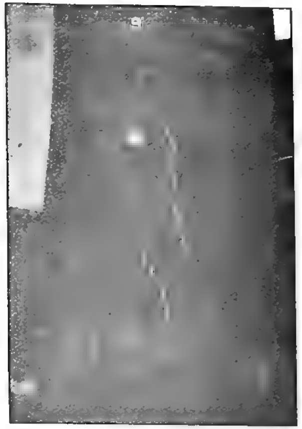
Barton & Browning

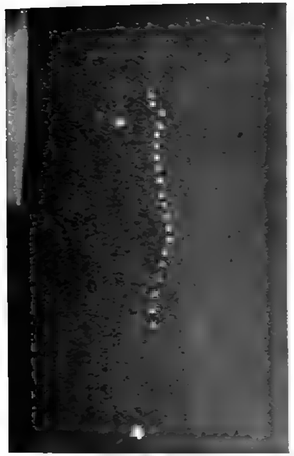
Phil. Mag. Ser. 6. Vol. 37. Pl. V.

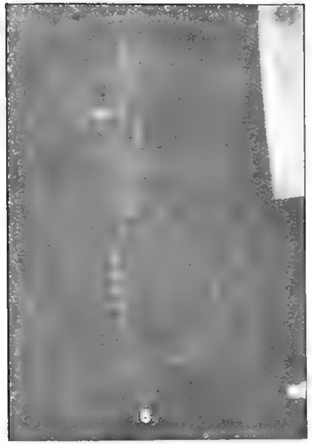


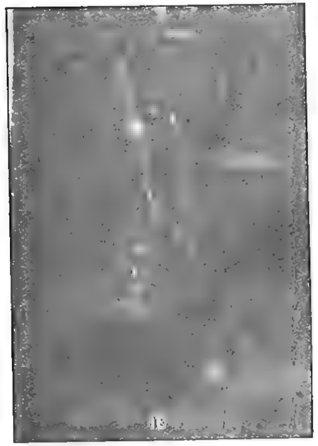


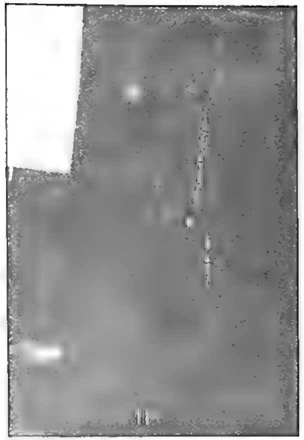


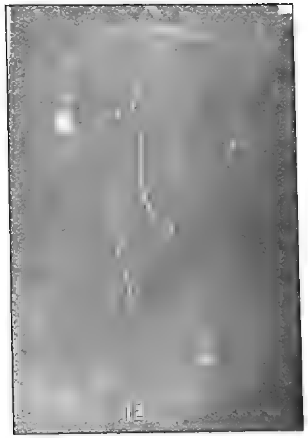


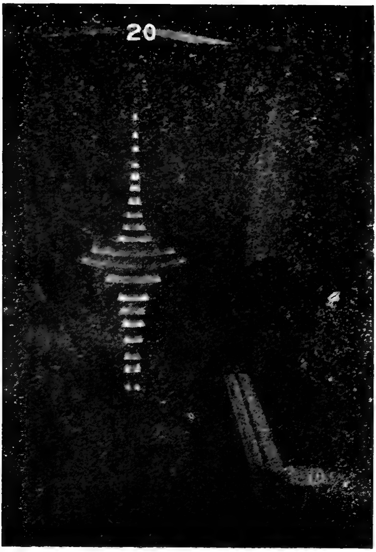




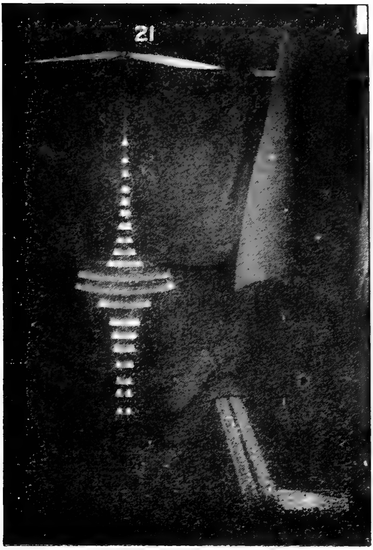




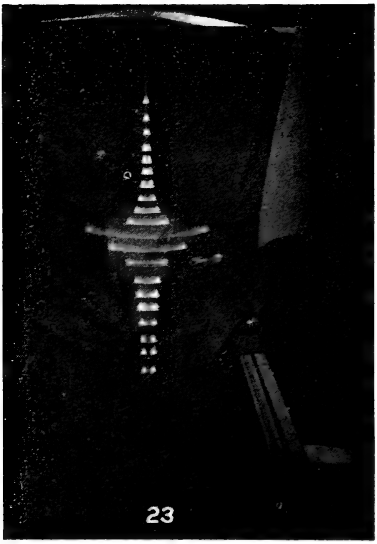




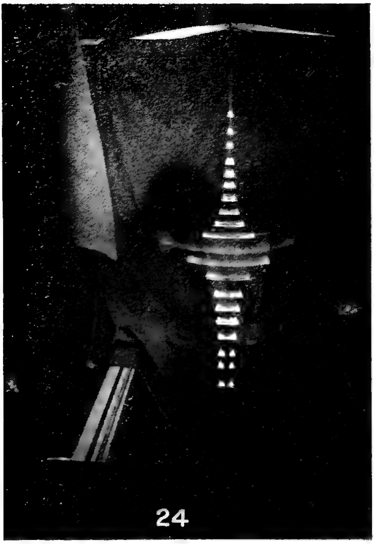
20



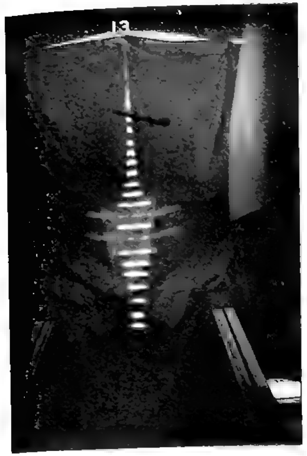
21

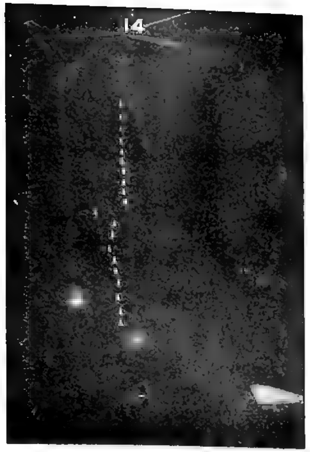


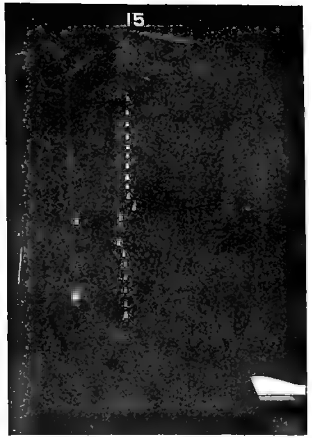
23



24

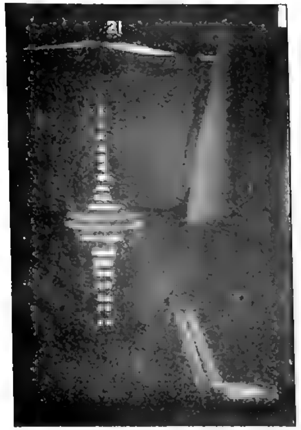

13


14


15

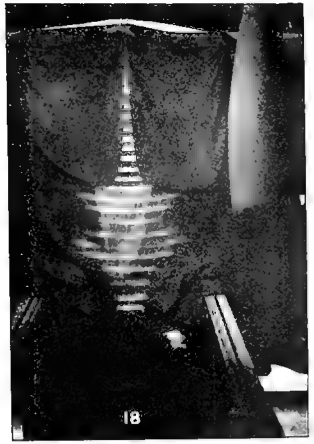

19


20


21


16


17


18


22


23


24

THE
LONDON, EDINBURGH, AND DUBLIN
PHILOSOPHICAL MAGAZINE
AND
JOURNAL OF SCIENCE.

[SIXTH SERIES.]

MAY 1919.

XLII. *On a possible means of determining the two characteristic Constants of the Æther of Space.* *By* Sir OLIVER LODGE, *F.R.S.**

[This elementary paper has for its object the eliciting of an opinion, not the communication of a result.]

THE two constants which regulate the properties and behaviour of the æther so far as known, whether the phenomenon studied be electric, magnetic, or optical, are simply and fundamentally defined thus, in terms of actual mechanical force :—

$$F = \frac{ee'}{Kr^2} \quad \text{and} \quad F = \frac{mm'}{\mu r^2}.$$

These definitions are independent of any system of units (*i.e.* are true in all), so the dimensions of an electric charge are $e = l\sqrt{(KF)}$ and of a magnetic pole $m = l\sqrt{(\mu F)}$.

We may also write magnetic moment $ml = \mu AI$, A being area and I being de/dt ; or we may equally well write $m/e = \mu v$; which happens to be of the same dimensions as the coefficient of electrical resistance.

The dimension of an induction coefficient is μl, and of a capacity is Kl.

In the case of *energy* all unknown dimensions disappear, since all forms of energy are interchangeable ; hence $\frac{1}{2}LI^2$ and $\frac{1}{2}SV^2$, or mI and eV, are truly dynamical expressions.

---

* Communicated by the Author.

There are also some expressions which are really *time*, namely RS, L/R, and $\sqrt{(\mathrm{LS})}$. And there is also the remarkable relation $\sqrt{(1/\mu\mathrm{K})}$ which dominates optics and is really a velocity. It follows that the product em is really angular momentum, or of the same dimension as Planck's constant h.

[It is even of interest to reckon the purely numerical relation between $e_1 m_1$ and h, where e_1 is the charge of an electron, and m_1 some corresponding unit magnetic pole which might be called a magneton. When an electron in an atom is revolving so fast that it flies off, these quantities are brought into relation with the *quantum*; and since $4\pi m$ is the total induction from a pole, and $4\pi e$ the total lines of force from a charge, and $h/2\pi$ the angular momentum of a revolving electron ready to escape from an atom so that its energy comes within our range of observation, it is natural to write

$$4\pi m_1 \times 4\pi e_1 = h/2\pi$$

or

$$h = 32\pi^3 e_1 m_1.$$

To see if the order of magnitude here involved is at all right, we can utilize the known relation $m = \mu ve$, and take v as practically the speed of light, c, for an escaping β ray.

Now h has been measured as

$$6{\cdot}547 \times 10^{-27} \text{ c.g.s. } (\mathrm{ML^2T^{-1}}),$$

while Professor Millikan's measure of

$$e \text{ is } 4{\cdot}774 \times 10^{-10} \text{ electrostatic unit.}$$

Hence $e_1 m_1 = \mu c e_1^2 = e_1^2/\mathrm{K}c = 7{\cdot}6 \times 10^{-30}$ c.g.s. $(\mathrm{ML^2T^{-1}})$ *.

It will be seen that $32\pi^3$ times this quantity gives an h of the right order of magnitude, though a little too big, namely $7{\cdot}5 \times 10^{-27}$. But the agreement is sufficiently near to be striking. To make it exact we should have to consider the electronic charge as

$$e_1 = 4{\cdot}36 \times 10^{-10} \text{ electrostatic unit.}$$

Or, if we take h and e, above, as both correctly measured, and $\mu v e_1^2 = h/32\,\pi^3$, we can reckon the velocity with which an electron is revolving before it escapes, as $v = 2{\cdot}61 \times 10^{10}$ cm. per sec.]

In passing we may note that electromotive force is m/t

---

* The above $e_1 m_1$ is the angular momentum of a sphere the size of electron and density 10^{-12} revolving on its own axis with an equatorial speed of light.

while magnetomotive force is e/t; so the ratio of the two potentials is the inverse ratio of the charges.

Every electrical expression which can be interpreted dynamically necessarily involves *two* of the electric or magnetic quantities,—one of each except when a mere ratio is involved,—otherwise the ætherial constants could not cancel.

If the constants are omitted and their unknown dimensions not attended to or allowed for, we get impossible equations expressing electric and magnetic quantities in terms of ordinary mechanical units; so that the relation between æther and matter is masked. If they are always retained, the equations are true in all respects, and it is easy to interpret them in any artificial and temporary system of units that may be convenient. The two recognized artificial systems are necessary until the nature of the constants μ and K is known: and even afterwards they will probably still be used for practical purposes.

All that we at present certainly know about these constants is derived from Maxwell's theory, and can be expressed thus:—that if the following operators are applied to any suitable electric vector this relation holds,

$$\nabla^2 + \mu \mathrm{K} \frac{d^2}{dt^2} = 0.$$

Whence it follows that the product μK is the reciprocal of the square of the velocity of wave propagation, or $\mu \mathrm{K} c^2 = 1$.

Well-known facts in electrostatics suggest that K has some kind of elastic significance; while μ, which dominates magnetic phenomena, simulates more the properties of inertia. Yet neither property can belong to ordinary matter: they both belong to vacuum: matter can only modify them, as it can only modify the velocity of light. Hence it becomes natural to suggest that $\frac{4\pi}{\mathrm{K}}$ may be the transverse elasticity or rigidity of the æther; while $4\pi\mu$ may be its density. (See 'Modern Views of Electricity,' Appendix *p*.)

Reasons are given in the Phil. Mag. for April 1907, p. 488, and in my book 'The Ether of Space,' for the probability—assuming æther to be the raw material out of which electrons are made and that it is incompressible—that the density of æther is excessively great: in fact possibly of the order 10^{12} grammes per c.c.

If that were true it would follow that the elasticity effective in radiation must be of the order 10^{33} c.g.s., so that the ratio of the two may be 9×10^{20}, the square of the velocity of light.

The problem before us is to find out if any experimental means can be devised for putting this hypothesis or assumption to an experimental test, so as either to confirm or correct it.

Now we know that magnetic lines of force, unlike electric lines of force, are always closed curves; hence it is possible for something material to flow along them continuously. Round every electric current there is magnetic circulation; is it possible that this may be ætherial circulation?

As I understand Larmor's theory of magnetism (see for instance pages 84 & 336 of his book, 'Æther and Matter.' See also the Phil. Mag. for April 1907, p. 500) it postulates a flow of something along magnetic lines. A magnetic field is a seat of kinetic energy, while an electric field is a seat of potential energy; the interaction of the two fields is responsible for wave propagation.

If any æther is really flowing along the lines of a magnetic field, and if it has a real density, it is natural to attribute the energy of the field to the kinetic energy of this flow; and to identify the self-induction of an electric current with the surrounding ætherial inertia.

That, at any rate, is the working hypothesis which I propose to make; and if we could determine the velocity of the flow in a given field, we should thereby determine the ætherial density. That is an experiment the possibility of which I propose to consider.

The obvious suggestion is to split a beam of light and send each half in opposite directions round a closed optical circuit in such a way that when they meet they shall form interference bands. Examine these bands in a micrometer, then magnetize the circuit in such a way that the lines of force run parallel to or coincident with the beam of light, and watch for a shift of the bands.

A circuit shaped like an equilateral triangle, with mirrors at the corners, would seem the best for this modification of Fizeau's classical experiment on the effect of moving water.

In 1892 & 1893 at Liverpool I performed carefully an experiment of this kind (see Phil. Mag. April 1907, p. 495), using a square not a triangle, (which had some advantages and some disadvantages: neither obvious). I tried filling the magnetic solenoids with water, bisulphide of carbon, and salts of iron, as well as with air: and sometimes I passed the light four times both ways round the square. The maximum drop of magnetic potential applied was nearly 2,000,000 c.g.s., and this did not affect the light by so much as 1/100 of a wave-length. I could have detected an æther-drift of 1 foot per second, or a thousand millionth of the velocity of light; but I did not press the experiment to further extremes such as would have decided against a *very slow* ætherial flow.

We must now examine reasonable laboratory data to see if by any possibility we could hope so to improve the arrangements as to make perceptible a drift of æther in a transparent magnetic field of practicable strength, even if ætherial density is very great.

Take then the simplest form of magnetic circuit, a closed solenoid or anchor-ring of sectional area A and mean periphery l, and let it be wound with n turns of wire; then its coefficient of self-induction is $L = 4\pi\mu n^2 A/l$ and the mass included in the ring is $M = Al\rho$.

A current circulating in the wire has the energy $\frac{1}{2}LI^2$; while the energy of a hypothetical ætherial flow round and round the ring, is $\frac{1}{2}Mv^2$.

Assuming that the two expressions are equal, or that one is the equivalent of the other, we get

$$\frac{v^2}{I^2} = \frac{L}{M} = \frac{4\pi\mu n^2}{\rho l^2},$$

or

$$v = n_1 I \sqrt{\left(\frac{4\pi\mu}{\rho}\right)}.$$

This is the drift to be observed.

If the current is expressed in electromagnetic units, μ will be 1; and if we *assume* the density ρ as 10^{12}, $\sqrt{(4\pi/\rho)}$ will be $3\frac{1}{2} \times 10^{-6}$; so in that case the velocity v will be very slow, and to make it perceptible we must contrive to make $n_1 I$ very large.

Magnetically the *size* of the solenoid does not matter, but we must arrange to crowd on it as many turns of wire to the centimetre as we can, and to send as big a current as possible through the wire without overheating it.

Let us see whether an extreme case can be thought of which shall come within the limits of experimental possibility. Take a closed magnetic circuit of any convenient sectional area and of length to be subsequently settled on optical considerations; wind it with wire so that if possible a thousand turns are piled up on every centimetre of its effective length; keep it so cold, say by liquid air, that it can carry 1000 amperes or 100 c.g.s. units of current; then $n_1 I = 10^5$ c.g.s. units. Wherefore v will be

$$\tfrac{1}{3} \text{ centimetre per second.}$$

That is to say in a magnetic field excited by a million ampere-turns per centimetre, the rate of æther flow along the lines of force will be only 3 millimetres per second or 7 inches a minute—about a snail's crawl. This is the speed which has to be compared with the velocity of light.

A considerable length of optical circuit becomes clearly necessary. Let each side of the supposed equilateral triangle constituting our magnetic circuit be a helix 10 metres long, and let the optical arrangements be so perfect that a shift of $\frac{1}{1000}$ of an interference band can be detected, then the ratio which we have available for comparing the velocities is the ratio of the total length of the circuit to one-thousandth of a wave-length; that is to say,

$$\frac{30 \text{ metres}}{6 \text{ tenth-metres}} \quad \text{or} \quad 5 \times 10^{10}.$$

The speeds to be compared are

$$\frac{3 \times 10^{10}}{\cdot 3} \quad \text{or} \quad 10^{11}.$$

Hence by taking advantage of the opposite transits of each half of the light beam, and taking advantage also of reversal of the magnetic field, the observation seems just barely feasible.

If the light were sent round a square circuit two or three times, as was done in my Æther experiment (Phil. Trans. vol. 184, A, 1893, page 757), it would seem possible to make a sort of measurement of the speed, and therefore of the ætherial density, even in the extreme case of its being as high as the estimate here suggested as a maximum. The density can hardly be greater: if it is less, the determination is proportionately easier. I expect it to be rather less, but not much: I think the order of magnitude is probably right. If there is no shift at all, and if the absence of drift can thus be certainly established, we shall learn that something is wrong with the hypothesis of ætherial flow along magnetic lines of force; for it is on that hypothesis that the whole suggested experiment has been founded.

If a distinct answer can be gained, the experiment is well worth while.

Perhaps it is not clear why I attach so much importance to a measurement of the ætherial constants and a determination of their dynamical nature. If a positive result could be secured it would be the first positive result which the æther, apart from matter, has yielded, since the fundamental fact of wave propagation and its definite velocity. This determined the product of the two constants, and was the first step in our knowledge of them. If however by some new phenomenon the two constants could be separately known, a second and even more important step would have been taken towards understanding the æther's structure and real nature. Until these constants are known, its relation to and interaction with ordinary matter must be largely guesswork. Radiation once excited obeys known laws, but of the emission and the absorption of radiation very little is really understood; and even the refraction or slowing of speed when passing through dense matter appears to be a subject of some difficulty, at least when anything more has to be apprehended than the bare fact and its elementary exposition.

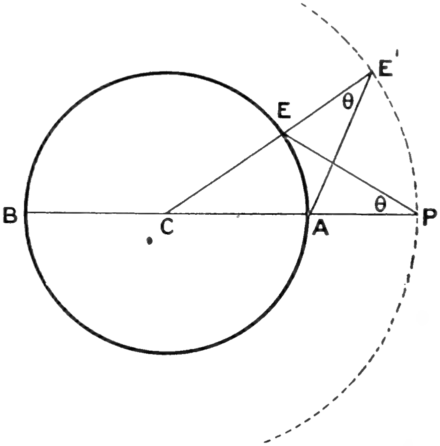
If the density were known, of course the elasticity would be known too, unless the dynamics of æther is not merely a variation on Newtonian dynamics but something utterly different. The only way to ascertain the truth on this subject is to try how far the æther can be treated as a substance amenable to ordinary laws. The principle of least action holds for light, and it seems possible that a developed turbulent or vortex sponge theory may account for the æther's elastic rigidity (*cf.* Appendix E and p. 124 of Larmor's 'Æther and Matter.' Also Phil. Mag. for April 1907, p. 503). It is essential however that we know the value of this rigidity. If it is kinetically explicable in the way originally suggested by Lord Kelvin (though afterwards abandoned by him) then the amount of energy locked up in the æther is something prodigious. Some day such a fact as this, when ascertained, may be found to have a bearing on really practical problems.

XLIII. *On a reciprocal relation between the Electrostatic Fields of certain Distributions of Electricity and the Magnetic Fields of corresponding Uniform Currents.* *By* Professor A. Gray, *F.R.S.*[*]

1. *A uniform circular linear distribution of electricity and a uniform circular current.*

THE heavy circle in fig. 1 represents the circular electrical distribution of line density ρ and radius a. P is a point external to the circle and its plane, at a distance a' from its centre, and A is the intersection of CP with the circle.

Fig. 1.



Consider an element of the circle at E of length *ds*. Project this element radially to E′ on the concentric circle (of radius a') described through P, and denote by ds' the length of the projection. Then $ds'/ds=a'/a$. Denote EP by r, ∠CPE by θ; we have then also $\text{AE}'=r$, and $\angle \text{CE}'\text{A}=\theta$.

The repulsion of the charge on ds exerted on a unit charge at P, that is the electric field intensity at P, is

$$\text{F}_e=\rho\frac{\cos\theta\, ds}{r^2}=\rho\frac{a}{a'}\frac{\cos\theta\, ds'}{r^2}\quad . \quad . \quad . \quad . \quad . \quad (1)$$

Now the second of these forms is, by the diagram, and

---

[*] Communicated by the Author.

according to the so-called law of Laplace, the *magnetic field intensity produced at* A by the element ds' of a circular conductor coincident with the concentric circle drawn through P, and carrying a current of strength $\gamma = \rho a/a'$. Thus the whole magnetic field intensity F_m at A, due to a current γ in the concentric circle through P, is given by

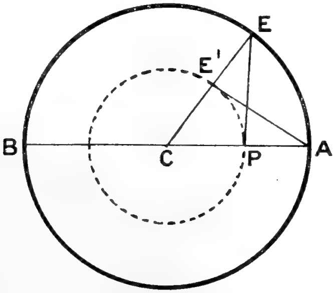
$$F_m = \gamma \int \frac{\cos\theta}{r^2}\,ds' = \rho \int \frac{\cos\theta}{r^2}\,ds = F_e, \quad . \quad . \quad (2)$$

where F_e is the electric (or gravitational) field intensity at P due to the circle of radius a and uniform line density ρ.

Of course F_e is directed radially outward, while F_m is normal to the plane of the circles.

The same result holds *mutatis mutandis* when the point P is within the given circle (fig. 2).

Fig. 2.



2. *The mutual inductance of two concentric circles is proportional to the electric (or gravitational) field intensity produced by a uniformly charged disk, the edge of which coincides with one circle, at a point on the circumference of the other.*

Imagine the charged disk divided into an infinite number of concentric narrow circular strips, and consider the outward repulsion of each of these on unit charge at P. These repulsions combine into a radially outward force at P equal to their sum. But each element of repulsion is equal to the product of the area of the strip producing it and the magnetic field intensity produced at any point of that strip by a fixed current flowing round the other circle. This conclusion seems very remarkable: I am not aware that it has been noticed before.

One direct mode of calculating the mutual inductance of two coplanar concentric circles, would therefore seem to be as follows. Imagine the circle AEB (fig. 1) replaced by a uniformly charged disk and calculate the repulsion of the disk on a unit charge placed at P. This will be most easily done by dividing the disk into narrow strips all at right angles to the diameter through A. Then the repulsion exerted on unit charge at P, by one of these strips at distance x from P, of breadth dx, and having D, F for its extremities, is equal to the repulsion of the incomplete circular strip of radius x, breadth dx, and intercepted between the lines PD and PF.

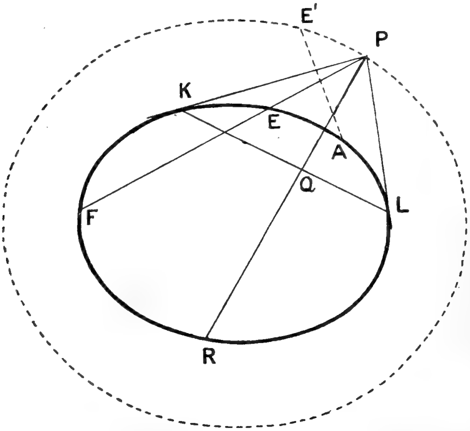
3. *A distribution of electricity on a plane conductor bounded by two close, similar, and similarly situated ellipses, and a uniform current in the confocal ellipse through the point considered.*

To generalize the theorem of § 1 consider the space between the ellipses of which the equations are

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=\kappa, \quad \frac{x^2}{a^2}+\frac{y^2}{b^2}=\kappa-d\kappa. \quad . \quad . \quad . \quad (3)$$

One of these is shown by the heavy curve in fig. 3. We take an element of elliptic arc at E of length ds, and suppose

Fig. 3.



that over the area of this length, which lies at E between the two curves, electricity is distributed with uniform surface density σ. If the length of the perpendicular from the centre on the tangent at E is p, the area is $\frac{1}{2}p\,ds\,d\kappa/\kappa$, and so the charge is $\frac{1}{2}\sigma p\,ds\,d\kappa/\kappa$.

Now through P let an ellipse confocal with the given ellipse LAK be described. A point A on the latter curve corresponds to P on the confocal, and a point E′ on the confocal corresponds to E on the given ellipse. Join E to P and A to E′. These lines have the same length, r. Let p' be the length of the perpendicular from the centre on the tangent at E′, and θ denote the angle between the line AE′ and that perpendicular. Let also p_0', θ_0 be the corresponding quantities for the point P and the line EP. [Care of course is to be taken that the lines are reckoned in the directions indicated by the letters, and that the perpendiculars are regarded as drawn both inward or both outward, so that there is no ambiguity as to the signs of the cosines.] In a former paper (Phil. Mag. April 1907) I have proved for two confocal ellipsoids the geometrical theorem (not, apparently, previously known)

$$p' \sec\theta = p_0' \sec\theta_0, \quad . \quad . \quad . \quad . \quad . \quad . \quad (4)$$

where p', θ and p_0', θ_0 refer to pairs of corresponding points on the ellipsoids, and have applied it to the complete and instantaneous evaluation of the integral for the force produced at an internal or external point by an elliptic homœoid, and hence to the solution of the problem of the attraction of a solid ellipsoid of uniform or of varying density.

The geometrical theorem as stated above asserts that the product $p' \sec\theta$ is invariant over the confocal ellipsoid: exactly the same theorem holds of course for the given ellipsoid. Moreover, since an ellipsoid is one of its own confocals, the theorem holds also for any two points P, Q on a given ellipsoid, the perpendiculars from the centre on the tangent planes at P, Q, and the chord joining these points. The theorem holds also for *any* confocal surfaces of the second degree.

In the present case let us take on the confocal ellipse through P, the element of arc ds', the points on which *correspond* to the points on ds; then since the equation of the confocal is

$$\frac{x^2}{a^2+u} + \frac{y^2}{b^2+u} = \kappa, \quad . \quad . \quad . \quad . \quad . \quad . \quad (5)$$

we have, as can easily be proved,

$$p\,ds = \frac{ab}{\{(a^2+u)(b^2+u)\}^{\frac{1}{2}}} p'\,ds'. \quad . \quad . \quad . \quad . \quad (6)$$

The electric field intensity at P, resolved along the normal to the confocal, is

$$F_e = \tfrac{1}{2}\sigma\frac{d\kappa}{\kappa}\int p\cos\theta_0\frac{ds}{r^2}, \quad . \quad . \quad . \quad . \quad . \quad . \quad (7)$$

where the integration is taken round the given elliptic strip. By the previous equation this can be written

$$F_e = \tfrac{1}{2}\sigma\frac{d\kappa}{\kappa}\frac{ab}{\{(a^2+u)(b^2+u)\}^{\frac{1}{2}}}\int p'\cos\theta_0\frac{ds'}{r^2}, \quad . \quad (8)$$

where the integration is taken round the confocal ellipse. But by the geometrical theorem stated above, the value of the component field intensity thus found becomes

$$F_e = \tfrac{1}{2}\sigma\frac{d\kappa}{\kappa}\frac{abp_0'}{\{(a^2+u)(b^2+u)\}^{\frac{1}{2}}}\int\cos\theta\frac{ds'}{r^2} \quad . \quad . \quad . \quad (9)$$

or, if we write

$$\gamma = \tfrac{1}{2}\sigma\frac{d\kappa}{\kappa}\frac{abp_0'}{\{(a^2+u)(b^2+u)\}^{\frac{1}{2}}}, \quad . \quad . \quad . \quad . \quad (10)$$

$$F_e = \int\gamma\cos\theta\frac{ds'}{r^2} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (11)$$

This is evidently the magnetic field intensity F_m produced at A by a current of strength γ flowing round the confocal ellipse. It is of course in the direction at right angles to F_e, that is perpendicular to the plane of the ellipse.

This relation between the normal component of the electric field intensity at P of the charged elliptic strip and the magnetic field intensity at the corresponding point A due to a current in the confocal ellipse, is curious and appears to be new. There is not, so far as I can see, any direct practical application of the theorem which can be made with advantage.

I may here recall that in the paper of April 1907, referred to above, the expression

$$F = \tfrac{1}{2}\rho\frac{d\kappa}{\kappa}\frac{abc}{\{(a^2+u)(b^2+u)(c^2+u)\}^{\frac{1}{2}}}p_0\int\frac{\cos\theta'}{r^2}dS' \quad (12)$$

[where p_0 is the perpendicular let fall from the centre on the tangent plane to the confocal at the point P, and θ' is the angle between the perpendicular from the centre to the tangent plane at the element dS', at E′, of the confocal surface, and the line AE′ (see fig. 3), and the integral is taken over the confocal] was found, by the geometrical theorem (4) quoted above, for the attraction of a homœoid at the external point P. It was remarked that this value of F at P is, to a constant factor, equal to the potential produced at any point internal to itself by a uniform magnetic shell coinciding with the confocal surface. The strength of this shell, though constant over the surface, is proportional to the length of the perpendicular from the centre on the tangent plane to the confocal at P, and therefore varies with the position of P on the surface.

The value of the integral $\int \cos\theta' dS'/r^2$ is of course 4π and so F the force at P due to the given elliptic homœoid, or its equivalent the magnetic potential within the confocal is obtained at once *.

4. *Evaluation of the field intensities for the case of* § 1.

The values of the electric and magnetic field intensities specified in § 1 in terms of elliptic integrals are of course well known, (see for example a paper by Dr. Alexander Russell, Phil. Mag. April 1870). But a process of integration, differing somewhat from the usual one, is of some interest and seems possibly capable of some extensions, which may be given in a subsequent note.

The direct process of evaluation of the electric field intensity F_e from

$$F_e = 2\rho\frac{a}{a'}\int_0^\pi \frac{\cos\theta\, ds'}{r^2} \quad . \quad . \quad . \quad . \quad . \quad . \quad (13)$$

in elliptic integrals is as follows. By fig. 1 we see at once that $\cos\theta = (a' - a\cos\phi)/r$ so that we get

$$F_e = 2\rho\frac{a}{a'}\int_0^\pi \frac{a' - a\cos\phi}{r^3} ds'. \quad . \quad . \quad . \quad (14)$$

This leads to

$$F_e = 2\rho\frac{a}{a'}\left\{\frac{1}{a'-a}E(k) + \frac{1}{a'+a}K(k)\right\}, \quad . \quad . \quad (15)$$

if $k = 2\sqrt{aa'}/(a + a')$, and we use the relation

$$\int_0^{\frac{\pi}{2}} \frac{d\psi}{(1-k^2\sin^2\psi)^{\frac{3}{2}}} = \frac{E}{1-k^2}, \quad . \quad . \quad . \quad (16)$$

which can be verified by direct integration. This is an elliptic integral of the third kind with parameter $-k^2$.

---

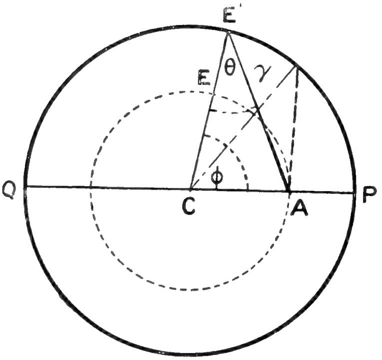
* In the particular case in which the confocals are concentric spherical surfaces, the attraction of a shell coincident with the inner sphere reduces at once to $4\pi\sigma a^2/a'^2$, since $\int \cos\theta\, dS'/r^2 = 4\pi$.

Similarly for the attraction at an internal point, the point A is external to the sphere radius a', and the solid angle is zero. Hence the internal attraction is zero.

But we may also, since by fig. 4 $r(d\phi+d\theta)/ds'=\cos\theta$, write

$$\int_0^\pi \frac{\cos\theta\,ds'}{r^2} = \int_0^\pi \frac{d\phi}{r} + \int \frac{d\theta}{r} \quad . \quad . \quad . \quad . \quad (17)$$

Fig. 4.



The first integral is obtained at once. It comes out

$$\int_0^\pi \frac{d\phi}{r} = \frac{2}{a+a'}\mathrm{K}(k), \quad . \quad . \quad . \quad . \quad . \quad (18)$$

and it is to be observed that multiplied by $2\rho' a'$ it is the potential at A due to a uniform distribution of electricity (or gravitating matter) of line density ρ' on the outer circle.

For the second integral $\int d\theta/r$ we take

$$a^2 = a'^2 + r^2 - 2a'r\cos\theta,$$

that is

$$r = a'\cos\theta \pm (a^2 - a'^2\sin^2\theta)^{\frac{1}{2}}. \quad . \quad . \quad . \quad . \quad (19)$$

Thus the largest value of θ is $\sin^{-1}(a/a')$, and is shown by the dotted lines in fig. 4, where a right angle at A is intended to be indicated. As successive positions of E and E′ are taken on the semicircles, AEB and PE′Q, θ varies from zero to $\sin^{-1}(a/a')$ and then falls off to zero.

For the first part of this range of variation we have to take the lower sign in the last equation, and for the second part the upper sign. Thus we get for the first part, putting

θ_1 for $\sin^{-1}(a/a')$,

$$\int \frac{d\theta}{r} = \int_0^{\theta_1} \frac{d\theta}{a' \cos\theta - (a^2 - a'^2 \sin^2\theta)^{\frac{1}{2}}}$$

$$= \frac{1}{a'^2 - a^2} \int_0^{\theta_1} d\theta \{a' \cos\theta + (a^2 - a'^2 \sin^2\theta)^{\frac{1}{2}}\},$$

or again, since $a' \sin\theta_1 = a$,

$$\int \frac{d\theta}{r} = \frac{1}{a'^2 - a^2} \left\{ -a + a' \frac{a^2}{a'^2} \int_0^{\frac{1}{2}\pi} \frac{\cos^2\chi \, d\chi}{\left(1 - \frac{a^2}{a'^2} \sin^2\chi\right)^{\frac{1}{2}}} \right\}, \quad (20)$$

where $\chi = \sin^{-1}\{(a'/a)\sin\theta\}$.

The second part of the integral $\int d\theta/r$ for the semicircle is found similarly to be

$$\int \frac{d\theta}{r} = \frac{1}{a'^2 - a^2} \left\{ a + a' \frac{a^2}{a'^2} \int_0^{\frac{1}{2}\pi} \frac{\cos^2\chi \, d\chi}{\left(1 - \frac{a^2}{a'^2} \sin^2\chi\right)^{\frac{1}{2}}} \right\}. \quad (21)$$

Thus the total value of $\int d\theta/r$ for the circle is

$$\frac{4a'k_1^2}{a'^2 - a^2} \int_0^{\frac{1}{2}\pi} \frac{\cos^2\chi \, d\chi}{(1 - k_1^2 \sin^2\chi)^{\frac{1}{2}}}$$

$$= \frac{4a'}{a'^2 - a^2} \{k_1^2 \mathrm{K}(k_1) + \mathrm{E}(k_1) - \mathrm{K}(k_1)\}, \quad (22)$$

where (with $a' > a$) $k_1^2 = a^2/a'^2$. Also

$$k = 2\sqrt{aa'}/(a + a') = 2\sqrt{k_1}/(1 + k_1).$$

Now k and k_1 are the moduli connected by Landen's transformation, and so the magnetic field intensity at A due to a current γ in the circle of radius a' is, as stated in § 1 above,

$$\mathrm{F}_m = \frac{4a'\gamma}{a'^2 - a^2} \mathrm{E}(k_1) = 2\gamma \left\{ \frac{\mathrm{E}(k)}{a' - a} + \frac{\mathrm{K}(k)}{a' + a} \right\}. \quad . \quad (23)$$

A comparison of the reduced form of (14), from which (15) was obtained by assuming (16), with (23) establishes (16), and so evaluates this simple form of the elliptic integral of the third kind.

5. *Field intensities not in the plane of the circular distributions.*

Returning to fig. 1 we may regard the points E′ and P as in a plane parallel to that of the circle AEB, and distant b from it, so that the two circles are now coaxial. The two distances EP and AE′, and the two angles (CPE and CE′A) marked θ are equal. This angle is not, however, any longer the complement of the angle between AE′ and the tangent to the circle at E′, drawn back towards P. We may calculate the components of the electric field intensity at P, denoting by X the component parallel to CA, and by Z the component parallel to the axis of the system.

If as before a, a' be the radii of the circles, and r be EP, so that now $r^2=a^2+a'^2+b^2-2aa'\cos\phi$, the components due to $a\,d\phi$ at E are, as we see at once,

$$dX_e=\rho\frac{a\,d\phi}{r^2}\frac{a'-a\cos\phi}{r}, \quad dZ_e=\rho b\frac{a\,d\phi}{r^3}.$$

Thus integrating round the circle AEB we get

$$X_e=2\rho a\int_0^\pi\frac{a'-a\cos\phi}{r^3}d\phi, \quad Z_e=2\rho ba\int_0^\pi\frac{d\phi}{r^3}. \qquad (24)$$

To pass from these to the components of magnetic field intensity at A due to a current of strength $\rho a/a'$ in the circle PE′, we have only to interchange X and Z, and take note of the directions of the components. As a little consideration will show, we have to replace X_e, which is parallel to CP, by a component along PC, and then suppose that turned normally out from the paper to give Z_m. It will be seen that the "vertical" component Z_e gives a component X_m parallel to EC, and that we have

$$X_m=-2\gamma a'b\int_0^\pi\frac{\cos\phi\,d\phi}{r^3}, \quad Z_m=2\gamma a'\int\frac{(a'-a\cos\phi)\,d\phi}{r^3}. \qquad (25)$$

These components can be at once expressed in elliptic integrals. The component Z_m is the more important as it enables the mutual inductance of the two circles to be found by integration over the circle AEB. The mutual inductance between the coaxial circles could be found by calculating the total X_m at P for each of a series of narrow concentric rings into which the circle of radius a, say, is divided, multiplying each by the area of the ring to which it belongs and calculating the sum of products thus obtained.

These reciprocal theorems of coaxial circles have not so far as I know been stated explicitly before; but Sir George Greenhill has pointed out to me that in a paper in the 'American Journal of Mathematics,' vol. xxxix. p. 439 (1917), he has given certain general reciprocal relations from which they may be deduced.

XLIV. *Notes on the "Break" of a Magneto or Induction-Coil. By* NORMAN CAMPBELL, *Sc.D.*[*]

NOTE.—The work described in these notes was carried out at the National Physical Laboratory under the direction of the Advisory Committee for Aeronautics. The results have been communicated in a confidential report to the Internal Combustion Engine Sub-Committee of that Committee, who have now given their consent to the publication of any portions which appear of pure scientific interest.

(1) *Introduction.*

IT is well known that if the greatest possible efficiency is to be obtained from a magneto or induction-coil, that is, the greatest possible ratio of the maximum secondary potential to the primary current broken, it is necessary to avoid sparking between the terminals at which the primary current is broken. No elaborate theory is necessary to explain the loss of efficiency due to sparking. In the first place the spark involves the dissipation of some of the electromagnetic energy originally present in the primary current which would otherwise have been available for conversion into electrostatic energy of the secondary. In the second place the passage of the spark, even if it involved no loss of energy, would prolong the time which elapses between the first decrease of the primary current, as the contact opens, and its total cessation; very general considerations will show that, if this time is so prolonged as to become an appreciable fraction of the period of the oscillations excited, a loss of efficiency will usually follow.

It is always possible to suppress the spark at the primary break by inserting a condenser of sufficient capacity in parallel with the separating terminals. Until an adequate theory of the induction-coil was developed, chiefly by the work of Prof. Taylor Jones and his collaborators[†], it seems to have been believed that the suppression of the spark was the only useful function of the primary condenser; and that if the suppression could be achieved by any other means, such as an increase in the speed of separation of the terminals, the addition of capacity to the primary circuit would be unnecessary. This view was, of course, largely based on the classical experiments of Lord Rayleigh[‡], who showed

---

[*] Communicated by the Author.

[†] Phil. Mag. Jan. 1909, p. 28; Nov. 1911, p. 706; April 1914, p. 580; Jan. 1915, p. 1; Aug. 1915, p. 224; April 1917, p. 322.

[‡] Phil. Mag. ii. p. 581 (1901).

that, in the absence of a primary condenser, the maximum secondary potential could be very considerably increased by increasing sufficiently the speed of the break. This conclusion was confirmed by Prof. Taylor Jones; but the further conclusion that, if the speed of the break were increased sufficiently, as great a secondary potential could be obtained without a condenser as with one, was not confirmed. Theory and experiment alike show that, even if the break is perfect and there is no sparking, there is an optimum value for the primary capacity, depending on the other constants of the inductive circuits; and that the use of a capacity either greater or less than this optimum value involves some sacrifice of efficiency.

Accordingly, in order that the greatest possible efficiency should be obtained, it is necessary that this optimum primary capacity should not be less than that which is necessary to prevent sparking; for otherwise, if a capacity is used large enough to prevent sparking, there will be a loss of efficiency, small but important, due to the use of a primary capacity greater than the optimum. In all magnetos which have been examined, and probably in all induction-coils with all forms of break, this necessary condition is not fulfilled; the optimum capacity is less than that required to prevent sparking. Most magneto manufacturers are aware that the secondary potential developed increases steadily as the primary capacity is diminished up to the point where sparking begins; the capacity of the primary condenser which it is necessary to use is determined wholly by the necessity for avoiding sparking.

It becomes of importance, therefore, to inquire whether the capacity necessary to avoid sparking might not be diminished by changing some other characteristic of the break. The only characteristic which seems to have been considered in this connexion is the speed of separation of the terminals. According to the view which seems generally accepted, the spark between the separating terminals occurs because the rate at which the potential difference between those terminals increases as a result of the interruption of the current is greater than the rate at which the spark potential of the gap between the terminals increases with the distance between them [*]. The primary condenser stops

---

[*] This theory is formally stated and supported by some experiments in a paper by H. Wiesinger, *Ann. d. Phys.* lv. 6, p. 401 (1918); but it must have occurred to everyone who has paid any attention to the matter. It will be seen later that the conditions of the experiments by which Wiesinger supported his theory were not those obtaining in the primary circuit of a magneto or induction-coil.

sparking because it decreases the rate of rise of the potential difference between the terminals. It is to be expected that sparking would also be stopped by an increase in the rate of rise of the spark potential, which would accompany an increase in the rapidity with which the terminals separate.

However, Lord Rayleigh's experiments seemed to indicate that any practical application of the second alternative is out of the question, because the necessary rate of separation is so great. He found that no advantage was gained by increasing the speed of separation up to that of a pistol bullet, and that it was only when a rifle bullet was employed to cut the primary circuit that any increase in the secondary potential was obtained. The critical velocity would doubtless depend on the constants of the circuits and on the primary current broken (for these determine the rate of rise of potential), but in the case investigated it was so much greater than anything which could be obtained with a mechanical break that research in this direction has not seemed promising.

It may be noted that, if we knew the rate of rise of potential and the relation between the spark potential and the distance between the contacts, the determination of the critical velocity would permit an experimental test of the theory. The rate of rise of potential can be determined with any arrangement for measuring the wave-form, but the relation between the spark potential and the distance between the contacts is less easy to determine satisfactorily for the very short distances which are here involved. It is known that, as the distance is reduced, the spark potential diminishes to the "minimum spark potential" of about 350 volts at a distance somewhat less than 0·1 mm., that it subsequently increases and then falls again very rapidly when the distance becomes about 0·001 mm. In the interval between ·001 mm. and 0·1 mm. the relation is so complicated that it is difficult to determine with the necessary accuracy.

(2) *Experiments.*—Some observations which have been made in the course of the work described in a recent paper * seem to throw some new light on the question; and although they are not sufficient to answer all the questions that can be raised, they seem to point to conclusions of sufficient practical importance to justify their record.

Observations that seem to indicate that the theory in its present form is not adequate to explain all the facts can be

---

* Phil. Mag. vol. xxxvii. ser. 6, p. 284, March 1919.

made very readily on any magneto. If we remove the magnets and bring the primary terminals outside the machine, we can supply from a battery a primary current which is independent of the speed [*]. With any given primary condenser and any given speed, it will be found that, if the current is below a certain limit, no sparking occurs, but that if it is increased above this limit, sparking appears. The limiting value can be fixed within comparatively narrow limits; in one experiment, for instance, it was found to lie certainly between 2·2 and 2·5 amp. Suppose now that we supply a current double the limiting value; then we shall very nearly (but not quite) double the rate of rise of potential across the terminals. Suppose now we double the speed of the magneto shaft and so double the speed of separation of the terminals; then at any moment during the rise of potential the relation between the spark potential and the potential difference between the terminals should be exactly the same as it was at some corresponding moment with the original values of the current and the speed. This conclusion involves only the assumption that the rate of rise of potential is proportional to the current broken; it does not involve any assumption concerning the relation between the spark potential and the distance. Accordingly the current which is now broken in the primary should again be the limiting current; in other words, by doubling the speed of break we should expect very nearly to double the limiting current. But as a matter of fact we find that the limiting current is not doubled or nearly doubled by doubling the speed; no certain evidence could be produced that increasing the speed of the break 20-fold produces a change in the limiting current which is greater than the error (10 or 20 per cent.) in the determination of that current. This conclusion appears absolutely inconsistent with the theory which has been given of the circumstances which determine the occurrence of sparking.

(3) *Effect of the electrodes.*—The clue to this apparent failure of a theory, so obvious that it has not generally been

---

[*] The same experiment can be tried without disconnecting the primary circuit by making use of the fact that the primary current in a magneto is almost independent of the speed over a considerable range; it is usually possible within this range to change the speed 100 per cent. without changing the primary current more than 10 per cent.; and if we start on one side of the maximum and decrease or increase the speed through the maximum to the other side, we can vary the speed more than 100 per cent. and regain the original value of the primary current. The value of the maximum current can be readily changed by changing the strength of the magnets.

thought worth while to state it, is readily found if the influence of the nature of the contacts at which the primary current is broken is examined. According to the theory we have been examining, the nature of the contacts should be almost without influence; for the spark potential is almost independent of the form and material of the terminals when the distance separating them is very small. But the crudest experiments will indicate that the nature of the contacts is a most important feature in determining the limiting current. In one matter indeed the influence is familiar to all who have used a magneto or induction-coil; we have only to roughen the surface of the contacts and sparking immediately occurs at the break which was sparkless before; the limiting current may be reduced 50 per cent. by scratching the surface and restored to its original value by polishing.

More systematic experiments indicated that, if the constants of the circuits remained unchanged, the limiting current which could be broken without sparking always increased as the resistance of the contacts decreased. Some observations on the resistance of the contacts are described in a note at the end of this paper, but in general it may be said that the resistance of a contact can be decreased either by increasing the area of the surfaces in contact or the pressure between them. Thus in a certain circuit the contact fitted in a magneto would break without sparking a current of 4 amp.; a contact consisting of a ball $\frac{1}{4}$ in. in diameter pressed with a force of 50 grams weight against a steel plate would break only 2·2 amp.; if force was reduced to 1 gm. weight, the limiting current was barely $\frac{1}{2}$ amp. In no case could any relation be found between the limiting current and the speed of break, which in these observations could be varied from 300 to 62 cm./sec. The influence of the material of which the contact was composed was not investigated.

(4) *Formation of an arc.*—The great variation of the limiting current with the resistance of the contact gives at once the clue to discrepancies which have been noted. The "spark" which is seen to occur at the broken contact is not a true spark, but an arc[*]. The discharge only occurs if the passage of the current between the contacts before the

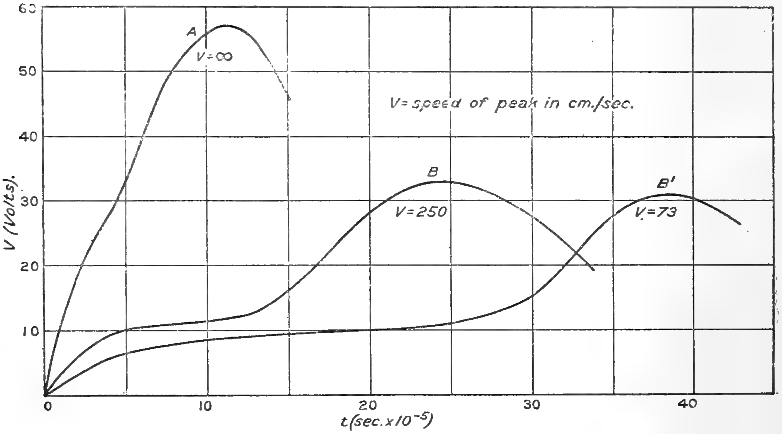
---

[*] This conclusion might almost have been deduced from the mere appearance of the "spark." So far as we have observed, a luminosity confined to a narrow path between the contacts more or less straight, is never seen; what appears when a primary circuit is broken is a "flash," such as occurs when steel strikes flint. It may be mentioned that several writers have called the spark at break an "arc," but they do not seem to have realized the implications of that term.

break has caused such intense local heating that portions of the terminals have been raised to the temperature at which notable thermionic emission begins, so that a discharge between them can be maintained by a potential difference quite inadequate to start a spark.

With the object of testing this explanation measurements were made on the rise of potential across the break as the terminals separated. The circuit broken was the primary of a magneto with a condenser of 1 mf. across the break; one terminal consisted of a $\frac{1}{4}$ in. steel ball attracted with a force of about 50 grams weight to the end of the magnet which formed the other terminal. The limiting current was 1·2 amp.; it was found that, so long as the current was less than this, the potential at any given moment after break was very nearly proportional to the current broken. Accordingly the rise of potential was first determined with a current less than the limiting value (1·0 amp.); if we multiply all the measured values of the potential by 1·5 we get a curve A in fig. 1, which represents the rise of potential which would follow when a current of 1·5 amp. is broken, if there were no arc formed. Curves B and B′ give the measured rise of potential when a current of 1·5 amp. is broken, and the speed of the separation of the terminals is 250 cm./sec. and 73 cm./sec. in B and B′ respectively. It will be seen that from the very first instant the rate of rise is much less rapid in B and B′ than in A; that a stage

Fig. 1.



is soon reached in which the potential is nearly constant for a considerable time, longer with the slower separation ; and that subsequently the potential rises rapidly and attains a maximum, nearly independent of the speed of separation, at a time after the beginning of the rise which is again nearly independent of the speed of separation and also nearly equal to the time that curve A takes to rise from zero to its maximum.

The form of the curves is exactly what we should expect if an arc is formed, and utterly unlike that which is characteristic of the passage of a spark (see p. 492). The initial rate of rise of potential is small because the current is not broken when the contacts separate but is maintained by the arc between the hot contacts. The constant potential which persists for some time is very nearly that of the battery supplying the primary current (10 volts). At the point where the second rise of potential begins the arc ceases, and the rest of the curve represents approximately that which would have been obtained if a current, equal to that flowing in the arc, had been broken at the first separation of the contacts.

Many more experiments would be necessary to establish the complete accuracy of all the statements that have been made and to work out in detail the quantitative relations which are suggested. But the observations which have been described are sufficient to show that the discharge which occurs when the "flash" is seen at the broken contact is of the nature of an arc rather than a spark, and that it is conditioned by the heating of the electrodes.

And it is now apparent why the "flash" may be prevented by decreasing the rate of rise of potential, but not by a corresponding increase in the rate of separation of the terminals. The potential of the battery (or, in the case of the magneto, that due to the rotation of the armature in the magnetic field) is not sufficient to maintain the arc permanently. If this potential is only 10 volts, then, however slowly the contact is opened, the current ceases a very small fraction of a second after the distance between the contacts has reached a value which is certainly less than 0·01 mm. The energy dissipated in the discharge is not sufficient to maintain the electrodes at the high temperature necessary for the arc-discharge. On the other hand, the passage of the discharge and the dissipation of energy which accompanies it delay the cooling of the electrodes, and they delay it the longer the greater is the potential difference maintaining the discharge. Accordingly the more slowly the potential rises after the separation of the contacts, the shorter will be the time which elapses before the arc is extinguished by the cooling of the terminals; indeed, if it rises sufficiently slowly the terminals will have cooled before the potential has reached a value which is sufficient to give a "flash," even if the electrodes were hot. In that case the break will be sparkless. On the other hand, it is known that the potential necessary to maintain an arc does not increase rapidly with the distance between the electrodes; and we should expect, therefore, as we find, that the maintenance of an arc for an appreciable time, which is probably necessary for the appearance of the "flash," would depend very greatly on the rate of rise of potential and very little on the rate of separation of the terminals.

A further examination of the curves produces evidence in support of the conclusion that the final cessation of the arc depends on the time which elapses since the break as well as on the distance to which the electrodes have separated. For though the time the arc lasts is longer for the slower separation, the distance of the electrodes when it ceases is greater for the quicker separation (0·30 mm. as against 0·19). Since the potential when the arc ceases is almost the same in the two cases, this difference can only mean that the time as well as the distance is important. The maintenance of the arc retards the cooling of the electrodes, but since the potential difference is not sufficient to keep the electrodes permanently hot, the instant when the arc ceases is determined in part by the losses due to radiation and conduction.

Again, we can obtain a minimum estimate of the speed of break which would be necessary to obtain a sparkless break simply by increasing the speed of separation. In order that this result should be attained, the separation must be so rapid that the distance between the electrodes becomes so great that, even if the electrodes are still hot, a visible discharge cannot be produced with the potential available. Now curve B shows that the arc can be maintained with a potential of 10 volts with the electrodes 0·30 mm. apart, if they are sufficiently hot; accordingly mere speed will not prevent sparking unless the electrodes have separated 0·30 mm. before curve A has reached 10 volts (or less). The minimum estimate of the speed which will prevent sparking is, therefore, 30,000 cm./sec.—a value very difficult to obtain in any mechanical break, though much less than the speed of Lord Rayleigh's pistol bullet. However, it must be remembered that the estimate is on all grounds merely a minimum estimate, and there is nothing to render it likely that the true limiting speed is not really much greater.

On the other hand, it should be noted that a sufficient increase in the speed of break may prevent arcing in another way; the heating of the electrodes may not begin until the separation begins, and a reduction of the time of separation may reduce the time available for heating them to the requisite temperature. A reduction of arcing due to this cause is most likely to occur when, as in Lord Rayleigh's experiments, the circuit is broken by cutting a solid conductor; but it will also occur with a contact such as that employed in a magneto, where the terminals are held together by considerable pressure.

It is probably for this reason that it has been found that if, by the use of special cams, the contacts of a magneto are opened very slowly, flashing can be made to occur when it will not occur with the normal rate of opening. In the contacts used in these experiments, the pressure between the contacts before opening was much less and the resistance of the contact much greater; heating began before the contacts separated. In such circumstances the limiting current appeared quite independent of the speed of separation within the range which could be observed (20 to 250 cm./sec.).

Again, it is possible that the increase in the secondary potential which Lord Rayleigh found may be merely due to the fact that the time that the arc lasted was greatly reduced; for if the interval between the first opening of the contacts and the total cessation of the current could be made very much less than the period of the oscillations excited, the loss of efficiency due to "sparking" would be greatly reduced. But this explanation is improbable. For, if it were correct, there would be a gradual increase in the efficiency as the speed of break was increased, whereas Lord Rayleigh found no increase whatever until the speed of a rifle bullet was reached. However, in the experiments described here there was some evidence that an increase in the speed of break, when there is sparking, produces some slight increase in the secondary potential developed; but when there is sparking the secondary potential varies so greatly at successive breaks that accurate measurements are difficult. In all the cases examined the secondary potential when there was sparking was at least 30 per cent. less than that obtained with a slightly smaller current which produced no sparking.

(5) *The Perfect Break.*—The conclusions to which these

observations are directed is that the speed of the separation of the terminals is a relatively unimportant matter in the prevention of sparking. The speed doubtless exerts some influence, and a reduction of the speed much below that usually employed would probably be harmful; on the other hand, an increase, if sufficiently large, would doubtless be beneficial. But there is a wide range of speeds, including those usually employed, over which the speed has practically no effect. If the use of a smaller primary capacity is to be made possible, the improvement must be sought by reducing the heating at the contact and preventing the development locally of high temperatures which permit the formation of an arc.

The exact conditions which tend to prevent arcing have not been studied in detail; it would be interesting, for example, to inquire whether the use of material of high thermal conductivity for the electrodes would be beneficial. But in the magneto the characteristics of the break are fixed rather by mechanical than electrical considerations, and it is likely that, even if alterations which would prevent arcing could be suggested, they would be found impracticable. However, it is important to realize that, if arcing can be prevented, the break, which will be sparkless even when there is no primary condenser, is attainable. It is often thought that, unless enormous velocities of separation of the contacts are available, it is quite impossible to break so highly inductive a circuit as the primary of a magneto or induction-coil without the occurrence of a spark at the break in the absence of a condenser of considerable capacity connected across the terminals. Such an impression is quite erroneous. It is quite easy to construct a break which will carry currents up to 1 amp. without any considerable heating of the electrodes at all; such a break will be sparkless if used for such currents in the primary of any magneto or induction-coil of ordinary construction, even if there is no primary condenser and the speed of separation of the contacts is as low as 20 cm. per sec.

But there is one point which needs further attention. Is the absence of a visible "flash" at the break evidence that the break is really perfect, and that the interruption of the current occurs instantaneously and without dissipation of energy?

A perfectly definite answer could be given to this question only if it could be shown that the oscillations excited by the break of the primary current agreed with those calculated on the assumption that the break was perfect. Unfortunately, however, there are circumstances (connected with the damping of the oscillations) which make an accurate calculation of the oscillations impossible, even on the assumption that the break is perfect. But there seems to be considerable indirect evidence that if the break is "sparkless" it is at least very nearly perfect. For it is found that, so long as no visible flash occurs, the potential developed in the secondary is very nearly proportional to the current broken in the primary; but as soon as the limiting current is overstepped and the flash occurs, the potential developed fails at least 30 per cent. The transition from one value of the secondary potential to the other is perfectly sharp; intermediate values are never observed and the lower potential only occurs when a flash is seen. Moreover, so long as no flashing occurs the peak potential obtained when a given current in the primary is broken is perfectly definite and consistent at successive trials; if there is flashing, the values obtained at successive trials are much less regular.

Such observations seem to show that the occurrence of a flash indicates the occurrence of some action which does not occur at all if there is no flash, and that it does not merely indicate an increase of some process which occurs even when a flash is not seen.

(6) *True sparking at the break.*—The observations seem also to show that if we suppress arcing we are not troubled in normal circumstances with the occurrence of a true spark, as suggested by the theory which we considered originally. But in order to make this conclusion more certain, it was thought of interest to inquire whether, in any circumstances, true sparking at the break could be observed.

Such circumstances could be obtained if, with a current so small that there is no possibility of arcing, the rate of increase of potential can be made greater than the rate of rise of the sparking potential of the gap; accordingly either the rate of rise of potential when a given primary current is broken must be greatly increased, or the rate of rise of sparking potential greatly reduced. The first is the more convenient alternative to adopt; the necessary conditions are readily secured by increasing greatly the self-inductance of the circuit in which the current is broken. Observations were made, therefore, when the contact to be broken was inserted in the secondary, instead of in the primary, circuit of he magneto.

It was now found that a visible spark occurred at the break when the current broken was about 0·02 amp., the speed of the break being about 250 cm./sec. The "spark" now looked like a spark, and did not take the form of the "flash" which appeared when the current was broken in the primary. Further, it could now be definitely established that the limiting current, which could be broken without the appearance of a spark, increased notably with the speed of the break, and, as far as could be ascertained, it was independent of the form of the terminals and of the pressure between them. All these differences are precisely those which are to be expected if the discharge is now a true spark due to the potential across the gap exceeding the normal spark potential. However, one discrepancy must be noted. Though the limiting current increased with the speed of separation, it did not increase as rapidly as was to be expected. When the speed of separation was doubled, the limiting current increased only some 50 per cent.; the reason for this discrepancy has not been determined.

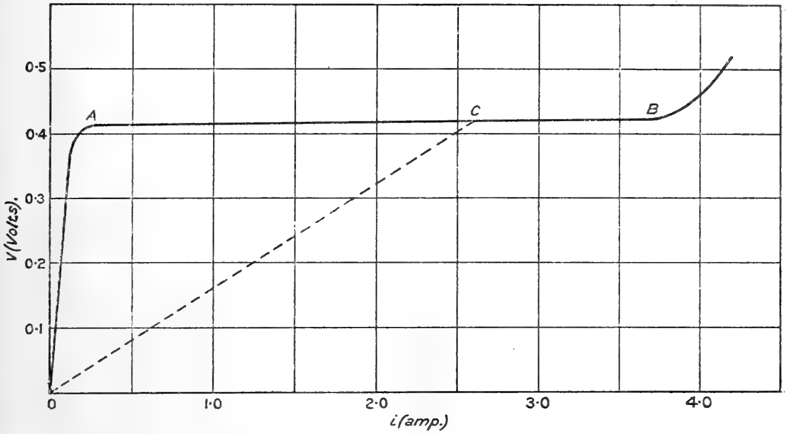
Observations similar to those shown in fig. 1 were made of the rise of the potential across the gap, both when the spark did occur and when it did not. It was found that, when the spark occurred, the initial part of the curve (corresponding to a time of the order of 10^{-5} sec. and a separation of the contacts of the order of ·03 mm.) was precisely the same as when the spark did not occur; but that, at some point on the curve, the readings became wholly irregular and no more consistent measurements could be obtained. The point at which this irregularity sets in is, of course, that at which the spark occurs.

The observations need not be described further. Those already mentioned are sufficient to prove the only fact which is of much importance in this connexion, that the observations when the spark occurs are wholly different from those in the conditions investigated in the earlier experiments and, accordingly, that in those conditions the discharge which is seen to occur is not a true spark. It may be asserted confidently that the conditions necessary for the passage of the true spark are never likely to occur when the primary of a magneto is broken, and that if arcing can be avoided, the break will be perfect.

(7) *Note on the Resistance of Contacts.*—In the course of these experiments some observations have been made which are of some interest and do not seem to have been described previously.

Suppose that we pass through a contact between two metal terminals a gradually increasing current i and measure the potential difference V between the terminals. Then we shall obtain a relation of the form shown by the full line in fig. 2. In the first stage, V increases with i,

Fig. 2.



it then becomes independent of i over a considerable range, and later begins to increase once more. The values of i_1 and i_2, corresponding to the points A and B on the curve, and the constant value of V (V_0) between A and B, vary greatly with the nature of the terminals and the pressure between them ; but the form of the curve is the same in all cases investigated.

But now suppose that, having increased i until the point C, lying between A and B, is reached, we begin to reduce i again, taking great care not to break or shake violently the contact ; then the values of V obtained will not lie on the full line but on the straight dotted line passing through the origin. That is to say, the contact has now a definite ohmic resistance, V_0/i_c, determined by V_0 and i_c the maximum value of the current which has been passed through the contact. If the contact is broken momentarily the relation between V and i becomes once more that given by the full line.

No explanation of these observations is offered. It has been recorded before that the resistance of a contact depends on the current which has been passed through it, and also on the time for which the current has passed*, but a definite relation does not seem to have been established. It should be mentioned that as the pressure between the terminals is increased, V_0 in general decreases and i_2 increases ; it is difficult to determine how i_1 changes. But no simple relation could be found between the pressure and the values of V_0 and i_2.

An attempt was made to establish a connexion between the form of the full line curve and the least current which would show arcing when the contact was broken. It was thought, for example, that i_2 might represent this least current. But no simple relation could be found.

Summary.

1. The notes deal with the question what conditions are necessary to prevent sparking at the break of the primary circuit of an induction-coil or magneto ; and, especially, how far the speed of separation of the contacts is important.

2. Experiments show that over a wide range of speeds, the greatest current which can be broken without sparking is independent of the speed of separation of the contacts.

3. On the other hand, the nature of the contacts separated influences greatly the limiting current.

4. The observations indicate that the "spark" at the break is really an arc: this conclusion is confirmed by measurements of the potential difference across the contact as it opens. If the discharge is an arc, the independence of the limiting circuit of the speed can be explained.

5. If arcing can be prevented, by preventing heating of the electrodes, the resulting break is perfect.

6. Conditions can be found in which the "spark" at the break is a true spark and not an arc; the limiting current is then dependent of the speed of separation of the contact. But these conditions are not likely to occur in any magneto or induction-coil.

7. Some observations on the resistance of contacts are recorded.

March 1919.

---

* In the observations of fig. 2 it is supposed that, before taking a measurement of V, i is kept constant for a time so long that V reaches a steady value; this time may amount to more than a minute.

XLV. *A Note on Latent Heats of Fusion and their relation to Molecular Composition.* *By* HARBORD GEORGE WAYLING, *M.Sc.**

SINCE Trouton in 1884 formulated a rule showing that, in the case of liquids which were not associated in the state of vapour, the ratio of the molecular heat of vaporization and the temperature of the boiling-point on the absolute scale is a constant, approximately 21, other investigators have applied the rule to the Latent Heats of fusion of elements and compounds. Crompton (J. C. S. 1895) introduced Valency as another factor; Robertson (J. C. S. 1902) with more success substituted the cube root of atomic or molecular volume for valency; Nernst (*Zeit. f. Electrochemie*, 1908) showed a relationship between specific cohesion and latent heats of fusion, and more recently E. Baud (*Compt. Rend.* 1911) studied variation in volume at the melting-point as a function of latent heats; de Guzman (*Anal. fis. quim.* 1913) investigated the relationship between latent heats of fusion and coefficient of viscosity; while H. S. Allen (Proc. J. Phys. Soc. 1916) identified this form of energy as being necessary to counterbalance that of a certain number of oscillators involved in the maintenance of crystalline structure. Honda (Sci. Rep. Imp. Univ. Tokio, 1918) put forward a connexion between latent heats and electronic frequency, the former being the energy of rotation gained by the electrons during the process of fusion; and Tsutsumi, another contributor to the same journal, has studied the specific resistance of several metals near their melting-point and found an abrupt increase in this factor.

If we go back to Trouton's rule and express it in this form

$$\frac{\text{Latent Heat of fusion} \times \text{Molecular number}}{\text{Melting-point on Abs. scale}},$$

the molecular number being the sum of Moseley's atomic numbers of the atoms in the molecule, we find that for several compounds the ratio is equal to the number of atoms in the molecule. Take the commonest of the compounds—water—

Latent heat of fusion	80
Sum of molecular numbers ...	2+8
Melting-point (absolute)	273

---

* Communicated by the Author.

Then

$$\frac{80 \times 10}{273} = 2{\cdot}93,$$

i. e. approximately equal to number of atoms in a molecule of water.

Twelve other examples are given to emphasize the relationship. The data are taken from Landolt-Börnstein's *Tabellen*, 1912 edition.

Compound.	$\frac{L \times N}{T}$.		Number of atoms in mol.
Thallium Bromide, TlBr	$\frac{12{\cdot}7 \times 116}{733}$	$= 2{\cdot}0$	2
Calcium Chloride, $CaCl_2$	$\frac{54{\cdot}6 \times 54}{1047}$	$= 2{\cdot}8$	3
Lead Iodide, PbI_2	$\frac{11{\cdot}5 \times 188}{648}$	$= 3{\cdot}3$	3
Lead Chloride, $PbCl_2$	$\frac{18{\cdot}5 \times 116}{761}$	$= 2{\cdot}8$	3
Arsenic Bromide, $AsBr_3$	$\frac{8{\cdot}93 \times 138}{304}$	$= 4{\cdot}0$	4
Antimony Bromide, $SbBr_3$	$\frac{9{\cdot}76 \times 156}{368}$	$= 4{\cdot}1$	4
Antimony Chloride, $SbCl_3$	$\frac{13{\cdot}3 \times 102}{346}$	$= 4{\cdot}0$	4
Silver Chloride, AgCl	$\frac{21{\cdot}3 \times 64}{728}$	$= 1{\cdot}9$	2
Phosphorus Oxychloride, $POCl_3$.........	$\frac{19{\cdot}8 \times 74}{276}$	$= 5{\cdot}3$	5
Stannic Bromide, $SnBr_4$	$\frac{7{\cdot}07 \times 190}{299}$	$= 4{\cdot}8$	5
Sodium Chlorate, $NaClO_3$..................	$\frac{49{\cdot}6 \times 52}{528}$	$= 4{\cdot}9$	5
Disodium Hydrogen Phosphate, $Na_2HPO_4\ 12H_2O$.	$\frac{66{\cdot}8 \times 190}{309}$	$= 41{\cdot}0$	44

The last compound is rather exceptional among those containing so-called water of crystallization, in that the molecules of water appear to keep the value found in the uncombined state.

When the majority of hydrated salts are studied, it is found that the value of 2 for each molecule of water leads to remarkably close approximation for the total number of atoms in the molecule. Thus, allowing 2 for each molecule of water and full value for all other atoms, the subjoined illustrate the relationship :—

Compound.	$\frac{L\times N}{T}$.		Total.
Sodium Thiosulphate, $Na_2S_2O_3\ 5H_2O$.	$\frac{37{\cdot}6\times 128}{283}$	= 17·0	17
Sodium Sulphate, $Na_2SO_4\ 10H_2O$.	$\frac{51{\cdot}2\times 170}{304}$	= 28·6	27
Calcium Chloride, $CaCl_2\ 6H_2O$.	$\frac{40{\cdot}7\times 114}{301}$	= 15·4	15
Calcium Nitrate, $CaNO_3\ 4H_2O$.	$\frac{33{\cdot}5\times 122}{315}$	= 13·0	13

The figures evidently show that the energy of the latent heats of fusion of these compounds is involved in overcoming the forces that exist between the negative electrons around the nuclear charge of the atom, and that in the preceding examples number rather than nature is the predominating factor. Among organic compounds formic acid is probably of a fairly simple structure and it conforms very well to the foregoing rule, and so does chloroform.

Compound.	$\frac{L\times N}{T}$.		No. of atoms.
Formic Acid, H_2CO_2	$\frac{57{\cdot}4\times 24}{281}$	= 4·9	5
Chloroform, $CHCl_3$	$\frac{19{\cdot}2\times 58}{212}$	= 5·3	5

Other compounds that are regarded as being derivatives of methane—CH_4, such as acetic acid, phenyl acetic acid, give the same ratio, viz. 5.

Compound.	$\frac{L\times N}{T}$.		Ratio.
Acetic Acid, CH_3 COOH	$\frac{45\times 32}{290}$	=	5
Phenyl Acetic Acid, $CH_2C_6H_5$ COOH...	$\frac{25\times 72}{348}$	=	5
Butyric Acid, C_3H_7 COOH	$\frac{28{\cdot}4\times 48}{273}$	=	5

Perhaps these relationships refer to the numbers of oscillators involved in the maintenance of the crystalline structure as suggested by H. S. Allen (Proc. of Phys. Soc. 1916). No such simple connexion seems to occur with the "closed-ring" or aromatic compounds.

XLVI. *On the Resultant of a Number of Unit Vibrations, whose Phases are at Random over a Range not limited to an Integral Number of Periods.* *By* Lord RAYLEIGH, *O.M., F.R.S.*[*]

A NUMBER (n) of points is distributed at random on a straight line of length a. When n is very great, the centre of gravity of the points tends to coincidence with the middle point of the line, which is taken as origin of coordinates. What is the probability that the error of position, that is its deviation from the origin, lies between x and $x+dx$?

Divide the length a into a large odd number $(2s+1)$ of parts, each equal to b. The number of points to be expected on each b is nb/a. This expectation would be fulfilled in the mean of a large number of independent trials, but in a single trial it is subject to error. If the actual number be $nb/a+\xi$, the chance that ξ lies between ξ and $\xi+d\xi$ is by Bernoulli's theorem

$$\frac{d\xi}{\sqrt{(2\pi nb/a)}}e^{-a\xi^2/2nb}, \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

in which it is assumed that while b/a is very small, nb/a is nevertheless very great[†]. In the language of the Theory of Errors, the modulus, proportional to "probable error," is $\sqrt{(2nb/a)}$.

The points which fall on any small part b may be treated as acting at the middle of the part. For instance, those which fall on the part which includes the origin are supposed to act at the origin and so make no contribution to the sum of the moments; while on other parts the moment is proportional to the distance between the middle of the part and the origin. Thus if

$$\xi_{-s}, \xi_{-s+1}, \xi_{-s+2}, \ldots \xi_{-1}, \xi_0, \xi_1, \ldots \xi_s$$

be the values of the various ξ's, the coordinate x of the centre of gravity is given by

$$x=\frac{b(\xi_1-\xi_{-1})+2b(\xi_2-\xi_{-2})+\ \ldots\ +sb(\xi_s-\xi_{-s})}{n+\xi_{-s}+\xi_{-s+1}+\ \ldots\ +\xi_s}. \quad (2)$$

If the whole number of the points be n exactly, the sum of the ξ's in the denominator of (2) must vanish exactly;

---

[*] Communicated by the Author.

[†] Compare Phil. Mag. vol. xlvii. p. 246 (1899).

but if we assume this beforehand, the various ξ's are not independent, as is required by the rules of the Theory of Errors. We may evade the difficulty by supposing the value of ξ on any part to be the result of an *independent* distribution of n points over the whole length. The total of the ξ's is then not necessarily zero, but if we select those cases in which n is zero, or nearly enough zero, the original requirement is fulfilled. In point of fact no selection is required, inasmuch as the probable error of the sum of ξ's is $\sqrt{(2s+1)}$ times the probable error of each and therefore proportional to $\sqrt{(2s+1)}\,.\sqrt{(2nb/a)}$, or $\sqrt{(2n)}$, so that no error of which there is a finite probability is comparable with n. We may accordingly take (2) in the simplified form

$$x=\frac{b}{n}\{\xi_1-\xi_{-1}+2(\xi_2-\xi_{-2})+\ldots\ldots+s(\xi_s-\xi_{-s})\}; \qquad (3)$$

and the (modulus)^2 for the composite error x is given by

$$\frac{\text{Mod}^2\,x}{\text{Mod}^2\,\xi}=\frac{2b^2}{n^2}(1^2+2^2+3^2+\ldots+s^2).$$

For our purpose the sum of the series may be identified with $\int_0^s s^2ds$, or $s^3/3$, or if we prefer it, $(2s+1)^3/24$, that is $a^3/24b^3$, and thus

$$\text{Mod}^2 \text{ for } x = a^2/6n, \quad . \quad . \quad . \quad . \quad . \quad (4)$$

s, as well as n, being regarded as infinitely great.

The probability of an error between x and $x+dx$ in the position of the centre of gravity of the n points is accordingly

$$\sqrt{\left(\frac{6n}{\pi}\right)}e^{-6nx^2/a^2}dx/a, \quad . \quad . \quad . \quad . \quad . \quad (5)$$

showing in what manner the probability of a finite x becomes infinitely small as n increases without limit.

The method hitherto employed requires that the total number (n) of points be very great. It is of interest also to inquire what are the various probabilities when n is small or moderate. In dealing with this problem it seems more convenient to reckon the distances from one end of the line a, and to calculate in the first instance the chances for the *sum* (σ) of the distances. We take $\phi_n(\sigma)d\sigma/a$ to represent the chance that for n points this sum lies between σ and $\sigma+d\sigma$, and we commence with a sequence formula connecting ϕ_{n+1} with ϕ_n. If for the moment we suppose

ϕ_n known and consider the inclusion of an additional point, we see that

$$\phi_{n+1}(\sigma)=\int_{\sigma-a}^{\sigma}\phi_n(\sigma)d\sigma/a. \quad . \quad . \quad . \quad . \quad (6)$$

By means of (6) the various functions may be built up in order.

We start from $\phi_1(\sigma)$. This is zero, unless $0<\sigma<a$, and then is unity. Hence between 0 and a

$$\phi_2(\sigma)=\int_0^{\sigma}\phi_1(\sigma)d\sigma/a=\sigma/a.$$

If σ lies between a and $2a$,

$$\phi_2(\sigma)=\int_{\sigma-a}^{a}\phi_1(\sigma)d\sigma/a=\frac{2a-\sigma}{a}.$$

Thus

$$\left.\begin{array}{l}\phi_2(\sigma)=0, \quad (\sigma<0); \quad \phi_2(\sigma)=\sigma/a, \quad (0<\sigma<a); \\ \phi_2(\sigma)=(2a-\sigma)/a, \quad (a<\sigma<2a); \quad \phi_2(\sigma)=0, \quad (2a<\sigma)\end{array}\right\}, \quad (7)$$

by which ϕ_2 is completely determined; and it will be seen that there is no breach of continuity in the values of ϕ_2 itself at the critical places. These values are symmetrical on the two sides of $\sigma=a$, and can be represented on a diagram by two straight lines passing through $\sigma=0$ and $\sigma=2a$, and meeting at $\sigma=a$. (See fig. 1.)

In like manner we can deduce ϕ_3 from ϕ_2. If $\sigma<0$, $\phi_3=0$, and indeed generally $\phi_n=0$. If $0<\sigma<a$,

$$\phi_3(\sigma)=\int_0^{\sigma}\frac{\sigma\,d\sigma}{a^2}=\frac{\sigma^2}{2a^2}.$$

If $a<\sigma<2a$,

$$\phi_3(\sigma)=\int_{\sigma-a}^{a}\frac{\sigma\,d\sigma}{a^2}+\int_a^{\sigma}\frac{2a-\sigma}{a^2}d\sigma=(-\tfrac{3}{2}a^2+3a\sigma-\sigma^2)/a^2.$$

From the symmetry it follows that when $2a<\sigma<3a$,

$$\phi_3(\sigma)=(3a-\sigma)^2/2a^2.$$

When $\sigma>3a, \quad \phi_3(\sigma)=0.$

It may be remarked that in this case not only is ϕ_3 continuous, but also the first derivative ϕ_3'. The representative curves for all three portions are parabolic. The maximum of ϕ_3, occurring at $\sigma=3a/2$, is $3/4$.

These problems might also be attacked in another and perhaps more direct manner by expressing the probabilities as multiple definite integrals. Thus in the case of two points the chance of distances x and y from the chosen end is $dx\,dy/a^2$; and what we require is the integral of this taken between the proper limits. If we treat x and y as rectangular coordinates of a point lying within the square whose side is a, the probability we seek is represented by the length of the line within the square which is drawn perpendicular to the diagonal through the origin, σ itself corresponding to the position of the line as measured along the diagonal.

For three points we have to consider a *cube* of side a, when the chance is represented in like manner by the *area* within the cube of a plane drawn perpendicularly to the diagonal through the origin. At first, that is near the origin, the area is triangular and increases as σ^2; afterwards it becomes hexagonal, and after passing through the form of a regular hexagon, when its area is a maximum, returns backwards through the same phases.

The calculations by the sequence formula present no difficulty of principle. When $n=4$, I find

$$(0<\sigma<a), \quad \phi_4(\sigma)=\sigma^3/6a^3;$$

$$(a<\sigma<2a), \quad \phi_4(\sigma)=\{\sigma^3-4(\sigma-a)^3\}/6a^3;$$

when $2a<s<4a$, the above values are repeated symmetrically. In this case there is no discontinuity either in ϕ_4, or ϕ_4', or ϕ_4''. When $\sigma=2a$, that is in the middle of the range,

$$\phi_4=2/3, \quad \phi_4'=0.$$

The calculations might be pursued to higher values of n without much trouble. In all cases there is symmetry with respect to the middle of the range. The functions ϕ_n are algebraic and rise in degree by a unit at each step. At the beginning of the range $\phi_{n+1}(\sigma)=(\sigma/a)^n/n\,!$, so that the contact at both ends of the representative curves with the line of abscissæ becomes of high order.

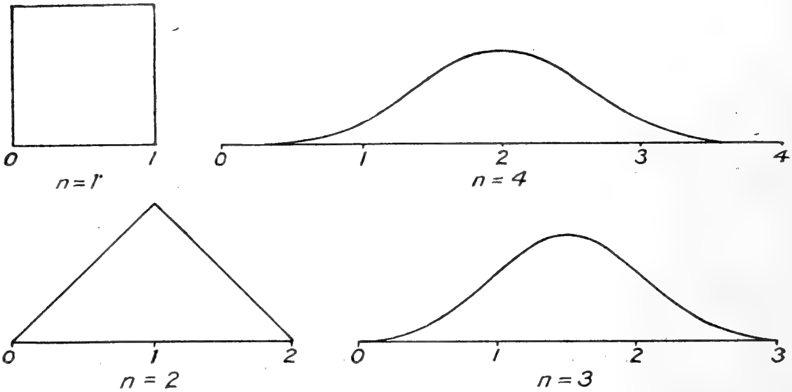
Again, since σ must lie somewhere between 0 and na, we must have

$$\int_0^{na} \phi_n(\sigma)d\sigma/a=1; \quad . \quad . \quad . \quad . \quad . \quad . \qquad (8)$$

from the above expressions we may test this in the cases of $n=2, 3, 4$.

A plot of the curves for these cases is given in fig. 1. The ordinate represents $\phi(\sigma)$ and the abscissa represents σ itself with a taken as unity, so that the area of each curve is unity.

Fig. 1.



In order to pass from these curves in which σ is the *sum* of the distances from one end to the representative curves for the *mean* distance, which must lie between 0 and a, we have merely to reduce the scale of the abscissæ in the ratio $n : 1$, and to increase the scale of the ordinates in the same ratio, so that the area is preserved. For instance, when $n=4$, the middle ordinate will be increased from 2/3 to 8/3.

The sequence formula (6) serves well enough for the derivation of the facility curves appropriate to moderate values of n, but it does not lend itself readily to examination of the passage towards the final form when n is great. This purpose is better attained by an adaptation of a remarkable method due to Laplace*, and employed by him and by Airy † for the derivation of the usual exponential formula for the facility of error. Here again it will be the *sum* of the distances of the points, now reckoned from the middle of the line, that we consider in the first instance.

The distances, instead of being continuously distributed are supposed to be limited to definite values, all equally probable,

$$-sb,\ (-s+1)b,\ (-s+2)b, \ldots . -b,\ 0,\ b,\ 2b, \ldots . sb,$$

---

* See Todhunter's 'History of the Theory of Probability,' p. 521.

† 'Theory of Error of Observation,' Macmillan, 1861, p. 8. In a comparison of the present notation with that of Laplace and Airy, the symbols n and s will be seen to be interchanged.

where $2sb=a$, and ultimately s will be made infinite. The question is—What is the chance that the sum of the distances of n points shall be equal to lb, where l is a positive or negative integer? On examination it appears that the combination follows the same laws "as the addition of indices in the successive multiplication of the polynomial

$$e^{-is\theta}+e^{-i(s-1)\theta}+e^{-i(s-2)\theta}+\ldots\ldots+e^{i(s-2)\theta}+e^{i(s-1)\theta}+e^{is\theta}$$

by itself, supposing the operation repeated $n-1$ times. And therefore the number of combinations required will be the coefficient of $e^{il\theta}$ (which is also the same as the coefficient of $e^{-il\theta}$) in the expansion of

$$\{e^{-is\theta}+e^{-i(s-1)\theta}+\ldots\ldots+e^{i(s-1)\theta}+e^{is\theta}\}^n.$$

The number of combinations required is therefore the same as the term independent of θ in the expansion of

$$\tfrac{1}{2}(e^{il\theta}+e^{-il\theta})\{e^{-is\theta}+e^{-i(s-1)\theta}+\ldots\ldots+e^{i(s-1)\theta}+e^{is\theta}\}^n,$$

or the same as the term independent of θ," when

$$\cos l\theta\{1+2\cos\theta+2\cos 2\theta+\ldots\ldots+2\cos s\theta\}^n$$

is expanded and arranged according to cosines of multiples of θ. By summing the series and application of Fourier's theorem this term is found to be

$$\frac{1}{\pi}\int_0^{\pi}\cos l\theta\left\{\frac{\sin\frac{1}{2}(2s+1)\theta}{\sin\frac{1}{2}\theta}\right\}^n d\theta. \quad . \quad . \quad . \quad (9)$$

This is the number of combinations which gives rise to a sum equal to l, and in order to obtain the probability of l it must be divided by the whole number of combinations equally probable, that is $(2s+1)^n$. What we have to consider is accordingly the value of

$$\frac{1}{\pi(2s+1)^n}\int_0^{\pi}\cos l\theta\left\{\frac{\sin\frac{1}{2}(2s+1)\theta}{\sin\frac{1}{2}\theta}\right\}^n d\theta. \quad . \quad (10)$$

In their discussion, Laplace and Airy regard both n and s as infinite. Here it is proposed to make s infinite, so as to attain a continuous distribution of the points, but without limitation upon the value of n, which may be any integer. If, as before, σ denote the sum of the distances,

$$\sigma=lb=la/2s.$$

When s is very great, $\sin s\theta$ alternates with great rapidity, as soon as θ becomes sensible, so that the integral comes to depend upon that part of the range where θ is very small. We may then replace $\sin\frac{1}{2}\theta$ by $\frac{1}{2}\theta$, and taking $\psi=\theta/s$, we find

$$\frac{1}{s\pi}\int_0^\infty \cos\frac{2\sigma\psi}{a}\,\frac{\sin^n\psi}{\psi^n}\,d\psi \quad . \quad . \quad . \quad . \quad (11)$$

as the equivalent of (10) when s becomes infinite. This is the probability which attaches to a single integral value of l, or to a change $d\sigma$, where $d\sigma=a/2s$. Thus the probability that σ lies between σ and $\sigma+d\sigma$ may be written

$$\frac{2d\sigma}{a\pi}\int_0^\infty \cos\frac{2\sigma\psi}{a}\,\frac{\sin^n\psi}{\psi^n}\,d\psi, \quad . \quad . \quad . \quad (12)$$

which is the required result for a continuous distribution and is applicable to any value of n. In our former notation,

$$\phi_n(\sigma)=\frac{2}{\pi}\int_0^\infty \cos\frac{2\sigma\psi}{a}\,\frac{\sin^n\psi}{\psi^n}\,d\psi, \quad . \quad . \quad (13)$$

in which, however, σ now represents the sum of the distances from the *centre* of the line, instead of from one *end* of it.

If $n=1$, (13) reduces to

$$\phi_1(\sigma)=\frac{1}{\pi}\int_0^\infty \frac{\sin(1+2\sigma/a)\psi+\sin(1-2\sigma/a)\psi}{\psi}\,d\psi,$$

which is unity when σ lies between $\pm\frac{1}{2}a$, but otherwise vanishes.

Again, if $n=2$, we find that $\phi_2'(\sigma)=\pm 1/a$, if σ lies between $\pm a$, and otherwise vanishes, and so on.

More generally, the sequence formula may be deduced from (11), but to obtain it in the original form (6), where the distances are measured from the end of the line, we must write $\sigma-\frac{1}{2}na$ for σ in (11). Then we have

$$\frac{2}{\pi}\int_0^\infty\int_{\sigma-a}^{\sigma} \cos\frac{2\psi}{a}(\sigma-\tfrac{1}{2}na)\,.\,\frac{\sin^n\psi}{\psi^n}\,d\psi\,d\sigma/a,$$

in which

$$\int_{\sigma-a}^{\sigma}\cos\frac{2\psi}{a}(\sigma-\tfrac{1}{2}na)d\sigma=\cos\frac{2\psi}{a}\left(\sigma-\frac{n+1}{2}a\right)\,.\,\frac{\sin\psi}{\psi},$$

so that (11) is verified.

We may now examine the form assumed by ϕ_n in (13), when n is very large. The process is almost the same as that followed in a recent paper [*]. By taking logarithms we find

$$\frac{\sin^n\psi}{\psi^n}=e^{-n\psi^2/6}\{1+nh_4\psi^4+nh_6\psi^6+\tfrac{1}{2}n^2h_4{}^2\psi^8\}, \qquad (14)$$

where

$$h_4=-\frac{1}{180}, \quad h_6=-\frac{1}{35\,.\,81}. \quad . \quad . \quad . \quad (15)$$

Retaining for the moment only the leading term, we get

$$\phi_n(\sigma)\,d\sigma/a=\frac{d\sigma}{a}\,\frac{2}{\pi}\int_0^\infty \cos\,(2\sigma\psi/a)\,e^{-n\psi^2/6}d\psi$$

$$=\sqrt{(6/n\pi)}e^{-6\sigma^2/na^2}d\sigma/a. \quad . \quad . \quad . \quad . \quad (16)$$

In comparing this with (5), we must observe that there x denotes the mean of the distances of which σ is the sum, so that $\sigma=nx$, and thus the two results are in agreement.

If we denote the leading term in ϕ_n by Φ, we obtain from (13) and (14)

$$\phi_n=\Phi+6^2nh_4\frac{d^2\Phi}{dn^2}-6^3nh_6\frac{d^3\Phi}{dn^3}+\tfrac{1}{2}6^4n^2h_4{}^2\frac{d^4\Phi}{dn^4}, \qquad (17)$$

by means of which the approximation in powers of $1/n$ can be pursued. The terms written would suffice for a result correct to $1/n^2$ inclusive, but we may content ourselves with the term which is of the order $1/n$ in comparison with the leading term. We have

$$\Phi=\sqrt{\left(\frac{6}{n\pi}\right)}e^{-6\sigma^2/na^2},$$

$$\frac{d^2\Phi}{dn^2}=\sqrt{\left(\frac{6}{n^5\pi}\right)}e^{-6\sigma^2/na^2}\left\{\frac{3}{4}-\frac{18\sigma^2}{na^2}+\frac{36\sigma^4}{n^2a^4}\right\},$$

and accordingly

$$\phi_n(\sigma)=\sqrt{\left(\frac{6}{n\pi}\right)}e^{-6\sigma^2/na^2}\left\{1-\frac{3}{5n}\left(\frac{1}{4}-\frac{6\sigma^2}{na^2}+\frac{12\sigma^4}{n^2a^4}\right)\right\}. \quad (18)$$

Here $\phi_n(\sigma)d\sigma/a$ expresses the probability that the sum of the distances, measured from the centre of the line, shall lie between σ and $\sigma+d\sigma$.

In terms of the mean (x) of the distances, we should have

$$\sqrt{\left(\frac{6n}{\pi}\right)}e^{-6nx^2/a^2}\left\{1-\frac{3}{5n}\left(\frac{1}{4}-\frac{6nx^2}{a^2}+\frac{12n^2x^4}{a^4}\right)\right\}dx/a, \quad (19)$$

as the probability that x shall lie between x and $x+dx$. It

---

[*] Phil. Mag. vol. xxxvii. p. 344 (1919), equations (65), (66), &c.

should be observed that in virtue of the exponential factor only moderate values of nx^2/a^2 need consideration.

As a check upon (19) we may verify that it becomes unity when integrated with respect to x between 0 and ∞. Starting from

$$\sqrt{\left(\frac{u}{\pi}\right)}\int_0^\infty e^{-ux^2}dx=1,$$

and differentiating with respect to u, we get

$$\frac{2u^{3/2}}{\pi^{1/2}}\int_0^\infty x^2e^{-ux^2}dx=1,$$

and differentiating again

$$\frac{4u^{5/2}}{3\pi^{1/2}}\int_0^\infty x^4e^{-ux^2}dx=1.$$

Using these integrals in (19) with $a=1$, $u=6n$, the required verification follows.

The above verification suggests a remark which may have a somewhat wide application. In many cases we can foresee that a facility function will have a form such as $\mathrm{A}e^{-ux^2}dx$, and then, since

$$\mathrm{A}\int_0^\infty e^{-ux^2}dx=1,$$

it follows that $\mathrm{A}=2\sqrt{(u/\pi)}$. According to this law, the expectation of x is zero, but the expectation of x^2 is finite. If we know this latter expectation, we may use the knowledge to determine u. For

$$\text{expectation of } x^2=2\sqrt{(u/\pi)}\int_0^\infty x^2e^{-ux^2}dx = 1/2u.$$

We may take an example from the problem, just considered, of the position of the centre of gravity of points distributed along a line. If $x_1, x_2, \ldots x_n$ be the coordinates of these points reckoned from the middle and x that of the centre of gravity,

$$\text{Mean } x^2=\iiint\ldots\frac{dx_1dx_2\ldots dx_n}{a^n}\frac{(x_1+x_2+\ldots+x_n)^2}{n^2},$$

the integrations being in each case from $-\frac{1}{2}a$ to $+\frac{1}{2}a$. Taking first the integration with respect to x_n, we find that

$$\text{Mean } x^2=\frac{a^2}{12n^2}+\text{the corresponding expression with } x_n \text{ omitted},$$

so that

$$\text{Mean } x^2 = a^2/12n.$$

Accordingly $u=6n/a^2$, as in (19).

A similar argument might be employed for the law of facility of various resultants (r) of n unit vibrations with phases entirely arbitrary, starting with $\mathrm{A}e^{-ur^2}r\,dr$, and assuming that the mean value of r^2 is n.

My principal aim in attacking the above problem was an introduction to the question of random vibrations when the phases of the unit components are distributed along a circular arc not constituting an entire circle. When the circle is complete the solution has already been given *, and the same solution obviously applies when the circular arc covers any number of complete revolutions. All phases of the resultant are then equally probable, and the only question relates to the probability of various amplitudes, or intensities. But if the arc over which the representative points are distributed is not a multiple of 2π, all values of the resultant phase are not equally probable and the question is in many respects more complicated.

There is an obvious relation between the question of the resultant of random vibrations and that of the position of the centre of gravity of the representative points of the components. For if θ denote the phase of a unit component, the intensity of the resultant is given by

$$\mathrm{R}^2 = (\Sigma \cos\theta)^2 + (\Sigma \sin\theta)^2.$$

If we suppose unit masses placed at angles θ round the circular arc of radius unity, the rectangular coordinates of the centre of gravity are

$$\bar{x} = (\Sigma \cos\theta)/n, \qquad \bar{\ } = (\Sigma \sin\theta)/n\,;$$

and r, the distance of the centre of gravity from the centre of the circle, is related to R according to $r=\mathrm{R}/n$. And in like manner the phase of the resultant corresponds with the angular position of the centre of gravity.

The analogy suggests that a mechanical arrangement might be employed to effect vector addition. A disk, supported after the manner of a compass-card, would carry the loads, and the resulting deflexion from the horizontal would be determined by mirror reading. Perhaps there would be a difficulty in securing adequate delicacy.

---

* Phil. Mag. vol. x. p. 73 (1880); Scientific Papers, vol. i. p. 491. See also Phil. Mag. vol. xxxvii, p. 344 (1919).

To return to the theoretical question, if we suppose the circular arc to be very small, we see that the probability of various phases of the resultant, within the narrow limits imposed, follows the laws determined for the centre of gravity of points distributed at random along a straight line. In this case the amplitude of the resultant is n to a high degree of approximation, n being the number of unit components.

But when the circular arc (α) is so large that $\sin\alpha$ deviates appreciably from α, the question is materially altered. We may, however, frame an argument on the lines followed in equations (1) and (2). Thus with α replacing a and β replacing b, we have for the resultant whose amplitude is R and phase (reckoned from the middle) Θ,

$$\begin{aligned} \mathrm{R}\sin\Theta = {} & \sin\beta\,.\,(\xi_1-\xi_{-1})+\sin 2\beta\,.\,(\xi_2-\xi_{-2}) \\ & + \ldots\ldots + \sin s\beta\,.\,(\xi_s-\xi_{-s}). \quad . \quad . \quad . \quad . \quad (20) \end{aligned}$$

$$\mathrm{R}\cos\Theta = \cos\beta\,.\,(\xi_1+\xi_{-1})+\ldots\ldots+\cos s\beta\,.\,(\xi_s+\xi_{-s}). \quad (21)$$

Here Θ is a small angle, whose probability is under consideration, but R is in general large and may then be reckoned as if the distribution were uniform. Thus

$$\mathrm{R} = \frac{2n}{\alpha}\int_0^{\frac{1}{2}\alpha}\cos\beta\,d\beta = (2n/\alpha)\sin\tfrac{1}{2}\alpha \;. \quad . \quad . \quad (22)$$

and

$$\Theta = \frac{\alpha}{2n\sin\frac{1}{2}\alpha}\left\{\sin\beta(\xi_1-\xi_{-1})+\ldots\ldots+\sin s\beta(\xi_s-\xi_{-s})\right\}. \quad (23)$$

By the rules of the Theory of Errors, we have

$$\frac{\mathrm{Mod}^2\,\Theta}{\mathrm{Mod}^2\,\xi} = \frac{\alpha^2}{2n^2\sin^2\frac{1}{2}\alpha}\left\{\sin^2\beta+\sin^2 2\beta+\ldots\ldots+\sin^2 s\beta\right\}. \quad (24)$$

In (24) $\mathrm{Mod}^2\,\xi = 2n\beta/\alpha$, as before, and the series of $(\sin)^2$ may be replaced by

$$\frac{1}{\beta}\int_0^{\frac{1}{2}\alpha}\sin^2\beta\,d\beta = \frac{\alpha-\sin\alpha}{4\beta}.$$

Thus

$$\mathrm{Mod}^2\,\Theta = \frac{\alpha(\alpha-\sin\alpha)}{4n\sin^2\frac{1}{2}\alpha}. \quad . \quad . \quad . \quad . \quad . \quad (25)$$

If α is small, this reduces to $\alpha^2/6n$, as in (4). If $\alpha=\pi$, that is if the distribution be over a semicircle, we get $\pi^2/4n$. If we make $\alpha=2\pi$ in (25), the result is indeterminate, since although $\sin\frac{1}{2}\alpha=0$, n is infinite. There is a like indeterminateness when α is any multiple of 2π, and this was to be expected. When the arc of distribution consists of entire revolutions, the phase of the resultant is arbitrary. But if the arc differs, even a little, from an integral number of revolutions, there is a definite phase favoured for the resultant, and $\text{Mod}^2\,\Theta$ diminishes as n increases.

The case where the arc consists of entire revolutions is exceptional also as regards the amplitude, or intensity, of the resultant. As we know, in that case no definite value is approached, however great n may be, and the *expectation* of intensity is n. But if there be a fractional part of a revolution outstanding, the intensity does tend to a definite value, that namely which corresponds to a uniform distribution over the arc, and this value is proportional to the *square* of n.

We may go further and calculate what exactly is the expectation of intensity. We have to evaluate

$$\iiint \ldots \frac{d\theta}{\alpha}\frac{d\theta'}{\alpha}\frac{d\theta''}{\alpha} \ldots \left[(\cos\theta + \cos\theta' + \cos\theta'' + \ldots)^2 + (\sin\theta + \sin\theta' + \sin\theta'' + \ldots)^2\right]$$

$$= \frac{1}{\alpha^n}\iiint \ldots d\theta\, d\theta'\, d\theta'' \ldots \left[n + 2\cos(\theta - \theta') + 2\cos(\theta - \theta'') + \ldots . + 2\cos(\theta' - \theta'') + \ldots\right], \quad (26)$$

the integration being in each case from $-\frac{1}{2}\alpha$ to $+\frac{1}{2}\alpha$. Taking first the integration with respect to θ, we have

$$\frac{1}{\alpha}\int_{-\frac{1}{2}\alpha}^{+\frac{1}{2}\alpha} d\theta\left[n + 2\cos(\theta - \theta') + 2\cos(\theta - \theta'') + \ldots . + 2\cos(\theta' - \theta'') + \ldots\right]$$

$$= 4\alpha^{-1}\sin\tfrac{1}{2}\alpha\{\cos\theta' + \cos\theta'' + \ldots\} + n + 2\cos(\theta' - \theta'') + \ldots .$$

On continuing the integration the first part yields finally

$$8(n-1)\alpha^{-2}\sin^2\tfrac{1}{2}\alpha\,;$$

while the remaining parts give the original terms over again with omission of those containing θ. Thus

Expectation of intensity

$$= n + 8\alpha^{-2}\sin^2\tfrac{1}{2}\alpha\{n-1+n-2+n-3+ \ldots . + 1\}$$

$$= n + 4n(n-1)\alpha^{-2}\sin^2\tfrac{1}{2}\alpha. \quad (27)$$

If $\alpha = 0$, this becomes n^2, as was to be expected. If $\alpha = 2\pi$, or any multiple of 2π, the expectation is n, as we knew. In general, when α becomes great, so as to include many complete revolutions, the importance of the n^2 part decreases. In (27) n may have any integral value.

In the case of $n=2$, we may go further and find the expression for the probability of a given amplitude (r) taken always positive, and phase (θ). The amplitude of the components is unity, and the phases, measured from the centre of the arc, θ_1 and θ_2. The probability that these phases shall lie between θ_1 and $\theta_1+d\theta_1$, θ_2 and $\theta_2+d\theta_2$ is $\alpha^{-2}d\theta_1 d\theta_2$. We have now to replace the two variables θ_1, θ_2 by r, θ, where

$$r=2\cos\tfrac{1}{2}(\theta_1-\theta_2), \quad \theta=\tfrac{1}{2}(\theta_1+\theta_2),$$

or

$$\theta_1=\theta\pm\cos^{-1}(\tfrac{1}{2}r), \quad \theta_2=\theta\mp\cos^{-1}(\tfrac{1}{2}r),$$

making

$$\frac{d\theta_1}{d\theta}=1, \quad \frac{d\theta_2}{dr}=\frac{\pm 1}{\sqrt{(4-r^2)}}, \quad \frac{d\theta_1}{dr}=\frac{\mp 1}{\sqrt{(4-r^2)}}, \quad \frac{d\theta_2}{d\theta}=1.$$

Accordingly

$$\frac{d\theta_1 d\theta_2}{\alpha^2}=\frac{\pm 2 d\theta\, dr}{\alpha^2\sqrt{(4-r^2)}}, \quad \ldots \ldots \quad (28)$$

The interchange of θ_1 and θ_2 makes no difference to r and θ, so that we may take

$$\frac{4 d\theta\, dr}{\alpha^2\sqrt{(4-r^2)}}, \quad \ldots \ldots \quad (29)$$

as the chance that the amplitude of the resultant shall lie between r and $r+dr$ and the phase between θ and $\theta+d\theta$. In (29) α is supposed not to exceed 2π.

As a check, we may revert to the case where $\alpha=2\pi$. The limits for θ are then independent of the value of r, and are taken to be $-\pi$ and $+\pi$. And

$$\frac{4dr}{\sqrt{(4-r^2)}}\int_{-\pi}^{+\pi}\frac{d\theta}{\alpha^2}=\frac{2}{\pi}\frac{dr}{\sqrt{(4-r^2)}}=\frac{2\pi r\, dr}{\pi^2 r\sqrt{(4-r^2)}}$$

represents the chance that r shall lie between r and $r+dr$ independently of what θ may be, in agreement with Pearson's expression *. Integrating again with respect to r, we find

$$\int_0^2\frac{2\,dr}{\pi\sqrt{(4-r^2)}}=1,$$

as should be, all cases being now covered.

In the general case the limits for r and θ are interdependent. The possible range for θ is from $-\tfrac{1}{2}\alpha$ to $+\tfrac{1}{2}\alpha$ $(\alpha<\pi)$, but we require the range when r is prescribed. In virtue of the symmetry it suffices to consider a positive θ,

---

* Compare Phil. Mag. vol. xxxvii. p. 328 (1919), equation (21).

and we begin by supposing α less than π, so that the extreme values of r are $2\cos\frac{1}{2}\alpha$ and 2. We proceed to consider the relations by which the limiting values of r and θ are connected.

For a given (positive) θ less than $\frac{1}{2}\alpha$ the upper limit of r is 2 and the lower limit is $2\cos(\frac{1}{2}\alpha-\theta)$. When $\theta>\frac{1}{2}\alpha$, there are no corresponding values of r. In fig. 2, where α is taken to be $\frac{1}{2}\pi$, the shaded area gives the possible values of r corresponding to any θ, or conversely the values of θ corresponding to a prescribed r.

In order to find the chances of a given θ, we integrate with respect to r in (29). We find

$$\frac{4d\theta}{\alpha^2}\int_{2\cos(\frac{1}{2}\alpha-\theta)}^{2}\frac{dr}{\sqrt{(4-r^2)}}=\frac{4d\theta}{\alpha^2}(\tfrac{1}{2}\alpha-\theta), \quad . \quad (31)$$

as the chance that θ, if positive, lies between θ and $\theta+d\theta$. If we integrate (31) again with respect to θ between 0 and $\frac{1}{2}\alpha$, we get $\frac{1}{2}$, the correct value, as there is an equal chance of θ being negative.

Again, in order to find the chance of a prescribed r, when θ is free to vary, we have to integrate (29) first with respect to θ. Referring to fig. 2, we see that when $r<2\cos(\frac{1}{2}\alpha)$, there are no corresponding values of θ, and that when r lies between $2\cos(\frac{1}{2}\alpha)$ and 2, the limits for θ are 0 and $\frac{1}{2}\alpha-\cos^{-1}(\frac{1}{2}r)$. In the first case there is no possibility of r lying between r and $r+dr$; in the second case the probability is

$$\frac{4dr}{\alpha^2\sqrt{(4-r^2)}}\{\tfrac{1}{2}\alpha-\cos^{-1}(\tfrac{1}{2}r)\}, \quad . \quad . \quad . \quad (32)$$

which must be doubled when we admit, as we must, negative values of θ. If we integrate (32) as it stands, again with respect to r, we find the correct value, since

$$\int_{2\cos\frac{1}{2}\alpha}^{2}\frac{4dr}{\alpha^2\sqrt{(4-r^2)}}\{\tfrac{1}{2}\alpha-\cos^{-1}(\tfrac{1}{2}r)\}=\tfrac{1}{2}.$$

We may regard (31) and (32) as the solution of the problem in the case where $\alpha<\pi$.

When $\alpha>\pi$, θ may lie outside the limits $\pm\frac{1}{2}\alpha$ applicable to θ_1 and θ_2, and the question becomes more complicated. It appears that we must distinguish two cases under this head, (i.) where $\pi<\alpha<3\pi/2$, and (ii.) where $3\pi/2<\alpha<2\pi$.

First for $\pi<\alpha<3\pi/2$, fig. 4, where α is supposed to be $5\pi/4$.

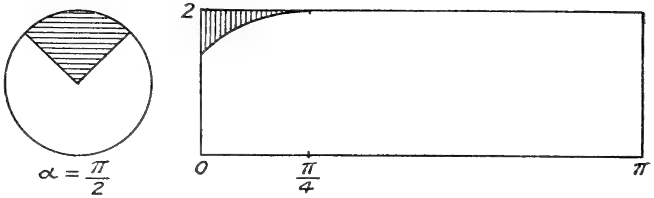


Fig. 2.

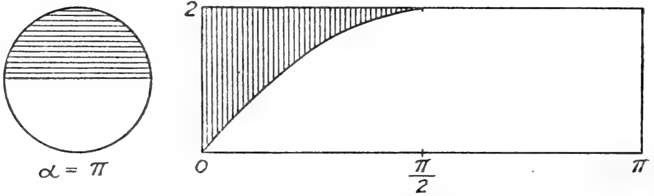


Fig. 3.

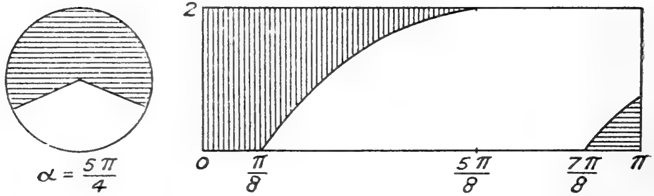


Fig. 4.

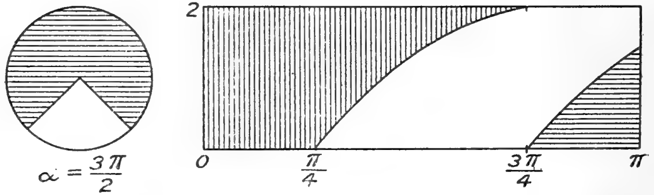


Fig. 5

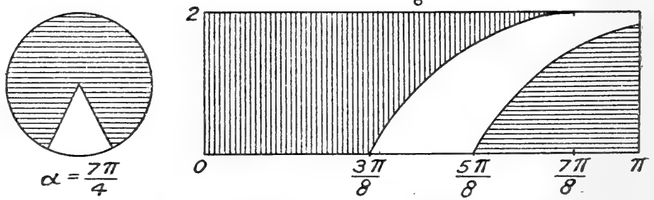


Fig. 6.

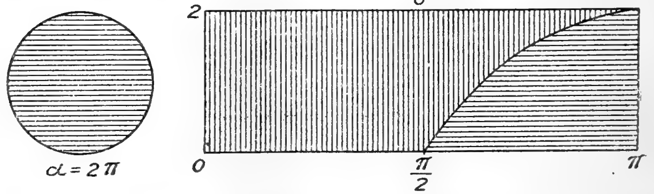


Fig. 7.

From $\theta=0$ to $\theta=\frac{1}{2}(\alpha-\pi)$, r ranges from 0 to 2. From $\theta=\frac{1}{2}(\alpha-\pi)$ to $\theta=\frac{1}{2}\alpha$, r ranges from $2\cos(\frac{1}{2}\alpha-\theta)$ to 2. At $\theta=\frac{1}{2}\alpha$ the lower and upper limits coincide. From $\theta=\frac{1}{2}\alpha$ to $\theta=3\pi/2-\frac{1}{2}\alpha$, there are no corresponding values of r. At the latter limit a zero value of r enters, and from $\theta=3\pi/2-\frac{1}{2}\alpha$ to $\theta=\pi$, r ranges from 0 to $2\cos(2\pi-\frac{1}{2}\alpha-\theta)$.

The whole range from $\theta=0$ to $\theta=\pi$ thus divides itself into four parts. In the first part from $\theta=0$ to $\theta=\frac{1}{2}(\alpha-\pi)$, we get as the chance of θ from (29)

$$\frac{4d\theta}{\alpha^2}\int_0^2 \frac{dr}{\sqrt{(4-r^2)}}=\frac{2\pi\,d\theta}{\alpha^2}, \quad . \quad . \quad . \quad (33)$$

In the second part from $\theta=\frac{1}{2}(\alpha-\pi)$ to $\theta=\frac{1}{2}\alpha$, the chance is

$$\frac{4d\theta}{\alpha^2}\int_{2\cos(\frac{1}{2}\alpha-\theta)}^2 \frac{dr}{\sqrt{(4-r^2)}}=\frac{4d\theta}{\alpha^2}(\tfrac{1}{2}\alpha-\theta). \quad . \quad (34)$$

For the third part, from $\theta=\frac{1}{2}\alpha$ to $\theta=3\pi/2-\frac{1}{2}\alpha$, there is no possibility.

For the fourth part, from $\theta=3\pi/2-\frac{1}{2}\alpha$ to $\theta=\pi$, the chance for θ is

$$\frac{4d\theta}{\alpha^2}\int_0^{2\cos(2\pi-\frac{1}{2}\alpha-\theta)} \frac{dr}{\sqrt{(4-r^2)}}=\frac{4d\theta}{\alpha^2}\left(\tfrac{1}{2}\alpha+\theta-\frac{3\pi}{2}\right). \quad (35)$$

If we integrate (33), (34), and (35) over the (positive) ranges to which they apply and add the results, we get the correct value, viz. $\frac{1}{2}$. This part of the question might be treated more simply without introducing r at all.

We have next to consider what in this case, viz. $\pi<\alpha<3\pi/2$, are the probabilities of various r's when θ is allowed to vary. When r is less than its value at $\theta=\pi$, viz. $2\cos(\pi-\frac{1}{2}\alpha)$, the corresponding range for θ is made up of two parts, the first from $\theta=0$ to $\theta=\frac{1}{2}\alpha-\cos^{-1}(\frac{1}{2}r)$, and the second from $\theta=2\pi-\frac{1}{2}\alpha-\cos^{-1}(\frac{1}{2}r)$ to $\theta=\pi$, so that the whole range of θ is

$$\tfrac{1}{2}\alpha-\cos^{-1}(\tfrac{1}{2}r)+\pi-\{2\pi-\tfrac{1}{2}\alpha-\cos^{-1}(\tfrac{1}{2}r)\}=\alpha-\pi.$$

Thus from $r=0$ to $r=2\cos(\pi-\frac{1}{2}\alpha)$ the chance of r lying between r and $r+dr$ is

$$\frac{4dr(\alpha-\pi)}{\alpha^2\sqrt{(4-r^2)}}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (36)$$

When r lies between $2\cos(\pi-\frac{1}{2}\alpha)$ and 2, the second part disappears and we have only the one range of θ, equal to

$\frac{1}{2}\alpha-\cos^{-1}(\frac{1}{2}r)$, so that the chance of r lying between r and $r+dr$ is

$$\frac{4dr\{\frac{1}{2}\alpha-\cos^{-1}(\frac{1}{2}r)\}}{\alpha^2\sqrt{(4-r^2)}}. \quad . \quad . \quad . \quad . \quad (37)$$

Expressions (36) and (37), obtained on the supposition that θ is positive, are to be doubled when we allow for the equally admissible negative values of θ.

When (36), (37), as they stand, are integrated over the ranges of r to which they apply and added, the sum is $\frac{1}{2}$, as it should be under the suppositions made.

It still remains to consider the case where $3\pi/2<\alpha<2\pi$. From $\theta=0$ to $\theta=\frac{1}{2}(\alpha-\pi)$, r (as before) ranges from 0 to 2. From $\theta=\frac{1}{2}(\alpha-\pi)$ to $\theta=\frac{1}{2}(3\pi-\alpha)$, r ranges from $2\cos(\frac{1}{2}\alpha-\theta)$ to 2. At this point (fig. 6) a second range enters for r. From $\theta=\frac{1}{2}(3\pi-\alpha)$ to $\theta=\frac{1}{2}\alpha$, the first range is, as before, from $2\cos(\frac{1}{2}\alpha-\theta)$ to 2, and the second range is from 0 to $2\cos(2\pi-\frac{1}{2}\alpha-\theta)$. Lastly, from $\theta=\frac{1}{2}\alpha$ to $\theta=\pi$, the first range of r disappears, while the second continues to be from 0 to $2\cos(2\pi-\frac{1}{2}\alpha-\theta)$.

The probabilities of various θ's being positive and lying within specified ranges can be obtained as before. For the range from $\theta=0$ to $\theta=\frac{1}{2}(\alpha-\pi)$ we get the expression (33), and from $\theta=\frac{1}{2}(\alpha-\pi)$ to $\theta=\frac{1}{2}(3\pi-\alpha)$ we get (34). For the third range from $\theta=\frac{1}{2}(3\pi-\alpha)$ to $\theta=\frac{1}{2}\alpha$, we get

$$\frac{4d\theta}{\alpha^2}\left[\int_{2\cos(\frac{1}{2}\alpha-\theta)}^{2}+\int_{0}^{2\cos(2\pi-\frac{1}{2}\alpha-\theta)}\right]\frac{dr}{\sqrt{(4-r^2)}}$$
$$=\frac{4d\theta}{\alpha^2}\left(\alpha-\frac{3\pi}{2}\right); \quad . \quad . \quad . \quad . \quad . \quad (38)$$

and from $\theta=\frac{1}{2}\alpha$ to $\theta=\pi$,

$$\frac{4d\theta}{\alpha^2}\int_{0}^{2\cos(2\pi-\frac{1}{2}\alpha-\theta)}\frac{dr}{\sqrt{(4-r^2)}}=\frac{4d\theta}{\alpha^2}\left(\tfrac{1}{2}\alpha+\theta-\frac{3\pi}{2}\right). \quad (39)$$

If the integrations with respect to θ are effected over the appropriate ranges and the results added, we get $\frac{1}{2}$, as was to be expected.

Finally, for the probabilities of various r's when θ is left open, we get for r between 0 and $2\cos(\pi-\frac{1}{2}\alpha)$ two ranges for θ, viz. from 0 to $\frac{1}{2}\alpha-\cos^{-1}(\frac{1}{2}r)$, and again, from $\theta=2\pi-\frac{1}{2}\alpha-\cos^{-1}(\frac{1}{2}r)$ to $\theta=\pi$, making altogether $(\alpha-\pi)$. Thus for these values of r the probability is that expressed in (36).

When r lies between $2\cos(\pi-\frac{1}{2}\alpha)$ and 2, we recover in like manner (37). And as before we may verify the results by showing that when the second integrations are carried out over the appropriate ranges and the integrals added, we recover $\frac{1}{2}$.

It may be remarked that the latter results may be applied to the complete circle by making $\alpha=2\pi$ (fig. 7). The second range for r then disappears, and for the whole range now extending for all values of θ from $r=0$ to $r=2$ we get

$$\frac{dr}{\pi\sqrt{(4-r^2)}}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (40)$$

which needs to be doubled in order to take account of negative values of θ.

This completes the investigation for an arbitrary α (less than 2π), when $n=2$. Since even for the complete circle ($\alpha=2\pi$) the case $n=3$ leads to elliptic integrals, there is no encouragement to try an extension to other values of α.

Terling Place, Witham.
March 31, 1919.

XLVII. *The Whirling of an Excentrically-loaded Overhung Shaft.* By S. Lees, *M.A., College of Technology, Manchester* [*].

§ 1. THE general problems arising in connexion with the whirling of shafts have been treated at some length by several investigators, prominent amongst whom may be mentioned Greenhill [†], Dunkerley [‡], and Chree [§]. The latter, in particular, has given a very elaborate treatment of the subject on mathematical lines, but does not give any attention to the practical problem of the whirling of shafts with excentric loads.

§ 2. In what follows, an attempt is made to deduce the dynamical equations giving the motion of such an overhung, excentrically-loaded shaft. The writer has taken into account the influence of the rotational inertia of the whirling load, but has made what seems a reasonable assumption, namely, that the whirling load can be taken as a perfectly true flywheel, mounted slightly excentrically. This simplifies the mathematical discussion somewhat. It is also further assumed that the weight of the shaft can be neglected.

---

[*] Communicated by the Author.

[†] Greenhill, Proc. Inst. Mech. Eng. 1883.

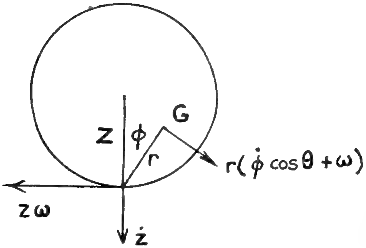
[‡] Dunkerley, Phil. Trans. Roy. Soc. A, 1894.

[§] Chree, Phil. Mag. May 1904.

To specify the position of the system of shaft and flywheel, three variables will be required. These will be taken as (i.) the deflexion z of the free end of the shaft from the undeflected position; (ii.) the angular slope θ which the deflected axis of the shaft at the free end makes with the axis at the constrained end; (iii.) the angle ϕ which the line joining the C.G. of the flywheel to the centre of the shaft makes with the line through this centre, perpendicular to the shaft and in the plane of bending. These coordinates are shown in the figures (figs. 1 and 2).

Fig. 1.

Fig. 2



The method we shall adopt will be to get expressions for the kinetic energy T and potential energy W of the system, and then apply Lagrange's Equations.

The shaft is supposed left to itself, *i. e.* no power is supplied and there are no frictional resistances.

§ 3. To get the expression for the kinetic energy T, we shall first consider the motions of the flywheel. The centre of the free end of the shaft has motions (fig. 1) given by

(*a*) $\dot{z}$ in the plane of the paper,

(*b*) $z\omega$ at right angles to the plane of the paper,

ω being the uniform speed of rotation of the shaft. The motion of the C.G. of the flywheel is now easily obtained on reference to fig. 2. Remembering that the plane of the diagram in fig. 2 is to be supposed at the small angle θ to the vertical plane of the paper, we easily see that the velocity of the C.G. has components :—

(i.) $\dot{z}+r(\dot{\phi}\cos\theta+\omega)\sin\phi$ vertically downwards,

(ii.) $z\omega-r(\dot{\phi}\cos\theta+\omega)\cos\phi$, right to left,

(iii.) $r\dot{\phi}\sin\theta\sin\phi$, parallel to undeflected axis of shaft.

The directions referred to are those in fig. 2, and r is the (small) distance which the flywheel is displaced excentrically. If M be the mass of the flywheel, the kinetic energy of translation of the wheel is accordingly :—

$$\frac{\mathrm{M}}{2}\left\{\dot{z}^2+z^2\omega^2+r^2(\dot{\phi}+\omega)^2+r^2\dot{\phi}^2\theta^2\sin^2\phi+2\dot{z}r(\dot{\phi}+\omega)\sin\phi\right.$$
$$\left.-2zr\omega(\dot{\phi}+\omega)\cos\phi\right\}. \quad . \quad . \quad . \quad (\text{i.})$$

Here $\cos\theta$ has been taken as unity, and $\sin\theta$ as θ.

We have now to discuss the kinetic energy of rotation of the flywheel, considered as rotating about an axis through its C.G. The resultant angular motion of the wheel is made up of (i.) an angular velocity ω about the axis of the shaft at the constrained end; (ii.) an angular velocity $\dot{\theta}$ about an axis perpendicular to the plane of bending ; (iii.) an angular velocity $\dot{\phi}$ about the axis of the shaft at the free end. The axes in (iii.), (ii.), and an axis at right angles to these may be taken as the principal axes at the C.G. If we denote the angular velocities about these axes by Ω_1, Ω_2, and Ω_3 respectively, we have

$$\Omega_1=\dot{\phi}+\omega\cos\theta\,; \quad \Omega_2=\dot{\theta}\,; \quad \Omega_3=\omega\sin\theta. \quad . \quad (\text{ii.})$$

Denoting the corresponding moments of inertia of the wheel by I_1, I_2, and I_3, we get for the kinetic energy of rotation about the C.G., by a well-known theorem,

$$\tfrac{1}{2}\mathrm{I}_1(\dot{\phi}+\omega\cos\theta)^2+\tfrac{1}{2}\mathrm{I}_2\dot{\theta}^2+\tfrac{1}{2}\mathrm{I}_3\omega^2\sin^2\theta. \quad . \quad (\text{iii.})$$

We have $\mathrm{I}_3=\mathrm{I}_2$, and if for simplicity we confine ourselves to the case of a circular disk, so that $\mathrm{I}_1=2\mathrm{I}_2$, expression (iii.) reduces to

$$\mathrm{I}_2\left\{\dot{\phi}^2+\omega^2+2\dot{\phi}\omega-\dot{\phi}\omega\theta^2+\dot{\theta}^2-\omega^2\theta^2\right\}. \quad . \quad . \quad (\text{iv.})$$

We have here replaced $\sin\theta$ by θ, and $\cos\theta$ by $1-\theta^2/2$.

From (i.) and (iv.) we get for the total kinetic energy T of the system,

$$2\mathrm{T}=\mathrm{M}\{\dot{z}^2+z^2\omega^2+r^2(\dot{\phi}+\omega)^2+r^2\dot{\phi}^2\theta^2\sin^2\phi$$
$$+2\dot{z}r(\dot{\phi}+\omega)\sin\phi-2zr\omega(\dot{\phi}+\omega)\cos\phi\}$$
$$+\mathrm{I}_2\{\dot{\phi}^2+\omega^2+2\dot{\phi}\omega-\dot{\phi}\omega\theta^2+\dot{\theta}^2-\omega^2\theta^2\}. \quad \text{(v.)}$$

§ 4. We have next to write down an expression for the potential energy of the system in the displaced position considered. Assuming that the shaft is of circular solid section of area A, moment of inertia of cross-section $\mathrm{A}k^2$, it is known * that the shape of the bent shaft is given by

$$\zeta=(3z-\mathrm{L}\theta)x^2/\mathrm{L}^2+(\mathrm{L}\theta-2z)x^3/\mathrm{L}^3, \quad . \quad . \quad \text{(vi.)}$$

and the potential energy, accordingly, by †

$$\mathrm{V}=2\mathrm{EA}k^2(3z^2-3z\mathrm{L}\theta+\mathrm{L}^2\theta^2/\mathrm{L}^3. \quad . \quad . \quad \text{(vii.)}$$

In these expressions, ζ is the deflexion of the shaft at a distance x from the constrained end, L is the length of the shaft, while E denotes the Young's Modulus of the shaft. We have neglected the effect of gravity on the potential energy. For our purposes, this will be small ; whilst if the shaft is vertical, the expression (vii.) is absolutely correct.

§ 5. We now utilize Lagrange's equations of the general form :—

$$\frac{d}{dt}\left(\frac{\partial\mathrm{T}}{\partial\dot{\psi}}\right)-\frac{\partial\mathrm{T}}{\partial\psi}+\frac{\partial\mathrm{V}}{\partial\psi}=0,$$

where ψ is any coordinate. Taking for ψ the coordinates respectively z, θ, and ϕ, we get, putting $2\mathrm{EA}k^2/\mathrm{L}^3=\mathrm{B}$,

$$\ddot{z}+r(\dot{\phi}+\omega)^2\cos\phi+r\ddot{\phi}\sin\phi-\omega^2z+3\mathrm{B}(2z-\mathrm{L}\theta)/\mathrm{M}=0, \quad . \quad . \quad . \quad \text{(viii.)}$$

$$\ddot{\theta}-\mathrm{M}r^2\dot{\phi}^2\theta\sin^2\phi/\mathrm{I}_2+(\dot{\phi}\omega+\omega^2)\theta+\mathrm{B}(-3z\mathrm{L}+2\mathrm{L}^2\theta)/\mathrm{I}_2=0, \quad . \quad . \quad . \quad \text{(ix.)}$$

$$r^2(\ddot{\phi}+2\dot{\theta}\dot{\phi}\theta\sin^2\phi+\ddot{\phi}\theta^2\sin^2\phi+\dot{\phi}^2\theta^2\sin\phi\cos\phi)$$
$$+r(\ddot{z}\sin\phi-2\dot{z}\omega\cos\phi-z\omega^2\sin\phi)+\mathrm{I}_2(\ddot{\phi}-2\omega\theta\dot{\theta})/\mathrm{M}=0. \quad . \quad . \quad \text{(x.)}$$

§ 6. Before going further, it will be necessary to discuss the possible states of steady motion. To get these, we have to equate to zero all differential coefficients of the variables

---

* Rayleigh, 'Sound,' § 183.

† Neglecting torsional oscillations.

in the last three equations. In this way we obtain :—

$$r\omega^2 \cos\phi - \omega^2 z + 3B(2z - L\theta)/M = 0, \quad . \quad . \quad \text{(xi.)}$$

$$+\omega^2\theta + B(-3zL + 2L^2\theta)/I_2 = 0, \quad . \quad . \quad \text{(xii.)}$$

$$-rz\omega^2 \sin^2\phi = 0. \quad . \quad . \quad \text{(xiii.)}$$

From (xiii.), we see at once that either

$$z = 0, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad \text{(xiv.)}$$

or $\sin\phi = 0$, which gives

$$\phi = 0 \text{ or } \pi. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad \text{(xv.)}$$

These values we now proceed to discuss. From (xiv.), (xii.), and (xi.), we find that if $z = 0$, then $\theta = 0$; whilst $\cos\phi = 0$, giving

$$\phi = \frac{\pi}{2} \text{ or } \frac{3\pi}{2}. \quad . \quad . \quad . \quad . \quad . \quad . \quad \text{(xvi.)}$$

Taking (xv.), (xii.), and (xi.), we find two equations for z and θ. Thus for z we get

$$z\{(6B/M - \omega^2)(2L^2B/I_2 + \omega^2) - 9L^2B^2/MI_2\}$$
$$\pm r\omega^2(2L^2B/I_2 + \omega^2 = 0. \quad \text{(xvii.)}$$

There will be two corresponding values for θ. If we take $\phi = 0$, we get z and θ negative. We therefore take $\phi = \pi$ as the value in (xv.), leading to a real steady value of z. The values of z and θ will be infinite if

$$(6B/M - \omega^2)(2L^2B/I_2 + \omega^2) - 9L^2B^2/MI_2 = 0. \quad \text{(xviii.)}$$

This is a quadratic equation for ω^2, and it will be found to be identical with the result given by Chree (*loc. cit.* eq. 10), for the whirling speed of an overhung shaft with a symmetrical flywheel load. Other things being equal, the steady values of z and θ are seen from (xvii.) to be proportional to r, as might have been expected.

§ 7. To examine possible oscillations about a state of steady motion for the case of $\phi = \pi$, we put $z = z_0 + \bar{Z}$, $\theta = \theta_0 + \bar{\Theta}$, $\phi = \phi_0 + \bar{\Phi}$, and substitute in (viii.), (ix.), and (x.). As we are to put $\bar{Z} = Ze^{ipt}$, $\bar{\Theta} = \Theta e^{ipt}$, $\bar{\Phi} = \Phi e^{ipt}$, with Z, Θ, and Φ small, we neglect terms of the second and higher orders in Z, Θ, and Φ, when substituting in (viii.), (ix.), and (x.). In this way we are led to

$$\ddot{Z} - \omega^2 Z + 3B(2Z - L\Theta)/M \qquad = 0. \quad . \quad . \quad \text{(xix.)}$$

$$\ddot{\Theta} + \omega^2\Theta + B(-3ZL + 2L^2\Theta)/I_2 = 0, \quad . \quad . \quad \text{(xx.)}$$

$$r^2\ddot{\Phi} - 2r\dot{Z}\omega + I_2\ddot{\Phi}/M \qquad = 0, \quad . \quad . \quad \text{(xxi.)}$$

or what is the same thing in effect,

$$Z(-p^2-\omega^2+6B/M)-3BL\Theta/M = 0, \quad . \quad \text{(xxii.)}$$

$$-3BLZ/I_2+\Theta(-p^2+\omega^2+2BL^2/I_2)=0, \quad . \quad \text{(xxiii.)}$$

$$-2ri\omega Zp - \Phi(r^2+I_2/M)p^2 = 0. \quad . \quad \text{(xxiv.)}$$

By eliminating Z, Θ, and Φ, we get an equation of the sixth degree for p.

The correct method of treating these equations is to solve for p from (xxii.) and (xxiii.), thus getting:—

$$\begin{vmatrix} -p^2-\omega^2+6B/M, & -3BL/M, \\ -3BL/I_2, & -p^2+\omega^2+2BL^2/I_2, \end{vmatrix} = 0. \quad \text{(xxv.)}$$

Φ is then got from (xxiv.), and we see that there is a difference of phase between Φ and Z of $\pi/2$.

On putting $p=0$ in (xxv.), we get the condition for unstable motion; this is readily seen to lead to (xviii.). We thus arrive at the result that the state of steady motion with the configuration $\phi=\pi$, is in general stable, but we get instability (*i. e.* whirling) when ω satisfies (xviii.).

§ 8. We now proceed to consider the possible oscillations about the steady state of motion defined by (xvi.).

Taking first the case of motion about $\phi=\pi/2$, we put $\phi=\pi/2+\psi$, where ψ is small. Putting z, θ, and ψ proportional to e^{ipt}, we get for p, from (viii.), (ix.), and (x.),

$$\begin{vmatrix} p^2+\omega^2-6B/M, & 3BL/M, & r(p^2+\omega^2), \\ 3LB/I_2, & p^2-\omega^2-2L^2B/I_2, & 0, \\ r(p^2+\omega^2), & 0, & p^2(r^2+I_2/M). \end{vmatrix} = 0. \quad \text{(xxvi.)}$$

This shows that p cannot be zero, in this type of vibration, whatever the values of the constants. The nature of the roots of (xxvi.) can be arrived at by remembering that the equation is a cubic in p^2, which has positive values for $p^2=\infty$, and $p^2=0$; and having negative values for $p^2=\omega^2+\dfrac{2L^2B}{I_2}$, and $p^2=-\infty$. Thus p^2 has two positive roots, and one negative root. The existence of the negative value of p^2 implies that the motion is unstable.

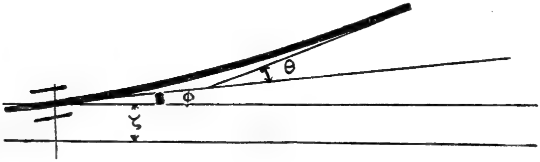
If we take the case of motion about $\phi=\dfrac{3\pi}{2}$, we are led to exactly the same equation for p^2. Hence the motion in this case also is unstable.

§ 9. We have now discussed at some length the motion of the excentrically loaded shaft, neglecting torsional oscillations and any yielding at the bearing. It may be remarked that the effect of torsional oscillations will be quite independent of the motions just considered, provided they are sufficiently small. At first sight, it might seem that ϕ would appear in equations taking into account torsional vibrations. A little reflection, however, will show that the torsional potential energy does not depend on ϕ, and for small displacements z and θ the torsional vibrations are quite independent of the vibrations treated above.

§ 10. It would here seem convenient to discuss the effect of yielding at the bearings. This may have an appreciable influence on affairs, if the framework is light. In the problem considered above, we have only one bearing to consider: this is taken for simplicity.

Referring to fig. 3, we shall now let θ denote the angular deviation of the flywheel end of the shaft from the direction of the shaft at the bearing end. This latter will, in general, not be quite constant in direction. Its variation (supposed to be in the same plane as θ) will be denoted by θ_1. Similarly the deflexion at the bearing will be taken (in the same plane as θ) as z_1, whilst the deflexion at the flywheel end will be taken as $z+z_1$. Neglecting the mass of the shaft itself, the kinetic energy will be the same as (v.), save that $\theta+\theta_1$ must be written for θ, and $z+z_1$ for z. As for the effect on the potential energy, we may represent it by adding a term

Fig. 3.



$$\tfrac{1}{2}(K_1z_1^2+2K_2z_1\theta_1+K_3\theta_1^2), \quad . \quad . \quad . \qquad \text{(xxvii.)}$$

to (vii.).

The net effect is to give five equations of motion, instead of the equations (viii.), (ix.), and (x.). A discussion of the possible steady motions gives the same alternatives as these equations led to. In particular, the stable motion is about $\phi=\pi$. The actual equations in this case reduce to:—

$$\ddot{z}+\ddot{z}_1-\omega^2(z+z_1)+3B(2z-L\theta)/M \qquad =0, \qquad \text{(xxviii.)}$$

$$\ddot{z}+\ddot{z}_1-\omega^2(z+z_1)+K_1z_1+K_2\theta_1 \qquad =0, \qquad \text{(xxix.)}$$

$$\ddot{\theta}+\ddot{\theta}_1+\omega^2(\theta+\theta_1)+B(-3zL+2L^2\theta)/I_2=0, \qquad \text{(xxx.)}$$

$$\ddot{\theta}+\ddot{\theta}_1+\omega^2(\theta+\theta_1)K_2z_1+K_3\theta_1 \qquad =0, \qquad \text{(xxxi.)}$$

$$r^2\ddot{\phi}-2r(\dot{z}+\dot{z}_1)\omega+I_2\ddot{\phi}/M \qquad =0. \qquad \text{(xxxii.)}$$

Disregarding the last of these for reasons which have been indicated earlier, and putting the variables proportional to e^{ipt}, we get

$$\begin{vmatrix} -p^2-\omega^2+6\mathrm{B}/\mathrm{M}, & -p^2-\omega^2, & -3\mathrm{BL}/\mathrm{M}, & 0, \\ -p^2-\omega^2, & -p^2-\omega^2+\mathrm{K}_1, & 0, & \mathrm{K}_2, \\ -3\mathrm{BL}/\mathrm{I}_2, & 0, & -p^2+\omega^2+2\mathrm{L}^2\mathrm{B}/\mathrm{I}_2, & -p^2+\omega^2, \\ 0, & \mathrm{K}_2, & -p^2+\omega^2, & -p^2+\omega^2+\mathrm{K}_3, \end{vmatrix}=0. \quad . \quad . \quad . \quad \text{(xxxiii.)}$$

This may be expressed in the form :—

$$\begin{vmatrix} 6\mathrm{B}/\mathrm{M}, & -p^2-\omega^2, & -3\mathrm{BL}/\mathrm{M}, & 0, \\ -\mathrm{K}_1-6\mathrm{B}/\mathrm{M}, & \mathrm{K}_1, & -\mathrm{K}_2+3\mathrm{BL}/\mathrm{M}, & \mathrm{K}_2, \\ -3\mathrm{BL}/\mathrm{I}_2, & 0, & 2\mathrm{L}^2\mathrm{B}/\mathrm{I}_2, & 0, \\ -\mathrm{K}_2+3\mathrm{BL}/\mathrm{I}_2, & \mathrm{K}_2, & -\mathrm{K}_3-2\mathrm{L}^2\mathrm{B}/\mathrm{I}_2, & -p^2+\omega^2+\mathrm{K}_3, \end{vmatrix}=0, \quad . \quad . \quad . \quad \text{(xxxiv.)}$$

a quadratic equation in p^2.

It should be noted that in (xxviii.) to (xxxi.), the values of z, z_1, θ, and θ_1 are not the absolute values, but give the oscillatory values about positions of steady motion defined by :—

$$\left.\begin{aligned} -r\omega^2-\omega^2(z+z_1)+3\mathrm{B}(2z-\mathrm{L}\theta)/\mathrm{M} &= 0, \\ -\omega^2(z+z_1)+\mathrm{K}_1z_1+\mathrm{K}_2\theta_1 &= 0, \\ \omega^2(\theta+\theta_1)+\mathrm{B}(-3z\mathrm{L}+2\mathrm{L}^2\theta)/\mathrm{I}_2 &= 0, \\ \omega^2(\theta+\theta_1)+\mathrm{K}_2z_1+\mathrm{K}_3\theta_1 &= 0, \end{aligned}\right\} . \quad \text{(xxxv.)}$$

These may be solved for z, z_1, θ, and θ_1, in the usual manner, by determinants. In particular, if

$$\Delta \equiv \begin{vmatrix} \dfrac{6\mathrm{B}}{\mathrm{M}}-\omega^2, & -\omega^2, & -3\mathrm{BL}/\mathrm{M}, & 0, \\ -\omega^2, & \mathrm{K}_1-\omega^2, & 0, & \mathrm{K}_2, \\ -3\mathrm{BL}/\mathrm{I}_2, & 0, & \dfrac{2\mathrm{BL}^2}{\mathrm{I}_2}+\omega^2, & \omega^2, \\ 0, & \mathrm{K}_2, & \omega^2, & \omega^2+\mathrm{K}_3, \end{vmatrix} =0, \quad \text{(xxxvi.)}$$

the values so obtained become infinite, and we get whirling. It will be found that this last equation is a quadratic in ω^2; in fact, is identical with (xxxiv.), when p is put equal to zero.

§ 11. The treatment we have given would not be accurately applicable if very high frequency vibrations were considered;

for we have disregarded the effect of the inertia of the shaft, *i. e.* we have neglected the effect of wave motion along the shaft. The effect of this may be gauged by the following considerations:—

The velocity propagation of a simple wave of tension or compression is $\sqrt{(E/\rho)}$, where ρ is the density of the material of the shaft. The time for such a wave to pass along a length L of shaft will accordingly be $L\sqrt{(E/\rho)}$. For the calculations given in this paper to hold then, this time must be quite small compared with the time of vibration of the shaft as given by (xxv.). It will be readily seen that this is not always the case, particularly when the shaft is running at very high speeds.

§ 12. The general conclusions to be drawn from the paper are as follows:—

The presence of the excentric load does not alter the fundamental character of the vibrations (natural) of the shaft. The periods obtained are the same as those with the symmetrical load. At the same time, there is a state of steady deflexion, given by (xvii.), about which the natural vibrations take place. This steady state of deflexion becomes impossible when the whirling speed is attained; such whirling speed being given by (xviii.). It may also be noted that (xvii.) and a corresponding result for θ, will enable one to assign an inferior limit to the stresses due to the bending that takes place at any speed.

In addition, the effect of yielding at the bearing will cause the critical speed to become lower. The practical effect of this it is difficult to gauge. A calculation is given, depending on unknown constants.

XLVIII. *The Possibility of separating Isotopes.*

By F. A. Lindemann, *Ph.D.*, *and* F. W. Aston, *M.A.*, *D.Sc.*[*]

As the existence of elements of different atomic weights which occupy the same position in the periodic table but appear to be inseparable by chemical means seems now to be generally accepted, it is of interest to consider the various methods by which there may be some hope of separating them. These appear to fall into four main groups:—Distillation, to which chemical separation is closely allied thermodynamically, Diffusion, Density, and Positive Rays.

---

[*] Communicated by the Authors.

The second law of thermodynamics $A-U=T\frac{dA}{dT}$ may be written $\frac{dp}{dT}=\frac{\lambda'}{(v-V)T}$, where p is the pressure of the saturated vapour, v its molecular volume, V the molecular volume of the liquid, λ' the latent heat of evaporation, and T the temperature. At reasonably low pressures $v=\frac{RT}{p}$ and v is large compared with V. One may therefore write

$$\frac{d\log p}{dT}=\frac{\lambda'}{RT^2}.$$

Now

$$\lambda'=\lambda+\int_0^T C_p\,dT-\int_0^T c_p\,dT,$$

where C_p is the molecular heat of the vapour at constant pressure, and c_p the molecular heat of the solid or liquid.

Therefore

$$\log p=-\frac{\lambda}{RT}+\int_0^T\frac{dT}{RT^2}\int_0^T(C_p-c_p)\,dT+\text{const.},$$

the latent heat of fusion of course being included in $\int_0^T(C_p-c_p)\,dT$ if the temperature is above the melting-point.

If p is small C_p is constant so that

$$\log p=-\frac{\lambda}{RT}+\frac{C_p}{R}\log T\cancel{+}\int_0^T\frac{dT}{RT^2}\int_0^T c_p\,dT+\text{const.}$$

Now $C_p=\cancel{C}_v+\frac{9\alpha^2vT}{\kappa}$, where v is the atomic volume, α the expansion coefficient, and κ the compressibility, whilst c_v, the atomic heat at constant volume, is given in the terms of the atomic frequency in the solid ν_m by

$$c_v=3R\left[\frac{12T^3}{(\beta\nu_m)^3}\int_0^{\frac{\beta\nu_m}{T}}\frac{\xi^3d\xi}{e^\xi-1}-\frac{3\beta\nu_m}{e^{\frac{\beta\nu_m}{T}}-1}\right]=f(\nu_m).$$

Therefore

$$\log p=-\frac{\lambda}{RT}+\frac{C_p}{R}\log T-\int_0^T\frac{dT}{RT^2}\int_0^T f(\nu_m)\,dT$$

$$+\int_0^T\frac{dT}{RT^2}\int_0^T\frac{9\alpha^2vT}{u}\,dT+\text{constant}.$$

It is clear that p cannot be identical over a wide range of T, *i. e.* that two substances must be separable by fractionation unless ν_m is identical for both as well as λ and C_p and $\frac{\alpha^2 v}{\kappa}$. All these constants may change with the atomic volume, *i. e.* they may and probably do depend upon the total pressure. Therefore if the theorem that the isotopes are not separable is mathematically true, at any rate λ and C_v must be identical over a finite range of atomic volumes. It has been shown by Soddy, at any rate in the case of lead, that the distances between the centres of the atoms in the solid state are identical. If this is so, it seems impossible for λ and ν_m to be identical.

Consider an atom at the surface of a plane of the isotope defined by $z=0$. Let the forces acting upon it have a component in the Z axis $\phi(z)$. The latent heat at the absolute zero λ is then proportional to $\int_r^\infty \phi(z)dz$. Since r is identical for two isotopes the condition that λ is identical means that $\int_r^\infty \phi(z).dz$ is identical. Further, they must be identical although r may be varied over a finite range.

Now imagine another plane of the same material placed in contact with the first. The force on the atom will be $\phi(z)-\phi(z)$, so that the quasi-elastic restoring force is by Taylor's theorem $2\Delta z\phi'(z)$. The frequency ν_m is therefore

$$\frac{1}{2\pi}\sqrt{\frac{2\phi'(z)}{M}}.$$

If this is to be identical in two isotopes $\phi'(z)$ must be proportional to the atomic weight M.

Therefore if two isotopes are to be inseparable by fractionation, $\phi(z)$ must be a function such that $\int_r^0 \phi(z)dz$ is identical in both cases, whereas $\phi'(z)$ must be proportional to the atomic weight.

If $\phi(z)$ may be represented as a power series, say $a_n z^n + a_{n-1}z^{n-1} + \ldots.$ in the case of one isotope and $b_n z^n + b_{n-1}z^{n-1} + \ldots.$ for the other, one has

$$\phi_1'(z) = n a_n z^{n-1} + (n-1)a_{n-1}z^{n-2} + \ldots.$$

$$= \frac{M_2}{M_1}\phi_2'(z) = n b_n z^{n-1} + (n-1)b_{n-2}z^{n-2} + \ldots)\frac{M_2}{M_1}.$$

This is only true over a finite range of z, if

$$b_n=\frac{M_1}{M_2}a_n,\ b_{n-1}=\frac{M_1}{M_2}a_{n-1}\text{ etc.}$$

On the other hand, if this is so,

$$\lambda_1=\int_r^\infty \phi_1(z)dz=\frac{a_n}{n+1}r^{n+1}+\frac{a_{n-1}}{n}r^n+\dots.$$

and

$$\lambda_2=\int_r^\infty \phi_2(z)dz=\frac{b_n}{n+1}r^{n+1}+\frac{b_{n-1}}{n}r^n+\dots.$$

$$=\frac{M_1}{M_2}\int_r^\infty \phi_1(z)\,dz=\frac{M_1}{M_2}\lambda_1.$$

Therefore if ϕ (z) is an analytical function, both λ and ν_m cannot be identical and the isotopes must be separable under appropriate conditions.

It is of course true that the separation may be minute. If, for instance, M only varies by 1 per cent. as in the case of lead, the percentage difference of pressure at the boiling-point would probably not exceed 1 per cent. and might be very much less if the first-order terms cancel one another.

A similar argument applies to the chemical separation of isotopes. For

$$A=U_0-T\int_0^T\frac{dT}{T^2}\int_0^T\Sigma c_p\,dT+\Sigma i,$$

so that complete identity of the affinity A implies the identity of U_0 and c_p over a finite range of values of r. It is almost inconceivable that the values of U_0 should be identical unless the values of λ are identical, for U_0 is made up of the heat of reaction of one atom of the isotope with one or more atoms in the gaseous state plus the algebraic sum of the latent heats of the combination and of the reacting substances at the absolute zero. The possibility that there is a difference between different isotopes in the heat of combination of one atom with one or more atoms of some other substance which exactly balances the difference in $\Sigma\lambda$ seems sufficiently remote to be ruled out without further discussion. But if one may conclude that the values of λ are identical the same difficulty arises in assuming the values ν_m to be identical as was experienced above. Since the values of $\frac{U_0}{RT}$

are usually large compared with

$$\int_0^T \frac{dT}{RT^2} \int_0^T \Sigma c_p \, dT$$

at all ordinary temperatures, the difference in the constant of the law of mass action would probably be imperceptible unless suitable equilibria were examined under favourable conditions. Thus isotopes could not be separated by an ordinary precipitation, *i. e.* what used to be called an irreversible reaction, any more than nitrogen and oxygen could be separated by dropping liquid air into a red-hot flask.

Fractional distillation was the first method used by Aston in attempting to separate the two hypothetical constituents of Neon ("Homogeneity of Atmospheric Neon," Brit. Assoc. Birmingham Meeting, 1913). The gas was fractionated over charcoal cooled in liquid air. The apparatus used is illustrated in the accompanying figure; the method of working was as follows:—

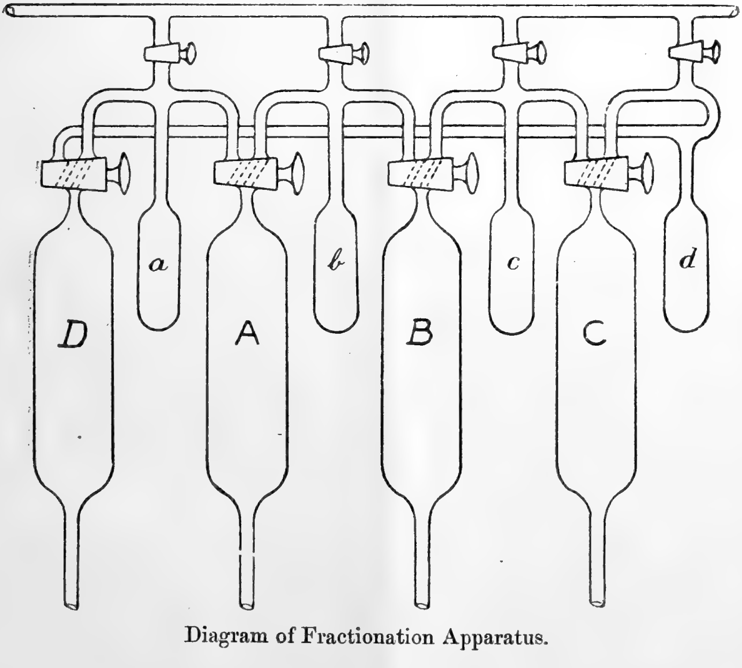


Diagram of Fractionation Apparatus.

The gas was admitted in *a*, one of the small charcoal bulbs *a b c d* all cooled in liquid air. After a reasonable time had elapsed the first fraction was pumped off by lowering mercury in gas-holder A and opening the connecting stopcock between it and *a*. After another interval the stopcock was turned, the mercury raised in A and the gas forced into bulb *b*. The mercury was next lowered in both A and B, the former receiving the second fraction from *a* while the latter withdrew the first fraction of the gas now in *b*. The fundamental assumption on which this arrangement was made was that at this stage, if the vapour-pressures of the gases are nearly the same, the gas in A would have the same composition as that left in *b*, and that they therefore might be mixed. This was done by raising the mercury, which not only drove the gas from A into *b* but also the lightest fraction from B into *c*, where it again fractionated, each process driving the lower boiling gas forward and keeping the higher back.

The apparatus may contain any number of units, the whole system being made cyclical and continuous by joining the charcoal bulb at one end with the gas-holder at the other. Four such units were actually employed, and after four operations the liquid air was removed from *a* and the residue it contained was pumped off completely with an Antropoff pump as the first contribution to the heaviest fraction, in the same way that in D was also pumped off as that of the lightest. The bulb *a* was then immersed again in liquid air and the process continued.

After about two-thirds of the gas had been collected in this way as light and heavy fractions, that remaining was all pumped out as the middle fraction. The process was next repeated with the light and heavy fractions in turn, the intermediate ones being combined by a definite rule.

By this arrangement, which does many operations at once, the small quantity of helium contained in the original gas was removed in a remarkably short time, after which the neon was subjected to continual fractionation for three weeks. The gas had now been through about 3000 fractionations and was divided into seven main fractions; the density of these was determined by the quartz micro-balance (F. W. Aston, Roy. Soc. Proc. A, vol. lxxxix. 1914), the figures for the pressures giving the same zero as oxygen at 76·35 were as follows:—

(1)	(2)	(3)	(4)	(5)	(6)	(7)
121·05	120·95	121·05	120·90	121·00	121·05	121·05

The mean of these, 121·05, gives a molecular weight of 20·19, which is identical within experimental error with the accepted one of 20·200 determined by Watson (J. C. S. 1910).

It is evident that no appreciable separation whatever had been achieved, hence if neon is not homogeneous the vapour-pressure of its constituents at the temperature employed must be very nearly the same. It is, however, impossible to draw any definite conclusion from the above experiment as to the homogeneity of the gas until further fractionations have been performed at other and lower temperatures, preferably without the use of charcoal.

Diffusion.

Lord Rayleigh (Phil. Mag. xlii. p. 493, 1896) has investigated the problem of the separation of gases of different density by diffusion. He obtained as an expression for the effect of a single operation

$$\frac{x+y}{X+Y}=\frac{X}{X+Y}\gamma^{v_1/v_2-v_1}+\frac{Y}{X+Y}\gamma^{v_2/v_2-v_1},$$

where (X, Y) (x, y) are the initial and final volumes of the gases, v_1 v_2 the velocities of diffusion, and γ the enrichment of the residue as regards the second constituent.

After the failure of fractional distillation described above, an attempt was made to separate the hypothetical constituents of neon by diffusion through pipeclay. In this case, as the molecular weights given by positive rays are apparently 20 and 22 respectively, ν and μ only differ by 5 per cent., so the above equation may be written in the simple form

$$\frac{x+y}{X+Y}=\gamma^{-20} \text{ or } \gamma=\sqrt[20]{\frac{\text{Initial Volume}}{\text{Residual Volume}}}.$$

The first apparatus was similar to that described by Ramsay and Collie (Roy. Soc. Proc. lx. p. 206, 1896). After an extremely laborious and prolonged series of diffusions a small but apparently real difference of density was detected amounting to about 0·7 per cent. A much more elaborate apparatus was designed and was started early in 1914. This apparatus is on the see-saw principle and the volume ratio is about 2000, so that given perfect mixing this should give an enrichment of 1·4. Rather more than 200 c.c. of enriched gas had been collected by the time the War interrupted the experiment. It is hoped that the apparatus will soon be started again, and when another 100 c.c. are collected the whole will be diffused again. This second operation should give a theoretical enrichment of 2, so that unless the mixing is very bad a definite increase of density might be obtained.

A very much simpler and better type of continuous diffusion apparatus would be one on the rotating principle used in the Gaede mercury pump, the difficulty being the manufacture of the rotating part of pipeclay or unglazed porcelain with a suitable device for withholding and collecting the small residual volume of less diffusible gas. Should diffusion methods prove sufficiently hopeful, an effort will be made to construct such an apparatus.

It will be seen from Rayleigh's equation above that there is not the least hope of getting approximately complete separation for a reasonable quantity of gas in a reasonable time by this diffusion method even under theoretically perfect conditions of mixing.

Another method that has been suggested consists in allowing the gas to diffuse through a column of heavier gas or liquid. Its practical difficulties are considerable, and its advantages over the other method by no means obvious.

Density.

There are two direct methods of separating isotopes by the effect of their different masses. The first of these, gravitational separation in the stratosphere, applies only to neon if this be really a mixture of two isotopes.

If M be the atomic weight, g the gravitational constant, p the pressure, and ρ the density, then if no mixing takes place $dp=-g\rho dh$, h being the height. In the isothermal layer convection is small. If it is small compared with diffusion the gases will separate to a certain extent. Since T is constant

$$p=\frac{\mathrm{RT}\rho}{\mathrm{M}} \text{ and } \frac{d\rho}{\rho}=\frac{\mathrm{M}g}{\mathrm{RT}}dh,$$

whence

$$\rho=\rho_0 e^{-\frac{\mathrm{M}g}{\mathrm{RT}}\Delta\mathrm{H}},$$

ρ_0 being the density at the height h_0 at which mixing by convection ceases, about 10 kilometres, and Δh the height above this level. If two isotopes are present in the ratio 1 to K_0, so that the density of one is ρ_0 and of the other $\mathrm{K}_0\rho_0$

at height h_0, then their relative density at height $h_0+\Delta h$ is given by

$$K=K_0\, e^{-\frac{g\Delta h}{RT}(M_1-M_2)}.$$

Putting T=220 as is approximately true in England,

$$\frac{K}{K_0}=e^{-5\cdot38\times10^{-3}\Delta h(M_1-M_2)},$$

Δh being measured in kilometres. If $M_1-M_2=2$, therefore

$$\frac{K}{K_0}=e^{-1\cdot075\times10^{-2}\Delta h}$$

It might be possible to design a balloon which would rise to 100,000 feet and there fill itself with air. In this case the relative quantity of hypothetical metaneon would be reduced from 10 per cent. to about 8·15, so that the atomic weight of neon from this height should be 20·163 instead of 20·2. If one could get air from 200,000 feet, *e.g.* by means of a long-range gun firing vertically upwards, the atomic weight of the neon should be 20·12. Though the quantities are measurable they do not appear sufficiently striking to warrant the outlay and labour such experiments would entail.

The second method which suggests itself for separating isotopes by means of their different masses is to centrifuge them. In this case the same equation holds as above except that g varies from the centre to the edge. In a gas therefore

$$\frac{d\rho}{\rho}=-\frac{Mv^2}{RT}\cdot\frac{dr}{r}=-\frac{M\omega^2}{RT}rdr,$$

whence

$$\rho=\rho_0 e^{-\frac{Mv_0{}^2}{2RT}},$$

v_0 being the peripheral velocity. Here again, if K_0 is the ratio of the quantities present at the centre, the ratio at the edge will be

$$K_0\, e^{-\frac{v^2}{2RT}(M_1-M_2)}.$$

A peripheral velocity of 10^5 cm./s. or perhaps even $1\cdot3\times10^5$ cm./s. might probably be attained in a specially designed contrifuge, so that $\frac{K}{K_0}$ might be made as great as

$$e^{-0{\cdot}205(M_1-M_2)}$$

or even
$$e^{-0{\cdot}37(M_1-M_2)}.$$

If M_1-M_2 is taken as 2 a single operation would therefore give fractions with a change of K of 0·65. Thus again, if neon is supposed to consist of 90 per cent. of gas of atomic weight 20 and 10 per cent. of a gas of atomic weight 22, the concentration of the heavier component at the edge would be 2·36 times that at the centre. The apparent atomic weight of the gas from the edge, however, would only be about 1·0065 greater than that of gas from the centre after one operation. By centrifuging several times, however, or by lowering the temperature the enrichment might be increased exponentially.

Centrifuging a liquid, *e.g.* liquid lead, would not appear to be so favourable, though it is difficult to form an accurate idea of the quantities without a knowledge of the equation of state. If compression is neglected and the one lead treated as a solution in the other, a similar formula to that given above holds. Provided the atoms occupy equal spaces, as they appear to do in the solid, the centrifugal force on the heavier component is $(M_1-M_2)\frac{v^2}{T}$, and, as above, its concentration should increase from the centre to the edge in the ratio

$$e^{-\frac{v^2}{RT}(M_1-M_2)}.$$

Again, putting $M_1-M_2=2$ (*e.g.* thorium lead and uranium lead), $v=10^5$ and $T=600$, one finds that the concentration of the thorium lead would be nearly 50 per cent. greater at the edge than at the centre, after one operation. Separation by this method therefore seems possible though difficult and costly.

Positive Rays.

This method consists in allowing the parabolas corresponding to the isotopes generated by the use of electric and magnetic fields in the usual way, to fall through two parabolic slits into separate vessels containing *e.g.* charcoal cooled in liquid hydrogen. In this manner theoretically complete separation would be achieved, so that it is worth while inquiring into the quantities of the gases obtainable in the case of neon.

The maximum separation of the parabolas corresponding to masses 20 and 22 (obtained when electric deflexion θ is half the magnetic) is approximately

$$2\frac{1}{\sqrt{2}}\frac{M_1-M_2}{M_1}\theta=\frac{\theta}{28}.$$

Taking a reasonable value of θ as ·3 the maximum angular width of the beam for complete separation =·01. If the canal-ray tube is made in the form of a slit at 45° to axes, *i. e.* parallel to the curves, the maximum angular length of the beam might be say 5 times as great, which would collect the positive rays contained in a solid angle of ·0005 sq. radian.

The concentration of the discharge at the axis of the positive ray bulb is considerable, and may be roughly estimated to correspond to a uniform distribution of the entire current over a $\frac{1}{4}$ sq. radian. One may probably assume that half the current is carried by the positive rays, and that at least half the positive rays consist of the gases desired. If neon is analysed by this method therefore the total current carried by the positive rays of mass 20 is

$$\cdot 0005\times 4\times \tfrac{1}{2}\times \tfrac{1}{2}\times i=\cdot 0005\, i.$$

If i is as large as 5 milliamperes this $=1\cdot 5\times 10^4$ E.S.U.

or $$\frac{1\cdot 5\times 10^4}{2\cdot 7\times 10^{19}\times 4\cdot 77\times 10^{-10}}=1\cdot 2\times 10^{-6}\text{ c.c./sec.},$$

i. e. one might obtain about one-tenth of a cubic millimetre of neon and $\frac{1}{100}$ cubic millimetre of metaneon per 100 seconds run.

The chief difficulty is the excessive cold necessary in the receiving vessels which must be sufficient to fix the molecules even at the extremely low pressure of 10^{-4} mm. in the camera. This could be obtained by the use of liquid helium, but whether charcoal cooled by liquid hydrogen would suffice is at present uncertain. No such difficulty occurs of course in the separation say of the lead isotopes, but so far it has not proved possible to obtain lead positive rays quite apart from the difficulty of separating the parabolas when obtained.

Conclusions.

The following conclusions may be drawn from the above considerations and results :—

(*a*) Thermodynamic theory indicates that isotopes must be separable in principle even chemically under some physical conditions, though the separation may be small and these conditions may not be obtainable without great difficulty.

(*b*) None of the physical methods considered give hope of easy separation even for gaseous isotopes. When the quantities dealt with are reasonably large the enrichment is extremely small, and when anything like complete separation is hoped for the yield is microscopic. The most promising method appears to be the use of a centrifuge, provided the engineering problems can be overcome.

(*c*) The experimental results quoted so far as they have gone cannot pretend to be conclusive either as to the homogeneity of neon or to the separability of isotopes.

Summary.

Theoretical considerations are advanced to show that elements of different atomic weights must be separable in principle though possibly not in practice.

Various possible methods of separating such substances are discussed.

Fractional Distillation or Chemical Separation may be possible under suitable conditions.

Diffusion must lead to a positive result, but the technique is necessarily laborious.

Gravitational Separation is small. Centrifugal Separation appears promising, but would involve heavy outlay and elaborate preparations.

Electrical Separation, the only method offering hopes of a pure product, will only yield microscopic quantities even if the very serious difficulties indicated can be overcome.

Experiments to separate the hypothetical constituents of Neon are described, in which neither by fractional distillation nor by diffusion have conclusive results yet been obtained.

Farnborough,
Feb. 15th, 1919.

XLIX. *Notices respecting New Books.*

VIBRATIONS OF BOWED INSTRUMENTS.

On the Mechanical Theory of the Vibrations of Bowed Strings and of Musical Instruments of the Violin Family, with Experimental Verification of the Results: Part I. By C. V. RAMAN, M.A., Life-Member and Vice-President, Indian Association for the Cultivation of Science. Calcutta: The Indian Association for the Cultivation of Science, 1918. Pp. iii+158. Price 3*s*. 4*d*.

THE investigations, mathematical and experimental, here recorded form a happy alternating blend and constitute a notable advance in this interesting subject.

Helmholtz, by his vibration microscope, obtained experimental data as to the motion of a well-bowed string, and upon this built up his theory. He also surmised that the bowed point moved forward at the speed of the bow and returned quicker. Among many other results obtained in the present work this surmise is shown to be true for ideal conditions. In other cases it may be approached but not reached.

After Helmholtz came the work of F. Krigar-Menzel and A. Raps, who directly photographed on a film the displacement-time curves of the vibrations of bowed strings under a variety of conditions. More recently E. H. Barton and his students worked upon the vibrations of bowed and plucked strings on the monochord and violin, obtaining photographically simultaneous displacement-time curves of string and some associated part of the instrument, bridge, belly, &c.

The present work strikes a new note in that the *vibrations* of the string under the *forcing* of the bow and *coupled* with the *yielding of the bridge* are all treated by the appropriate differential equations which are solved and discussed.

The wolf-note pitch of the 'cello is specially dealt with and simultaneous curves obtained for string and bridge showing cyclical alternations of amplitude. The theory of the bridge motion, in general and when muted, is also tested by experimental curves with the bridge loaded in various ways.

A kinematical theory of the bow's action is worked out, which leads to the possibility of vibrations represented by graphs consisting of two-step zig-zags, or more complicated forms of similar types.

Bowing under various conditions as to pressure and speed are treated.

Besides 28 text-figures, the work contains 26 full-page plates giving excellent photographic reproductions of vibration curves many of them being of striking beauty and interest.

The present instalment is but a portion of Part I. of the whole work, and the succeding portions will be eagerly awaited by those interested in the distinct contribution to a fascinating subject which is being made by this distinguished scientist. E. H. B.

L. *Intelligence and Miscellaneous Articles.*

DR. LINDEMANN'S CONSTRUCTION FOR RECTIFYING ANY ARC OF A CIRCLE.

To the Editors of the Philosophical Magazine.

6 Addison Road, W. 4.
Dec. 8th, 1918.

GENTLEMEN,—

MAY I be allowed to point out that the construction given by Dr. Lindemann in the current number of the Philosophical Magazine (vol. xxxvi. Dec. 1918, p. 472) follows the lines of a note entitled "A Geometrical Construction for π," in the Mathematical Gazette for 1910 (vol. v. p. 188).

The demonstration given with Dr. Lindemann's construction is analytical in character, and does not bring out the important fact that the lengths AE, AH in his figure are the sums of the lengths of the chords joining the points which divide the arc AB into two and four equal parts respectively, whilst the length of AI is the sum of the lengths of the tangents at A and B and at the middle point of AB. The construction is calculated to give the length of the arc as the common limit of the sums of the sides of the inscribed and circumscribed polygons. Dr. Lindemann's last step, the trisection of the interval KI, makes the approximation to the limit more rapid (theoretically), but sacrifices symmetry, and necessitates the introduction of trigonometrical analysis; otherwise the Third Book of Euclid suffices for the proof.

I am, Gentlemen,
Yours faithfully,
F. J. W. WHIPPLE.

THE
LONDON, EDINBURGH, AND DUBLIN
PHILOSOPHICAL MAGAZINE
AND
JOURNAL OF SCIENCE.

[SIXTH SERIES.]

JUNE 1919.

LI. *Collision of α Particles with Light Atoms.* I. *Hydrogen.*

By Professor Sir E. Rutherford, *F.R.S.**

§ 1. On the nucleus theory of atomic structure, it is to be anticipated that the nuclei of light atoms should be set in swift motion by intimate collisions with α particles. From consideration of impact, it can be simply shown that as a result of a head-on collision, an atom of hydrogen should acquire a velocity 1·6 times that of the α particle before impact, and should possess ·64 of the energy of the incident α particle. Such high speed "H" atoms should be readily detected by the scintillation method. This was shown to be the case by Marsden †, who found that the passage of α particles through hydrogen gave rise to numerous faint scintillations on a zinc sulphide screen placed far beyond the range of the α particles. The maximum range of the H particles, set in motion by the α particles from radium C, was over 100 cm. in hydrogen or about four times the range of the colliding α particles in that gas. This range agreed well with the value calculated by Darwin ‡ from Bohr's § theory of the absorption of α particles by matter.

In most of the experiments of Marsden, a thin glass α-ray tube, containing purified radium emanation, was used as an intense source of rays. This was placed in a closed vessel at a suitable distance from a zinc sulphide screen, and the space

---

* Communicated by the Author.

† Marsden, Phil. Mag. xxvii. p. 824 (1914).

‡ Darwin, Phil. Mag. xxvii. p. 499 (1914).

§ Bohr, Phil. Mag. xxv. p. 10 (1913).

between filled with compressed hydrogen. It was found that the number of H scintillations fell off approximately according to an exponential law when absorbing screens of matter were interposed, and the relative absorption of metal foils was in good accord with the square root law observed by Bragg for α particles.

In a second paper, Marsden * showed that the α-ray tube itself gave rise to a number of scintillations like those from hydrogen. Similar results were observed with an α-ray tube made from quartz instead of glass, and also with a nickel plate coated with radium C. The number of H scintillations observed in all cases appeared to be too large to be accounted for by the possible presence of hydrogen in the material, and Marsden concluded that there was strong evidence that hydrogen arose from the radioactive matter itself. Further experiments were interrupted by the departure of Mr. Marsden to New Zealand early in 1915 to fill the Professorship of Physics in Victoria College, Wellington. The quantity of radium available there was too small to continue observations, while the possibility of further work was precluded by the return of Professor Marsden to Europe on Active Service.

We have seen that Marsden in his second paper had some indications that the radioactive matter itself gave rise to swift H atoms. This, if correct, was a very important result, for previously the presence of no light element except helium had been observed in radioactive transformations.

It was thought desirable to continue these experiments in more detail, and during the past four years I have made a number of experiments on this point and on other interesting problems that have arisen during the progress of the work. The experiments recorded in this and subsequent papers have been carried out at very irregular intervals, as the pressure of routine and war-work permitted, and in some cases experiments have been entirely dropped for long intervals.

§ 2. *Source of the scintillations from active matter.*

Marsden had observed that the number of H scintillations from a nickel plate, coated with radium C, was considerably greater than for a corresponding quantity of emanation—measured by γ rays—from an α-ray tube. It thus seemed possible that H atoms might arise from the disintegration of radium C, for it is well known that this product is transformed in an anomalous manner. In order to test this point, observations were made on the variations of the

---

* Marsden, Phil. Mag. xxx. p. 240 (1915).

number of H scintillations from an α-ray tube immediately after it was filled with emanation. It is well known that the amount of radium C in such a tube increases at first very slowly. For example, after filling a tube with emanation, the fraction of the final amount of radium C present after 10 minutes is only 2 per cent., but reaches 9 per cent. after 20 minutes*. Consequently, observations made on the number of scintillations within 10 minutes after filling should decide definitely whether the scintillations arise from radium C alone and not from the other α-ray products present, viz. the emanation and radium A. In the latter case, the number of scintillations after 10 minutes should be only 2 per cent. of the final number reached about three hours later when radium C is in transient equilibrium with the emanation.

A number of α-ray tubes were kindly made and filled for me by Mr. N. Tunstall, B.Sc. The whole process of filling and removal for testing was done as rapidly as possible, and the counting of scintillations was usually begun within four minutes after filling. The α-ray tube was placed between the poles of a strong electromagnet in order to reduce the luminosity due to β rays on the zinc sulphide screen, placed 2 centimetres beyond the range of the α rays. After every precaution had been taken to avoid radioactive contamination, the number of scintillations observed between 4 and 10 minutes was greatly in excess of the number to be expected if they had their origin in the transformation of radium C alone. The actual ratio of the maximum number varied with the thickness of the α-ray tube, but the fraction observed initially was from 20 to 40 per cent. of the maximum reached three hours later.

These results showed conclusively that, if the H atoms from a glass α-ray tube were a product of radioactive disintegration, they arose not only from radium C but also from radium A or the emanation or both. It is hoped to discuss in a later paper the results of a number of experiments to test whether hydrogen is a product of radioactive change. It is not easy to give a decisive answer to this important problem on account of the numerous factors involved. It will be seen later that the number of scintillations from hydrogen is much greater than is to be expected on the simple theory, and it is difficult to be sure of the absence of hydrogen as a contamination in the source and absorbers of the radiation. In addition, both nitrogen and oxygen atoms are set in such swift motion by collision with α particles that they cause

---

* 'Radioactive Substances and their Radiations,' Rutherford, p. 499.

scintillations outside the range of the α particles. It seems probable that the large number of scintillations observed by Marsden (*loc. cit.*) from a nickel plate coated with radium C were mainly due, not to H atoms, but to high-velocity N and O atoms produced from the air between the source and the screen.

§ 3. *Source of radiation.*

While the use of α-ray tubes as an intense source of radiation has many advantages, it has the drawback that the α radiation is heterogeneous arising from the three products radium A, radium C, and the emanation. In addition, it is difficult to make α-ray tubes of uniform thickness whose stopping power is less than two centimetres of air. For these reasons, I have discarded the use of α-ray tubes and have conducted the majority of the experiments with a homogeneous source of radiation, consisting of the active deposit of radium. Twenty minutes after removal from the emanation, the α radiation arises entirely from radium C and is homogeneous with a range in air of 7 cm. A brief account will now be given of the method for obtaining an intense source of radiation of convenient dimensions. The source usually consisted of a circular bevelled brass disk which was screwed on the lower end of a glass stopcock. The emanation, after removal from the radium solution, was sparked with oxygen to remove excess of hydrogen until the volume was reduced to about 0·5 c.c. This emanation was introduced by means of a mercury trough into a small transfer pump and the mercury raised until its level was 1 or 2 mm. below the disk to be activated. The disk was connected through the stopcock with the negative pole of the lighting circuit and the mercury with the positive pole, in order to concentrate to some extent the active matter on the surface of the disk. After two hours' exposure, the emanation was pumped out and the active disk removed. Theoretically, in order to obtain the maximum activity, the exposure to the emanation should be more than three hours, but in practice it is found that an exposure of two hours gives more activity, while an exposure of twenty-four hours gives much less than an exposure of two hours. This anomalous effect had been previously observed by Ratner [*], and is apparently due to the loss of active matter from the disk through the intermediary of the electric wind.

Using a large quantity of emanation, it is possible to

---

[*] Ratner, Phil. Mag. xxxiv. p. 429 (1917); xxxvi. p. 397 (1918).

obtain in this way a disk, coated on one side with radium C. which has a gamma-ray activity equal to 80 mg. of radium, In most experiments, sources were employed of activity between 5 and 80 mg. of Ra.

The active disk after removal was washed in alcohol and then heated for a minute in an exhausted tube inside an electric furnace at about 300° C. As Ratner (*loc. cit.*) has pointed out, the treatment with alcohol reduces greatly the loss of active matter by so-called volatilization, while the heating tends to remove the surface gases and the emanation occluded in the disk during its exposure. The quantity of active matter on the disk was determined with the aid of a standardized gamma-ray electroscope. The decrease of intensity with time is known from the well-known curve of decay.

§ 4. *Counting scintillations.*

As the systematic counting of H scintillations under varied conditions is a rather difficult and trying task, it may be of some value to mention the general arrangements found most suitable and convenient in practice. Using the excellent zinc sulphide screens, specially prepared by Mr. Glew, the scintillation due to a high-speed H atom appears as a fine brilliant star or point of light, very similar in appearance and intensity to that produced by an alpha particle about 3 mm. from the end of its range. Near the end of the range of the H atom, the scintillation becomes very feeble, and can only be observed on a dark background. Consequently, in a heterogeneous beam of H atoms, the actual number counted per minute is to some extent dependent on the luminosity of the background seen in the microscope. It is important to adjust and keep the luminosity of the screen to the right amount throughout the whole interval of an experiment. This is most simply done by means of a small "pea"-lamp fixed in a metal tube in which the current is varied. While weak scintillations are readily counted on a dark background, it is difficult under such conditions to keep the eye focussed on the microscope image and the eye rapidly becomes fatigued and counting becomes erratic. The microscope employed had a magnification of about 40 and covered a field of 2 mm. diameter. This in practice was found to be a very convenient magnification. In later experiments, special zinc sulphide screens were prepared in which the smaller crystals were sifted through a fine gauze on to a glass plate covered with a thin layer of adhesive material. These fine crystals completely covered the plate several crystals deep. With such a screen, the H scintillations appeared larger and more diffuse, probably due to the scattering of the light in passing through the thick layer of crystals, and were more easily counted, while weak scintillations could be counted on a brighter background than with the ordinary screen. At the same time, the layer of crystals was so uniform, that each incident H atom produced a scintillation.

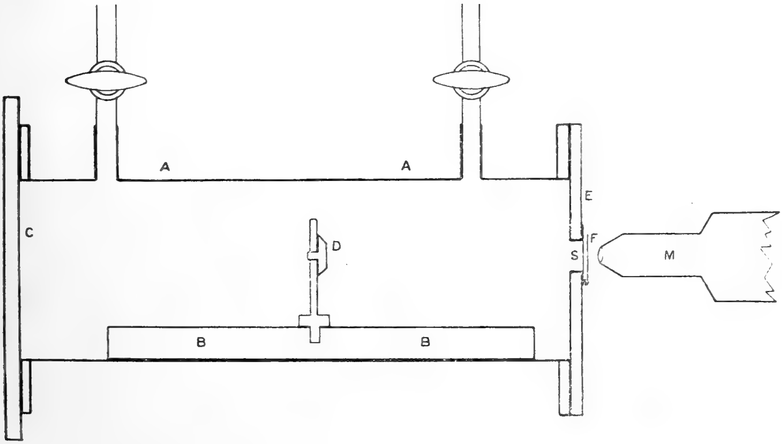
In these experiments, two workers are required, one to remove the source of radiation and to make experimental adjustments, and the other to do the counting. Before beginning to count, the observer rests his eyes for half an hour in a dark room and should not expose his eyes to any but a weak light during the whole time of counting. The experiments were made in a large darkened room with a small dark chamber attached to which the observer retired when it was necessary to turn on the light for experimental adjustments. It was found convenient in practice to count for 1 minute and then rest for an equal interval, the times and data being recorded by the assistant. As a rule, the eye becomes fatigued after an hour's counting and the results become erratic and unreliable. It is not desirable to count for more than 1 hour per day, and preferably only a few times per week.

Under good conditions, counting experiments are quite reliable from day to day. Those obtained by my assistant Mr. W. Kay and myself were always in excellent accord under the most varied conditions. It was usually arranged that the number of scintillations to be counted varied between 15 and 40 per minute.

§ 5. *Experimental arrangement.*

For experiments with hydrogen and other gases, the active disk D (fig. 1) was mounted at a convenient height parallel to the screen on a metal bar B which slid into a rectangular brass box A, 18 cm. long, 6 cm. deep, and 2 cm. wide, with metal flanges at both ends fitting between the rectangular poles of a large electromagnet. One end was closed by a ground glass plate C, and the other by a waxed brass plate E, in the centre of which was cut a rectangular opening 1 cm. long and 3 mm. wide. This opening was covered by a thin plate of metals of silver, aluminium or iron, whose stopping power for α particles lay between 4 and 6 cm. of air. The zinc sulphide screen F was fixed opposite the opening and distant 1 or 2 mm. from the metal covering. By means of two stopcocks, the vessel was filled with the gas to be examined either by exhaustion or displacement. It is a great advantage to have the zinc sulphide screen outside the apparatus, in order to avoid contamination due to volatilized active matter, and for the easy introduction of absorbing material between the end plate and the screen.

Fig. 1.



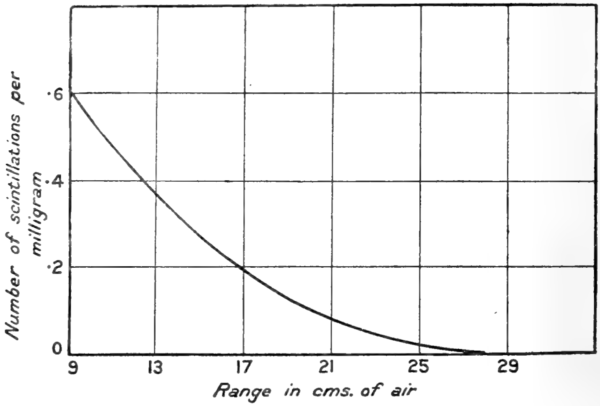
In practice, the source was introduced into the brass vessel at a convenient distance from the screen, and the air exhausted. The α rays after traversing the end plate fell on the screen, and the marked luminosity due to them was a guide in fixing the microscope M in the centre of the opening. The diameter of the field of view (2 mm.) was less than the width of the opening (3 mm.).

Since the number of H atoms observed under ordinary conditions is less than one in a hundred thousand of the number of α particles, H atoms, projected in the direction of the α particles, can only be detected when the α rays are stopped by the absorbing screens. It was not found possible to bring an intense source closer than 3 cm. from the screen on account of the luminosity excited in it by the γ rays and swift β rays, which prevented counting of weak scintillations. A strong magnetic field was necessary to bend away the β rays which caused a very marked luminosity on the screen. A field of 6000 gauss was generally employed for this purpose.

§ 6. *Scintillations due to source and absorbing screens.*

When the containing vessel was exhausted of air, scintillations were always observed on the screen proportional in number to the activity of the source. The number fell off rapidly between 7 and 12 cm. air absorption and then more slowly, but a few could be observed nearly to 28 cm. The variation of number with amount of absorption in terms of cms. of air is shown in fig. 2. This refers to a heated brass source, 3·3 cm. from the screen, with a heated silver plate of stopping power 6 cm. of air just before the screen.

Fig. 2.



These scintillations appear to be due mainly to H atoms excited partly in the source and partly in the absorbing screens. Thin foils of aluminium, for example, placed close to the source increase the number of scintillations. This is due to the occlusion of hydrogen, which can be removed by heating the aluminium in an exhausted furnace just below the melting-point. Similar effects were observed with silver but not with gold. In practice, all screens to be used in the path of the *a* rays were heated to drive off occluded gases as far as possible. This is very necessary when small numbers of scintillations have to be counted. Usually a silver plate was used to absorb the *a* rays. Gold was found to be very free from hydrogen, but it could not be used in place of silver close to the screen on account of the marked luminosity set up on the screen well beyond the range of the *a* particles. This peculiarity of gold had been previously noted by Marsden, but I was surprised to observe the magnitude of the effect with strong sources of radiation. A fuller account of the nature and cause of this luminosity will be postponed till a later paper. In a similar way, mica was found to cause a good deal of luminosity, apparently due to gamma rays. In addition, as is to be expected, mica gives rise to numerous H atoms and swift oxygen atoms. For these reasons, mica is unsuitable for an absorbing screen for α particles in this type of experiment.

§ 7. *Theory of Collision of α particles with light atoms.*

It will be seen later that the number of H atoms and their distribution with velocity differ markedly from the results to be expected theoretically. It is consequently desirable to consider first with some detail the results to be anticipated on simple theoretical grounds, before discussing the experimental results.

The effect of collision of swift α particles with light atoms has been worked out by C. Darwin *.

α particle: M mass, E charge, v initial velocity, ϕ angle of scattering from original direction.

Light atom: m mass, e charge, u velocity after impact, θ angle of deflexion from original direction of α particle.

From considerations of simple impact, it follows that

$$u = 2v\frac{\mathrm{M}}{\mathrm{M}+m}\,.\cos\theta, \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

$$\tan\phi = \frac{m\sin 2\theta}{\mathrm{M}-m\cos 2\theta}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

If there is no loss of energy in the impact we should consequently expect $u = \frac{8}{5}v\cos\theta$ for the hydrogen atom, quite independently of any assumption as to the nature and magnitude of the forces involved in the impact. In order, however, to calculate the number of H atoms scattered within a given angle θ, it is necessary to make special assumptions as to the magnitude and direction of the forces. Assuming that the forces arise from the charges carried by the atomic nuclei which are to be regarded as points, and that the forces vary as the inverse square, Darwin has shown that

$$p = \mu\tan\theta, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

where p is the perpendicular distance from the atom on the initial direction of motion of the α particle and $\mu = \frac{\mathrm{E}e}{v^2}\left(\frac{1}{m}+\frac{1}{\mathrm{M}}\right)$.

---

* C. Darwin, Phil. Mag. (*loc. cit.*).

If Q α particles pass normally through a layer of gas thickness dx, which contains N atoms per c.c. at N.T.P., then the number dn of H atoms projected between the angles 0 and θ is given by

$$\begin{aligned} dn &= QN\pi p^2 dx \\ &= \pi NQ\mu^2 \tan^2\theta \, . \, dx. \end{aligned}$$

Since the reduction of velocity of the α particle in passing through 1 cm. of hydrogen is small, the number n of H atoms produced per cm. of path is given by

$$n/Q = \pi N\mu^2 \tan^2\theta. \quad . \quad . \quad . \quad . \quad . \quad . \quad (4)$$

In this case n/Q is the fraction of α particles which give rise to an H atom between 0 and θ.

Taking $e = \frac{1}{2}E = 4{\cdot}77 \times 10^{-10}$ e.s. unit, $v = 1{\cdot}922 \times 10^9$ cm. per sec., $N = 5{\cdot}41 \times 10^{19}$, and $e/m = 9570$ for hydrogen,

then $\mu = 9{\cdot}27 \times 10^{-14}$

and $n/Q = 1{\cdot}46 \times 10^{-6} \tan^2\theta. \quad . \quad . \quad . \quad . \quad . \quad (5)$

It was found experimentally that the swiftest H atoms due to an α particle from radium C had a range corresponding to 28 cm. of air or four times the range of the α particle. Generally it was found that the maximum range of the H atom was four times the range of the α particle producing it. Since the range of α particles varies as the cube of their velocity, it follows that the range of H atoms is proportional, at any rate approximately, to the cube of their velocity. Since the velocity of an H atom projected at an angle θ with the α particle is $u_0 \cos\theta$ where u_0 is the maximum velocity of the H atom, the range R of an H atom projected at angle θ is given by $R/R_0 = \cos^3\theta$ where R_0 is the maximum range. Since, however, the α particles fall nearly normally on the screen, the H atoms deflected at an angle θ travel a distance $R\sec\theta$. Consequently the range R in the direction of the α particles is given by $R/R_0 = \cos^4\theta$. Substituting the value of θ in equation (5),

$$n/Q = 1{\cdot}46 \times 10^{-6} \left(\sqrt{\frac{R_0}{R}} - 1 \right).$$

This equation only applies to α particles of velocity v_0 emitted by radium C. Since $p \propto 1/v^2$, it is seen that the number of H atoms varies as $1/v^4$. Remembering that the range of the α particle varies as v^3, it is easily seen that for α particles of range r

$$n/\mathrm{Q} = 1{\cdot}46\times10^{-6}(r_0/r)^{\frac{4}{3}}\left(\sqrt{\frac{\mathrm{R}_{\text{max.}}}{\mathrm{R}}}-1\right),$$

where r_0 is range of α particle from radium C, viz. 7·0 cm., and $\mathrm{R}_{\text{max.}}$ is the maximum range of the H atom for a range r, viz. $4r$.

The values of n/Q for different values of R are given in Table I. for α particles of range 7, 5, and 3 cm. in air respectively. It should be noted that Q/n represents the number of α particles required to produce on the average in traversing one centimetre of the gas one H atom, which has a range equal to or greater than R cms. of air.

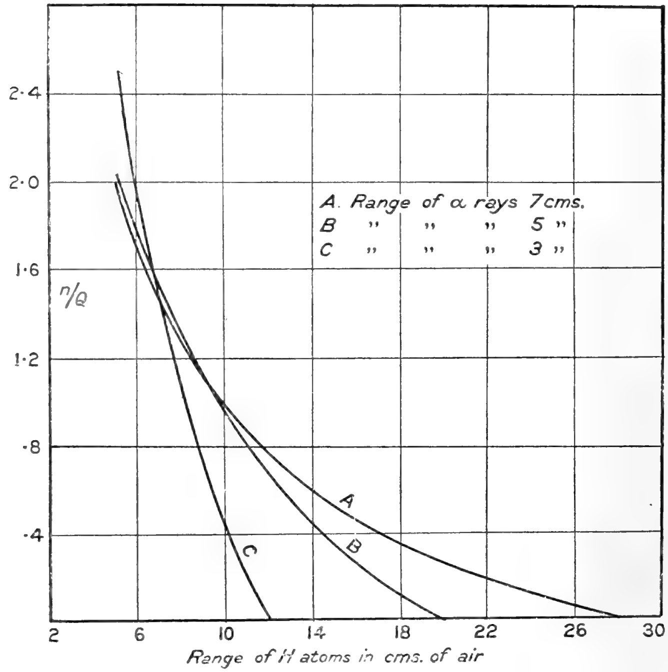
TABLE I.

Range of α particles =7 cms.		Range of α particles =5 cms.		Range of α particles =3 cms.	
Range R of H atoms.	n/Q.	Range R of H atoms.	n/Q.	Range R of H atoms.	n/Q.
1 cm.	$6{\cdot}3\times10^{-6}$	1 cm.	$7{\cdot}9\times10^{-6}$	1 cm.	$11{\cdot}1\times10^{-6}$
2 ,,	4·0 ,,	2 ,,	4·9 ,,	2 ,,	6·5 ,,
4 ,,	2·8 ,,	4 ,,	2·8 ,,	3 ,,	4·5 ,,
7 ,,	1·46 ,,	7 ,,	1·53 ,,	4 ,,	3·3 ,,
10 ,,	·98 ,,	10 ,,	·95 ,,	5 ,,	2·5 ,,
14 ,,	·60 ,,	14 ,,	·45 ,,	6 ,,	1·9 ,,
18 ,,	·34 ,,	16 ,,	·27 ,,	8 ,,	1·0 ,,
22 ,,	·19 ,,	18 ,,	·12 ,,	10 ,,	·4 ,,
24 ,,	·12 ,,	20 ,,	0 ,,	12 ,,	0 ,,
26 ,,	·05 ,,				
28 ,,	0 ,,				

These results are shown graphically in fig. 3, curves A, B, and C respectively, for ranges of the H atoms from 5 to 28 cm. It is seen that the curve A is approximately exponential between 8 and 18 cm., falling to half value in about 5·3 cm. This holds equally for curves B and C over corresponding ranges. These curves give the theoretical variation in number of the H atoms with range such as would be observed if the numbers of H atoms were counted for different thicknesses of absorber.

As the value of n/Q is less than 1/100000, it is not feasible with the present arrangement to detect H atoms within the range of the much more numerous α particles. For this reason, it is not possible to compare theory with experiment in the region of short ranges for which some of the values are calculated. It is seen that while the number of H atoms for

Fig. 3.



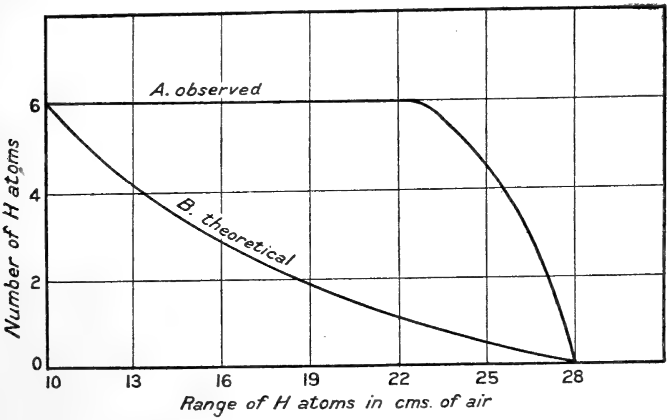
short ranges increases rapidly with the reduction of range of the α particle, the three curves show approximately the same ordinates for 6·5 cm. range of the H atoms.

§ 8. *Absorption of* H *atoms.*

The source of α rays was a brass disk coated with radium C only in the central part, in order to reduce the area emitting α rays. The initial γ-ray activity of the disk was equivalent to about 10 mg. Ra. The zinc sulphide screen was mounted parallel to the disk in the apparatus shown in fig. 1 at 3·3 cm. distance from the source. An opening in the end of the box was covered with a heated silver plate, whose absorption for α particles was equivalent to 5·8 cm. of air, and the whole apparatus was filled by exhaustion with dry hydrogen, at atmospheric pressure. Suitable absorbing screens of aluminium foil were interposed between the silver plate and the screen, and the scintillations counted.

The results obtained are shown in fig. 4, curve A, where the ordinates represent number on an arbitrary scale and the abscissæ the thickness of absorbing material measured in terms of cms. of air for α particles. The equivalent absorption in the silver plate and in the hydrogen is included. The latter was taken as equivalent to 8 mm. of air or one quarter the length of the path of the α particles in hydrogen. The correction due to the natural scintillations from the source and silver plate was small.

Fig. 4.



It is seen that there is no diminution in the number of scintillations for absorptions between 9 and 19 cm. of air*. After 19 cm., there was a slow decrease followed by a rapid fall near the end of the curve. No scintillations were observed beyond 28 cm., *i. e.* for a range four times that of the α particles from radium C.

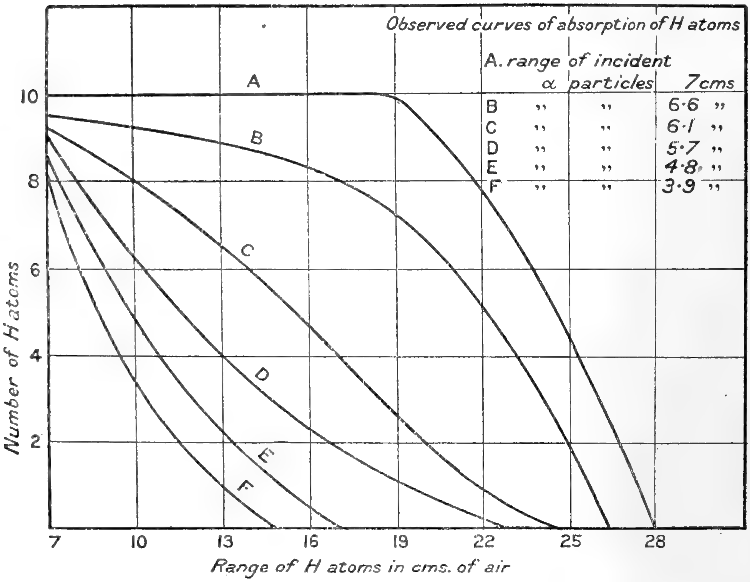
The shape of the absorption curve is entirely different from that to be expected theoretically. The latter is shown in curve B, calculated from the data given in § 7, the same ordinate being taken for an absorption of 10 cm. Between 9 and 19 cm. absorption, the number of scintillations according to theory should fall from 100 to 28.

---

* It should be remarked that, for the distances employed, the width of the testing vessel (fig. 1) was sufficient to give the correct average distributions of H atoms with velocity, corresponding to a source at the centre of a sphere.

This peculiarity of the absorption curve is only marked for long-range α particles. In fig. 5 the absorption curves for initial ranges of the α particles 7 cm., 6·6, 6·1, 5·7, 4·8, 3·9 cm. are shown. The range was reduced by interposing gold or aluminium screens of known stopping power for the α particles close to the source. Even when the range of the α particles is reduced from 7 to 6 cm., the absorption curve already shows an evident decrease of number with thickness of absorber, and this decrease becomes much more marked for decreasing ranges between 6 and 3 cm.

Fig. 5.

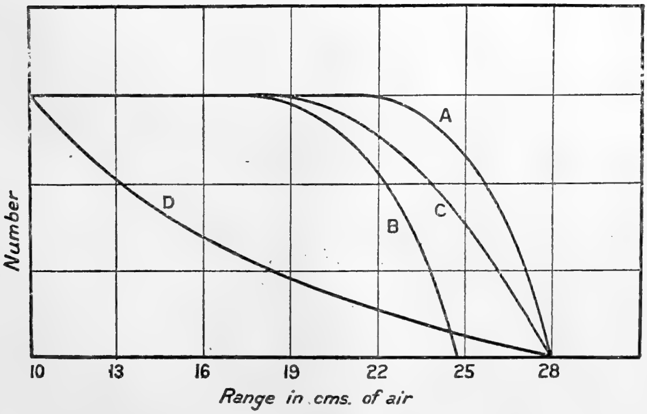


The absorption curves for ranges between 5·7 and 3·9 cm. are very similar in shape to the theoretical curves. For example, in curve F for an initial range of α particles of 3·9 cm. the number of H particles is reduced to $\frac{1}{2}$, $\frac{1}{4}$, $\frac{1}{8}$ for increase of absorption of 2·4, 4·0, 5·5 cm. respectively. The numbers are in good agreement with the calculated values, viz. 2·8, 5·0, 6·4 cm. respectively. The numbers are in still closer agreement if we take the *average* range of the α particles acting on the hydrogen column, viz. 3·9—·4=3·5 cm. The corresponding numbers are then 2·3, 4·2, 5·0 cm. respectively.

We shall now consider the interpretation of the anomalous absorption curve for a range of 7 cm. shown in curve A. The curve is very similar to that to be expected if the hydrogen atoms were thrown forward mainly in the direction of the α particles, and all with the same velocity; in fact, the absorption curve for a pencil of H atoms is very similar in shape to that for a pencil of homogeneous α rays from radium C.

It is well known that the number of α particles counted by the scintillation method in a homogeneous pencil of α rays from radium C remains constant from 0 to 6 cm. of the range, and then rapidly falls to zero in the last centimetre of the range. This end effect is usually ascribed to the scattering of the α particles in their passage through the absorbing material. Now if the H particles consist of H atoms carrying unit positive charge e and projected with a velocity $u=1{\cdot}6\,v$, where v is the velocity of the α particle, the average angular scattering per cm. should be proportional to e/mu^2 and should thus be ·78 of that suffered by the α particle for an equal range. Since the H atom has four times the range of the α particle, the average angular scattering of the H atoms before absorption should be approximately $2\times{\cdot}78=1{\cdot}56$ that of the α particle. It follows, therefore, that the decrease in the number for a homogeneous beam of H atoms should begin about 6 cm. from the end of the maximum range 28 cm.

Fig. 6.



This theoretical curve is shown in fig. 6, curve A. Remembering that the stopping power of the hydrogen column for α rays corresponds to 8 mm. of air, the absorption curve of H atoms due to α particles of range 6·2 cm. has a maximum range 24·8 cm. The corresponding absorption curve is given in curve B. The intermediate curve C shows the distribution to be expected for the hydrogen column, supposing the H atoms are all projected in the direction of the α particles with a velocity proportional to the velocity of the α particle at each point of the hydrogen column.

This theoretical curve C is very similar in all respects to the experimental curve (fig. 4, curve A), showing that the H atoms produced in a thin film of hydrogen are nearly homogeneous in velocity and are thrown forward in the direction of the colliding α particles.

It does not follow that the direction of the H atom coincides with the direction of the α particles, but the average deflexion cannot be much more than 10° or 15°. For an angle of deflexion of θ, the range of the H atom in the direction of the α particles is $R_{\text{max.}} \times \cos^4\theta$. The value of $\cos^4\theta$ is 0·94 for 10°, 0·87 for 15°, and 0·78 for 20°. An average value of θ of 20° would make the decrease in the number begin about 13 cm. instead of 19 cm.

It is to be anticipated that the average angle of deflexion should increase rapidly with decrease of the velocity of the α particle. The rapid changes in shape of absorption curve with change of velocity of α particle are at any rate partly due to this cause.

It is difficult to determine directly the actual average angle of deflexion of H atoms, since the H atoms are scattered considerably in passing through the minimum 7 cm. of air or other absorbing material required to stop the α particles.

There seems to be little doubt that if a film of hydrogen were exposed to α particles of greater initial velocity than those from radium C, a nearly homogeneous beam of H rays would be obtained, all of which would travel nearly in the direction of the α particles.

§ 9. *Variation of number of* H *particles with velocity of α particles.*

In order to reduce the velocity of the α particles, the vertical source in the apparatus shown in fig. 1 was completely covered with different thicknesses of gold foil whose stopping power in terms of air was accurately determined. The distance between the source and screen was 3·3 cm., and the apparatus filled with hydrogen at atmospheric pressure. The gold foils were pressed tightly against the source to prevent production of H atoms between the source and foils. The number of H particles was determined after passing through absorbers of known stopping power. The results are given in the table below.

Absorption of gold foil in terms of cms. of air.	Issuing range of α rays in cms. of air	Absorption in terms of air between source and screen including the hydrogen.	Observed number of H atoms.	Calculated number of H atoms.
0 cm.	7·0 cm.	8·3 cm.	100	100
1·7 ,,	5·3 ,,	8·3 ,,	77	103
2·5 ,,	4·5 ,,	7·5 ,,	51	119
3·3 ,,	3·7 ,,	6·6 ,,	25	139
4·0 ,,	3·0 ,,	6·6 ,,	5	128

In the last column are given the relative numbers of H atoms to be expected on the simple theory given in § 7, when account is taken of the maximum range of the H atoms and the thickness of absorber traversed. Since the number of H atoms for corresponding ranges varies as the inverse fourth power of the velocity of the α particles, *i. e.*, as the inverse four thirds power of the effective range of the α particles, the number of H atoms should increase with lowering of the velocity of the incident α particles. The observed numbers, however, instead of increasing with reduction of velocity of α particles, fall off slowly at first and then very rapidly for ranges between 3·5 and 3 cm.

In these experiments, the intensity of the H radiation was reduced by this passage through absorbing material equal to 6·6 cm. of air. In order to reduce this absorption, another series of experiments was made in which the silver plate of stopping power 5·8 cm. was replaced by an aluminium plate of stopping power 3·7 cm. The velocity of the α particles was reduced by aluminium foil instead of gold foil, placed close to the source. The aluminium foils used in these experiments were freed as far as possible from hydrogen by heating in a vacuum, and the results obtained with aluminium as absorber were similar to those with gold. The presence of numerous H atoms was observed for α particles of range 2·5 cm., but the number was small and just measurable with certainty for α particles of range 2·0 cm. The actual number in the latter case was small compared with that observed for α particles of range 3 cm. Experiments at low ranges are rendered somewhat difficult by the necessity of taking into account the H scintillations which arise from the absorbing material and source. These are always present, and in number comparable with those produced by the admission of hydrogen. While the general results show that, under the experimental conditions, the number of H atoms becomes relatively small for ranges of α particles between 2 and 3 cm., it is not possible to say with certainty whether the number falls to zero for still smaller velocities of the incident α particles. We are unable to continue observations with this experimental arrangement for absorptions less than 7 cm. of air, so that no information is available of the number of H atoms of range less than 7 cm.

In § 8, the absorption curves for H atoms produced by α particles of different velocities have already been given.

§ 10. *Number of* H *atoms.*

We have already mentioned that the number of H atoms is considerably greater than that to be expected on the simple theory. It is important to determine the number as accurately as possible, as it gives us important information on the nature of the collision. The apparatus of fig. 1 was employed. A thick copper plate with a hole 1·02 mm. diameter was placed over the end silver plate of stopping power about 6 cm., and the zinc sulphide screen placed about 1 mm. away. Even allowing for possible scattering, all the H atoms passing through the opening were counted by the microscope, which had a field of view of diameter 2 0 mm. The source was part of a small hemisphere whose outer surface was active, placed 2·85 cm. from the end of the vessel. The space between was filled with hydrogen at atmospheric pressure. The initial γ-ray activity of the source was about 10 mg. Ra.

The zinc sulphide screen was specially made for the purpose and was estimated to have about 90 per cent. efficiency in giving scintillations. As a result of three separate concordant determinations, it was found that the number of H atoms for hydrogen at N. T. P. falling on the screen corresponded to 5·1 per minute per milligram of activity, including an allowance of 10 per cent. for inefficiency of the screen.

If l = length in cms. of path of α particles in hydrogen,
A = area of opening in sq. cms.,
n = number of α particles emitted per second by one milligram of radium,
ρ = fraction of α particles which produce an H atom per centimetre of path in hydrogen at N. T. P.

Then obviously

number of H atoms per second on area $A = \frac{5 \cdot 1}{60} = \frac{\rho A l n}{4\pi l^2}$.

Taking $l = 2 \cdot 85$ cm., $A = 0 \cdot 84$ sq. mm., $n = 3 \cdot 72 \times 10^7$,

then $\rho = 9 \cdot 7 \times 10^{-6}$,

or in round numbers $\rho = 10^{-5}$.

This number was obtained for a total absorption in path of H atom corresponding to about 15 cm. of air, but we have already seen that the number under conditions of experiment does not vary sensibly between 9 and 19 cm. absorption. We have seen in § 7 that the number of H atoms to be expected on the simple theory is $\cdot 98 \times 10^{-6}$ for 10 cm. absorption and $\cdot 31 \times 10^{-6}$ for 19 cm. We thus see that for an absorption of 10 cm., the observed number of H atoms is 10 times the theoretical value and for 19 cm. 31 times.

Using the observed result that 1 in 10^5 of the α particles produces one H atom per centimetre of path of hydrogen, it is easy to calculate the maximum distance of the direction of flight of the α particles from the centre of the hydrogen atom in order to produce a high speed atom.

If p = this perpendicular distance,

N = number of atoms of H per c.c. at N.T.P.

Then $\pi p^2 N = 10^{-5}$.

Taking $N = 2 \times 2 \cdot 705 \times 10^{19}$,

then $p = 2 \cdot 4 \times 10^{-13}$ cm.,

or, on an average, each α particle of radium C of range 7 cm. produces an H atom when the perpendicular distance of its path from the centre of the H atom is equal or less than $2 \cdot 4 \times 10^{-13}$ cm. It should be remembered that this calculation deals with the α particles of range 7 cm. when the H atoms are projected mainly in the direction of the incident α particles and with a range not less than 19 cm. of air, *i. e.*, with a velocity comparable with the maximum velocity of the H atom. As already shown, the distribution of velocity is very different for α particles of shorter range, although the actual number in all cases exceeds considerably the value calculated on the simple theory.

§ 11. *Closeness of approach of α particles to* H *nucleus.*

The experimental results considered show that the number and distribution of H particles are very different from those calculated on the assumption that the α particle and H atom are to be regarded as point nuclei carrying charges $+2e$ and $+e$ respectively, and indicate that the forces involved in a close collision differ considerably in magnitude and probably in direction from those to be expected on the simple theory.

In order to throw light on the magnitudes involved, consider the following case. Assume that for distances greater than D between the centres of the colliding atoms, the forces are given by the simple theory but for decreasing distances the forces between the nuclei augment rapidly according to other laws, and that all the collisions of closer approach than D result in the production of a high-speed H atom which for α particles of range about 7 cm. tends to be projected approximately in the line of flight of the α particles.

Darwin (*loc. cit.*) has shown that the apsidal distance D between an α particle and H atom is given on the simple theory by

$$D=\frac{\mu v_0^2}{v^2}(1+\sec\theta),$$

where $\mu=\frac{Ee}{v_0^2}\left(\frac{1}{m}+\frac{1}{M}\right)=9{\cdot}27\times10^{-14}$ for α particles of maximum range 7 cm., where θ is the angle of deflexion of H atom and v_0 the velocity of α particles from radium C. In the same notation (§ 7)

$$p=\mu\frac{v_0^2}{v^2}\tan\theta.$$

Eliminating θ from these two equations, $p^2=D\left(D-2\mu\frac{v_0^2}{v^2}\right)$.

We have seen (§ 9) that for α particles of range about 7 cm. the value of $p=2{\cdot}4\times10^{-13}$. Substituting this value of p and putting $v=v_0$ we find the corresponding value of $D=3{\cdot}5\times10^{-13}$ cm. It will be seen later that all collisions for which D on the simple theory is greater than this value, should give rise to H atoms of velocity too small for detection. We may consequently conclude that all collisions for which D is equal or less than $3{\cdot}5\times10^{-13}$ cm. give rise to a high-speed H atom.

It is of interest to consider, on these assumptions, how the number of H atoms should vary with the velocity of the incident α particles. From the above equation, it is seen that $p=0$ when $D=2\mu\frac{v_0^2}{v^2}$. Substituting the value $D=3{\cdot}5\times10^{-13}$, $\mu=9{\cdot}27\times10^{-14}$, we find $v_0/v=1{\cdot}89$. The range of α particles of this velocity v is 2·7 cm. This result means that α particles of range less than 2·7 cm., acted on by the forces given by the simple theory, are unable to approach within the critical distance D of the nucleus of the hydrogen atom.

Since the number of H atoms produced is proportional to the value of p^2 given in equation

$$\frac{\text{Number of H atoms of velocity } v}{\text{,, \quad ,, \quad ,, \quad ,, } \quad v_0} = \frac{D - 2\mu\frac{v_0^2}{v^2}}{D - 2\mu}.$$

Substituting the values of D, μ, v_0/v, the relative number of H atoms to be expected for different values of v are given below :—

Range of incident α particles in cms.	7	6	5	4	3·5	3·0	2·7
Relative number of H atoms	100	88	72	50	35	15	0

It has been previously pointed out that the observed number of H atoms shows a rapid decrease for ranges between 3 and 2 cm., a result in general accord with these calculations. It is, however, not to be expected that there would be any close agreement between theory and experiment, for the theory supposes that there is an abrupt variation in the magnitude and direction of the forces for an apsidal distance D, a condition which is physically improbable. We may, however, conclude that the variations of number of H atoms with velocity is not inconsistent with the view that the forces between colliding atoms augment rapidly for values of $D < 3{\cdot}5 \times 10^{-13}$ cm.

From the known values of D and μ, we are able to calculate the value of θ, *i. e.*, the angle of deflexion of the H atom for a collision of apsidal distance $D = 3{\cdot}5 \times 10^{-13}$ cm. For α rays of range 7 cm., $\theta = 69°$; the corresponding effective range of the H atom is $28 \cos^4 \theta$ or 4·6 mm. It is thus clear that, on the assumptions made, no atoms for which $D > 3{\cdot}5 \times 10^{-13}$ should be detected under the experimental conditions.

The general results are consistent with the view that the field of force between the α particle and hydrogen nucleus undergoes rapid changes in magnitude and probably also in direction when the nuclei approach within $3{\cdot}5 \times 10^{-13}$ cm. of each other.

§ 12. *Summary.*

1. The production of high-speed hydrogen atoms due to close collisions between α particles and atoms of hydrogen has been studied using the α particles from radium C as a homogeneous source of radiation. In such close collisions, where the nuclei approach within a distance of about 3×10^{-13} cm., the number and distribution of the H atoms are entirely different from those calculated on the assumption that the nuclei are to be regarded as point charges repelling each other according to the law of inverse squares.

2. The H atoms produced by swift α particles of range 7 cm. are shot forward mainly in the direction of the α particles and are nearly uniform in velocity.

3. The distribution with velocity of H atoms becomes more and more heterogeneous with decrease of velocity of the α particles. For α particles of range less than 4 cm. of air, the distribution and absorption of H atoms are in fair accord with the simple theory although the observed numbers are greater than those calculated on the theory.

4. The number of swift H atoms produced by α particles of range 7 cm. is 30 times greater than the theoretical number. The number falls off rapidly for ranges of α particles between 3 and 2 cm. On an average 10^5 α particles give rise to one swift hydrogen atom in traversing one centimetre of hydrogen.

5. It has been calculated that all α particles of range 7 cm. projected within a perpendicular distance $p = 2{\cdot}4 \times 10^{-13}$ cm. of the centre of the hydrogen nucleus give rise to swift H atoms. The corresponding apsidal distance is about $3{\cdot}5 \times 10^{-13}$ cm.

6. As observed by Marsden, hydrogen atoms are emitted by the radioactive source. The number observed is small, and it is difficult to decide whether these H atoms arise from the radioactive transformation or from occluded hydrogen in the source.

Discussion of results.

On the nucleus theory of the atom, the charged nucleus is supposed to be of such small dimensions that it may be regarded as a point charge for distances of the order of 10^{-11} cm. The correctness of this point of view in the case of hydrogen is strongly supported by the remarkable success of Bohr and those who have followed him in explaining by its aid the finer points of the hydrogen spectrum. The experiments of Geiger and Marsden [*] on the large angle scattering of heavy atoms like those of gold showed that the nucleus of the gold atom could be regarded as a point charge for distances of the order of 3×10^{-12} cm., and that the law of inverse squares held up to that distance within the limits of experimental error. In the present experiments on the collision of particles with hydrogen atoms, the atomic nuclei approach still closer, viz. to a distance of the order of 3×10^{-13} cm. It is to be anticipated that for such small distances of the order of the diameter of the electron, the structure of the helium nucleus can no longer be regarded as a point, and this is borne out by experiment. Such a conclusion in no way invalidates the nucleus theory as ordinarily understood; but a study of the forces close to the nucleus is of great importance in throwing light on its actual dimensions.

It is clear from the results given in this paper that a close collision between an α particle and a hydrogen nucleus is an exceedingly rare occurrence. Only 1 in 100000 of the α particles passing through 1 cm. of hydrogen at normal pressure and temperature gives rise to a high-speed H atom, while in the same distance each α particle on an average passes through the sphere of action of about 10000 hydrogen molecules. Thus for every 10^9 collisions with the molecules, in only one case does the α particle pass close enough to the nucleus to give rise to a swift H atom. No doubt a much greater number of H atoms are set into comparatively swift motion by less direct collisions, but these do not give rise to H atoms which can be detected beyond the range of the α particle.

It is clear that for such close collisions, each hydrogen atom in any complex molecule acts as an independent unit, so that swift H atoms should be liberated by α particles from every substance containing free or combined hydrogen. This is fully borne out by experiment.

In seeking for an explanation of these anomalous results, there are two salient facts to bear in mind, viz., that (1) the H atoms produced by α particles of range greater than 6 cm. are projected mainly in the direction of the α particles and over a narrow range of velocity, and (2) the number of such swift H atoms is far in excess of the number on the simple theory of point charges.

If we consider the nuclei of the atoms in collision to act

---

[*] Geiger and Marsden, Phil. Mag. xxv. p. 604 (1913).

as point charges, no advantage in explanation is gained by supposing that the free charges carried by the nuclei are greater than those usually supposed; for while such an assumption gives an increased number of H atoms of all velocities, it fails to account for (1) above.

If we suppose the central forces fall off more rapidly than the inverse square law, the proportion of swift atoms increases relatively. This can be deduced from consideration of the calculations given by Darwin* for the case of the inverse cube law, and it is not difficult to see that this relative increase of high speed particles becomes more marked the more rapid the law of variation of the central force. In all cases, however, the pencil of H atoms should be widely heterogeneous for all velocities of the colliding α particle. It thus seems clear that no theory of single central forces can account for the experimental facts.

This is not unexpected, for we have every reason to believe that the α particle has a complex structure consisting probably of four hydrogen nuclei and two negative electrons†. If we assume, for simplicity, that the hydrogen nucleus acts as a point charge for the distances under consideration, we still have a complicated system of forces near the nucleus of the α particle.

Now we have seen that the anomalous effects in hydrogen manifest themselves when the two nuclei approach within about 3×10^{-13} cm. of each other. Geiger and Marsden have shown that the scattering of α particles in passing through atoms of a heavy element like gold, is consistent within experimental error with an inverse square law of repulsion, and in the case of a head-on collision, the closest distance of approach is about 3×10^{-12} cm. or about 10 times the distance in the case of a close collision between the α particle and the hydrogen atom. It appears significant that, in the latter case, the closest distance of approach is about the same as the accepted value of the diameter of the negative electron, viz. $3{\cdot}6 \times 10^{-13}$ cm. The observed effects are similar to those to be expected if the helium nucleus, for example, consisted of a charged disk of radius about 3×10^{-13} cm. with its plane perpendicular to the direction of motion, and it seems clear that the helium nucleus must have dimensions of this order of magnitude.

If the helium nucleus is composed of two electrons and four hydrogen nuclei, we should expect a complicated field

---

* Darwin (*loc. cit.*).

† Rutherford, Phil. Mag. xxvii. p. 488 (1914).

of force round the nucleus and rapid variations in direction and magnitude of the forces for distances of the order of the diameter of the electron. In our ignorance of the detailed structure of the nucleus, we can only speculate as to the magnitude and direction of the forces close to it. Considering, however, the enormous repulsive force between two positive nuclei in collision at a distance of 3×10^{-13} cm.—about five kilograms weight on the inverse square law,—it is to be anticipated that not only the structure of the complex helium nucleus should be much deformed, but that the electron itself may suffer strong deformation under the intense electric forces. If such deformation of the electron be possible, it is not difficult to see in a general way that the forces between the nuclei in collision may vary exceedingly rapidly close to the nucleus, and may even change rapidly from one of repulsion to one of attraction.. It may be possible in this way to explain the experimental effects observed, including both the projection in the direction of the α particle and the increase over the number to be expected on the simple theory.

It is of course possible to suppose that the actual law of force, apart from deformation, does not follow the inverse square for very small distances ; but since the inverse square law appears to hold at any rate approximately for positive charges up to a distance 3×10^{-12} cm., it seems simpler to suppose that the rapid alteration in magnitude and direction of the force close to the nucleus is due rather to a deformation of its structure and of its constituent parts. Taking into account the intense forces brought into play in such collisions, it would not be surprising if the helium nucleus were to break up. No evidence of such a disintegration, however, has been observed, indicating that the helium nucleus must be a very stable structure.

It will be shown in a later paper that the anomalous effects observed in hydrogen are shown also by collision of swift α particles with nitrogen and oxygen atoms and for about the same distance between the nuclei.

My thanks are due to Mr. W. Kay for his assistance in counting and in all the experimental work.

University of Manchester,
April, 1919.

LII. *Collision of α Particles with Light Atoms.* II. *Velocity of the Hydrogen Atom.* *By* Professor Sir E. RUTHERFORD, *F.R.S.**

IN the first paper giving an account of the number of H atoms produced by α particles and their absorption by matter, it has been implicitly assumed that the long-range scintillations observed in hydrogen are due to swift hydrogen atoms set in motion by close collisions with α particles. This is supported by the observations that the range of the atoms is in good accord with the value calculated by Darwin from Bohr's theory of absorption of charged particles.

Taking into account, however, the intense forces developed in such collisions and the possibility of the disruption of the structure of the nuclei involved in the collisions, it was thought desirable to determine experimentally the mass and velocity of these flying atoms, and to compare the values with those deduced from the collision theory. Such a determination was rendered the more necessary by certain apparent anomalies observed in connexion with the brightness and distribution of H atoms, an account of which will be given later in this paper.

To determine the mass and velocity of the H atom, it was necessary to measure the deflexions of a stream of H atoms both in a magnetic and in an electric field. The experiments were somewhat tedious and difficult on account of the small number of H scintillations present under the experimental restrictions.

Magnetic deflexion of H *atoms.*

In these experiments it was necessary to produce the H atoms at a definite point, and for this purpose a film of paraffin wax of convenient thickness, exposed to an intense beam of α rays, was used. The method finally adopted was to compare directly the deflexion of a pencil of H atoms produced from the film of paraffin wax, with the deflexion of a pencil of α rays using the same source of α rays in both cases.

The experimental arrangement is shown in fig. 1.

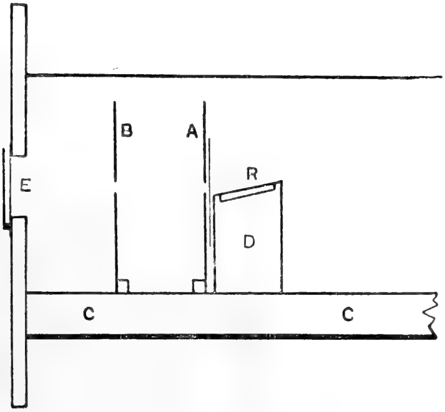
The horizontal slits A and B, about 1 cm. broad and 1 mm. wide, were mounted on a rectangular brass bar C. The source R, consisting of a circular brass disk coated on one side with radium C, was mounted on a vertical block D, close to the slit A and making a small angle with the horizontal.

---

* Communicated by the Author.

This carrier was then introduced into the rectangular brass vessel shown in fig. 1 of the previous paper, and the whole apparatus was placed between the poles of a strong electromagnet. A vertical slit 1·5 cm. long and 3 mm. wide, cut in the end plate of the box, was covered with a thin sheet of iron E whose stopping-power for α particles corresponded to 4 cm. of air. Close to this was placed the zinc sulphide screen S. The distance between the slits A and B was 2·85 cm., and between B and the iron screen 3·25 cm. On exhausting the apparatus of air a well-defined band of α rays about 2 mm. broad was observed on the screen. The distance measured by the microscope between the centres of the two bands on reversing the field of about 9000 gauss was about 3·9 mm. A film of paraffin wax about 30 μ thick mounted on a frame was then placed close to the slit A between it and the source. This gave a band of H scintillations on the screen, which was of about the same width as the beam of α rays. Aluminium screens were introduced between the iron plate and zinc sulphide screen, so that the total absorption between the source and screen was equivalent to 14·4 cm. of air. Under these conditions the two bands of H scintillations obtained by reversal of the field were carefully determined by the microscope, and the centres of the bands were found to be about 6 mm. apart.

Fig. 1.



As a result of three concordant determinations, it was found that under the experimental conditions the average deflexion of the pencil of H atoms was 1·45 times the deflexion of the pencil of α rays from radium C. Since the value of $\frac{\text{M}v}{\text{E}}$ for α particles from radium C has been accurately determined in other experiments and found to be $3 \cdot 98 \times 10^5$ *, the average value of $\frac{mu}{e}$ for the pencil of H atoms is $2 \cdot 74 \times 10^5$. Now, on passing the slit A, the maximum range of the H atom is $28 - 3 \cdot 4 = 24 \cdot 6$ cm., while the minimum range for H atoms to be observed on the screen is $28 - 14 \cdot 4 = 13 \cdot 6$ cm. Since the range of the H atom is proportional to the cube of its velocity, the velocity of the H atoms observed which passed through the magnetic field lies between $\cdot 96 u_0$ and $\cdot 79 u_0$, where u_0 is the maximum velocity of the H atom due to an α particle from radium C. Since the relation between number and velocity was approximately linear over this range, the average velocity of the H atoms was $\cdot 87 u_0$. Consequently the value of $\frac{mu_0}{e}$ for the swiftest H rays produced by α particles from radium C is $1/\cdot 87 \times 2 \cdot 74 \times 10^5 = 3 \cdot 15 \times 10^5$. On the collision theory, the velocity u of the H atoms of mass m is given by

$$u = \frac{2\text{M}}{\text{M}+m} \cdot v \cos\theta,$$

and the maximum value

$$u_0 = 1 \cdot 6 \, v_0.$$

The value of $\frac{mu_0}{e}$ for H atoms carrying unit charge should consequently be $3 \cdot 2 \times 10^5$. The agreement between theory and experiment is closer than we should expect considering the difficulty of the measurements. In these experiments it was found that all the H atoms carried a positive charge, and no sign of scintillations was observed indicating the presence of negatively charged or neutral atoms.

Electrostatic deflexion of H *atoms.*

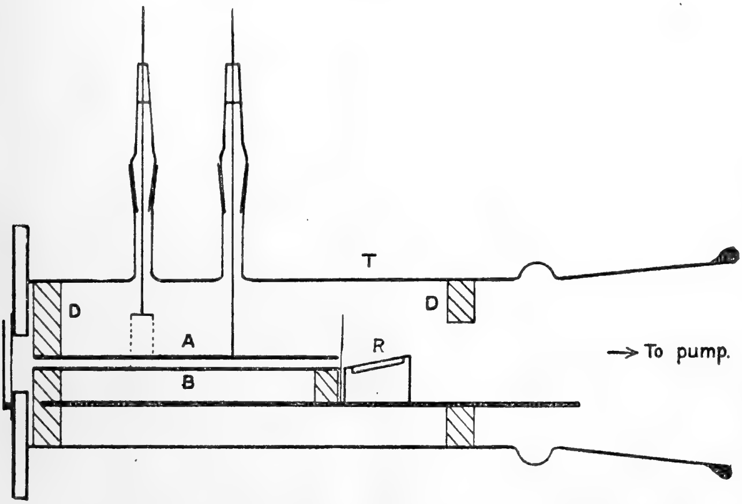
The determination of the deflexion of H atoms in an electric field was a much more difficult and lengthy process. The experimental arrangement finally adopted is shown in

---

* Rutherford and Robinson, Phil. Mag. xxviii. p. 552 (1914).

fig. 2. The α rays from a slanting source R passed through a film of paraffin wax about 30 μ thick placed at the end of two parallel insulated brass plates A and B, 6·02 cm. long, and 0·155 cm. apart. These were mounted on an ebonite frame DD with circular ebonite ends which slipped into a glass tube T. A brass plate with a slit 1 cm. long and 3 mm. wide covered with a thin silver plate of stopping power 6 cm. of air, was fixed at the end of the tube T. The zinc sulphide screen was mounted outside close to the silver plate. The electric connexions with the plates A and B were made through ground-glass stoppers shown in the figure.

Fig. 2.

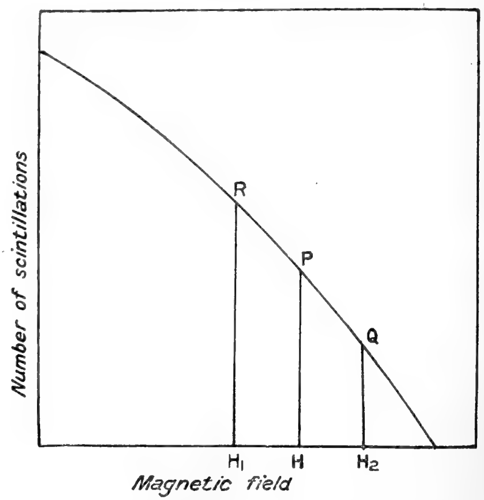


As radium C was employed as a source, it was necessary to arrange for rapid exhaustion of the apparatus to stand 5000 volts between the plates after a short interval. After preliminary evacuation by a Fleuss and Gaede mercury-pump a Langmuir pump was used, and the process was so rapid that the necessary vacuum was reached and held within two minutes of introducing the apparatus into the glass tube.

In order to deflect completely the H atoms in passing between the parallel plates, it was calculated that about 30,000 volts would be required. Apart from the difficulty of obtaining rapidly a vacuum sufficient to support and maintain such a voltage, a steady supply of not more than 7000 volts was available in the Laboratory. To overcome this difficulty, it was arranged to compare the deflexions of

H atoms due to a magnetic field with that due to a combined magnetic and electric field. The glass tube carrying the source and parallel plates was placed between the poles of a strong electromagnet, the plane of the plates being parallel to the direction of the magnetic field. The microscope was fixed in the centre line of the plates A and B so as to count the scintillations emerging from the plates, and the variation of the number with strength of the magnetic field was determined. The reduction of the number with increase of the magnetic field depended on two causes :—(1) the removal of H atoms bent to the sides of the plates, and (2) the bending of the H atoms emerging from the plates in the magnetic field in the short distance between the end of the plates and the zinc sulphide screen. These two effects were difficult to separate, but (2) was made as small as possible by reducing to a minimum the distance between the end of the plates and the screen. The relation between the number of scintillations and strength of field with no electric field acting, is shown diagrammatically in fig. 3. Suppose the magnetic field to be of a strength H corresponding to a point P on the curve. If a voltage be now applied so as to bend the H atoms in the same direction as the magnetic field, the number of scintillations on the screen decreases corresponding to a point Q on the curve of field H_2. On reversing the voltage the two fields oppose each other, and the number of scintillations correspond to the point R of field H_1. Suppose

Fig. 3.



for simplicity that the number of H atoms are counted as they emerge from the plates A and B. Let H be the steady magnetic field and X the electric field applied between A and B. Then it is clearly seen that, if u be the velocity of the H atom,

for assisting fields, $\quad Heu+Xe=H_2eu,$

for opposing fields, $\quad Heu-Xe=H_1eu.$

Subtracting $\quad 2eX=(H_2-H_1)eu,$

and $\quad u=\dfrac{2X}{H_2-H_1},$

so that the velocity of the H atoms can be determined directly. In practice, it was found that the curve PQR over the experimental range was nearly a straight line. The initial field H was varied in different experiments, but was usually about 4000 gauss. The steady voltage employed was about ±4500 volts. The ratio of the number of scintillations observed on reversal of the electric field varied in different experiments according to the magnetic field H and the voltage, but lay between 1·8 and 3 in the various experiments. Each experiment was complete in itself, for not only were the scintillations counted on reversal of the electric field, but also the number for two magnetic fields on either side of the fixed field which give nearly the same ratio of number of scintillations as that obtained by reversal of the electric field.

The paraffin film had a stopping power of 3·2 cm. of air, and the stopping power of the silver plate together with the aluminium screens was 11·4 cm. The range of H atom, which passed between the parallel plates and produced scintillations on the screen, thus lay between 28 − 3·2 = 24·8 cm. and 28 − 14·6 = 13·4 cm.

The corresponding velocities are $\cdot 96u_0$ and $\cdot 78u_0$ where u_0 is the maximum velocity of an H atom due to an α particle from radium C. As in the experiments on the magnetic deflexion the average velocity was found to be $\cdot 87u_0$. In calculating the relative effect due to a magnetic and electric field, a small correction is necessary to allow for the fact that the electric field was only effective the length of the plates, while the magnetic field acted on the H rays from the slit A to the zinc sulphide screen. Making this correction, estimated to be about 12 per cent., the deflexion due to 1000 volts between the plates was found to correspond in five different experiments to 235, 227, 260, 235, 221 gauss respectively, with an average value of 238 gauss.

Thus,

$$u=\frac{X}{H}=\frac{10^{11}}{\cdot 155}\times\frac{1}{238}=2\cdot 71\times 10^9 \text{ cm. per sec.}$$

The maximum value

$$u_0=\frac{1}{\cdot 87}\times 2\cdot 71\times 10^9=3\cdot 12\times 10^9 \text{ cm. per sec.}$$

The calculated value of $u_0=1\cdot 6\times 1\cdot 92\times 10^9$

$$=3\cdot 07\times 10^9 \text{ cm. per second.}$$

The experimental and calculated values agree well within the probable error of experiments. From the magnetic deflexion we found that

$$\frac{mu_0}{e}=3\cdot 15\times 10^5,$$

and from the electric deflexion

$$u_0=3\cdot 12\times 10^9,$$

consequently, $e/m=10^4$ e.m. units.

The value of e/m for the hydrogen atom in the electrolysis of water is 9570. The agreement is sufficiently close to show that the long-range scintillations produced by α particles in hydrogen are due to hydrogen atoms carrying a unit positive charge. The agreement between the calculated and observed velocities shows that, within the margin of experimental error, the conservation of momentum and energy hold for close collisions between the atomic nuclei and that there is no sensible loss of energy due to radiation.

Brightness of scintillations.

The maximum energy communicated to an H atom is ·64 of the energy of the colliding α particle. After passing through 12 cm. of air, for example, the energy of the H atom is reduced to ·44 of the energy of the particle. Supposing that the H atoms are produced by α particles of radium C of range 7 cm., the energy of the H atom after passing through 12 cm. of air, corresponds to an α particle of range about 2 cm. In practice, the brightness of the corresponding H scintillations is much less than is to be expected from its energy, and is not greater than that produced by an α particle of range 5 mm. This relative lack of brightness of H scintillations compared with α particles of corresponding energy holds for all velocities of the H atoms. Since we have seen that we can rely on the calculations of the energy of the H atom, it seems clear that the H atom is less effective in producing light on a zinc sulphide screen than an α particle of equal energy. This may be a consequence of the much weaker ionization along the path of the H atom, for since its range is four times that of the α particle and energy ·64, the energy spent per unit path is only 1/6 of that due to an α particle.

In this connexion it is of interest to note, that nitrogen atoms set in motion by α particles from radium C have a range in air of about 9 cm. Although the energy of the nitrogen atoms after traversing 7 cm. of air is less than that of the H atoms after traversing 12 cm., the nitrogen atom gives a much brighter scintillation than the H atom.

Probability distribution of H *scintillations.*

In the course of counting H scintillations, it was often noted that a number of the scintillations appeared as *instantaneous* doubles, *i. e.* two points of light of about equal brightness appeared in the field of view at the same instant. Some preliminary experiments seemed to show that the number of these doubles was greater than was to be expected from probability considerations. For example, in counting bright scintillations due to the active deposit of thorium, on an average, about 1·5 doubles per minute were counted for an average of 30 scintillations per minute, while for a similar number of H scintillations the number of doubles was about 5. If these "doubles" from hydrogen were instantaneous doubles, it was obviously a matter of great importance, possibly indicating the disruption by collision of one of the nuclei into two parts.

A large number of experiments were made to test this question, using both hydrogen and paraffin wax as a source of H atoms, but very similar results were obtained under all conditions of experiment. The most favourable theoretical conditions were chosen to increase the number of such doubles if they existed. For example, the H atoms were liberated in a thin film of paraffin covering an opening 1 mm. in diameter, placed near the zinc sulphide screen. The distance of the screen from the paraffin film and the nature of the absorber between was adjusted, so that even if two atoms were shot forward nearly in the same direction, the scattering would separate them on an average a convenient distance in the field of view of the microscope, which included a field of view 2 mm. diameter. No apparent advantage as regards the number of doubles was gained by this arrangement.

I was fortunate, in January of this year, to obtain for a short time the skilled assistance of Professor E. Marsden before his return to New Zealand. Systematic observations were undertaken to record electromagnetically the time of appearance of each scintillation on a chronograph tape while the number of doubles was separately recorded. Mr. Marsden and Mr. Kay counted alternately for a minute interval, and the counts of each observer were separately analysed by the former at leisure. On the probability theory, the number of intervals between t_1 and t_2 seconds is given by $Ne^{-\mu(t_1-t_2)}$ where $1/\mu$ is the average interval between each scintillation and N the total number of intervals. Marsden and Barratt* had previously verified the correctness of this theory, which shows that short intervals are more probable than long ones. If the average number of scintillations is 30 per minute, $\mu=1/2$, and if the eye fails to distinguish an interval less than 1/10 of a second, the average number of doubles to be expected is 1·5 per minute. In practice, under favourable conditions, the eye is just able to detect 1/10 second intervals for bright α ray scintillations.

Comparisons were made to test the probability distribution of α particles from polonium, whose range was adjusted to give a scintillation of about the same average brightness as the H atom.

The results of a typical series of counts both for α rays and H atoms are included in the following table. The theoretical and observed number of intervals $<1/10$, $<1/2$, and <1 second are given in the table below :—

Observer.	Average number scintillations per min.	Total number of scintillations.	Number of doubles.	Calculated number of intervals. <1/10 sec.	Observed number of intervals. <1/2 sec.	Calculated number of intervals. <1/2 sec.	Observed number of intervals. <1 sec.	Calculated number of intervals. <1 sec.
α particles from polonium.								
M ..	28·0	280	13	12·9	53	60	106	105
K ...	25·4	229	10	9·6	45	45	83	79
Hydrogen atoms.								
M ...	24·3	243	15	9·6	50·5	46	84	81
K ...	22·7	250	25	9·2	58·5	45	92	79
M ...	31·0	216	24	11	60·5	50	95	88
K ...	29·6	148	18	7	33	35	59	58

---

* Marsden and Barratt, Proc. Phys. Soc. xxiii. p. 367 (1911); xxiv. p. 50 (1913).

The calculated numbers are the sum of each observation worked out separately.

It will be seen that while there is a very satisfactory agreement between theory and experiment for the α rays from polonium, the agreement is not so good for the H atoms. In the case of the α rays, the number of doubles shows that the eye cannot distinguish an interval less than 1/10 second; while in the case of H atoms the number of doubles is nearly twice the theoretical number calculated on this power of distinction. Whether this difference is apparent or real is difficult to decide, for it must be remembered that counting such weak scintillations and at the same time distinguishing time intervals make a difficult task.

It is clear that under the experimental conditions, only a small fraction of the number of scintillations can be regarded as possible instantaneous doubles, and the effect is too small and uncertain to draw any very definite conclusions. It may be urged that a question of this kind could be settled more definitely by arranging that a small number of scintillations fell on the screen per minute when the probability of short intervals becomes very small. On the other hand, it takes a long time to count a sufficient number to compare theory with experiment, and it is very fatiguing to the eye and unreliable to count for long under such conditions.

I am much indebted to Professor Marsden for his valuable help in obtaining and analysing data for me on this important point.

LIII. *Collision of α Particles with Light Atoms. III. Nitrogen and Oxygen Atoms.* *By* Professor Sir E. RUTHERFORD, *F.R.S.**

BOHR † has worked out a general theory of the absorption of electrified atoms in passing through matter, and has verified his conclusions by consideration of the absorption of α particles. On this theory, Darwin ‡ has shown that the range of a swift hydrogen atom in hydrogen can be calculated, and the value so found is in good accord with experiment. It is not difficult to deduce by the same method that the range x in hydrogen of an electrified atom of charge e and mass m moving with a speed equal to an

---

* Communicated by the Author.

† Bohr, Phil. Mag. xxv. p. 10 (1913).

‡ Darwin, Phil. Mag. xxvii. p. 499 (1914).

α particle of range R in hydrogen is given by

$$x/\mathrm{R}=\frac{m}{\mathrm{M}}\cdot\frac{\mathrm{E}^2}{e^2}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

where M is the mass and E the charge on the α particle.

It is to be expected that this relation would hold approximately for the passage of electrified atoms through light substances like air and aluminium. Since $\mathrm{M}=4$ and $\mathrm{E}=2e$ where e is the unit charge, the range x of a particle carrying a single charge is obviously $x=m\mathrm{R}$. The velocity u acquired by an atom of mass m due to a close collision with an α particle of velocity v is given by

$$u=\frac{2\mathrm{M}}{\mathrm{M}+m}\cdot v\cos\theta,$$

where θ is the angle of deflexion of the atom after the collision. Assuming that the range of electrified atoms in general like the range of α particles varies as the cube of the velocity, the range x after collision of an atom carrying unit charge is given by

$$x=m\mathrm{R}\left(\frac{2\mathrm{M}}{\mathrm{M}+m}\right)^3\cos^3\theta.$$

Applying this result to H atoms, the maximum velocity should be $(8/5)^3\mathrm{R}=4{\cdot}1\mathrm{R}$, while the observed value is about 4R. As a further test of this relation, consider the range to be expected for the recoil atom of radium B of mass m resulting from the expulsion of an α particle of range 4·75 cm. from radium A. By the principle of momentum $\mathrm{M}v=mu$ and the velocity of recoil $u=\frac{\mathrm{M}}{m}v$ where $m=214$. Consequently the range in air x

$$=214\,.\left(\frac{4}{214}\right)^3\times 4{\cdot}75={\cdot}067 \text{ cm.}$$

The value found by Wertenstein * is ·12 mm., but, considering the very wide range of velocity, the agreement is fairly satisfactory. If it be assumed that the range is proportional to the power 2·85 instead of 3, this is a good agreement both for the hydrogen and recoil atoms.

If the atom after collision with an α particle carries a charge of two units, its range from (1) should be only about 1/4 of the same atom carrying a single charge. For

---

* Wertenstein, *C. R.* cl. p. 869 (1910); cli. p. 469 (1910).

example, in a collision of an α particle with the helium nucleus of equal mass, the range of the helium atom should be the same as the α particle before the collision if it carries two charges, but four times this range if it carries one charge.

We have collected in the following table data connected with the collision of α particles with the lighter atoms of matter. The maximum velocity, momentum, and energy of the atom after collision are given as fractions of that of the incident α particle. The range is calculated on the assumption that it is proportional to the power 2·9 of the velocity and that the atom carries unit charge.

For convenience, the data for hypothetical atoms of mass 2 and 3 times that of hydrogen are included.

Table I.

Element.	Atomic weight.	Ratio of velocity to that of *a* particle.	Ratio of momentum to that of *a* particle.	Ratio of energy to that of *a* particle.	Ratio of range to that of incident *a* particle.
Hydrogen	1	1·6	·4	·64	3·91
?	2	1·33	·66	·89	4·6
?	3	1·14	·85	·98	5·05
Helium	4	1·00	1·00	1·00	4·00
Lithium	7	·727	1·27	·925	2·78
Beryllium	9	·615	1·38	·85	2·20
Boron	11	·533	1·46	·78	1·77
Carbon	12	·500	1·50	·75	1·61
Nitrogen	14	·444	1·55	·69	1·33
Oxygen	16	·400	1·60	·64	1·12
Fluorine	19	·348	1·65	·575	·89
Neon	20	·333	1·67	·55	·82
Sodium	23	·296	1·70	·50	·67
Magnesium	24	·286	1·71	·49	·64
Aluminium	27	·258	1·74	·45	·53
Iron	56	·133	1·86	·25	·19
Silver	108	·071	1·92	·136	·05
Gold	197	·040	1·92	·079	·017

It is seen that, on the assumption of unit charge, all the atoms of atomic weight up to oxygen should be detected beyond the range of the α particle. Supposing that α particles of range 7 cm. are used, the maximum range to be expected for unit charge are for He 28·0, Li 19·6, Be 15·4, Bo 12·4, C 11·2, N 9·3, O 7·8 cm.

Some preliminary experiments have been made with helium, using the apparatus similar to that employed for hydrogen and described in paper I. but on a smaller scale.

The results show that if the collisions of α particles with helium atoms give any long-range scintillations of the order of 28 cm. range, their number is very small compared with that produced in hydrogen under similar conditions. We may consequently conclude that the swift helium atoms produced by collision carry a double charge like the α particle.

A few experiments have been made to test whether the atoms of lithium, boron, or beryllium have the range to be expected if they carry a single charge. The salts Li_2CO_3, B_2O_3, BeO were spread in a thin layer over the active source which was inclined at a small angle with the horizontal, and determinations made of the number of scintillations beyond the range of the α particle. The air was exhausted and the α particles absorbed in aluminium and silver foils. No certain evidence was obtained of the presence of appreciable numbers of scintillations at the ranges to be expected if the atoms carry a single charge. Experiments of this kind are not easy on account of the difficulty of obtaining thin uniform films of the salts or metal under examination, and of the necessity of getting rid of all traces of hydrogen and water vapour, which give rise to numerous H atoms. It is intended later to make a systematic examination of these elements to determine the range of the atoms produced by close collisions with α particles.

Experiments in Air and Oxygen.

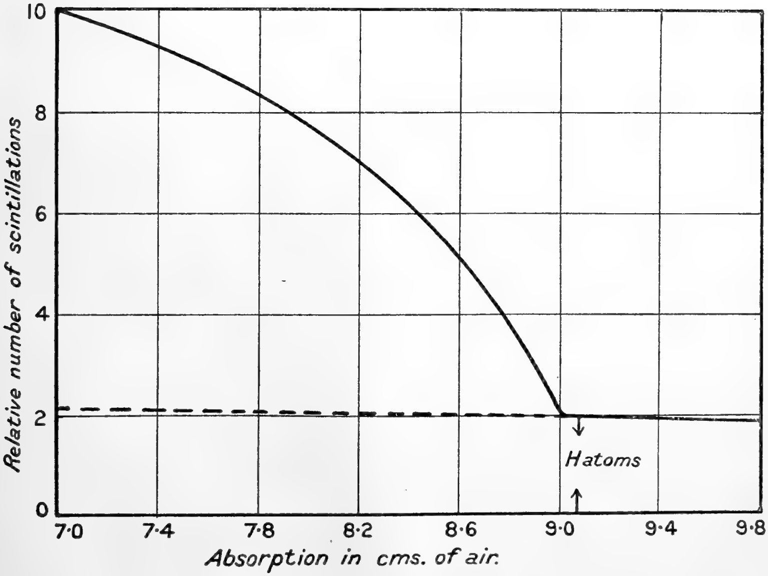
Experiments on the range of swift atoms become much easier and more certain when the elements are in the gaseous state, for there is then no uncertainty with regard to the uniformity of the absorbing column and usually no difficulty in ensuring absence of hydrogen and water vapour. Thin films of rolled metals like aluminium, silver, or gold are usually very irregular in thickness. This irregularity comes out very obviously when intense sources of radiation are employed under conditions when one in a million of the incident particles can be detected. It is not unusual in these cases to find that α particles can be detected at a distance 10 per cent. beyond the average range of the α particles as determined by ordinary methods. Mica films are very uniform and show none of these irregularities, but unfortunately mica contains both hydrogen and oxygen and gives rise to numerous H and O atoms beyond the range of the bombarding α particles.

We have seen that both N and O atoms carrying a single charge should be detected beyond the range of the α particles, and this is borne out by experiment. In the case of air, the active disk coated with radium C of γ-ray activity about 30 mg. Ra, was mounted with its plane vertical at a distance of about 7 cm. from a zinc sulphide screen in the open air. Both the source and screen were placed between the poles of a large electromagnet to deflect the β rays. The vertical convection currents due to the heated electromagnet prevented any contamination of the screen by active matter escaping from the source.

The end of the range of the α particles was sharply defined, but numerous bright scintillations were observed for distances nearly 2 cm. beyond the range of the α particles. There was a steady decrease both in number and brightness up to 9 cm. of air, and beyond that distance the small number of scintillations observed, due to H atoms from the source and from the water vapour in the air, fell off slowly.

The range of these atoms was best determined by placing the screen just outside the range of the α particles (7·1 cm. at 15° C.) and then adding thin screens of aluminium foil close to the zinc sulphide screen. The variation of number with absorption in terms of cms. of air is shown in fig. 1.

Fig. 1.



It will be seen that the scintillations fall off at first slowly with increase of absorption and more rapidly near the end of their range, which was equal to 9·0 cm. of air at normal pressure and 15° C. The scintillations, presumably due to swift N and O atoms, are bright and easily counted for a total absorption corresponding to about 7·5 cm. of air. At this stage they appear equal in brightness to those given by an α particle of range about 1 cm.

In other experiments with air, nitrogen, and oxygen, and carbon dioxide, the screen and source were placed in a rectangular box and a slow current of the dried gas passed through during the experiment. This prevented contamination of the screen by diffusion of active matter from the source, and the range was determined by altering the distance between source and screen.

The scintillations in pure oxygen and carbon dioxide were about the same brightness for corresponding ranges, and had nearly the same equivalent ranges in air as those due presumably to N atoms from the air.

This was rather surprising, as we should expect the O atoms to have considerably less range than the lighter N atom. The calculated ranges (see table above) are 7·8 and 9·3 cm. respectively. This suggested the possibility that the scintillations might be due not to N or O atoms but to actual α particles of range 9 cm. which were expelled from the radioactive source. If this were the case, the total range of the α particles should not be altered by placing an absorbing screen of aluminium or gold of known stopping power close to the source in the path of the α rays. On the other hand, if the scintillations were due to swift N or O atoms from the air, the range should be diminished. For example, if a screen of stopping power equal to 3·5 cm. of air were placed in the path of the α rays of range 7·0 cm., the resulting range of the α particles acting on the gas is 3·5 cm., and the total range of the N or O atoms measured from the source should be $3{\cdot}5+\frac{9}{7}\times3{\cdot}5=8{\cdot}0$ cm. instead of 9·0 cm. Experiments of this kind were made with an aluminium and a gold screen of stopping powers 3·7 and 4·2 cm. respectively, but were not altogether satisfactory on account of the inequalities of the films already referred to. They showed, however, that no appreciable number of scintillations could be detected beyond 8 cm. The results indicated that the scintillations were due to atoms of N and O and not to α particles from the source. This was further confirmed by experiments with mica screens of stopping power 7·0 cm. The number of bright scintillations which resembled α particles were less than half the number observed in air or oxygen gas under similar conditions, but the presence of numerous H atoms from the mica interfered with an accurate determination. Since mica contains oxygen as well as hydrogen we should obtain swift O atoms, and the number of scintillations observed was about that to be expected from the amount of oxygen present, but was less than the number observed in air. There appears to be no doubt that the scintillations observed in air between the ranges 7 and 9 cm. arise from collision of α particles with N and O atoms. The observation that the range of the swift atoms, produced by α particles in their passage through carbon dioxide, is equivalent to the range of O atoms, indicates that there are no carbon atoms carrying a single charge, for in that case bright scintillations should have been observed for ranges up to 12 cm. of air (see Table I.).

It will be remembered that in the beautiful photographs of Mr. C. T. R. Wilson * showing the trails of α particles, an example is given where the α particle in air shows a sudden deflexion of 43°, and there is clear evidence of a well-marked spur presumably showing the trail of the N or O recoil atom. It is of interest to compare the length of this spur with the range to be expected for a collision with an O atom. If ϕ be angle of deflexion of the α particle and θ the deflexion of the O atom,

$$\tan\phi = \frac{m \sin 2\theta}{M - m\cos 2\theta},$$

where m=mass of O atom and M=mass of α particle.

Putting $M=4$, $m=16$, $\phi=43°$, then $\theta=63°{\cdot}55$.

If v = velocity of the α particle before the collision, the velocity of the O atom

$$= \frac{2M}{M+m} \cdot v\cos\theta = {\cdot}178\,v,$$

while the velocity of the α particle after the collision is $\cdot 934\,v$.

$$\frac{\text{Range of recoil O atom}}{\text{Range of } \alpha \text{ particle after collision}} = 16 \times \left(\frac{\cdot 178}{\cdot 934}\right)^{2\cdot 9} = \cdot 13.$$

This is based on the calculation that the maximum range of O atoms due to α particles from radium C is 7·8 cm., while the observed range is 9·0 cm. Making this correction, the value ·13 becomes ·15.

It is possible to compare only roughly the ranges of the α particle and recoil atom in the photograph, but the results are in fair accord with the calculation.

---

* C. T. R. Wilson, Proc. Roy. Soc. A. lxxxvii. p. 277 (1912).

In the same photograph the α particle shows another sudden bend of 10°·5. In this case, the range of the recoil O atom should be only about 1/800 of the α particle and could not be distinguished on the photograph.

Number of N *atoms.*

In a previous paper we have calculated the number and distribution of H atoms produced by α particles on the assumption that the nuclei may be regarded as point centres of force repelling according to the law of the inverse square. When these calculations are applied to the collision of α particles with nitrogen or oxygen nuclei, the distribution with velocity of the N and O atoms is very similar to that for H atoms. We should consequently expect on the simple theory that the number of N and O atoms should fall off very rapidly between 7 and 9 cm., and that the number of short-range atoms should greatly preponderate. Quite the contrary is observed in the experiments (fig. 1), where it is seen that the number of scintillations fall off quite gradually with range.

There seems to be no doubt that the effects produced by the collision of α particles with N and O atoms are very similar to those observed in hydrogen. The observations only receive an explanation on the assumption that the N and O atoms like the H atoms are thrown forward mainly in the direction of the α particles and, at any rate for swift α particles, the velocities of the recoil atoms are nearly uniform for a given velocity of the α particles. It should be pointed out that the experiments with air and oxygen differ in one respect from those with hydrogen. In the case of air the α particles are completely absorbed in the column of gas, while in the case of hydrogen the stopping power was usually equivalent to less than 1 cm. of air. Consequently in the air experiments, the scintillations observed are due to N and O atoms which are produced by α particles of all ranges between 7 and 0 cm., and thus have a wide range of velocities.

A number of experiments were made by the use of absorbing-screens of aluminium and gold in order to determine the number of N and O atoms produced by α particles of different range. The result as a whole showed that, for example, the number produced in the first 3·5 cm. of the range of the α particle from radium C was greater than in the last 3·5 cm., but accurate deductions were vitiated by the lack of uniformity in thickness of the metal films.

A number of concordant measurements were made to fix the total number of scintillations observed in air for a known activity of the source. The number of scintillations per minute due to N and O atoms at a distance of 7·5 cm. in air at 15° C. was 2·2 on an area of the zinc sulphide screen equal to 3·14 sq. mm. Referring to curve 1, it is seen that the number corresponding to an absorption of 7 cm. should be 2·6 and the number for 8 cm. absorption 1·5.

All those atoms of range equal to or greater than 8 cm. must be produced in the first 3·5 cm. of the path of the α rays; for the O atoms produced by α particles of range 3·5 cm. cannot travel further than 8 cm. from the source, and probably only a small fraction reach this distance owing to scattering and straggling.

For the purpose of calculation, suppose that the production of swift atoms is uniform over the first 3·5 cm. of the range and that ρ is the ratio of the number of swift atoms produced per cm. of path to the number of α particles passing through the gas.

The number Q of recoil atoms falling per second on the screen of area A after passing through l cm. of gas is given by

$$Q=\rho \cdot \frac{AlN}{4\pi r^2},$$

where N is the total number of α particles emitted by the source per second ($3·7 \times 10^7$ per second per mg. Ra of activity) and r is the distance of the source from the screen.

Putting

$$Q=\frac{1·5}{60}, \quad A=·0314 \text{ sq. cm.}, \quad l=3·5 \text{ cm.}, \quad r=7·5 \text{ cm},$$

then the average value of $\rho=4·3/10^6$.

When we take into consideration the well-known way in which the α particles fall off near the end of the range in consequence of scattering, it is obvious that the true value of ρ is considerably greater than the above and is probably about $7/10^6$.

In the experiments with hydrogen, it was shown that $\rho = 1/10^5$ about—a value not very different from that observed in these experiments. We may consequently conclude that about the same number of swift atoms are produced per centimetre of path by the passage of α particles through air, oxygen, and hydrogen. As in the case of hydrogen, it can be shown that all α particles, shot within a perpendicular distance $p=2·4 \times 10^{-13}$ cm. of the atomic nucleus, give rise to swift atoms of nitrogen and oxygen.

It is clear from these results that the nuclei under consideration can no longer be regarded as point charges for distances of approach of the order of the diameter of the electron. As far as experiment has so far gone, it is difficult to fix with certainty the distance at which the forces between the nuclei become abnormal, but a rough estimate can be made. Regarding the nuclei as point charges, the closest distance of approach in a collision is $1{\cdot}9 \times 10^{-13}$ cm. in the case of a hydrogen atom and $3{\cdot}8 \times 10^{-13}$ cm. in the case of the oxygen atom. Taking into account the close similarity of the effects produced by α particles in hydrogen and oxygen and the greater repulsive forces between the nuclei in the latter case, it seems probable that the abnormal forces in the case of oxygen manifest themselves at about twice the distance observed in the case of hydrogen. This would mean that the rapid variation in the magnitude and direction of the forces between the nuclei which lead to the recoil of swift atoms mainly in the direction of the α particle should begin at a distance about 7×10^{-13} cm. Such a result is to be anticipated on general grounds, for presumably the oxygen nucleus is more complex and has larger dimensions than that of helium.

In a paper published three years ago Mr. A. B. Wood and the writer * described experiments which showed that the active deposit of thorium gave rise to a few α particles of range 11·3 cm. in addition to the main group of ranges 5·0 and 8·6 cm. In these experiments, the α rays of range 8·6 cm. were absorbed in mica. In the light of the present experiments, the oxygen present in the mica should give rise to scintillations like α particles of range

$$8{\cdot}6 \times \frac{9{\cdot}0}{7{\cdot}0} = 11{\cdot}1 \text{ cm.}$$

This range is nearly the same as that observed in the thorium experiment, and raises the question whether these long range α particles are not in reality due to collision of α particles with the oxygen atoms in the mica. A fraction of the scintillations must undoubtedly have been due to this cause, but on the other hand the number of scintillations observed, about 1/10000 of the number of α particles, is considerably greater than is to be expected from the experiments with radium C. Further experiments to clear up this important point have been undertaken by Professor Marsden in New Zealand.

---

* Rutherford and Wood, Phil. Mag. xxxi. p. 379 (1916).

LIV. *Collision of α Particles with Light Atoms. IV. An Anomalous Effect in Nitrogen.* *By* Professor Sir E. Rutherford, *F.R.S.*[*]

IT has been shown in paper I. that a metal source, coated with a deposit of radium C, always gives rise to a number of scintillations on a zinc sulphide screen far beyond the range of the α particles. The swift atoms causing these scintillations carry a positive charge and are deflected by a magnetic field, and have about the same range and energy as the swift H atoms produced by the passage of α particles through hydrogen. These "natural" scintillations are believed to be due mainly to swift H atoms from the radioactive source, but it is difficult to decide whether they are expelled from the radioactive source itself or are due to the action of α particles on occluded hydrogen.

The apparatus employed to study these "natural" scintillations is the same as that described in paper I. The intense source of radium C was placed inside a metal box about 3 cm. from the end, and an opening in the end of the box was covered with a silver plate of stopping power equal to about 6 cm. of air. The zinc sulphide screen was mounted outside, about 1 mm. distant from the silver plate, to admit of the introduction of absorbing foils between them. The whole apparatus was placed in a strong magnetic field to deflect the β rays. The variation in the number of these "natural" scintillations with absorption in terms of cms. of air is shown in fig. 1, curve A. In this case, the air in the box was exhausted and absorbing foils of aluminium were used. When dried oxygen or carbon dioxide was admitted into the vessel, the number of scintillations diminished to about the amount to be expected from the stopping power of the column of gas.

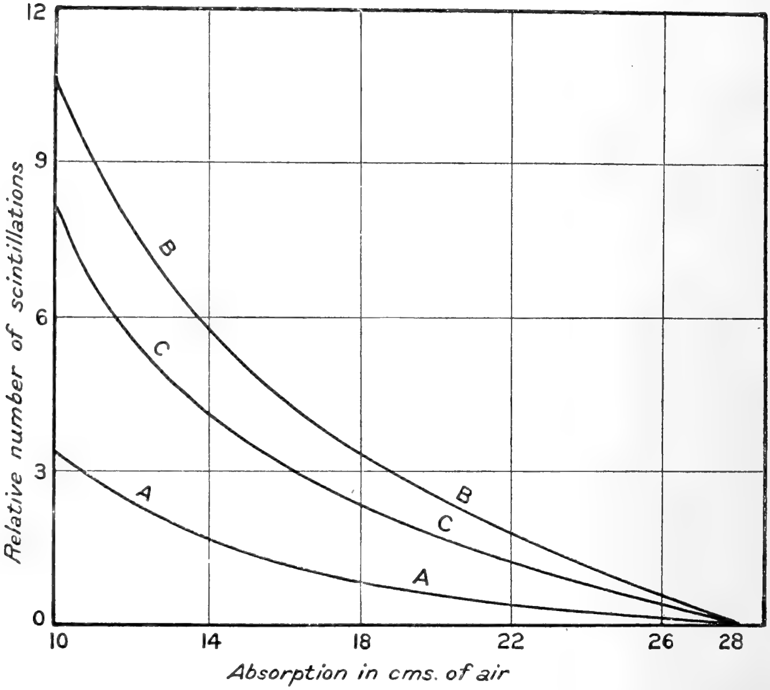
A surprising effect was noticed, however, when dried air was introduced. Instead of diminishing, the number of scintillations was increased, and for an absorption corresponding to about 19 cm. of air the number was about twice that observed when the air was exhausted. It was clear from this experiment that the α particles in their passage through air gave rise to long-range scintillations which appeared to the eye to be about equal in brightness to H scintillations. A systematic series of observations was undertaken to account for the origin of these scintillations. In the first place we have seen that the passage of α particles through nitrogen and

---

[*] Communicated by the Author.

oxygen gives rise to numerous bright scintillations which have a range of about 9 cm. in air. These scintillations have about the range to be expected if they are due to swift N or O atoms, carrying unit charge, produced by collision with α particles.

Fig. 1.



All experiments have consequently been made with an absorption greater than 9 cm. of air, so that these atoms are completely stopped before reaching the zinc sulphide screen.

It was found that these long-range scintillations could not be due to the presence of water vapour in the air; for the number was only slightly reduced by thoroughly drying the air. This is to be expected, since on the average the number of the additional scintillations due to air was equivalent to the number of H atoms produced by the mixture of hydrogen at 6 cm. pressure with oxygen. Since on the average the vapour pressure of water in air was not more than 1 cm., the effects of complete drying would not reduce the number by more than one sixth. Even when oxygen and carbon dioxide saturated with water vapour at 20° C.

were introduced in place of dry air, the number of scintillations was much less than with dry air.

It is well known that the amount of hydrogen or gases containing hydrogen is normally very small in atmospheric air. No difference was observed whether the air was taken directly from the room or from outside the laboratory or was stored for some days over water.

There was the possibility that the effect in air might be due to liberation of H atoms from the dust nuclei in the air. No appreciable difference, however, was observed when the dried air was filtered through long plugs of cotton-wool, or by storage over water for some days to remove dust nuclei.

Since the anomalous effect was observed in air, but not in oxygen, or carbon dioxide, it must be due either to nitrogen or to one of the other gases present in atmospheric air. The latter possibility was excluded by comparing the effects produced in air and in chemically prepared nitrogen. The nitrogen was obtained by the well-known method of adding ammonium chloride to sodium nitrite, and stored over water. It was carefully dried before admission to the apparatus. With pure nitrogen, the number of long-range scintillations under similar conditions was greater than in air. As a result of careful experiments, the ratio was found to be 1·25, the value to be expected if the scintillations are due to nitrogen.

The results so far obtained show that the long-range scintillations obtained from air must be ascribed to nitrogen, but it is important, in addition, to show that they are due to collision of α particles with atoms of nitrogen through the volume of the gas. In the first place, it was found that the number of the scintillations varied with the pressure of the air in the way to be expected if they resulted from collision of α particles along the column of gas. In addition, when an absorbing screen of gold or aluminium was placed close to the source, the range of the scintillations was found to be reduced by the amount to be expected if the range of the expelled atom was proportional to the range of the colliding α particles. These results show that the scintillations arise from the volume of the gas and are not due to some surface effect in the radioactive source.

In fig. 1 curve A the results of a typical experiment are given showing the variation in the number of natural scintillations with the amount of absorbing matter in their path measured in terms of centimetres of air for α particles. In these experiments carbon dioxide was introduced at a pressure calculated to give the same absorption of the α rays as ordinary air. In curve B the corresponding curve is given when air at N.T.P. is introduced in place of carbon dioxide. The difference curve C shows the corresponding variation of the number of scintillations arising from the nitrogen in the air. It was generally observed that the ratio of the nitrogen effect to the natural effect was somewhat greater for 19 cm. than for 12 cm. absorption.

In order to estimate the magnitude of the effect, the space between the source and screen was filled with carbon dioxide at diminished pressure and a known pressure of hydrogen was added. The pressure of the carbon dioxide and of hydrogen were adjusted so that the total absorption of α particles in the mixed gas should be equal to that of the air. In this way it was found that the curve of absorption of H atoms produced under these conditions was somewhat steeper than curve C of fig. 1. As a consequence, the amount of hydrogen mixed with carbon dioxide required to produce a number of scintillations equal to that of air, increased with the increase of absorption. For example, the effect in air was equal to about 4 cm. of hydrogen at 12 cm. absorption, and about 8 cm. at 19 cm. absorption. For a mean value of the absorption, the effect was equal to about 6 cm. of hydrogen. This increased absorption of H atoms under similar conditions indicated either that (1) the swift atoms from air had a somewhat greater range than the H atoms, or (2) that the atoms from air were projected more in the line of flight of the α particles.

While the maximum range of the scintillations from air using radium C as a source of α rays appeared to be about the same, viz. 28 cm., as for H atoms produced from hydrogen, it was difficult to fix the end of the range with certainty on account of the smallness of the number and the weakness of the scintillations. Some special experiments were made to test whether, under favourable conditions, any scintillations due to nitrogen could be observed beyond 28 cm. of air absorption. For this purpose a strong source (about 60 mg. Ra activity) was brought within 2·5 cm. of the zinc sulphide screen, the space between containing dry air. On still further reducing the distance, the screen became too bright to detect very feeble scintillations. No certain evidence of scintillations was found beyond a range of 28 cm. It would therefore appear that (2) above is the more probable explanation.

In a previous paper (III.) we have seen that the number of swift atoms of nitrogen or oxygen produced per unit path by collision with α particles is about the same as the corresponding number of H atoms in hydrogen. Since the number of long-range scintillations in air is equivalent to that produced under similar conditions in a column of hydrogen at 6 cm.

pressure, we may consequently conclude that only one long-range atom is produced for every 12 close collisions giving rise to a swift nitrogen atom of maximum range 9 cm

It is of interest to give data showing the number of long-range scintillations produced in nitrogen at atmospheric pressure under definite conditions. For a column of nitrogen 3·3 cm. long, and for a total absorption of 19 cm. of air from the source, the number due to nitrogen per milligram of activity is ·6 per minute on a screen of 3·14 sq. mm. area.

Both as regards range and brightness of scintillations, the long-range atoms from nitrogen closely resemble H atoms, and in all probability are hydrogen atoms. In order, however, to settle this important point definitely, it is necessary to determine the deflexion of these atoms in a magnetic field. Some preliminary experiments have been made by a method similar to that employed in measuring the velocity of the H atom (see paper II.). The main difficulty is to obtain a sufficiently large deflexion of the stream of atoms and yet have a sufficient number of scintillations per minute for counting. The *α* rays from a strong source passed through dry air between two parallel horizontal plates 3 cm. long and 1·6 mm. apart, and the number of scintillations on the screen placed near the end of the plates was observed for different strengths of the magnetic field. Under these conditions, when the scintillations arise from the whole length of the column of air between the plates, the strongest magnetic field available reduced the number of scintillations by only 30 per cent. When the air was replaced by a mixture of carbon dioxide and hydrogen of the same stopping power for *α* rays, about an equal reduction was noted. As far as the experiment goes, this is an indication that the scintillations are due to H atoms; but the actual number of scintillations and the amount of reduction was too small to place much reliance on the result. In order to settle this question definitely, it will probably prove necessary to employ a solid nitrogen compound, free from hydrogen, as a source, and to use much stronger sources of *α* rays. In such experiments, it will be of importance to discriminate between the deflexions due to H atoms and possible atoms of atomic weight 2. From the calculations given in paper III., it is seen that a collision of an *α* particle with a free atom of mass 2 should give rise to an atom of range about 32 cm. in air, and of initial energy about ·89 of that of the H atom produced under similar conditions. The deflexion of the pencil of these rays in a magnetic field should be about ·6 of that shown by a corresponding pencil of H atoms.

Discussion of results.

From the results so far obtained it is difficult to avoid the conclusion that the long-range atoms arising from collision of α particles with nitrogen are not nitrogen atoms but probably atoms of hydrogen, or atoms of mass 2. If this be the case, we must conclude that the nitrogen atom is disintegrated under the intense forces developed in a close collision with a swift α particle, and that the hydrogen atom which is liberated formed a constituent part of the nitrogen nucleus. We have drawn attention in paper III. to the rather surprising observation that the range of the nitrogen atoms in air is about the same as the oxygen atoms, although we should expect a difference of about 19 per cent. If in collisions which give rise to swift nitrogen atoms, the hydrogen is at the same time disrupted, such a difference might be accounted for, for the energy is then shared between two systems.

It is of interest to note, that while the majority of the light atoms, as is well known, have atomic weights represented by $4n$ or $4n+3$ where n is a whole number, nitrogen is the only atom which is expressed by $4n+2$. We should anticipate from radioactive data that the nitrogen nucleus consists of three helium nuclei each of atomic mass 4 and either two hydrogen nuclei or one of mass 2. If the H nuclei were outriders of the main system of mass 12, the number of close collisions with the bound H nuclei would be less than if the latter were free, for the α particle in a collision comes under the combined field of the H nucleus and of the central mass. Under such conditions, it is to be expected that the α particle would only occasionally approach close enough to the H nucleus to give it the maximum velocity, although in many cases it may give it sufficient energy to break its bond with the central mass. Such a point of view would explain why the number of swift H atoms from nitrogen is less than the corresponding number in free hydrogen and less also than the number of swift nitrogen atoms. The general results indicate that the H nuclei, which are released, are distant about twice the diameter of the electron (7×10^{-13} cm.) from the centre of the main atom. Without a knowledge of the laws of force at such small distances, it is difficult to estimate the energy required to free the H nucleus or to calculate the maximum velocity that can be given to the escaping H atom. It is not to be expected, *a priori*, that the velocity or range of the H atom released from the nitrogen atom should be identical with that due to a collision in free hydrogen.

Taking into account the great energy of motion of the α particle expelled from radium C, the close collision of such an α particle with a light atom seems to be the most likely agency to promote the disruption of the latter ; for the forces on the nuclei arising from such collisions appear to be greater than can be produced by any other agency at present available. Considering the enormous intensity of the forces brought into play, it is not so much a matter of surprise that the nitrogen atom should suffer disintegration as that the α particle itself escapes disruption into its constituents. The results as a whole suggest that, if α particles—or similar projectiles—of still greater energy were available for experiment, we might expect to break down the nucleus structure of many of the lighter atoms.

I desire to express my thanks to Mr. William Kay for his invaluable assistance in counting scintillations.

University of Manchester,
April 1919.

LV. *The Rotational Oscillation of a Cylinder in a Viscous Liquid.* *By* D. Coster[*].

THIS problem has been dealt with by Stokes[†] for the purpose of numerical calculations to determine the viscosity of the air. Still, I think it interesting to publish another solution of the problem which gives more opportunity of discussing the different cases, though it is perhaps less adapted to precise calculations.

The method to be followed will be in the main the same as that used by Prof. Verschaffelt in the analogous case of the sphere[‡]. We consider the rotational swings about its axis of an infinitely long cylinder which executes a forced vibration. Our object will be to ascertain the motion in the liquid which will establish itself after an infinite time (in practice after a relatively short time[§]) in order to compute the frictional moment of forces exerted on the cylinder by the liquid. The calculations will be referred to a height of 1 cm.

The motion of the cylinder may be represented by $\alpha = a\cos pt$ where α is the angle of rotation. An obvious assumption to be made is that the liquid will be set in motion in coaxial cylindrical shells each of which will execute its oscillations as a whole. On this assumption it is not difficult

---

[*] Communicated by Prof. G. N. Watson, M.A., D.Sc. First published in the Amsterdam Proc. May 1918, vol. xxi. p. 193.

[†] Math. Papers, vol. v. p. 207.

[‡] *Cf.* Amst. Proc. vol. xviii. p. 840; Comm. Leiden, 148 C.

[§] *Cf.* Comm. Leiden, p. 22, footnote.

to establish the differential equation of the motion of the liquid.

Let ρ be the density,

μ the viscosity of the liquid.

ω the angular velocity of a cylindrical shell.

r the radius of the shell.

The frictional force per unit area of one of the shells will then be $F=r\mu\frac{\partial\omega}{\partial r}$ and the frictional couple on a cylindrical surface of radius r will be $2\pi r^3\mu\frac{\partial\omega}{\partial r}$.

Taking a shell of thickness dr its equation of motion will be

$$2\pi r^3 dr\rho\frac{\partial\omega}{\partial t}=\frac{\partial}{\partial r}\left\{2\pi r^3\mu\frac{\partial\omega}{\partial r}\right\}dr,$$

which reduces to

$$\frac{\rho}{\mu}\frac{\partial\omega}{\partial t}=\frac{\partial^2\omega}{\partial r^2}+\frac{3}{r}\frac{\partial\omega}{\partial r}. \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

For an infinitely long time of vibration, *i. e.* for uniform rotation, (1) simplifies to

$$0=\frac{d^2\omega}{dr^2}+\frac{3}{r}\frac{d\omega}{dr}. \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

The solution of (2) is $\omega=\frac{c_1}{r_2}+c_2$, c_1 and c_2 being constants of integration. If the solid cylinder (radius R) rotates with uniform speed Ω in an infinite liquid, the result will be $\omega=\frac{R^2\Omega}{r^2}$, giving for the frictional couple, as is well known, the expression

$$-4\pi\mu R^2\Omega. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2')$$

In order to arrive at a possible solution of (1) we have to make our assumption regarding the motion of the liquid a little more definite by assuming that the angular displacement of each shell is represented by

$$a_r=f(r)\cos\{pt-\phi(r)\}. \quad . \quad . \quad . \quad . \quad . \quad (3)$$

We may also consider (3) as the real part of the complex function ue^{ipt}, where u is a function of r the modulus of which gives the amplitude of the oscillation and the argument the phase-shift $\phi(r)$. Remembering that $\omega=\frac{\partial\alpha}{\partial t}$ equation (1) may be reduced to

$$\frac{d^2u}{dr^2}+\frac{3}{r}\frac{du}{dr}-\frac{ip\rho u}{\mu}=0. \quad . \quad . \quad . \quad . \quad . \quad (4)$$

Equation (4) is closely related to the differential equation of the cylindrical functions. Indeed by the substitution $y=zv$, Bessel's equation of the first order

$$\frac{d^2y}{dz^2}+\frac{1}{z}\frac{dy}{dz}+\left(1-\frac{1}{z^2}\right)y=0$$

changes to

$$\frac{d^2v}{dz^2}+\frac{3}{z}\frac{dv}{dz}+v=0.$$

It follows that the general solution of equation (4) is

$$u=\frac{1}{r}\{\mathrm{A}\mathrm{J}_1(cr)+\mathrm{B}\mathrm{N}_1(cr)\}, \quad . \quad . \quad . \quad . \quad (5)$$

where $c=\sqrt{\frac{-i\rho p}{\mu}}$, A and B being complex constants of integration. J_1 is the cylindrical function of the first kind and first order, N_1 that of the second kind and first order *.

As regards c an agreement must be come to. We shall choose the root with the negative imaginary part, i. e., $c=ke^{-\frac{i\pi}{4}}$, where $k=|c|=\left|\sqrt{\frac{\rho p}{\mu}}\right|$.

As a first boundary-condition we have $\lim\limits_{r=\infty} ra_r=0$. As this relation must hold for all values of t, it follows that $\lim\limits_{r=\infty} ru=0$.

The cylindrical functions with complex argument all become infinite at infinity with the exception of the so-called functions of the third kind, or Hankel's functions $\mathrm{H}_p^{(1)}$ and $\mathrm{H}_p^{(2)}$. Of these $\mathrm{H}_p^{(1)}$ disappears at infinity in the positive imaginary half-plane and on the contrary becomes infinite in the negative half, whereas the opposite is true for $\mathrm{H}_p^{(2)}$. By our choice of c in the negative imaginary half we are led to the function $\mathrm{H}_1^{(2)}$. For the constants of integration in equation (5) this gives the relation † $\mathrm{B}=-i\mathrm{A}$, so that (5) becomes

$$u=\frac{\mathrm{A}}{r}\mathrm{H}_1^{(2)}(cr). \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (6)$$

For the determination of A we have to use the second

---

* *Cf.* Gray and Mathews, 'Bessel Functions.' Nielsen, *Cylinderfunktionen*; Jahnke und Emde, *Funktiontafeln.* Instead of N, Gray and Mathews use the symbol Y.

† Between J, N, and H a linear relation holds. *Cf.* Jahnke u. Emde, p. 95.

boundary-condition $a_R = a\cos pt$, R being the radius of the cylinder. We therefore assume that there is no slipping along the wall.

Hence
$$A = \frac{aR}{H_1^{(2)}(cR)},$$

so that
$$a_r = \mathbf{R}\,\frac{aR}{H_1^{(2)}(cR)}\,\frac{H_1^{(2)}(cr)}{r}\,e^{ipt}. \quad . \quad . \quad . \quad . \quad . \qquad (7)$$

The symbol **R** means that the real part has to be taken of the function which stands after it.

If we had chosen for c the root with the positive imaginary part, we should have had to utilize the function $H_1^{(1)}$. It is quite easy to verify that this would not have made any essential change in the solution (7).

For large values of x (real and positive) $H_1^{(2)}(x\sqrt{-i})$ approaches asymptotically to
$$-\frac{e^{-\frac{x}{\sqrt{2}}}}{\sqrt{\frac{1}{2}\pi x}}\,e^{-i\left(\frac{x}{\sqrt{2}}-\frac{\pi}{8}\right)};$$

therefore for $(k\mathrm{R})$ sufficiently large:
$$a_r \approx -\frac{aR}{|H_1^{(2)}(cR)|}\,\frac{e^{-\frac{kr}{\sqrt{2}}}}{\sqrt{\frac{1}{2}\pi k}\,r^{1\frac{1}{2}}}\cos\left(pt-\frac{kx}{\sqrt{2}}+\frac{\pi}{8}-\phi\right), . \qquad (8)$$
where $\phi = \arg H_1^{(2)}(cR)$.

From (8) it appears that damped waves are propagated from the cylinder to infinity, the velocity of propagation being
$$v = \frac{p}{k/\sqrt{2}} = \frac{p\sqrt{2}}{k} = \sqrt{\frac{2p\mu}{\rho}},$$

and the wave-length
$$\lambda = vT = \frac{2\pi v}{p} = \frac{2\pi\sqrt{2}}{k} = 2\pi\sqrt{\frac{2\mu}{\rho p}}. \quad . \quad . \quad . \quad . \qquad (8')$$

The frictional moment on the wall of the vibrating cylinder is $2\pi\mu R^3\left[\frac{\partial\omega}{\partial r}\right]_R$ where $\omega = \frac{\partial\alpha}{\partial t}$. First we determine $\left[\frac{\partial\alpha_r}{\partial r}\right]_R$ from (7):
$$\left[\frac{\partial\alpha_r}{\partial r}\right]_R = \mathbf{R}\left[-\frac{a}{R}e^{ipt} + ac\,\frac{H_1^{(2)\prime}(cR)}{H_1^{(2)}(cR)}\,e^{ipt}\right]. \quad . \qquad (9)$$

For the reduction of the second part on the right-hand side of (9) we make use of the well-known recurrence-formula of the cylindrical functions:

$$\frac{d\mathrm{H}_1^{(2)}(z)}{dz} = \mathrm{H}_0^{(2)}z - \frac{1}{z}\,\mathrm{H}_1^{(2)}(z).$$

By its application (9) assumes the form:

$$\left[\frac{\partial \alpha_r}{\partial r}\right]_{\mathrm{R}} = \mathbf{R}\left[-\frac{2a}{\mathrm{R}}\,e^{ipt} + ac\,\frac{\mathrm{H}_0^{(2)}(c\mathrm{R})}{\mathrm{H}_1^{(2)}(c\mathrm{R})}\,e^{ipt}\right] \quad . \quad (10)$$

giving for the frictional couple:

$$\mathrm{K} = 2\pi\mu\mathrm{R}^3\left[\frac{\partial \omega}{\partial r}\right]_{\mathrm{R}} = -4\pi\mu\mathrm{R}^2\omega + \mathbf{R}\,\frac{d}{dt}\left[2\pi\mu\mathrm{R}^3 ac\,\frac{\mathrm{H}_0^{(2)}(c\mathrm{R})}{\mathrm{H}_1^{(2)}(c\mathrm{R})}\,e^{ipt}\right]. \quad (11)$$

For an infinite time of swing, i.e., $p=0$, but with a rotational velocity differing from 0, $|c| = \sqrt{\frac{\rho p}{\mu}}$ becomes 0. In that case the second term on the right of (11) disappears on two grounds: first, because $c=0$, secondly, $\lim\limits_{c\mathrm{R}=0} \frac{\mathrm{H}_0^{(2)}(c\mathrm{R})}{\mathrm{H}_1^{(2)}(c\mathrm{R})} = 0$; only the first term then remains, which agrees with (2′).

Moreover*:

$$\lim_{c\mathrm{R}=\infty} \frac{\mathrm{H}_0^{(2)}(c\mathrm{R})}{\mathrm{H}_1^{(2)}(c\mathrm{R})} = -i.$$

It appears from the accompanying graphs † of the modulus and argument of $\frac{\mathrm{H}_0^{(2)}(c\mathrm{R})}{\mathrm{H}_1^{(2)}(c\mathrm{R})}$ that this limiting value is practically reached at

$$|c\mathrm{R}| = k.\mathrm{R} = 10, \quad . \quad . \quad . \quad . \quad . \quad . \quad (12)$$

where

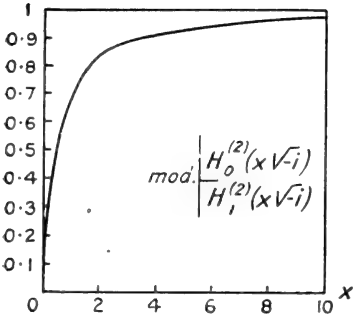
$$|c| = k = \frac{2\pi\sqrt{2}}{\lambda}\ (cf.\ 8').$$

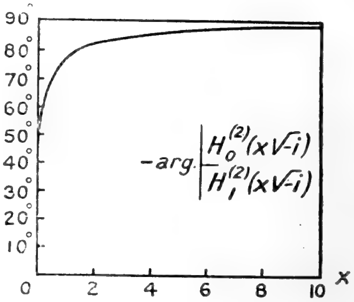
---

* *Cf.* Jahnke u. Emde, l. c.

† Tables for $\mathrm{H}_0^{(1)}$ and $\mathrm{H}_0^{(2)}$ will be found Jahnke u. Emde, pp. 139, 140.

The condition $cR \geqq 10$ means that the radius of the cylinder must be about equal to or larger than the wave-length. If R is small compared with λ the second part of





the frictional couple is negligible. For $|cR| \geqq 10$ the second term on the right-hand side of (10) becomes:

$$-acie^{ipt} = -ake^{i\left(pt+\frac{\pi}{4}\right)} \left(\text{since } c = ke^{-\frac{i\pi}{4}}\right).$$

Hence equation (11) now becomes:

$$K = -4\pi\mu R^2\omega - 2\pi\mu k R^3 \frac{d}{dt}\left(a\cos\left(pt+\frac{\pi}{4}\right)\right) \quad . \quad (13)$$

where $$\omega = \frac{d}{dt}(a\cos pt).$$

The frictional couple thus divides into two parts, one of which does not contain the density of the liquid, and another in which it occurs and which therefore refers to the emission of waves. In the transition to the limit of uniform rotation the first part only remains.

In the discussion of the second part of the frictional moment the quantity $k = \sqrt{\frac{p\rho}{\mu}}$ is an important factor. If we take a time of oscillation of 2π seconds, so that $p=1$, we have $k = \sqrt{\frac{\rho}{\mu}}$.

This gives the following values for k:

	ρ.	μ.	$k=\sqrt{\frac{\rho}{\mu}}\,(p=1)$.
Water 16°	1	0·011	9·5
Atm. air 0°	0·0013	0·000171	2·8
Air 0·01 atm.*			0·28
Air 0·001 atm.*			0·09
Hydrogen 1 atm. 0°	0·0000898	0·000085	1

* At these pressures μ has not become much smaller. *Cf.* Kundt u. Warburg, Pogg. *Ann.* 1875, Band clv.

From this table it appears that, except for dilute gases, R has to be relatively small in order that the second part may be neglected with respect to the first. For instance, for atmospheric air with $R=0{\cdot}5$ cm. $kR=1{\cdot}4$ and $\dfrac{H_0^{(2)}(cR)}{H_1^{(2)}(cR)}=0{\cdot}80$, so that the amplitude of the second term of the frictional couple is still 56 per cent. of that of the first (see equation (11)), in which everything is calculated for a time of oscillation of 2π seconds.

There is a further special limiting case of equation (13), which is of some interest. Let R become infinite, and let a at the same time disappear, in such a manner that Ra converges to a finite limit b. We thus approach the one-dimensional problem of the oscillation of an unlimited flat plate in its own plane in an infinitely extended liquid. The frictional force per unit of area is found from (13) to be

$$F=-\mu k\frac{d}{dt}\left(b\cos\left(pt+\frac{\pi}{4}\right)\right), \quad . \quad . \quad . \quad . \qquad (14)$$

a formula which is well known from hydrodynamics *. A term analogous to $-4\pi\mu R^2\omega$ does not occur in the one-dimensional problem, the reason evidently being that with a uniform translation of the plate a condition of equilibrium does not arise, until the whole liquid away to infinity proceeds with the velocity of the plate.

Finally it is of importance to ascertain for what frequency the amplitude of the forced vibration becomes a maximum, in other words, to what frequency the system cylinder-liquid resounds, if the cylinder is urged back to the position of equilibrium by a quasi-elastic force.

The differential equation for the forced oscillation in complex notation is as follows:

$$\theta\frac{d^2a}{dt^2}+L\frac{da}{dt}+Ma=Ee^{ipt}. \quad . \quad . \quad . \qquad (15)$$

Here in our case L is a complex quantity $L=L'+iL''$, where

$$L'=(4\pi\mu R^2+\surd 2\pi\mu kR^3)$$

$$L''=\surd 2\pi\mu kR^3.$$

If we only concern ourselves with the particular solution

---

* *Cf.* Lamb, 'Hydrodynamics,' 3rd edition, 1905, p. 559.

of (15) which gives the forced oscillation, we can also write (15) in the form :

$$\left(\theta+\frac{L''}{p}\right)\frac{d^2a}{dt^2}+L'\frac{da}{dt}+Ma=Ee^{ipt}. \quad . \quad . \quad (16)$$

We see, therefore, that in consequence of the motion of the liquid an apparent increase of the moment of inertia arises.

Putting

$$\theta+\frac{L''}{p}=\theta'$$

the particular solution of (16) becomes :

$$a=\frac{E}{\sqrt{(M-\theta'p^2)^2+L'^2p^2}}\,e^{i(pt-\phi)}$$

in which the phase-angle ϕ is determined by the constants of the differential equation.

Resonance occurs for $M-\theta'p^2=0$

or

$$\theta p^2+L''p-M=0 \quad . \quad . \quad . \quad . \quad . \quad . \quad (17)$$

Now L'' is proportional to k and $k=\sqrt{\frac{p\rho}{\mu}}$, so that we may conveniently write $L''=Np^{\frac{1}{2}}$, N being a constant.

(17) is now replaced by

$$\theta p^2+Np^{\frac{3}{2}}-M=0. \quad . \quad . \quad . \quad . \quad . \quad . \quad (18)$$

This equation, which is bi-quadratic in $\sqrt{p}$, determines the frequencies to which the system resounds. On closer examination there appears to be but one resonance-frequency. Naturally we are only concerned with the real roots p of equation (18). There are found to be two such roots, one for which $\sqrt{p}$ is positive, and another for which $\sqrt{p}$ is negative. Now it follows from our calculation that we have assumed $\sqrt{p}$, which occurs in k, to be essentially positive. For if we substitute a negative value for $\sqrt{p}$ in our equations, we obtain a system of waves which moves from infinity towards the cylinder. But the amplitude of this system is infinite at infinity, so that our first boundary-condition would not be satisfied.

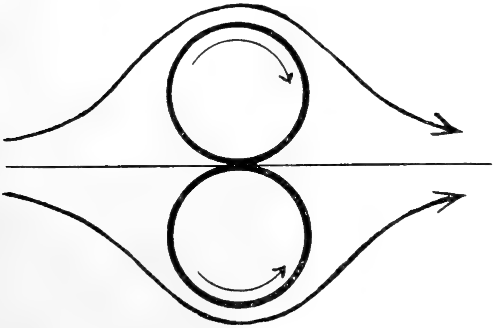
Delft (Holland),
March, 1919.

LVI. *A proposed Hydraulic Experiment.*

To the Editors of the Philosophical Magazine.

GENTLEMEN,—

IN the issue of this Journal for October 1918 (p. 315) Lord Rayleigh has proposed an experiment on the flow of a liquid between two cylinders, standing side by side, and on the influence of a rotation of these cylinders on the form of the stream-lines. May I draw your attention to a remark by Prof. F. Prandtl of Göttingen, put forward in a discussion at a meeting of November 1911 and published in the *Zeitschrift für Flugtechnik und Motorluftschiffahrt*, iii. p. 32 (1912), on an experiment which is only slightly different from that proposed by Lord Rayleigh? It is stated there that no vortices (eddies) arise if care has been taken that everywhere the parts of the walls go faster than the adjacent fluid. If two cylinders, standing side by side, very near to each other, rotate in opposite directions, it is possible to make the stream-lines close perfectly behind the cylinders. The arrangement differs from that as proposed by Lord Rayleigh only as far as the flow is directed along the exterior sides of the cylinders, and not between them. Photographs seem to have been taken of the form of the stream-lines; however, they have not been published.



Delft (Holland),
4 Dec. 1918.

Yours truly,
J. M. BURGERS.

LVII. *On the Numerical Integration of Differential Equations.*

By H. T. H. Piaggio, *M.A.*, *D.Sc.*, *University College, Nottingham* *.

I. *Introduction and Summary.*

MANY physical problems lead to differential equations which cannot be integrated in finite terms by any of the usual devices. In such cases some form of approximation must be used. The earliest method of approximating was by an infinite series, but this is often very tedious when accurate numerical values are required. Runge (*Math. Ann.* vol. xlvi. 1895) has given a formula for calculating approximately the increment of y corresponding to a small increment of x, when x and y are connected by the differential equation

$$\frac{dy}{dx}=f(x, y),$$

and it is given that $y=b$ when $x=a$.

Other approximate formulæ have been given by Heun and Kutta (*Zeitschrift f. Math. u. Physik*, vols. xlv. and xlvi.).

In applying any of these methods to actual examples, it is important to know how far wrong the result may be. Runge assumes that when two steps of his approximations come fairly close together, the error in the final result will be of about the same order of magnitude as one-third of their difference. But he does not give any definite upper limit for this error.

The object of the present paper is to supply this omission. Four simple formulæ are found which give four numbers, between the greatest and least of which the required increment of y must lie. A new approximate formula is derived from these.

As an illustration this new formula is applied to the example given by Runge. The result is more accurate than those given by the methods of Runge, Heun, or Kutta.

II. *Limits between which the value of a definite integral lies.*

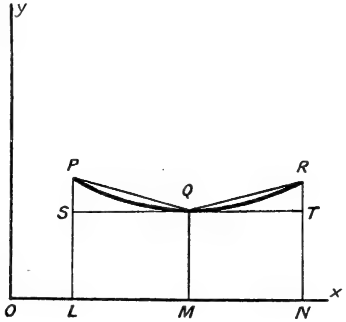
Let $F(x)$ be a function which, together with its first and second differential coefficients, is continuous (and therefore finite) between $x=a$, and $x=a+h$. Let $F''(x)$ be of constant sign in the interval. In the figure this sign is taken as

---

* Communicated by the Author.

positive, making the curve concave upwards. LP, MQ, NR are parallel to the axis of y, M is the middle point of LN, and SQT is the tangent at Q. $OL = a$, $LN = h$.

Fig. 1.



Then the area PLNR lies between that of the trapezium SLNT and the sum of the areas of the trapezia PLMQ, QMNR.

That is, $\int_a^{a+h} F(x)dx$ lies between

$$hF(a+\tfrac{1}{2}h) = A \text{ say},$$

and $$\tfrac{1}{4}h\{F(a) + 2F(a+\tfrac{1}{2}h) + F(a+h)\} = B \text{ say}.$$

In the figure $F''(x)$ is positive and A is the lower limit, B the upper. If $F''(x)$ were negative, A would be the upper limit and B the lower.

As an approximation to the value of the integral it is best to take, not the arithmetic mean of A and B, but $\tfrac{2}{3}B + \tfrac{1}{3}A$, which is exact when PQR is an arc of a parabola with its axis parallel to the axis of x. It is also exact for the more general case when $F(x) = a + bx + cx^2 + ex^3$, as is proved in most treatises on the Calculus in their discussion of Simpson's Rule.

III. *Extension of preceding results to functions defined by differential equations.*

Consider the function defined by

$$\frac{dy}{dx} = f(x, y) \; ; \; y = b \text{ when } x = a,$$

where $f(x, y)$ is subject to the following limitations in the range of values a to $a+h$ for x and b to $b+h$ for y. It will be seen from what follows below that the increment of y is less than h, so that all values of y will fall in the above range. The limitations are :

(1) $f(x, y)$ is finite and continuous, as are also its first and second partial differential coefficients.

(2) It never numerically exceeds unity. If this condition is not satisfied, we can get a new equation in which it is satisfied by taking y instead of x as the independent variable.

(3) Neither d^3y/dx^3 nor $\partial f/\partial y$ changes sign.

Let m and M be any two numbers, such that

$$m < f < \mathrm{M} \leqq 1.$$

Then if the values of y when x is $a+\frac{1}{2}h$ and $a+h$ are denoted by $b+j$ and $b+k$ respectively,

$$\tfrac{1}{2}mh < j < \tfrac{1}{2}\mathrm{M}h \leqq \tfrac{1}{2}h, \quad \ldots \ldots \quad (1)$$

and

$$mh < k < \mathrm{M}h \leqq h. \quad \ldots \ldots \quad (2)$$

We shall now apply the formulæ of the last section, taking y to be the same function as that defined by

$$y = b + \int_a^{a+x} \mathrm{F}(x)dx,$$

so that

$$k = \int_a^{a+h} \mathrm{F}(x)dx.$$

We have to express the formulæ in terms of f instead of F.

Now, $\mathrm{F}(a)$ = the value of dy/dx when $x=a$,

so

$$\mathrm{F}(a) = f(a, b).$$

Similarly,

$$\mathrm{F}(a+\tfrac{1}{2}h) = f(a+\tfrac{1}{2}h, b+j),$$

and

$$\mathrm{F}(a+h) = f(a+h, b+k).$$

Now, if $\partial f/\partial y$ is positive, so that f increases with y, the inequalities (1) and (2) lead to

$$f(a+\tfrac{1}{2}h, b+\tfrac{1}{2}mh) < f(a+\tfrac{1}{2}h, b+j) < f(a+\tfrac{1}{2}h, b+\tfrac{1}{2}\mathrm{M}h), \quad (3)$$

and

$$f(a+h, b+mh) < f(a+h, b+k) < f(a+h, b+\mathrm{M}h) ; \quad (4)$$

while if $\partial f/\partial y$ is negative,

$$f(a+\tfrac{1}{2}h, b+\tfrac{1}{2}mh) > f(a+\tfrac{1}{2}h, b+j) > f(a+\tfrac{1}{2}h, b+\tfrac{1}{2}\mathrm{M}h), \quad (5)$$

and

$$f(a+h, b+mh) > f(a+h, b+k) > f(a+h, b+\mathrm{M}h). \quad (6)$$

Thus if $F''(x)=dy/dx$ is positive and $\partial f/\partial y$ is also positive, the result of section II.,

$$A<k<B,$$

may be replaced by

$$p<k<Q, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (7)$$

where $p=hf(a+\tfrac{1}{2}h,\ b+\tfrac{1}{2}mh)$

and $Q=\tfrac{1}{4}h\{f(a,\ b)+2f(a+\tfrac{1}{2}h,\ b+\tfrac{1}{2}Mh)+f(a+h,\ b+Mh)\}$;

while if $F''(x)$ is positive, and $\partial f/\partial y$ is negative,

$$P<k<q, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (8)$$

where $P=hf(a+\tfrac{1}{2}h,\ b+\tfrac{1}{2}Mh)$

and $q=\tfrac{1}{4}h\{f(a,\ b)+2f(a+\tfrac{1}{2}h,\ b+\tfrac{1}{2}mh)+f(a+h,\ b+mh)\}$.

Similarly, if $F''(x)$ and $\partial f/\partial y$ are both negative,

$$p>k>Q, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (9)$$

while if $F''(x)$ is negative and $\partial f/\partial y$ positive,

$$P>k>q. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (10)$$

These results may be summed up by saying that *in every case* (subject to the limitations on f stated at the beginning of this section) *k lies between the greatest and least of the four numbers p, P, q, and Q.*

As an approximate formula we use $k \doteqdot \tfrac{2}{3}B+\tfrac{1}{3}A$, replacing B by Q or q, and A by p or P.

IV. *Application to a numerical example.*

Consider the example selected by Runge and Kutta to illustrate their methods

$$\frac{dy}{dx}=\frac{y-x}{y+x};\ y=1 \text{ when } x=0.$$

It is required to find the increment k of y when x increases by 0·2. Here $f(x,y)=(y-x)/(y+x)$. This function satisfies the conditions laid down in the last section.

We take $M=1$, $m=(1-0{\cdot}2)/(1{\cdot}2+0{\cdot}2)=4/7$.

Then
$$p=0{\cdot}1654321,$$
$$P=0{\cdot}1666667,$$
$$q=0{\cdot}1674987,$$
$$Q=0{\cdot}1690476.$$

Thus k lies between p and Q.

	Errors.
$\frac{2}{3}Q+\frac{1}{3}p=0{\cdot}1678424$,	0·0000007
Kutta's value 0·1678449,	0·0000032
Runge's value 0·1678487,	0·0000070
Heun's value 0·1680250,	0·0001833

The second, third, and fourth of these were calculated by Kutta. They are also given in Bateman's 'Differential Equations,' p. 229. Now this particular example admits of integration in finite terms, giving

$$\log(x^2+y^2)-2\tan^{-1}(x/y)=0.$$

Hence we may find the accurate value of k.

Accurate value = 0·1678417.

Thus in this example our result is the nearest to the accurate value, the errors being as stated above.

We may also test the method by taking a larger interval $h=1$. Of course a more accurate way of obtaining the result would be to take several steps, say $h=0{\cdot}2$, $0{\cdot}3$, and finally $0{\cdot}5$, as Runge does.

Still, it is interesting to see how far wrong the results come for the larger interval.

We take $M=1,\ m=(1-1)/(2+1)=0.$

Then $\frac{2}{3}Q+\frac{1}{3}p=0{\cdot}50000.$

	Errors.
True value = 0·49828,	
Kutta's value = 0·49914,	0·00086
Our value = 0·50000,	0·00172
Heun's value = 0·51613,	0·01785
Runge's value = 0·52381,	0·02553

This time Kutta's value is the nearest, and ours is second.

University College, Nottingham,
February 22nd, 1919.

LVIII. *Precision-measurements in the X-Ray Spectra.*

By MANNE SIEGBAHN[*].

1. *Introduction.*

THE systematical measurements of the X-ray spectra, discovered by G. G. Barkla, with the help of crystal-lattices, taken up by W. H. Bragg, H. G. J. Moseley, de Broglie, and others[†], have given a great many results of the utmost interest. The theories of the constitution of the atom and its radiation have received empirical basis which has greatly furthered this branch of physical knowledge. Through the theoretical investigations of Debye, Kroo, Sommerfeld and others, the empirical material has, however, in some respects been distanced. It will be seen that the spectral formulæ of the authors just mentioned require more precise measurements of the wave-lengths than those which have hitherto been attained, to allow a desirable control.

For this purpose the author has worked out methods for more precise measurements of wave-lengths in the X-ray spectra. The results of this investigation, which extends over the longer waves (greater than 1 Å.E.) show that the new values have about a hundred-fold accuracy compared with the older. In a following paper a method for accurate measurements of the short waves will be given.

2. *Method and Apparatus.*

In all the older methods of wave-length measurements the reflexion angle (ϕ) in the Bragg formula

$$n\lambda = 2d \sin \phi$$

was in some way or other *geometrically* determined, which highly limited the attainable accuracy. In the following method the angle of the reflected rays is determined as the angle through which the same photographic plate has to be turned in order to receive impressions of the nth order reflexion on both sides of the direct ray. These angles were simply read off on an accurate circle-scale. When the plate was turned accurately 4ϕ (and the crystal, nearly $2\phi + 180°$), the spectral lines on both sides would cover one another. When the angle was nearly 4ϕ the lines on the plate fell at some little distance apart. By measuring this distance the small correction to 4ϕ could readily be determined with desirable accuracy.

The method requires that plate and slit have the same distance from the rotating axis of the reflecting face of

---

[*] Communicated by the Author.

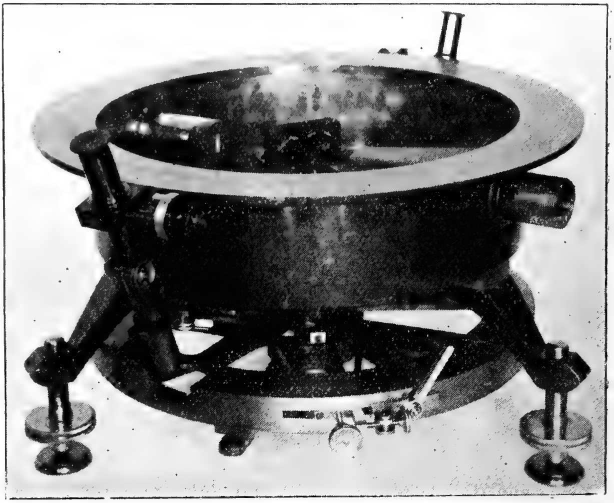
[†] Siegbahn, *Jahrb. d. Rad. und Elektronik*, 1916.

crystal (and of the plate-holder)*. Only in this case the exact angle of the crystal is without influence on the place of the spectral lines on the plate. In this case also do the same results come out when the crystal during the exposure is continually turned over a little range of angles or remains still. In using bad crystals small irregularities may in this way be eliminated, if necessary. All the following measurements are carried out with selected crystals where such precautions were superfluous.

For the determining of the lattice-constant of potassium ferrocyanide only a bad specimen was obtainable. These results are therefore not of the same accuracy as the others, but are sufficient for the purpose in question.

As the long waves which this investigation ranges over are considerably if not wholly absorbed in atmospheric air, the spectrograph was built for vacuum (0·1–0·01 mm. Hg). Its construction may be seen in fig. 1. To the plate-holder is attached a circle-scale of great accuracy which is read off with two microscopes. Another circle-scale of smaller accuracy allows the approximate turning of the crystal in the desired position.

Fig. 1.

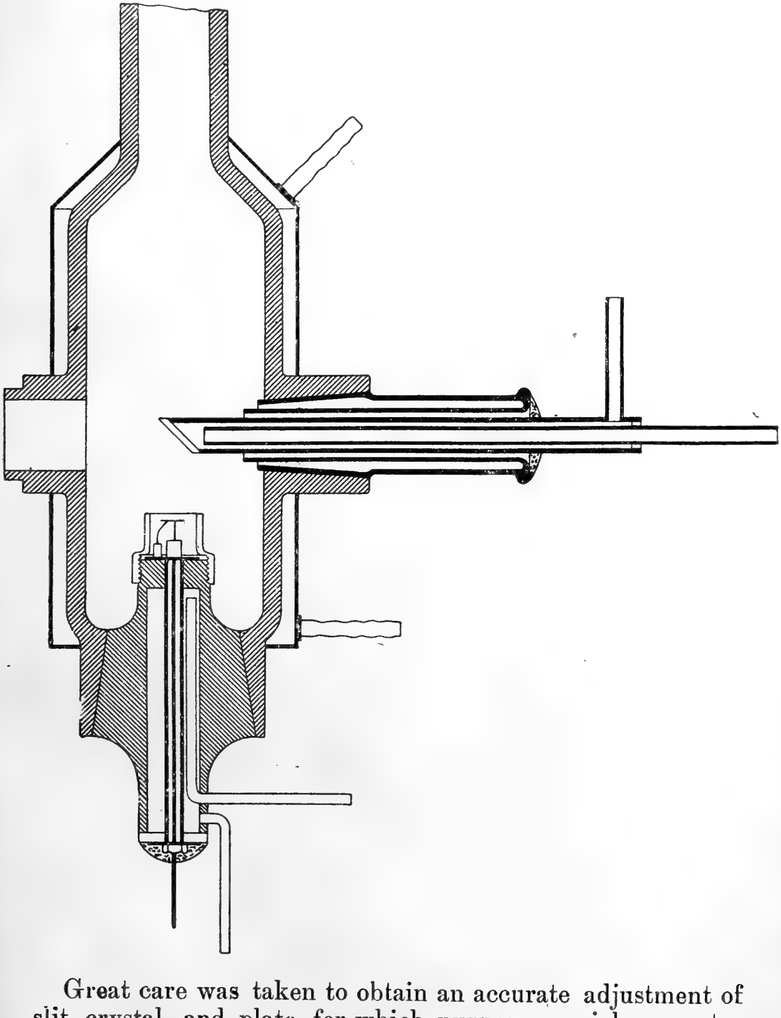


---

* Bragg, 'X-Rays and Crystal Structure.'

To separate the high vacuum in the X-ray tube from the low vacuum in the spectrograph a gold-beaters' skin with red paper, an aluminium foil or some other foil was pressed over the slit.

Fig. 2.



Great care was taken to obtain an accurate adjustment of slit, crystal, and plate, for which purpose special apparatus were constructed and built.

A great deal of the work has been devoted to the construction of an effective X-ray tube for spectral purposes in the region of the long waves. Many types have been built and tried, of metal and glass and combinations of both. Tubes with cold and with hot cathodes (Wehnelt and hot wire) have been compared. The definite type which resulted from these investigations is to be seen in fig. 2.

This tube is entirely of metal with a wolfram spiral in a cylindrical cathode. Only the anticathode is isolated by means of a glass tube. Cathode and anticathode can easily be detached when necessary. The wolfram spiral is substituted in a moment. By screwing the cylindrical cathode up or down the focus-spot can be varied from the size of a millimetre to the entire anticathode face.

To allow continuous work both cathode, anticathode, and tube are cooled by running water. In some cases a tube of this type has worked steadily 10 to 15 hours with 40–50 milliamp. 10–15 k.v. without any interruption. Such long exposures were naturally unnecessary for the *line*-spectra with which the following will exclusively deal.

3. *Experimental Results.*

A. *The Wave-length of* Cu K α.

In many experimental researches on X-rays the characteristic radiation of Cu has been employed as standard. From this point of view it seems of the greatest interest to give an exact determination of its wave-length. The value given by Moseley for Cu K α was $1{\cdot}549 \,.\, 10^{-8}$, whereas our* value was $1{\cdot}539 \pm 0{\cdot}003$.

The wave-length of Cu K α has been determined with the above-described apparatus on different occasions with new adjustments of the whole instrument. The measurements are therefore independent of one another, and give a good illustration of the accuracy attained.

Plate.	Order of Reflexion.	Wave-length.	Reflecting Crystal.
4 a. 5	1	$1537{\cdot}24 \,.\, 10^{-11}$ cm.	Rock-salt.
10	2	·34	,,
10	3	·52	,,
23	1	·40	,,
84	1	·21	,,
90	2	·44	,,
112	1	·44	,,
28	1	·22	Calcite.
28	2	·35	,,
26	1	·49	,,
...	1	·29	,,

On an average we obtain

$$1537{\cdot}358 \pm 0{\cdot}033 \,.\, 10^{-11} \text{ cm.}$$

---

* Siegbahn u. Stenström, *Phys. Zs.* xvii. p. 48 (1916).

where the lattice-constant of rock-salt has been taken at $2{\cdot}81400 \,.\, 10^{-8}$, $\log 2d = 0{\cdot}7503541$, and of calcite at $\log 2d = 0{\cdot}7823347$ (see below).

B. *The lattice-constant of calcite.*

Bragg in his eminent work 'X-Rays and Crystal Structure' recommends the use of calcite instead of rock-salt for X-ray investigations on account of its more regular reflexions. A curve of the intensity of reflexion produced by rock-salt given in the book mentioned clearly shows what bad images are obtained in some cases by these crystals. In spite of this fact, experience from many hundred X-ray spectrograms has taught me that with selected specimens (I have used rock-salt from Stassfurt) perfectly good images with spectrographs of this kind may be obtained. Only at very large angles do somewhat better results with calcite seem to come out, probably owing to the greater absorption in calcite which reduces the thickness of the reflecting layer.

The extending use of the calcite for X-ray spectrography makes it desirable for one to know the relation between the lattice-constant of calcite and that of rock-salt with sufficient accuracy. By means of the above method for determining the reflexion-angles for monochromatic X-rays this is easily effected. The determination has been carried out with the following three characteristic radiations:—

Cu K α,
Fe K α,
Sn L α.

With each of them a number of spectrograms with repeated adjustments have been taken. The results are given in the following table:—

	λ.	$\log 2d$.	$d \,.\, 10^{-11}$.
Cu	1537·36	0·7823339	3029·03
Fe	1932·39	0·7823386	3029·07
Sn	3592·94	0·7823327	3029·02

which gives us the most probable value for the lattice-constant of calcite (cleavage-face)

$$\log 2d = 0{\cdot}7823347,$$

where the value for rock-salt is taken as

$$\log 2d = 0{\cdot}7503541,$$

as above. For the sake of comparison we may mention the following values of other authors: Millikan* from his precise measurements on fundamental physical constants deduces the value

$$d=3{\cdot}030+0{\cdot}001\,.\,10^{-8};$$

S. Gorton† gives the value

$$d=3{\cdot}028\,.\,10^{-8};$$

and, finally, A. H. Compton‡ finds

$$d=3{\cdot}0279.$$

C. *The lattice-constant of potassium ferrocyanide.*

Potassium ferrocyanide has a special interest on account of the fact that Moseley, in his classical investigations on the X-ray spectra of the elements, used this crystal. The systematical determinations which have been made at the laboratory of Lund show, when compared with the measurements of Moseley, a constant deviation of about 0·5 per cent., which considerably exceeds the limits of experimental error. As a probable cause of this deviation I have suggested that the lattice-constant which Moseley used in the calculations of the wave-lengths was not in accordance with the assumed value for rock-salt. The latter crystal has been used in our investigations, and the value $d=2{\cdot}814$ given by Moseley in his earlier paper was taken as its lattice-constant. For the longer waves selenite was used, the lattice-constant of which was referred to that of rock-salt.

For the control measurement of the lattice-constant for potassium ferrocyanide a home-made crystal specimen was used. The crystal was not very good, but good enough to allow sufficiently accurate measurements. From a spectrogram on Cu K α of the second order the reflexion angle was determined at:

$$\mathrm{Cu\,K}\,\alpha_1:\ \phi_2=10^\circ\ 32'\ 14'',$$

which gives as value for d,

$$d=8{\cdot}407\,.\,10^{-8}\ \mathrm{cm}.$$

---

* Millikan, 'The Electron.'

† S. Gorton, Phys. Rev. Feb. 1906.

§ A. H. Compton, Phys. Rev. June 1916.

Another spectrogram of Pt L α gives in the third order the angle

$$\mathrm{Pt\,L}\,\alpha_1:\ \phi_3=13^\circ\ 30'\ 58'',$$

while the angle of the same line by rock-salt in the first order is

$$\phi_1=13^\circ\ 27'\ 47''.$$

As a result of these two values we find

$$d=8{\cdot}409\,.\,10^{-8},$$

or as an average of the lattice-constant of potassium ferrocyanide

$$d=8{\cdot}408\,.\,10^{-8}.$$

The value which Moseley used was

$$d=8{\cdot}454\,.\,10^{-8}.$$

This implies that the wave-length tables of Moseley must be corrected with $-0{\cdot}54$ per cent. in order that the value may be referred to the assumed lattice-constant of rock-salt.

D. *The wave-lengths of the* K *group in the region* Cu–Cl.

The above methods for accurate wave-length measurements have been employed for the longer waves in the K group, where the elements Cu–Cl have been investigated. As the spectrograms show, the reflexion of these long waves is approximately a pure surface effect, where the thickness of the reflecting layer may be neglected. In fact, the breadth of the lines on the plate is the same as that of the slit (about 0·1 mm.). The lines have also approximately sharp edges on both sides. The values in the following table are the means of several spectrograms taken after repeated adjustments, and in some cases with different crystals (rock-salt and calcite). The weaker β-lines are in general referred to the resp. α-line.

As will be seen from the next paragraph, the β-line is not single, and that has possibly had no little influence on the result.

		$\lambda \,.\, 10^{-11}$ cm.	
	N.	K α_1.	K β_1.
Cl	17	4718·70	—
K	19	3733·86	3447·37
Ca	20	3351·86	3087·89
Sc	21	3025·26	2774·54
Cr	24	2285·17	2081·44
Fe	26	1932·39	1753·97
Co	27	1785·24	1617·58
Ni	28	1654·67	—
Cu	29	1537·36	1389·53

E. *The fine structure of the lines in the* K-*group.*

In an exceedingly interesting paper Sommerfeld [*] has shown that the α-doublet of the K-group is to be ascribed to the double character of the second electron ring of the Bohr-Rutherford atom. In reality there are two possible ways of distributing the two quanta assigned to the second ring, which either gives a circular or an elliptical orbit with definite excentricity, the mass of the electron being taken as a constant. The replacing of the electron from the one or the other of these orbits, to the first ring, then gives the α line in two ways. When the variable mass of the electron is taken into consideration there will come out two somewhat different frequencies. As this difference can be theoretically deduced it gives a very good opportunity to empirically prove this hypothesis.

The theoretical value given by Sommerfeld in his above-mentioned paper is:

$$\frac{\Delta\nu}{(\mathrm{N}-3{\cdot}5)^4}=\Delta\nu_{\mathrm{H}}\left[1+\frac{5\alpha^2}{2}\frac{(\mathrm{N}-3{\cdot}5)^2}{2^2}+\frac{53\alpha^4}{8}\frac{(\mathrm{N}-3{\cdot}5)^4}{2^4}+\ldots\right] \quad . \quad . \quad . \quad (1)$$

where $\Delta\nu_{\mathrm{H}}$ is the analogous frequency-difference by hydrogen in the optical spectrum, N is the atomic number, and α a constant:

$$\alpha=\frac{2\pi e^2}{ch}.$$

The best value of $\Delta\nu_{\mathrm{H}}$ determined from the spectrum of He by Paschen [†] in his beautiful research on the fine structure of some He-lines is:

---

[*] Sommerfeld, *Ann. d. Phys.* li. (1916).

[†] Paschen, *Ann. d. Phys.* l. (1916).

$$\Delta\nu_{\mathrm{H}} = 0{\cdot}3645 \pm 0{\cdot}0045$$

with $$\alpha^2 = 5{\cdot}3146 \,.\, 10^{-5}.$$

The value of the wave-length-difference in the X-ray spectrum of Cu as determined from the plates 4 and 5 is

$$\Delta\lambda = [0{\cdot}00379 \pm 0{\cdot}00004]\,10^{-8},$$

which gives $$\frac{\Delta\nu}{(\mathrm{N}-3{\cdot}5)^4} = 0{\cdot}379 \pm 0{\cdot}004$$

with $\mathrm{N}=29$. The right member of the above equation (1) has the value

$$\Delta\nu_{\mathrm{H}}'\,[\,\ldots\,] \qquad = 0{\cdot}3645 + 0{\cdot}0079 + 0{\cdot}0003$$
$$= 0{\cdot}373 \pm 0{\cdot}005,$$

which shows a complete agreement with the empirical value. The further examination of this question with other elements has led to surprising results, which require further investigations. These are now going on and will soon be published.

Another interesting consequence of the theory of Sommerfeld has been briefly pointed out in a note * to the Bavarian Academy of Science. It deals with fine structure of the β-line, which had previously been regarded as simple. This line arises when an electron runs from the third to the first ring. But as the intermediate second ring will be somewhat affected through the variation of the system, the change in its energy will also have a little influence on the frequency of the line. The double nature of this ring will therefore necessarily cause the β-line to appear as a doublet. The theory also shows that the frequency-difference between the two lines of this doublet will be more remarkable in the case of elements with smaller atomic weights, and that the weaker component will have the greater wave-length.

All these predictions of the theory were in the best accordance with the empirical results. In fact, the author has, without any knowledge of these theoretical results, on the spectrograms of Ni and Co, observed the expected effect (see the concluding remarks in the cit. note).

Some preliminary values for the frequency-difference in this doublet $[\beta, \beta']$ are given in the following table, which will soon be completed :—

---

* Sommerfeld, *Sitz. Ber. d. Bayr. Akad. d. Wiss.* (June 1918).

	N.	$\Delta\lambda \,.\, 10^{-11}$.	$\Delta\nu$.
Fe	26	3·55	·116
Mn	25	4·55	·126
Cr	24	4·87	·113

The weakness of the lines makes an accurate measurement very difficult.

4. *The Formulæ of the* K α *series.*

Moseley, in his first paper on the X-ray spectra, suggested the following formula to represent the K α series:

$$\frac{\nu}{R} = (N-1)^2 \left[\frac{1}{1^2} - \frac{1}{2^2}\right] \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

where R is the Rydberg constant and N the atomic number of the element. The theoretical basis which Moseley gave this equation can no longer be maintained. The great importance of this formula lies in the fact that Moseley introduced the atomic numbers which have shown themselves to be so fundamental for the X-ray spectra.

The extension of the measurements has shown that there are systematical deviations from this simple formula. To find a better representation of the numerical results Sommerfeld * suggested, in his first paper on this subject, the formula

$$\frac{\nu}{R} = \frac{(N-1{\cdot}6)^2}{1^2} - \frac{(N-3{\cdot}5)^2}{2^2} \quad . \quad . \quad . \quad . \quad . \quad (2)$$

where the two terms, which constitute the spectral lines in the Bohr theory, have their respective nucleus charges, N − 1·6 and N − 3·5. The values of these charges were determined from the empirical data. With correction for the variability of the electron-mass Sommerfeld deduced the formula

$$\frac{\nu}{R} = \frac{2}{\alpha^2}\left[\sqrt{1 - \frac{\alpha^2}{4}(N-3{\cdot}5)^2} - \sqrt{1-\alpha^2(N-1{\cdot}6)^2}\right], \quad . \quad (3)$$

$$\alpha^2 = 5{\cdot}3 \,.\, 10^{-5}.$$

A considerable improvement was made by Debye †, who succeeded in giving a quantitative explanation of the effective nucleus charges. His formula was

$$\frac{\nu}{R} = \frac{1}{\alpha^2}\left[4\sqrt{1-\alpha^2(N-S_2)^2} + 2\sqrt{1-\frac{\alpha^2}{4}(N-2)^2} - 6\sqrt{1-\alpha^2(N-S_3)^2}\right], \quad . \quad . \quad (4)$$

---

* Sommerfeld, *loc. cit.*

† Debye, *Phys. Zs.* 1917, p. 276.

where S_p represents the Bohr function

$$S_p = \frac{1}{4} \sum_{1}^{i=p-1} \frac{1}{\sin i \frac{\pi}{p}}.$$

The basis of this formula may be briefly related :—

Debye considers an atom where the first inner electron ring of one quantum holds n electrons. When one of these n electrons is brought to an outer orbit with two quanta and then falls back to the inner ring the α-line will be the result. The best agreement with the empirical values is obtained when n is taken as 3. In the normal state all the atoms (from Na) would therefore have an inner electron ring with this number of electrons

A somewhat better representation is arrived at when the above assumptions are slightly modified, as has been shown by J. Kroo *. If the two innermost electron rings before replacing are characterized by

The first ring (1 quantum) with 2 electrons,
and the second ring (2 quanta) ,, 9 ,,

and after replacing by

the first ring (1 quantum) ,, 3 ;,
and the second ring (2 quanta) ,, 8 ,,

The calculation of the radiated frequency accompanying this change gives the formula

$$\frac{\nu}{R} = \frac{2(k-1)}{\alpha^2} \sqrt{1-\alpha^2 F_1} + \frac{2(l+1)}{\alpha^2} \sqrt{1-\alpha^2 F_2} - \frac{2k}{\alpha^2} \sqrt{1-\alpha^2 F_3} - \frac{2l}{\alpha^2} \sqrt{1-\alpha^2 F_4}, \quad . \quad . \quad . \quad . \quad (5)$$

where $k=3, \quad l=8,$

$$F_1 = (N - S_{k-1})^2,$$
$$F_2 = \tfrac{1}{4}(N - k + 1 - S_{l+1})^2,$$
$$F_3 = (N - S_k)^2,$$
$$F_4 = \tfrac{1}{4}(N - k - S_l)^2.$$

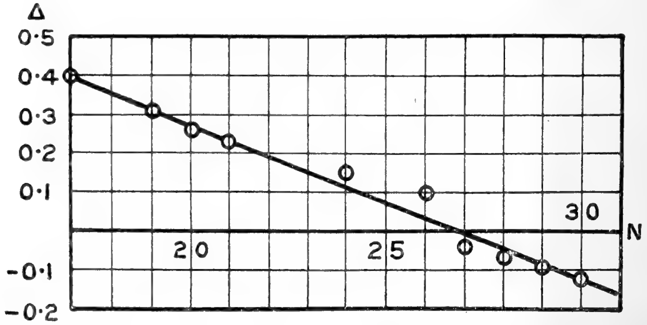
There is only a slight difference between the equations (4) and (5), but the comparison with the measurements speaks decidedly in favour of the Kroo formula.

---

* J. Kroo, *Phys. Zs.* 1918, p. 307.

An elegant treatment of the same problem has been given by Sommerfeld[*] in intimate connexion with the investigations of Kroo. In the paper of Sommerfeld the disturbances are taken into consideration and their influences are numerically deduced. The results show the effect of the disturbances to be without influence in comparison with the older measurements where the accuracy is of the order 0·1–0·2 per cent. only. By increasing the precision of the measurements there seems to be some possibility of obtaining at least a qualitative conception of the disturbances in the atom-planetary system.

Fig. 3.



The following table gives an illustration of the congruence between the formulæ 3, 4, 5 and the empirical values. In the last columns are the deviations from the Kroo formula also given; these differences are represented in fig. 3.

N.	Sommer-feld (1916).	Debye.	Kroo.	Exp.	Kroo-Exp.	Per cent.
17	192·38	192·53	192·73	193·12	−0·39	−0·20
19	243·89	244·11	243·76	244·06	−0·30	−0·12
20	272·00	272·15	271·60	271·86	−0·26	−0·10
21	301·62	301·77	300·99	301·22	−0·23	−0·08
24	399·92	400·09	398·55	398·70	−0·15	−0·03
26	473·32	473·57	471·49	471·58	−0·09	−0·02
27	512·49	512·68	510·34	510·29	−0·05	−0·010
28	553·13	553·36	550·80	550·73	−0·07	−0·013
29	595·47	595·72	592·85	592·75	−0·10	−0·017
30	639·43	639·66	636·54	636·42	−0·12	−0·019

Lund Physical Laboratory,
December 31, 1918.

---

[*] Sommerfeld, *Phys. Zs.* 1918, p. 297.

INDEX TO VOL. XXXVII.

END OF THE THIRTY-SEVENTH VOLUME.

Printed by Taylor and Francis, Red Lion Court, Fleet Street.