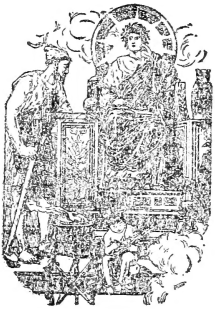

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CONDUCTED BY
SIR OLIVER JOSEPH LODGE, D.Sc., LL.D., F.R.S.
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RICHARD TAUNTON FRANCIS
AND
WILLIAM FRANCIS, F.L.S.

> "Nec aranearum sane textus ideo melior quia ex se fila gignunt, nec noster vilior quia ex alienis libamus ut apes." Just. Lips. *Polit.* lib. i. cap. 1 Not.

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"Meditationis est perscrutari occulta; contemplationis est admirari perspicua .... Admiratio generat quæstionem, quæstio investigationem, investigatio inventionem."—*Hugo de S. Victore.*

——" Cur spirent venti, cur terra dehiscat,
Cur mare turgescat, pelago cur tantus amaror,
Cur caput obscura Phœbus ferrugine condat,
Quid toties diros cogat flagrare cometas,
Quid pariat nubes, veniant cur fulmina cœlo,
Quo micet igne Iris, superos quis conciat orbes
Tam vario motu."

J. B. Pinelli ad Mazonium.

ALERE FLAMMAM.

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[SIXTH SERIES.]

JANUARY 1921.

I. *The Properties of Plastic Crystals of Ammonium Nitrate.*

By W. N. BOND, *M.Sc., A.R.C.Sc., A.Inst.P.**

[Plates I. & II.]

THIS paper consists of a description of some experiments performed with crystals of Ammonium Nitrate that exhibit plasticity. The paper may be divided into three parts. In the first, an account is given of experiments on the bending of these crystals; in the second, an account of observations on bent crystals with polarized light. The third consists of a discussion of the results, and of the internal changes accompanying bending.

PART I.

The specimens originally used were fine needle-like crystals, produced by Messrs. Brunner, Mond and Company. Later some Ammonium Nitrate of French manufacture, from Messrs. Baird and Tatlock, was used for re-crystallizing.

The long needle-like crystals can be bent and twisted in the fingers (see Pl. I. figs. 1 & 2). If this be done carefully and not too rapidly, small loops of diameters of the order of half a centimetre or less can be made. Too rapid bending

---

* Communicated by Professor A. W. Porter, F.R.S.

results in fracture at the part most strained, the crystal being thus both plastic and brittle. The thinnest needles seem to bend better when fresh from the solution they have been crystallized from. In connexion with this, it may be noted that, according to Kleinhanns*, rock salt is bent more easily when immersed in liquids that dissolve it.

The following account of the crystalline forms of Ammonium Nitrate is based on that of Groth†.

At ordinary temperatures, it crystallizes from an aqueous solution as rhombohedral and pseudo-tetragonal crystals. These change on warming, as first noticed by Lehmann ‡, to a second form, which is apparently also rhombic ; then to an optically uniaxal form, and finally to a single-refracting fourth form. This fourth form is stable at 124° C. ; the third from that temperature to 83° C. ; the second from 83° C. to 32° C., and the first from 32° C. downwards.

Wallerant, however, says that the third form is not, as Lehmann supposed, trigonal, but tetragonal and optically positive. This passes into the more strongly double-refracting second form mentioned above, of which the prisms have their length parallel to the second axis of the tetragonal crystals. By adding traces of potassium nitrate to the molten sample, the 32° transition point is lowered, so that the second form still remains stable at ordinary temperatures. We are then able to produce twinning lamellæ. The transition from the third (tetragonal) form to the less dense second form cannot be produced if the former is under pressure; the change to the first (rhombic) form (which has the greater density) takes place instead.

This first modification is stable at normal temperature ; according to Wallerant, if cooled to −16° C. it undergoes a change to another form, less dense, uniaxal, optically positive, with double refraction somewhat weaker than that of the third, tetragonal, form. Wallerant concludes that the form below −16° C. is the same as that stable from 124° C. to 83° C.

The tetragonal form arising from the cubic at 124° C. remains stable even at the lowest temperature if a little $CsNO_3$ is added ; but changes into the rhombic form under pressure. At 20° C. a very slight pressure is sufficient. If the amount of $CsNO_3$ is very slight, the crystal changes back

---

* K. Kleinhanns, *Phys. Zeitschr.* xv. pp. 362–363, April 1st, 1914.

† Groth, *Chemische Krystallographie,* Zweiter Teil, p. 66.

‡ O. Lehmann, *Zeitschr. f. Krystal.* 1877, i. p. 106.

after removing the pressure. (This is not the case if there is a rather greater proportion of $CsNO_3$.)

The usual or "first" form consists of thin pseudo-tetragonal prisms without end planes (but Grossner also found a tabular form). This crystal is a strongly double-refracting biaxal crystal, with 2E = 59° 30′ (Grossner), and the dispersion of the axes is weak ($r < v$), according to Lang.

All the crystals experimented on, whether obtained by re-crystallization or not, were long and prismatic or needle-like in form; though occasionally flat tabular crystals formed. Observations showed that the value of 2E was 59° to 60°. Also $r < v$, the difference being fairly pronounced. The crystals are optically negative.

Prolonged grinding of the crystals, dry or in oil, on a smooth stone by hand, produced a white cloudiness throughout the crystal. This appeared to be due to the combined action of the heat generated by friction and the pressure. On warming a crystal slowly in a test-tube, it was found to become similarly white and opaque at a temperature between 30° and 40° C. This is probably due to the transition from the first form to the second form mentioned by Groth. This transition did not, as far as could be seen, take place in any of the bending experiments, or in any mounted slides, except a little in one or two cases of repeated bending under large loads.

Ammonium nitrate crystals are distinctly hygroscopic; but fortunately the room used for the experiments was so dry that no trouble was experienced on this account.

Since the ammonium nitrate tends to crystallize with what appears to be multiple twinning, it is difficult to get single crystals for bending experiments. It is impossible to measure the sectional area with any great precision, and, since a crystal may behave differently if used for a second test, the effect of different stresses cannot be determined with any high degree of accuracy.

Preliminary experiments were performed on needle-like crystals arranged horizontally, clamped at one end, and loaded at the other. (The clamp consisted of a hole in a wooden block, with small wooden wedges forced in above and below the crystal.) This method has the disadvantage that there is an objectionable local stress at the clamp, where the total stress is a maximum.

Fig. 3 shows the rate of depression of the tip of the crystal for constant load. For the small amounts of bending measured, this is equivalent, approximately, to constant stress at any one point.

Fig. 3.

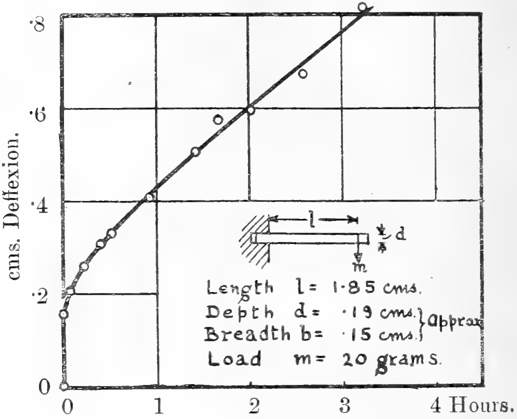


Fig. 4 shows the result of an experiment in which the load was changed for an equal one acting in the opposite direction, after a certain amount of bending had taken place. Finally the crystal was loaded as at the start.

Fig. 4.

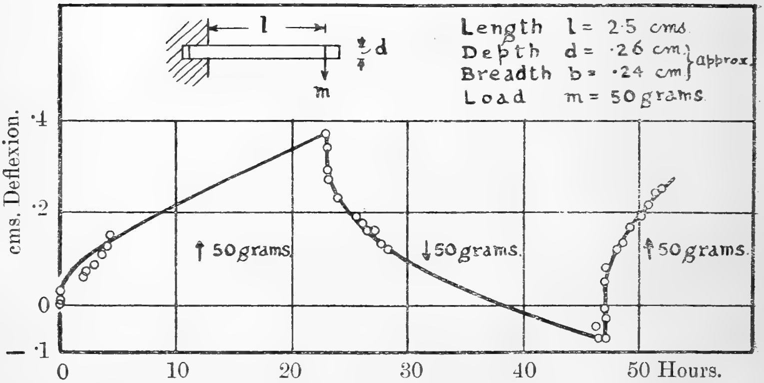
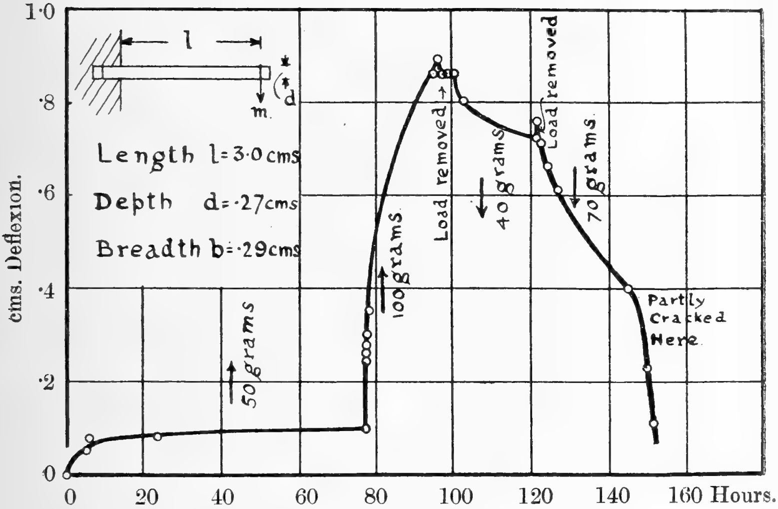


Fig. 5 shows the effect of varying the sense and the magnitude of the load on another crystal clamped, as before, at one end. We see that for small stresses the deflexion tends to a limiting value. The rate of bending is not directly proportional to the stress, but increases more rapidly with increase of stress. Indeed, in subsequent experiments, loads that were not very much smaller than those which produced fairly rapid deflexion, produced exceedingly slow motion.

Fig. 5.



There is a slight unbending when the load is removed, showing that the crystal is slightly elastic, but most of the bending is permanent.

The 40 grams load produced more rapid bending than the 50 grams first applied. This must be due to the previous history being different in the two cases. The 40 grams, it will be noticed, is applied just after the 100 grams, and there may be a slow return still taking place. On again removing the load, it is seen that there is an elastic return. The 70 grams load produced a rate of deflexion of about the value that might be expected ; but produced fracture in the crystal at the clamp, the point of maximum strain.

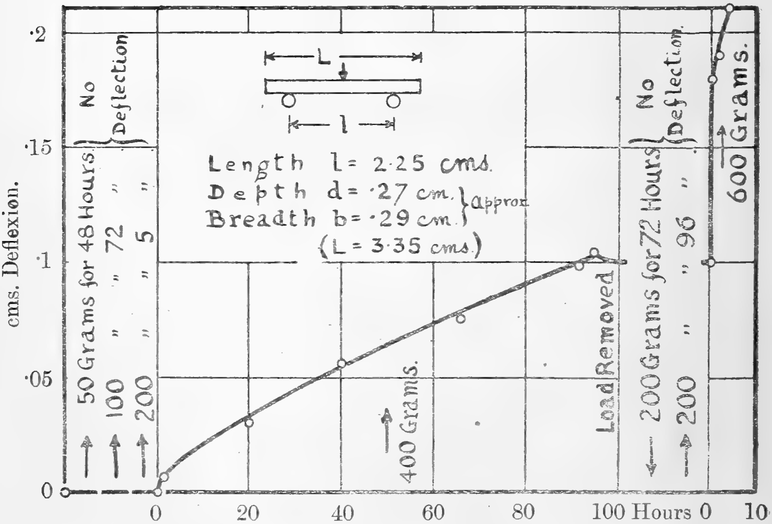
The results of these experiments show that the bending is at first rapid after applying a load. Then it approaches more nearly a constant rate ; finally either rapidly breaking, or, perhaps in some cases of small load, tending to a limiting value. The rate of bending increases rapidly with increased load, being by no means proportional to the load. Further, a small elastic return is indicated.

Since the clamping of the crystal must produce unknown local stresses, experiments were next made by supporting a crystal near its two ends on small steel pegs, and loading it in the middle by means of a small loose wire loop (see Pl. I. fig. 1). This loop would produce much less local disturbance than the wedge-form of end-clamp. This fresh method of support is discussed more fully later in the paper.

The crystal used for the test shown in fig. 5 was broken near one end as a result of the test, and was but little bent in the remaining portion. It was, therefore, used again for this new method of experiment. It was supported on points 2·25 cm. apart, and loads of 50, 100, and 200 grams were applied seriatim. The deflexion was exceedingly slow, and a load of 400 grams was, therefore, applied. This experiment gave results shown in fig. 6. On removing the load, the elastic return could not be detected with certainty. On applying a load of 200 grams in the opposite sense (*i. e.* by inverting the crystal) no deflexion could be detected. On again reversing and applying 200 grams still no effect was observed. When 600 grams load was applied, rapid bending took place and fracture followed.

Thus again we find that small loads produce a very small deflexions and large loads considerable change. The 400 gram load in fig. 6 ought to give the same maximum stress as about 75 grams in fig. 5. And, if the bending took

Fig. 6.

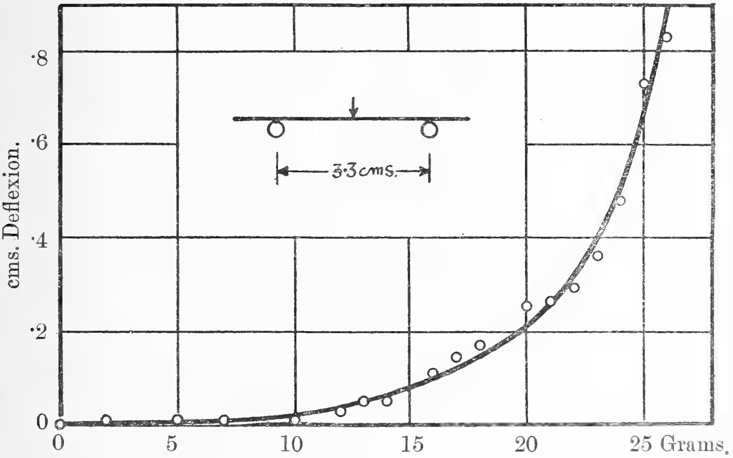


place chiefly at the point of maximum bending moment in each case (as it appeared to), the deflexion produced by this 75 grams ought to have been about twice that produced by the 400 grams in an equal time in the other experiment. It will be seen that this is not the case. If, however, in place of the load of 75 grams, one of about 55 or 60 grams had been applied, the deflexions would have corresponded.

When, in place of crystals of about 2 mm. thickness, considerably thinner ones were placed on the two supports, the non-recoverable immediate bend, that can be detected in figs. 3 to 6, was found to be much more pronounced.

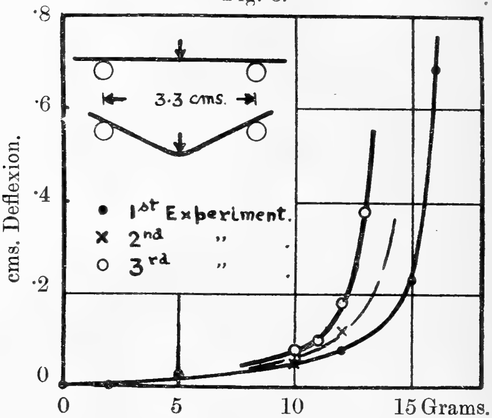
If the load be removed after this initial bend, and then replaced carefully, no further bending will ensue. If, however, a larger load be applied a larger initial bend is produced. Fig. 7 shows the deflexion produced by this method. A small load is first applied, and the deflexion noted. The load is removed, leaving the crystal bent. A rather larger load is applied; this causes an increased deflexion. The new deflexion is measured and the load again increased. That this effect was not due to friction at the supports was shown by repeating the experiments using the single support method. Fig. 8 shows three curves obtained with a second crystal. For the second of the three curves, the crystal had been straightened by hand; and yet again for the third. The difference is thus an indication of the effect bending has on the plastic properties. In this case, it has caused

Fig. 7.



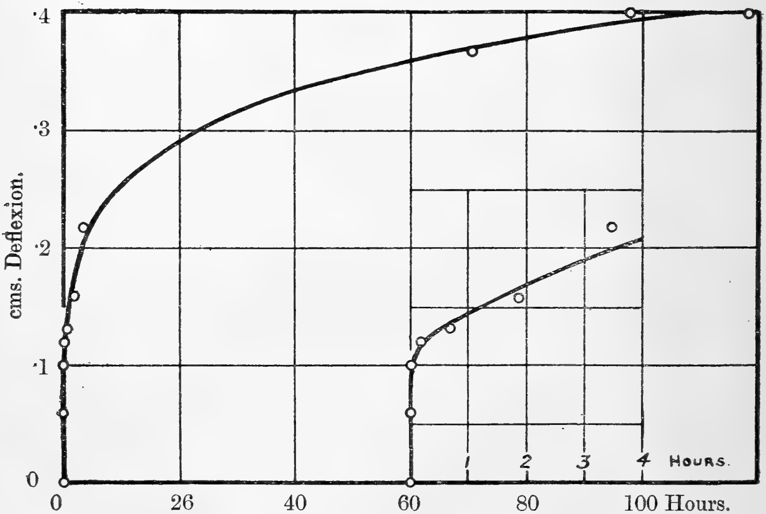
subsequent bending to be slightly larger for the same load. In these experiments on the initial bend, the crystals were by no means ideal single ones, and it is possible that, at any rate in part, this effect was due to the complexity of the specimens.

Fig. 8.



The amount of bending at different times after loading was next investigated in the case of a crystal supported

Fig. 9.



near both ends and loaded at the middle. Then crystals were used in order to observe both the "initial" bend and the subsequent bending. Fig. 9 shows the result of one of the experiments. The part that is shown with a larger scale along the time axis, shows the portion of the bend which may be called initial. In this case, ·05 cm. depression resulted at once on loading, and ·10 cm. within two minutes. After thirteen minutes this had increased to ·12 cm., which represents the initial bend. In obtaining the measurements plotted in figs. 7 and 8, the successive loads were put on at as nearly equal intervals as possible, and thus fair values proportional to the initial bend were obtained. It will be seen, however, that, though there is a large initial bend, it is not possible accurately to discriminate between it and the subsequent bending, which tends to become slower and slower as time goes on.

It will be seen that the bending curves for a crystal supported near its ends are similar to those obtained when one end was clasped and the load applied at the other.

An attempt was made to observe the amount of stretching of a crystal clamped vertically and supporting a considerable weight from its lower end. Many difficulties, however, were met with. Thick crystals could only be obtained of an inch or two in length. It was necessary to observe marks on their surface with a cathetometer. On each occasion, before any definite increase in length could be observed, fracture occurred. In one case of breakage under considerable load, the stress was calculated and the same crystal arranged to have the same maximum stress in a bending experiment. The rate of deflexion was exceedingly slow (·01 cm. in one and a half months). Thus the stress that resulted in fracture in the direct tension experiments was hardly large enough to give observable deflexions by the bending method. It seems, at first, that this means the failure of the method of experiment most likely to lead to results the meaning of which can be understood. And, no doubt, if this method were successful, important and easily interpreted results would ensue. However, it must be remembered that we have no direct evidence that a simple tension will result in a plastic flow. This point will be discussed more fully later in the paper.

The change of form is not due to a simple viscous flow, for the rate of bending for a given crystal and load is not constant, but generally decreases from the time of loading onwards; this is the characteristic of a plastic change. Thus, there is not a unique value for the viscosity of the substance. It may, however, be of interest to proceed as if the flow were viscous and to calculate from the slope of the portion of the curve after the initial rapid bending, a pseudo-viscosity, which will give some means of comparing the rate of bending with that of viscous solids under similar conditions.

TABLE I.

Method of Experiment.	Maximum Stress in Dynes/cm^2.	λ.
Cantilever	$4{\cdot}175 \times 10^7$	89×10^{14}
	4·54 ,,	2·09 ,,
	5·85 ,,	·34 ,,
	6·03 ,,	·093 ,,
	8·35 ,,	·30 ,,
Beam	3·13 ,,	over 32 ,,
	4·13 ,,	·033 ,,
	6·20 ,,	7·04 ,,
	9·39 ,,	·44 ,,
	14·55 ,,	·106 ,,
Stretching	2·38 ,,	over 4·09 ,,
Viscosity of Ice	$6{\cdot}0 \times 10^{12}$ to $147{\cdot}7 \times 10^{12}$	(Deeley & Parr, 1913.)

In the above table, λ is what Trouton* terms the coefficient of viscous traction, and is, according to him, three times the viscosity. It will be seen that the values of the pseudo-viscosity vary considerably with the stress.

If we assume the viscosity of ice to be as given by Deeley and Parr †, the above table shows that the rate of bending of the crystals of ammonium nitrate under stresses of the order of those employed in these experiments is about equal to that observed in glacier ice under similar conditions.

Fig. 10

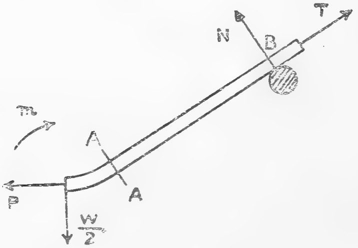


Fig. 10 shows the forces that are acting on one half of the crystal when bent by the second method. At any section of

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* Prof. Trouton, Coeff. of Viscous Traction, etc., Proc. R. Soc. A, vol. lxxvii. 1906.

† R. M. Deeley & P. H. Parr, Phil. Mag. xxvi. pp. 85–111, July 1913, and xxvii. pp. 133–176, Jan. 1914.

the crystal AA, if we consider the forces acting on the right hand side, we see that the unknown frictional force T does not take any appreciable part in the bending. Further, the moment of the force N is almost directly proportional to the distance measured along the crystal, since with even considerable bending each half of the crystal is almost straight. (See fig. 12 *b*.)

For the elastic bending of a uniform beam we have :—

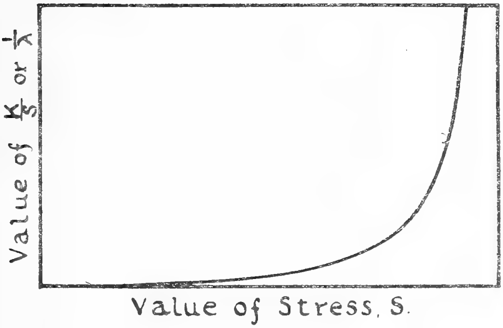
$\frac{1}{\rho} = \frac{M}{EI}$, where $\rho =$ radius of curvature of beam.

M = applied bending moment at the point,
E = Young's modulus for the beam,
I = moment of inertia of a transverse section (constant).

For simple viscous flow, the equation for calculating bending may be written

$$\frac{d}{dt}\left(\frac{1}{\rho}\right) = \frac{M}{\lambda I}.$$

Fig. 11.



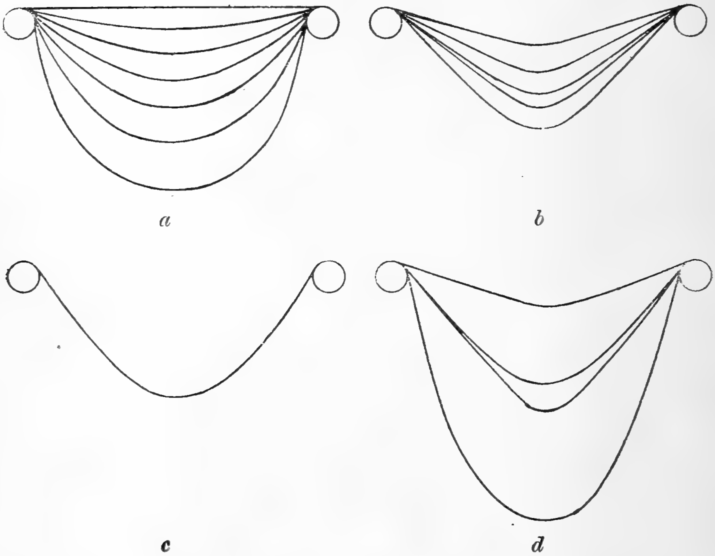
Thus for moderate deflexions, the curved shapes for elastic and simple viscous bending are of approximately the same form, being identical for very small amounts of bending.

It should be noticed that, as the crystal bends, its ends slide over the pegs inwards. Thus the parts now forming the ends of the curve have not been subject to bending for the full time. They should thus be less bent than the corresponding elastic curve.

On comparing these (see figs. 12 *a* and 12 *b*), we find that in either case the part concerned is sensibly straight, so that no appreciable alteration is caused by this sliding over.

This fig. 12 shows for comparison the forms of the following specimens when subjected to a central load: (*a*) the elastic curves (theoretical), (*b*) the curved forms of some bent crystals, (*c*) the curve of a celluloid strip, and (*d*) some curves obtained with strips of lead.

Fig. 12.



It will be seen that the celluloid is more curved at the central parts than the corresponding elastic curves. This is due, partly at least, to there being a plastic bend of the celluloid at the central portion (as was noticed on removing the load). In the case of the lead, two of the curves shown have almost equal depression. Of these, the one with the sharper bend was bent more rapidly than the less pointed one (*i.e.* a larger load was applied). The specimens were made from the same sheet and of the same dimensions. In the case of more rapid bending where the stress was everywhere greater than in the slower case, the bending is more pronounced where the bending moment is greatest. Thus the rate of bending increases more rapidly than in direct proportion to the stress. This is in agreement with the results obtained by Andrade*. If we do not consider his initial

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* Andrade, "On the Viscous flow of Metals," Proc. R. Soc. A, vol. lxxxiv. 1910; & vol. xc. 1914.

" β " flow, he showed that for lead $\frac{1}{\lambda}$ (or $\frac{K}{S}$ in his notation) increases very rapidly as the stress reaches a certain value (see fig. 11).

Since the curves of the bent crystals are more curved at the centre than the corresponding elastic curves, which approximate to the simple viscous curves, it is evident that also in the case of the crystals, the rate of bending increases more rapidly than in direct proportion to the stress or bending moment. This agrees with the results of the tests on bending crystals, described earlier in the paper.

When bending has taken place, the inner edge of the crystal seems to have suffered a compression, and the outer an extension, near the bend (just as in the case of elastic bending), as observation indicates that there is no longitudinal slip detectable at the ends.

PART II.

Let us now turn to the optical properties of the bent crystals. Thin crystals were bent into loops by the fingers. These loops were then ground on a smooth stone or fine ground glass in oil (since water dissolves the crystal). After both faces of the loop had been ground, it was examined between crossed nicols. If no interference effects were noticed, it was bent to a smaller radius. Then, if thin enough and smoothly enough ground, it was mounted in Canada balsam dissolved in xylol.

Straight crystals of ammonium nitrate show straight extinction, when examined between crossed nicols. Then, if the crystals were bent, we might expect extinction lines consisting of four dark transverse bands, equally distributed round the circular loop of the crystal. Fig. 13 (Pl. I.) shows the result that was obtained.

It will be seen that the transverse dark bands are not radial, but slope in this case at about 45° to the radius. If the crystal is of the full orthorhombic symmetry, there is no reason to expect the slope to be in one sense more than in the other. And it is found that in some slides the sense changes. The sense is probably determined by the exact method of bending, though in what way has not been ascertained. Local irregularities may produce sudden changes in sense, as shown in fig. 14 (Pl. II.), which shows three positions of the same slide, obtained by a small rotation between each figure and the next. The sense of the line changes, and then it returns to its original direction.

The angle of slope is, however, as nearly as can be estimated, of constant value. When the sense is changing, an intermediate portion may show a double dark extinction-line or cross. Or the sense may change half-way radially across the crystal, causing a "V" shaped band.

To a first approximation, the band is at 45° to the radius of the arc formed by the crystal. It is difficult to measure this angle accurately, and it is possible that it is not quite constant.

On the other hand, it will be seen, on looking at the figures in the following table, that for all radii of curvature and widths of crystal examined, the slope is not greatly removed from 45°.

Table II.

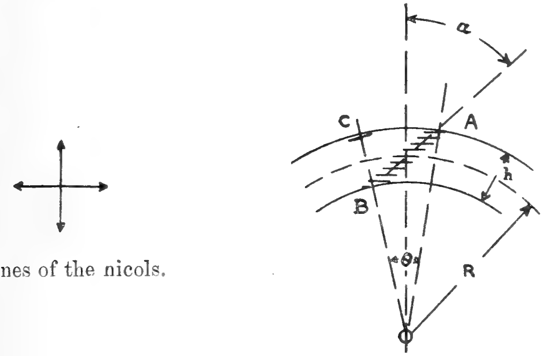
Radius of Curvature.	Width of Crystal.	$\frac{\text{Radius}}{\text{Width}}$.	Slope of Extinction-line in degrees.
17	2·1	8·1	44
30	2·0	15	64
17	1·8	9·45	60
10	1·6	6·25	53
7·5	2	3·75	55 to 64
10	2	5	50
7·5	1·6	4·69	40

If these measurements be used to plot the slope of the extinction-line against $\frac{\text{Radius}}{\text{Width}}$, no connexion between them can be found.

Some of the crystals were bent in such a direction that the plane containing the optic axes was at right angles to the plane of bending.

The plane containing the optic axes is parallel to the principal section of one of the nicols at all points where extinction is occurring on any one dark extinction-line AB (fig. 15). Thus the angle between this plane at any point and the radius to that point would vary with the distance of the point from the inner edge of the curve. Let α be the slope of the dark line as indicated in fig. 15. Then the change in direction of the plane containing the optic axes on passing radially from B to C is equal to the change in direction on passing from A to C, and is thus also equal to the angle subtended by the extinction-line at the centre of curvature of the crystal (if α is constant for this part of the crystal).

Fig. 15.



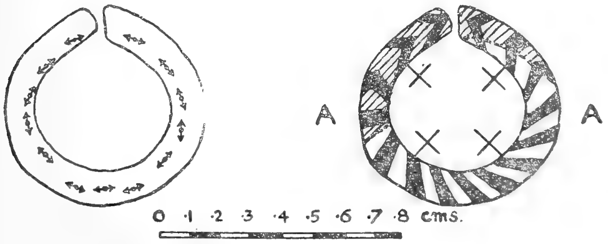
Planes of the nicols.

Or θ, this change in direction in crossing the crystal radially from B to C, is given by

$$\theta = \frac{h}{R} \tan \alpha. \qquad \text{(Approximately.)}$$

Experiments were carried out with the slide shown in fig. 13 (Pl. I,) in order to determine the direction of the plane containing the optic axes at different points. The "eyes" were observed by employing convergent polarized light and an auxiliary lens in the microscope tube which enabled the back focal plane of the objective to be viewed. Fig. 16 *a*

Fig. 16.



a. b.

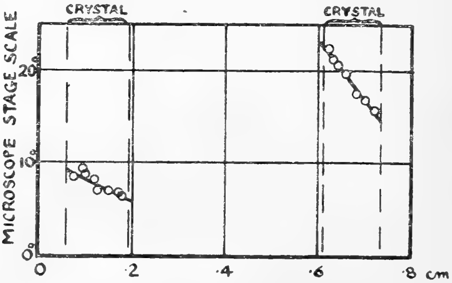
shows the direction of the plane at different points. This was determined by moving the slide over some graph paper to the desired positions ascertained by means of a scale drawing of the slide prepared previously. A high-power objective was used throughout, in order to examine small regions only of the crystal.

It will be seen that, to a first approximation at any rate, the plane containing the axes keeps tangential to the curve. That is to say, that its direction relative to the crystal is little changed by bending.

Fig. 16 *b* is a composite diagram showing the form of the dark bands exhibited by the specimen (Pl. I. fig. 13) in different successive orientations, the portion shaded being irregular. (For some of the positions of the extinction-lines, the corresponding direction of the planes of the nicols is indicated by a small cross.)

An experiment was next undertaken to see in what way the plane changes direction as we pass across the crystal. The microscope stage was slightly rotated till the two branches of the dark brush observable in convergent polarized light united to form a cross. The reading on the stage rotation scale was then taken. This was repeated for different points lying on a transverse line AA, fig. 16 *b* (crossing the bent crystal twice). The results indicate a value of the change in direction of the plane that is in moderate agreement with the predicted results. The differences are probably due to the dark line not having a quite constant value of α at different adjacent points along the length of the crystal, or to the difficulty in making observations exactly along the desired line AA. These would both affect the predicted results. The measured results are plotted in fig. 17. The points lie along straight lines, which is to be expected, since the dark extinction-lines are straight. Also the left-hand side yields an almost horizontal line, corresponding to the dark line sloping very little on this side (see Pl. I. fig. 13). But the slope of the extinction-line on the right-hand side,

Fig. 17.



as deduced from these results, is about 20°. A microscope having both rotating and travelling stage was not available; so in this, as in the previous experiment, the position of the slide had to be determined by fine graph paper.

The angle between the apparent optical axes (2 E) was measured in these and other experiments, and showed no change from the normal value that could not be accounted for by experimental error.

PART III.

In considering the internal changes that accompany bending, it is clear from the optical results that the crystalline structure is by no means destroyed, nor is it greatly altered. This result may be contrasted with those obtained by Terada* in the case of rock salt. The crystals were bent between two parallel cylinders whilst being heated (at temperatures of at least 150° C.), one axis of the crystal being parallel to that of the cylinders. When examined by the X-ray method, the crystal was found to have been changed into a number of lengths having the space lattice unaltered, with the spaces between these lengths filled by mixtures of broken crystal. When the bending was carried out at 100° C., the original structure was quite destroyed. Later work has been done by Carmak† on this subject.

The bending is probably plastic in its nature. It seems difficult to imagine a purely viscous flow taking place in a crystal. Unless the atoms could migrate to cause this change in shape, the space lattice would have to be distorted to an indefinite extent (in the case of a longitudinal pull, the spaces in one direction becoming longer and longer). Nor is it possible to regard the structure as a set of small crystals originally arranged at random in a matrix and later becoming aligned‡.

The slight change in the direction of the plane containing the optic axes, on crossing the width of the bent crystal, corresponding to the slope of the extinction-lines, might be supposed due to the effect of a residual strain, superposed on the other normal optic properties. Since the axes of the optical ellipsoid are coincident with the crystallographic axes, we see by symmetry that this strain would have to be oblique

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* T. Terada, *Mathematico-Physical Soc.*, Tokyo, Proc. vii. pp. 290–291, May 1914.

† P. Carmak, *Phys. Zeits.* xvii. p. 556, Nov. 15, 1916.

‡ Andrade, *loc. cit.*, where a full discussion of the distinction between plasticity and viscosity is given.

to the axes of the crystal, or no rotational change of the plane containing the optic axes would result. This strain would also have to vary gradually in amount from the inner to the outer curved faces. It is, however, desirable, in the case of a crystal, to attempt to explain the results in terms of changes in the space lattice. Some internal change may take place. This seems, perhaps, more likely, since ammonium nitrate crystallizes in different forms at different temperatures. Also its molecule is fairly complex, giving a complex lattice and opportunity for internal change. The optical results would suggest that the internal change might consist of a rotation.

If the space lattice is subject to slight forces, we should expect a slight deformation to result, which would disappear when the forces were removed. If the forces were such that the crystal was bent into an arc of a circle, we should have to suppose that the distance between adjacent atoms was smaller at the inner curved face than at the outer. This would appear an impossible configuration for equilibrium, when the external forces are removed.

When a long crystal is bent (as in fig. 1, Pl. I.) near its centre, the ends remain almost unaffected. Thus a sharp "V" shape may be reached by the crystal. If simple slipping is assumed, in order to account for the bending shown in figs. 1, 13, and 14 (Pls. I. & II.), the slip must be supposed parallel to the length of the crystal.

The change on bending would then exactly correspond to folding a book, as supposed by McConnel*. This means that slip would be supposed to occur along the whole length in order to produce the sharp and purely local bend at the centre. This seems exceedingly unlikely; and attempts to determine whether there has been slip at the ends, by observing the relative positions of marks on the surface of the crystal both before and after bending, indicate that probably the slip is not appreciable. If such slip does not occur, it is necessary to conclude that the spacing of the lattice is greater at the outside of the curve than at the inside, unless some atoms have migrated across the crystal from the inner to the outer edge.

If we assume that simple slip has not occurred, but that the lattice spacing is greater at the outer edge, we have to explain how there can be stability after the bending forces have been removed. We have also to explain the observed slight rotation of the plane containing the optic axes, as we

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* McConnel, Proc. R. Soc. vol. xlviii. p. 259 (1890); and vol. xlix. p. 323 (1891).

cross the bent crystal radially. This, it should be remembered, would also have to be accounted for if we assumed that the plasticity was due to slip.

When a crystal of ammonium nitrate is bent, probably causing the atomic spacing to be greater on the outer edge of the curve, we may suppose that a rotational change occurs in the structure of the crystal near the bent portion, causing the atoms to keep in equilibrium at their new distance apart. This rotational change would have to vary steadily in extent from the inner to the outer curved faces, as the spacing would vary continuously. Since refraction apparently depends on the atomic structure, this rotational change would probably cause a change in the direction of the plane containing the optic axes, which would also vary steadily from the inner to the outer curved faces. If the rotational change were proportional to the fractional elongation or contraction, the slope of the extinction-line would be almost independent of the curvature, as appears to be the case (provided the bending is enough to render the line quite distinct). Such a rotation or orientation of the atomic groups would explain the very rapid bending which takes place at first when a load is applied.

Thus the hypothesis of a rotation in the atomic groups seems, on the whole, the most plausible one; and it is in fair agreement with all the observations. On the other hand, it is still possible that some slight longitudinal slip may occur (though attempts to detect it failed). But even if this were the case, the optical properties show, almost beyond doubt, that a rotation forms at least part of the change accompanying the plastic bending of these crystals.

In conclusion, I wish to thank Dr. Porter of University College, London, for his kindness and the help that he gave me whilst I was carrying out this investigation, which consists of a continuation and extension of an unpublished preliminary investigation which he had made.

SUMMARY.

When Ammonium Nitrate was crystallized from an aqueous solution at laboratory temperatures, long needle-like crystals formed, these being often complex in cross-section. Sometimes flat or tabular crystals formed instead. It was found that both had 2 E about 59° 30′ as given by Groth; also that $r<v$ and the crystal is optically negative.

The needle-like crystals could be bent or twisted in the fingers. There is a slight elastic return. For rapid motion the crystals are brittle. It was not found possible to observe the effect of a direct tension on the long crystals.

Prolonged grinding, or heating in air to 30°–40° C. produces a white opaqueness. This probably indicates the transition to Groth's second form, said to occur at 32° C.

On applying a load to a crystal supported horizontally, the initial rate of bending is the greatest (except when fracture is occurring). The rate of bending increases much more rapidly than in direct proportion to the stress (for the loads that were employed). This result, deduced from bending experiments, is verified by the pointed shape of the curve assumed by the crystal when bent.

A pseudo-viscosity $\left(\frac{\lambda}{3}\right)$ (calculated as if the bending were viscous) gave the results of the order of that of glacier ice.

Curves showing how the "initial" bend (obtained immediately on applying a load) varied with this load were obtained. This initial bend may possibly be due to the narrow crystals which exhibited it being complex in structure.

Attempts to determine whether slip occurred parallel to the length of a bent crystal indicated that if there were any, it was not large. Simple slip in other directions could not account for all the bent shapes obtained. The form consisted in some cases of a sharp "V" shape, the bending being confined to a very small central portion of the long crystal.

In bent crystals, ground and mounted, the extinction-lines seen on viewing the specimen between crossed Nicol prisms, are not normal to the curved crystal. These lines appear always to be inclined at roughly 45° in either direction to the radius of curvature of the crystal. The sense probably depends on irregularities and the exact mode of bending. It may change rapidly. There is a slight progressive change in the direction of the plane containing the optic axes, on crossing the crystal radially. To a first approximation, however, the plane remains parallel to the length of the crystal. The angle between the optic axes is little, if at all, changed by the bending.

On the whole, the structure, as indicated by the optical properties, is little changed by the bending. This may be compared with the experiments of T. Terada and P. Carmak.

It seems probable that (whether or not there be any simple slip) a local rotational atomic adjustment occurs, causing the crystal to remain in equilibrium, though bent. This rotation would vary steadily in amount as the crystal is crossed radially; and it would probably be accompanied by a rotation of the plane containing the optic axes, which would also vary gradually in extent, as is observed. If the rotation were proportional approximately to the fractional change in length at each point, the dark extinction-band would be a straight line; and its slope would vary but little with the radius of curvature and width of the crystal. Further, the rapidity of the initial bend on applying a load might be expected on a theory involving rotational local change. And, on the other hand, any theory based on a simple slip would have to explain the rotational effect shown by the optical properties.

II. *Surface Energy, Latent Heat, and Compressibility.*

By D. L. HAMMICK, *Oriel College, Oxford*[*].

ACCORDING to the classical researches of Young, the relation between surface energy and cohesion or internal pressure takes the form

$$\rho = \frac{K \cdot r}{3},$$

where ρ = surface energy,
K = internal pressure,
r = radius of action of the cohesive attraction.

Young's result is independent of molecular views as to the nature of liquids; if, however, we adopt the view that internal pressure is a molecular phenomenon and that the radius of action of the inter-molecular attraction is most probably equal to the radius of the molecule itself, we have

$$\rho = K \cdot \frac{d}{6},$$

where d = molecular diameter.

If K can be represented as a function of the specific volume v of a liquid by $\frac{a}{v^2}$, we get

$$\rho = \frac{a}{v^2} \cdot \frac{d}{6} \cdot \qquad \text{(i.)}$$

It has been shown (Phil. Mag. xxxix. p. 32) that the van der Waals expression for the internal pressure in the form $\frac{a_T}{v^2}$, a being taken to be a function of the temperature, gives

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[*] Communicated by the Author.

satisfactory values for the latent heat of vaporization when used in Bakker's equation

$$\lambda = \int_{v}^{v\,\text{vap.}} \frac{a_T}{v^2} . dv + \frac{RT}{M}$$

$$= a_T \left(\frac{1}{v} - \frac{1}{v_{\text{vap.}}} \right) + \frac{RT}{M} . \quad . \quad . \quad . \quad . \quad \text{(ii.)}$$

In other words, λ_1, the internal latent heat, is equal to $\frac{a_T}{v}$ at low temperatures, where $-\frac{1}{v_{\text{vap.}}}$ may be neglected, and we have :

$$\rho = \frac{a_T}{v} . \frac{1}{v} . \frac{d}{6}$$

$$= \frac{\lambda_1}{v} . \frac{d}{6},$$

$$\text{or} \qquad \lambda_1 = \frac{6\rho v}{d} . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad \text{(iii.)}$$

This result has already been arrived at on general grounds (Phil. Mag. xxxviii. p. 240), and it would therefore appear that the internal pressure of a liquid at any temperature T is $\frac{a_T}{v^2} = \lambda_1 . \frac{1}{v} = \lambda_1 s$ (s=sp. gr. of liquid). In words, the internal pressure of a liquid is equal to the internal latent heat per unit volume, in accordance with the original view of Dupré.

When vapour pressure is negligible in comparison with internal pressure, we may take the pressure in a liquid as $K = \frac{a_T}{v^2} = \lambda_1 s$, whence $V\frac{\partial K}{\partial V} = -2\frac{a}{v^2} = -2\lambda_1 s$ (in mechanical units). Now $V\frac{\partial K}{\partial V} = \frac{1}{\beta}$, where β is the coefficient of variation of the volume of the liquid with internal pressure; this may reasonably be taken as equal to the compressibility of the liquid. Hence

$$\frac{1}{\beta J} = -2\lambda_1 s \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad \text{(iv.)}$$

(J = mechanical equivalent of heat).

The above result involves the assumption that the term a_T is independent of the volume. If a_T is a function of v, we should have

$$\frac{\partial K}{\partial v} = \frac{\partial K}{\partial T} \cdot \frac{\partial T}{\partial v}$$

$$= \left(- \frac{2a}{v^3} \cdot \frac{\partial v}{\partial T} + \frac{1}{v^2} \cdot \frac{\partial a}{\partial T} \right) \cdot \frac{\partial T}{\partial v}$$

$$= \left(- 2\lambda_1 s \, . \, \alpha + \lambda_1 s \, . \, \frac{1}{a} \cdot \frac{\partial a}{\partial T} \right) \cdot \frac{\partial T}{\partial v},$$

where $\alpha = \frac{1}{v} \frac{\partial v}{\partial T}$ = coefficient of cubical expansion.

$$\therefore \qquad v \frac{\partial K}{\partial v} = \lambda_1 s \left(- 2\alpha + \frac{1}{a} \cdot \frac{\partial a}{\partial T} \right) . \, v \, . \frac{\partial T}{\partial v}$$

$$= \lambda_1 s \left(- 2 + \frac{1}{a} \frac{\partial a}{\partial T} \cdot \frac{1}{\alpha} \right) . \qquad . \quad . \quad . \quad . \quad . \qquad (v.)$$

From (ii.) and (iii.) above, we have

$$\frac{a_T}{v} = \lambda_1 = \frac{6 \rho v}{d},$$

$$\therefore \qquad a_T = \frac{6 \rho v^2}{d}$$

$$\text{and} \qquad \frac{1}{a} \cdot \frac{\partial a}{\partial T} = \frac{1}{\rho} \cdot \frac{\partial \rho}{\partial T} + 2\alpha.$$

In (v.) we get :—

$$v \frac{\partial K}{\partial v} = \frac{1}{\beta} = \lambda_1 s \, . \, J \left(\frac{1}{\rho} \frac{\partial \rho}{\partial T} \cdot \frac{1}{\alpha} \right),$$

$$\text{or} \qquad \frac{1}{\beta J} = \lambda_1 s \, . \frac{\bar{\rho}}{\alpha} \qquad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (vi.)$$

$$\left(\bar{\rho} = \frac{1}{\rho} \frac{\partial \rho}{\partial T} \right).$$

In short, if a_T is independent of volume (and hence pressure) we expect to find

$$\frac{1}{\beta J} = - 2 \lambda_1 s, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (iv.)$$

whereas if a_T is a function of v,

$$\frac{1}{\beta J} = \lambda_1 s \,.\, \frac{\bar{\rho}}{\alpha}\text{*} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad \text{(vi.)}$$

By substituting in (iv.) and (vi.) the value $\frac{6\rho}{d}\cdot\frac{1}{J}$ for $\lambda_1 s$ (of (iii.)), we get:

$$\frac{1}{\beta} = -2\,.\,\frac{6\rho}{d} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad \text{(vii.)}$$

and

$$\frac{1}{\beta} = \frac{6\rho}{d}\cdot\frac{\bar{\rho}}{\alpha} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad , \qquad \text{(viii.)}$$

In order to test the above conclusions, data are needed for surface energy, latent heat, specific gravity, and compressibility. For the latter, Tyrer's values (Trans. Chem. Soc. 1914, p. 2534) for pressures between 1–2 atmospheres have been used; his paper contains data for the evaluation of α, the coefficient of expansion. Tyrer's results extend over the range 0° C.–boiling-point; hence the latent heat data available are limited. Those used are due to Young unless otherwise stated. The surface energy data have been taken from Jaeger (*Zeit. Anorg. Chem.* ci. pp. 1–314 (1917)) and from Ramsay, Shields, and Aston's results. It is noticeable that though Jaeger's results, obtained by the "gas bubble" method, agree very well with the Ramsay values at low temperatures, marked divergencies occur at higher temperatures with the more volatile liquids. In most cases therefore the two sets of results have been plotted and those values for the surface energy used that lie on the straight line (or smoothed curve) that runs through Jaeger's low temperature and Ramsay's high temperature values.

In the following Tables, (iv.) and (vi.) are applied to some liquids and (vii.) and (viii.) to others. The number of liquids available is necessarily limited by the paucity of data. In the column headed ϵ in each table is recorded either the ratio $-\frac{\frac{1}{\beta J}}{\lambda_1 s} = \epsilon$ or $-\frac{\frac{1}{\beta}}{\frac{6\rho}{d}} = \epsilon.$ It will be seen from (iv.), (vi.), (vii.), (viii.) that the ratio ϵ should be equal to $\frac{\bar{\rho}}{d}$ if $a_T = f(v)$ and equal to -2 if $a_T \neq f(v)$. In the cases where d, the molecular diameter, is required, it has been calculated from the latent heat at the boiling-point from the relation

$$\lambda_1 \,.\, J = \frac{6\rho v}{d}.$$

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* The sign of $\bar{\rho}$ will clearly be negative.

TABLES.

Ether.

T °C.	$\beta\times10^{12}$ dynes×cm.^{2}	$\lambda_1 s$ cals.÷c.c.	ρ ergs×cm.^{2}.	$-\bar{\rho}$.	α.	$\epsilon=\frac{\frac{1}{\beta J}}{-\lambda_1 s}$.	$\frac{\bar{\rho}}{\alpha}$.
−30	108 (extr.)	72·4 (extr.)	28·5	...	...	−3·04	...
−20·5	119·5 ,,	68·5 ,,	21·5	$\cdot 0_2581$	$\cdot 0_2139$	2·92	−4·18
0	153	64·6	19·2	$\cdot 0_2658$	$\cdot 0_2151$	2·41	4·36
10	170·3	61·7	...	$\cdot 0_2704$	$\cdot 0_2157$	2·27	4·48
20	189	59·9	17·1	$\cdot 0_2758$	$\cdot 0_2162$	2·10	4·68
30	211·8	56·2	...	...	...	2·01	...
35	224 (extr.)	52·9	...	...	...	2·02	...

Benzene.

T.	$\beta\times10^{12}$.	$\lambda_1 s$ *.	ρ.	$-\bar{\rho}$.	α.	$\epsilon=\frac{\frac{1}{\beta J}}{-\lambda_1 s}$.	$\frac{\bar{\rho}}{\alpha}$.
5·4	81·95	92·2	30·6	$\cdot 0_2435$	$\cdot 0_2117$	−3·15	−3·72
20	95·65	88·4	28·4	$\cdot 0_2469$	$\cdot 0_2120$	2·83	3·90
40	111·4	88·2	25·6	$\cdot 0_2521$	$\cdot 0_2125$	2·60	4·17
60	130·0	76·3	23·0	$\cdot 0_2580$	$\cdot 0_2131$	2·41	4·43
70	143·2	73·5	21·6	$\cdot 0_2617$	$\cdot 0_2134$	2·27	4·60
80	156·5	70·0	20·4	$\cdot 0_2653$	$\cdot 0_2138$	2·17	4·73

* From Young (Kaye & Laby) and from Griffiths & Marshall (Phil. Mag. [5] xlvi. p. 1 (1896).

Carbon Tetrachloride.

T.	β.	$\lambda_1 s$ *.	ρ.	$-\bar{\rho}$.	α.	$\epsilon=\frac{\frac{1}{\beta J}}{-\lambda_1 s}$.	$\frac{\bar{\rho}}{\alpha}$ †.
−20	81·0 (extr.)	83·2	31·3	$\cdot 0_2426$	$\cdot 0_2112$ (extr.)	−3·55	−3·80
0	91·2	79·1	...	...	...	3·31	...
20	105·0	74·8	...	...	...	3·05	...
40	124·5	70·3	...	...	...	2·73	...
60	147·5	66·3	20·8	$\cdot 0_2529$	$\cdot 0_2132$	2·44	4·00
80	172·4	62·1	18·7	$\cdot 0_2596$	$\cdot 0_2135$	2·24	4·28
100	...	...	16·48	$\cdot 0_2676$	$\cdot 0_2143$	...	4·73

* Calculated from $\lambda s=\frac{6\rho}{d}$: $d=4{\cdot}35\times10^{-8}$ cm.

† Bakker (*Zeit. Phys. Chem.* lxxxvi. p. 129 (1914)) gives $\frac{\rho}{\alpha}=3{\cdot}72$ at 10° C.

Carbon Bisulphide.

T.	β.	$\lambda_1 s$.*	ρ.	$\bar{\rho}$.	a.	$\epsilon = \dfrac{\frac{1}{\beta J}}{-\lambda_1 s}$.	$\dfrac{\bar{\rho}}{a}$.
−20	71·0 (extr.)	122·4	36·4	$\cdot 0_2 381$	$\cdot 0_2 112$	−2·71	−3·40
0	81·5	114·9	33·6	$\cdot 0_2 413$	$\cdot 0_2 114$	2·56	3·62
10	87·5	110·1	32·2	$\cdot 0_2 431$	$\cdot 0_2 116$	2·48	3·71
20	93·8	105·9	31·0	$\cdot 0_2 447$	$\cdot 0_2 119$	9·41	3·76
30	100·6	100·8	29·5	$\cdot 0_2 470$	$\cdot 0_2 121$	2·36	3·88
40	107·9	96·0	28·1	$\cdot 0_2 494$	$\cdot 0_2 123$	2·31	4·01
50	116·3	91·2	26·7	$\cdot 0_2 520$	$\cdot 0_2 126$	2·26	4·13

* Regnault.

Chloroform.

T.	β.	$\lambda_1 s$.	ρ.	$\bar{\rho}$.	a.	$\epsilon = \dfrac{\frac{1}{\beta J}}{-\lambda_1 s}$.	$\dfrac{\bar{\rho}}{a}$.
−20	72·0 (extr.)	107·5	32·3	$\cdot 0_2 443$	$\cdot 0_2 115$ (extr.)	−3·09	−3·85
0	85·9	95·3	28·5	$\cdot 0_2 479$	$\cdot 0_2 121$	2·92	3·95
20	101·2	89·6	26·8	$\cdot 0_2 534$	$\cdot 0_2 127$	2·64	4·20
30	110·0	86·8	25·3	$\cdot 0_2 565$	$\cdot 0_2 130$	2·51	4·34
50	130·7	81·1	22·5	$\cdot 0_2 635$	$\cdot 0_2 138$	2·26	4·60
60	142·5	78·2	...	...	...	2·15	...
70	157·0 (extr.)	75·3	...	...	...	2·02	...

Ethyl Acetate.

T.	β.	$\lambda_1 s$.	ρ †.	‾.	a.	$\epsilon = \dfrac{\frac{1}{\beta J}}{-\lambda_1 s}$	$\dfrac{\bar{\rho}}{a}$.
−20	82 (extr.)	97·5 *	25·8	$\cdot 0_2 457$	$\cdot 0_2 123$	−2·99	−3·52
0	96·3	89·7	...	...	$\cdot 0_2 127$	2·77	3·77 ‡
10	105·0	85·9	22·2	$\cdot 0_2 531$	$\cdot 0_2 131$	2·65	4·06
30	125·3	79·1	20·0	$\cdot 0_2 590$	$\cdot 0_2 139$	2·42	4·24
50	149·8	72·7	17·5	$\cdot 0_2 674$	$\cdot 0_2 148$	2·20	4·56
60	164·2	69·7	16·3	$\cdot 0_2 724$	$\cdot 0_2 152$	2·09	4·76
70	181·4	66·9	...	...	$\cdot 0_2 156$	1·97	..
80	187·5 (extr.)	64·1	...	...	$\cdot 0_2 152$	1·99	...

* From vapour-pressure at −20° C.
† Bakker (*loc. cit.*). ‡ Ramsay & Shields.

Ethylene Dichloride.

T.	β.	ρ.	$\bar{\rho}$.	a.	$\epsilon = \frac{\frac{1}{\beta}}{-\frac{6\rho}{d}}$.	$\frac{\bar{\rho}}{a}$.
−20	63 (extr.)	37·2	$\cdot0_2391$	$\cdot0_2106$ (extr.)	−3·27	−3·67
0	70·1	34·1	$\cdot0_2427$	$\cdot0_2111$	3·20	3·85
10	75·2	32·6	$\cdot0_2446$	$\cdot0_2114$	3·12	3·91
20	80·7	31·2	$\cdot0_2466$	$\cdot0_2116$	3·04	4·02
40	93·8	28·4	$\cdot0_2512$	$\cdot0_2121$	2·88	4·23
60	109·7	25·6	$\cdot0_2568$	$\cdot0_2127$	2·72	4·47
80	129·6	23·2	$\cdot0_2627$	$\cdot0_2133$	2·54	4·72
90	140 (extr.)	22·2	...	...	2·46	...

$d=4{\cdot}59\times10^{-8}$ cm. (from $\rho=23{\cdot}7$, $v={\cdot}864$, $\lambda_1=64$ (Regnault) at b.p.).

Ethyl Iodide.

T.	β.	ρ.	$\bar{\rho}$.	a.	$\epsilon = \frac{\frac{1}{\beta}}{-\frac{6\rho}{d}}$.	$\frac{\bar{\rho}}{a}$.
−20	76 (extr.)	32·5	$\cdot0_2339$	·00109 (extr.)	−2·83	−3·11
0	85·6	30·4	$\cdot0_2363$	$\cdot0_2112$	2·69	3·24
10	92·4	29·2	$\cdot0_1377$	$\cdot0_2115$	2·59	3·28
20	99·5	28·1	$\cdot0_2392$	$\cdot0_2118$	2·50	3·32
40	115·9	25·9	$\cdot0_2426$	$\cdot0_2123$	2·33	3·46
60	135·7	23·7	$\cdot0_2465$	$\cdot0_2128$	2·17	3·63
70	146·8	22·6	$\cdot0_2488$	$\cdot0_2131$	2·10	3·73
80	157·5 (extr.)	21·5	$\cdot0_2513$	$\cdot0_2134$	2·06	3·83

$d=4{\cdot}19\times10^{-8}$cm. (from $\rho=22{\cdot}5$, $\lambda_1=42{\cdot}4$ (Andrews) at b.p.).

Chlor-benzene.

T.	β.	ρ.	$\bar{\rho}$.	a.	$\epsilon = \frac{\frac{1}{\beta}}{-\frac{6\rho}{d}}$.	$\frac{\rho}{a}$.
−20	60 (extr.)	38·0	$\cdot0_2322$	$\cdot0_3918$	−3·19	−3·50
−10	63·2	36·8	$\cdot0_2332$	$\cdot0_3940$	3·12	3·53
0	67·0	35·7	$\cdot0_2342$	$\cdot0_3949$	3·05	3·60
20	75·2	33·3	$\cdot0_2366$	$\cdot0_3972$	2·91	3·77
40	85·0	31·0	$\cdot0_2394$	$\cdot0_2100$	2·74	3·94
60	96·3	28·6	0_2427	$\cdot0_2103$	2·65	4·14
80	109·3	26·3	$\cdot0_2464$	$\cdot0_2106$	2·54	4·38
100	127 (extr.)	24·0	...	...	2·39	...

$d=4{\cdot}37\times10^{-8}$ cm. (from $\rho=20{\cdot}2$ (mean) at b.p. and $\lambda_1=67{\cdot}06$).

Toluene.

T.	β.	ρ.	$\bar{\rho}$.	α.	$\epsilon = \dfrac{\frac{1}{\beta}}{-\frac{6\rho}{d}}$.	$\dfrac{\bar{\rho}}{\alpha}$.
-20	72·5 (extr.)	34·2	$\cdot 0_2 353$	·00100	$-3{\cdot}05$	$-3{\cdot}53$
0	79·3	31·6	$\cdot 0_2 380$	$\cdot 0_2 103$	3·03	3·69
20	90·8	29·1	$\cdot 0_2 412$	$\cdot 0_2 107$	2·86	3·85
40	104·7	26·7	$\cdot 0_2 450$	$\cdot 0_2 113$	2·35	3·98
60	121·6	24·3	$\cdot 0_2 494$	$\cdot 0_2 119$	2·52	4·15
80	140·5	22·2	...	...	2·43	...
100	160·5 (extr.)	20·2	...	...	2·33	...

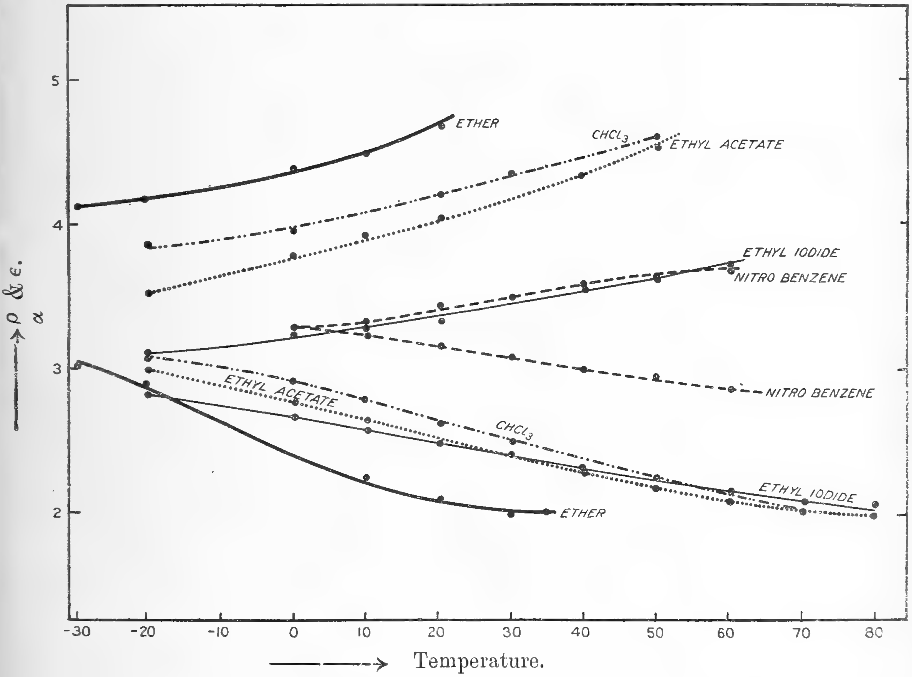
$d=4{\cdot}53\times10^{-8}$ cm. (from $\rho=19{\cdot}5$, $\lambda_1=79{\cdot}0$ at b.p.).

Nitro-benzene.

T.	β.	ρ.	$\bar{\rho}$.	α.	$\epsilon = \dfrac{\frac{1}{\beta}}{-\frac{6\rho}{d}}$.	$\dfrac{\bar{\rho}}{\alpha}$.
0	44·7	44·3	$\cdot 0_2 273$	$\cdot 0_3 83$	$-3{\cdot}29$	$-3{\cdot}29$
10	47·3	43·1	$\cdot 0_2 279$	$\cdot 0_3 834$	3·24	3·34
20	50·0	41·8	$\cdot 0_2 288$	$\cdot 0_3 837$	3·16	3·44
30	52·8	40·6	$\cdot 0_2 297$	$\cdot 0_3 847$	3·08	3·50
40	55·6	39·5	$\cdot 0_2 305$	$\cdot 0_3 856$	3·00	3·56
50	58·4 (extr.)	38·3	$\cdot 0_2 314$	$\cdot 0_3 863$	2·95	3·64
60	61·4 ,,	37·7	$\cdot 0_2 319$	$\cdot 0_3 869$	2·86	3·67

$d=4{\cdot}16\times10^{-8}$ cm. (from $\rho=21{\cdot}2$, $\lambda_1=71{\cdot}2$ at b.p.).

It is apparent from the above tables that the ratio $\epsilon = \dfrac{\frac{1}{\beta J}}{\lambda_1 s} = \dfrac{\frac{1}{\beta}}{\frac{6\rho}{d}}$ tends to approximate to $\dfrac{\bar{\rho}}{\alpha}$ at lower temperatures, and to the value -2 at higher temperatures (near the boiling-point in many cases). The course of the variation of ϵ with temperature is shown for a few examples in the accompanying figure. For each liquid two curves have been drawn. The upper curve in each example shows the variation of the ratio $\dfrac{\bar{\rho}}{\alpha}$ (vertical axis) with temperature (horizontal axis); the lower curve represents the variation of ϵ with temperature. In the case of nitrobenzene, the two curves touch at 0° C., at which temperature those for ethyl iodide are also very close. With ethyl iodide and acetate, chloroform and ether, the ϵ curve touches the -2 axis in the neighbourhood of the boiling-points.



It would thus appear that from (roughly) the boiling-point downwards it becomes progressively necessary to regard "a" as a function of the volume.

Latent Heat and Heat of Expansion.

It will be noticed that at 0° C. the value of ϵ is roughly -3 (mean value for the liquids dealt with is $-2{\cdot}94$, excluding ether). In other words, at 0° C. we have

$$\frac{1}{\beta J} = -3\lambda_1 s \quad \text{approximately,}$$

or

$$\frac{v}{\beta J} = -3\lambda_1.$$

If v_0 is the specific volume at the absolute zero, we have approximately $v=v_0(1+\alpha\mathrm{T})$, where α=coefficient of expansion and T=absolute temperature.

Hence

$$\frac{v_0(1+\alpha\mathrm{T})}{\beta\mathrm{J}}=-3\lambda_1,$$

or

$$\frac{v(1+\alpha\mathrm{T})}{\beta\mathrm{J}}=-3\lambda_1\cdot\frac{v}{v_0},$$

or

$$\frac{v}{\mathrm{J}}\left(\frac{1}{\beta}+\frac{\alpha}{\beta}\mathrm{T}\right)=-3\lambda_1\cdot\frac{v}{v_0}.$$

Now the ratio of critical volume to volume at absolute zero, $\frac{v_c}{v_0}$, is a constant and approximately 4. Putting the ratio $\frac{v_c}{v}=n$, we have

$$\frac{v}{v_0}=\frac{v_c}{v_0}\cdot\frac{v}{v_c}=\frac{4}{n},$$

and from above :

$$\frac{1}{\mathrm{J}}\left(\frac{1}{\beta}+\frac{\alpha}{\beta}\mathrm{T}\right)=-3\times\frac{4}{n}\times\lambda_1 s. \quad . \quad . \quad . \quad \text{(ix.)}$$

$$\left(s=\frac{1}{v}\right).$$

The ratio $n=\frac{v_c}{v}$ has been evaluated for 21 liquids (using Young's data for v_c). Excluding chlor-benzene ($n=2{\cdot}0$) and acetonitrile ($n=3{\cdot}4$), the values of n range between 3·1 and 2·8, the mean value being 3·0. We have, therefore, in (ix.) above :

$$\frac{1}{\mathrm{J}}\left(\frac{1}{\beta}+\frac{\alpha}{\beta}\mathrm{T}\right)=-4\lambda_1 s.$$

But we have found $\frac{1}{\beta\mathrm{J}}=-3\lambda_1 s$ approximately (at 0° C.).

Hence

$$\frac{1}{\mathrm{J}}\cdot\frac{\alpha}{\beta}\mathrm{T}=-\lambda_1 s \text{ approximately at } 0^\circ\ \mathrm{C}. \quad . \quad \text{(x.)}$$

Now $-\frac{\alpha}{\beta}\mathrm{T}.\frac{1}{\mathrm{J}}$ is the "work of expansion" or the heat that must be withdrawn from a body in compressing it isothermally by unit volume. In other words, we should expect to find that at 0° C., the work of expansion of a liquid should be roughly equal to the latent heat per unit volume. That this is the case was discovered empirically by Lewis (*Zeit. Phys. Chem.* lxxviii. p. 24 (1911)).

Surface Energy and Compressibility.

In conclusion, it may be pointed out that since, as shown above, $\frac{1}{\beta}=-\epsilon\lambda_1 s$ (in mechanical units), we have, from (iii.):

$$\frac{1}{\beta}=-\frac{\epsilon \, . \, 6\rho}{d},$$

or

$$\beta\rho=-\frac{d}{6}\cdot\frac{1}{\epsilon} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad \text{(xi.)}$$

Since d is approximately the same for the common organic liquids (about $4{\cdot}5\times10^{-8}$ cm.) and ϵ at 0° C. is not very different from -3, we may expect to find the product $\beta\rho$ approximately constant at 0° C. and equal to

$$\frac{4{\cdot}5\times10^{-8}}{18}={\cdot}25\times10^{-8}.$$

The mean value for $\beta\rho$ from the data used above is ${\cdot}252\times10^{-8}$ at 0° C. It is, of course, well known that the product $\beta\rho$ is by no means constant at other temperatures; this follows at once from (xi.), since ϵ varies with temperature and differently for different liquids. It is to be expected that at higher temperatures, where the ratio ϵ has become -2, $\beta\rho$ would again become approximately constant at the value $\frac{4{\cdot}5}{12}\times10^{-8}={\cdot}39\times10^{-8}$. The empirical relations of Richards and Matthews (*Zeit. Phys. Chem.* lxi. p. 449) and particularly Tyrer (*Zeit. Phys. Chem.* lxxxvii. p. 169), who found respectively that the expressions $\beta\rho^{4/3}$ and $\frac{\beta\rho^{4/3}}{\mathrm{T}^{1/3}}$ are approximately constant over the ordinary temperature ranges, obviously take into account the variation of ϵ with temperature.

Summary.

(1) If the van der Waals constant "a" is independent of volume, it is shown that the compressibility β of a liquid should be connected with the internal latent heat by the relation

$$\frac{1}{\beta}\cdot\frac{1}{J}=-2\lambda_1 s. \quad . \quad . \quad . \quad . \quad . \quad . \qquad \text{(xii.)}$$

If a is a function of v, then

$$\frac{1}{\beta}\cdot\frac{1}{J}=\epsilon\lambda_1 s, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad \text{(xiii.)}$$

where $\epsilon=\frac{\bar{\rho}}{\alpha}$ $\left(\bar{\rho}=\frac{1}{\rho}\frac{\partial\rho}{\partial T}\right.$: coefficient of variation of surface energy with temperature :

$\alpha=\frac{1}{v}\frac{\partial v}{\partial T}$: coefficient of cubical expansion).

It is found, for the limited number of liquids for which data are available, that (xii.) is approached at higher temperatures (the relation holding as such for some liquids at the boiling-point), and that (xiii.) becomes progressively applicable at low temperatures.

(2) Lewis's empirical approximation, that heat of compression is very nearly equal to latent heat of vaporization per unit volume at 0° C., is deduced.

(3) The approximate constancy of the product of compressibility and surface energy at 0° C. is deduced, and the observed lack of constancy at other temperatures accounted for.

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III. *On the Effective Inductance, Effective Resistance, and Self-Capacity of Magneto Windings.* *By* N. W. McLachlan, *D.Sc. Eng., M.I.E.E.* (From the National Physical Laboratory.) *

Synopsis.

(1) Introduction.
(2) Measurement of the Self-Inductance and Effective Resistance of the primary winding by Undamped Oscillations.
(3) Measurement of the Self-Inductance and Effective Resistance of the secondary winding by Undamped Oscillations.
(4) Measurement of the Self-Capacity of and Dielectric Loss in Secondary winding.
(5) Bridge method of Measuring the Self-Capacity of Secondary winding.
(6) Results of Experiments:—
  (*a*) Armature in Housing.
  (*b*) Armature out of Housing.
  (*c*) Effect of Polarization.
  (*d*) Effect of Armature Brass End-plates.
  (*e*) Effective Permeability.
  (*f*) Self-Capacity of Secondary.
  (*g*) Dielectric Loss in Secondary.
(7) Calculation of Energy Loss prior to the passage of the spark.
(8) Alternative method of calculating the Energy Loss.
(9) Comparison of methods used in (7) and (8).
(10) Comparison of Loss with armature in and out of Housing.
(11) The effect of an Air-core Inductance in the Primary circuit on the Damping.
(12) Self-Resonance of the Secondary winding.
(13) Effective Inductance and the Effective Resistance of iron-cored coils under Damped Oscillations.
(14) Measurement of Effective Inductance and Effective Resistance of iron-cored coils under Damped Oscillations.

(1) *Introduction.*

IN the Phil. Mag. March 1919 Dr. N. Campbell published some results of experiments on the high-tension magneto. On p. 286 of this paper we find that a method of obtaining the self-inductances of the primary and secondary windings is described. This yields the results given in fig. 3 and on p. 293. From these it appears that the inductances of the primary and secondary windings for damped oscillations remain constant over a certain range of frequency up to about 2500 ~ per sec. This is shown to hold good for a Thomson-Bennett rotating armature magneto and for a B.T.-H. polar inductor magneto. Now the inductance of a magneto

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* Communicated by the Author.

winding of fixed dimensions depends chiefly on the magnetic quality and behaviour of the iron. Provided the current is constant and the oscillations are damped or undamped sine waves, we may write approximately that $L = A_0\mu_e$, where, for a given position of the rotor, L is the inductance, A_0 is a constant, and μ_e is the effective permeability of the magnetic circuit, including air-gaps. The value of A_0 is approximately constant for the range of frequencies in a magneto say from $2 \times 10^3 \sim$ to $10^4 \sim$, but the value of μ_e varies with the frequency. It is well known that the apparent permeability of iron (μ_a) found from measurements with a coil wound on an iron core in which there is no air-gap or in which the demagnetizing coefficient and leakage are very small, decreases with the frequency whether the oscillations are damped or undamped. The magneto circuit is one in which the magnetic path is not completely wound with copper, especially in the inductor type, and one in which there are several air-gaps. Although the air-gaps are only a small proportion of the length of the magnetic circuit, they are equivalent to a length of iron μ_a times that of the gap. The air-gaps do not, however, alter the general behaviour of the iron qualitatively. This can easily be shown approximately neglecting leakage as follows:—

Let l = mean length of magnetic circuit containing air-gaps, assumed equal to length of magnetizing coil.
l_1 = mean length of iron of circuit.
A = cross-sectional area of circuit assumed constant.
N = total flux.
B = N/A = apparent flux density under alternating current in circuit having air-gaps.
H = apparent magnetizing force under alternating current.
B/H = μ_e.
μ_a, μ_e = apparent permeability * and effective permeability under alternating current.

Then
$$Hl = \frac{Nl_1}{A\mu_a} + \frac{N}{A}(l - l_1)$$

or
$$\frac{Hl}{B} = \frac{l_1 - \mu_a(l - l_1)}{\mu_a}$$

or
$$\frac{B}{H} = \mu_e = \frac{\mu_a l}{l_1 + \mu_a(l - l_1)}. \quad . \quad . \quad . \quad . \quad . \quad (1)$$

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* When magnetic circuit is of uniform cross-section and there is no air-gap.

Taking

$$\mu_a = 500$$

$$l = 10 \text{ cm.},$$

length of air-gap $l - l_1 = 0{\cdot}03$ cm.,

we have

$$\mu_e = \frac{5 \times 10^3}{10 + (500 \times 0{\cdot}03)}$$

$$= 200.$$

At a higher frequency assume

$$\mu_a = 100\ ;$$

then

$$\mu_e = \frac{10^3}{10+3} \doteqdot 77.$$

The effect of the air-gaps is clearly greater at the lower frequency owing to the higher apparent permeability of the iron, or in other words the air-gap at the lower frequency is equivalent to a greater length of iron than at the higher frequency. As the value of μ_a decreases, the proportionate effect of the air-gaps decreases also. Thus the effect of the air-gaps in decreasing the effective permeability of the magnetic circuit is greatest when the current is growing in the primary before break, since μ_a is greatest then. The gaps cause a diminution in the self-inductance of the primary winding. In the above case it is of interest to observe that little advantage would be gained by using an iron core in which the value of μ_a was 1000, since this would only increase μ_e to 250 ; *i. e.*, 100 per cent. increase in μ_a produces only 25 per cent. increase in μ_e. On the other hand, increasing μ_a from 100 to 200 increases μ_e from 77 to 125, *i. e.* 62 per cent. increase. When the air-gap in the magnetic circuit is proportionately large there is a considerable leakage and the flux density in the gap is far from uniform. In this case we may write approximately

$$\mu_e = \mu_a l / \{l_1 + g\mu_a(l - l_1)\}, \quad . \quad . \quad . \quad . \quad . \quad (2)$$

where "g" is a variable parameter depending on the frequency. When l_1 is small compared with $g\mu_a(l-l_1)$, the expression for the effective permeability reduces to $\mu_e \doteqdot l/\{(l-l_1)g\}$. Since l and l_1 are constant it follows that μ_e is approximately constant and independent of the frequency, provided g is constant. The latter condition will obtain at acoustic frequencies. This case corresponds approximately to that of a magneto in which the armature is removed from the housing. It is found by experiment that the above conclusion is correct. (See fig. 10, curve 3.)

It is clear that the air-gaps cause the effective permeability to attain an approximately constant value at a much lower frequency than would otherwise be the case. Moreover, the primary and secondary inductances will diminish with the frequency until they are nearly constant. This is an undesirable feature for high peak-voltage in the magneto. Hence the air-gaps should be reduced to a minimum, and the apparent permeability of the iron should—at the higher frequencies—be as large as possible. This latter condition can be fulfilled by using very fine laminations—*i. e.* as thin as conditions of economical manufacture will allow. It is also necessary that the brand of iron used should have a low remanence, small hysteresis, and high resistivity to ensure small loss. There is one point to be noted, however, in connexion with fine laminations. It can be shown * that with a given brand of iron, for a given total cross-sectional area of the magnetic circuit, *i e.* iron + insulation between sheets, there is a certain thickness which gives optimum peak-voltage, and, therefore, maximum secondary energy $\frac{1}{2}C_2V_2^2$.

It is well known that at frequencies of the order of 50 ∼ per sec. the primary inductance of a magneto depends on the position of the armature with reference to the pole-shoes. This can be accounted for in two ways: (1) the variation in length of the iron and air-gaps completing the magnetic circuit through the armature core, *i. e.* the variation in reluctance; (2) the variation in polarization of the iron due to the flux from the magnet. An increase in either the reluctance of the *magnetic circuit of the armature* or the polarization, entails a decrease in the inductance. This also applies to the secondary inductance, since both primary and secondary are wound on the same core. In the inductor type magneto the polarization and reluctance have minimum values just before break, when the timing lever is set at full advance. In this position the greater part of the magnetic circuit through the primary is unpolarized, since the flux through the primary winding is zero, and the circuit is completed across the inductor. We should expect, therefore, the inductance to be a maximum in this position, and this is confirmed by experiment. In any other position of the timing lever, the magnetic circuit through the primary winding is completed partially, at least, through the magnet, and the reluctance of the circuit is greater owing to the

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* See McLachlan, 'Wireless Year Book,' 1918, p. 898.

additional air-gaps introduced. In the rotating armature magneto the maximum inductance also occurs before advance break, for in this position the polarization and reluctance of the armature magnetic circuit have minimum values.

Owing to diminution in the value of μ_a at high frequencies, it follows from above that the inductance would not be expected to vary so much with the angular position of the armature. This also is corroborated by experiment.

When the magneto is in operation and connected to a peak-voltage measuring apparatus, the damping in the secondary circuit is very large owing to the high effective resistance of the iron under oscillations of audio frequency. The effective inductance and effective resistance in this case must be viewed in a different manner from that when the oscillations are undamped, as will be shown later.

The measurements described hereafter have been carried out with undamped oscillations as accurately as experimental conditions would permit. None of the methods was susceptible of a high degree of accuracy. Data were required which showed the "order of magnitude" of different quantities and which exhibited the general behaviour of the magneto under certain experimental conditions. These conditions are different from those which obtain in practice. The chief object of the research was to obtain some knowledge regarding the loss due to iron, etc. Using the values of the coefficients of the circuits thus obtained, it is possible to get an estimate of the efficiency of the magneto. A B.T-H. polar inductor magneto was used in all experiments, as it is well suited to research work.

(2) *Measurement of the Self-Inductance and Effective Resistance of the Primary of a Magneto by Undamped Oscillations.*

Before describing the method of measurement adopted it is necessary to define the quantities it is intended to obtain. If a sinusoidal current is passed through an iron-cored coil, the voltage on the coil is nearly sinusoidal provided the frequency is fairly high and the magnetizing force low. At low frequencies, *e. g.* 50 ~ per sec., the voltage wave form is not sinusoidal* owing to the variation in the permeability of the iron with the current. The sinusoidal voltage can be split up into two components R_eI and ωL_eI

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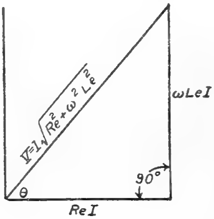
* See Journ. I. E. E. June 1915.

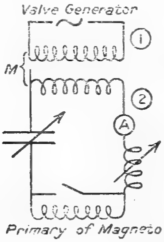
in quadrature, as shown in fig. 1. These represent, respectively, the ohmic drop due to the effective resistance caused chiefly by iron loss but also by dielectric loss and eddies in metal fittings, and the drop due to the effective inductance.

Fig. 1.—Vector diagram illustrating definition of effective inductance and effective resistance.



The apparatus used is shown diagrammatically in fig. 2. A triode is employed to generate oscillations in circuit (1).

Fig. 2.—Diagrammatic arrangement of apparatus used to determine the primary inductance and effective resistance.



A second circuit contains an inductance coupled to the inductance in circuit (1), a variable condenser, variable inductance, and the primary of the magneto under test. It is necessary to *remove the secondary winding* and to arrange that the total inductance in circuit 2 (apart from that of the magneto) is as small as convenient. This is essential so that a small change in inductance will produce a perceptible change in the current measured on the Duddell milliammeter A, since the high effective resistance of the magneto causes the resonance curve to be rather flat-topped. The variable inductance is adjusted so that circuit (2) is brought into resonance. The resonance point is found by obtaining the inductances on either side of the maximum current for which the current has equal values. The mean of the two values of the inductance is taken as the inductance at resonance, and the variometer is set accordingly. The magneto is then disconnected and the inductance again varied until resonance is obtained. The difference in the readings is the effective inductance of the primary winding of the magneto.

The effective resistance is found in the well-known manner by obtaining (*a*) the current at resonance, (*b*) the current when the circuit is out of resonance. The latter condition can be obtained by varying either the condenser or the inductance. In these experiments the latter, which had a constant resistance, was varied. The difference in current should be appreciable to obtain accurate results. It is important that the current in the primary oscillating circuit should be the same for cases (*a*) and (*b*). If it varies, a correction can be made, provided the two values are known.

Let I_1 = current at resonance with magneto in circuit.
I_2 = current when circuit is out of resonance.
L = variation in inductance to put circuit out of resonance.
ω = pulsatance = $2\pi f$.
R_{e1} = effective resistance of primary winding at frequency f.
r_e = effective resistance of circuit without primary of magneto.

Then

$$r_e + R_{e1} = \omega L\left(\frac{I_2^2}{I_1^2 - I_2^2}\right)^{\frac{1}{2}} = \omega L\left(\frac{1}{(I_1/I_2)^2 - 1}\right)^{\frac{1}{2}} . \quad (3)$$

r_e is found in a similar manner to that of $r_e + R_{e1}$.

A more accurate method, if the apparatus is available, is to employ a bridge as used by Hund *. In this way it is possible to find the inductance and resistance for the fundamental and various harmonics. This method should be useful where currents of the same magnitude as the maximum primary oscillatory current under working conditions are employed, since in this case there would be harmonics owing to the apparent permeability of the iron not being constant over the range of values of H employed.

In the present experiments, it was found that the inductance and effective resistance varied slightly with the current at the lower frequencies when the armature was situated in the housing. Throughout the experiments, the root mean square current used in measuring the inductance was about

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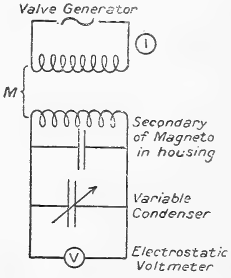
* Electrician, vol. lxxv. p. 78 (1915).

40 milliamperes. In finding the effective resistance it is necessary to vary the current. This causes a variation in R_{e1} at the lower frequencies. The value obtained for R_{e1} increased with decrease in the current through the coil when the circuit was thrown out of resonance—*i. e.*, R_{e1} increased with decrease in I_2 for a given value of I_1. Owing to time limitations, the inductances with currents larger than 40 milliamperes were not ascertained. The primary current in a magneto is much greater than this (see Proc. Phys. Soc. Dec. 1919). Since the peak-voltage is proportional to the current broken in the primary from 0·25 amp. to 4 amp., it is very probable that the inductances are constant throughout this range.

(3) *Measurement of the Self-Inductance and Effective Resistance of the Secondary by Undamped Oscillations.*

The methods of measurement adopted for the primary winding are inapplicable to the secondary, owing to the magnitudes of the quantities to be obtained. The effective resistance of the secondary is of the order of 5×10^4 ohms at a frequency of 1700 ∼ per sec., and it is quite clear that resonance could not be obtained on a milliammeter with this resistance in circuit (2) of fig. 2 without the aid of extremely large currents in circuit (1). The above quantities can be found, however, by modifying the resonating circuit in fig. **2.**

Fig. 3.—Diagrammatic arrangement of apparatus used to determine the secondary inductance and effective resistance.



The arrangement of the modified circuit is shown diagrammatically in fig. 3. The magneto itself is used to obtain a coupling between circuits (1) and (2). In this case the aluminium end plates are removed and the primary is on open circuit. The variable condenser is adjusted until the voltmeter-reading is a maximum. Under this condition, circuit (2) is in resonance. Knowing the frequency of the oscillations in this circuit, the effective inductance can be calculated from the expression

$$f = \frac{1}{2\pi}\left(\frac{1}{L_{e2}C}\right)^{\frac{1}{2}},$$

where C is the self-capacity* of the secondary winding plus the capacity of the variable condenser plus the capacity of the voltmeter for the reading shown thereon. It is evident that this method cannot be employed for frequencies exceeding that at which the coil has its first self-resonance. In fact the self-resonance cannot be obtained by direct measurement owing to the capacity of the voltmeter.

Although it is not possible to measure the secondary inductance at all frequencies by the preceding method, its value can be calculated approximately from that of the primary, and the results at low frequencies can be compared with those found experimentally. The agreement is about 5 per cent. at frequencies between 1500 ∼ and 2500 ∼, the calculated values being in defect.

Let L_1 = Inductance of primary winding, out of housing, with air core (measured with secondary removed).

L_2 = Inductance of secondary winding, out of housing, with air core.

A_i = Cross-sectional area of iron core.

A_1 = Cross-sectional area of primary with air core (mean).

A_2 = Cross-sectional area of secondary with air core (mean).

L_{e1} = Effective inductance of primary in housing.

L_{e2} = Effective inductance of secondary in housing.

μ_e = Effective permeability of magnetic circuit including air gaps.

Then L_{e1} = Additional inductance due to iron + inductance with air core

$$= L_1\frac{A_i}{A_1}(\mu_e - 1) + L_1$$

$$= L_1\{(\mu_e - 1)K_1 + 1\}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (4)$$

where $K_1 = A_i/A_1$.

Similarly

$$L_{e2} = L_2\{(\mu_e - 1)K_2 + 1\}, \quad . \quad . \quad . \quad (5)$$

where $K_2 = A_i/A_2$.

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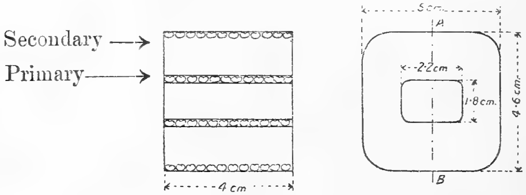
* This can be measured, as shown below.

If all the factors in (3) but μ_e are known, then μ_e can be calculated. Substituting this value in (4) it is possible to calculate L_{e2}.

The mean areas of the primary and secondary windings used in formulæ (3) and (4) are obtained by measuring the self-inductances experimentally when the iron core is removed. The inductance of the primary is found after the secondary has been unwound. The value of a, the equivalent mean radius in Lorenz's equation $L=an^2Q$ *, is then ascertained by plotting an "a" and "L" curve and reading off the value of "a" corresponding to the value of "L" found by experiment, or by trial and error. The above formulæ are only intended to apply when the magnetic circuit is nearly closed and the leakage is not very large. Approximate overall dimensions of the armature windings are given in fig. 3 *a*. The armature used for these measurements was slightly different from that mentioned in section 4.

Fig. 3 *a*.

Section AB.



Primary = 150 turns. Secondary = 10^4 turns.

Primary res. (d.c.) $\doteqdot$ 0·55 ω. Secondary res. (d.c.) $\doteqdot$ 1900 ω.

The method of measuring the effective resistance is to find the voltage at resonance, reduce the capacity so that the circuit is thrown out of resonance, and again find the voltage. In this case as in that of the primary winding, the effective resistance includes iron, dielectric, and stray losses in the metal framework.

Let V_1 = voltmeter reading at resonance.

C_1 = condenser reading at resonance including voltmeter.

I_1 = current reading at resonance.

V_2 = voltmeter reading when circuit is out of resonance.

C_2 = condenser reading out of resonance including voltmeter.

I_2 = current reading when circuit is out of resonance.

ω = pulsatance.

R_{e2} = effective resistance of secondary.

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* Bulletin Bureau of Standards, vol. viii. No. 1.

Then it can be shown that provided R_{e2} is constant,

$$R_{e2}=\frac{C_1-C_2}{\omega C_1 C_2}\left(\frac{I_2^2}{I_1^2-I_2^2}\right). \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (6)$$

Since

$$I_1=\omega C_1 V_1$$

$$I_2=\omega C_2 V_2$$

we obtain

$$R_{e2}=\frac{C_1-C_2}{\omega C_1 C_2}\left(\frac{C_2^2 V_2^2}{C_1^2 V_1^2-C_2^2 V_2^2}\right)^{\frac{1}{2}}. \quad . \quad . \quad . \quad . \qquad (7)$$

In this case the effective resistance of the remainder of the circuit is negligible.

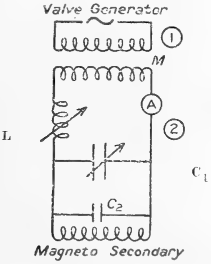
In carrying out experiments with the armature in the housing, it was found that the inductance and resistance varied with the current for the range of frequencies obtainable, viz. 1000 to 2500 ∼ per sec. The inductances were measured with a current of about 6×10^{-4} ampere, since this produced approximately the same ampere-turns as the current used in the measurement of the primary circuit. Owing to the necessity for varying the current to obtain the effective resistance, the results are not very accurate, but merely serve to indicate the order of magnitude. Using the above formula for R_{e2}, its value increased slightly with decrease in C_2V_2, when C_1V_1 was constant.

Measurements of the inductance and effective resistance for various conditions have been made. These will be dealt with in discussing the experimental results.

(4) *Measurement of Self-Capacity of and Dielectric Loss in Secondary winding.*

The circuit is arranged as shown in fig. 4. Circuit 2 is loosely coupled to circuit 1, and brought to resonance by varying either C_1 or L when the magneto is *completely disconnected*. This latter precaution must be observed, owing to the capacity effect due to the housing. The

Fig. 4.—Diagram showing apparatus for determination of self-capacity of secondary winding by substitution method.

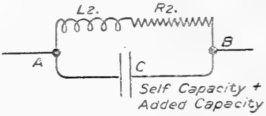


condenser reading is observed (C_1 should be small, L large, and the frequency fairly high); also the reading of the milliammeter. The magneto is then connected in parallel with C_1, and the latter is varied until resonance is again obtained. The difference in the condenser readings (the condenser should have an open scale) is the self-capacity of the coil. It was found that the self-capacity did not vary much with the frequency (2×10^4 to $8 \times 10^4 \sim$), but it was less when the armature was out of the housing than when it was in, as one would naturally expect. When the armature was out of the housing, the iron core, which was a little longer than the armature coil, made little difference in the self-capacity. With the armature in the housing and the primary winding not connected thereto, the self-capacity was less than that with the primary connected to the housing but greater than that out of the housing.

Theory of method.—To simplify matters we shall assume the secondary coil with parallel condenser to be approximately represented by an inductance L having resistance R in parallel with a condenser C equal to the self-capacity of the coil, plus the condenser in the secondary circuit. Neglecting dielectric loss, the admittances of the two paths between A and B are respectively (see fig. 5)

$$\frac{1}{R_2 + j\omega L_2} \quad \text{and} \quad j\omega C.$$

Fig. 5.—Equivalent (assumed) circuit of secondary coil for determination of self-capacity.



The combined impedance between A and B is therefore

$$Z = \frac{1}{\dfrac{1}{R_2 + j\omega L} + j\omega C}$$

$$= \frac{R_2 + j\omega L_2}{1 - \omega^2 L_2 C + j\omega C R_2}$$

$$= \frac{(R_2 + j\omega L_2)(A - j\omega C R_2)}{A^2 + \omega^2 C^2 R_2^2},$$

$$\text{where} \quad A = (1 - \omega^2 L_2 C)$$

$$= \frac{(1 - \omega^2 L_2 C) R_2 + \omega^2 L_2 C R_2 + j\omega L_2 (1 - \omega^2 L_2 C) - j\omega C R_2^2}{\omega^4 L_2^2 C_2^2 + \omega^2 C^2 R_2^2}.$$

If $\omega^2 L_2 C$ is large compared with unity, we get

$$Z = \frac{R_2}{\omega^2 C^2(R_2^2+\omega^2 L_2^2)} - \frac{j\omega\, C(R_2^2+\omega^2 L_2^2)}{\omega^2 C^2(R_2^2+\omega^2 L_2^2)}$$

$$= \frac{R_2}{\omega^2 C^2(R_2^2+\omega^2 L_2^2)} + \frac{1}{j\omega C} = R' + \frac{1}{j\omega C}\,; \quad . \quad . \quad (8)$$

i. e., the combination is equivalent to the condenser C in series with a non-inductive resistance

$$R' = \frac{R_2}{\omega^2 C^2(R_2^2+\omega^2 L_2^2)},$$

as shown in fig. 6.

Fig. 6.—Circuit which behaves in the same manner as that of fig. 5, when certain conditions (stated in text) are fulfilled.

This is due to the fact that by far the greater portion of the current passes into the coil as displacement current and not as a circulating current passing through the coil. In order to obtain this condition, it is essential that $\omega^2 L_2 C$ should be large compared with unity. Since C must be small so that a small difference in C* affects the tuning of the circuit, ω^2 and therefore f must be correspondingly large.

In carrying out experiments using the above method, it was found that the resonance currents were different according as the magneto or a condenser of equal capacity was connected in circuit. It was greater in the latter case. This difference must be due to either copper, iron, or dielectric loss, or all of them. In order to investigate this point, the resonance currents were measured (*a*) with the armature in the housing, (*b*) with the armature out of the housing, (1) with the iron core in place, (2) with the iron core withdrawn. The current was smaller in case (*a*) than in case (*b*). In the latter case it was greater with the iron than without it. The frequency was so high that $\omega^2 C^2 R_2^2+\omega^2 L_2^2)$ was much greater than R_2 except when the iron was removed. Thus, since L_2 is much less without

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* If the capacity in parallel with the coil is large compared with the self-capacity, the distribution of current in the coil will be fairly uniform.

the iron core than with it*, the value of R′ in expression (8) was not negligible. By increasing C and keeping ω constant, there was no measurable difference with and without the iron, whether the latter was very thick or very thin. Thus the reduction in current is in this case brought about by dielectric and copper loss. The dielectric loss can be represented in the usual manner by a high resistance in parallel with the condenser. This is equivalent to a series resistance of small magnitude. Since the current passing through the inductance (fig. 5) is extremely small, it can be neglected. The conditions that

$$\frac{R_2}{\omega^2C^2(R_2^2+\omega^2L_2^2)}$$

is negligible, and that ω^2L_2C is much greater than unity, must be observed. Under these conditions the coil with the condenser in parallel is, therefore, equivalent to the circuit of fig. 7.

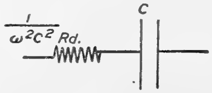
Fig. 7.—Representation of secondary winding as a condenser shunted by a high resistance, when certain conditions (specified in text) are fulfilled.

When $\omega^2C^2R_d^2$ is large compared with unity, the arrangement of fig. 7 is equivalent to a condenser C in series with a resistance of magnitude

$$R'_{eq}=\frac{1}{\omega^2C^2R_d} \quad \text{(see fig. 8),}$$

where R_d represents the resistance of the dielectric to alternating currents under the special conditions in the magneto

Fig. 8.—Circuit equivalent to that of fig. 7.



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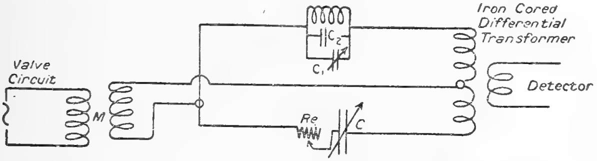
* By calculation and measurement, L_2 without the iron was of the order of 3 henries for the coil used in these experiments.

or $1/R_d$ is the conductance. This of course includes dielectric hysteresis. In conducting experiments on dielectric loss the armature should be taken out of the housing and the iron removed. The primary circuit should be open and the condenser absent. The separation of the dielectric and copper losses is dealt with later.

(5) *Bridge Method of measuring Self-Capacity.*

The self-capacity at frequencies greater than 20,000 ~ per second can be found by using a bridge of the form shown in fig. 9, instead of the substitution method already outlined.

Fig. 9.—Diagram showing bridge connexions for the measurement of self-capacity of secondary winding.



The circuit of fig. 9 consists of an iron-cored differential transformer having two primaries with equal numbers of turns and a secondary. The primaries are connected in opposition. The coil under test is connected in parallel with a variable condenser in one arm, and a variable non-inductive resistance and variable condenser in the other. When balance is obtained we have $C_1+C_2=C$, provided $\omega^2L_2(C_1+C_2)$ is very large compared with unity, $\omega L_2 \overset{>}{\underset{>}{>}} 1/\omega C_2$, and $R_2/\omega^2C^2(\omega^2L_2^2+R_2^2)$ is very small: also $R_{eq}=\dfrac{1}{\omega^2C^2R_d}+\left(\dfrac{C_2}{C}\right)^2R_2$, where R_2 is the ohmic resistance of the copper of the secondary winding. Thus the dielectric resistance for the particular conditions which obtain in the magneto can be found. Since C_1 is known, C_2 is found by subtraction. In order to obtain accurate results, the value of C_2 should not be too small compared with C_1. At high frequencies C_1 could be zero. The detector may be a telephone* with interrupter, tikker or other device to provide

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* The telephone might be troublesome owing to the noise produced when the current passes through the magneto winding, also the noise from the valve generating the oscillations.

audible frequency, or a valve amplifier may be used in conjunction with a rectifying valve, smoothing out condenser, and a microammeter. Care must be taken, however, with regard to self-resonance effects in the differential transformer and the valve-amplifier circuit. The current in the bridge can be measured by using a transformer and amplifying if the current is small, or by a thermoammeter reading to 100 milliamperes, if the current is large enough.

(6) *Results of Experiments.*

(a) *Armature in Housing.*

The values of the coefficients of the primary and secondary circuits obtained with the armature in the housing are shown in figs. 10 to 14. It will be seen that the inductances decrease rapidly until the frequency is 15,000 ~ per second. Beyond that value the decrease in inductance is much more gradual, owing to the fact that the effective permeability of the magnetic circuit is approximately constant. This is in agreement with the argument at the beginning of the paper. The inductance, especially at low frequencies, depends on the position of the rotor, being greatest just before advance break and least after retard break. The difference in the values after advance and after retard break was only a small percentage. The same holds good with reference to the effective resistance. This variation is due to the change in the reluctance and polarization of the magnetic circuit with the position of the rotor. As the frequency increases the difference between the inductances and resistances for various rotor positions decreases, owing to diminution in the apparent permeability of the iron. The ratio of the secondary to the primary inductance is shown in Table I., and it will be seen that the variation with frequency up to 5000 ~ is small.

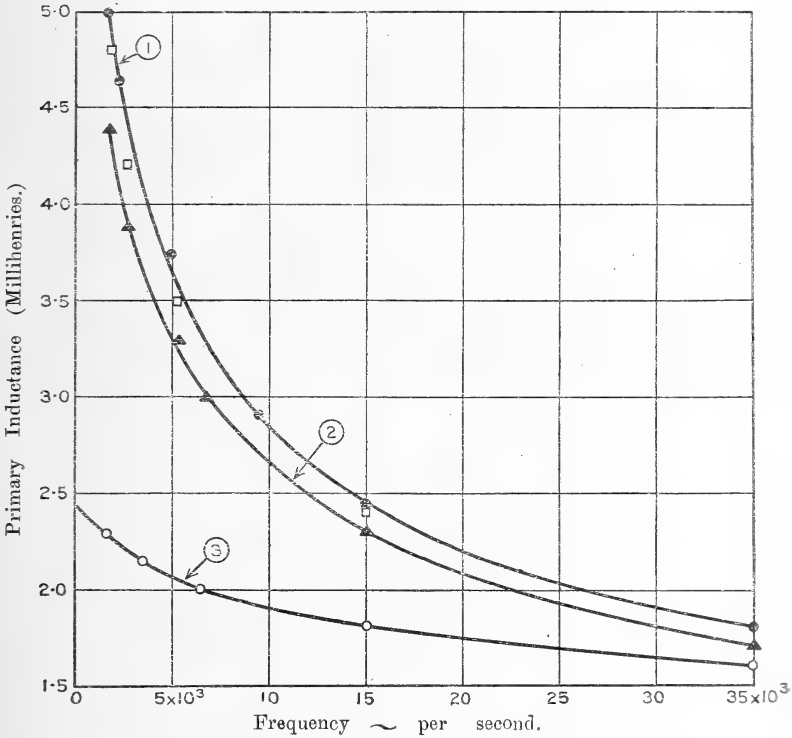
Table I.

Showing Ratio L_{e2}/L_{e1}. Armature in housing; rotor position after full advance break. Values of L_{e2} obtained by calculation.

f ~ per sec.	L_{e2}/L_{e1} with iron core.	L_2/L_1 with air core.
$1·7 \times 10^3$	$4·4 \times 10^3$	$11·5 \times 10^3 = \frac{2·7}{0·234 \times 10^{-3}}$
3·5 ,,	4·5 ,,	
5 ,,	4·55 ,,	
10 ,,	4·66 ,,	
15 ,,	4·76 ,,	
20 ,,	4·85 ,,	
35 ,,	5·35 ,,	

It will be observed from fig. 11 that the rate of increase of the primary effective resistance diminishes slightly with the frequency, although it is nearly constant up to 10,000 ~ per second. This is due, of course, to the behaviour of the iron resulting from the screening effect of eddy currents. The power factor of the primary winding is illustrated by the curve of fig. 13. It increases rapidly until $f=5000$ ~ and afterwards at a much slower rate, until at $f=15{,}000$ ~ it is

Fig. 10.—Curves showing variation in primary inductance, under different conditions, with frequency.



Curve 1 = Primary Inductance with iron unpolarized, *i. e.* magnet removed, after full advance. Coil with brass end-plates, and in housing.

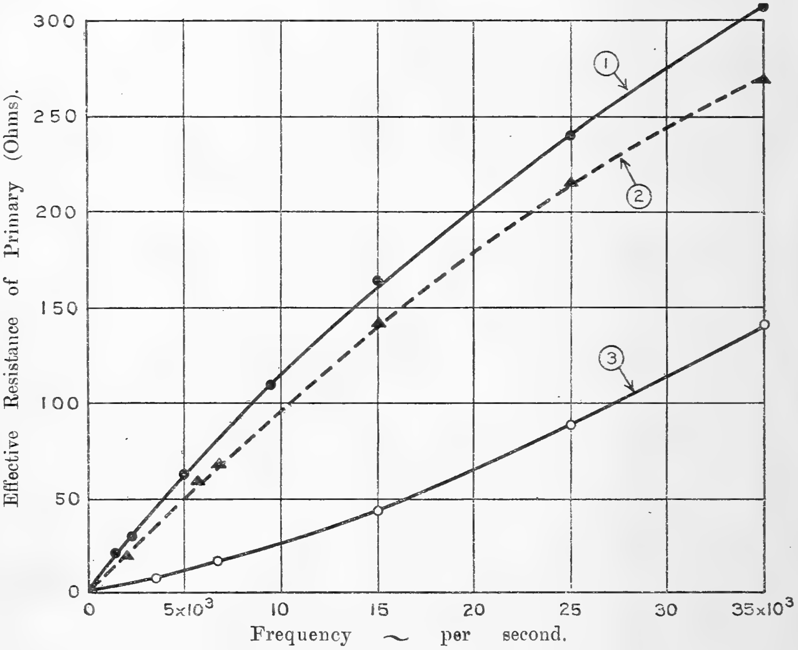
Curve 2 = As in Curve 1 but with magnet in place, *i. e.* iron polarized.

Curve 3 = Primary Inductance with armature removed from housing.

Points thus : ☐ = As in Curve 1, but *before* advance break.

nearly constant. This is again attributable to eddy currents and to the effect of the air-gaps. A similar curve is obtained in testing an iron circuit completely wound and without air-gaps. In this latter case the power factor of iron sheets of the

Fig. 11.—Curves showing variation in primary effective resistance, under different conditions, with frequency.



Curve 1 = Effective Resistance of Primary with iron unpolarized, *i. e.* magnet removed after full advance. Coil with brass end-plates, and in housing.

Curve 2 = As in Curve 1 but with magnet in place, *i. e.* iron polarized.

Curve 3 = Effective Resistance with armature removed from housing.

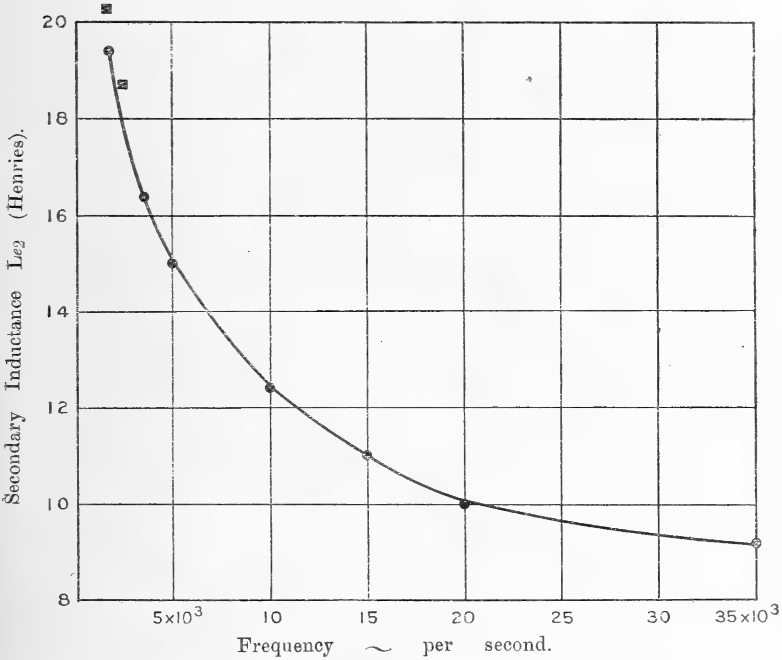
Intercept on vertical axis = res. of primary at zero frequency ≑ 0·55 ω.

same thickness as those in the magneto is greater at 10,000 ∼ than that in fig. 13. The power factor of the primary winding cannot be considered as "small." Its magnitude is due to (1) the increase in loss and (2) the decrease in inductance with frequency. The use of thin sheets of iron of high permeability, high resistivity, and low hysteresis loss would improve matters considerably.

(b) *Armature out of Housing.*

The various coefficients obtained with the armature out of the housing are plotted in figs. 10, 11, 14. From the curves it is evident that the results agree closely with those foreshadowed in the remarks at the beginning of the paper. There is little variation in the inductance of either primary or secondary with frequency. The variation of the former is greater than the latter, since it is wound nearer to the iron core and leakage effects are smaller. The effective resistance is reduced very considerably and it increases at a more rapid rate than the frequency. This is due to the absence of large

Fig. 12.—Curve showing variation in secondary self-inductance as obtained by calculation, using the curve of fig. 16.



Out of housing, secondary inductance ≑ 9·7 henries from 1200 ~ to 3000 ~.

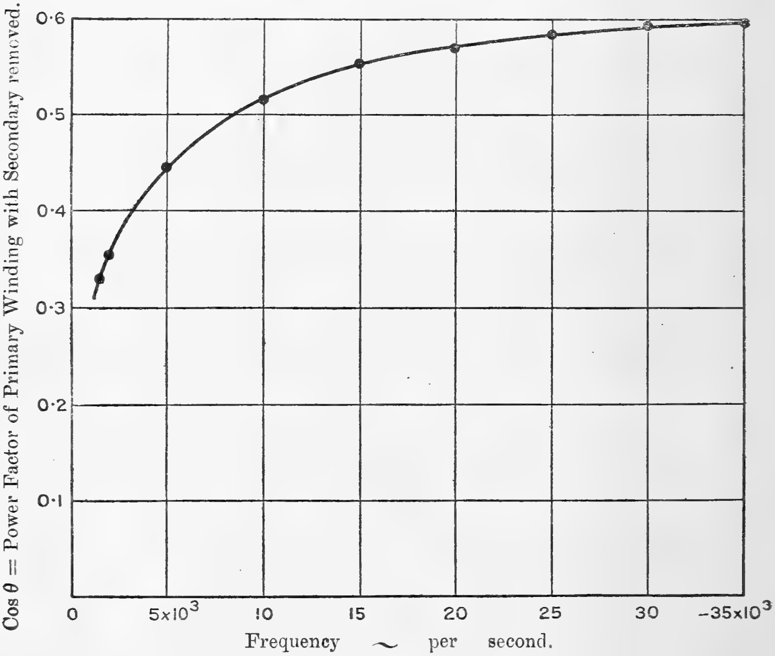
□ = Experimental points.
⊙ = Points obtained by calculation from fig. 16.

Armature with brass end-plates; in housing.
Rotor position = after full advance break.

masses of solid iron from the magnetic circuit, and can be verified by using a solid core or a core of thick strips.

The effective resistance was also measured, using cores of 0·38 mm. stalloy and 0·06 mm. pure iron, of the same dimensions as that supplied with the magneto. Both cores gave slightly smaller values of the effective resistance than the ordinary core. Stalloy was the best of the three cores, owing to its high resistivity. It may seem peculiar that stalloy 0·38 mm. thick should give a lower effective resistance than pure iron of $\frac{1}{6}$ the thickness, since a reduction in eddy-current loss would be anticipated. This, however, is due to the fact that for the same ampere-turns or magnetizing force due to the oscillatory current, the flux density, and therefore the loss, is greater with the thinner material. With equal flux densities the loss in the 0·06 mm. iron would be less than that in the 0·38 mm. iron.

Fig. 13.—Variation of Power factor of primary winding with frequency.

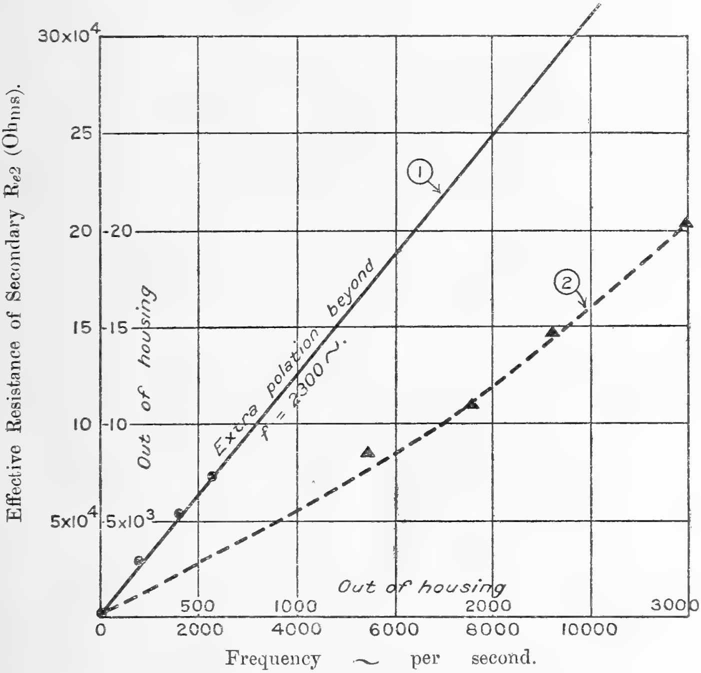


Armature with brass end-plates ; in housing.
Rotor position = after full advance break.

(c) *Effect of Polarization.*

In Phil. Trans. A, vol. 184 (1893), Ewing and Klaassen showed when a sample of iron was subjected to a magnetizing force produced by a constant current, and therefore causing polarization, that the hysteresis loss resulting from a superposed force varying between definite limits, decreased with increase in the polarizing force. This is due to the constraint of the molecular magnets and the consequent diminution in the variation of the flux density. Owing to the latter effect the eddy-current loss is also diminished.

Fig. 14.—Variation in effective resistance of secondary with frequency.



Curve 1 = Effective resistance of secondary in housing, with magnet and brass end-plates; after full advance break.

Curve 2 = Effective resistance with armature removed from housing. Intercept on vertical axis = res. of secondary at zero frequency $\doteqdot 1900\,\omega$.

In the magneto the effect of polarization is clearly exhibited by curves (1) and (2) in figs. 10 and 11 [*]. These were

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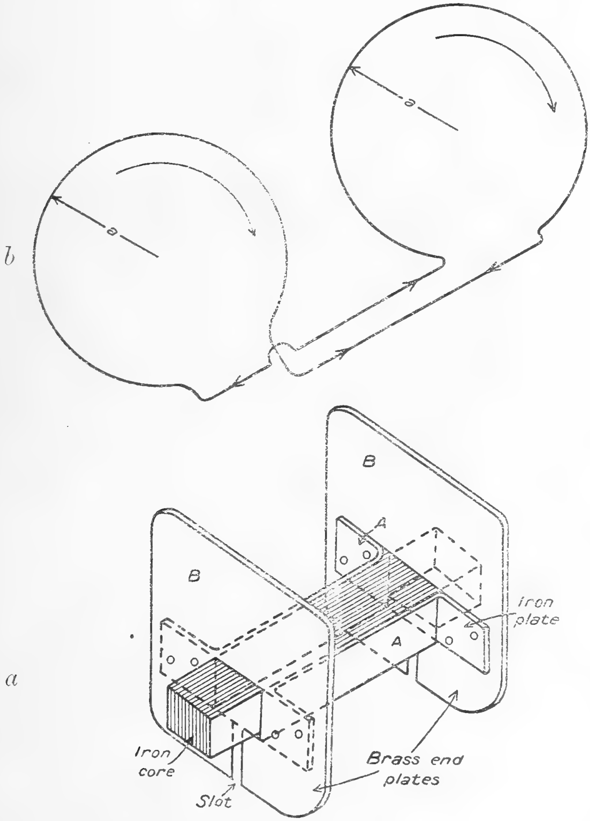
[*] During the experiments the rotor was moved slightly beyond its position at advance break in order to show the polarization effect more clearly.

obtained using a definite current, *i. e.* the variation in H was nearly constant. The reduction in the hysteresis and eddy-current loss results in a diminution in the effective resistance. The decreased variation in the flux density causes a diminution in the inductance. The same effects were observed in connexion with the secondary winding. Owing to the fact that the iron forms a greater proportion of the mean cross-sectional area of the primary than of the secondary, the polarization effect was more marked in the first case than in the second. Just before advance break the polarization is nearly zero and the influence of the magnet should, therefore, be zero. This was found to be true, since any variation in inductance was within the limits of experimental error. The points obtained are shown in fig. 10 thus : □. It is evident that they follow nearly the same curve as those obtained with the magnet removed. The slight difference is due to the slightly altered configuration of the magnetic circuit before and after break. In the absence of polarization, the variation of inductance with the position of the rotor is considerably reduced. In order to determine the influence of the metal of the magnet, apart from polarization, a spare magnet was demagnetized and fitted to the magneto. There was an increase in the secondary inductance, when measured after advance break, of about 5 per cent.

(d) *Effect of Armature Brass End-Plates.*

The brass end-plates are fixed to the core as shown in fig. 15 by bending two iron plates, A A, at each end and riveting over to the end-plates, B B. Considering each end to be replaced by a single turn of thin wire of radius a, it is clear that no current would flow in the system, since the E.M.F.'s in the two coils at any instant would be equal and opposite. Owing, however, to the fact that there are large masses of metal above and on each side of the slots in the end-plates, eddy currents are induced therein and the circuits of these currents act as tertiary circuits when the magneto is in operation, and also when undamped oscillations are passed through the primary or the secondary winding. The effect of the end-plates can be studied by means of an endless coil of copper wire having several turns of 18 s.w.g. The end-plates are removed ; the primary of the armature with core is taken from its housing and connected with the circuit of fig. 2. The circuit is brought to resonance, and the coil is then slipped into the position occupied by one of the end-plates. It is found that the effective external inductance and effective resistance of the primary are decreased. Both of these effects are due to the demagnetizing influence of the current induced in the short-circuited coil. The "decrease" in effective resistance is due to the decreased iron loss caused by the smaller variation in flux density resulting from the demagnetizing action of the coil—*i. e.*, transformer action. With an air-cored coil the external inductance decreases, but the copper loss increases owing to ohmic loss due to the currents in the secondary exploring coil. Thus with the iron core in place, although the total effective resistance (iron

Fig. 15.—(*a*) Pictorial representation of armature before primary and secondary are wound.
(*b*) Illustrating electric circuits completed by the brass end-plates and the iron plates riveted thereto.



loss + copper loss) decreases, the portion due to copper loss increases. These effects are more pronounced the greater the coupling between the exploring coil and the primary winding.

In the actual case of the magneto the effects are not nearly so marked as those obtained with the exploring coil, owing to the brass end-plates being slotted. At 10,000 ∼ the diminution in the inductance and effective resistance was of the order of 5 per cent. with the armature in or out of housing *. The effect of the aluminium end-plates carrying the outer ball races was examined, but it was extremely small. The influence of the other non-magnetic fittings surrounding the iron circuit was not examined owing to lack of time. It will be shown later that the dielectric loss is proportionately small. It is probable, therefore, that the greatest loss is due to the iron, although the eddy currents in the non-magnetic fittings in which the laminations are fixed must contribute to the total loss.

One method of investigating the latter source of loss would be to build a magneto with the laminations held in non-metal fittings, and compare the results with those obtained for the standard machine. Care would have to be taken to ensure small dielectric loss in the non-metal fittings. Tests could be conducted with the rotor at rest or in motion.

(e) *Effective Permeability.*

The effective permeability of the complete magnetic circuit, not merely the iron *per se*, has been defined by equation (3). It is an index of the usefulness of the iron in increasing the flux through the windings. The curve of fig. 16 shows the variation with frequency. Its general shape is in accordance with the deductions on pp. 35, 39. On comparison with a static permeability curve, it will be seen that μ_e is greatly reduced owing to the air-gaps which are of paramount importance at low frequencies. This will be clear on inspection of curves (1) and (2) of fig. 10. Since the inductances decrease with increase in polarization, it is evident that μ_e decreases too.

(f) *Self-Capacity of Secondary winding.*

The self-capacity of the secondary was measured at frequencies from 2×10^4 ∼ to 7×10^4 ∼ per second by the

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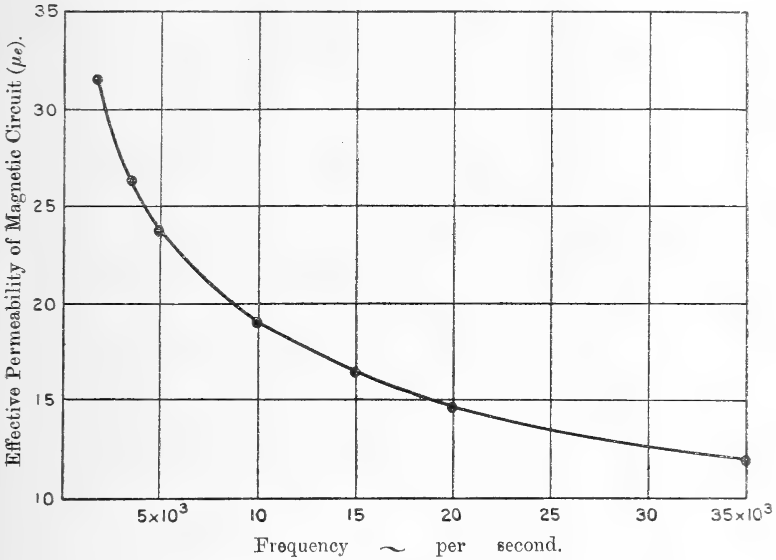
* From a peak-voltage test on full advance and retard the effect of the brass end-plates was negligible. If the slots in the end-plates are closed, the peak-voltage is roduced considerably.

substitution method outlined herein. Any variations were within the limits of error of observation and experiment.

Fig. 16.—Curve showing variation in Effective Permeability of magnetic circuit with frequency.



Rotor position = after full advance break.
Armature with brass end-plates. In housing.

The results obtained with two different armatures for the polar inductor magneto are tabulated below.

Table II.

Self-Capacity of Secondary winding of B.T.-H. polar inductor Magneto.

Armature (1) without brass end-plates.

Frequency (~ per sec.).	Self-capacity (Picofarads).	Experimental conditions.
$3{\cdot}5 \times 10^4$	65	Armature in housing. Loose end of primary connected to housing.
,,	55	As above, but end of primary free.
,,	50	Armature out of housing, with iron core inserted.
$7{\cdot}2 \times 10^4$	50	Ditto.
,,	50	Ditto, but iron core removed.

Armature (2) with or without brass end-plates.

$3{\cdot}5 \times 10^4$	50	In housing with primary connected thereto.
,,	40	Out of housing with iron core.

(g) *Dielectric Loss.*

The problem of obtaining an accurate measurement of the dielectric loss in the magneto is rather difficult owing to the peculiar conditions therein. This is partly due to the fact that in using the coil as a condenser, the charging current has to traverse the copper winding. This results in an ohmic loss, which is rather indefinite owing to the distribution of the current. All that can be done in the present instance is to obtain a rough approximation, and to show that even if the ohmic effects are included, the loss is not large compared with that due to the iron.

When the iron core and the brass end-plates are removed from the armature coil and the inductive reactance is very much greater than the capacity reactance, *i.e.* $\omega L_2 \gg \frac{1}{\omega C_2}$, the coil acts as a condenser, and the circulating current passing through the winding can be neglected in comparison with the charging current. Thus, when the coil is connected as shown in fig. 4, the loss which occurs is that indicated above. The current at resonance is obtained using the coil, as before. The coil is removed and the resonance current of equal magnitude obtained by inserting a non-inductive resistance in series with the thermoammeter A. The total loss in the coil is equal to the loss in this resistance. In separating out the dielectric loss the procedure given below is adopted.

Let I be the current at resonance.

Let R_{eq} be the resistance inserted to give the same resonance current as the coil.

Total loss in Coil = Dielectric loss + Ohmic loss due to charging current $= I^2 R_{eq}$.

Assuming charging current uniformly distributed in the winding,

$$\text{the Ohmic loss} = (\text{Charging current})^2 \times \text{res. of winding}$$

$$= (\omega C_2 E)^2 R_2$$

$$= I^2 R_2 \left(\frac{C_2}{C}\right)^2,$$

where C_2 is the self-capacity of the coil and C is the total capacity in circuit.

$$\underset{\text{(charging current)}}{\text{Equivalent series resistance}} = \left(\frac{C^2}{C}\right)^2 R_2 = r_{eq}.$$

In an experiment using a certain armature coil, without brass end-plates or other metal parts, the following data were obtained :—

$$\begin{aligned} f &= 6{\cdot}2 \times 10^4 \sim \text{per sec.} \\ \mathrm{I} &= 7{\cdot}85 \times 10^{-2} \text{ amp.} \\ \mathrm{R}_{eq} &= 2{\cdot}5 \text{ ohms.} \\ \mathrm{R}_2 &= 2200 \text{ ohms (without iron core).} \\ \mathrm{C} &= 2100 \text{ picofarads.} \\ \mathrm{C}_2 &= 50 \quad ,, \end{aligned}$$

Thus equivalent series resistance for charging current $r_{eq}=1{\cdot}25$ ohm, and therefore the series resistance equivalent to dielectric loss, $\mathrm{R}'_{eq}=\mathrm{R}_{eq}-r_{eq}=1{\cdot}25$ ohm. Equivalent dielectric resistance *for particular case of magneto secondary winding*

$$\begin{aligned} \mathrm{R}_d &\doteq \frac{1}{\omega^2 \mathrm{C}^2 \mathrm{R}'_{eq}} \\ &= 1{\cdot}2 \times 10^6 \text{ ohms.} \end{aligned}$$

Loss at $f=6{\cdot}2\times10^4 \sim$ and V max.=140 volts=$7{\cdot}7 \times 10^{-3}$ watt. Assuming loss $\propto f\mathrm{V}^2$ max.*, the amount dissipated during one quarter period prior to the passage of the spark when V max.=10^4 volts, is $1{\cdot}57\times10^{-4}$ joule, *i. e.* only 6 per cent. of $\frac{1}{2}\mathrm{C}_2\mathrm{V}_2^2$. Inclusion of the loss arising from the charging current in the winding increases this percentage to 7.

(7) *Calculation of Energy Loss in Primary and Secondary of Magneto prior to the passage of the spark.*

In calculating the energy loss in a magneto, we are only concerned with the energy which is lost during the time which elapses between the breaking of the primary current and the passage of the spark. The following method of calculation is approximate, but it serves to show the order of magnitude of the loss.

Let V_2 = secondary voltage when spark passes.
C_2 = self-capacity of secondary + capacity of leads to spark gap.
R_{e2} = effective resistance of secondary.
$\mathrm{I}_{\mathrm{RMS}}$ = root mean square of current.

Then current in secondary at any instant is $i_2=\mathrm{C}_2\dfrac{d\mathrm{V}_2}{dt}$.

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* See G. E. Bairsto, Proc. Roy. Soc. A. vol. xcvi. pp. 363–382 (1920).

Assuming for simplicity that the voltage follows a sine law from zero to its first maximum, we have

$$i_2 = \omega C_2 V_2 \cos \omega t \quad \text{and} \quad I_{RMS} = \frac{\omega C_2 V_2}{\sqrt{2}}.$$

The loss in joules is $W = I_{RMS}{}^2 R_{e2} \times t$, where t = time for the voltage to attain its maximum value.

The energy stored in the condenser when the voltage is a maximum $= \frac{1}{2} C_2 V_2{}^2$, and this is dissipated chiefly in the spark. Hence total energy in secondary is $W + \frac{1}{2} C_2 V_2{}^2$. Thus efficiency $= \dfrac{\frac{1}{2} C_2 V_2{}^2}{W + \frac{1}{2} C_2 V_2{}^2}$.

The maximum or peak voltage which would have occurred if there had been no loss, is obtained from the relationship

$$\tfrac{1}{2} C_2 V'_2{}^2 = W + \tfrac{1}{2} C_2 V_2{}^2$$

or

$$V'_2 = \left[2 \frac{W}{C_2} + V_2{}^2 \right]^{\frac{1}{2}}.$$

As an example take the following data :—

$V_2 = 10^4$ volts.
$C_2 = 55$ picofarads, including leads.
$R_{e2} = 2 \times 10^5$ ohms.
$f = 6 \times 10^3 \sim$ (mean value for the two oscillations after break)*.

Then

$$\begin{aligned} W &= I_{RMS}{}^2 R_{e2} t \\ &= \omega^2 C_2{}^2 V_2{}^2 R_{e2} t/2 \\ &= 1{\cdot}8 \times 10^{-3} \text{ joule.} \end{aligned}$$

Actual energy obtained from magneto in electrostatic form $\left.\right\} = \frac{1}{2} C_2 V_2{}^2 = 2{\cdot}8 \times 10^{-3}$ joule.

Treating the primary circuit in a similar manner and assuming as a first approximation that the ratio of the peak voltages is equal to the ratio of the turns, we find the loss is $1{\cdot}3 \times 10^{-3}$ joule. Hence, neglecting the loss in the primary condenser, the total loss prior to the passage of the spark is $3{\cdot}1 \times 10^{-3}$ joule.

If this energy had been transferred to the secondary, the

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* This frequency is assumed for the period prior to sparking. Its magnitude is suggested by the curve of fig. 2 in Dr. Campbell's paper *(loc. cit.)*.

total electrostatic energy would have been $5{\cdot}9\times10^{-3}$ joule, and the peak voltage 14,600 volts. Thus the effect of the energy loss in the primary and secondary windings is to reduce the peak voltage 32 per cent.

(8) *Second method of calculating Energy Loss.*

(*a*) The energy loss in a magneto can be approached from a different point of view using the formulæ given by Taylor-Jones *. He has shown that when (1) there is no loss in the primary and secondary circuits (the whole of the energy in the primary would not necessarily be transformed into the spark in the secondary if there were no loss), (2) the two sets of oscillations which occur after break are damped sine waves, and (3) the interruption of the primary circuit is perfect, the peak voltage is given by the expression

$$V_2=\frac{L_{e1}+L_{e21}}{\sqrt{L_{e2}C_2}}\,i_0\,U\sin\phi. \quad . \quad . \quad . \quad . \quad . \quad . \qquad (9)$$

Taking $L_{e21}=M=k\sqrt{L_{e1}L_{e2}}$, this expression can be written

$$V_2=\left[\frac{L_{e1}}{\sqrt{L_{e2}}}+k\sqrt{L_{e1}}\right]\frac{U\sin\phi}{\sqrt{C_2}} \text{ per ampere}$$

broken in the primary circuit. . (10)

U is given by the expression

$$\frac{1}{[u+s-2\{(1-k^2)u\}^{\frac{1}{2}}]^{\frac{1}{2}}}, \quad . \quad . \quad . \quad . \quad . \qquad (11)$$

where $u=\dfrac{L_{e1}C_1}{L_{e2}C_2}$.

ϕ is given by $\dfrac{2\pi}{1+n^2/n_1}$, where n_2 and n_1 are the frequencies of the two oscillations, and n_2/n_1 lies between 1 and 5. If n_2/n_1 is greater than 5, the expression for ϕ is found by multiplying by some power of 2.

The ratio n_2/n_1 is obtained from the relation

$$\left(\frac{n_2}{n_1}\right)^2=\frac{s+u+[(s+u)^2-4(1-k^2)u]^{\frac{1}{2}}}{s+u-[(s+u)^2-4(1-k^2)u]^{\frac{1}{2}}}. \quad . \quad . \qquad (12)$$

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* Phil. Mag. vol. xxxvi. Aug. 1918.

From $f=2000 \sim$ to $f=10,000 \sim$ the value of L_{e1}/L_{e2} remains approximately constant. (See Table I.) Since C_1/C_2 is constant, it follows that u is constant. We will also assume that k^2 is constant. For a certain B.T.-H. inductor type magneto we have

$C_2 = 55$ picofarads (including leads).

$C_1 = 0{\cdot}18$ microfarad.

$u = 0{\cdot}7$.

Taking s as unity, the following values of n_2/n_1 are found for different values of k^2 (see Table III.).

TABLE III.

n_2/n_1.	k^2.
5·0	0·84
4·3	0·8
3·4	0·7

It is very probable, judging from the experiments of Taylor-Jones, that the value of k^2 lies between 0·7 and 0·84. Assuming its value to be 0·8, the foregoing expression can be utilized to calculate ϕ. Thus we find that $\sin\phi=0{\cdot}93$, and from (11) the value of U is 1. In order to obtain V_2 we must substitute in (10). The individual magnitudes of n_1 and n_2 not being known, a difficulty arises with regard to the values to be assigned to L_{e1} and L_{e2}, since both of the latter vary with the frequencies n_1 and n_2. As a first approximation we will take the values of L_{e1} and L_{e2} at a frequency of $6000 \sim$ per second. This gives $L_{e1} \doteqdot 3{\cdot}14\times10^{-3}$ henry and $L_{e2} \doteqdot 14{\cdot}4$ henries. Substituting the above values in (10) we find that the peak voltage in the absence of loss is $V_2=12,600$ volts for a current of 2 amperes. With 2 amperes broken in the primary at full advance, the peak voltage was about 10,000. Thus due to the losses, (*a*) the reduction in peak voltage is 21 per cent., (*b*) the loss in energy is 60 per cent. of the actual energy given by $\frac{1}{2}C_2V_2^2=2{\cdot}8\times10^{-3}$ joule. The reduction in peak voltage found from the calculation in the preceding section worked out at 33 per cent., and the loss was 110 per cent. of $\frac{1}{2}C_2V_2^2$.

Consider the conversion of the electromagnetic energy in the primary winding into electrostatic energy in the secondary. Apart from the loss which occurs in both circuits, we must take the coupling and other factors into account. In a magneto or an induction-coil with a primary winding only, all the electromagnetic energy is dissipated in iron, copper, and stray losses. The efficiency of conversion is zero. Suppose the secondary to have the same number of turns as the primary. The secondary peak voltage and the self-capacity are extremely small, and therefore the electrostatic energy represented by the expression $\frac{1}{2}C_2V_2^2$ is very small. As the number of secondary turns is increased, the electrostatic energy is also increased (apart from losses) and with it the efficiency of conversion, until it reaches a maximum value. With a given number of primary and secondary turns and a fixed primary condenser, Taylor-Jones has shown that for a certain induction-coil the peak voltage and, therefore, the electrostatic energy prior to the passage of the spark has an optimum value, when the coupling is about 0·57. This is due to the phase relationships of the two sets of oscillations, *i. e.*, their combined effect in enhancing the peak potential is a maximum for this degree of coupling. In the magneto it is not practicable to vary the coupling by withdrawing the primary core, but it can be varied by adding external air-core inductance to the secondary. Experiments by Morgan * show that the peak voltage can be increased in this manner.

If at the moment at which the spark commenced, the energy of the primary was zero, the electrostatic energy obtained from the magneto *per se* would be the maximum possible. By reducing the losses this maximum would increase until finally the electromagnetic energy in the primary would be wholly converted into electrostatic energy in the secondary.

There is another method of increasing the secondary electrostatic energy apart from that of altering the coupling. If a condenser is connected in parallel with the secondary, the product $\frac{1}{2}C_2V_2^2$ increases with the capacity up to a certain point. For example, with a total capacity of 600 picofarads the peak voltage of a magneto was reduced from 10,000 to 5,000 at a certain speed. The self-capacity of the magneto was 65 picofarads, so that the energy was increased 150 per cent. Unfortunately, however, the peak voltage, which appears to be the prime factor in ignition, was decreased 50 per cent. Since the primary current was the same in both cases, it is clear that the conversion efficiency was also increased in the same proportion as the energy.

Returning to the example cited in section (7), the initial energy in the primary winding was not taken into account.

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* Morgan, 'Electric Spark Ignition' (Crosby Lockwood).

This can be done in the following way. The initial primary energy can be conveniently expressed in the form $\frac{1}{2}LI^2$, where L is some factor, usually termed the self-inductance, and I is the current at break. It is difficult to give a perfectly definite meaning to L owing to the peculiar conditions which obtain in the magneto. The usual procedure in measuring the value of L is to pass an alternating current through the primary. The conditions which then obtain, however, are very different from those just before the primary circuit is broken. It is impossible to state whether the value of L found by means of an alternating current is equal to that in the expression $\frac{1}{2}LI^2$. L in this expression would probably be found most accurately if the actual energy in the primary for given values of the current and a definite position of the timing lever could be ascertained. In the absence of data with respect to L, we will take the value as found by measurements with low frequency alternating current.

Here again there is a discrepancy since the inductance varies with the current, so that we can only approximate by taking the inductance of the armature at the point of break for a current whose maximum value is equal to the current broken in the primary. Knowing L_1 and I_1, at break*, also the peak voltage, we can approach the energy problem and determine the overall conversion efficiency from primary to secondary, *i. e.*, the ratio $\frac{1}{2}C_2V_2^2/\frac{1}{2}L_{e1}I_1^2$.

In the B.T.H. magneto used for these tests, $L_{e1} \doteqdot 6{\cdot}5 \times 10^{-3}$ henry at full advance break and $I_1 \doteqdot 2$ amperes when $V_2 = 10^4$ volts. Thus $\frac{1}{2}L_{c1}I_1^2 = 12{\cdot}5 \times 10^{-3}$ joule. From the calculation in the preceding section we have

Energy lost in primary and secondary prior to sparking + energy in spark

$$= 3{\cdot}1 \times 10^{-3} + 2{\cdot}8 \times 10^{-3} = 5{\cdot}9 \times 10^{-3} \text{ joule.}$$

Hence the energy unaccounted for is

$$(12{\cdot}5 - 5{\cdot}9)\,10^{-3} = 6{\cdot}6 \times 10^{-3} \text{ joule.}$$

This is unconverted into electrostatic energy prior to the passage of the spark. Part of it is transferred after the spark commences, the remainder being dissipated in the primary and secondary windings. Thus we may, following Morgan (*loc. cit.*), consider that the energy, apart from that derived from rotation, is supplied from the

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* McLachlan, Proc. Phys. Soc. Dec. 1919.

primary in the electrostatic form up to the time of sparking and thereafter in the inductive form as a current in the secondary circuit. It is the former component which determines the ignition of an explosive mixture.

The overall conversion efficiency of the magneto is

$$\frac{280}{12\cdot 5}=22 \text{ per cent.}$$

Taking now the figures obtained by using the formulæ of Taylor-Jones, we find that the secondary energy, if there were no loss, is $4\cdot 4\times 10^{-3}$ joule, whereas the former method gave $5\cdot 9\times 10^{-3}$.

(*b*) *Effect of Variation in Inductance on Peak Voltage and Energy Loss.*

Apart from the reduction in peak voltage due to dissipation of energy in the primary and secondary windings, there is also a reduction caused by the diminution in inductance. It has already been shown that $u=\frac{L_{e1}C_1}{L_{e2}C_2}$ is nearly constant. The value of k does not undergo any very pronounced variation up to $f=6000\sim$; it decreases with f, owing to the reduction in apparent permeability of the iron, but the rate of decrease is not very large. For simplicity k will be taken as constant. Moreover $U\sin\phi$ is constant. Thus at full advance the ratio of the peak voltage with unvarying inductance to that in the case cited above where $f=6000\sim$, is approximately equal to

$$\frac{\sqrt{L_{e1}\text{ to steady currents}}}{\sqrt{L_{e1}\text{ at } f=6000\sim}}$$

$$=\sqrt{\frac{6\cdot 5}{3\cdot 14}}=1\cdot 44.$$

Hence the diminution in inductance causes a drop in the theoretical peak voltage of 31 per cent., provided of course the value of n_2/n_1 due to increase in " k " does not increase beyond 5 and so alter the phase relations of the two oscillations which occur after break.

Substituting the direct-current or low-frequency values of the coefficients in (10), we find that the peak voltage is 18,200. In practice it is 10,000, *i. e.* 55 per cent. of the ideal.

Finally, if all the primary energy at break were transferred to the secondary in the form $\frac{1}{2}C_2V_2^2$, the peak voltage would be 21,300.

(9) *Comparison of foregoing methods of calculating Loss.*

We are now in a position to compare the values of the loss in a magneto obtained by the two methods of computation. The first method yields $3{\cdot}1 \times 10^{-3}$ joule and the second $1{\cdot}6 \times 10^{-3}$ joule. It is evident, therefore, that one or both of the methods or the assumed frequency is in error. It is possible of course to select a *frequency for which the methods yield identical results.*

In the experiment with the short-circuited exploring coil (see 6 *d*) it was shown that the effective resistance of the primary winding was decreased owing to mutual action. Now in a magneto under working conditions there is a mutual action between the primary and secondary currents. The oscillations of lower frequency are almost in phase, whereas those of higher frequency are almost in opposition. Thus the magnetization will be increased in the first case but decreased in the second. This affects the effective resistances accordingly, and introduces an indefiniteness with regard to their magnitude under working conditions. It appears, therefore, that without a knowledge of the precise conditions, the data obtained in the experiments is not of much value in the prediction of the loss. Owing to the variation with frequency the requisite values for substitution in expression (10) are unknown and can only be assumed. Hence on this basis both methods of computation are unreliable.

The interaction of primary and secondary suggests a third way in which the problem of ascertaining the loss can be attacked. The effective resistance of the primary is found by current variation as before, but with the secondary winding in position. In this way the reaction* of the secondary is obtained. The results are exhibited in fig. 17, which has the same appearance as the usual type of resonance curve. This curve may be termed a resistance-resonance curve, since the peak value of the resistance occurs at approximately the same frequency as that at which the secondary coil has its first self-resonance. If the mean frequency of the oscillations in the magneto is assumed, we can calculate the total primary and secondary

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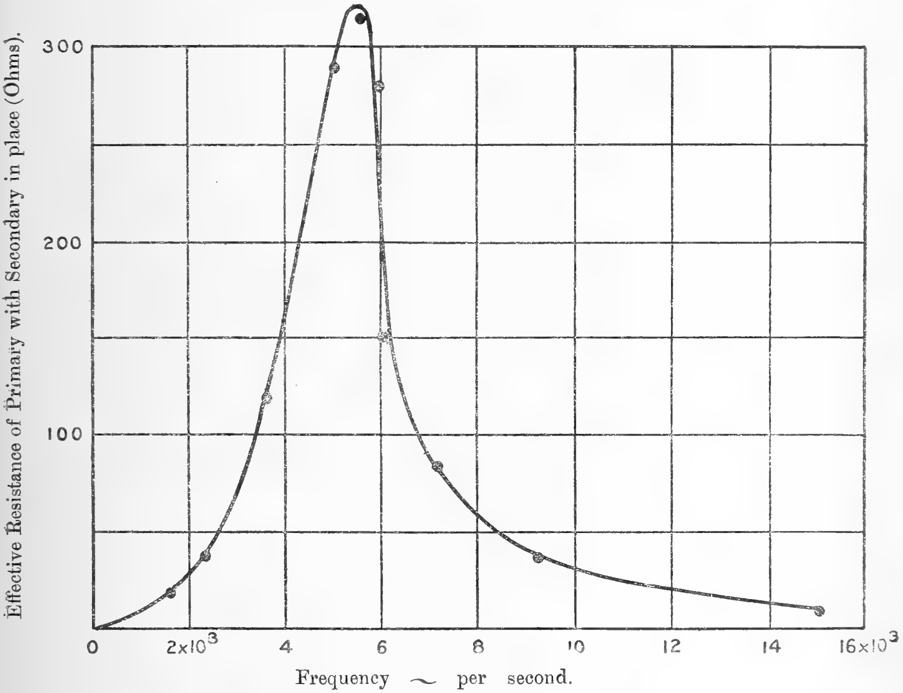
* The reaction under damped oscillations is different from that obtained with undamped oscillations.

loss prior to sparking by method 1. The difference is simply the altered value of the resistance. In this way the loss works out at 5×10^{-3} joule, a figure which is larger than either of those obtained from the previous calculations. The high value is evidently a result of choosing a frequency in the neighbourhood of the self-resonance of the secondary coil.

Fig. 17.—Curve showing effective resistance of primary winding at various frequencies, measured with secondary in position.



Resonance of Secondary Winding occurs when $f \doteqdot 5500 \sim$ per sec.
Armature in housing. Rotor position = after full advance break.

With damped oscillations in the magneto the conditions are not the same as those outlined above. However, some experiments have been carried out which seem to indicate that a phenomenon similar to that described already occurs in the magneto when certain conditions are fulfilled. This will be treated separately below. Since no consistency was discovered among the three methods of estimating the loss, it was imperative to elucidate the matter from another point of view. In Table IV. figures are given showing the ratio of the first maximum and minimum values of the voltage wave when the magneto is connected to a peak-voltage measuring apparatus [*]. If we assume an oscillation of the form $e^{-\lambda t}\sin\omega t$, it is possible to calculate the loss due to damping. Thus for the magneto used in these experiments $\lambda=0{\cdot}92\,\omega/\pi$, and the ratio of the actual peak voltage to that when $\lambda=0$ is 1 : 1·5. The value of λ depends of course on the effective resistance and inductance under damped oscillations. It is increased by increase of the former and diminution of the latter. Hence λ includes these two effects and is not merely a measure of the total energy loss alone. The energy loss calculated from the above figures is $3{\cdot}6\times10^{-3}$ joule, a value which is of the same order of magnitude as that obtained by method 1.

It will be evident from the foregoing computations that little advantage would have been gained by obtaining the various coefficients of the circuits to a high degree of accuracy. The above results amply demonstrate that in its present form the magneto (at least the particular type used in this research) is an inefficient apparatus for converting electromagnetic energy into electrostatic energy of high peak potential. The same conclusion was formulated by Taylor-Jones using a rotating armature type magneto, who attacked the problem from a slightly different aspect.

(10) *Comparison of Loss with Armature in and out of Housing.*

With the armature out of the housing we should expect the loss prior to the passage of the spark to be a good deal smaller than that which occurs in the housing when the iron circuit is almost complete. Some approximate data are given in Table IV. illustrating this point.

$$\text{The damping} = \text{the ratio } \frac{\text{1st maximum of voltage wave}}{\text{1st minimum of voltage wave}}.$$

The values of A are calculated from equation (10) using the low-frequency inductances.

It is of interest to observe that, using the same primary condenser, the iron only increases the peak voltage about threefold. The conversion efficiency, however, is trebled.

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[*] For method of measurement see McLachlan, Proc. Phys. Soc. Feb. 1920.

TABLE IV.

Rotor position = after full advance break.

Peak Volts per ampere broken in primary.				Ratio B/A.		Conversion Efficiency. $\frac{C_2V_2^2}{L_{e1}I_1^2}$		Damping.	
Theory. A.		Experiment. B.							
In housing.	Out of housing.	In housing.	Out of housing.	In housing.	Out of housing.	In housing. %.	Out of housing. %.	In housing.	Out of housing.
9100	5400	5000	3900	0·55	0·72	24	32	2·5	1·2

Damping with air-cored armature .... = 1·1

Peak voltage per ampere broken in primary with air-core = 1250

From these figures it will be evident that the iron loss is a greater proportion of the initial primary energy with the armature in the housing than out of it. The most striking contrast is that of the damping, which is halved by removal of the armature from the housing. It is of interest to note that with an air core, the damping is not much greater than that with the iron core. This indicates that the iron loss with the armature out of the housing is not very large.

(11) *The Effect of an Air-Core Inductance in the Primary Circuit on the Damping.*

Prof. Taylor-Jones * and Dr. Campbell † have carried out experiments on the peak potential obtainable from a magneto when an air-core inductance is put in series with the primary winding and a definite direct current supplied from an external source is broken. The secondary peak voltage under these circumstances varies with the primary capacity, and for a certain capacity it has a minimum value. This can be predicted from the theory of Taylor-Jones.

It is the purpose in what follows to describe the effect on the damping of varying the inductance when the primary

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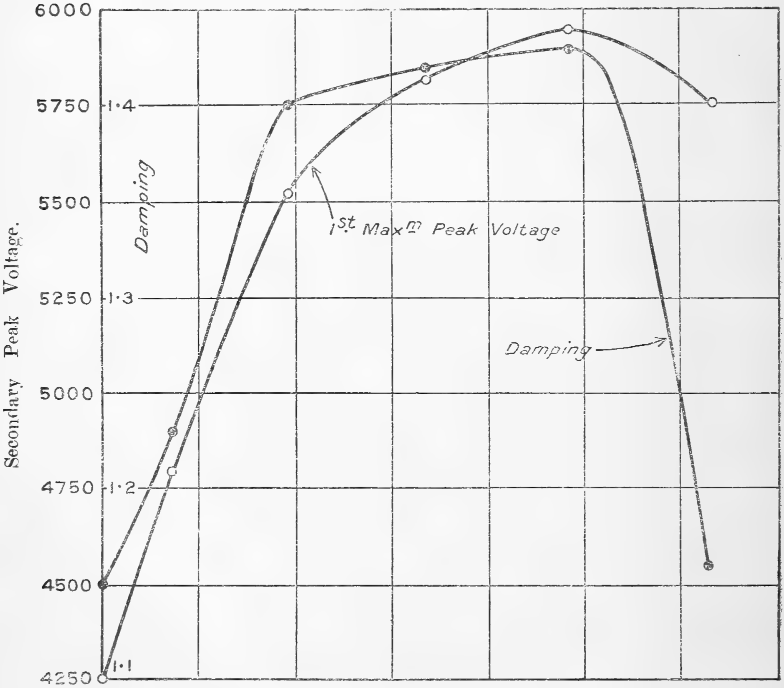
* *Loc. cit.*

† *Loc. cit.*

capacity is fixed. There are two cases which arise: (*a*) the armature out of the housing, (*b*) the armature in the housing. The measurement of the damping, *i. e.* the ratio of the amplitude of the first positive to that of the first negative oscillation, is conducted in the manner described in a former paper *. It should be mentioned that both the wave form and the separate frequencies of the two oscillations alter with the added inductance. We are at the moment, however, only interested in the above ratio.

Fig. 18.—Showing the effect on the Peak Voltage and the Damping of adding air-core inductance to the primary winding. A current of 1 ampere broken by using a separate contact-breaker.



Armature out of housing, with core of stalloy sheets 0·38 mm. thick.
Current broken in primary ≑ 1·0 ampere.

Some experiments of the above nature resulted in the curves of fig. 18. It is seen at once that not only do

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* McLachlan, Proc. Phys. Soc. Feb. 1919.

the peak voltage and the damping attain maximum values, but these occur at the same value of the added inductance. In the absence of wave forms, it is impossible to explain the phenomenon fully; but there is a similarity between the results and those of fig. 17.

It appears, therefore, that when the phase relation of the two oscillations is such that the peak voltage is a maximum the effect on the iron of the core is to magnetize it in such a way that the damping is a maximum. Although the peak voltage is increased, it will be found that the conversion efficiency is practically the same with an added inductance of 2·2 millihenries as it is without this inductance. Similar curves were obtained with the armature *in the housing*, but the damping was much higher. By increasing the inductance sufficiently, more than one peak was obtained.

Oneof the most important factors in ignition with a sparking plug is the "rate of rise of voltage." Although the peak voltage for a given current is augmented by adding uncoupled inductance to the primary, it does not follow that this would be beneficial, since it is possible that it would cause a reduction in the rate of rise of voltage. Dr. Campbell has shown that, apart from the foregoing consideration, it would be impracticable to add uncoupled inductance to the primary of a magneto *.

General Conclusions on Losses.

(1) The evidence with regard to iron loss caused by hysteresis and eddy-currents is such that its occurrence reduces the efficiency of the magneto considerably.

(2) The reduction in the effective permeability of the magnetic circuit caused (*a*) by air-gaps, (*b*) by the screening effect of eddy currents, (*c*) by polarization due to the magnet especially at retard break, gives rise to an appreciable drop in peak voltage.

(3) Apart from modifications arising from alteration in the coefficient of coupling of the two circuits, the detrimental effects commented on in (1) and (2) can be partially remedied by the use of thin laminations of the form specified at the beginning of the paper.

(12) *Self-Resonance of Secondary Winding.*

Owing to the large effective inductance and self-capacity of the secondary coil, its first resonance occurs at a frequency well within the acoustic range. Two methods of

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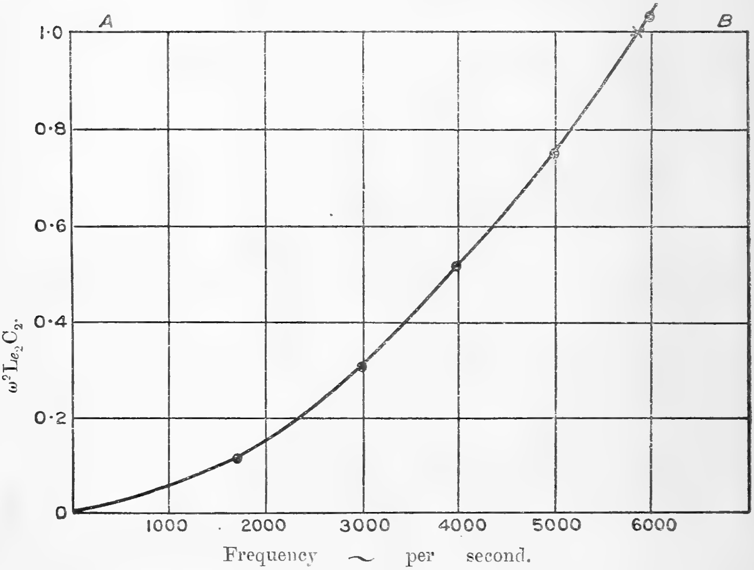
* *Loc. cit.*

obtaining the self-resonance were used, one of which has already been outlined. The other method of obtaining the value of the resonant frequency is a combination of experiment and calculation. The inductance can be measured up to a frequency of 2500 ~ per second, and the self-capacity is measured by the method described above in (4). The value of the latter when the coil is shunted by a comparatively large condenser (say three or four times the self-capacity of the coil) is assumed to be the same as that at frequencies in the neighbourhood of the self-resonance. Beyond $f=2500$ ~ the inductance is obtained by the method of calculation outlined previously in (3).

Fig. 19.—Curve used to obtain self-resonance of secondary winding.



× = Resonance point.

$f \doteqdot 5900$ ~ per sec.

Armature in housing; brass end-plates in place.

Rotor position after full advance break.

Thus, since values of L_{e2} and C_2 are known for frequencies from 2,500 ~ to 35,000 ~, the corresponding values of $\omega^2 L_{e2} C_2$ can be found. Plotting $\omega^2 L_{e2} C_2$ against frequency, the curve shown in fig. 19 is obtained. The self-resonance frequency occurs when $\omega^2 L_{e2} C_2 = 1$, *i. e.* where the curve meets the horizontal line AB.

Referring once more to the curve of fig. 17, its utility is not confined merely to the magneto. The method of procedure can be adopted to gain some knowledge of the self-resonance of any iron-cored coil. Since the resonance curve (obtained by varying the inductance in circuit (2) of fig. 2) is extremely flat-topped in the neighbourhood of the self-resonant frequency of the coil, the values of the primary effective resistance can be obtained much more accurately by a bridge method. It is highly probable that the curve obtained would show subsidiary peaks indicating minor resonance points at frequencies greater than the first self-resonance. Some evidence of this was obtained during the experiments, since the apparent inductance of the primary was alternately positive and negative. It does not follow, of course, that with an iron-cored coil the frequencies at which these peaks occurred would bear any definite relation to one another.

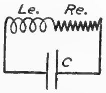
The self-resonant frequency obtained by the methods given in this section and in section (5) differs by 7 per cent. This is due doubtless to the calculated value of L_{e2} being low, since it was 5 per cent. in defect at $f=2500$.

As a matter of interest, the self-resonance of the secondary without an iron core was investigated using the apparatus of fig. 2, and the value so obtained agreed with that calculated from separate inductance and capacity measurements to about 1 per cent.

(13) *Effective Inductance and Resistance under Damped Oscillations.*

Consider an oscillatory circuit such as that shown in fig. 20, in which the coil has an iron core. When the condenser discharges there will be oscillations, provided the usual well-known condition is satisfied, *i.e.* $\frac{1}{L_e C} > \frac{R_e^2}{4L_e^2}$.

Fig. 20.—Oscillatory circuit with iron-cored inductance and condenser.



Now the apparent permeability of the iron varies with the current flowing through the coil, and this causes a variation in the effective resistance, the periodic time, and also the shape of the voltage and current waves. It is essential, therefore, that in order to treat the subject at all, some assumptions must be made with regard to the above variables. The simplest method is to assume the permeability and the effective resistance to be constant, with a given capacity in circuit, from which it follows that the oscillation is of the form $e^{-\lambda t}\sin(\omega t+\theta)$.

Let
L_e = effective inductance, assumed constant.
R_e = effective resistance, assumed constant.
C = capacity of condenser.
$\lambda = R_e/2L_e$.
v_c = voltage on condenser at any instant.
v_L = ,, ,, inductance ,,
v_r = ,, ,, resistance ,,
i = current at any instant.
V_0 = maximum voltage on condenser.

Then it can be shown in the usual manner that the

Instantaneous voltage on condenser,

$$v_c = V_0\left(\frac{\lambda^2+\omega^2}{\omega}\right)^{\frac{1}{2}} e^{-\lambda t}\sin(\omega t+\theta) \quad . \quad . \quad . \quad . \quad . \quad (13)$$

Instantaneous voltage on resistance

$$v_r = V_0CR_e\left(\frac{\lambda^2+\omega^2}{\omega}\right)e^{-\lambda t}\sin(\omega t+\pi) \quad . \quad . \quad (14)$$

Instantaneous voltage on inductance

$$v_L = L_e(di/dt) = V_0CL_e\left(\frac{\lambda^2+\omega^2}{\omega}\right)^{\frac{3}{2}} e^{-\lambda t}\sin(\omega t-\theta)\ ; \quad (15)$$

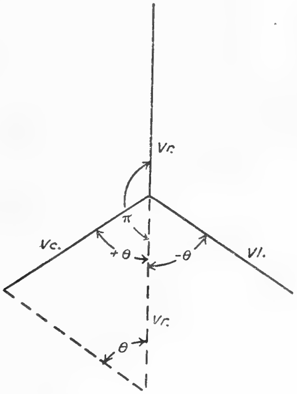
where

$$\theta = \tan^{-1}\omega/\lambda \quad \text{and} \quad \omega^2 = \frac{1}{L_eC} - \frac{R_e^2}{4L_e^2}.$$

Now the vector sum of the e.m.f.'s round a closed circuit is zero: hence we can represent the e.m.f.'s of the circuit of fig. 20 at any instant as shown in fig. 21.

As the time increases from the instant the condenser discharges, the vectors are to be imagined to rotate with angular speed ω and shrink in accordance with the damping factor $e^{-\lambda t}$. They also preserve the same relative angular displacements as those shown in fig. 21.

Fig. 21.—Vector diagram showing phase relations of voltages in circuit of fig. 20. $V_L = V_c$ in magnitude and the vector sum of the two is V_r. When V_r is zero or very small, $\theta = \pi/2$ and V_L and V_c are equal and opposite.



It is clear that there is a difference in the definition of the terms L_e and R_e, if not in actual value, from that for undamped oscillations of the same frequency (see fig. 1). The experimental work to obtain the values of the above quantities for a magneto under actual conditions is extremely difficult, owing to the small amount of energy available and the very high damping.

(14) *Measurement of the Effective Inductance and Effective Resistance under Damped Oscillations.*

We will now examine a peculiar result obtained by Dr. Campbell and mentioned in his paper (see p. 385, *loc. cit.*). Using an oscillation method (damped oscillations) of determining the primary inductance, he obtained a value of $6{\cdot}23 \times 10^{-3}$ henry. On inserting an air-cored coil of 1×10^{-3} henry in series with the primary, the self-inductance was reduced to $5{\cdot}78 \times 10^{-3}$. This is, of course, a physical impossibility since the total inductance should be $7{\cdot}23 \times 10^{-3}$ henry. The apparent paradox can be explained if we consider the effect of the iron loss on the natural period of the circuit.

Consider the relation

$$\omega^2 = \frac{1}{L_e C} - \frac{R_e^2}{4L_e^2}.$$

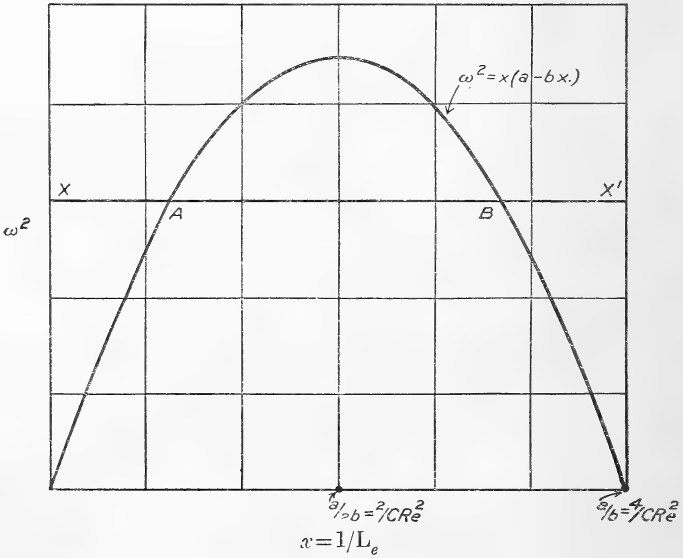
This can be written

$$\omega^2 = \left(\frac{1}{C} - \frac{R_e^2 x}{4}\right) x, \text{ where } x = 1/L_e$$

$$= (a - bx)x. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (16)$$

If we plot ω^2 and x, the curve obtained is of the form shown in fig. 22, *i. e.* a parabola. Draw any line xx' parallel to the x axis. This cuts the curve in two points A, B. The value of ω^2, and therefore of ω and f, is the same at A and B, but the value of x and therefore of L_e is quite different. Thus, so long as $1/L_e > 4/CR_e^2$ or $CR_e^2 < 4L_e$ and the resistance and condenser are fixed, there are two values of L_e which give the same frequency. When $L_e > CR_e^2/4$ but

Fig. 22.—Curve showing relation between ω^2 and $1/L$ in an oscillatory circuit having resistance of appreciable magnitude such that it affects the natural period.



$<\mathrm{CR}_e^2/2$, the value of ω^2 increases with the inductance; when $\mathrm{L}_e > \mathrm{CR}_e^2/2$, ω^2 decreases with increase in inductance. Thus increasing the inductance will increase or decrease ω according as we are at B or A.

Let $\mathrm{L}_e =$ effective inductance of primary winding.
$\mathrm{L} =$ inductance of air-core coil.
$\mathrm{C} =$ primary capacity.
$\omega_0 =$ pulsatance for primary alone.
$\omega_1 =$ pulsatance for primary + air-core coil.

Then from (10) we have

$$\mathrm{R}_e = 2\mathrm{L}_e\left(\frac{1}{\mathrm{L}_e\mathrm{C}} - \omega_0^2\right)^{\frac{1}{2}}, \quad \ldots \ldots \quad (17)$$

$$\mathrm{R}_e = 2(\mathrm{L}_e + \mathrm{L})\left(\frac{1}{(\mathrm{L}_e + \mathrm{L})\mathrm{C}} - \omega_1^2\right)^{\frac{1}{2}}. \quad \ldots \quad (18)$$

Since the values of ω_0 and ω_1 are not very different, the values of R_e from (17) and (18) are practically identical, provided the max. currents and the damping are equal in both cases*. Hence, equating (17) and (18) and squaring both sides, we obtain

$$\mathrm{L}_e^2\left(\frac{\mathrm{L}}{\mathrm{L}_e\mathrm{C}} - \omega_0^2\right) = (\mathrm{L}_e + 2\mathrm{LL}_e + \mathrm{L}^2)\left(\frac{1}{(\mathrm{L}_e + \mathrm{L})\mathrm{C}} - \omega_1^2\right),$$

from which we get

$$\mathrm{L}_e = \frac{-\mathrm{L}\omega_1^2 \pm \left\{\mathrm{L}^2\omega_1^4 + \mathrm{L}\left(\frac{1}{\mathrm{C}} - \mathrm{L}\omega_1^2\right)(\omega_1^2 - \omega_0^2)\right\}^{\frac{1}{2}}}{\omega_1^2 - \omega_0^2}. \quad . \quad (19)$$

In the case mentioned above, the capacity employed in determining the inductance is not stated, but we will assume this to be 1 microfarad. Neglecting the effective resistance as Dr. Campbell did, we have

$$\omega_0^2 = \frac{1}{\mathrm{L}_e\mathrm{C}} \text{ for the primary winding}$$
$$= 1/(6{\cdot}23 \times 10^{-9}) = 1{\cdot}61 \times 10^{+8},$$
$$\omega_1^2 = \frac{1}{(\mathrm{L}_e + \mathrm{L})\mathrm{C}} \text{ (for primary + air-core coil)}$$
$$= 1{\cdot}73 \times 10^{+8}.$$

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* The maximum current should (if possible) be such that the inductance, resistance, and periodic time do not vary appreciably as the oscillations die away.

Substituting in (19) for L, C, ω_0, ω_1, we find $L_e=2\cdot 34 \times 10^{-3}$ henry. Substituting in (14) for R_e we obtain $R_e=76$ ohms. Thus it is quite evident that the effective resistance cannot be disregarded. If C is taken as 0·1 microfarad, L_e still has the same value but R_e is now 240 ohms. The variation in R_e is a result of the assumption regarding the value of C, or of the fact that the correct value of ω is not known. If the value used in the experiments were known, we should be able to find the correct value of R_e.

From the preceding analysis we can immediately suggest a method of obtaining R_e and L_e experimentally. The method is to find ω or f using the primary winding, preferably with the secondary removed, and a condenser. The latter can be varied to vary f. f is then found when an air-cored coil, of such a value that the former is not increased more than about 10 per cent., is put in series. L_e and R_e are obtained by substitution in the formulæ given above. In order to secure accuracy the measurement of f requires careful attention *. The above rests on the assumption that the oscillations are damped sine waves, and that there is no variation of L_e and R_e due to the slight change in frequency. If we assume $R_e \propto \omega$ and $L_e \propto 1/\omega$, we can obtain formulæ for L_e and R_e of the same nature as those already given. This refinement is hardly justifiable.

It is possible to examine these experiments from another point of view. We have

$$\omega^2 = \frac{1}{L_e C} - \frac{R_e^2}{4L_e^2},$$

$$\text{or} \qquad 4\omega^2 C L_e^2 - 4L_e + CR_e^2 = 0. \quad . \quad . \quad . \quad (20)$$

Assuming that R and L are constant, we have

$$4\omega_0^2 C_0 L_e^2 - 4L_e + C_0 R_e^2 = 0,$$

$$4\omega_1^2 C_1 L_e^2 - 4L_e + C_1 R_e^2 = 0,$$

where C_0 and C_1 are two values of the primary capacity as used in experiments in the above paper (*loc. cit.*).

Solving these equations we obtain

$$L_e = \frac{R_e}{4\pi}\left(\frac{C_1 - C_0}{C_0/T_0^2 - C_1/T_1^2}\right)^{\frac{1}{2}}. \quad . \quad . \quad . \quad . \quad (21)$$

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* See Taylor-Jones and Campbell (*locc. citt.*).

If T^2 is directly proportional to C we have

$$\frac{C_0}{T_0^2} = C_1/T_1^2,$$

and since R_e is not zero L_e is infinite, which is impossible. This, however, does not occur actually, since the line through the points obtained by plotting C and T^2 does not pass through the origin.

From (20) we obtain

$$\frac{T^2}{C} = 4\pi^2 L_e\left(1+\frac{R_e^2 T^2}{16\pi^2 L_e^2}\right).$$

Thus if T^2/C is constant we must have approximately (neglecting the intercept on the T^2 axis)

$$4\pi^2 L_e\left(1+\frac{R_e^2 T^2}{16\pi^2 L_e^2}\right) = \text{a constant}$$

$$= \text{slope of line in fig. 3 of above paper.}$$

Thus if L_e is assumed constant and found from the slope of the line, as in Dr. Campbell's experiments, its value is approximately $\left(1+\frac{R_e^2 T^2}{16\pi^2 L_e^2}\right)$ times too large. Using the values of R_e, ω, and L_e obtained previously, we find that the above factor is 2·6. Hence the value of L_1, viz. $6{\cdot}23 \times 10^{-3}$ henry as found by Dr. Campbell, is 2·6 times too large. It appears, therefore, that in finding the inductance of a magneto by methods in which damped oscillations are employed, the effective resistance of the winding cannot be neglected.

The effective inductance and the effective resistance can also be obtained by varying the capacity in series with the iron-cored cell, instead of varying the inductance.

Using the same symbols as before, we obtain

$$\omega_0^2 = \frac{1}{L_e C_0} - \frac{R_e^2}{4L_e^2}, \quad \dots\dots\dots \quad (22)$$

$$\omega_1^2 = \frac{1}{L_e C_1} - \frac{R_e^2}{4L_e^2}. \quad \dots\dots\dots \quad (23)$$

From (16) and (17) we find that

$$L_e = \frac{C_0 - C_1}{C_0 C_1} \cdot \frac{1}{\omega_1^2 - \omega_0^2}$$

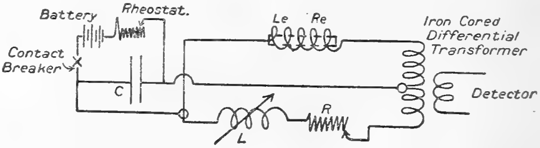
and

$$R_e = 2\left(\frac{L_e}{C_0} - \omega_0^2 L_e^2\right)^{\frac{1}{2}}.$$

The difference in capacity $C_0 - C_1$ should be small, so that R_e and L_e do not change appreciably, and the maximum current should be the same in each case. The frequency corresponding to the values of L_e and R_e may be taken as $(f_0+f_1)/2$, *i.e.* the mean of the two frequencies.

It is not suggested that either of the methods outlined will give particularly accurate results. They will at least indicate the order of magnitude. In measuring inductances and resistances of iron-cored coils, the conditions are invariably complex, especially at low frequencies where the apparent permeability is susceptible to pronounced variation, and also at high frequencies if the magnetizing forces are large. More accurate results could probably be obtained by using the method shown in fig. 23. Damped oscillations are obtained using a battery and a contact-breaker. R and L are adjusted to obtain a balance. Then $R = R_e$ and $L = L_e$, and the frequency of the system is given by

Fig. 23.—Diagram of apparatus used for bridge method of determining self-inductance and effective resistance under damped oscillations.



$$\omega^2 = \left(\frac{2}{LC} - \frac{R^2}{4L^2}\right).$$

By varying C, different frequencies can be obtained. If the detector is selective, the effect of harmonics can be examined. In place of the battery and condenser, a source of undamped waves can be used to determine R_e and L_e instead of the methods used herein, as has been stated before.

November 1919.

IV. *The Transverse Vibrations of Beams and the Whirling of Shafts supported at Intermediate Points.* *By* E. R. Darnley*.

§ 1. THIS paper investigates the equations giving the periods of the transverse vibrations of uniform beams and the whirling speeds of uniform shafts, when the beams or shafts are simply supported at any number of intermediate points (§§ 4–6). The remainder of the paper is confined to the case where the ends also are simply supported. New functions denoted by ϕ and ψ are tabulated and graphed. The use of these tables and graphs considerably extends the class of cases in which numerical solutions can be obtained, and suggests that some of Dunkerley's results need revision (§§ 7–10). A general theorem is given relating to symmetrical arrangements of the supports (§ 11). This theorem is similar to one relating to the critical loading of a strut, recently published by Cowley and Levy †. The question of the distribution of the supports to give the greatest value of the slowest period or whirling speed is discussed; and it is shown that this period or speed is a maximum when the supports are equidistant (§ 13).

The general theory of the vibrations of beams of one section or bay has been treated by Rayleigh in chapter VIII. of his book on Sound, and by Love in chapters XII. and XX. of his book on Elasticity. The former gives a detailed account of the vibrations of a beam in the six possible cases of terminal conditions, arising according as either or both of the ends are supported, clamped, or free, with very exact numerical solutions for the periods, and a study of the shape of the vibrating beam and of the position of the nodes and loops in certain of the graver modes of vibration. The latter gives an account of the flexural vibrations of a circular cylinder from the point of view of the mathematical theory of elasticity, leading to a complicated frequency equation containing Bessel's functions, which reduces, when the radius of the cylinder is supposed small, to the equation derived from Kirchhoff's theory of thin rods or beams, which is used by Rayleigh and in this paper.

The theory of the transverse vibrations of thin beams is analytically identical with that of the whirling of shafts, regarding which reference may in particular be made to the papers by Dunkerley and Chree, quoted in the footnote.

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* Communicated by Prof. A. N. Whitehead, F.R.S.

† Proc. Roy. Soc. vol. xciv. p. 405 (1918).

In this connexion it has found important practical applications.

Dunkerley[*] and Chree[†] do not apply the strict Kirchhoff theory to beams or shafts of more than two bays or sections, but they deal with three bays and with many cases of loaded shafts by means of an approximate method suggested by Rayleigh, in which the form of the vibrating shaft is assumed to be defined by various types of algebraic formulæ.

§ 2. For an account of the method of forming the equations of motion reference may be made to the books by Rayleigh and Love.

Let m be the mass of the beam per unit length,
E its modulus of elasticity,
I the moment of inertia of the cross-section about a diameter.

Take the origin at one end of the beam, measure x along the beam, and let y be the lateral displacement, supposed small.

Then, neglecting the effect of the rotation of elements about an axis perpendicular to the beam, the equation of motion is

$$\mathrm{EI}\frac{d^4y}{dx^4}+m\frac{d^2y}{dt^2}=0,$$

which must be satisfied at all points along the beam, and the following are the conditions at the ends and at supported intermediate points.

At a simply supported end, the ordinate and the bending moment are zero, *i. e.* $y=0$, and $\frac{d^2y}{dx^2}=0$.

At an end fixed in direction, the ordinate and inclination are zero, *i. e.* $y=0$, and $\frac{dy}{dx}=0$.

At a free end, the bending moment and the shear are zero, *i. e.* $\frac{d^2y}{dx^2}=0$, and $\frac{d^3y}{dx^3}=0$.

At a simply supported intermediate point the ordinate is zero, and both the inclination and the bending-moment are continuous, *i. e.* $y=0$, and $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$ are continuous.

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[*] Dunkerley, "Whirling and Vibrations of Shafts," Phil. Trans. A. vol. clxxxv. pt. 1 (1894).

[†] Chree, "Whirling and Transverse Vibrations of Rotating Shafts," Phil. Mag. May 1894.

§ 3. If it is assumed that y is of the form $u\cos(pt+\epsilon)$, we have

$$\mathrm{EI}\frac{d^4u}{dx^4}=p^2mu,$$

and, putting $\frac{p^2m}{\mathrm{EI}}=\mathrm{K}^4$,

$$\frac{d^4u}{dx^4}=\mathrm{K}^4u.$$

The solution of this equation is

$$u=a\cos\mathrm{K}x+b\cosh\mathrm{K}x+c\sin\mathrm{K}x+d\sinh\mathrm{K}x,$$

where a, b, c, d are constants to be determined by the end conditions and the conditions at the intermediate supports.

The differential equation for u is satisfied at all points of the beam, but the constants of the solution will be different for each bay.

§ 4. Consider the case of a continuous beam of n bays simply supported at the ends and at $(n-1)$ intermediate points. Let l_1, $l_2 \ldots l_n$ be the lengths of the bays. Take the origin successively at the left-hand end of each bay. The equation giving the value of u for the rth bay will be of the form

$$u=a_r(\cos\mathrm{K}x-\cosh\mathrm{K}x)+c_r\sin\mathrm{K}x+d_r\sinh\mathrm{K}x,$$

since $a_r+b_r=0$, because $u=0$ when $x=0$.

The further equations expressing the facts that $u=0$ and $\frac{du}{dx}$, $\frac{d^2u}{dx^2}$ are continuous at the supports are of the form

$$a_r(\cos\mathrm{K}l_r-\cosh\mathrm{K}l_r)+c_r\sin\mathrm{K}l_r+d_r\sinh\mathrm{K}l_r=0 \quad . \quad . \quad . \quad (1)$$

$$-a_r(\sin\mathrm{K}l_r+\sinh\mathrm{K}l_r)+c_r\cos\mathrm{K}l_r+d_r\cosh\mathrm{K}l_r=c_{r+1}+d_{r+1}. \quad (2)$$

$$a_r(\cos\mathrm{K}l_r+\cosh\mathrm{K}l_r)+c_r\sin\mathrm{K}l_r-d_r\sinh\mathrm{K}l_r=2a_{r+1}. \quad . \quad . \quad (3)$$

From (1) and (3) by addition and subtraction

$$a_r\cos\mathrm{K}l_r+c_r\sin\mathrm{K}l_r=a_r\cosh\mathrm{K}l_r-d_r\sinh\mathrm{K}l_r=a_{r+1}, \quad . \quad . \quad (4)$$

whence, if $\sin\mathrm{K}l_r$ is not zero,

$$c_r+d_r=a_r(\coth\mathrm{K}l_r-\cot\mathrm{K}l_r)-a_{r+1}(\mathrm{cosech}\,\mathrm{K}l_r-\mathrm{cosec}\,\mathrm{K}l_r). \quad (5)$$

Let $\coth\mathrm{K}l_r-\cot\mathrm{K}l_r=\phi(\mathrm{K}l_r)=\phi_r,$

$$\mathrm{cosech}\,\mathrm{K}l_r-\mathrm{cosec}\,\mathrm{K}l_r=\psi(\mathrm{K}l_r)=\psi_r.$$

Then $$c_r+d_r=a_r\phi_r-a_{r+1}\psi_r, \quad . \quad . \quad . \quad . \quad . \quad . \qquad (6)$$

and, similarly, if $\sin Kl_{r+1}$ is not zero,

$$c_{r+1}+d_{r+1}=a_{r+1}\phi_{r+1}-a_{r+2}\psi_{r+1}. \quad . \quad . \quad . \qquad (7)$$

By (4)

$$c_r\cos Kl_r+d_r\cosh Kl_r=a_{r+1}\cot Kl_r-a_r\frac{\cos^2 Kl_r}{\sin Kl_r}$$
$$-a_{r+1}\coth Kl_r+a_r\frac{\cosh^2 Kl_r}{\sinh Kl_r};$$

and by (2) and (7)

$$a_{r+1}\phi_{r+1}-a_{r+2}\psi_{r+1}+a_r(\sin Kl_r+\sinh Kl_r)$$
$$=c_r\cos Kl_r+d_r\cosh Kl_r=-a_{r+1}\phi_r-a_r\frac{\cos^2 Kl_r}{\sin Kl_r}+a_r\frac{\cosh^2 Kl_r}{\sinh Kl_r},$$

whence

$$a_{r+1}(\phi_r+\phi_{r+1})-a_{r+2}\psi_{r+1}+a_r\left\{\left(\sin Kl_r+\frac{\cos^2 Kl_r}{\sin Kl_r}\right)\right.$$
$$\left.+\left(\sinh Kl_r-\frac{\cosh^2 Kl_r}{\sinh Kl_r}\right)\right\}=0,$$

or $$a_r\psi_r-a_{r+1}(\phi_r+\phi_{r+1})+a_{r+2}\psi_{r+1}=0.$$

§ 5. If the end of the first bay be free, the equation for this bay will be

$$u=a_1(\cos Kx+\cosh Kx)+c_1(\sin Kx+\sinh Kx),$$

since at the origin $\frac{d^2u}{dx^2}$ and $\frac{d^3u}{dx^2}$ are zero.

The conditions at the junction of the first two bays are

$$a_1(\cos Kl_1+\cosh Kl_1)+c_1(\sin Kl_1+\sinh Kl_1)=0,$$
$$a_1(-\sin Kl_1+\sinh Kl_1)+c_1(\cos Kl_1+\cosh Kl_1)=c_2+d_2,$$
$$a_1(\cos Kl_1-\cosh Kl_1)+c_1(\sin Kl_1-\sinh Kl_1)=2a_2,$$

whence

$$(c_2+d_2)\phi_1=a_2\left\{\frac{2(1+\cosh Kl_1\cos Kl_1)}{\sinh Kl_1\sin Kl_1}\right\}=a_2\chi_1, \text{ say.}$$

But $c_2+d_2=a_2\phi_2-a_3\psi_2$.

Hence

$$a_2\left(\frac{\chi_1}{\phi_1}+\phi_2\right)-a_3\psi_2=0.$$

§ 6. Equations giving the periods or whirling speeds can now be obtained as determinants. These determinants are similar to continuants. When expanded they contain only even powers of the ψ's, so that the negative signs of the latter may be changed to positive. They are obtained on the hypothesis that $\sin Kl_r$ is not zero for any value of r. The solutions when that hypothesis is not fulfilled can, however, be obtained by considering the limiting ratios of the functions involved when they become infinite, and these solutions are important in the case where all the bays are equal.

Case I.—Ends simply supported.

The period equation is

$$\triangle_n \equiv \triangle(l_1, l_2 \ldots l_n) \equiv \left|\begin{array}{ccc:ccc} \phi_1+\phi_2, & \psi_2, & 0 & & & \\ \psi_2, & \phi_2+\phi_3, & \psi_3 & & & \\ 0, & \psi_3, & \phi_3+\phi_4 & & & \\ \hdashline & & & \phi_{n-3}+\phi_{n-2}, & \psi_{n-2}, & 0 \\ & & & \psi_{n-2}, & \phi_{n-2}+\phi_{n-1}, & \psi_{n-1} \\ & & & 0, & \psi_{n-1}, & \phi_{n-1}+\phi_n \end{array}\right| = 0,$$

which can be evaluated by the formula

$$\triangle_n - (\phi_{n-1}+\phi_n)\triangle_{n-1} + \psi^2_{n-1}\triangle_{n-2} = 0.$$

Particular cases : two bays, $\phi_1+\phi_2=0$;

three bays, $(\phi_1+\phi_2)(\phi_2+\phi_3)-\psi_2^2=0$;

four bays,

$$(\phi_1+\phi_2)(\phi_2+\phi_3)(\phi_3+\phi_4)-\psi_2^2(\phi_3+\phi_4)-\psi_3^2(\phi_1+\phi_2)=0.$$

Case II.—One or both ends fixed in direction.

The period equation may be derived from that in Case I. by regarding an end fixed in direction as the limit of an extra bay when the length of that bay is indefinitely diminished.

When both ends are so fixed the equation is

$$\left|\begin{array}{ccc:ccc} \phi_1, & \psi_1, & 0 & & & \\ \psi, & \phi_1+\phi_2, & \psi_2 & & & \\ 0, & \psi_2, & \phi_2+\phi_3 & & & \\ \hdashline & & & \phi_{n-2}+\phi_{n-1}, & \psi_{n-1}, & 0 \\ & & & \psi_{n-1}, & \phi_{n-1}+\phi_n, & \psi_n \\ & & & 0, & \psi_n, & \phi_n \end{array}\right| = 0.$$

Case III.—One or both ends free.

The equation for the periods is obtained from that in Case I. by replacing ϕ_1 by $-\frac{\chi_1}{\phi_1}$, or ϕ_n by $-\frac{\chi_n}{\phi_n}$, or both.

When there are two bays, one with a free end, otherwise simply supported, the equation is $\phi_1\phi_2=\chi_1$ or χ_2 according to which end is free.

§ 7. The general nature of the functions $\phi(x)$ and $\psi(x)$ will be seen from the tables and graph at the end of the paper.

$\phi(x)$ is zero when x is zero, and has poles (or infinities) when $x=\pi$ or any multiple of π. $\phi(x)=\frac{2}{3}x$ nearly when x, expressed in radians, is small. $\phi(x)$ exceeds this value by less than one per cent. up to $x=\frac{\pi}{3}$ or 60°. Near $x=\pi$, $\phi(x)$ tends to the value $\frac{1}{\pi-x}$ and passes through infinity from + to −. When x is greater than π, $\phi(x)$ tends to the value $1-\cot x$, exceeding that value by less than ·004 when $x=\pi$. $\phi(x)$ vanishes when $x=3{\cdot}9266$ radians, or 224° 56′ 58″. At this point it exceeds $(1-\cot x)$ by ·0008.

Similarly, $\psi(x)=\phi\left(\frac{x}{2}\right)-\phi(x)$, and is nearly $-\frac{1}{3}x$ when x is small. At $x=\pi$, $\psi(x)$ passes through infinity from − to +, and $\psi(x)=\phi(x)$ near $x=\frac{3\pi}{2}$ where both functions are nearly unity.

§ 8. The case of two bays simply supported is easily solved approximately from the figure. A more exact solution can then be obtained from the tables.

The period equation is $\phi_1+\phi_2=0$, and it is convenient to draw the graph of the function $-\phi$ for values of the argument from 180° to 225°. The problem then reduces itself to finding, by trial and error, a parallel to the axis of x which cuts the graphs of ϕ and $-\phi$ in points whose abscissæ are in the ratio of the lengths of the bays.

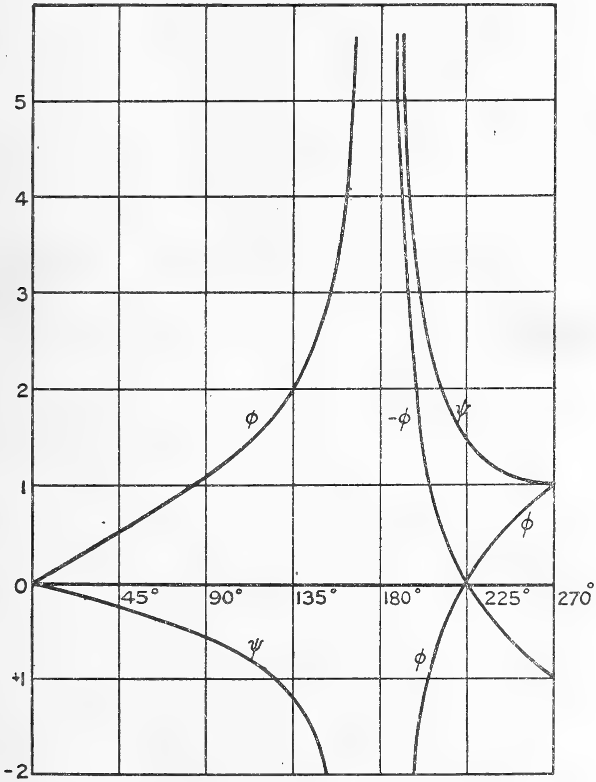
Thus, if $l_1=6$ feet, $l_2=4{\cdot}5$ feet, we see from the graph that Kl_2 is about 145°, corresponding to

$$Kl_1=\frac{6\times145}{4{\cdot}5}\text{ degrees}=203\tfrac{1}{3}^{\circ}.$$

From the tables we find $\phi(147°)=2{\cdot}5518$, $\phi(196°)=-2{\cdot}4853$, and a closer result, say $195\frac{3}{4}°$, may be arrived at by double interpolation. Taking $Kl_1=195\frac{3}{4}°$, the result in radians is 3·416.

For the next higher frequency, it appears from the graph that Kl_2 is a little more than 200°. From the tables $\phi(206^\circ) = -1{\cdot}0488$, and from a table of circular functions $\phi\ (274^\circ\ 40') = 1 - \cot 274^\circ\ 40'$, $= -1{\cdot}0816$. Interpolation gives $205\frac{3}{4}^\circ$, $274\frac{1}{4}^\circ$.

Fig. 1.



The third frequency is given approximately by $Kl_1 = 287\frac{1}{2}^\circ$, $Kl_2 = 383\frac{1}{3}^\circ$.

The actual frequencies are in the ratio of the squares of the corresponding angles, or 1 : 1·96 : 3·82.

Dunkerley gives, on page 297 of his paper already quoted, solutions of the problem of two bays for various integral values of the ratio $\frac{l_1}{l_2}$. The author's results in these cases differ considerably from those of Dunkerley, as will be seen from the following table :—

TABLE I.

Value of $\frac{l_1}{l_2}$.	Least value of Kl, in radians.	
	Dunkerley's results.	Author's results.
$\frac{1}{4}$	3·6056	3·695 3·6923
$\frac{1}{3}$	3·5101	3·643
$\frac{1}{2}$	3·3282	3·565

The results in the last column, except the result 3·6923, have been obtained by double interpolation from the annexed tables. The result 3·6923 has been obtained in a more accurate way by the use of the large tables of hyperbolic functions issued by the Smithsonian Institution, and indicates the amount of error to be expected in the use of the short tables. It appears that the method of approximation adopted by Dunkerley is accurate only when $\frac{l_2}{l_1}$ is very small.

Dunkerley's experimental results in these cases differed from his theoretical results by percentages $-2·4$, $+0·5$, and $+6·2$, or an average of 3·0 per cent. The new results would apparently give differences of $+2·4$, $+7·1$, and $+7·6$, or an average of 5·7 per cent.

When the bays tend to equality, it will be seen from the graph that the least solution tends to 180°.

§ 9. The equation for three bays is

$$(\phi_1+\phi_2)(\phi_2+\phi_3)=\psi_2^2,$$

and it will be easy to find the least value of K in any particular case.

Take the case of $l_1 : l_2 : l_3 = 1 : 2 : 3$.

If any solution exists for which Kl_2 is less than π, ϕ_1 will be positive, and ϕ_2 will be numerically greater than ψ_2 Hence, to satisfy the above equation, ϕ_3 must be negative and Kl_3 must lie between π and $\frac{5\pi}{4}$ approximately. A few trials show that $Kl_3=202°$ is a close solution.

When $l_1=l_3$, the equation becomes

$$\phi_1+\phi_2=\pm\psi_2$$

$$\text{or} \qquad \left.\begin{array}{l}\phi_1+\phi_2=\psi_2 \\ \phi(Kl_1)+\phi(\tfrac{1}{2}Kl_2)=0\end{array}\right\}. \qquad \text{and}$$

The first equation corresponds to symmetrical vibrations.

The second equation corresponds to skew symmetrical vibrations with a node at the centre of the middle bay, being the same as that for two bays of lengths l_1, $\frac{1}{2}l_2$.

These are special cases of a general theorem which is given in § 11.

For three bays, the symmetrical case gives the lower whirling speed.

§ 10. The equation for four bays simply supported is

$$(\phi_1+\phi_2)(\phi_2+\phi_3)(\phi_3+\phi_4)-\psi_2^2(\phi_3+\phi_4)-\psi_3^2(\phi_1+\phi_2)=0.$$

When l_1, l_2, l_3, l_4 are unequal, it appears that this equation could be solved by a somewhat tedious process of tabulation, in the form

$$\frac{\psi_2^2}{\phi_1+\phi_2}+\frac{\psi_3^2}{\phi_3+\phi_4}=\phi_2+\phi_3,$$

unless $(\phi_1+\phi_2)$, $(\phi_3+\phi_4)$ can vanish simultaneously.

If, however, the beam is symmetrical about its middle point, so that $l_1=l_4$ and $l_2=l_3$, the equation reduces to the two equations

$$\phi_1+\phi_2=0$$

$$\text{and} \qquad (\phi_1+\phi_2)\phi_2-\psi_2^2=0.$$

The first of these equations is that for two bays simply supported, and the second is that for two bays fixed in direction at one extremity. This is a particular case of a general theorem which will now be given.

§ 11. Consider the case of $2n$ bays symmetrical about the middle point, so that $l_1=l_{2n}$, $l_2=l_{2n-1}$, etc. Then by a property of reciprocal continuants,

$$\triangle_{2n}=2\triangle_n(\phi_n\triangle_n-\psi^2_{n-1}\triangle_{n-1})$$

and $$\phi_n\triangle_n-\psi^2_{n-1}\triangle_{n-1}=0$$

is the period equation for n bays fixed in direction at one extremity. Thus the periods for the $2n$ bays are the same as those for the system divided at its middle point when (1) the middle point is simply supported and (2) the beam is fixed in direction at its middle point.

In the first case the modes of vibration are skew symmetrical about the middle point, and in the second case they are symmetrical.

When the number of bays is odd and equal to $2n-1$, by a theorem in continuants

$$\triangle_{2n-1}=\triangle_n^2-\psi_n^2\triangle^2_{n-1},$$

and the period equations are

$$\triangle_n\pm\psi_n\triangle_{n-1}=0.$$

Taking the upper sign, a single determinant is obtained which differs from $\triangle_n$ only in replacing the constituent $(\phi_{n-1}+\phi_n)$ by $(\phi_{n-1}+\phi_n+\psi_n)$. Since $\phi_n+\psi_n=\phi(\frac{1}{2}\mathrm{K}l_n)$, this equation reduces to

$$\triangle(l_1, l_2, \ldots l_{n-1}, \tfrac{1}{2}l_n)=0,$$

and represents skew symmetrical vibrations, in which, of course, the middle point is a node.

Taking the lower sign, the equation gives the symmetrical vibrations. It has been shown in special cases that the lowest period occurs among the symmetrical vibrations.

§ 12. The case when all the bays are equal, the supports remaining simple, requires to be considered separately. Let there be n bays each of length l, so that nl is the length of the whole beam.

The results for one and two bays indicate that solutions may be expected in the neighbourhood of the poles. Now the limit of $\dfrac{\phi}{\psi}$ when $\mathrm{K}l=s\pi$ is $(-1)^s$, so that the period equation can evidently be satisfied at all the poles, and the least of these solutions is $\mathrm{K}=\dfrac{\pi}{l}$.

This is the well-known case in which the beam vibrates in the same form as a string would vibrate. It will now be shown that all the other solutions are greater than $\frac{\pi}{l}$.

The period equation reduces to

$$\psi^n \frac{\sin n\alpha}{\sin \alpha} = 0,$$

where $\cos\alpha = \frac{\phi}{\psi}$.

Hence other solutions are given by $\sin n\alpha = 0$, $\sin\alpha \neq 0$, leading to

$$\phi = \psi \cos\frac{s\pi}{n},$$

where s is an integer, not zero or a multiple of n.

When the argument is less than π, $\frac{\phi}{\psi}$ is always numerically greater than unity. Thus these solutions are all greater than $\frac{\pi}{l}$.

When the argument lies between π and the point near $\frac{5\pi}{4}$ where $\phi = 0$, $\frac{\phi}{\psi}$ is negative and decreases numerically from unity to zero. Hence the next gravest period is given by

$$\phi = -\psi \cos\frac{\pi}{n}.$$

Thus for three bays

$$\phi = -\psi\cos\frac{\pi}{3} = -\tfrac{1}{2}\psi,$$

or

$$3\phi(\mathrm{K}l) = \phi(\tfrac{1}{2}\mathrm{K}l).$$

This solution is about 204°.

These solutions tend to the limit π when n is large.

§ 13. The question what is the best distribution of supports so as to make the whirling speed of a shaft of given length as great as possible is of special interest to engineers. It will be shown that when all the supports are simple the whirling speed is a maximum when the supports are equidistant.

Consider the function

$$f(\mathrm{K}) = \triangle_n \sin \mathrm{K}l_1 \sin \mathrm{K}l_2 \ldots \sin \mathrm{K}l_n.$$

The determinant $\triangle_n$ is of the second degree in ϕ^r and ψ_r, but the terms of the second degree are of the form of $(\phi_r^2-\psi_r^2)$, and when $\mathrm{K}l_r\rightarrow\pi$ the terms of the second degree in cosec $\mathrm{K}l_r$ are of the form $\cot^2\mathrm{K}l_r-\operatorname{cosec}^2\mathrm{K}l_r$ or unity.

Hence, when $\mathrm{K}l_r\rightarrow\pi$, $\triangle_n$ is of the first degree in cosec $\mathrm{K}l_r$ and $f(\mathrm{K})$ remains finite. $f(\mathrm{K})$ is thus a continuous function without poles.

Let $\mathrm{K}l=\pi$ where $nl=l_1+l_2+\ldots+l_n$, and denote by $f(\mathrm{K})_\pi$ the value of $f(\mathrm{K})$ when $\mathrm{K}l_1$, $\mathrm{K}l_2,\ldots\mathrm{K}l_n$ all tend to π, and by $f(\mathrm{K})_0$ the value of $f(\mathrm{K})$ when k tends to zero. Then if $f(\mathrm{K})_\pi$ and $f(\mathrm{K})_0$ have opposite signs, the equation $f(\mathrm{K})=0$ will have a root between $\dfrac{\pi}{l}$ and zero.

Let $\qquad \mathrm{K}l_1=\pi+a_1,\ \mathrm{K}l_2=\pi+a_2,\ldots\mathrm{K}l_n=\pi+a_n,$

where $a_1, a_2\ldots a_n$ are small and $a_1+a_2+\ldots+a_n=0$.

Let $\cosh\pi=x$, cosech $\pi=y$.

Then, as far as the second order of small quantities,

$$f(\mathrm{K})_n=a_1,\ a_2\ldots a_n\left|\begin{array}{lllll} 2x+\frac{1}{a_1}+\frac{1}{a_2}, & y-\frac{1}{a_2}, & 0, & & \\ y-\frac{1}{a_2}, & 2x+\frac{1}{a_2}+\frac{1}{a_3}, & y-\frac{1}{a_3}, & & \\ 0, & y-\frac{1}{a_3}, & 2x+\frac{1}{a_3}+\frac{1}{a_4}, & & \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ & & & y-\frac{1}{a_{n-1}}, & 2x+\frac{1}{a_{n-1}}+\frac{1}{a_n}. \end{array}\right.$$

Denote by ${}_1\mathrm{D}_n$ the value of this determinant if x and y were zero, and by ${}_r\mathrm{D}_s$ the value of the similar determinants where only a_r, $a_{r+1},\ldots a_s$ are retained. Then by a property of determinants

$${}_r\mathrm{D}_s=\frac{a_r+a_{r+1}+\ldots+a_s}{a_ra_{r+1}\ldots a_s}.$$

The terms of the first order of small quantities in $f(\mathrm{K})_\pi$ are $a_1a_2\ldots a_n\,.\,{}_1\mathrm{D}_n$, and vanish since $a_1+a_2+\ldots+a_n=0$.

To find the terms of the second order denote summation from $r=p$ to $r=q$ $(p\not> q)$ by $\sum\limits_p^q$.

The determinant may be expanded by Laplace's method in a manner analogous to that given by Muir for continuants (Proc. Roy. Soc. Edinb. 1874, p. 230).

The terms of the second order are thus found to be

$$a_1a_2\ldots a_n\left\{2x\sum_1^{n-1}{}_1\mathrm{D}_r\cdot{}_{r+1}\mathrm{D}_n-2y\sum_2^{n-1}{}_1\mathrm{D}_{r-1}\cdot{}_{r+1}\mathrm{D}_n\cdot\frac{1}{a_r}\right\}$$

$$=2x\sum_1^{n-1}(a_1+\ldots+a_r)(a_{r+1}+\ldots+a_n)$$

$$-2y\sum_2^{n-1}(a_1+\ldots+a_{r-1})(a_{r+1}+\ldots+a_n)$$

$$=-2x\sum_1^{n-1}(a_1+\ldots+a_r)^2$$

$$+2y\sum_2^{n-1}\{(a_1+\ldots+a_r)^2-(a_1+\ldots+a_r)a_r\}$$

$$=-2(x-y)\sum_1^{n-1}(a_1+\ldots+a_r)^2$$

$$-2y\left\{\sum_1^{n-1}a_r^2+\sum_1^{n-1}a_ra_s\ (\text{all unequal suffixes})\right\}$$

$$=-2(\coth\pi-\operatorname{cosech}\pi)\sum_1^{n-1}(a_1+\ldots+a_r)^2$$

$$-\operatorname{cosech}\pi(a_1^2+a_2^2+\ldots+a_n^2),$$

and this quantity is always negative.

Now let k tend to zero, then

$$\phi(\mathrm{K}l_r)=\tfrac{2}{3}\mathrm{K}l_r \text{ and } \psi(\mathrm{K}l_r)=-\tfrac{1}{3}\mathrm{K}l_r \text{ nearly.}$$

The sign of $f(\mathrm{K})_0$ will depend on the sign of the determinant

$${}_1\mathrm{D}_n'\equiv\left|\begin{array}{ccc:cc} l_1+l_2, & -\frac{1}{2}l_2, & 0, & & \\ -\frac{1}{2}l_2, & l_2+l_3, & -\frac{1}{2}l_3, & & \\ \hdashline & & & -\frac{1}{2}l_{n-1}, & l_{n-1}+l_n. \end{array}\right|$$

With a notation similar to that adopted for D, the terms of ${}_1\mathrm{D}_n'$ containing l_r^2 or l_r are

$$\tfrac{3}{4}{}_1\mathrm{D}_{r-1}'\cdot{}_{r+1}\mathrm{D}_n'l_r^2+({}_1\mathrm{D}_{r-1}\cdot{}_r\mathrm{D}_n'+{}_1\mathrm{D}_r'\cdot{}_{r+1}\mathrm{D}_n')l_r.$$

Hence, if all the determinants in this expression are positive, ${}_1\mathrm{D}_n'$ increases when l_r increases. When all the l's are zero ${}_1\mathrm{D}_n'$ is zero. Hence ${}_1\mathrm{D}_n'$ is positive if ${}_r\mathrm{D}_s'$ is positive for all values of r and s for which ${}_r\mathrm{D}_s'$ is of lower order than ${}_1\mathrm{D}_n'$.

Now ${}_1\mathrm{D}_2'$, ${}_1\mathrm{D}_3'$, ${}_2\mathrm{D}_3'$ are positive.

Hence by induction ${}_1\mathrm{D}_n'$ is positive, and $f(\mathrm{K})_0$ is positive.

Tables of the functions ϕ and ψ.

(The values between 5° and 44°, which are omitted, may be obtained by simple addition, *e. g.* $\phi(12^\circ)=\phi(10^\circ)+\phi(2^\circ)$. Owing to the manner in which the tables have been constructed, errors of one or two units may occur in the last decimal place.)

Deg.	ϕ.	ψ.	Deg.	ϕ.	ψ.
1	+0·0116	−0·0058	88	+1·0623	−0·5491
2	0·0233	0·0116	89	1·0762	0·5572
3	0·0349	0·0174	90	1·0903	0·5654
4	0·0465	0·0232	91	1·1046	0·5739
5	0·0582	0·0290	92	1·1188	0·5822
10	0·1164	0·0582	93	1·1333	0·5907
15	0·1745	0·0873	94	1·1480	0·5995
20	0·2327	0·1164	95	1·1628	0·6084
25	0·2909	0·1454	96	1·1778	0·6175
30	0·3493	0·1747	97	1·1928	0·6267
35	0·4077	0·2041	98	1·2081	0·6361
40	0·4662	0·2335	99	1·2236	0·6456
42	0·4896	0·2453	100	1·2392	0·6553
44	0·5132	0·2572	101	1·2551	0·6653
45	0·5249	0·2631	102	1·2711	0·6754
46	0·5366	0·2690	103	1·2873	0·6856
47	0·5484	0·2750	104	1·3038	0·6961
48	0·5603	0·2810	105	1·3205	0·7069
49	0·5720	0·2870	106	1·3374	0·7179
50	0·5839	0·2930	107	1·3546	0·7291
51	0·5958	0·2991	108	1·3721	0·7406
52	0·6077	0·3052	109	1·3898	0·7524
53	0·6196	0·3112	110	1·4079	0·7645
54	0·6315	0·3172	111	1·4263	0·7769
55	0·6434	0·3233	112	1·4449	0·7895
56	0·6554	0·3294	113	1·4640	0·8026
57	0·6674	0·3356	114	1·4833	0·8159
58	0·6794	0·3418	115	1·5031	0·8296
59	0·6914	0·3479	116	1·5232	0·8438
60	0·7035	0·3542	117	1·5438	0·8584
61	0·7156	0·3605	118	1·5647	0·8734
62	0·7278	0·3668	119	1·5862	0·8888
63	0·7400	0·3732	120	1·6082	0·9047
64	0·7522	0·3796	121	1·6306	0·9211
65	0·7645	0·3860	122	1·6536	0·9380
66	0·7768	0·3924	123	1·6771	0·9554
67	0·7891	0·3989	124	1·7012	0·9734
68	0·8014	0·4054	125	1·7260	0·9920
69	0·8138	0·4119	126	1·7514	1·0133
70	0·8262	0·4185	127	1·7776	1·0315
71	0·8387	0·4252	128	1·8045	1·0523
72	0·8513	0·4319	129	1·8322	1·0739
73	0·8640	0·4387	130	1·8607	1·0962
74	0·8767	0·4455	131	1·8901	1·1195
75	0·8895	0·4524	132	1·9205	1·1437
76	0·9023	0·4594	133	1·9519	1·1690
77	0·9151	0·4665	134	1·9844	1·1953
78	0·9281	0·4736	135	2·0181	1·2229
79	0·9411	0·4808	136	2·0530	1·2516
80	0·9542	0·4880	137	2·0893	1·2817
81	0·9674	0·4953	138	2·1269	1·3131
82	0·9807	0·5027	139	2·1660	1·3460
83	0·9939	0·5102	140	2·2069	1·3807
84	1·0074	0·5178	141	2·2496	1·4172
85	1·0210	0·5255	142	2·2941	1·4554
86	1·0347	0·5333	143	2·3407	1·4957
87	+1·0484	−0·5412	144	+2·3896	−1·5383

Deg.	φ.	ψ.	Deg.	φ.	ψ.
145	+2·4409	−1·5833	208	−0·8793	+2·1831
146	2·4949	1·6309	209	0·8026	2·1147
147	2·5518	1·6815	210	0·7307	2·0512
148	2·6118	1·7351	211	0·6630	2·0919
149	2·6754	1·7923	212	0·5991	1·9365
150	2·7427	1·8532	213	0·5387	1·8847
151	2·8143	1·9185	214	0·4815	1·8361
152	2·8906	1·9884	215	0·4270	1·7903
153	2·9721	2·0535	216	0·3754	1·7475
154	3·0596	2·1445	217	0·3260	1·7069
155	3·1535	2·2319	218	0·2790	1·6688
156	3·2547	2·3266	219	0·2340	1·6329
157	3·3642	2·4296	220	0·1908	1·5987
158	3·4832	2·5421	221	0·1495	1·5667
159	3·6129	2·6653	222	0·1097	1·5360
160	3·7550	2·8008	223	0·0716	1·5072
161	3·9114	2·9506	224	−0·0347	1·4796
162	4·0847	3·1173	225	+0·0008	1·4536
163	4·2776	3·3036	226	0·0342	1·4298
164	4·4939	3·5132	227	0·0682	1·4054
165	4·7383	3·7510	228	0·1002	1·3831
166	5·0169	4·0230	229	0·1314	1·3618
167	5·3374	4·3368	230	0·1615	1·3416
168	5·7103	4·7029	231	0·1908	1·3223
169	6·1500	5·1369	232	0·2193	1·3039
170	6·6766	5·6556	233	0·2470	1·2865
171	7·3188	6·2910	234	0·2740	1·2698
172	8·1204	7·0857	235	0·3003	1·2539
173	9·1491	8·1079	236	0·3260	1·2387
174	10·5190	9·4706	237	0·3510	1·2244
175	12·434	11·379	238	0·3755	1·2105
176	15·305	14·243	239	0·3995	1·1977
177	20·085	19·016	240	0·4231	1·1851
178	29·640	28·564	241	0·4461	1·1733
179	58·294	57·211	242	0·4687	1·1619
180	∞	∞	243	0·4909	1·1510
181	−56·286	+57·383	244	0·5127	1·1409
182	27·632	28·737	245	0·5340	1·1313
183	18·078	19·190	246	0·5551	1·1220
184	13·298	14·417	247	0·5758	1·1132
185	10·427	11·553	248	0·5963	1·1049
186	8·5114	9·6447	249	0·6164	1·0968
187	7·1414	8·2820	250	0·6363	1·0897
188	6·1136	7·2616	251	0·6560	1·0826
189	5·3111	6·4814	252	0·6754	1·0760
190	4·6687	5·8315	253	0·6946	1·0698
191	4·1420	5·3123	254	0·7136	1·0640
192	3·7022	4·8800	255	0·7322	1·0588
193	3·3292	4·5145	256	0·7509	1·0537
194	3·0086	4·2014	257	0·7693	1·0490
195	2·7298	3·9301	258	0·7876	1·0447
196	2·4853	3·6934	259	0·8058	1·0405
197	2·2688	3·4841	260	0·8239	1·0368
198	2·0758	3·2294	261	0·8418	1·0335
199	1·9023	3·1337	262	0·8596	1·0305
200	1·7457	2·9849	263	0·8773	1·0279
201	1·6033	2·8504	264	0·8950	1·0255
202	1·4733	2·7284	265	0·9126	1·0234
203	1·3541	2·6172	266	0·9302	1·0217
204	1·2444	2·5155	267	0·9477	1·0204
205	1·1429	2·4221	268	0·9652	1·0192
206	1·0488	2·3362	269	0·9826	1·0185
207	−0·9611	+2·2566	270	+1·0001	+1·0180

Thus $f(\mathrm{K})$ will have a root less than $\frac{\pi}{l}$ unless $a_1, a_2 \ldots a_n$ are all zero.

Hence the whirling speed is a maximum when all the bays are equal.

In the case of two bays the restriction that the a's are small may be removed, so that it can rigorously be proved that the greatest possible whirling speed is reached when the bays are equal, but this theorem does not appear to have been proved for any greater number of bays.

The introduction of the function $f(\mathrm{K})$ is due to Mr. Berry, and the author has extended its use to any number of bays.

V. *Electronic Energy and Relativity.*

By Prof. FREDERICK SLATE [*].

THE *modus operandi* characteristic of relativity proves to be working in disguised agreement with what Newton's equations also teach, when these are properly written to include variable inertia [†]. The older problem of motion resisted proportionally to the square of speed has furnished the principal key in this reconciling dissection, after its equation of motion was put into a form inspired by a prospect of making it a model for electrons. That problem is surprisingly interwoven with our attempted analysis. It allows aspects to be rated normal under its general type that had once seemed exceptional. Especially is this true of vital relations for energy, where correspondences show that are of first importance to recognize.

The equation of rectilinear motion, and the recasting of it with the aid of terminal velocity (v_1) referred to are

$$(a)\ \mathrm{P}-\mathrm{R}=m_1\frac{dv}{dt}\,;\quad (b)\ \mathrm{P}=\left(\frac{m_1 v_1^2}{v_1^2-v^2}\right)\frac{dv}{dt}\,.\ [\mathrm{R}\equiv m_1 k v^2]\ \ddagger. \quad . \quad . \quad . \quad (1)$$

In (b) that factor occurs directly, whose different powers run through all the calculations of relativity. Much can be made to grow out of its striking property that links an arithmetical and an harmonic mean :

$$\frac{1}{2}\left[\frac{v_1}{v_1+v}+\frac{v_1}{v_1-v}\right]=\left(\frac{v_1}{v_1+v}\right)\left(\frac{v_1}{v_1-v}\right)=\frac{v_1^2}{v_1^2-v^2}. \quad . \quad (2)$$

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[*] Communicated by the Author.

[†] Slate, Phil. Mag. April 1920, p. 433; July 1920, p. 31. This is a continuation of those papers, carrying on their plan and their notation consistently. They are cited as (I.), (II.) To avoid wasteful repetition, free reference is made to results already recorded there.

[‡] See (II.), eq. (8).

The first member is on the track of the root-idea in Einstein's substitutions; while the equality leads to the connexions of sums and products, or of quantities and their logarithms, which control the mathematics at many points. Therefore it is not astonishing to find relativity at one with the present argument in building the force (P) and the equation of motion (1) into the foundations. Einstein's velocities actually secure for (P) invariance in form and magnitude throughout the group of frames (U), so long as their derivatives based on a common time-variable are expressed*. Only at the transition to "local time" are the reduction factors combined with this invariant nucleus, which are disentangled and discussed in the previous papers. The task for physics is to trace the simpler elements here plainly indicated through the shifting patterns of a rather kaleidoscopic algebra.

Retaining at first the general symbol (v_1) for a terminal velocity, let us notice the scope of equation (1), some peculiar consequences of it, and its limitations. It implies a total (gross) flux of energy due to the propelling force (P), this being partly diverted from the kinetic energy of (m_1) by passive resistances summed in (R). Complete diversion (and conversion) can be entertained as a limit. Secondly, remark how the mass (m_1), "weight-mass" and constant in (a), is replaced as a factor of the same acceleration by a variable effective inertia in (b). But the physics of the original form is not modified by this merely mathematical step. In the third place, since the transformation involves a terminal velocity essentially, it is intrinsically invalid outside the range $(0, v_1)$ thus marked off, whatever magnitude (v_1) may have. Imaginary combinations which occur beyond those logical limits argue nothing against greater speeds attainable under other physical conditions. Specifically, this holds for light-speed (c) if that happens to enter as a terminal velocity.

For an interval taken conveniently between rest and any velocity (v), the equation of work in the standard frame (F) leads through equation $(1(b))$ to the value

$$\mathrm{W} \equiv \int_0^v \left(\frac{m_1 v_1{}^2}{v_1{}^2 - v^2}\frac{dv}{dt}\right) v\,dt = \frac{m_1 v_1{}^2}{2}\log\left(\frac{v_1{}^2}{v_1{}^2 - v^2}\right)$$

$$= \frac{m_1 v_1{}^2}{2}\log\left(\frac{\mathrm{P}}{m_1 \dfrac{dv}{dt}}\right) = \frac{m_1 v_1{}^2}{2}\log\left(\frac{z\mathrm{P}}{z m_1 \dfrac{dv}{dt}}\right). \quad . \quad (3)$$

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* See (II.), note to eq. (25).

Any dimensions and magnitude are allowable for (z). The above expression covers any constant inertia (m_1); and its eliminations have brought it to include any propelling force (P), provided that the conditions preserve the same terminal velocity *. Writing next

$$\int_0^v \mathrm{P}v\,dt = \int_0^v \left(m_1\frac{dv}{dt}\right)v\,dt + \int_0^v \mathrm{R}v\,dt$$

$$= \frac{m_1v^2}{2} + \left[-\frac{m_1v^2}{2} + \frac{m_1{v_1}^2}{2}\log\left(\frac{{v_1}^2}{{v_1}^2 - v^2}\right)\right], \quad . \quad (3\,a)$$

we see the same cancellation of explicit kinetic energy operating as in the work equation for the "deformable electron" of Lorentz †. Therefore that circumstance does not *by itself require* electronic inertia to be wholly electromagnetic. The differences

$$\mathrm{W} - \frac{m_1v^2}{2} \equiv \mathrm{D} = \frac{m_1}{2}\left[{v_1}^2\log\left(\frac{{v_1}^2}{{v_1}^2 - v^2}\right) - v^2\right];$$

$$\frac{d\mathrm{W}}{dt} - m_1v\frac{dv}{dt} = \frac{d\mathrm{D}}{dt} = v\mathrm{R}, \quad . \quad . \quad . \quad . \quad (4)$$

record the diverted (converted) energy and the activity of (R).

As a mathematical device, introduce now a variable inertia (m'). Let it depend upon (v, m_1), have for $(v=0)$ the magnitude (m_1), and a rule of change making

$$\frac{dm'}{dv} = \frac{m'v}{{v_1}^2 - v^2}; \quad \log\left(\frac{m'}{m_1}\right) = \frac{1}{2}\log\left(\frac{{v_1}^2}{{v_1}^2 - v^2}\right);$$

$$m' = \frac{m_1v_1}{\sqrt{{v_1}^2 - v^2}}. \quad . \quad . \quad . \quad . \quad (5)$$

Then work and activity restated by means of (m') become

$$\mathrm{W} = m_1{v_1}^2\log\left(\frac{m'}{m_1}\right); \quad v\mathrm{P} = \frac{\sqrt{{v_1}^2 - v^2}}{v_1}\left({v_1}^2\frac{dm'}{dt}\right);$$

$$v\mathrm{R} = \frac{\sqrt{{v_1}^2 - v^2}}{v_1}\left(v^2\frac{dm'}{dt}\right). \quad . \quad . \quad (6)$$

Consequently, for arbitrary (n),

$$n\left[\frac{v}{v_1}\left(\mathrm{P}\frac{v_1}{\sqrt{{v_1}^2 - v^2}}\right) - v_1\frac{dm'}{dt}\right] = 0;$$

$$n\left[\mathrm{R}\frac{v_1}{\sqrt{{v_1}^2 - v^2}} - v\frac{dm'}{dt}\right] = 0. \quad . \quad . \quad (7)$$

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* See (II.), comment on eq. (9).

† See (I.), p. 435.

The identifications $v_1 \equiv c$, $m_1 \equiv m_0$, bring the first of equations (7) into formal parallel with results adopted for the Lorentz electron and enlarged for legitimate frames in relativity, with $(m \equiv \gamma(v) m_0)$ replacing (m'). One basic proposition of our treatment is thereby corroborated; but also the question is opened, in how far (m) is a *physical* quantity. Conceding the physics of equation (1 (b)) to be artificial, an admixture of fiction in the subsequent dynamics is unavoidable. Nor is it dangerous, unless pseudo-values cause misconception of the physical process *.

Return to equation (3), and observe the possible general reading of this work as a kinetic energy, measurable by a changing multiple of (m_1) at a standard speed (v_1). The multiple is invariant under all choices of (z); yet the influence of (z) elsewhere may leave its determination an important detail for each separate application resting on equations (1, 3, 4). The vogue of the contractile electron has led to particularizing

$$z \equiv \frac{m_0}{m_1} \frac{c}{\sqrt{c^2 - v^2}} \equiv \gamma(v) \frac{m_0}{m_1}; \quad z\mathrm{P} \equiv m_0 \gamma^3(v) \frac{dv}{dt} \equiv \mathrm{T}_o$$

$$z m_1 \frac{dv}{dt} \equiv m \frac{dv}{dt} \quad . \quad . \quad . \quad . \qquad (8)$$

But the unique advantage of that choice remains a matter for deeper inquiry; at least one alternative will be considered presently, as well as the foundation on which decision must stand.

The permanent factor (v_1) is a resource in making certain forms valid indifferently, whether inertia be constant or variable, after standardizing all momentum and kinetic energy at the terminal velocity by appropriate reductions. Define two "reduced inertias" (μ, μ') corresponding to any (m_2, v_2) by

$$\mu v_1 \equiv m_2 v_2; \quad \mu' v_1^2 \equiv m_2 v_2^2; \quad \text{showing } \mu' = \frac{v_2}{v_1} \mu. \quad . \qquad (9)$$

These reductions are similar to those of force-moment and moment of inertia in relation to axial distance. Further, it is evident how they expand the idea of equations (5, 6), and move toward making broadly representative what was invented more specially for the Lorentz electron.

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* See (II.), p. 36. The same thought recurs at equations (22, 27) below. The reminder is in place, perhaps, that no pretension to close any such issue is here intended. On the contrary, the aim everywhere is to open some escape, where earlier conclusions might be fallaciously confident.

Then tangential force and power are expressible generally as

$$T \equiv \frac{d}{dt}(m_2 v_2) = v_1 \frac{d\mu}{dt}; \quad 2\frac{dE}{dt} \equiv \frac{d}{dt}(m_2 v_2{}^2) = v_1{}^2 \frac{d\mu'}{dt} \quad . \quad . \quad (10)$$

Also activity enters into the forms, since $\mu v_1 \equiv m_2 v_2$,

$$A = v_2 T = v_1 \left(v_1 \frac{d\mu'}{dt} - \mu \frac{dv_2}{dt} \right); \quad v_1{}^2 \frac{d\mu'}{dt} - 2\mu v_1 \frac{dv_2}{dt}$$

$$= v_2 \left(T - m_2 \frac{dv_2}{dt} \right) = v_2{}^2 \frac{dm_2}{dt} \quad . \quad . \quad . \quad (11)$$

The last member, vanishing for constant inertia, affords some general measure for that rate of conversion—mechanical energy into other modes. The first of equations (11) is the plain equivalent of

$$A = 2\frac{dE}{dt} - \frac{\partial E}{\partial v_2}\frac{dv_2}{dt}; \quad \frac{dE}{dt} = \frac{1}{2}\left(A + \frac{\partial E}{\partial v_2}\frac{dv_2}{dt} \right), \quad . \quad (12)$$

the partial derivative being taken with (m_2) stationary at its epoch-value. When (m_2) is constant, this partial becomes total ; the principle of *vis viva* appears. Under a fictitious supposition of such constancy, the last member may be viewed as containing contribution from some (pseudo) potential to the work of other external force. For any epoch when $(\mu = 0)$ there are particular coincidences which, like equations (23) below, have bearing upon the use of rest-frames.

An additional suggestion from the algebra associated with terminal velocity is to utilize a kinetic estimate of future action as potential energy, according to the conditions, for intervals beginning at rest and at (v) :

$$\int_0^{\infty} \left(m_1 \frac{dv}{dt} \right) v\, dt = \frac{m_1 v_1{}^2}{2} \equiv U_0 \; ; \quad \int_v^{v_1} m_1 v\, dv = \frac{m_1}{2}(v_1{}^2 - v^2) \equiv U_1 . \quad . \quad . \quad . \quad (13)$$

Then

$$\frac{m_1 v^2}{2} + U_1 = U_0 \; ; \quad -\frac{\partial U_1}{\partial s} = m_1 \frac{dv}{dt} \equiv T_1 \; ; \quad vT_1 = -\frac{\partial U_1}{\partial s}\frac{ds}{dt}; \quad . \quad . \quad . \quad (14)$$

which prepares the way for a possible consolidation with electrostatic potential energy.

The specification that (P) should account completely for the influx of energy makes equations like

$$\frac{d\mathrm{W}}{ds}=\mathrm{P}\,;\quad \mathrm{W}=\int \mathrm{P}\,ds\,;\quad .\quad .\quad .\quad .\quad .\qquad (15)$$

one a total derivative and the other integrable. But after having recognized that forces (P) can present themselves through partial derivatives of momentum, a discrimination about time-integral is seen to be needed there. Before clear physical indications have fitted together acceptable values of momentum and of energy, tentative factorings of (P) place it in either class as a derivative of momentum. Note how the complete activity (vP) is compatible with these groupings and others :

$$\mathrm{P}=m_1\left(\frac{v_1{}^2}{v_1{}^2-v^2}\,\frac{dv}{dt}\right)=\left(\frac{m_1v_1{}^2}{v_1{}^2-v^2}\right)\frac{dv}{dt}$$

$$=\left(\frac{m_1v_1}{\sqrt{v_1{}^2-v^2}}\right)\left(\frac{v_1}{\sqrt{v_1{}^2-v^2}}\,\frac{dv}{dt}\right)=\left(\frac{m_1v_1}{v_1-v}\right)\left(\frac{v_1}{v_1+v}\,\frac{dv}{dt}\right)$$

$$=v_1\frac{d}{dt}\left[\frac{m_1}{2}\log\left(\frac{v_1+v}{v_1-v}\right)\right].\quad .\quad .\quad .\quad .\quad .\quad .\quad .\quad .\qquad (16)$$

The last member is guided by equations (10) ; the third, fourth, and fifth members make (P) a partial derivative. The supplement to (P) changes with the possible pair (m_2, v_2) segregated ; that is, according to the more plausible selection of momentum (or quasi-momentum). This flexibility is added to that afforded by the zero-parenthesis and arbitrary factor in equations (7) and elsewhere. In effect, a liberal leeway in quantitative adjustment to experimental data is permitted, while equation (1) is retained as prototype *. One turning-point is found to be the discovery of a "scale-factor" (z) that meets some condition connectable with the energetics of equations (11) ; when (v_2, m_2, T) are furnished otherwise ; for example, making

$$v_2\left(\mathrm{T}-m_2\frac{dv_2}{dt}\right)=vz\left(\mathrm{P}-m_1\frac{dv}{dt}\right).\quad .\quad .\quad .\quad .\qquad (17)$$

In determining the physical elements for a cycle of problems to which the sequence of equations (1) to (16) can be adapted, the unifying feature being the terminal velocity (v_1), some margin of independent choice at equations (1, 16) and at equations (3, 4, 7) can be foreseen.

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* Compare previous remark on this matter, (II.), p. 35.

Rigid dynamics has used us to the couple, which affects (rotational) energy but not momentum. Moment of momentum there bridges the break; its factor (ω) appears in kinetic energy also. Both (ω, v_1) can be employed to standardize (the former at unit axial distance). That thought falls into line with equations (9, 10, 11, 12), and with several obtrusive analogies to symptoms of a hidden spin about the axis of advance, which the electron's dynamics reveals.

The foregoing statements have been kept on general ground (for any (v_1)), with a definite purpose of insisting upon the probably unrealized range within which that is feasible; but we take up finally the electron for which ($v_1 \equiv c$). The dominant criterion on the physical side is yielded by a predetermined (or available) electromagnetic energy (M). The Einstein velocities have been brought under similar control by considerations based on activity *. The dynamical schemes will look towards devising a momentum (Q′) reducible to consistency with (M) and its time-rate, when associated with the actual working-speed (v). The remark following equation (17) enlarges somewhat the limits of that consistency; but the points of agreement to contrive and their companion conventions may be put thus:

$$\left.\begin{aligned} &\mathrm{A}' \equiv \frac{d}{dt}(\mathrm{M}) \equiv v\mathrm{T}'\,;\ \ \mathrm{Q}' \equiv m_2 v\,;\ \ \mathrm{T}' \equiv \frac{d}{dt}(\mathrm{Q}')\,;\ \ 2\mathrm{E}' \equiv v\mathrm{Q}'\,; \\ &\qquad v\mathrm{T}' = \left[\frac{d}{dt}(\mathrm{E}') + \frac{1}{2}v^2\frac{dm_2}{dt}\right] = m_2 v\frac{dv}{dt} + v^2\frac{dm_2}{dt}. \end{aligned}\right\} . \quad (18)$$

Adding an assumption about effective inertia (m_2) to our knowledge of (v) fixes the other (mechanical) quantities; hence any attempted adjustment of this sort stands or falls mainly on that as a cardinal choice. The last equation enforces the need, where (m_2) is variable, of upholding the distinction between (A′) and power ($d\mathrm{E}'/dt$) which equations (12) formulate. Turning first to Abraham's electron, we shall illustrate how the plan outlined above can be adapted to his original development. That assigns a leading place to "electromagnetic mass," as we know; also it proves to conform to the present train of ideas, though as a whole they are not made explicit in it. Particularly for that reason, the comparison is profitable.

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* See (I.), p. 437; (II.), eq. (18), etc.

Begin with Abraham's value *

$$\left.\begin{aligned} \mathrm{M} &= \tfrac{3}{4} m_0 c^2 \left[\frac{c}{v} \log\left(\frac{c+v}{c-v}\right) - 1\right]; \text{ and therefore} \\ \frac{1}{v}\frac{d}{dt}(\mathrm{M}) &\equiv \mathrm{T}' = \tfrac{3}{4} m_0 c^2 \left[\frac{1}{v^2}\frac{2c^2}{c^2-v^2} - \frac{c}{v^3}\log\left(\frac{c+v}{c-v}\right)\right]\frac{dv}{dt}. \end{aligned}\right\} \quad (19)$$

Establish a junction with equation (16) through its first or its last form; either shows (P) as the total derivative of an auxiliary momentum

$$\mathrm{Q}'' = \int_0^v \mathrm{P}dt \equiv m''v = m_1\left[\frac{c}{2}\log\left(\frac{c+v}{c-v}\right)\right] = c\left[\frac{m_1}{2}\log\left(\frac{c+v}{c-v}\right)\right]. \quad \ldots \quad (20)$$

The last member being already standardized at (c), the plan of equations (9, 10, 11) gives

$$\left.\begin{aligned} \mu &= \frac{m_1}{2}\log\left(\frac{c+v}{c-v}\right); \quad \mu' \equiv \frac{m_1 v}{2c}\log\left(\frac{c+v}{c-v}\right); \\ 2\mathrm{E} &\equiv \frac{m_1 cv}{2}\log\left(\frac{c+v}{c-v}\right); \quad 2\left(\frac{d\mathrm{E}}{dt} - \mu c\frac{dv}{dt}\right) \\ &= \left(\frac{m_1 c^2 v}{c^2-v^2}\right)\frac{dv}{dt} - \frac{m_1 c}{2}\log\left(\frac{c+v}{c-v}\right)\frac{dv}{dt} = v^2\frac{dm''}{dt}. \end{aligned}\right\} \quad \ldots \quad (21)$$

Mark how the last member returns to one leading idea of equations (4, 6, 7).

The scale-factor (z'') that equalizes this rate of work-conversion with what equation (19) demands is then conditioned by

$$z''\left(v^2\frac{dm''}{dt}\right) = v\mathrm{T}'; \quad z'' = \left(\frac{3m_0}{2m_1}\right)\frac{c^2}{v^2}; \quad v\mathrm{T}' = c^2\frac{dm''}{dt} \; [2m_1 = 3m_0]. \quad \ldots \quad (22)$$

In the last equation, due choice of the arbitrary ratio (m_0/m_1) has reduced one coefficient to unity. The corresponding activities in equations (6, 7, 22) exhibit, then, as sole essential distinction the effective inertia whose time-rate occurs in each one—a remarkably significant symmetry. Why should it include the Lorentz electron without particularized values beyond ($v_1 \equiv c$; $m_1 \equiv m_0$)? The answer

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* Translated into our notation, but unaltered otherwise. The quotation is from his 'Theorie der Elektrizität,' vol. ii. (1908). The chief equations referred to are (113, 113 *c*, 113 *d*, 117, 117 *a*, 117 *b*). The numbering of equations is fortunately unchanged through a series of editions. Rest-mass (m_0) coincides with the usage of Lorentz-Minkowski, allowing for "rational units," etc.

can be traced from the fact that this electron appropriates the advantage of the general relation in equations (5), in combination with equations (10).

Since (μ) vanishes with (v), (m'') depends upon an excess above initial magnitude; and because the final result involves time-rate alone of (m''), any special value (m_0'') can be used in expressions like

$$m''=m_0''+m_2''\ ;\ \text{giving correctly}\ \frac{dm''}{dt}=\frac{dm_2''}{dt}. \quad (23)$$

The algebraic link with Abraham's treatment may now be closed by simple verifications. First, that the "longitudinal mass" (m_s) and the "transverse mass" (m_r) satisfy the relation

$$m_s\frac{dv}{dt}=\left(m_r+v\frac{d}{dv}(m_r)\right)\frac{dv}{dt}=\frac{d}{dt}(m_rv)\equiv\mathrm{T}', \quad . \quad (24)$$

and thus stand squarely upon the completed second law of Newton. Secondly, that the separate time-rates of magnetic energy and of electrostatic energy, as Abraham allots them, match exactly the *bracketed* segregation of activity in equations (18), when ($m_2\equiv m_r$). And finally, that Abraham's "Lagrange function" ((L) of his equation (113)) belongs to the plan of equations (12), contributing, in fact, the partial derivative there.

A second auxiliary momentum, also derivable from equation (16), shunts the calculations towards the Lorentz electron. Using (m') consistently with equations (5), define

$$\mathrm{Q}_1''\equiv[m_1\gamma(v)][v\gamma(v)]\equiv m'v'=[m_1\gamma^2(v)]v. \quad . \quad (25)$$

In respect to (Q_1''), (P) is a partial derivative; from either factoring it proves that

$$\begin{aligned}\frac{d}{dt}(\mathrm{Q}_1'')=\mathrm{T}_1''=\frac{c^2+v^2}{c^2-v^2}\mathrm{P}&=\gamma^2(v)\mathrm{P}\left(1+\frac{v^2}{c^2}\right)\\&=\gamma^2(v)\left(2\mathrm{P}-m_1\frac{dv}{dt}\right). \quad . \quad (26)\end{aligned}$$

By simple reduction this yields the forms

$$\gamma(v)\frac{m_0}{m_1}\left(\mathrm{P}+\frac{m_1}{2}\frac{dv}{dt}\right)=m\frac{dv}{dt}+\frac{m}{2}\frac{c^2+v^2}{c^2-v^2}\frac{dv}{dt}=\mathrm{T}+\frac{m}{2}\frac{dv}{dt}. \quad (27)$$

Since it is true that

$$\begin{aligned}v\frac{dm}{dt}=\frac{mv^2}{c^2-v^2}\frac{dv}{dt}&=\frac{m}{2}\cdot\frac{c^2+v^2}{c^2-v^2}\frac{dv}{dt}+\left(-\frac{m}{2}\frac{dv}{dt}\right)\\&\equiv v\left(\frac{dm}{dt}\right)_1+v\left(\frac{dm}{dt}\right)_2, \quad . \quad (28)\end{aligned}$$

equations (8, 26, 27) are in one aspect mere algebraic variations. Nevertheless, it should arrest attention that the last term in equation (27) is now transferred among positive contributors to energy-flux, instead of figuring as a *negative quota* of resistance. The type prescribed for the first member of equation (1 (*b*)) is better maintained; separate record is now entered, however, for each branch of a composite process *.

Equating the alternative mated pairs of terms in the total derivative of equation (25) provides another direct connexion with equations (6, 7), and the possible rearrangements of the four items present fair substitutes for some of relativity's energetics. Omitting those details as nearly obvious, follow up the more vital comparison which parallels that of equations (19, 22). On usual assumptions, electromagnetic energy, its time-derivative, and the assignment of activity to magnetic energy and to electric energy are † :

$$\left.\begin{aligned} &\mathrm{M}_1=\tfrac{3}{4}mc^2+\tfrac{1}{4}mv^2\,;\ \mathrm{T}_1{}'=\frac{1}{v}\frac{d}{dt}(\mathrm{M}_1)=\mathrm{T}_o+\frac{m}{4}\frac{dv}{dt}\,;\\ &\frac{d}{dt}(\mathrm{M}_1)=v\left[\frac{3m}{4}\frac{dv}{dt}+\frac{m}{4}\left(\frac{c^2+v^2}{c^2-v^2}\right)\frac{dv}{dt}\right]+v\left[\frac{m}{4}\left(\frac{c^2+v^2}{c^2-v^2}\right)\frac{dv}{dt}\right]\\ &=v\left[m\frac{dv}{dt}+\frac{m}{2}\left(\frac{c^2+v^2}{c^2-v^2}\right)\frac{dv}{dt}\right]-\frac{mv}{4}\frac{dv}{dt} \end{aligned}\right\} \quad \ldots\ldots\ (29)$$

Examine more narrowly the second member and the third of the last equation. The former is co-ordinated instructively with the subdivisions in the bracketed member of equations (18) which Abraham's electron fits. One stabilized association of magnetic activity and power is common ; and so is the type of the (electric) remainder, if $((dm/dt)_1)$ displaces (dm/dt). The last member of equations (29) adheres to the primary conception of equations (1) to (7) ; the final subtractive term harmonizes with their supposition of *partial* conversion into other energy-forms of work done by (P). The kinetic energy of any "weight-mass" (m_1) must be rated mechanical. Perhaps the current doctrine that the

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* The last two members of equation (28) and the verifications under equation (24) give the foundation for a previous claim (I.), first lines of p. 436.

† See, for example, Richardson's 'Electron Theory of Matter' (1914) pp. 217 *sqq*.

inertia of the Lorentz electron is wholly electromagnetic may be insecure—at least if equation (1) be made fundamental.

In certain respects, the perfect conversion that Abraham's equation of motion predicates (and that our test does not disallow) may be said to introduce the limit-case ($m_1=0$). The consequent perpetual balance of the forces replacing (P, R), not uniquely at the terminal velocity but through all stages of approach to it, was at its date novel *. Nevertheless the main issue is not decided there, because we deal with excess only beyond an initial value. A nucleus of mass for an electron is not yet barred until some inherent contradiction due to it has been clearly located.

The disguise of mechanical energy as electromagnetic is favoured through equations (13, 14). The effective deduction of ($m_0/4$) that appears in the developments for both electrons can be reconciled numerically with a composite rest-mass†. Any "weight-mass" is presumably external to that conversion of energy in the electronic problem, which the scale-factor must seek to cover. A preliminary inclusion of associated mass must be made inoperative in the rectified estimate of electromagnetic activity. The elastic conformity of one set of equations to varied physical suppositions is strongly marked in all these phases, and in other combinations left unmentioned that correct algebra permits. Such uncertainties must continue until new evidence decides them. One pertinent inquiry will concern itself with the quantity ($\gamma(v)$), whose integral powers figure so prominently. Is this series naturally commensurate with intervals that the physics determines?

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* Abraham, *loc. cit.* p. 129, eq. VI.; p. 144, eq. (95 *a*). Also, on grounds of formal resemblance, it was sometimes misunderstood to be nothing else than d'Alembert's idea.

† The additional "potential energy" required by Abraham's argument agrees in magnitude, within the velocity-range (0, *c*), with what the particular scheme of equations (27, 28, 29) would set aside as "pseudo-conversion." Compare Abraham, *loc. cit.* pp. 189–95. Also the suggestion added to equation (12) above.

VI. *An Electromagnetic Theory of Radiation.*
By H. BATEMAN *.

1. SOME interesting suggestions with regard to the nature of radiation have been made recently by Sir Joseph Thomson † and Leigh Page ‡. According to the first writer, radiation of energy seems to be generally associated with the formation of closed lines of electric force which travel away from a source with the velocity of light. In Page's work radiation of energy is supposed to be caused by a rotation of the field of an electron, and it follows from his expressions for the field vectors in a rotating field that this field may be obtained by superposing on the non-rotating field of an electron a radiant field in which circular lines of electric force travel outwards with the velocity of light.

Page's work is particularly interesting because the amount of energy radiated in one complete revolution of the field is almost exactly $\frac{1}{32}h\nu$, where ν is the frequency of the rotation and h is Planck's constant. In order that ν may be identified with the frequency of the emitted light, it seems necessary either to assume the existence of discrete tubes of force or to extend Page's analysis to the case in which an electron revolves or oscillates with its frequency of rotation.

A peculiar feature of the radiant field is that there is a radiation of magnetic charges which travel along straight lines with the velocity of light §. There is, however, no magnetic charge associated with the electron, and so it seems likely that the magnetic charges arise from a distribution of magnetic doublets which in turn arise from moving electric doublets. Whether this is the case or not, the mathematical analysis suitable for the formal development of this theory of radiation is akin to that which has been given in the present author's speculations regarding the electrical nature

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* Communicated by the Author.

† 'Engineering,' vol. 101. p. 381 (1916); Phil. Mag. June 1920, p. 679.

‡ Proc. Nat. Acad. Sci. March 1920, p. 115.

§ A class of radiant fields in which there is a radiation of magnetic particles in *particular* directions was found by the present author, Proc. Nat. Acad. Sci. March 1918. It is probable that Page's radiant field can be built up by superposing a number of fields of this older type. It may be remarked, too, that circular lines of electric force can be obtained in fields of the older type, and in this case it is possible for magnetic charges which are equal but of opposite sign to be emitted in consecutive directions. Circular lines of electric force of opposite senses are then generated in approximately the same region like vortices having different senses of rotation.

of the æther, and so it may be worth while to present the analysis in a compact form.

2. In a former paper * an attempt was made to describe the motion of the lines of force of an electric pole by means of a succession of infinitesimal transformations which transform light-particles into light-particles. If for simplicity we consider only transformations which correspond to rigid body-displacements in Minkowski's four-dimensional space, the path curves or trajectories of the transformations may be written in the form

$$\left.\begin{aligned} dx&=d\tau\,[\xi'(\tau)+c(t-\tau)p-(y-\eta)h+(x-\zeta)g],\\ dy&=d\tau\,[\eta'(\tau)+c(t-\tau)q-(x-\zeta)f+(x-\xi)h],\\ dz&=d\tau\,[\zeta'(\tau)+c(t-\tau)r-(x-\xi)g+(y-\eta)f],\\ cdt&=d\tau\,[c+(x-\xi)p+(y-\eta)q+(x-\zeta)r], \end{aligned}\right\} \quad (1)$$

where $\xi, \eta, \zeta, f, g, h, p, q, r$ are functions of τ. The infinitesimal transformation associated with a particular value of τ is supposed to be applied to space-time points (x, y, z, t) which satisfy the relation

$$[x-\xi(\tau)]^2+[y-\eta(\tau)]^2+[z-\zeta(\tau)]^2=c^2(t-\tau)^2, \quad (2)$$

and also to certain space-time points specified by equations of type

$$\left.\begin{aligned} x&=F_1(X, Y, Z, \tau), & z&=F_3(X, Y, Z, \tau),\\ y&=F_2(X, Y, Z, \tau), & t&=F_0(X, Y, Z, \tau), \end{aligned}\right\} \quad . \quad (3)$$

which may be regarded as solutions of the differential equations (1), X, Y, and Z being constants of integration or invariants for the sequence of transformations.

The equations (3) may be supposed to give the co-ordinated motions of the different points of an electron of which (ξ, η, ζ, τ) is a particular point which we shall call the focus. The lines of force of the electron are supposed to be generated by light-particles fired out from the different positions of this focus S, the direction of projection of the light-particles associated with one line of force varying in a manner indicated by the succession of infinitesimal transformations. In fact, if (l, m, n) are the direction cosines of the line of projection of the light-particles emitted at time τ we have

$$\frac{dl}{d\tau}=p-mh+ng-l(lp+mq+nr). \quad . \quad . \quad . \quad (4)$$

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* Proc. London Math. Soc. (2) vol. xviii. p. 95 (1919).

3. Let us now consider the electromagnetic field specified by the vectors **E** and **H** whose components are given by equations of type

$$E_x = \frac{1}{c}\frac{\partial(\sigma,\tau)}{\partial(x,t)} + \frac{\partial(\theta,\tau)}{\partial(y,z)}, \qquad H_x = \frac{\partial(\sigma,\tau)}{\partial(y,z)} - \frac{1}{c}\frac{\partial(\theta,\tau)}{\partial(x,t)}, \quad (5)$$

$$\cdots\cdots\cdots\cdots \qquad \cdots\cdots\cdots\cdots$$

$$\left.\begin{aligned} M\sigma &= (x-\xi)\alpha + (y-\eta)\beta + (z-\zeta)\gamma + \delta, \\ M\theta &= (x-\xi)\lambda + (y-\eta)\mu + (z-\zeta)\nu, \\ M &= (x-\xi)\xi' + (y-\eta)\eta' + (z-\zeta)\zeta' - c^2(t-\tau), \end{aligned}\right\} \quad (6)$$

where α, β, γ, δ, λ, μ, ν are functions of τ, and c is the velocity of light. Writing

$$\frac{x-\xi}{l} = \frac{y-\eta}{m} = \frac{z-\zeta}{n} = \frac{c(t-\tau)}{1},$$

we see that if

$$\delta\frac{dl}{d\tau} = c\alpha + \mu\zeta' - \nu\eta' + m(c\nu - \alpha\eta' + \beta\xi') - n(c\mu - \gamma\xi' + \alpha\zeta')$$
$$- l[l(c\alpha + \mu\zeta' - \nu\eta') + m(c\beta + \nu\xi' - \lambda\zeta') + n(c\gamma + \lambda\eta' - \mu\xi')]. \quad . \quad . \quad (7)$$

we may write

$$\mathbf{E} = \frac{1}{c}\frac{\partial\tau}{\partial t}\frac{\delta}{M^2}[c\mathbf{s} - \mathbf{v} - c(t-\tau)\mathbf{s}'], \qquad \mathbf{H} = \mathbf{s}\times\mathbf{E}, \quad . \quad (8)$$

where **a**, **w**, **v**, and **s** denote the vectors with components (α, β, γ), (λ, μ, ν), (ξ', η', ζ'), (l, m, n) respectively.

The rate of radiation of energy in the direction **s** at a very great distance from S is approximately

$$\frac{c}{4\pi}\frac{\delta^2}{M^4}\left(\frac{\partial\tau}{\partial t}\right)^2 (t-\tau)^2 (l'^2 + m'^2 + n'^2),$$

and this is positive except when **s** does not vary with τ. Both **a** and **w** must be zero for there to be no radiation.

The equation (7) is identical with (4) if

$$c\alpha + \mu\zeta' - \nu\eta' = \delta p, \qquad c\lambda - \beta\zeta' + \gamma\eta' = -\delta f.$$

These equations generally determine the ratios of α, β, γ, δ, λ, μ, ν uniquely in terms of f, g, h, p, q, r. To make the electric charge associated with S a constant quantity, e, we write

$$4\pi\delta = e(c^2 - \xi'^2 - \eta'^2 - \zeta'^2),$$

and the field is then completely determined when the motion of the electron is known.

4. The part of the field depending on the function θ represents the radiant field added by Page to allow for the effect of rotation. A moving line of electric force in this field is given by the equations $\theta=$constant, $\tau=$constant. When t is constant the lines of electric force associated with a given value of τ are cut out on the sphere (2) by the planes through the polar line of a line which meets the sphere in two real points A and B. The directions SA and SB are "associated" directions of projection, the points A and B being collinear with the point T which would be reached by S at time t if it continued to move with the velocity which it has at the instant of time τ. As t and τ vary the points A and B trace out two associated lines of electric force. These may be regarded as examples of the guiding lines of force of which Sir Joseph Thomson speaks. The other lines of force circle round one of these lines.

The lines of magnetic force being orthogonal to the lines of electric force are circles through pairs of points such as A and B. It should be noticed that the field vectors in the radiant field may also be expressed in the form

$$H_x=\psi\frac{\partial(\phi,\tau)}{\partial(y,z)},\qquad E_x=\frac{1}{c}\psi\frac{\partial(\phi,\tau)}{\partial(x,t)},$$

where

$$\phi=\log\frac{P(x-\xi)+Q(y-\eta)+R(z-\zeta)-S(t-\tau)}{U(x-\xi)+V(y-\eta)+W(z-\zeta)}\equiv\log\frac{A}{B},$$

$$SU=\eta'\nu-\mu\zeta',\qquad SV=\zeta'\lambda-\xi'\nu,\qquad SW=\xi'\mu-\eta'\lambda,$$

$$\lambda=VR-WQ,\qquad \mu=WP-UR,\qquad \nu=UQ-VP,$$

$$\psi=\frac{AB}{M^2}.$$

The lines of magnetic force are given by $\phi=$constant, $\tau=$constant, and are traced out by particles which travel along the radii for which $\frac{A}{M}$, $\frac{B}{M}$, and τ are constant. There are two such radii for each set of values of ϕ, ψ, and τ. The line joining the positions at time t of the magnetic particles which travel along these radii passes through a fixed point $Q(t, \tau)$ given by the equations $A=0$, $B=0$, $M=0$. The polar plane of Q with respect to the sphere (2) divides the positive magnetic charges on the sphere from the negative magnetic charges. The volume density of the magnetic charge may be written in the form

$$\epsilon = \frac{\partial(\psi, \phi, \tau)}{\partial(x, y, z)} = -\frac{2(t-\tau)}{\mathrm{M}^4}\left[\mathrm{M}(\mathbf{v}\,.\,\mathbf{w}) + (c^2 - v^2)\theta\right],$$

and this is equivalent to the expression given by Page when we write

$$\mathbf{w} = -\frac{ec}{4\pi}\left(1 - \frac{v^2}{c^2}\right)\omega,$$

where ω is a vector representing the angular velocity of the electron or its field in the case when $v=0$.

5. The rate of radiation of energy in a field of type (5) is found to be

$$\frac{32}{3}c\pi^2\left[\frac{a^2 + w^2 - \frac{1}{c}\{\mathbf{v}\,.\,(\mathbf{a}\times\mathbf{w})\}}{(c^2-v^2)^2} + \frac{(\mathbf{v}\,.\,\mathbf{a})^2 + (\mathbf{v}\,.\,\mathbf{w})^2}{(c^2-v^2)^3}\right].$$

The vectors **a** and **w** are at present at our disposal. If we take $\mathbf{a}=0$ and choose the above value of **w**, assuming also that $(\mathbf{v}\,.\,\mathbf{w})=0$, we obtain for the rate of radiation

$$\frac{2}{3}\frac{e^2\omega^2}{c},$$

and the amount of energy radiated in an interval $\frac{2\pi}{\omega}$ is *

$\frac{4\pi e^2\omega}{3c}$, which is approximately $\frac{\omega}{64\pi}6{\cdot}4(10)^{-27}$, when e is assumed to be $4{\cdot}77 \times 10^{-10}$ E.S.U.

6. The possibility of radiation in quanta having been established the next step must be to justify our assumption that $\mathbf{a}=0$. When the electron is stationary or moving uniformly in a straight line this assumption seems quite natural, but there must be some occasion when an electron moves in a curve, and so we must consider the possibility that **a** may be zero in a general type of motion.

In the case of a moving electric pole the expression for σ is †

$$\sigma = \frac{e}{4\pi}\frac{\xi''(x-\xi) + \eta''(y-\eta) + \zeta''(z-\zeta) + c^2 - \xi'^2 - \eta'^2 - \zeta'^2}{\xi'(x-\xi) + \eta'(y-\eta) + \zeta'(z-\zeta) - c^2(t-\tau)} + f(\tau).$$

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* A similar result may be obtained without the introduction of magnetic particles by putting $w=0$ and regarding a as a multiple of an angular velocity about an axis perpendicular to v.

† *Cf.* R. Hargreaves, Proc. Camb. Phil. Soc. 1915.

Now by making a suitable choice of $f(\tau)$ this may be written in the form

$$+\left\{\frac{d}{d\tau}\left(\frac{\partial L}{\partial \zeta'}\right)\right\}(z-\zeta)+\frac{d}{d\tau}\left(\frac{c^2 L}{c^2-v^2}\right)(t-\tau)-L$$

$$\sigma=\frac{e}{4\pi}\frac{\left\{\frac{d}{d\tau}\left(\frac{\partial L}{\partial \xi'}\right)\right\}(x-\xi)+\left\{\frac{d}{d\tau}\left(\frac{\partial L}{\partial \eta'}\right)\right\}(y-\eta)}{\frac{\partial L}{\partial \xi'}(x-\xi)+\frac{\partial L}{\partial \eta'}(y-\eta)+\frac{\partial L}{\partial \zeta'}(z-\zeta)+\frac{c^2 L}{c^2-v^2}(t-\tau)},$$

where $L=m(c^2-v^2)^{\frac{1}{2}}$ reminds us of a Hamiltonian function used in the theory of relativity [*], m being a constant of the nature of mass.

If **a** is zero in the case of an electron this may be in consequence of the equations of motion of the electron and it may be necessary to extend the expression for σ by adding to $\frac{d}{d\tau}\left(\frac{\partial L}{\partial \xi'}\right)$ a term representing the effect of the external electric field, and possibly also a term $-\frac{\partial L}{\partial \xi}$ to allow for a variation of m with position. The complete coefficient of $x-\xi$ is then zero on account of the equations of motion, and the complete coefficient of $t-\tau$ should be zero on account of the energy equation.

This modification of σ makes the introduction of a radiant field depending on θ seem fairly reasonable, because the coefficients in the numerator of θ may be regarded as components of angular momentum when the denominator has been suitably modified. It is possible, too, that there should also be terms to represent the effect of impulsive couples which act only at isolated intervals of time. These impulsive couples may, perhaps, be produced by magnetic particles emitted from other sources.

7. It is, perhaps, a little hard to understand why the electron should turn completely round a finite number of times when emitting radiation. It may be that a particular line of force is naturally associated with some particle at a very great distance from S and that the line of force is eventually recaptured by this particle after it has been once torn away. This would account for the return of the line of force to approximately the original direction.

A further difficulty in the present theory is that when an

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[*] The corresponding expression appropriate in Einstein's generalized theory of relativity is easily written down, but the expression for τ in terms of x, y, z and t is not given by such a simple equation as (2).

electron both rotates and revolves, different lines of force generally revolve at different rates when $\mathbf{a}=0$ and there may be only one line of force whose angular velocity corresponds with the frequency of the light. This may, however, be the particular line of force to which we have just referred. To make all the lines of force revolve with the same angular velocity it seems necessary to make $\mathbf{a}$ equal to the acceleration as in Page's analysis, but then a non-radiating orbit is an impossibility.

8. According to the present theory the idea that all electromagnetic fields occurring in nature can be built up from the fields of electric poles travelling with velocities less than c seems to be untenable ; it seems almost necessary, in fact, to adopt a more general type of electromagnetic field as fundamental, and it may be that the simple radiant fields described by the present author will turn out to be the most suitable.

VII. *The Angular Momentum and some Related Properties of the Ring Electron.*

By H. STANLEY ALLEN, *M.A., D.Sc., University of Edinburgh* [*].

The Angular Momentum of the Ring Electron.

SHORTLY before the war the late Prof. S. B. McLaren was engaged in writing on the magneton [†], and it is greatly to be regretted that in consequence of his death on the Western front on 14 August, 1916, the applications of his work to the theory of complete radiation, spectral series, and the asymmetrical emission of electrons in ultra-violet light were never published. Rejecting entirely the idea of magnetic or electric substance, he regarded the magneton as an inner limiting surface of the æther, formed like an anchor ring. The tubes of electric induction which terminate on its surface give it an electric charge, the magnetic tubes linked through its aperture make it a permanent magnet. He found that the angular momentum of any such system, whatever its shape or dimensions, about its axis of symmetry is $(8\pi^2 c)^{-1}\mathrm{N}_e\mathrm{N}_m$. Here c is the velocity of light, N_e is the number of tubes of electric induction terminating on the surface, and N_m is the number of tubes of magnetic induction passing through the aperture.

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[*] Communicated by the Author.

[†] Phil. Mag. vol. xxvi. p. 800 (1913); 'Nature,' vol. xcii. p. 165 (1913), vol. xcvii. p. 547 (1916).

No proof of McLaren's theorem appears to have been published, and it may be of interest to show that the result can be obtained in a comparatively simple way.

Let the position of any point in the æther be referred to cylindrical coordinates, taking the axis of symmetry of the magneton as the axis of z. Since each Faraday tube is rotating about this axis, the motion of any portion of a tube is at right angles to its length, and the equivalent mass per unit volume of a tube is $4\pi\mu N^2$, where N is the electric polarization or displacement at the point. The angular momentum for unit volume of the tube is consequently $4\pi\mu N^2 r^2 \omega$, where ω is the angular velocity with which the system is rotating about the axis. The motion of the Faraday tubes produces a magnetic field in a direction at right angles to their length and to the direction of motion of magnitude $H=4\pi N r\omega$. Hence the angular momentum for unit volume of the tube can be expressed in the form $\mu H^2/4\pi\omega$ or

$$\frac{2}{\omega}\times\frac{\mu H^2}{8\pi}.$$

Thus the total angular momentum $=\frac{2}{\omega}\Sigma\frac{\mu H^2}{8\pi}$, where the summation extends over the whole space external to the magneton. But $\Sigma\mu H^2/8\pi$ represents the amount of energy associated with the magnetic field, and it is easy to show by considering the energy as distributed in the magnetic tubes that this is equal to $\frac{1}{2}Li^2$, where L is the coefficient of self-induction of the magneton and i the current flowing round it. Thus the total angular momentum $=Li^2/\omega$.

But $Li=N_m$, the number of magnetic tubes linked with the magneton, and $i/\omega=e/2\pi=N_e/2\pi$, where e is the charge on the magneton. So we obtain finally as the total angular momentum of the magneton

$$\frac{1}{2\pi}\mathbf{N}_m\mathbf{N}_e.$$

The difference between this expression and that given by McLaren arises from the fact that he employed rational units.

It is easily shown from a consideration of the dimensions of N_m and N_e that the product has the same dimensions as angular momentum both in the electrostatic and in the electromagnetic system.

The proof given being perfectly general applies to any magneton of the type considered, whatever may be the shape of the cross-section, or the size of the cross-section or of the ring. Further, it would seem that the same result would hold not only for a surface distribution of electrification, but also for a volume distribution as in the case of the ring electron, which is usually looked upon as a circular anchor-ring of negative electricity rotating about its axis with large velocity [*].

Sir Joseph Larmor [†] has pointed out that one or more classical electrons constrained to move round a channel would be like an amperean current. The same method of proof might be applied in such a case, leading to the above expression for the angular momentum.

Angular Momentum and Planck's Constant.

In his paper on the constitution of the solar corona, Prof. J. W. Nicholson [‡] first introduced the concept of a natural unit of angular momentum, finding such a unit in the quantity $h/2\pi$, where h is Planck's constant. This is the quantity which appears in Bohr's theory as the angular momentum of a "bound" electron. If we identify this unit of angular momentum with the angular momentum of the magneton, we find

$$\text{Unit angular momentum} = h/2\pi = \mathrm{N}_m\mathrm{N}_e/2\pi.$$

This identification gives the remarkable result

$$\mathbf{h} = \mathbf{N}_m\mathbf{N}_e,$$

or *Planck's constant is equal to the product of the number of tubes of magnetic induction and the number of tubes of electric induction associated with the magneton.*

Assuming $\mathrm{N}_e = \epsilon$, the natural unit of electric charge, which is equivalent to identifying the magneton with the electron, we find

$$h/\epsilon = \mathrm{N}_m.$$

Taking $\epsilon = 4{\cdot}774 \times 10^{-10}$ E.S.U. and $h = 6{\cdot}558 \times 10^{-27}$, this gives $\mathrm{N}_m = 1{\cdot}374 \times 10^{-17}$ E.S.U. or $4{\cdot}120 \times 10^{-7}$ E.M.U., assuming $c = 2{\cdot}999 \times 10^{10}$ cm. per sec.

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[*] Parson, "A Magneton Theory of the Structure of the Atom," Smithsonian Misc. Coll. vol. lxv. No. 11 (1915).

[†] Proc. Phys. Soc. vol. xxxi. p. 68 (1919).

[‡] Monthly Notices, R.A.S., June 1912.

The Relation of Lewis and Adams.

From their theory of ultimate rational units Lewis and Adams [*] deduced a relation between Planck's constant, h, and the electric charge, ϵ, of the form

$$15h^3c^3=8\pi^5(4\pi\epsilon)^6,$$

or

$$hc=\sqrt[3]{\left(\frac{8\pi^5}{15}\right)}\times(4\pi\epsilon)^2,$$

where ϵ is in electrostatic units.

Using the notation employed in an earlier paper [†], in which I discussed certain numerical relations between electronic constants,

$$hcq=2\pi\epsilon^2,$$

where q is a pure number and $=7{\cdot}28077\ldots\times10^{-3}$.

It is worthy of remark that the numerical factor $8\pi^5/15$, which occurs under the cube root in the expression above, may be written as $48\pi\alpha$, where

$$\alpha=1+\frac{1}{2^4}+\frac{1}{3^4}+\frac{1}{4^4}+\ldots.=1{\cdot}0823\ldots=\frac{8\pi^4}{4\,!}\,B_2,$$

B_2 being the second Bernoulli number $=1/30$ [‡].

The relation given by Lewis and Adams receives very strong support from the experimental determinations of the constants involved, as has been shown by R. T. Birge [§] in a paper on the most probable value of Planck's constant. He gives the mean value of h as $6{\cdot}5543\pm0{\cdot}0025\times10^{-27}$ erg. sec., assuming $\epsilon=4{\cdot}774\times10^{-10}$ and $c=2{\cdot}9986\times10^{10}$. The calculated value he gives as $6{\cdot}560\times10^{-27}$.

By combining McLaren's result with the relation of Lewis and Adams we find

$$N_m\epsilon c=\sqrt[3]{\left(\frac{8\pi^5}{15}\right)}\times(4\pi\epsilon)^2\quad\text{or}\quad\frac{2\pi\epsilon^2}{q}$$

in electrostatic units.

Hence

$$N_mc=\sqrt[3]{\left(\frac{8\pi^5}{15}\right)}\times16\pi^2\epsilon\quad\text{or}\quad\frac{2\pi\epsilon}{q}.$$

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[*] Phys. Rev. vol. iii. p. 92 (1914).

[†] Proc. Phys. Soc. vol. xxvii. p. 425 (1915). In Sommerfeld's papers on the fine structure of spectrum lines in the *Annalen der Physik* for 1916, the constant which I termed q is denoted by α.

[‡] Planck's 'Heat Radiation' (Masius) § 160.

[§] R. T. Birge, Phys. Rev. vol. xiv. p. 361 (1919).

The left-hand side of this equation gives the magnetic flux through the aperture of the magneton in maxwells, and we see that the flux in maxwells is equal to the electron charge (in E.S.U.) multiplied by a numerical factor.

Let L′ denote the self-inductance of the ring in electrostatic units, and n the number of revolutions per second of the electricity in circulation. Then $N_m = L'n\epsilon$, and

$$L'nc = \sqrt[3]{\left(\frac{8\pi^5}{15}\right)} \times 16\pi^2 \quad \text{or} \quad \frac{2\pi}{q}.$$

Or, if L denote the self-inductance in electromagnetic units,

$$Ln/c = \sqrt[3]{\left(\frac{8\pi^5}{15}\right)} \times 16\pi^2 \quad \text{or} \quad \frac{2\pi}{q}.$$

It would be interesting to know whether a toroidal surface could be found which would require a factor of the form $\sqrt[3]{\left(\frac{8\pi^5}{15}\right)}$ in the formula for its self-inductance.

It is usual to assume that the ring electron is of the form of an anchor-ring of circular cross-section. Webster * has pointed out that, in order to account for the observed mass of the electron, it is necessary to suppose that ρ, the radius of the section, is very small compared with a, the radius of the circular axis. The self-induction of electric currents in a thin anchor-ring of this type has been investigated by the late Lord Rayleigh †. When terms involving the square of ρ/a are neglected, the formula for the self-inductance is

$$L = 4\pi a\left[\log\frac{8a}{\rho} - k\right],$$

where k has the value 2 when the current is limited to the circumference of the anchor-ring, and the value 7/4 when the current is uniformly distributed over the cross-section. In the present problem the term $\log 8a/\rho$ is so large that the difference between the values of k is inappreciable.

Substituting this value for L in the previous equation, we find

$$\frac{4\pi na}{c}\left[\log\frac{8a}{\rho} - k\right] = \sqrt[3]{\left(\frac{8\pi^5}{15}\right)} \times 16\pi^2 \quad \text{or} \quad \frac{2\pi}{q}.$$

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* Webster, Phys. Rev. vol. ix. p. 484 (1917).

† Rayleigh, Roy. Soc. Proc. vol. lxxxvi. p. 562 (1912).

Now $2\pi na = v$, the velocity of the electricity as it travels round the ring. Hence

$$\frac{v}{c} = \frac{\sqrt[3]{\left(\frac{8\pi^5}{15}\right)} \times 8\pi^2}{\log\frac{8a}{\rho} - k} \quad \text{or} \quad \frac{\pi}{q\left[\log\frac{8a}{\rho} - k\right]}.$$

Since the peripheral velocity, v, must be less than c, the velocity of light, $\log\frac{8a}{\rho} - k$ must be greater than

$$\sqrt[3]{\left(\frac{8\pi^5}{15}\right)} \times 8\pi^2,$$

which gives $\log\frac{8a}{\rho}$ greater than 433, showing that ρ is excessively minute compared with a.

The Mass of the Parson Magneton, and the Radius of the Ring.

The electromagnetic mass of the Parson magneton has been investigated by Webster, who found an expression of the form

$$m = \frac{\epsilon^2}{\pi c^2 a} \log\frac{8a}{\rho},$$

in which the small term k has been omitted. This leads as before to a very small value for the radius of the cross-section of the ring, and it follows that most of the energy and momentum of the field are concentrated very closely around the ring.

To obtain further quantitative information as to the properties of the ring electron, one further quantity must be known, and we shall assume for this purpose the value of the radius a given by A. H. Compton *. He finds $a = (1{\cdot}85 \pm {\cdot}05) \times 10^{-10}$ cm. on the basis of an investigation of the scattering of X-rays or γ-rays by the ring. Taking $m = 8{\cdot}999 \times 10^{-28}$ gm., we find $\log\left(\frac{8a}{\rho} - k\right) = 2064$, which satisfies the condition that $\log\frac{8a}{\rho}$ should be greater than 433. The value of v/c is 433/2064, so that the peripheral velocity is about one-fifth of the velocity of light.

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* Phys. Rev. vol. xiv. pp. 20, 247 (1919). It is well to keep in mind the possibility that the radius of the ring may assume various values, as in the case of the radius of an electron orbit in Bohr's theory.

By combining Rayleigh's formula for the self-inductance with Webster's formula for the mass, we find

$$L=\left(\frac{2\pi c a}{\epsilon}\right)^2 m,$$

which, on substituting numerical values, gives

$$L=4{\cdot}80\times10^{-6}\ \text{E.M.U.}$$

Quantitative Results.

In the table following are collected the most important numerical constants for the ring electron. The results in the earlier part of the table are based on Millikan's value for the electron charge, and on the value $c=2{\cdot}999\times10^{10}$ cm. per sec. for the velocity of light. The mass is calculated on the assumption that $\epsilon/m=5{\cdot}305\times10^{17}$ E.S.U. The later results assume the estimate for the radius of the ring $a=(1{\cdot}85\pm{\cdot}05)\times10^{-10}$ cm. given by A. H. Compton to be correct. It will be noticed that the strength of the current flowing round the ring is relatively enormous, being nearly one ampere. In consequence, the strength of the magnetic field at the centre of the ring is nearly 3×10^9 gauss. At a point on the axis at a distance equal to the radius of the ring the strength of the field is about 10^9 gauss, and at a distance equal to the diameter is about $2{\cdot}6\times10^8$ gauss. Such intense local fields have been postulated in certain theories of atomic structure.

TABLE.—*Properties of the Ring Electron.*

Property.	Numerical Value.		Basis for Estimate.
Charge, ϵ	$4{\cdot}774\times10^{-10}$	E.S.U.	Millikan's experiments.
Planck's constant, h	$6{\cdot}558\times10^{-27}$	C.G.S.	Lewis and Adams.
Angular momentum	$1{\cdot}044\times10^{-27}$	C.G.S.	$h/2\pi$.
Magnetic flux	$1{\cdot}374\times10^{-17}$ $4{\cdot}120\times10^{-7}$	E.S.U. E.M.U.	h/ϵ.
Mass, m	$8{\cdot}999\times10^{-28}$	gm.	ϵ/m.
Magnetic moment	$9{\cdot}232\times10^{-21}$	E.M.U.	$\frac{1}{2}(\epsilon/m)(h/2\pi)$.
Radius of ring, a	$1{\cdot}85\times10^{-10}$	cm.	A. H. Compton.
Area of ring	$1{\cdot}075\times10^{-19}$	sq. cm.	πa^2.
Flux density (average)	$3{\cdot}83\times10^{12}$	E.M.U.	$(h/\epsilon)\div(\pi a^2)$.
Self-inductance	$4{\cdot}80\times10^{-6}$	E.M.U.	Rayleigh's formula.
Peripheral velocity, v	$6{\cdot}29\times10^{9}$	cm./sec.	v/c.
Frequency, n	$5{\cdot}41\times10^{18}$	sec.$^{-1}$.	$n=v/2\pi a$.
Current in ring, i	$8{\cdot}61\times10^{-2}$	E.M.U.	$i=\epsilon n$.
Field at centre	$2{\cdot}93\times10^{9}$	gauss.	$F=2\pi i/a$.

The case for and against the ring electron has been presented in a discussion held at a meeting of the Physical Society on October 25, 1918 *.

Summary and Conclusion.

A proof is given of a theorem due to S. B. McLaren, according to which the angular momentum of a magneton of any shape or dimensions about its axis of symmetry is $N_m N_e/2\pi$, where N_e is the number of tubes of electric induction terminating on the surface, and N_m is the number of tubes of magnetic induction passing through the aperture. Combining this result with J. W. Nicholson's natural unit of angular momentum, $h/2\pi$, where h is Planck's constant, we find that h is equal to the product of the number of tubes of magnetic induction and the number of tubes of electric induction, or $N_m = h/\epsilon$, where ϵ is the electron charge.

A further relation between h and ϵ is given by Lewis and Adams. This leads to a simple expression for the self-inductance of the ring, which may be combined with Lord Rayleigh's formula for the inductance of an anchor ring of circular cross-section. By employing Webster's expression for the mass of the Parson magneton and A. H. Compton's estimate for the radius of the ring, it is possible to calculate the most important numerical constants for the ring electron. A table of these constants is given, from which it appears that intense local magnetic fields must exist in the neighbourhood of the magneton.

VIII. *The Beta-Ray Emission from Thin Films of the Elements exposed to Röntgen Rays* †. *By* LEWIS SIMONS, *B.Sc.(London), Senior Lecturer in Physics in the University of Cape Town* ‡.

[Plate III.]

Introductory.

THE emission of corpuscles from metal plates exposed to Röntgen rays has been studied by Sadler § and Beatty ‖. Whiddington ¶ coupled the results of these observers with his own and showed that the maximum range, in air, of the

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* Proc. Phys. Soc. vol. xxxi. pp. 49–68 (1919).

† The expenses of this research are partly covered by a Government grant through the Research Committee of the Advisory Board of Science and Industries of the Union of South Africa.

‡ Communicated by Prof. O. W. Richardson, F.R.S.

§ Phil. Mag. ser. 6, vol. xix. p. 337 (1910).

‖ Phil. Mag. vol. xx. p. 320 (1910).

¶ Proc. Roy. Soc. A. vol. lxxxvi. p. 370 (1912).

corpuscles varied as the fourth power of the velocity with which they emerged from the metal.

In the later work of Barkla and Shearer *, reference was made to the fact that the electrons emitted from metal plates had different associations according to whether the X-radiation to which the plate was exposed caused it to emit the K, L, M, etc. series, or only the L, M, etc. series of tertiary X-radiations. This would depend primarily upon the relationship between the wave-length of the exciting radiation and that of the K, L, M, etc. emission lines in the radiation of the screen † acted upon. In spite of the different associations, it was found that the *maximum* velocity of emergence did not depend upon the type of X-radiation set up in the screen, the controlling factor being the wave-length of the incident X-rays; nevertheless, the authors significantly remarked that "although the exact shape of the curve depended upon the substance of the screen, the pressure at which the curve became straight was identical for all screens exposed to a particular radiation" ‡.

The work of Moseley and others on the X-ray spectra of the elements has shown the subject to be of greater complexity. There must be a distinct β-ray emission corresponding to each of the X-ray spectral lines of a given element. Interest lies in the study, not only of the maximum speed of emission, but also of any real sub-speeds that may exist (such sub-speeds *not* arising from any diminution in energy through the electrons having emerged from deep layers of material).

There is supporting evidence for the existence of a β-ray "spectrum" from the non-radioactive elements. Hahn § produced some interesting photographs of the β-ray "spectrum" from a thin film of radioactive matter. Reference has already been made to the statement of Barkla and Shearer that their absorption-curves varied slightly in shape along their curved portions. O. W. Richardson ‖ pointed out that

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* Phil. Mag. ser. 6, vol. xxx. p. 745 (1915).

† The elements emitting the electrons will be referred to throughout as "the screens."

‡ *Ibid.* p. 748. The curves referred to are those connecting pressure and ionization in a flat ionization chamber backed with the various screens. The ionization is due to the effect of the β-rays from the screens upon the gas in the chamber. The pressure at which the curve becomes straight measures the range and therefore the fourth power of the velocity of the particles.

§ Reproduced in Rutherford's 'Radioactive Substances and their Radiations' (1913), figs. 70 A and B.

‖ Proc. Roy. Soc. A. vol. xciv. p. 272 (1918).

the paths of the electrons emerging from the gaseous atoms, shown in C. T. R. Wilson's cloud experiments, do not vary continuously from a maximum downwards, whilst to be completely satisfied, the photo-electric equation requires that each X-ray spectral line shall be accompanied by a corresponding β-ray spectral line.

The X-ray spectrum of an element is invariable except in that the lines of shorter wave-length may be entirely absent if the incident waves are of longer wave-length; on the contrary, the β-ray spectrum will vary completely with the wave-length of the incident radiation. Assuming that the photo-electric equation does apply, we have

$$\tfrac{1}{2}mv^2 = hv - w,$$

where $\frac{1}{2}mv^2$ is the kinetic energy of ejection of a β-particle from an atom within which the potential energy of the particle is w and hv is the quantum of incident energy. The energy w is that which was radiated by the atom during the previous binding of the electron, and equals * $(hv_K + hv_L + \text{etc.})$ if the electron occupied a "K" ring, or $(hv_L + \ldots)$ if it occupied an "L" in the parent atom. Assuming that the X-rays are absorbed in quanta and that it is not a necessary condition that every atom taking part in the absorption should return at once to the stream the maximum possible quantity of X-radiation, we should obtain groups of electrons of speeds v_1, v_2, v_3 etc., emitted from an element S when X-rays of frequency v fall upon it, given by

$$\begin{aligned}\tfrac{1}{2}mv_1{}^2 &= hv - {}_S(hv_K + hv_L + hv_M + \ldots)\\ \tfrac{1}{2}mv_2{}^2 &= hv - {}_S(hv_L + hv_M + \ldots)\\ \tfrac{1}{2}mv_3{}^2 &= hv - {}_S(hv_M + \ldots),\end{aligned}$$

etc.

These groups constitute the β-ray spectrum varying with v.

$$v_3 > v_2 > v_1.$$

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* Bohr, Phil. Mag. ser. 6, vol. xxx. p. 412 (1915) gives a discussion of what should constitute the quantity in the brackets. Apparently his conclusions lead him to one, viz., the first term in the brackets, but he produces evidence from the separate work of Moseley, Kossel, and Barkla for the inclusion of all the terms as they appear here and in subsequent expressions. Throughout this paper I have followed the latter method, but at the conclusion it is shown that a better agreement with the theory can be obtained amongst the results if we assume that the energy required to remove a "K" electron to a point of zero potential with no kinetic energy $=hv_K$, the atom readjusting itself subsequent to the removal. (See also O. W. Richardson, 'Electron Theory of Matter,' p. 509 (1916).)

According to this scheme it must be noted that the so-called "K" electron is the slowest one, whilst the most rapidly ejected electron is the one that comes from the periphery of the atom, *i. e.* the so-called photo-electron[*].

In a previous paper[†] I pointed out that the absorption coefficients of these particles in various gases derived by Beatty and Sadler must be regarded as mean values. If the above scheme is correct, not only has the mean to be taken over the various speeds, but the matter is complicated further by our not knowing the relative numbers possessing the respective speeds. As one might say, we do not know the relative intensities of the respective β-ray spectral lines. I also produced experimental evidence in support of the above views.

Although it has often been stated that there is a definite "K" group of electrons associated with "K" X-radiation, and an "L" group associated with "L" X-radiation, yet I have been unable to find any experimental determination as to which group will have the greater velocity if the parent atoms are emitting simultaneously both "K" and "L" X-radiations. The nearest approach to any definite statement was made by Sadler, who pointed out that under certain conditions intense electronic emission from a screen was not necessarily associated with the emission of any (tertiary) X-radiations that could be detected. "For instance, when the secondary exciting beam from silver itself falls upon silver as tertiary radiator, no homogeneous Röntgen radiation is produced, and yet a considerable emission of corpuscular radiation occurs"[‡].

Finally, some extremely important deductions have been made from the work quoted above, *e. g.* it is deduced that the maximum velocity of the β-rays is never greater than that of the parent cathode ray within the discharge-tube.

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[*] It must be pointed out that this statement is not contrary to the experimental results of Barkla and Shearer, who excited first K radiation in a given screen by the incidence of a still harder radiation and then L radiation in a screen of higher atomic weight, by the same radiation (too soft to produce the K radiation in this latter screen). They then studied the maximum velocity of emergence of the β-rays from the two screens and found them to be equal. These experiments support this conclusion, but an attempt has been made to push the subject farther. Concomitantly with K emission, a given screen must be emitting L, M, etc. X-radiations, and what one calls the "K" electron is probably that emerging from the atom with the smallest velocity of the groups of electrons associated with the emission of the various X-ray spectral lines.

[†] Trans. Roy. Soc. S. Africa, vol. viii. pt. 1, p. 82 (1919).

[‡] *Ibid.* p. 354.

In the experiments described below some attempt has been made to find a solution to the following problems :—

1. Is the maximum speed of ejection of an electron from an atom entirely independent of the nature of that atom and dependent only upon the wave-length of the incident radiation?
2. Apart altogether from the question of the diminution in speed of emergence of those β-rays which have their origin in the deeper layers of the material, what is the precise nature of the distribution of speeds amongst the electrons emitted from the parent atoms when these are exposed to X-rays? How is the distribution affected by (*a*) the nature of the parent atom, (*b*) the wave-length of the incident radiation?
3. If the existence of a β-ray "spectrum" can be proved, what is the interrelation between each β-ray spectral line and the corresponding X-ray spectral line?

Apparatus.

The method employed throughout was that due originally to Beatty in his study of the absorption coefficients of the β-rays. A few modifications were introduced. It was thought that the emission would be simplified by using only very thin films of the various elements. These were As, Se, Zr, Ag, Sn, Sb, Ba, Au, Pb, and Bi. The X-rays which were incident upon these screens in turn were the so-called homogeneous X-radiations, first from silver and secondly from barium *. The primary beam of X-rays was produced by a standard Coolidge tube, and every endeavour was made throughout the whole of the work, a period of about eleven weeks, to keep matters constant. The parallel spark, between point and plate, backed up by the tube measured four inches, and the current in the heating spiral was adjusted so that the point and plate were always just on the point of sparking. This current was four amperes throughout. The barium secondary radiator was a flat cell of BaO_2, the front of the cell being tissue-paper, the back card. When this was used an extra aluminium sheet 1 mm. thick was interposed between it and the ionization chamber and other apparatus in order to eliminate the characteristic L radiation

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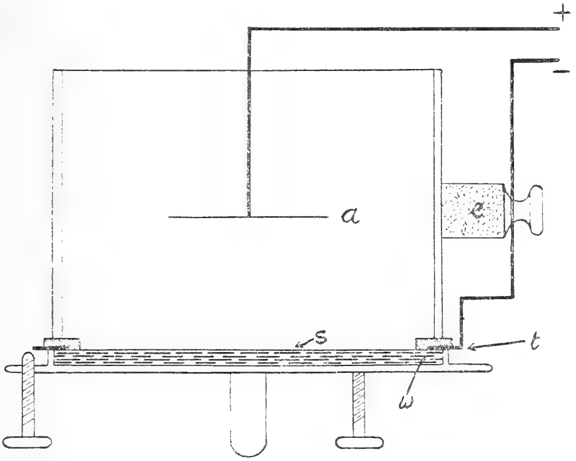
* The fact that the so-called homogeneous K radiation consists of a series of lines of different wave-lengths, whilst rendering impossible accurate measurements by this method, does not seriously interfere with the main conclusions drawn herein. An attempt is being made to employ really homogeneous radiations and to overcome the necessary mechanical difficulties.

from Ba. When the silver secondary radiator was used, the front of the ionization chamber, being aluminium ·75 mm. in thickness, was sufficient to absorb the L radiation from silver.

The Screens.—These were prepared in the following manner. A brass disk, 12·7 cm. in diameter and 0·17 cm. thick, had a brass rim soldered centrally upon it. The rim was 10·2 cm. in diameter and 0·4 cm. deep. In this manner a flat cell was produced 10·2 cm. in diameter and 0·4 cm. deep. Eleven such cells were prepared. These cells were filled with paraffin-wax, the surface being scraped down flat and level with the upper edge of the rim and polished with natural graphite. To obtain films of Au, Sn, and Ag electro-deposition from the double cyanide was employed. A brass cylinder, open at both ends, about 10 cm. in diameter and about 7 cm. high, was stood upon the graphited wax plate *w* (fig. 1). The inside of this cylinder was waxed so as to prevent contamination, *e* is an ebonite block. By an arrangement of an indiarubber ring sealed round the edge of the cylinder, and an annular tinfoil electrode *t* stuck on the indiarubber ring, good electric contact could be made with the graphite surface. The electroplating solution was then poured into the cell, a suitable anode *a* dipped in, and deposition started on S and judged by the eye. In the case of these three substances, the films were uniform and so

Fig. 1.



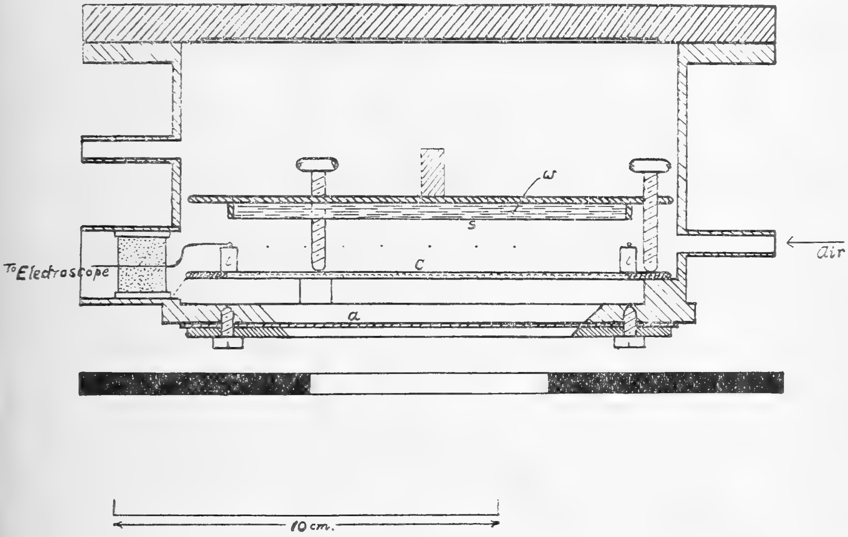
thin that the graphite backing could easily be seen through them.

Films of the other substances were obtained in the following manner. It was necessary to obtain an adherent conducting surface in each case. The surface had to be conducting in order that the electrostatic capacity of the ionization chamber, of which the surface really formed the back, should remain unaltered when the various screens were substituted for one another. This conducting surface had also to be made of some material of low atomic weight from which there is no electronic emission. To obtain a degree of uniformity, the graphited wax surface was placed horizontally, face uppermost, and an inverted electric hot-plate was brought down for a few moments to within a distance of about 0·5 cm. from the surface. The graphite surface cracks and the paraffin-wax comes up through the fissures of retreat. If the process is watched carefully and is stopped at the right moment, the whole surface is conducting, as tested by an electroscope, and it will take a powder in a way that polished graphite cannot. It also remains quite plane. Pure As, Se, Zr, Bi, and Sb were very finely powdered in a mortar and lightly dusted over the prepared plates, the larger grains being wiped off and the plates being vigorously tapped in order to get rid of any particles not properly adhering to the wax. Microscopic examination showed that each plate was covered with a uniform layer of minute grains. No estimate of the size of the grains was made. Each plate was tested for electrical conductivity. Barium peroxide was similarly dusted on a plate for the Ba screen and red-lead for the Pb screen. An eleventh screen was made containing a flat sheet of carbon (cut from a dynamo brush) instead of the usual paraffin-wax. Finally, each disk was provided with three screw-legs round the edge outside the rim, and all the parts except the prepared screen and the back of the disk were coated with a thin layer of wax and rubbed with graphite.

A small brass conical gauge-piece was made 1·45 cm. high, which was placed on a sheet of glass and the screen placed over it. The screw-legs were now adjusted until each part of the surface of the screen just touched the apex of the gauge-piece. The screens were then ready to be placed in the ionization chamber, but before doing so each was placed for a short while under the receiver of an air-pump in which the pressure was maintained at about 2 mm.

The ionization chamber is shown in fig. 2. To the bottom of the brass cylindrical case a sheet of aluminium (*a*) ·75 mm. thick was sealed on with Chatterton compound and fixed with a screwed-on brass ring. The heavy top was ground flat for easy removal, the sealing material in this case being a soft mixture of bees'-wax and vaseline. A pure carbon plate (*c*), about 11 cm. in diameter and 1·5 mm. thick, was mounted flush within a flat brass ring; this ring with its plate rested on three small chairs round the bottom of the case, and when a screen was put in, its legs rested on the brass ring. On the ring there were three small sulphur beads (*i*), which supported an electrode connected to a Wilson electroscope, the electrode being made of radial carbon filaments. Finally, all the brass parts within the case were painted with a mixture of sugar-water and artificial graphite. Lead diaphragms were arranged so as to keep the X-radiation from falling anywhere but within the prepared surface of the screen.

Fig. 2.



The only gas used throughout the experiments was air, and the pressure within the chamber could be varied from a few millimetres up to atmospheric pressure. Before passing into the chamber the air was roughly dried by bubbling it through sulphuric acid. A fresh supply of air constantly passed through the chamber during the experiments.

Throughout the first set of experiments the silver plate was used as the secondary radiator. First the carbon screen was placed in position, the pressure within the ionization chamber adjusted, and the mean ratio obtained of the ionization in it to that in a standardizing electroscope 12 to 14 points on the curves being obtained within the range from zero up to atmospheric pressure. Another screen was substituted and the process repeated. The screens were not used in any definite order. The barium secondary radiator (with its accessory aluminium plate for cutting out the L radiation) was now substituted for the silver one, and the whole process repeated, again using the eleven screens in no definite order.

Experimental results.

Fig. 3 (Pl. III.) gives the points obtained in using the eleven screens successively with silver as radiator, and fig. 4 with barium as radiator. No corrections whatever have been made in putting down the points, the air effect and the normal leak of the instruments being so small that they could be neglected.

The ordinates of the carbon curve were subtracted from those of the other curves. Each resulting curve would represent the ionization produced in air at various pressures by the β-particles emerging from the respective screens. There is a critical pressure at which the range of the β-particle is just equal to the thickness of the ionization chamber: viz., 1·45 cm. Above this pressure the curves become horizontal straight lines, for the β-particles are now completely absorbed in the air. Below this pressure some of the energy of the particles will be lost by their being absorbed in the carbon front of the chamber.

The resulting ten curves, say with silver as secondary radiator, differ widely from one another in their maximum ordinate. In other words, the total ionization by the β-particles in these experiments shows a considerable variation over the range of substances used. It must be remarked at once that this variation here has but little meaning. It is known that the intensity of the β-ray emission increases with increasing atomic weight of the screen, but apart from this, as the films of substances radiating were so exceedingly thin to a certain extent of variable thickness, but by no means thick enough to absorb to any extent the β-particles produced from the side of the material in contact with the graphite surface, no importance can therefore be attached here to the fact that the total ionization observed differs widely for the various screens.

Figs. 5 and 6 (Pl III.).—For the purpose of comparing the curves along their non-linear portions, each was scaled up to the same maximum ordinate *. The result is shown in fig. 5 (Pl. III.), which might be called the cathode ionization curves from the ten screens, silver as radiator, and in fig. 6 (Pl. III.), barium as radiator.

In the paper previously referred to, the writer has shown fully how the first derivatives of the curves shown in figs. 5 and 6 (Pl. III.) give the density of the ionization in a chamber containing air at 0° C and 76 cm. pressure at a distance $\frac{p}{76} \times \frac{273}{273+\theta} \times 1{\cdot}45$ cm. from the screen, where p is the pressure at which an ordinate such as AB is drawn, 1·45 cm. the thickness of the ionization chamber used, and θ the mean temperature of the experiment, about 24° C. in this case. This expression has been used in order to convert values of p into corresponding values of the distance from the screen as measured in air at 0° C. and 76 cm. pressure. The ordinate AB $(=N_x)$ measures the total ionization by β-rays in air (at 0° C. and 76 cm. pressure) in the region from the screen up to an imaginary layer 0·1 cm. from the screen.

Fig. 7 (Pl. III.).—When the curves of figs. 5 and 6 (Pl. III.) were first drawn, there seemed to be little order about their sequence, except that in fig. 5 (Pl. III.) the curve for the silver screen was lowermost and in fig. 6 (Pl. III.) the barium curve lowermost. In order to show up any sequence, fig. 7 (Pl. III.) was drawn. This figure has for abscissæ the atomic numbers of the elements of the screens, whilst the ordinates are taken directly from figs. 5 and 6 (Pl. III.) as follows. The length CB (fig. 5) represents the total ionization by β-rays in air (at 0° C. and 76 cm. pressure) included within the region from an imaginary plane 0·1 cm. from the screen up to the front face of the chamber ; it is therefore a measure of the β-ray energy crossing this plane, and equals (N_0-N_x). The line CB, at the 0·1 cm. mark, is drawn right across the 10 curves, and its various lengths raised above the respective elements

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* The simplest physical interpretation of such an adjustment is to imagine the number of atoms at the surface of each screen so packed that the β-ray emission from each screen gives rise to equal total ionization in the air of the chamber for complete absorption of the particles.

placed in order of atomic number along the horizontal axis of fig. 7 (Pl. III.). This process is repeated for other distances from the screens. Thus, since the curves figs. 5 and 6 (Pl. III.) refer to equal total ionizations, a high ordinate on fig. 7 (Pl. III.) at any point indicates a larger proportion of higher velocity electrons passing that plane, and *vice versâ*.

The question arises that the marked variations in the shapes of the curves of figs. 5 and 6 (Pl. III.) might be due entirely to the ionization in the chamber by the variable tertiary X-radiation from the successive screens, superposed on the ionization by the β-radiations which has a uniform function for each of the screens. Some of the reasons for this view being untenable are set down below. Though this tertiary X-radiation must be present, the ionization produced by it, in comparison with that produced by the β-radiations from the screens, must be negligibly small: Barkla and Shearer neglected this effect in their work already cited.

(1) The total ionization in the air in the chamber with carbon ends produced by the incident beam of X-rays is quite small, even at atmospheric pressure, in comparison with that produced by the β-radiations from the screens, and it is presumed that the effect of the tertiary X-radiation from the screen would only be a fraction of that due to the secondary X-radiation.

(2) The linear portions of the curves in fig. 3 (Pl. III.) could not be parallel to each other if the effect of tertiary radiation were appreciable. The greater the effect of tertiary radiation, the more sloping would these lines become.

(3) For a given substance, the thinner the radiating film, the smaller will be the ratio of ionization by tertiary X-rays to that by β-rays in a comparatively thin layer of air. The absorption coefficient of the β-rays in the radiating film itself is extremely great, whilst that of the tertiary X-rays is comparatively small. If a film were prepared of such a thickness as just to absorb the β-rays generated at its face in contact with the graphite, increasing the thickness would not increase the tertiary X-ray emission.

(4) The apparent independence of the shape of the curves in fig. 7 (Pl. III.) on the relative dimensions of the curves in figs. 3 and 4. If the ionization by the tertiary X-rays from the screens were finite, it would, of course, be included in the dimensions of the curves figs. 3 and 4 (Pl. III.).

(5) Sadler's remark that the emission of corpuscular radiation is not necessarily associated with the emission of tertiary X-radiation. This statement will be examined in more detail later.

There are many points of interest in fig. 7 (Pl. III.). First, that ordered curves are obtained for the screens used. Again, if the degree of scattering is the same for all screens, and the phenomenon does not depend upon the actual number of β-particles involved, then, of the screens used, the silver atom emits the greatest proportion of high-speed electrons when excited by silver X-radiation, and similarly for the barium atom when excited by barium X-radiation.

Consider for the moment the effect of silver X-rays on the screens successively. There are two important effects that have to be distinguished. The first is that the K, L, M, etc. radiations are excited in those elements below, and only L, M, etc. radiations are excited in those elements above the atomic weight of silver. The second effect is much more obscure. We do not know the relative numbers of atoms radiating each of the spectral lines. With regard to the latter effect, it appears in photographic images of the X-ray spectra and in ionization work generally that the lines of longer wave-length are the more intense. This can be understood, for the waves of shorter wave-length would soon pass over the atom, and, reacting on an outer ring, be transformed partly into longer waves, and part of the energy would go into the new corpuscles produced. The complete interpretation of the curves in fig. 7 (Pl. III.) would necessitate a knowledge of the relative intensity of the various X-ray spectral lines from a thin film of the element.

There is still another point of very great interest in fig. 7. In 1912, J. C. Chapman, working on homogeneous X-rays, arrived at the formula [*]

$$W_K = \tfrac{1}{2}(W_L - 48),$$

where W_L is the atomic weight of an element emitting L radiation, W_K that of an element emitting K radiation of the same absorbability in aluminium.

This formula yields the result that the K frequency from arsenic equals the L frequency from gold. Moseley expressed practically the same result in a different form [†]. He gave

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[*] Proc. Roy. Soc. A. vol. lxxxvi. p. 447 (1912).

[†] Phil. Mag. ser. 6, vol. xxvii. p. 712 (1914).

the K_α frequency of the elements as

$$v_{K_\alpha}=\left(\frac{1}{1^2}-\frac{1}{2^2}\right)(N-1)^2 n_0,$$

and also

$$v_{L_\alpha}=\left(\frac{1}{2^2}-\frac{1}{3^2}\right)(N-7\cdot4)^2 n_0,$$

where N is the atomic number of the element and n_0 the fundamental frequency of line spectra ($=109720\times$ velocity of light). These formulæ give the K_α frequency of arsenic equals the L_α frequency of lead. The result is approximately the same as Chapman's.

Reference to the curves in fig. 7 (Pl. III.) will show that this result is applicable not only to X-ray emission but analogously to β-ray emission. This is clearer in the dotted curves, the distribution in speeds of the electrons about an arsenic screen being similar to that about a lead screen. There are appearances, too, that the effect is carried on through the range of the elements.

An attempt to detect groups of β-particles possessing sub-speeds and to determine the numerical relations between them. (Figs. 8 and 9, Pl. III.)

The range of the β-particles and the law of absorption that they follow are matters of some obscurity. Whiddington set himself to determine whether the range was proportional to E or E^2, where E is the energy of the particle. W. Wilson's * results give $E^{3/2}$ for the rays from radioactive substances. J. J. Thomson † and Bohr ‡ have both deduced complex formulæ involving E^2 as one of their terms. The difficulty arises from the impossibility of determining the diminution in energy along the complex path of the β-ray. Wilson showed that only a complex distribution of velocities amongst the β-rays from radioactive substances would give rise to an exponential law of absorption, whilst Sadler stated that his slow-moving homogeneous β-rays were absorbed according to an exponential law.

In a previous paper § I deduced tentatively a result based upon an exponential law of absorption. The major speed v_1

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* Proc. Roy. Soc. A. lxxxiv. p. 141 (1910).

† 'Conduction of Electricity through Gases,' p. 381 (1906).

‡ Phil. Mag. ser. 6, vol. xxv. p. 28 (1913).

§ *Loc. cit.*

of the β-rays from gold (produced by the incidence of silver X-rays) was calculated from the expression

$$\tfrac{1}{2}mv_1^2 = {}_{\mathrm{Ag}}(hv)_{\mathrm{K}},$$

and the minimum speed v_2 from

$$\tfrac{1}{2}mv_2^2 = {}_{\mathrm{Ag}}(hv)_{\mathrm{K}} - {}_{\mathrm{Au}}(hv_{\mathrm{L}} + hv_{\mathrm{M}} + \ldots).$$

The ratio of v_1 to v_2 agreed well with the experimental result obtained from the logarithmic absorption coefficients of the two sets of rays and by assuming Whiddington's fourth-power law $\lambda v^4 = \text{constant}$, where λ is the logarithmic absorption coefficient.

I do not think it safe to assume an exponential or any theoretically derived law * in connexion with these experiments. The problem of the range of the sub-groups has therefore been attacked in a totally different manner. Unfortunately, it has to presume the existence of the sub-group, but the result obtained points strongly to the accuracy of this assumption and also to the general agreement with the photo-electric hypothesis of the sub-groups.

Figs. 8 and 9 (Pl. III.).—A close examination of figs. 5 and 6 (Pl. III.) shows that the curves cannot be members of the same family, but that the irregularities might be due to the superposition on curves belonging to the same family of other curves varying among themselves. Figs. 8 and 9 represent the results of an attempt to analyse geometrically the curves of figs. 5 and 6 (Pl. III.).

Suppose these curves, figs. 5 and 6 (Pl. III.), to be represented by $N_x = N_0 f(x)$, where N_0 is the ionization due to the total absorption of the β-ray energy in the air of the chamber, N_x the total ionization in air at 0° C. and 76 cm. pressure between the illuminated screen and the imaginary plane x cm. from the screen. The variation of the density of ionization with distance from the screen is given by $N_0 f'(x)$, and the variation of a logarithmic absorption coefficient of the β-particles with the distance from the screen by $\dfrac{d}{dx}\log_e N_0 f'(x)$.

If the curves of figs. 5 and 6 (Pl. III.) could be represented by $N_x = N_0(1 - e^{-\lambda x})$, this operation would yield $-\lambda$ constant. The value of λ so obtained gives a measure

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* The theoretically derived laws deal only with radiators without a boundary; in practice a boundary is necessary. The condition of absorption in the gas near the boundary of the screen will be different from that obtaining in the central region. Unless a guard-ring method were employed, an empirical formula would be required for each chamber of different dimensions.

of the speed of the particles from Whiddington's law, $\lambda v^4 = \text{constant}$ *.

But the operation

$$\frac{d}{dx}\log_e N_0(1-f(x))$$

performed on the exponential curves would also yield $-\lambda$, meaning that only if the original curves were exponential, plotting the slope of the logarithms of residues such as BC, fig. 5 (Pl. III.), would result in the same constant, represented in figs. 8 and 9 (Pl. III.) by a horizontal straight line.

In this work involving unknown functions, in which the operations have to be performed geometrically, the latter process involving one differentiation is possible, although the physical interpretation of the resulting curves is not so clear as if the former process had been adopted. Figs. 8 and 9 (Pl. III.) show the results of operating by the latter method on the curves of figs. 5 and 6 (Pl. III.). As these curves are roughly logarithmic, it was thought that this shorter method would show up just as accurately as the method involving two differentiations where the function represented by the one part of the curve changed, if at all, into that represented by the remainder of the same curve. Figs. 8 and 9 show, of course, that none of the curves of figs. 5 and 6 are exponential. Portions of the curves seem to be represented by the same function, definite changes taking place at the points marked.

It is not claimed that the resulting lines are true in their smaller detail. A large portion of the end has been omitted in consequence of the impossibility of performing the graphical operation, whilst there can be little accuracy at the beginning because of the difficulty of measuring the small ionization currents. It is claimed that the largest features of the curves are fairly correct. It is presumed that the positions of the minima on these curves give the range of the sub-group of electrons †.

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* Proc. Roy. Soc. A. vol. lxxxvi. p. 375 (1912).

† A little consideration will show that if there were two homogeneous groups of electrons emitted from a given screen and superposed, each following an exponential law of absorption, the curve in this case, drawn in the manner of figs. 8 and 9 (Pl. III.), would fall gradually to a minimum, marking the range of the slower group, after which it would run on horizontally. If, on the other hand, the logarithmic absorption coefficient, as it appears in these experiments, increases with distance from the screen, the resulting curve, drawn in the same manner, would still show a minimum or a decided change in direction approximately at the range of the slower group.

We can employ the expressions for the photo-electric equation given in the introduction in order to calculate what should be the theoretical value of the energies of the electrons of the K, L, or M type sent out from the various screens when each is subjected, first to silver and second to barium X-rays. In the absence of any definite knowledge of the relative intensities of the α, β, γ, etc. lines of the K, L, M X-rays we must confine ourselves throughout to the α lines.

TABLE I.

		Values of v/n_0.			Incidence of Silver (K) X-rays.			Incidence of Barium (K) X-rays.		
Element (S).	At. No.	K_α.	L_α.	M_α.	(*a*).	(*b*).	(*c*).	(*d*).	(*e*).	(*f*).
As	33	783	94	15	*741*	1524	*1618*	*1491*	2274	*2368*
Se	34	835	101	16	*681*	1516	1617	*1431*	2266	2367
Zr	40	1156	150	23	*304*	1460	1610	*1054*	2210	2360
Ag	47	1633	220	36	...	1377	1597	*494*	2127	2347
Sn	50	1871	254	43	...	1336	1590	*215*	2086	2340
Sb	51	1950	265	46	...	1322	1587	*122*	2072	2337
Ba	56	2383	328	60	...	*1245*	1573	...	1995	2323
Au	79	...	717	156	...	*760*	1477	...	*1510*	2227
Pb	82	...	777	173	...	*683*	1460	...	*1433*	2210
Bi	83	...	797	180	...	*656*	1453	...	*1406*	2203

TABLE I.—Columns 3 and 4 give the values of v/n_0 for the K_α and L_α X-ray spectral lines from the various elements S. These values have been obtained by interpolation from the observed values quoted by Vegard * ; the values for the M_α line are interpolated from the formula †

$$v_{M_\alpha}/n_0 = \frac{7}{144}N^2 - 2{\cdot}37N + 40.$$

Column (*a*) is obtained from the expression

$$_{\mathrm{Ag}}(v/n_0)_{K_\alpha} - {}_{\mathrm{S}}\left[(v/n_0)_{K_\alpha} + (v/n_0)_{L_\alpha} + (v/n_0)_{M_\alpha} + \cdots\right],$$

column (*b*) by omitting the first of the terms within the square brackets, and column (*c*) the first and second terms from the square brackets. Columns (*d*), (*e*), and (*f*) in a similar manner when $_{\mathrm{Ba}}(v/n_0)_{K_\alpha}$ is substituted for $_{\mathrm{Ag}}(v/n_0)_{K_\alpha}$ in the first part of the expression.

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* L. Vegard, Phil. Mag. ser. 6, vol. xxxv. pp. 293, 301, 316 (1918).

† *Loc. cit.*

According to the theory outlined in the introduction, the numbers in these six columns, if multiplied by the constant n_0h, give respectively the energies of the K, L, and M electrons emitted by the various screens when they are subjected first to Ag X-rays and then to Ba X-rays, the K electrons being the least rapid and the M electrons the most rapid ones.

This close study by Beatty's "pressure variation" method has led me to the conclusion that it is extremely difficult to distinguish any variation in the maximum range in the β-rays emitted by the various screens all exposed to rays of one type; the practical reason is that the beam of the β-rays is so attenuated, both by the transformation of its energy into ions and by diffusion near its extreme range, and theoretically from the fact that the numbers in column (*c*) or in column (*f*), Table I., are so close together, that the electrons, whose range according to Whiddington's fourth-power law should be proportional to the squares of these numbers, are almost indistinguishable. Or better, the electrons of maximum energy are the peripheral electrons whose potential energy within the parent atom is negligibly small, whatever the atom may be, in comparison with the incident quantum. They emerge with practically the whole of the energy of the incident quantum, and their range will be almost independent of the nature of the parent atom, as Barkla and Shearer have already shown. Those numbers referring to the ranges which these experiments might be able possibly to distinguish from each other or from the mean maximum range are italicised in Table I. These are taken out in Table II. One further point worth mention is that when Silver X-rays (K) type are incident on silver or on elements just above silver in atomic weight, there is no K emission of X-rays, but groups of electrons possessing a large amount of energy. A similar statement is true for Barium (K) X-rays on barium. This accounts for Sadler's observation.

The distances in the two columns marked "range" are obtained from the positions of the minima of the curves in figs. 8 and 9 (Pl. III.) respectively. According to this work, these are approximately the distances traversed in air at 0° C. and 76 cm. pressure by the groups of electrons having ranges which could be distinguished by this method from those of the fastest groups. From Whiddington's fourth-power law, which must be regarded as approximate, it follows that $E/d^{\frac{1}{2}}$ should be constant, where E is the kinetic energy of the electrons of range d in air. Apart from the three results, which are obviously in error, it will be seen that there is a rough agreement with this law for the minor groups.

TABLE II.

	Incidence of Silver (K) X-rays.			Incidence of Barium (K) X-rays.		
Element (S).	Kinetic energy (E) of sub-group $\div n_0 h$.	Range (d) in air at 0° & 760 mm. from fig. 8.	$E/d^{\frac{1}{2}}$. $\div n_0 h$.	Kinetic energy (E) of sub-group $\div n_0 h$.	Range (d) in air at 0° & 760 mm. from fig. 9.	$E/d^{\frac{1}{2}}$. $\div n_0 h$.
		cm.			cm.	
As	741	·228	1550	1491	·595	1930
Se	681	·193	1550	1431	·455	2120
Zr	304	·157	*770?	1054	·438	1590
Ag	...	...	...	494	·140	1320
Sn	...	...	...	215	·128	*600?
Sb	...	...	...	122	·070	*460?
Ba	1245	·333	2160	...	...	...
Au	760	·175	1820	1510	·595	1960
Pb	683	·164	1690	1433	·560	1920
Bi	656	·164	1620	1406	·427	2150
As	1618	·60 †	2090	2368	1·0 †	2368

* These numbers are got from the small ranges where the error must be considerable.

† Obtained from figs. 5 and 6 (Pl. III.). These are the maximum ranges for the incidence of Ag and Ba X-rays respectively.

Some further considerations on the potential energy of the electrons.

There seem to be two distinct points of view: (1) If an electron be removed from a "K" ring an "L" electron falls into its place, an "M" electron into the place of the "L" electron, and so forth. During this process the one atom would yield a series of line spectra. This is the view adopted up to the present, and also that all the atoms of a given substance exposed to X-rays of a given type are not necessarily depleted originally of K electrons—a neighbouring atom may be depleted of an L electron or an M electron. (2) If an electron be removed from a K ring, this ring readjusts itself from one stable state to the next, during which there is an emission of K radiation. It seems to me that according to (1) the potential energy of the K electron should be ($h\nu_K + h\nu_L +$ etc.), and according to (2), only ($h\nu_K$).

Table III. has been prepared according to this latter scheme, and the results show a very much better agreement than is shown by Table II. prepared from the former scheme.

TABLE III.

Screen (S).	Values of v/n_0.		Incidence of Silver (K) X-rays.			Incidence of Barium (K) X-rays.		
	K_α.	L_α.	Kinetic energy (E) of sub-group $\div n_0h$.	Range (d). cm.	$E/d^{\frac{1}{2}}$. $\div n_0h$.	Kinetic energy (E) of sub-group $\div n_0h$.	Range (d). cm.	$E/d^{\frac{1}{2}}$. $\div n_0h$.
As ...	783	...	850	0·228	1780	1600	0·595	2070
Se ...	835	...	798	0·193	1820	1548	0·455	2300
Zr ...	1156	...	477	0·157	1200?	1227	0·438	1860
Ag ...	1633	...	...	...	...	750	0·140	2010
Sn ...	1871	254	1379	...	...	512	0·128	1430
Sb ...	1950	265	1368	...	...	433	0·070	1640
Ba ...	2383	328	1305	0·333	2260	...	...	...
Au ...	...	717	916	0·175	2190	1666	0·595	2160
Pb ...	...	777	856	0·164	2110	1606	0·560	2150
Bi ...	...	797	836	0·164	2070	1586	0·427	2430
As ...	...	...	1618	0·60	2090	2368	1·0	2370

Mean of all the values of $E/d^{\frac{1}{2}} \div n_0h = 2000$.

TABLE III.—Columns 2 and 3 are taken from Table I. Column 4 is obtained from the expression

$${}_{\mathrm{Ag}}(v/n_0)_{K_\alpha} - {}_{\mathrm{S}}(v/n_0)_{K_\alpha \text{ or } L_\alpha},$$

the value for K_α being employed in the latter term for those screens of atomic number lower than that of Silver. Similarly, column 7 is obtained by substituting ${}_{\mathrm{Ba}}(v/n_0)_{K_\alpha}$ for the former term, and employing the values for K_α in the latter term for those screens below the atomic number of Barium, and L_α for those above that of Barium. The values of As at the end are taken from Table I.

It will be seen that the values for the constant agree very much better among themselves than is shown in Table II.

Taking $m = 8{\cdot}8 \times 10^{-28}$ gm., $a = 2{\cdot}1 \times 10^{40}$ in air at 0° C. and 76 cm. pressure (within an accuracy of 15 per cent. according to Whiddington), $n_0 = 3{\cdot}3 \times 10^{15}$ sec.$^{-1}$ (Rydberg's constant), $h = 6{\cdot}6 \times 10^{-27}$ erg. sec. (Planck's constant), we obtain for the constant * observed in Tables II. and III. for $E/n_0 h a^{\frac{1}{2}}$ the value 2930. Bohr †, however, calculates from theoretical considerations that Whiddington's "a" should have the value $1{\cdot}1 \times 10^{40}$ for air at a pressure of 76 cm. and at 15° C. for slowly moving electrons. Using Bohr's value corrected to 0° C., we obtain the result 2160 for the theoretical value of the constant tabulated in Tables II. and III.

It will be seen that the values for the constant in Table III. show better agreement amongst themselves and with the theoretical value than is shown in Table II. The mean value in Table III. is 2000, whilst the separate values in Table II. differ so widely that it does not appear legitimate to take a mean value at all. The direct conclusion is that each atom is associated with the emission of one quantum only, and that entirely different atoms of a homogeneous substance emitting various spectral lines concomitantly are associated with the K, L, or M radiations; but further experiment must decide this important conclusion.

Finally, it must be clearly stated that the method by which these curves have had to be analysed precludes all possibility of an accurate determination of the ranges of the sub-groups as it turns on the discrimination of those points in the curves of figs. 5 and 6 (Pl. III.), where the function represented by the one portion of each curve changes into that represented by the proximate position.

To me, the surest evidence for the existence of the sub-groups rests in figs. 8 and 9 (Pl. III.), roughly conforming as they do to the requirements of the photo-electric equation. The ten curves of fig. 8 (Pl. III.) divide themselves into two distinct sections: those of the one section being below, and those of the other section above the silver curve, the positions of the minima, representing the range of the sub-group, approaching nearer and nearer to the screen as we pass up from As to Se to Zr, starting out again at Ba and approaching once more as we pass up through Au to Pb to Bi. In fig. 9 (Pl. III.) the division into sections is at the Ba, the sub-group having a smaller and smaller speed as we pass up from As to Sb, starting out again at Au and diminishing to Bi. All this is exactly as is required by the photo-electric equation.

Because of the difficulties involved in obtaining experimental curves which will bear differentiation with any

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* $\dfrac{E}{n_0 h a^{\frac{1}{2}}} = \dfrac{\frac{1}{2} m v^2 a^{\frac{1}{2}}}{n_0 h v^2} = \dfrac{\frac{1}{2} m a^{\frac{1}{2}}}{n_0 h}$, from $v^4 = ad$.

† Phil. Mag. ser. 6, vol. xxv. p. 28 (1913).

accuracy, the results of this part of the work must be taken qualitatively rather than quantitatively.

I desire to express my thanks to Professor O. W. Richardson for his kind interest in this work.

The general conclusions may be summarized as follows:—

(1) It is improbable that the maximum speed of ejection of electrons from different substances under the influence of X-rays of definite wave-length is exactly constant independently of the nature of the substance, and dependent only upon the frequency of the incident X-rays. These experiments have shown that throughout the whole range of motion the distribution of electrons depends fundamentally upon the substance from which they are emitted, but the difference, if any, in the maximum velocities is too small to be demonstrable by this method.

(2) The experimental results point to the conclusion that there may be speeds of emission of electrons from an atom (S) when X-rays of frequency v fall upon it given by either

$$\tfrac{1}{2}mv^2 = hv - {}_{\mathrm{S}}(hv_{\mathrm{K}} + hv_{\mathrm{L}} + hv_{\mathrm{M}} + \ldots),$$

each successive speed being given by the removal of a term from the bracket, commencing with the term, or

$$\tfrac{1}{2}mv^2 = hv - {}_{\mathrm{S}}(hv)_{\mathrm{K\ or\ L\ or\ M}},$$

the expression in brackets representing the potential energy of the electron from whichever atomic ring it was ejected, v_{K}, v_{L} etc. being the K, L, etc. frequencies of the X-ray spectral lines of the parent atom (S). The experimental data agree rather better with the latter expression.

(3) There is a type of electronic emission fundamentally associated with each type of X-ray emission. Assuming that it is not a necessary condition that each of the radiating atoms of one substance should be emitting all possible types of spectral lines, from those of highest energy downwards, then the energy of electronic emission of any one of these atoms is complementary to that of its wave-emission; in other words, the "K" electron is the slowest on emergence, the concomitant "L" electron faster, and so on.

IX. *Inertial Frame given by a Hyperbolic Space-time.*

To the Editors of the Philosophical Magazine.

SIRS,—

IT is interesting to modify de Sitter's inertial frame to a hyperbolic form. Referring to Professor Eddington's valuable 'Report' on Relativity, re-write his (51.1) thus

$$ds^2 = -\mathrm{R}^2\{d\Theta^2 + \sinh^2\Theta(d\theta^2 + \sin^2\theta d\phi^2)\} + \cosh^2\Theta dt^2 \,. \,. \quad (51.1)$$

His (51.2) and (51.3) become

$$i \sinh\Theta = \sin\zeta \sin\omega, \quad \tan(t/\mathrm{R}) = \cos\zeta \tan\omega, \quad . \quad . \quad (51.2)$$

$$ds^2 = \mathrm{R}^2(d\omega^2 + \sin^2\omega(d\zeta^2 + \sin^2\zeta(d\theta^2 + \sin^2\theta d\phi^2))), \quad (51.3)$$

the only change here being in the sign of R^2. His (52.2) changes in the same way, or

$$ds^2 = \frac{-dr^2}{(1-\epsilon r^2)^2} - \frac{r^2}{1-\epsilon r^2}(d\theta^2 + \sin^2\theta d\phi^2) + \frac{dt^2}{1-\epsilon r^2}, \quad . \quad (52.2)$$

where $\epsilon = 1/\mathrm{R}^2$. [In the work below I alter the scale of this three-dimensional map, putting $r = \tanh\Theta$, $\epsilon = 1$ in place of $r = \mathrm{R}\tanh\Theta$, $\epsilon = 1/\mathrm{R}^2$.]

The effects of the modification are first that there is no time-barrier; to a fixed observer the converse of what occurs in de Sitter's space will happen—things will seem to move uncannily fast in the distant parts of the universe and distant spiral nebulæ should show a spurious tendency to approach by a spectrum shift towards the violet.

The relative motion of two particles undisturbed by gravitation is excessively simple but, to my mind at least, most extraordinary and unexpected. Let one of the particles be taken as fixed at the origin and let the motion of the other be mapped after the manner of (52.2). It will not move in a straight line but in the ellipse

$$\frac{x^2}{a^2} + \frac{z^2}{b^2} = 1,$$

for which t/R is the eccentric angle. Thus all such undisturbed particles retrace their sinusoidal motions for ever and all have the common period $2\pi\mathrm{R}$ (velocity of light $=1$). Moreover, every observer is the centre of all the elliptic orbits.

In the last paragraph t is coordinate time, but there is no great difference when s the proper time of a fixed observer is used, because tan (s/R) varies (for a given orbit) as $\tan(t/\mathrm{R})$. In fact,

$$\tan(t/\mathrm{R}) = \tan(s/\mathrm{R}) \,.\, \cosh\alpha/\cosh\beta,$$

where $\tanh\alpha = a$, $\tanh\beta = b$; that is, $\mathrm{R}\alpha$ and $\mathrm{R}\beta$ are the true hyperbolic magnitudes of the semi-axes of the ellipse.

These results may be proved thus. Introduce five direction-cosines $l_1, \ldots, l_5$ to indicate the direction of a radius of the spherical surface in five dimensions specifying the space-time continuum of four dimensions, thus

$$\begin{aligned}
l_1 &= \cos\omega &&= \cosh\Theta\cos\frac{t}{\mathrm{R}},\\
l_2 &= \sin\omega\cos\zeta &&= \cosh\Theta\sin\frac{t}{\mathrm{R}},\\
l_3 &= \sin\omega\sin\zeta\cos\theta &&= i\sinh\Theta\cos\theta,\\
l_4 &= \sin\omega\sin\zeta\sin\theta\cos\phi &&= i\sinh\Theta\sin\theta\cos\phi,\\
l_5 &= \sin\omega\sin\zeta\sin\theta\sin\phi &&= i\sinh\Theta\sin\theta\sin\phi.
\end{aligned}$$

Then s being the geodesic between $(l_1, \ldots, l_5)$ and $(l_1' \ldots, l_5')$ we have

$$\cos(s/\mathrm{R}) = l_1 l_1' + \ldots + l_5 l_5'.$$

For the motion of any undisturbed particle relative to an observer fixed at the origin we may take $\phi = 0$ and also

$$\sinh\Theta\sin\theta = a\cosh\Theta\sin(t/\mathrm{R}),$$
$$\sinh\Theta\cos\theta = b\cosh\Theta\cos(t/\mathrm{R}).$$

These at once give

$$z = a\sin(t/\mathrm{R}), \quad z = b\cos(t/\mathrm{R}),$$
$$\cos(s/\mathrm{R}) = l_1 l_1' + l_3 l_3' = \cosh\Theta\cos(t/\mathrm{R}) \,.\, \surd(1-b^2),$$

where l_1' and l_3' are the initial values of l_1 and l_3. The three conditions are the equivalents of the equations

$$l_5 = 0, \;\; l_4 = ial_2, \;\; l_3 = ibl_1,$$

all three being linear and homogeneous in $l_1, l_2, \ldots l_5$. It will be seen that the a and b are the same as the a and b above, but t/R and s/R (s being taken zero when t is zero) are the complements of s/R and t/R previously used.

To see that the above results are really extraordinary, consider how the same moving particle will appear to two different observers who are situated on a radius vector of the ellipse but on opposite sides of the curve. P and Q are two observers a few metres or kilometres apart and an undisturbed particle passes between them in a direction perpendicular to PQ. They remain fixed at their observation posts watching the particle, for many million years, and at last face one another, and P reports to Q that the particle is now for the first time directly behind him. It seems difficult to believe or to disbelieve that what Q has to report to P at the same instant is that it is directly behind himself also.

The relative rest of Q with regard to P is a case of constrained motion, for if Q has the minutest velocity relative to P, and is undisturbed, he will rotate (in the belief of P) about P in the same period as does the particle. In such case our paradox disappears, but it still remains if we suppose that a constraint bringing about relative rest is so much as possible.

Yours, &c.,
ALEX. M^cAULAY.

University of Tasmania,
June 1st, 1920.

X. *Newton-Einstein Planetary Orbit.*

By Sir GEORGE GREENHILL *.

THE modification in the Problem of Two Bodies of the elliptic planetary orbit under ordinary Newtonian gravity to the Sun, due to the additional term introduced by Einstein into the attraction, varying as the square of the angular velocity round the Sun, has engaged the attention of many writers in the recent numbers of the Philosophical Magazine.

The result may be stated as a change of Newton's ellipse into Einstein's orbit, from

$$\frac{1}{\mathrm{SP}}=\frac{\cos^2\frac{1}{2}\theta}{\mathrm{SA}}+\frac{\sin^2\frac{1}{2}\theta}{\mathrm{SA}'},\quad\text{into}\quad\frac{\mathrm{cn}^2\,\frac{1}{2}p\theta}{\mathrm{SA}}+\frac{\mathrm{sn}^2\,\frac{1}{2}p\theta}{\mathrm{SA}'},\qquad(1)$$

and so provides an Elliptic Function application to a Central Orbit. It is proposed here to standardize this problem in the Dynamics of a Particle, and to examine closely the units employed.

According to Einstein, a term he denotes by $3m\omega^2$ must be added to Newton's term μu^2 to give the total central acceleration to the Sun; here ω denotes the angular velocity of the planet, and in a central orbit $\omega=hu^2$, with u the

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* Communicated by the Author.

inverse radius vector ; so that we may write the central acceleration

$$P=\mu u^2+3m\omega^2=\frac{\mu}{h}\omega+3m\omega^2=\mu u^2+3mh^2u^4. \quad . \quad (2)$$

In his treatment of the Problem of Two Bodies in Matter and Motion, Maxwell expresses the Newtonian attraction as proportional to ω. This follows from the property of the Hodograph ; turned through a right angle, the hodograph of an elliptic orbit is a circle with centre at S. the pole of the velocity vector HU being at the other focus H.

Thus the velocity of P, perpendicular and proportional to HU, can be resolved into two constant components ; one perpendicular to SP and proportional to SU, and the other perpendicular to the major axis and proportional to SH.

A steamer P for instance, circling past a lightship S in a tideway, keeping the light always abeam, will describe an elliptic orbit over the ground in true planetary style, the minor axis being in the direction of the tidal current.

Crossing the road in the same way in front of the headlight of an advancing tramcar, the path relative to the light S will be described as a planetary orbit.

The velocity of the velocity vector being at right angles to SPU, the force and acceleration is directed to S, and varies as ω, the angular velocity of SPU.

Then in Maxwell's notation, $\mu/h=h/l$ is a velocity, V, the velocity at the point P where $p=l$, $p'=a$, $pp'=al=b^2$, where p, p' denote the perpendiculars SY, HZ from S, H on a tangent, and l the semi-latus rectum.

At this point P, SP is parallel to CZ, and HZ=CZ ; also if X is the foot of the directrix to the focus S, and the tangent at P cuts the major axis in T, ST=2SX, HT=2CX ; and with T also the periodic time of the ellipse

$$VT=\frac{hT}{l}=\frac{2\pi ab}{l}=2\pi\frac{a^2}{b}, \quad . \quad . \quad . \quad . \quad (3)$$

circumference of the circle of radius a^2/b, the radius of curvature at the end of the minor axis.

Then S denoting the mass of the Sun, P of the planet in grammes (g), and G the gravitation constant, $G=666\times10^{-10}$, in C.G.S. units ; and a denoting the mean distance in centimetres (cm),

$$G(S+P)=\mu=n^2a^3=\frac{4\pi^2a^3}{T^2}, \quad . \quad . \quad . \quad . \quad (4)$$

the expression of Kepler's Law III, and implying that G is a constant for all matter throughout the Universe.

In these C.G.S. units, Einstein's m must denote a length, in centimetres. It is mysterious then that Einstein is quoted as calling m the mass of the Sun, as if a mass could be measured in centimetres, by a metre rule, and not in grammes; some mysterious unexplained astronomical units must have been employed, and writers should enlighten us on this point of the theory.

The differential equation in Particle Dynamics of the Central Orbit, analytical expression of the normal component of the acceleration, becomes when Einstein's term is added,

$$\frac{d^2u}{d\theta^2}+u=\frac{\mathrm{P}}{h^2u^2}=\frac{\mu}{h^2}+3mu^2, \quad . \quad . \quad . \quad . \qquad (5)$$

and integrating, with $h^2=\mu l$,

$$\left(\frac{du}{d\theta}\right)^2+u^2=\mathrm{C}+\frac{2u}{l}+2mu^3, \quad . \quad . \quad . \quad . \quad . \qquad (6)$$

$$\left(\frac{du}{d\theta}\right)^2=2m(u-\alpha\,.\,u-\beta\,.\,u-\gamma), \quad . \quad . \qquad (7)$$

and α, β, γ are the roots of a cubic in u, where

$$\alpha+\beta+\gamma=\frac{1}{2m},\quad \beta\gamma+\gamma\alpha+\alpha\beta=\frac{1}{lm},\quad \alpha\beta\gamma=-\frac{\mathrm{C}}{2m}. \qquad (8)$$

In a closed orbit, α, β, γ are positive; and taken in the sequence so that

$$\alpha>\beta>u>\gamma,$$

$$\left.\begin{aligned}\tfrac{1}{2}pd\theta&=\frac{\sqrt{(\alpha-\gamma)}du}{\sqrt{(4\,.\,\alpha-u\,.\,\beta-u\,.\,u-\gamma)}},\\ p^2=2m(\alpha-\gamma)&=\frac{\alpha-\gamma}{\alpha+\beta+\gamma},\end{aligned}\right\} \quad . \quad . \qquad (9)$$

bringing in an elliptic integral, which is to be reversed in Abel's manner; and then, expressed by the inverse elliptic functions,

$$\left.\begin{aligned}&\tfrac{1}{2}p\theta=\mathrm{sn}^{-1}\sqrt{\frac{u-\gamma}{\beta-\gamma}}=\mathrm{cn}^{-1}\sqrt{\frac{\beta-u}{\beta-\gamma}}=\mathrm{dn}^{-1}\sqrt{\frac{\alpha-u}{\alpha-\gamma}}\\ &k^2=\frac{\beta-\gamma}{\alpha-\gamma};\end{aligned}\right\} \qquad (10)$$

equivalent in the direct notation, as in (1), to

$$u=\gamma\,\mathrm{cn}^2\,\tfrac{1}{2}p\theta+\beta\,\mathrm{sn}^2\,\tfrac{1}{2}p\theta. \quad . \quad . \quad . \quad . \quad . \qquad (11)$$

In the degenerate case of pure Newtonian gravity, $m=0$; and then $\alpha=\infty$, β and γ finite, $p=0$, $k=0$; and the orbit reduces, as in (1), to

$$u=\gamma\cos^2\tfrac{1}{2}\theta+\beta\sin^2\tfrac{1}{2}\theta. \quad . \quad . \quad . \quad . \quad . \qquad (12)$$

The apsidal angle ϖ in (1) is changed from π to $2\mathrm{K}/p$; and the importance of Einstein's theory lies in the application to Astronomy, where the term m is considered small, in the endeavour to account for the anomalous amount of the advance of the apse in the orbit of Mercury.

In that case where m is small, α is very large compared with β, γ; and we take

$$\left.\begin{aligned} &m\rightarrow 0, \quad 2m\alpha=1-2m(\beta+\gamma)\rightarrow 1,\\ &p^2\rightarrow 1-\frac{\gamma}{\alpha}, \quad k^2\rightarrow\frac{\beta-\gamma}{\alpha}, \quad \frac{\mathrm{K}}{\frac{1}{2}\pi}\rightarrow 1+\tfrac{1}{4}k^2, \end{aligned}\right\} \quad (13)$$

$$\left.\begin{aligned} &\frac{2\mathrm{K}}{\pi p}\rightarrow\frac{1}{p}(1+\tfrac{1}{4}k^2)\rightarrow\frac{1}{\sqrt{\left(1-\frac{\gamma}{\alpha}\right)}}\left(1+\frac{\beta-\gamma}{4\alpha}\right)\rightarrow 1+\frac{\beta+\gamma}{4\alpha},\\ &\qquad\qquad \frac{\Delta\varpi}{\pi}=\frac{2\mathrm{K}}{\pi p}-1\rightarrow\frac{\beta+\gamma}{4\alpha}; \end{aligned}\right\} (14)$$

$$\frac{\beta+\gamma}{4\alpha}=\frac{1}{4lm\alpha^2}-\frac{\beta\gamma}{4\alpha^2}\rightarrow\frac{1}{4lm\alpha^2}\rightarrow\frac{m}{l}, \quad \frac{\Delta\varpi}{\pi}\rightarrow\frac{m}{l}, \quad . \qquad (15)$$

making the advance of the apse in one revolution 360° (m/l).

The addition of a term varying as u^3 or $\omega^{\frac{3}{2}}$ to the central attraction would cause the differential equation of Newton's orbit to change from

$$\frac{d^2u}{d\theta^2}+u=\frac{\mu}{l}, \qquad \text{into} \quad \frac{\mu}{l}+nu, \quad . \quad . \quad . \quad . \quad . \quad . \qquad (16)$$

$$\left(\frac{du}{d\theta}\right)^2=\mathrm{C}+\frac{2\mu u}{l}-u^2, \quad \text{into} \quad \mathrm{C}+\frac{2\mu}{l}-(1-n)u^2, \quad . \qquad (17)$$

$$\left(\frac{du}{d\theta}\right)^2=\beta-u\,.\,u-\gamma, \qquad \text{into} \quad (1-n)(\beta-u\,.\,u-\gamma), \qquad (18)$$

$$\left.\begin{aligned} &u=\gamma\cos^2\tfrac{1}{2}\theta+\beta\sin^2\tfrac{1}{2}\theta, \quad \text{into}\\ &\qquad\qquad \gamma\cos^2\tfrac{1}{2}q\theta+\beta\sin^2\tfrac{1}{2}q\theta, \quad q=\sqrt{(1-n)}, \end{aligned}\right\} (19)$$

the character of the integral and of the equation of the orbit is not altered essentially, except by an expansion or contraction, in a fan-like manner, of the vectorial angle θ.

Then in G. C. Darwin's investigation in the Phil. Mag. May 1920, the influence of both additional terms together is considered, by taking physical quantities α and β, making

$$P=\mu u^2+\beta h^2 u^3+\tfrac{3}{2}\alpha h^2 u^4, \quad . \quad . \quad . \quad . \quad (20)$$

so that his α is our $2m$; and then, changing his β into n,

$$\frac{d^2u}{d\theta^2}+u=3mu^2+nu+\frac{1}{l}, \quad . \quad . \quad . \quad . \quad (21)$$

$$\left(\frac{du}{d\theta}\right)^2+u^2=2mu^3+nu^2+\frac{2u}{l}+C, \quad . \quad . \quad (22)$$

$$\left(\frac{du}{d\theta}\right)^2=2m(u-\alpha\,.\,u-\beta\,.\,u-\gamma), \quad . \quad (23)$$

making the orbit, as before, in (11),

$$u=\gamma \operatorname{cn}^2 \tfrac{1}{2}p\theta+\beta \operatorname{sn}^2 \tfrac{1}{2}p\theta, \quad . \quad . \quad . \quad . \quad (24)$$

but with

$$\alpha+\beta+\gamma=\frac{1-n}{2m}, \quad \beta\gamma+\gamma\alpha+\alpha\beta=\frac{1}{lm}. \quad . \quad (25)$$

Here with m small, but n unrestricted,

$$m\rightarrow 0, \quad 2m\alpha=1-n-2m(\beta+\gamma)\rightarrow 1-n,$$

$$p^2=2m(\alpha-\gamma)\rightarrow\left(1-\frac{\gamma}{\alpha}\right)(1-n), \quad \frac{\beta+\gamma}{4\alpha}\rightarrow\frac{1}{4lm\alpha^2}\rightarrow\frac{m}{l(1-n)^2}, \quad . \quad . \quad . \quad (26)$$

$$\frac{\varpi}{\pi}=\frac{2K}{\pi p}\rightarrow\frac{1}{\sqrt{(1-n)}}\left(1+\frac{\beta+\gamma}{4a}\right)\rightarrow\frac{1}{\sqrt{(1-n)}}+\frac{m}{l(1-n)^{\frac{5}{2}}}, \quad . \quad . \quad . \quad (27)$$

reducing to the (15) above for $n=0$.

Add a further term to P, $2\delta h^2 u^5$, varying as $\omega^{\frac{5}{2}}$,

$$\frac{d^2u}{d\theta^2}+u=2\delta u^3+3mu^2+nu+\frac{1}{l}, \quad . \quad . \quad . \quad (28)$$

$$\left(\frac{du}{d\theta}\right)^2+u^2=\delta u^4+2mu^3+nu^2+\frac{2u}{l}+C, \quad . \quad (29)$$

$$\left(\frac{du}{d\theta}\right)^2=\delta(u,1)^4; \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (30)$$

and further progress in the elliptic integral requires some knowledge of the factors of the quartic.

Then there is the memoir of W. J. Harrison in the Proc. Cambridge Phil. Society (C. P. S.) Nov. 1919, on the pressure in a radiating current of viscous liquid, and a further Note on Vibration, C. P. S. May 1920, where the complete solution requires the elliptic function, as explained above. These are the vibrations in the ideal case where terms varying as x^2. x^3 are added to the leading term n^2x of simple rectilinear vibration.

But these vibrations can be visualized and studied in reality in the various projections of pendulum motion, in the extension to the more general case where the axle is fixed on a whirling arm, as in Watt's governor, the gyrocompass, or the Gilbert barogyroscope; and a complete solution has been given in the Report to the Aeronautical Committee on Gyroscopic Theory, 1914, Chapter VIII; while the verification by differentiation and integration has provided exercises in 'Applications of the Elliptic Function,' 1892.

Stroboscopic vision will convert these rectilinear vibrations into the central orbits considered above, but with $\theta = mt$, so that the shape of the curve is seen, but not the velocity in the orbit.

Staple Inn.
Nov. 5, 1920.

XI. *The Genesis of the Law of Error*[*].
By Prof. F. Y. Edgeworth, *F.B.A.*[†]

PROFESSOR SAMPSON'S courteous reference to some remarks of mine upon this subject deserved an earlier acknowledgment. I would have sooner expressed appreciation of his criticisms, but that I wished to finish the article which has appeared in the September number of this Magazine. I hoped to be able to speak more to the purpose about the law of error[‡] after considering its application to a leading branch of Physical Science.

The points at which new light may be expected from Prof. Sampson's original reflections may be introduced by a

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[*] Communicated by the Author.

[†] Referring to the article on this subject by Prof. R. A. Sampson in the Philosophical Magazine for October 1919.

[‡] The term "error" designates according to the context either error proper, *i. e.* deviations of an observation from the physical quantity which is observed, or generally deviation from an average. The law of error (or the normal law) designates the function $(1/c\sqrt{\pi})\exp - x^2/c^2$ considered as approximately representing the frequency-distribution of an aggregate; the first approximation, further approximations being furnished by a series descending in powers of $1/\sqrt{n}$, where n is the number of independent items in the aggregate.

statement of propositions which I should regard as well established, but for his seeming dissent.

I. A. (1)* The origin of the law of error is to be sought in the theory of Probabilities. (2) It is obeyed by random aggregates of variable constituents. Games of chance afford the simplest examples. (3) Thus if, a batch of n balls having been taken at random from an immense medley of black and white balls, the number of white balls in the batch is recorded, and the operation is repeated many times, the series of numbers presented by the record will conform to the law-of-error; more or less approximately according as n is larger, and as the proportion of white to black in the medley is nearer to equality. This correspondence between the binomial series and the law of error was happily employed by Quetelet to illustrate the fulfilment of the law in various kingdoms of nature. In games of chance and organic nature there does not often arise a difficulty which the exact sciences may present; namely, that (4) the aggregation of constituents which is treated as fortuitous is known to be determinate. The most familiar instance is the sequence of decimal places in the evaluation of a natural constant, such as π †. Each figure is what it is in virtue of determinate law; and yet the *ensemble* presents the arrangement which is an outcome of chance. A more important instance is furnished by the distribution of velocities in a molecular chaos; they obey at once the laws of Dynamics and the laws of Probabilities. The Philosophers ‡ have exercised themselves about this paradox; but it remains a mystery §. One rule emerges that (5) we should not seek for law (other than statistical uniformity) as the explanation of chance. We must not do as the fatuous gambler who scrutinizes the records of the roulette table in the hope of eliciting a rule to guide his future stakes. In the weighty words of De Morgan: "No *primary* considerations connected with the subject of Probability can or ought to be received if they depend upon the results of a complicated mathematical analysis ‖. To use Dr. Venn's ¶ phraseology,

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[*] The passages headed by bracketed numerals in Section I. are referred to in Section II.

[†] *Cp.* Venn, 'Logic of Chance,' p. 112 *et seq.*, ed. 3, and Nixon, Journal of the Royal Statistical Society, vol. lxxvi. p. 702 (1913).

[‡] Renouvier, Venn, von Kries, etc.

[§] *Cp.* Poincaré, "There is here something mysterious inaccessible to the mathematician."

[‖] *Encyclopædia Metropolitana,* vol. ii.: article on the "Theory of Probability," S. 1.

[¶] 'Empirical Logic.'

uniformities of statistics cannot be explained by other kinds of uniformity.

Of course, other kinds of uniformity, propositions other than those proper to Probabilities, may be employed to ascertain whether and how far the conditions necessary for the genesis of the law-of-error are fulfilled in any concrete case. The gambler may reasonably inquire whether a roulette table is faked, or the dice loaded. The student of Probabilities may properly expect that a long set of figures forming the period of a recurring decimal will approximately conform to the law-of-error; and that the same will be true of pairs formed at random from these figures. And yet he may learn from theory or observation that pairs selected in a particular manner from certain periods—*e. g.* by adding the first place to the $(n+1)$th, the second place to the $(n+2)$th, and so on in a period of $2n$ decimals—will not fulfil the conditions necessary for the genesis of the law of error *. The principal conditions are three. (6) First, the number of constituents must be large—how large depends on the degree in which the other conditions are fulfilled. Secondly, the variation of the constituents must be *independent*, a condition which is, perhaps, never perfectly filled in concrete nature, not even in games of chance †. Thirdly, the constituents are to be aggregated by simple summation. This condition is sometimes perfectly fulfilled, as in games of chance; but often only approximately, as where the compound is some *function* other than the sum of the constituents, but such that, when expanded in ascending powers of the constituents, it is equateable approximately to their sum. (7) According as these conditions are more or less perfectly satisfied the law-of-error is more or less perfectly fulfilled. So the path of a projectile is more nearly a parabola the less dense the resisting medium ‡.

There is a *fourth* condition which is, indeed, essential, but hardly requires to be stated since it is always fulfilled in concrete nature. (8) The *standard deviation* and generally the mean powers of deviation for each and all of the constituent variables must not be *infinite*. The definition does not exclude the possibility of an infinite deviation provided that it is sufficiently rare. Thus a group fulfilling the law-of-error is qualified to be a constituent in the genesis

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* The property is illustrated by the recurring decimal representing 1/19, viz. ·052631578. 947368421. 1/1861 furnishes a better instance. See "Law of Error," Transactions of Cambridge Philosophical Society, p. 129 (1905).

† See Venn, 'Logic of Chance,' ed. 3, p. 77 *et seq.*

‡ *Cp.* Phil. Mag. vol. xxxv. p. 422 (1918).

of that law, and yet a member of that group may conceivably deviate to infinity. Moreover (9) "infinite" in the definition must be understood literally, not simply as an equivalent to "immense." Thus, considering any frequency-function with infinite mean powers of deviation, *e. g.* $1/\pi(1+x^2)$, if we have to do only with values of x which, however large, are finite, we can always take n (the number of the constituents) so large that the aggregate, or average, will fulfil the law-of-error. (10) As to the functions which extend to infinity without violating the condition (8), it may be doubted whether outside the law of error such frequency-groups have any concrete existence. Even for the law-of-error considered as resulting from a finite number of finite deviations the infinite deviation is a *limit* never attained, probably not even by the velocities in a molecular medley [*].

B. A right understanding of the theory is promoted, or, at least, evidenced, by a correct appreciation of the writers to whom it is due. The foundation of the theory is to be sought in Laplace's classical treatise [†]. (11) The first section which he devotes to the subject exhibits the essential features of the law-of-error: namely, that, when the three conditions above specified are present, "the peculiarities of the (constituent) functions efface themselves in the final result" [‡]. True, he does not advert to the fourth condition. But it was not his wont to dilate upon conditions which might ordinarily be taken for granted. Thus he repeatedly assumes that a function may be expanded in powers of the variable, and powers above the first neglected. In the next section, for instance, he thus obtains a linear equation for the *correction* of an observation. Of course he knew that there might occur singular points. A *caveat* was the less necessary in the present case, because in no application of the law which he contemplated could an infinite deviation occur. He could not suppose—nor forsee that anyone would suppose—a really infinite error-of-observation.

It is, perhaps, remarkable that Laplace did not extend the law which he demonstrated to account for the prevalence of the law throughout Nature, as shown by Quetelet. (12) It is still more remarkable that Laplace should not have applied the law to demonstrate that the normal error-function would " emerge from the mere superposition of the definite numbers

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[*] That the sum-total energy is finite is not *conclusive* proof..

[†] *Théorie Analytique des Probabilités*, Liv. II. cap. iv.

[‡] The words are those used by Professor Sampson in *denying* this conclusion (Congress of Mathematicians 1912, Proceedings, p. 168, par. 1).

of small errors which had *arbitrary laws of frequency of their own*" [*]. "Laplace does not seem to have regarded an error in this light" [†], as Glaisher states correctly: "Nowhere does he assume that if one observation only is made its law of facility is $e^{-h^2x^2}$." [‡] Laplace evidently did not contemplate this deduction when in the important section dealing with the application of inverse probability to errors of observation he speaks of "the complete ignorance in which we are with respect to law of errors for each observation" [§].

(13) Laplace's proof of the law-of-error derives strong support from Poincaré's theorem that the frequency-function pertaining to the sum of numerous variable elements is such that all its mean powers of deviation are approximately equal to those of the normal error-function [||].

Further confirmation is obtained from other proofs. To some of them, in particular the proof by way of partial differential equations, it may be objected that they assume the existence of a final state, an ultimate law of frequency, to which the continued superposition of elements must tend [¶]. But (14) the assumption is surely not very arbitrary. It is of a piece with the assumption countenanced by Tait [**] that the velocities in a molecular medley tend to a final distribution.

It is pleasant to believe that these views are not so much

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[*] The aptly-worded *predicate* of this proposition is borrowed from Prof. Sampson (Phil. Mag. p. 349); but not the proposition itself.

[†] Memoirs of the Royal Astronomical Society, vol. xxxix. p. 106 (1872). *Cp.* Monthly Notices of the Society, vol. xxxiii. p. 397: "he (Laplace) did not himself so apply it (the law)."

[‡] Memoirs, *loc. cit.* p. 108.

[§] *Théorie Analytique*, Liv. II. cap. iv. art. 23.

[||] See Phil. Mag. vol. xxxv. p. 426 (1918). This proof is confirmed by the solution of Stieltfes' *problème des moments*, as presented by Prof. G. H. Hardy in the 'Messenger of Mathematics,' vol. xlvi. (1917) p. 175; regard being had to the rapidity with which the error-function tends to zero (as the variable increases).

[¶] Article on "Probability," *Encyclopædia Britannica*, 9th edition, p. 393. The objection does not apply to the proofs given by Morgan Crofton, Phil. Trans. (1870).

[**] "Everyone therefore who considers the subject from either of these points of new (ordinary) statistics or the theory of probabilities, must come to the conclusion that continued collisions among our set of elastic spheres will, *provided they are all equal*, produce a state of things in which the percentage of the whole which have at each moment any distinctive property must (after *many* collisions) tend towards a definite numerical value." The proposition is presently extended to sets of spheres "no one of which is overwhelmingly more numerous than another, nor in a hopeless minority as regards the sum of others."—"Kinetic Theory of Gases." Royal Society of Edinburgh, vol. xxxiii. p. 225.

at variance with those of Professor Sampson as seemed before his conciliatory explanations. There was, however, and, indeed, still is, an *appearance* of difference which may excuse my previous expressions of dissent. It may be of more than personal interest to exhibit the seeming difference.

II. A. Wishing to "discard as far as possible the language of probabilities" (Congress [*], p. 163), Professor Sampson runs the risk of obscuring the conceptions proper to the science (1) [†]. I do not recognize the doctrines of Probabilities in the following description of the Laplace-Poisson theory of error :— "It is a theorem of convergence and must be judged so. It is either true or false. Such phrases as '*à très peu près*' are not in the first place admissible. If they are required to help the demonstration out, that means the theorem is false ; for Poisson in particular seems to have held that no conditions were necessary to impose upon the frequencies of the elementary contributing errors '*la fonction aura telle forme que l'on voudra*'" [‡]. But if the view above presented—(6) and (7)—is correct, the conditions for the genesis of the law-of-error are, in general, only fulfilled approximately ; the phrase of Laplace "*à très peu très*" is generally, not to say, universally [§] appropriate. On the same view the words of Poisson are perfectly correct; convergence towards the normal law will set in, whatever the frequency-function of the contributory elements—with one exception, indeed, formed by the fourth condition (8), an exception which proves the rule as it has no concrete existence (9) and (10).

Discarding the language of Probabilities, Prof. Sampson writes (Congress, p. 168) :—"The term accidental error has come to carry with it an undefined suggestion of a peculiar quality, but there seems no reason to treat an error otherwise than as a disregarded unknown" . . . It appears to me that in accounting for the presence of the law of error we cannot discard the peculiar quality of accident or Probability (1) and (2). It is the attempt to get rid of that peculiarity which I described as Professor Sampson's "peculiar notion

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[*] The reference is to Professor Sampson's paper on the "Law of Distribution of Errors" read before the 5th Congress of Mathematicians (1912) at a joint meeting of the sections on Astronomy and Statistics ; published in the 'Proceedings of the Congress,' vol. ii. p. 163.

[†] The numerals in Section II. refer to passages numbered in Section I.

[‡] Phil. Mag. p. 349. The references hereinafter thus made are to Prof. Sampson's article on the "Genesis of the Law of Error" in the Philosophical Magazine for October 1918, vol. xxxvi.

[§] Even in games of chance perfect *independence* is probably only an ideal (6) ; also the number of trials being finite, there can only be an approximation, *e. g.* (3) to the normal law of frequency.

of the nature of an error of observation" (Phil. Mag. p. 347). Whether we are arguing that the *average* of errors-of-observation, or that the errors themselves (as sums of contributory elements), are normally distributed, in both cases, the peculiar character of Probability must be postulated.

In the sequel of the passage just quoted (Congress, p. 168) Professor Sampson's treatment of the disregarded unknowns appears somewhat peculiar. My interpretation of his formula $y = \sin t$ for the frequency-function of an observation was, I think, very natural (Phil. Mag. vol. xxxv. p. 431). But, of course, I accept the explanation which he has given (Phil. Mag. p. 351): "All values within the limits $\pm a$ would be equally likely, and that is what the frequency-graph described would imply" *. Yet I fail to see in what respect this conception has any advantage over Laplace's theory. Laplace also in the very first section of his path-breaking chapter (11) supposes that all values between the limits $+a$ and $-a$ would be equally likely. Laplace believed that the value of each observation occurred according to determinate laws †; but he did not profess to know what these laws were.

I am encouraged to surmise that nothing very paradoxical is intended by the passage just quoted illustrating one of the points for which novelty is claimed (Phil. Mag. p. 350), when I consider the passages relating to another point. Here, too, there occur difficulties of interpretation. They are largely due, I dare say, to the obtuseness of the interpreter. Yet the author has candidly taken to himself blame for having thrown out collaterally one misleading statement (Congress, p. 170; Phil. Mag. p. 347). It is not the only puzzling statement in the context. What are we to think of the two statements with respect to the reproduction of form (by the superposition of two functions of the same form) made on the same page? (Congress, p. 170):—"So far as I can determine it, (reproduction) belongs with any generality only to the functions $\exp(x)$ and $\exp(x^2)$" (*sic*); and a few lines below, "it will suffice to show that $\exp(-h^2x^2)$ and, of course, also $(\exp - h^2x^2)\cos(kx+\gamma)$ reproduce themselves."

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* The explanation continues (*loc. cit.*): "If then we suppose that errors are not of mysterious character, *sui generis*, but are simply the mass of numberless neglected disturbances, each according to regular law and order of its own, it is seen that we obtain the approximation to Gauss's law which is necessary to begin with by the operation of neglecting the circumstances and order of their origin and scheduling merely in sequence of magnitude the number of times that each particular value occurs."

† The determinism of Laplace has often been noticed, especially by Renouvier, *Essais de Critique générale.*

Both these statements cannot be true. But they may both have been thrown out collaterally. For, firstly, the property belongs, with perfect generality, to any number of functions, all the members of the family

$$y=\int_{-\infty}^{\infty} e^{-\alpha t}\cos\alpha t d\alpha\ ^*;$$

and, secondly, it does not seem to belong in any obvious, or useful, sense of the term to $\exp(-h^2x^2)\cos(kx+\gamma)$. It is true that the superposition of the last written functions results in convergence to the normal law. Of course it does, if they comply with the fourth condition. And they comply with that condition, since $\exp(-h^2x^2)$ does; and every element of the integral

$$\int_{-\infty}^{\infty} x^t\exp(-h^2x^2)\cos(kx+\gamma)dx$$

is *less* than the corresponding element of the integral

$$\int_{-\infty}^{\infty} x^t\exp(-h^2x^2)dx.$$

Accordingly no great addition is made to our knowledge when the author concludes : "If we go on piling error upon error, provided each has the fluctuating character indicated above, we shall, as a limit, converge to the pure law of Gauss" (Phil. Mag. p. 350). But why drag in the "fluctuating character"? The proposition remains true when the proviso is omitted. It is true, for instance (for the reason just given), of the function $(\exp-h^2x^2)/(1+x^2)$.

No doubt there is some interest in verifying the fact of convergence by actual integration. But this interest is not very great when the deviations, extending to infinity, are such as never could occur in concrete nature. The verifications given by Dr. Burton in the Philosophical Magazine for December 1889 are much more interesting.

The author describes his contribution "as a view by which we can *see the law coming into existence*, which I submit the other forms of proof, one and all, fail to supply." Is not the desiderated view supplied by regarding the frequency-function pertaining to an aggregate as of the form $e^{-h^2x^2}(1+R)$, where R is the continuation of Poisson's expansion in ascending powers of $1/\sqrt{n}$; and observing, say, on the lines of Morgan Crofton's method, that when a new observation is taken in, n is changed into $(n+1)$?† Is the

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* See Camb. Phil. Trans. vol. xiv. pp. 142, 158 (1885).

† Todhunter, "History of Probabilities," Art. 1002; Camb. Phil. Trans. vol. xx. p. 47 (1905).

difficulty of apprehending the relations of chance and law (4) and (5) lightened by the introduction of the "holomorphic function" (Phil. Mag. p. 350), or other novel point?

B. So far, perhaps, I may say with Professor Sampson, "the points of difference" between us appear "unsubstantial" (Phil. Mag. p. 347). But with respect to his main criticism of Laplace's proof, I cannot regard the difference as unsubstantial. I cannot retract, but would rather emphasize what I have written on the point (Phil. Mag. vol. xxxv. p. 425 (1918)). Professor Sampson argues: "The conclusion is, therefore, unwarranted, and there is no proof at all that peculiarities of the functions efface themselves in the final result. I do not believe that the theorem is true" (Congress, p. 167). I can only agree with these statements in the case where the fourth condition is not complied with; that is, in a case which never occurs in fact. I cannot retract the statement that the attack on the proof given by Poisson and Laplace strikes at all the applications of the law (Phil. Mag. May 1918, p. 423); for the objection to the reasoning of Laplace and Poisson can only be admitted on a supposition which would be fatal to every other proof of the law of error [*]. The negative of that supposition, the axiomatic fourth condition (8), forms the corner-stone of the edifice, whether built according to the design of Laplace and Poisson or some more modern plan. That fundamental support being rejected, the whole edifice would collapse. All the applications of the theory—the Method of Least Squares, both as a good method where the frequency-function for the errors of observation is unknown, and as the best method [†] when the frequency-function for the errors-of-observation is believed by inference from the law-of-error to be normal [‡]; the representation of statistics by a normal curve (or surface) in the manner of Quetelet, or more exactly by the employment of the second approximation given by Poisson [§]; the test of Sampling as practised in social investigations by a Bowley Kïar [‖]; the splendid and useful results deduced from normal

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[*] The law being defined as above. Note ‡ to p. 148.

[†] See as to this distinction, Journal of the Statistical Society, vol. 71, pp. 509, 393 (1908), and references there given. Note too (*loc. cit.* p. 499) that the law-of-error is required to determine the precision of a result obtained by *inverse probability* from a set of observations for which the error-function is known, but not normal.

[‡] Above, p. 149.

[§] As employed with success by Bowley, 'Elements of Statistics,' ed. 2; and Journal of the Royal Statistical Society *passim*.

[‖] Referred to in the Journal of the Royal Statistical Society, vol. 76, p. 192 (1912–13).

correlation by Pearson and other mathematical statisticians *—all these and many other applications of the law to physics and social science, no longer based on reasoning from Probabilities, would fall in ruins.

With reference to Laplace, may be noticed a statement which, though not seriously misleading, adds to the perplexity into which the student of Probabilities is sometimes thrown by Professor Sampson's treatment of the subject. He attributes to Laplace the following argument:—"Laplace first offered a demonstration that the error-function possessed a definite form and might be derived from the combination of an unlimited number of small errors individually following any arbitrary laws of distribution" (Congress, p. 166; Phil. Mag. p. 349). But, as above pointed out (12), it is remarkable, and has been remarked by a leading authority, that Laplace did *not* make this use of his theory, but applied it only to *averages* of observations. Was the demonstration attributed to Laplace first employed by Morgan Crofton in the Transactions of the Royal Society, 1870 †?

Professor Sampson's reference to Poincaré (Congress, p. 168) was calculated, on a first hearing at least, to confirm the impression that justice was not being done to the law of error. It seemed as if an advocate referred to the terms of a statute as in favour of his case, omitting mention of a clause which afforded strong support to the case against him (13). It may be observed that the Poincaré proof is free from the objection which Professor Sampson makes against other proofs, including Morgan Crofton's, which "begin by formulating the existence of an error-function" (Phil. Mag. p. 348) ‡.

That objection might be formidable if we confined the theory to errors proper. We could not *à priori* assert that

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* See Yule, 'Outlines of Statistics'; Pearson, 'Mathematical Theory of Evolution,' Royal Society *passim*, and *Biometrika*.

† The argument is well stated by Glaisher in the 'Memoirs of the Astronomical Society,' cited above, dated 1872.

‡ Laplace's proof is not open to this objection. The observations whose averages he proves to be obedient to the law of error might be like the "causes of error" (the contributory elements whose aggregate make up an error of observation) according to Morgan Crofton (Phil. Trans. p. 180 (1870)) such that "a function or curve does not assist our conceptions, and we shall do better to consider the points or dots themselves." (Consider the version of Laplace's proof, Camb. Phil. Trans. p. 51 (1905). The expression $\phi(x)$ employed for the locus of the frequency need not be a continuous differentiable function.) And if the method is employed (though Laplace himself did not so employ it) to deduce the form of the frequency-function pertaining to errors of observation from the fact that an error is approximately the sum of numerous contributory elements, it is not necessary to postulate that the sought function is continuous and differentiable.

errors of observation are distributed according to a continuous differentiable function (Congress, p. 164). But has the objection much weight *in general*; with reference, say, to a molecular medley (14), or to an average of statistics prolonged indefinitely under stable conditions?

On the whole it appears from the explanations which Professor Sampson has offered in the Philosophical Magazine that he designed to repair or reconstruct the edifice founded by Laplace. But, according to the view here taken, the operation was calculated to weaken the foundation and to damage the structure at other points. Nor were there compensating advantages in the additions proposed as improvements. On a *first* view the danger to the structure was more apparent than the intention to make repairs. It was natural, then, that one who was deeply impressed by the use and beauty of the edifice should protest strongly against an attack upon it, though made by an expert whose authority on mathematical subjects commands great respect.

XII. *Proceedings of Learned Societies.*

GEOLOGICAL SOCIETY.

[Continued from vol. xl. p. 826.]

February 20th, 1920.—Mr. G. W. Lamplugh, F.R.S., President, in the Chair.

The President in his Anniversary Address discussed Some Features of the Pleistocene Glaciation of England, dealing principally with the changes brought about by the ice in the surface-features of our country. With the aid of a sketch-map he showed that over 5000 square miles of English land, or about one-tenth of the whole country, would vanish if the drifts were removed, as the 'solid' rocks lie below sea-level in tracts of this extent. A further area of about 10,000 square miles is overspread by drift of sufficient thickness wholly to mask the 'solid' land-forms, so that rather more than one-quarter of the country owes its present shape to Glacial and Postglacial deposits. Another 20,000 square miles was glaciated, and more or less modified, but without losing the dominating features of its rocky framework. The remainder of the country was affected only by the intensification of the atmospheric agencies, whereby its original features were accentuated. In a general sense, the hill-districts have not been greatly changed, but the lowlands have been in most parts completely altered.

The source of the huge mass of material contained in certain of the lowland drift-sheets was next considered, and the opinion was expressed that a large portion of this was an addition to the land, brought in by the ice from outside our present coast-line. The position and extent of these drift-sheets could be explained by regarding them as the broad terminal belts of débris concentrated where the ice from the basins thinned off towards its periphery, and where also its motion was checked by the rising slope of the ground. The débris-choked outer margin of the ice may be supposed to have become stagnant after its final forward spurt, and in its waning phase most of its thaw-water probably escaped backward into the basins, leaving wide stretches of bare boulder-clay unencumbered with water-washed material. Many peculiarities of the drift-features were explicable on the supposition that the ice-movement was not continuous and regular, but proceeded, at the margin of the ice-sheets, by alternations of quick advance with longer intervals of stagnation or relative quiescence, such as have been observed now in existing glaciers and ice-sheets in many parts of the world.

The unequal distribution of the Glacial deposits in the area of scanty drift was then discussed, along with some local peculiarities in the shape assumed by the deposits in several places, and it was shown that the difference between the aspect of the main drift-sheets and the scattered drift could be accounted for by the difference of local conditions, which led to original irregularity of deposition and to early exposure of certain tracts to exceptionally vigorous erosion. It was also pointed out that the local incidence of glaciation may often have been an important factor, as it is known from existing conditions in Arctic lands that great stretches of moving ice may leave bare land, aside from its path, at lower levels on its flanks.

Comment was then made on the curious rarity of peat or other land-detritus in boulder-clay known to have been derived entirely from the land, and this was thought to indicate that the conditions for a long period before the actual glaciation had been unfavourable for the growth of timber or peat-producing vegetation.

A brief review was given of the minor changes and new erosion-features produced in the hills as a result of the glaciation. The effect of Postglacial erosion and deposition in modifying the Glacial features was also referred to.

February 25th.—Mr. R. D. Oldham, F.R.S., President,
in the Chair.

The following communication was read :—

'The Lower Carboniferous Chert-Formations of Derbyshire.' By Henry Crunden Sargent, F.G.S.

The chert-formations occurring in the Carboniferous Limestone and associated rocks of Derbyshire may be classified under two heads :—

(1) Those which owe their silica to gaseous or aqueous emanations from igneous rocks.

(2) Those which derived their silica from the land by means of chemical denudation.

The author considers that in both cases the silica was precipitated direct, and did not, to any considerable extent, pass through an intermediate stage of secretion by organisms with subsequent solution and redeposition.

Field-evidence shows that the chert is a contemporaneous deposit, though with possible rare minor exceptions, as noticed in the paper.

Field-evidence also shows that the silica was rapidly deposited, and that it was consolidated before the underlying limestone. Even if siliceous organisms existed on the sea-floor in sufficient numbers to form the chert-beds, which field and microscopic evidence shows to be extremely doubtful, there are no known agencies likely to be present in the sea capable of bringing their skeletons and tests into solution with the rapidity shown to have taken place in the deposition of the chert.

Cherts of magmatic origin differ in field-relations and associations, and often in structure, from the cherts which derived their silica from the land. The latter occur mainly in the upper, generally thin-bedded, dark limestone which was laid down near a shore-line. It is believed that, when a sufficient concentration of silica in the sea was attained, rapid flocculation and precipitation would result from contact with the bivalent ions of lime in the presence of carbon dioxide.

The author adduces evidence to show that simultaneous deposition of silica and calcium carbonate often took place, and it is believed that, in such cases, segregation ensued, and sometimes resulted in the formation of nodules and lenticular masses of chert.

It is suggested that the bedded cherts of terrestrial origin resulted from heavier precipitation of silica, comparatively free from calcium carbonate, and spread over the sea-floor by gentle currents.

Metasomatic replacement of limestone and calcareous organisms by silica has taken place at their contact with the chert. Impurities in the silica have tended to limit such replacement. The freer the chert was from impurity the greater was the replacement, but the chert is not a pseudomorph after limestone. Impurities have also had an important effect on the crystallization of the silica.

Organisms existing in the sea or on the sea-floor would be entangled in the precipitated silica, and their presence in the chert is thus explained.

The blackness of some chert is shown to be due to the presence of carbonaceous matter. Ferrous iron may possibly have operated sometimes in the same way.

The banding frequently seen in chert is believed to be due to a segregation of impurities by diffusion and rhythmic precipitation in the course of dehydration.

THE
LONDON, EDINBURGH, AND DUBLIN
PHILOSOPHICAL MAGAZINE
AND
JOURNAL OF SCIENCE.

[SIXTH SERIES.]

FEBRUARY 1921.

XIII. *An Asymptotic Formula for the Hypergeometric Function* $_0\Delta_4(z)$. *By* DOROTHY WRINCH, *M.Sc., Fellow of Girton College, Cambridge, and Member of the Research Staff, University College, London* *.

THE present communication is concerned with the asymptotic forms to which certain generalized hypergeometric functions tend as their argument which may be real or complex is increased. The expansions obtained include as special cases most of the more familiar asymptotic expansions of the more special functions of hypergeometric type. The investigation originally arose in relation to certain functions which occur in physical problems, more especially those of elasticity, and the need for asymptotic values of these functions has been pointed out †. The particular instance which gave rise to the investigation is concerned with the lateral vibrations of bars whose cross-section is a function of the distance from one end. The more important functions which thus arise can be included in a hypergeometric type analogous to the Bessel functions. We consider in this paper the general function

$$
{}_0\Delta_4[1+\alpha,\,1+\beta,\,1+\gamma,\,1+\delta\,;\,z]
$$

$$
=1+\sum_{v=1}^{\infty}\frac{z^v}{\prod\limits_{\alpha\beta\gamma\delta}(1+\alpha)(2+\alpha)\ldots(v+\alpha)},
$$

where the parameters $\alpha, \beta, \gamma, \delta$ can take all values except

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* Communicated by the Author.

† *Vide* e. g. Nicholson, Proc. Roy. Soc. 1917.

negative integer values. The expansion of this function is developed for all values of z, real or complex, and the procedure is indicated by which we can determine the linear combination of solutions of a characteristic differential equation of the fourth order which takes any of the individual forms

$$e^{4x^{1/4}}x^{-\frac{1}{4}(\alpha+\beta+\gamma+\delta+3/2)}, \qquad e^{-4x^{1/4}}x^{-\frac{1}{4}(\alpha+\beta+\gamma+\delta+3/2)},$$

$$x^{-\frac{1}{4}(\alpha+\beta+\gamma+\delta+3/2)}\cos\left[4x^{1/4}-\tfrac{1}{4}(\alpha+\beta+\gamma+\delta+3/2)\pi/2\right],$$

$$x^{-\frac{1}{4}(\alpha+\beta+\gamma+\delta+3/2)}\sin\left[4x^{1/4}-\tfrac{1}{4}(\alpha+\beta+\gamma+\delta+3/2)\pi/2\right],$$

near infinity.

Approximation to a Contour Integral.

Consider the integral

$$\frac{1}{2\pi i}\int\left(1+\frac{t}{1+\alpha}+\frac{t^2}{1+\alpha\,.\,2+\alpha}\cdots\right)$$
$$\left(1+\frac{z/t}{1+\beta}+\frac{(z/t)^2}{1+\beta\,.\,2+\beta}+\cdots\right)\frac{dt}{t}.$$

z being a complex number with principal argument ϕ and modulus ρ, round a circle enclosing the origin. Since both the series

$${}_0\Delta_1(t,\ 1+\alpha)=1+\frac{t}{1+\alpha}+\frac{t^2}{1+\alpha\,.\,z+\alpha}\cdots,$$

$${}_0\Delta_1(z/t,\ 1+\beta)=1+\frac{z/t}{1+\beta}+\frac{z^2/t^2}{1+\beta\,.\,2+\beta}\cdots,$$

are regular within the contour, the integral is equal to the residue of the integrand at the point $t=0$, viz.:

$${}_0\Delta_2(z;\ 1+\alpha,\ 1+\beta)=1+\frac{z}{1+\alpha\,.\,1+\beta}$$
$$+\frac{z^2}{1+\alpha\,.\,2+\alpha\,.\,1+\beta\,.\,2+\beta}+\cdots$$

Thus in order to approximate to ${}_0\Delta_2(z;\ 1+\alpha,\ 1+\beta)$ as $\rho\longrightarrow\infty$, we may consider the asymptotic behaviour of the integral

$$\mathrm{I}=\frac{1}{2\pi i}\int_c {}_0\Delta_1(t,\ 1+\alpha){}_0\Delta_1(z/t,\ 1+\beta)\frac{dt}{t}$$

as $\rho \longrightarrow \infty$ where C is a circular contour round the point $t=0$. Let δ be the radius of the circle C, then

$$\mathrm{I}=\int_{-\pi}^{\pi} {}_0\Delta_1(\rho/\delta e^{i\phi-i\theta},\ 1+\beta)\,{}_0\Delta_1(\delta e^{i\theta},\ 1+\alpha)\,d\theta.$$

Let χ stand for the argument of z, which is such that $-\pi<(\chi-\theta)<\pi$. We can approximate to the value of I, if δ is such that δ and ρ/δ tend to infinity as $\rho\longrightarrow\infty$, since

$${}_0\Delta_1(z,\ 1+v) \sim \Gamma(1+v)e^z z^{-v} - \sum_{n=1}^{\infty} \frac{v \,.\, v-1 \ldots v-n+1}{z^n}.$$

If $-\pi<\chi<\pi$ and $\arg z^{-v}=-v\chi$,

$$\mathrm{I}\sim \mathrm{I}_1-\mathrm{I}_2-\mathrm{I}_3+\mathrm{I}_4,$$

where

$$\mathrm{I}_1=\Gamma(1+\alpha)\Gamma(1+\beta)\int_{-\pi}^{\pi} e^{\rho/\delta e^{i(\chi-\theta)}}\left[\rho/\delta e^{i(\chi-\theta)}\right]^{-\beta} \times e^{\delta e^{i\theta}}(\delta e^{i\theta})^{-\alpha}d\theta,$$

$$\mathrm{I}_2=\int_{-\pi}^{\pi} e^{\rho/\delta e^{i(\chi-\theta)}}(\rho/\delta e^{i(\chi-\theta)})^{-\beta}\sum_{n=1}^{\mathrm{N}}\frac{\alpha(\alpha-1)\ldots\alpha-n+1}{(\delta e^{i\theta})^n}\,d\theta,$$

$$\mathrm{I}_3=\int_{-\pi}^{\pi} e^{\delta e^{i\theta}}(\delta e^{i\theta})^{-\alpha}\sum_{n=1}^{\mathrm{N}}\frac{\beta(\beta-1)\ldots(\beta-n+1)}{\left[\rho/\delta e^{i(\chi-\theta)}\right]^n}\,d\theta,$$

$$\mathrm{I}_4=\int_{-\pi}^{\pi}\sum_{n=1}^{\mathrm{N}}\frac{\alpha(\alpha-1)\ldots(\alpha-n+1)}{(\delta e^{i\theta})^n}\sum_{n=1}^{\mathrm{N}}\frac{\beta(\beta-1)\ldots(\beta-n+1)}{\left[\rho/\delta e^{i(\chi-\theta)}\right]^n}.$$

The first is an integral of an oscillatory exponential type, and we may approximate to its value by Kelvin's method [*]. "Critical points" of the integrand occur where the derivate of the exponent is zero, or

$$\frac{d}{d\theta}\left(\frac{\rho}{\delta}e^{i(\chi-\theta)}+\delta e^{i\theta}\right)=0,$$

and therefore are easily seen to exist when and only when $\delta=\rho^{\frac{1}{2}}$. Taking this value for δ, critical values of θ are given, μ being an integer, by

$$\chi-\theta=\theta+2\mu\pi$$

or

$$\theta=\frac{\chi}{2}-\mu\pi.$$

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[*] A more rigorous investigation can be made by the "method of steepest descents." But this would be very long, and is not necessary for our purposes.

Suppose χ_r is a critical point such that $-\pi<\chi_r<\pi$, then $\chi_r-\theta$ and θ both lie between $\pm\pi$, and since they differ by zero or a multiple of 2π they must be equal.

Now as $|x|\longrightarrow\infty$,

$$\int_0^x e^{-z^2}dz\sim+\tfrac{1}{2}\sqrt{\pi} \quad \text{if} \quad -\frac{\pi}{2}<\arg x<\frac{\pi}{2}.$$

Hence as $|x|\longrightarrow\infty$, if a is positive and real,

$$\int_0^a e^{-x\epsilon^2}d\epsilon\sim\tfrac{1}{2}\sqrt{\pi}, \quad \text{if} \quad -\frac{\pi}{2}<\arg\sqrt{x}<\frac{\pi}{2}.$$

Then

$$\rho^{-\frac{1}{2}(\alpha+\beta)}e^{-i\chi_r(\alpha+\beta)}\int_{-\pi-\chi_r}^{\pi-\chi_r} e^{\rho^{\frac{1}{2}}e^{i\chi_r}(2-\epsilon^2)}d\epsilon$$

$$=\rho^{-\frac{1}{2}(\alpha+\beta+\frac{1}{2})}e^{-i\chi_r(\alpha+\beta+\frac{1}{2})}e^{2\rho^{\frac{1}{2}}e^{i\chi_r}}\int_{-(\pi+\chi_r)\rho^{\frac{1}{4}}e^{\frac{i\chi_r}{2}}}^{(\pi-\chi_r)\rho^{\frac{1}{4}}e^{\frac{i\chi_r}{2}}} e^{-z^2}dz$$

$$\sim\sqrt{\pi}\rho^{-\frac{1}{2}(\alpha+\beta+\frac{1}{2})}e^{-i\chi_r(\alpha+\beta+\frac{1}{2})}e^{2\rho^{\frac{1}{2}}e^{i\chi_r}}.$$

Next take the case when π and therefore also $-\pi$ is a critical point; then the integrals

$$\rho^{-\frac{1}{2}(\alpha+\beta)}\int_{-2\pi}^{0} e^{-2\rho^{\frac{1}{2}}\cos\epsilon}d\epsilon,$$

and

$$\rho^{-\frac{1}{2}(\alpha+\beta)}\int_{0}^{2\pi} e^{-2\rho^{\frac{1}{2}}\cos\epsilon}d\epsilon,$$

are negligible as $\rho\longrightarrow\infty$, after the manner of the asymptotic theory of the function $I_n(x)$. Hence $\pm\pi$ as critical points contribute nothing to the asymptotic value of I_1. Therefore

$$I_1\sim\Gamma(1+\alpha)\Gamma(1+\beta)\Sigma_r\sqrt{\pi}\rho^{-\frac{1}{2}(\alpha+\beta+\frac{1}{2})}e^{-i\chi_r(\alpha+\beta+\frac{1}{2})}e^{2\rho^{\frac{1}{2}}e^{i\chi_r}},$$

the summation being taken over all χ_r's satisfying the conditions

$$-\pi<\chi_r<\pi\,; \quad -\chi_r=\frac{\chi}{2}-\mu\pi.$$

Hence, as $\rho\longrightarrow\infty$ if $-2\pi<\phi<2\pi$

$$I_1\sim\frac{\Gamma 1+\alpha\Gamma 1+\beta}{2\sqrt{\pi}}\rho^{-\frac{1}{2}(\alpha+\beta+\frac{1}{2})}\left[e^{-i\frac{\phi}{2}(\alpha+\beta+\frac{1}{2})}\right]$$

$$\times\left[e^{2\rho^{\frac{1}{2}}e^{\frac{i\phi}{2}}}+e^{-2\rho^{\frac{1}{2}}e^{\frac{i\phi}{2}}}e^{\pm\pi i(\alpha+\beta+\frac{1}{2})}\right],$$

according as ϕ is positive or negative, the second term being evanescent when $\phi=0$.

Applying Kelvin's method to I_2, I_3, I_4, we see that no "critical points" exist. Hence the asymptotic equivalence stands in the form

$$1+\frac{z}{1+\alpha\,.\,1+\beta}+\frac{z^2}{1+\mu\,.\,2+\alpha\,.\,1+\beta\,.\,2+\beta}+\,..$$

$$\sim\frac{\Gamma(1+\alpha)\Gamma(1+\beta)}{2\sqrt{\pi}}\rho^{-\frac{1}{2}(\alpha+\beta+\frac{1}{2})}e^{-\frac{i\phi}{2}(\alpha+\beta+\frac{1}{2})}$$

$$\times\left[\rho^{2\rho^{\frac{1}{2}}e^{\frac{i\phi}{2}}}+e^{-2\rho^{\frac{1}{2}}e^{\frac{i\phi}{2}}}e^{\pm\pi i(\alpha+\beta+\frac{1}{2})}\right],$$

according as ϕ is positive or negative, the second term being evanescent when $\phi=0$.

In particular when ρ is real

$$1+\frac{\rho}{1+\alpha\,.\,1+\beta}+\,..\sim\frac{\Gamma 1+\alpha\Gamma 1+\beta}{2\sqrt{\pi}}\rho^{-\frac{1}{2}(\alpha+\beta+\frac{1}{2})}e^{2\rho^{\frac{1}{2}}},$$

$$1-\frac{\rho}{1+\alpha\,.\,1+\beta}\cdots\sim\frac{\Gamma 1+\alpha\Gamma 1+\beta}{\sqrt{\pi}}\rho^{-\frac{1}{2}(\alpha+\beta+\frac{1}{2})}$$

$$\cos\left[2\rho^{\frac{1}{2}}-\frac{\pi}{2}(\alpha+\beta+\tfrac{1}{2})\right],$$

$$1+\frac{i\rho}{1+\alpha\,.\,1+\beta}\cdots\sim\frac{\Gamma 1+\alpha\Gamma 1+\beta}{\sqrt{\pi}}\rho^{-\frac{1}{2}(\alpha+\beta+\frac{1}{2})}e^{\frac{3}{8}\pi i(\alpha+\beta+\frac{1}{2})}$$

$$\cos\left[2\rho^{\frac{1}{2}}-\frac{5}{8}\pi(\alpha+\beta+\tfrac{1}{2})\right],$$

$$1-\frac{i\rho}{1+\alpha\,.\,1+\beta}\cdot\cdot\sim\frac{\Gamma 1+\alpha\Gamma 1+\beta}{\sqrt{\pi}}\rho^{-\frac{1}{2}(\alpha+\beta+\frac{1}{2})}e^{-\frac{3}{8}\pi i(\alpha+\beta+\frac{1}{2})}$$

$$\cos\left[2\rho^{\frac{1}{2}}+\frac{5}{8}\pi(\alpha+\beta+\tfrac{1}{2})\right];$$

while, in general, if ϕ and ρ are the principal argument and modulus of z respectively :

$$1+\frac{z}{1+\alpha\,.\,1+\beta}\cdots\sim\frac{\Gamma 1+\alpha\Gamma 1+\beta}{\sqrt{\pi}}e^{2\rho^{\frac{1}{2}}\cos\frac{\phi}{2}}$$

$$\times\;e^{\pm\frac{\pi i}{2}(\alpha+\beta+1)-\frac{i\phi}{2}(\alpha+\beta+1)}\cos\left[2\rho^{\frac{1}{2}}\sin\frac{\phi}{2}\mp\frac{\pi}{2}(\alpha+\beta+\tfrac{1}{2})\right].$$

the upper or lower signs being adopted according as ϕ is positive or negative, and

$$1+\frac{z}{1+\alpha\,.\,1+\beta}\cdots\sim\frac{\Gamma 1+\alpha\Gamma 1+\beta}{2\sqrt{\pi}}\rho^{-\frac{1}{2}(\alpha+\beta+\frac{1}{2})}e^{2\rho^{\frac{1}{2}}},$$

when ϕ is zero.

[It may be remarked that the form of this result contains an apparent ambiguity: for there are two values of ϕ such that

$$-2\pi<\phi<2\pi,$$

and

$$\arg(z)=\phi.$$

Suppose ϕ_1 and ϕ_2 are values of ϕ, satisfying these two conditions.

Then $\phi_1\sim\phi_2=2\pi$, and

$$e^{-i\phi_1/2(\alpha+\beta+\frac{1}{2})}\left[e^{2\rho^{\frac{1}{2}}e^{i\phi_1/2}}+e^{-2\rho^{\frac{1}{2}}e^{i\phi\frac{1}{2}}}e^{\pm\pi i(\alpha+\beta+\frac{1}{2})}\right]$$

$$=e^{-\frac{i\phi_2}{2}(\alpha+\beta+\frac{1}{2})}\left[e^{2\rho^{\frac{1}{2}}e^{\frac{i\phi}{2}}}+e^{-2\rho^{\frac{1}{2}}e^{i\phi_2/2}}e^{\mp\pi i(\alpha+\beta+\frac{1}{2})}\right],$$

the upper signs or lower signs being taken according as ϕ_1 is positive and ϕ_2 negative (when $\phi_1-\phi_2=2\pi$), or ϕ_1 is negative and ϕ_2 positive (when $\phi_2-\phi_1=2\pi$). The same result is then obtained whichever is taken of the two values of ϕ, and the ambiguity is only apparent.]

The more general series.

We may adopt the same method in order to obtain the asymptotic equivalent of the convergent series

$${}_0\Delta_4[z;\ 1+\alpha,\ 1+\beta,\ 1+\gamma,\ 1+\delta]$$
$$=1+\frac{z}{1+\alpha\,.\,1+\beta\,.\,1+\gamma\,.\,1+\delta}+\ldots,$$

where the principal argument of z is ϕ and its modulus is ρ.

For

$${}_0\Delta_4[z;\ 1+\alpha,\ 1+\beta,\ 1+\gamma,\ 1+\delta]$$
$$=\frac{1}{2\pi i}\int_c {}_0\Delta_2[z/t;\ 1+\alpha,\ 1+\beta]\,{}_0\Delta_2[t;\ 1+\gamma,\ 1+\delta]\frac{dt}{t},$$

C being a circle with its centre at the origin. We can again approximate to this integral as $|z|\rightarrow\infty$ by Kelvin's method. Let δ be the radius of the circle, then

$${}_0\Delta_4(z;\ 1+\alpha,\ 1+\beta,\ 1+\gamma,\ 1+\delta)$$
$$=\frac{1}{2\pi}\int_{-\pi}^{\pi}{}_0\Delta_2[\rho/\delta e^{i(\phi-\theta)};\ 1+\alpha,\ 1+\beta]\,{}_0\Delta_2[\delta e^{i\theta};\ 1+\gamma,\ 1+\delta]d\theta.$$

If δ is so chosen that ρ/δ and $\delta \longrightarrow \infty$ with ρ, and χ stands for an argument of z which makes $-2\pi < \chi - \theta < 2\pi$, then

$$
{}_0\Delta_4(z ;\, 1+\alpha, 1+\beta, 1+\gamma, 1+\delta) \sim \frac{\Gamma 1+\alpha \Gamma 1+\beta \Gamma 1+\gamma \Gamma 1+\delta}{8\pi^2}
$$

$$
\int_{-\pi}^{\pi} \Big[\rho/\delta^{-\frac{(\alpha+\beta+\frac{1}{2})}{2}} \delta^{-\frac{\gamma+\delta+\frac{1}{2}}{2}} (e^{i\theta})^{-\frac{\gamma+\delta+\frac{1}{2}}{2}} (e^{i(\chi-\theta)})^{-\frac{\alpha+\beta+\frac{1}{2}}{2}}
$$

$$
\times \left(e^{2\delta^{\frac{1}{2}} e^{i\theta/2}} + e^{-2\delta^{\frac{1}{2}} e^{i\theta/2}} e^{\pm i\pi(\gamma+\delta+\frac{1}{2})} \right)
$$

$$
\times \left(e^{2(\rho/\delta)^{\frac{1}{2}} e^{i(\chi-\theta)/2}} + e^{-2(\rho/\delta)^{\frac{1}{2}} e^{i(\chi-\theta)/2}} e^{\pm i\pi(\alpha+\beta+\frac{1}{2})}\right.
$$

Either value of χ will give the same result, the order of the terms being interchanged. Let us take the smaller in absolute value, viz. ϕ itself.

In this expression the lower or upper sign in $e^{\pm(\gamma+\delta+\frac{1}{2})i\pi}$ is to be taken in the range in which θ is negative and that in which θ is positive respectively, and the upper or lower sign of $e^{\pm \pi i(\alpha+\beta+\frac{1}{2})}$ is to be taken in the range in which $\chi-\theta$ is positive and that in which $\chi-\theta$ is negative respectively. When $\theta=0$ the term $e^{-2\delta^{\frac{1}{2}} e^{i\theta/2}} e^{\pm\pi i(\gamma+\delta+\frac{1}{2})}$ is to be omitted. Similarly, when $\chi-\theta=0$, the term $e^{-2\delta^{\frac{1}{2}} e^{i\theta/2}} e^{\pm i\pi(\alpha+\beta+\frac{1}{2})}$ is to be omitted.

Hence

$$
{}_0\Delta_4(z ;\ 1+\alpha, 1+\beta, 1+\gamma, 1+\delta)
$$

$$
\sim \frac{\Gamma 1+\alpha \Gamma 1+\beta \Gamma 1+\gamma \Gamma 1+\delta}{8\pi^2} [{}_1\mathrm{N}_1 + {}_1\mathrm{N}_2 + {}_2\mathrm{N}_1 + {}_2\mathrm{N}_2],
$$

where

$$
{}_1\mathrm{N}_1 = \int_{-\pi}^{\pi} (\delta e^{i\theta})^{-\frac{\gamma+\delta+\frac{1}{2}}{2}} (\rho/\delta\ e^{i\overline{\chi-\theta}})^{-\frac{\alpha+\beta+\frac{1}{2}}{2}}
$$

$$
\times e^{2\left[\delta^{\frac{1}{2}} e^{i\theta/2} + (\rho/\delta)^{\frac{1}{2}} e^{\frac{i\overline{\chi-\theta}}{2}}\right]} d\theta,
$$

$$
{}_2\mathrm{N}_1 = \int_{-\pi}^{\pi} e^{\pm i\pi(\gamma+\delta+\frac{1}{2})} (\delta e^{i\theta})^{-\frac{\gamma+\delta+\frac{1}{2}}{2}}
$$

$$
\times (\rho/\delta\ e^{i(\chi-\theta)})^{-\frac{\alpha+\beta+\frac{1}{2}}{2}} e^{-2\delta^{\frac{1}{2}} e^{i\theta/2} + 2(\rho/\delta)^{\frac{1}{2}} e^{i(\chi-\theta)/2}} d\theta.
$$

$$
{}_1\mathrm{N}_2=\int_{-\pi}^{\pi} e^{\pm i\pi(\alpha+\beta+\frac{1}{2})}(\delta e^{i\theta})^{-\frac{\gamma+\delta+\frac{1}{2}}{2}}
$$

$$
\times(\rho/\delta\, e^{i(\chi-\theta)})^{-\frac{\alpha+\beta+\frac{1}{2}}{2}}\, e^{2\delta^{\frac{1}{2}}e^{i\theta/2}-2(\rho/r)^{\frac{1}{2}}e^{i\overline{\chi-\theta}/2}}\, d\theta,
$$

$$
{}_2\mathrm{N}_2=\int_{-\pi}^{\pi} e^{\pm\pi i(\gamma+\delta+\frac{1}{2})}e^{\pm\pi i(\alpha+\beta+\frac{1}{2})}(\delta e^{i\theta})^{-\frac{\gamma+\delta+\frac{1}{2}}{2}}
$$

$$
\times(\rho/\delta\, e^{i(\chi-\theta)})^{-\frac{\alpha+\beta+\frac{1}{2}}{2}}\, e^{-2\delta^{\frac{1}{2}}e^{i\theta/2}-2(\rho/\delta)^{\frac{1}{2}}e^{i(\chi-\theta)/2}}\, d\theta.
$$

We are still at liberty to choose δ as we please, provided only that ρ/δ and δ tend to infinity with ρ, the factors $e^{\pm\pi i(\gamma+\delta+\frac{1}{2})}$ being adopted according as θ is positive or negative, and the factors $e^{\pm\pi i(\alpha+\beta+\frac{1}{2})}$ being adopted according as $\phi-\theta$ is positive or negative.

We shall therefore, if possible, choose δ with a view to ensuring the existence of critical points for the integrals ${}_1\mathrm{N}_1$, ${}_1\mathrm{N}_2$, ${}_2\mathrm{N}_1$, ${}_2\mathrm{N}_2$.

Critical values of θ will exist if there exist roots of the two equations

$$
\frac{d}{d\theta}\left[\delta^{\frac{1}{2}}e^{\frac{i\theta}{2}}\pm\left(\frac{\rho}{\delta}\right)^{\frac{1}{2}}e^{i\frac{\phi-\theta}{2}}\right].
$$

We therefore choose $\delta=\rho^{\frac{1}{2}}$, which gives the only possible case.

Consider first the integrals ${}_1\mathrm{N}_1$ and ${}_2\mathrm{N}_2$. Critical values are given by the equation

$$
\theta=\frac{\phi}{2}+2\mu\pi.
$$

There will be one, and only one, for each integral, and it will be

$$
\theta=\frac{\phi}{2}.
$$

Hence, using Stokes's result as before,

$$
{}_1\mathrm{N}_1\sim\rho^{-\frac{1}{4}(\alpha+\beta+\gamma+\delta+1)}e^{-i\frac{\phi}{4}(\alpha+\beta+\gamma+\delta+1)}
$$

$$
\times\int_{-\pi-\frac{\phi}{2}}^{\pi-\frac{\phi}{2}} e^{2\rho^{\frac{1}{4}}e^{\frac{i\phi}{4}}\left(2-\frac{\epsilon^2}{4}\right)}\, d\epsilon
$$

$$
=\sqrt{2\pi}\,.\,\rho^{-\frac{1}{4}(\alpha+\beta+\gamma+\delta+3/2)}e^{-i\frac{\phi}{4}(\alpha+\beta+\gamma+\delta+3/2)}e^{4\rho^{\frac{1}{4}}e^{i\phi/4}}
$$

$$ {}_2\mathrm{N}_2 \sim \rho^{-\frac{1}{4}(\alpha+\beta+\gamma+\delta+1)}\left(e^{i\frac{\phi}{2}}\right)^{-\frac{\alpha+\beta+\gamma+\delta+1}{2}\pm\pi i(\gamma+\delta+\alpha+\beta+1)} $$

$$ \times\int_{-\pi-\frac{\phi}{2}}^{\pi-\frac{\phi}{2}} e^{-2\rho^{\frac{1}{4}}e^{i\frac{\phi}{4}}\left(2-\frac{\epsilon^2}{4}\right)}d\epsilon $$

$$ =\sqrt{2\pi}\,.\,\rho^{-\frac{1}{4}(\alpha+\beta+\gamma+\delta+3/2)}e^{-\frac{i\phi}{4}(\alpha+\beta+\gamma+\delta+3/2)} $$

$$ \times e^{-4\rho^{\frac{1}{4}}e^{\frac{i\phi}{4}}}e^{\pm\pi i(\alpha+\beta+\gamma+\delta+3/2)}, $$

according as ϕ is positive or negative.

If $\phi=0$ the integral does not occur.

Consider next the integrals ${}_2\mathrm{N}_1$ and ${}_1\mathrm{N}_2$. Critical values are given by the equation

$$ \theta=\frac{\phi}{2}+(2\mu+1)\pi;\quad \phi-\theta=\frac{\phi}{2}-(2\mu+1)\pi. $$

There will be one, and one only, for each integral, viz.,

$$ \theta=\frac{\phi}{2}\pm\pi, $$

according as ϕ is positive or negative, except in the case $\phi=0$, when there will be critical values π and $-\pi$ if

$$ \theta=\frac{\phi}{2}\mp\pi,\quad \chi-\theta=\frac{\phi}{2}\pm\pi. $$

Hence, according as ϕ is positive or negative,

$$ {}_2\mathrm{N}_1 \sim \rho^{-\frac{1}{4}(\alpha+\beta+\gamma+\delta+1)}\int_{-\pi-\left[\frac{\phi}{2}\mp\pi\right]}^{\pi-\left(\frac{\phi}{2}\mp\pi\right)} e^{\mp(\gamma+\delta+\frac{1}{2})\iota\pi} $$

$$ \times e^{-i\left(\frac{\phi}{2}\mp\pi\right)\frac{\gamma+\delta+\frac{1}{2}}{2}}e^{-i\left(\frac{\phi}{2}\pm\pi\right)\frac{\alpha+\beta+\frac{1}{2}}{2}} $$

$$ \times e^{+2\rho^{\frac{1}{4}}e^{\frac{1}{2}\left(\frac{\phi}{2}\pm\pi\right)}\left[2-\frac{\epsilon^2}{4}\right]}d\epsilon $$

$$ =e^{4\rho^{\frac{1}{4}}e^{\frac{1}{2}\left(\frac{\phi}{2}\pm\pi\right)}}\rho^{-\frac{1}{4}(\alpha+\beta+\gamma+\delta+1)}e^{-i\left(\frac{\phi}{2}\pm\pi\right)\frac{(\alpha+\beta+\gamma+\delta+1)}{2}} $$

$$ \int_{-\pi-\left(\frac{\phi}{2}\mp\pi\right)}^{\pi-\left(\frac{\phi}{2}\mp\pi\right)} e^{-\frac{1}{2}\epsilon^2\rho^{\frac{1}{4}}e^{\frac{1}{2}\left(\frac{\phi}{2}\pm\pi\right)}}d\epsilon, $$

$$ \sim\sqrt{2\pi}\left[\rho e^{i(\phi/2\pm\pi)}\right]^{-\frac{1}{4}(\alpha+\beta+\gamma+\delta+3/2)}e^{4\rho^{\frac{1}{4}}e^{i(\phi/4\pm\pi/2)}}, $$

if $\phi\neq0$.

The values of ϕ lie between $-\pi$ and $+\pi$ ($+\pi$ not excluded), and the result therefore holds in all cases under consideration, for Stokes's result holds for all integrals of the form $\int_0^b e^{-az^2}dz$, when $-\pi < \arg a < \pi$.

Similarly,

$$
{}_1\mathrm{N}_2 \sim \sqrt{2\pi}\rho^{-\frac{1}{4}(\alpha+\beta+\gamma+\delta+3/2)} \times e^{-i\left(\frac{\phi}{2}\mp\pi\right)\frac{\alpha+\beta+\gamma+\delta+3/2}{2}} e^{4\rho^{\frac{1}{4}}e^{i\left(\frac{\phi}{4}\mp\frac{\pi}{2}\right)}}.
$$

If $\phi=0$,

$$
\begin{aligned}
{}_1\mathrm{N}_2 = {}_2\mathrm{N}_1 \sim \tfrac{1}{2}\sqrt{2\pi}\rho^{-\frac{1}{4}(\alpha+\beta+\gamma+\delta+3/2)} & \times e^{-\frac{i\pi}{2}(\alpha+\beta+\gamma+\delta+3/2)} e^{4\rho^{\frac{1}{4}}e^{+\frac{\pi i}{2}}} \\
+\tfrac{1}{2}\sqrt{2\pi}\rho^{-\frac{1}{4}(\alpha+\beta+\gamma+\delta+3/2)} & \times e^{+\frac{i\pi}{2}(\alpha+\beta+\gamma+\delta+3/2)} e^{4\rho^{\frac{1}{4}}e^{-\frac{\pi i}{2}}}.
\end{aligned}
$$

Hence

$$
{}_1\mathrm{N}_2 + {}_2\mathrm{N}_1 \sim \sqrt{2\pi}\rho^{-\frac{1}{4}(\alpha+\beta+\gamma+\delta+3/2)} \times \cos\left[4\rho^{\frac{1}{4}} - \frac{\pi}{2}(\alpha+\beta+\gamma+\delta+3/2\right].
$$

Thus if $-\pi < \phi \leq \pi$,

$$
\begin{aligned}
{}_1\mathrm{N}_1 + {}_1\mathrm{N}_2 + {}_2\mathrm{N}_1 + {}_2\mathrm{N}_2 \sim \sqrt{2\pi}\rho^{-\frac{1}{4}(\alpha+\beta+\gamma+\delta+3/2)} & \Big[e^{-\frac{i\phi}{4}(\alpha+\beta+\gamma+\delta+3/2)} e^{4\rho^{\frac{1}{4}}e^{\frac{i\phi}{4}}} \\
& + e^{\pm\frac{i\pi}{2}(\alpha+\beta+\gamma+\delta+3/2)} e^{4\rho^{\frac{1}{4}}e^{-\frac{i\pi}{2}+\frac{i\phi}{4}}} \\
& + e^{-\frac{i\pi}{2}(\alpha+\beta+\gamma+\delta+3/2)} e^{4\rho^{\frac{1}{4}}e^{\frac{i\pi}{2}+\frac{i\phi}{4}}} \\
& + e^{\pm\pi i(\alpha+\beta+\gamma+\delta+3/2)} e^{-4\rho^{\frac{1}{4}}e^{\frac{i\phi}{4}}}\Big],
\end{aligned}
$$

according as ϕ is positive or negative, the last term being omitted if ϕ is zero.

Hence $\quad {}_0\Delta_4(z\,;\ 1+\alpha,\ 1+\beta,\ 1+\gamma,\ 1+\delta)$

$$\sim\frac{\Gamma 1+\alpha\Gamma 1+\beta\Gamma 1+\gamma\Gamma 1+\delta}{2^{5/2}\pi^{3/2}}$$

$$\rho^{-\frac{1}{4}(\alpha+\beta+\gamma+\delta+3/2)}\Big[\mathrm{B}\Big]$$

where B is the preceding large bracket, according as ϕ is positive or negative, the last term being omitted if ϕ is zero.

Hence in particular, ρ being real,

$${}_0\Delta_4(\rho\,;\quad 1+\alpha,\ 1+\beta,\ 1+\gamma,\ 1+\delta)$$

$$=1+\frac{\rho}{1+\alpha\,.\,1+\beta\,.\,1+\gamma\,.\,1+\delta}\ldots$$

$$\sim\frac{\Gamma 1+\alpha\Gamma 1+\beta\Gamma 1+\gamma\Gamma 1+\delta}{2^{5/2}\pi^{3/2}}\rho^{-\frac{1}{4}(\alpha+\beta+\gamma+\delta+3/2)}$$

$$\left[e^{4\rho^{\frac{1}{4}}}+2\cos\left\{4\rho^{\frac{1}{4}}-\frac{\pi}{2}(\alpha+\beta+\gamma+\delta+3/2)\right\}\right]$$

and

$$1-\frac{\rho}{1+\alpha\,.\,1+\beta\,.\,1+\gamma\,.\,1+\delta}+\ldots$$

$$\sim\frac{\Gamma 1+\alpha\Gamma 1+\beta\Gamma 1+\gamma\Gamma 1+\delta}{2^{5/2}\pi^{3/2}}\left(e^{4\rho^{\frac{1}{4}}\cos\frac{\pi}{4}}\right.$$

$$\left.\times\cos\left[4\rho^{\frac{1}{4}}\sin\frac{\pi}{4}-\frac{\pi}{4}(\alpha+\beta+\gamma+\delta+3/2)\right]\right).$$

Special forms near infinity.

Series of the type $_0\Delta_4$ occur as solutions of equations of the type

$$\vartheta(\vartheta+a)(\vartheta+b)(\vartheta+c)\omega=z\omega,$$

which are of frequent occurrence in problems of lateral vibrations of bars whose section is not uniform.

If we put

$$\omega=z^{-v}y,$$

the equation becomes

$$(\vartheta-v)(\vartheta+a-v)(\vartheta+b-v)(\vartheta+c-v)y=zy.$$

The general solution of the original equation—if we formally exclude the cases when logarithmic solutions are involved—

is therefore, P, Q, R, S being constants,

$$\begin{aligned} &\mathrm{P}_0\Delta_4(1+a,\ 1+b,\ 1+c,\ 1\ ;\ z) \\ &\qquad +z^{-a}\mathrm{Q}_0\Delta_4(-a,\ 1+b-a,\ 1+c-a,\ 1\ ;\ z) \\ &\qquad +z^{-b}\mathrm{R}_0\Delta_4(-b,\ 1+c-b,\ 1+a-b,\ 1\ ;\ z) \\ &\qquad +z^{-c}\mathrm{S}_0\Delta_4(-c,\ 1+a-c,\ 1+b-c,\ 1\ ;\ z). \end{aligned}$$

Let $y_1\, y_2\, y_3\, y_4$ represent these four solutions, which together form the general solution. Then the results obtained with respect to the asymptotic behaviour of

$$_0\Delta_4(1+\alpha,\ 1+\beta,\ 1+\gamma,\ 1+\delta\ ;\ z)$$

give

$$y_1 \sim \frac{\Gamma 1+a\,\Gamma 1+b\,\Gamma 1+c}{2^{5/2}\pi^{3/2}}\, z^{-\frac{a+b+c+3/2}{4}} \Big[e^{4z^{\frac{1}{4}}} + e^{4iz^{\frac{1}{4}}} e^{-\frac{\pi i}{2}(a+b+c+3/2)}$$

$$+ e^{-4iz^{\frac{1}{4}}} e^{+\frac{\pi i}{2}(a+b+c+3/2)} + e^{-4z^{\frac{1}{4}}} e^{\mp \pi i(a+b+c+3/2)} \Big],$$

$$y_2 \sim \frac{\Gamma 1-a\,\Gamma 1+b-a\,\Gamma 4c-a}{2^{5/2}\pi^{3/2}}\, z^{-\frac{a+b+c+3/2}{4}} \Big[e^{4z^{\frac{1}{4}}} + e^{4iz^{\frac{1}{4}}} e^{-\frac{i\pi}{2}(-3a+b+c+3/2)}$$

$$+ e^{-4iz^{\frac{1}{4}}} e^{+\frac{i\pi}{2}(-3a+b+c+3/2)} + e^{-4z^{\frac{1}{4}}} e^{\mp \pi i(-3a+b+c+3/2)} \Big],$$

$$y_3 \sim \frac{\Gamma 1-b\,\Gamma 1+c-b\,\Gamma 1+a-b}{2^{5/2}\pi^{3/2}}\, z^{-\frac{a+b+c+3/2}{4}} \Big[e^{4z^{\frac{1}{4}}} + e^{4iz^{\frac{1}{4}}} e^{-\frac{i\pi}{2}(c+3/2+a-3b)}$$

$$+ \ldots \Big],$$

$$y_4 \sim \frac{\Gamma 1-a\,\Gamma 1+a-c\,\Gamma 1+b-c}{2^{8/2}\pi^{3/2}}\, z^{-\frac{a+b+c+3/2}{4}} \Big[\ldots . \Big].$$

Linear combinations of $y_1\, y_2\, y_3\, y_4$ therefore exist which behave asymptotically near infinity like

$$z^{-\frac{(a+b+c+3/2)}{4}} e^{4z^{\frac{1}{4}}}, \qquad z^{-\frac{a+b+c+3/2}{4}} e^{-4z^{\frac{1}{4}}},$$

$$z^{-\frac{a+b+c+3/2}{4}} \cos 4z^{\frac{1}{4}}, \text{ or } \ z^{-\frac{a+b+c+3/2}{4}} \sin 4z^{\frac{1}{4}}.$$

Let us consider the linear combination which is asymptotically equivalent to

$$z^{-\frac{a+b+c+3/2}{4}} e^{-4z^{\frac{1}{4}}}.$$

Writing

$$y_1' = \frac{2^{5/2}\pi^{3/2}}{\Gamma(1+a)\Gamma(1+b)\Gamma(1+c)} z^{\frac{a+b+c+3/2}{4}} y_1,$$

$$y_2' = \frac{2^{5/2}\pi^{3/2}}{\Gamma(1-a)\Gamma(1+b-a)\Gamma(1+c-a)} z^{\frac{a+b+c+3/2}{4}} y_2,$$

$$y_3' = \frac{2^{5/2}\pi^{3/2}}{\Gamma(1-b)\Gamma(1+c-b)\Gamma(1+a-b)} z^{\frac{a+b+c+3/2}{4}} y_3,$$

$$y_4' = \frac{2^{5/2}\pi^{3/2}}{\Gamma(1-c)\Gamma(1+a-c)\Gamma(1+b-c)} z^{\frac{a+b+c+3/2}{4}} y_4;$$

it is clear that the linear combination

$$\begin{vmatrix} y_1' & 1 & 1 & 0 \\ y_2' & 1 & \cos \pi a & \sin \pi a \\ y_3' & 1 & \cos \pi b & \sin \pi b \\ y_4' & 1 & \cos \pi c & \sin \pi c \end{vmatrix}$$

will behave asymptotically like $e^{-4z^{\frac{1}{4}}}$. A simpler form of this expression is, on development,

$$\begin{aligned} 4y_1' \sin\left(\pi\overline{b-c}\right) \sin\left(\pi\overline{c-a}\right) \sin\left(\pi\overline{a-b}\right) & \\ -4y_2' \sin \pi c \sin \pi b \sin \overline{\pi(b-c)} & \\ -4y_3' \sin \pi a \sin \pi c \sin \overline{\pi(c-a)} & \\ -4y_4' \sin \pi b \sin \pi a \sin \overline{\pi(a-b)} & \end{aligned}$$

$$\begin{aligned} = -\frac{4}{\pi^3} \sin\left(\pi\overline{b-c}\right) \sin\left(\pi\overline{c-a}\right) \sin \overline{\pi(a-b)} \sin \pi a \sin \pi b & \\ \sin \pi c\, [y_1'\Gamma(-a)\Gamma(-b)\Gamma(-c)\Gamma(1+a)\Gamma(1+b)\Gamma(1+c) & \\ +y_2'\Gamma(a)\Gamma(1-a)\Gamma(a-c)\Gamma(a-b)\Gamma(1+c-a)\Gamma(1+b-a) & \\ +y_3'\Gamma(b)\Gamma(1-b)\Gamma(b-a)\Gamma(b-c)\Gamma(1-b+a)\Gamma(1-c+b) & \\ +y_4'\Gamma(c)\Gamma(1-c)\Gamma(c-b)\Gamma(c-a)\Gamma(1-c+b)\Gamma(1-a+c). & \end{aligned}$$

Thus

$$\begin{aligned} y_1\Gamma(-a)\Gamma(-b)\Gamma(-c) + y_2\Gamma(a)\Gamma(a-c)\Gamma(a-b) & \\ + y_3\Gamma(b)\Gamma(b-a)\Gamma(b-c) + y_4\Gamma(c)\Gamma(c-a)\Gamma(c-b) & \\ \sim \lambda z^{-\frac{a+b+c+3/2}{4}} e^{-4z^{\frac{1}{4}}}, & \end{aligned}$$

where λ is a function only of a, b, and c.

In the same manner we can isolate the solutions which tend to the other individual asymptotic forms, and the value of λ is obtained readily in each case. Such determinations are, however, apart from our main object, and this instance will suffice.

XIV. *A Generalized Hypergeometric Function with* n *parameters.* *By* DOROTHY WRINCH, *M.Sc., Fellow of Girton College, Cambridge, and Member of the Research Staff, University College, London* *.

THE behaviour of the series

$$1+\frac{z}{1\,!\prod\limits_{r=1}^{n-1}(1+\alpha_r)}+\frac{z^2}{2\,!\prod\limits_{r=1}^{n-1}(1+\alpha_r)(2+\alpha_r)}+\dots$$

as $|z|$ tends to infinity has been investigated in several particular cases. In the case when $n=1$ the series is easily to be derived from the Bessel function, and its behaviour on the circle $|z|=\infty$ is well known. In a recent paper by the present author, the case when $n=3$ was worked out. The asymptotic expansion of the series in the general case has gradually become apparent, and is established in this paper.

We may obtain the *form* of the asymptotic expansion of the series

$${}_0\mathrm{F}_{n-1}(1+\alpha_1,\ 1+\alpha_2\dots 1+\alpha_{n-1};\ z)$$
$$=1+\sum_{r=1}^{\infty}\frac{z^v}{v\,!\prod\limits_{r=1}^{n-1}(1+\alpha_r)(2+\alpha_r)\dots(v+\alpha_r)},$$

when $|z|$ is large after the usual manner.

${}_0\mathrm{F}_{n-1}(z)$ satisfies the differential equation

$$\vartheta(\vartheta+\alpha_1)\dots(\vartheta+\alpha_{n-1})y=yz,$$

which becomes

$$\vartheta(\vartheta+n\alpha_1)\dots(\vartheta+n\alpha_{n-1})y=yt^n,$$

if by analogy with the treatment of Bessel functions we put

$$z=(t/n)^n,$$

t/n being any one of the nth roots of z. Putting $y=e^tt^xy$, the equation becomes

$$(\vartheta+t-x)(\vartheta+t+n\alpha_1-x)\dots(\vartheta+t+n\alpha_{n-1}-x)y_1=t^ny_1.$$

If x be chosen in such a way that the coefficients of the leading power of t on both sides are equal, the asymptotic expansion for ${}_0\mathrm{F}_{n-1}\{(t/n)^n\}$ will be

$$\Sigma f(n)e^tt^x\left(1+\frac{\alpha_1}{t}+\frac{\alpha_2}{t^2}\dots\right),$$

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* Communicated by the Author.

where $f(n)$ is independent of t and the summation is taken over the various values of t, which make

$$(t/n)^n = z,$$

the series $y_1 = 1 + \frac{\alpha_1}{t} + \frac{\alpha_2}{t^2} \ldots$ being obtained by solving in descending powers of t, the equation resulting from the equation

$$(\vartheta + t - x)(\vartheta + t + n\alpha_1 - x) \ldots (\vartheta + t + n\alpha_{n-1} - x) y_1 = t^n y_1,$$

by the substitution of the appropriate value for x. It is evident that the coefficients α_1, $\alpha_2 \ldots$ are independent of the particular nth root of z chosen. Clearly the relation between the coefficients α_1, $\alpha_2 \ldots$ will involve more than two consecutive ones; we shall therefore not attempt to find any but the first two.

The equation may be written in the form

$$\begin{aligned} y_1(t^n + {}_1a_1 t^{n-1} + {}_1a_2 t^{n-1} \ldots + {}_1a_{n-1} t + {}_1a_n) + \vartheta(y_1)\,({}_2a_1 t^{n-1} \\ + {}_2a_2 t^{n-2} \ldots + {}_2a_n) + \ldots + \vartheta^{n-1}(y_1)\,({}_{n-1}a_{n-1} t + {}_{n-1}a_n) \\ + \vartheta^n(y_1) = t^n y_1, \end{aligned}$$

and the coefficients ${}_ra_s$ can be found by induction and inspection. In order to find only the first few coefficients in the series of descending powers of v, only a selection of the ${}_ra_s$ need be determined.

Equating the coefficients of t^{n-1}, t^{n-2}, t^{n-3}, on either side to each other, we get the equations

$${}_1a_1 = 0,$$

$${}_1a_1 \alpha_1 + {}_1a_2 - {}_2a_1 \alpha_1 = 0,$$

$${}_1a_1 \alpha_2 + {}_1a_2 \alpha_1 + {}_1a_3 - ({}_2a_2 \alpha_1 + 2\,{}_2a_1 \alpha_2) + {}_3a_1 \alpha_1 = 0.$$

These equations determine x, α_1 and α_2. As to the value of the ${}_ra_s$ coefficients involved, it is evident that

$${}_2a_1 = n, \quad {}_3a_1 = \frac{n(n-1)}{2}, \ldots$$

The value of ${}_1a_1$ is $\frac{n(n-1)}{2} = \sum_1$, where $\sum_r$ represents the sum of the terms

$$-x, \quad n\alpha_1 - x, \ \ldots \ n\alpha_{n-1} - x,$$

taken r at a time. Since ${}_1a_1 = 0$ this gives

$$-x = \sum_{s=1}^{n-1} \alpha_s + \frac{n-1}{2}.$$

The values of the other relevant ${}_r a_s$ coefficients found by induction are as follows :

$$ {}_1a_2 = \sum_2 + \frac{(n-1)(n-2)}{2}\sum_1 + \frac{n(n-1)(n-2)(3n-5)}{24}, $$

$$ {}_1a_3 = \sum_3 + \frac{(n-2)(n-3)}{2}\sum_2 + \frac{(n-1)(n-2)(n-3)(3n-8)}{24}\sum_1 $$
$$ + \sum_{s=1}^{n} \frac{(s-1)(s-2)(s-3)^2(3s-8)}{24}, $$

$$ {}_2a_2 = \frac{n(n-1)(n-2)}{2} + (n-1)\sum_1. $$

α_1 and α_2 are then found to be

$$ {}_1a_2/{}_2a_1, \quad \left[{}_1a_2^{\,2} + ({}_3a_1 - {}_2a_2)\,{}_1a_2 + {}_2a_1 \cdot {}_1a_3\right]/2\,{}_2a_1^{\,2}, $$

respectively. Subsequent coefficients α_3, α_4 ... can be found in the same way, but they will be increasingly cumbrous. They are not, however, of the same practical importance, as α_1 and α_2 will, in general, alone be required in applications.

The form of the asymptotic expansion of ${}_0F_{n-1}(z)$ is then

$$ |z|^{-\frac{s_n}{n}} \sum_r f_r(n) e^{n\,{}_nz_r}\left(1 + \frac{\alpha_1}{n\,{}_nz_r} + \frac{\alpha_2}{n^2\,{}_nz_r^2} + \ldots\right), $$

where

$$ s_n = \sum_{s=1}^{n-1} \alpha_s + \frac{n-1}{2} $$

and ${}_nz_r$ represents an nth root of z, the summation being taken over the nth roots of z.

Now, in the case of a value of ${}_nz_r$ which has a negative real part, whatever the value of $f_r(n)$ the corresponding part of the asymptotic expansion, namely

$$ |z|^{-\frac{s_n}{n}} f_r(n) e^{n\,{}_nz_r}\left(1 + \sum_{v=1}^{\infty} \frac{\alpha_v}{n_v({}_nz_r)^v}\right), $$

is asymptotically equivalent to zero. Hence, although it is true that

$$ {}_0F_{n-1}(z) \sim |z|^{-\frac{s_n}{n}} \sum f_r(n) e^{n\,{}_nz_r}\left(1 + \sum_{v=1}^{\infty} \frac{\alpha_v}{n^v({}_nz_r)^v}\right) \quad . \quad (1) $$

$f_r(n)$ corresponding to a value of ${}_nz_r$ with a negative real part, can take any value whatever, and there is therefore no sense in which any one set of values of $f_r(n)$ corresponding to such values of ${}_nz_r$ gives the asymptotic behaviour of ${}_0F_{n-1}(z)$ when $|z|$ is large rather than any other. It would therefore be misleading to leave (1) as it stands, where certain functions of n are substituted as the values of $f_r(n)$; for the asymptotic equivalence persists, whatever value may be given to those of the set of $f_r(n)$ functions corresponding to nth roots of z with negative real parts. There is, indeed, no sense in which one can talk, for example in the case of n even, of a "sudden jump"* in the value of *the* function of n multiplying the exponential term $e^{-n|z|^{1/n}}$ as arg (z) changes from being positive to being negative, since the product of $e^{-n|z|^{1/n}}$ and any function of n whatever is asymptotically equivalent to zero, whatever the value of arg (z). We therefore proceed to find the *unique* set of functions of n, whose existence is already plain, which makes the asymptotic expansion of ${}_0F_{n-1}(z)$ equal to

$$|z|^{-s_n/n} \sum_{R({}_nz_r) \geqslant 0} f_r(n)\, e^{n\,{}_nz_r} \left(1 + \sum_{v=1}^{\infty} \frac{\alpha_v}{n^v({}_nz_r)^v}\right),$$

when $|z|$ is large, s_n being $\sum_{s=1}^{n-1} \alpha_s + \frac{n-1}{2}$.

If C is a closed contour containing the origin

$$\begin{aligned} {}_0F_n(1+\alpha_1,\ 1+\alpha_2, \dots 1+\alpha_{n-1}\ ;\ z) &= \frac{1}{2\pi i}\int_C {}_0F_{n-1}(1+\alpha_1\dots 1+\alpha_{n-1}\ ;\ t)\left[1 + \frac{z/t}{1+\alpha_n}\right. \\ &\qquad \left. + \frac{(z/t)^2}{1+\alpha_n\,.\,2+\alpha_n}\dots\right]\frac{dt}{t}\,..(2) \end{aligned}$$

Further, if the contour is so chosen that $|t|$ and $|z/t|$ tend to infinity as $|z|$ tends to infinity (and it is plain that such a choice is possible), it will be possible to obtain the leading terms of the asymptotic expansion of ${}_0F_n$ by approximating to the value, as $|z|$ tends to infinity, of the integral in which the leading terms of the asymptotic expansion of ${}_0F_{n-1}(t)$ and of

$$1 + \frac{z/t}{1+\alpha_n} + \frac{(z/t)^2}{1+\alpha_n\,.\,2+\alpha_n} + \dots$$

have been put in the place of these series.

---

* *Cp.* E. W. Barnes, Proc. Camb. Phil. Soc. 1906.

Now it has been established * that

$$1+\frac{u}{1+\alpha}+\frac{u^2}{1+\alpha\,.\,2+\alpha}+\ldots\sim e^u u^{-\alpha}\Gamma(1+\alpha)-\alpha/u-\frac{\alpha(\alpha-1)}{u^2}\ldots$$

$$=\left[e^u u^{-\alpha}-\frac{1}{\Gamma\alpha\,.\,u}-\frac{1}{\Gamma(\alpha-1)\,.\,u^2}-\frac{1}{\Gamma\alpha-2\,.\,u^3}\ldots\right]\Gamma(1+\alpha)$$

if $|\arg u|<\pi$, $u^{-\alpha}$ being interpreted as $e^{-\alpha\log u}$, $\log u$ having its imaginary part between $\pm i\pi$.

It remains to choose the correct hypothesis for the asymptotic expansion of ${}_0\mathrm{F}_{n-1}(z)$ and to prove its correctness by induction.

The relevant data are

(1) If $|z|$ is large and $|\arg z|<\pi$,

$$\mathrm{J}_n(z)\sim\left(\frac{2}{\pi z}\right)^{\frac{1}{2}}\Big[\cos\left(z-\tfrac{1}{2}n\pi-\tfrac{1}{4}\pi\right)\left[1+\frac{\beta_2}{z^2}+\ldots\right]$$

$$-\sin\left(z-\tfrac{1}{2}n\pi-\tfrac{1}{4}\pi\right)\left[\frac{\beta_1}{z}-\frac{\beta_3}{z^3}\ldots\right]\Big]$$

$$=\left(\frac{2}{\pi z}\right)^{\frac{1}{2}}\Big[e^{i\left(z-\frac{1}{2}s_2\right)}\left[1+\frac{\alpha_1}{iz}+\frac{\alpha_2}{i^2z^2}+\ldots\right]$$

$$+e^{-i(z-\frac{1}{2}s_2)}\left[1-\frac{\alpha_1}{iz}+\frac{\alpha_2}{i^2z^2}\ldots\right]\Big],$$

where $s_2=n+\frac{1}{2}$.

(2) If $|z|$ is large,

$${}_0\mathrm{F}_3(1+\alpha_1,\ 1+\alpha_2,\ 1+\alpha_3\ ;\ z)$$

$$\sim\frac{\Gamma 1+\alpha_1\,\Gamma 1+\alpha_2\,\Gamma 1+\alpha_3}{2^{5/2}\pi^{3/2}}z^{-\frac{1}{4}(\alpha_1+\alpha_2+\alpha_3+3/2)}\Big[e^{4z^{\frac{1}{4}}}$$

$$+e^{+4iz^{\frac{1}{4}}}e^{-\frac{\pi i}{2}(\alpha_1+\alpha_2+\alpha_3+3/2)}+e^{-4iz^{\frac{1}{4}}}e^{+\frac{\pi i}{2}(\alpha_1+\alpha_2+\alpha_3+3/2)}\Big]$$

$$=\frac{\Gamma 1+\alpha_1\,\Gamma 1+\alpha_2\,\Gamma 1+\alpha_3}{(2\pi)^{3/2}\sqrt{4}}z^{-\frac{s_4}{4}}\Big[e^{4z^{\frac{1}{4}}}+e^{4iz^{\frac{1}{4}}}e^{-\frac{\pi i s_4}{2}}$$

$$+e^{-4iz^{\frac{1}{4}}}e^{+\frac{\pi i s_4}{2}}\Big].$$

---

* Barnes, *loc. cit.*

Both of these results fall under the hypothesis that

$$
{}_0F_{n-1}(1+\alpha_1,\ 1+\alpha_2,\ \ldots\ 1+\alpha_{n-1}\ ;\ z)
$$

$$
\sim \frac{\Pi\Gamma(1+\alpha_s)}{(2\pi)^{\frac{n-1}{2}}\sqrt{n}} \sum_{R({}_nz_r)\geq 0} e^{n\,{}_nz_r}({}_nz_r)^{-s_n}\left[1+\sum_{v=1}^{\infty}\frac{\alpha_v}{n^v({}_nz_r)^v}\right],
$$

where $({}_nz_r)^{-s_n}$ means the complex number whose argument is $-s_n \times \arg({}_nz_r)$, and the summation is taken over those values of ${}_nz_r$, the nth root of z whose real parts are not negative.

We may now proceed to discuss the value as $|z| \longrightarrow \infty$ of the integral in which the integrand of the integral (1) has been replaced by the leading terms of the asymptotic expansions of the series.

We have

$$
2\pi i\, {}_0F_n(z) \sim \frac{\prod_1^n \Gamma(1+\alpha_s)}{(2\pi)^{\frac{n-1}{2}}\sqrt{n}} \int_{C\,r} \sum e^{n\,{}_nt_r}({}_nt_r)^{-s_n}
$$

$$
\times\left(e^{z/t}(z/t)^{-\alpha_n} - \sum_{v=1}^{\infty}\frac{t^v}{z^v\Gamma(\alpha_n-1+v)}\right)\frac{dt}{t}.
$$

The only limitations on C, which is a contour in the t-plane, are, that it shall enclose the origin and be such, that on it $|t|$ and $|z/t|$ are large when $|z|$ is large. We may therefore choose as the contour a circle with its centre at the origin and with radius δ, provided δ and $|z|/\delta$ are large when $|z|$ is large. Then

$$
{}_0F_n(z) \sim \frac{\prod_1^n \Gamma(1+\alpha_s)}{(2\pi)^{\frac{n+1}{2}}\sqrt{n}} \int_{-\pi}^{\pi} d\theta \sum_s e^{n\delta^{1/n} e^{i\frac{\theta+2s\pi}{n}}}\, \delta^{-\frac{s_n}{n}}\, e^{-i\frac{\theta+2s\pi}{n}s_n}
$$

$$
\times\left(e^{i(\phi-\theta)}\left(\frac{|z|}{\delta}\right)^{-\alpha_n} e^{-i\alpha_n(\phi-\theta)} - \sum_{v=1}^{\infty}\frac{\delta^v e^{iv\theta}}{z^v\Gamma(\alpha_n-1+v)}\right),
$$

the s-summation being taken over integer values of s, such that

$$
-\frac{\pi}{2} \leq \frac{\theta+2s\pi}{n} \leq \frac{\pi}{2},
$$

$\phi-\theta$ being the argument of z/t (requiring sometimes a different value of ϕ in different parts of the range), which lies between $-\pi$ and $+\pi$. [The special case when

$\phi - \theta = +\pi$ or $-\pi$ will not arise.] Let I_1 and I_2 represent

$$\int_{-\pi}^{\pi} \sum_s e^{n\delta^{1/n} e^{i\frac{\theta+2s\pi}{n}}} \delta^{-\frac{s_n}{n}} e^{-i\frac{\theta+2s\pi}{n}s_n} \times e^{\frac{\delta}{|z|}e^{i(\phi-\theta)}} \left(\frac{|z|}{\delta}\right)^{-\alpha_n} e^{-i\alpha_n(\phi-\theta)}\,d\theta$$

$$\int_{-\pi}^{\pi} \sum_s e^{n\delta^{1/n} e^{i\frac{\theta+2s\pi}{n}}} \delta^{-\frac{s_n}{n}} e^{-i\frac{\theta+2s\pi}{n}s_n} \times \sum_{v=1}^{\infty} \frac{\delta^v e^{iv\theta}}{z^v\Gamma(\alpha_n - 1 + v)}\,d\theta$$

respectively. I_1 is made up of a set of integrals of an oscillatory exponential type and we may approximate to their values by Kelvin's method. "Critical points" of the integrands of these integrals occur where the derivative of the exponent is zero, or

$$\frac{d}{d\theta}\left(n\delta^{1/n} e^{i.\frac{\theta+2t\pi}{n}} + \frac{\delta}{|z|}e^{i(\phi-\theta)}\right) = 0,$$

and are easily seen to exist when, and only when, $\delta = |z|^{n/n+1}$. Taking this value of δ, we satisfy the condition that on the contour C, $|t|$ and $|z/t|$ are large when $|z|$ is large.

The integrals to be evaluated for I_1 are then of the form

$${}_tI_1 = \int_{-\pi}^{+\pi} e^{|z|^{1/n+1}\left[ne^{i\frac{\theta+2t\pi}{n}} + e^{i(\phi-\theta)}\right]} \times |z|^{-\frac{s_n+\alpha_n}{n+1}} e^{-i\left(\frac{\theta+2t\pi}{n}s_n + (\phi-\theta)\alpha_n\right)}\,d\theta.$$

"Critical points" are those values of θ for which

$$\frac{\theta+2t\pi}{n} - (\phi-\theta) = 2\nu\pi$$

or
$$\frac{\theta+2t\pi}{n} - 2\nu\pi = \phi - \theta = \frac{\phi+2t\pi}{n+1} - 2\nu\pi \,.\, \frac{n}{n+1}. \quad . \quad (3)$$

Suppose $\theta = \phi - \zeta$ is a critical point of the integrand of ${}_tI_1$, that is to say $\phi - \zeta$ satisfies (3) and is less than or equal to π in absolute value. Then

$${}_tI_1 \sim \sum_{\zeta}\operatorname*{lt}_{\epsilon\to 0} |z|^{-\frac{s_n+\alpha_n}{n+1}} e^{-i\zeta(\alpha_n+s_n)} \int_{\phi-\zeta+\pi.}^{\phi-\zeta-\pi} e^{|z|^{1/n+1}e^{i\zeta}\left(n+1-\frac{n+1}{2n}\epsilon^2\ldots\right)}\,d\epsilon$$

$$\sim \sum_{-\frac{\pi}{2}\leq\zeta\leq+\frac{\pi}{2}} |z|^{-\frac{s_n+\alpha_n+\frac{1}{2}}{n+1}} e^{-i\zeta(\alpha_n+s_n+\frac{1}{2})} e^{(n+1)|z|^{1/n+1}e^{i\zeta}} \times \sqrt{\frac{2n\pi}{n+1}},$$

since as x tends to infinity

$$\int_0^x e^{-z^2}\,dz \sim \tfrac{1}{2}\sqrt{\pi}, \text{ if } -\frac{\pi}{2} < \arg x < \frac{\pi}{2}.$$

From the equations (3) it is plain that every value of ζ is such that

$$|z|^{1/n+1}e^{i\zeta}$$

is an $(n+1)$th root of z, and the summation is taken over those values of ζ alone which are less than or equal to $\pi/2$ in absolute value. Thus the asymptotic equivalent for ${}_t\mathrm{I}_1$ is the sum of certain terms of the form

$$\sqrt{\frac{2n\pi}{n+1}}\left({}_{n+1}z_r\right)^{-(s_n+\alpha_n+\frac{1}{2})}e^{n+1}{}_{n+1}z_r,$$

or since

$$s_{n+1}=s_n+\alpha_n+\tfrac{1}{2}$$

of the terms of the form

$$\sqrt{\frac{2n}{n+1}}\left({}_{n+1}z_r\right)^{-s_{n+1}}e^{n+1}{}_{n+1}z_r.$$

The question remains as to whether, when all the integrals of the term ${}_t\mathrm{I}_1$ are taken into account, any of the $(n+1)$th roots of z with a non-negative real part are excluded (it is of no importance whether those with negative real parts are excluded or not) and as to how many times terms corresponding to each of these roots occur. A slight consideration of the equations (3) shows that the first possibility is cut out. As to the second point, it is clear that only values of ζ less than or equal to $\pi/2$ in absolute value need be considered. In this restricted class of cases, it is plain that the same value of ζ cannot occur in the case of integrals corresponding to two different values of t. Hence

$$\frac{\prod\limits_1^n\Gamma(1+\alpha_s)}{(2\pi)^{\frac{n+1}{2}}\sqrt{n}}\mathrm{I}_1\sim\frac{\prod\limits_1^n\Gamma(1+\alpha_s)}{(2\pi)^{\frac{n}{2}}\sqrt{n+1}}\Sigma\left({}_{n+1}z_r\right)^{-s_{n+1}}e^{(n+1)}{}_{n+1}z_r$$

the summation being taken over values of ${}_{n+1}z_r$ satisfying the conditions

$$\left({}_{n+1}z_r\right)^{n+1}=z$$

$$-\frac{\pi}{2}\leq\arg{}_{n+1}z_r\leq\frac{\pi}{2},$$

$\left({}_{n+1}z_r\right)^{-s_{n+1}}$ being interpreted to mean the complex number whose argument is

$$-s_{n+1}\times\arg{}_{n+1}z_r.$$

The integral I_2 has no "critical points" in the Kelvin sense, and therefore by considerations which are sufficiently well known in applications of this method, the important "groups of errors" in the divergent development of the functions all arise from I_1. We shall, however, show independently that I_2 is negligible compared with I_1.

In considering I_2 we again choose a circle with centre $t=0$ and radius δ as the contour, but are free to choose δ as we please provided only that δ and $|z|/\delta$ are large when $|z|$ is large. Then

$$\left|\int_c e^{{}_nt_r}\, {}_nt_r{}^{-s_n} \times \frac{t^r}{z^r}\, dt\right|,$$

$$\leq \frac{1}{|z^r|}\int_c e^{n\delta^{1/n}} \left|\delta^{-s_n/n}\, e^{\frac{2\pi s}{n} is_n}\right| \delta^r\, dt,$$

$$< \frac{2\pi}{|z|^r} e^{n\delta^{1/n}}\, \delta^{-\frac{R(s_n)}{n}}\, e^{-\frac{2\pi s}{n} is_n}\, \delta^r.$$

But let δ be so chosen that

$$\delta = |z|^{n/2(n+1)}.$$

Then

$$\left|\int_c e^{{}_nt_r}\, t_r{}^{-s_n} \times \frac{t^r}{z^r}\, dt\right| \leq \mathrm{K} e^{n|z|^{\frac{1}{2(n+1)}}}\, |z|^{\frac{nr}{2(n+1)} - \frac{R(s_n)}{2(n+1)} - r}.$$

Whatever the value of r, this is negligible compared with any of the terms

$$e^{{}_{n+1}z_r} ({}_{n+1}z_r)^{-s_{n+1}}$$

in which $R(z_r) > 0$. Hence I_2 is negligible compared with I_1 and the asymptotic expansion for ${}_0F_n$ results in the form

$${}_0F_n(1+\alpha_1, \ldots 1+\alpha_n\,;\, z) \sim \frac{\prod\limits_{s=1}^{n} \Gamma(1+\alpha_s)}{(2\pi)^n \sqrt{n+1}} \sum_{|\arg({}_{n+1}z_r)| \leq \frac{\pi}{2}} e^{{n+1}\,{}_{n+1}z_r} ({}_{n+1}z_r)^{-s_{n+1}}.$$

This result is of a similar form to the one assumed for ${}_0F_{n-1}$ and already proved for ${}_0F_3$, and is therefore now established in the general case. The complete form is then

$$
{}_0F_n(1+\alpha_1, \ldots 1+\alpha_n\,;\ z) \sim \frac{\prod\limits_{s=1}^{n} \Gamma(1+\alpha_s)}{(2\pi)^n \sqrt{n+1}}
$$

$$
\times \sum_{|\arg {}_{n+1}z_r| \leq \frac{\pi}{2}} e^{(n+1)\,{}_{n+1}z_r} \left({}_{n+1}z_r\right)^{-s_{n+1}} \left(1 + \sum_{v=1}^{\infty} \frac{\alpha_v}{(n+1)^v ({}_{n+1}z_r)^v}\right).
$$

It is worth while to notice the forms which result for ${}_0F_{n-1}(z)$ and for ${}_0F_{n-1}(-z)$ when z is real and positive. If x is real, positive, and large, and $s_n = \sum\limits_1^{n-1} \alpha_r + \frac{n-1}{2}$

$$
1 + \frac{x}{1!\prod\limits_1^{n-1}(1+\alpha_r)} + \frac{x^2}{2!\prod\limits_1^{n-1}(1+\alpha_r)(2+\alpha_r)} + \ldots
$$

$$
\sim \frac{\prod\limits_1^{n-1}\Gamma(1+\alpha_r)}{(2\pi)^{\frac{n-1}{2}}\sqrt{n}}\, x^{-s_n/n} \left[e^{nx^{1/n}} \left[1 + \sum \frac{\alpha_v}{n^v x^{v/n}}\right] \right.
$$

$$
\left. + 2 \sum_{t=1}^{2t/n \leq \pi/2} e^{nx^{1/n} \cos \frac{2\pi t}{n}} \frac{\alpha_v}{n^v x^{v/n}} \cos\left[nx^{1/n} \sin \frac{2\pi t}{n} - \frac{2\pi t}{n}(s_n + v)\right]\right].
$$

As a first approximation we may record the result

$$
1 + \frac{x}{1!\prod\limits_1^{n-1}(1+\alpha_r)} + \ldots \sim \frac{\prod\limits_1^{n-1}\Gamma(1+\alpha_r)}{(2\pi)^{\frac{n-1}{2}}\sqrt{n}}\, x^{-s_n/n}
$$

$$
\times \left\{ e^{nx^{1/n}} + 2 \sum_{t=1}^{\frac{2t}{n} \leq \frac{1}{2}} e^{nx^{1/n} \cos \frac{2\pi t}{n}} \cos\left(nx^{1/n} \sin \frac{2\pi t}{n} - \frac{2\pi t s_n}{n}\right) \right\},
$$

or

$$
1 + \frac{(u/n)^n}{1!\prod\limits_1^{n-1}(1+\alpha_r)} + \ldots \sim \frac{\prod\limits_1^{n-1}\Gamma(1+\alpha_r)}{(2\pi)^{\frac{n-1}{2}}\sqrt{n}}\, n^{-s_n}
$$

$$
\times \left\{ e^u + 2 \sum_{t=1}^{\frac{2t}{n} \leq \frac{1}{2}} e^{u \cos \frac{2\pi t}{n}} \cos\left(u \sin \frac{2\pi t}{n} - \frac{2\pi t s_n}{2}\right) \right\},
$$

giving the familiar result

$$
I_\alpha(u) \sim \frac{e^u}{\sqrt{2\pi u}},
$$

and useful results in other cases such as

$$1+\frac{(u/r)^r}{1!\prod\limits_1^{r-1}1+\alpha_s}+\ldots\sim\frac{\prod\limits_1^{r-1}\Gamma 1+\alpha_s}{(2\pi)^{\frac{r-1}{2}}\sqrt{r}}(u/r)^{-s_r}$$

$$\times\left[e^u+2e^{u\cos\frac{2\pi}{r}}\cos\left(u\sin\frac{2\pi}{r}-\frac{2\pi s_r}{r}\right)\right],$$

when r is 4, 5, 6, or 7, and in the case of $r=8$

$$1+\frac{(u/8)^8}{1!\prod\limits_1^{7}1+\alpha_s}+\ldots\sim\frac{\prod\limits_1^{7}\Gamma 1+\alpha_s}{(2\pi)^{7/2}2\sqrt{2}}(u/8)^{-s_8}$$

$$\times\left[e^u+2e^{u\cos\frac{\pi}{4}}\cos\left(u\sin\frac{\pi}{4}-\frac{\pi s_8}{4}\right)+2\cos\left(u-\frac{\pi s_8}{4}\right)\right],$$

and so on.

Similar results are available in the case when z is negative and real. If x is large, real, and positive

$$1-\frac{x}{1!\prod\limits_1^{n-1}1+\alpha_r}+\ldots\sim\frac{\prod\limits_1^{n-1}\Gamma(1+\alpha_r)}{(2\pi)^{\frac{n-1}{2}}\sqrt{n}}x^{-s_n/n}$$

$$\times\sum_{t=1}^{\frac{2t-1}{n}\leq\frac{1}{2}}e^{nx^{1/n}\cos\frac{(2t-1)\pi}{n}}\left[\cos nx^{1/n}\sin\frac{2t-1}{n}\pi-\frac{2t-1}{n}\pi s_n\right.$$

$$\left.+\sum_{v=1}^{\infty}\frac{\alpha_v}{n^v x^{v/n}}\cos\left(nx^{1/n}\sin\frac{2t-1}{n}\pi-\frac{2t-1}{n}\pi(s_n+v)\right)\right]$$

and again, as a first approximation, we may put

$$1-\frac{x}{1!\,\Pi\,1+\alpha_r}+\ldots\sim 2\frac{\prod\limits_1^{n-1}\Gamma 1+\alpha_r}{(2\pi)^{\frac{n-1}{2}}\sqrt{n}}x^{-\frac{s_n}{n}}\sum_{t=1}^{\frac{2t-1}{n}\leq\frac{1}{2}}e^{nx^{1/n}\cos\frac{(2t-1)\pi}{n}}$$

$$\times\cos\left(nx^{1/n}\sin\frac{2t-1}{n}\pi-\frac{2t-1}{n}\pi s_n\right).$$

or

$$1-\frac{(u/n)^n}{1!\,\Pi\,1+\alpha_r}\;\ldots\sim 2\frac{\prod\limits_1^{n-1}\Gamma 1+\alpha_r}{(2\pi)^{\frac{n-1}{2}}\sqrt{n}}(u/n)^{-s_n}\sum_{t=1}^{\frac{2t-1}{n}\leq\frac{1}{2}}e^{u\cos\frac{(2t-1)\pi}{n}}$$

$$\times\cos\left(u\sin\frac{2t-1}{n}\pi-\frac{2t-1}{n}\pi s_n\right),$$

which gives the corresponding result in the case when $n=2$ to the one already pointed out for $I_n(v)$, viz.

$$J_n(v) \sim \sqrt{\frac{2}{\pi v}} \cos\left(v - \frac{\alpha}{2}\pi - \tfrac{1}{4}\pi\right),$$

and useful results such as

$$1 - \frac{(v/r)^r}{1\,!\,\prod\limits_1^{r-1} 1+\alpha_s} \ldots \sim \frac{\Pi\Gamma 1+\alpha_s}{(2\pi)^{\frac{r-1}{2}}\sqrt{r}} (v/r)^{-s_r} e^{u\cos\frac{2t-1\,.\,\pi}{n}}$$

$$\times \cos\left(u \sin\frac{\pi}{r} - \frac{\pi s_n}{r}\right),$$

for $r=1, 2, 3, 4, 5$; and for $r=6$,

$$1 - \frac{(u/6)^6}{1\,!\,\prod\limits_1^{5} 1+\alpha_s} \ldots$$

$$\sim \frac{\Pi\,\Gamma 1+\alpha_s}{(2\pi)^{5/2}\sqrt{6}} (u/6)^{-s_6}\Big[e^{u\cos\pi/6} \cos\left(u\sin\pi/6 - \pi/6(\Sigma\alpha + 5/2)\right)$$

$$+ \cos\left(u - \pi/6(\Sigma\alpha + 5/2)\right)\Big];$$

and so on.

The other $(n-1)$ solutions of the equation

$$\vartheta(\vartheta+\alpha_1)(\vartheta+\alpha_2)\ldots(\vartheta+\alpha_{n-1})y = zy,$$

satisfied by

$$y = {}_0F_{n-1}(1+\alpha_1, 1+\alpha_2, \ldots 1+\alpha_{n-1}; z),$$

are the series

$$z^{-\alpha_r}{}_0F_{n-1}(1+\alpha_1-\alpha_r, \ldots 1+\alpha_{r-1}-\alpha_r, 1-\alpha_r,$$

$$1+\alpha_{r+1}-\alpha_r, \ldots 1+\alpha_{n-1}-\alpha_r; z)$$

with $r=1, 2, \ldots n-1$. Calling these solutions $y_1, y_2, \ldots y_{n-1}$, since the sum of the parameters for y_r is $\Sigma\alpha - n\alpha_r$, it is plain that the asymptotic expansion of y_r is of the form

$$\Sigma f_r'(n) z^{-\alpha_r} . z^{-\frac{\Sigma\alpha - n\alpha_r + \frac{n-1}{2}}{n}} e^{n(z)^{1/n}}$$

or

$$\Sigma f_r'(n) z^{-\left(\Sigma\alpha + \frac{n-1}{2}\right)/n} e^{nz^{1/n}},$$

so that we have

$$y=\Sigma f_r(n)z^{-s_n/n}e^{nz^{1/n}},$$

$$y_1=\Sigma\,{}_1f_r(n)z^{-s_n/n}e^{nz^{1/n}},$$

$$y_2=\Sigma\,{}_2f_r(n)z^{-s_n/n}e^{nz^{1/n}},$$

$$y_{n-1}=\Sigma\,{}_{n-1}f_r(n)z^{-s_n/n}e^{nz^{1/n}},$$

where $s_n=\Sigma\alpha+\frac{n-1}{2}$.

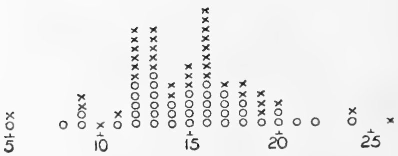
Linear combinations of $y, y_1, \ldots y_{n-1}$, therefore exist which behave, when $|z|$ is large, like $z^{-s_n/n}e^{nz^{1/n}}$, or like

$$z^{-s_n/n}e^{nz^{1/n}\cos\frac{2\pi t}{n}}\cos\left(nz^{1/n}\sin\frac{2\pi t}{n}-\frac{2\pi ts_n}{n}\right).$$

XV. *A Statistical Survey of the Colour Vision of* 1000 *Students. By* R. A. Houstoun, *D.Sc., Lecturer on Physical Optics, and* Margaret A. Dunlop, *Thomson Experimental Scholar in the University of Glasgow* *.

UNDER the title of "A Statistical Survey of Colour-Vision," there appeared a paper by Dr. R. A. Houstoun in the Roy. Soc. Proc. A, vol. xciv. p. 576 (1918). This paper described the results of a test made on the colour vision of 79 observers by means of an apparatus similar to the Edridge-Green colour perception spectrometer. Each observer determined the number of monochromatic patches he saw in a continuous spectrum; this number varied from five in the case of the colour-blind to twenty-six in the case of an observer with exceptionally good colour vision. The results were exhibited in the diagram (fig. 1) where the

Fig. 1.



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* Communicated by the Authors.

circles represent men and the crosses women. This diagram is a frequency curve of the kind used by statisticians, and the object of the investigation was to find whether this frequency curve was of the type represented by fig. 2 or fig. 3, *i. e.* whether the colour-blind were merely the "outliers of a homogeneous population" or whether they were an independent or independent homogeneous populations of their own. The second alternative is required by the Young-Helmholtz and Hering theories. The conclusion arrived at was that normal colour vision has quite enough "scatter" to include colour-blindness as an outlying portion of itself, but that it would require a very much greater number of observers than 79 to positively disprove the existence of separate maxima for the dichromats and monochromats.

Fig. 2.

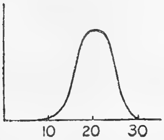
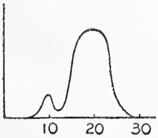


Fig. 3.

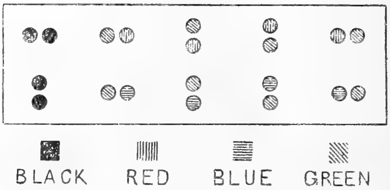


The present investigation is an attempt to get more certainty by using a greater number of observers. At the same time an entirely different experimental arrangement has been employed.

It was stated on p. 586 of the former paper that one had to trust to the judgment of the observer as to when the difference of colour appeared and that, while nothing occurred to indicate carelessness on the part of the observer, still it would be better if a method could be employed in which the readings of the observer were checked. Consequently attempts were made to devise a new apparatus, and after a little trouble the following very simple and satisfactory arrangement was adopted.

A low-power microscope was taken, one previously employed in the elementary laboratory course for determining the index of refraction of glass slabs. The focal length of its object-glass was 4·0 cm. and the height of the object-glass above the stage could be determined from a millimetre scale; when the microscope was focussed its tube moved up and down this scale. The microscope magnified about 25 times. On the stage was placed a slide carrying ten test objects; the test objects are represented in fig. 4. The test objects consisted of pairs of circular disks, which might either be black or red or green or blue. These disks were

Fig. 4.

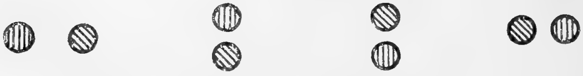


each slightly less than one millimetre in diameter. They were drawn with a pen all on one piece of cardboard, the black ones in Indian ink and the coloured ones in Dove's water-proof inks. The piece of cardboard measured 42 mm. by 15 mm., was protected by a glass cover, and was mounted on a strip of wood. The red ink, Dove's carmine, and the green ink, Dove's deep green, were used at the same strength as they came out of the bottle; the blue, Dove's Prussian blue, was too dark when used at full strength, and was consequently diluted with water so as to have the same luminosity as the other two colours. The observer's colour vision was measured by the distance out of focus at which he could just distinguish the different colours.

Before commencing the test the observer was required to read the following instructions which were typewritten on a piece of cardboard with the disks entered in their appropriate colours:—

(1) First focus the two vertical black spots ⁚ as sharp as possible. Next focus the two horizontal black spots as sharp as possible ●●.

(2) Screw down until you can just say whether the object is the two horizontal or the two vertical black spots. Repeat. In this and the following tests the microscope is to be as far out of focus as possible consistent with your distinguishing the one case from the other. The distance from focus is to measure the excellence of your vision.



(3) Screw down until you can just see which of the above four objects you have, then say whether the red is above, below, to the right or to the left. Repeat.




(4) Screw down until you can just see which of the above four objects you have, then say whether blue is above, below, to the right or to the left.

Perhaps the nature of the test will be understood best from the figures for a particular case. The vertical black spots were focussed sharp at 34·5 and the horizontal black spots at 34·3 mm. The mean, 34·4, was taken as the observer's focus. He was then given the horizontal black disks very far out of focus, so that there was nothing visible at all in the field, and screwed slowly down until he could just recognize them. This he did at 48. Similarly he could just recognize the vertical black disks at 49·4. The differences of these two numbers from 34·4, *i. e.* 13·6 and 15, were taken as a measure of his ability to distinguish black from white. The supervisor next placed a red-green pair in the field far out of focus. The observer looking into the microscope saw nothing at all in the field; he screwed down and recognized the object as "red left" at 49·7. He was then given another red-green pair and recognized it as "red right" at 52·5. Consequently his ability to distinguish red from green was measured by 15·3 and 18·1. He was next given a green-blue pair and recognized it as "blue above" at 49. Finally he recognized "blue below" at 45·9, so that his ability to distinguish blue from green was measured by 14·6 and 11·5. Very often, when he was first given the black disks out of focus, owing to having not fully understood the instructions the observer screwed down much too far and asked permission to repeat the determination. This was always granted in the case of the first test with the black disks, but not in the case of the other tests. The test with the black disks was arranged principally for giving familiarity with the apparatus before the colour tests were entered upon.

As the coloured disks were upon a white background, and as they merged with the background when they were out of focus, the effect of screwing out of focus is simply to mix more and more white with the colour in question, until we have finally, in the case of "red above" for example, merely a white field, with a reddish tinge near its centre and immediately below this a greenish tinge. Each observer took about seven minutes in all. The tests were made between January 1919 and March 1920. Of the observers, one was a member of the University staff, 336 were arts or science men students, 84 were arts or science women students, 261 were first year men medical students, 79 were first year women medical students, and the remainder 239, 237 men and 2 women, were first year Technical College students, whom we were able to test through the courtesy of Prof. Muir and the staff of the Natural Philosophy Department of the Technical College. All the tests were made in the course of the student's work in the physical laboratory; each man simply left the optical bench or spectrometer or whatever experiment he was working at, and then returned to it again when the test was completed. Only owing to the large number of demobilised men in the laboratory courses was it possible to carry through the tests with so little trouble; the students were older and, especially the medicals, more capable than usual, and such a favourable opportunity for making such a survey will probably not occur again. The numbers are not the full numbers in the laboratory courses. Some students were omitted through absence or through their not reaching the optical laboratory; the observers were not selected in any way, but just taken as they came. One student declined when asked to make the test, as his experiment was at a critical stage at the time, and as it was the last week in the session no opportunity occurred to test him again, but his was a solitary case. No one else declined.

The observer could not see the test object except through the microscope. In the tests with the black disks the horizontal spots were sometimes given twice and the vertical spots not at all, or the vertical spots might be given twice and the horizontal not at all, but in the case of the coloured disks it was always a different object each time.

After the tests were completed we determined the average distances from focus at which each of the ten test objects had been recognized. They were: black horizontal 13·00 mm., black vertical 12·83, red left 15·24, red above 14·89, red below 15·52, red right 16·13, blue left 15·70, blue above 15·44, blue below 13·59, blue right 13·76. Thus all the objects had not been equally difficult: to allow for this the readings for red right were decreased in the proportion 16·13 to 15·2, those for blue below increased in the proportion 13·59 to 15·6, and blue right in the proportion 13·76 to 15·6.

This does not make the average distance the same in each case, but it reduces the inequality sufficiently to make it negligible. Thus to return to the example already cited by way of illustration, the observer's values for the black disks 13·6 and 15 were left as they were, his values for the red-green were corrected to 15·3 and 17·06 and his values for the blue-green to 14·6 and 13·2. This gives a mean value of 14·3 for the black, 16·18 for the red-green, and 13·9 for the blue-green. These three numbers were assumed to specify the observer's ability to distinguish black from white, red from green, and blue from green respectively. Similar

Table.

Range.	Men.			Women.		
	Black.	Red-green.	Blue-green.	Black.	Red-green.	Blue-green.
0 & < 1	...	3				
1 & < 2	...	2	...	...	1	
2 & < 3	...	2	...	...	0	
3 & < 4	1	2	...	...	0	
4 & < 5	0	1	1	...	0	
5 & < 6	0	2	1	...	0	
6 & < 7	2	1	3	2	1	
7 & < 8	7	5	4	1	0	2
8 & < 9	8	5	4	1	1	1
9 & <10	16	13	13	5	2	0
10 & <11	68	23	17	21	7	3
11 & <12	139	40	46	25	5	6
12 & <13	202	68	57	46	18	10
13 & <14	177	95	75	35	21	21
14 & <15	104	120	109	14	16	20
15 & <16	51	115	132	6	24	27
16 & <17	30	114	142	6	20	16
17 & <18	13	71	90	2	18	16
18 & <19	11	61	64	0	16	13
19 & <20	4	49	37	0	6	13
20 & <21	1	30	17	0	4	10
21 & <22	0	6	10	0	3	0
22 & <23	0	3	3	1	1	2
23 & <24	0	1	3	...	1	2
24 & <25	1	2	1	...	...	3
25 & <26	...	1	5			
26 & <27	..	...	0			
27 & <28	...	...	0			
28 & <29	...	...	1			

numbers were determined for all the observers. The results of the tests are given in the preceding table and are represented in figs. 5 to 10. We learn, for example, from the table that 68 men recognized the black disks between 10 and 11 mm. from focus, the range including 10 mm. itself but stopping short of 11 mm., and that 21 women recognized the same disks within the same range.

The question naturally arises, as to how far the readings taken with the microscope can be regarded as a reliable test of colour vision. They were checked to some extent by Dr. Edridge-Green's bead test. In this test the observer is provided with a number of coloured beads and four little boxes labelled red, yellow, green, and blue, and is asked to put the beads into the appropriate boxes: once the bead goes through the hole in the box, it is lost from view. There is no doubt that this is a very effective test, not so much by reason of what the observer does eventually, as by how he does it; if he hesitates long with a pink over the mouth of the hole marked blue, his colour vision cannot possibly be normal. Generally speaking the two methods agreed: if an observer could not distinguish red from green with the microscope except close up, he put some beads in the wrong holes and *vice versa*. But there were exceptions: one man whose mean distance for distinguishing red from green was 2·99 mm. appeared quite normal under the bead test, and another who distinguished red from green at 12·5 mm. put a bright red into the green hole. This man was examined by the bead test at his own request.

Prof. Karl Pearson has established a system of formulæ for representing frequency distributions. A short account of these formulæ is given in his "Tables for Statisticians and Biometricians," p. lx, and a fuller account together with the method of deriving their constants by means of the moments of the distributions together with worked examples is given in Elderton's "Frequency Curves and Correlation." In figs. 5 to 10 the area of each rectangle represents the number of observers in one group. If the number of observers was increased to many thousands and the number of groups also increased, the rectangles would become narrower, until finally in the limit the steps would disappear altogether, and the frequency distribution would be represented simply by a smooth curve. The frequency distributions are not smooth curves because the observations are not numerous enough. Are they reasonable approximations to Pearson's types, considering the limitations of the material? It is possible to answer this question by comparing the observed distribution with one calculated by the formula and determining a function P from a table given in the Tables above referred to. If for example P comes out ·20, in 20 out of 100 trials we should get in random sampling a fit as bad or worse than the observations actually give. It is not possible to specify exactly when a good fit ends and a bad one begins, but apparently above ·3 is good and above ·1 not unreasonable. The final value of P depends considerably on how the distribution is grouped.

When P is worked out for the observations recorded in the former paper, assuming a Gaussian distribution, the value for the 43 men is ·73 and for the 38 women ·14, but by taking the women in five groups the value for P becomes as high as ·6. Consequently the Gaussian distribution is a good fit, and no significance is to be attributed to the minimum at 14 referred to on p. 584 of the former paper: it was due simply to the number of observers being so small, and would have vanished, had there been more of them.

The results of the red-green test for the men are shown by fig. 5. One of the observers could not distinguish the red disks from the green disks even when perfectly focussed, another could just distinguish them when focussed, and a third distinguished them one millimetre out of focus on the other side. These three are grouped together in the rectangle from 0 to 1. Some doubt might be expressed as to whether this is justifiable, but it does not affect the estimate of goodness of fit appreciably, as for this purpose the first ten observers were grouped together. The broken line represents the graduation of the data by the formula

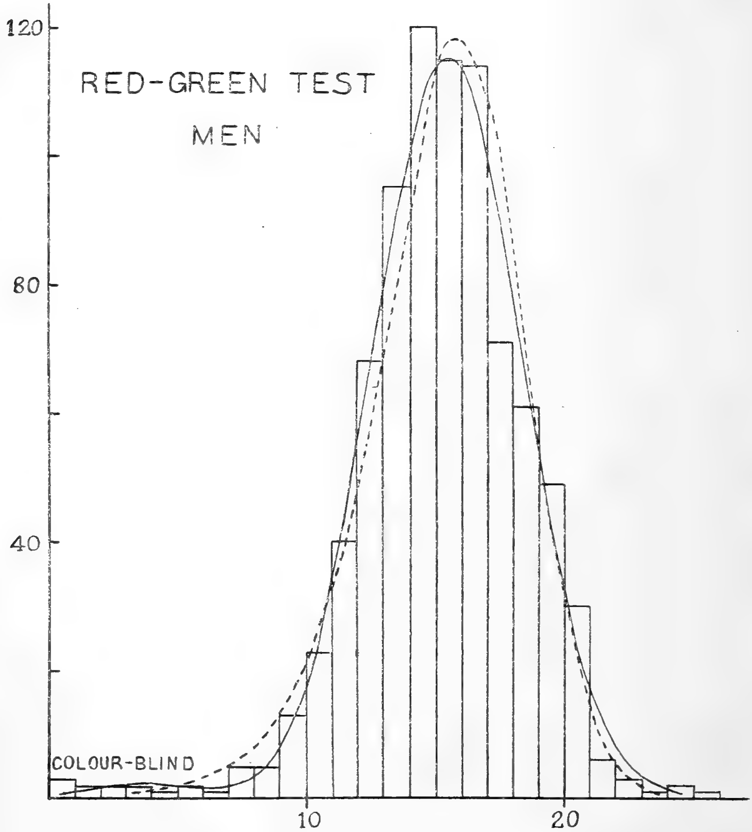
$$y_0 \frac{e^{-\nu \tan^{-1} x/a}}{(1+x^2/a^2)^m},$$

where $y_0 = 30·82$, $\nu = 5·695$, $m = 5·799$, $a = 7·946$, and the origin is taken at 19·942. This is the most suitable formula according to the classification, since $\beta_1 = ·6203$ and $\beta_2 = 4·981$. It gave $P = ·008$. An attempt was also made to represent the data by the formula $y_0 x^{-p} e^{-\gamma/x}$, but it gave a value of P less than one ten-millionth. It is not so much the presence of the tail on the left as the fact that the central maximum is too symmetrical, that makes it difficult to represent the whole distribution by these formulæ.

We next tried to represent the results as the superposition of two Gaussian distributions, a large one with its mean at

15·47, $\sigma=2·839$ and its area $=821·9$, and a small one with its mean at 3·7, $\sigma=2·5$ and its area $=13·8$. These values were obtained by trial and error. The curve thus obtained is represented by the full line in fig. 5 and gives $P=·71$, *i. e.* in 71 out of 100 trials we should get in random sampling a fit as bad or worse. Since the first ten observations were grouped, this ·71 is no guarantee for the presence or shape of the small curve, only for the main distribution being a very good fit to the Gaussian formula. If the calculated

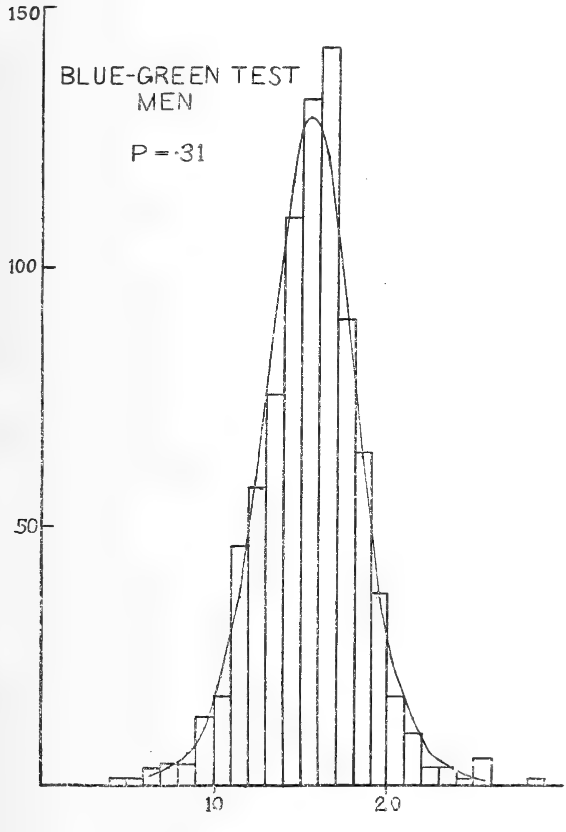
Fig. 5.



Abscissæ represent distance from focus in millimetres, ordinates number of observers.

main distribution is subtracted from the observations, and the remainders on the left compared with the small curve, taking the groups singly, P comes out ·08; we do not know, however, whether the theory can be applied to such a small number of cases, and in any case the influence of the somewhat dubious 3 in the first group now becomes very great.

Fig. 6.

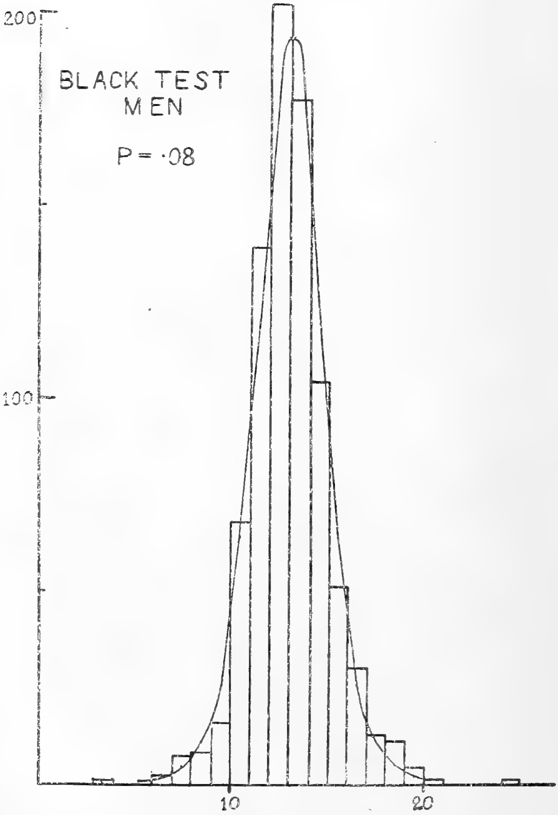


Abscissæ represent distance from focus in millimetres, ordinates number of observers.

Fig. 6 gives the results of the blue-green test for the men. The curve shows their graduation by $y_0/(1+x^2/a^2)^m$, where $y_0=128{\cdot}0$, $a=7{\cdot}244$, $m=4{\cdot}624$, and the origin is taken at 15·55. P works out at ·31, so the fit is a good one.

Fig. 7 gives the results of the tests with the black disks for the men. The curve shows the graduation by $y_0/(1+x^2/a^2)^m$, where $y_0=193{\cdot}5$, $a=4{\cdot}162$, $m=3{\cdot}671$, and the origin is taken at 13·022. P works out at ·03. If, however, the formula $y_0e^{-\nu\tan^{-1}x/a}/(1+x^2/a^2)^m$ is used, P becomes ·08.

Fig. 7.



Abscissæ represent distance from focus in millimetres, ordinates number of observers.

Figs. 8, 9, and 10 give the results for the black, red-green, and blue-green tests for the women. The smooth curve in the case of fig. 8 is given by $y_0/(1+x^2/a^2)^m$, where $y_0=39{\cdot}27$, $a=3{\cdot}800$ $m=3{\cdot}319$, and the origin is at 12·63. The smooth

Fig. 8.

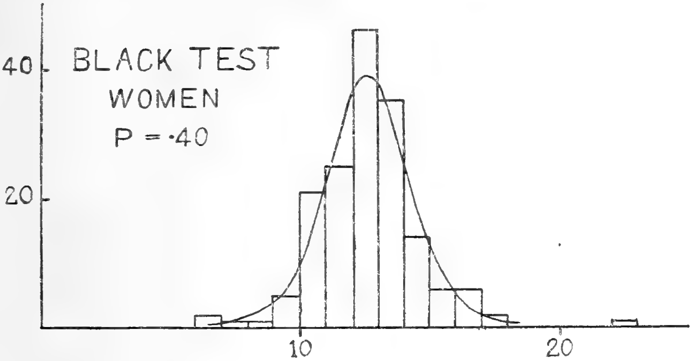


Fig. 9.

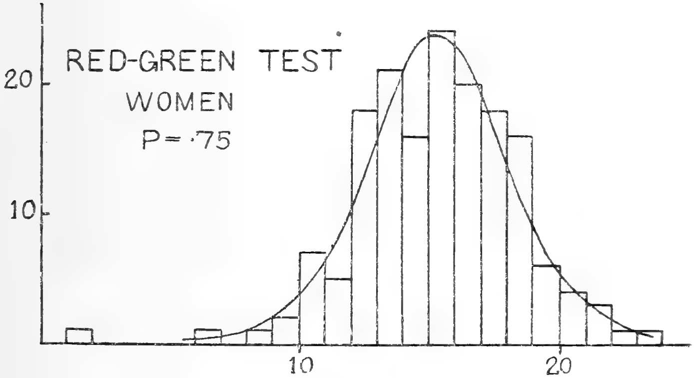
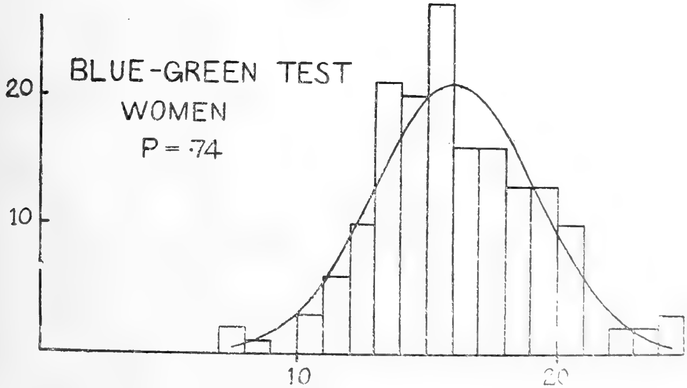


Fig. 10.



Abscissæ represent distance from focus in millimetres, ordinates number of observers.

curve in the case of fig. 9 is given by the same expression, where $y_0=24{\cdot}29$, $a=7{\cdot}020$, $m=4{\cdot}101$, and the origin is at 15·33. The smooth curve in the case of fig. 10 is a Gaussian one, σ being $=3{\cdot}129$, while the origin is at 16·06 and $y_0=21{\cdot}04$. The observer shown at 1·5 in fig. 9 was examined by the bead test and appeared normal. P has the values ·40, ·75, and ·74 for figs. 8, 9, and 10 respectively. Sheppard's adjustments were not used except for fig. 5 and ordinates were taken instead of areas except for fig. 7, so some of the values of P obtained may be improved upon.

Thirteen men obtained a value less than 7 in the red-green test. The mean value of their readings for the black is 10·95 with a probable error of ±1·83. The mean value of all the men for the black was 13·022. The mean value of the same thirteen observers for the blue-green test was 10·61 with a probable error of ±1·97. The mean value for all the men was 15·547. Hence the red-green colour blind are decidedly below the average in their ability to distinguish blue from green, and somewhat below the average in their ability to distinguish black from white. The numbers obtained by these thirteen observers for the black test were plotted against their numbers for the red-green test, as were also the numbers obtained for the blue-green test. But there was no correlation visible from the diagram.

The following observations were obtained by some of the colour-blind, all men. The number in brackets is the observer's number, then come in order the final corrected figure for the black, red-green, and blue-green test.

(200)	13·55	0	15·5	Failed in bead test. Could not distinguish red from green at focus.
(211)	14	2·95	15·23	Failed in bead test.
(246)	13·25	2·99	10·15	Normal in bead test.
(269)	11·05	1·99	13·59	Mistakes in bead test.
(316)	14·4	6·2	10·7	Normal in bead test.
(387)	18·1	12·5	16·24	Failed in bead test, put bright red in green hole.
(476)	12·15	10·25	13·86	Said he had difficulty in distinguishing blue from green.
(554)	9·45	4·01	12·33	Had been rejected by Admiralty lantern test.
(579)	11·2	1·2	10·83	Said at first both red and green spots were green.
(634)	9·6	−1·11	8·54	Said red spot was black.
(694)	15	11·79	14·6	Said he recognized red as the darker colour.

From previous tests made by Dr. Houstoun it has been found that quite 4 per cent. of men have difficulty in distinguishing blues from greens, not doubtful cases like peacock-blue, but quite decided tints. This would include the observers in fig. 6 up to 11 mm.

When we come to sum up the results of our investigation, we are faced at the outset with the difficulty that there is apparently no satisfactory definition of a homogeneous population. Perhaps the best definition is a distribution with a frequency curve fitting one of Pearson's formulæ. If we take this definition and reject the fourteen men on the left of fig. 5, all our distributions may be regarded as homogeneous, as variations about a mean: all the blue-green colour-blindness and a part of the red-green colour-blindness is thus an outlying part of a homogeneous distribution. But we cannot think of a simple analogue for the fourteen rejected observers.

The striking fact brought out by the investigation is that fig. 5 is a very good Gaussian distribution, if the tail of fourteen observers on the left is rejected. This seems to suggest a difference in kind between the colour blindness of the rejected observers and the colour blindness of those in the outlying portion of the main curve.

We have not yet followed out all the consequences of the investigation, but from the standpoint of Hering's theory figs. 6 and 7 show that none of the men were wanting in the blue-yellow or the black-white processes. The fourteen rejected observers must consequently be deficient in the red-green process. From the standpoint of the Young-Helmholtz theory fig. 6 shows that none of the men lacked the blue or green sensations. The fourteen rejected observers must consequently lack the red sensation. But 14 out of 835 is a smaller percentage than these theories require, and the distribution of the 14 does not suggest a well-defined class. Also the bases of these theories deserve reconsideration in view of the large rôle normal variation has now been shown to play in the matter.

Perhaps it is not superfluous to say a word here on the position of those who do not accept the Young-Helmholtz theory, as their views are often misunderstood. The fact that normal colour vision is trichromatic was really discovered by Newton ('Opticks,' Book I. Prop. VI. Problem II.). It is implied in the somewhat fanciful colour diagram he gives, though not expressly stated in words. His diagram makes it clear, that almost all colours can be produced by mixing a red, a green, and a blue. This fact is accepted by all theorists, and forms the basis of the three-colour process of photography. It was not until two centuries later that Helmholtz took the additional step of assuming that there were three elementary colour sensations, and that colour blindness was accounted for by the absence of one or more of these sensations. In this he had been anticipated by Young. It is desirable in the interests of clearness, that the work of Newton and Helmholtz should not be confused and the name trichromatic theory not employed, when the Young-Helmholtz theory is referred to.

We are indebted to Mr. Alex. H. Gray for assistance with some preliminary work involved in the investigation, and to Prof. Karl Pearson for two letters of advice and criticism, which were of great help in planning the investigation. We are also indebted to Sir Oliver Lodge for criticism, which enabled us to effect a considerable improvement in the latter part of the paper.

XVI. *Bessel Functions of small Fractional Order and their application to problems of Elastic Stability.* *By* JOHN R. AIREY, *M.A., D.Sc.*[*]

MANY problems of elastic stability depend for their solution upon the roots of Bessel functions of small fractional order, *e. g.* the stability of a flat bar or blade bent in its own plane[†], or of a long thin rod placed vertically and clamped at the lower end[‡]. For a deep horizontal beam fixed at one end and supporting a load at the other, the condition of equilibrium is given by the roots of $J_{-\frac{1}{4}}(x)=0$; and for a beam carrying a uniform load, the roots of $J_{-\frac{1}{6}}(x)=0$. Generally, if the bending-moment is proportional to x^n, where x is the distance from the free end of the beam, the condition of stability is found from the roots of $J_{-p}(z)=0$, where $p=\frac{1}{2(n+1)}$ and $\left(\frac{z}{2p}\right)^{2p}=x$.

When the clamped vertical rod is of uniform circular section, the height consistent with stability is subject to the condition that $J_{-\frac{1}{3}}(x)=0$, and in the general case, where the rod is a solid of revolution, if r the section radius at a depth x below the top $=\lambda x^m$, and W the weight of the rod above the section $=\mu x^n$, the solution of the problem depends upon the roots of $J_p(x)=0$ where $p=\frac{4m-1}{n-4m+2}$. This

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[*] Communicated by the Author.

[†] "Elastic Stability of Long Beams under transverse forces." A. G. M. Michell. Phil. Mag 5th series, vol. xlviii. Sept. 1899, pp. 298–309.

"The Buckling of Deep Beams." J. Prescott. Phil. Mag. 6th series, vol. xxxvi. Oct. 1918, pp. 297–314.

[‡] "Height consistent with Stability." Sir George Greenhill. Proc. Camb. Phil. Soc. vol. iv. Feb. 1881, pp. 65–73.

reduces to $J_{-\frac{1}{3}}(x)=0$ when $m=0$ and $n=1$, *i. e.*, for a homogeneous rod of uniform section.

The roots ρ_s of $J_n(x)$ where the order n of the function is fractional. viz. $\pm\frac{1}{3}$, $\pm\frac{2}{3}$, $\pm\frac{1}{4}$, and $\pm\frac{3}{4}$, have been calculated by Dinnik[*] to two places of decimals from tables computed to four places for $x=0{\cdot}0$ to $8{\cdot}0$ by intervals of $0{\cdot}2$.

n.	ρ_1.	ρ_2.	n.	ρ_1.	ρ_2.	ρ_3.
$\frac{3}{4}$ ...	3·49	6·65	$-\frac{3}{4}$ ...	1·06	4·29	7·44
$\frac{2}{3}$ ...	3·38	6·53	$-\frac{2}{3}$ ...	1·25	4·43	7·58
$\frac{1}{3}$ ...	2·90	6·03	$-\frac{1}{3}$ ...	1·88	4·99	8·12
$\frac{1}{4}$ ...	2·78	5·91	$-\frac{1}{4}$ ...	2·01	5·12	8·25

Tables of $J_{\pm\frac{1}{3}}(x)$ are given by G. N. Watson[†] to five places of decimals for values of x from $0{\cdot}00$ to $2{\cdot}00$ by intervals of 0 05 and from $2{\cdot}0$ to $8{\cdot}0$ by intervals of $0{\cdot}2$.

Roots of $J_n(x)$.

The first root of $J_n(x)$ is the most important in its application to physical problems. When n is small, the value of this root can be found by a method introduced by Girard[‡] in 1629 and employed by Lord Rayleigh[§] in calculating the roots ρ_1 of $J_0(x)$ and $J_1(x)$.

The sum of the powers of the reciprocals of $\rho_1, \rho_2, \rho_3 \ldots$, the roots of $J_n(x)$ are expressed in terms of the coefficients of the expansion from the identity

$$1-\frac{x^2}{2(2n+2)}+\frac{x^4}{2\,.\,4(2n+2)(2n+4)}-\ldots\ldots$$

$$=\left(1-\frac{x^2}{\rho_1{}^2}\right)\left(1-\frac{x^2}{\rho_2{}^2}\right)\left(1-\frac{x^2}{\rho_3{}^2}\right)\ldots\ldots \quad . \quad (1)$$

by taking logarithms of both sides, expanding and equating like powers of x. Thus

$$\Sigma\rho_s{}^{-10}=\frac{7n+19}{2^9(n+1)^5(n+2)^2(n+3)(n+4)(n+5)} \cdot\cdot \quad (2)$$

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[*] "Tafeln der Besselschen funktionen." A. Dinnik. *Archiv der Math. und Physik,* Band 18, p. 338 (1911); Band 21, p. 326 (1913).

[†] "The Zeros of Bessel Functions." G. N. Watson. Proc. Royal Soc. A. vol. xciv. pp. 190–206 (1918).

[‡] *Calcul des Dérivations,* Arbogast, 1800, pp. 56–57.

[§] "The Numerical Calculations of the Roots of Fluctuating Functions." Lord Rayleigh. Proc. Lond. Math. Soc. vol. v. pp. 119–124 (1874); or Collected Works, vol. i. pp. 190–195.

For small values of n, the approximate value of

$$\rho_2=\pi(2n+7)/4$$

only is required to give the value of ρ_1 to five or six places of decimals. The first root ρ_1 of $J_n(x)=0$ for $n=-\frac{1}{2}$, $-\frac{3}{8}\ldots+\frac{7}{8}$, $+1$ by intervals of $\frac{1}{8}$, thirteen values in all, were computed in this manner and intermediate values obtained by central difference interpolation. The second root, ρ_2, is correctly given to four places by MacMahon's formula * derived from Jacobi's asymptotic expansion † of $J_n(x)$. The values of ρ_2 of $J_n(x)$ for the foregoing values of n were calculated to six places of decimals from the expansion of the function in ascending powers of the variable. In the following tables, the colon **:** indicates that the fifth place of decimals is approximately 5; thus ρ_1 of $J_{-\frac{3}{8}}(x)$ may be written 1·79465, the exact value to five places being 1·79463.

n.	ρ_1.	ρ_2.	n.	ρ_1.	ρ_2.
$-\frac{1}{2}$ ...	1·5708	4·7124	$\frac{1}{4}$ ...	2·7809	5·9061 :
	1·6167 :	4·7541 :		2·8175 :	5·9442
	1·6620 :	4·7958		2·8541	5·9822
	1·7068	4·8372		2·8905	6·0201
	1·7509 :	4·8785		2·9267 :	6·0579 :
$-\frac{3}{8}$ ...	1·7946 :	4·9196 :	$\frac{3}{8}$ ...	2·9628 :	6·0957
	1·8378	4·9606		2·9988 :	6·1333 :
	1·8805 :	5·0014 :		3·0347	6·1709
	1·9228 :	5·0421		3·0704 :	6·2084
	1·9647 :	5·0826 :		3·1061	6·2458 :
$-\frac{1}{4}$ ...	2·0063	5·1230 :	$\frac{1}{2}$ ...	3·1416	6·2832
	2·0475	5·1633		3·1770	6·3204 :
	2·0883	5·2034 :		3·2123	6·3576 :
	2·1288 :	5·2434 :		3·2474 :	6·3947 :
	2·1690 :	5·2833 :		3·2825 :	6·4318
$-\frac{1}{8}$ ...	2·2090	5·3231	$\frac{5}{8}$ ...	3·3175	6·4688
	2·2486 :	5·3627		3·3524	6·5057
	2·2880 :	5·4022		3·3872	6·5425 :
	2·3272	5·4416		3·4219	6·5793
	2·3661 :	5·4809		3·4565	6·6160
0 ...	2·4048	5·5200 :	$\frac{3}{4}$ ...	3·4910	6·6526 :
	2·4433	5·5591 :		3·5254	6·6892
	2·4815 :	5·5981		3·5597 :	6·7257
	2·5196	5·6369		3·5940	6·7621 :
	2·5574 :	5·6757		3·6282	6·7985 :
$\frac{1}{8}$ ...	2·5951	5·7143 :	$\frac{7}{8}$ ...	3·6623	6·8348 :
	2·6326	5·7529		3·6963	6·8711 :
	2·6699	5·7913 :		3·7302 :	6·9073 :
	2·7070 :	5·8297		3·7641 :	6·9435
	2·7440 :	5·8679 :		3·7979 :	6·9795 :
$\frac{1}{4}$ ...	2·7809	5·9061 :	1 ...	3·8317	7·0156

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* "The Roots of Bessel and other related functions." Gray and Mathews, 'Bessel Functions,' p. 241.

† *Astron. Nachrichten*, Band 28, p. 94 (1848).

The roots ρ_1 of $J_n(x)$ for values of n between $-\frac{1}{2}$ and -1 are tabulated in the last section. From the five place table, the following values of ρ_1 have been found :—

$J_{-\frac{1}{3}}(x)$, 1·8663 : $J_{-\frac{1}{6}}(x)$, 2·1423

$J_{\frac{1}{3}}(x)$, 2·9026 $J_{\frac{1}{6}}(x)$, 2·6575

Roots of $J_n'(x)$.

Although of less importance in the solution of problems in physics and applied mathematics than the roots of $J_n(x)=0$, the values of the argument making $J_n(x)$ a maximum or a minimum are required in some cases. When the order n is negative, the function $\frac{d}{dx}\cdot J_n(x)$, where $-1<n<0$, has a pair of imaginary roots. From the identity

$$1-\frac{n+2}{n}\cdot\frac{x^2}{4(n+1)}+\frac{n+4}{n}\cdot\frac{x^4}{4\,.\,8(n+1)(n+2)}-\ldots\ldots$$

$$=\left(1-\frac{x^2}{\sigma_1{}^2}\right)\left(1-\frac{x^2}{\sigma_2{}^2}\right)\left(1-\frac{x^2}{\sigma_3{}^2}\right)\ldots\ldots \quad . \quad (3)$$

where σ_1, σ_2, σ_3... are the roots of $J_n'(x)=0$, we find that

$$\Sigma\,.\,\sigma_s{}^{-6}=\frac{n^3+16n^2+38n+24}{4\,.\,8\,.\,n^3(n+1)^3(n+2)(n+3)}\cdot \quad . \quad . \quad . \quad (4)$$

As in the case of $J_n(x)$, only the approximate values of $\sigma_2=\pi(2n+9)/4$, $\sigma_3=\pi(2n+13)/4$... are required to give σ_1 with considerable accuracy. A table of the roots of $J_n'(x)$, especially the imaginary roots, cannot be conveniently used over the full range from $n=-\frac{1}{2}$ to $+\frac{1}{2}$ for determining intermediate values by interpolation as the second and third differences are considerable. This difficulty can be removed by tabulating $\delta=\sigma_1/\sqrt{2n}$, where linear interpolation only is needed. The root σ_1 is then easily calculated from the value $\delta\sqrt{2n}$.

For small positive or negative values of n, between $-\frac{1}{8}$ and $+\frac{1}{8}$, (4) gives the following expression for σ_1 :

$$\sigma_1=\left(1+\frac{9}{4}n+\frac{23}{16}n^2+\frac{35}{192}n^3\ldots\right)^{\frac{1}{6}}\cdot\sqrt{2n}.$$

For the second root of $J_n'(x)$, MacMahon's formula gives the values to about four places of decimals, *e.g.* σ_2 of $J'_{-\frac{1}{4}}(x)=0$ is 3·4183, the correct value to five places from the ascending series being 3·41839.

n.	$\delta=\sigma_1/\sqrt{2n}$.	σ_2.	n.	$\delta=\sigma_1/\sqrt{2n}$.	σ_2.
$-\frac{1}{2}$ …	0·7717	2·9751	0 …	1·0000	3·8317
	0·7859	3·0212		1·0093	3·8718 :
	0·7997	3·0669		1·0184 :	3·9117 :
	0·8131 :	3·1121 :		1·0275	3·9515 :
	0·8262	3·1570		1·0364 :	3·9911
$-\frac{3}{8}$ …	0·8389 :	3·2014	$\frac{1}{8}$ …	1·0452 :	4·0305
	0·8513 :	3·2455		1·0539 :	4·0697 :
	0·8634	3·2892		1·0625	4·1088 :
	0·8752 :	3·3326		1·0710	4·1477 :
	0·8867 :	3·3756 :		1·0793 :	4·1865 :
$-\frac{1}{4}$ …	0·8980	3·4184	$\frac{1}{4}$ …	1·0876	4·2251 :
	0·9090 :	3·4608 :		1·0958	4·2636 :
	0·9199	3·5030 :		1·1038 :	4·3020
	0·9305 :	3·5449 :		1·1118 :	4·3402
	0·9410	3·5866		1·1198	4·3783
$-\frac{1}{8}$ …	0·9512	3·6280	$\frac{3}{8}$ …	1·1276	4·4162 :
	0·9613	3·6692		1·1353 :	4·4541
	0·9712	3·7101 :		1·1430	4·4918
	0·9809 :	3·7508 :		1·1506	4·5294
	0·9905	3·7914		1·1581	4·5668 :
0 …	1·0000	3·8317	$\frac{1}{2}$ …	1·1655 :	4·6042

Roots of $\frac{d}{dx}\,.\,z^p\,.\,J_p(z)$.

If a long deep beam is supported at its ends and prevented from turning about its axis and loaded with a single transverse load at its middle point, the condition of critical stability is determined by the roots of $\frac{d}{dx}\,.\,x^{\frac{1}{2}}J_{\frac{1}{4}}\left(\frac{x^2}{2}\right)$ and for a beam with uniform load, by the roots of $\frac{d}{dx}\,.\,x^{\frac{1}{2}}J_{\frac{1}{6}}\left(\frac{x^3}{3}\right)$. Taking the general case, to find the roots of

$$\frac{d}{dx}\,.\,x^{\frac{1}{2}}J_p(2px^{\frac{1}{2p}})=0, \quad . \quad . \quad . \quad . \quad . \quad (5)$$

where p has been written for $\frac{1}{m+2}$, change the variable by substituting $x=\left(\frac{z}{2p}\right)^{2p}$; then (5) becomes

$$\frac{d}{dz}\,.\,z^p\,.\,J_p(z)=0.$$

But

$$\frac{d}{dz} \cdot z^p \cdot J_p(z) = z^{p-1}(zJ_p' + pJ_p) = z^p \cdot J_{p-1}(z).$$

Hence, the roots of (5) are simply related to those of $J_{p-1}(z)$. Since $\frac{1}{m+2} = p$, for values of m from -1 to ∞, $p-1=n$ ranges from 0 to -1. To solve equation (5) a short table of the roots of $J_n(z)=0$ is required, n having the limits 0 and -1. A table of this kind, however, like that of $J_n'(z)$ is inconvenient for calculating other roots by linear interpolation, especially when the order n is near the latter limit. This difficulty does not present itself if the values of $\rho_1^2/4(n+1)$ are tabulated. In constructing the following table, the first roots of $J_n(z)$ were calculated in nine cases, when the order n increases from -1 to 0 by intervals of $\frac{1}{8}$, and from these the values of $\rho_1^2/4(n+1)$ were found. The table was completed by interpolation to fifths:—

n.	$\rho_1^2/4(n+1)$.	n.	$\rho_1^2/4(n+1)$.	n.	$\rho_1^2/4(n+1)$.	n.	$\rho_1^2/4(n+1)$.
-1 ...	1·0000	$-\frac{3}{4}$ ...	1·1204:	$-\frac{1}{2}$ ...	1·2337	$-\frac{1}{4}$ ...	1·3417:
	1·0124:		1·1320:		1·2447		1·3523
	1·0248		1·1436		1·2557		1·3628:
	1·0370:		1·1550:		1·2666		1·3733:
	1·0492		1·1664:		1·2775		1·3838
$-\frac{7}{8}$ ...	1·0613	$-\frac{5}{8}$ ...	1·1778	$-\frac{3}{8}$ ...	1·2883	$-\frac{1}{8}$ ...	1·3942
	1·0733		1·1891		1·2990:		1·4046
	1·0852		1·2003:		1·3098		1·4149:
	1·0970		1·2115		1·3205		1·4252:
	1·1087:		1·2226:		1·3311:		1·4355:
$-\frac{3}{4}$ ...	1·1204:	$-\frac{1}{2}$ ...	1·2337	$-\frac{1}{4}$ ...	1·3417:	0 ...	1·4458

Sir George Greenhill * has recently called attention to the practical importance of the Bessel-Clifford function, $C_n(x)$. Since this function can be expressed in terms of the J functions, $x^{\frac{n}{2}} \cdot C_n(x) = J_n(2\sqrt{x})$, the roots of $C_n(x)$ are easily calculated from those of $J_n(z)=0$ and the roots of $\frac{d}{dx} \cdot C_n(x)$ from those of $J_{n+1}(z)=0$. This readily follows from the above relation.

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* "The Bessel-Clifford Function and its applications." Sir George Greenhill. Phil. Mag. vol. xxxviii. Nov. 1919, pp. 501–528.

XVII. *The Determination of Values of Young's Modulus and Poisson's Ratio by the Method of Flexures.* *By*

H. Carrington, *B.Sc.*, *M.Sc.*, *Tech.A.M.I.Mech.E.*[*]

THE determination of Young's Modulus and Poisson's Ratio by the method of flexures involves the accurate measurement of the principal curvatures of the anticlastic neutral surface of a beam or rod of suitable cross-section when it is bent by couples applied to its ends. The couples should be applied so that the curvature of the longitudinal axis is a principal curvature of the surface. When this is the case, the curvature of the neutral axis of any normal cross-section is the other principal curvature. If the longitudinal curvature and the values of the corresponding couples and dimensions of the cross-section are known, Young's Modulus corresponding with the length of the beam can be calculated. Also Poisson's Ratio corresponding with longitudinal strain in the direction of the length, and lateral strain in the direction of the breadth, is numerically equal to the ratio $\frac{\text{Lateral Curvature}}{\text{Longitudinal Curvature}}$.

Values of Poisson's Ratio for glass were obtained by the method of flexures by M. A. Cornu[†] (1869), who explored the anticlastic surface of a beam of glass by means of interference fringes produced between the surface of the beam and a glass plate laid upon it.

The method was also used by A. Mallock[‡] (1879), who obtained values for a considerable number of materials, including three for white pine and two for box and beech. Mallock placed four short wire pillars in the beams: one pair in the plane of the longitudinal curvature and the other pair in the plane of the lateral curvature. By measuring, by means of a microscope, the distance between the ends of each pair of pillars before and after a beam was bent, he was able to determine the principal curvatures of the surface, and hence the values of Poisson's Ratio. The writer is not aware of any record of the use of the method subsequent to 1879.

The method described in this paper differs essentially from the above in the manner in which the curvatures were measured. The method will be explained by reference to

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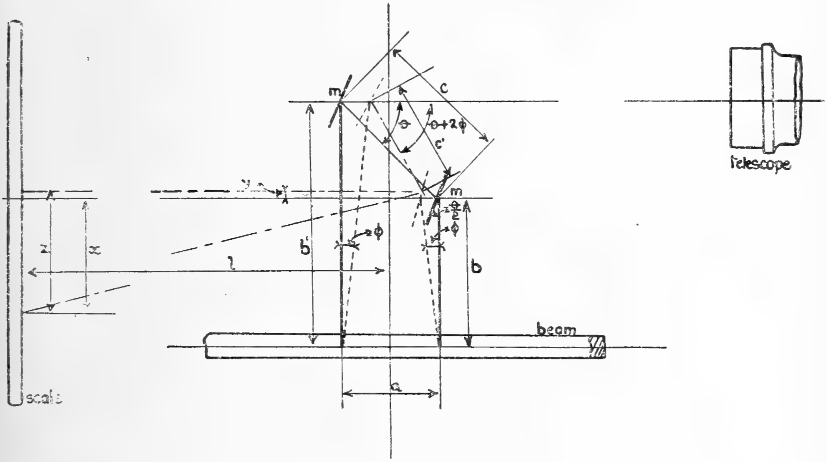
[*] Communicated by the Author.

[†] *Comptes Rendus*, p. 333 (1869).

[‡] Proc. Royal Society, vol. xxix. (1879).

fig. 1, which refers to the measurement of the longitudinal curvature. A pair of pillars with small mirrors, *m*, *m*, pivoted to their upper ends, were fixed to the beam. The lower ends of the pillars were bent in order to make point-contact in the plane of the couples on the upper and lower surfaces of the beam. For a beam 1 in. wide and $\frac{1}{4}$ in. thick the distance *a* in fig. 1 was about 1 in. and was symmetrical with respect to the central normal cross-section. The mirrors were adjusted so that the scale was reflected by them along the telescope shown, and the adjustment was made so that the axis of the telescope, axes of the pillars, and the scale were all in the plane of the couples.

Fig. 1.



The full lines show diagrammatically the position of the line of sight before the beam was bent. The chain and dotted lines show their positions when the beam was bent concave upwards.

The correct scale-reading from which the curvature should be calculated is $x \pm y = z$, and it was found that by suitably choosing the lengths *a*, *b*, and *b'*, the value of *y* could be made sensibly to vanish, for

$$y = c \sin\theta - c' \sin(\theta + 2\phi),$$

$$c = \frac{a}{\cos\theta}, \quad c' = \frac{a - b' \sin\phi - b \sin\phi + y\left(\tan\frac{\theta}{2} - \phi\right)}{\cos(\theta + 2\phi)}.$$

Hence

$$y=\frac{a\tan\theta-\{a-(b'+b)\sin\phi\}\tan(\theta+2\phi)}{1+\tan\left(\frac{\theta}{2}-\phi\right)\tan(\theta+2\phi)},$$

or

$$y=\frac{a\{\tan\theta-\tan(\theta+2\phi)\}+\{(b'+b)\sin\phi\}\tan(\theta+2\phi)}{1+\tan\left(\frac{\theta}{2}-\phi\right)\tan(\theta+2\phi)}.$$

Neglecting the square and higher powers of the small quantity ϕ in the numerator and writing

$$1+\tan\left(\frac{\theta}{2}-\phi\right)\tan(\theta+2\phi)=1+\tan\frac{\theta}{2}\tan\theta=\sec\theta,$$

then $$y\sec\theta=\phi\{(b'+b)\tan\theta-2a\sec^2\theta\}.$$

$$\therefore\ y=\phi\{(b'+b)\sin\theta-2a\sec\theta\}.$$

Hence if $\sin 2\theta=\frac{4a}{b+b'}$, then $y=0$.

This relation is satisfied if, for instance, $b=1\cdot5$ in., $b'=2\cdot5$ in., $a=1$ in., and $\theta=45°$.

The above analysis and the diagram in fig. 1 will also apply to the lateral curvature; the axes of the pillars, axis of the telescope, and the scale then being in the plane of the central normal cross-section of the beam.

When using the method, the beam was supported in a horizontal position on knife-edges 12 in. apart, and symmetrical with respect to the pillars and the central normal cross-section. Two other knife-edges were placed on the beam at points 8 in. apart, which were also symmetrical with respect to the central cross-section, and the couples were applied by weights suspended from the inner knife-edges. The weights were increased by small amounts, and after every increase both the scales were read. The readings were then plotted against the weights, and the slopes of the resulting straight lines were proportional to the principal curvatures. Thus, if the values a and l in fig. 1 were the same for both pairs of pillars and scales, then Poisson's Ratio was given by the ratio of the slopes of the two lines. Also if M was the bending moment and x the corresponding scale-reading for the longitudinal curvature, then Young's Modulus (E) was given by

$$E=\frac{24a\left(l+\frac{c}{2}+\frac{a}{2}\right)}{b_1d^3}\left(\frac{M}{x}\right),$$

where b_1=breadth and d=depth of the cross-section.

The writer used the above method extensively during the war to determine values of Young's Modulus and Poisson's Ratio for timber. It was found capable of measuring very small changes of curvature. For instance, the value of Poisson's Ratio for spruce corresponding with lateral strain in the direction of the grain is about 0·01, so that the lateral curvature is only about $\frac{1}{100}$ of the longitudinal curvature. It was, however, possible to measure the lateral curvature in such cases by using a micrometer instead of a scale, and placing it about 150 in. from the mirrors, instead of about 50 in. as was usually the case. By this means it was found possible to measure changes in curvature corresponding with strains of about 1 in 5×10^6.

There are a few evident precautions necessary when conducting an experiment, such as avoiding longitudinal thrust on the specimen by properly suspending the knife-edges, and preventing bodily rotation of the specimen during test. Also the knife-edges were slightly curved, so that lateral curvature was free to take place.

In Table I. are given two sets of values of Young's Modulus (E) and Poisson's Ratio (σ) obtained for five

TABLE I.

Elastic Coefficient.	Type of Experiment.	Steel.	Wrought Iron.	Brass.	Copper.	Aluminium.
E (10^6 lb./sq. in.).	Flexure.	28·1	27·9	14·2	17·6	9·23
E (10^6 lb./sq. in.).	Tensile (Marten's Instrument).	28·3	28·3	14·0	17·9	9·30
M (10^6 lb./sq. in.).	Torsion.	11·2	10·8	5·41	6·80	3·49
σ	Flexure.	0·236	0·245	0·333	0·305	0·313
σ	Calculated from rows 1 and 3.	0·255	0·290	0·310	0·295	0·320

E=Young's Modulus. σ=Poisson's Modulus. M=Modulus of Rigidity.

different metals. The values of E in the first row were deduced by the method of flexures, and those in the second row from tensile experiments on the strips, using Marten's Extensometer. The values of σ in the fourth row were obtained by the method of flexures, and those in the fifth row were calculated from the equation $\sigma = \frac{E}{2M} - 1$, where M is the modulus of rigidity—the values of M being deduced from torsion experiments on the strips. It should be noted that an error of 1 per cent. in the ratio $\frac{E}{2M}$ will result in an error of 4 per cent. or 5 per cent. in the calculated value of σ.

The largest difference in the values of σ by the two methods occurs for wrought iron. Since this material is fibrous, it is quite possible that the difference may be caused by the assumption of isotropy, which was made when calculating σ from values of E and M.

It is important to notice that, according to the theory of flexures, certain conditions must be fulfilled in order that the method should yield accurate results. These are:—"That the greatest diameter of the cross-section, and the third proportional to the diameters in and perpendicular to the plane of flexure, should not be great compared with the radius of curvature of the flexure" *. Since the strips of metal were 0·750 in. wide and 0·130 in. thick the third proportional to the thickness and breadth was about 4·3 in.

The flexure experiments were continued until the radius of curvature of flexure was about 40 in., and no indications of failure of the method were then noted. In the case of some of the timber specimens, the experiments were continued until the ratio of the radius of curvature of flexure to the third proportional was as low as 6, but even in these cases the method was found quite satisfactory.

The dimensions of the cross-sections of the beams used by Mallock † were 1 in. wide and $\frac{1}{4}$ in. thick, so that the third proportional to the thickness and breadth was 4 in. The minimum radius of flexure was usually about 200 in., but he notes that the method gave accurate results when the radius of curvature of flexure was much less than this.

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* Thomson and Tait's 'Natural Philosophy,' art. 718.

† Mallock, *loc. cit.*

XVIII. *On Talbot's Bands and the Colour-Sequence in the Spectrum.* *By* NIHAL KARAN SETHI, *M.Sc., Assistant Professor of Physics in the Benares Hindu University* [*].

1. *Introduction.*

THE theory of the very remarkable system of bands in the spectrum discovered by Talbot which is seen on covering half the aperture of the dispersing system by a retarding plate has been discussed by a number of writers, notably by Airy [†], Stokes [‡], Rayleigh [§], Schuster [||], Walker [¶], and Wood [**]. The treatments given make it clear that a spectrum in which the bands are seen is less pure than it would be in the absence of the retarding plate—in other words, that its introduction results in a re-distribution of energy in the different parts of the spectrum as actually formed. But the effect of such re-distribution of energy on the sequence of colours as seen by the eye in a spectrum showing Talbot's bands, does not so far appear to have been considered [††]. Some observations made by the present writer show that the colour-sequence in Talbot's bands presents some very remarkable features, which become particularly striking and noticeable when the total number of bands in the visible spectrum is not large. It is proposed in this paper to describe the methods of observation used and the results obtained, and incidentally also to discuss certain other aspects of the theory of formation of Talbot's bands.

2. *Observation of the Colours of Talbot's Bands.*

The human eye is in reality a very sensitive instrument for detecting changes in colour, and, as was pointed out by the late Lord Rayleigh [‡‡], it is capable under favourable conditions of detecting as small a difference of tint as that which exists between the two D-lines of sodium. The eye fails,

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[*] Communicated by Prof. C. V. Raman, M.A.

[†] Phil. Trans. i. p. 1 (1841).

[‡] Phil. Trans. ii. p. 227 (1848); also Math. & Phys. Papers, vol. ii. p. 14.

[§] Scientific Papers, vol. iii. p. 123.

[||] Phil. Mag. January 1904, p. 1.

[¶] Phil. Mag. April 1906, p. 531.

[**] Phil. Mag. November 1909, p. 758.

[††] For a full bibliography of the literature on Talbot's bands, see Appendix to a paper by T. E. Doubt, Phys. Rev. Oct. 17, p. 332.

[‡‡] Scientific Papers, vol. v. p. 621.

however, to perform this important function in an adequate manner when a large number of different colours are presented to it simultaneously without sharp lines of separation, as, for instance, in a prismatic spectrum. But if the colours are presented to the eye separately, and occupying a fairly large area in the field of view, its power in this respect becomes surprisingly great. These facts have to be borne in mind in attempting to discriminate between the sequence of colours in an ordinary spectrum and the sequence as seen in a spectrum showing a considerable number of Talbot's bands. The following method of observation has been found to be suitable. The eyepiece of the observing telescope is removed and a narrow slit is placed in the focal plane of the objective. On putting the eye immediately behind the slit, the whole of the effective portion of the prism face is seen to be of one uniform colour, and the field of view is also considerably broadened by the diffraction of the light entering the eye through the slit. On moving the telescope by slowly turning the tangent screw, the colour of the different bands and even of the different portions of a band can be easily examined.

Having chosen, then, a mica plate of such thickness that it gave about 25 or 30 bands in the whole spectrum, the aperture of the beam was carefully adjusted till, with the help of a nicol, the visibility of the bands became about the best. The eyepiece was now replaced by the slit mentioned above, and it was observed on turning the tangent screw of the telescope that the colour seen changed in a remarkable manner. It remained almost unaltered except in intensity throughout the width of a bright band, but in crossing a minimum, there was practically a sudden jump to another colour quite distinct from the first. This again persisted unchanged in tint until the next minimum was reached, when there was another sudden change, and so on. The gradation of colour and the almost imperceptible passage from one tint to the next observed in an ordinary spectrum was entirely lost. The spectrum showing Talbot's bands was thus seen to consist not of an infinite variety of tints as found in the ordinary spectrum, but of a limited number only. It is needless to add that this remark applies strictly to that part of the spectrum only for which the adjustment for this purpose is perfect.

3. *Observation of Talbot's Bands with small Retardations.*

In spite of the convincing nature of the experiment described above, it was felt that the effect would certainly be much more striking if it could be observed without any special aid to the eye and so as to show the different colours side by side at one glance. This would be possible only if the changes were much more sudden, which evidently means that the total number of bands in the whole spectrum must be considerably reduced—to say 5 or 10, and the thickness of the mica plate must be correspondingly diminished. This in turn involves the reduction of the aperture of the beam which beyond a certain limit becomes inconvenient. But the relation between the width of the diffraction pattern and the dispersion of the prism can also be adjusted by reducing the latter instead of increasing the former by making the aperture narrow. A very suitable arrangement for this purpose was found in the employment of an achromatic combination of two prisms which could be made to give the small dispersion necessary by simply altering the inclination of one to the other. Fairly wide apertures could now be used, and the process of adjustment for best visibility became extremely simple, consisting not in the somewhat inconvenient adjustment of the width of the aperture bisected by the edge of the mica, but in the comparatively easy alteration of the dispersion by changing the relative inclination of the two prisms. A Nicol prism was also unnecessary with the thin mica plates now employed. The method was quite successful even with mica giving a retardation of about 5 or 6 wave-lengths in the yellow region of the spectrum, corresponding to only *three* Talbot's bands in the entire visible spectrum.

With about the best adjustment, the uniformity of the colour within a band and the sudden change in passing from one band to another was strikingly evident. The colour of the bands in a few typical cases is given below :—

5 Bands.

1. Red.	Orange.	Green.	Blue.	Violet.
2. Scarlet.	Yellow.	Bluish green.	Blue.	Violet.

3 Bands.

Orange.	Blue.	Violet.

It will be seen that this method easily lends itself to the study of the colours at the various stages of adjustment, and it is very instructive to watch the gradual disappearance of the gradation of colour as the dispersion is diminished until the above-mentioned abrupt changes are fully established. If the dispersion is reduced still further, the sequence of colour within a band becomes actually reversed. The edge of an orange band towards the green side becomes distinctly redder, and that towards the red side distinctly more yellow. It is needless to point out that, although the bands retain excellent visibility within a wide range of adjustment, the minima are not absolutely black except at a particular stage which naturally varies from point to point in the spectrum. And it was found that this stage was not identical with the stage which showed the best uniformity of colour within a band, but was slightly different—the dispersion required for uniform colour being *less* than that necessary for giving absolutely black minima.

4. *Talbot's Bands and the Maxwell Colour-Triangle.*

A very simple explanation of the above facts is furnished by the usual method of regarding Talbot's bands as being produced by the dispersion of the laminary diffraction pattern formed by the mica plate and the aperture, the patterns in different wave-lengths suffering relative shifts depending on the dispersion of the prism.

The curves in fig. 1 show the intensity of light at various points in the diffraction pattern for wave-lengths, for which the mica plate produces a retardation specified on them. They have been plotted from the well-known expression *

$$\mathrm{I} = \frac{2}{z^2}\sin^2 h_1 z\left[1 + \cos\left(\rho - 2h_1 z\right)\right]. \quad . \quad . \quad . \quad (1)$$

If the prism shifts each successive curve to the left by proportionate amounts, it is easy to see that the minima A_1, A_2, A_3, A_4, etc. may all be made to coincide. This adjustment will give absolutely dark minima. But it will be observed that in this case the maxima at B_1, B_2 B_3, B_4, etc. will not coincide because the intensity curves are unsymmetrical and the maxima for shorter wave-lengths will lie more towards the left and those for larger wave-lengths more towards the right. But if the dispersion is a little less (about $\frac{4}{5}$ of the former), it is possible to arrange so that the maxima B_1, B_2, B_3, B_4, etc. may coincide. In this case the minima will not coincide, but between one minimum and another the proportions of the various wave-lengths contributing most of the light will remain fairly uniform, and will give rise to a

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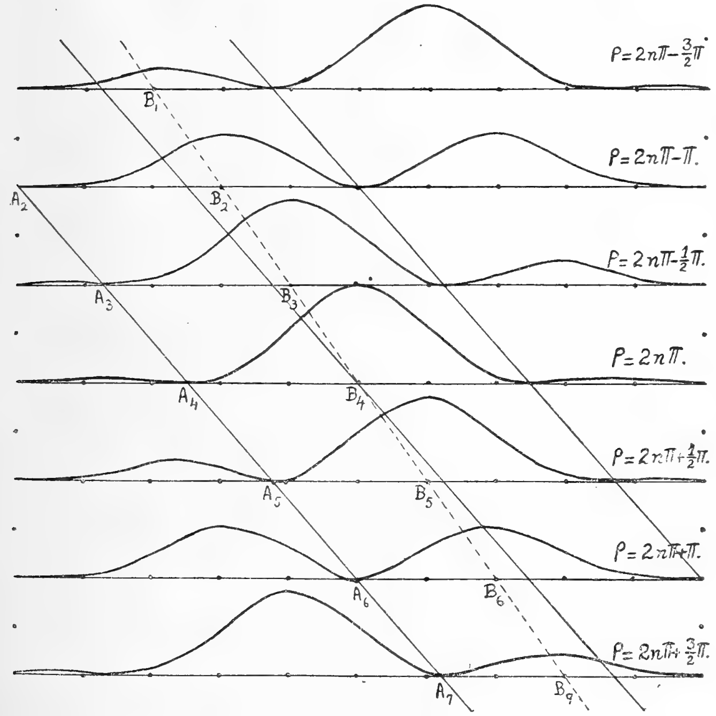
* Rayleigh, *loc. cit.*

uniform colour of the band except perhaps near the edges where the total illumination will be very small. If, however, the dispersion is still less, the maxima for smaller wave-lengths will lie more to the right and produce the reversal of the sequence of colour within a band.

Fig. 1.



The colours at different points in a spectrum showing Talbot's bands have been worked out in an actual case with five bands, the retardation of the plate being nine wave-lengths for $\lambda = 5870$, the spectrum being normal, and the dispersion adjusted so that the condition for uniform colour stated above is satisfied in the neighbourhood of this wave-length. The assumed law of dispersion gives

$$\lambda = 5870 - 162{\cdot}4x,$$

the unit of abscissa being such as to make

$$h_1 z = x \cdot \frac{\pi}{5} \quad \text{for} \quad \lambda = 5870.$$

To perform the calculation it is necessary first of all to find the intensity of the various wave-lengths at the different points in the spectrum. This was done with the aid of the fuller expression

$$I=\frac{1}{\lambda}\cdot\frac{1}{(h_1z)^2}\sin^2 h_1z\left[1+\cos(\rho-2h_1z)\right], \quad . \quad . \quad (2)$$

which does not omit λ in the coefficient, by first drawing graphs for a number of wave-lengths, so chosen for convenience that the retardation for them increased successively by $\frac{\pi}{2}$, then changing the origin of each according to the law of dispersion assumed and reading off the ordinate at the point in question. Table I. shows the wave-lengths used in calculating the colour. As will be seen, the values chosen are sufficiently representative of all parts of the spectrum.

Table I.

7042	6037	5282	4695
6816	5870	5140	4593
6603	5710	5030	4494
6401	5559	4914	4402
6215	5418	4801	

Now it is well known from the work of Maxwell * that all the colours in the spectrum can be formed by the combination in proper proportions of three primary colours—red, green, and blue. An extended table was compiled by Lord Rayleigh † from the more trustworthy set of the observations given in Maxwell's paper, in which the proportions of these components are given for a number of wave-lengths in the spectrum as observed by an eye with normal vision. Using that table and graphical interpolations, the red, green, and blue components for the wave-lengths in question were determined. Multiplying the intensity at a point due to a particular wave-length (as described above) by these numbers, the contributions of that wave-length to the red, green, and blue components of the colour at that

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* Phil. Trans. 1860, and Scientific Papers, vol. i. p. 410.

† Scientific Papers, vol. ii. p. 503.

point are determined, and finally adding up the contributions of all the wave-lengths, the three components of the resultant colour at that point become known.

Fig. 2 *a* is a graphical representation of these components for the various points. In fig. 2 *b* are given for comparison the intensity curves for the three components in the case of a *pure spectrum* showing the same number of bright and dark bands observed by the method described in the next section.

Fig. 2 *a*.

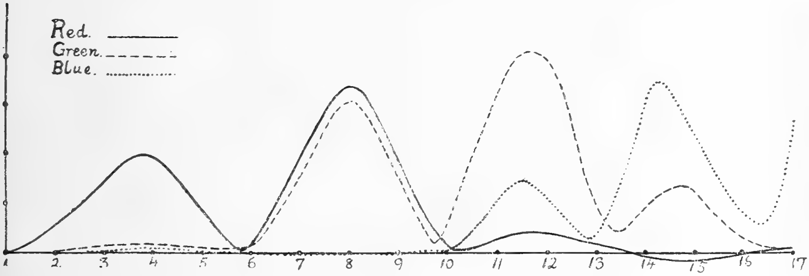
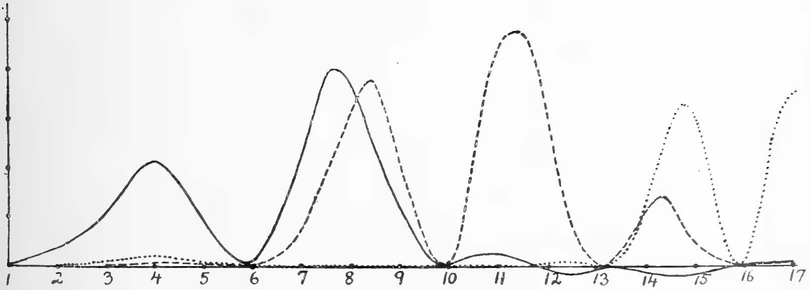


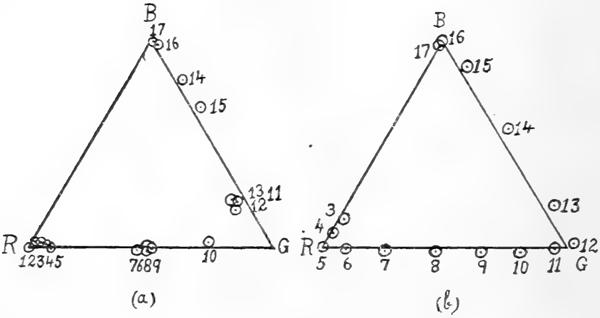
Fig. 2 *b*.



In the latter case no re-distribution of energy has taken place, and the sequence of colours is exactly the same as in a continuous spectrum, although the intensity varies from point to point. But in fig. 2 *a* it will be observed that the proportions of the components within a band are fairly constant in the first three bands. In the last two bands on the violet side of the spectrum the sequence of colour has been reversed owing to the dispersion being less than that required. For the same reason, the minima in this part of the spectrum are not so sharp.

The above results are much more strikingly evident in figs. 3 *a* and 3 *b*, which exhibit the resultant colours in the two cases as points on Maxwell's Colour-Triangle. It will be observed that the points in fig. 3 *a* cluster together in groups considerably separated from each other, while in fig. 3 *b* they are more evenly distributed, exactly as in the case of a normal spectrum.

Fig. 3.

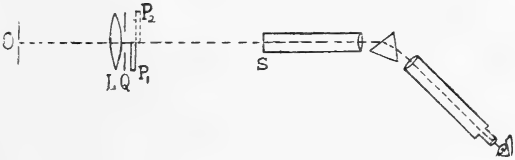


5. *A New Method of observing Talbot's Bands.*

In the usual method of forming Talbot's bands by covering half the aperture of the dispersing system by a retarding plate, the object-glass of the observing telescope serves the dual function of forming the diffraction-pattern of the light passing through the restricted aperture and of focussing the spectrum formed by the dispersing system. Though this has the advantage of simplicity, it is for certain purposes more convenient and instructive to observe Talbot's bands with an arrangement in which the two processes are separated, *i. e.* the diffraction-pattern is first formed in white light and then analysed into a spectrum.

Fig. 4 represents the disposition of the apparatus required for observing Talbot's bands in the manner indicated above. O is a narrow vertical slit whose image is focussed by the lens L on the vertical slit S of the spectroscope. P is the retarding plate with its edge also vertical and covering half the aperture Q. Two alternative positions of this plate are shown by heavy and dotted lines at P_1, P_2. For observing Talbot's bands the slit S should be removed or *opened wide.*

Fig. 4.



It will then be found that, with the plate in position P_1, the spectrum appears channelled by dark and bright bands, while with the plate in the position P_2 the spectrum is uninterrupted. One advantage of the arrangement described here is that Talbot's bands can be observed under the condition of best visibility for very small retardations of the order of 4 or 5 wave-lengths without unduly restricting the aperture through which the light has to pass, or reducing the dispersive power of the spectroscope employed. All that is necessary for this purpose is to increase the distance L S, which can be done by bringing the lens L and the plate P nearer to the slit O, so that the diffraction-pattern at S may be formed on a larger scale. The colour-sequence in Talbot's bands may be very well observed with this arrangement, and is best seen when a diffraction-grating of a few thousand lines per inch is used as the dispersing apparatus, instead of a prism as shown in the figure. If the aperture Q is small, the lens L may be dispensed with, but this is not so satisfactory.

Another interesting feature of the arrangement described above is that it serves admirably to illustrate the theory of the formation of Talbot's bands. This is shown by studying the effect of narrowing the slit S of the spectroscope so that the whole of the diffraction-pattern formed at the focus of the lens L cannot enter the dispersing apparatus. In this case, if the slit be very narrow, the interference bands in the spectrum are seen equally well in either of the two positions of the plate, the colours seen being those of the pure spectrum (figs. 2 *b* and 3 *b*). On gradually opening the slit S, however, the difference in the appearance of the spectrum in the two positions of the plate becomes at once evident. In the position P_1 of the plate the bands remain as clear as ever, their brightness increasing with the width of the slit, assuming, of course, that the condition for maximum visibility has been satisfied, and the colours of the bands change in the manner already considered in the preceding sections. With the plate in the position P_2, on the other hand, the visibility of the bands in the spectrum at first rapidly diminishes to zero ; the bands then reappear, but with the maxima and minima of illumination interchanged in position and with greatly diminished visibility. With further widening of the slit their distinctness again falls to zero, reappears again very faintly, falls again to zero, and finally vanishes.

The reason why, with the plate in one position, the visibility of the bands is unaffected by opening the slit wide, while with the plate in the other position it is so rapidly destroyed, can be readily understood if the ordinary slit of the spectroscope is replaced by one which can be given a slow lateral motion while the rest of the spectroscope remains fixed. It will be observed then, that with the slit narrow and the plate in the position P_1 the interference bands seen remain stationary as the slit is gradually moved, evidently because their drift in the diffraction-pattern due to change in wave-length is completely compensated by the dispersion of the prism. On the other hand, with the plate in position P_2, the bands are seen to shift laterally in the field of view, because now the shift and the dispersion are in the same direction, and the latter only helps to increase the former. When the slit is opened wide, the successive appearances seen with a narrow slit in different positions are presented to the eye simultaneously, so that in one position of the plate the bands remain visible, and in the other position they vanish as the result of superposition after one or two faint reappearances and disappearances.

6. *Mathematical Theory.*

The theory of the phenomena described in the preceding section may be worked out on exactly the same lines as in the case of Talbot's bands, except that in the present case the diffraction-pattern admitted into the spectroscope is limited by the width of the slit, and the integration of expression (1) for the intensity in the diffraction-pattern has consequently to be confined to finite limits $-z_1$ and $+z_1$, where $2z_1$ is the width of the slit S. With the notation of Lord Rayleigh, this gives

$$\mathrm{I}=\int_{-z_1}^{+z_1}\frac{2}{(h_1z)^2}\sin^2 h_1z\left[1+\cos\{\rho'+(\varpi-2h_1)z\}\right]dz. \quad . \quad (3)$$

In the special case when $\varpi=2h_1$ corresponding to the best thickness of the retarding plate for observing Talbot's bands, this reduces to

$$\mathrm{I}=(1+\cos\rho')\int_{-z_1}^{+z_1}\frac{2}{(h_1z)^2}\sin^2 h_1z\,dz. \quad . \quad . \quad (4)$$

When $\rho'=(2n+1)\pi$, this will give zero illumination whatever z_1 may be. So that in this case we shall have Talbot's bands with maximum visibility irrespective of the width of the slit.

But when the same plate is turned over to the position P_2, $\varpi=-2h_1$ and

$$\mathrm{I}=\int_{-z_1}^{+z_1}\frac{2}{(h_1z)^2}\sin^2 h_1z\left[1+\cos(\rho'-4h_1z)\right]dz. \quad . \quad . \quad (5)$$

This integration can be easily performed with the aid of the tables of the Si function given by Glaisher *, for it can be easily put in the form :—

$$\begin{aligned}\mathrm{I}&=2\left[\mathrm{Si}\,2x_1-\frac{\sin^2x_1}{x_1}\right]-\cos\rho'\Big[4\mathrm{Si}\,4x_1-3\mathrm{Si}\,6x_1\\&\qquad\qquad-\mathrm{Si}\,2x_1+\frac{2}{x_1}\sin^2x_1\cos4x_1\Big]\\&=\mathrm{A}+\mathrm{B}\cos\rho', \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (6)\end{aligned}$$

where $x_1=h_1z_1$.

When x_1 is very small, $\mathrm{Si}\,x_1=x_1$ and

$$\begin{aligned}\mathrm{I}&=2\left[2x_1-x_1\right]-\cos\rho'\left[16x_1-18x_1-2x_1+2x_1\right]\\&=2x_1(1+\cos\rho'),\end{aligned}$$

a value identical with that obtained from (4).

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* Phil. Trans. 1870, p. 367.

When x_1 is very large

$$\mathrm{Si}\, x_1 = \frac{\pi}{2}; \quad \frac{\sin^2 x_1}{x_1} = 0.$$

The equations (4) and (6) reduce in this case respectively to

$$I = 2(1 + \cos \rho')\frac{\pi}{2}$$

and

$$I = 2 \cdot \frac{\pi}{2} - \cos \rho' [4 - 3 - 1]\frac{\pi}{2} = \pi,$$

so that when the slit is opened very wide we obtain dark and bright bands in position P_1 of the plate, but uniform illumination in position P_2.

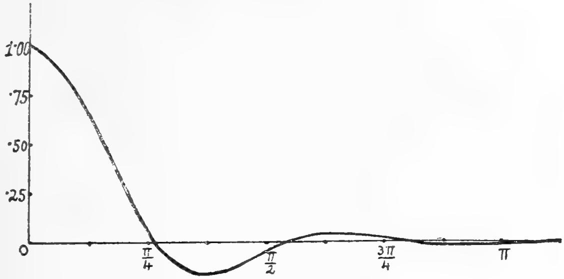
The values of A and B in (6) and the corresponding visibility $\frac{B}{A}$ for various widths of the slit are given below:—

TABLE II.

x_1.	A.	B.	Visibility $= \frac{B}{A}$.
$\frac{\pi}{16}$	·28	·23	·84
$\frac{\pi}{8}$	·76	·50	·66
$\frac{\pi}{4}$	1·47	·07	·05
$\frac{3\pi}{8}$	2·04	−·33	−·16
$\frac{\pi}{2}$	2·43	−·07	−·03
$\frac{5\pi}{8}$	2·68	·10	·04
$\frac{3\pi}{4}$	2·81	·017	·006
π	2·83	·004	·001
$\frac{3\pi}{2}$	2·93	·001	·000
2π	2·98	·001	·000

Fig. 5 is a graphical representation of the visibility for various slit widths. From this it is clear that the visibility of the bands at first diminishes and becomes zero when the width of the slit corresponds to $x_1 = \frac{\pi}{4}$ nearly, *i. e.* when it admits about $\frac{1}{4}$ of the central bright fringe in the diffraction-pattern. Then it reappears with the positions of the dark

Fig. 5.



and bright bands interchanged and increases till the slit admits about $\frac{3}{8}$ of the central portion. The maximum of the visibility in this reappearance is about 16 per cent. of the full visibility. The visibility again vanishes when about $\frac{1}{2}$ of the central fringe of the diffraction-pattern enters the analysing spectroscope. With further increase of slit width we have another reappearance followed by a disappearance, but these changes can only be seen with difficulty. Actual measurement of the slit width at the different stages showed fair agreement with the results of theory.

7. *Summary and Conclusion.*

1. The present paper contains some observations and a theoretical discussion of the remarkable difference in the colour-sequence as observed in a normal spectrum and in a spectrum showing Talbot's bands. In the case of a spectrum showing a fairly large number of the bands, these differences may be studied by placing the eye behind a very narrow slit placed in the focal plane of the telescope and moving it over the spectrum. With a spectrum showing only a small number of Talbot's bands in the whole visible region the phenomena may be easily seen without such aid. Talbot's bands may be satisfactorily observed with such small retardations by using a prism combination of small adjustable dispersion.

2. It was found in all cases that when adjustments were properly made, the colour changed from band to band in a fairly abrupt manner—remaining nearly uniform within the bright regions, but changing suddenly in passing across the

minima of illumination. The spectrum with Talbot's bands is thus shown to consist not as usual of an infinite variety of tints, but of a limited number only, depending on the number of bands present.

3. It is found that the condition for observing this phenomenon is not the same as that required for observing Talbot's bands at best visibility, *i. e.* with perfectly black minima, but is slightly different, the dispersion required in the former case being about $\frac{4}{5}$ of what is needed in the latter. If the dispersion is less than even this the sequence of colour within a band is actually reversed.

4. The colour at every point in a spectrum showing five Talbot's bands has been calculated and the result shown on a graph and plotted on Maxwell's Colour-Triangle.

5. A new method for observing Talbot's bands is described, which consists briefly in first forming the laminar diffraction-pattern in white light and then analysing it with a spectroscope having a wide slit. This method is very convenient when Talbot's bands with small retardations have to be observed, as it is then unnecessary to unduly restrict the aperture or to use very small dispersive powers.

6. With the foregoing arrangement it is possible also to study the manner in which the visibility of Talbot's bands is influenced by admitting more or less of the laminar diffraction-pattern into the dispersing apparatus, and thus to actually trace the successive stages of the process of superposition which results in the interference bands having full visibility when the plate is put on the thinner side of the dispersing prism, and zero visibility when it is on the thicker side.

7. It would be interesting to see if the phenomenon of the colour-sequence in Talbot's bands could be explained on the simple theory of their formation which has been given by Schuster (Phil. Mag. Jan. 1904).

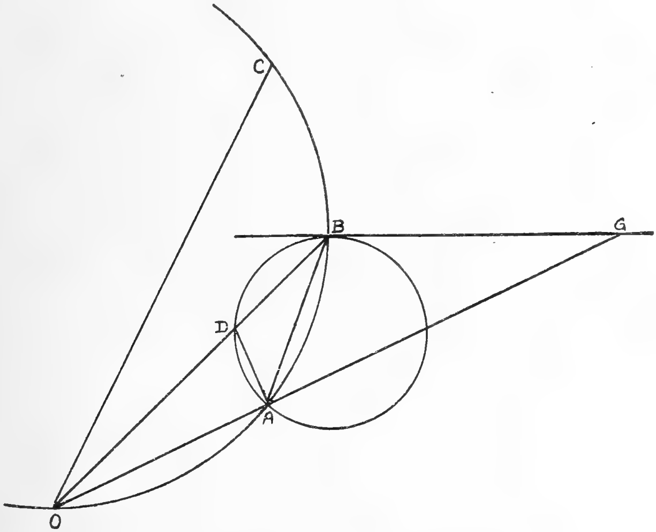
In conclusion, the writer has much pleasure in recording his sense of deep gratitude to Prof. C. V. Raman for his valuable and helpful interest during the progress of the work which was carried out in the laboratory of the Indian Association for the Cultivation of Science.

Calcutta, 8th July, 1920.

XIX. *Notes on Times of Descent under Gravity, suggested by a proposition of Galileo's.* *By* W. B. MORTON, *M.A.*, *and* T. C. TOBIN, *M.A.*[*]

IN the series of propositions which form the subject-matter of the third day of his "Dialogues" Galileo established, by geometrical methods, all the well-known properties of uniformly accelerated rectilinear motion, and, in addition, a number of theorems, many of them of great interest and beauty, which are not included in modern text-books. One of these is Prop. 36, which states that a particle which slides to the lowest point O of a vertical circle (fig. 1) starting from rest at any point B of the circumference below the level of the centre, will make the journey in shorter time if it moves along two successive chords BA, AO of the circle than if it goes directly along the one chord BO. This proposition is noteworthy for two reasons. In the first place its geometrical proof, as given by Galileo, is quite simple, but if one attempts it by the straightforward algebraic methods of the present day the work is unexpectedly complicated. And, secondly, the theorem has an historical interest, for it was the first step towards the solution of the problem of the brachistochrone.

Fig. 1.



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[*] Communicated by the Authors.

It brings out clearly the fact that a path longer in distance may be shorter in time.

Accordingly Galileo, in a Scholium to the proposition, states that it appears to follow ("colligi posse videtur") that the most rapid passage from one fixed point to another is not by the shortest geometrical path, the straight line, but by an arc of a circle. The argument which follows is limited to the proof that the time of descent is continually shortened as more and more chords are taken between the starting point and the bottom of the circle, approaching the arc as a limit. In order to establish this Galileo has to make an unproved assumption, which he introduces by the words "verisimile est" to the effect that the superiority of the two-chord path continues to hold when the particle starts with an initial velocity due to a fall from a level which is not above the centre of the circle.

The "Dialogues" were printed in 1638. In 1696 John Bernoulli discovered the cycloid to be the true curve of quickest descent, and published the problem as a challenge to the learned world. He does not appear to have been aware of Galileo's earlier attempt until his attention was called to it by Leibnitz. In a paper which the latter contributed to the 'Acta Eruditorum' in 1697[*], it was pointed out that Galileo had, through lack of the methods of the differential calculus, gone wrong on two questions, viz., the form of the catenary which he identified with the parabola, and the curve of quickest descent which he supposed to be a circle. To this at a later date Bernoulli added a rather curious comment[†]. The catenary, as he had himself shown, can be constructed by the rectification of a parabola, and the cycloid by the rectification of a circle, therefore, in both instances, "Galilée a deviné quelque chose d'approchant."

To return to the proposition, Galileo's method of proof, slightly simplified, may be stated as follows. B is the point of departure on the circle OABC whose lowest point is O. The circle BDA is described with B as its highest point. Then the times BD and BA from rest at B are equal, and the velocity at A is the same whether the particle slides along BA or GA. Therefore we have to show that the time along DO from rest at B is greater than that along AO from rest at G. The proof turns on the fact that DO is longer than AO. This is proved by Galileo, as a lemma, in a rather clumsy way. It is most readily seen by expressing the angles ODA

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[*] Leibnitz, *Math. Schriften* (Gerhardt's edition), v. p. 333.

[†] John Bernoulli's works, i. p. 199.

and OAD in terms of the inclinations, α β say, of the chords OA OB to the horizontal. It is easy to show that

$$\text{ODA}=\alpha+\beta,$$

$$\text{OAD}=180^\circ-2\beta.$$

Since $$\alpha<\beta \quad \text{and} \quad \beta<45^\circ,$$

$$\therefore \quad \alpha+\beta<2\beta<180^\circ-2\beta,$$

$$\text{ODA}<\text{OAD}.$$

The inclined planes OB, OG having the same height, the times of fall along them are proportional to their lengths and may be represented by these lengths. Take GX the mean proportional between GA, GO and BY that between BD, BO. Then BY, GX represent on the same scale the times of sliding from rest down BD, GA, and OY, OX the times for the remaining parts DO, AO. It remains to show that OY > OX.

We have $$\text{OB}<\text{OG} \quad \text{and} \quad \text{OD}>\text{OA},$$

$$\therefore \quad \text{OB}:\text{OD}<\text{OG}:\text{OA},$$

$$\therefore \quad \text{OB}:\text{DB}>\text{OG}:\text{AG}.$$

But $$\text{OB}:\text{YB}=\text{YB}:\text{DB}=\text{OY}:\text{YD},$$

$$\therefore \quad \text{OB}:\text{DB}=\text{OY}^2:\text{YD}^2$$

$$\text{OG}:\text{AG}=\text{OX}^2:\text{XA}^2,$$

$$\therefore \quad \text{OY}:\text{YD}>\text{OX}:\text{XA}.$$

In other words, OY is a larger fractional part of the longer line OD: so OY > OX, and the proposition is proved.

In connexion with the proposition and scholium the following questions suggest themselves, and in spite of the well-worn character and the comparative unimportance of the subject, some of the results obtained are perhaps not devoid of interest.

(1) The range of validity of the proposition as the upper point rises out of the lower quadrant of the circle and the truth or falsehood of the assumption made in the scholium. This suggests a comparison of the times down one chord and down two chords for the general case when the particle starts with an initial velocity and leads, incidentally, to an analytical proof of the proposition.

(2) Extension to the case where the intermediate point A is not on the circle. This includes an examination of the locus of A for a given time of descent down the two lines

BAO, and the position A for which the time is a minimum, or what may be called the two-line brachistochrone.

(3) Range of validity of the statement in the Scholium that the arc is quicker than the chord: involving comparison of arc and chord-times in the general case where there is initial velocity.

(4) It seems obvious that Galileo's circular arc, having a horizontal tangent at the lowest point, will not in general give a shorter time than any other circular arc which can be drawn. So we may examine the question of the circular brachistochrone.

(5) Comparison of the actual times taken by the different routes: the books give no idea as to how much quicker the cycloid is than, say, the direct line.

(1) It is required to compare the times taken by a particle to describe the paths BO and BAO, when its initial velocity at B is that due to a fall from the level of C.

Let the inclinations to the horizontal of the chords OA, OB, OC be $\alpha\,\beta\,\gamma$, then the inclination of BA is $\alpha+\beta$. If a particle starts from rest at the level of C its velocity at the levels of BAO will be proportional to $(\sin^2\gamma-\sin^2\beta)^{\frac{1}{2}}$, $(\sin^2\gamma-\sin^2\alpha)^{\frac{1}{2}}$, and $\sin\gamma$. To get the time along any chord divide the gain of velocity by the acceleration. The condition that the path BAO should be quicker than BO is

$$\operatorname{cosec}\beta\{\sin\gamma-(\sin^2\gamma-\sin^2\beta)^{\frac{1}{2}}\}-\operatorname{cosec}\alpha\{\sin\gamma-(\sin^2\gamma-\sin^2\alpha)^{\frac{1}{2}}\}$$
$$-\operatorname{cosec}(\alpha+\beta)\{(\sin^2\gamma-\sin^2\alpha)^{\frac{1}{2}}-(\sin^2\gamma-\sin^2\beta)^{\frac{1}{2}}\}>0.$$

Multiply across by the positive quantity $\sin\alpha\,\sin\beta\,\sin(\alpha+\beta)$ and get rid of the roots by writing

$$\sin\alpha=\sin\gamma\sin\theta,\quad \sin\beta=\sin\gamma\sin\phi.$$

So that $\quad \cos\alpha=\Delta\theta,\quad \cos\beta=\Delta\phi\ (\text{mod. }\sin\gamma).$

Dropping the factor $\sin^3\gamma$ we arrive at

$$(\sin\theta\Delta\phi+\sin\phi\Delta\theta)\{\sin(\phi-\theta)-(\sin\phi-\sin\theta)\}$$
$$-\sin\theta\sin\phi(\cos\theta-\cos\phi)>0.$$

From this the factor $\sin\frac{1}{2}\theta\sin\frac{1}{2}\phi\sin\frac{1}{2}(\phi-\theta)$ can be removed and the inequality reduces to

$$\sin\phi\{2\Delta\theta-(1+\cos\theta)\}+\sin\theta\{2\Delta\phi-(1+\cos\phi)\}>0,$$

or finally

$$\operatorname{cosec}\theta\{2\Delta\theta-(1+\cos\theta)\}+\operatorname{cosec}\phi\{2\Delta\phi-(1+\cos\phi)\}>0.$$

In this the two variables $\theta\ \phi$, or $\alpha\ \beta$, are separated and the discussion now turns on the values of the function

$$f(\theta\,\gamma)=\operatorname{cosec}\theta\{2\Delta\theta-(1+\cos\theta)\},$$

or, reinstating α,

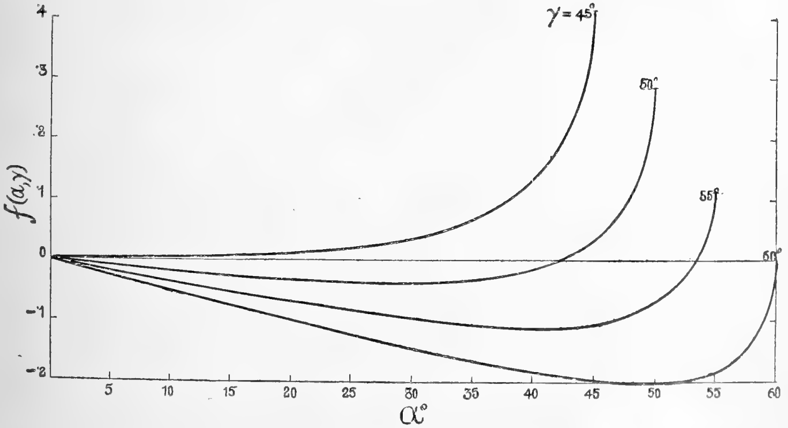
$$f(\alpha\,\gamma)=\operatorname{cosec}\alpha\{\sin\gamma(2\cos\alpha-1)-(\sin^2\gamma-\sin^2\alpha)^{\frac{1}{2}}\}.$$

When $f(\alpha\,\gamma)+f(\beta\,\gamma)$ is positive the two-chord path is the quicker.

Now it is easy to verify that if $\gamma<45^\circ$ the function f is positive for all values of the other angle. So the two chords are quicker than the one provided the level to which the velocity is due is below the centre of the circle, and this is precisely the limitation imposed by Galileo. On the other hand if $\gamma>60^\circ$ then, for all values of α and β which are less than γ, f is negative and the single-chord path is the quicker.

What happens for values of γ between 45° and 60° can be seen by examining the graphs of the function f, drawn on fig. 2 for $\gamma=45^\circ$, 50°, 55°, 60° and $\alpha<\gamma$. The curve for $\gamma=45^\circ$ touches the axis at the origin. For higher values of γ it sinks below the axis and then rises, crossing at a point given by $\sec\alpha=4\sin^2\gamma-1$. This zero of the function moves to the right as γ increases and coincides with the end-point $\alpha=\gamma$ when $\gamma=60^\circ$. For still larger values of γ, as has already been said, the part of the curve which concerns us lies wholly below the origin.

Fig. 2.



The special points we consider on the 50° and 55° curves are those at which the positive ordinate is equal to the greatest depth to which the curve sinks below the axis. Call the abscissa here β_1, and the corresponding point on the circle B_1. If $\beta > \beta_1$, then, no matter what value α has (α being less than β), $f(\alpha\,\gamma) + f(\beta\,\gamma)$ is positive. If β is between β_1 and the zero of the graph, say β_0, then the sum of the ordinates will only be positive provided α is sufficiently near the ends of the range 0 to β_0. The point β_1 coincides with the end of the graph when γ is about 55°·9. We have therefore the following results for the critical region $45° < \gamma < 60°$.

$\underline{45° < \gamma < 55°{\cdot}9.}$ There are two critical positions B_1 B_0 for the upper point. If B is between C and B_1, then the two chord path is quicker for all positions of A. If B lies between B_1 and B_0, then on the arc OB there are two points A_1 A_2 such that the times along BA_1O, BA_2O, and BO are all equal. If A lies between A_1 A_2, BO is quicker, if A lies outside A_1 A_2 it is slower, than the way viâ A. When B is below B_0, then the single-chord is always quicker.

$\underline{55°{\cdot}9 < \gamma < 60°.}$ There is now only the point B_0. When B lies above this the two-chord path is quicker when A is sufficiently close to B or O. Below B_0 it is always slower.

It is easy to make the modifications necessary for the case $\beta = \gamma$ which corresponds to Galileo's original proposition. Evidently its validity extends further than the quadrant to which he confined his proof; it holds up to $\gamma = 55°{\cdot}9$, *i. e.*, through an arc of about 111°·8 from the lowest point of the circle.

(2) Particles move under gravity from rest at an upper fixed point B to a lower point O along a rectilinear broken path BAO. It is required to find the locus of the intermediate point A when the time of descent is assigned. The algebraic equation of this locus can, of course, be written down but its expression is complicated. We may plot the required curves by another method, identical with that used by Clerk Maxwell in his diagrams of equipotentials of point charges. Let the difference of level between the upper and lower points be h. We take as unit of time the time of vertical fall through this height. If a particle slides from rest at B along any straight line its position after p units of time will be on a circle of diameter p^2h having its highest point at B. On the other hand, if particles approach O along straight lines, having velocity due to a fall from B, the locus of points from which time p is taken to reach O has, with

reference to O as origin, the polar equation

$$r = h(2p - p^2 \sin\theta),$$

where θ is the inclination to the horizontal. This locus is a limaçon derived from the circle of diameter p^2h. Let a set of circles be drawn below B and a set of limaçons round O, each curve being marked with the corresponding value of p Then a series of points on the locus sought can be obtained by marking intersections of circles and limaçons whose p's have a constant sum. By drawing one set of curves on tracing-paper and moving this sideways over the other set, placed in its proper relative position, it is possible to get the curves for different slopes of the line OB, keeping the same value of h.

The method fails when the line OB is horizontal, but in this case the algebraic equation for the locus simplifies to a manageable form. Taking the origin at the centre of the line, the axis of x horizontal and y vertically downwards, the equation is

$$x^2 = q^2y(4y^2 - q^2ly + l^2)/(l - q^2y),$$

where l is the length of the line and q is the time expressed as a multiple of that required for vertical fall through l.

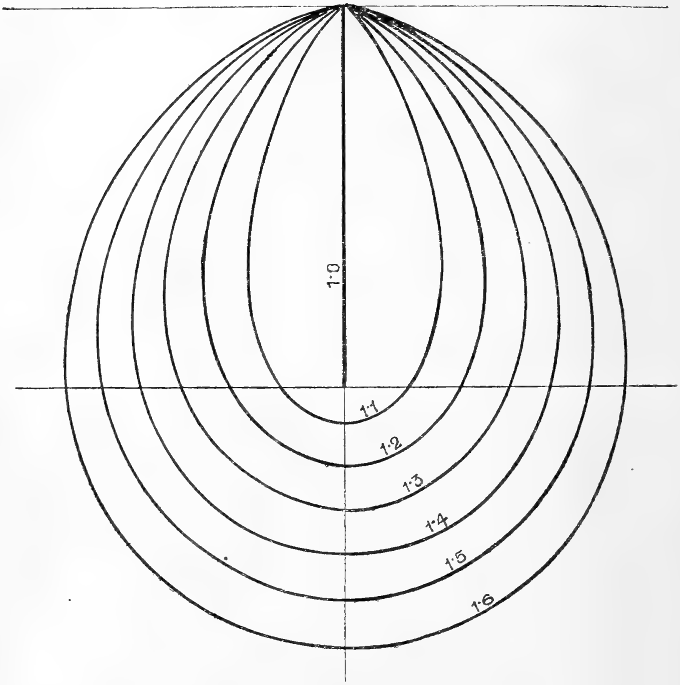
Figs. 3, 4, and 5 show the forms of the loci when the line joining the terminal points is vertical, sloping, and horizontal. The length of this line is made the same in each case, and the numbers attached to the curves give the time as a multiple of that required for vertical fall through this length. The slope in fig. 4 is such that the time of sliding down the inclined line is 1·5 times the unit taken. The line itself forms part of the locus for this time. It will be seen that the loci for longer times of descent pass through the upper end of the line. We can deduce a simple expression for their curvature at this point. Let B′ be a near point on the locus for time t' and let t be the time down the straight line BO. Then

$$(t' - t) = \text{time along BB}'\text{O} - \text{time along BO}.$$

But in the limit the time along B′O becomes equal to that along BO. Therefore $(t' - t) =$ limit approached by the time of sliding along BB′ as B′ approaches B. Since this has to be finite the tangent at B must be horizontal and $(t' - t) =$ time down any chord of the circle of curvature $= (4\rho/g)^{\frac{1}{2}}$,

$\therefore\ \rho = \frac{1}{4}g(t'-t)^2$, vanishing, as it obviously should, for the locus of which the straight line forms part.

Fig. 3.



For times of descent shorter than that along the line the loci are detached from B and form loops which contract to a point corresponding to the minimum time from B to O along a two-line path. In the case where the two end-points are on the same level (fig. 5), it is obvious from elementary considerations that the two lines are inclined at 45° to the horizontal.

We now examine the general case of the "two-line brachistochrone."

Let the inclinations of OA, OB be $\alpha\ \beta$ and their lengths $a\ b$, and let c, γ be length and inclination of AB. Keeping $b\ \beta$ fixed we require two equations satisfied by $\alpha\ \gamma$ when the time along BAO is a minimum. These can be got by equating to zero the partial differentials of the expression for the time in terms of $\alpha\ \beta\ \gamma$ and b. But they are obtained much more neatly by considering geometrically the effect of displacing A in two special directions.

Fig. 4.

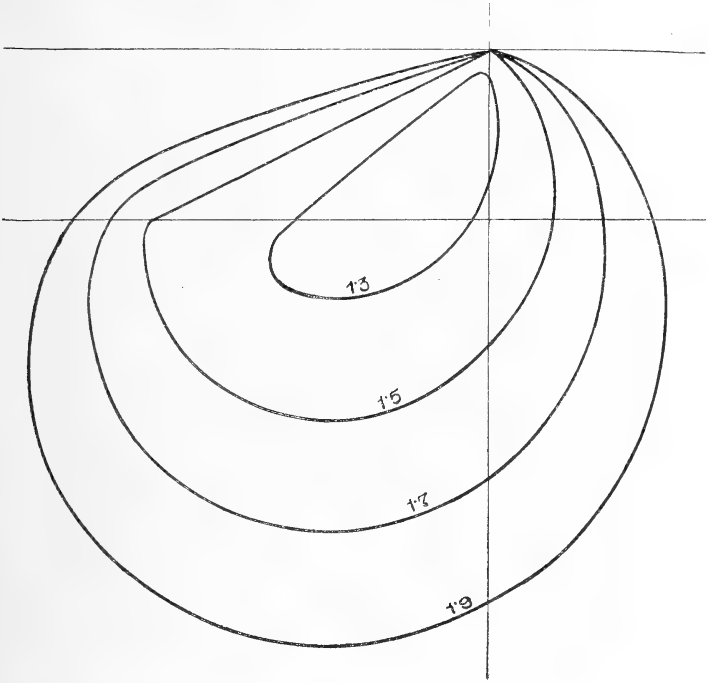
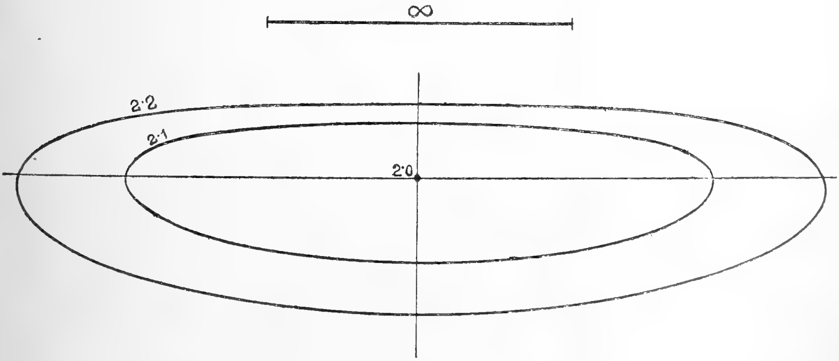


Fig. 5.



(1) In the first instance make the displacement along OA (fig. 6). Make AK=AA,′ then if times along BA′O and BAO are equal, on taking away the equivalent pieces KAO and A′O we find the same time along BK, BA′.

Therefore these are chords of a circle with highest point at B and in the limit A′K touches the circle, Therefore KA′B = inclination of BA′ to the horizontal tangent at B $=\gamma$,

$$AKA' = AA'K = \pi - \gamma - (\gamma - \alpha).$$

Fig. 6.

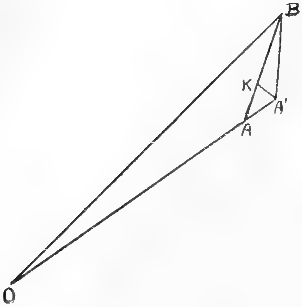
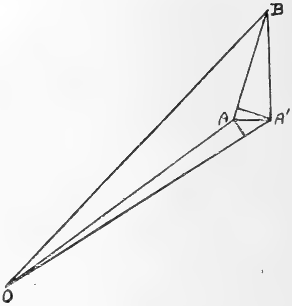


Fig. 7.



Therefore in the triangle AKA′

$$2(\pi - 2\gamma + \alpha) + (\gamma - \alpha) = \pi$$

or

$$3\gamma - \alpha = \pi,$$

a simple relation connecting the inclinations of the two parts of the path.

(2) Secondly displace A horizontally (fig. 7), then the speed at A is unchanged in the varied path. Call this v_1, and the speed at O, v_2. The time is $2c/v_1 + 2a/(v_1+v_2)$.

The variation of this vanishes, so

$$\delta c/v_1 + \delta a/(v_1 + v_2) = 0.$$

But
$$\delta c = -AA' \cos\gamma, \quad \delta a = AA' \cos\alpha,$$

$$\therefore \quad \cos\gamma(v_1 + v_2) = \cos\alpha \, . \, v_1$$

and
$$v_1{}^2 : v_2{}^2 = c \sin\gamma : b \sin\beta$$
$$= \sin\gamma \sin(\beta - \alpha) : \sin\beta \sin(\gamma - \alpha),$$

$$\therefore \cos\alpha \sec\gamma = 1 + \{\sin\beta \operatorname{cosec}\gamma \sin(\gamma - \alpha) \operatorname{cosec}(\beta - \alpha)\}^{\frac{1}{2}}.$$

If $\alpha = \pi - 3\gamma$ be put into this, it gives

$$\{2 \sin\beta \cos\gamma \operatorname{cosec}(3\gamma - \beta)\}^{\frac{1}{2}} = -2\cos 2\gamma,$$

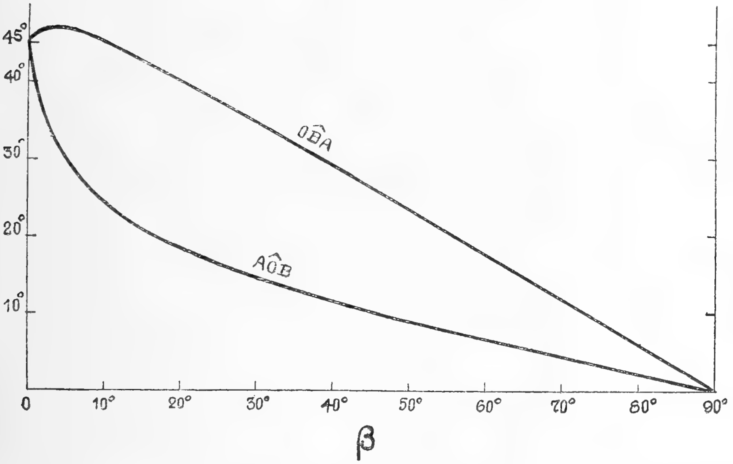
which can be transformed into

$$\tan\beta = \tan\gamma\,(1 + \cos 4\gamma)/(\cos 4\gamma - 2\cos 2\gamma + 2).$$

So starting with γ we may calculate β and α, and then plot against the inclination β the corresponding values of

$(\gamma-\beta)$ and $(\beta-\alpha)$, the angles which the upper and lower parts of the path make with the direct line OB. The results are shown on the upper and lower curves of fig. 8. When

Fig. 8.



the line is horizontal both angles are 45°, as has been already mentioned. For inclined positions the upper angle is the greater, and as the vertical is approached this ratio approaches three. Thus the intermediate point for the quickest path between two points nearly on the same vertical approaches a position one quarter way down the line, in other words *the times of describing the two parts of the path are equal.*

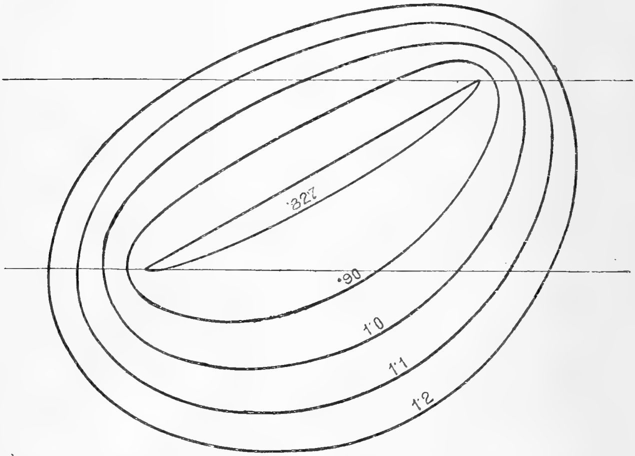
On examination the rather remarkable result is found that this is a general property of the minimum path. The condition for equality of times is $c/v_1 = a/(v_1+v_2)$,

$$\text{or} \quad \sin(\gamma-\beta)\operatorname{cosec}(\beta-\alpha) = 1 + \{\sin\beta \operatorname{cosec}\gamma \sin(\gamma-\alpha) \operatorname{cosec}(\beta-\alpha)\}^{\frac{1}{2}}.$$

If this be combined with $\alpha = 3\gamma - \pi$ it will be found to lead to the equation already found to connect $\beta\gamma$.

The method used for drawing the loci for constant times can easily be modified to meet the case in which the particle has an initial velocity at the upper point. The points are got as the intersections of two sets of limaçons. The curves break off from the upper point, and as the initial velocity is increased it is evident they become loci for equal *distances*, *i. e.* ellipses with foci at the terminal points. The tendency in this direction is seen in fig. 9, which is drawn for the case in which the level of zero velocity is at a height above the upper point equal to the difference of level between upper and lower points. (The times marked on the curves are given as multiples of the time of fall through this vertical distance, and not, as in the former set, through the length of the line.)

Fig. 9.



The investigation of the path for minimum time in this more general case has proved intractable. It is easy to show that the times along the two lines are not in general equal, although this holds in the limiting case when the initial velocity is very great. The intermediate point then approaches the middle point of the line.

(3) We now come to the comparison of the times of descent along arc and chord when the particle starts with initial velocity. It is convenient to take separately the cases where the level of zero velocity is below and above the highest point of the circle, $h<2a$ and $h>2a$, h being the height of the starting level above the lowest point and a the radius of the circle.

In the former case put $h=2a\sin^2\gamma$, then γ has the same meaning as in the first note, viz. the inclination of OC, the chord from the lowest point to the intersection of the circle with the starting level. β is again the inclination of the chord OB along which the descent takes place. The times are expressible in terms of an auxiliary angle ϕ, defined by $\sin\phi=\sin\beta/\sin\gamma$,

$$\text{arc-time}=\sqrt{(a/g)}\,\mathrm{F}(\phi)\quad \text{mod.}\ \sin\gamma,$$

$$\text{chord-time}=\sqrt{(a/g)}\ .\ 2\tan\tfrac{1}{2}\phi.$$

For $h > 2a$ write $h = 2a \operatorname{cosec}^2\gamma'$

and $\sin\phi' = \sin\beta \sin\gamma'$,

then we find arc-time $= \sqrt{(a/g)} \sin\gamma' \mathrm{F}(\beta)$ mod. $\sin\gamma'$,

chord-time $= 2\sqrt{(a/g)} \tan\frac{1}{2}\phi'$.

When the particle starts from rest at the highest point of arc and chord $\gamma = \beta$, and the times become $\sqrt{(a/g)}\mathrm{K}$ and $2\sqrt{(a/g)}$ respectively, where K is the complete elliptic integral to modulus $\sin\beta$. These are equal when β has the value 53°·35 approximately. For smaller values of β the time down the arc is less than that down the chord.

Consider first a value of β lying in this region and let the starting level be raised continuously. It is evident that when the initial velocity is large the times will be nearly proportional to the lengths of arc and chord, and so the arc-time ultimately becomes the longer of the two. There is therefore, some definite starting-level giving equal times. Examination of the formulæ shows that the corresponding value of γ ranges from 45° to 53°·35 as β increases from 0° to 53°·35. It thus appears that Galileo was correct in the assumption on which he founded his proof of the "scholium," seeing he expressly limited himself to a lower quadrant of the circle.

For $\beta > 53°{\cdot}35$ the ratio (arc-time)/(chord-time) has a value greater than unity when the particle starts from rest. Its mode of variation as the starting-level is raised may be summarized here. There is a second special value of β given by $\mathrm{K} = \sec\beta$, $\beta = 63°{\cdot}9$ approximately. For values of β less than this the ratio in question increases steadily as the starting-level rises to infinite distance. For greater values the ratio first decreases, reaches a minimum, and then increases to its limiting value of (length of arc)/(length of chord). For example, if $\beta = 70°$ the ratio begins at value 1·252, drops to minimum of 1·236, and has value 1·239 when the starting-level is at the highest point of the circle and approaches the limiting value 1·240. As β approaches 90° the level giving the minimum ratio recedes to infinity. For motion along the semi-circle and the vertical diameter ($\beta = 90°$) the ratio decreases continuously from ∞ to $\frac{1}{2}\pi$.

(4) The circular arc of quickest descent one would expect to be determined by a compromise between the shortness of path on one hand, and on the other the steepness of fall at the upper point. The feature by which Galileo's arc is characterized, viz., tangent at lower point horizontal, does not play any essential part in the matter. We may take the inclination to the horizontal of the tangent to the circle at

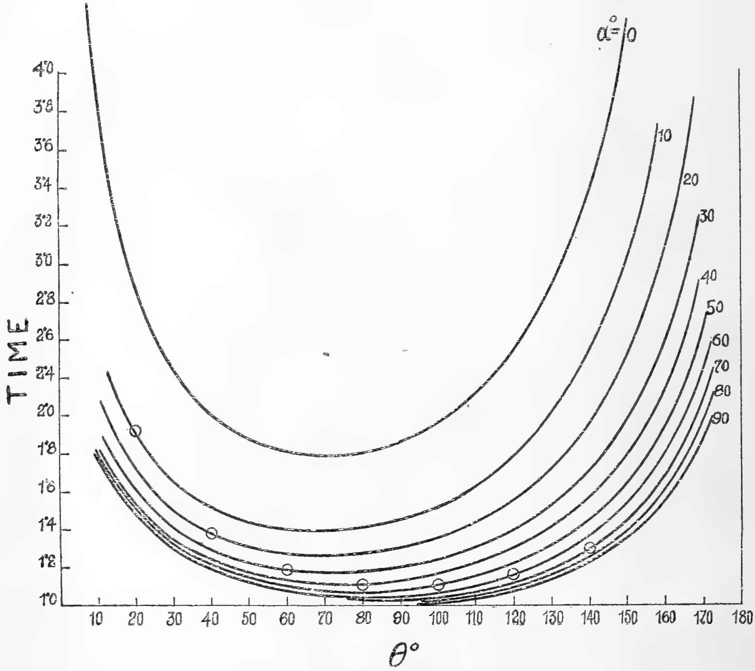
the upper point as fixing the arc. Call this θ and the inclination of the chord, as before, β. The radius of the circle must be replaced in the formulæ by the length of the chord l which is to be regarded as constant. When this is done the expression for the time, expressed as a multiple of that required for vertical fall through l, is

$$\{\mathrm{K}-\mathrm{F}(\phi)\}/2\sin^{\frac{1}{2}}(\theta-\beta), \quad \text{mod. } \sin\tfrac{1}{2}\theta,$$

where $$\sin\phi=\sin(\beta-\tfrac{1}{2}\theta)\operatorname{cosec}\tfrac{1}{2}\theta.$$

This is shown plotted against θ on fig. 10. The points marked with a ring correspond to Galileo's arc. It will be seen that the direction of initial motion for the arc of minimum time is inclined to the horizontal at an angle which varies from about 70° (in the case of a horizontal chord) to 90°.

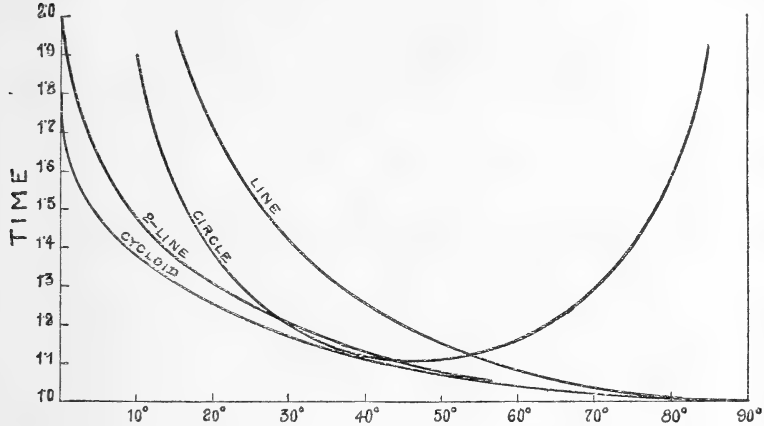
Fig. 10.



(5) The comparative times along the various routes which have been discussed are shown in fig. 11. The initial and final points are taken at a fixed distance apart and the abscissa is the inclination to the horizontal of the line joining them. The unit of time is that required for the vertical fall

through the distance between the points. The ordinates of the curves show the amounts by which the corresponding times exceed this.

Fig. 11.



The curve marked "line" gives the motion along the straight line of given length at different slopes, in accordance with the value of $\operatorname{cosec}^{\frac{1}{2}}\beta$. That marked "circle" gives the times along what Galileo supposed to be the brachistochrone, viz. a circular arc with horizontal tangent at the lowest point.

The "two-line" curve corresponds to the quickest path along two straight lines, as discussed in § 2.

The curve marked "cycloid" shows the true minimum time along a cycloid with vertical cusp at the upper point. This is calculated by the formula $\psi \sin^{\frac{1}{2}}\beta \operatorname{cosec} \psi$, where ψ is connected with the slope β by the relation

$$\tan\beta = \sin^2\psi/(\psi - \sin\psi\cos\psi).$$

Of course it is easiest to plot the graph by beginning with assumed values for ψ.

The graph for the "circular arc of quickest descent" is got by picking out the minima of the different curves on fig. 10. When this is done it is found that the resultant curve lies so close to the "cycloid" curve as to make it impossible to separate them on the scale of our figure. This is an unexpected result. For $\beta=0$ the circle time is about 1·78, the cycloid time is $\sqrt{\pi}=1{\cdot}77$, and the curves at other points are closer than this.

It will be noticed that Galileo's curve approaches very closely to the true minimum at $\beta=37\frac{1}{2}^{\circ}$ or thereabouts, the difference in the times amounts to about $\frac{1}{2}$ per cent.

XX. *The Thermal Effect produced by a slow Current of Air flowing past a Series of fine heated Platinum Wires, and its Application to the Construction of Hot-Wire Anemometers of great Sensitivity, especially applicable to the Investigation of slow Rates of Flow of Gases.* *By* J. S. G. THOMAS, *D Sc. (Lond.), B.Sc. (Wales), A.R.C.S., A.I.C., Senior Physicist, South Metropolitan Gas Company, London*[*].

IN a recent communication [†] a type of directional hot-wire anemometer was described, consisting of two fine platinum wires inserted in a pipe or channel parallel to one another, and one behind the other in close juxtaposition, transversely to the direction of flow of gas in the pipe or channel. It was shown that the sensitivity of such a hot-wire anemometer for the measurement of the velocity of a slow-moving stream of gas was considerably greater than that of the Morris type of hot-wire anemometer[‡]. It may be remarked that the Pitot tube is most satisfactorily employed for purposes of anemometry, for the measurement of large velocities, as the measured effect is proportional to the *square* of the velocity of the stream in which the tube is inserted. The hot-wire anemometer, on the other hand, in which the observed effect is proportional to the *square root* of the velocity to be measured, appears *especially* suitable for use in the region of low velocities. Nevertheless, it has been shown by King [§] and others that the hot-wire anemometer is an instrument of very high precision, even in the region of high velocities. The difficulties encountered in the measurement of low velocities arise principally owing to the existence of the free convection current of gas in the neighbourhood of the wire due to the heated condition of the wire. In the directional type of hot-wire anemometer the effect of the free convection current is largely eliminated. The directional type of instrument possessing the desirable characteristics referred to, experiments were undertaken to determine whether the sensitivity of this type of anemometer could be still further increased in the region of low velocities, and the present communication details some results obtained in the course of such an investigation.

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[*] Communicated by the Author.

[†] Phil. Mag. vol. xxxix. pp. 525-527 (1920); see also Proc. Phys. Soc. vol. xxxii. pp. 196-207 (1920).

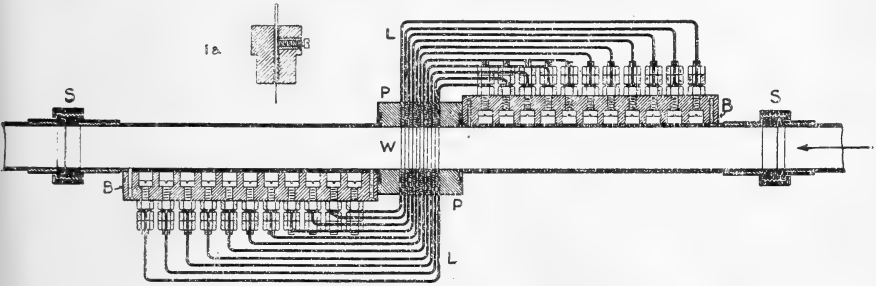
[‡] Eng. Pat. 25,923/1913. See also B.A. Reports, 1912 and 1920; 'Electrician,' Oct. 4, 1912, p. 1056, and Aug. 27, 1920, p. 227.

[§] King, Phil. Mag. vol. xxix. p. 564 (1915).

Experimental.

The flow-tube employed in the present series of experiments had a diameter of 2·0534 cm., the same as that used in previous work referred to above. Fig. 1 illustrates the manner in which the heated platinum wires employed—eleven in number—were mounted, and their mode of insertion in the flow-tube. The respective ends of each of the fine platinum wires W were affixed, by means of the smallest amount of silver solder affording a secure junction, to portions of considerably thicker copper wires represented by L, L. These copper wires passed tightly through holes in plugs of ebonite P, P, which fitted accurately into rectangular slots furnished by projections joined to the main tube at right angles, as shown. Precautions were taken that the slots were diametrically disposed with respect to the cross-section of the flow-tube, and after insertion of the plugs the continuity of the inner surface of the tube was restored by carefully filing and polishing the inner surfaces of the respective inserted ebonite plugs. The resistances of the several platinum wires were made as nearly equal as possible in the manner previously explained* ; and by appropriate adjustment of the copper wires to which the platinum wires were affixed, it was possible to secure that the several wires were initially coplanar, and remained equally spaced in the initial diametral plane of the flow-tube on being heated by means of an electric current. The copper wires passing through the ebonite plugs were, after adjustment of the platinum wires as above, secured in position by means of small screws passing through fine brass bushes inserted

Fig. 1.



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* Phil. Mag. vol, xl. pp. 641, 642 (1920).

into the ebonite in the manner shown at 1 a. Alternate bushes were inserted from opposite sides of each ebonite block. The ends of the copper wires were connected to binding screws affixed to ebonite blocks B secured to the anemometer tube as shown, so that each of the platinum wires inserted in the tube was connected with very approximately the same length of lead. The several terminals on the ebonite blocks were connected separately by means of thick stranded copper wire with terminals dipping into mercury cups, so that any or all of the heated platinum wires could be inserted in any external circuit as desired. The insertion of the tube in the main flow tube was effected by means of the spigot unions shown at S, S, a device affording a smooth junction between the tubes, with the prevention of eddies. The anemometer tube was wrapped in three layers of felt, affording efficient thermal insulation, and the remainder of the flow-tube wound with asbestos cord. The general method of calibration of the heated wires for purposes of anemometry is given in the papers by the author already referred to*. The fixed ratio arm in the Wheatstone bridge employed was throughout adjusted to 1000 ohms. For purposes of measuring the respective temperatures to which the several wires were raised by the heating current and the air stream employed, the respective values of R_0 for the individual wires were determined from their respective resistances, measured at atmospheric temperature, using the value of the temperature coefficient of the sample of wire from which the several wires were cut, determined by means of a Callendar and Griffiths bridge. The current used for this purpose was 0·005 amp., and allowance was made for the lead resistance. When the platinum wires were heated by the current (1·1 amp.) employed in the series of experiments detailed below, their respective temperatures were deduced from their resistances as calculated from the voltage drop occurring across the respective wires. The platinum-scale temperatures so calculated were in all cases corrected to the gas scale. The Weston voltmeter employed enabled readings to be made to 0·001 volt, and it was calibrated before use. Readings of current were made either on a calibrated Weston or Siemens & Halske milliammeter. In the present series of experiments the anemometer wires were used in a horizontal position, and the flow-tube was likewise placed horizontally.

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* Phil. Mag. *loc. cit.* Proc. Phys. Soc. *loc. cit.*

Results and Discussion.

Diameter of flow-tube	2·0534 cm.
Diameter of platinum wires	0·101 mm.
Temperature coefficient of resistance of platinum wires	0·003335
Mean distance between successive wires	1·3 mm.
Number of platinum wires employed	11
Heating current	1·1 amp.
Fixed ratio arm in Wheatstone bridge	1000 ohms.

The respective resistances of the several wires at 0° C., and the temperatures to which they are raised by the heating current of 1·1 amp. subsequently employed, are set out herewith in Table I, the wires being numbered consecutively in the direction in which the flow was established in the tube :—

TABLE I.

Wire No.	1.	2.	3.	4.	5.	6.	7.	8.	9.	10.	11.
Resistance at 0° C. (ohm).	0·2454	0·2479	0·2507	0·2498	0·2470	0·2470	0·2470	0·2498	0·2545	0·2500	0·2535
Initial temperature to which wires were raised by current of 1·1 amp. in absence of flow (°C.)	560°	674°	690°	688°	695°	690°	705°	663°	672°	635°	526°

The mean value of the resistances at 0° C. of the several wires was found to be 0·2484 ohm. The individual resistances, except Nos. 1, 9, and 11, were all within 1 per cent. of this mean value. The percentage variation from the mean value in the case of the 1st, 9th, and 11th wires respectively were 1·2 per cent., 2·4 per cent., and 2 per cent. The distribution of temperatures amongst the several wires for zero flow may be roughly described as a condition in which the temperature increases considerably from the respective outer wires of the series to the adjacent ones, the remaining wires contained within the four outer wires being all raised to an approximately uniform temperature of 680° C. When the motion due to the slow current of air in the flow-tube was impressed upon the free convection currents arising from the wires heated by a current of 1·1 amp., it was found that, except in the case of the first wire of the series, the resistance of the individual wires was increased. An indication of the constancy of the air-flow was secured by employing the leading pair of wires, in conjunction with a Wheatstone bridge, as a hot-wire anemometer in the manner already described. A comparison of the respective thermal effects experienced by the several wires due to the air-flow is most readily made by determining in each case the change of temperature of the individual wires accompanying the establishment of the air-flow in the tube. The results given in figs. 2 and 3 indicate how the temperatures

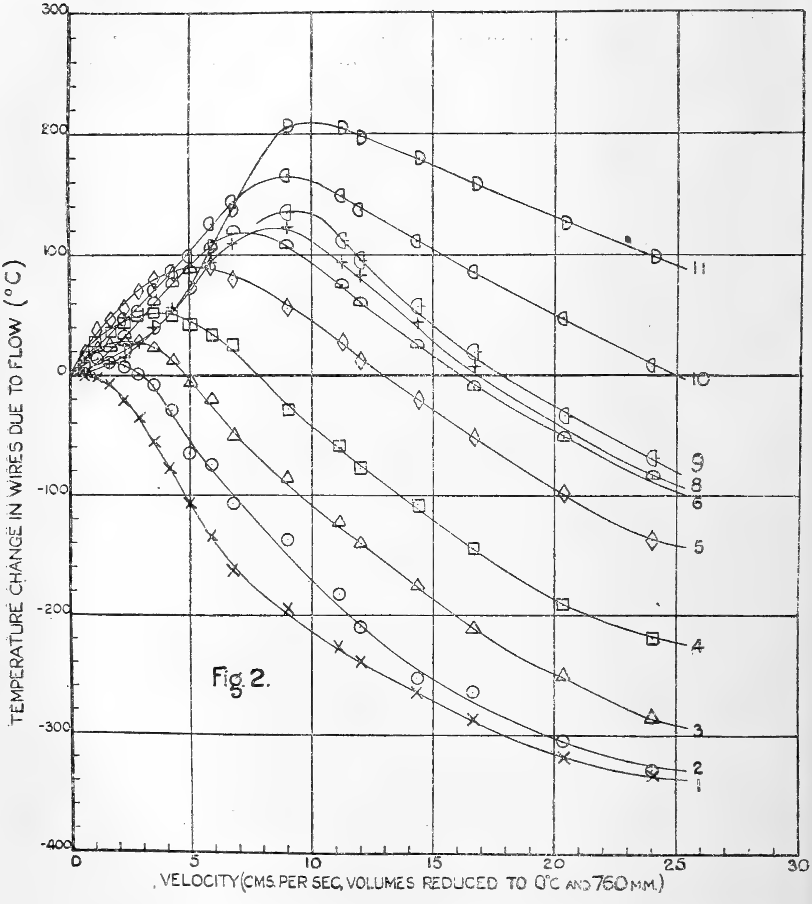

Fig. 2.

to which the respective wires were raised by the heating current of 1·1 amp. employed depended upon the magnitude of the mean velocity of the air-flow established in the tube. The ordinates in every case represent the increase or decrease of temperature of the wire above or below the normal temperature of the wire in the absence of flow. The curve representing the behaviour of No. 7 wire is omitted from fig. 2 for the sake of clearness. Its descending portion lies between the corresponding portions of the curves representing the behaviour of wires Nos. 6 and 8. The initial portion of the curve for No. 9 wire is omitted from

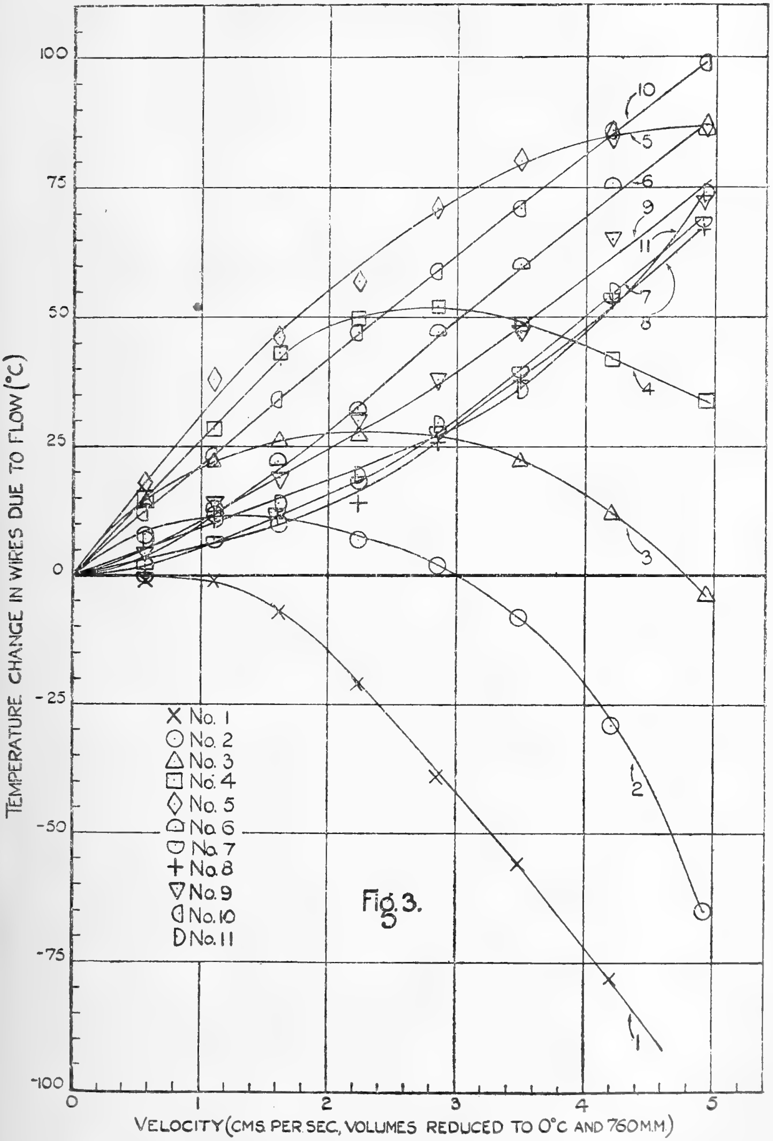

Fig. 3.

fig. 2 for the same reason. Fig. 3 shows on an enlarged scale the form of the various curves contained in fig. 2 for values of the average impressed velocity of the air stream flowing in the tube, ranging between 0 and 5 cms. per sec. Briefly, the characteristics of the several curves may be summarised as follows :—

(1) From fig. 2 it is seen that, except in the case of No. 1 wire, the temperature of each wire of the series increases with increase in the mean velocity of the air stream, from its initial zero value, until a maximum value of the temperature increase is attained in the case of each wire, the temperature increase thereafter decreasing continuously, the temperature of each wire ultimately falling continuously below its normal value with increase in the impressed velocity of the air stream.

(2) The temperature of No. 1 wire falls off continuously with increase of the impressed velocity of the air stream.

(3) The maximum rise of temperature of the various wires increases progressively in the direction of flow of the impressed air stream, and, moreover, the maximum rise in the case of each wire occurs at a value of the impressed velocity which is progressively greater the further the wire in question is removed from the first wire of the series. Thus the maximum rise of temperature in the case of No. 9 wire is greater than that of, say, No. 6 wire, and occurs at a greater value of the impressed velocity of the air stream.

(4) The variation in the values of the velocity at which the maximum temperature rise is attained in the case of the successive wires, is seen to be large in the case of the the leading wires of the series (wires Nos. 1–6, fig. 2) compared with the variation in the case of later members of the series.

(5) The thermal behaviour of the several wires in the present case enables the wires to be grouped together as follows :—

> (*a*) Wires Nos. 1, 2, 3, 4, and 5, for which the initial increase of temperature, where it occurs, progressively increases the later the wire occurs in the sequence in the direction of the air-flow.
>
> (*b*) Wires 6 and 7, for which the initial rate of increase of temperature is approximately constant and less than that for class (*a*).
>
> (*c*) Wires 8, 9, and 10, for which the initial increase of temperature progressively increases as in group (*a*).
>
> (*d*) Wires 10 and 11, in which the initial increase of temperature progressively decreases as in group (*b*).

Note.—In this classification the first member of each class after the first may be regarded as affording a transition between successive classes of wires.

With regard to (1) above, Table II. gives the maximum rise in temperature, and the maximum temperature attained by each wire in the case of the present series of wires.

TABLE II.

Wire No.	1.	2.	3.	4.	5.	6.	7.	8.	9.	10.	11.
Maximum increase of temperature. (°C.)	0	12	28	52	91	120	120	125	130	166	198
Maximum temperature attained. (°C.)	560	686	718	740	786	810	825	788	802	801	724

It is seen that the maximum increments of temperature are by no means small, attaining values equal to about 200°C. in the case of the later members of the sequence of wires. This fact is of some little consequence in connexion with the use of electrical heating coils of fine wire for the purpose of heating a stream of air or other gas. Contrary to what might be anticipated, it is easily possible for the temperature of the coil, in the absence of an impressed flow, to be such that actual fusion of one or more turns of the coil occurs when a slow stream of air or other gas is established past the coil. In the case of the wires employed in the present series of experiments, the highest temperature is attained by No. 7.

The various peculiarities—see (1) to (5) above—of the thermal effects experienced by the respective wires can be readily discussed in the light of the characteristics of the heat transfer by convection, etc., to which attention has been directed in a previous paper*. Owing, however, to the multiplicity of wires employed in the present instance, such a mode of presentation could afford, at best, but a hazy mental picture of the phenomena appropriate to the present case. The following alternative graphical method of presentation, while possessing no claims to quantitative accuracy, has been found of value in the discussion of the heat transference occurring in the present and similar instances :—

In fig. 4 the several wires of the system are denoted by A, B, C, D, E, F, G, H, I, K, L. The full lines extending into the region above the respective wires represent the heated convection currents arising therefrom. The broken lines in the region below the several wires represent the comparatively colder convection currents ascending towards

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* Proc. Phys. Soc. vol. xxxii. pp. 199–206 (1920).

the wires. The figure represents the condition of affairs in the absence of any impressed flow of air in the tube. The lengths of the various full lines are drawn so as to indicate roughly the relative temperatures of the respective wires, and consequently of the respective convection currents arising therefrom (see Table I.). The colder convection currents below the wires are similarly represented by the broken lines, but are somewhat shorter in length, owing to their necessarily lower temperature. Fig. 5 represents the altered conditions ruling when the heated wires are subjected to the influence of an impressed flow of air moving with small velocity. For the sake of simplicity, consider first of all the two wires A and B. It has been shown elsewhere* that the heat-transfer from the first of the two wires A, say, to B, owing to an increasing impressed air-flow, is characterised by two opposing tendencies, viz. (1) the initial decreasing temperature of the resultant convection current, and (2) the decreasing fall in temperature occurring during transfer from the first to the second wire. The resultant thermal

Fig. 4.

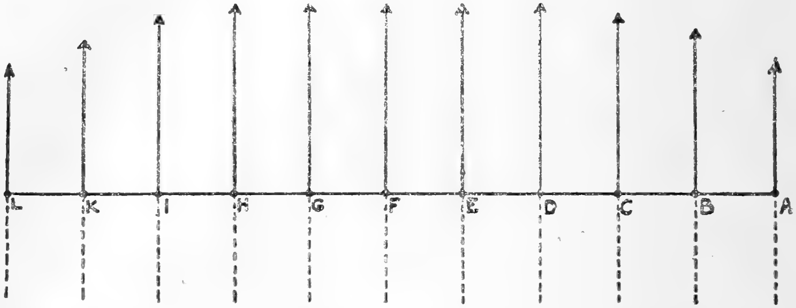
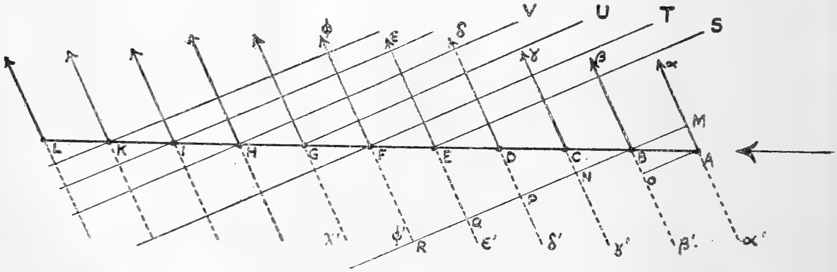


Fig. 5.



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* Proc. Phys. Soc. *loc. cit.*

effect experienced by the second wire is determined by the relative magnitudes of these opposing tendencies. The effect of an impressed flow of air past the wire is, as shown in fig. 5, to deflect the convection currents in such manner to cause an approach to B of the hot ascending from A, and to cause the cold convection current ascending towards B to approach nearer to A. In fig. 5, BM is drawn at right angles to Bβ. It will be readily seen that with increase of the velocity of the impressed stream of air, Aα and Bβ rotate in a counter clockwise direction, and the triangle ABM goes through a series of values corresponding to the variation of the thermal effect experienced by the second wire under these circumstances *. If, in like manner, AO be drawn at right angles to Bβ', a *decrease* in AO is to be interpreted as an *increase* in the cooling effect experienced by the wire A, owing to the impressed air-flow, and *vice versa*. It has been pointed out elsewhere†, and the result will be clearly seen, from a comparison of the initial portions of curves (1) and (2) in fig. 3, that for small velocities of the impressed air stream the heat gained by the second wire is considerably larger than that lost by the first. For this, and other reasons, the broken lines in figs. 4 and 5 are shown of shorter length than the full lines in the upper region of the diagram. It is clear that the thermal effects experienced by the respective wires A and B, due to a small impressed flow of air, can be roughly represented by the area of the triangle AMB and the reciprocal of AO. The introduction of a third wire C may now be readily seen by drawing BN perpendicular to Cγ'. It is clear that, owing to the impressed flow of air, the *cold* convection current rising towards C approaches B (*i. e.* $BN < BC$). We conclude, therefore, that by the introduction of a third wire C, the initial rise of temperature of the wire B produced by a small impressed air-flow is less than when the wire C is absent. It follows, therefore, that the initial sensitivity of a directional hot-wire anemometer constituted by the wires A and B for small impressed velocities is greater in the absence than in the presence of the third heated wire (see fig. 7, later). Attention has been previously directed to the significance of this phenomenon in connexion with increased sensitivity of the directional type of hot-wire anemometer (see Proc. Phys. Soc. *loc. cit.* p. 204). The effect of the respective wires D, E, F, etc. subsequent to C upon the temperature rise experienced by B is clearly to

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* Proc. Phys. Soc. *loc. cit.* p. 202.

† Phil. Mag. vol. xl. pp. 652, 653 (1920).

decrease further at a decreasing rate the temperature rise experienced by B, due to an impressed air stream. The effects of the respective wires upon the temperature of B may be taken as represented by the reciprocals of NP, NQ, and NR, the effect of subsequent wires being negligible, this being graphically represented by the fact that the line BNPQR does not meet the finite line Gχ'.

The characteristics of the initial parts of the curves contained in fig. 3, and referred to under 5 (*a*), (*b*), (*c*), and (*d*) above, can now be readily explained, thus:—

Group (*a*). For the wires A, B, C, D, and E in fig. 5 the positive triangular areas drawn above the wires and representing the thermal effect experienced by the respective wires clearly increases as the wire in question occupies the successive positions in the direction of flow of the stream. The total cooling or negative effect, on the other hand, is the same for successive wires from A to E. The initial rise in temperature of successive wires in this group therefore increases from A to E.

Group (*b*). For these the initial rise of temperature is less than that in the case of No. 5 wire, and is approximately constant. Graphically, this is illustrated in the case of wire F in this group by the fact that under conditions such that ES meets four of the lines representing the hot convection currents ascending from the wires, the line FT would meet three only*. The wire A is, in fact, so far away from the wires included in this group that its relatively small free convection current is without influence upon the wires in this group. For succeeding wires of this group the effect of the free convection current arising from B is in like manner negligible. Thus FT, GU alike cut three of the lines representing the hot convection currents arising from the wires. Obviously, owing to the skew nature of the diagram fig. 5, the wires constituting this group will be unsymmetrically placed with respect to the central wire of the series, and will be grouped about a point to the left of that wire.

Group (*c*). For this group, constituted of wires 8, 9 and 10, the initial rise of temperature progressively increases as in group (*a*). This is to be attributed, as shown in fig. 5, to the decreasing number of cold convection currents rising towards the wires to the left of the respective wires. The

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* It is clear, that if the diagram were correctly drawn with special reference to this case, Aα would be shown so that angle αAB < angle βBC, on account of lower temperature of wire A. Aα would then meet ES, although FT would not meet Bβ.

effect of the hot ascending currents upon members of this group is the same throughout. An increase of the initial rise of temperature of successive members of this group is therefore to be anticipated.

Group (*d*). In this group, constituted of wires K and L in fig. 5, it will be seen from the figure that, whereas the perpendiculars from K upon the lines representing the various hot convection currents meet four *full-length* lines, those cut by the perpendicular from L include the line of shorter length drawn from K. As already explained, the effect of the cold convection current rising to L upon the temperature of K is small. The temperature of L, being therefore principally conditioned by the relatively small free convection current rising from K, while the temperature of the latter is determined principally by the relatively larger convection currents arising from J, in accordance with principles enunciated above, it may be anticipated that in the presence of an impressed air-flow the initial rise of temperature of the wire L will be less than that in the case of wire K.

The thermal effect experienced by the individual wires consequent upon an increase in the velocity of the impressed stream of air can be readily seen by suitable modification of fig. 5. Such modification is to be effected, bearing in mind that a slight increase in the impressed velocity of the stream tends to cause the lines representing the respective convection currents to take up positions which are reached by a small anti-clockwise rotation from their previous positions. In like manner, an increase in temperature of any wire tends to cause the appropriate line to take up a position reached by a suitable clockwise rotation, a decrease of temperature having a contrary tendency. The resultant positions of the lines representing the convection currents are to be determined from the joint consideration of such tendencies. The magnitude of the free convection current appropriate to any temperature of the wire may be found from the results given previously*.

Application of above results to the construction of sensitive directional hot-wire anemometers.

The results contained in figs. 2 and 3 may be utilized in a variety of manners for the construction of hot-wire anemometers of the directional type, which, as already pointed out, have a special sphere of usefulness in the region of low

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* Phil. Mag. vol. xxxix. p. 523 (1920).

velocities of the impressed stream of air or other gas. A number of results employing various members of the series of wires for this purpose are given in figs. 6 and 7.

The deflexion-velocity calibration curves given in fig. 6 were obtained by inserting various members, or groups of members, of the wires adjacent to one another in opposite arms of a Wheatstone bridge, as explained in previous papers. The remaining wires were not heated, so that the temperature conditions of the various wires were not those indicated in fig. 1. The constant ratio arm was throughout adjusted to a constant value of 1000 ohms, and the sensitivity of the galvanometer was maintained constant throughout, the galvanometer being shunted with 10 ohms. The current employed was 1·1 amp. The curve marked 1 was obtained by inserting the first pair of wires, one in each appropriate arm of the bridge. The hot-wire anemometer so constituted is a simple hot-wire directional instrument similar to that described in a previous paper. It has been shown that the sensitivity of such an instrument is, in the region of low velocities, considerably greater than that of the corresponding Morris type of instrument. The curve marked 2 is the calibration curve obtained when the wires 1 and 2 are inserted in one arm of the bridge, and wires 3 and 4 in the opposite arm. A greater sensitivity of the bridge arrangement is to be anticipated for the latter arrangement, as, employing the same total current in the bridge with zero flow, the out-of-balance current through the galvanometer is approximately doubled on doubling the resistances in the respective arms of the bridge. Except in the neighbourhood of the origin, it will be seen that the sensitivity of the arrangement employing two wires, as described, in each arm of the bridge is considerably more than twice that when one wire alone is inserted in the respective arms. A similar result was found in the cases where 3, 4, or 5 wires, made up of an appropriate number of consecutive wires, were inserted in each arm of the bridge. Thus from fig. 6 it will be seen that, corresponding to the use of 1, 2, 3, 4, and 5 wires in the appropriate arms of the bridge, the deflexions corresponding to a mean impressed velocity of the air stream equal to 8 cms. per sec. were respectively 57, 185, 355, 485, and 527 divisions respectively. The results are readily explainable from a consideration of the increase or decrease of the temperature of the respective wires corresponding to an impressed velocity of 8 cms. per sec. Thus, in the case where all the wires are heated, under the influence of this impressed velocity, as seen in fig. 2, the respective changes of temperature of wires Nos. 1, 2, 3, and 4 would be approximately −184°, −128°, −72°, and −4° respectively. Thus, while wires 1 and 2 would differ in temperature by 56°, the mean temperature of wires 1 and 2 would differ from the mean of the temperature of wires 3 and 4 by 118°. The increased sensitivity of the device employing two wires in each arm above that to be anticipated from a mere doubling of the effective resistance, is attributable to an increase of this nature in the effective temperature difference between the pairs of wires. The increased sensitivities of the arrangements when three, four, and five wires are introduced into the respective arms are similarly explained. Analogous results were obtained when the whole series of wires were heated. The sensitivities in these cases were, however, slightly different from those indicated by fig. 6, but the relative sensitivities were of the same order as those just discussed. In the region of very low impressed

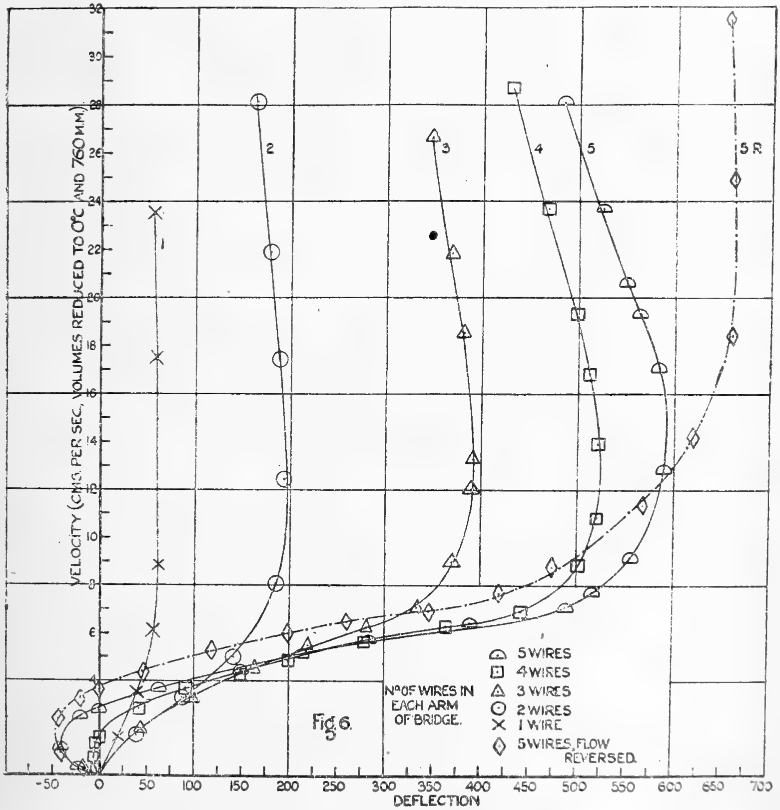

Fig. 6.

velocities of the air stream it will be seen from fig. 6 that in the cases where 4 or 5 wires were employed in the respective arms of the bridge, the deflexion is initially negative and increases numerically as the magnitude of the impressed velocity of the air stream is increased, until a maximum value of the deflexion is attained, the deflexion thereafter decreasing, attaining a zero value and thereafter increasing in the manner already described. A ready explanation of this apparently anomalous result is readily afforded by a consideration of the curves in fig. 3. It will be seen that, *initially*, the temperatures of wires Nos. 2 to 5 increase more rapidly than those of succeeding members of the series. The temperatures of Nos. 6, 7, 8, and 9, it will be noted, increase initially very slowly. It will be seen, therefore, that whereas for low values of the impressed velocity the rise in temperature of the wire No. 2 is always greater than that of wire No. 1, and the rise in temperature in the group of wires Nos. 3 and 4 greater than that of the group 1 and 2, by the introduction of wire No. 6 the mean rise of temperature of the group containing it may be initially less than that occurring in the corresponding group inserted in the bridge. A similar argument holds in the case of groups of wires in which subsequent wires are included. This reversal of the initial direction of deflexion is shown in fig. 6 in the respective cases where four and five wires are employed in each arm of the bridge, as already explained. That the effect was not attributable to any abnormality of the setting of any of the wires in the flow-tube, was readily proved by reversing the order of the wires in the flow-tube. The calibration curve so obtained, employing five wires in the respective arms of the bridge, is shown by the broken curve 5R in fig. 6. This, while not identical with the corresponding full-line curve—a result due to want of exact equality of the respective wires—reproduces the characteristic feature of reversal of the initial deflexion. Zero deflexion corresponds, of course, to the attainment of the same mean rise of temperature, due to the impressed air stream, in the two arms of the bridge. Fig. 6 indicates the attainment of such a state of affairs, in the case of five wires being employed in each arm of the bridge, for a value of the impressed velocity of the air stream equal to about 2·7 cms. per sec. The curves given in fig. 3, which are based on results obtained about two months prior to those shown in fig. 6, the wires having been intermittently employed during this period, indicate that zero difference in the mean rise of temperature corresponds to a value of the impressed velocity of the air stream equal to about 2·2 cms. per sec. Exact agreement is not to be anticipated, as wire No. 11 was not heated in the determination of the curve 5 in fig. 6. Finally, in this connexion it may be remarked that, as seen from fig. 6, the initial sensitivity of the arrangement employing three wires in each arm, as already explained, is practically identical with that of the two-wire arrangement. This initial decrease in the anticipated sensitivity is, of course, attributable to the fact that No. 6 wire, with its small initial rate of increase of temperature with increase of the impressed velocity, is included in the second group of wires constituting such a device.

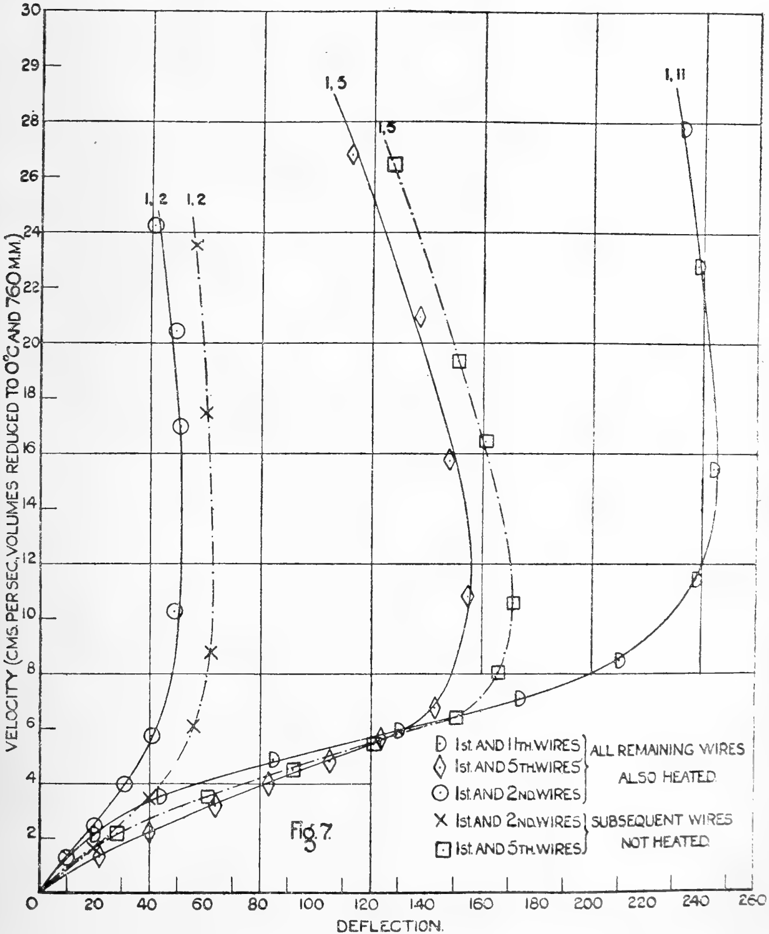

Fig. 7.

The calibration curves given in fig. 7 were obtained by inserting the single wires indicated in the arms of the bridge, intermediate wires functioning as a heating coil. In the case of the full-line curves in fig. 7, all remaining wires of the series were also heated, and in the case of the broken-line curves, wires subsequent to the pair inserted in the bridge were not heated. The heating current, both in the anemometer wires and in the heating coil, were throughout adjusted to 1·1 amp., and the galvanometer shunted with 10 ohms. The curves in fig. 7 are therefore, strictly comparable with those given in fig. 6. Considering first the full-line curves, which were obtained with all wires of the series heated, it will be seen that, while the ultimate sensitivity of the anemometer device for comparatively large velocities is greater, the later the wire employed in conjunction with the first is situated in the sequence of wires, yet this is not so for small values of the impressed velocity. Thus, whereas the respective deflexions corresponding to an impressed velocity of 8 cms. per sec. are 48, 150, and 200 when the 2nd, 5th, and 11th wires are used in the bridge in conjunction with the 1st wire, the deflexions corresponding to an impressed velocity of 4 cms. per sec. are respectively 32, 82, and 57. It is clearly seen that for low values of the impressed velocity the arrangement employing the 5th wire in conjunction with the 1st is more sensitive than that employing the 11th in conjunction with the 1st. The explanation of this phenomenon is clear from a consideration of the curves in fig. 3, wherein it will be seen that initially the temperature of No. 5 wire rises more rapidly than that of No. 11 wire, although, as seen from fig. 2, wire No. 11 attains the highest ultimate rise of temperature of the whole sequence of wire. It is clear from fig. 3 that, under the conditions specified, wire No. 5 used in conjunction with wire No. 1 affords the hot-wire anemometer employing one wire in each appropriate arm of the bridge possessing the maximum sensitivity in the region of very low velocities. From fig. 7 it will be likewise seen that, in accordance with anticipations advanced on page 249, the sensitivity of the anemometer device employing wires Nos. 1 and 2 is greater when subsequent wires are not heated than when such wires are heated. It will also be noticed that while this is also true of the anemometer device employing wire No. 5, in conjunction with wire No. 1, for values of the impressed velocity greater than about 6 cms. per sec., a reversal of the relative sensitivities occurs below this velocity, the greater sensitivity being then shown by the device in which subsequent wires are heated. The case of wires Nos. 1 and 5 being employed differs essentially from the case in which wires Nos. 1 and 2 are employed. As already pointed out, the extreme wires in a sequence are subject, as shown in figs. 4 and 5, to what may be termed an "end effect." The use of wires Nos. 1 and 2 alone, results in what is tantamount to the abolition of the end effect, as the end effect in this case is the whole effect, there being only two wires. The effect of the subsequent heated wires upon the temperature rise experienced by No. 2 wire owing to an impressed air-stream moving with slow velocity has already been discussed. The case of wire No. 5 being employed in conjunction with wire No. 1 can be readily discussed as follows:—When wires subsequent to No. 5 in the sequence are not heated, wire No. 5 is one of the end pair of wires, and is subject to what has been termed the "end effect." Its initial temperature and that of No. 4 wire are both less than that of No. 3 wire. When wires subsequent to No. 5 in the sequence are heated, wires 4 and 5 no longer experience the end effect. Their temperatures are considerably higher than was previously the case. With the establishment of a slow impressed stream of air, wire No. 5 now experiences a thermal effect, due principally to the approach towards it of the hot convection current arising from No. 4 wire. The convection current being warmer than was previously the case, the rise of temperature of wire No. 5 is materially greater, and the sensitivity of the anemometer device employing wires Nos. 1 and 5 consequently initially greater when wires subsequent to No. 5 are heated than is the case when these are not heated electrically. Possibly the matter may be made clearer by reference to fig. 3. There it will be seen that initially the temperature rise experienced by wire No. 11 the last of the sequence, is very much less than that experienced by the adjacent wire No. 10. In like manner, it is to be anticipated that when No. 5 wire is the last of the sequence of heated wires, its rise of temperature due to an impressed air-stream will be initially small compared with what it would be if wire No. 6 were heated, and similarly for subsequent wires of the whole sequence. In fig. 2 it will also be seen that the last wire of the sequence ultimately attains the greatest rise of temperature. Similarly, it is to be anticipated that when wire No. 5 is the last heated wire of the sequence, with increase in the velocity of the impressed stream, its temperature rise will ultimately be greater than when subsequent wires of the sequence are heated. Under these circumstances it is to be expected that ultimately the sensitivity of the anemometer device employing wires Nos. 1 and 5 will be greater if subsequent wires in the series are not heated than would be the case if these latter were heated.

In conclusion, it may be remarked that the sequence of wires illustrated in fig. 1 may be used, after the manner employed in platinum thermometry, for purposes of anemometry. Thus wires Nos. 1 and 11 might be inserted in a bridge, employing a bridge current of, say, 0·01 amp. Intervening wires would be employed as a heating coil. The deflexion-velocity calibration curves obtained in this manner present the same main features as those discussed in the present paper. With increase in the impressed velocity, the deflexion increases until an upper limit is reached, and thereafter decreases in the manner already described. The sensitivity of the device is, however, owing to the small current employed in the arms of the bridge, very considerably smaller than that of the type of anemometer described in the main part of this paper. Thus, using the same electrical apparatus, with its sensitivity equal to that employed throughout this work, in conjunction with what may be termed the thermometric anemometer just described, the heating current being 1·1 amp., the maximum deflexion obtained was 10 scale divisions. The thermometric type of hot-wire instrument has been introduced by C. C. Thomas* for the measurement of gas-flow. The electric energy requisite to maintain a constant difference of temperature of 2° C. in a pair of differential platinum thermometers situate one on each side of a heating-coil arranged across the section of the tube, is measured. It is clear that with certain dispositions of the thermometers and heating coil, for very small values of the velocity of the gas stream, the considerations advanced in the present paper become of importance. Thus, provided the heating coil and thermometers are suitably disposed, for low velocities of the gas stream, the electrical energy necessary to maintain a constant difference of temperature between the two thermometers may decrease with increase in the impressed velocity of the gas stream.

The work detailed herein was carried out in the Physical Laboratory of the South Metropolitan Gas Company. The author desires to express his sincere gratitude to Dr. Charles Carpenter, C.B.E., for his unfailing and inspiring interest in the research, and for the ready provision of all facilities necessary for the prosecution of the work.

Physical Laboratory,
South Metropolitan Gas Company,
709 Old Kent Koad, S.E.
20 Oct., 1920.

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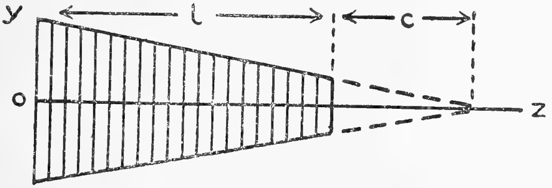
* Journ. Franklin Inst. 1911, pp. 411–460; Trans. American Soc. Mech. Eng. 1909, p. 655; Proc. American Gas Inst. 1912, p. 339.

XXI. *On the Period of Vibration of the Gravest Mode of a Thin Rod, in the form of a Truncated Wedge, when in Rotation about its Base*[*]. *By* LORNA M. SWAIN, *Lecturer in Mathematics, Newnham College, Cambridge*[†].

THIS problem was suggested to the author, in connexion with work undertaken at the Royal Aircraft Establishment, under Mr. R. V. Southwell, to estimate the effect of centrifugal force on the periods of vibration of airscrew blades. The effect was worked out rigorously for the case of a uniform rod by Mr. Arthur Berry[‡] and for the case of a rod tapering to a knife-edge by Mr. H. A. Webb and the author[§]. As explained in the latter report, the mathematical work for a rod tapering to an arbitrary depth would appear to be somewhat involved, and it seemed worth while to consider if the problem could not be tackled by some other method, likely to give results sufficiently accurate to be of use.

This paper contains an attempt to apply Lord Rayleigh's method for the calculation of the period of vibration of the gravest mode[||] to this case and also gives some estimation of the accuracy likely to be attained.

Fig. 1.



Consider a thin rod in the form of a truncated wedge, symmetrical about an axis (oz), encastred at the base, which contains the origin o, and rotating with uniform angular velocity ω about a perpendicular axis (oy). The breadth, measured perpendicular to the plane yoz, is uniform,

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[*] For the case of no rotation, see also a paper by J. Morrow in Phil. Mag. vol. x. p. 124 (1905).

[†] Communicated by the Author.

[‡] Advisory Committee for Aeronautics. Reports and Memoranda, No. 488.

[§] Advisory Committee for Aeronautics. Reports and Memoranda, No. 626.

[||] Rayleigh, 'Theory of Sound,' vol. i., §§ 88, 89.

and the depth, measured parallel to oy, decreases uniformly, remaining finite at the free end. The taper is defined by taking the length to be l and the additional length, which would have to be added to make it taper to a knife-edge, to be c.

The case of a uniform rod is then obtained by making c tend to ∞ and that of a rod tapering to a knife-edge by putting $c=0$*.

The problem is to find the period of the gravest mode for vibrations in the plane yoz.

Let σ be the area of any cross-section of the rod, perpendicular to oz, at distance z from o,

let I be the moment of inertia of the section about its mean line perpendicular to the plane yoz,

let σ_0, I_0 be the values of σ, I respectively for $z=0$.

Then $$\sigma=\sigma_0\frac{l+c-z}{l+c}, \qquad \mathrm{I}=\mathrm{I}_0\left(\frac{l+c-z}{l+c}\right)^3.$$

The rotation of the rod causes a tension in it and we proceed to evaluate this tension in the steady motion.

Let E be Young's modulus of elasticity,
F be the shear,
M be the bending moment,
T be the tension, assumed to act along the axis of the rod,
ρ be the density.
} at distance z from o,

These quantities are measured in gravitational in.-lb.-sec. units.

Resolving along the axis of the rod,

$$\frac{d\mathrm{T}}{dz}+\frac{\rho\omega^2}{g}\sigma z=0,$$

$$\frac{d\mathrm{T}}{dz}=-\frac{\rho\omega^2}{g}\sigma_0\frac{l+c-z}{l+c}z.$$

Integrating this equation and expressing the condition that $\mathrm{T}=0$, when $z=l$, we find

$$\mathrm{T}=\frac{\rho\omega^2}{g}\sigma_0 l^2\left\{\frac{1}{2}\left[1-\left(\frac{z}{l}\right)^2\right]-\frac{1}{3}\frac{l}{l+c}\left[1-\left(\frac{z}{l}\right)^3\right]\right\}.$$

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* For this method of treatment the author is indebted to Mr. H. A. Webb.

Now to apply Lord Rayleigh's method*, we require to assume some form for the deflexion curve of the axis of the rod. We shall make two different assumptions and compare the results obtained.

Let y be the displacement of the axis at distance z from o.

First assume

$$y=\eta\frac{z^2}{l^4}(z^2-4lz+6l^2),$$

where 3η is the displacement of the free end and for small vibrations in the plane yoz is a function of t.

This is the simplest algebraic curve that satisfies the end conditions for a uniform rod, encastred at one end and free at the other—viz.,

$$\left.\begin{aligned} y&=0 \\ \frac{dy}{dz}&=0 \end{aligned}\right\}\text{for } z=0,$$

$$\left.\begin{aligned} \mathrm{EI}\frac{d^2y}{dz^2}&=0 \\ \frac{d}{dz}\left(\mathrm{EI}\frac{d^2y}{dz^2}\right)&=0 \end{aligned}\right\}\text{for } z=l.$$

The last two equations become indeterminate for the rod tapering to a knife-edge, as I is itself zero for $z=l$, and hence we should not expect this assumption for the deflexion curve to give a very accurate value for this case. This will in fact emerge in the course of the work.

We require to calculate the kinetic and potential energies of the rod. For the small vibrations in the plane yoz, we have kinetic energy due to $\dot{y}$ and potential energy due both to the bending and to the tension.

$$\begin{aligned} \text{K.E}&=\tfrac{1}{2}\int_0^l \rho\sigma\dot{y}^2dz, \\ &=\tfrac{1}{2}\dot{\eta}^2\int_0^l \rho\sigma_0\left(\frac{l+c-z}{l+c}\right)\frac{z^4}{l^8}(z^2-4lz+6l^2)^2dz, \\ &=\tfrac{1}{2}\rho\sigma_0 l\dot{\eta}^2\int_0^l\left(1-\frac{lx}{l+c}\right)x^4(x^2-4x+6)^2dx, \text{ where } lx=z, \\ &=\rho\sigma_0 l\dot{\eta}^2\left[\frac{52}{45}-\frac{l/c}{1+l/c}\,\frac{292}{315}\right]. \end{aligned}$$

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* *Loc. cit.*

P.E. due to bending

$$=\tfrac{1}{2}\int_0^l \mathrm{EI}g\left(\frac{d^2y}{dz^2}\right)^2 dz,$$

$$=\frac{\mathrm{EI}_0 g}{2l^3}\eta^2\int_0^l\left(\frac{l+c-z}{l+c}\right)^3(12z^2-24lz+12l^2)^2dz,$$

$$=\frac{72\mathrm{EI}_0 g}{l^3}\eta^2\int_0^l\left(1-\frac{lx}{l+c}\right)^3(1-x)^4dx,$$

$$=\frac{72\mathrm{EI}_0 g}{l^3(1+l/c)^3}\eta^2\left(\frac{1}{5}+\frac{1}{2}\frac{l}{c}+\frac{3}{7}\frac{l^2}{c^2}+\frac{1}{8}\frac{l^3}{c^3}\right).$$

P.E. due to the tension

$$=\tfrac{1}{2}\int_0^l g\mathrm{T}\left(\frac{dy}{dz}\right)^2 dz,$$

$$=8\rho\omega^2\sigma_0 l\eta^2\int_0^l(x^3-3x^2+3x)\left(\frac{1}{2}(1-x^2)-\frac{1}{3}\frac{l(1-x^3)}{l+c}\right)dx,$$

$$=\rho\omega^2\sigma_0 l\eta^2\left(\frac{61}{45}-\frac{341}{315}\frac{l/c}{1+l/c}\right).$$

We have made two assumptions in calculating the kinetic and potential energies. In the first place we have neglected rotatory inertia and in the second we have followed Clebsch in calculating the potential energy due to bending; this is justifiable in most practical cases[*].

Assuming periodic motion of period $2\pi/p$, we finally obtain the equation for the period

$$\mu^4=\frac{72\left(\dfrac{\frac{1}{5}\frac{c^3}{l^3}+\frac{1}{2}\frac{c^2}{l^2}+\frac{3}{7}\frac{c}{l}+\frac{1}{8}}{(1+c/l)^3}\right]+\lambda\left[\dfrac{122}{45}-\dfrac{682}{315}\dfrac{1}{1+c/l}\right]}{\dfrac{52}{45}-\dfrac{292}{315}\dfrac{1}{1+c/l}}$$

where $\lambda=\frac{1}{2}\dfrac{\rho\sigma_0\omega^2l^4}{g\mathrm{EI}_0}$, $\mu^4=\dfrac{\rho\sigma_0p^2l^4}{g\mathrm{EI}_0}$.

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[*] Clebsch, *Theorie der Elasticität fester Körper*, p. 253. Rayleigh, 'Theory of Sound,' vol. i. § 188. *Enc. der Math. Wissenschaften*, vol. iv. 4 p. 251.

The neglect of one term in the potential energy due to the tension and the retention of another is legitimate, since it will be found that the ratio of these terms depends on the ratio of a radius of gyration of a section to the length of the rod, which must be small, when the approximate theory for thin rods is used.

This gives for the uniform rod, $\mu^4 = 12{\cdot}46 + 2{\cdot}35\lambda$ . . . (i *a*)

" " " " rod, tapering
to a knife-edge, $\mu^4 = 39{\cdot}37 + 2{\cdot}39\lambda$ . . . (i *b*)

Secondly, let us assume $y = \eta \dfrac{z^2}{l^2}$, where η is the displacement of the free end.

This is likely to give better results for the rod tapering to a knife-edge than the first assumption, as only one of the end conditions now becomes indeterminate.

Proceeding in the same way as before, we find

$$\text{K.E.} = \tfrac{1}{2}\rho\sigma_0 l\dot{\eta}^2 \left[\frac{1}{5} - \frac{1}{6}\frac{l}{l+c}\right],$$

P.E. due to bending

$$= \tfrac{1}{2}\frac{g\mathrm{EI}_0}{l^3}\eta^2\frac{l+c}{l}\left[1 - \left(\frac{c}{l+c}\right)^4\right],$$

P.E. due to the tension

$$= \rho\omega^2\sigma_0 l\eta^2 \left[\frac{2}{15} - \frac{1}{9}\frac{l}{l+c}\right].$$

Hence

$$\mu = \frac{\tfrac{1}{2}\dfrac{l+c}{l}\left[1 - \dfrac{1}{(1+l/c)^4}\right] + \lambda\left[\dfrac{4}{15} - \dfrac{2}{9}\dfrac{l/c}{1+l/c}\right]}{\dfrac{1}{10} - \dfrac{1}{12}\dfrac{l/c}{1+l/c}},$$

$$= \frac{\left(1+\dfrac{c}{l}\right)\left[1 - \left(\dfrac{c/l}{1+c/l}\right)^4\right]}{\dfrac{1}{5} - \dfrac{1}{6}\dfrac{1}{1+c/l}} + \tfrac{8}{3}\lambda.$$

This gives for the uniform rod, $\mu^4 = 20 + 2{\cdot}66\lambda$ . . . (ii *a*)

" " " " rod, tapering
to a knife-edge, $\mu^4 = 30 + 2{\cdot}66\lambda$ . . . (ii *b*)

To obtain some idea of the closeness of the approximation, (i *a*), (i *b*), (ii *a*), (ii *b*) have been graphed in figures 2 *a*, 2 *b*, and compared with the more accurate results obtained in the two papers already referred to. It should perhaps be pointed out in passing, that for the uniform rod when $\lambda = 7$, μ^4 should be 29·2 and not 32·1 as given in Mr. Berry's paper. With this correction, we see that the lower Rayleigh curve for the uniform rod is that corresponding to the first assumption and lies almost along the accurate value curve. It should of course by the general theory lie above it, whereas for the larger values of λ it appears to lie just below. This may be accounted for by the fact, that the values obtained in Mr. Berry's paper are only accurate within about 1 per cent. This shows that the values obtained from the Rayleigh curve should be accurate within about $\frac{1}{2}$ per cent. For the rod tapering to a knife-edge, the Rayleigh curve corresponding to the second assumption is the lower and gives results correct within about 6 per cent.

Fig. 2 *a*.—Uniform rod.

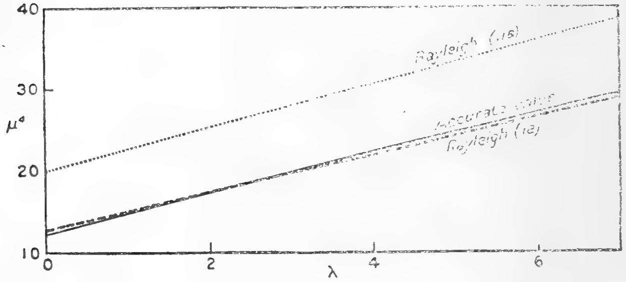
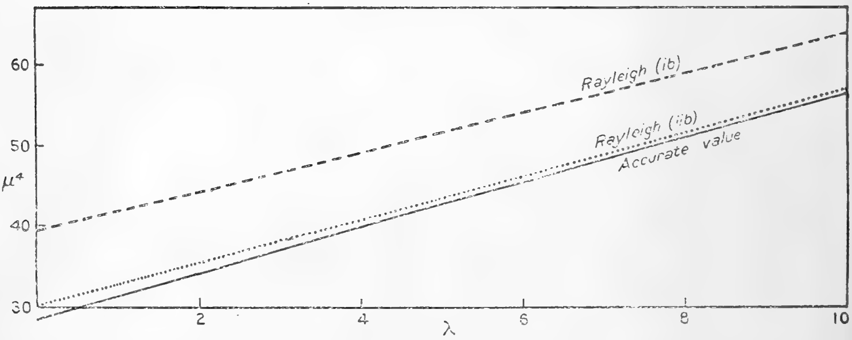


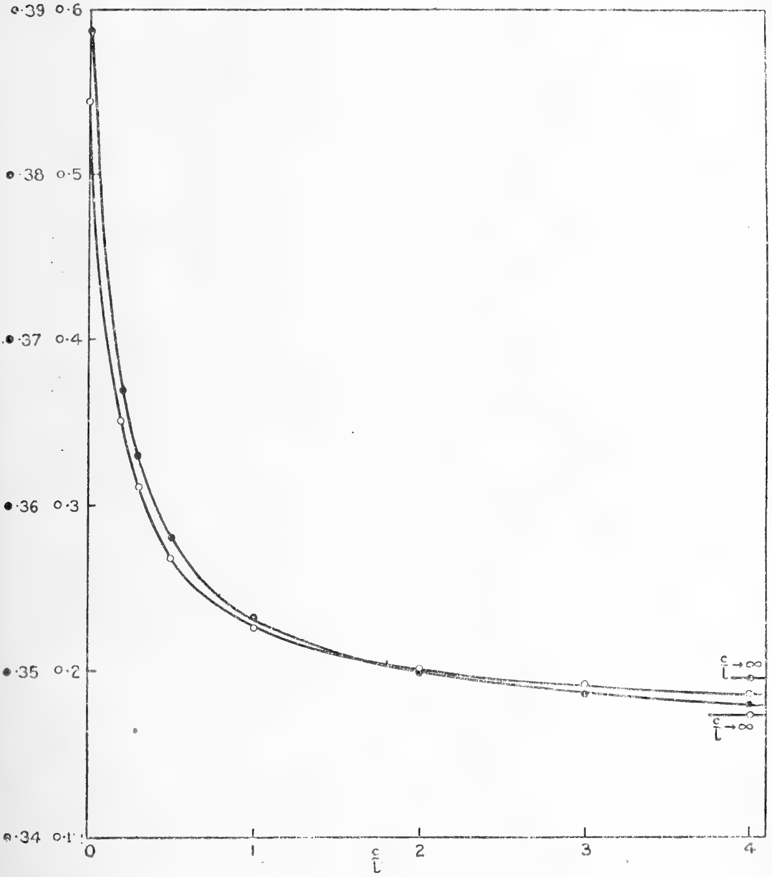
Fig. 2 *b*.—Rod tapering to a knife-edge.



Rayleigh (i *a*), (i *b*) correspond to the displacement curve $y=\eta\frac{z^2}{l^2}\left(1-4\frac{z}{l}+6\frac{z^2}{l^2}\right)$.

(ii *a*), (ii *b*) " " " " $y=\eta\frac{z^2}{l^2}$.

Fig. 3.

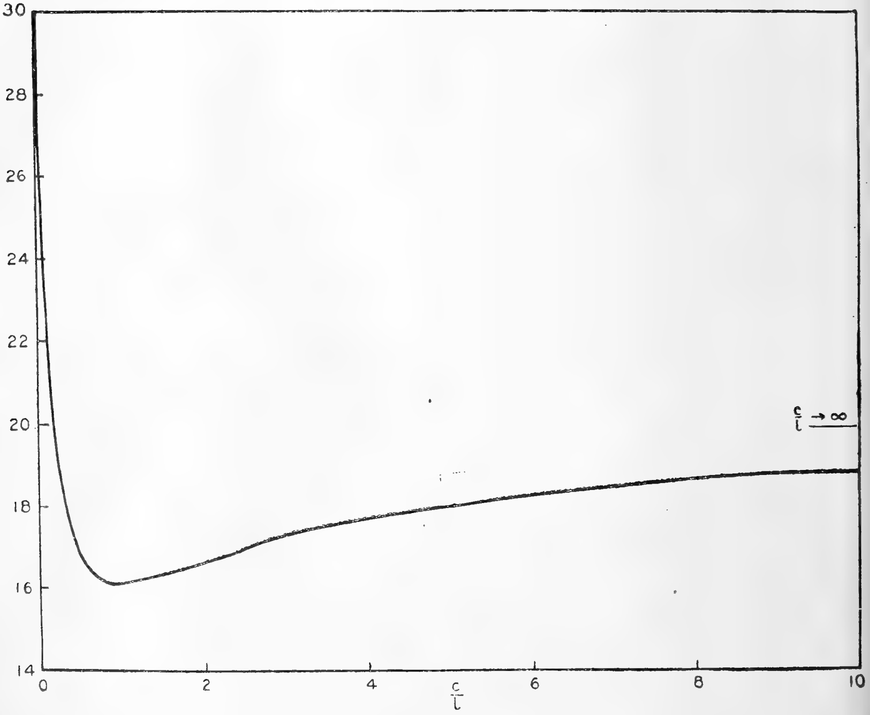


Curve marked o is graph of $\dfrac{\frac{1}{8}+\frac{3}{7}\frac{c}{l}+\frac{1}{2}\frac{c^2}{l^2}+\frac{1}{5}\frac{c^3}{l^3}}{\left(\frac{52}{45}+\frac{292}{315}\frac{1}{1+\frac{c}{l}}\right)\left(1+\frac{c}{l}\right)^3}$.

Curve marked ● is graph of $\left\{\dfrac{\frac{122}{45}-\frac{682}{315}\frac{1}{1+\frac{c}{l}}}{\frac{52}{45}-\frac{292}{315}\frac{1}{1+\frac{c}{l}}}-2.\right\}$

Thus for any rod, tapering to a section of finite depth, its period of vibration should be given, with a possible error of about 6 per cent., by the Rayleigh curve corresponding to that assumption, which gives the lower result for the value of c/l under consideration.

Fig. 4.



Graph of $\left(1+\frac{c}{l}\right)\left[1-\left(\frac{\frac{c}{l}}{1+\frac{c}{l}}\right)^4\right]\Big/\left[\frac{1}{5}-\frac{1}{6}\frac{1}{1+\frac{c}{l}}\right]$.

The term in μ^4 independent of ω and the coefficient of λ have be engraphed against c/l for the two different assumptions as to the deflexion curve (see figures 3, 4), so that it should be easy to see which assumption will give the better result for any particular value of c/l or λ.

In conclusion the author would like to express her gratitude to Mr. Arthur Berry for much kind advice and helpful criticism.

XXII. *On the Problems of Temperature Radiation of Gases.* (Paper C.) *By* MEGH NAD SAHA, *D.Sc., Lecturer in Physics and Applied Mathematics, University College of Science, Calcutta* *.

§ 1.

THE object of the present paper is to discuss and examine the present-day position of the question of temperature radiation of gases. The problem before us is, whether by simply heating a quantity of gas confined within a closed vessel (say, a silica or a graphite tube), it is possible to make the gas emit its characteristic line-radiation. The experimental results on this subject are somewhat conflicting, and for different elements are widely divergent. While Pringsheim and others † found that permanent gases like H_2, He, Ne, A, N_2, O_2, etc., remain non-luminous even at the highest temperatures which can be commanded in the laboratory, it is known that vapours of many elements easily become luminous at moderate temperatures. Such, for example, are the vapours of I, Br, As, S, Se, Sb, and other metalloids. If we take the tube-furnace spectra of King ‡ to be cases of pure temperature-radiation, we have to admit that at temperatures varying from 2000° to 3000° K, most of the alkalies, the alkaline earths, thallium, iron, vanadium, etc., can be made to emit their line-spectra. Gibson § obtained the green line of thallium by simply heating the element contained in a quartz tube. But the conclusion drawn by him—that the intensity of the green line is the same as that of the black body-spectrum at this particular wave-length—is entirely wrong. He placed the quartz tube within a black-body chamber heated to about 1200° C., and observed that the continuous spectrum from the black body was crossed by a black absorption line corresponding to the green emission line. But this black line faded away as soon as the thallium vapour took up the temperature of the enclosure. From this he concluded that the emission of the green line had just become as intense as that of the black body at the same part of the spectrum. But the conclusion is erroneous, for substances in temperature equilibrium within a black body enclosure would all emit like a black body, and the experiment proves nothing but this simple property of a black-body enclosure.

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* Communicated by the Author.

† Pringsheim, *Verh. d. Deutsch. Phys. Gesellschaft*, 1903. Wood, *Phys. Zeits.* viii. (1907), and other papers.

‡ King, *Astro. Journal*, vols. xxviii., xxxiv., xxxv., xxxvii.

§ Gibson, *Phys. Zeits.* xii. pp. 1145–1148 (1911).

According to many physicists, the flame, the arc, and the spark represent gradually increasing stages of temperature, viz. 2000° K, 4000° K, and 5000° K. It is well known that spectra produced under these conditions are widely different in their general characteristics, but the hypothesis that these variations can mainly be attributed to the varying values of a single physical variable, viz. the temperature, is not generally accepted.

I wish to point out that the value of the ionization potential, as obtained experimentally by Franck and Hertz, McLennan, and others, or theoretically from the quantum relation $eV = h(1, s)$, has a great bearing on the problem. As a rule, *the higher the Ionization-potential of an element, the greater is the difficulty with which it can be excited to emit its line-spectrum.* This will be apparent from the following tables[*]:

Table I.

Element..	Mg	Ca	Sr	Ba	Na	K	Rb	Cs
Ionization potential.	7·65	6·12	5·70	5·21	5·11	4·32	4·16	3·88
Element..	Zn	Cd	Hg	H	He	Ne	A[†]	N[‡]
Ionization potential.	9·4	9	10·45	13·6–17·1	20·5–25·6	17–23[†]	16	17–18

The line or lines $\nu = (1, s) - (2, p)$ form the most important lines of an element, and experiments on the ionization potential have shown that when the vapour of an element is bombarded by electrons, this is the line which is the first to be excited, other lines appearing only when the stimulus is substantially greater. The potential $V = \frac{h}{e}[(1s) - (2p)]$ is therefore called the resonance potential, and may be taken to be a measure of the stimulus which is required so that an element may be just excited to radiation of its fundamental line. A better name would probably be "Radiation-Potential."

We may give a number of interesting examples. It is well known that generally it is very difficult to excite helium, the smallest trace of a foreign gas tending to quench the He-lines. According to the present theory, this is due to the fact that helium has the highest ionization and radiation potential of all elements, so that when it is subjected to a

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[*] McLennan, Proc. Lond. Phys. Soc.—Guthrie Lecture, Dec. 1918. Franck and Hertz, *Verh. d. Deutsch. Ges.* vol. xx. (1919).

[†] Rentschler, *Phys. Rev.* vol. xiv. p. 503 (1919). Horton and Davies, Proc. Roy. Soc. Lond., vol. xcviii. p. 124.

[‡] Davis & Goucher, *loc. cit.* Jan. 1919; also Smyth, *loc. cit.* (1919).

stimulus, this, by preference, passes through the more easily excitable impurities, leaving the He atoms unaffected. Franck (*Zeitschrift für Physik*, 1920, vol. i.) describes an interesting experiment on the excitation, by his well-known electron-bombardment method, of helium-lines contained in a tube into which a trace of mercury (less than 1 in 1000 parts) was purposely introduced. Mercury has an I.P. of 10·45 volts and R.P. of 4·9 volts, while the corresponding numbers for helium are 25·6 and 20·5. "We have then," in Franck's own words, "a strong flashing out of the lines of the element with the lower I.P., at the cost of the lines with the larger I.P." This also explains qualitatively why, with the range of temperatures available in the laboratory, it is not possible by purely thermal means to excite the permanent gases. In the case of the alkalies and the alkaline earths, the value of the ionization potential is low and gradually decreases as we proceed to elements with higher atomic weight in the same group, and it is found that the difficulty with which the spectrum can be excited lessens in a parallel manner. Thus under all conditions, if we classify the alkaline earths according to the ease with which they can be excited, the order is Ba, Sr, Ca, and Mg. The same can be said of the other elements.

In the present paper I have used the word "Stimulus" to denote, in a general manner, all physical agencies tending to make the atoms luminescent. We shall discuss how a high temperature alone can bring about this state. The question can best be approached from the theoretical side.

§ 2.

Modern spectroscopic works have shown that the lines of an element may be grouped under the following headings*:—

(1) *Lines due to the normal atom.*—In this case the lines are produced by the quantum-changes of orbit of the outermost electron, the nucleus and the remaining electrons behaving as a single charge. The characteristic Rydberg number in the series-formula is $N\left(=\frac{2\pi^2e^4m}{h^3}\right)$. These lines are produced under a comparatively low stimulus.

(2) *Lines due to the atom which has lost one electron (the outermost one).*—In this case the lines are produced by the quantum changes of orbit of the now outermost electron, the nucleus and the remaining charges behaving as a net

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[*] For example, compare Fowler's work on the "Emission Spectrum of Magnesium," Phil. Trans. vol. 214.

double charge. The characteristic Rydberg number is $4N\left(=\frac{2\pi^2e^2m}{h^3}(2e)^2\right)$. These lines are produced at a higher stimulus, and generally, but not always [*], lie in the ultra-violet.

(3) The atom may lose two electrons, and now a new series of vibrations may begin with the new outermost electron. The Rydberg number is 9N.

The lines of any of these groups are as distinct from each other as if they belong to different elements altogether. According to Sommerfeld [†], when an atom loses one electron, its spectral properties become similar to those of the element just preceding it in the Periodic Tables. Thus the system of lines of Ca^+ are constituted in the same manner as those of K; similarly, the relation between Sr^+ and Rb, Ba^+ and Cs.

Lines of these different groups may be simultaneously present, but generally one group gains in intensity at the expense of the other. Thus in the flame Ca^+ lines (the H, K) are very faint, the Ca-line (*g*-line) is very strong. Now "*g*" begins to lose relatively in intensity as the (H, K) are strengthened with rise of temperature. At the spark-conditions, the most intense lines are the H, K, while the "*g*" is almost evanescent.

Let us now examine how the transition from the neutral state to the ionized states (from Ca to Ca^+) takes place, under the influence of heat alone. In Phil. Mag. Oct. 1920 ("Ionization in the Solar Chromosphere," called henceforth paper A), it has been shown that the problem can be attacked with the aid of the "New Thermodynamics" of Planck and Nernst [‡], and the statistical equilibrium between neutral atoms and ionized atoms can be calculated in terms of temperature and pressure, when the energy of ionization is known. The calculations for alkaline earths will be found in paper A, and those for the alkalies are given in paper B [§].

According to the mechanical theory, the outermost electron of the neutral atom revolves in a stable orbit when the atom is not subject to any stimulus. Ionization means the transference of this electron to infinity. But the process

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[*] For example, in case of alkaline earths. All alkaline earths are distinguished by having large (2, *p*) terms, which causes the principal lines to occur in the visible region.

[†] Sommerfeld, *Atombau und Spektralanalyse*, Chap. 4, § 6 and Appendix.

[‡] Planck, *Wärmestrahlung*. Nernst, *Das Neue Wärmesatz*, etc.

[§] "Elements in the Sun," Phil. Mag. Dec. 1920.

is not an abrupt one, for, according to the quantum theory of spectral emission, the electron may have an infinite number of orbits distinguished by different quanta-numbers.

The theory of the stable orbits has been formulated by Sommerfeld [*] in the following manner. Let n_1, n_2 denote the rotational quantum numbers for an orbit, and n' the radial quantum number, *i. e.* r, θ, ϕ being the coordinates of the electron ; then,

$$\int\left(\frac{\partial T}{\partial r}\right)dr=n'h, \quad \int\left(\frac{\partial T}{\partial \dot{\theta}}\right)d\theta=n_1h, \quad \int\left(\frac{\partial T}{\partial \dot{\phi}}\right)d\phi=n_2h,$$

the integration extending over the whole orbit. The energy of the system is now given by the expression

$$A-h\left[\frac{N}{\{n+n'+f(n_1,\ n_2,\ n')\}^2}\right], \quad n_1+n_2=n,$$

where A = a constant, $f(n, n')$ is a function, the value of which decreases with increasing values of n and n'.

The possible orbits can now be thus classified by assigning different sets of values to n and n'.

TABLE II.

Rotational quantum-no. n.	Radial quantum-no. n'.	$\frac{\text{Energy of the System}}{h}$ *i. e.* $\frac{N}{\{n+n'+f(n_1, n_2, n')\}^2}$.	Remarks.
1	0	$-(1, s)$	This is the orbit which the electron possesses when it is subject to no stimulus. $m>1$, the s-orbits.
1	$m-1$	$-(m, s)$	
2	0	$-(2, p)$	The p-orbits. $m>2$.
2	$m-2$	$-(m, p)$	
3	0	$-(3, d)$	The d-orbits. $m>3$.
3	$m-3$	$-(m, d)$	
4	0	$-(4, f)$	The f-orbits. $m>4$.
4	$m-4$	$-(m, f)$	
5	0	$-(5, k)$	The k-orbits. $m>5$.
5	$m-5$	$-(m, k)$	

$(1, s)$, $(2, p)$, $(3, d)$, .. (m, s), .. are the familiar expressions which, in Paschen's notation, denote the terms of which a

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[*] Sommerfeld, *Verh. d. Deutsch. Phys. Ges.* vol. xxi. (1919).

series-formula is composed. Thus, for the principal series, the series-formula is

$$\nu=(1,s)-(m,p), \quad m \geqq 2.$$

When the atom is in a free gaseous condition, and is not subject to any stimulus, it has the energy $A-h(1,s)$. The higher orbits are produced only when a stimulus is applied. The lines are emitted as the electron changes its habitat from one stable orbit to another with less energy. Thus the line $(1,s)-(2,p)$ is emitted when the electron changes its habitat from the p-orbit $(2,p)$ to the s-orbit $(1,s)$, etc. The law according to which these changes take place has been thus formulated by Rubinowicz * and Bohr. Let n_0 and n denote the rotational quantum numbers of the initial and final orbits. Then

$$n-n_0=1,\ 0,\ \text{or}\ -1.$$

We can thus distinguish among the following different types of combination :—

GROUP I.—Positive combination, $n-n_0=1$.

Combination	$(1,s)-(m,p)$ $m \geqq 2$	$(2,p)-(m,d)$ $m \geqq 3$	$(3,d)-(m,f)$ $m \geqq 4$
Accepted Appellation.	Principal Series.	Diffuse Series.	Bergmann Series.
	$(2,s)-(m,p)$ $m \geqq 3.$	$(3,p)-(m,d)$ $m \geqq 4.$	etc.

GROUP II.—Negative combination, $n-n_0=-1$.

Combination	$(2,p)-(m,s)$ $m \geqq 1.$	$(3,d)-(m,p)$ $m \geqq 2.$	
Accepted Appellation.	Sharp Series.		etc.

GROUP III.—Neutral combination.

Combination	$(1,s)-(m,s)$	$(2,p)-(m,p)$	$(3,d)-(m,d)$

Each of the terms (m,s), (m,p), (m,d) may have double or multiple values owing to the different possible orientations of the vibrating electron with regard to the remaining atom †.

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* Rubinowicz, *Phys. Zeits.* vol. xix. pp. 441–465 (1918). Sommerfeld, *Atombau und Spektralanalyse*, pp. 390–403.

† For a theoretical treatment of the case, see S. N. Basu, "On the Deduction of Rydberg's Law from the Quantum Theory of Spectral Emission," Phil. Mag. Nov. 1919.

Thus far we have dealt with the electrodynamics of the atomic system, *i. e.* the possible stable orbits of the vibrating electron calculated according to quantum-mechanics. We have now to deal with the statistics of the case, for the higher stable orbits are produced in sufficient proportion only with increasing stimulus, which we obtain only at higher temperatures. The intensity of a line will depend upon the product of the numbers which show the relative proportions of orbits in the initial and final stages at any instant, and also upon the chance of changing from one orbit to another. Thus the intensity of $(1, s)-(2, p)$ will be proportional to $\lambda\lambda'\epsilon$, where λ is the number of orbits in state $(1, s)$, λ' is the number in state $(2, p)$, and ϵ' is the chance that the orbit will change from the state $(2, p)$ to $(1, s)$.

When the stimulus is sufficiently great, some of the electrons will pass off to infinity, and we shall have partial ionization. This problem can be treated thermodynamically, for here we have to consider a sort of chemical equilibrium between three distinct phases—the neutral atom, the ionized atom, and the electron. But radiation cannot be so treated, for this is a case of internal change of orbits only, not involving any phase-changes.

The problem before us can therefore be thus stated:—"How to find out the statistical distribution of atoms into different possible stable orbits when the mass of the gas is subjected to a given thermal stimulus?" According to the theory of ionization sketched in paper A, if Ca-vapour is enclosed in a vessel, such that the pressure is always maintained at 10^{-3} atm., we have seen that with increasing temperature the proportion ionized varies as follows:—

Temperature ...	2000°	3000°	4000°	5000°	6000°	7000°
Percentage of Ionization ...	3×10^{-4}	2×10^{-1}	6	47	91	100

The non-ionized atoms cannot all be with the primitive orbit $(1, s)$, but a good proportion will be found with the other possible stable orbits, for the electron, while detaching itself from the neutral atom, has a chance of taking its habitat in some of these stable orbits, and hence some must be found in these states. The phenomena of line-radiation therefore comes before the ionization becomes complete.

§ 3.

At the present time, the electrodynamical part,—*i. e.*, the manner in which the lines of an element originate from the quantum-vibrations of the constituent electrons of the atomic system—has been worked up in the case of the H-atom alone. For the other atoms, it is only in the process of making [*]. But the thermodynamical part of the problem—that is, how the proportion of different possible stable orbits varies according to temperature, and how the orbits change from one into the other giving rise to line-radiations—has not even been clearly formulated. In his interesting development of the quantum theory applied to systems with more than one degree of freedom, Planck [†] has laid the foundations of a new method for dealing with the statistical aspect of the question. The second aspect—namely radiation following as a result of mutual interchange of stable orbits—has been dealt with by Einstein [‡]. But there are still many difficulties to overcome. When this is done, a new and fruitful chapter—that of line radiation of gases—will be added to thermodynamics.

We may provisionally proceed along the following lines:—Let us suppose that the orbits having the rotational quanta 1, 2, ... m ... vary as a geometric progression

$$(1-f)[1, f, f^2, f^3 \dots],$$

where f is a fraction and a function of temperature and concentration. Similarly, let us suppose that the orbits having the radial quanta 0, 1, 2, 3, etc., vary according to the terms of the geometric progression

$$(1-g)[1, g, g^2, g^3 \dots],$$

where g is a similar fraction.

Then at any instant the proportion of orbits with the rotational quantum number $n+1$ and the radial quantum number n' is

$$f^n g^{n'}(1-f)(1-g),$$

since at low temperatures and high concentrations almost the whole number of atoms is in the state $(1, s)$ corresponding to $n=1$, $n'=0$.

We see that f and g are very small quantities under these

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[*] Sommerfeld, *Atombau und Spektralanalyse.*

[†] Planck, *Verh. d. D. Phys. Ges.* vol. xvii. p. 407 (1915); *Ann. d. Phys.* vol. 50. p. 385 (1916).

[‡] Einstein, *Verh. d. D. Phys. Ges.* vol. xviii. p. 318 (1916); *Phys. Zeit.* 1918, p. 124.

conditions and gradually increase as the temperature is raised or the concentration is lowered. As an example, we may take that the distribution of the Ca atoms in the orbits at 2000° K follow according to the scheme (we are considering only f here)

$$(1-\tfrac{1}{10})[1,\ 10^{-1},\ 10^{-2},\ 10^{-3}\ldots\ldots],$$

while at 4000° K, the distribution is

$$(1-\tfrac{1}{2})[1,\ \tfrac{1}{2},\ (\tfrac{1}{2})^2,\ (\tfrac{1}{2})^3\ldots\ldots],$$

so that the relative intensity of the lines

$$(1,s)-(2,p),\quad (2,p)-(3,d),\quad (3,d)-(4,b)$$

will be $$1 : 10^{-1} : 10^{-2},\ \text{etc.};$$

but fully 80 per cent. of the atoms will remain inactive, while at 4000° K, the ratio will be

$$1 : \tfrac{1}{2} : (\tfrac{1}{2})^2,$$

but now only 25 per cent. of the non-ionized atoms remain inactive.

The above considerations are not based upon any theoretical argument, but are given here as a sort of *construibar vorstellung* of the statistics of line-radiation of gases. We can say that, under all physical conditions, a very small concentration of radiant atoms suffices for the production of the series $\nu=(1,s)-(m,p)$, especially the fundamental line $\nu=(1,s)-(2,p)$. The $(2,p)-(m,d_i)$, $(2,s)-(m,p)$, $(3,d)-(m,f)$ lines require gradually increasing concentrations of radiant matter in addition to the condition that the stimulus should be sufficiently great. For example, if we take sodium gas at a temperature of 1500° K [an ordinary Bunsen flame tinged with sodium vapour], and gradually decrease the amount of vapour in the flame, the order in which the lines disappear are $(3,d)-(m,b)$, $(2,s)-(m,p)$, $(2,p)-(m,d)$; the $(1,s)-(m,p)$ lines, of which the leading members are the D_1 and the D_2, are the last to disappear, a fact which was recognized by Du Gramont, when he conferred the appellation "*raies ultimes*" upon this class of lines[*]. This state of affairs persists when the temperature is raised and gradually increasing percentages of atoms are ionized. If a line is represented by the series formula $\nu=(n,f)-(n',f'')$,

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[*] According to Bunsen and Kirchhoff, 7×10^{-12} gm. of sodium in the Bunsen flame is quite sufficient to show the D_1 and the D_2 line (Pringsheim, *Physik der Sonne*, p. 121).

the difficulty of detection of the line will be greater the larger are the values of n and n'.

In the following table we give the temperature of complete ionization of a few elements, with the temperatures at which luminescence just begins and attains its maximum intensity. But it will be clear from what has been said that the ordinary way of speaking—"the gas is heated to incandescence or luminescence just begins"—has no meaning in itself unless we say which particular line is emitted, or which orbit is produced. The orbits which are produced are specified in column 3. Under the heading "Remarks," the manner in which these temperatures have been estimated are briefly described. In this connexion, the following section on absorption should be consulted. The pressure has been taken equal to one atmosphere unless otherwise stated.

Table III.

Element.	Temperature of Ionization.	Orbit generated.	Luminescence begins at	Remarks.	Luminescence is maximum at	Remarks.
H	24,000	$(2, p)$	5000° K	Temp. of Ma stars	12,000° K	Temp. of Ao stars
He	32,000	$(2, p)$	11,000° K	Temp. of A2 class	17,000° K	Temp. of B2A class
Mg^+ ...	24,000	$(3, d)$	7500° K	Temp. of Go class	11,000° K	Temp. of A2F class
Ca, Sr, Ba ...	10,000–8000° K (Pressure 10^{-1} atm.)	$(2, p)$	1500° K	Bunsen flame	4000	The open arc

The elements N_2, O_2, A, Ne, etc., resemble H_2 and He in having large values for the ionization-potential, and therefore they fail to respond to the temperatures which can be commanded in the laboratory. The alkali metals (particularly K, Rb, Cs) are more prominent in the flame-spectra and less in the arc than the alkaline earths. Mg-lines are rather faint in the flame, but come out very prominently in the open arc. Zn, Cd, Hg, Fe, Ti lie between the alkaline earths and the permanent gases in their spectral properties. All this is in very good qualitative agreement with the hypothesis sketched in the present paper.

§ 4. *Absorption.*

In this connexion, we may consider the puzzling question of reversal of lines. According to Kirchhoff's law, we expect that the emission-lines of an element should be reversed when a strong beam of white light is sent through cooler layers of the vapour. But this expectation is not always fulfilled. Wood [*] has found that if a white light be sent through a column of sodium vapour, only the lines of the principal pair-series $(1, s)-(m, p_1)$, $(1, s)-(m, p_2)$, can be obtained as absorption-lines. None of the lines of the diffuse or the sharp series are reversed. Bevan [†] has extended the method to the other alkali metals, *i. e.* Potassium, Rubidium, and Cæsium, and arrived at identical results. Recently Dobbie and Fox (Proc. Roy. Soc. vol. xcviii. p. 147) studied the absorption-spectra of Hg, Zn, and Cd vapour, and found no absorption up to $\lambda=3200$. But this is due to the fact that the $(1, s)-(m, p)$ lines of these elements lie below 3000 Å.U. In fact, Wood found in 1913 (*Phys. Zeit.* pp. 191–195) that ordinary Hg vapour absorbs the fundamental line $\lambda=2536$ $(1, s)-(2, p_2)$.

The explanation easily follows from our theory. The condition for absorption is that in the atoms present, there should be a fairly large number with orbits corresponding to the first term of the pulse of radiation to be absorbed. Thus, in order that a line $(2, p)-(m, d)$ may be absorbed, we must have a sufficient number of atoms with $(2, p)$ orbits. At low temperatures only atoms with $(1, s)$ orbits are present. Hence only the lines corresponding to the combination $(1, s)-(m, p)$ are absorbed. The lines represented by the positive combinations $(2, p)-(m, d)$, $(3, d)-(m, f)$, or the negative combinations $(2, p)-(m, s)$, can only occur when atoms with $(2, p)$ or $(3, d)$ orbits are present. This can happen only at high temperature or under electrical stimulus.

The temperature required for this purpose is very high—much higher than the temperatures used by Wood and Bevan for all elements. In fact, the atoms begin to absorb the lines $(2, p)-(m, d)$ only when they are hot enough to emit the leading terms of the principal series. A line of the Bergmann series will begin to be absorbed at even a higher temperature, viz. at the temperature of emission of the diffuse series $(3, d)-(4, b)$.

If the views presented here be correct, we may probably obtain the reversal of the lines of the diffuse or the Bergmann

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[*] Wood, The Astrophys. Journal, vol. xxix. pp. 97–100.

[†] Bevan, Proc. Roy. Soc. vol. lxxxiii. pp. 423–428; vol. lxxxv. pp. 58–76.

series of the alkali metals by heating the absorbing column of vapour to about 2000° to 3000° K. The most favourable element to start with is cæsium, which has the lowest ionization-potential of all elements.

In many cases confusion may arise about the proper identification of the $(1, s)-(m, p)$ terms. Thus, what are usually called the Principal series of helium and parhelium (viz. the series beginning with the line 20,587 for parhelium and 10,834 for helium) do not really correspond to the combination $(1, s)-(m, p)$, but to the combination $(2, s)-(m, p)$. The $(1, s)$ term for helium is still unknown, and the series $(1, s)-(m, p)$ lie far down in the ultraviolet *. Hence, according to our theory, none of the lines belonging to the combinations $(2, s)-(m, p)$, $(2, p)-(m, d)$ can be absorbed by a layer of helium gas.

But if by heating or some other means we can convert a good proportion to the states $(2, s)$ or $(2, p)$, then and then only can these lines appear as absorption-lines. But at a pressure of 10^{-1} atm. helium becomes incandescent, *i. e.* emits the lines $(1, s)-(m, p)$, and absorbs the lines $(2, p)-(m, d)$ at probably not less than 11,000° or 12,000° K, *i. e.* only in stars of the B-class.

But instead of a high temperature we may think of other means. The spark produces mechanically the very same conditions which can be realized at very high temperatures. This is exactly what Paschen † has done. He found that the lines of the combination $(2, s)-(m, p)$ for helium and parhelium cannot be absorbed by an ordinary layer of the helium gas. But when a spark is sent through the absorbing layer the lines are strongly absorbed, the absorbed energy being again re-emitted in all directions.

The paper thus suggests more problems than it attempts to solve. A critical examination and further development of the hypothesis advanced here requires an overhauling of the whole data on the line-radiation of gases—such as are contained in Kayser's *Handbuch der Spektroskopie* and further works. But this programme requires much more time and more extensive study, both practical and theoretical.

University College of Science,
Calcutta,
May 25, 1920.

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* It is quite possible that some of the $(1, s)-(m, p)$ lines for helium and parhelium may be identical with the lines discovered by Lyman in the ultra-violet, and some with the lines discovered by Richardson and Bazzoni in the region of 300 to 400 A.U. by the photo-electric method (*vide* Richardson and Bazzoni, Phil. Mag. 1918).

† Paschen, *Ann. d. Physik*, vol. xlv. p. 625 (1914).

XXIII. *Possible Magnetic Polarity of Free Electrons.*

By Arthur H. Compton, *Ph.D.*[*]

MY attention has recently been called by Mr. Shimizu to the fact observed by Mr. C. T. R. Wilson that the paths of beta and secondary cathode rays excited by X-rays in air usually terminate in converging helices. The comparatively uniform character of these curves shows itself clearly in certain of Mr. Wilson's beautiful stereoscopic photographs which have not been published but which he has very kindly allowed me to examine. For example, one photograph, which shows the complete tracks of 66 secondary cathode particles, reveals 52 tracks of helical form, only two or three tracks showing no general curvature of this type, the remaining 12 rays having paths too irregular to detect with certainty any helical curvature that may exist. These paths may have the form of either a right- or left-handed helix, and the axes of the different helices have a nearly random orientation. A detailed examination of a large number of tracks to prove that the observed curvature of the paths is not a random one, will necessarily involve much time and labour. In default, however, of a complete proof, it is of interest to see whether an explanation can be offered of the apparent tendency of the beta particles to move in a type of spiral orbit.

Though the spiral form of the paths would suggest a motion of the particles in a magnetic field, the chance orientation of the axes of the different helices shows that these axes are not determined by any external magnetic field, but are rather characteristic of the individual beta rays. More specifically, the axis of the helix must be parallel with some polarity of the beta particle which is relatively permanent in direction. It is apparent that a simple electric charge can possess no polarity whose orientation will remain constant in spite of numerous collisions with other charges unless it is in rapid rotation. Mr. Shimizu accordingly suggested that Mr. Wilson's photographs may be explicable on the assumption that the electron has a definite magnetic polarity which on account of gyroscopic action does not change rapidly in direction.

It is clear that a magnetic field whose direction is determined by the electron passing through it, is capable of producing the type of spiral track that is observed. But a beta particle which acts as a magnetic doublet as well as an electric charge is capable of producing such a magnetic field

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[*] Communicated by Professor Sir E. Rutherford, F.R.S.

if the medium through which it passes is susceptible to magnetization. For the introduction of such a doublet will induce magnetization in the surrounding medium just as a bar magnet induces magnetization in a neighbouring mass of iron. If the atmosphere acts paramagnetically, the magnetic field at the doublet due to the induced magnetization has the same direction as the magnetization of the doublet. Conversely, the induced magnetic field due to a diamagnetic atmosphere is opposite in direction to the doublet's axis. This induced magnetic field will clearly have the same effect on the motion of the electron as would an externally applied field of the same intensity. That is, the beta ray will move in a helical path whose axis is parallel with the magnetic axis of the beta particle. The path may have the form of either a right- or left-handed helix, according as the north or south pole of the beta particle is foremost. This is in accord with observation, which shows paths of both kinds. Furthermore, if the induced magnetic field does not decrease while the velocity of the beta ray diminishes, the tracks of the particles will be *converging* helices such as appear in the photographs.

The intensity of the magnetic field induced at a beta particle due to the effect of its magnetic moment on the part of the medium at a distance greater than r from the doublet may be shown to be

$$H_r = \frac{8\pi\mu s}{3r^3},$$

where μ is the magnetic moment of the doublet and s is the effective susceptibility of the medium. If the cathode particle is moving at a speed corresponding to a drop through 10,000 volts, the minimum distance of its approach to an electron at rest is according to usual theory about 10^{-10} cm. Taking this as the value of r, using for s the value 3×10^{-8} of the magnetic susceptibility of air, and for μ the moment 10^{-20} of an electron with an angular momentum $h/2\pi$, this expression indicates that the intensity of the magnetic field induced at the electron should be of the order of 3000 gauss. This is approximately the field which would be required to produce the observed curvature.

The impulsive torque exerted by the beta ray doublet on a magnetic electron at rest when passing it at a distance of 10^{-10} cm. has, however, a period corresponding to X-ray frequency. It is clearly necessary, therefore, to take into account the inertia of the elementary magnetic doublets of which the medium is composed. Furthermore, the magnetic forces acting on an electron due to a beta particle at this close range will greatly exceed the restoring forces due to the other electrons in the atom which for weaker fields may make the atom diamagnetic. In fact, there is no reason for supposing that the effective susceptibility of the medium when subjected to such highly intense magnetic pulses of very short duration has any intimate relation to its susceptibility in steady and comparatively weak fields. It seems rather that the problem must be treated as a statistical one of encounters of a rapidly moving, electrically charged, magnetic doublet with a random distribution of similar doublets at rest. A preliminary investigation of this problem indicates, however, that the average motion of the beta ray should be rather similar to its motion when passing through a medium of uniform susceptibility. The above numerical discussion is therefore of value as showing that it is not unreasonable to expect a magnetic particle to induce in the surrounding medium a magnetization of the magnitude required to account for the observed helical paths.

If the obvious explanation of these spiral tracks is the correct one, their interpretation yields very valuable results. We have seen that the beta ray seems to act as a tiny gyroscope with a magnetic moment, which is capable of giving rise to torques on other electrons of a duration corresponding to the frequency of X-rays. The reaction, according to classical dynamics, must result in mutational oscillations of the spinning beta particle. This obviously supplies a mechanism for the production of high frequency radiation by a free electron, which has been suggested by Webster to account for Doppler effects at the target of an X-ray tube. The possibility that the electrons are magnetic doublets is also of great importance in connexion with our ideas of the structure of the atom and the nature of chemical combination.

Cavendish Laboratory,
Cambridge University.
August 16, 1920.

XXIV. *The Constitution of Atoms.* *By* Professor Orme Masson, *F.R.S.*, *University of Melbourne* *.

SINCE it must now be conceded that all material atoms are compounded of positive and negative electrical atoms, it is surely time that each of these fundamental and universal constituents were known by some distinctive name. This compliment has been paid to the one, but not,

* Communicated by Professor Sir E. Rutherford, F.R.S.

as yet, to the other. For convenience of reference and notation, if for nothing else, it is just as necessary to have a name for positive electrical atoms as for the electrons.

Though the hydrogen nucleus has been identified with the positive particle, it would not be well to adopt a name specially indicative of the fact; for hydrogen has no monopoly in these particles, which are also present in the nucleus of every other atom. Moreover, the electron is just as essential a constituent of hydrogen itself, though not of its nucleus, as is the positive particle.

The outstanding characteristic of the electrons is that they mainly determine the electro-chemical characters of the atom; so they are well named. The outstanding characteristic of the positive particles is that they mainly determine the mass of the atom. I therefore suggest that they should be called *barons* (βάρος, weight)*.

If this name be adopted, we can conveniently symbolize the baron as b, using e for the electron. We thus have, in what follows,

b = one baron (charge +1, mass 1),
e = one electron (charge −1, mass negligible).

A may stand for the mass of any elementary atom with the atomic number N.

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* *Footnote by Professor Rutherford*:—

At the time of writing this paper in Australia, Professor Orme Masson was not aware that the name "proton" had already been suggested as a suitable name for the unit of mass nearly 1, in terms of oxygen 16, that appears to enter into the nuclear structure of atoms. The question of a suitable name for this unit was discussed at an informal meeting of a number of members of Section A of the British Association at Cardiff this year. The name "baron" suggested by Professor Masson was mentioned, but was considered unsuitable on account of the existing variety of meanings. Finally the name "proton" met with general approval, particularly as it suggests the original term "protyle" given by Prout in his well-known hypothesis that all atoms are built up of hydrogen. The need of a special name for the nuclear unit of mass 1 was drawn attention to by Sir Oliver Lodge at the Sectional meeting, and the writer then suggested the name "proton."

Professor Orme Masson sent the present paper for publication through the writer, and in order to avoid the long delay involved in correspondence, his paper is printed in its original form. If the name "proton" is generally approved, it is merely necessary to change the symbol "b" into "p" in the chemical equations given in the paper.

It should be pointed out that a somewhat similar type of nomenclature for the constituents of atoms has been suggested in the interesting paper of Professor W. D. Harkins, entitled "The Nuclei of Atoms and the New Periodic System" (Phys. Review, xv. p. 73, 1920).

The inference from radioactivity work, that most of our elements are not "pure" but consist of mixtures of isotopes, and that the A value of any pure element, or single isotope, is appreciably integral ($O=16$ being taken as standard) has been splendidly confirmed by Aston's recent work. Of the eighteen elements already examined by his mass-spectrum method, only hydrogen gives an atomic weight (1·008), which is not integral to within one in a thousand. The cause of this exception requires further investigation, but in the meantime it may be set aside as related in some way (as suggested by Aston) to the fact that hydrogen is unique in containing no electrons in its nucleus.

It is unique also in other ways, and especially in the value of the ratio A/N, which in H is 1 and in no other atom is less than 2 (nor more than 2·6).

If we write $A-2N=n$, we find from Aston's work that $n=0$ in pure elements, He, C, N, O, S, and also in the lower isotopes of B, Ne, Si, and Ar. There can be no doubt that this also holds for the lower isotopes of Li, Mg, Ca, and, perhaps, some of the other light atoms. It follows, therefore, that the group (b_2e) may be regarded as a secondary unit of positive charge, with mass 2, and that $(b_2e)_N$ expresses the composition of the nucleus of any of those atoms.

In the higher isotopes of B, Ne, Si and Ar, in the pure elements F and P, and in both the isotopes of Cl, n has small values, ranging from 1 to 4; in As it is 9, in the two Br isotopes it is 9 and 11, and in the six isotopes of Kr it ranges between 6 and 14. The values are higher still in Xe (20 to 27); and in Hg it apparently ranges between 37 and 44. The numerous isotopes of elements with N values from 81 to 92, contained in the three radioactive series, have all n values from 42 to 54. There is thus a general tendency for n to increase with N, modified by the fact that it may vary considerably among isotopes with the same N.

If the b_2e group be still taken as the unit of positive charge, there must be added to it in most cases n electrically neutral couplets (be), each having unit mass. We thus can express the nuclei of all atoms from He to U by the general formula $(b_2e)_N(be)_n$, n having any integral value from 0 to 54; and even the unique case of H is included if n be given the special value of -1.

If we distinguish the nucleus from the shell electrons by enclosing the former within square brackets, $[(b_2e)_N(be)_n]e_N$ becomes the perfectly general formula for any electrically neutral atom, while positively or negatively charged ions are similarly indicated with the appropriate decrease in the number of electrons in the shell.

We can now express in general terms (which, of course, may be made specific by substitution of the proper numbers for N and n) any action that occurs within the nucleus. Thus an α-ray change depends on the intranuclear action $2(b_2e) = (b_2e)_2^{\nearrow}$ and is expressed by the sub-chemical equation

$$[(b_2e)_N(be)_n]e_N = [(b_2e)_{N-2}(be)_n]e_N + (b_2e)_2^{\nearrow}$$

where $(b_2e)_2$ is the expelled He nucleus and the other product is the ion of the new element, carrying a double negative charge till, by discharge, it becomes the atom $[(b_2e)_{N-2}(be)_n]e_{N-2}$. This clearly expresses the characteristics of α-ray action—that the atomic number is lowered by 2 and the mass by 4, to which may be added the statement that there is no change in the number of neutral couplets (n). A β-ray change, on the other hand, depends on another intranuclear action, $2(be) = (b_2e) + e^{\nearrow}$, thus subtracting 2 from n and adding 1 to N, while the mass is unchanged and the main product is the single charged positive ion of an isobare. In full, the equation is

$$[(b_2e)_N(be)_n]e_N = [(b_2e)_{N+1}(be)_{n-2}]e_N + e^{\nearrow},$$

The results obtained by Sir E. Rutherford by the bombardment of light atoms with swift α-particles can be expressed similarly. Thus the expulsion of a particle with the mass 3 and a charge of $+2$ (the nucleus of a lower isotope of helium) can obviously be attributed to the change $2(b_2e) = (b_3e)^{\nearrow} + (be)$, while the expulsion of a hydrogen nucleus (a single baron) results from the action $(b_2e) = b^{\nearrow} + (be)$. Sir E. Rutherford has shown that the nitrogen atom gives both these particles, apparently by actions which occur independently. The equations for them are

$$[(b_2e)_7]e_7 = [(b_2e)_5(be)]e_7 + (b_3e)^{\nearrow},$$

and
$$[(b_2e)_7]e_7 = [b_2e)_6(be)]e_7 + b.$$

In the first case the main product is the double charged negative ion of Aston's higher isotope of boron. In the second case it is the single charged negative ion of a previously unknown isotope of carbon.

The suggested notation * does not indicate any reason why nitrogen atoms should emit both b_3e and b ions under α-ray bombardment, while oxygen atoms emit b_3e but no b ions. Sir E. Rutherford seems inclined to infer that barons exist, as such, in the nitrogen nucleus and not in that of oxygen, just as it has been usual to take the emission of α-rays as evidence that the radioactive atoms contain helium nuclei as such. Of course it may be so, but the inference may not be justified. A similar inference, long ago proved incorrect, is embalmed in the term carbohydrate. The sugar molecules do not contain water molecules as such, though they do, under certain conditions, emit water molecules and leave a residue of carbon.

It need hardly be said that no claim is made that the formulæ suggested in this paper express the atomic constitutions in the full sense of the term. There are really as yet no data to justify such an attempt, so far at least as the nuclear part of the atom is concerned. They do, however, express correctly the nuclear charge (N) and mass $(2N+n)$ and the shell charge, whether of atoms or of ions; they locate the difference between isotopes in the numerical value of n (and there is no other difference); and they serve to correlate the whole system of atoms and their proved processes of disintegration by means of a comparatively simple notation, which may be employed to illustrate either general rules or special instances.

If the chart of radioactive transformations, as given by Soddy (J. C. S. Jan. 1919, p. 16) be re-drawn, making the N scale horizontal and the A scale vertical and one unit of the former equal in length to two units of the latter, the n values of all the atoms can be read off on a scale drawn diagonally from N.E. to S.W. across the centre of the chart. A succession of α-ray changes is thus marked by arrows running along a line of equal n, pointing from S.E. to N.W., and such lines of equal n may pass through the symbols of atoms belonging to the same or different series, just as do the vertical isotopic lines or the horizontal isobaric lines.

The University of Melbourne,
8th October, 1920.

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* It may be pointed out here that the (b_3e) particle, isotopic with the He nucleus, may be formulated as $[b_2e)_2(be)_{-1}]$, just as the baron may be writen $[(b_2e)(be)_{-1}]$. These formulæ are, of course, merely a repetition of the equations already given for intranuclear changes by which the particles are generated; but the negative value of n serves to classify them together and apart from other ions. H is no longer quite unique in this respect.

XXV. *Fluid Discharges as affected by Resistance to Flow.* By Wm. John Walker, *B.Sc., Ph.D., Lecturer in Engineering, College of Technology, Manchester*[*].

TYPICAL instances of fluid discharges, greater than appear to be possible from theoretical considerations of non-viscous flow, are those of the Venturi meter for low heads, and also, under certain conditions, the discharge of steam through nozzles. The latter case is explained by H. W. Callendar's "supercooling theory"[†]. The following analysis shows, however, that the discharge of a viscous fluid may actually be greater, under otherwise similar conditions, than that of a non-viscous one. This result appears to be paradoxical in the extreme, and it is with the greatest diffidence that the writer advances it here. His excuse lies in the fact that the deduction is the result of the inclusion of a resistance expression which holds in certain cases of fluid motion.

The Venturi meter, used principally in water-flow measurement, consists essentially of a converging circular tube, the pressures at the entrance and exit of which are measured, the difference between these serving as the measure of discharge. The theoretical formula for non-viscous flow is

$$v=\sqrt{\frac{2gh}{m^2-1}}, \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad (1)$$

where $m=\dfrac{\text{area at entrance}}{\text{area at exit}}$,

$v=$ velocity of fluid in feet per second,

and $h=$ pressure difference in feet of water.

In practice this is written

$$v=C\sqrt{\frac{2gh}{m^2-1}}, \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad , \quad (2)$$

where C is known as the "meter coefficient." Naturally this is less than unity, having values from ·96 to ·99. At low velocities, however, values as high as 1·36[‡] and as low

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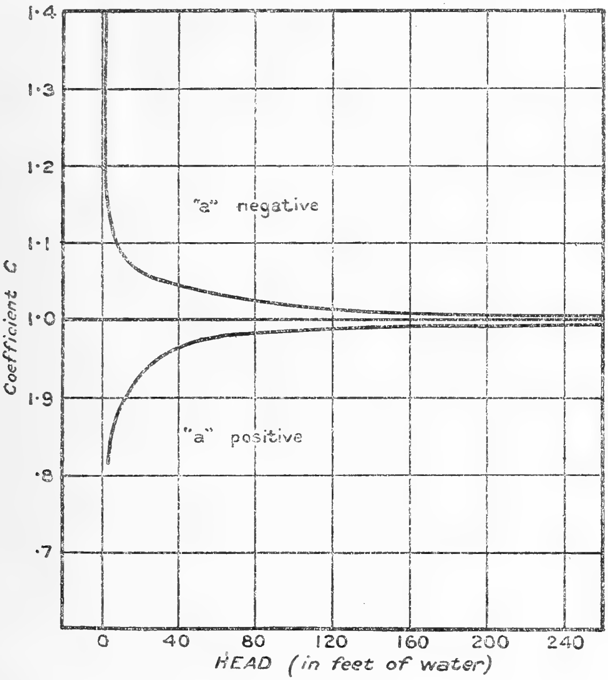
[*] Communicated by the Author.

[†] "On the Steady Flow of Steam through a Nozzle or Throttle," Proc. I. M. E. p. 53 *et seq.*, Jan. 1915.

[‡] Coker, Canadian Soc. C.E. March 1902. See also Gibson, Proc. I. C.E. p. 392.

as 0·75 * have been obtained. The curves of fig. 1 showing C against V are typical of such results, each curve tending to either high or low values of C, for low heads.

Fig. 1.



Resistance to fluid motion may be written †

$$R=\rho v^2\left\{A\frac{\nu}{vl}+K\right\}$$

or $$=av^2+bv^2, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

where "*a*" and "*b*" are constants. Generally, the value of "*a*" is positive, but cases occur frequently ‡ in which it is found to have a negative value. The presence of this negative coefficient is undoubtedly difficult to explain, but leaving any such explanation aside for the present, it

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[*] Herschel, Trans. Am. Soc. C.E. vol. xvii. p. 228. See also Gibson, *loc. cit.*

[†] Advisory Committee Aeronautics, 1910–11.

[‡] *Ibid.* See also Villamil, 'Motion of Liquids,' p. 178.

can be shown that curves such as those shown in fig. 1 are an immediate consequence of equation (3).

By including the resistance head R in equation (1) we get

$$h=\frac{v^2}{2g}(m^2-1)+av+bv^2. \quad . \quad . \quad . \quad . \quad (4)$$

The discharge coefficient C is given by

$$C=\frac{V}{\sqrt{\frac{2gh}{m^2-1}}}. \quad . \quad . \quad . \quad . \quad . \quad (5)$$

Solving (4) for v, the equation for C then becomes

$$C=-\frac{M}{\sqrt{h}}+\sqrt{\frac{M^2}{h}+N},$$

where M and N are constants.

Assuming $N=\cdot 98$ and $\frac{M}{\sqrt{h}}=-\cdot 02$ when $h=256$ feet of water, the higher graph of fig. 1 is the curve obtained.

Assuming, however, $\frac{M}{\sqrt{h}}=+\cdot 02$, the lower curve is obtained.

Resistance to fluid motion, therefore, of the kind denoted generally by equation (3), may give discharges greater than would be obtained in non-viscous flow. Apart from any practical value, this fact is interesting enough in itself. It implies, of course, that in cases where "a" is negative, regeneration of energy must be taking place. It does not appear that such energy regeneration (or rather recuperation) is impossible. The conditions necessary may be present in certain cases, which in the present state of knowledge are not well defined.

The coincidence in the form of the curves of fig. 1 with those experimentally obtained is considered too close to be accidental, and it is for this reason that the writer has ventured to submit the results (paradoxical though they appear) of the foregoing analysis. The assumption of a resistance function of the form $R \propto v^n$ does not give similar results.

XXVI. *A Quantum Theory of Vision.*
By J. Joly, *F.R.S.*[*]

THE Theory of Vision herein described originated in views respecting the origin of the latent image which formed the subject of an address to the Photographic Convention of the United Kingdom in 1905 ('Nature,' vol. lxxii. p. 308). Some couple of years before the war I returned to the subject and made experiments on the retinas of oxen and sheep, believing that it might be possible to detect electrons liberated by visible light falling on the retina or on the black pigment. The results were negative. The war interrupted further experiments, but more recently Mr. J. H. J. Poole, using more sensitive apparatus, examined the black pigment of sheep and oxen as well as the fresh retinas of frogs in a state of dark adaptation. These experiments also gave negative results. Further consideration of the whole matter has convinced me that such a surface emission of electrons was hardly probable under the conditions attending the experiment: conditions which involve the unavoidable presence of surface impurities. The failure to detect liberated electrons by no means invalidates the theory herein discussed.

Dr. H. Stanley Allen, writing to 'Nature' (Oct. 30, 1919), refers to a theory of colour vision which on January 7th, 1919, he communicated to the Röntgen Society. In this theory he supposes "that photoelectric action takes place in the rods or cones, so that we have a separation of electrons resulting in electrification of the nerve-cells which set up the nervous impulse to the brain." An essentially similar suggestion was made by Sir Oliver Lodge at the meeting of the British Association in 1919.

(1) I assume that the origin of luminous vision and of colour vision is to be sought in the liberation of electrons under light stimulus within a photoelectric substance or substances existing in the retina. The rhodopsin is such a photosensitive substance. In the case of the rods (in which rhodopsin is found) this substance acts as the basis of vision. In the case of the cones the same substance is very probably responsible. A strong argument for this view is to be found in the fact that in the fovea, where only cones exist, the spectral range of vision is in fair agreement with the spectral

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[*] Communicated by the Author.

absorption of the visual purple or rhodopsin (see (8)). Further, Kühne states that the maximum spectral absorption of this substance is at that part of the spectrum which seems brightest to the eye, and which is most active in bleaching the rhodopsin. It seems therefore very probable that all over the retina it is this substance which forms the intermediary between the light and the nerve ; translating the quanta of light-energy into nerve-stimulus.

It is an important feature of the theory herein advocated that the sensitiser in the case of the cones—as the organs of colour vision—should lie outside the cone and should not exist within it as in the case of the rod. That it does not exist within the cones is agreed by all observers. Dr. Edridge Green ('The Physiology of Vision,' 1920, p. 43) claims to have seen the unbleached rhodopsin between but not in the cones of the fovea. He states that when the retina was first examined the fovea was the reddest part of the whole retina. He also calls attention to a confirmatory observation of Kühne's in the case of a shark's retina. Mr. J. Herbert Parsons more recently points out that Hering had recognized such a distribution of rhodopsin as possible (Brit. J. Ophthal., July 1920)*.

I assume that in the case of the rods the sensitiser is operative within the nerve. In the case of the cones it is operative from without.

(2) The belief is gaining ground that photo-chemical and photo-electric processes are fundamentally alike (Lewis, Physical Chemistry, iii. p. 134). In the case of photographic actions the view that the movement of electrons within the light-sensitive film is responsible for the phenomena observed is supported by facts regarding the various modes in which the plate may be stimulated and by the formation of the latent image. No other theoretical basis affords so general an explanation of the effects of the light. (See 'Photo-Electricity,' by H. Stanley Allen: Longmans Green, 1913.) The range of photographic "vision" may be controlled by the use of a sensitiser. This substance is one which absorbs vigorously in one or more special regions of the spectrum. It sensitises the plate to the same range of wave-lengths as it absorbs. The photographic sensitisers are rich in light-absorbing molecules:—chromogens. They are photo-electric: emitting electrons over the range of frequencies which they absorb.

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* Dr. Edridge Green believes that vision is due to photo-chemical action progressing in the rhodopsin surrounding the cones. He considers that the rods are not percipient.

(3) Among the well ascertained facts of photo-electric science the following concern the present theory. (*a*) The electron is liberated with a velocity which, normally, depends on the frequency of the light only: increasing as the wave-length diminishes in such a way as to render the kinetic energy a linear function of the frequency. (*b*) The velocity is independent of the intensity (amplitude) of the light. (*c*) For equal intensities of light of different frequency, the light of highest frequency liberates most electrons. (*d*) For lights of the same frequency, the number of liberated electrons increases with the intensity. (*e*) The electron in most cases absorbs one quantum, the value of this quantum depending on the frequency according to the well-known equation $\epsilon = h\nu$, where h is Planck's constant ($=6{\cdot}57 \times 10^{-27}$ erg.sec.) and ν is the frequency. In virtue of the absorbed energy, the electron acquires a certain velocity and pursues a certain free path in the medium till diverted by collision.

(4) The value of the mean free path of electrons taking part in photo-electric emission from platinum has been determined by Robinson (Phil. Mag. 1912, 1913). He concludes that it is of the order 10^{-7} cm. Partzsch and Hallwachs (*Ann. d. Phys.* xli. p. 247, 1913) concluded that 99 per cent. of the photo-electrons from platinum emerge from a layer thinner than 28×10^{-7} cm.; from which we may conclude that the maximum range is about of this value. Patterson (Phil. Mag. 1902, iii. p. 643, iv. p. 652), dealing with the electrical conductivity of thin metallic films, arrives at the conclusion that the mean free path in various metals, including carbon, is of the order 10^{-6} cm. He cites a result by Vincent (*Ann. d. Chem. et d. Phys.* xix. p. 421, 1900) that for silver the mean electronic free path is 6×10^{-6} cm. Vincent's result also is derived from measurements of resistance.

Such of the above determinations as are based on direct photo-electric measurement, using ultra-violet light, require correction for the less value of the quantum associated with visible wave-lengths: that is, if we assume the penetrating power of the electron is dependent on its velocity. On the other hand, judging by the influence of density on the coefficient of absorption of β electrons, a correction for density may be necessary when we venture to so far exterpolate as to apply results on platinum to aqueous solutions of low-density molecules such as the fluid of the retina. Applying these corrections to the deductions of Robinson and to those of Partzsch and Hallwachs, we find the mean free path in water for electrons supposed to carry the quantum for yellow light to be $5{\cdot}5 \times 10^{-7}$ cm., and the maximum free path to be 154×10^{-7} cm. These are, of course, only approximations and must be regarded as only admissible in the absence of more secure results. The quantities arrived at by Patterson and by Vincent for the mean free path would assign to it a value from five to ten times the above, even if we make no assumption as to the influence of density. A correction on the score of initial velocity is not called for, as the initial velocity affecting Patterson's deductions is about $7{\cdot}6 \times 10^6$ cm.: a value not greatly different from that acquired by electrons carrying quanta associated with visible light.

Respecting these figures, we must of course bear in mind that the "collisions" refer in general to a deviation of path, not an arrest of motion and loss of energy.

(5) The velocity just referred to as acquired by the electron when a quantum of energy is imparted to it by visible light is very great—of the order 10^7 cm. per sec. The course of the electron before its kinetic energy is given up in work of ionization, in thermal agitation, or otherwise, is a brief one: probably less than the billionth of one second.

(6) The rods contain the sensitiser in the form of rhodopsin. I assume that this substance emits electrons in the same manner as other light-absorbing and optically unstable substances *. The electrons set free from the sensitiser by the rays absorbed expend their kinetic energy in stimulating the nerve, and, perhaps, establish an electronic current into the ganglion cell with which the nerve makes connexion. I assume that the electronic emission in the rod constitutes an intimate and generally copious source of stimulus—the light being very completely absorbed. The conditions are therefore very favourable to the appreciation of feeble illumination.

In this association of the photosensitive substance with the nerve, the quantitative value of the stimulus is developed at the expense of its qualitative value. Colour will not be interpreted to the brain, or only defectively. The liberated electrons are in no degree selectively presented to the nerve. All that escape from the rhodopsin molecules contribute to the total stimulus. In the case of some the energy of the quantum is all but spent; others reach the nerve possessed of the maximum kinetic energy. The stimuli may overlap

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* Kühne dwells on the remarkable instability of rhodopsin towards light and its great stability towards chemical reagents ('Photochemistry of the Retina and on the Visual Purple,' 1878).

in time and space. It is probable that some quanta are expended in conferring purely thermal movements on electrons, and along with the destructive effects of quanta upon the sensitiser the regeneration of this must all the time be progressing within the nerve. A confused flow of stimuli, too impure and too crowded for analysis, is the outstanding character of the sensory contribution of the rod. It corresponds to noise in the case of audition. Nevertheless, the conservation and integration within the rod of the stimuli arising from the interaction of light and sensitiser render it a most sensitive exponent of luminosity reaching the retina.

Its sensitivity is, indeed, extraordinary. Henri and des Bancels have shown that the retina is sensitive to an amount of light energy of the value of 5×10^{-12} erg. Now the quantum for green light is 4×10^{-12} erg. We may assume, therefore, that one quantum is sufficient to excite vision (Bayliss, General Physiology, p. 512). That is to say, the liberation of a single electron by green or blue light will excite visual sensation.

Again, consider the following relatively commonplace case. A standard candle removed to a distance of 3000 metres projects on each square centimetre luminous energy of the amount 4×10^{-7} erg per sec. This luminosity will evoke vision; and such feeble radiants are known to be best appreciated when the image falls on parts of the retina rich in rods. How many quanta are involved in this excitation of vision? The pupil admits, say, the luminosity reaching one-half a square centimetre or 2×10^{-7} erg per second. This corresponds to about 7×10^{4} quanta per second; such quanta as would be associated with yellow light. The number of rods which receive this energy is considerable. The size of the image is indefinite, but it will not be a point image. Suppose it covers one-tenth of a square millimetre, we can roughly estimate the number of rods involved. The total area of the retina is about 1000 square millimetres, and the total number of rods has been estimated as 130 millions, and again as half this number. We shall take the number to be 100 million. There will be about 10,000 rods illuminated. The quanta are in fact distributed over this number of rods: that is, 7 quanta enter each rod per second. We may translate this into 7 electrons liberated in each rod per second. It is evident that these small individual stimuli must be so far conserved as to make their way to the optic nerve. If they did not do so but died out within the nerve, there would be no vision. We have it then, however wonderful, however incredible it may seem, that the stimulus arising from one quantum must constitute an appreciable fraction of the threshold stimulus.

(7) The cones are structurally different from the rods in that they contain no visible quantity of rhodopsin. Further, they differ in that each cone is connected through a separate ganglion cell to the optic nerve. This prevails in the fovea where only cones exist and where colour vision is at its best. Throughout the retina, on the other hand, the rods are grouped; several individuals contributing their stimuli to the one ganglion cell. This fact is often referred to as accounting for the sensitivity of the foveal area. It is very significant. It reveals an effort of Nature to husband and conserve the cone stimuli and to convey them undiluted to the brain. It suggests that these stimuli are more delicate than those coming from the rods, and are of such a character as to bear no intermingling with other stimuli.

It is also noteworthy that in the central fovea the cones lose their characteristic conical form. They attain a remarkable length, at the same time diminishing in diameter till the latter sinks as low as one micron. Taking the diameter of the outer segment of the cone at this latter figure, and the length as 45 microns *, we find the surface amounts to 180 times that of the cross section.

To what is this remarkable effort after surface to be ascribed? If we assume the light entering the cone at its inner extremity to be uniformly distributed throughout the cone, it must escape laterally with a luminous intensity reduced 180 times. Plainly there is some advantage on the score of sensitivity in this diffusion of the light. In the central fovea we are told that the cones are so closely packed as to take on a prismatic form where the inner segments approximate one to another. The light must, therefore, at least for the greater part, move as I have indicated.

I assume that the outer segment of the cone is bathed in a photosensitive fluid, probably—almost certainly—rhodopsin. In this the light is absorbed: either directly at the surface of the cone or within the thin layer which separates cone from cone. At the meeting of the cone surface with the sensitiser electrons are emitted. They will also be emitted at such distance from the cone surface as the light can penetrate. Some of those freed at the surface enter the nerve with maximum velocity and kinetic energy. Those liberated more deeply enter the nerve with diminished velocity.

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* Greeff's drawing, according to Schafer, underestimates the length of the cone. On the drawing it is 38 microns. See Quain's 'Anatomy.'

Some do not escape from the sensitiser. Others fail to travel beyond the delicate covering of neurokeratin which is believed to invest the nerve. The fastest electrons carry into the nerve almost the full quantum of energy which is characteristic of the frequency giving rise to them. These are the most effective in exciting a nerve stimulus. If the intensity of the light is considerable there are many such. If feeble, there are only a few; but the speed, trajectory, and energy of these electrons remain characteristic of the frequency. Such stimuli are too few and too brief to confuse one with another by overlapping. They are appreciated at their cerebral destination as would be successive notes heard in music.

The penetration of the faster electrons into the nerve must be considerable. A maximum free path of 154×10^{-7} cm. may be assumed. The radius of the cone is about 5×10^{-5} cm. It follows that electrons moving in a radial direction may traverse one third of the radius before they become deflected. One half the total cross sectional area of the nerve is traversed by these direct movements. Deviated electrons may be supposed to reach the centre or travel beyond it.

As to the nature of the stimuli arising from electronic bombardment and as to the manner and form in which the energy of the electron is transmitted outwards from the retina to the optic nerve, we have much to learn. The transmission is probably electric in character according to many physiologists. It seems, however, to be probable that the velocity of the absorbed electron is not that which is quantitatively appreciated by the nerve, the time interval involved is far too small. But we may assume that the disturbance set up by the shock is not so short-lived. It travels relatively slowly from its point (or rather line) of origin. Probably what the nerve appreciates is the energy value of the individual stimuli, and this depends on the quantum of energy associated with the electron; which in turn is determined by the frequency of the light, *i. e.* by its "colour." That each individual electronic stimulus must possess a certain sensory value appears from the figures cited above respecting threshold vision by the rods.

It is necessary to consider a little more fully the specialized nature of the cone functions.

In the central foveal area, covering about 0·16 square millimetres, there appear to be some 2×10^4 cones *. Suppose such an amount of light as would certainly excite

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* Rather more according to Golding-Bird's drawing, if this is intended to depict these organs numerically.

vision in the rods fell upon this area. I assume that the image of a candle flame, which is 1500 metres distant from the eye, is focussed upon the central foveal area and effectively covers this. We now have twelve electrons formed per cone per second. It seems safe to conclude that at this distance the fovea would appreciate the candle-flame.

Now it is very evident that enormously greater numbers of electrons might be generated in the sensitiser bathing the cone without risk of overlapping of the stimuli. For not only must we assume that but a small percentage of the received quanta is restored to the cone in the form of what we may call "characteristic" electrons (*i.e.* those possessing speeds near the maximum speed; in other words, carrying the quantum proper to the wave-length), but we must also bear in mind that the work of each electron occupies at most but a very small fraction of a second.

We have, indeed, arrived at the weak point of the arrangement—its prodigality. Accordingly, we find that Nature, driven to adopt the external disposition of the sensitiser in order to avoid the confusion prevailing in the rods, proceeds now to improve on her design and to develop the cone so as to obtain the maximum number of high-speed electrons. This is effected by increasing the area of the cone wetted by the sensitiser. For the more the activating light is reduced in intensity per unit area (within limits) the less the penetration of the ray into the sensitiser, and the greater the number of electrons released at the immediate cone surface.

Colour vision is in abeyance at very low luminosities. The explanation is that ultimately there are insufficient characteristic electrons to excite the colour sensation. On the other hand, colour vision cannot be excited without the diluting effects of the slower moving electrons appearing. For there must always be many (probably a majority of) slow-moving electrons stimulating the nerve. The sensation arising from these is not characteristic of the frequency which gives rise to them, and a sensation of white light is the result. The colour fails to be "saturated." White sensation is always added to the colour sensation excited by the characteristic electrons. We never experience quite saturated colour sensation.

It will be gathered that the present theory ascribes quantitative sensitivity to the rods; qualitative sensitivity to the cones. The difference being mainly referable to the fact that in the one case the sensitiser is located within the nerve, in the other it is located without the nerve. The sensory stimulus emanating from the rods is compounded of many sources of stimuli, *i. e.*, such as may originate in electrons possessing every velocity and kinetic energy up to the maximum proper to the frequency of the activating light, such as may arise in electronic movements associated with the regeneration of the sensitiser and in thermal electronic agitation excited by the quanta taking part in these operations. The stimulus emanating from the cones, on the other hand, is purified of all stimuli save those arising from the kinetic energy of electrons, which are activated by the energy of absorbed quanta. The electrons are, in fact, selectively presented to the nerve: all other sources of stimuli take place outside the cone and are cut off from it by the filament which invests it. In the cone the more intense stimuli tap out to the brain the sensation of colour which we associate with the intensity of the quanta involved. To this succession of characteristic nerve impulses there is added an underlying accompaniment: the white or luminous sensation made up of all the feebler electrons which impart to the nerve but a fraction of that which is characteristic of the frequency of the light entering from the world without. The cone is the more highly specialized organ of the two and is probably a more recent product of evolution.

(8) The higher luminosity of the colour threshold arises out of the conditions affecting the stimulation of the cones; the sensitiser being external and hence a part only of the evoked electrons producing visual sensation. In the rods all absorbed radiations are expended on exciting the sensation of luminosity; the electrons being liberated within the rod. There will be a colourless interval attending foveal stimulation of low intensity for the reason that the characteristic electrons constitute only a fraction of the stimulus and it requires a certain density of such electrons before colour vision is experienced. In other words, the achromatic effect of very low light intensities or of very brief exposures is due to commencing stimulation by non-characteristic electrons.

(9) One consequence of the different disposition of the sensitiser respecting the rod and the cone is that the characteristic quanta stimulating the cone cannot possess the full energy value proper to the originating frequency. That is to say, it reaches the nerve with a quantity of energy such as would be associated with a frequency ν', less than ν. This is because the electron in the case of the cone must part with some of its energy in penetrating the outer sheath of the cone. We may expect from this that the entire luminosity curve of

photopic vision will be shifted towards the red end of the spectrum. This, according to the present theory, is the explanation of why the photopic luminosity curve does not quite coincide with the scotopic luminosity curve; the maximum of the first being at D, nearly, and that of the second being at E (Abney, Colour Vision, p. 103). Working from these wave-lengths it is easy to show that the electron must lose about $0{\cdot}3 \times 10^{-12}$ erg in penetrating the sheath; *i. e.*, almost 10 per cent. of its energy. It is, according to the Quantum Theory, quite unnecessary to seek for any other sensitiser than rhodopsin as the basis of vision.

(10) It is probable, according to the present views, that both rods and cones functionate by transmitting electrons from the sensitiser into the optic nerve. The observed current from fundus to cornea attending the light-stimulation of the excised eye finds explanation in the present theory. Bayliss (*loc. cit.* p. 522) reviewing the researches of Waller, Einthoven and Jolly, and others, says:—"Respecting the results of these researches the main fact is that, in the uninjured eye of the vertebrate, the incidence of light causes an electrical change in such a direction that the nervous layer of the retina becomes electrically positive to the rod and cone layer." It will be seen that this points to the liberation of free negative electricity in rods and cones attending the light stimulus; *i. e.*, to the presence of free electrons in these terminal organs.

(11) Chemical effects have been observed as taking place in the retina when adaptation is changed from dark to light: *i. e.*, there is a change from alkalinity to acidity. This may be involved in the loss of an electron by the ion $\overline{\text{H}}\text{O}$ and the formation of the ion H^+.

(12) The movement of the cones attending light stimulus, which in the case of most animals certainly occurs, and which occurs in the case of Man, may be a mechanism designed to bring the cones into un-exhausted sensitiser; the bleaching of the immediate layer touching the cone being fatal to its full and proper activity. The rods carrying the sensitiser within would not profit by such a movement, and accordingly do not exhibit it.

(13) Mechanical effects, such as pressure, appear to liberate electrons and produce the latent image on a photographic plate. The luminous sensation attending pressure of the eye-ball may arise in the same way: *i. e.*, by the mechanical liberation of electrons. It can be referred to what is known as triboluminescence.

(14) The degree of spectral analysis attainable according

to the foregoing theory of colour vision must be limited. The interpretation of colour is referred to the appreciation by the nerve of the value of the quantum. A complete detailed analysis of the whole gamut of wave-lengths between red and violet on such a basis is, probably, unattainable, even if it was any benefit to the organism. And if attainable it might result in badly differentiated colour sensations. The evolutionary growth of three highly developed colour sensations corresponding to the central and mean quanta of the spectrum is the result. It is Nature's compromise with her limitations. It is one which is, in part, cerebral in character: the light-sensitive part of the brain accepting as interpretive of the many separate frequencies a commingling of sensations excited by the central and end frequencies. These views do not preclude the possibility of more than three primary colour sensations existing. I assume, however, that red, green, and violet are alone primary. If, now, rays of the wave-length 5893 Å.U., say, are received on the retina, no sensation special to this wave-length arises, although electrons having velocities quite peculiar to it activate the nerve. It is more efficient for the organism to develop special sensitivity towards three representative stimuli, produced by widely differentiated quanta. Thus we feel so much red sensation according to the proximity of λ 5893 to λ 6563, and so much green sensation according to its proximity to λ 5461. The combined sensations we call yellow, and yellow becomes a distinct sensation, although it may be really compounded of two other sensations. It is a sort of unconscious memory: the one stimulus "reminding" the colour-visual centre of the stimuli which evoke the representative sensations, red and green.

Of course it would be easy to imagine a purely objective explanation of colour vision if rhodopsin exhibited appropriate absorption bands. But, on the contrary, its absorption spectrum is remarkably uniform over the range of the visible spectrum.

(15) A colour-blind individual is one whose foveal nerves respond feebly to certain quanta. The same abnormality affects the cones all over his retina. Thus if he is violet- and green-blind, the quanta proper to E or to F produce only a feeble stimulus. But those proper to C and D are fairly normal in the stimulus they excite. His brain has developed no more than the one sensation, the maximum luminosity of which lies between E and D. The abnormality is fundamentally a physical deficiency; and this leads to mental deficiency, as commonly happens in similar cases.

Colour vision curves constructed from the examination of abnormal sight show the curves as overlapping. As I have already pointed out, the entry of spent electrons into the normal nerve—*i. e.* electrons possessing less kinetic energy than is proper to the wave-length illuminating the retina—introduces indeterminate stimuli which result in luminous or white sensation; for such, received in the sensory colour-centre, could not be differentiated from the sensation arising when white light falls on the retina, and quanta exciting all three colour sensations stimulate the nerve. Such spent electrons affect the sensations of the colour-blind also, according to his limitations. They can give no new sensation to one possessed of monochromatic vision. The subject is full of obscurities and difficulties, and I shall not enter upon it. I see nothing in the present theory to accentuate or add to those difficulties. It is to be hoped it may contribute to clearing them up.

(16) The spectral limits of the colour-sensation curves find a simple physical explanation in the failing absorption of rhodopsin for these wave-lengths. This matter is, of course, bound up with the limitations imposed by the absorption of quanta (associated with the higher and lower frequencies) by the media through which the light has to pass before it reaches the retina.

(17) The light-sensitive substance, rhodopsin, is probably extremely complex. It is said to be related to Anthocyanins which are rich in chromogens and are held accountable for a wide range of colour in flowering plants. Evidence for the richness in chromogens of rhodopsin is, I think, to be found in its sustained absorption of visible spectral rays. "Fatigue" may evidently be referred to the exhaustion of such chromogens as have been deprived of an electron. This appears to be the natural explanation.

(18) When a bright object is looked at for some time and the eyes then closed, "after-images" are seen. These are both positive and negative.

The phenomena of positive after-images suggest that something of the nature of the latent photographic image is formed in the nerve-substance. Indeed, *à priori*, one would expect this to happen, the conditions having much in common in the two cases. On this view electrons entering the nerve would in some cases remain attached to atoms within its substance and an electrostatic field would prevail between them and positive ions in contact with the cone. This system must break down ultimately—probably is continually breaking down and being rebuilt. The electrons attached within the nerve would be distributed at radial distances from the surface proportional to the frequency of the light which gave rise to them. The "red" electrons (*i. e.* those exciting pre-eminently the red sensation) outermost, then the "green" and lastly the "violet." But in the case of very intense light stimulus the disturbance due to the passage of very many electrons might result in permitting only very few of the red and green, but allowing abundant violet, to collect in this manner; the latter attaining the outer limits of the field of disturbance. Now in the colour succession of after-images there is found evidence for both these modes of distribution. We must suppose that when the light stimulus is withdrawn the electrostatic field gradually breaks down. We have the red electrons going first; for they are the most strongly attracted; the green following, and finally the violet. We assume that in the act of reverting to the sensitising molecule the electron creates fresh colour sensation. But the energy available must be less than that originally possessed by the electron when entering the nerve. Einstein's explanation of the law of Stokes respecting fluorescence (Allen, *loc. cit.* p. 190) may be invoked. In the present case there should be a diminution of the intensity of the stimulus; in other words a shift of the sensation towards the red end.

Now it is agreed by many observers that amidst many variants the after-images appear very generally in the order red, green, blue (Parsons, *loc. cit.* pp. 111, 261). This is for moderate to bright light. For long continued excitation by more intense light, blue takes precedence of all. There is also evidence for a lowering of the spectral sensation. Thus McDougall writes: "An important feature of the after-images of bright white light is that, after a first short period in which two colours fuse to give yellow, or, as is the case after the brighest lights, all three fuse to give white, the colours that in turn occupy the area of the after-image, alone and unchanging for considerable periods, are red green and blue only. The red is a rich crimson red, decidedly less orange than the red of the solar spectrum, the blue is a rich ultramarine, and the green a pure green having no inclination towards blue or yellow." He goes on to describe the high saturation of these colours. As regards the repetitional effect generally observed, that is the recurrence of the three colours in the like order, we again find a photographic counterpart in recurrent reversal. This is, according to the electronic theory of photography, due to the break-down of a succession of latent images;

accumulations of electrons occurring till a point is reached when these revert to the parent molecules. But in what manner in the case of the nerve could such effects be stored and saved from immediate degradation? A possible explanation suggests itself. The cone during light stimulus contracts; subsequently it again elongates. When contracting we may suppose the cone moves towards parts of the sensitiser still unacted upon. [And this is probably the primary object of the movement.] The effect will be to remove the cone from the field of positive ionization and so free the internal, fixed electrons from electrostatic attraction. But when the retina is again darkened the cone moves back into the exhausted sensitiser; a region rich in free positive ions. Hence as it elongates an electrostatic positive field accumulates till there is break-down and discharge during which the fundamental sensations are successively evoked. If the successive colour cycles attending the movement of the cones overlap there will ensue the irregular sequence often perceived. On the other hand, if the cones move under the influence of a common stimulus and advance, not uniformly, but with pauses of quiescence, then the repetitional colour cycles find complete explanation; each cycle corresponding to the latent "image" breaking down over one short segment of the cone. That the movement of the cones is general and not due to a stimulus local to each cone is, I suggest, shown by the fact that in the case of the frog, where the cone-movement is very marked, "light on one eye causes reaction on both as also light on the skin so long as the brain is intact" (Engelmann, 'Nahmacher.' See Parsons, *loc. cit.* p. 12).

It is consistent with the view that after-images are of the nature of the latent photographic image that they may persist for very considerable time intervals.

The negative after-image is probably explained by fatigue. It appears a little after the positive image, and when the retina is re-exposed to feeble illumination. What was bright now appears as dark, and the colours change to the complementary hues.

A discontinous motion of the cones will naturally arise if the nervous actions involved are reflex in character; as the observations on the frog very surely indicate. In this case electronic stimulus of the cone initiates its retraction and the cessation of the stimulus initiates its extension. Hence when, on the extension of the nerve, the luminous after-image begins to be formed the electronic movements act as the afferent stimulus and extension ceases or contraction may ensue. Only when the after-image dies out is the extension of the cone continued. But now a fresh part of the "latent image" becomes involved and again there is arrestment: and so on.

(19) The momentary electrical response which is noticed in the retina when light is cut off and which is the same in direction as the light response (Bayliss, *loc. cit.* p. 522) is not difficult to explain on the present theory. It is due to the break-down partially or completely of the latent "image" in the cones; that is to the stimulus which arises when the anchored electrons are attracted back to the positive field surrounding the cone; the electrostatic effects of inflowing electrons attending light stimulus having ceased to affect them.

(20) Simultaneous contrast effects are, according to my own observations, largely due to imperfect fixation. There is probably a psychological factor also involved. The tissue paper increases the effects of adjacent colour-patches because it renders fixation inaccurate.

(21) The Purkinje Effect has been explained by Von Kries on the assumption that the cones are sensitive to colour, possess a maximum sensitiveness in the yellow, and are responsible for vision at high luminous intensities; the rods being responsible for vision at low intensities. Into this matter the dark adaptation of the retina enters, for with it scotopic values rise. There is nothing here inconsistent with the Unitary Theory of vision.

(22) The dependence of the colour-sensitivity of the cone upon its surface area is well shown by the increasing colour-blindness of the retina towards the periphery; the active area of the cone diminishing as retinal colour-blindness increases.

The study of colour vision is hampered by many difficulties, chief among which is the elusive and variable nature of the effects under observation. On this account we find disagreement among high authorities as to many phenomena of vision. I shall not here pursue the matter into further details.

The foregoing theory is founded on the conception of the quantum. The nerve is supposed to discriminate between the quanta of three or more representative spectral centres. And should not the quantum be regarded as a *vera causa*, when we find that a single one of them acting on the retinal nerve suffices to stimulate the sense of vision? What alternatives have we to a quantum theory at the present time? One thing seems certain. No interpretation of colour in terms of the frequency seems possible, whether primary or forced vibrations be appealed to. The late Lord Rayleigh, in a letter to 'Nature' (May 21, 1918) questions the possibility of sound-frequencies of 256 vibrations per second being directly conveyed by the nerves to the brain. "It is rather difficult to believe it," he adds, "especially when we remember that frequencies to 10,000 per sec. have to be dealt with. Even if we could accept this, how deal with light-processes in action along the nerve repeated 10^{15} times per second?"

I have received kind help from many friends. At the time of my pre-war experiments Professor John Mallet Purser gave me much valuable instruction, and since has continually assisted me. Prof. H. H. Dixon, Dr. O'Sullivan, S.F.T.C.D., and the late Sir Henry Thompson also advised me. More recently, I have to acknowledge much assistance in experimental work from Prof. Pringle and his Assistant, Dr. Fearon. Dr. Euphan Maxwell has been so good as to place valuable histological specimens at my disposal. I owe much to my discussions with Mr. J. H. J. Poole.

Trinity College, Dublin.
Jan. 4, 1921.

XXVII. *Electric Discharge in Hydrogen.*

By Dr. D. N. Mallik, *F.R.S.E.*[*]

THE peculiar behaviour of an electric discharge through Hydrogen has been the subject of study for a long time. That it is somewhat erratic, for reasons which are altogether obscure, has been noticed by several experimenters. In particular, in a paper in the 'Philosophical Transactions' published as long ago as June 1907, Prof. H. A. Wilson and Mr. Martyn described the peculiar behaviour of discharge through hydrogen in a De La Rive tube. They found that the rotation effects were not at all well marked when the contained gas was hydrogen; although when the tube was filled with air or N_2O, the phenomenon was well defined and capable of quantitative measurement. Working with an induction-coil, as well as storage cells and a tube which had given satisfactory results with several other gases and vapours, I was unable to detect any rotation at all with hydrogen. I was, accordingly, at first disposed to regard this as an illustration of the generally erratic behaviour of hydrogen, which previous experimenters had remarked upon.

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[*] Communicated by the Author.

On repeating the experiments, however, with tubes of various lengths and with induction coils giving various E.M.F.'s, I found that the rotatory phenomenon is observable in hydrogen as in other gases, but the conditions have to be carefully adjusted for the purpose, for reasons which will appear from the theoretical discussion which is given in the present paper. This discussion, however, is necessarily incomplete, in view of our ignorance of the intricate mechanism of the processes which obtain in a discharge-tube.

We know that the equation of continuity in a discharge-tube is [J. J. Thomson, 'Conduction of Electricity through Gases,' art. 137]

$$\frac{\partial n}{\partial t}+\frac{\partial nu}{\partial x}=(\alpha-\beta)\frac{n\bar{c}}{l},$$

where $\alpha\frac{n\bar{c}}{l}$ = no. of corpuscles generated by collision per unit length of the tube per unit time,

$\beta\frac{n\bar{c}}{l}$ = no. that disappear through recombination,

n = no. of corpuscles per unit length of tube, having average velocity $\bar{c}$,

l = length of mean free path.

For steady rotation we must have

$$\alpha-\beta=0,$$

and, moreover,

$$\alpha=\mathrm{F}(\mathrm{X}el)=f\left(\frac{\mathrm{X}e}{p}\right),$$

where F, f are unknown functions, and X = electric intensity and p the pressure.

The condition accordingly depends not merely on the pressure of the gas, but also on X and β.

Now, both experiment and theory have amply shown that X depends not merely on the voltage of the induction-coil, but *also on the nature of the gas.* We must, therefore, admit that β also depends on the nature of the gas; it follows accordingly, that the rotation stage will depend on this quantity, as well as on other factors (pressure, length of the tube, and the voltage of the induction-coil).

That β depends on the nature of the gas is *à priori* evident. The following investigation, further, shows that it ought to be so.

If n=number of electrons in unit volume, then the number of electrons which have speeds lying between c and $c+dc$

$$\propto \frac{n}{\alpha^3} e^{-\frac{c^2}{\alpha^2}} c^2 dc\,;$$

where α is the most probable speed and varies as $\bar{c}$.

Then the corresponding number of collisions

$$\propto \frac{c}{l} \cdot \frac{n}{\alpha^3} e^{-\frac{c^2}{\alpha^2}} c^2 dc.$$

Hence the total number of collisions corresponding to the range of velocities 0 to c

$$\propto \frac{n}{l\alpha^3} \int_0^c e^{-\frac{c^2}{\alpha^2}} c^3 dc$$

$$\propto \frac{n\alpha}{l} \cdot \frac{c^4}{\alpha^4} \text{ nearly ; } i.\ e. \propto \frac{n\bar{c}}{l} \cdot \frac{c^4}{\bar{c}^4}.$$

If, therefore, only the collisions lying within the range 0 and c (supposed, small) result in recombination, $\beta \propto \frac{c^4}{\bar{c}^4}$.

Now, we may reasonably suppose $\bar{c} \propto \frac{1}{p}$ for any gas, and therefore $\bar{c} \propto \frac{\rho}{p}$ for different gases.

Also $\bar{c} \propto Xe$, *i. e.* we may take $\bar{c} \propto \frac{Xe}{p}\rho$.

Moreover, if the kinetic energy corresponding to c is supposed to arise in accordance with the equation

$$\tfrac{1}{2}mc^2 = \frac{e^2}{l'},$$

where l' is the distance [supposed, constant] at which the velocity of corpuscle is zero, then $\frac{\bar{c}^4}{c^4}$

$$\propto \left(\frac{X\rho}{p}\right)^4 \propto \frac{1}{\beta}.$$

In order to satisfy the condition for steady rotation, therefore, we must have

$$f\left(\frac{Xe}{p}\right) = \beta = \frac{1}{\left(\frac{X}{p}\right)^4} \frac{1}{\rho^4}.$$

Also, since $\frac{X}{p}$ = const. for any gas, at the rotation stage we have, at this stage,

$$\frac{X}{p} \propto \frac{1}{\rho}.$$

In a previous paper I have shown that $\frac{X}{p} = p\omega$, where ω = the angular rotation of electric discharge under electric intensity X. We have accordingly $p\omega \propto \frac{1}{\rho}$, a result which has been experimentally verified.

Moreover, experiment has shown that (at the rotation stage) $\frac{X}{p}$ varies as the effective voltage of the induction-coil producing the discharge. This voltage, therefore, must vary inversely as the density of the gas used. It stands to reason, therefore, that in the case of hydrogen, the effective voltage required will be much higher than in air, and accordingly, the effect will be much more difficult to observe.

XXVIII. *On the Collision of α particles with Hydrogen Atoms.*

To the Editors of the Philosophical Magazine.

Gentlemen,—

In two papers in your Journal (June 1919) I gave an account of some experiments on the collision of α particles with hydrogen atoms, using the scintillation method for determining the number of H atoms set in swift motion. With the microscope used at that time the counting of scintillations was a difficult and trying experience, but during the last year by the use of holoscopic lenses of large aperture and suitable eyepieces, recently constructed in this country, it has been found possible to improve greatly the ease and certainty of counting such weak scintillations. With the old arrangement, using a complex beam of H atoms of different velocities, the observer was very liable to miss the weaker scintillations due to lower velocity H atoms in the presence of the brighter scintillations due to swifter ones.

The accurate determination of the relation between the number and angle of ejection of the H atoms for different speeds of the α particles is of great importance in order to throw light on the nature and distribution of the forces in such close collisions when the nuclei must approach within 3×10^{-13} cm. of each other. This problem is being attacked in the Cavendish Laboratory by two methods. The electric balance method, as devised by Mr. McAulay (v. Phil. Mag. Dec. 1919), has the great advantage that it does not involve the eyestrain of counting, and, if the electrical effect is sufficiently large, is practically independent of the probability variations which make it necessary to count so many particles in the scintillation method. This electrical method should yield approximate quantitative data and has certain advantages over the scintillation method for examining the effect of H atoms near the end of their range where their ionization per unit path is increased due to their reduction of velocity. The relation between the number of H atoms shot out at different angles with the primary beam of α rays for different velocities of the particle is under direct examination by Mr. Chadwick and Mr. Bieler using the scintillation method. The general results so far obtained by these new methods indicate that the number of H atoms liberated is much larger than the theoretical number to be expected from point nuclei and possibly even greater than my original estimate. They also confirm the observation that the H atoms tend, for the high velocities of the α particle, to be shot forward in the direction of the α particle but not to the same extent as the preliminary experiments showed. The direct method used by Chadwick and Bieler indicates that even for swift α particles there is present a certain proportion of lower velocity H atoms. The determination of the relation between the number of H atoms and angle of ejection should afford valuable data for calculation in order to throw light on the structure of the nuclei and forces between them. Unfortunately such counting experiments involve a large amount of time, so that there will be some delay before the necessary data are available for adequate comparison of theory with experiment.

I would like to take this opportunity of drawing attention to a slip in my original paper (*loc. cit.* p. 546) which fortunately is of little importance to the main argument. In calculating the emergent range for H atoms scattered at an angle θ with the direction of the α particle, the correcting factor for absorption under the experimental conditions was taken as $\cos \theta$. As Mr. C. Darwin pointed out to me some time ago, this is only a rough approximation and the average value for the correcting factor is a complex function of θ which can readily be calculated.

E. RUTHERFORD.

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JOURNAL OF SCIENCE.

[SIXTH SERIES.]

MARCH 1921.

XXIX. *The Intensity of Reflexion of X-Rays by Rock-Salt.*

By W. LAWRENCE BRAGG, *M.A., Langworthy Professor of Physics, The University of Manchester;* R. W. JAMES, *M.A., Senior Lecturer in Physics, The University of Manchester; and* C. H. BOSANQUET, *Balliol College, Oxford*[*].

Introduction.

1. ACCURATE comparisons of the intensity of reflexion of X-rays by crystal-faces were first made by W. H. Bragg[†], who measured by the ionization method the energy of the X-rays reflected by various faces of rock-salt. He showed that if X-rays of definite wave-length are reflected at a glancing-angle θ by a face with one or more even crystallographic indices (*e. g.* (100) (110)), the intensity of the reflected beam can be expressed approximately by the formula

$$I_\theta = \frac{A}{\sin^2\theta} e^{-B\sin^2\theta},$$

where A and B are constants and I_θ measures the intensity of the reflected beam of rays. In the case of even orders of reflexion from faces with odd indices, the intensities of reflexion may be expressed by the same formula. When the observed intensities for various faces and orders are plotted

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[*] Communicated by the Authors.

[†] W. H. Bragg, Phil. Mag. vol. xxvii. p. 881 (May 1914).

against sin θ, the points lie on a smooth curve, showing that the intensity is a function of θ and does not otherwise depend on the indices of the face or the order of the reflexion.

Theoretical expressions for the intensity of the reflected beam have been deduced by Darwin * and Compton †. Any theoretical formula for the intensity of reflexion or diffraction by a crystal must contain a factor whose value depends upon the number and arrangement of the electrons in the atoms. In the case of reflexion, the intensity falls off more quickly with increase of glancing-angle than the theoretical formula would indicate, and this falling off must be accounted for by the factor referred to above. In his paper, Compton makes use of the relative measurements of intensity made by W. H. Bragg, and assuming the theoretical formula for the intensity of reflexion to be correct, he obtains the relative value of the factor at different angles, and thence deduces certain possible arrangements for the electrons in the atom.

A very interesting paper by Debye and Scherrer ‡ deals with the same question from a slightly different standpoint. The relative intensities of the diffraction haloes obtained by Debye's powder method are measured photometrically. The authors come to the same conclusion as had been drawn from the reflexion measurements §, that the falling-off in the intensity with increasing angle of scattering must be partly ascribed to the variation in the factor we are considering.

The experiments described in the present paper have been made with the object of extending the measurements of intensity over a larger range of glancing-angles. Further, a direct comparison has been made between the energy of an incident homogeneous beam, and its reflexion by the crystal. The results so obtained have been compared with those given by the theoretical formula; and it will be shown that there is strong evidence that the formula is accurate. From the observations it is possible to calculate not only the relative values of the factor, which depends on the arrangement of the electrons in the atom, but also its absolute value over a range of angles, so that a direct comparison may be made between the observed value and the value calculated for various models of the atomic structure.

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* C. G. Darwin, Phil. Mag. vol. xxvii. pp. 315 & 675 (Feb. and April 1914).

† A. H. Compton, Phys. Rev. ix. p. 1 (Jan. 1917).

‡ *Phys. Zeitschr.* xix. pp. 474–483 (1918).

§ *Cp.* W. H. Bragg, Trans. Roy. Soc. A, ccxv. pp. 253–274 (1915).

Comparison of the Intensity of Reflexion by different Faces.

2. The method employed is fundamentally the same as that described by W. H. Bragg. Homogeneous rays are emitted from the focal spot on the target, and are limited to a narrow beam by a slit termed the bulb-slit. The beam falls on the crystal, and the reflected beam is received by an ionization-chamber through a second slit. If the chamber-slit and the bulb-slit are equidistant from the axis of the instrument with which the crystal face coincides, the chamber-slit can be set so that it receives all rays of any particular wave-length reflected by the crystal face, although the reflexion may take place at various points on the face owing to irregularity of the crystal structure. This focussing effect has been described by W. H. Bragg and one of the authors *.

The chamber may therefore be placed so as to receive all reflected homogeneous rays of any required wave-length—for instance, those corresponding to the K_α-line of Rhodium. If this is done, and the crystal is slowly rotated about the axis of the spectrometer, very little effect is observed until the crystal planes come into the position where the equation for reflexion is satisfied. The ionization then rapidly rises to a maximum, and falls away again as the crystal passes beyond this position. The more perfect the crystal, the narrower the range over which reflexion takes place. No crystals are perfect; in all cases the rays are reflected by a number of facets making a small angle with each other. As the crystal turns, these come, one after the other, into the correct position for reflexion, the reflected beam from each falling on the chamber-slit in virtue of the focussing effect described above.

The intensity of the reflexion cannot be measured by the effect observed when the crystal is set at the position which gives the most intense reflected beam, for the strength of this beam is dependent on the degree of perfection of the crystal face. It is measured by sweeping the crystal with uniform angular velocity through the entire range over which it reflects, and by observing the total ionization produced in the chamber during this process. In this way, every part of the pencil of homogeneous rays will fall at some time on a portion of crystal which reflects it, and will contribute its share to the whole effect. Experiments made with different crystals show that the intensity, measured in this way, is not dependent on the degree of perfection of the crystal,

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* 'X-Rays and Crystal Structure,' p. 31.

that it is the same for any one face and order, however the crystal may be distorted, and is, in fact, a definite physical quantity on which theoretical calculations may be based.

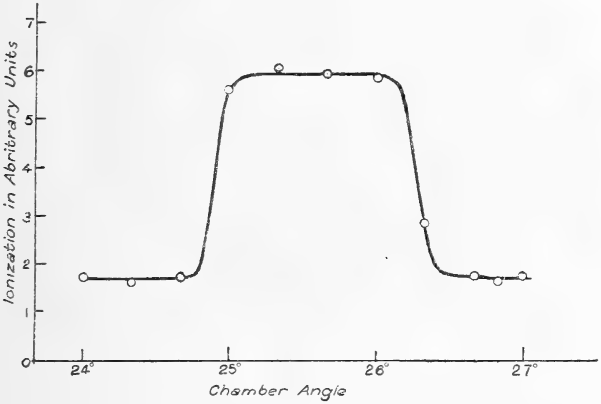
3. The X-ray spectrometer is of the type devised by W. H. Bragg, which has been described in former papers. The ionization-chamber is filled with methyl bromide, and a potential of 320 volts is applied between the outer walls of the chamber, and the inner electrode which is connected to the electrometer. This potential is sufficient to prevent appreciable recombination of ions with the strongest ionization produced. A Lutz-Edelmann string electrometer is used to observe the charge communicated to the electrode, its sensitivity being adjusted to about 100 divisions to a volt. A null method is employed to measure the charge. The inner rod of a small cylindrical condenser is connected to the electrometer, the outer cylinder being raised to any desired potential by a potential divider and battery. When the crystal is swept through the reflecting angle, a charge is communicated to the electrometer. This charge is neutralized by adjusting the potential of the outer cylinder of the condenser until the string in the electrometer returns to its zero. The potential applied to the condenser is then proportional to the total charge which has passed through the ionization-chamber.

4. It is necessary that the incident beam of rays should remain constant in intensity. A Coolidge bulb is used, in which the anticathode consists of a button of rhodium embedded in a tungsten block. A large induction-coil and Sanax break supply a current of 1·5 milliamperes at a potential of about 50,000 volts. It is possible to keep the intensity of the rays constant to within 2 or 3 per cent. and with the Coolidge tube it is also possible to repeat the conditions of the experiment on successive occasions in a satisfactory manner. Such variations in intensity as do occur are probably due to the irregular action of the break.

5. Superimposed on the homogeneous rays, there is a general radiation of all wave-lengths which is also reflected by the crystal. In making a measurement of intensity it is necessary to allow for this general radiation. When comparing the intensity for two faces or orders, a preliminary survey is made in each case to enable a measurement to be made of the effect of the general radiation. The chamber is set at a series of angles over a range including the angle at which it receives the homogeneous beam. At each position of the chamber, the crystal is swept through the corresponding reflecting angle and the total ionization measured.

A series of readings plotted in this way is shown in fig. 1. The readings are at first approximately constant, being due to the general radiation. As the position at which the chamber is set approaches that at which homogeneous rays are received, the ionization rises rapidly, remains constant again as long as the whole pencil of homogeneous rays enters the chamber, and then falls to a value approximately equal to its former steady value when the homogeneous rays are no longer received.

Fig. 1.



6. When comparing two crystal faces, this survey is made in each case. One of the faces is then mounted in the spectrometer, the chamber set so that it receives the homogeneous beam, and a series of readings taken by sweeping the crystal backwards and forwards. The crystal faces are interchanged, the chamber reset, and a series of readings taken for the other face. This process is repeated several times, and the means of the intensities for the faces are compared. The preliminary survey indicates what fraction of the total intensity observed must be subtracted, for each face, in order to allow for the general radiation; and when this has been done, the ratio of the corrected readings gives the ratio of the intensity of reflexion by the two faces. A series of readings obtained in this way is given below. It is a comparison of the reflexion by the (311) face of NaCl, mounted so as to face left on the spectrometer, of the same face turned through 180° so that it faces right, and of the third-order reflexion from the face (100) mounted so as to face right. The difference between the values for (311) L and (311) R is due to inaccurate grinding of the crystal

surface, the effect of which will be discussed later. It can be shown that, although they differ greatly, their mean represents accurately the strength of reflexion if the face were cut true. In taking the readings, the crystal was turned 5 minutes of arc for every beat of a metronome, beating 100 to the minute.

Comparison of (311) L, (311) R, and (300) R.

Face.	Sweep of crystal.	Chamber angle.	Potentiometer scale.	Readings.	Mean of readings.
	° ′ ° ′	° ′			
(311) L	8 50–11 20	20 50	2	(71, 73, 73, 72)	72·2
(311) R	10 05–12 35	21 00	2	(57, 55, 58, 57, 56, 56)	56·5
(311) L	8 50–11 20	20 50	2	(73, 74, 74, 75)	74·0
(300) R	17 30–20 00	38 25	3	(77, 78, 78, 78)	77·8
(311) L	8 50–11 20	20 50	2	(72, 71, 70, 72, 71, 70)	71·0
(300) R	17 30–20 00	38 25	3	(78, 78, 79, 80, 80)	79·0

A survey of the three reflexions showed that the homogeneous radiation was responsible for 76·9 per cent. of the total effect in the case of the (300) R reflexion, 33·0 per cent. for the (311) R, and 32·2 per cent. for the (311) L reflexion. Since the intensity is very much greater for the (300) face than for the (311) face, different scales on the potentiometer were used. A reading of 72·2 on the second scale represents 72·2 per cent. of a total voltage of 15·72, the corresponding voltage for the third scale being 22·79.

Taking this into account and allowing for the general radiation, one gets a ratio

$$\frac{\text{Mean intensity, face (311)}}{\text{Intensity, face (300) R}}=\frac{3{\cdot}22}{13{\cdot}45}=0{\cdot}2395.$$

In another experiment, (300) R and (300) L were compared, and in this way the relative mean intensities of (311) and (300) measured.

In order to have a uniform system of indicating both the order of reflexion and the face at which it is taking place, the convention of multiplying the indices of the face by the order has been adopted. Thus, by the reflexion from (622) is meant the second order of reflexion from the face (311).

The crystal is not turned continuously during each reading; its setting is altered five minutes of arc at each beat of a metronome by means of a series of spokes on the tangent screw. It would be preferable to turn the crystal with a uniform angular velocity, but it is unlikely that any appreciable error was caused by the method used. In order to make certain that this was the case, the crystal was turned slightly between each reading, in order to ensure that the halting-places did not occur at exactly the same angles.

7. The faces used in this experiment were prepared by grinding, and were of sufficient area to intercept the whole of the incident beam of rays. In general, faces were prepared 3 or 4 centimetres in length and breadth. The perfection of the crystal structure may be judged from the range of angles at which reflexion takes place. In most cases the greater part of the effect was observed to take place within less than a degree of arc as the crystal was turned, the faces being prepared from large blocks of rock-salt which were very little distorted.

The face should be cut so that it is as nearly parallel to the planes of the crystal structure as possible. If this is not the case, there will be a difference in the intensities of reflexion when the crystal is mounted facing right and left on the spectrometer table. This effect is described and explained in the paper by W. H. Bragg referred to above (Phil. Mag. *loc. cit.* p. 888). When the crystal face is not parallel to the planes of the structure, the incident and reflected beams do not make equal angles with the face of the crystal. If the glancing-angle of incidence is less than that of reflexion, the rays suffer less absorption in the crystal than when the reverse is the case. The smaller the angle of incidence, the greater is this effect.

As an example of this effect, the following table gives a comparison of the 1st-, 2nd-, and 3rd-order reflexions from a natural face $(1\bar{1}0)$ of a ruby crystal (Al_2O_3). The crystal had the form of a six-sided prism bounded by the faces $(1\bar{1}0)$, but had so developed that this prism tapered towards one end. The angle between the face of the prism and the crystal planes could be measured by comparing the angles at which reflexion took place on the right-hand and left-hand sides. In this case it was 1° 49′. It will be observed what a large effect is produced by the small deviation from truth in the orientation of the face.

Comparison of $(1\bar{1}0)$, $(2\bar{2}0)$, $(3\bar{3}0)$ Al_2O_3.

Face	$1\bar{1}0$	$2\bar{2}0$	$3\bar{3}0$
Right-hand side	100	61·0	41·5
Left-hand side	52	38·9	39·5

The corresponding figures for the face (100) of NaCl are given below :—

	(100).	(200).	(300).
Right-hand side	100	21·3	5·08 *
Left-hand side	116·6	21·8	

The error in the orientation of the (100) face was in this case too small to be measured with accuracy; it was less than 30 minutes of arc.

Since the effect of inaccurate grinding of the face is so much less for the second order than for the first order, the intensity of reflexion from (200) NaCl was taken as standard, and all other intensities compared with it. On account of the difficulty of grinding the face accurately, there was generally a difference in the intensities on the two sides. It can be proved, however, that the mean of the intensities on the two sides can be taken as the right value without making an appreciable error so long as the difference in the value does not exceed about 25 per cent. The intensities were in all cases measured on both sides and the mean taken. In most cases the difference between them was small; for the higher orders it did not exceed 5 per cent.

8. It is necessary to use faces which have been ground. The strength of the reflexion is very different, especially for small glancing-angles, when a cleavage surface is compared with one which has been ground. As an example, a comparison is given below of the reflexion from a very perfect cleavage face of a rock-salt crystal (A) with a similar cleavage face on a crystal (D), which was afterwards ground until a layer 1 millimetre thick had been removed.

The intensity of reflexion from crystal (D) was measured with two orientations of the face. In the first, the crystal was set so that the edge, on which the knife was pressed in cleaving the crystal, was horizontal. In the second position, the edge was vertical and therefore at right angles to the horizontal beam of X-rays.

Face.	Intensity.
A (100). (Cleavage face)	50·8
D (100) before grinding :—	
First position	25·4
Second position	12·9
D (100) after grinding	100

Hence D (100), after grinding, reflected eight times as well as in the second position before grinding.

* The difference between right-hand and left-hand sides was less than the error of determination.

Comparison of 1st, 2nd, and 3rd orders :—

	(100).	(200).	(300).
Crystal D (after grinding)	100	18·4	4·1
Crystal A (cleavage face)	50·8	18·1	4·7

These comparisons were made for reflexion on the right-hand side only, and must be regarded as approximate.

In this case, as in the case of the error due to inaccurate grinding, the effect is much greater for the first-order reflexion. On account of this effect, intensity measurements of the (100) reflexion are doubtful, and this provides an additional reason for using the reflexion (200) as standard.

The difference in intensity for the two positions of the D cleavage face indicates that the reason for the imperfect reflexion may be due to a distortion of a freshly-cleaved surface. A cleaved surface has a rippled appearance, the ripples being parallel to the line on which the knife-edge was pressed in cleaving the crystal. The range of angles, however, over which the crystal reflects is no greater for a cleavage surface than for a ground surface, so that it would not appear that the imperfect reflexion is due to small variations in orientation of the face. Measurements made at various stages in the grinding down of a crystal face indicate that the effect is deep seated, and a depth of a millimetre at least was removed from the face D (100) before it was used in obtaining the results given in this paper.

9. Certain precautions must be taken in order to ensure an accurate result.

The crystal must be swept through a range of angles sufficiently great to ensure that all the facets add their share to the total effect. A range of three degrees is generally used. To check whether this was sufficient, a larger range was used in certain cases. This increased the amount of general radiation, but when this was subtracted, the intensity due to the homogeneous rays was found to be the same as when the smaller range was used.

The crystal planes must be parallel to the axis of the instrument. The crystal is mounted on a table which can be rocked about a horizontal axis parallel to the faces, and the intensity of a high-order spectrum is measured for various tilts of the face. The crystal is fixed at that setting which gives a maximum effect. Unless this precaution is taken, the beam on reflexion may be thrown upwards or downwards, and not be completely received by the ionization-chamber.

A simple calculation shows what the width of the slit of the ionization-chamber must be in order that all the homogeneous rays may enter it. It is advisable to limit this as much as possible, as the amount of general radiation is directly proportional to its width. The breadth of the homogeneous beam may be found by a survey with a very fine chamber-slit set at a series of angles. For the higher orders, the chamber-slit must be wider in order to include the two components of the K_α doublet.

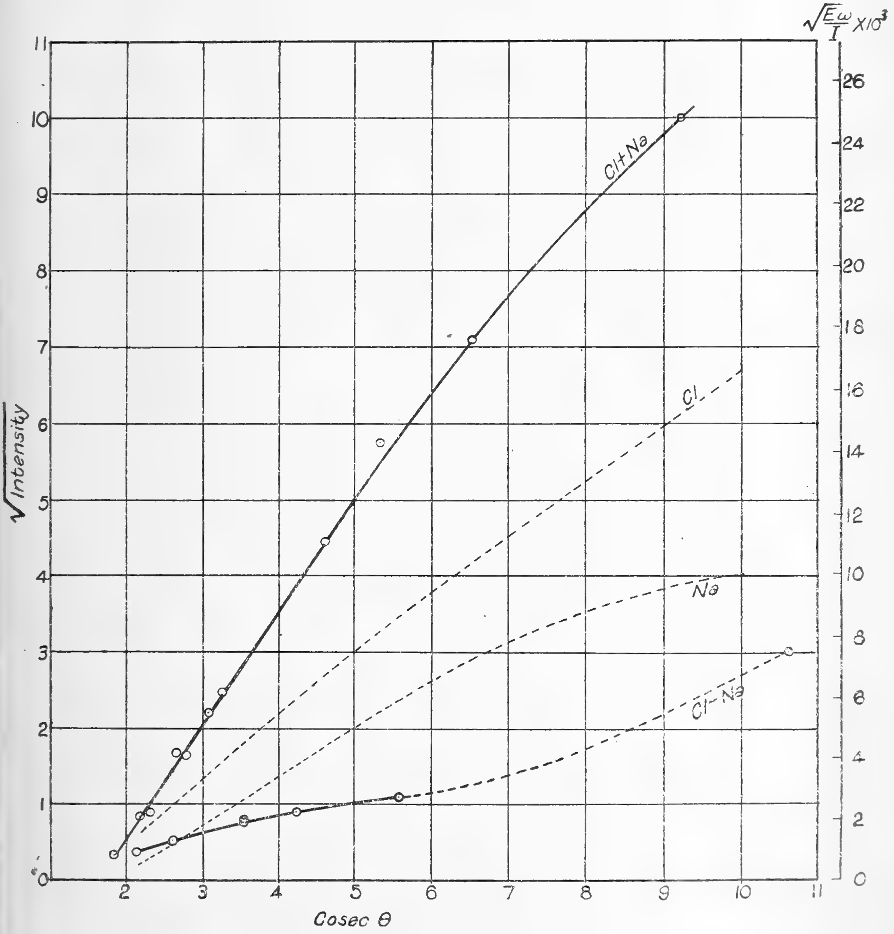
10. The comparisons which have been made are tabulated below, and the results are plotted in the form of a graph in fig. 2.

Plane.	Intensity.	$\sqrt{\text{Intensity.}}$	Cosec θ.	$\frac{E\omega}{I}\times 10^6$.	$\sqrt{\frac{E\omega}{I}}\times 10^3$.
100	100	10	9·21	612	24·80
200	19·90	4·46	4·60	122	11·05
300	4·87	2·21	3·07	29·8	5·49
400	0·79	0·89	2·30	4·85	2·20
500	0·116	0·34	1·84	0·71	0·84
110	50·4	7·10	6·50	310	17·60
220	6·10	2·47	3·25	37·3	6·12
330	0·71	0·84	2·17	4·35	2·08
111	9·00	3·00	10·62	55·1	7·45
222	33·1	5·75	5·31	202	14·25
333	0·58	0·76	3·54	3·55	1·89
444	2·82	1·68	2·65	17·2	4·16
555	0·137	0·37	2·12	0·84	0·92
311	1·17	1·09	5·56	7·22	2·70
622	2·69	1·64	2·78	16·40	4·06
331	0·81	0·90	4·23	4·95	2·23
511	0·61	0·78	3·54	3·74	1·93
711	0·302	0·55	2·58	1·87	1·37

The figures in the column headed "Intensity" were initially expressed in terms of the (100) reflexion, which was put equal to 100. Since it was discovered later that the (200) reflexion was a more reliable standard of intensity, all the other intensities have been determined relatively to it. In order to facilitate comparison with figures given by other authors, its value has been fixed at 19·90, since the ratio 100 : 19·90 was the most reliable value for the ratio

(100) : (200). The values for (100) and (111) are, however, difficult to measure accurately, since the nature of the crystal face has so great an effect on the intensity, and the figures given here must be regarded as approximate.

Fig. 2.



For the other faces, it is believed that the average error does not exceed 2 per cent., except in the case of the very small intensities of high order. The smallest intensity measured is that denoted by (500), and is little more than one thousandth of the (100) reflexion.

The figures obtained by W. H. Bragg are given below for the sake of comparison :—

Plane.	Intensity.
100	100
200	18·7
300	6·25
110	41·0
220	7·05
222	24·4
444	4·20

11. In fig. 2 the square-root of the relative intensity has been plotted against the cosecant of the glancing-angle. By plotting the intensities in this way, the approximate relation found by W. H. Bragg—that the intensity varies inversely as $\sin^2 \theta$—is made evident.

All the points lie on two smooth curves, showing that they form two groups within each of which the intensity is a function of the glancing-angle alone. For instance, the reflexions from the faces (511) and (333) occur at the same angle, and the corresponding intensities 0·74 and 0·72 are identical within the error of observation.

The points which lie on the lower curve are those for which all the indices are odd—the faces (111), (311), (331), (333), (511), (711), (555). These reflexions are from planes which contain alternately sodium and chlorine atoms. The wave-train reflected from the planes containing sodium atoms is 180° out of phase with that reflected by the plane containing chlorine atoms. The other reflexions are either from planes which contain both sodium and chlorine atoms and are identical in their nature, or are reflexions of an even order from planes containing sodium and chlorine atoms alternately. In both cases the sodium and chlorine atoms reflect wave-trains which are in phase with each other.

Since the square-root of the intensity has been plotted, this may be taken as being proportional to the amplitude of the reflected wave-trains. The upper curve, therefore, represents the sum of the amplitudes due to sodium and chlorine atoms, the lower, the difference of these two amplitudes.

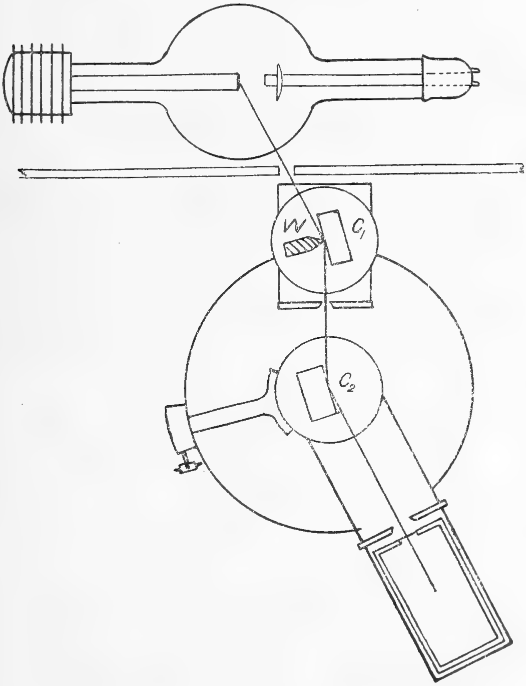
Comparison of Incident and Reflected Beams.

12. The rays from the bulb consist of heterogeneous radiation of all wave-lengths over a certain range, superimposed on the homogeneous radiation whose intensity of

reflexion has been measured. In order to compare directly the energy in the incident and reflected homogeneous beams, it is necessary to obtain a homogeneous beam by reflexion from a crystal face, and observe the total amount of radiation reflected by a second crystal, turning with constant angular velocity, on which this homogeneous beam is allowed to fall.

Fig. 3 shows the arrangement of the apparatus to effect this. The rays from the anticathode were reflected by the crystal C_1 so as to pass after reflexion through the collimator slits of the spectrometer. The incident beam was not limited by slits, but by being forced to pass on reflexion around the lead wedge W [*], the edge of which was pressed against the crystal face. The position of the bulb and the orientation of the crystal face were adjusted until the reflected beam passed truly through the axis of the spectrometer. This beam fell on the second crystal C_2, which was rotated with uniform velocity ω, and the total amount of radiation

Fig. 3.



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[*] *Cp.* Seeman, *Phys. Zeit.* xv. p. 795 (1914).

reflected was measured in the ordinary way. In this case, since the rays are homogeneous, no allowance for general radiation need be made.

The amount of energy reflected is proportional to the intensity of the incident beam, and inversely proportional to the angular velocity of rotation. The quantity $\frac{E\omega}{I}$ is therefore a constant characteristic of any one face and order, where

E = Total amount of energy reflected when the crystal is rotated with angular velocity ω radians per second.
I = Total amount of energy passing into ionization-chamber when the incident beam enters it for one second.

This constant $\frac{E\omega}{I}$ will be defined as the "Reflecting Power" of the crystal face for the wave-length λ.

The chamber was placed so as to receive the whole of the incident beam (*i.e.* that reflected from C_1), and the effect measured when the rays entered the chamber for a known time. It was then turned so as to receive the beam reflected from C_2 when the second crystal was mounted on the spectrometer table and turned with a known angular velocity.

This was done for the face (100) of NaCl. The reflexions from other faces are so much weaker that it was not convenient to compare them directly with the incident beam. As a check, an absolute measurement was made of the reflecting power for (222), which was found to be in agreement with that calculated by a comparison with (100).

A series of measurements gave for the constant

$$\frac{E\omega}{I} = 0\cdot000612 \text{ for NaCl (100)*.}$$

Since the reflecting powers of the other faces have been determined in terms of that from the face (100), their absolute reflecting powers may now be calculated. These values are given in the fifth column of Table I.

[The reflecting power of a face can only be defined satisfactorily in this way. It may be of interest, however, to give approximately the proportion of homogeneous radiation reflected when the crystal face is set at the most favourable

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* More recent determinations of this value have shown that the figure 0·000612 is too high. The value varies somewhat with the nature of the crystal face, and a better mean value is ·00055.

angle, although this depends on the state of perfection of the face as explained above. A direct comparison showed that when a narrow beam of X-rays falls on the face (100) set so as to reflect it, the intensity of the reflected beam is about one twenty-fifth of the incident beam.]

Theoretical Formula for the Intensity of Reflexion.

13. Formulæ for the intensity of reflexion have been deduced by Darwin and Compton (*loc. cit.*). The formula given by Compton is directly applicable, for he calculates the total amount of energy reflected when the crystal is turned at a uniform rate through the reflecting-angle. That given by Darwin may be extended to this case, and is in agreement with Compton's formula.

These formulæ are based on the amount of radiation scattered by a free electron when set in oscillation by rays of given intensity. It has been shown by J. J. Thomson * that the amount of energy S radiated per second by a single electron is given by

$$\frac{S}{P}=\frac{8\pi}{3}\frac{e^4}{m^2c^4},$$

where P is the energy of the incident radiation falling on 1 sq. cm. per second, e and m are the charge and mass of the electron respectively, and c is the velocity of light. This expression is confirmed by Barkla's work † on the total amount of radiation scattered by elements of low atomic weight, from which he deduced that the number of electrons in the atoms of these elements is approximately equal to one-half the atomic weight. If the incident radiation is plane polarized, the relation between the amplitude of the electric vector of the incident radiation, and that of the radiation scattered in any direction perpendicular to the direction of the electric vector, is given by

$$\frac{A'}{A}=\frac{e^2}{mc^2}\cdot\frac{1}{R},$$

where R is the distance from the electron. To simplify matters, we will consider that the radiation reflected from the crystal face is plane polarized in such a manner that the electric vector is perpendicular to the plane of incidence, and allow for the "polarization factor" at a later stage of the calculation.

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* J. J. Thomson, 'Conduction of Electricity through Gases,' p. 321.

† C. G. Barkla, Phil. Mag. vii. p. 543 (1904), and xxi. p. 648 (1911).

The following is a brief summary of the calculation, treated in a slightly different manner from that in Darwin's and Compton's papers, where it is worked out more completely.

If an atom contains Z electrons, and the waves scattered by these electrons are in phase, the amplitude of the scattered wave will be

$$\mathrm{A}' = \frac{\mathrm{A}}{\mathrm{R}} \cdot \mathrm{Z} \frac{e^2}{mc^2} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

If the spatial distribution of the electrons is such that the scattered waves are not in phase, the factor Z must be replaced by a function F, which depends on the angle of scattering and the positions of the electrons. F tends to its maximum value Z at small angles of scattering.

Let rays from a source S fall at a glancing-angle θ on a plane containing n atoms per unit area, and be reflected. The amplitude at any point P is equal to one-half the total effect due to the scattering by the atoms lying in the first Fresnel zone around the corresponding point of incidence I. The area of the zone is equal to

$$\frac{\pi\lambda}{\sin\theta} \cdot \frac{r_1 r_2}{r_1 + r_2},$$

where $r_1 = \mathrm{SI}$, $r_2 = \mathrm{IP}$. The number of atoms it contains is therefore

$$\frac{n\pi\lambda}{\sin\theta} \cdot \frac{r_1 r_2}{r_1 + r_2},$$

and the amplitude at P is equal to

$$\frac{1}{2} \cdot \frac{2}{\pi} \frac{n\pi\lambda}{\sin\theta} \cdot \frac{r_1 r_2}{r_1 + r_2} \cdot \mathrm{A}' = \frac{n\lambda}{\sin\theta} \cdot \frac{r_1 r_2}{r_1 + r_2} \cdot \frac{\mathrm{A}}{r_2} \cdot \mathrm{F} \cdot \frac{e^2}{mc^2}.$$

If r_1 is great compared with r_2, so that the incident rays may be considered as a parallel beam, we get the relation

$$\frac{\text{Amplitude of reflected beam}}{\text{Amplitude of incident beam}} = \frac{\mathrm{D}'}{\mathrm{D}} = \frac{n\lambda}{\sin\theta} \cdot \mathrm{F} \cdot \frac{e^2}{mc^2}. \quad (2)$$

Considering now a thin slip of crystal consisting of p planes at a distance d apart, the reflexion will be most intense when

$$m\lambda = 2d \sin\theta.$$

At a glancing-angle $(\theta+\epsilon)$ the amplitude of the reflected beam will be

$$D'p\frac{\sin\left(p\,.\,\pi\,.\,\epsilon\,.\,\frac{2d\cos\theta}{\lambda}\right)}{p\,.\,\pi\,.\,\epsilon\,.\,\frac{2d\cos\theta}{\lambda}}.$$

If we put $\phi=p\,.\,\pi\,.\,\epsilon\,.\,\frac{2d\cos\theta}{\lambda}$,

Amplitude $= D'p\,.\,\frac{\sin\phi}{\phi}$,

and the energy of the reflected beam is proportional to

$$(D')^2p^2\frac{\sin^2\phi}{\phi^2}.$$

If, now, the crystal is rotated with constant angular velocity ω, the total amount of radiation reflected is proportional to

$$(D')^2\,.\,p^2\,.\int_{-\infty}^{+\infty}\frac{\sin^2\phi}{\phi^2}\,dt$$

$$=(D')^2p^2\int_{-\infty}^{+\infty}\frac{\sin^2\phi}{\phi^2}\frac{d\epsilon}{\omega}$$

$$=(D')^2p^2\int_{-\infty}^{+\infty}\frac{\sin^2\phi}{\phi^2}\frac{\lambda}{2p\pi d\cos\theta}\cdot\frac{d\phi}{\omega}.$$

Since $\int_{-\infty}^{+\infty}\frac{\sin^2\phi}{\phi^2}\,d\phi=\pi$,

this becomes

$$\frac{(D')^2p\,.\,\lambda}{2d\cos\theta\,.\,\omega}=D^2\,.\,\frac{n^2\lambda^2}{\sin^2\theta}\cdot F^2\frac{e^4}{m^2c^4}\cdot\frac{p\lambda}{2\omega d\cos\theta}.$$

If N = number of atoms per unit volume,

t = thickness of crystal slip,

we have $n=N\,.\,d$, $t=pd$.

The energy in both the reflected and incident beams is proportional to the square of the amplitude of the electric vector. If the energy of the incident beam falling on the crystal per second is I, and it is all intercepted by the slip of crystal, the total energy reflected will be given by

$$\frac{\mathrm{E}}{\mathrm{I}}=\frac{n^2\lambda^2}{\sin^2\theta}\cdot\mathrm{F}^2\frac{e^4}{m^2c^4}\cdot\frac{p\lambda}{2\omega d\cos\theta},$$

or

$$\frac{\mathrm{E}\omega}{\mathrm{I}}=\frac{\mathrm{N}^2\lambda^3t}{2\sin^2\theta\cos\theta}\cdot\mathrm{F}^2\frac{e^4}{m^2c^4}\quad.\quad.\quad.\quad.\quad.\quad.\quad(3)$$

In this calculation it has been assumed that the absorption of the radiation is inappreciable.

As a corollary, we can calculate the reflecting power of a homogeneous fragment of crystal of volume V. The volume of the slip irradiated by a narrow pencil of rays is equal to $\frac{\mathrm{S}}{\sin\theta}\cdot t$, where S is the area of cross-section of the pencil. From the above formula

$$\frac{\mathrm{E}\omega}{\mathrm{I}}=\frac{\mathrm{N}^2\lambda^3}{2\sin\theta\cos\theta}\cdot\mathrm{F}^2\cdot\frac{e^4}{m^2c^4}\cdot\frac{t}{\sin\theta}.$$

Now, $\mathrm{I}=\mathrm{SI}_0$ when I_0 is equal to the intensity of the beam irradiating the crystal, defined as the amount of energy falling on one square centimetre per second, whence

$$\frac{\mathrm{E}\omega}{\mathrm{I}_0}=\frac{\mathrm{N}^2\lambda^3}{\sin 2\theta}\cdot\mathrm{F}^2\frac{e^4}{m^2c^4}\cdot\mathrm{V}.\quad.\quad.\quad.\quad.\quad.\quad(4)$$

This result shows that the "Reflecting power" of a homogeneous fragment of the crystal is proportional to its volume, if the fragment be so small that absorption in it is inappreciable.

We will now assume that the crystal consists of a number of such homogeneous crystalline particles, set approximately parallel to each other, but not exactly so. When the rays are reflected from the face of a crystal, the reflexion by particles below the surface is diminished by absorption. It will be assumed that the linear coefficient of absorption μ is a constant. (This assumption will be discussed more fully below.)

Rays reflected by a particle at a depth z beneath the crystal surface suffer absorption by passing through a distance $\frac{2z}{\sin\theta}$ of the crystal. They are therefore reduced in intensity in the ratio

$$1:e^{-\frac{2\mu z}{\sin\theta}}.$$

By equation (3) the reflecting power of a thickness t of the crystal is given by the formula

$$\frac{E\omega}{I}=\frac{N^2\lambda^3}{2\sin^2\theta\cos\theta}\cdot F^2\,\frac{e^4}{m^2c^4}\cdot t.$$

The total reflecting power of the crystal face is therefore equal to

$$\frac{N^2\lambda^3}{2\sin^2\theta\cos\theta}\cdot F^2\,\frac{e^4}{m^2c^4}\int_0^\infty e^{-\frac{2\mu z}{\sin\theta}}\,dz$$

$$=\frac{N^2\lambda^3}{2\sin^2\theta\cos\theta}\cdot F^2\,\frac{e^4}{m^2c^4}\cdot\frac{\sin\theta}{2\mu}$$

$$=\frac{N^2\lambda^3}{2\mu\sin 2\theta}\cdot F^2\,\frac{e^4}{m^2c^4}\cdot \qquad (5)$$

Compton gives his formula in the form

$$R=\frac{E_r}{E_i}=\frac{N^2\lambda^3\phi^2\psi^2}{2\mu\sin 2\theta}\cdot\frac{1}{\Delta\theta},$$

which agrees with this, since his factors $N^2\phi^2\psi^2$ have the same meaning as $N^2F^2\,\frac{e^4}{m^2c^4}$ defined as above. Compton derives the equation by a more complete mathematical treatment, and has discussed very fully the effect of imperfection of the crystal and of the length of the wave-train. He arrives at the same formula, whatever assumptions are made.

This expression must now be multiplied by a "polarization factor" $\frac{1+\cos^2 2\theta}{2}$ and a "Debye factor" $e^{-B\sin^2\theta}$. The complete expression for the reflecting power R is therefore

$$\frac{E\omega}{I}=R=\frac{N^2\lambda^3}{2\mu\sin 2\theta}\cdot F^2\,\frac{e^4}{m^2c^4}\cdot\frac{1+\cos^2 2\theta}{2}\cdot e^{-B\sin^2\theta}. \qquad (6)$$

The Debye Factor for Rock-salt.

14. Debye * gives the formula for the factor which expresses the diminution of the intensity of reflexion with rise of temperature in the form

$$e^{-B\sin^2\theta},$$

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* P. Debye, *Ann. der Phys.* (4) xliii. p. 49 (1914).

where the constant B is a function of the temperature, the wave-length λ, the atomic weight, and the characteristic temperature θ of the crystal.

W. H. Bragg [*] made a series of measurements of the effect of temperature in reducing the intensity of reflexion by rock-salt. Intensities were compared at 288° K and 643° K. The results were, within the errors of experiment, consistent with the ratios given by Debye's formula. The latter gives different values for B according to the assumption or otherwise of the existence of the "Nullpunktsenergie." Compton (*loc. cit.* p. 47) gives, as the two values for B in NaCl, 4·6 and 3·6 respectively.

The mean value for B at 288° C., calculated from W. H. Bragg's results, is equal to 4·12, and this will be assumed in the calculations which follow. To assume that the effect of temperature on both curves of fig. 2 is the same, is equivalent to supposing that the average amplitude of vibration of sodium and chlorine atoms is the same. This is very probably not the case, and the authors intend to measure the effect of cooling down the crystal in order to obtain an empirical law over a wider range expressing the temperature effect. However, the factor $e^{-B\sin^2\theta}$ does not affect very greatly any but the smallest intensities measured, and will therefore not make much difference to the conclusions to be drawn from the curves.

The Linear Coefficient of Absorption "μ."

15. The coefficient of absorption by rock-salt of the homogeneous radiation was measured in the usual way by interposing plates of rock-salt of various thickness in the path of the direct beam (fig. 3) and measuring the diminution in energy of the beam. Experiments were made with plates from 0·05 cm. to 0·15 cm. thick. The linear coefficient of absorption μ was found to be 10·7.

In the theoretical formula it has been assumed that μ is constant. Now, W. H. Bragg has shown that in the case of the diamond, when the crystal is set so as to reflect the radiation, the absorption-coefficient is abnormally large. It was therefore interesting to try whether such an effect is observable in the case of rock-salt. A slip of crystal 0·92 mm. in thickness with faces parallel to (100) was set on the spectrometer table at right angles to the homogeneous beam from C_1 in fig. 3 and the absorption measured. It was then turned through an angle of about 66° until the (100)

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[*] W. H. Bragg, Phil. Mag. *loc. cit.* p. 897.

planes at right angles to the crystal face reflected the radiation, the reflexion being observed in the usual way. On redetermining the absorption-coefficient it was found to have increased by about 15 per cent. This effect is discussed in Darwin's paper referred to above, and will reduce the intensity of the reflected beam. Its effect will be smaller for higher orders of reflexion, since the increase in absorption is due to multiple reflexion within the crystal interfering with the primary beam, and reflexion is so much weaker in the higher orders.

The effect will not be taken into account in the calculations, since it is not obvious what allowance should be made for it. It is to be remembered, however, that the reflexions from (100) and (110) must be diminished by the increase in the absorption-coefficient.

It may possibly be the case that the effect of grinding a cleavage face, which increases so greatly the intensity of reflexion, is due to the fact that grinding breaks the crystal up into a number of small homogeneous crystals oriented in slightly different directions, so that absorption at the reflecting angle plays a less important part in diminishing the intensity of reflexion.

The Comparison of the Theoretical and Observed Results.

16. The formula for the reflecting-power of a face states that

$$\frac{E\omega}{I}=\frac{N^2\lambda^3}{2\mu\sin 2\theta}\cdot F^2\,\frac{e^4}{m^2c^4}\cdot\frac{1+\cos^2 2\theta}{2}\cdot e^{-B\sin^2\theta}.$$

Since all the quantities have been measured except F, we can calculate the absolute value of F for a range of values of θ.

If the effect of the electrons in the chlorine atoms be represented by F_{Cl}, and of those in the sodium atom by F_{Na}, then for reflecting-powers corresponding to points on the upper curve of fig. 2 we have

$$F=F_{Cl}+F_{Na};$$

for those corresponding to points in the lower curve

$$F=F_{Cl}-F_{Na}.$$

From the formula

$$F=\sqrt{\frac{E\omega}{I}\,\frac{mc^2}{e^2}}\cdot\frac{2\mu^{\frac{1}{2}}}{N\lambda^{\frac{3}{2}}}\sqrt{\frac{\sin 2\theta}{1+\cos^2 2\theta}}\cdot e^{+\frac{B}{2}\sin^2\theta},$$

where N is the number of molecules of NaCl in unit volume of the crystal.

Taking

$$\frac{e}{m}=5\cdot30\times10^{17}, \qquad \mu=10\cdot7,$$

$$e=4\cdot77\times10^{-10}, \qquad \mathrm{N}=\tfrac{1}{2}\left(\frac{1}{2\cdot81\times10^{-8}}\right)^{3},$$

$$c=3\times10^{10}, \qquad \lambda=0\cdot615\times10^{-8},$$

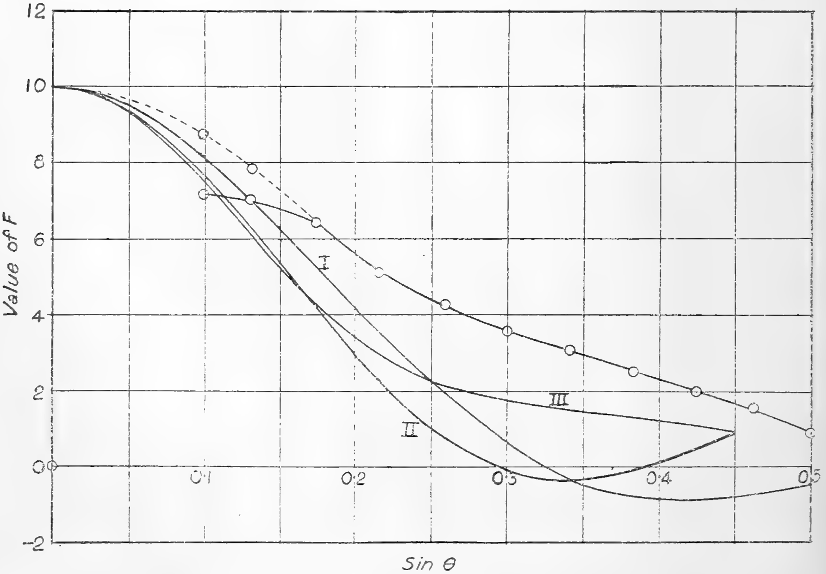
$$\mathrm{B}=4\cdot12,$$

this reduces to

$$\mathrm{F}=2143\sqrt{\frac{\mathrm{E}\omega}{\mathrm{I}}}\cdot\sqrt{\frac{\sin 2\theta}{1+\cos^2 2\theta}}\cdot e^{+2\cdot06\sin^2\theta}.$$

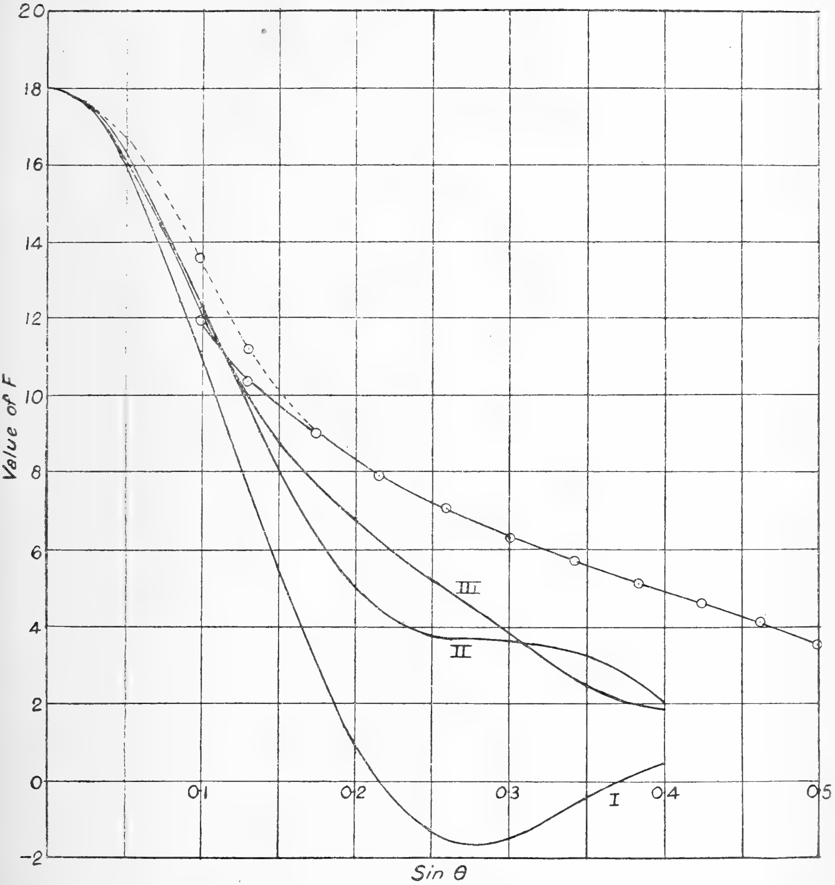
The dotted curves in fig. 2 represent one-half the sum of the ordinates, and one-half the difference between the ordinates, of the upper and lower curves. From these dotted curves the absolute value of F_{Cl} for chlorine, and F_{Na} for sodium, can be calculated directly. They are tabulated below, and the values are plotted against $\sin\theta$ in figs. 4 and 5.

Fig. 4.



Values of F for Sodium.
The small circles indicate the observed values.

Fig. 5.



Values of F for Chlorine.
The small circles indicate the observed values.

The values of F for Chlorine and Sodium are:—

Glancing-angle θ.	sin θ.	F_{Cl}.	F_{Na}.
° ′			
(5 44)	(0·100)	(11·67)	(6·90)
7 30	0·1305	10·11	6·88
10 0	0·1736	8·78	6·26
12 30	0·2164	7·72	4·98
15 0	0·2588	6·88	4·18
17 30	0·3007	6·14	3·47
20 0	0·3420	5·56	2·95
22 30	0·3827	5·00	2·41
25 0	0·4226	4·50	1·91
27 30	0·4617	4·01	1·49
30 0	0·5000	3·43	0·83

The angle between the scattered and incident beams is twice the glancing-angle θ.

It will be seen at once that the values of F are of the right order of magnitude. F should tend to a value 18 for chlorine, and 10 for sodium, as $\sin\theta$ approaches zero, assuming the atoms in the crystal to be ionized. The greatest value of F_{Cl} is 11·67, and of F_{Na} is 6·90, when $\sin\theta = 0·10$.

17. It now remains to take various models of the atom and see how the form of the function F calculated for these models agrees with that actually observed.

It is not intended here to lay much stress on the agreement between the calculated and observed forms of F for all values of θ. The object of the comparison is to demonstrate that any probable arrangement of electrons gives a close agreement between theory and experiment at small glancing-angles, and therefore to prove that the formula for the intensity of reflexion is very probably the true one.

The first atom model is one in which the electrons are supposed to be distributed uniformly throughout a sphere whose radius is $1·02 \times 10^{-8}$ cm. in the case of chlorine, $0·67 \times 10^{-8}$ cm. in the case of sodium.

In the second model the electrons are supposed to be arranged in a series of spherical shells. It is also assumed that, in considering the average effect of the atom, we may take the effect of the electrons in each shell to be equivalent to a uniform distribution of diffracting particles over the whole surface of the shell. The radii of these shells and the number of electrons in each are as follows :—

Chlorine.	No. of electrons.	Radius.	Sodium.	No. of electrons.	Radius.
1st shell ...	2	0·12	1st shell ...	2	0·40
2nd shell ...	8	0·41	2nd shell ...	8	0·67
3rd shell ...	8	1·02			

The diameters of the outer shells are those calculated by one of the authors * from crystal data.

In the third model the electrons are supposed to be arranged on shells of the same diameters as in the second model, but to be in oscillation about their mean positions along a line joining them to the centre of the atom with a total amplitude equal to their distance from the centre. This extreme case has been chosen to illustrate the effect of such an oscillation of the electrons on the form of the curve.

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* W. L. Bragg, Phil. Mag. xl. p. 169, August 1920.

The values of F for chlorine and sodium calculated for these three types of atom model are plotted against $\sin\theta$ in figs. 4 and 5, the curves corresponding to the first, second, and third models being numbered I., II., and III. The measured values of F are shown for comparison as a series of small circles.

Of the three models chosen, the third type is the only one which gives diffraction curves of the same general shape as those actually observed. Both of the other models yield curves which have maxima and minima. The actual values of F fall off more slowly than do any of the theoretical curves, indicating that the distances of the electrons from the centre of the atom have been taken to be greater than their true value, both for sodium and chlorine.

The curve for sodium becomes nearly horizontal at some distance from the vertical axis, and the curve for chlorine shows a similar tendency. This must be ascribed to the fact that the reflecting power for small glancing angles is diminished by the increase of the absorption-coefficient, and that our values for (100) and (110) are too small. We know that this effect must exist, since measurement has shown that the absorption-coefficient increases by 15 per cent. in the neighbourhood of the angle corresponding to the (100) reflexion, and the increase at the exact angle of reflexion may be far greater than this. It is difficult to allow for this effect. In fig. 4 the circles on the dotted curve have been plotted so as to give that curve a more probable form passing through a maximum at 10. The actual values for $\sin\theta=0{\cdot}1$ and $\sin\theta=0{\cdot}13$ lie well below this dotted curve. In fig. 5 the two greatest values of F have been increased by the same amounts as for the other curve. This increase is of the order to be expected from the variation which was found to exist in the coefficient of absorption μ, but its exact value is, of course, merely conjectural.

There is another striking feature of the curves for the observed values of F. The curve for chlorine approaches the axis more slowly than that of the sodium. This is just the reverse of what would be expected from the relative dimensions of the two atoms, since we would expect the electrons to be on the whole at a greater distance from the centre in chlorine than in sodium. Here, again, the third type of atom chosen as a model gives results which agree most closely with those actually observed.

Although it is necessary to check the form of the curves for F by measurements on other crystals before drawing any definite conclusions as to the arrangement of the electrons in the atom, the results so far obtained indicate that :—

(*a*) The formula for the amplitude of the wave scattered by each electron holds good, at any rate for small glancing-angles. The values for F when $\sin\theta = 0{\cdot}1$ are of the order to be expected if the curves for chlorine and sodium have maxima at about 18 and 10 respectively.

(*b*) A uniform distribution of the diffracting points throughout a sphere cannot explain the form of the curve, for the theoretical form of the F function for such a model, which would coincide with the actual curve at small glancing-angles, meets and crosses the horizontal axis, whereas the observed curve falls away far more gradually with increasing θ.

(*c*) The general form of the curve makes it probable that as θ increases, the outer electrons for some reason become less and less effective in diffracting the X-rays. The result of assuming them to be in vibration is that the corresponding theoretical curve for F falls rapidly at first, and then very slowly, with increasing θ. It would seem necessary to make some such assumption, in order to obtain a theoretical curve approximating in form to that actually observed.

The electrons have been taken to be in radial vibration in the model, but this has only been done in order to obtain a theoretical expression in which the effect of the outer electrons falls away with increasing θ. It may be due to the electrons having a form such as that of the ring-electron assumed by Coster [*] in discussing diffraction by rings of connecting electrons in diamond.

Unless we suppose that the outer electrons become less effective with increasing θ, we must conclude, in order to explain the very gradual falling away of both curves, that the electrons are within a sphere of diameter smaller than is probable. By analogous reasoning, Debye (*Phys. Zeitschr. loc. cit.* p. 10) comes to the conclusion that in diamond the electrons are within a sphere of $0{\cdot}43 \times 10^{-8}$ cm. radius. This must not be excluded as impossible, or even very improbable ; but if this is assumed, the difficulty remains of explaining why the curve for sodium approaches the horizontal axis more rapidly than that for chlorine, as if the former atom were the larger.

A uniform distribution of electrons throughout the volume of the atom will not account for the observed curves. A

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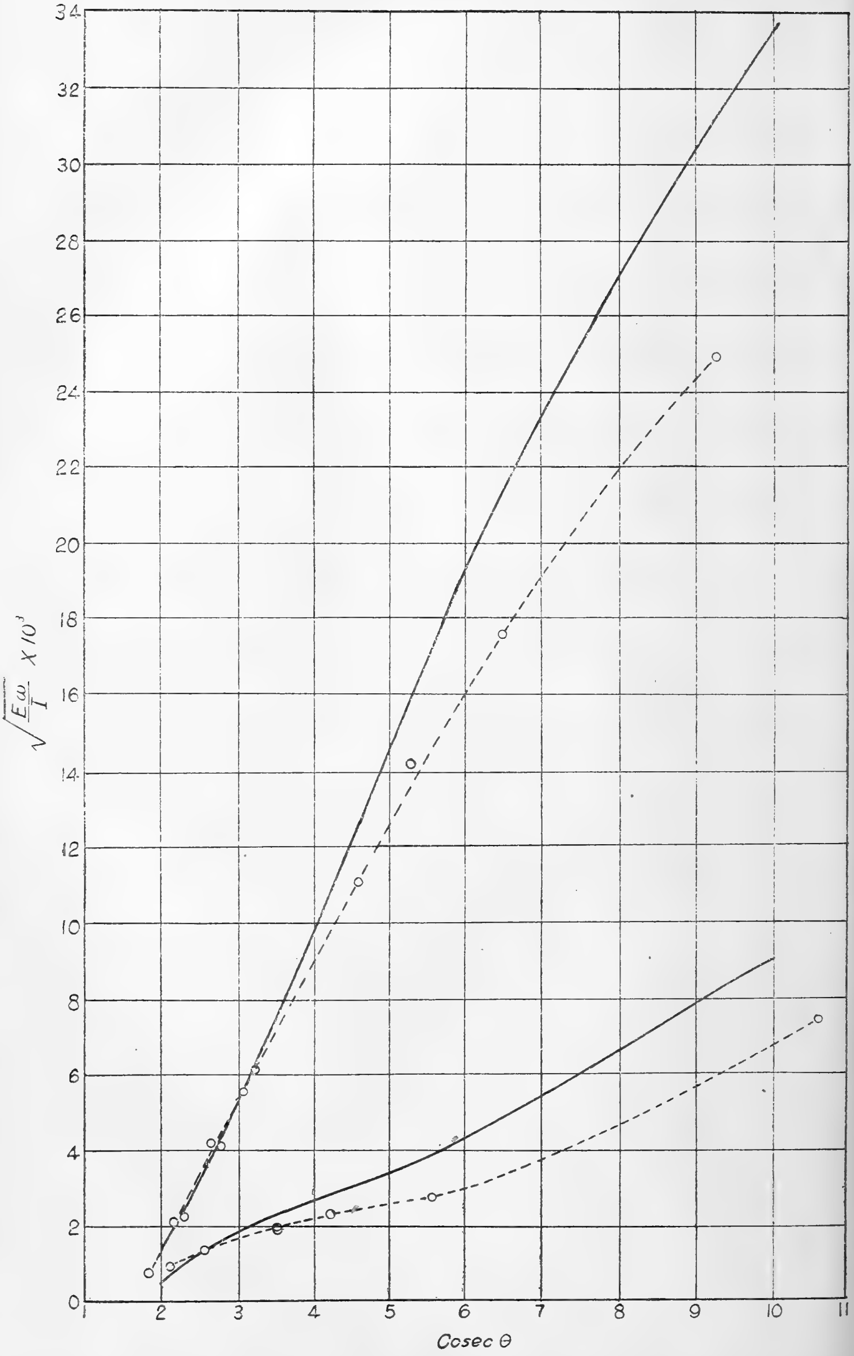
[*] D. Coster, Proc. Roy. Acad. Sci. Amsterdam, xxi. No. 6, Oct. 1919.

model must be taken which has a greater concentration of electrons near the centre than that in the case of a uniform distribution.

An important distinction must be made between the diffraction of X-rays by a crystal and the scattering of X-rays by an amorphous mass of material. In the formula for the intensity of reflexion, the quantity $F\frac{e^2}{mc^2}$ represents the amplitude of a polarized wave diffracted by a single atom in various directions. This amplitude must not be supposed to be necessarily the same as that which determines the amount of radiation scattered in various directions by an amorphous mass of the same atoms in a random arrangement. It is justifiable to consider the scattering by an amorphous substance as the summation of the intensities due to the separate atoms. If the electrons are in vibration, as has been supposed in the third atom model, their movements will be slow as compared with the frequency of the X-radiation. In the case of a single atom which is scattering the radiation, the arrangement of electrons in the atom at any one moment may be a random one, and the displacements may be so large and arbitrary that we may simply consider the scattering as due to a random arrangement of Z electrons, Z being the atomic number. If the amplitude of the scattered wave is $F'\frac{e^2}{mc^2}$, F' may be nearly equal to $\sqrt{Z}$ for all angles of scattering, except for very small angles where all the electrons are in phase and "excess scattering" comes into play.

The factor F' will not be the same as the factor F in the case of the atoms of a crystalline substance. When examining the reflexion from a crystal, we have a large number of atoms diffracting waves which are exactly in phase with each other. $F\frac{e^2}{mc^2}$ is now the amplitude scattered by what we may term the "statistical" atom. In this case the movements of the electrons are allowed for by supposing that diffraction takes place, not at single electron points displaced from their mean positions, but from all over a certain region for each electron in which all its possible positions lie, due weight being given to each element of the region. This has been done in calculating F for the third atom model. This region may be so large that the effect of the outer electrons is practically zero for the higher orders, and this illustrates the essential difference between the two

Fig. 6.



Circles indicate observed values.

cases. F may become zero for certain values of θ. F′, depending on a random arrangement of electrons in each individual atom, will have no zero values.

18. The model of the atom which agrees most closely with the observed values of F for the large values of θ would be one of type 3, in which, however, all the figures given above for the distances of the electrons from the centre of the atom must be reduced in the ratio 3:2. Assuming this type of atom, it is possible to calculate from formula (6) the reflecting powers for any plane of the rock-salt crystal. The figures so calculated are shown by the continuous curves in fig. 6, and can be compared with those actually obtained (represented by points on the dotted curve). It is to be emphasized again that the comparison is not relative, but an absolute comparison of the reflecting powers actually observed and those calculated from the formula.

Summary.

The absolute values of the reflecting power for different faces of rock-salt have been measured. The reflecting power has been determined for eighteen glancing-angles over a range between 5° 30′ and 30° 0′.

The values obtained have been compared with those calculated from the theoretical formulæ for reflexion deduced by Darwin and Compton, and it has been shown that they afford strong confirmation of the accuracy of these formulæ.

The greatest care has been taken to make the measurements as accurate as possible, in order that they may serve as a basis for an analysis of the arrangement of the electrons in the atom. Possible arrangements are discussed.

In order to confirm the results, the effect of temperature on the intensity of reflexion must be more fully determined. The authors intend to make a series of determinations at liquid-air temperature, in order to be able to extrapolate to the values at absolute zero. It is hoped to extend the measurements over a wider range of angles at low temperatures.

It is further intended to repeat the experiments with KCl in order to check the formula in this case. Sylvine affords a simpler case for investigation than rock-salt, since the ions of potassium and chlorine will, in all probability, have a very similar structure.

The authors wish to acknowledge very gratefully the kind assistance given them by Dr. W. D. Coolidge, of the General Electric Company, Schenectady, to whom they are indebted for the gift of the Coolidge tube with which the investigations were carried out.

XXX. *On the Colours of Mixed Plates.*—Part I. *By* C. V. Raman, *M.A., Palit Professor of Physics, and* Bhabonath Banerji, *M.Sc., Assistant Professor, University of Calcutta*[*].

[Plates IV. & V.]

1. Introduction.

THE colours exhibited by a mixed plate or film consisting of two interspersed transparent media were investigated by Thomas Young[†], and were ascribed by him to the interference of the coherent streams of light passing through the media and emerging from the film having suffered different retardations. Later, it was pointed out by Brewster[‡] that the colours were due to laminar diffraction; but his treatment was not very complete, and apparently did not convince later writers such as Verdet and Mascart, who dealt with this case in their treatises as one of simple interference, and ignored the part played by diffraction. More recently Profs. Charles Fabry[§] and R. W. Wood[||] have attempted to give an explanation of the colours of mixed plates from the standpoint of elementary diffraction theory. On investigating the subject, however, we have found that there are several features in the observed phenomena which the treatments proposed fail to explain. For instance, when a mixed plate consisting of a uniform film of liquid enclosing a large number of air-bubbles of widely varying radii is held in front of the eye, and a distant source of white light is viewed through it, we should expect, according to the theory given by Wood[¶], that the halo seen surrounding the source should be throughout of a more or less uniform colour complementary to the tint of the regularly transmitted light, and fluctuate as a whole when the thickness of the film is varied. As will be seen in the following section, this is very far indeed from being in agreement with what is actually observed. Further, the experimental examination of the subject which we have carried out has brought to light a number of interesting features which appear hitherto to have been overlooked. It is proposed, in Part I. of this paper, to give a general description of the observed effects.

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[*] Communicated by the Authors.

[†] Phil. Trans. Roy. Soc. 1802, p. 390, and 'Elements of Natural Philosophy,' vols. i. & ii.

[‡] Phil. Trans. Roy. Soc. 1838, p. 73.

[§] *Journal de Physique*, viii. p. 595 (1899).

[||] Phil. Mag. April 1904, and 'Physical Optics,' 2nd Edition, p. 252.

[¶] *Loc. cit.*

The subsequent instalments of the paper will contain a description of further observations and experiments on the subject and the detailed discussion of the theory of the phenomena.

2. THE DIFFRACTION-HALOES DUE TO A MIXED PLATE.

The most instructive case is that referred to above—namely, that of a mixed plate which is held *normally* in front of the eye, a distant point-source of light being observed through it. The plate should have a thickness as uniform as possible, and to minimise the effects of slight unavoidable variations of thickness it should be brought up very close to the eye. In order to observe the phenomenon at its best, it is useful to work with a fairly powerful source of illumination, which should be completely enclosed except for a small aperture through which the light issues and falls upon the plate held at a sufficient distance from the source. The observations should be made in a darkened room so that the fainter extensions of the diffraction-halo surrounding the source can be easily seen. A tungsten-filament lamp of, say, 400 c.p. is a suitable source for observations in white light. For observations in monochromatic light a quartz-mercury vapour lamp with green ray filter is the most convenient source to use, though good results may also be obtained with a monochromator illuminated by the electric arc, or with a bead of salt held on a platinum wire in the hottest part of the flame of a Meker burner.

A mixed plate of uniform thickness may easily be obtained by spreading a few drops of saliva or of egg-albumen between two plates of glass 5 in. × $2\frac{1}{2}$ in. in size and $\frac{1}{4}$ in. thick, and working up the material into a film of uniform consistency by circular sliding movements of the plates over each other. Examined under the microscope, a film of this description shows a thin layer of liquid enclosing a large number of air-bubbles widely varying in size, irregularly arranged and of shape often departing considerably from circularity (see fig. 1 in Plate IV.), but showing no bias towards elongation in any particular direction[*]. The air-bubbles can be distinguished from the liquid by the slightly diminished intensity of the transmitted light, and also by the presence of very minute drops of liquid on the surface of the plates enclosed within the bubbles. (These can be seen in the micro-photograph,

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[*] Cases in which the bubbles have distorted forms elongated specially in one direction will be considered below in a separate section.

fig. 1 in Plate IV.) The laminar boundaries appear as black lines in transmitted light under the microscope.

Using the method of observation described above and a *freshly prepared* plate, some remarkable effects are observed which will now be described.

Observations in Monochromatic Light.

Surrounding the source is seen a brilliant halo, which, if the source is sufficiently small and distant, shows especially near the centre a finely mottled or granular structure. The halo consists of a succession of dark and bright circular rings of which the number and position depend on the thickness of the film. With thick films fifteen or twenty rings may easily be obtained, with thin films a proportionately smaller number. The most remarkable features of the ring-system are: *firstly*, that the successive rings are closer together near the centre of the halo and wider apart towards the margin of the halo, where they are very broad and faint; *secondly*, the dark rings in the halo are all more or less perfectly black, *except the two or three rings nearest the centre of the halo* which are not so dark as the rest, and fluctuate in sharpness and intensity with their exact position in the halo. Figs. 2 and 3 in Plate IV. show the ring-system above described for a moderately thin film. The photographs of the halo here reproduced were secured by using a short-focus wide-angle lens (Zeiss-Tessar F/3·4 lens of focal length 5 cm.), the mixed plate being placed as close to the lens as possible. Fig. 2 is a light print showing the faint and broad outer rings, and fig. 3 a deep print showing more clearly the part of the halo near the centre.

When the mixed plate is moved in its own plane so as to bring a thinner part of the film in front of the eye, the rings seen in the diffraction halo move *inwards,* closing up at the centre so that the number visible in the field decreases and the rings appear wider apart. As each of the dark rings in the inner part of the halo contracts and moves inwards, it undergoes a periodic fluctuation of intensity, becoming alternately broad and diffuse, and then sharp and black, just as it is about to close up at the centre. Further away from the centre, however, the rings merely contract and move inwards without any noticeable change in their appearance. In fact, what is seen distinctly suggests that near the centre of the halo there is a second and fainter ring-system superposed on the first, with the result that in this region the dark rings vary in sharpness and intensity according as, at a given point, the two sets of rings are in or out of step. A similar closing in of the rings is observed if an electric arc with monochromator is used as the source and the wave-length of the light incident on the plate is gradually increased.

Observations in White Light.

In this case the halo has a fibrous radial structure, and shows a series of coloured rings surrounding the source. The *outermost* ring in the halo is very broad and practically *colourless* or *achromatic*. The first few rings subsequent to this achromatic ring within the halo are strongly coloured even if a thick film be used, the succession of colours being not very dissimilar to that observed in passing from the achromatic centre to the coloured bands in the diffraction pattern of the Fraunhofer class due to a rectilinear slit. The colours near the centre of the halo are very weak and impure, unless the film is so thin that the total number of coloured rings in the halo is rather small. In the latter case the halo shows vivid colours, even near the centre.

In all cases the source of light as seen through the mixed plates appears enfeebled, as is evidently to be expected in view of the fact that part of the incident energy appears in the scattered light forming the halo. When the observations are made in monochromatic light, and the plate is moved in front of the eye so as to pass from a thick to a thinner part of the film, the intensity of the source suffers periodic fluctuations, being greatest when the innermost dark ring in the halo is just about to close in at the centre and least when one of the bright rings just surrounds the source. When similar observations are made in white light, the source appears to fluctuate both in intensity and colour as the plate is moved so as to alter the thickness of the film in front of the eye. The part of the halo immediately surrounding the source and the source itself appear to be of complementary colours.

3. Haloes due to Obliquely-held Plates.

If the plate instead of being held normally to the light is gradually tilted, some very striking results are obtained, especially when the observations are made in monochromatic light. As the obliquity of the plate to the incident light is increased, new rings appear at the centre of the halo and move outwards: when a dark ring first appears it is very clear and well-defined, but in moving out it becomes faint and broad, and then again sharper and blacker, and so on. These fluctuations, however, occur only when the dark ring is near the centre, and become imperceptible when it has moved to the position of the third or fourth ring from the centre, beyond which the dark rings are all more or less perfectly black. At the same time, the form of the rings undergoes alteration, becoming elliptical near the centre and of an oval form in the outer parts of the halo. With increasing obliquity these ovals assume unsymmetrical shapes, the curvatures at the two ends of the minor axis (which lies in the plane of incidence) being unequal. This effect is shown in fig. 7 in Plate V. Indeed, on the flatter side the ovals may actually straighten out and even reverse their curvature—that is, become concave outwards; while on the other side, which corresponds to directions more nearly parallel to the plate, the rings remain convex outwards. At the same time, the different parts of the rings in the halo appear very unequal in their illumination, the light becoming more or less completely concentrated in the plane of incidence and the flatter side of the rings appearing brighter than the more strongly curved side (fig. 7 in Plate V.).

4. Films containing Distorted Boundaries.

Mixed plates freshly prepared by rubbing up saliva or white of egg between glass plates generally show, when examined under the microscope, that the air-bubbles tend to take up a circular form, or at any rate do not show a bias towards elongation in any particular direction. But if the glass plates enclosing a mixed film of white of egg and air are pressed together and continuously moved over each other in any one direction, the bubbles in the film become distorted, assuming elliptic or oval shapes of which the major axis is in a direction *perpendicular* to that of movement of the plates. Fig. 6 in Plate IV. is a micro-photograph of a film obtained in this manner. When the diffraction-halo due to such a plate (normally held) is observed, it is found that the rings in the halo retain their relative position and circular shape, the only conspicuous effect of the altered form of the boundaries being to increase the luminosity of the halo in directions transverse to the preponderating direction of the boundaries. This is illustrated in fig. 5. This observation, taken along with those described in the preceding section, clearly shows that the essential feature of the halo, namely, the succession of dark and bright rings of widths increasing from the centre outwards, does not depend for its formation on the position, size, or shape of the air-bubbles in the film, and is determined only by the thickness of the film containing the bubbles.

When glycerine, turpentine, fats, or oils are used for forming the mixed plates, the tendency towards elongation of the air-liquid boundaries in the film becomes excessive, in these cases occurring parallel to the direction of movement at any instant of the enclosing glass plates. With such liquids it requires some dexterity to prepare a film showing the complete circular haloes. Generally only a diametrical streak is obtained which shows different colours at different parts of its length, and turns round and round as the glass plates enclosing the film are slid over each other with a circular movement in front of the eye.

5. Effect of Partial or Complete Drying of an Albumen Film.

When a mixed plate is prepared with white of egg between parallel glass plates and is allowed to stand for an hour or two, some very remarkable changes occur in the structure of the film and in the optical effects produced by it. The bubbles of air in the film which at first lie about indiscriminately (fig. 1 in Plate IV.) soon draw together, coming into contact over a considerable portion of their edges (fig. 8 in Plate V.), and this process continues gradually till the edges everywhere touch each other. (With very thin films, especially those formed between curved surfaces, this process may be indefinitely retarded.) Ultimately, the edges which have joined up straighten out, and when after a day or two the film has completely dried up, it is found on examining the film under a microscope that the albumen is confined to a number of very fine ridges holding the two glass plates together, the form of these ridges being that of a number of irregular hexagons, pentagons, or quadrilaterals forming a network (fig. 9 in Plate V.). The diffraction-halo seen round a distant source of white light when such a completely dried plate is held normally in front of the eye is entirely different in character from that due to a mixed plate freshly prepared; in fact, the relative position of the achromatic and coloured portions and the spacing of the rings are completely reversed with the dried film. The halo in this case is much fainter. It shows a broad central area which is achromatic, followed *outside* by rings of gradually *decreasing* width which are strongly coloured. The radial fibrous structure of the halo is exceedingly well marked (fig. 11 in Plate V.). In fact, the diffraction-halo seen round the source when viewed through the completely dry film is remarkably similar in appearance to the well-known diffusion-rings observed around the focus of a thick concave mirror with a dusted surface. The resemblance extends also to the case in which the plate is held obliquely in front of the eye. As the plate is gradually tilted, the halo runs out on one side, fresh fringes appearing on that side of the source; and ultimately, when the plate is held at a moderate obliquity, the halo consists (at least in its brightest part) of a system of circular arcs which are unequally spaced, the arc which passes through the source being *achromatic* and those on either side of it being strongly coloured in white light (see fig. 12 in Plate V.). [The oblique streak seen in the figure running across the circular arcs was due to an accidental circumstance.] In fact, the halo in its brightest part is very similar to the diffusion rings due to a dusted mirror when the latter is tilted. It should be mentioned, however, that in the outer fainter parts of the halo (which do not appear in the photograph reproduced) certain more complex effects are observed. The detailed description of these may be deferred for the present.

When the film is only partially dry (as, for instance, in fig. 8 in Plate V.), both sets of rings, that is those due to a freshly prepared film and those shown by a completely dry film, appear simultaneously, and owing to their superposition, the phenomena appear somewhat confused, especially near the centre of the halo. The effect of partial drying of the film is also clearly noticeable in the diffraction-halo due to an obliquely held plate. The elliptic or oval rings appear traversed by a system of circular arcs running transverse to the plane of incidence. These are seen (somewhat faintly) in the photograph reproduced in fig. 10 in Plate V. They become more and more prominent as the drying of the film progresses, and ultimately remain alone in the field (fig. 12, Plate V.) when the elliptic rings have disappeared.

6. Non-Uniform Films.

Standing in close relation with the phenomena described above due to mixed plates of uniform thickness, are the effects observed with mixed plates of variable thickness. The difference between the two cases is principally as regards the method of observation. In the former case the film is placed close to the eye, and the diffraction-halo surrounding a distant source is observed. With mixed films of variable thickness, on the other hand, the most suitable method of observation is to focus the eye on the film itself, the latter being held at a suitable distance from the observer. As in the former case, it is necessary, if the effects are to be studied critically, to use a light-source of small dimensions and to place the film at a sufficient distance from it. A film of variable thickness may be readily formed between the surfaces of two lenses similar to those used for observation of Newton's Rings. The thickness of the film at its centre need not necessarily be zero; and, in fact, if egg-albumen is used, there is considerable difficulty felt in forcing the lenses into actual contact at the centre. We shall confine our attention here to the phenomena observed with *freshly prepared* films.

Observations in White Light : Normal Incidence.

When the mixed plate is held at a sufficient distance in the line of sight between the eye and a distant source of light, vividly coloured rings localized on the film are seen (provided its thickness is not too great), these rings being the lines of equal thickness on the film. In this case the light reaching the eye is that diffracted through small angles by the air-liquid boundaries which the film contains; and, indeed, it is these boundaries which appear luminous to the eye and *not* the whole continuous film *. The source of light itself appears coloured, and is of a complementary tint to the part of the film through which it is seen. When the eye is moved a *little* out of the direct line between the plate and source of light, some remarkable changes occur in the appearance of the film. The colours become feeble and impure, and on moving the eye further out of the direct line, the colours reappear again vividly, the rings having simultaneously expanded and moved outwards. With still further movement of the eye the phenomenon repeats itself, but with much less marked fluctuations in the vividness of the colours ; and if the film were initially so thick as to show a coloured centre, fresh rings also appear and move outwards from the centre, until finally an achromatic centre develops and expands so as to cover the whole area of the film when viewed sufficiently obliquely. When the source of light is fairly powerful (as, for instance, when the film is held normally in the track of a

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* Fig. 4 in Plate IV. is a photograph (much enlarged) of these luminous boundaries in a dark field as observed by the method of the "Foucalt test." Each of the laminar boundaries appears as a pair of brilliantly coloured lines running parallel to each other and separated by a perfectly black line coinciding with the exact outline of the boundary. The theory of this effect will be more fully considered in Part II. of this paper.

parallel beam of light from an optical lantern), this expansion of the rings with increasing obliquity of observation may be followed up till the direction in which the film is seen makes an angle up to 90° with the direction of the incident light. It is a noteworthy fact, that a film too thick to show colours when observed nearly in the direction of the transmitted light will show the coloured rings vividly when viewed at a moderate obliquity. Ultimately all films, whether thick or thin, appear practically achromatic in a sufficiently oblique direction. The thinner the film, the smaller the angle of diffraction necessary for this. The thinnest films, which may be obtained by forcing the glasses together till they nearly come into contact, scatter much less light than the thicker films. Hence the film which is achromatic when viewed obliquely shows a darker area near its centre.

Observations in Monochromatic Light: Normal Incidence.

In this case, if the mixed plate is held directly in the line of sight between the eye and the source of light at a sufficiently great distance from both of them, a series of *perfectly black* rings (alternating with bright rings) may be seen on the film, even if this be fairly thick. Bringing the film nearer the source of light or increasing the dimensions of the latter has a very deleterious effect on the perfect blackness and sharpness of the rings. Placing the eye a little out of the direct line also results in the rings becoming blurred in appearance; and, indeed, on merely bringing the film nearer the observer in the line of sight so as to increase the angle it subtends at the eye, the rings may appear broken up and blurred in parts. (The effect is the more striking the thicker the film and the larger its area.) If a particular dark ring be watched as the eye is gradually moved out of the line through film and source, it will be noticed that the ring becomes blurred and broadened, then again sharp and perfectly black, but with its position shifted outwards in the film; this process further repeats itself, but with rapidly diminishing fluctuations in the intensity and sharpness of the ring. Viewed at a moderately large obliquity, the dark rings are always perfectly black, and their sharpness is much less dependent upon the use of a light-source of restricted area. The rings expand with increasing obliquity of observation and move out of the film, until finally no rings are visible at all. A darker patch at the centre, where the film is thinnest, can be seen, as in the case of white light.

Obliquely Incident Light.—When the film is tilted relatively to the direction of the incident rays, the rings seen on the film contract and move inwards. As in the case of normal incidence, perfectly black rings may be seen on the film, if it be viewed in monochromatic light very nearly in the direction of the transmitted rays, and the appearance of these rings alters with the obliquity of observation in much the same way. It should be remarked, however, that in the present case the effects observed vary not only with the angle between the transmitted pencil and the direction of observation, but also with the particular plane in which the latter direction lies. The maximum permissible angle of observation varies with this plane. In the plane of incidence it is $\pi/2-\alpha$ and $\pi/2+\alpha$ respectively on the two sides of the transmitted pencil, where α is the angle of incidence. If α is considerable, the rings continue to be visible when the film is viewed nearly along the surface of the plate on one side of the transmitted pencil, while on the other side the rings move out and disappear from the film at a moderate obliquity, so that in white light the film appears achromatic over a wide range of angles of observation.

Calcutta, India,
18th September, 1920.

XXXI. *The Photo-Electric Theory of Vision.*
By J. H. J. Poole, *M.A.*[*]

IN his address to the British Association in 1919, Sir Oliver Lodge suggested that the light sensation in the eye might be caused by the action of photo-electrons excited by the incident light. He thought that, if experiments were carried out on the various substances present in the retina, some of them might be found to exhibit photo-electric properties when exposed to visual light. The same idea had occurred to Dr. Joly some years previously, and in 1915–16 some experiments were actually carried out in this laboratory on the possible photo-electric properties of the black pigment No evidence of the black pigment possessing any photo-electric powers, when exposed to visual light, could however be obtained, and as thus no direct experimental evidence in favour of the theory could be given, Dr. Joly refrained from publishing this view of the mechanism of

[*] Communicated by Prof. J. Joly, F.R.S.

vision. When, however, Sir Oliver Lodge independently proposed the same theory, it was thought worth while repeating the experiments more fully, and seeing if any definite evidence in favour of the theory could be obtained.

It seems highly probable, it might almost be said certain, that the effect of light on the eye must ultimately be due to a photo-electric action of some kind; but the real point of issue is whether it can be shown that the photo-electron excited by the light ever gets free of its parent molecule as it does when a metal is illuminated with ultra-violet light, or whether the action is less violent and may only result in disturbing the equilibrium of the electron inside the molecule, and hence altering the chemical properties of some light-sensitive compound in the eye. The latter view would favour such a theory of vision as the Young-Helmholtz theory, which postulates the existence of three definite and independent colour-sensations in the eye, which might be caused by three different light-sensitive compounds; but if it could be shown that the former view were correct, a very much more simple and direct explanation of colour-vision can be given, at least from the physical point of view. This explanation rests on the fact that it has been shown that the velocity of emission of a photo-electron is a simple function of the wave-length of the incident light—in fact, that the energy of the electron is given by the equation $E = h\nu - p$, where h is Planck's constant, ν is the frequency of the light, and p is the work done in getting the electron free from the molecule and will thus be a constant for any particular material. Thus for every different colour in the spectrum an electron of a definite velocity would be emitted, and it is only necessary to suppose that the manner in which the cones are excited depends on the velocity of these electrons, to obtain a possible view of the mechanism of colour-vision. It might perhaps be urged against such a theory as this, that on this view the eye ought always to be able to distinguish between a pure single frequency light and an impure mixture of two different frequency lights, which, as is well known, the eye is in certain cases incapable of doing: *e. g.*, it cannot distinguish between spectrum yellow and a suitable mixture of green and red. However, we could meet this objection by supposing that the resolving power of the cones for photo-electrons of various velocities was not perfect, and that they were only capable of separating the electrons into three main groups, which would correspond to the three primary sensations of red, green, and blue. Thus in the case of spectrum yellow, which lies between red and green, the resultant electrons would excite both the red and green sensations and hence such a colour would be visually indistinguishable from a mixture of red and green. In fact, if we adopt this view of the resolving power of the cones for photo-electrons of different velocities, we could explain all the facts which agree with the Young-Helmholtz theory of three primary colour sensations.

Before making any direct tests of the possible photo-electric powers of the retina, some preliminary indirect trials were made. It seems probable that the formation of the latent image in a photographic plate is really based on a photo-electric action due to the silver halides which are found to be vigorously photo-electric. Sensitisers are also photo-electric, their activity being dependent on the absorption of light of some particular colour. It is, therefore, perhaps permissible to infer that if either the black pigment or rhodopsin were really photo-electric to visual light, they would also be capable of acting as sensitisers for a photographic plate, if treated in a suitable manner. Experiments on this point were accordingly made by exposing a plate which was insensitive to the red end of the spectrum to intense red light. One half of the plate was treated with the preparation from the eye, which it was desired to test. The untreated portion acted as a standard for comparison purposes, and by putting a slight scratch with a penknife on the treated half it could always be subsequently identified. An ordinary carbon glow-lamp was used as a source of light. Before falling on the plate, the light passed through a picric-acid screen, ordinary photographic ruby glass, and also a special red screen, as it was found that the ruby glass allowed quite a large proportion of the blue end of the spectrum to pass through.

The first material to be dealt with was the black pigment. Various methods of applying it to the plate were tried, but in no case was definite positive evidence of any sensitizing action obtained. In the first attempts the black pigment was simply removed from a freshly dissected bull's eye, and placed directly in contact with the sensitive film. The plate was exposed through the glass, as owing to the opacity of the black pigment no effect could be expected if the black pigment coating was between the film and the incident light. After exposure, the pigment was carefully washed off and the plate given a thorough rinsing with water to remove as far as possible any trace of grease. As already stated however, no sign of any sensitising action could be discovered on development, and in fact sometimes the reverse effect was observed, when a sufficiently long exposure to slightly fog the plate was tried. This slight fogging of the plate with long exposures was probably due to a very small amount of actinic light getting through the screens, and the retarding action of the pigment observed may be caused by its slightly greasy nature impeding development. Other attempts were also made by treating the pigment with various solvents, such as water, alcohol, chloroform, etc., and treating a plate with the resultant fine suspension of the black pigment, as it did not appear to be soluble in any of the solvents used. No result was obtained. Another method adopted was to reduce the pigment to a very fine powder by dehydration and subsequent grinding in an agate mortar, the resulting powder being then dusted on to the plate; but again no sensitising effect was shown. The difficulty of obtaining good enough contact between the pigment and the sensitive film of the plate would probably be sufficient to account for the negative result in all these cases, as it has been found that for ordinary sensitising dyes to be effective, the dye must actually dye the silver-halide grain, and not merely stain the plate.

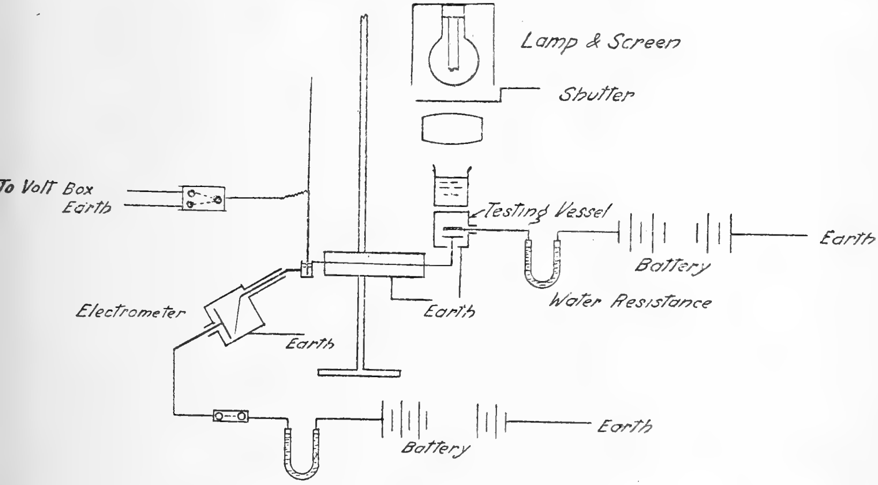
It is more difficult to deal with the visual purple or rhodopsin in the retina, both on account of the much smaller quantity available, and also because the rhodopsin is only present in dark-adapted eyes. For this reason the animal from which the eye is to be obtained must be kept in the dark for some time before being killed, to allow the rhodopsin to form in the retina, and all subsequent dissections have to be carried out in the dark or in a very subdued red light, as the rhodopsin quickly fades when exposed to light. Professor Pringle was, however, kind enough to supply us with a solution of rhodopsin in bile salts which had been prepared from dark-adapted frogs' eyes. This solution was tested on the plate in the usual manner. Again no sensitising action was detected: in fact, with very long exposures which slightly fogged the plate, a small retarding effect was noticed. This effect was found to be due to the bile salts,—a solution of them with no rhodopsin having the same slightly retarding effect on the plate. It is unfortunate that bile salts have this action on the plate, as it would completely mask any slight sensitising effect which the very small amount of rhodopsin present might produce, and it is apparently not easy to extract the rhodopsin from the retina by other means.

These sensitising experiments thus lead to no direct evidence in favour of the photo-electric theory of vision, but, on the other hand, it certainly would not be justifiable to conclude from them that the materials tested have no photo-electric properties, since the difficulties of getting intimate contact with the sensitive film are, as already mentioned, probably quite sufficient to account for the nul results obtained.

After the failure of these experiments to show any positive effects, a direct method of testing the black pigment etc. was used.

The usual plan of doing this was employed. The material to be tested is placed on an insulated conducting plate, and the rate of leak of negative electricity from the plate when it is illuminated measured with an electrometer. To facilitate the escape of the negative electrons, an accelerating electric field is generally used. A sketch of the various electrical connexions is given. The Wilson Kaye Tilted Leaf Electrometer was used to measure the electronic current. Its high sensibility to voltage changes and small capacity render it peculiarly suitable for this purpose. Fortunately it was found unnecessary to work with the instrument in its most sensitive condition, as the natural leak in the ionization-chamber limits the sensitivity at which it is desirable to work. Usually it was found that a sensitivity of about 30 or 40 scale-divisions to the volt was sufficient, though

Fig. 1.



on occasion sensitivities up to 130 divisions per volt were used. To maintain the charged plate of the electrometer at the requisite voltage, a battery of small Ever-Ready dry cells was employed. This battery was carefully insulated from earth, the individual cells being stood on a layer of paraffin-wax, and its voltage was found to remain constant over long periods so long as no current was taken from it. To protect it in case of an accidental short-circuit due to the gold-leaf touching the charged plate, a water-resistance is inserted between it and the latter. A switch is also inserted for convenience in working with the instrument.

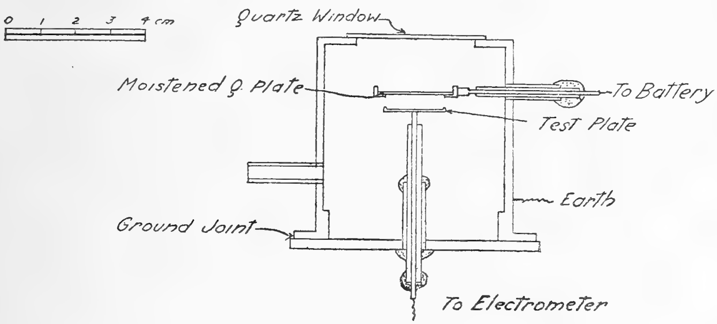
For earthing the gold-leaf when requisite, a copper rod dipping into a strong $CaCl_2$ solution contained in the small copper cup supplied with the instrument was used. When it was desired to isolate the gold-leaf, the copper rod could be lifted by a silk thread passing over a suitable arrangement of pulley-wheels. On the whole this key proved fairly satisfactory, but at times it was inclined to give too big an initial charge to the gold-leaf. Usually, however, after two or three attempts, a small enough displacement could be obtained.

In order to conveniently measure the sensitivity of the electrometer, the copper rod of the earthing key was not directly connected to earth, but was connected to a two-way switch so that a known voltage could be applied to it by means of a volt-box. Thus the sensitivity of the electrometer can be quickly determined at any time, which is essential as any small accidental shake of the instrument may alter it considerably, and for this reason it is necessary to test it fairly frequently.

A separate sketch of the final form of testing-chamber is shown. The chamber was constructed of brass, and made in two parts so that it could be easily taken apart to insert the material under test. The joint was carefully ground with carborundum-powder and tallowed, in order to insure the vessel being air-tight when assembled. It could be exhausted through a side tubulure, which was connected to a Geryk pump and manometer. Pressures down to about 3 mm. of mercury could be maintained in the testing-chamber.

The black pigment or other material is placed on the small brass testing-plate. This plate is carried on a copper rod which projects slightly from the bottom of the vessel and is connected to the electrometer by a fine wire. The wire is surrounded by an earthed shield to prevent any stray electrostatic effects due to the observer etc. affecting the electrometer. The copper rod itself is insulated from the main body of the chamber, which is earthed, by sealing-wax and a quartz tube as depicted. The sealing-wax serves to make the joint air-tight. About 3 mm. above the testing-plate, a small quartz plate carried by a brass ring is fitted. The latter is supported by a small steel rod, which passes through the side of the vessel and is also insulated with a glass tube and sealing-wax. This plate replaces the ordinary grid of copper gauze or other form of grating usually employed to collect the negative ions escaping from the photo-electric surface. By applying a thin film of a fairly strong solution of either H_2SO_4 or P_2O_5 to the lower surface of the plate, it can be made sufficiently conducting for working with the high voltages and small currents met with in these experiments. The outer brass ring which supports the plate also serves to make contact with the liquid film. To facilitate the escape of the electrons from the test plate, the upper quartz plate is connected through its steel supporting rod to the positive pole of a high-voltage battery. At low pressures the potential gradient was high enough to largely increase the electronic current by ionization due to collision, but in experiments carried out at air-pressure this would not be the case.

Fig. 2.



The use of the moistened quartz plate, instead of the usual form of wire grid, has some great advantages. The most obvious one lies in its superior transparency, but it also possesses a great virtue in that its shape is much more constant than that of the wire grid. The importance of this fact has been pointed out by H. H. Dixon and H. H. Poole in a paper on "Photo-Synthesis and the Electronic Theory" (Proc. Roy. Dublin Soc. vol. xvi. N. S. No. 5), in which they show that the change in capacity due to the sagging of a wire-gauze grid when heated by the incident light may produce apparently quite large photo-electric effects, which are really quite spurious. The quartz plate, however, will manifestly not suffer from this complaint on at least three accounts. These are: (*a*) The thermal expansion of quartz is practically zero; (*b*) Owing to its transparency it will absorb very little energy from the incident beam; and (*c*) Even if the plate did expand slightly it would do so in its own plane and not alter the capacity of the isolated system appreciably. The chief drawback of the arrangement is that it cannot obviously be used at very low pressures owing to the evaporation of the liquid film.

The test-plate was illuminated through a quartz window at the top of the testing vessel. A quartz window was employed, and it is convenient to be able to use ultra-violet light on occasion to test whether the arrangement is functioning properly. As a source of light a 200-watt ½-watt lamp was employed, and a suitable train of lenses used to concentrate the light. A cardboard screen was placed round the lamp to protect the observer's eyes, and the electrometer was completely shielded from the light by a large wooden screen. It was found that switching on this light had a slight effect on the electrometer, although the leads to the lamp were kept as far distant as possible. In consequence, a shutter was used by which the illumination could be controlled without turning the lamp off.

In order to obtain some idea of the absolute current to which a known rate of deflexion of the gold-leaf corresponds, it is necessary to measure the capacity of the gold-leaf and testing plate taken in conjunction. This capacity was determined by comparing it with a second capacity whose value could be calculated approximately. The method adopted was that described by H. H. Dixon and H. H. Poole in their paper on Photo-Synthesis (*loc. cit.*). The capacity was found to be about 6 cms., and as the sensitivity is also known we can easily determine the value of the currents obtained. The value obtained for the capacity is probably not very accurate, but great accuracy is not requisite, as really all we require to know in these experiments is the order of magnitude of the currents dealt with.

The method of procedure adopted was first to spread a thin film of the substance to be tried on a disk of pure lead-foil, which was then placed on the testing plate of the ionization chamber. The chamber was re-assembled, and the pump worked till the desired pressure had been attained. The rate of leak of the electrometer with the light off and on was then measured in the usual way. The advantage of using lead disks to place the active material on is that it makes the manipulation of the latter easier, and also, when one preparation has been tested, the disk can be quickly removed from the testing vessel and a second one substituted. Lead is a convenient metal to use, as it is easily cut to the required shape, and is also very inert photo-electrically.

The experiments carried out on the black pigment with this apparatus completely confirmed the nul results previously obtained with it. Various methods of preparing the pigment were tried. It was tested moist and after being dried in a desiccator, immediately after dissection and after the lapse of some time, but in no case was any photo-electric effect obtained. A test was also made on a complete retina, which was spread entire on one of the lead disks, but it also showed no photo-electric powers. All these experiments were conducted at a pressure of about 5 mm. of mercury. There can, it seems, be very little doubt that the black pigment of the eye is not photo-electric when exposed to visual light.

As regards the rhodopsin in the retina, it was thought best to test it *in situ* in the retina. This was done by testing a freshly prepared retina obtained from a frog's eye. The frog, before the experiment, had been kept in the dark, so that there was a certain amount of rhodopsin present in the retina. Professor Pringle kindly did the requisite dissections which were carried out in feeble red light, the retina not being exposed to ordinary illumination at all till it was tested. As it was considered that perhaps the vacuum in the testing vessel might tend to render the retina inactive owing to its drying action, these trials were all carried out at air pressure.

The result of these experiments showed that while the frog's retina certainly possesses no permanent photo-electric effect, there is usually a very small effect for a few minutes after the retina is first placed in the testing vessel. In all cases such an effect was found, but its size was very small and irregular, varying from about 6×10^{-6} E.S.U. to $0{\cdot}5 \times 10^{-6}$ E.S.U.

In view of the small size of the currents obtained, it is perhaps of interest to consider what would be the smallest number of electrons per second which the human eye could be expected to detect. If we assume that a candle is visible at night at a distance of 3000 metres, and that the effective aperture of the eye in such a case would be about 1 sq. cm., we find that the minimum visible energy-flow is equal to about 4×10^{-7} erg per second, taking the energy emitted by the unit candle as 5×10^{5} ergs per second. Let us also assume that the average wave-length is about 5000 Å.U. (this is probably too short, but is sufficiently accurate for this purpose), and we find that the energy of the corresponding photo-electron on the quantum theory will be equal to $h\nu = 3{\cdot}9 \times 10^{-12}$ erg.

If all the energy of the incident light is converted into energy of the photo-electrons, this would lead to about 10^{5} electrons per second being the minimum number visible, and a current of 5×10^{-5} E.S.U. being the minimum electronic current.

In comparing this latter figure with the actual small current obtained from the frog's retina, there are several considerations to be taken into account: *i. e.*, (*a*) a frog's eye is very much smaller than a human eye; (*b*) the illumination was not the minimum, but was intensely bright. Also, in any case, we could not expect to get anything approaching the theoretically possible current owing to absorption of the electrons in the surface, loss of energy by reflexion, and also the fact that even under ideal conditions the total incident energy may not be entirely converted into that of the emitted electrons. As regards (*a*) and (*b*), the actual energy falling on the frog's retina per second was approximately 10^{4} ergs per second, or $2{\cdot}5 \times 10^{10}$ times that required to excite the human eye. This would give a theoretically possible current of $1{\cdot}25 \times 10^{6}$ E.S.U., or about 2×10^{11} times the actual current obtained. It is possible that absorption in the surface might account for the enormous discrepancy between these two figures, but certainly no other consideration could do so. In this connexion some results obtained by Elster and Geitel (*Phys. Zeits.* xiii. p. 468, 1912) are of interest. By the use of a very sensitive potassium cell, they were able to detect an amount of blue light imparting 3×10^{-7} erg per second per sq. cm., and they found that under these conditions the energy of the emitted electrons accounted for $\frac{1}{2300}$ part of the incident energy. As it has been found that the photo-electric current for a substance is usually a linear function of the light intensity, it would from this appear probable that in all cases of photo-electric action a sensible proportion of the incident energy would be represented in the energy of the emitted electrons. Thus we are driven to conclude that since in the case of the retina only 5×10^{-12} part is so represented, it cannot be considered to be photo-electric. The current obtained is so small that it may possibly be due to some extraneous effect due to heating etc.

From these experiments we may therefore safely conclude that neither the black pigment nor the retina as a whole is photo-electric to visual light. It is possible however that, as previously mentioned, the nul effects obtained with the retina may be due to absorption of the electrons in the surface. It remains quite conceivable that the rhodopsin in the eye is actually photo-electric, but so immersed in inactive material that the electrons cannot escape. Thus the photo-electric theory of vision, while still presenting many attractions, cannot be said to be in any way confirmed by these experiments, but neither can it be actually refuted by them. To obtain really conclusive evidence it would be necessary to isolate sufficient rhodopsin in a pure state to be able to test it directly.

In conclusion I wish to express my thanks to Professor Pringle and Dr. Fearon for their kind aid in the preparation of the frogs' retinas ; also to Dr. Joly, to whose suggestion and assistance the research is mainly due.

Iveagh Geological Laboratory,
November 1920.

XXXII. *The Distribution of the Active Deposits of Radium, Thorium, and Actinium in Electric Fields.* *By* G. H. Briggs, *B.Sc., Lecturer in Physics at the University of Sydney* *.

I. *Introduction.*

THE experiments to be described in this paper were begun with the object of deciding whether, as E. M. Wellish † had concluded, there is a definite limiting fraction of the recoil atoms from radium emanation, positively charged at the end of the recoil path, or whether, as G. H. Henderson concluded, all the recoil atoms from thorium and radium emanations are positively charged at the end of the recoil path. As the work progressed, it was found necessary to extend its scope.

In his experiments Wellish used a cylindrical vessel with a central electrode. He obtained the following results for the recoil atoms of radium emanation : 88·2 per cent.

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* Communicated by the Author.

† Wellish, Phil. Mag. xxviii. p. 417 (1914).

initially positively charged in air and hydrogen, 78·9 in carbon dioxide, the remainder being neutral, and in ether vapour all the deposit atoms were neutral. The term "initial" refers to the instant when the recoil atom has reached the end of its recoil path, and the values quoted represent the fraction of the recoil atoms which possess a positive charge at the end of this path before either volume or columnar recombination has had a chance to become operative. The chief advantage of the type of vessel used by Wellish is that a delicate test of the presence of negative deposit atoms can be made if the central electrode is positive. In his first paper Henderson * described the results of experiments on thorium active deposit. The electrodes were parallel plates with a guard ring, and were 3 cm. apart. In the air the cathode activities at 120 and 12,000 volts were 98·6 and 99·8 per cent. As the latter value differed from 100 by less than the experimental error, Henderson concluded that all the deposit atoms from thorium emanation are initially positively charged in air. When ether vapour was added to the air the percentage decreased, becoming zero for pure ether vapour. In two succeeding papers, however, Henderson † described experiments on radium emanation, and concluded from the results he obtained when using a new form of parallel plate exposure vessel and strong uniform electric fields, that all the deposit particles from radium emanation in air, carbon dioxide, and sulphur dioxide are positively charged. In column D of Table I. are given values calculated from those of Henderson, who expressed his results as the ratio of the cathode activity to the total activity.

As the values were still increasing at the highest potential used, he concluded that there is no limit to the percentage cathode activity, and hence that initially all the deposit atoms are positively charged.

II. *Experiments with Radium Emanation in Strong Fields.*

Fig. 1 shows an apparatus designed by the author for use with strong fields. A and B are two circular brass disks cemented to an ebonite ring P. The inner surface of the ring was curved, as shown, to prevent sparking across this surface, and it projected beyond the brass disks to prevent brush discharge and sparking from their edges.

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* Henderson, Trans. Nova Scotian Inst. Sc. xiv. p. 1 (1914–15).

† Henderson, Trans. Nova Scotian Inst. Sc. xiv. p. 123 (1916); and Trans. Roy. Soc. Canada, x. p. 151 (1916).

The central portions, C and D, of the disks were removable, and could be made air-tight by running soft wax into channels formed by brass rings on either side of the junction-line. The diameter of the portion of the disks exposed on the inside of the vessel was 10·6 cm., and that of the removable disks 5·6 cm. The plates were 2 cm. apart, and their internal surfaces were cleaned with fine emery paper. Radium emanation was introduced mixed with the gas under observation, or was allowed to accumulate over-night from a small quantity of radium placed inside the vessel. A Wimshurst machine was used for high potentials. The gases were dried and filtered. At the end of an exposure, which was always longer than three hours, the alpha-ray activities of the central disks were compared by means of a Dolezalek electrometer. Wellish[*]

Fig. 1.

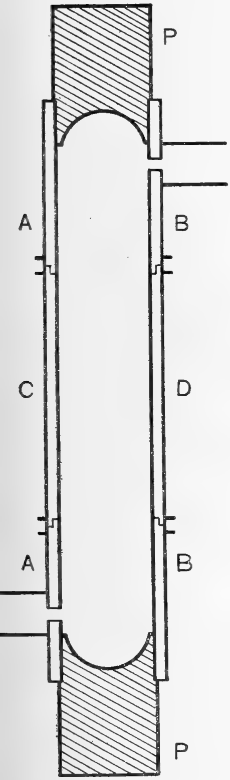
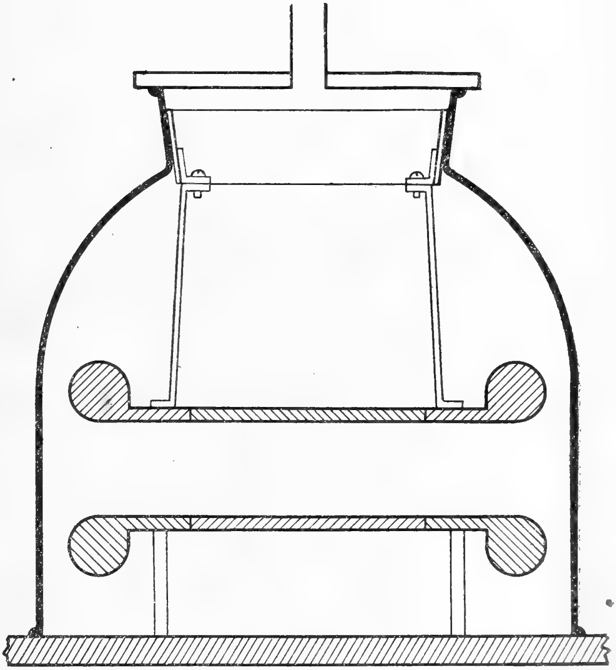


Fig. 2.



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[*] *Loc. cit.*

showed that the anode activity was due to the diffusion of neutral deposit atoms: hence if c and d are the cathode and anode activities respectively, $\frac{c-d}{c+d}$ would give the fraction of the RaA atoms positively charged, if, as assumed by Wellish, the recoil of RaB from the electrodes is negligible. This assumption will be shown later to be incorrect. In column A of Table I. are given the values of $\frac{c-d}{c+d}$ calculated from the anode and cathode activities for radium emanation mixed with air. No evidence is shown of a continual increase with voltage, the limiting value being practically reached at 60 volts per cm. The mean value 89·6 is in good agreement with that found by Wellish when using a different type of testing vessel.

TABLE I.

Air. $\frac{c-d}{c+d}\times 100.$

Pressure.	Volts per cm.	A.	B.	C.	D.
20 cm.	60	89·2			
„	„	89·5			
„	„	89·4			
Atmospheric.	150	...	...	...	89·8
„	1000	...	...	...	92·4
„	2000	...	...	...	92·8
„	4000	...	...	...	93·8
„	6000	90·7	...	94·7	
„	7000	88·9	...	95·4	
„	„	89·0			
„	8000	89·3			
„	„	88·7			
„	9000	...	90·5		
„	„	...	90·6		
„	10,000	90·6	91·9		
„	12,000	89·4	91·3		
„	14,000	91·0	91·8	...	95·6
	Mean ...	89·6			

To determine the reason for the discrepancy in the results obtained by the author and by Henderson, experiments were made with an apparatus similar as far as possible to that used by Henderson. The two results shown in column C sufficed to show that his experimental results could be reproduced. However, when this apparatus was slightly modified, the increase of the experimental values with the voltage practically disappeared. Fig. 2 shows the modified form of Henderson's apparatus used by the writer. The electrodes were two brass plates 10·4 cm. in diameter, held, as indicated, at a distance of 2 cm. apart in a bell-jar of 11·5 cm. internal diameter. The central portions, whose diameter was 5·1, were removable, and the activities on these were measured. Henderson used flat brass plates separated by three vertical glass rods, the rods and the edges of the plates being covered with paraffin wax to stop brush discharge. He used cotton wadding to prevent diffusion of active deposit from above or below the plates into the region between them. The following modifications were made by the writer to eliminate to a much higher degree than was done by Henderson brush discharge near the edges of the plates:—(1) The edges were rounded off as shown in the figure, (2) a roll of silk was used instead of cotton-wool, and (3) the upper plate was supported from above. The results obtained with this apparatus are given in column B. The continual increase found by Henderson at high voltage has practically disappeared. The difference between the values of column B and the mean of column A is to be ascribed possibly to some brush discharge still being present, and in the case of the last three readings to the fact that for these readings the plates were very highly polished. The importance of keeping the state of polish of the surfaces always the same had not been recognized at this stage of the experiments.

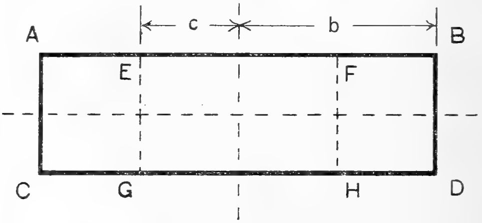
The high values given by the apparatus used by Henderson were due to the action of brush discharge from the edges of the plates or from cotton-wool fibres. If cotton-wool is held against a pole of a Wimshurst machine in the dark, it glows quite brightly, whereas good silk cloth does not. From any point near the edges of the disks where a brush discharge occurs, the gas is blown on to the opposite electrode, or more probably on to the wall of the bell-jar, to which any neutral deposit atoms carried by the gas will adhere. In a short time all the gas between the electrodes will be entrained in these so-formed jets, and only a fraction of the neutral deposit will settle on the central portions of the plates. As the voltage is increased so will the intensity of the brush discharge, and with it the apparent percentage cathode activity. This action of the brush discharge was illustrated by filling a vessel similar to Henderson's with smoke. At places on the glass wall opposite points on the electrodes where brush discharge occurred a deposit of smoke was observed.

Further confirmation of Wellish's conclusion that there is a definite limiting value to the percentage of radium active deposit positively charged was given by the results found for ethylene. At 720 volts and 20 cm. pressure, using the vessel shown in fig. 1, the value for ethylene was 23·4. Higher values were not obtained at voltages as high as 12,000, using the vessel shown in fig. 2.

III. *The Diffusion of the Neutral Deposit Atoms.*

The experiments were then continued, using the ebonite ring type of vessel, in an endeavour to discover how the percentage depended on the nature of the gas. The positively charged atoms which reach the central disks EF or GH (fig. 3) are drawn from gas in the cylindrical volume EFHG. A portion, however, of the neutral atoms produced in this volume will diffuse out of it towards the walls AC and BD. Unless the number that so diffuse is negligible, this type of vessel will yield values for the percentage positively charged that are too high. Wellish * showed that the number of neutral deposit atoms at any point in a cylinder is given by the following equation:

Fig. 3.



$$n = \frac{q}{4\mathrm{D}}(b^2 - r^2) - \frac{2q}{b\mathrm{D}} \sum_{n=1}^{\infty} \frac{1}{\lambda_n^3} \frac{e^{\lambda_n z} + e^{-\lambda_n z} \mathrm{J}_0(\lambda_n r)}{e^{\lambda_n l} + e^{-\lambda_n l} \mathrm{J}_1(\lambda_n b)},$$

where r and z are cylindrical co-ordinates, the origin being at the centre of the cylinder and z being measured along its axis.

b is the radius of the cylinder, and $2l$ its length. q is the number of neutral atoms produced per c.c. by the decay of the emanation, and D their diffusion coefficient.

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* *Loc. cit.*

$\lambda_1, \lambda_2, \ldots \lambda_n$ are the real positive roots of the equation

$$J_0(\lambda b) = 0.$$

Let c be the radius of each of the central disks EF and GH.

The number of neutral atoms reaching EF or GH per second is

$$-2\pi D\left(\frac{dn}{dz}\right)_l \int_{r=0}^{r=c} r\,dr = \frac{4\pi qc}{b} \sum_{n=1}^{\infty} \frac{1}{\lambda_n^3} \frac{J_1(\lambda_n c)}{J_1(\lambda_n b)} \tanh(\lambda_n l).$$

The total number of neutral atoms generated in the cylinder EFHG is $2\pi c^2 lq$, and the ratio of the number reaching the central plates to the number generated between them is

$$\frac{4}{bcl} \sum_{n=1}^{\infty} \frac{1}{\lambda_n^3} \frac{J_1(\lambda_n c)}{J_1(\lambda_n b)} \tanh(\lambda_n l),$$

an expression which depends only on the dimensions of the containing vessel.

This series converges rather slowly. From the first twelve terms the result found for the vessel with the ebonite ring was 0·993. A second determination, made by giving c a very slightly different value, was 0·989. Hence all but about one per cent. of the neutral atoms formed in the volume EFHG reach the plates EF and GH. The error introduced is therefore negligible, and has been neglected in all subsequent calculations.

In the case of the apparatus described by Henderson, for which $2b = 11{\cdot}5$ cm., $2c = 7{\cdot}0$ cm., and $2l = 3$ cm., values calculated were 0·965 and 0·952. Values of J_1 were obtained from a table given by Wilson and Peirce *.

IV. *Experiments with Radium B.*

Previous investigators on the distribution of radium and thorium active deposits in electric fields have neglected the part played by the disintegration of RaA into RaB and ThA into ThB. It will be shown subsequently that if the percentage of RaB which is initially positively charged at the end of the recoil path in a particular gas, and also the efficiency of recoil of RaB from RaA from the surfaces of the electrodes are known, then the percentage of RaA initially positively charged in that gas may be calculated from the results of experiments in which the gas is mixed with radium emanation.

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* Wilson and Peirce, Bull. Amer. Math. Soc. iii. p. 153 (1896–7).

Also, by making use of the fact that RaA, ThA, and AcA are isotopes, as also are RaB, ThB, and AcB, the difference in the distribution of radium, thorium, and actinium emanations in an electric field may be quantitatively explained. The experiments by which the percentage of RaB initially positively charged in various gases, and the efficiency of recoil of RaB from the electrodes were measured, will now be described.

The experiments consisted in collecting RaB by recoil from a plate made active with RaA, first in the gas under observation and then in air. This gave the ratio of the fraction of RaB initially positively charged at the end of the recoil path in the gas, to the corresponding fraction for air. In a second experiment the fraction for air was found.

Two metal plates 6·7 cm. in diameter were separated by a vertical ebonite plate 1·5 cm. thick, having a hole 4·8 cm. in diameter in the centre. Stop-cock grease was used to make the contact surfaces air-tight. Through a three-way tap, air or the gas under observation having passed through drying and filtering tubes could be admitted. Another three-way tap led to a pump and manometer. When the active plate was negatively charged, or if there was no field between the plates, and the vessel filled with a dry dust-free gas, the opposite plate received a negligible amount of RaB, which was probably chiefly due to the initial disturbance caused by exhausting and filling the vessel, and not to the presence of negative RaB atoms. All the gases used in these experiments were tested for negative atoms by means of exposures in cylindrical vessels, in which the gas was mixed with radium emanation, and in all cases the activity on the central rod when positive was no greater than the amount that should have reached it by the diffusion of neutral atoms. There is no evidence, therefore, of the presence of either negative RaA or RaB atoms. In the experiment with the plates uncharged a layer of neutral RaB atoms is formed close to the active plate, and the result obtained shows that practically all the neutral atoms return to this plate. A large fraction, depending on the gas in the vessel, of the total activity, was deposited on the cathode when the active plate was positive. It was found that this amount with the pressure of the gas at 15 cm. did not increase perceptibly when the voltage was increased from 160 to 720.

In an experiment three plates were used. Plate (1) was activated with RaA by exposure to radium emanation for about half a minute, the portion activated being an area

2 cm. in diameter in the centre of the plate. Plate (2) having been placed on one side of the ebonite plate and charged positively, (1) was placed in position opposite it and charged negatively. The air was pumped out and the gas under observation was admitted and pumped out five times, and the pressure finally adjusted to about 15 cm. The field was then reversed and a stop-watch started. The activation of plate (2) was continued for five minutes, at the end of which time the field was again reversed so that (2) ceased to receive RaB from (1). Plate (2) was quickly removed and plate (3) substituted in its place, and the gas pumped out and replaced by dry filtered air at about 15 cm. pressure. This change could be made within one minute, and when the stop-watch registered six minutes the field was reversed and plate (3) activated for five minutes. When exhausting the vessel, the pressure was not allowed to fall below 1 cm., in order that RaB should not recoil directly on to (2) or (3). The maximum alpha-ray activities of plates (2) and (3) were compared. Since the amount of RaA on (1) falls to one-quarter in six minutes,

$$\frac{\text{Percentage of RaB positively charged in the gas}}{\text{Percentage of RaB positively charged in air}} = \frac{\text{maximum activity of (2)}}{4 \times \text{maximum activity of (3)}}.$$

The results obtained for various gases are given in column 2 of Table II. No alteration in the results was found when air was admitted in the first half of the experiment and the gas in the second half, or when the activity of plate (1), and hence the ionization between the plates, was increased by long exposure to the emanation to ten times its usual amount. In these experiments brass plates were used, except for ammonia, acetylene, and hydrogen-sulphide, for which the plates were of mild steel. Their surfaces were cleaned with fine emery paper.

In experiments on the distribution of the active deposits of radium and thorium emanations the transference from the anode to the cathode of positively charged RaB or ThB atoms, which were originally neutral RaA or ThA atoms, must be considered. It was accordingly necessary to measure the efficiency of recoil from surfaces similar to those used in the experiments with radium emanation. These surfaces had been cleaned with fine emery paper, except in the case of the bell-jar apparatus, in which, as noted previously, they were highly polished.

A clean plate having been placed on the ebonite plate and charged negatively, a second plate was exposed for ten seconds to radium emanation and quickly placed opposite the first, the action of a spring immediately connecting it to the positive pole of the battery (720 volts). Dry filtered air was drawn through the apparatus for about one minute, the final pressure being about 15 cm. The time that elapsed between removing the active plate from the emanation and putting it in position was usually about five seconds. In this case 0·033 of the RaA has decayed before the active plate is in position. At the end of 27 minutes the two plates were removed, and their activities compared by means of a Dolezalek electrometer of sensitivity 2000 mm. per volt. The results obtained, after correcting for the initial decay of the RaA, for the efficiency of recoil in air of RaB from plates of brass, steel, and zinc, cleaned with emery paper, were respectively 81·7, 73, and 73·9 per cent. The values given by different readings remained constant to within about 2 per cent.

The Percentage of RaB positively charged in Air.

The percentage of RaB initially positively charged in air was measured by modifying the method described above for comparing gases with air. It was arranged that the RaB atoms leaving plate (1) during the first 5 minutes' exposure should be received by a disk fastened to plate (2), so that it was separated from plate (1) by a distance of 0·06 mm. Mica on the edge of the disk prevented contact with (1). A potential of 160 volts was maintained between plate (1) and the disk, the latter being negative. Plate (2) and the disk were placed in position first, and the pump started. Plate (1), with its deposit of RaA, was then placed in position, and the stop-watch started. The pump was kept running throughout the exposure and rapidly produced a vacuum of 0·01 mm. At the end of the five minutes air was admitted and the disk quickly removed. An exposure was then made with plate (3) opposite plate (1) from the 6th to the 11th minute in air at 15 cm. pressure at the original distance of 1·5 cm., using 720 volts. Since the range of recoil of RaB at 0·01 mm. pressure is several metres, all the RaB atoms shot off from the plate should be collected during the first exposure. The mean result found by this method was 93 ± 0·6 for the percentage of RaB recoil atoms initially charged in air. By multiplying the values given in column 2 by 0·93, the corresponding values for the other gases are obtained. These are given in column 3. When the values given above for the efficiency of recoil in air are divided by 0·93, the efficiency of recoil is given in a form which is independent of the gas into which the atoms are projected and which depends only on the nature and surface of the metal, the values so obtained are: for the brass plates 87·8, for the steel 78·5, and for zinc 79·5.

It has been assumed in the foregoing that the total amount of RaB recoiling from a plate is independent of the gas in contact with it. It is difficult also to estimate the reliability of the result 93 per cent. for RaB in air, as it is unknown whether reflexion of RaB from the cathode or from residual gas occurs to any large extent during the first exposure. However, it appears from the results found later that the percentage for RaB in any gas is always much greater than that for RaA, and therefore that the value in air for RaB is much greater than 82.

V. *The Distribution of the Active Deposits of Radium, Thorium, and Actinium Radiations in Electric Fields.*

The distribution of the activity on the two electrodes is the result of several distinct processes.

1. The disintegration of the emanation gives rise to A-atoms, definite fractions of which are positive and neutral at the end of the recoil path.

2. The positive A-atoms are drawn by the field towards the cathode, and a fraction of them transform on the way into positive or neutral B-atoms. Of the former, owing to the comparatively long life of the B-atoms of the three radioactive series, practically all reach the cathode as B-atoms. The neutral B-atoms reach the electrodes by diffusion.

3. The neutral A-atoms produced by process (1) diffuse in equal numbers towards each electrode, and a fraction of them transform on the way into neutral or positive B-atoms, the former going to the cathode and the latter diffusing in equal numbers to each electrode.

The A atoms which have reached the electrodes by process (2) or (3) transform into B-atoms, and a portion of these, depending on the efficiency of recoil from the surfaces, are projected into the gas, a fraction being positive and the remainder neutral at the end of the recoil path. The former are drawn to the cathode, and since the recoil path is a small fraction of a millimetre in these experiments, it is held that all the neutral B-atoms return by diffusion to the surface from which they are projected.

Direct recoil on to the electrodes is negligible in the experiments of Wellish, Henderson, and the author. The magnitude of the effect produced by processes (2) and (3) depends on the transformation constants of the A-atoms of the three radioactive series of elements. No transformations after that of A- into B-atoms need be considered, since β-recoil of C-atoms is entirely negligible in these experiments, and the products of the C-atoms contribute a negligible amount to the ionization by which the activity of the electrodes is measured.

In a theory which will be given later, it is held that the last few collisions at the end of the recoil path determine whether the recoil atom will be positive or neutral. If this is so, the percentage initially positively charged in a gas will be the same for RaA, ThA, and AcA, since they are isotopes, although their initial velocities are different. In the same way, the percentages for RaB, ThB, and AcB will be identical and independent of whether the transformation takes place in the body of the gas or if the atom recoils into the gas from an electrode.

The fraction of radium and thorium positive A-atoms which transform into B-atoms before reaching the cathode is negligible in the experiments described here. The fraction of the neutral A-atoms which so transform in a parallel plate vessel may be found as follows:—In section III. it was proved that the diffusion of neutral atoms between the central disks is very approximately linear along the z-axis.

Let D be the coefficient of diffusion of the A atoms, and λ their transformation constant. q and z have the same meanings as before.

Let n be the number of neutral A-atoms per c.c. at a distance z along the z-axis.

We have

$$\mathrm{D}\frac{d^2n}{dz^2} = \lambda n - q.$$

Writing $y = \lambda n - q$, the equation becomes

$$\mathrm{D}\frac{d^2y}{dz^2} = \lambda y,$$

the solution of which is

$$\cosh\left\{\frac{\lambda^{\frac{1}{2}}}{\mathrm{D}^{\frac{1}{2}}}(z+a_1)\right\} = a_2 y,$$

where a_1 and a_2 are constants.

It is readily found from the boundary condition, viz. $n=0$ when $r=\pm l$, that

$$a_1 = 0 \quad \text{and} \quad a_2 = \frac{-\cosh\left(\frac{\lambda^{\frac{1}{2}}}{D^{\frac{1}{2}}}l\right)}{q}.$$

Let s be the fraction of the neutral A-atoms produced in the gas that reaches the central electrodes before transforming into B-atoms. Then

$$\begin{aligned} s &= -\frac{D\left(\frac{dn}{dz}\right)}{ql} \\ &= \frac{D^{\frac{1}{2}}}{l\lambda^{\frac{1}{2}}}\tanh\left(\frac{l\lambda^{\frac{1}{2}}}{D^{\frac{1}{2}}}\right). \end{aligned}$$

Wellish* found 0·045 for the coefficient of diffusion of the neutral deposit atoms of radium emanation in air at atmospheric pressure. He neglected, however, the effect of the recoil of RaB during his experiments. On taking this into account, the writer has found that Wellish's results lead to the value 0·056. Using this value, for RaA in air at atmospheric and at 20 cm. pressure, in the vessel shown in fig. 1, $s = 0·978$ and 0·994 respectively. The corresponding values for ThA are 0·107 and 0·207, and for AcA 0·018 and 0·025. At the latter pressure the amount of RaA transforming before reaching the electrodes may be neglected.

VI. *Radium A.*

The percentage of RaA initially positively charged in a gas may be deduced from the value of $\frac{c-d}{c+d}$ found in an experiment in which the gas is mixed with radium emanation, and from the percentage of RaB initially positively charged at the end of a recoil path in that gas.

Let a = the fraction of RaA atoms initially positively charged.
b = the fraction of RaB atoms initially positively charged. (Column 3, Table II.)
r = the efficiency of recoil of RaB from the electrodes. This number includes both the positive and neutral atoms.
$p_1 = \frac{c-d}{c+d}$, c and d being the cathode and anode activities in an exposure in which the gas is mixed with radium emanation.

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* *Loc. cit.*

TABLE II.

Gas.	Radium B.		Radium emanation.	p.c. of RaA positive.	Thorium emanation. $\frac{c-d}{c+d}\times 100$.		Dissociation of gas by heat.
	$\frac{\text{p.c. positive in gas}}{\text{p.c. positive in air}}$.	p.c. positive in gas.	$\frac{c-d}{c+d}\times 100$.		Observed.	Calculated.	
Oxygen	—	—	92·9	—	—	—	
Air	100	93·0	89·6	82·4	96·6	96·9	
Carbon dioxide	89·5	83·2	82·1	71·8	92·8	92·7	1 % dissociated at 1300° C.†
Carbon monoxide......	—	—	78·5	—	—	—	Not sensible at 1400° C. (white heat) ‡.
Nitrous oxide	85·8	79·8	75·2	61·8	89·6	89·2	1·5 % dissociated at 520° C.§
Methane	84·0	78·1	46·3	15·9	74·2	71·6	Not appreciable below 700° C.‖
Acetylene	80·5	74·9	31·3	2·2 *	64·1	64·9	Below 480° C.‖
Ammonia	74·6	69·4	27·9	0·05 *	57·1	59·2	Begins 450° to 500° C.¶
Ethylene	66·5	61·8	23·4	−1·6 *	52·0	54·3	Slowly at 570° C.‖
Hydrogen sulphide ...	59·0	54·9	22·4	0·8 *	48·3	49·1	2·3 % dissociated at 627° C.**

* Zero, to within experimental error.

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† Nernst and Wartenberg, *Zeit. Phys. Chem.* lvi. p. 548 (1906).

‡ Gautier, *Bull. Soc. Chim.* vii. p. 764 (1910).

§ Berthelot, *C. R.* lxxvii. p. 1448,

‖ Bone and Coward, J. Chem. Soc. Trans. p. 1197 (1908).

¶ White and Melville, J. Amer. Chem. Soc. xxvii. p. 363 (1905).

** Preuner, *Zeit. Anorg. Chem.* lv. p. 279 (1907).

Let us consider the activity accumulating on the anode. It receives by diffusion half of the neutral A atoms, *i. e.* $\frac{1-a}{2}$. Of these, $\frac{r}{4}(1-a)$ free themselves by recoil, and $\frac{br}{4}(1-a)$ of them are initially positively charged and are drawn to the cathode. The neutral atoms return to the anode by diffusion. Hence the total number of atoms left on the anode is $\frac{1}{2}\left(1-\frac{br}{2}\right)(1-a)$. The rest of the atoms resulting from the decay of the emanation reach the cathode. It is readily found that

$$a=\frac{2p_1-br}{2-br}.$$

Since b is obtained by multiplying the experimental values given in column 3 by 0·93, and r is obtained by dividing the value found for the efficiency of recoil found in air by the same number, then br and hence the above equation are independent of the experiment in which 93 per cent. was obtained for RaB in air. The values of a calculated from this equation are given in column 5.

VII. *Thorium Emanation.*

To test the theory given for the distribution of the active deposits, exposures were made with thorium emanation mixed with various gases, using the vessel shown in fig. 1. The relative difference in mass of the isotopes of RaA or of RaB is so small that we may assume that for each group of these isotopic atoms the percentage positively charged, the coefficient of diffusion, and the efficiency of recoil are identical. Hence it is easily shown that p_2, the value of $\frac{c-d}{c+d}$ for thorium emanation, is given by

$$p_2=a+b(1-a)\left(1-s+\frac{sr}{2}\right).$$

It is to be noted that p_2, owing to the terms involving s, depends on the size and the type of testing vessel used and also on the pressure of the gas. The values of p_2 found by experiment and by means of this equation are given in columns 6 and 7. It was assumed that the coefficient of diffusion of the A-atoms is inversely proportional to the square root of the molecular weight of the gas through which diffusion takes place. The experiments with radium and thorium emanations were made at a pressure of 20 cm., the time of exposure being generally greater than 16 hours, and at a voltage of 720, except in the case of hydrogen sulphide, for which consistent results for either emanation or in the experiments with RaB were only obtained when voltages as high as 10,000 were used. Zinc electrodes were used for the gases ammonia, acetylene, and hydrogen sulphide. Each result given in Table II. is the mean of several experiments. It was found necessary to liquify and fractionate the gases methane and ethylene to free them from the vapours of organic compounds produced during their preparation. Hydrogen sulphide was obtained by heating a solution of magnesium hydrosulphide, and was also fractionated.

VIII. *Actinium Active Deposit.*

The experiments of Kennedy*, Lucian †, or of McKeehan ‡ might be used to test the theory given above for the case of actinium emanation. Lucian's experiments are by far the most suitable for this purpose, since direct recoil on to the electrodes was negligible in his but not in the experiments of the other two. Lucian's method was very similar to that used by Wellish for radium active deposit. In section V. it has been shown that the number of neutral atoms of AcA diffusing to the walls of a vessel with parallel plates 2 cm. apart, before transforming into AcB, is 1·8 per cent. at atmospheric pressure. The number so diffusing, in the vessel used by Lucian, which was a cylinder of 4·9 cm. diameter with a central rod as cathode, will be of the same order of magnitude and may be neglected. By means of a formula given by Lucian, the amount of AcA positively charged which reaches the cathode before transforming into AcB may be calculated. Taking 1·2 for the mobility of AcA, the fraction of the AcA positively charged reaching the cathode in Lucian's vessel, before transforming into AcB, is 14 per cent. at 600 volts, 17 at 980 volts, and 20 at 1700 volts. Let us consider 100 molecules of actinium emanation. The first transformation produces 100−82·4 or 17·6 initially neutral atoms of AcA. Practically all of these break up before diffusing to the walls, and 100−93 or 7 per cent. of them, *i. e.* 1·23, become neutral atoms

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* Kennedy, Phil. Mag. xviii. p. 744 (1909).
† Lucian, Phil. Mag. xxviii. p. 761 (1914).
‡ McKeehan, Phys. Rev. x. p. 473 (1917).

of AcB. Of the original 82·4 positive A-atoms, 86 per cent. at 600 volts break up before reaching the cathode, and therefore 82·4 × 0·86 × 0·07 or 4·96 become neutral B-atoms. The total number of neutral B-atoms is then 6·19, and since the half-transformation period of AcB is 36 minutes, they all diffuse to the walls and remain there at the β-transformation to AcC. Lucian found, for the smallest quantity of emanation used, 6·3 per cent. neutral, volume recombination being apparently not entirely absent. At 980 he found values ranging from 5·3 to 5·0, while the value calculated as above is 5·9; at 1700 he found 5·1 per cent. neutral, while the calculated value is 5·8. It is to be noted that the neutral B-atoms resulting from the disintegration of positive A-atoms have been carried some distance towards the cathode, and hence the cathode receives a greater percentage of the neutral atoms while the field is on than when there is no field. The values found by Lucian are calculated on the assumption that the cathode receives the same percentage of neutral atoms in both cases, and hence they are too small. Lucian's results are therefore consistent with the general explanation advanced for the distribution of the active deposits.

IX. *Discussion of the Results.*

These experiments confirm Wellish's conclusion that there is a definite limiting value to the percentage of the active deposit initially positively charged in a gas, and that the anode activity is due to the diffusion of neutral uncharged deposit. The latter conclusion is at variance with that of Ratner [*], who maintained that the anode activity is due to the carriage to it by an *electric wind* of deposit particles which have lost their charge. It is difficult, however, to see how there can be any *wind* involving the motion of neutral gas molecules or neutral deposit atoms in an enclosed parallel plate vessel such as used in the present experiments. The gas-pressure acting on the two plates is certainly increased as a result of the electric force acting on the positive and negative ions; but there will be no wind unless these forces are unbalanced, as, for example, if the plates were perforated, or as in the case of discharge from a point. While there is no doubt that many of Ratner's experimental results were due to an electric wind, quite a different explanation is suggested for those made in the glass tube with disk electrodes, and which have most bearing on the present subject. It is possible that the large amount of emanation used per c.c., which was

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[*] Ratner, Phil. Mag. xxxiv. p. 429 (1917).

about twenty times that at which Wellish * showed that aggregation set in, together with the additional strong source of alpha-ray ionization, resulted in the production of aggregates of deposit atoms which are known to be able to take positive or negative charges. They are likely then to assume a positive or negative charge according as to whether they are in a region in which there is an excess of positive or of negative ions. The additional source of alpha-ray ionization, together with the separation of the ions by the electric field, would produce a distribution of positive and negative ions in the gas which would account for the results on this theory.

On searching for a general law connecting the results obtained and the properties of the various gases, it was seen that the percentage of either RaA or RaB positively charged was generally high for those gases whose heat of formation was large. However, a closer relationship appeared to exist between the results and the stability of the gases for increase of temperature: this latter quality, of course, depends partly on the heat of formation. In the table, data estimated from the results of various observers are given concerning the temperatures at which the various gases begin to dissociate. For the gases investigated, the percentage of recoil atoms positively charged decreases with the temperature of dissociation of the gas. It is well known that alpha-rays decompose many gases, and as the initial energy of a recoil atom is of the order 10^{10} times the average energy of a gaseous molecule at 0° C., it is to be expected that a recoil atom will dissociate many of the molecules in its path, even in the case of elementary gases such as hydrogen and oxygen, and that a part at least of the observed dissociation is produced in this way. The dissociation of gases by heat is usually a bimolecular process; whereas if dissociation by recoil atoms occurs, it will probably consist in the disruption of single particles. Hunter †, for example, has shown that the dissociation of nitrous oxide by heat is represented by the equation

$$2N_2O = 2N_2 + O_2 ;$$

and Wourtzel ‡ gives reasons for believing that in the case of alpha-rays some of the gas is dissociated according to the equation

$$N_2O = NO + N.$$

If a bi-atomic molecule, for instance one of carbon monoxide,

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* Wellish, Phil. Mag. xxviii. p. 417 (1914).

† Hunter, *Zeit. Phys. Chem.* liii. p. 441 (1905).

‡ Wourtzel, *Le Radium*, xi. p. 333 (1919).

is dissociated by a recoil atom, a free atom of oxygen and one of carbon will be produced. The heat of formation of bimolecular oxygen from atomic oxygen has been estimated by Lewis * to be about 14×10^4 calories per gram-molecule. The work done, therefore, in dissociating single molecules will be much greater than when the dissociation is bimolecular. It is considered, therefore, that the dissociation of a gas by heat is only an approximate guide to the stability of the gas under the bombardment of recoil atoms.

A qualitative explanation of the results obtained may be given if the following assumptions are made :—

1. At a collision between a recoil atom and a molecule the former may become positively charged if neutral before the collision, if the relative velocity of the two is greater than a certain value, which depends on the nature of the recoil atom and of the gas.

2. If the relative velocity is greater than this amount, the recoil atom has a large chance of being positively charged immediately after a collision in which the gas molecule is not dissociated. If positive before such a collision, it generally retains its charge.

3. At a collision which dissociates the gas molecule, the recoil atom has a large chance of becoming neutral.

According to these assumptions, the last collision causing dissociation may leave the recoil atom neutral, and may reduce its velocity to zero or to an amount which is too small to allow the recoil atom to ionize itself and become positively charged. Hence some neutral atoms are likely to be present in all gases except those which are monatomic. If the minimum velocity at which dissociation can be effected is much greater than that at which the recoil atom can ionize itself, then its chance of being positive at the end of its path is large. If, however, the former velocity is less than the latter, all the atoms would be neutral if we were to assume that dissociation occurs at every collision, provided the velocity is sufficient and that the atom is never positive after such a collision. These two assumptions are not at all likely to be correct; but if the former velocity is very much less than the latter, values approaching zero would be expected from the assumptions previously made. In support of assumption (3), it is suggested that on the disruption of a molecule one or more electrons may be freed which are liable to be captured by a positively charged recoil atom. Also, a head-on dissociating collision will probably be inelastic, the recoil atom penetrating the molecule and

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* Lewis, J. Chem. Soc. Trans. cxv. p. 182 (1919).

thus having a large chance of neutralizing itself. Assumption (1), besides appearing necessary in order to explain the present results, is a natural deduction from Wertenstein's [*] observation that RaD recoiling from RaC in very low vacuo is uncharged at the moment of expulsion, and that the rate of acquisition of a positive charge increases with the pressure.

The ionization potential of the elements increases generally from left to right in the periodic table. From this it is to be expected that the ionization potential of RaA is greater than that of RaB. If a similar relationship holds for ionization by collision with molecules, then the theory given would require that the percentages for RaA should be less than for RaB, as is found in all the gases examined, though it is quite possible that the effect is due to a difference in dissociating power of the two.

Summary.

1. Wellish's conclusion that a definite percentage of the active deposit atoms from radium emanation is initially positively charged at the end of the recoil path, the remainder being neutral, has been verified; and the error in Henderson's experiments, from which he concluded that all the deposit atoms are initially positive, pointed out.

2. The importance of the part played by the recoil of RaB from the electrodes has been indicated, and the percentages of RaB initially positively charged in various gases have been measured; and from this the percentages of RaA positively charged calculated from experiments with radium emanation

3. An explanation of the difference in distribution on the anode and cathode of the active deposits of radium, thorium, and actinium has been given on the assumption that the percentages of ThA and AcA and of ThB and AcB initially positive in a gas are identical with those of RaA and RaB respectively, as is to be expected from their isotopic nature. The distribution of thorium active deposit was experimentally determined for eight gases in a parallel plate vessel, and found to be consistent with the theory, as also were Lucian's results for the distribution of actinium active deposit in air in a cylindrical vessel.

4. An apparent connexion between the percentage of recoil atoms initially positive in a gas and the temperature

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[*] Wertenstein, *C. R.* clxi. p. 696 (1915).

at which the gas begins appreciably to dissociate has been indicated, and a theory, based on the assumption that the molecules of a gas are decomposed by the deposit atoms during their recoil, has been given to explain the behaviour of the deposit atoms.

I have to thank very sincerely Mr. E. M. Wellish, a suggestion of whose led to this work being undertaken, for his continual interest and advice.

The Physical Laboratory,
The University,
Sydney, N.S.W.,
October 13, 1920.

XXXIII. *Magnetic Field of Circular Currents.* *By* H. NAGAOKA, *Professor of Physics, Imperial University, Tokyo*[*].

[Plate VI.]

MANY years ago I[†] have shown how ϑ-functions are suited for calculating the strength of the magnetic field due to a circular current. By means of the same functions, the inductances of circular coils can be easily expressed by formulæ developed in q-series, which are rapidly convergent. In the present paper the expansion in q-series is applied to obtain expressions for the magnetic force of a single coil and of Gaugain-Helmholtz coils, at points not far distant from the axis, in powers of the coordinates.

It is usual to have recourse to expansion in spherical harmonics for expressing the magnetic force of circular currents; but as each harmonic contains different powers of coordinates calculated from the centre of a single coil, or from the middle point of the axis of double coils, the calculation of the deviations of the magnetic force from that at the middle point cannot be easily expressed in power series of the coordinates. By the expansion in q-series, we can arrange terms giving the deviations according to their importance in disturbing the field, and map out the distribution of these perturbations. The study of the curves of equal deviations will be of practical use for finding to what degree of accuracy the field can be assumed as uniform.

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[*] Communicated by the Author.

[†] Phil. Mag. vi. p. 19 (1903); Bull. Bureau Stand. xiii. pp. 269-393 1911); Phil. Mag. xxxv. p. 13 (1918).

It is found that deviations of different orders are positive in one sector starting from the mean point, followed by that of negative sign in alternate steps, and the curves of constant deviation resemble hyperbolas.

Recent advance in accurate measurement of current calls forth the necessity of deducing some formulæ, which will easily give the magnetic field accurate to about a millionth part. For this purpose, the ordinary method of expansion in spherical harmonics is not suitable, as the convergence is not sufficiently rapid. The following calculation was made with the view of meeting such needs in practical problems.

Let U be the Newtonian potential of a homogeneous body of rotation about the y-axis; referred to the axial coordinate y and the radial coordinate $\rho=\sqrt{x^2+z^2}$, Laplace's equation can be written

$$\frac{\partial^2 U}{\partial y^2}+\frac{\partial^2 U}{\partial \rho^2}+\frac{1}{\rho}\cdot\frac{\partial U}{\partial \rho}=0, \quad . \quad . \quad . \quad . \quad . \quad (1)$$

or

$$\frac{\partial^2 U}{\partial y^2}=-\frac{1}{\rho}\cdot\frac{\partial}{\partial \rho}\left(\rho\frac{\partial U}{\partial \rho}\right)$$

and

$$\frac{\partial^2 U}{\partial \rho\,\partial y}=\frac{1}{\rho}\cdot\frac{\partial}{\partial y}\left(\rho\frac{\partial U}{\partial \rho}\right).$$

If we put

$$\phi=\frac{\partial U}{\partial y} \quad \text{and} \quad \psi=\rho\frac{\partial U}{\partial \rho}, \quad . \quad . \quad . \quad . \quad . \quad (2)$$

they satisfy the equation

$$\frac{\partial \phi}{\partial \rho}\cdot\frac{\partial \psi}{\partial \rho}+\frac{\partial \phi}{\partial y}\cdot\frac{\partial \psi}{\partial y}=0,$$

showing that the families of surfaces $\phi=$ const. and $\psi=$ const. are orthogonal, and the sections through the y-axis of these surfaces represent diagrams of equipotentials and lines of force.

Referred to polar coordinates r and θ, the potential of a uniform circular disk of radius a is given by

$$U=\int_0^a\int_0^{2\pi}\frac{r\,dr\,d\theta}{\sqrt{\rho^2-2r\rho\cos\theta+r^2+y^2}}. \quad . \quad . \quad . \quad (3)$$

Putting $R^2=\rho^2-2r\rho\cos\theta+r^2$, we have by the addition theorem of cylinder functions

$$J_0(\lambda R)=J_0(\lambda\rho)J_0(\lambda r)+2\sum_1^\infty J_n(\lambda\rho)J_n(\lambda r)\cos\theta,$$

and by the well-known integral

$$\int_0^\infty e^{-\lambda y} J_0(R\lambda)\,d\lambda = \frac{1}{\sqrt{R^2+y^2}},$$

$$U = 2\pi \int_0^\infty \int_0^a e^{-\lambda y} r\, J_0(\lambda\rho) J_0(\lambda r)\, d\lambda\, dr,$$

which by the relation

$$rJ_0(\lambda r) = \frac{1}{\lambda} \cdot \frac{\partial}{\partial r}\left(rJ_1(\lambda r)\right),$$

gives

$$U = 2\pi a \int_0^\infty \frac{e^{-\lambda y}}{\lambda} J_0(\lambda\rho) J_1(\lambda a)\, d\lambda. \quad . \quad . \quad . \quad (4)$$

The magnetic potential of a circular current is easily derivable from the above result by simple differentiation

$$\phi = -\frac{\partial U}{\partial y} = 2\pi a \int_0^\infty e^{-\lambda y} J_0(\lambda\rho) J_1(\lambda a)\, d\lambda, \quad . \quad (5)$$

and the function which represents the magnetic lines of force is

$$\psi = -\rho \frac{\partial U}{\partial \rho} = 2\pi a \rho \int_0^\infty e^{-\lambda y} J_1(\lambda\rho) J_1(\lambda a)\, d\lambda. \quad . \quad (6)$$

These formulæ were already obtained in my former paper; the present investigation aims at the development of formula (5) for expressing the magnetic field due to different coils.

By a simple differentiation of (5) with respect to ρ and y, we obtain y- and ρ-components Y and P of the magnetic field due to a unit current:

$$\left.\begin{aligned} Y &= -\frac{\partial \phi}{\partial y} = 2\pi a \int \lambda e^{-\lambda y} J_0(\lambda\rho)\, J_1(\lambda a)\, d\lambda, \\ P &= -\frac{\partial \phi}{\partial \rho} = 2\pi a \int \lambda e^{-\lambda y} J_1(\lambda\rho)\, J_1(\lambda a)\, d\lambda. \end{aligned}\right\} \quad . \quad (7)$$

From the addition theorem, we obtain

$$J_0(\lambda\rho)\, J_1(\lambda a) = \frac{1}{\pi} \int_0^\pi \frac{J_1(\lambda R)}{R} (a - \rho\cos\theta)\, d\theta,$$

$$J_1(\lambda\rho)\, J_1(\lambda a) = \frac{1}{\pi} \int_0^\pi J_0(\lambda R)\cos\theta\, d\theta,$$

and

$$\int_0^\infty e^{-\lambda y} J_1(\lambda R)\, d\lambda = \frac{1}{R}\left(1 + \frac{y}{\sqrt{R^2+y^2}}\right);$$

whence by differentiation with respect to y, we have

$$\int_0^\infty \lambda e^{-\lambda y} J_1(\lambda R)\, d\lambda = \frac{R}{(R^2+y^2)^{\frac{3}{2}}}$$

and

$$\int_0^\infty \lambda e^{-\lambda y} J_0(\lambda R)\, d\lambda = \frac{y}{(R^2+y^2)^{\frac{3}{2}}}.$$

Consequently

$$Y = -\frac{\partial\phi}{\partial y} = 2a\int_0^\infty\int_0^\pi \lambda e^{-\lambda y} J_1(\lambda R)\left(\frac{a-\rho\cos\theta}{R}\right) d\lambda\, d\theta$$

$$= 2a\int_0^\pi \frac{(a-\rho\cos\theta)}{(R^2+y^2)^{\frac{3}{2}}}\, d\theta\,;$$

$$P = -\frac{\partial\phi}{\partial\rho} = 2a\int_0^\infty\int_0^\pi \lambda e^{-\lambda y} J_0(\lambda R)\cos\theta\, d\lambda\, d\theta$$

$$= 2ay\int_0^\pi \frac{\cos\theta\, d\theta}{(R^2+y^2)^{\frac{3}{2}}}.$$

These expressions can be evaluated in terms of elliptic integrals, and have been first deduced by Russell*, following an entirely different method, which is more direct than that here given.

The evaluation of the elliptic integrals in (7) leads easily to the component Y in the axial direction and X perpendicular to it at the point x, y, o. Thus

$$\left.\begin{aligned} X &= \frac{2}{r_1}\left\{\frac{2ay}{r_1'^2}E - \frac{y}{x}(K-E)\right\}, \\ Y &= \frac{2}{r_1}\left\{\frac{2a(a-x)}{r_1'^2}E + (K-E)\right\}, \end{aligned}\right\} \quad . \quad . \quad (8)$$

where

$$r_1^2 = (a+x)^2+y^2,$$
$$r_1'^2 = (a-x)^2+y^2,$$

and the modulus of the elliptic integrals is given by

$$k^2 = \frac{4ax}{r_1^2}.$$

The case of two coaxial circles of the same radius, at distance equal to the radius, is of great importance in obtaining uniform magnetic field, and applies to Gaugain-Helmholtz coils. As the field near the mean point is

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* Proc. London Phys. Soc. xx. p. 476 (1907).

of practical importance, we shall confine our attention to the case of small values of the coordinates ξ, η referred to the mean point as origin, so that

$$\xi = x\,; \quad \eta = y - \frac{a}{2}\,; \quad . \quad . \quad . \quad . \quad . \quad . \qquad (9)$$

where x, y are referred to axes with the origin at the centre of the first circle. The magnetic force at ξ, η is the sum of the forces arising from the first and the second circle. Evidently the expressions (8) can be used in calculating the component forces; as to the modulus of the elliptic integrals K and E, we notice that

$$k_1^2 = \frac{4a\xi}{r_1^2}, \text{ where } r_1^2 = (a+\xi)^2 + \left(\frac{a}{2}+\eta\right)^2 \quad . \quad . \qquad (9')$$

for the first coil, and for the second

$$k_2^2 = \frac{4a\xi}{r_2^2}, \text{ where } r_2^2 = (a+\xi)^2 + \left(\frac{a}{2}-\eta\right)^2 \quad . \quad . \qquad (9'')$$

The expressions for X and Y due to the circle carrying unit current are

$$\left.\begin{aligned} \mathrm{X} &= \frac{2}{r_1}\left\{\frac{2a\left(\frac{a}{2}+\eta\right)}{(a-\xi)^2+\left(\frac{a}{2}+\eta\right)^2}\mathrm{E} - \frac{\left(\frac{a}{2}+\eta\right)}{\xi}(\mathrm{K}-\mathrm{E})\right\}; \\ \mathrm{Y} &= \frac{2}{r_1}\left\{\frac{2a(a-\xi)}{(a-\xi)^2+\left(\frac{a}{2}+\eta\right)^2}\mathrm{E} + (\mathrm{K}-\mathrm{E})\right\}. \end{aligned}\right\} \qquad (10)$$

This may be conveniently written under the form

$$\begin{aligned} \mathrm{X} &= \frac{\left(\frac{a}{2}+\eta\right)}{2\sqrt{a\xi^3}}k\left\{\frac{k^2}{k'^2}\mathrm{E} - 2(\mathrm{K}-\mathrm{E})\right\} \\ &= \frac{\left(\frac{a}{2}+\eta\right)k^2}{2\sqrt{a\xi^3}}\left\{\frac{k^2}{k'^2}\mathrm{E} + 2\frac{d\mathrm{E}}{dk}\right\}; \quad . \quad . \quad . \quad . \\ \mathrm{Y} &= \frac{k}{2\sqrt{a\xi}}\left[\frac{a}{\xi}\frac{k^2}{k'^2}\mathrm{E} - \left\{\frac{k^2}{k'^2}\mathrm{E} - 2(\mathrm{K}-\mathrm{E})\right\}\right] \\ &= \frac{k^2}{2\sqrt{a\xi}}\left\{\frac{k}{k'^2}\left(\frac{a}{\xi}-1\right) - 2\frac{d\mathrm{E}}{dk}\right\}. \quad . \quad . \quad . \quad . \qquad (11') \end{aligned}$$

For the first coil we have to put the value of k_1 in (9′), and for the second coil k_2 in (9″) for calculating K and E; the value of η is positive for the first coil, but negative for the second. This can be easily distinguished by adding the suffix + or − to X and Y. Thus we may conveniently write for these components

$$X = X_+ + X_- ; \quad Y = Y_+ + Y_- . \quad . \quad . \quad . \quad (12)$$

We shall in the first place confine our attention to the evaluation of the values of X and Y for the single coil. Evidently k, K, and K−E can be expressed by means of ϑ-functions as given in every text-book on elliptic functions, so that by expansion in terms of q we obtain easily

$$\left.\begin{aligned} k\left\{\frac{k^2}{k'^2}E - 2(K-E)\right\} &= 192\pi q^{\frac{5}{2}}(1+20q^2+225q^4+\ldots); \\ \frac{k^3}{k'^2}E &= 32\pi q^{\frac{3}{2}}(1+30q^2+423q^4+3986q^6+\ldots). \end{aligned}\right\} (13)$$

These expressions are of great importance in calculating the value of the magnetic force. Thus for a single coil,

$$\left.\begin{aligned} X &= 96\pi\frac{\eta}{\sqrt{a\xi^3}}q^{\frac{5}{2}}(1+20q^2+225q^4+\ldots); \\ Y &= 16\pi\frac{1}{\sqrt{a\xi}}q^{\frac{3}{2}}\left\{\frac{a}{\xi}(1+30q^2+423q^4+3986q^6+\ldots)\right. \\ &\qquad \left. -6q(1+20q^2+225q^4+\ldots)\right\}; \end{aligned}\right\} \text{(I.)}$$

and for a double coil referred to the middle point as origin,

$$\left.\begin{aligned} X_\pm &= 96\pi\frac{\left(\frac{a}{2}\pm\eta\right)}{\sqrt{a\xi^3}}q_\pm^{\frac{5}{2}}(1+20q_\pm^2+225q_\pm^4+\ldots); \\ Y_\pm &= 16\pi\frac{1}{\sqrt{a\xi}}q_\pm^{\frac{3}{2}}\left\{\frac{a}{\xi}(1+30q_\pm^2+423q_\pm^4+3986q_\pm^6+\ldots)\right. \\ &\qquad \left. -6q_\pm(1+20q_\pm^2+225q_\pm^4+\ldots)\right\}. \end{aligned}\right\} \text{(II.)}$$

Putting $x=\xi$, $y=\eta$ in (8), we find for the case of a single coil

$$k^2 = \frac{4a\xi}{(a+\xi)^2+\eta^2};$$

whence we find, by expansion and the known relation

$$q = \frac{k^2}{16} + \frac{k^4}{32} + \frac{21}{1024} k^6 + \dots ,$$

the following powers of q expressed in terms of the coordinates. As they are of great convenience in various calculations of similar nature, it will not be out of place to give them to the 7th power of ξ and η.

$$\left.\begin{aligned}
q &= \frac{\xi}{4a} + \frac{\xi^3}{16a^3} - \frac{\xi\eta^2}{4a^3} + \frac{17\xi^5}{512a^5} - \frac{7\xi^3\eta^2}{16a^5} + \frac{\xi\eta^4}{4a^5} + \frac{8045\xi^7}{2048a^7} \\
&\qquad - \frac{309\xi^5\eta^2}{512a^7} + \frac{9\xi^3\eta^4}{8a^7} - \frac{\xi\eta^6}{4a^7} + - \dots , \\
q^2 &= \frac{\xi^2}{16a^2} + \frac{\xi^4}{32a^4} - \frac{\xi^2\eta^2}{8a^4} + \frac{21\xi^6}{1024a^6} - \frac{\xi^4\eta^2}{4a^6} + \frac{3\xi^2\eta^4}{16a^6} + - \dots , \\
q^3 &= \frac{\xi^3}{64a^3} + \frac{3\xi^5}{256a^5} - \frac{3\xi^3\eta^2}{64a^5} + \frac{75\xi^7}{8192a^7} - \frac{27\xi^5\eta^2}{256a^7} + \frac{3\xi^3\eta^4}{32a^7} + - \dots , \\
q^4 &= \frac{\xi^4}{256a^4} + \frac{\xi^6}{256a^6} - \frac{\xi^4\eta^2}{64a^6} + - \dots , \\
q^5 &= \frac{\xi^5}{1024a^5} + \frac{5\xi^7}{4096a^7} - \frac{5\xi^5\eta^2}{1024a^7} + - \dots , \\
q^6 &= \frac{\xi^6}{4096a^6} + - \dots , \\
q^7 &= \frac{\xi^7}{16384a^7} + - \dots , \\
\sqrt{q} &= \frac{1}{2}\sqrt{\frac{\xi}{a}} \left\{ 1 + \frac{\xi^2}{8a^2} - \frac{\eta^2}{2a^2} + \frac{15\xi^4}{256a^4} - \frac{13\xi^2\eta^2}{16a^4} + \frac{3\eta^4}{8a^4} \right. \\
&\qquad \left. + \frac{16075\xi^6}{2048a^6} + \frac{551\xi^4\eta^2}{512a^6} + \frac{115\xi^2\eta^4}{64a^6} - \frac{5\eta^6}{16a^6} + - \dots \right.
\end{aligned}\right\} \quad (14)$$

From these expressions we easily find, by substituting in the expansion in terms of q, the following formulæ for X and Y :

$$\mathrm{X} = \frac{\pi}{a^3}\xi\eta\left\{3 + \frac{45}{8}\frac{\xi^2}{a^2} - \frac{15}{2}\frac{\eta^2}{a^2} + \frac{525}{64}\frac{\xi^4}{a^4} - \frac{525}{16}\frac{\xi^2\eta^2}{a^4} + \frac{105}{8}\frac{\eta^4}{a^4} + \dots\right\}; \quad \text{(A)}$$

$$\begin{aligned}
\mathrm{Y} = \frac{\pi}{a}\bigg\{ 2 &+ \frac{3}{2}\frac{\xi^2}{a^2} - 3\frac{\eta^2}{a^2} + \frac{45}{32}\frac{\xi^4}{a^4} - \frac{45}{4}\frac{\xi^2\eta^2}{a^4} + \frac{15}{4}\frac{\eta^4}{a^4} + \frac{6175}{128}\frac{\xi^6}{a^6} \\
&- \frac{1575}{64}\frac{\xi^4\eta^2}{a^6} + \frac{525}{16}\frac{\xi^2\eta^4}{a^6} - \frac{35}{8}\frac{\eta^6}{a^6} + \dots \bigg\}. \quad \text{(B)}
\end{aligned}$$

Thus the values of the magnetic force, as we proceed outward from the plane of the circle by ξ, η, are given by the formulæ (A) and (B). The process seems at first rather tedious, but when once the value of q is known in terms of the coordinates ξ, η, the expansion in ascending powers of the variables is but a simple process of addition and multiplication. The usual process of expansion in zonal harmonics of different orders is looked upon as a practical guide to the solution of such problems. If we try to treat the problem such that the magnetic force at the point ξ, η is expanded in different powers of ξ, η, the solution in spherical harmonics becomes very tedious as each harmonic involves different powers of ξ, η. A practical calculation will convince the reader of the facility of the q-series. Of course the knowledge of the expansions of the integrals in q-series must be presupposed.

The above calculation was checked by expanding the component forces X and Y in zonal harmonics. The advantage of expansion in power series of the coordinates referred to the centre and the plane of the circle lies in finding the deviations of the magnetic force from the value at the mean point as we recede from it. We can group terms of different powers and discuss their influence in disturbing the field, just as we discuss the aberrations of a lens, by expanding them according to powers of the so-called Seidel's coordinates, and class the deviations according to the orders of the coordinates involved in the aberrations.

Radial Component X *for a Single Coil.*

Reverting to the expression of magnetic force X given in (A), we find, by assuming the radius of the coil a and the axial component of force at the centre to be both unity, that the deviation of the second order is given by

$$\tfrac{3}{2}\xi\eta = c, \quad . \quad . \quad . \quad . \quad . \quad . \qquad (15)$$

(Pl. VI. Fig. 1.)

which is represented by rectangular hyperbolas. By considering c as a parameter, and giving small fractional values to it, we obtain loci of points at which the radial component is c times the force at the centre. Thus the distribution of the radial deviations of second order is given in fig. 1; they are mapped out for equal difference of c except for curves near the centre, in a square of sides equal to the radius, in a plane through the centre perpendicular to the plane of the circle.

Since the deviations of odd orders vanish for a circular coil, we have next to discuss the deviation of fourth order, which is given by

$$\tfrac{15}{16}(3\xi^3\eta - 4\xi\eta^3) = c, \quad . \quad (16) \text{ (Fig. 2.)}$$

as is evident from (A). The curves of constant deviation have different branches, all of which resemble hyperbolas, with asymptotes inclined to the axis of ξ at angles of ±40° 53′ 48″ and the normals at the vertices inclined at ±20° 58′ 32″ and ±66° 7′ 18′.. The distribution of the deviation about the centre is shown in fig. 2.

The deviation of sixth order is given by

$$\tfrac{105}{128}(5\xi^5\eta - 20\xi^3\eta^3 + 8\xi\eta^5) = c, \quad (17) \text{ (Fig. 3.)}$$

The curves of constant deviation are more complicated than the preceding, but have branches resembling hyperbolas, of which the asymptotes are inclined to the axis of ξ at angles of ±27° 57′ 34″ and ±56° 7′ 8″. The normals at the vertices are inclined to ξ at angles of ±14° 8′ 56″, ±42° 39′ 11″, ±73° 37′ 44″. The inspection of fig. 3 will at once show how the deviation of this order is very small near the centre, but increases very rapidly as we recede from it. For practical purposes the deviation of higher order will be generally negligible, and if necessary the calculation will not entail much labour.

Axial Component Y *of a Single Coil.*

As to the axial component Y, the discussion of the deviations can be made in the same manner as already indicated, so that it will be only necessary to tabulate them. The common characteristics of the curves of constant deviation are that they all resemble hyperbolas and are separated by asymptotes, which form the boundaries of positive and negative deviations, as is evident from the figures at a glance.

Deviation of second order for Y-component :—

$$\tfrac{3}{4}(\xi^2 - 2\eta^2) = c. \quad . \quad . \quad . \quad (18) \text{ (Fig. 4.)}$$

Asymptotes : ±35° 15′ 52″.

Normals at the vertices : 0° ; ±90° ; 180°.

Deviation of fourth order for Y-component :—

$$\tfrac{15}{64}(3\xi^4-24\xi^2\eta^2+8\eta^4)=c. \quad . \quad (19)\ \text{(Fig. 5.)}$$

Asymptotes : ±19° 52′ 33″; ±59° 26′ 40″.
Normals at the vertices : 0° ; 180°; ±40° 53′ 36″; ±90°.

Deviation of sixth order for Y-component :—

$$\tfrac{5}{256}(1235\xi^6-630\xi^4\eta^2+840\xi^2\eta^4-112\eta^6)=c. \quad (20)\ \text{(Fig. 6.)}$$

Asymptotes : ±69° 41′ 34″.
Normals at the vertices : 0°; 180°; ±90°.

Gaugain-Helmholtz Coils.

For the calculation of the double coils at distance equal to the radius of the coil, we have

$$k^2=\frac{4a\xi}{(a+\xi)^2+\left(\frac{a}{2}\pm\eta\right)^2}, \quad . \quad . \quad . \quad . \quad . \quad (21)$$

where + sign refers to the first coil and − sign to the second.

Expanding k^2 and expressing q according to the formula connecting q and k, we obtain the following sets of formulæ :—

$$\left.\begin{array}{l}
q_{\pm}=\dfrac{\xi}{5a}\left\{1\mp\dfrac{4}{5}\dfrac{\eta}{a}-\dfrac{4}{25}\dfrac{\eta^2}{a^2}+(\mp5\xi^2\eta\pm6\eta^3)\dfrac{16}{125a^3}\right. \\
\qquad\qquad \left.+(-31\xi^4+280\xi^2\eta^2-152\eta^4)\dfrac{2}{625a^4}+\ldots\right\}, \\
q^2_{\pm}=\dfrac{\xi^2}{25a^2}\left\{1\mp\dfrac{8\eta}{5a}+\dfrac{8\eta^2}{25a^2}+(\mp5\xi^2\eta\pm7\eta^3)\dfrac{32}{125a^3}+\ldots\right\}, \\
q^3_{\pm}=\dfrac{\xi^3}{125a^3}\left\{1\mp\dfrac{12\eta}{5a}+\dfrac{36\eta^2}{25a^2}+\ldots\right\}, \\
q^4_{\pm}=\dfrac{\xi^4}{625a^4}\left\{1\mp\dfrac{16\eta}{5a}+\ldots\right\}, \\
q^5_{\pm}=\dfrac{\xi^5}{3125a^5}+\ldots, \\
\sqrt{q_{\pm}}=\sqrt{\dfrac{\xi}{5a}}\left\{1\mp\dfrac{2\eta}{5a}-\dfrac{4\eta^2}{25a^2}+(\mp5\xi^2\eta\pm5\eta^3)\dfrac{8}{125a^3}\right. \\
\qquad\qquad \left.+(-31\xi^4+200\xi^2\eta^2-80\eta^4)\dfrac{1}{625a^4}+\ldots\right\}.
\end{array}\right\} \quad (22)$$

whence we notice that

$$q_+ + q_- = \frac{2\xi}{5a}\left\{1 - \frac{4}{25}\frac{\eta^2}{a^2} + (-31\xi^4 + 280\xi^2\eta^2 - 152\eta^4)\frac{1}{625a^4} + \dots\right\},$$

$$q_+^2 + q_-^2 = \frac{2\xi^2}{25a^2}\left\{1 + \frac{8}{25}\frac{\eta^2}{a^2} + \dots\right\},$$

$$q_+^3 + q_-^3 = \frac{2\xi^3}{125a^3}\left\{1 + \frac{36}{25}\frac{\eta^2}{a^2} + \dots\right\},$$

$$q_+^4 + q_-^4 = \frac{2\xi^4}{625a^4} + \dots,$$

$$\sqrt{q_+} + \sqrt{q_-} = 2\sqrt{\frac{\xi}{5a}}\left\{1 - \frac{4}{25}\frac{\eta^2}{a^2} + (-31\xi^4 + 200\xi^2\eta^2 - 80\eta^4)\frac{1}{625a^4} + \dots\right\}.$$

It will not be out of place to illustrate the advantages of q-series (II.) in the calculation of X and Y in the case of double coils. If we use the formulæ (II.) for the calculation of X_+, X_-, and Y_+, Y_-, the second term in X contains $q^{9/2}$, and the third term in Y $q^{11/2}$. Take for example the case $\xi = 0{\cdot}2$, $\eta = 0{\cdot}2$; then

$$q_+ = 0{\cdot}031 \qquad \text{and} \qquad q_- = 0{\cdot}041 \text{ nearly :}$$

so that

$$q_+^{9/2} = 0{\cdot}000\,00016 \quad \text{and} \quad q_-^{9/2} = 0{\cdot}000\,000057,$$

and $$q_+^{11/2} = 0{\cdot}0000\,00005 \quad \text{and} \quad q_-^{11/2} = 0{\cdot}0000\,00024.$$

Thus three terms are generally sufficient to attain the limit of practical accuracy. q must, however, be calculated to six or seven significant figures.

Substituting the values of q_+, q_- in the expression for X_+, X_-, and Y_+, Y_-, as given for a single coil in (A) and (B), we easily find that

$$X = \frac{2304\pi}{625\sqrt{5}a^5}\xi\eta(4\eta^2 - 3\xi^2) + \dots \qquad \text{(C)}$$

and

$$Y = \frac{32\pi}{5\sqrt{5}a}\left\{1 - \frac{18}{125a^4}(3\xi^4 - 24\xi^2\eta^2 + 8\eta^4)\right\} + \dots, \qquad \text{(D)}$$

showing that at the middle point of the coils

$$X = 0 \text{ ; and } Y = \frac{32\pi}{5\sqrt{5}a}. \qquad (23)$$

The deviations of the second order vanish for the double coils here considered, so that we have only to discuss those of the fourth order upward. The beauty of the arrangement lies chiefly in the elimination of this most important deviation, and we at once notice how terms of higher order are insignificant and make the use of the arrangement for obtaining uniform field very important in some practical applications.

In the following, deviations of fourth order for X and Y components are given in tables.

Deviation of fourth order for X-component :—

$$\tfrac{72}{125}(4\eta^3\xi - 3\eta\xi^3) = c. \quad . \quad . \quad . \qquad (24)\ (\text{Fig. } 7.)$$

Asymptotes : ±40° 53′ 48″.
Normals at the vertices : ±20° 58′ 32″; ±66° 7′ 18″.

Deviation of fourth order for Y-component :—

$$-\tfrac{18}{125}(3\xi^4 - 24\xi^2\eta^2 + 8\eta^4) = c. \quad . \quad . \qquad (25)\ (\text{Fig. } 8.)$$

Asymptotes : ±19° 52′ 30″ ; ±59° 26′ 40″.
Normals at the vertices: 0°; 180°; ±90°; ±40° 53′ 48″.

The comparison of the figures 7 and 8 with the corresponding deviations of fourth order for a single coil in figures 3 and 4 shows how the deviations in double coils are small, and especially the gradients. The use of double coils in exact measurements of magnetic field, as for example of the terrestrial magnetic force, is strongly to be recommended.

The discussion here given refers to single layer coils. By winding the coils in different layers, we have to choose a proper section of winding layers. For a rectangular section, Maxwell shows that when the ratio of the depth to the breadth is as 6 to 5·57, an important correction due to the finite section of the winding is made to disappear.

As before discussed, the deviations of magnetic force due to terms of different orders have an analogy to the aberrations of lenses and mirrors, and will be of great advantage in designing galvanometers and other electric instruments, in which high order of accuracy and the uniformity of the field are a principal aim.

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XXXIV. *Integration Theorems of Four-Dimensional Vector Analysis.* *By* H. T. FLINT, *Lecturer in Physics, King's College, London* *.

Introduction.

THE four-dimensional vector analysis developed by Sommerfeld (*Ann. der Physik*, vols. xxxii. p. 749 *sqq.* and xxxiii. p. 649 *sqq.*) evidently bears the same relation to a more general analysis as do scalar and vector products and the theorems of Gauss and Stokes to vector or quaternion analysis.

In attempting to work out some of the details of the restricted principle of Relativity in terms of the more general notation, it was observed that such expressions as the scalar and vector products of four vectors, combinations of four and six vectors, together with the generalized div and curl and the operation lor (Lorentz operator) of Minkowski's Calculus are parts of more general expressions.

It is the object of the following pages to set forth this analysis. It will be obvious to the reader how much is owed to quaternion analysis, in particular to Joly's 'Manual of Quaternions,' ch. xviii., of which the notation is adopted here, and to Sommerfeld's papers mentioned above. The application of the notation to the theory of Relativity has already been pointed out by Professor Johnston and discussed by Sir J. Larmor †.

In the first few sections the notation is explained before the main object of developing the integral theorem is reached.

§ 1. In treating space of four dimensions it is usual and convenient to adopt a language and notation similar to that with which we are familiar in three, though it is difficult and perhaps impossible to form corresponding geometrical pictures.

Thus a straight line through the points $(a_1\, b_1\, c_1\, d_1)$ $(a_2\, b_2\, c_2\, d_2)$ is the locus :

$$\frac{x-a_1}{a_1-a_2}=\frac{y-b_1}{b_1-b_2}=\frac{z-c_1}{c_1-c_2}=\frac{u-d_1}{d_1-d_2}, \quad . \quad . \quad (1{\cdot}1)$$

$(x\, y\, z\, u)$ being current coordinates, axes Ox, Oy, Oz, Ou being assumed as axes of reference.

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* Communicated by the Author.

† Proc. Roy. Soc. Dec. 1919.

A vector is a quantity which may be represented by a straight line in this way.

We assume four directions in space and denote them by $i_1\ i_2\ i_3\ i_4$ and any other vector ρ may then be expressed in terms of them :

$$\rho = xi_1 + yi_2 + zi_3 + ui_4.$$

If $i_1\ i_2\ i_3\ i_4$ are unit vectors, $x\ y\ z\ u$ are the projections of ρ along them.

§ 2. If two vectors ρ_1 and ρ_2 are multiplied an expression results containing terms in $i_m i_n$ and $i_m i_m$, where $m \neq n$.

We define the product of the two like vectors $i_m i_m$ to be -1, and the product of $\rho_1\rho_2$ then contains terms in two vectors and other terms from which the i's have disappeared.

We write :

$$\rho_1\rho_2 = \mathrm{V}_2\rho_1\rho_2 + \mathrm{V}_0\rho_1\rho_2. \quad . \quad . \quad . \quad . \quad (2{\cdot}1)$$

$\mathrm{V}_2\rho_1\rho_2$ contains i_2i_3, i_1i_2 etc. and $\mathrm{V}_0\rho_1\rho_2$ denotes the remainder of the product not containing any i's. $\mathrm{V}_0\rho_1\rho_2$ with a minus sign prefixed is called the scalar product of $\rho_1\rho_2$.

If this vanishes ρ_1 and ρ_2 are said to be perpendicular. We complete our definition of the product $\rho_1\rho_2$ by writing

$$\rho_2\rho_1 = -\mathrm{V}_2\rho_1\rho_2 + \mathrm{V}_0\rho_1\rho_2. \quad . \quad . \quad . \quad . \quad (2{\cdot}2)$$

This gives a rule of multiplication known in the case of ordinary vector analysis. It is not commutative but is defined to be associative and distributive.

In choosing the unit vectors $i_1\ i_2\ i_3\ i_4$ as fundamental directions we make them mutually perpendicular,

$$\text{i. e.} \qquad \mathrm{V}_0 i_2 i_3 = 0 \quad \text{etc.}$$

The rule of multiplication then shows that

$$i_m i_n = -i_n i_m, \qquad m \neq n.$$

If we write :

$$\left.\begin{aligned} \rho_1 &= a_1 i_1 + b_1 i_2 + c_1 i_3 + d_1 i_4, \\ \rho_2 &= a_2 i_1 + b_2 i_2 + c_2 i_3 + d_2 i_4, \end{aligned}\right\} \quad . \quad . \quad . \quad (2{\cdot}3)$$

we have on expansion :

$$\mathrm{V}_0\rho_1\rho_2 = -(a_1a_2 + b_1b_2 + c_1c_2 + d_1d_2)$$

and

$$\begin{aligned} \mathrm{V}_2\rho_1\rho_2 = {} & i_2i_3(b_1c_2 - b_2c_1) + i_3i_1(c_1a_2 - c_2a_1) + i_1i_2(a_1b_2 - a_2b_1) \\ & + i_1i_4(a_1d_2 - a_2d_1) + i_2i_4(b_1d_2 - b_2d_1) + i_3i_4(c_1d_2 - c_2d_1.). \end{aligned}$$

$$. \quad . \quad (2{\cdot}4)$$

The vector $V_2\rho_1\rho_2$ is the vector area of the parallelogram of which the two adjacent sides are ρ_1 and ρ_2. It can be split up into six components, which we may denote by A_{yz}, A_{zx}, etc., these being the coefficients of i_2i_3, i_1i_2, etc., and which are the projections of the area on the six planes which may be chosen out of the four fundamental directions. The vector area although it has six coefficients is defined by five quantities, for a relation exists between the coefficients, viz.:

$$A_{yz}A_{xu}+A_{zx}A_{yu}+A_{xy}A_{zu}=0, \quad . \quad . \quad . \quad (2\cdot5)$$

as can be seen by expansion of the determinants.

The elementary area bounded by two short lines $d\rho$ and $\delta\rho$ may be written:

$$i_2i_3(dy\,\delta z-\delta y\,dz)+i_3i_1(dz\,\delta x-\delta z\,dx)+\dots \quad . \quad (2\cdot6)$$

Minkowski uses the terms four- and six-vectors in a particular sense, the vectors are subject to a particular transformation, but the comparison of these vectors with (2·3) and (2·4) is clear, *e. g.* (2·4) is a six-vector without the limitations regarding the transformation.

Thus the scalar and vector products of Minkowski's calculus are component parts of the complete quaternionic product $\rho_1\rho_2$. We may not speak of a unique perpendicular to a plane in four dimensions, for just as in three dimensions a perpendicular to a line is not determinate, so in three the perpendicular to a plane is not definite.

We have seen that a plane may be defined by two lines α and β, and it is denoted by $V_2\alpha\beta$.

The magnitude of the area, or the tensor as it is called, is defined to be the square root of the expression:

$$A_{yz}^2+A_{zx}^2+A_{xy}^2+A_{xu}^2+A_{yu}^2+A_{zu}^2.$$

If this is unity the plane is a unit plane.

We may now show that the plane normal to $V_2\alpha\beta$ is definite if we define it as the locus of lines perpendicular both to α and to β and consequently to any line in their plane. For let the unit plane normal to $V_2\alpha\beta$ be $V_2\gamma\delta$.

Then by definition:

$$V_0\alpha\gamma=V_0\alpha\delta=V_0\beta\gamma=V_0\beta\delta=0,$$

and the tensor of $V_2\gamma\delta=1$

or $TV_2\gamma\delta=1$.

These five equations determine the five quantities necessary to fix the plane, which is therefore definite if $V_2\alpha\beta$ is known.

It is easy to show that $V_2\gamma\delta$ written in full is :

$$i_2i_3A_{xu}+i_3i_1A_{yu}+i_1i_2A_{zu}+i_1i_4A_{yz}+i_2i_4A_{zx}+i_3i_4A_{xy},$$

where the A's denote the components of the original plane. It is to be noted that the coefficients A_{yz} and A_{xu}, and similarly for the other pairs, change their places in passing from a plane to its normal. We have tacitly assumed in this expansion of $V_2\gamma\delta$ that $V_2\alpha\beta$ is also a unit plane. If it is not, we have only to multiply throughout by a constant—the reciprocal of the tensor.

Thus if a plane is represented by kV_2, where k is a scalar constant, the tensor of the plane and V_2 is the unit vector defining it vectorially, any plane normal to it is $k'V_2'$, where k' is the tensor of the latter and V_2' is obtained from V_2 by interchanging the coefficients in the above manner.

Functions enter into analysis which have the structure of a vector area but do not satisfy the condition (2·5). They are defined by 6 quantities, and we may write such a function as

$$P_2=i_2i_3P_{yz}+i_3i_1P_{zx}+i_1i_2P_{xy}+i_1i_4P_{xu}+i_2i_4P_{yu}+i_3i_4P_{zu}, \quad (2·7)$$

and there corresponds to this the reciprocal function :

$$P_2'=i_2i_3P_{xu}+\ldots\ldots+\ldots\ldots+i_1i_4P^{yz}+\ldots\ldots+\ldots\ldots \quad (2·8)$$

But in this case the relation

$$V_2P_2P_2'\neq 0,$$

as in the case of the product of vector areas when these are normal.

We shall return to this point later.

It is to be noted that we can always express P_2 in the form :

$$P_2=kV_2\alpha\beta+k'V_2'\alpha\beta.$$

k' gives the additional arbitrary constant.

This should be compared with Sommerfeld's discussion, § 1 of vol xxxii., already referred to.

If a vector area be split up into elements so that each element projects into rectangles on the coordinate planes, we write for it :

$$dv=i_2i_3dy\,dz+i_2i_1dz\,dx+i_1i_2dx\,dy$$
$$+i_1i_4dx\,du+i_2i_4dy\,du+i_3i_4dz\,du. \quad . \quad (2·9)$$

This element is a vector like P_2.

§ 3. The laws of multiplication lead to certain relations between products of the i's of which the following are illustrative:

$$\left.\begin{aligned} i_1 i_2 i_1 &= -i_1 i_1 i_2 = i_2, \\ i_2 i_2 i_1 &= -i_1. \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad (3{\cdot}1)$$

A change of position of any vector by an odd number of places in any product changes the sign, unless the product is scalar, or contains a scalar part.

Thus

$$\alpha\beta\gamma = -\beta\alpha\gamma = \beta\gamma\alpha.$$

Products of the type $i_1 i_2 i_3$ are not reducible.

In multiplying three vectors it is convenient to regard the product as composed of reducible and irreducible parts. Thus

$$\rho_1\rho_2\rho_3 = \mathrm{V}_3\rho_1\rho_2\rho_3 + \mathrm{V}_1\rho_1\rho_2\rho_3 \,; \quad . \quad . \quad . \quad (3{\cdot}2)$$

$\mathrm{V}_3\rho_1\rho_2\rho_3$ contains terms in $i_2 i_3 i_4$, $i_3 i_4 i_1$, etc., and $\mathrm{V}_1\rho_1\rho_2\rho_3$ contains terms in i_1 i_2 i_3 and i_4 which are the result of such reduction as is illustrated in (3·1).

If ρ_1 and ρ_2 have the values in (2·3) and

$$\rho_3 = a_3 i_1 + b_3 i_2 + c_3 i_3 + d_3 i_4$$

the expansion of the product $\mathrm{V}_3\rho_1\rho_2\rho_3$ gives:

$$\begin{aligned} \mathrm{V}_2\rho_1\rho_2\rho_3 = {} & i_2 i_3 i_4 (b_1 c_2 d_3) + i_3 i_4 i_1 (c_1 d_2 a_2) \\ & + (i_4 i_1 i_2)(d_1 a_2 b_3) + i_1 i_2 i_3 (a_1 b_2 c_3), \quad . \quad . \quad (3{\cdot}3) \end{aligned}$$

where $$(\mathrm{A}_1\,\mathrm{B}_2\,\mathrm{C}_3) \equiv \begin{vmatrix} \mathrm{A}_1 & \mathrm{B}_1 & \mathrm{C}_1 \\ \mathrm{A}_2 & \mathrm{B}_2 & \mathrm{C}_2 \\ \mathrm{A}_3 & \mathrm{B}_3 & \mathrm{C}_3 \end{vmatrix}$$

The determinants are recognized as three-dimensional volumes. They are the components of the directed volume $\mathrm{V}_3\rho_1\rho_2\rho_3$ on the various sets of three dimensions that may be chosen from the four.

The coefficients of these irreducible products are minors of the determinant $(a_1 b_2 c_3 d_4)$, and in writing the components $\mathrm{V}_1\,\mathrm{V}_2\,\mathrm{V}_3\,\mathrm{V}_4$ we shall adopt the rule of signs:

$$\mathrm{V}_1 = (b_1 c_2 d_3), \qquad \mathrm{V}_2 = -(c_1 d_2 a_3), \qquad \mathrm{V}_3 = (d_1 a_2 b_3),$$
$$\mathrm{V}_4 = -(a_1 b_2 c_3).$$

Thus for any directed volume V we have :

$$V = i_2 i_3 i_4 V_1 - i_3 i_4 i_1 V_2 + i_4 i_1 i_2 V_3 - i_1 i_2 i_3 V_4. \quad . \quad (3\cdot4)$$

The volume element $d\tau$ may be divided into elementary components. So that :

$$d\tau = i_2 i_3 i_4\, dz\, dy\, du - i_3 i_4 i_1\, dy\, du\, dx + i_4 i_1 i_2\, du\, dx\, dy - i_1 i_2 i_3\, dx\, dy\, dz. \quad . \quad . \quad . \quad (3\cdot5)$$

The remaining part of the complete product $\rho_1\rho_2\rho_3$ is $V_1\rho_1\rho_2\rho_3$, and on expansion it is seen to be :

$$-i_1 a_1 \Sigma a_2 a_3 + i_1\{b_1(a_2 b_3) + c_1(a_2 c_3) + d_1(a_2 d_3)\} + \text{similar terms.} \quad . \quad . \quad . \quad (3\cdot6)$$

The coefficient of the second i_1 is the OX component of the vector product of a four- and six-vector employed in Minkowski's Calculus. It may be represented in our notation as :

$$V_1(\rho_1 V_2 \rho_2 \rho_3). \quad . \quad . \quad . \quad . \quad . \quad . \quad (3\cdot7)$$

§ 4. The product of four vectors will consist of terms in $i_1 i_2 i_3 i_4$, others in $i_2 i_3$ which have arisen from such products as $i_2 i_4 i_3 i_4$ and so on, and finally there will be terms independent of i's arising from such products as $i_2 i_3 i_2 i_3$ which reduce to

$$i_2 i_3 i_2 i_3 = -i_2 i_2 i_3 i_3 = -1.$$

Thus

$$\rho_1\rho_2\rho_3\rho_4 = V_4\rho_1\rho_2\rho_3\rho_4 + V_2\rho_1\rho_2\rho_3\rho_4 + V_0\rho_1\rho_2\rho_3\rho_4. \quad (4\cdot1)$$

On expansion we find :

$$V_4\rho_1\rho_2\rho_3\rho_4 = i_1 i_2 i_3 i_4 (a_1 b_2 c_3 d_4). \quad . \quad . \quad . \quad . \quad (4\cdot2)$$

Thus the tensor of $V_4\rho_1\rho_2\rho_3\rho_4$, or $TV_4\rho_1\rho_2\rho_3\rho_4$ as it will be written, is the volume of the four-dimensional parallelepiped bounded by the four vectors.

It is not possible to write $i_1 i_2 i_3 i_4 = \pm 1$ by analogy with the corresponding $ijk = -1$, for this would lead to inconsistencies.

Suppose we write $i_1 i_2 i_3 i_4 = -1.$

Then $i_1 i_1 i_2 i_3 i_4 = -i_1$

and $i_1 i_2 i_3 i_4 i_1 = -i_1.$

The first gives : $i_2 i_3 i_4 = i_1,$

the second $i_2 i_3 i_4 = -i_1.$

Again we should have :

$$i_1 i_4 i_1 i_2 i_3 i_4 = -i_1 i_4$$

or

$$i_2 i_3 = \quad i_1 i_4.$$

The product $i_1 i_2 i_3 i_4$ will be denoted by I.

When I enters into any expression it obeys the ordinary rule of sign, *i. e.* an odd number of displacements of I in a vector product changes the sign, while an even number leaves the sign unchanged.

$$\mathrm{I}\alpha\beta\gamma = -\alpha\mathrm{I}\beta\gamma = \alpha\beta\mathrm{I}\gamma = -\alpha\beta\gamma\mathrm{I}.$$

Also

$$\mathrm{I}i_2 i_3 = -i_1 i_4$$

and

$$\mathrm{I}i_2 i_3 i_4 = \quad i_1 \qquad \text{etc.}$$

Thus the factor $-\mathrm{I}$ will change a product V_2 into its reciprocal and V_3 is changed into a vector of the type V_1.

We have evidently

$$\mathrm{I}\mathrm{V}_3\alpha\beta\gamma = \mathrm{V}_1\mathrm{I}\alpha\beta\gamma. \quad . \quad . \quad . \quad . \quad (4{\cdot}3)$$

§ 5. The definition of the component of a vector ρ along any direction denoted by the unit vector n is similar to that in ordinary vector analysis. We define the component to be $-\mathrm{V}_0\rho n$.

In the case of a plane we define the component in the plane of a vector of the type V_2 in a similar way.

The component is $-\mathrm{V}_0\mathrm{P}_2\nu$, where ν denotes a unit plane.

Thus the component of P_2 in the plane yz is

$$-\mathrm{V}_0\mathrm{P}_2 i_2 i_3 = \mathrm{P}_{yz}.$$

We cannot define a component along a vector normal to the plane since the normal is indefinite ; we may, however, define a component in the reciprocal plane.

Let $\mathrm{V}_2\alpha\beta$ denote a unit plane, and $\mathrm{V}_2'\alpha\beta$ its reciprocal. The component normal to V_2 may be defined as

$$-\mathrm{V}_0\mathrm{P}_2\mathrm{V}_2'\alpha\beta,$$

or what is the same thing :

$$\mathrm{V}_0\mathrm{P}_2(\mathrm{I}\mathrm{V}_2\alpha\beta).$$

This is equal to $\quad \mathrm{V}_0\mathrm{I}\mathrm{P}_2\mathrm{V}_2\alpha\beta = -\mathrm{V}_0\mathrm{P}_2'\mathrm{V}_2\alpha\beta.$

Thus the component of a vector in the reciprocal (or normal) plane is equal to the component of the reciprocal n the plane.

It is to be noted that $V_0(V_2\alpha\beta)(V_2'\alpha\beta)=0$ on account of the relation (2·5), so that the two planes may be said to be normal in the same way that two vectors ρ_1 and ρ_2 are normal if $V_0\rho_1\rho_2=0$.

But
$$V_0P_2P_2'\neq 0 \qquad \text{although the relation}$$
$$V_0P_2P_2'=V_0P_2'P_2 \quad \text{still holds.}$$

Sommerfeld has defined the components of a six-vector with respect to a plane. We can very easily obtain his result in our notation. The component is $-V_0P_2(V_2\alpha\beta)$.

We must make $V_2\alpha\beta$ a unit plane and naturally shall choose α and β as two unit perpendicular vectors.

Write
$$\alpha=l_1i_1+m_1i_2+n_1i_3+p_1i_4,$$
$$\beta=l_2i_1+m_2i_2+n_2i_3+p_2i_4.$$

Since α and β are unit perpendicular vectors,
$$\Sigma l_1^2=\Sigma l_2^2=1$$
$$\text{and} \qquad \Sigma l_1l_2=0.$$

Then $-V_0P_2V_2\alpha\beta$
$$=-V_0(\Sigma i_2i_3P_{yz})(\Sigma i_2i_3(m_1n_2))$$
$$=\quad P_{yz}(m_1n_2)+P_{zx}(n_1l_2)+P_{xy}(l_1m_2)+P_{xu}(l_1p_2)$$
$$+P_{yu}(m_1p_2)+P_{zu}(n_1p_2) \quad . \quad (5·1)$$

This should be compared with Sommerfeld's result (*Ann. d. Phys.* xxxii. p. 760).

For a vector A of the directed volume type we have:
$$A_3=i_2i_3i_4A_x-i_3i_4i_1A_y+i_4i_1i_2A_z-i_1i_2i_3A_u$$
$$=I(i_1A_x+i_2A_y+i_3A_z+i_4A_u)=IA_1 \quad \text{(say).}$$

The component of A along any direction n may be written:
$$-TV_4A_3n \quad \text{or} \quad -V_0(IA_3)n \quad \text{or} \quad -V_0A_1n.$$

The component of A_3 along Ox is by this definition A_x, and similarly $A_y\,A_z\,A_u$ are the components along the other axes.

A_3 is normal to any direction n if $V_0(IA_3)n=0$.

If four vectors $\rho_1\,\rho_2\,\rho_3\,\rho_4$ bound a four-dimensional volume it is convenient to adopt a rule of signs for the three-dimensional volumes that may be chosen from them. The rule adopted is to define the volumes as :

$$+\mathrm{V}_3\rho_2\rho_3\rho_4, \quad -\mathrm{V}_3\rho_3\rho_4\rho_1, \quad +\mathrm{V}_3\rho_4\rho_1\rho_2, \quad -\mathrm{V}_3\rho_1\rho_2\rho_3.$$

This is to be associated with the rule of § 3. The volumes corresponding to the vectors i_1dx, i_2dy, i_3dz, i_4du are :

$$i_2i_3i_4\,dy\,dz\,du, \quad -i_3i_4i_1\,dz\,du\,dx, \quad i_4i_1i_2\,du\,dx\,dy,$$
$$\text{and} \quad -i_1i_2i_3\,dx\,dy\,dz.$$

§ 6. A vector operator D_1 defined as

$$\dot{i}_1\frac{\partial}{\partial x}+\dot{i}_2\frac{\partial}{\partial y}+\dot{i}_3\frac{\partial}{\partial z}+\dot{i}_4\frac{\partial}{\partial u}$$

plays an important part in four-vector analysis just as the Hamiltonian ∇ does in three dimensions.

The suffix D_1 serves to show that it is of the type of a vector V_1.

Any function σ of ρ will undergo a small change $d\sigma$, on account of a variation $d\phi$ given by the equation

$$d\sigma=-\mathrm{V}_0 d\rho\,\mathrm{D}_1\,.\,\sigma. \quad . \quad . \quad . \quad . \quad (6{\cdot}1)$$

(6·1) is the same result as is expressed in Cartesians by :

$$d\sigma=\frac{\partial\sigma}{\partial x}dx+\frac{\partial\sigma}{\partial y}dy+\frac{\partial\sigma}{\partial z}dz+\frac{\partial\sigma}{\partial u}\,.\,du.$$

This method of introducing D_1 appears to be the simplest. ∇ is sometimes defined by the operation :

$$\operatorname*{Lim}_{v\to 0}\frac{\int d\nu\,.\,\mathrm{F}(\rho)}{v}=\nabla\mathrm{F}(\rho),$$

where $d\nu$ denotes a divided element of area in two dimensions bounding a small volume v, and F is any function of ρ, the vector from the origin to $d\nu$.

If this method of defining an operator be extended to four dimensions we shall have :

$$\operatorname*{Lim}_{v\to 0}\frac{\int d\tau\,\mathrm{F}(\rho)}{v}=\mathrm{DF}(\rho), \quad . \quad . \quad . \quad (6{\cdot}2)$$

where $d\tau$ is the directed element of three-dimensional volume and v a small four-dimensional volume.

This mode of arriving at D gives to it the inconvenient canonical form :

$$i_2 i_3 i_4 \frac{\partial}{\partial x} - i_3 i_4 i_1 \frac{\partial}{\partial y} + i_4 i_1 i_2 \frac{\partial}{\partial z} - i_1 i_2 i_3 \frac{\partial}{\partial u} . \quad . \quad (6 \cdot 3)$$

§ 7. The circuital theorems of ordinary vector analysis

$$\int d\rho \, \mathrm{F}(\rho) = \iint \mathrm{V}(d\nu \nabla) \, . \, \mathrm{F}(\rho)$$

and

$$\iint d\nu \, \mathrm{F}(\rho) = \iiint \nabla \, \mathrm{F}(\rho) \, . \, d\nu$$

are generalizations of Stokes's and Gauss's theorems. We proceed to establish the corresponding circuital theorems for four-vectors. They are three in number and the operator ∇ is now replaced by D_1.

§ 8. In order to establish the theorem for integration along a line we require the relation :

$$\alpha_1 \mathrm{V}_0 \alpha_2 \alpha - \alpha_2 \mathrm{V}_0 \alpha_1 \alpha = \mathrm{V}_1(\mathrm{V}_2 \alpha_1 \alpha_2 \, . \, \alpha). \quad . \quad . \quad (8 \cdot 1)$$

If we write :

$$\begin{aligned} \alpha_1 &= i_1 a_1 + i_2 b_1 + i_3 c_1 + i_4 d_1, \\ \alpha_2 &= i_1 a_2 + i_2 b_2 + i_3 c_2 + i_4 d_2, \\ \alpha \; &= i_1 a \;\; + i_2 b \;\; + i_3 c \;\; + i_4 d, \end{aligned}$$

we find that the coefficient of i_1 on the left is

$$a_2(aa_1 + bb_1 + cc_1 + dd_1) - a_1(aa_2 + bb_2 + cc_2 + dd_2),$$

while on the right the coefficient is

$$-b(a_1 b_2) + c(c_1 a_2) - d(a_1 d_2),$$

and these are of course identical.

§ 9. Let M be at the extremity of the vector ρ and

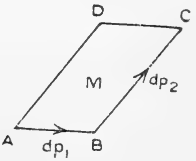


Fig. 1.

construct about it as centre a small parallelogram bounded by the vectors $d\rho_1$ and $d\rho_2$ as in the figure.

If the value of a function be $F(\rho)$ at M then along the side marked $d\rho_2$ the value will be

$$F(\rho)-\tfrac{1}{2}V_0 d\rho_1 D_1 . F(\rho), \quad . \quad . \quad . \quad . \quad (9{\cdot}1)$$

in accordance with § 6 ; while along the opposite side we shall have the value

$$F(\rho)+\tfrac{1}{2}V_0 d\rho_1 D_1 . F(\rho), \quad . \quad . \quad . \quad . \quad (9{\cdot}1)$$

and there will be similar expressions for the other two sides.

Thus on making the summation $\Sigma d\rho F(\rho)$ and making the circuit in the positive direction, we find

$$\begin{aligned} \Sigma d\rho F(\rho) &= d\rho_1 V_0 d\rho_2 D_1 . F(\rho) - d\rho_2 V_0 d\rho_1 D_1 . F(\rho) \\ &= V_1(F_2 d\rho_1 d\rho_2 . D_1) . F(\rho) \\ &= V_1(d\nu\, D_1) . F(\rho). \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (9{\cdot}2) \end{aligned}$$

If $F(\rho)$ is regular throughout a finite surface we have by adding up for each element of which the surface is composed :

$$\int d\rho\, F(\rho) = \iint V_1 (d\nu\, D_1)\, F(\rho). \quad . \quad . \quad (9{\cdot}3)$$

In the summation all the edges of the elementary area except those round the boundary contribute twice to the summation with opposite signs in each case, so that finally the integral is taken round the boundary.

A particular case is obtained by choosing $F(\rho)$ in the form $i_1 X + i_2 Y + i_3 Z + i_4 U$, when we obtain :

$$\begin{aligned} &\int (X\,dx + Y\,dy + Z\,dz + U\,du) \\ &\quad = \iint \left(\frac{\partial Z}{\partial y} - \frac{\partial Y}{\partial z}\right) dy\,dz + \left(\frac{\partial X}{\partial z} + \frac{\partial Z}{\partial x}\right) dz\,dx \\ &\qquad + \left(\frac{\partial Y}{\partial x} - \frac{\partial X}{\partial y}\right) dx\,dy + \left(\frac{\partial U}{\partial x} - \frac{\partial X}{\partial u}\right) dx\,du \\ &\qquad + \left(\frac{\partial U}{\partial y} - \frac{\partial Y}{\partial u}\right) dy\,du + \left(\frac{\partial U}{\partial z} - \frac{\partial Z}{\partial u}\right) dz\,du, \end{aligned}$$

by writing $d\nu$ in the form (2·9) and taking the scalar product of each side.

If F is of the special form given above since

$$d\nu\,\mathrm{D}_1 = \mathrm{V}_3 d\nu\,\mathrm{D}_1 + \mathrm{V}_1 d\nu\,\mathrm{D}_1$$

and $$\mathrm{V}_0(\mathrm{V}_3 d\nu\,\mathrm{D}_1)\,.\,\mathrm{F} = 0,$$

we have : $$\mathrm{V}_0\{\mathrm{V}_1 d\nu\,\mathrm{D}_1\,.\,\mathrm{F}\} = \mathrm{V}_0(d\nu\,\mathrm{D}_1\mathrm{F}) = \mathrm{V}_0(d\nu\mathrm{V}_2\mathrm{D}_1\mathrm{F}),$$

$$\therefore\quad \mathrm{V}_0\int d\rho\,\mathrm{F}(\rho) = \iint \mathrm{V}_0\{(\mathrm{V}_2\mathrm{D}_1\mathrm{F})\,.\,d\nu\}.$$

The curl of a vector is defined by the equation

$$\mathrm{V}_0\int d\rho\,\mathrm{F}(\rho) = \iint \mathrm{V}_0(\text{curl F}\,.\,d\nu).$$

Thus $$\text{curl F} = \mathrm{V}_2\mathrm{D}_1\mathrm{F}.$$

§ 10. The second theorem obtained by performing the operation $\Sigma\,d\tau\mathrm{F}(\rho)$ through a small four-dimensional element *dv* connects a three-dimensional to a four-dimensional integral. By a similar treatment in this case as in the last :

$\Sigma d\tau\,.\,\mathrm{F}(\rho) =$

$$-\mathrm{V}_3 d\rho_2 d\rho_3 d\rho_4\,.\,\mathrm{V}_0 d\rho_1\mathrm{D}_1\,.\,\mathrm{F} + \mathrm{V}_3 d\rho_3 d\rho_4 d\rho_1\,.\,\mathrm{V}_0 d\rho_2\mathrm{D}_1\,.\,\mathrm{F}$$
$$-\mathrm{V}_3 d\rho_4 d\rho_1 d\rho_2\,.\,\mathrm{V}_0 d\rho_3\mathrm{D}_1\,.\,\mathrm{F} + \mathrm{V}_3 d\rho_1 d\rho_2 d\rho_3\,.\,\mathrm{V}_0 d\rho_4\mathrm{D}_1\,.\,\mathrm{F}.$$

This is simplified by observing that :

$$\mathrm{V}_4\alpha_1\alpha_2\alpha_3\alpha_4\,.\,\alpha = -\mathrm{V}_3\alpha_2\alpha_3\alpha_4\mathrm{V}_0\alpha_1\alpha + \mathrm{V}_3\alpha_3\alpha_4\alpha_1\mathrm{V}_0\alpha_2\alpha$$
$$-\mathrm{V}_3\alpha_4\alpha_1\alpha_2\,.\,\mathrm{V}_0\alpha_3\alpha + \mathrm{V}_3\alpha_1\alpha_2\alpha_3\,.\,\mathrm{V}_0\alpha_4\alpha.$$

$$\therefore\qquad \Sigma\,d\tau\,.\,\mathrm{F}(\rho) = \mathrm{V}_4 d\rho_1 d\rho_2 d\rho_3 d\rho_4\,.\,\mathrm{D}_1\mathrm{F}(\rho)$$
$$= \mathrm{I}dv\,.\,\mathrm{D}_1\mathrm{F}(\rho).$$

$$\therefore\qquad \iiint d\tau\,.\,\mathrm{F}(\rho) = \iiiint \mathrm{I}\,\mathrm{D}_1\mathrm{F}(\rho)\,.\,dv, \qquad . \quad . \quad . \quad (10{\cdot}1)$$

or $$\iiint \mathrm{I}\,d\tau\,.\,\mathrm{F}(\rho) = \iiiint \mathrm{D}_1\mathrm{F}(\rho)\,dv.$$

The same limitations as before are to be imposed on F throughout the finite region.

By taking the scalar product of each side and employing again the special form for F(ρ) we have :

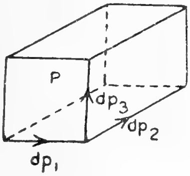
$$\iiint (\mathrm{X}dy\,dz\,du + \mathrm{Y}\,dz\,du\,dx + \mathrm{Z}du\,dx\,dy + \mathrm{U}\,dx\,dy\,dz)$$
$$= \iiiint \left(\frac{\partial \mathrm{X}}{\partial x} + \frac{\partial \mathrm{Y}}{\partial y} + \frac{\partial \mathrm{Z}}{\partial z} + \frac{\partial \mathrm{U}}{\partial u}\right) dx\,dy\,dz\,du.$$

Sommerfeld's extended Div F is equal to $-\mathrm{V}_0\mathrm{D}_1\mathrm{F}$, or to:

$$\lim_{dv\to 0}\frac{-\mathrm{V}_0\int \mathrm{I}\,d\tau\,.\,\mathrm{F}(\rho)}{\delta v}.$$

§ 11. The last of the three theorems occupies a place intermediate between the other two and connects an integral in two dimensions with one in three dimensions.

Fig. 2.



Proceeding as in the other two cases, if ρ defines the centre of the element bounded by the three vectors $d\rho_1$, $d\rho_2$, and $d\rho_3$:

$$\Sigma d\nu \,.\, \mathrm{F}(\rho) =$$

$$\mathrm{V}_2 d\rho_2 d\rho_3 \,.\, \mathrm{V}_0 d\rho_1 \mathrm{D}_1 \,.\, \mathrm{F}(\rho) - \mathrm{V}_2 d\rho_3 d\rho_1 \,.\, \mathrm{V}_0 d\rho_2 \mathrm{D}_1 \,.\, \mathrm{F}(\rho)$$
$$- \mathrm{V}_2 d\rho_1 d\rho_2 \,.\, \mathrm{V}_0 d\rho_3 \mathrm{D}_1 \,.\, \mathrm{F}(\rho).$$

It is easy to show that

$$\mathrm{V}_2(\mathrm{V}_3 \alpha_1\alpha_2\alpha_3) \,.\, \alpha = + \mathrm{V}_2 \alpha_2\alpha_3 \,.\, \mathrm{V}_0 \alpha_1 \alpha + \mathrm{V}_2 \alpha_3\alpha_1 \,.\, \mathrm{V}_0 \alpha_2 \alpha$$
$$+ \mathrm{V}_2 \alpha_1\alpha_2 \,.\, \mathrm{V}_0 \alpha_3 \alpha \qquad (11\cdot1)$$

so that

$$\Sigma d\nu \,.\, \mathrm{F}(\rho) = -\{\mathrm{V}_2(\mathrm{V}_3 d\rho_1 d\rho_2 d\rho_3) \,.\, \mathrm{D}_1\}\mathrm{F}(\rho)$$
$$= + \mathrm{V}_2(d\tau \mathrm{D}_1) \,.\, \mathrm{F}(\rho).$$

Thus by extension as in the previous cases to regions of finite extent :

$$\iint d\nu\, \mathrm{F}(\rho) = \iiint \mathrm{V}_2(d\tau \mathrm{D}_1) \,.\, \mathrm{F}(\rho). \quad , \quad . \quad (11\cdot2)$$

If the summation $\Sigma d\nu' \mathrm{F}(\rho)$ is required, since

$$d\nu' = -\mathrm{I}\, d\nu,$$

we have :

$$\iint d\nu' \mathrm{F}(\rho) = -\iiint \mathrm{I}\, \mathrm{V}_2(d\tau \mathrm{D}_1)\, \mathrm{F}(\rho). \quad . \quad (11\cdot3)$$

Sommerfeld defines the component of the vector divergence normal to $d\tau$ of a six-vector P_2 by the operation :

$$\underset{\mathrm{T}\delta\tau \to 0}{\mathrm{Lim}} \frac{-\iint \mathrm{V}_0(d\nu' \mathrm{P}_2)}{\mathrm{T}\delta\tau}.$$

where $\mathrm{T}\delta\tau$ is the magnitude of the three-dimensional volume $\delta\tau$. This expression is called the component normal to $\delta\tau$.

We may also define Minkowski's lor P_2 from (11·3), this expression denoting the Lorentz operator, equivalent to the vector divergence.

Lor P_2 is defined by the equation :

$$\left.\begin{aligned} +\iint V_0(dv' P_2) &= -\iiint V_0(\text{lor } P_2 \,.\, I d\tau) \\ &= -\iiint V_0 I \,.\, V_2(d\tau D_1) \,.\, P_2 \\ &= -\iiint V_0\{I d\tau D_1 P_2\}, \end{aligned}\right\} (11\cdot4)$$

since the part $V_4 d\tau D_1$, which with $V_2 d\tau D_1$ makes up the product $d\tau D_1$, cannot add to the scalar product.

Now $I d\tau$ is a product of the type V_1 so that

$$V_0(I d\tau \,.\, D_1 \,.\, P_2) = V_0(I d\tau \,.\, V_1 D_1 P_2),$$

$$\therefore \qquad \text{lor } P_2 = V_1 D_1 P_2\,^{*}.$$

Since by § 4, $V_0(dv P_2) = V_0(dv' P_2')$, we have by (12·4) and (12·2)

$$\iint V_0(dv P_2) = \iint V_0(dv' P_2') = -\iiint V_0(\text{lor } P_2' \,.\, I\, d\tau). \quad (11\cdot5)$$

Thus in integrating the normal components of P_2 over a surface, the operation lor P_2 occurs, while for tangential components (*i. e.* components of P_2 along dv) lor P_2 is replaced by lor P_2'.

This is important in the electromagnetic equations.

§ 12. The field equations of the electron theory as expressed by Minkowski may now be regarded as a generalization of Gauss's integral of intensity over a surface.

Putting these integrals in their usual form we have :

$$\int D_n dS = -\int \rho\, dv \quad \text{or} \quad \int S\, D\, dv = \int \rho\, dv,$$

$$\int B_n dS = 0 \qquad \text{or} \quad \int S\, B\, dv = 0.$$

If now we form two six-vectors :

$$P_2 = (-iD_x) i_2 i_3 + (-iD_y) i_3 i_1 + (-iD_z) i_1 i_2 + H_x i_1 i_4 + H_y i_2 i_4 + H_z i_3 i_4,$$

$$Q_2 = iB_x i_2 i_3 \quad + \dots\dots \quad + E_x i_1 i_4 + \dots\dots$$

where $i = \sqrt{-1}$. D_x etc. denote the components of displacement as usual, and H, B, and E have their usual significance.

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* *Cf.* Cunningham, 'Theory of Relativity,' pp. 101–2.

If S is the current four-vector and equal to

$$\rho u_x i_1 + \rho u_y i_2 + \rho u_z i_3 + \rho i\, i_4,$$

where ρ is the density of charge and the u's components of velocity of the charge, the general equations may then be written :

$$\text{Lor } P_2 = S,$$

$$\text{Lor } Q_2 = 0.$$

and for the case when the permeability and dielectric constant are both unity $Q_2 = P_2'$.

Thus from (11·4) and (11·5) we may summarize the theory by stating that the normal component of the six-vector P_2 over the surface of a three-dimensional volume is numerically equal to the amount of S within the volume, while the total tangential component vanishes over the surface. By the amount of S is meant the numerical value of

$$V_0 \int I\, d\tau \,.\, S.$$

It is the value of the component of S along the four-vector $I\,d\tau$.

§ 13. The four-vectors of Minkowski's Calculus are limited in that they are subject to a linear transformation. The vector ρ becomes a new vector σ where

$$\sigma = \phi\rho.$$

ϕ is a linear vector operator and can be expressed as:

$$\phi\rho = -(\alpha_1 V_0 \alpha_2 \rho + \beta_1 V_0 \beta_2 \rho + \gamma_1 V_0 \gamma_2 \rho + \delta_1 V_0 \delta_2 \rho). \quad (13{\cdot}1)$$

Thus ϕ is a dyadic and associated with it are certain invariants just as in the case of three dimensions, and whereas in the simpler case ϕ satisfies a certain cubic relation so here ϕ satisfies a quartic.

In three dimensions the ratio of the two scalar products $S\phi\lambda\phi\mu\phi\nu$ to $S\lambda\mu\nu$ is independent of λ, μ, and ν; to this corresponds the invariance of the ratio $TV_4\phi\lambda\phi\mu\phi\nu\phi\pi$ to $TV_4\lambda\mu\nu\pi$, which means that the ratios of the four-dimensional volumes after and before the operation of ϕ are a constant. When ϕ denotes the Einstein transformations this ratio is unity.

The linear transformation referred to is analogous to that occurring in the theory of elasticity in the case of homogeneous strain, and since a further condition to be satisfied by the four-vector is that of unchanging tensor such a transformation is analogous to a rotation.

Gibbs has shown that a rotation may be expressed by

$$\phi\rho = i'(i\rho) + j'(j\rho) + k'(k\rho),$$

where $(i\rho)$ = the scalar product of i and ρ; and we may also express a rotation in four-vectors similarly,

$$\phi\rho = -(\alpha' \mathrm{V}_0 \alpha\rho + \beta' \mathrm{V}_0 \beta\rho + \gamma' \mathrm{V}_0 \gamma\rho + \delta' \mathrm{V}_0 \delta\rho),$$

$i'\, j' k'$, $i\, j\, k$, are unit vectors and each group is mutually perpendicular. Similarly $\alpha'\, \beta'\, \gamma'\, \delta'$ and $\alpha\; \beta\, \gamma\, \delta$ are unit mutually perpendicular vectors.

The Einstein transformation is a particular case of this in which

$$\alpha' = k(i_1 - iv i_4), \qquad \alpha = i_1,$$

$$\beta = i_2 = \beta',$$

$$\gamma = i_3 = \gamma',$$

$$\delta' = k(iv i_1 + i_4),$$

where $k = (1 - v^2)^{-\frac{1}{2}}$, $i = \sqrt{-1}$; v the arbitrary constant in ϕ is the velocity and may be said to define the strain.

The notation suggests that the restricted principle of Relativity may be summed up by stating that a fundamental four-dimensional medium exists which may be subjected to a strain of the type (13·1) with the values of the vectors given above and that phenomena described by the unstrained vectors will be described by the strained vectors in exactly the same way.

It is natural to generalize ϕ and give up the demand that it should be linear.

The strain is then heterogeneous and the principle becomes more general.

For small regions there is still a linear relation, for if σ is a function of ρ we have:

$$d\sigma = -\mathrm{V}_0 d\rho\, \mathrm{D}_1 \,.\, \sigma = \phi\, d\rho\;;$$

but ϕ now contains ρ in its constitution and the linear relation is true only for small changes $d\rho$ in the neighbourhood of ρ.

This appears to correspond to the "naturalness" of small regions in the theory of Relativity. Space-time is Galilean for infinitesimal regions. In addition $(d\sigma)^2$ is invariant or

$$(d\sigma)^2 = (d\rho)^2.$$

XXXV. *Does an Accelerated Electron* necessarily *Radiate Energy on the Classical Theory?* By S. R. Milner, *D.Sc.*, *Acting Professor of Physics, The University, Sheffield* *.

THIS question is of fundamental importance in modern theory, and it would, I imagine, at the present time receive an affirmative answer from most physicists. It is true that certain adverse experimental results, such as the normal absence of radiation from electronic motions in the atom, call urgently for theoretical explanation; but it seems to be accepted that the necessity in the classical theory for radiation from accelerated charges is so firmly based that it can only be removed by far-reaching and revolutionary changes, such as the quantum theory supplies. Some apology seems necessary for attempting to open the question again at this date; and I should not have ventured to do so, but for the result of the consideration of a certain concrete case of accelerated electronic motion which is amenable to an accurate mathematical treatment. This example shows that it is possible to obtain, even on the classical theory, a solution for a particular case of the accelerated motion of charges, which satisfies completely both Maxwell's equations and the mechanical laws which characterize a conservative system, without any irreversible radiation of energy. A study of it enables us, I think, to prove that a certain step made in the deduction on the classical theory of the general necessity for radiation is not invariably a valid one, and to show that a comparatively minor modification of the boundary conditions of the solution is sufficient to do away with radiation in at any rate one case of accelerated motion.

A remarkable solution of Maxwell's equations for a particular type of accelerated electronic motion has been given by Schott †. It is remarkable in that it is the only case, other than that of uniform motion, which up to the present has been solved in finite terms.

Consider a point-charge moving along the (positive) axis of x, and, ξ being its distance from the origin at time t, let it move in such a way that

$$\xi^2 = k^2 + c^2t^2, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

where k is constant. At $t = -\infty$ the charge is at $\xi = +\infty$ moving towards the origin with the velocity of light c, it

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* Communicated by the Author.

† 'Electromagnetic Radiation,' pp. 63–69.

comes to rest at $\xi=+k$ when $t=0$, and moves back towards $\xi=+\infty$, acquiring again at $t=+\infty$ the velocity of light. The motion is that which would be produced, according to the ordinary principles of mechanics, by the action, on a particle of mass

$$m=\frac{m_0}{(1-\beta^2)^{\frac{1}{2}}}, \text{ where } \beta=\frac{1}{c}\frac{d\xi}{dt},$$

of a constant force F, such that

$$\mathrm{F}=\frac{c^2m_0}{k}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

To express the electromagnetic field associated with a point-charge e moving in this way, at any point let χ be the angle included between two lines, lengths r_1 and r_2, respectively drawn from it to the instantaneous position of the point-charge and to that of its image in the plane $x=0$, and let $\psi=\log r_2/r_1$. Then χ, ψ are related to the cylindrical co-ordinates x, y by the equations

$$x=\frac{\xi\sinh\psi}{\cosh\psi-\cos\chi}, \quad y=\frac{\xi\sin\chi}{\cosh\psi-\cos\chi}. \quad . \quad (3)$$

The third co-ordinate ϕ, the angle through which the plane containing the point has rotated about the axis of x from a fixed position, is the same in both systems.

The electric and magnetic forces at any point of the field are given by

$$\mathrm{E}=\frac{e(1-\beta^2)(\cosh\psi-\cos\chi)^2}{\xi^2(1-\beta^2\sin^2\chi)^{\frac{3}{2}}}, \quad \mathrm{H}=\beta\sin\chi\,.\,\mathrm{E}. \quad (4)$$

The nature of the field will be made clear by a reference to figs. 1 and 2. These show meridian sections at the moments $t=\mp\frac{k}{c}$ and $t=0$, when the charge is at the distance $\sqrt{2}k$ and k respectively from the origin. The lines of force in each case form the arcs of circles ($\chi=$const.) passing through the charge and through its image in the median plane $x=0$; in the figures they are drawn so that $\frac{1}{8}$ of the total flux of induction is enclosed by adjacent lines. The changes which the field undergoes can be pictured by supposing that each line of force in fig. 1 at $t=-\frac{k}{c}$ is moving normally inwards with a velocity $\beta c\sin\chi$. The velocity gradually decreases until the line comes to rest momentarily in the position of the corresponding line of fig. 2 at $t=0$; it then moves outwards, passing the position of fig. 1 again at $t=+\frac{k}{c}$. The magnetic force is directed along the parallel of latitude, and, passing through zero value, is reversed in sign at the moment when the electric lines change their direction of motion. The dotted circles, $\psi=\text{const.}$, orthogonal to the lines of force, are the lines of the Poynting energy-flux.

Fig. 1.

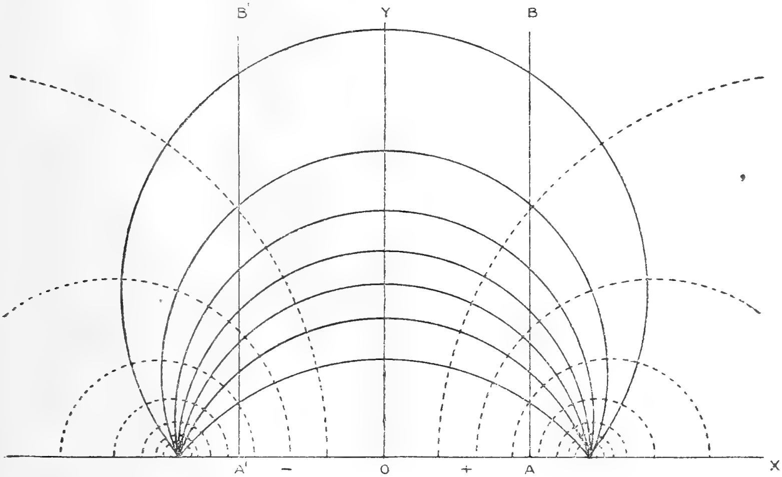
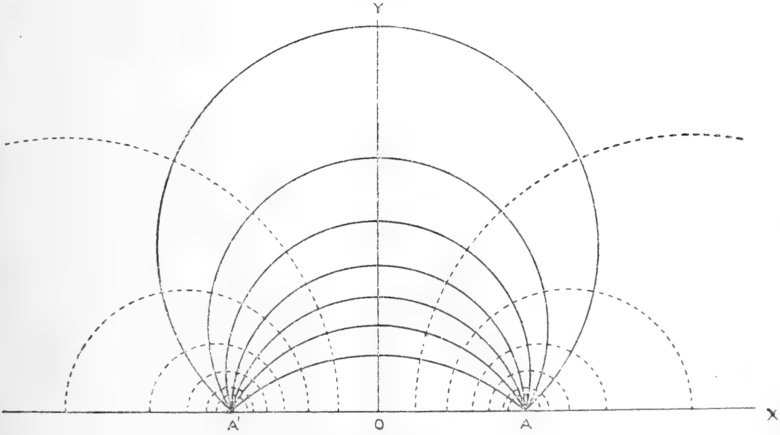


Fig. 2.



The field, as given by Schott, is limited by a moving boundary, formed by the plane $x+ct=0$ (AB at $t=-k/c$, A$'$B$'$ at $t=+k/c$, fig. 1 ; OY, fig. 2). The boundary is a transition layer (of a thickness comparable with the radius of the electron) in which the electric and magnetic forces vary from the values given in (4) on the right of the boundary to zero beyond it. For a point-charge it forms a layer of discontinuity, in which the lines of force emanating from the charge suddenly change their direction, and thenceforth lie in the plane.

It will readily be seen that this field gives an irreversible radiation of energy. The direction of the energy-flux shows that, except at the moment $t=0$ when the moving boundary crosses it, no energy ever passes through the median plane. Thus the field energy which, at positive values of t, exists to the left of the origin, must, along with the boundary, constitute a permanent loss by radiation from the system.

A very simple modification of Schott's solution eliminates from it the presence of irreversible radiation. Consider the equations (4) for E and H, and let them now be valid over all space and all time, the moving boundary being dispensed with. The electromagnetic field thus expressed possesses the following properties :—

(1) It satisfies Maxwell's equations

$$\dot{\mathbf{E}}=c \operatorname{curl} \mathbf{H}, \quad \dot{\mathbf{H}}=-c \operatorname{curl} \mathbf{E},$$
$$\operatorname{div} \mathbf{E}=4\pi\rho, \qquad \operatorname{div} \mathbf{H}=0,$$

at every point of space and time from $-\infty$ to $+\infty$.

(2) The third equation is satisfied in the sense that div $\mathbf{E}=0$ everywhere except at the points $x=\pm\xi$, where it becomes ∞ in such a way that $\int \mathbf{E}_n d\mathrm{S}$ round any closed surface surrounding the point is equal to $\pm 4\pi e$. The solution thus forms throughout all space and time an electromagnetic field which can be associated with two point-charges, $+e$ and $-e$, moving with the particular type of accelerated motion defined by (1).

The two charges start with the velocity of light at $t=-\infty$ from positive and negative infinity of x respectively; they move symmetrically along the axes inwards towards the origin, come to rest at $t=0$ at the points $x=\pm k$, and then move outwards to infinity, ultimately acquiring again the velocity of light. The lines of force are as in figs. 1 and 2, except that the image is now a real charge and there is no boundary, the lines extending from one charge right up to the other.

(3) There is no irreversible radiation of energy from the system. This is evident, for the field at $t = +t_1$ is identical throughout all space with that at $t = -t_1$ except that the sign of H is reversed. Hence, whatever flux of energy outwards from either charge may have occurred at $t = -t_1$, it will be exactly annulled by an equal inward flux at $t = +t_1$.

(4) It will be useful to express the energy and the momentum of the system. The volume of the orthogonal element comprised between the spheres χ and $\chi + d\chi$, ψ and $\psi + d\psi$, and the planes ϕ and $\phi + d\phi$ is

$$dV = \frac{\xi^3 \sin\chi \, d\chi \, d\psi \, d\phi}{(\cosh\psi - \cos\chi)^3},$$

and the total energy of the field is

$$\int \frac{1}{8\pi}(E^2 + H^2)\,dV.$$

The integration extends from $\phi = 0$ to 2π, $\chi = 0$ to π, but with regard to that for ψ, the space occupied by the nucleus or charged surface of the electron must be excluded from the integration. The result will depend on the shape assumed for the nucleus. If we integrate from $\psi = 0$ to the surface given by

$$\frac{\sinh\psi}{\xi} = \frac{1}{a}\left(\frac{1-\beta^2}{1-\beta^2\sin^2\chi}\right)^{\frac{1}{2}}, \quad . \quad . \quad . \quad . \quad (5)$$

we get for the field energy of either charge in the region external to the surface

$$\frac{2}{3}\frac{e^2}{a}(1-\beta^2)^{-\frac{1}{2}} - \frac{1}{6}\frac{e^2}{a}(1-\beta^2)^{\frac{1}{2}}. \quad . \quad . \quad . \quad (6)$$

This is identical with the field energy of a Lorentz electron of the same e and a in uniform motion with the same velocity.

The surface (5) is not precisely the same as the spheroid which forms the surface of the Lorentz electron in a state of uniform motion; but it reduces to it in two cases: when the acceleration is zero, and when the radius of the electron a is indefinitely small. In each of these cases $\xi/\sinh\psi$ (the radius of the ψ sphere passing through a given point) reduces to identity with the radius vector to the point drawn from the centre of the nucleus. There is a difficulty in dealing with the problem for an electron of finite size, partly mathematical if the Lorentz spheroid is assumed, and partly because we cannot say *à priori* what is the exact shape which must theoretically be ascribed to the nucleus of an electron in this type of accelerated motion. The theory of the Lorentz spheroid only applies strictly to uniform motion. There is no need, however, to discuss this difficulty here, as for the present purpose it can be set aside by our assuming, as is now done, that the electrons in the problem are of infinitely small size. The surface (5) then becomes identical with a Lorentz spheroid bounding the charge, and it can be taken to represent the surface of the nucleus without any difficulties being encountered. It is true that the assumption that a is infinitely small makes the energy and the momentum of the system formally infinite; nevertheless, they are definitely evaluated in the limit, and the essential feature of the solution, the absence of radiation from the system, is not affected by the assumption in any way.

The expression (6) now represents the total electromagnetic energy in the external field of one of a pair of Lorentz electrons moving with the given type of motion, in the limit when they are of infinitely small size. In order to produce agreement between the electromagnetic and the mechanical schemes, the nucleus, precisely like that of the uniformly moving electron, must be supposed to possess a store of internal energy of the amount

$$\frac{1}{6}\frac{e^2}{a}(1-\beta^2)^{\frac{1}{2}}. \qquad (7)$$

Then the total energy of each electron in the system becomes

$$w=\frac{2}{3}\frac{e^2}{a}(1-\beta^2)^{-\frac{1}{2}}. \qquad (8)$$

(5) The electromagnetic momentum parallel to x is given by

$$\int\frac{\mathrm{EH}\sin\theta}{4\pi c}d\mathrm{V},$$

where θ is the angle made by the line of force, $\chi=$const., with the positive axis of x. On substituting

$$\sin\theta=\frac{\sinh\psi\sin\chi}{\cosh\psi-\cos\chi}. \qquad (9)$$

and integrating with respect to ϕ and χ as before, but with respect to ψ up to the surface

$$\frac{\cosh\psi}{\xi}=\frac{1}{a}\left(\frac{1-\beta^2}{1-\beta^2\sin^2\chi}\right)^{\frac{1}{2}}, \qquad (10)$$

we find the x-momentum associated with the positive electron external to this surface to be

$$g=\frac{2}{3}\frac{e^2}{ac}\beta(1-\beta^2)^{-\frac{1}{2}}. \quad . \quad . \quad . \quad . \quad (11)$$

This is precisely the same as the momentum of the Lorentz electron of the same e and a in uniform motion with the same velocity.

It must be noted that this result has only been obtained by making the surface (10), to which the integration in ψ was extended, a slightly different one from that (5) used for the determination of the energy. The difference, however, disappears when, as is the case here, the surfaces are of indefinitely small size, as in this case (10) as well as (5) reduces to identity with the Lorentzian spheroid.

There is a distinction between what we may call "fictitious" and "real" cases of electronic motion, which is brought into evidence by this example. Maxwell's equations, as is well known, are of great generality, and can be satisfied by cases of motion which cannot really exist. Theoretically a solution for the field of a charged particle in an arbitrary state of motion can be obtained. But from a physical standpoint the purely arbitrary motion of a charge is a condition impossible to produce. We know of no way in which an electron can actually be set into motion except by the application to it of an electromagnetic field (excluding possible effects of gravitational fields). Not only is this not entirely arbitrary, being subject to the fundamental equations, but also its inclusion alters the problem in an important respect. The problem becomes now, not that of finding the field of an electron whose charge is imagined to be moved about in a given way by external non-electromagnetic means, but that of finding a solution of the fundamental equations which will represent the real time-history of a given electromagnetic field which contains an electron. But for this problem Maxwell's equations are insufficient. Being satisfied by the simple superposition of two electromagnetic fields, they do not reveal the way in which, when an electron is present in a given field, the two fields interact. An additional theory is required for this, such as that of the Lorentz equation.

Leaving this aside for the moment, we can consider in a general way the conditions which a system of two superposed fields must be expected to satisfy for it to represent a real case of electronic motion. There is undoubtedly the law of the conservation of energy. The electromagnetic system must, in fact, be a conservative one, capable of existing without the introduction of energy by imaginary processes from outside. (Radiation, if present, is of course part of the electromagnetic system.) We may, perhaps, further expect that the energy and momentum of each part of the system which is in motion should be related to each other in accordance with the laws of mechanics.

From this point of view the solution of electronic motion which has been discussed is a fictitious one, as it does not obey the conservation of energy. As the electrons are approaching each other the energy of the system gets smaller, and it increases as they move apart. By a simple modification, however, the system can be made conservative.

Superpose on the field (4), which forms one solution of Maxwell's equations, another solution in the form of a uniform electric field X of infinite extent and parallel to the axis of x. Let $Xe = F$, so that X is the field which is required, on the basis of Lorentz's equation, to produce the actual motion which the electrons possess. Then, since for a Lorentz electron we have $m_0 = \frac{2}{3}\frac{e^2}{c^2 a}$, we have by (2)

$$X = \frac{2}{3}\frac{e}{ak}. \qquad \qquad (12)$$

The resulting field is now given by

$$E_x = E\cos\theta + X, \quad E_y = E\sin\theta,$$

$$H = \beta\sin\chi \, . \, E,$$

where E and X are given by (4) and (12), and it forms of course a solution valid over all space and time.

These equations represent a system of two electrons of opposite sign and Lorentz mass moving in a prescribed way in a uniform electric field. We shall, however, as before consider only the case in which a is infinitely small, when the electrons become point-charges, still of Lorentz mass, and the field X, though of infinite strength, is the limit of the definite value required to cause in the point-charges a finite acceleration.

Although the total energy of the system is infinite, the question of its conservation can be examined by considering the flux of energy in the field. For mathematical purposes we can regard such a field as a solution of Maxwell's equations which extends over all space and time, except that it is limited by certain moving boundaries, which form the surfaces of the electrons, and inside which **E** and **H** are prescribed to be permanently zero. We know by Poynting's theorem that, if the quantity $\frac{1}{8\pi}(\mathbf{E}^2+\mathbf{H}^2)$ be identified with the energy density, the conservation of energy will hold throughout the whole field except for the space included within the boundaries; for the Poynting flux is equivalent to the transfer of this quantity from one part of the field to another without net loss or gain. The only places, therefore, where energy can enter or leave the electromagnetic system are the boundaries, so that if we calculate the total flux of energy which is passing through them at any time, we shall obtain the energy which is being introduced into or is disappearing from the electromagnetic system per unit of time. It must be noted that this is not necessarily zero for a conservative system. Even in the simplest case, that of the uniformly moving electron, considerations of the continuity of the flux *, as well as those based on dynamics and on relativity †, necessitate the postulation of a definite store of internal energy, of the amount given by (7), within the nuclear boundary. When the internal energies of the electrons are included in the scheme, conservation in the system will be characterized by the net flux of energy outwards across the boundaries being equal to the net rate of decrease of the internal energies of the electrons.

Confining attention to the positive electron and to the positive x half of the field, we shall take the boundary to be the spheroid given by the limit of (5) when a is infinitely small. Let **P** be the Poynting flux outwards through a fixed boundary momentarily coinciding with (5), and let **v** be the velocity outwards of any point of the boundary surface, then the net flux outwards from the whole moving surface is given by

$$\int(\mathbf{P}-\mathrm{W}\mathbf{v})\mathbf{n}\,d\mathrm{S},$$

where **n** is a unit vector outwards normal to the surface, and

$$\mathrm{W}=\frac{1}{8\pi}(\mathrm{E}^2+\mathrm{H}^2+2\mathrm{E}\cos\theta\,.\,\mathrm{X}+\mathrm{X}^2)\quad.\quad.\quad(13)$$

is the energy density of the field at the point. Expressed in

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* Milner, Phil. Mag. Oct. 1920, p. 494.

† *Cf.* Lorentz, 'Theory of Electrons,' § 180.

Cartesians, the flux becomes

$$\int\left\{\frac{c}{4\pi}(\mathrm{E}\cos\theta+\mathrm{X})\mathrm{H}+\mathrm{W}\frac{dy}{dt}\right\}2\pi y\,dx$$

$$+\int\left\{\frac{c}{4\pi}\mathrm{E}\sin\theta\,.\,\mathrm{H}-\mathrm{W}\frac{dx}{dt}\right\}2\pi y\,dy. \quad . \quad . \quad . \quad (14)$$

Here x and y are the co-ordinates of a point of constant χ on the surface (5), which is bodily moving through space and at the same time altering its shape since ξ and β are functions of t. To effect the integration we substitute

$$dx=\left(\frac{\partial x}{\partial\chi}+\frac{\partial x}{\partial\psi}\frac{\partial\psi}{\partial\chi}\right)d\chi, \quad \frac{dx}{dt}=\left(\frac{\partial x}{\partial\xi}+\frac{\partial x}{\partial\psi}\frac{\partial\psi}{\partial\xi}\right)\frac{d\xi}{dt},$$

etc., and use the transformation equations (3). On writing for E and H, X, W, and θ their values (4), (12), (13), and (9), and substituting for ψ, $\frac{\partial\psi}{\partial\chi}$, $\frac{\partial\psi}{\partial\xi}$ everywhere their values in terms of χ and ξ obtained from (5), and then expressing by (1) ξ in terms of k and β, (14) becomes a determinate function of χ, which, integrated from 0 to π, gives the net flux outwards from the boundary. On finally taking the limiting value when a is indefinitely small, it reduces to

$$\frac{\beta ce^2(1-\beta^2)^2}{4ak}\int_0^{\pi}\frac{(1+2\cos^2\chi+\beta^2\sin^2\chi)\sin\chi\,d\chi}{(1-\beta^2\sin^2\chi)^{\frac{5}{2}}}$$

$$-\frac{\beta ce^2(1-\beta^2)}{3ak}\int_0^{\pi}\frac{\sin\chi\,d\chi}{(1-\beta^2\sin^2\chi)^{\frac{3}{2}}}.$$

The first term represents the flux corresponding to the electronic field E, H alone, the second the additional flux due to the superposition of the field X. On integration the net flux outwards becomes

$$\frac{1}{6}\frac{\beta ce^2}{ak}(1-\beta^2).. \quad . \quad . \quad . \quad . \quad . \quad (15)$$

The internal energy is given by (7), and its rate of decrease with the time, in the existing state of motion (1), is identical with the expression (15) for the net outward flux.

It follows that the total energy of the system is conserved. As the electrons are being brought to rest, a continual process of conversion of magnetic energy into electric is going on in the field, and at the same time field energy is being transferred into the nucleus, where it shows itself as the increase in the internal energy of the electron. When the electrons, having been brought to rest, begin to move away from each other, the stored internal energy comes out into the field, and at the same time the excess of electric energy which has accumulated in the field is reconverted into the magnetic form.

If we distinguish that part of W given by $\frac{E^2+H^2}{8\pi}$ as the "electronic" from the "mutual," $\frac{E\cos\theta \,.\, X}{4\pi}$, and the "external field," $\frac{X^2}{8\pi}$, energy densities, the equations (8) and (11) show that the total electronic energy w and momentum g fit into a mechanical scheme. In fact, since

$$\frac{dg}{dt}=Xe \text{ and } \frac{dw}{dt}=Xe\,.\,\beta c,$$

their rates of increase are identical with those which would be created by a mechanical force Xe acting on the moving nucleus. This shows that the system formally satisfies the Lorentz equation

$$\mathbf{F}=\rho\left(\mathbf{E}+\frac{1}{c}[\mathbf{vH}]\right),$$

as applied to a point-charge e in a uniform electric field X *.

There seem to be only two respects in which the solution fails to represent a real case of electronic motion. (1) It refers to a limiting case in which the electrons are concentrated to a point and are moving in a correspondingly infinite field. (2) Extending over all space and time, it does not represent the initiation of the motion, and does not correspond substantially with a practical case, such as is given when an electron, passing through a hole in a charged plate, comes into an ultimately uniform field. Neither of these points, I imagine, has any significance with regard to the question at issue. (1) clearly does not affect the radiation from the system; and with regard to (2), electromagnetic solutions, like those of other differential equations, in general only represent steady states under idealized conditions, and without reference to the manner of their initiation (*cf.* the solution for the uniformly moving electron).

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* It may be noted that the example does not reveal whether a term of the order $e^2/4\xi^2$, expressing a mutual action of the charges, should be included, as, if present, it would be negligible compared with the infinite Xe.

In the light of this solution it seems desirable to examine with some care the proof of the presence of radiation as a necessary accompaniment to the accelerated motion of charges. The proof, as given by Larmor in 'Æther and Matter' (Chapter XIV.), and by Lorentz in 'The Theory of Electrons' (§ 39), is based on the solution in terms of retarded potentials for the field of a point-charge in a prescribed state of motion. This solution shows that the alterations in the field at any point can be considered as due to disturbances which emanate from the charge at each point of its path, and are propagated outwards with the velocity of light. Outside a moving boundary, marking the farthest points to which the disturbances have travelled, the previously existing field is unaffected by the motion of the charge. It is now found that in the resulting field, at a sufficiently great distance from the charge, there is an outward flux of energy proportional to the square of the acceleration, which clearly seems to represent an irreversible loss by radiation.

Let us first test this general conclusion by applying the method to Schott's solution, as originally limited by the moving boundary. The point law of retarded potentials gives for the field of a point-charge moving arbitrarily *

$$\mathbf{E}=e\left[-\frac{\dot{\mathbf{v}}}{c^2\mathrm{K}^2\mathrm{R}}+\frac{\left(\mathbf{R}_1-\frac{\mathbf{v}}{c}\right)\{(\dot{\mathbf{v}}\mathbf{R})+c^2-v^2\}}{c^2\mathrm{K}^3\mathrm{R}^2}\right],$$

$$\mathbf{H}=[\mathbf{R}_1\mathbf{E}],\quad \text{where } \mathrm{K}=\left(1-\frac{\mathbf{v}\mathbf{R}_1}{c}\right).$$

$\mathbf{E}$, $\mathbf{H}$ are here the electric and magnetic forces which exist at the time $t=\tau+\frac{\mathrm{R}}{c}$ at any point which is at a distance R from the point where the charge was at the time τ. For simplicity take $\tau=0$, so that the disturbances considered are those emitted at the turning-point A, $(k, 0)$, of the charge, whence in the formula we shall have $\mathbf{v}=0$ and $\dot{\mathbf{v}}=c^2/k$ parallel to x. Then at the time $t=\mathrm{R}/c$, and on the spherical surface of radius R described about A as centre, $\mathbf{E}$ and $\mathbf{H}$ will be given by

$$\mathbf{E}=e\left(-\frac{\sin\alpha}{k\mathrm{R}}\mathbf{P}_1+\frac{1}{\mathrm{R}^2}\mathbf{R}_1\right),\quad \mathbf{H}=[\mathbf{R}_1\mathbf{E}],$$

where α is the angle between the axis of x and $\mathbf{R}$, and $\mathbf{P}_1$, $\mathbf{R}_1$

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* Schott, 'Electromagnetic Radiation,' p. 23.

are unit vectors perpendicular to and parallel to **R**. This is consistent with (4) as it should be. When R is so large that the second term is negligible compared with the first, **E** is perpendicular to **R**, and there is a flux of energy outwards through the surface of amount

$$\frac{c}{4\pi}\int_0^\pi \mathbf{EH}\,.\,2\pi\mathrm{R}^2\sin\alpha\,d\alpha=\frac{2}{3}\frac{e^2c}{k^2}=\frac{2}{3}\frac{e^2}{c^3}\dot{\mathbf{v}}^2. \quad . \quad (16)$$

This agrees with the usual formula, and apparently verifies the presence of radiation in the solution. Nevertheless, an examination of the lines of energy-flow in figs. 1 and 2 shows that, as regards more than half of this flux—namely, that through the part of the surface on the positive side of the origin,—the energy concerned is not being radiated to infinity irreversibly at all, but is travelling along curved paths which in the course of time inevitably bring it back again to the nucleus of the electron; and, in fact, it forms a permanent part of the electronic field. The reason for this is that the proof of the existence of the flux (16) does not apply to *any* large sphere surrounding the point A, but only to a specified sphere at a given time. If we apply the calculation to a sphere of different radius R′ at the same time t, this new sphere is not centred on A, but on the position of the charge at the time $t-\mathrm{R}'/c$. The instantaneous lines of energy-flow, which are normal to both spheres, are therefore necessarily curved lines, however far out they may be traced, and in this case their curvature is such as to keep the energy permanently in the electronic system.

In the negative half of the field, the energy, although here also it is not travelling outwards indefinitely, but in curves related in a similar way to the image, is apparently true radiation, since, as is shown by the direction of the flux-lines, no energy ever comes back to the electron through the median plane. Nevertheless, a curious point may be noticed in this connexion. If we investigate the flux of energy from the field into the moving plane, $x+ct=0$, we find that it is invariably negative; consequently the boundary, considered as the limit of a thin transverse field separating the electronic field from zero, is always engaged in laying down field energy behind it as it advances, and never, even at $t=-\infty$, in receiving any energy from the electron. The solution therefore premises an initial intrinsic energy in the boundary, apart from that of the electronic field. Now the field energy on the negative side of the median plane is clearly laid down by the boundary, since no energy ever passes through the plane except at the moment when the boundary crosses it. We consequently see that the only radiation which the solution gives is, strictly speaking, not from the electron at all, but is to be attributed ultimately to the moving boundary which is postulated to be the limit of the field.

The limitation of the solution by a moving boundary seems to be included, although perhaps tacitly, in the proof of radiation; but a comparison with the unlimited solution discussed in this paper raises the question whether the boundary is essential in every case for the representation of real motions. The arbitrary motion discussed in the proof makes the case considered there fictitious in the sense previously explained; but this point is only material in that it makes it clear that an additional conception is involved in the complete proof, which must necessarily concern itself with a charge set in motion by an electromagnetic field. Lorentz's equation is applied, and interpreted by assuming that the electron will move in the same way as it would do if it were a particle of given mass acted on by determinate mechanical force. The retarded potential solution corresponding to the resulting motion of the charge, superposed on the external field which gives the motion, then forms the complete solution of the problem, and the boundary is present in it as before.

The question whether the boundary is necessary or not seems to be largely a question of the physical interpretation made of the point law and of Lorentz's equation. The conclusion that it is necessary is based on the conception that the charges or nuclei of the electrons are first set into motion by the operation of the field in their immediate neighbourhood, and that the resulting changes in the field are *actually* propagated outwards from them. But it does not follow, from the mathematical fact that the changes in the field at a point are the same as if the disturbances were propagated from the charge, that the propagation is a physical fact. The field variations at a point can also be described in terms of the differential coefficients of the field at a point in a way which does not bring in the charge at all. Moreover, if we regard the nucleus of the electron from a mathematical standpoint as a small closed surface limiting the field and characterized by the constancy of the flux of force over it, once the field is known at all points, not only the field variations but also the motion of the electrons is uniquely determined. It seems from first principles as logical to consider the motion of the charges and the changes in the field to be two connected aspects of a single solution regarded as a whole, as to suppose that one is antecedent to the other. The adoption of this point of view would not affect the character of the field of being expressible in terms of retarded potentials, but it would permit other boundary conditions than those of a simple moving boundary to be applicable for the representation of real motions. The example which has been discussed in this paper seems to show, in one case at least, the legitimacy of a solution with less prescribed boundary conditions.

October 10th, 1920.

XXXVI. *On a Graphical Method for determining the Frequencies of Lateral Vibration, or Whirling Speeds, for a Rod of Non-Uniform Cross-Section.* By R. V. SOUTHWELL, *M.A., Fellow and Lecturer of Trinity College, Cambridge*[*].

[Plate VII.]

THE determination of the normal modes and frequencies of lateral vibration for a rod of varying cross-section is a problem which has attracted the attention of many elasticians, and several papers on the subject have appeared in this Magazine[†]. It would seem that a complete solution,

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[*] Communicated by the Author.

[†] References to the investigations of Bernouilli, Euler, Kirchhoff, Sturm, Liouville, and others will be found in Lord Rayleigh's 'Theory of Sound,' or in the 'Dynamical Theory of Sound' of H. Lamb. The following is believed to be a fairly complete list of recent papers bearing on the problem:—

W. J. M. Rankine, 'The Engineer,' vol. xxvii. (1869), p. 219.
S. Dunkerley, Phil. Trans. Roy. Soc. (A), vol. clxxxv. (1894), p. 279.
C. Chree, Phil. Mag., vol. vii. (1904), p. 504, and vol. ix. (1905), p. 132.
C. A. B. Garrett, Phil. Mag., vol. viii. (1904), p. 581.
J. Morrow, Phil. Mag., vol. x. (1905), p. 113; vol. xi. (1906), p. 354; and vol. xii. (1907), p. 233.
A. Morley, 'Engineering,' July 30th and August 13th, 1909.
P. F. Ward, Phil. Mag., vol. xxv. (1913), p. 85.
A. Fage, 'Engineering,' July 20th, 1917.
J. W. Nicholson, Proc. Roy. Soc. (A), vol. xciii. (1917), p. 506.
H. H. Jeffcott, Proc. Roy. Soc. (A), vol. xcv. (1918), p. 106.
W. L. Cowley and H. Levy, Advisory Committee for Aeronautics, R. & M. 485 (1918).
J. Morris, Advisory Committee for Aeronautics, R. & M. 551 (1918).
G. Greenhill, Advisory Committee for Aeronautics, R. & M. 560 (1918).
F. B. Pidduck, Lond. Math. Soc. Proc., vol. xviii. (1920), p. 393.

applicable to rods of *any* given shape, is beyond the range of exact analysis: at all events, mathematical investigations have in general been restricted to rods of fairly simple form, the case most frequently discussed being that of a rod with one free and one clamped end, in which the flexural rigidity at any section varies as some power of the distance from the free end.

If it be conceded that the interest of this problem lies principally in the quantitative results obtained, then a reasonably accurate method of solution which is unrestricted in its application would appear to possess advantages over any isolated analytical solution, however rigorous. No great mathematical interest can attach to exact results when these are based upon a theory (of thin rods) which admittedly is only an approximation to the truth, and a slight decrease in accuracy will be more than compensated by ability to take account of any specified end conditions, and to deal with rods of which the cross-sections vary in any specified way, or with continuous rods supported at several points. This is more especially true in relation to the engineering applications of the theory. The "whirling speeds" of a rotating shaft, as has frequently been pointed out*, are identical with its natural frequencies of vibration, and in many cases in which their values are of practical importance the variation of flexural rigidity along the length of the shaft is incapable of expression in mathematical form, whilst the bearings impose constraint upon the direction of the central line at two or more points: no great accuracy is required in the result, but it is essential that the method employed shall not break down in any particular instance by reason of purely mathematical difficulties.

The graphical processes now to be described constitute a simple extension of methods which I have recently propounded † for finding the critical load of a strut of varying cross-section. They seem preferable, as regards ease and quickness of application, to any general method of solution which is based upon the use of infinite series ‡; and the accuracy with which they reproduce the results of exact calculation in examples which can be treated analytically

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* *Cf.*, *e.g.*, C. Chree, Phil. Mag. vol. vii. (1904) p. 504; the expressions obtained in this paper for the frequencies of lateral vibration in a *rotating* shaft have, however, been shown by Pidduck (*loc. cit.*) to be erroneous. I hope shortly to publish a paper dealing with this point at greater length.

† 'Aircraft Engineering,' April 1920, pp. 113–114.

‡ *Cf.*, *e.g.*, the discussions of W. L. Cowley and H. Levy, and of J. Morris, quoted above.

suggest that—given accurate draughtsmanship, and assuming the validity of the theory of thin rods—errors will not exceed 1 or 2 per cent. in any practical problem.

Stated mathematically, our object is to determine the forms of certain curves of deflexion (the "normal modes" of vibration) which are associated with any given shape of rod. These curves form a family which is defined by a certain differential equation, and they are subject to certain "conditions of constraint" (expressing the effects of clamps, journals, etc., in the system under consideration), which have the common feature that they leave the *magnitude* of the deflexion unrestricted, whilst defining the manner of its variation along the length of the rod. In the problem of "whirling," each curve of deflexion has the property that it can just be maintained, when the shaft is rotating at some definite and appropriate speed, by the centrifugal forces acting against the elastic restoring forces: in the vibrating bar, the instantaneous deflexion at every section of the rod is compounded of one or more "normal modes of vibration," any one of which may be regarded as a curve of deflexion which varies, in absolute magnitude, as some appropriate simple harmonic function of time.

The differential equation which governs the curve of deflexion may be obtained as follows:—

Let

ρ be the line-density of the rod at any section, in pounds per foot run,

$\mathfrak{B}(=\mathrm{EI})$ be the flexural rigidity at the same section, in pound- (foot)^2 units,

and x be the distance of the section from one end of the rod, measured in feet

(so that ρ and $\mathfrak{B}$ vary in some specified manner with x);

and let y be the instantaneous deflexion of the shaft, in feet, at the section x,

and **n** be the number of vibrations—or of revolutions, in the problem of the whirling shaft—per second (so that our problem is to determine the value of **n** and the manner in which y varies with x):

then the elastic restoring couple M, which acts at any section of the deflected shaft, is given by the usual equation

$$\mathrm{M} + \mathfrak{B}\frac{\partial^2 y}{\partial x^2} = 0, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (1)$$

and its restoring effect is equivalent to an intensity of lateral loading w which is given by the equation

$$w+\frac{\partial^2 \mathrm{M}}{\partial x^2}=0, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

the direction of w being opposite to that of the deflexion (and so tending to restore the straight configuration), and its intensity, as given by (2), being measured in pounds per foot run.

In the vibrating bar, this "effective lateral loading" produces the required acceleration of the bar towards the central position: in the "whirling" shaft it produces the normal acceleration towards the axis of rotation. In either case, the magnitude of the acceleration is given (in foot-second units) by the equation

$$\alpha=4\pi^2\mathbf{n}^2 y, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

the instantaneous deflexion y being a function of x only in the case of the whirling shaft, and represented in the case of the vibrating bar by the expression

$$y=\mathbf{Y}\sin 2\pi\mathbf{n}t, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (4)$$

where $\mathbf{Y}$ is a function of x only.

We have therefore, in both problems, the equation

$$w=\frac{\rho\alpha}{g}=\frac{4\pi^2\mathbf{n}^2\rho}{g}y, \quad . \quad . \quad . \quad . \quad . \quad . \quad (5)$$

and by eliminating w and M from (1), (2), and (5) we obtain the differential equation

$$\frac{4\pi^2\mathbf{n}^2\rho}{g}y=\frac{\partial^2}{\partial x^2}\left(\mathfrak{B}\frac{\partial^2 y}{\partial x^2}\right),$$

which reduces to

$$\frac{d^2}{dx^2}\left(\mathfrak{B}\frac{d^2\mathbf{Y}}{dx^2}\right)-\frac{4\pi^2\mathbf{n}^2\rho}{g}\mathbf{Y}=0, \quad . \quad . \quad . \quad (6)$$

both when $y=\mathbf{Y}$ simply, and when y is given by (4).

Equation (6) governs $\mathbf{Y}$ as a function of x, and thus defines the curve of deflexion. The arbitrary constants in its complete solution are determined in any particular instance by the special conditions which define the constraints: thus, at an end which is "simply supported" (as by a swivelling bearing, in the problem of the whirling shaft) we have

$$y=0\,; \quad \mathrm{M}=0: \quad . \quad . \quad . \quad . \quad . \quad . \quad (7)$$

at a completely free end (where the resultant shear, as well as the resultant bending-moment on the section, is zero) we have

$$\mathrm{M}=0\,; \quad \frac{\partial \mathrm{M}}{\partial x}=0: \quad . \quad . \quad . \quad . \quad . \quad . \qquad (8)$$

and at an end which is "clamped" (as by a fixed bearing in the problem of the whirling shaft) we have

$$y=0\,; \quad \frac{\partial y}{\partial x}=0. \quad . \quad . \quad . \quad . \quad . \quad . \qquad (9)$$

It will be convenient at this point to introduce a new quantity $\mathbf{M}$, defined by the equation

$$\mathbf{M}+\mathfrak{B}\frac{d^2\mathbf{Y}}{dx^2}=0, \quad . \quad . \quad . \quad . \quad . \quad . \qquad (10)$$

—so that M is equal to $\mathbf{M}$, simply, in the problem of the whirling shaft, and to $\mathbf{M}\sin 2\pi \mathrm{n}t$ in the problem of the vibrating bar. Then we can express the foregoing conditions of constraint in terms of $\mathbf{Y}$ and $\mathbf{M}$, as follows:—

$$\left.\begin{aligned} &\text{At a "simply supported" end,} && \mathbf{Y}=\mathbf{M} &&=0\,; \\ &\text{at a completely free end,} && \mathbf{M}=\frac{d\mathbf{M}}{dx} &&=0\,; \\ &\text{and at a clamped end,} && \mathbf{Y}=\frac{d\mathbf{Y}}{dx} &&=0. \end{aligned}\right\} \quad . \quad . \quad (11)$$

Corresponding relations can be written down to represent the conditions imposed by other types of constraint, *e. g.*, the effects of large concentrated masses, spinning masses which introduce gyroscopic couples, etc. For present purposes it is only necessary to add that the existence of "simple supports" at intermediate sections of a continuous rod requires that the values of $\mathbf{Y}$, of $\frac{d\mathbf{Y}}{dx}$, and of $\mathbf{M}$ shall be continuous at such sections.

Whatever be the specified conditions in the problem under consideration, it will be found that a solution of (6) is obtainable by graphical methods, when we have assumed any value for the frequency n, which satisfies all but one of them; but that they cannot all be satisfied simultaneously unless n has certain definite values, known as the "natural frequencies" of whirling or of vibration. It is these frequencies with which we are primarily concerned in practice, and the method of determination which is now to be described proceeds by constructing a curve of deflexion which satisfies equation (6), and by modifying this (on "trial-and-error" principles) until it satisfies all the constraint conditions simultaneously : to each modification corresponds a change in the frequency $\mathbf{n}$ for which (6) is satisfied, and thus we arrive finally at the required solution.

Our graphical construction employs the principle of the "funicular polygon." We construct two diagrams (figs. 1 and 3), the first representing the deflexions **Y**, and the second representing the corresponding bending-moments **M**, defined by equation (10). Either end of the rod may be taken as a starting-point, and constraint conditions of the type (11) will determine the values at this end of some pair of the quantities **Y**, $\frac{d\mathbf{Y}}{dx}$, **M** and $\frac{d\mathbf{M}}{dx}$. By assuming values for the remaining two quantities, we are enabled to proceed with the construction of the curves, making use for this purpose of equation (10) above, and of equation (6) rewritten in the equivalent form

$$\mathrm{Y}+\mathrm{K}\frac{d^2\mathrm{M}}{dx^2}=0, \quad . \quad . \quad . \quad . \quad . \quad . \quad (12)$$

where the symbol K is substituted for $g/4\pi^2\mathbf{n}^2\rho$, a positive quantity of dimensions $[\mathrm{L}][\mathrm{T}]^2/[\mathrm{M}]$, which varies in some specified manner with x.

In the engineering problem at any rate, it will often be desirable to vary the length of our diagrams, and we must therefore give careful consideration to the question of *scale*. Let us for the moment replace x by the non-dimensional variable z, defined by the equation

$$x=lz,$$

in which l denotes the distance between any specified pair of sections (for example, two adjoining sections of constraint). Then equation (6) may be rewritten in the form

$$\frac{d^2}{dz^2}\left(\mathfrak{B}\frac{d^2\mathrm{Y}}{dz^2}\right)-\frac{l^4\mathrm{Y}}{\mathrm{K}}=0, \quad . \quad . \quad . \quad . \quad . \quad (13)$$

whence it is evident that the natural frequencies will be obtainable from expressions of the type

$$\frac{4\pi^2\mathbf{n}^2}{g}\rho_0=\frac{1}{\mathrm{K}_0}=\lambda\frac{\mathfrak{B}_0}{l^4}, \quad . \quad . \quad . \quad . \quad . \quad (14)$$

—ρ_0, $\mathfrak{B}_0$ and K_0 denoting the values of ρ, $\mathfrak{B}$ and K at some specified section, and λ being a non-dimensional coefficient. From this point of view, our problem is to determine possible values for λ, and we are free to change our scale of length in any convenient manner, whilst the results will be applicable to a whole family of rods, differing in actual length, and specified simply by the variation of the ratios $\mathfrak{B}/\mathfrak{B}_0$ and ρ/ρ_0 with the ratio x/l: the natural frequencies for different members of the family will vary inversely as the square of the length of the rod.

In the light of these remarks we may now proceed to develop our construction. In fig. 1 (Pl. VII.) we choose axes of $\mathbf{Y}$ and x, as shown, and take OA to represent l, the length of the rod (or the distance from O to the next section of constraint, if the rod is continuous); and we draw vertical lines to divide OA at the points 1, 2, 3, ... into any convenient number, n, of equal parts ($n=10$ in the diagram shown). The values of $\mathfrak{B}$ are assumed to be known at the points of subdivision, and lengths b_1, b_2, ... are taken, as shown in fig. 2, proportional to $\mathfrak{B}_1$, $\mathfrak{B}_2$, ... etc. Similarly, in fig. 3 we choose axes O′$\mathfrak{m}$ and O′x for a curve of which the ordinates $\mathfrak{m}$ are to be made proportional to $\mathbf{M}$, and continue the vertical subdividing lines; and in fig. 4 we take lengths k_1, k_2, ... etc. proportional to the values of K which correspond to the points of subdivision. It should be noted that the actual scale on which $\mathfrak{m}$ represents $\mathbf{M}$ is for the moment left undetermined.

The construction then proceeds as follows:—Oa and O′a' are drawn to represent the values of $\mathbf{Y}$, $\frac{d\mathbf{Y}}{dx}$, $\mathbf{M}$, and $\frac{d\mathbf{M}}{dx}$ at the end $x=0$ (figs. 1–4, Pl. VII. are drawn for an example in which it is specified that both $\mathbf{Y}$ and $\mathbf{M}$ vanish at this end: the slopes of Oa and O′a' are for the moment immaterial); and the polar diagrams (figs. 2 and 4) are begun by drawing $p\mathrm{O}_1$ parallel to Oa and $p'\mathrm{O}_1'$ parallel to O′a', the points O_1 and O_1' being distant by amounts b_1 and k_1 respectively from the verticals through p and p'. We then make pq in fig. 2 equal to $a'1'$ in fig. 3, and $p'q'$ in fig. 4 equal to $a1$ in fig. 1; join $q\mathrm{O}_1$ and $q'\mathrm{O}_1'$, producing them to O_2 and O_2' as shown, and continue figs. 1 and 3 by drawing ab and $a'b'$ parallel to $q\mathrm{O}_1$ and $q'\mathrm{O}_1'$ respectively*.

The processes just described are repeated at every section of subdivision: viz., qr is made equal to $b'2'$ and $q'r'$ to $b2$, bc and $b'c'$ are drawn parallel to O_2r and $\mathrm{O}_2'r'$ respectively,

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* Dashes have been accidentally omitted from the letters f in fig. 3 and O_1, O_2, . . . etc. in fig. 4.

and so on : it is not easy to explain the method in few words, but its principles should be easy to grasp, if the reader will construct a diagram for himself by means of the foregoing directions. We have now to show that the curves which would be obtained by drawing continuous lines through the points O*abc* .... in fig. 1 and O′*a*′*b*′*c*′ .... in fig. 3 will constitute the solution of the differential equations (10) and (12). Considering, for example, the portion *abc* of fig. 1, we see that the tangents at the middle points of the arcs *ab* and *bc* will, if these arcs are small, be very nearly parallel to the chords *ab*, *bc* : hence, to a first approximation, the value of $\frac{d^2\mathrm{Y}}{dx^2}$ at the point *b* will be given by the expression

$$\frac{\mathrm{OA}}{n}\left[\frac{d^2\mathrm{Y}}{dx^2}\right]_2 = (\text{slope of } bc) - (\text{slope of } ab),$$

$$= (\text{slope of } r\mathrm{O}_2) - (\text{slope of } q\mathrm{O}_2),$$

by construction ;

$$= -\frac{qr}{\mathbf{b}_2} \text{ in fig. 2},$$

$$= -\frac{\mathbf{m}_2}{\mathbf{b}_2}, \text{ since } qr = b'2', \text{ by construction.}$$

A similar relation obtains at every section, whence it is evident that we may drop the suffixes, and write

$$\frac{l}{n}\frac{d^2\mathrm{Y}}{dx^2} + \frac{\mathbf{m}}{\mathbf{b}} = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad (15)$$

(since OA in fig. 1 is the quantity denoted by l) ; and by a similar investigation we see that the ordinates $\mathbf{m}$ of a smooth curve drawn through O′*a*′*b*′*c*′ .... in fig. 3 would satisfy the equation

$$\frac{l}{n}\frac{d^2\mathbf{m}}{dx^2} + \frac{\mathrm{Y}}{\mathbf{k}} = 0. \quad . \quad . \quad . \quad . \quad . \quad . \quad (16)$$

Eliminating $\mathbf{m}$ from (15) and (16), we obtain the relation

$$\frac{d^2}{dx^2}\left(\mathbf{b}\,\frac{d^2\mathrm{Y}}{dx^2}\right) - \frac{n^2}{l^2}\,\frac{\mathrm{Y}}{\mathbf{k}} = 0, \quad . \quad . \quad . \quad . \quad (17)$$

and comparing this with (6) we can see that the ordinates of fig. 1 (Pl. VII.) will satisfy the differential equation of our problem if the scales of the diagrams are such that

$$\frac{\mathrm{b}_0}{\mathfrak{B}_0}=\frac{n^2\mathrm{K}_0}{l^2\mathrm{k}_0},$$

or—by (14)—if

$$\lambda=\frac{l^4}{\mathfrak{B}_0\mathrm{K}_0}=\frac{l^2n^2}{\mathrm{b}_0\mathrm{k}_0}, \quad . \quad . \quad . \quad . \quad . \quad (18)$$

b_0 and k_0 being the actual lengths in figs. 2 and 4 which represent $\mathfrak{B}_0$ and K_0, the values of $\mathfrak{B}$ and K at the specified section.

Equation (18) therefore gives us the value of λ—and so, by (14), of the frequency $\mathbf{n}$—for which the ordinates of fig. 1 represent the deflexions $\mathbf{Y}$. Comparing (15) with (10), we see further that the ordinates $\mathbf{m}$ of fig. 3 will represent the corresponding values of $\mathbf{M}$ on a scale given by

$$\frac{\mathbf{M}}{\mathbf{m}}=\frac{n\,.\,\mathfrak{B}}{l\,.\,\mathrm{b}}:$$

this last result, however, has little practical importance.

Having thus shown that the ordinates $\mathbf{Y}$ of fig. 1, *for some value of* $\mathbf{n}$, satisfy the fundamental equation (6), we have now to consider how far the conditions of constraint at the section A may be satisfied: it is evident that we shall not in general satisfy either of them at our first attempt. For purposes of illustration, we shall assume that the specified conditions are that $\mathbf{Y}$ and $\mathbf{M}$ also vanish at A; *i. e.*, that A is the other end of the rod, and is simply supported. At the beginning of the instructions given above for constructing figs. 1 and 3, it was stated that Oa and $O'a'$ could be drawn at any convenient angles, their actual slopes being immaterial: if we now change the slope of one of these lines, and repeat the construction, we shall obtain different values for $\mathbf{Y}$ and $\mathbf{M}$ at the end A. Then by synthesis of the two solutions thus obtained—*i. e.*, by writing

$$\left.\begin{aligned}\mathbf{Y}&=\mathbf{Y}_1+\alpha\mathbf{Y}_2\\ \text{and}\qquad \mathbf{M}&=\mathbf{M}_1+\alpha\mathbf{M}_2,\end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad . \quad (19)$$

where $\mathbf{Y}_1$ and $\mathbf{M}_1$ correspond to the first, and $\mathbf{Y}_2$ and $\mathbf{M}_2$ to the second solution,—and by suitably choosing the constant α, we can evidently obtain a third solution of the fundamental equations which will satisfy both of the imposed conditions at the end O, and one of the imposed conditions (say, the condition $\mathbf{M}=0$) at the end A. The second condition will not also be satisfied, unless by a lucky accident, because we shall not in general have assumed for the frequency $\mathbf{n}$ one of its possible values.

It is at this point that we have to introduce "trial-and-error" methods. Equation (6) shows that the curvature of figs. 1 and 3 will be increased by an increase in the value of $\mathbf{n}$, and hence we can see that if the result of the synthesis expressed by (19) has been to make $\mathbf{Y}$ positive throughout the span OA when α is so chosen that $\mathbf{M}$ vanishes at the end A, then a higher value of $\mathbf{n}$ must be assumed in our second attempt. The necessary procedure, in the case of a rod constrained in any specified way at two sections only, should now be obvious:—Assuming any value of $\mathbf{n}$, we can obtain two graphical solutions of the differential equations, each of which satisfies the imposed conditions at the starting end: by combining these solutions, we obtain a solution satisfying all but one of the imposed conditions of constraint, and we can calculate the "error," or amount by which it fails to satisfy this last condition (*e. g.*, in the example just considered, the "error" is the value of $\mathbf{Y}$ at the end A, since in the correct solution $\mathbf{Y}_A=0$): repeating the process for a second assumed value of $\mathbf{n}$, we obtain a second resultant error, and by plotting the "error" against $\mathbf{n}$, as in fig. 5, we can construct a curve which enables us very quickly to make a correct guess at the required value of $\mathbf{n}$.

In practice, labour may be saved by making the length scale the variable quantity in our successive attempts at constructing the correct diagrams which correspond to figs. 1 and 3: the lengths b_1, b_2, ... etc. in fig. 2 and k_1, k_2, ... etc. in fig. 4 are then determined once for all, and only the spacing of the equidistant vertical lines in figs. 1 and 3 needs to be varied. Equation (18) shows that taking these lines further apart is equivalent to assuming a greater value for λ (or $\mathbf{n}$), and hence we can conveniently proceed by plotting the resultant error against the assumed value of AB (or of l), instead of against $\mathbf{n}$ as in fig. 5; then, having determined by interpolation the value of l which will result in zero error, we can calculate the corresponding value of $\mathbf{n}$ directly by means of (14) and (18).

It remains to discuss the procedure which is required for dealing with *continuous* rods or shafts. We have seen that at sections where these are simply supported there must be continuity in the values of $\mathbf{Y}$, $\frac{d\mathbf{Y}}{dx}$, and $\mathbf{M}$, but not necessarily in the value of $\frac{d\mathbf{M}}{dx}$, since the resultant shear will in general have values differing by a finite quantity on opposite sides of the support. If therefore we are given that A is a section of support, but no longer an end section of the rod, and that Y vanishes at A, then this condition can be realized by combining two solutions as before, and we obtain definite values of $\frac{dY}{dx}$ and of **M** wherewith to begin the portion of the curve which lies to the right of A. In our first attempt at determining the curves for the next unsupported span we may assume that $\frac{d\mathbf{M}}{dx}$ does not undergo a change of value at A: if, then, at our second attempt we start from A, taking zero values for Y, $\frac{dY}{dx}$, and **M**, but a finite slope $\frac{d\mathbf{M}}{dx}$ in fig. 3, and proceed for the span on the right of A by the methods which have been described above, the curves so obtained can be combined in any proportion with our previously determined curves for the double span, and thus we can satisfy the conditions at the next point of support. It is not until we reach the other terminal section that trial-and-error methods (in which n is varied) have to be introduced. The labour is naturally increased by more frequent supports, but need not be regarded as prohibitive.

The diagrams (Pl. VII.) which accompany this paper have been prepared from actual drawings, wherein the foregoing construction was applied to a rod of variable cross-section for which the mathematical solution was known. It is easily verified that if $\mathfrak{B}$ and ρ are given by the expressions

$$\left.\begin{aligned} \mathfrak{B} &= 2\mathfrak{B}_0 \Big/ \left\{ 1 + \frac{1}{3}\,\frac{\sin\frac{3\pi x}{l}}{\sin\frac{\pi x}{l}} \right\} \\ \text{and}\qquad \rho &= \frac{10}{9}\rho_0 \Big/ \left\{ 1 + \frac{1}{27}\,\frac{\sin\frac{3\pi x}{l}}{\sin\frac{\pi x}{l}} \right\} \end{aligned}\right\} \quad . \quad . \quad . \quad (20)$$

—so that $\mathfrak{B}_0$ and ρ_0 are terminal values,—then equation (6) will be satisfied by assuming that

$$Y \propto \left(\sin\frac{\pi x}{l} + \frac{1}{27}\sin\frac{3\pi x}{l} \right),$$

provided that

$$\frac{9g\pi^2\mathfrak{B}_0}{20n^2l^4\rho_0}=1,$$

whence we obtain

$$\lambda=9/5\times\pi^4=175\cdot36 \text{ approximately.}$$

To compare with this result, a solution was obtained graphically. The length of the rod was divided into ten parts, and after values of $\mathfrak{B}$ and ρ had been calculated for the points of sub-division, suitable lengths were chosen for b_1, b_2, b_3, ... etc., and k_1, k_2, k_3, ... etc., as shown in the table below * :—

TABLE I.

x/l	=	0	0·1	0·2	0·3	0·4	0·5
$\mathfrak{B}$	=	639	682	831	1134	1611	1920
ρ	=	·668	·676	·700	·732	·7595	·770
K($\propto 1/\rho$)	$\propto$	1·498	1·480	1·429	1·366	1·317	1·299
k (in inches)	=	5·992	5·920	5·716	5·464	5·268	5·196
b (in inches)	=	3 195	3·41	4·155	5·67	8·055	9·60

A value for λ was then guessed roughly, by estimating mean values of $\mathfrak{B}$ and ρ, and thence calculating the whirling speed from the ordinary formula. Substituting this value for λ in (18), it was found that the corresponding assumption for the length OA in the diagrams would be about 5·71 inches, and this length was adopted at the first application of the construction. Figs. 1 & 3 were then constructed with initial slopes of 30°, as shown, for a first attempt, while in the second attempt fig. 1 was started with zero slope, and fig. 3 again with a slope of 30° : this second attempt gave diagrams of which the general nature is indicated in the left-hand bottom corner of the Plate. The purpose of starting fig. 1 with zero slope in the second attempt was to enable the boundary conditions for **M** to be more easily satisfied by synthesis, without altering the initial slope of the corresponding **Y**-diagram : it is clear that the error will be proportional to the initial slope, and hence, when constructing

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[*] The values here given for $\mathfrak{B}_0$ and ρ_0 are representative of a steel shaft of diameter $\frac{1}{2}$ in. at its terminal sections. For finding λ, only the *relative* values of $\mathfrak{B}$ and ρ at different sections are, of course, material.

the "error diagram" (Pl. VII. fig. 5) it is necessary to have all errors referred to some standard value for this slope.

The diagrams illustrate an obvious simplification which can be made when, as in the present instance, a shaft is symmetrical about its centre. It is only necessary to continue the construction as far as the first point of subdivision past the centre, since the slope of the lines *df* and *d′f′* gives a measure of the error involved: obviously, if the complete diagrams had zero error they would be symmetrical about their centres, and so *df* and *d′f′* would be horizontal. It is convenient to take as a definite measure of the "error" the total rise of these lines over the horizontal distance OA, so that when the diagrams obtained in the first and second attempts have been combined in such a way that *d′f′* is horizontal, the total rise of *df* over this length may be taken as the "error," and plotted against the assumed length of OA.

In the first application of the construction (length OA taken as 5·71 inches) the line *df* was found, after correcting the bending-moment diagram, to *fall* by 0·044 inch over the total length. A fall indicates that the figure assumed for l (or $\mathbf{n}$) was too small, and hence at the second application a length of 6 inches was assumed: this gave an error of the opposite sign and of amount 0·28 inch. Plotting the corresponding points on the error diagram (fig. 5) and joining them by a straight line, a length of 5·75 inches was obtained as the estimated length corresponding to zero error. This was tried, and an error of the same sign as in the first attempt was obtained, the amount of the error being now reduced to 0·004 inch. Plotting this as a third point in the error diagram, and estimating again, the value of l for zero error was given as 5·755 inches, and the corresponding value of λ was obtained from equation (18) as 173. The correct answer, as shown above, is 175·4, approximately, and hence the method has resulted in this instance in an error of 2·4 in 175—slightly under 1·4 per cent.

From (14) it follows that the whirling speed $\mathbf{n}$ is given correctly to well within 1 per cent., and it is clear that practically as accurate a solution could have been obtained by dispensing with the third application of the construction, and accepting the estimated value of 5·75 inches for l.

Trinity College, Cambridge,
June 4th, 1920.

XXXVII. *On Systems with Propagated Coupling.* By ALFRED W. PORTER, *D.Sc., F.R.S., F.Inst.P., and* REGINALD E. GIBBS, *B.Sc., A.Inst.P.*[*]

IN the majority of electrically coupled circuits, the two circuits are so close together that the time of propagation of the mutual action can be neglected. For great distances of separation or for very rapid frequencies this would no longer be true. When the coupling is of a mechanical nature (instead of being electrical), and the actions are propagated with the velocity of sound, the entry of time would have to be considered for even quite small frequencies at moderate distances. For example, with a frequency of 1000 per second, through air, even 30 cm. separation would change the phase of action through a complete period.

A fairly simple case in which acoustic coupling comes into play as mentioned above, is that of a microphone transmitter and a telephone receiver coupled electrically through a valve set, and simultaneously acoustically through the intervening air. It is well known that a transmitter and receiver can maintain each other in vibration as in the case of the singing telephone ; the introduction of the valve set is merely to amplify the action and allow of its study at greater distances of separation.

To maintain continuous oscillations in any system, definite relations must exist between the phases of the forced and forcing systems, as is often stated by saying that they must be "in step" with one another. In the case under consideration the electric coupling in the valve is constant, whilst the acoustic coupling between the transmitter and receiver is a variable quantity dependent on their separation. Thus at some distances this "phase condition" will be realized, at others it will not, so that a periodic variation of intensity with separation is to be expected, while experimentally it is found that if the receiver be moved gradually away from the transmitter in any direction, a series of maxima and minima of mutual action can be obtained.

The whole of the room in which the experiments are conducted can be mapped out in this way into surface-loci of maxima and minima alternating with one another every quarter wave-length. The first impression, that these surfaces represent nodal and ventral surfaces of a system of stationary waves, cannot be true, for the conditions were not suited

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[*] Communicated by the Authors.

to the creation of such waves. Experimentally it was found easy to follow the variations for separations of over twelve yards, whilst diffraction effects could be illustrated using a plane edge (a door) or by reflexion from, or transmission through suitable zone plates.

The system used was too complicated to deal with satisfactorily when treated mathematically, but other simpler cases can be considered in illustration. Case (i.) below is given merely as an illustration of the propagation of mutual induction; but no importance in practice can be attributed to it.

(i.) Electrical transformer allowing for propagation.

Assuming the two circuits to be separated by a distance x and an alternating e.m.f. in the primary, using the ordinary notation, we have

$$\left(L_1\frac{d}{dt}+R_1\right)C_1+Me^{-iqx}\frac{dC_2}{dt}=Ee^{ipt},$$

$$Me^{-iqx}\frac{dC_1}{dt}+\left(L_2\frac{d}{dt}+R_2\right)C_2=0.$$

The variation of the phase of the mutual action with x is adequately expressed by the introduction of the factor $\exp(-iqx)$.

Hence

$$\left[L_1ip+R_1+\frac{M^2e^{-2iqx}p^2}{R_2+L_2ip}\right]C_1=Ee^{ipt}.$$

Let

$$R'=R_1+\frac{M^2p^2[R_2\cos 2qx-L_2p\sin 2qx]}{R_2{}^2+L_2{}^2p^2},$$

$$L'=L_1-\frac{M^2p^2[R_2\sin 2qx+L_2p\cos 2qx]}{R_2{}^2+L_2{}^2p^2},$$

$$\tan\alpha_2=\frac{L_2p}{R_2},\qquad \tan\alpha_1=\frac{L_1p}{R_1},\qquad \tan\alpha=\frac{L'p}{R'}.$$

Then

$$C_1=\frac{Ee^{ipt}}{R'+L'ip}$$

$$=\frac{E\cos(pt-\alpha)}{\sqrt{R'^2+L'^2p^2}}.$$

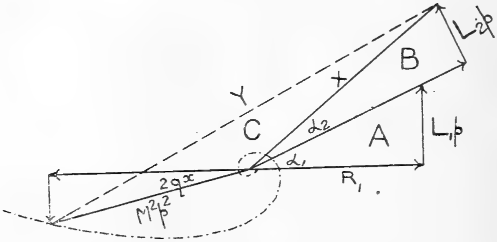
The denominator passes through maxima and minima as x increases. The value of M itself decreases with increase of distance. At near points the change through the distance of a wave-length is very great, but at distant points its variation produces a small effect compared with that due to change of phase. Attention may therefore at first be concentrated on the latter change alone.

Differentiating $R'^2+L'^2p^2$ with regard to x, one obtains finally

$$2qx=\alpha_1+\alpha_2+P\pi$$

as the condition for maxima and minima: the maxima are separated by *half* a wave-length and the minima are midway between them.

Fig. 1.



The above result can be shown diagrammatically as a vector diagram (fig. 1). The triangle A represents the operation $L_1\frac{d}{dt}+R_1$, the triangle B the operation $L_2\frac{d}{dt}+R_2$, whilst C is an allowance for the mutual action. The maxima and minima of Y occur when M^2p^2 (which may first be considered constant) is in line with X, *i. e.* when $2qx=\alpha_1+\alpha_2+P\pi$. If M is not constant, then with centre O a helix (shown dotted) is constructed such that its radius vector represents M^2p^2 for each value of x. Y is a maximum now, for those values of x for which it cuts the helix perpendicularly.

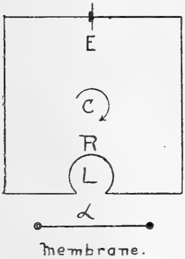
(ii.) Simple Dynamical case illustrating maintenance of vibrations.

Whereas in the previous case the frequency was fixed, in that of the reacting telephones it is not so: the problem, in fact, becomes one of the stability of chance vibrations that may be excited. Such problems have been discussed by the late Lord Rayleigh who, taking an ordinary dynamic equation of the second order

$$m\frac{d^2y}{dt^2}+r\frac{dy}{dt}+p^2y=0,$$

inquired the effect of disturbing terms arising in phase with either y or $\frac{dy}{dt}$. In order that a small disturbance if excited shall increase with time the resulting coefficient r must be negative; the motion will then increase until the equations (which are only approximate) no longer hold good and would need to be modified. A much more complete treatment is to take the disturbance as itself arising from a mechanical system for which the full equations can be written down and then to solve the equations for the two systems simultaneously.

Fig. 2.



As an example, consider a telephone membrane placed at an adjustable distance x from the coil which is connected to a battery. The current at any instant is C, and the displacement of the membrane y. An alteration of C will alter y and vice versa. If a chance alteration is produced, will it persist?

The equations including a mutual action proportional to the current and to $\frac{dy}{dt}$ are,

$$\left(\mathrm{L}\frac{d}{dt}+\mathrm{R}\right)\mathrm{C}+\alpha\frac{dy}{dt}=\mathrm{E},$$

$$\alpha\mathrm{C}+\left(m\frac{d^2}{dt^2}+r\frac{d}{dt}+\mathrm{K}\right)y=0,$$

which on elimination of y gives

$$\left\{\mathrm{L}m\frac{d^3}{dt^3}+(r\mathrm{L}+\mathrm{R}m)\frac{d^2}{dt^2}+(\mathrm{KL}+r\mathrm{R}-\alpha^2)\frac{d}{dt}+\mathrm{RK}\right\}\mathrm{C}=\mathrm{EK}.$$

Putting $\frac{d}{dt}=\xi,\quad \frac{r}{m}=a,\quad \frac{\mathrm{R}}{\mathrm{L}}=b,\quad \frac{\mathrm{K}}{m}=c,\quad \frac{\alpha^2}{\mathrm{L}m}=f,$

the auxiliary equation is

$$\xi^3+(a+b)\xi^2+(c+ab-f)\xi+bc=0,$$

or

$$\xi^3+g\xi^2+h\xi+j=0,$$

where g and j are essentially positive, whilst h may be either positive or negative.

If the roots are ξ_1, ξ_2, ξ_3, then

$$\xi_1+\xi_2+\xi_3=-g,$$

$$\xi_1\xi_2+\xi_2\xi_3+\xi_3\xi_1=h,$$

$$\xi_1\xi_2\xi_3=-j.$$

We are only concerned with the case of two roots complex, say

$$\xi_2 = A + iB,$$
$$\xi_3 = A - iB.$$

Hence

$$\xi_1 + 2A = -g, \quad \ldots \ldots \quad \text{(i.)}$$
$$A^2 + B^2 + 2A\xi_1 = h, \quad \ldots \ldots \quad \text{(ii.)}$$
$$\xi_1(A^2 + B^2) = -j. \quad \ldots \ldots \quad \text{(iii.)}$$

From (iii.) ξ_1 is of the opposite sign to j and is therefore essentially negative. $\xi_1 = -P_1$,

$$\therefore \quad 2A \text{ is positive when } P_1 > g.$$

From (i.) A passes through zero when $\xi_1 = -g$, and at the same time $B^2 = h$ from (ii.), thus h is then positive.

Further, $\xi_1 B^2$ is then equal to $-j$ from (iii.) and therefore $B^2 = \frac{j}{g} = h.$

If $A > 0$, ξ_1 is more negative than $-g$. In terms of a and b, for A to be positive

$$-\xi_1 = P_1 > a + b > \frac{r}{m} + \frac{R}{L}.$$

It follows that if $A = 0$, $\xi_1 = -g$, and $j = gh$, so that h is positive.

Hence the solution for the current is,

$$C = \frac{E}{R} + Fe^{-gt} + G\cos(Bt - \eta)$$

where

$$B = \sqrt{h} = \sqrt{j/g}.$$

Also

$$y = -\frac{Ex}{RK} + F'e^{-gt} + G'\cos(Bt - \eta').$$

The value of $\sqrt{h}$ is $\sqrt{\dfrac{R/L \, . \, K/m}{\dfrac{r}{m} + \dfrac{R}{L}}}$.

Hence a disturbance such as in the above critical case if once excited will be maintained constant. It will be noticed that $\sqrt{\frac{K}{m}}$ is the frequency-constant of the free undamped vibrations of the membrane, say n_0, and therefore

$$\sqrt{h} = n_0 \sqrt{\frac{1}{1+\frac{rL}{Rm}}},$$

$$= n_0 \sqrt{\frac{1}{1+t_e/t_0}}.$$

where t_e = time-constant for electrical circuit,

t_0 = ,, ,, ,, ,, membrane ;

in each case when isolated from the other vibrating systems.

If ξ_1 is more negative than $(-g)$ the vibration will grow, whilst the equation for y is then

$$y = -\frac{E\alpha}{RK} + F'e^{-(g+P)t} + G'e^{\frac{Pt}{2}}\cos(B't-\eta'),$$

where P = positive quantity.

The value of B′ is given by

$$B'^2 + A^2 = \frac{j}{g+2A};$$

hence for a small positive value of A it is less than j/g and will always be less than the free frequency of the membrane. For all positive values of A, y will increase up to a value at which the assumed equations are no longer valid owing to the higher order terms (which have been ignored) acquiring increasing importance and ultimately controlling the maximum attainable value.

So far the time of propagation of the mutual action has been taken as zero ; the change necessary to allow for it is, as in the previous example, to introduce into the value of α a factor e^{-iqx}.

The auxiliary equation becomes

$$mL\xi^3 + (rL+Rm)\xi^2 + (KL + rR - \alpha^2 e^{-2iqx})\xi + RK = 0,$$

or $$\xi^3 + g\xi^2 + h\xi + j = 0,$$

where $$h = \lambda + i\mu$$

and $$\lambda = \frac{KL + rR - \alpha^2\cos 2qx}{mL},$$

$$\mu = \frac{\alpha^2 \sin 2qx}{mL}.$$

The critical case for which maintenance will begin is that for which two of the roots are given by $\pm ip$. This condition gives the two equations

$$p^2=\lambda,$$

$$gp^2+\mu p-j=0,$$

whence eliminating p,

$$\lambda+\frac{\mu}{g}\sqrt{\lambda}=\frac{j}{g}.$$

Whatever value of x satisfies this equation, other values will also satisfy it, provided that $2qx$ is greater than at first by 2π times an integer, at least for points so distant that α^2 can be treated as constant.

$$p^2=\lambda=j/g-(\mu/g)\sqrt{\lambda}.$$

But in any usual case α^2 and therefore μ will be very small compared with $\mathrm{KL}+r\mathrm{R}$, so that

$$p^2 \simeq j/g-(\mu/g)\sqrt{j/g},$$

$$=n_1{}^2-\frac{\mu}{g}n_1,$$

where the second term on the right is small ; and

$$\mu=\frac{\alpha^2}{m\mathrm{L}}\sin 2qx,$$

and therefore p^2 is subjected to a small harmonic variation (dependent on x) about its average value $n_1{}^2$.

The equations become somewhat modified but at the same time more symmetrical if the electrical circuit possesses a capacity S.

Each of the systems is now "elastic" and possesses a free vibratory period of its own.

The equations are, for the circuit

$$\left(\mathrm{L}\frac{d^2}{dt^2}+\mathrm{R}\frac{d}{dt}+\frac{1}{\mathrm{S}}\right)\mathrm{Q}+\alpha\frac{dy}{dt}=\mathrm{E},$$

and for the membrane

$$\alpha\frac{d\mathrm{Q}}{dt}+\left(m\frac{d^2}{dt^2}+r\frac{d}{dt}+\mathrm{K}\right)y=0,$$

leading to the auxiliary equation

$$\left(\mathrm{L}\xi^2+\mathrm{R}\xi+\frac{1}{\mathrm{S}}\right)(m\xi^2+r\xi+\mathrm{K})-\alpha^2\xi^2=0.$$

If any solution of this is a pure imaginary ip, then

$$\left(-Lp^2+\frac{1}{S}\right)(-mp^2+K)-rRp^2+\alpha^2p^2=0,$$

and
$$R(-mp^2+K)+r\left(-Lp^2+\frac{1}{S}\right)=0,$$

whence for the critical case at which self-maintenance just begins

$$-mp^2+K=\pm p\sqrt{\alpha^2\frac{r}{R}-r^2}$$

with the condition $\frac{\alpha^2}{R}>r$ if p is to be real, which is therefore one of the conditions for maintenance.

Thus p^2 differs from $\frac{K}{m}$ by a quantity proportional to p which is in practice small. If the factor of p is denoted by F, then

$$p=\sqrt{\frac{K}{m}+\frac{F^2}{4m^2}}\mp\frac{F}{2m}$$

$$\bumpeq\sqrt{\frac{K}{m}}\mp\frac{F}{2m}.$$

It should be noted that if α^2 is sufficiently small it is impossible for self-maintained oscillations to exist.

As before, to pass to the case of propagation α^2 is replaced by α^2e^{2iqx}, which (if α^2 is eliminated instead of $\left(-Lp^2+\frac{1}{S}\right)$ as before) leads to the equation

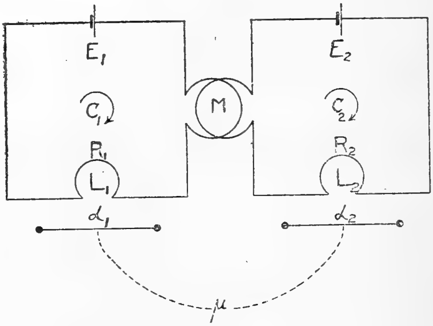
$$p\sin 2qx\left[\left(-Lp^2+\frac{1}{S}\right)(-mp^2+K)-rRp^2\right]$$
$$=p^2\cos 2qx\left[R(-mp^2+K)+r\left(-Lp^2+\frac{1}{S}\right]\right).$$

It would be unprofitable to consider this fully; special cases can be worked out when desired.

(iii.) The above cases differ widely from the actual conditions of the experiments referred to in the first paragraph. They are given simply to illustrate a general mode of examining the maintenance of vibrations, both with and without allowance for the influence of time of propagation upon the coupling.

The experimental case—except for the presence of the valves—can be represented by the diagram following (fig. 3).

Fig. 3.



In this diagram M is the mutual inductance between the electric circuits, α_1 and α_2 are the mutual coefficients representing the magnetic coupling between the coil and diaphragm in the respective telephones, and μ the corresponding coefficient for the acoustic coupling between the diaphragms.

The equations are

$$\left[\mathrm{L}_1\frac{d}{dt}+\mathrm{R}_1\right]\mathrm{C}_1+\mathrm{M}\frac{d\mathrm{C}_2}{dt}+\alpha_1\frac{dy_1}{dt}=\mathrm{E}_1,$$

$$\left[\mathrm{L}_2\frac{d}{dt}+\mathrm{R}_2\right]\mathrm{C}_2+\mathrm{M}\frac{d\mathrm{C}_1}{dt}+\alpha_2\frac{dy_2}{dt}=\mathrm{E}_2,$$

$$\alpha_1\mathrm{C}_1+\left[m_1\frac{d^2}{dt^2}+r_1\frac{d}{dt}+\mathrm{K}_1\right]y_1+\mu\frac{dy_2}{dt}=0,$$

$$\alpha_2\mathrm{C}_2+\left[m_2\frac{d^2}{dt^2}+r_2\frac{d}{dt}+\mathrm{K}_2\right]y_2+\mu\frac{dy_1}{dt}=0,$$

leading to an auxiliary equation of the sixth degree. This case is too complicated to justify further account being given of it here.

The general nature of the effect of propagation is sufficiently well represented by case ii.

XXXVIII. *On Vapour Pressures and the Isothermals of Vapours.* By J. H. SHAXBY, *B.Sc., Lecturer in Physics and Director of the Viriamu Jones Research Laboratory, University College, Cardiff*[*].

§ 1. A LARGE number of empirical formulæ have been proposed to express the vapour pressure of a fluid in terms of the temperature and certain "constants" for the substance in question. Many of them, *e. g.* those of Dalton, Roche, Biot, Kirchhoff, and van der Waals, are of logarithmic or exponential form.

This at once recalls Dieterici's Equation of State

$$p = \frac{RT}{v-b} e^{-\frac{A}{vRT}},$$

in which p is the external pressure upon a fluid of specific volume v at absolute temperature T, R being the gas constant and A and b constants for the particular substance.

In obtaining this equation Dieterici makes use of a quantity A′, the work done per unit mass against the forces of cohesion in removing molecules from the fluid ; A′ is proportional to the density of the fluid and is replaced by $\frac{A}{v}$. On similar lines, let us examine the relation between the total internal pressure, Π_1, in the liquid phase of any substance, and that, Π_2, in its vapour phase. The work done in the transfer of unit mass from liquid to vapour may be equated, following Dieterici, to twice the kinetic energy lost during this transformation of high speeds (of the liquid molecules which escape) into average speeds of these same molecules on their arrival in the vapour. This loss of kinetic energy per unit mass is $\frac{1}{2}s^2$, where s is the critical value of the component normally towards the surface of the velocity of a liquid molecule, *i. e.* the smallest value compatible with escape into the vapour. Now, $s^2 = 2RT \log_e \frac{N_1}{N_2}$, where N_1, N_2 are the numbers of molecules per unit volume in liquid and vapour respectively. Thus, in the case of a substance which is not dissociated either in liquid or in vapour form,

$$s^2 = 2RT \log_e \frac{d_1}{d_2},$$

where d_1 = density of liquid, d_2 of vapour.

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[*] Communicated by Prof. A. W. Porter, F.R.S.

Thus $$\Pi_2=\Pi_1 e^{-\frac{s^2}{\mathrm{RT}}}=\Pi_1\frac{d_2^2}{d_1^2}.$$

But a fundamental relation connecting Π_1 and Π_2 is

$$\Pi_1(v_1-b)=\mathrm{RT}=\Pi_2(v_2-b),$$

where v_1 and v_2 are the volumes of unit mass in the liquid and in the vapour.

So $$\frac{d_2^2}{d_1^2}=\frac{v_1-b}{v_2-b}=\frac{\frac{1}{d_1}-b}{\frac{1}{d_2}-b},$$

whence $$b=\frac{1}{d_1+d_2}.$$

The quantity (d_1+d_2) diminishes linearly, and fairly slowly, with rise of temperature: this fact is well known and is embodied in the so-called Cailletet-Mathias rule. The rule is nearly exact for substances which are not associated, but does not hold for such substances as water, for which the changes in (d_1+d_2) are less regular. For most substances the rate of decrease of (d_1+d_2) is of the order of ·001 per 1° C. Hence b is not a true constant, and this is known to be the case from comparison of Dieterici's equation with experimental data.

We find, then, that

$$\frac{v_1-b}{v_1}=\frac{d_2}{d_1+d_2} \quad\text{and}\quad \frac{b}{v_1}=\frac{d_1}{d_1+d_2},$$

while $$\frac{v_2-b}{v_2}=\frac{d_1}{d_1+d_2} \quad\text{and}\quad \frac{b}{v_2}=\frac{d_2}{d_1+d_2}.$$

That is to say, the "free" space per unit volume in a liquid at any temperature is equal to the "occupied" space per unit volume in its saturated vapour, and conversely. Equal volumes of liquid and vapour are, as it were, complementary in this respect.

§ 2. We may write Dieterici's equation in the form

$$\frac{p(v-b)}{\mathrm{RT}}=e^{-\frac{\mathrm{A}}{v\mathrm{RT}}}.$$

Thus for a liquid and its saturated vapour at temperature T, if p is the saturation pressure,

$$\frac{p(v_1-b)}{\mathrm{RT}}=e^{-\frac{\mathrm{A}}{v_1\mathrm{RT}}}$$

and $$\frac{p(v_2-b)}{\mathrm{RT}}=e^{-\frac{\mathrm{A}}{v_2\mathrm{RT}}}.$$

Hence $$\left(\frac{p(v_1-b)}{\mathrm{RT}}\right)^{v_1}=\left(\frac{p(v_2-b)}{\mathrm{RT}}\right)^{v_2}$$

or $$\left(\frac{p}{\mathrm{RT}}\right)^{v_2-v_1}=\frac{(v_1-b)^{v_1}}{(v_2-b)^{v_2}}.$$

Substituting $\frac{1}{d_1}$ for v_1, $\frac{1}{d_2}$ for v_2, and $\frac{1}{d_1+d_2}$ for b, this becomes

$$p=\mathrm{RT}(d_1+d_2)\left(\frac{d_2}{d_1}\right)^{\frac{d_1+d_2}{d_1-d_2}}. \quad . \quad . \quad . \quad . \quad (1)$$

We thus obtain an equation for the saturated vapour pressure in terms of the densities of the liquid and its saturated vapour, the absolute temperature, and the gas constant. The equation contains no adjustable constants whatever.

The constant b of Dieterici's equation is here replaced by $\frac{1}{d_1+d_2}$, which varies slowly with the temperature. Similarly, Dieterici's A now becomes

$$\frac{2\mathrm{RT}}{d_1+d_2}\log_e\frac{d_1}{d_2},$$

also a function of temperature. For the critical temperature, when $d_1=d_2=d_c=\frac{2}{b_c}$, we find

$$p_c=2\mathrm{RT}_c d_c e^{-2}.$$

The equation, like Dieterici's, thus gives the satisfactory value $\frac{1}{2}e^2=3{\cdot}695$ for $\frac{\mathrm{RT}}{pv}$ at the critical point.

The agreement of the values given by the equation (1) with these directly observed has been tested by calculation for the substances * whose densities in liquid and vapour states are given in the tables of L. Graetz's article on vapour pressures in Winkelmann's *Handbuch der Physik*, 2nd ed. vol. iii. pt. 2, pp. 962–1086. The approximation to actual values is about the same for all these substances, whether associated or non-associated. Table I. is typical;

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* Ammonia, Carbon Dioxide, Water, Pentane, Hexane, Heptane, Octane, Isopentane, Di-isopropyl, Di-isobutyl, Hexamethylene, Ethyl Ether, Methyl Formate, Ethyl Formate, Propyl Formate, Methyl Acetate, Ethyl Acetate, Propyl Acetate, Methyl Propionate, Ethyl Propionate, Methyl Butyrate, Methyl Isobutyrate.

the first seven columns are as follows:—(1) temperature, t° C.; (2) density of liquid, d_1; (3) density of vapour, d_2; (4) b, i. e. $\frac{1}{d_1+d_2}$; (5) A, *i. e.* $\frac{2RT}{d_1-d_2}\log_e\frac{d_1}{d_2}$; (6) calculated saturated vapour pressure; (7) observed saturated vapour pressure. It will be seen that the equation represents the facts with fair accuracy. The calculated value is usually too high at low pressures, then becomes somewhat low until the temperature is within a few degrees of the critical; at the latter temperature it is again too high.

The corresponding Equation of State thus becomes

$$p\left(v-\frac{1}{d_1+d_2}\right)=RTe^{-\frac{2\log\frac{d_1}{d_2}}{(d_1-d_2)v}}/(d_1-d_2)v=RT\left(\frac{d_2}{d_1}\right)^{\frac{2}{(d_1-d_2)v}}/(d_1-d_2)v. \quad . \quad . \quad . \quad (2)$$

The Isothermals of Isopentane, Hexane, Ethyl Ether, and Water have been calculated from this equation for the specific volumes quoted in Winkelmann, vol. iii. pt. 2, pp. 1114–1135.

Table II. gives the result for Water and Isopentane; those for the other substances are quite similar. Column 1 gives the Specific Volume, column 2 the calculated pressures, and the *last* column (5) the actual pressures, as given by experiment. It will be noted that the agreement is fair for low temperatures.

At the critical temperature the equation assumes the form

$$p\left(v-\frac{1}{2d_c}\right)=RT_ce^{-\frac{2}{d_cv}}.$$

For temperatures above the critical this same form has been used, except that d_c has been replaced by $\frac{d_1+d_2}{2}$ and the value of the latter obtained by extending the linear relation of Mathias to the given temperature. For the present we cannot assign any physical meaning to (d_1+d_2) for temperatures above the critical; we return to this point later. The table shows that the equation still represents the facts moderately well for the smaller pressures, but completely breaks down at the higher, *i. e.* when the specific volumes are small.

Since this paper was written (and in part read at the Cardiff meeting of the British Association, 1920), Professor Porter has called my attention to a paper by F. H. MacDougall (J. Amer. Chem. Soc. xxxviii. no. 3, 1916, p. 528) on the Equation of State, in which Dieterici's equation is studied. In this paper MacDougall develops the same equation as (1) above for vapour pressures. He reaches it by assuming that $\frac{1}{v_1}+\frac{1}{v_2}=\frac{2}{v_3}$, where v_1 and v_2 have the same meanings as in the present paper and v_3 is the volume corresponding to the same pressure (the vapour pressure) on the unstable (James Thomson) part of the isothermal. Thus $\frac{v_3}{2}$ is equal to b, and the equation $b=\frac{1}{d_1+d_2}$ of course follows immediately. The grounds for the assumption are given:—(1) $v_1=v_2=v_3=2b$ at the critical temperature, and v varies slowly as the temperature falls, so that it is plausible to assume $v=2b$ for all temperatures. (2) The Cailletet-Mathias relation becomes $d_3=d_0(1-\alpha T)$; a small and linear temperature coefficient suggests a real significance for the quantity d_3. (3) The saturation pressure is the geometric mean of the maximum and minimum pressures on the isothermal, if $v=2b$.

MacDougall also notes and illustrates in tables the approximate constancy of the product Ab (see § 3 below), but draws no further conclusion from this. He does not appear to have noticed the remarkable relation (§ 1 above) that liquid and saturated vapour are complementary as regards free and occupied volumes per c.c.

The thermodynamic relations of Dieterici's equation and the calculation of the latent heat of vaporization are discussed in detail, and the whole paper is of great interest.

§ 3. The matter may usefully be examined from a slightly different point of view. The work done in transferring unit mass of a substance isothermally from liquid to vapour is

$$\int_{v_1}^{v_2}\Pi dv=\mathrm{RT}\int_{v_1}^{v_2}\frac{dv}{v-b}=\mathrm{RT}\log\frac{v_2-b}{v_1-b}=2\mathrm{RT}\log\frac{d_1}{d_2},$$

the symbols having the same meanings as before. This work may be equated to A_1-A_2, where A_1 is the work required to overcome the cohesive forces of the liquid, per unit mass, and A_2 is the similar quantity for the saturated vapour. These quantities, A_1 and A_2, as mentioned in § 1, are proportional, at any assigned temperature, to the densities d_1 and d_2 at that temperature.

Hence we have

$$2\mathrm{RT}\log\frac{d_1}{d_2}=A_1\left(1-\frac{d_2}{d_1}\right) \text{ or } A_1=2\mathrm{RT}d_1\frac{\log\frac{d_1}{d_2}}{d_1-d_2}.$$

Similarly,

$$A_2 = 2RTd_2 \frac{\log \frac{d_1}{d_2}}{d_1 - d_2},$$

from either of which equations it immediately follows that

$$A = 2RT \frac{\log \frac{d_1}{d_2}}{d_1 - d_2},$$

as before.

Before we can proceed further it becomes necessary to consider the meaning of the quantities A and b.

A may be termed the Cohesive constant of a substance, and is the quotient of the work per unit mass against the forces of cohesion divided by the density

$$A = \frac{A_1}{d_1} = \frac{A_2}{d_2}.$$

If A_1 and A_2 depended simply on the density, we should have A the same for all temperatures, which is not the case. A diminishes with rise of temperature, indicating, what one might expect, a decrease in the forces of cohesion as the temperature is raised.

The quantity $(d_1 + d_2)$ which plays such a fundamental part in the equations deduced above, may be regarded as a measure of the closeness of molecular packing which would result from the cohesive forces alone in the absence of molecular motions; for, if there were no "free" space, $v = b$. Thus the density would be $(d_1 + d_2)$, and we may regard "free" space, in the sense of the equations of the kinetic theory, as a result of the molecular movements, which prevent the molecules from being as closely packed as they would be under their mutual attractions if they did not possess kinetic energy. If this be so, we can regard the quantity $(d_1 + d_2)$, which we will term the Cohesive Density, as a measure of the intensity of the forces of cohesion. We should thus expect $(d_1 + d_2)$ to depend in some simple manner upon A. The two quantities, as a matter of fact, prove to be almost exactly proportional to one another. That is to say, the product Ab is nearly constant, as is shown in column 8 of Table I. There is a clear tendency of the product to increase slightly with rise of temperature to a maximum value, and then to decrease again as the critical temperature is approached in all non-associated substances. This lack of exact constancy in Ab may be regarded as due to the fact that the relation $\frac{A_1}{d_1} = \frac{A_2}{d_2}$ is only approximately true for any assigned temperature.

If we write $\frac{A_1}{d_1}$ or $\frac{A_2}{d_2}$ for A, we have

$$\frac{A}{d_1+d_2} = \frac{A_1}{d_1(d_1+d_2)} = \frac{A_2}{d_2(d_1+d_2)} = \frac{A_1-A_2}{d_1^2-d_2^2},$$

and these expressions should also be independent of temperature. For an actual substance, let us suppose that $A_1 = rd_1$, and that $A_2 = r(1+\alpha)d_2$, where r and α are of course not independent of temperature.

Then

$$\frac{A}{d_1+d_2} = \frac{A_1-A_2}{d_1^2-d_2^2} = r\frac{d_1-d_2-\alpha d_2}{d_1^2-d_2^2} = r\left(\frac{1}{d_1+d_2} - \frac{\alpha d_2}{d_1^2-d_2^2}\right).$$

With rise of temperature the first term inside the bracket increases slowly; the second, apart from α, also becomes greater, and very rapidly so as the critical temperature is approached. Hence, if we may regard r and α as not varying very rapidly with temperature, it is easy to see that for small positive values of α the quantity Ab will at first increase slowly to a maximum and then decrease, as the temperature rises.

§ 4. For an "ideal" substance, then, we may suppose that A is proportional to (d_1+d_2), *i. e.* that $\frac{RT\log\frac{d_1}{d_2}}{d_1^2-d_2^2}$ is independent of temperature. For actual substances this is not exactly true, but we may none the less suppose that there is for them also a relation, Ab=constant, expressing the constant relation of the cohesive forces to the cohesive density. In this, however, either A or b or both differ slightly from the values $\left(\frac{2RT\log\frac{d_1}{d_2}}{d_1-d_2} \text{ and } \frac{1}{d_1+d_2} \text{ respectively}\right)$ which we have hitherto assigned to them. To test this point we must modify our original equation for the saturation pressure

$$p = RT(d_1+d_2)\left(\frac{d_2}{d_1}\right)^{\frac{d_1+d_2}{d_1-d_2}},$$

and the most simple way of doing this in the required way is to substitute a constant K for the quantity

$$\frac{2\mathrm{RT}\log\frac{d_1}{d_2}}{d_1^2-d_2^2}.\quad \text{Thus}\quad \frac{\log\frac{d_1}{d_2}}{d_1-d_2}=\frac{\mathrm{K}(d_1+d_2)}{2\mathrm{RT}}$$

$$\text{and}\qquad p=\frac{\mathrm{RT}}{b}e^{-\frac{\mathrm{C}}{b^2\mathrm{T}}}, \quad \ldots\ldots \quad (1\,a)$$

where C is a new constant (independent of density and temperature), and b has its modified value, nearly but no longer exactly equal to $\frac{1}{d_1+d_2}$.

In the same way the Equation of State becomes

$$p=\frac{\mathrm{RT}}{v-b}e^{-\frac{2\mathrm{C}}{vb\mathrm{T}}}. \quad \ldots\ldots \quad (2\,a)$$

In applying the equation ($1a$) to actual substances, we may make our adjustments most readily as follows:

$$\log p=\log \mathrm{RT}-\log b-\frac{\mathrm{C}}{\mathrm{T}b^2}.$$

The term $\log b$ is small and not very different from $\log\frac{1}{d_1+d_2}$, so we may replace it by the latter without serious error and so evaluate $\frac{\mathrm{C}}{\mathrm{T}b^2}$, using experimentally determined values of p.

Again substituting $(d_1+d_2)^2$ for $\frac{1}{b_2}$, we obtain the general magnitude of C, and finally select from the array of slightly varying C's a value which is seen to give good agreement (for its own particular temperature) with experimental fact, *i. e.* the C for a temperature at which the original unmodified formula (1) gave a satisfactory value of p.

From the adopted value of C the quantity b is recalculated for each temperature, and the new value of p then obtained from the equation ($1\,a$). The results of this semi-empirical method are given in column 10 of Table I., column 9 giving the recalculated values of b, upon which they are based.

It can be seen that a much closer approximation to experimental fact is thus attained. The lack of agreement now seldom amounts to much more than 1 per cent., and in the best "fitting" part of each table is very slight indeed, but a systematic deviation is still apparent. The closeness of fit for any particular substance depends on the selected value of C, and no very great care was taken in most cases to secure the best value ; a likely value was "spotted," and the resulting magnitudes of b and p worked out. It was evident, in several cases, that slightly better agreement could be obtained; but this was not worth the time it would have taken, in view of the facts that the formula can be modified and made somewhat less empirical (see the next section), and that a systematic error exists which requires a re-examination of the premises ; this has been done, and will form the subject of a further paper.

The new values of b determined as above have also been applied to the recalculation of the isothermals of Table II. Column 3 of this table shows the resulting pressures. Here also greatly improved agreement with experimental data is found. At temperatures above the critical the agreement is as good as at lower temperatures, whereas the former equation broke down in these cases.

§ 5. We have thus found that equation (1 a) for saturation pressure, and equation (2 a) for the isothermal relation between pressures and the corresponding volumes yield close agreement with the results of experiment. In these equations, however, C is an adjustable constant, and b, though it has a physical interpretation, is not directly given by independent data, whereas in the original equations (1) and (2) only separately measurable quantities occurred, viz. temperatures and densities.

We may to some extent restore a physical meaning to C, and thereby give it a determinate magnitude for each substance. At the critical temperature equation (1 a) becomes

$$p_c = \frac{\mathrm{RT}_c}{b_c} e^{-\frac{\mathrm{C}}{\mathrm{T}_c b_c^2}},$$

while equation (1) reduces to $p_c = \frac{\mathrm{RT}_c}{b_c} e^{-2}$.

If, then, we identify these two values of p_c, we have

$$\mathrm{C} = 2\mathrm{T}_c b_c^2,$$

and therefore

$$p = \frac{\mathrm{RT}}{b} e^{-\frac{2\mathrm{T}_c b_c^2}{\mathrm{T}b^2}} = \frac{\mathrm{RT}}{b} e^{-\frac{\mathrm{T}_c}{2d_c^2\mathrm{T}b^2}} . \quad . \quad . \quad . \qquad (1\,b)$$

The equation of state similarly becomes

$$p(v-b) = \mathrm{RT}e^{-\frac{4\mathrm{T}_c b_c^2}{vb\mathrm{T}}} = \mathrm{RT}e^{-\frac{\mathrm{T}_c}{d_c^2 b\mathrm{T}v}} . \quad . \quad . \qquad (2\,b)$$

The values, calculated from (2 b), of the pressures along the various isothermals are given in column 4 of Table II.

§ 6. The equation $p=\frac{\mathrm{RT}}{b}e^{-\frac{2\mathrm{T}_c b_c{}^2}{\mathrm{T}b}}$ can readily be thrown into a "Reduced" form: at the critical temperature

$$p_c=\frac{\mathrm{RT}_c}{b_c}e^{-2}.$$

$$\therefore \qquad \frac{p}{p_c}=\frac{\mathrm{T}b_c}{\mathrm{T}_c b}e^{-2\left(\frac{\mathrm{T}_c b_c{}^2}{\mathrm{T}b}-1\right)}.$$

Write π for $\frac{p}{p_c}$, θ for $\frac{\mathrm{T}}{\mathrm{T}_c}$, and ψ for $\frac{b}{b_c}$.

Then $$\pi=\frac{\theta}{\psi}e^{-2\left(\frac{1}{\psi^2\theta}-1\right)}.$$

Replacing b by $\frac{1}{d_1+d_2}$ and b_c by $\frac{1}{2d_c}$, and writing δ for $\frac{d_1+d_2}{2d_c}$, we obtain

$$\pi=\theta\delta e^{-2\left(\frac{\delta 2}{\theta}-1\right)}.$$

§ 7. The critical density of a substance is difficult to measure accurately, and its magnitude is commonly estimated by using the Cailletet-Mathias linear relation. It may also be calculated as follows:

$$\frac{\log_e \frac{d_1}{d_2}}{d_1{}^2-d_2{}^2}=\frac{\mathrm{T}_c}{2\mathrm{T}d_c{}^2}.$$

Let $d_1=qd_2$. Therefore

$$\frac{\log_e q}{d_2{}^2(q^2-1)}=\frac{\mathrm{T}_c}{\mathrm{T}}\frac{1}{2d_c{}^2}$$

or $$\frac{d_2}{d_c}=\sqrt{\frac{\mathrm{T}}{\mathrm{T}_c}}\sqrt{\frac{2\log_e q}{q^2-1}} \quad \theta^{\frac{1}{2}}\sqrt{\frac{2\log_{10} q}{\cdot 4343(q^2-1)}}.$$

Thus $d_c=\frac{d_2}{\phi\theta^{\frac{1}{2}}}$, writing ϕ for the last square root. ϕ is a purely numerical quantity; to facilitate calculation its logarithm can be tabulated for a series of values of $\log\frac{d_1}{d_2}$, say from 0 to 3 in steps of ·1, with difference columns allowing of estimation for intervals of ·001. The values thus found for the critical densities of Water and Isopentane are given in the last column (11) of Table I. The Cailletet-Mathias method gives 0·2344 for Isopentane and fails for Water.

TABLE I.

Water.

t° C.	d_1.	d_2.	b.	A×10^{-9}.	p.		Ab×10^{-9}.	C=3035.		d_c.
					Calc.	Obs.		Recalc. b.	Recalc. p.	
0	·9999	$\cdot 0_5 4737$	1·000	30·89	4·478	4·51	30·89	·952	4·736	......
60	·9833	$\cdot 0_3 1303$	1·016	27·91	149·8	148·7	28·38	1·009	149·7	.. ...
120	·9441	·001138	1·058	25·86	1529	1496	27·36	1·069	1485	......
144·2	·923	·002187	1·081	25·27	3089	3047	27·24	1·093	3086	......
180	·8881	·005216	1·119	24·33	7754	7520	28·09	1·132	7437	·3289
202·2	·861	·007976	1·151	24·07	12140	12160	28·16	1·157	12090	·3261
240	·8117	·01632	1·208	23·26	25210	25093	27·23	1·205	25130	·3239
270	·7701	·02766	1·253	22·47	42080	42235	27·71	1·251	42300	·3236

Isopentane.

t° C.	d_1.	d_2.	b.	A.	p (mm. Hg).		Ab×10^{-10}.	C=4287.		d_c.
					Calc.	Obs.		Recalc. b.	Recalc. p.	
20	·6196	·0024	1·608	6·085	584·8	572·6	·978	1·614	570·6	0·2332
40	·5988	·0045	1·658	5·943	1141	1131·1	·985	1·659	1130	·2324
60	·5769	·0078	1·710	5·811	2025	2035·6	·994	1·707	2039	·2314
80	·5540	·0128	1·764	5·671	3346	3400·8	1·001	1·758	3414	·2304
100	·5278	·0203	1·824	5·526	5235	5354·5	1·008	1·812	5389	·2297
120	·4991	·0311	1·886	5·379	7766	8039·9	1·014	1·873	8098	·2288
140	·4642	·0473	1·955	5·222	11090	11620	1·021	1·940	11700	·2384
160	·4206	·0729	2·027	5·037	15380	16285	1·021	2·018	16350	·2283
170	·3914	·0932	2·064	4·919	18030	19094	1·015	2·062	19110	·2290
180	·3498	·1258	2·103	4·881	20250	22262	1·027	2·110	22190	·2338
185	·3142	·1575	2·123	4·659	23420	23992	·989	2·135	23850	·2320
187	·2857	·1834	2·131	4·597	24500	24713	·980	2·146	24550	·2329
187·8	·2344	·2344	2·133	4·516	25300	25005	·964	2·150	24800	

TABLE II.

Isothermals of Water.

$T_c=365°$ C. $b_c=1\cdot536$.

v.	p_1.	p_2.	p_3.	Exp. p.
14°·91 C.	$d_1+d_2=\cdot9991$.		Corrected $b=\cdot967$.	
39941	7·12	7·12	7·12	7·12
80311	12·41	12·41	12·41	12·34
99°·6 C.	$d_1+d_2=\cdot9597$.		Corrected $b=1\cdot045$.	
7986·45	161·3	161·3	161·3	159·61
3953·80	325·1	325·1	325·1	322·26
2421·19	529·7	529·8	529·7	525·47
1981·36	646·3	646·5	646·3	640·78
1690·46	756·5	756·7	756·5	749·12
182°·9 C.	$d_1+d_2=\cdot8891$.		Corrected $b=1\cdot131$.	
2204·262	742·3	744·2	738·7	708·7
1021·215	1589	1598	1572	1518·1
531·649	3009	3041	2949	2903·1
256·348	6046	6180	5802	5990·7
218·314	7021	7204	6689	6952·2
195·316	7784	8009	7374	7888·8
187·622	8074	8320	7633	7971·4
202°·21 C.	$d_1+d_2=\cdot8688$.		Corrected $b=1\cdot157$.	
1725·812	947·9	947·9	947·9	943·6
941·536	1729	1729	1729	1714·8
515·864	3130	3129	3130	3111·4
226·418	6958	6955	6958	6910·1
133·169	11480	11480	11480	11592·6
125·372	12130	12130	12140	12181·1

Isothermals of Isopentane.

$T_c=187°\cdot5$ C. $b_c=2\cdot150$.

v.	p_1.	p_2.	p_3.	Exp. p.
50° C.	$d_1+d_2=\cdot5941$.		Corrected $b=1\cdot683$.	
330	811·9	811·7	812·2	814
227·2	1157	1156	1157	1158
170·3	1511	1511	1512	1514
140° C.	$d_1+d_2=\cdot5115$.		Corrected $b=1\cdot940$.	
417·1	839·0	839·5	839·5	858
176·5	1925	1928	1928	1933
85·78	3753	3764	3768	3808
52·16	5772	5798	5806	5914
36·16	7720	7771	7785	7977
26·69	9588	9677	9699	9993
24·34	10180	10280	10310	10640
21·22	11070	11200	11230	11595

TABLE II. (*continued*).

187°·5 C. (Crit. temp.). $d_1+d_2=\cdot 4688$. Corrected $b=2{\cdot}150$.

v.	p_1.	p_2.	p_3.	Exp. p.
19·41	14870	14800	14844	15190
14·40	17970	17840	17920	18430
10·67	20980	20810	20900	21480
6·970	24240	23910	24080	24430
3·872	25310	24780	25110	25030

280° C. $d_1+d_2=\cdot 3840$. Corrected $b=2{\cdot}49$.

540·8	872·2	879·0	879·0	876
263·0	1766	1795	1795	1786
125·70	3579	3698	3698	3680
52·34	7884	8525	8531	8459
24·43	14310	16900	16930	16870
11·40	21810	31110	31210	31240
5·810	24860	49380	49680	49530
5·172	24880	53560	53930	53760

Summary.

1. Consideration of the internal pressure in fluids leads to the relation that in "ideal" substances the ratio of the occupied volume (co-volume) to the total volume of a liquid is equal to the ratio of the unoccupied volume to the total volume of its saturated vapour. The two phases, in equilibrium with each other, are complementary in this respect. The relation is nearly true for actual fluids.

2. The equation

$$p=\mathrm{RT}(d_1+d_2)\left(\frac{d_2}{d_1}\right)^{\frac{d_1+d_2}{d_1-d_2}} \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

is obtained for the saturation pressure of a vapour in terms of the temperature and the densities of the two phases at that temperature. This leads to an Equation of State

$$p\left(v-\frac{1}{d_1+d_2)}\right)=\mathrm{RT}\left(\frac{d_2}{d_1}\right)^{\frac{2}{(d_1-d_2)v}} . \quad . \quad . \quad . \quad (2)$$

3. It is shown that the quantity (d_1+d_2) may be regarded as a density factor which is a measure of the cohesive forces per unit mass.

4. On this hypothesis that for actual substances there is a "density" proportional to the cohesive forces, nearly but not exactly equal to (d_1+d_2), modified forms of equations (1) and (2) are obtained. These equations express the experimental facts with considerable accuracy for all temperatures and pressures.

5. "Reduced" forms of the vapour-pressure equation are obtained and an equation for the calculation of the critical density of a substance. This gives consistent values, even in cases such as that of water when the Mathias rule does not hold.

I wish to thank Professor Porter for his kindly criticism and for the interest he has taken in this paper.

XXXIX. *On the Determination of "H."* By Walter A. Jenkins, *M.Sc., Professor of Physics, Dacca College*[*].

In the Philosophical Magazine of October 1913, a new method of determining the Horizontal Component of the Earth's Magnetic Field was described by the writer. The method, suggested by Dr. Hicks, was the creation of an artificial magnetic field exactly equal in intensity to twice the Earth's field. Equality of the two fields was determined by the equality of the times of swing of a suspended magnet placed in both fields. It was there shown that the method was capable of giving an accuracy of one part in 10,000 and was as efficient a method as the Kew Magnetometer one. But in both the Kew Magnetometer method and the Solenoid method previously described, the chief part of the experiment is the determination of a time of swing, and the limit of accuracy of the methods is the accuracy with which the time of swing can be determined. In the previous method difficulty was experienced in obtaining a suspension fibre sufficiently fine, strong, and short enough to allow the magnet to oscillate for the period required for making accurate observations and at the same time to conform to the rest of the apparatus. Two methods will be described in the present paper, both of which obviate this difficulty and make the determination of "H" a short and reliable experiment.

First Method.—The apparatus used is essentially the same as that designed for the former experiment and the principle of the method much the same. In this method, however, the equality of the fields is determined by measuring the angle of deflexion of the suspended magnet when under the influence of both the Earth's field and the Solenoid field. The Solenoid field is placed exactly at right angles to the

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[*] Communicated by the Author.

Earth's field, so that if θ is the angle of deflexion,

$$2\mathrm{H}l \sin\theta = 2\mathrm{F}\,l \cos\theta,$$

$$\mathrm{H} = \mathrm{F} \cot\theta.$$

A determination of F and θ therefore gives the value of "H." As before F is equal to $4\pi n\mathrm{C}\cos\alpha$, where n and α are constants of the Solenoid and the measurement of F reduces itself to the measurement of the current C.

Diagram 1.

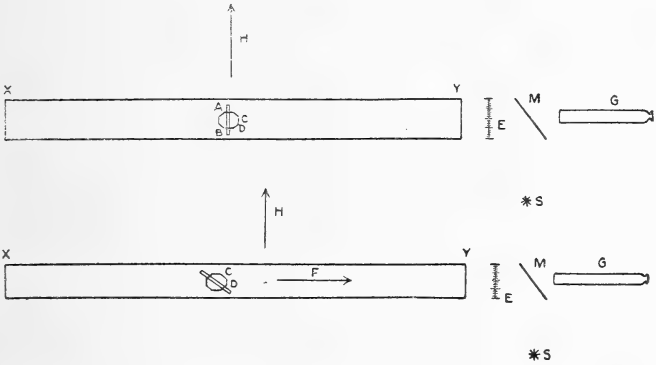


Diagram 1 illustrates the method adopted for measuring the angle of deflexion.

X Y is the coil accurately placed at right angles to the Earth's field.

A B is the undeflected magnet.

An image of the scale E is reflected from the mirror surface C into the telescope G. Suppose the zero of the scale to be at the intersection of the telescope cross-wires. The current is now switched on and the magnet deflected as shown in the lower part of the diagram. The strength of the current is then adjusted until the zero of the scale is reflected from surface D to the intersection of the cross-wires of the telescope. The angle of deflexion is then the angle between the mirrors C and D. This is a constant quantity and can be accurately determined. C and D are silvered microscopic cover-glass mirrors attached to the sides of an octagonal aluminium framework which is hollowed out for the sake of lightness. Similar mirrors are attached to all sides of the octagon in order to preserve a symmetrical distribution of the weight. The framework is attached to the underside of the magnet-holder.

Details of the Experiment.—The tube, mounted as before on a board capable of rotation, was adjusted until it was exactly parallel to the Earth's field. The method used for this—that of adjusting the position of the tube until, when a large current is sent round the solenoid, no deflexion of the magnet occurs—was described in the previous paper.

The solenoid tube was then rotated through an angle of 90°. This was done by means of a telescope and scale and two mirrors at right angles to each other, mounted in a suitable position on the board carrying the solenoid. The actual angle between the mirrors was 89° 25′ 30″, but so long as the angle is definitely known the fact that it is not 90° makes no difference to the experiment.

The distance of the telescope and scale from the mirrors was 210 cm., and as a rotation of 1° gave a motion of 75 divisions of the scale, an accuracy in the determination of the angle of rotation of 12″ was obtainable, for a motion of $\frac{1}{4}$ of a division could easily be followed.

The source of light, plane reflecting glass, scale and telescope were then arranged as shown in diagram 1 and an image of the scale reflected from face C obtained. A current was now sent through the tube and adjusted until the deflexion of the magnet was such that the mirror-face D reflected the same mark of the scale on to the cross-wires of the telescope as did the mirror C. The current strength was then measured by means of a Kelvin Balance which was in the circuit, and the current was found to be ·01945 ampere.

The balance will measure the current to ·00001 ampere, and is a very convenient instrument for carrying out the experiment to a fair degree of accuracy. When high accuracy is desired, an electrical method similar to the one described in the previous paper can be used.

The angle between the mirrors D and C had previously been found by mounting the system on the table of an accurately calibrated spectroscope. It was found to be 44° 29′ 15″. Thus we get $H = F \cot \theta$, or allowing for the fact that the angle of rotation was not 90°,

$$H = F \frac{\cos 44^\circ\ 29'\ 15''}{\sin 44^\circ\ 56'\ 15''}$$

$$= F \cdot 99209.$$

where $F = 4\pi n C \cos \alpha$, α being half the angle subtended at the centre of the solenoid by the diameter of the end of the helix. Substituting the values $n = 13{\cdot}362$, $\cos \alpha = {\cdot}99986$, we get $H = {\cdot}3236$.

Accuracy obtainable.—The following sources of error were discussed in the previous paper :—

1. The magnetic axis of the solenoid not being horizontal.
2. The magnetic axis of the solenoid not being parallel to the direction of "H."
3. Irregularity in the winding.
4. The longitudinal displacement of the magnet from the centre of the tube.
5. Residual torsion of the fibre.
6. Inaccurate determination of the period of oscillation.
7. Inaccurate determination of the current.
8. An error in the determination of n.
9. Heating of the coil by passage of the current.

It was there shown that with the exception of 6, 7 and 9, the sources of error were negligible.

In the present experiment 6 is not present at all, while as the current passing is only one half of what it was in the previous method, the heating effect is diminished to $\frac{1}{4}$ of its previous value. As by taking proper precautions it was previously negligible, it is more so now. The determination of the magnitude of the current still remains one of the chief factors limiting the accuracy obtainable, but with standard resistances and a good galvanometer an accuracy of 1 part in 10,000 is not difficult to obtain. Error 5, due to the torsion of the fibre, is negligible when an extremely fine fibre is used. In the present experiment, however, when prolonged oscillation of the magnet is not only not required, but is undesirable, a thicker suspension fibre possesses considerable advantages. Consequently a moderately thick suspension fibre was used. A stronger, heavier magnet system was also inserted in the holder, and on putting into the fibre 1440° of torsion a deflexion of $\sin^{-1} {\cdot}012$ was observed. The following equations give the maximum allowable value of the angle of deflexion due to the residual torsion for an accuracy of 1 in 10,000.

Suppose θ of torsion remain in the fibre and n be the coefficient of the torsion.

$2Hl \sin \alpha = n(\theta - \alpha)$, where α is the deflexion due to torsion. When the tube is rotated through 90° and the magnet system deflected 45°,

$$2\ \mathrm{H}l \sin(45+\alpha) = n\ (\theta-\alpha) + 2\ \mathrm{F}l \sin 45,$$

$$\mathrm{H} \sin(45+\alpha) - \mathrm{H}\sin\alpha = \mathrm{F}\sin 45$$

$$\mathrm{H} = \mathrm{F}\frac{\sin 45}{\sin(45+\alpha) - \sin\alpha}$$

$$= \frac{\mathrm{F}}{\cos\alpha + \sin\alpha - \sqrt{2}\sin\alpha}$$

$$= \frac{\mathrm{F}}{\cos\alpha - (\sqrt{2}-1)\sin\alpha}.$$

If $\sin\alpha$ is of the order $\frac{1}{10{,}000}$ then $\cos\alpha$ is $1-\frac{1}{2}\,10^{-8}$, so that $\cos\alpha$ can be called 1.

Therefore $\mathrm{H}=\mathrm{F}\,(1-{\cdot}41\sin\alpha)$, and in order that the torsion can be neglected $\sin\alpha$ must be less than $\frac{1}{4000}$.

The torsion which will give a deflexion of this order is 36°.

The torsion was eliminated, as in the previous experiment, by substituting a brass bar equal in weight to the magnet and allowing the system to come to rest. It is not probable that in such a case 36° of the torsion would remain in the fibre.

The following additional errors are possible :—

(*a*) Error in the determination of the angle between the mirrors attached to the magnet.

(*b*) Error in the determination of the angle through which the solenoid is rotated.

(*c*) Error in the determination of the coincidence of the zero of the scale with the cross-wires of the telescope.

(*a*) The angle is an invariable quantity and can easily be measured to 10 seconds. Approximately $\mathrm{H}=\mathrm{F}\cot 45$ An error of 10 seconds in 45 degrees gives an error of 1 in 10,000 in the calculated value of H. If desired the angle could be measured with greater accuracy than that indicated.

(*b*) Is similar to (2).

(*c*) The distance of the scale from the mirror was 75 cm. ; with care the coincidence could be determined to ·1 of a division, *i. e.* ·01 cm. This corresponds to an angle of rotation of the mirror of approximately 15 seconds.

The actual angle through which the mirror is rotated is about 45°. Therefore we get

$$\mathrm{H}=\mathrm{F}\cot(45^\circ \pm 15'').$$

This gives an accuracy of 1 in 6666, or say 1 in 7000.

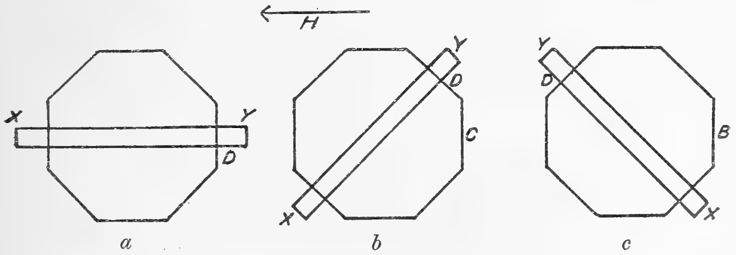
The accuracy of the experiment therefore reduces itself to the accuracy with which the angle of deflexion can be measured. A similar measurement is involved in the Kew Magnetometer method, and the accuracy of the two methods may therefore be said to be the same.

SECOND METHOD.—The principle of this method is very similar to that described in the previous paper. An artificial field is produced in the solenoid opposite in direction to, and in intensity twice that of the Earth's field. The equality of the two fields is tested by means of the deflexions caused by a subsidiary magnet placed in the Tangent B position of Gauss.

Details of the Experiment.—The tube was first set accurately parallel to the Earth's field. The octagonal shaped framework suspended underneath the magnet had mirrors attached to all its faces and the adjustments were made so that the scale was reflected from face D (diagram 2 *a*).

The external magnet sliding in a groove at right angles to the solenoid was then adjusted until the scale was reflected from face C, diagram 2 *b*, into the telescope. Without moving the external magnet the solenoid field was switched on and adjusted in strength until the position of the suspended magnet was as in diagram 2 *c*. Then the image of the scale was reflected from face B into the telescope.

Diagram 2.



a. Position under influence of Earth's field alone.
b. Position under influence of Earth's field + that of external magnet.
c. Position under influence of Earth's field + external magnet + solenoid field.

If the octagonal framework is accurately made and the angles between the adjacent faces are 135° exactly, then obviously the solenoid field is exactly twice the Earth's field, and we get

$$2H = 4\pi n C \cos \alpha.$$

The measurement of H is now simply a measurement of C. In practice, however, the angles are not exactly 135°, and it is necessary to make an allowance for this in the calculations. The angles between the mirror faces can be accurately measured and, from the values found, the angles of deflexion θ_1 and θ_2 in the two parts of the experiment calculated. Then the following equations will give the value of H :—

$$F \cos \theta_1 = H \sin \theta_1,$$
$$F \cos \theta_2 = (X - H) \sin \theta_2,$$

where X is the field of the solenoid.

$$H = \frac{X}{\tan \theta_1 \cot \theta_2 + 1.}$$

In the experiment actually carried out,

$$\theta_1 = 44^\circ\ 26'\ 30''$$
$$\theta_2 = 46^\circ\ 12'\ 15''$$
$$n = 13{\cdot}362$$
$$\cos \alpha = {\cdot}99986$$
$$C = {\cdot}037547 \text{ ampere}$$
$$X = {\cdot}62955$$
$$H = \underline{{\cdot}3245}$$

The experiment was carried out ten days after the former one, and hence coincidence of results could not be expected.

Errors.—Possibilities of errors not previously described arise in this experiment owing to the following causes :—

(1) A lateral displacement of the subsidiary magnet from its true position opposite the centre of the suspended one.
(2) An angular displacement of the subsidiary magnet from its true position perpendicular to the solenoid.
(3) Alteration of magnetic moment of the subsidiary magnet.

(1) The following calculation shows the error likely to arise from a lateral displacement of the subsidiary magnet.

Suppose the magnet is displaced 1 cm. from its correct position.

Then a simple calculation shows that an error of 1 part in 3000 is introduced if, as was the case in the actual experiment, d is about 50 cm. and l about 18 cm. If the displacement is not more than 3 mm. the error is not more than 1 part in 10,000. The subsidiary magnet can easily be placed to within 3 mm. of its correct position.

(2) An angular displacement of $1\frac{1}{2}^\circ$ would introduce an error of 1 in 10,000. The magnet can be placed in its correct position to a greater degree of accuracy than this.

(3) The only factor likely to cause a sudden change of

magnetic moment of the subsidiary magnet in the interval elapsing between the two parts of the experiment is a temperature change. The temperature coefficient of magnetic moment is ·001. Thus in order to obtain the required accuracy the temperature must be kept constant to within $\frac{1}{10}$° C. Suitable precautions can easily be devised to ensure this.

In both methods described in the present paper the magnet when in a deflected position lies in a field of strength $H\sqrt{2}$ approximately, and hence owing to induced magnetism its magnetic moment will be slightly greater than when in a field H. This, however, introduces no error, for the magnetic moment of the suspended magnet is a factor which does not enter into the problem except in so far as it alters the angle of deflexion due to residual torsion. At the most the change introduced into α is a small fraction, and hence as α is negligible the change owing to altered magnetic moment is also negligible.

Comparison of the two methods.—Both methods give an accurate result, and both are easier to carry out than the Kew Magnetometer method. Once the constants n, cos α, the angles between the mirrors, and the direction of H are determined the only determination in the first method is that of the current required to give the necessary deflexion. When the solenoid is rotated through 90° the optical apparatus must also be rotated through the same angle, but in the apparatus which is being designed for permanent installation the optical system, consisting of eyepiece-scale and reflecting glass, is to be fitted into the end of the tube and will therefore rotate with it.

In the second method no rotation of the tube is necessary, but it involves the use of a subsidiary magnet. Moreover, two adjustments are necessary.

(1) The adjustment of the position of the magnet to give the requisite deflexion under the influence of the Earth's field.

(2) The adjustment of the current to create an oppositely directed field which gives the same deflexion.

Provided that the solenoid is mounted upon a large circular table whose rotation can be measured, as that of a spectroscope table is, the first method is extremely simple and accurate, and in the author's opinion provides the best method of determining H with accuracy. The fact that the determination of H by both methods gives substantially the same result when allowance is made for the variation of H from day to day, is an indication of the reliability of the method. Check experiments with a not very reliable Kew Magnetometer instrument gave the same value of H.

In conclusion I again wish to express my thanks to Dr. Hicks, who first of all suggested the use of the principle involved in the Solenoid method.

Dacca College.
7th August, 1920.

XL. *Impulsive Sparking Voltages in small gaps.*

By J. D. Morgan, *B.Sc.* *

In a paper entitled "Time Lag in the Spark Discharge" (Phil. Mag. vol. xxxvii., August 1919) Dr. Norman Campbell discusses the fact that the impulsive sparking voltage is in certain gaps greater than the static sparking voltage. As explained in that paper, when the voltage applied to a gap is caused to rise rapidly from zero, the value it will attain before sparking occurs is greater than that reached by a voltage which is applied gradually. The ratio of these two values has been termed by Peek the "impulse ratio," and this expression is now generally adopted. It is commonly believed that two conditions other than voltage are involved in the process of spark production, namely, time and initial ionization. Campbell recognizes these two conditions, and shows how they can be used to explain why the impulsive sparking voltage of a gap is greater than the static voltage, or in other words, why the gap has an impulse ratio greater than unity. Arguing from his own and Peek's investigations, the conclusion reached by Campbell is that there are two kinds of lag, a regular and an irregular one, in the process of spark production, though it is apparent that he sees a probable connexion between them.

Taking Campbell's mode of presenting the facts, there would appear to be justification for distinguishing between regular and irregular lags, but it is questionable whether this manner of regarding the subject is likely to lead to the most useful practical results. In the writer's opinion it is more convenient to assume only one lag, and to regard the sparking voltage of a given gap as dependent jointly on the three variable conditions which have hitherto been recognized. These conditions are (1) rate of rise of voltage,

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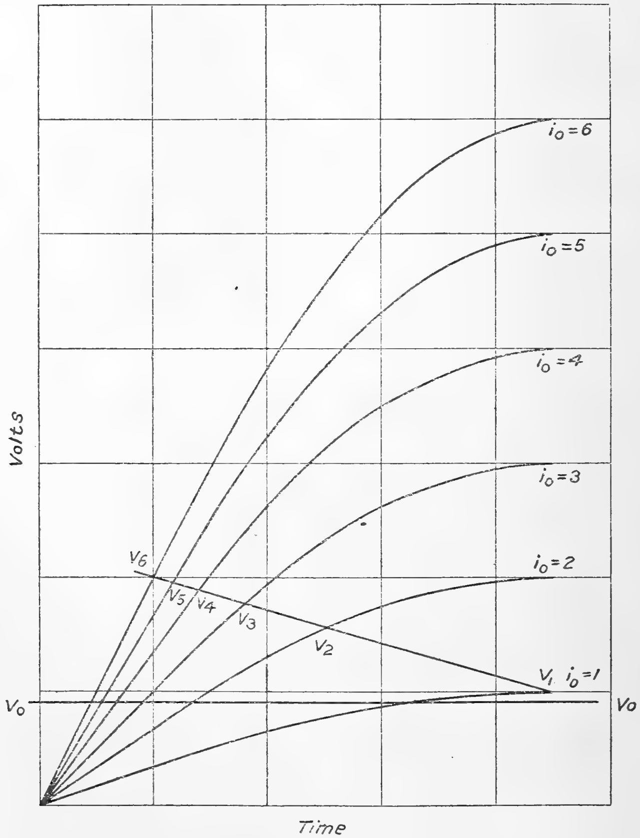
* Communicated by the Author.

(2) the time for which the voltage must continue to act after it reaches a certain value, and (3) the initial ionization of the gap. When the voltage rises rapidly, variation of any one of these will affect the sparking voltage. By considering these conditions together an explanation of the variations found in impulsive sparking voltages in a constant gap can be given in a direct and simple manner.

Let it be supposed that the sparks are to be produced in a small gap between, say, small spherical electrodes, by means of an induction-coil. Assuming first that the gap is removed from connexion with the coil, and that the high tension terminals of the coil are so far apart that no spark can pass when the primary current is interrupted, then the maximum voltage which can be attained in the secondary of the coil is variable in a regular manner by varying the primary current (i_0) at break. Disregarding, for the sake of simplicity, the double oscillation which occurs in the secondary, the rise of voltage can be represented by a sine curve, its period being constant, and its amplitude proportional to the primary current at break. The rate of rise of voltage from zero will vary with i_0. In figure 1 a number of sine curves are drawn corresponding to the arbitrary current values noted in the graph. The ordinates represent Volts and the abscissæ Time. A small gap is now connected to the coil and adjusted so that with $i_0=1$ in the primary, sparking just occurs. It will be supposed that the spark appears when V reaches the crest of the curve $i_0=1$. It will also be supposed that the static sparking voltage for the same gap is a slightly smaller value V_0 such as is represented by the horizontal line. Assuming that there is ample initial ionization in the gap between the electrodes, then the time interval after the intersection of the curve $i_0=1$ and the line V_0 is that required for the voltage to act before sparking can occur. During that time the voltage rises to V_1. As it is supposed that $i_0=1$ is the smallest primary current by which sparking can be obtained, the effect of variation in the initial ionization is to alter the regularity of occurrence of the spark.

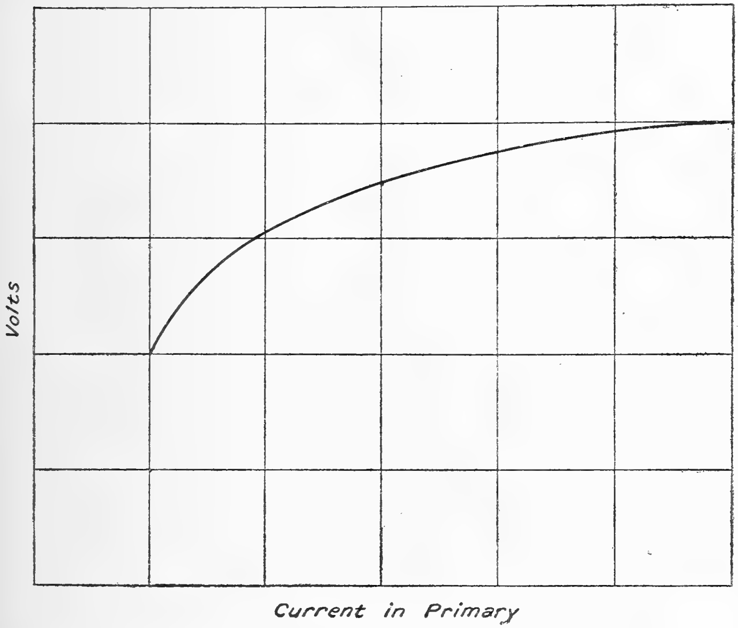
From the crest of $i_0=1$, draw any straight line as shown at an inclination to the horizontal. The inclination of this line will be different for different gaps. Thus for a gap between large spheres it will be small, but for a gap between points it will be relatively large. The horizontal distance or time between the intersection of this line on any curve and that of the line V_0 on the same curve decreases regularly from curve to curve. This may be taken to represent in a simple manner the usual assumption (shared by Campbell) that the time during which the voltage acts on the gap after passing the line V_0 decreases as the voltage at which the spark appears increases. Considering the curve $i_0=2$, then with sufficient initial ionization the sparking voltage is given by V_2. If the initial ionization is diminished the voltage must be maintained for a longer time and the sparking voltage will reach some value higher than V_2. Likewise with the other values of i_0 the sparking voltages are represented by $V_3 \ldots\ldots V_6$.

Fig. 1.



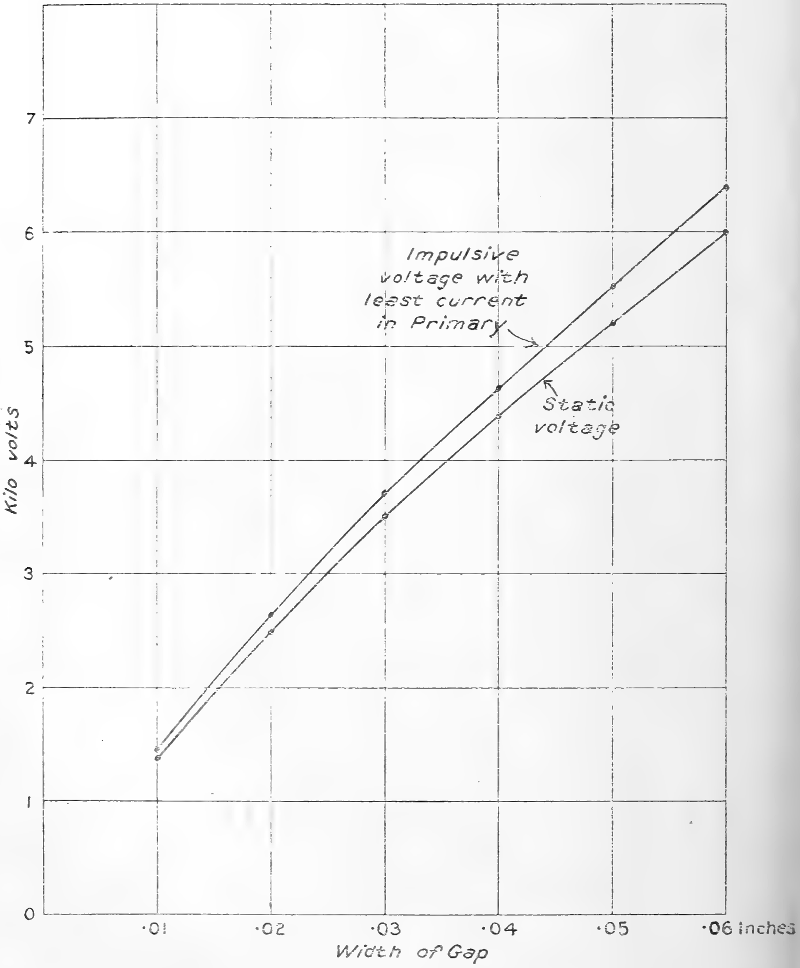
The sparking voltage conditions for a given gap as indicated by figure 1 are purely hypothetical. What is there attempted is to associate definitely the rate of rise of voltage with time, and it has been mentioned that any of the sparking values found (except V_1) will vary with changes in the initial ionization of the gap. It is necessary now to test the matter by experiment. Two obvious checks present themselves. In the first place, if the argument is true, the sparking voltage for a given gap should vary with the primary current at break in a manner shown at figure 2 which is obtained from figure 1. In the second place the instant of occurrence of the spark should advance as i_0 is increased, and the amount of advance should progressively diminish as i_0 increases. This latter is easily tested. Starting with i_0 as a minimum and focussing the cross wires of a telescope on the image of the spark in a rapidly rotating mirror, it was found that the image shifted relatively to the cross-wires in the manner anticipated. Regarding the variation of V with i_0, this was tested by an experiment on a small gap formed between polished brass balls of ·5 inch diameter, both electrodes being insulated from earth. First the static sparking voltages for different gap settings were found. The results are shown by figure 3. Then using impulsive voltages given by a 10 inch induction-coil, the sparking voltage was found with the least current in the primary that

Fig. 2.



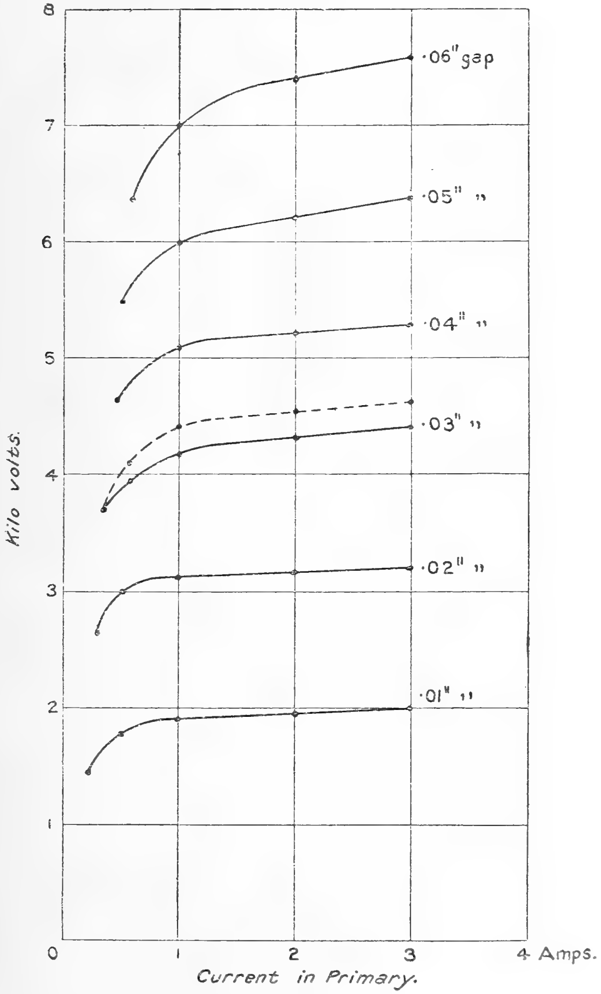
would produce a spark at a number of different gap settings. For measuring the voltages a thermionic valve was used in conjunction with an electrostatic voltmeter. The results are also shown by figure 3. Finally, with the same measuring apparatus, the variations of V with i_0 were found for each setting. These are given in figure 4. It will be observed that the curves obtained all have the same general character as the hypothetical curve given in figure 2. From the

Fig. 3.



experimental results there would appear to be good reason for believing that the assumptions on which figure 1 is based are substantially true.

Fig. 4.



Before the curves shown in figure 4 can be fully appreciated it is necessary to mention certain supplementary details observed in the experiments. Taking any one of the curves, the first point (*i. e.* the one at which sparking just occurred with the least current in the primary) was always the most definite. The same experiments were repeated many times at intervals extending over several weeks, and always the same voltage value was found for the first point. The contact-breaker in the primary circuit of the induction-coil was operating at a speed of about four breaks per second, and with the primary current adjusted until sparking at the test gap was just possible, the irregularities noticed at different times were not in the spark voltage but in the frequency with which the sparks occurred. The lag at the first point appears to correspond with what Campbell terms the regular lag, and the lag observed by Peek in his experiments. Proceeding now along any one of the curves, two changes were observed. In the first place, the sparking became more regular until eventually a spark followed each break of the primary circuit, and in the second place the sparking voltage gradually became less regular. But in none of the experiments recorded by the curves was the voltage so irregular as to make the drawing of a curve unjustifiable (as is the case with some gaps). These irregularities appear to correspond with Campbell's irregular lag, and are mainly if not entirely due to variations in the ionization of the gap immediately prior to the passage of the spark. Judging from his book (' Dielectric Phenomena in High Voltage Engineering,' New York, 1915), Peek does not appear to have investigated sparking voltages for a given gap higher than those corresponding to the first point above mentioned, and this accounts in the writer's opinion for the absence (commented on by Campbell) of any reference by Peek to irregular sparking voltages. In considering the above-mentioned irregularities it is not overlooked that they may be due, at least in part, to irregularities in the operation of the contact-breaker causing the rate of rise of voltage to vary. Such a condition does undoubtedly account for some irregularity (a condition which would become more accentuated as the primary current is increased), but it does not account for all. It is more probable that the irregularities observed were due mainly to variations in the initial ionization of the gap. As bearing on this point it is necessary to remark that on different days under different atmospheric conditions the same curves could not always be repeated. The dotted line curve in figure 4 shows for example one variation which was found. Here the first point was very definite and the same as before, but with increase of primary current the voltage readings became not only rather irregular as before but also quite different.

Subsequently the tests were extended to a variety of gaps and in all cases the general relationship shown in figures 2 and 4 was found, the differences always being one of degree and not of kind. Where the gap is formed by electrodes of relatively large diameter or flat plates, the rise of sparking voltage with increase of primary current is usually small, and, when artificial means are employed to produce copious initial ionization of the gap, may be very small. But when pointed electrodes are used the variations are usually large, and with the higher current values are very irregular. The use with pointed gaps of a third insulated point reduces the sparking voltage and regularizes the action of the gap. It also diminishes variation of sparking voltage with primary current. Instances were noted with a flat-plate, an annular, and a three-point gap, in which the sparking voltage was apparently constant and independent of the primary current, but these results were occasional and could not be reproduced at will. No explanation of this exceptional behaviour can be given. The gaps in which the effect was observed behaved normally (*i. e.* showed an increase of voltage with primary current) at other times.

It follows from such considerations as those outlined in this paper that the impulsive sparking voltage for a given gap is a quantity depending on three variable conditions. The rate of rise of voltage varies with the kind of apparatus used for producing the spark. With the same apparatus, keeping the circuit conditions constant, it can be varied by varying the primary current. The initial ionization of the gap varies with atmospheric conditions, but can be made more or less constant by artificial means. The element of time varies with the other two conditions and is different for different gaps, as Campbell rightly points out. When the potential gradient in the gap is uniform, the time for which a certain minimum potential must be imposed on the gap may be very small, but it may become relatively large when the potential gradient varies greatly (as in a point-gap). Due to the latter condition two gaps of different form but having the same static sparking potential and the same initial ionization will have different impulsive sparking voltages, as is well known.

The subject is one of considerable importance, and has direct practical application in connexion with *e. g.* research on the ignition of gases by induction-coil sparks, ignition apparatus for internal combustion engines, protective gaps for certain kinds of electrical machinery, and overhead transmission systems subject to rapid electrical impulses.

XLI. *On the Design of Soft Thermionic Valves.*

By G. Stead, *M.A.*[*]

[Plate VIII.]

Preliminary Note.—The experiments described in this paper were undertaken on behalf of H.M. Signal School, Portsmouth, and were carried out in the Cavendish Laboratory, Cambridge, at various times during 1916–17. They are now published with the consent of the Admiralty. The writer wishes to acknowledge his great indebtedness to the members of the W/T staff of H.M. Signal School, and in particular to Professor C. L. Fortescue and Mr. B. S. Gossling, with whom he was in close touch throughout the work.

I. *Introduction.*—A general account of the development of thermionic valves for Naval purposes has been given in a recent paper by B. S. Gossling [†], and in section (7) of the paper the evolution of the "soft" or gas-filled valve is discussed. It is thought that a more detailed account of some of those portions of the work with which the present writer has been directly associated may be of general interest. The experiments described deal in the main with the effect on the valve characteristics of the pressure and nature of the contained gas, and of the position and form of the electrodes. Chronological order has been followed as far as possible, but has not been adhered to where rearrangement seemed likely to render the account more logical and connected.

II. *Position and Form of Electrodes.*—In 1916, when the work described in this section was undertaken, little was known about the effect on valve characteristics of changing the position and form of the three electrodes, and experimental work in this direction seemed urgently required. Two kinds of experiment were carried out, one to determine the effect of altering the distance between the grid and the filament, and the other to ascertain the effect of altering the spacing of the grid wires.

(i.) *Experiments with a movable grid.*—A special valve was designed in which it was possible to move the grid whilst the valve was exhausted. The arrangement is shown diagrammatically in fig. 1. The anode was an aluminium disk A, and the filament was in the form of a V with its

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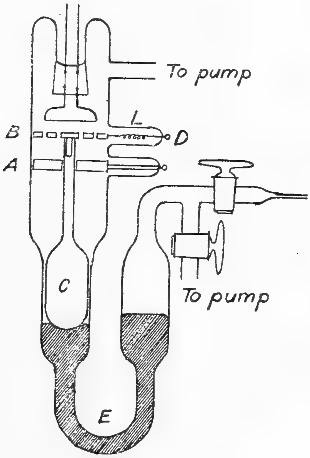
[*] Communicated by Professor Sir J. J. Thomson, F.R.S.

[†] Journal I. E. E. lviii. p. 670 (1920).

plane parallel to that of the anode. Between them was a grid B, made of copper wire and mounted on a thin glass tube, whose lower end was sealed to a float C. The grid was connected to the source of grid potential by means of the flexible lead L sealed into the glass at D. The float rested on mercury contained in the U-tube E, and the height of the float, and therefore the position of the grid, could be varied by altering the pressure of the air in the other limb of the U-tube. The distance between the planes of the anode and filament was approximately 2 cm., and the grid could move freely in this space, except that it was prevented by a stop from coming nearer to the filament than about 1 mm. A valve of this type necessarily contained mercury vapour at the saturation pressure corresponding to room temperature (about ·001 mm.) and was thus fairly soft. The valve was exhausted as thoroughly as possible by a rotary Gaede pump followed by charcoal cooled by liquid air, and was baked during the process of exhausting so far as the presence of the mercury permitted, *i. e.* to about 350° C.

Fig. 1.



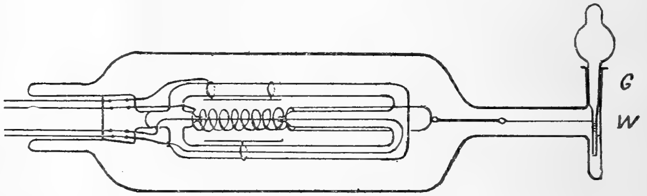
Characteristics were plotted for various positions of the grid, the filament current and the temperature of the mercury being kept the same throughout. Some specimen characteristics are shown in fig. 2 (Pl. VIII.), for three different distances between grid and filament. The general effect of increasing this distance is to diminish the slope of the anode current-grid voltage characteristic, and to diminish the ratio of the grid current to the anode current. This is shown by the numbers in the following table, which were obtained by measurements on the original characteristics.

Distance from grid to filament.	Maximum slope. (V=40).	Grid current. / Anode current.
3 mm.	0·6 milliamp. per volt.	3·42
7 „	0·22 „ „	1·05
10 „	0·20 „ „	0·89

The second column gives the maximum slope of the anode current-grid voltage curve for an anode voltage of 40. The third column gives the ratio of the grid and anode currents for the case when the anode voltage is 40 and the grid voltage 15.

(ii.) *Experiments with a grid of variable spacing.*—A diagram of the special valve used for these experiments is shown in fig. 3. It was of cylindrical type and the grid consisted of a steel spring, of helical form, which could be stretched by means of a thread attached to a small winch W operated from the outside through the ground glass joint G. The anode was a brass cylinder about 3·5 cm. in length and 2 cm. in diameter. Small holes were drilled through the anode cylinder so that the grid and filament were visible from the outside. The filament was a straight tungsten wire about 2 cm. long and ·0082 cm. in diameter. The tube connecting the winch with the valve was made long, so that the ground glass joint could be outside the oven during the baking process, and the conduction of heat from the oven to the ground glass joint was prevented by coiling round the glass tube a thin lead pipe through which cold water was allowed to flow.

Fig. 3.



The valve was exhausted in the usual way and was rendered as hard as possible by means of charcoal and liquid air, mercury vapour being also frozen out by liquid air. Hydrogen was then admitted. The hydrogen was prepared by electrolysis of a solution of barium hydroxide, and was dried by phosphorus pentoxide, followed by the freezing action of liquid air. Characteristics were then plotted for a series of grid spacings. Specimens of these characteristics are shown in fig. 4 (Pl. VIII) for grid spacings of 1, 2, 3, and 4 mm. (By grid spacing is meant the width of the openings between the grid wires, not the pitch of the helix.) It will be noticed that for a given anode and grid voltage, the anode current first increases and then diminishes as the grid helix is pulled out, and the characteristics attain maximum steepness for a particular value of the grid spacing. The manner in which the anode and grid currents alter as the grid spacing is increased is shown in fig. 5 (Pl. VIII.). The curves reproduced represent anode current (or grid current) plotted against grid spacing for a fixed anode potential, and for several different grid potentials. In this particular valve the diameter of the grid helix was about 0·6 cm. and the diameter of the grid wire 0·5 mm. The grid spacing which gives the maximum anode current increases from 2·3 to 2·85 mm. as the grid potential increases from 1·6 to 15 volts.

III. *Preliminary experiments in Mercury Vapour, Residual Air, and Argon.*—The effect of the pressure and nature of the contained gas on the characteristics of a valve of audion type (with plane anodes and zig-zag grids) was examined in some detail, the atmospheres employed being mercury vapour, residual air, and argon. For the experiments on mercury vapour a small tube containing a drop of mercury was attached to the valve, and this tube was placed in a vacuum flask containing water, or methylated spirit, at a known temperature. In this way any desired vapour pressure could be obtained. The range of temperature employed was from −13° C. to +56° C., corresponding to a range of vapour pressure from ·00004 to ·019 mm. "Residual air" was obtained by simply pumping down the valve to the required pressure, and doubtless consisted chiefly of electrode gas. *i.e.*, hydrogen and oxides of carbon. Pressures between ·008 mm. and ·00025 mm. were employed. The argon was made from liquid air residues by prolonged sparking over potash with excess of oxygen, the latter being finally removed by phosphorus. The purity of the argon was tested spectroscopically. Characteristics were obtained for a range of pressures from ·032 mm. to ·00005 mm.

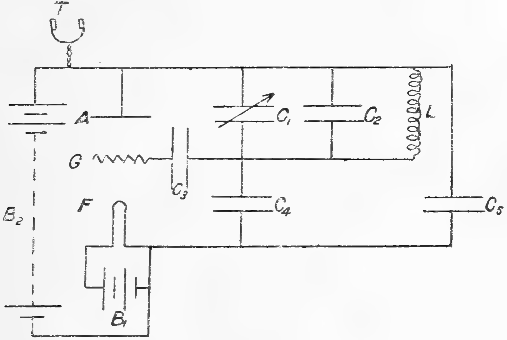
These preliminary experiments showed that the general effect on the characteristics of variation of gas pressure was the same for all the gases studied, but that the pressure at which an audion works most satisfactorily is at least five times as great in residual air and argon as in mercury vapour.

IV. *General remarks on Soft Valves.*—The chief difficulty to be overcome in the design of a satisfactory soft valve is that caused by variations of gas pressure whilst the valve is in operation. It seemed likely, at first sight, that changes of pressure would be less marked if larger amounts of gas could be employed, but in valves of the audion type, with plane anodes, a general glow readily occurs throughout the bulb if the anode voltage is too high, and the valve is at all soft This glow, which is accompanied by rapid disintegration of the tungsten filament, is probably associated with ionization along the tracks of electrons which travel from the filament to the anode by long paths outside the electrodes. The probability of such paths is greatly diminished by the adoption of a cylindrical construction for the electrodes. Hence a higher gas pressure should be possible in cylindrical valves than in those of the audion type. Experiment showed this to be the case. The cylindrical construction has the further advantage that all four leads can be carried by the same stem ; the electrodes can therefore be completely assembled outside the bulb, and then sealed into the bulb at one operation.

In order to test the theory of the general glow given above, a special cylindrical valve was constructed in which one end of the anode was permanently closed, and the other end could be closed when required by means of a small brass cup which was carried on a stem attached to a glass bulb floating on mercury. It was found that when the end of the cylinder was closed by the cup the general glow was entirely eliminated, and "kinks" in the characteristics could be avoided at pressures high enough to produce them if the anode were open.

V. *Soft Valves as Oscillators.*—A detailed investigation was made in order to determine the pressures at which various types of valve would generate oscillations in two different circuits. A diagram of one of these circuits is shown in fig. 6. Nitrogen was the gas first selected for these experiments, and five different valves were employed. Two of them were of audion type, and the other three were cylindrical valves closely resembling the French type. The grid wires of the three cylindrical valves were of different diameters, and so possessed different projected areas, but the three grids were of the same effective degree of closeness, *i.e.*, possessed the same value of the quantity m in the formula

Fig. 6.



A =anode.
G =grid.
F =filament.
B_1=filament battery.
B_2=anode battery.
T =telephones.
L =inductance.
c_1, c_2, c_3, c_4, and c_5=condensers.

$$m=\frac{\pi N d' \log (d/d')}{\log [1/(\pi N d_g)]}$$

where d and d' are the anode and grid diameters, respectively, d is the diameter of the grid wires, and N is the number of turns of grid helix per centimetre. This formula was communicated to H.M. Signal School by Professor Sir J. J. Thomson and is an adaptation to the cylindrical case of a formula given by Maxwell * for a plane anode, grid and cathode. (See also the paper by Gossling, already cited.) Of the two audion valves employed in this investigation, one had a normal type of grid and the other a very open grid.

Experiment showed that under given conditions a soft valve would not generate oscillations unless the anode potential lay between certain limits. Both limits rose as the pressure of the gas was diminished, but the upper limit rose much more rapidly than the lower limit, so that the oscillating range increased as the gas was pumped out. For high gas pressure the range of anode voltage over which oscillations would take place was very limited, *e. g.*, less than one volt, so

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* 'Electricity and Magnetism,' i. p. 312 (1892).

that the adjustment of the anode potential to obtain oscillations was very critical. The upper limit of pressure, however, depends very much on the nature of the gas and also on the design of the valve. Thus the amount of helium which can be introduced into a valve is about ten times the amount of nitrogen which is practicable, and this again several times as great as the amount of mercury vapour which is possible.

In these experiments, in order to find the lower limit of the oscillation range the anode potential was gradually raised until the anode current showed a sudden drop. This indicated that oscillations had started. The upper limit was determined in a similar manner by starting with an anode potential so large that no oscillations were generated and gradually diminishing this until a sudden drop of current occurred. As regards the determination of the potential of the grid it is to be observed that the circuit shown in fig. 6 has an insulated grid, and for this reason the method adopted was to observe the anode current and then connect the grid to a potentiometer and adjust this until the anode current was the same as before. The range of anode voltage over which oscillations could be produced under the given conditions was determined for each of the five valves for a series of pressures lying between the limits 0·6 mm. and ·0074 mm. The oscillations were usually produced under somewhat difficult conditions, with an inductance of 1000 microhenries and a capacity of ·005 microfarad, and an additional resistance of 2 ohms in the oscillating circuit. Some observations were also made under different conditions.

In each case small portions of the valve characteristics were drawn for the region containing the oscillation range. These experiments showed clearly that considerably higher gas pressures are practicable in cylindrical valves than in those of the audion type. Observations for two different valves are shown in the following table :—

Pressure.	Cylindrical type valve.			Audion type valve.		
	Minimum voltage.	Maximum voltage.	Range.	Minimum voltage.	Maximum voltage.	Range.
0·6 mm.	19·6	19·8	0·2	—	—	—
0·2 ,,	20·1	21·2	1·1	20·0	20·2	0·2
·05 ,,	24·2	26·1	1·9	21·9	23·9	2·0
·015 ,,	32·1	37·5	5·4	28·5	33·2	4·7
·0074 ,,	45·0	65·6	20·5	—	—	—

One of the five valves was finally selected as giving the most satisfactory results in the circuits employed. This was a cylindrical valve having a grid made of wire of diameter 0·4 mm. coiled into a helix of mean diameter 4·5 mm. and having 5·5 turns per cm. Full characteristics were drawn for this valve for a range of pressures from 0·5 mm. to ·0045 mm. Finally a pressure of about ·06 mm. was selected as giving the most satisfactory characteristics. This valve was the Naval valve designated by the number R2.

For reasons which will appear later (see Section VIII. below) nitrogen did not prove a satisfactory gas for valve purposes, and the next gas tried was argon, which was chosen as being the most easily obtained inert gas. Experiments on oscillation ranges in argon were carried out for pressures between 0·25 and ·042 mm. The results given by argon were very similar to those obtained when nitrogen was employed, but it was soon found that there was a fatal objection to the use of argon in valves. When the filament of an argon-filled valve was heated up there was no indication of a change of pressure, or of filament resistance, after 31 hours, but as soon as a potential of about 20 volts was applied to the anode the filament resistance began to rise rapidly, and after 30 hours had increased by nearly 50 per cent., although there was still no alteration in pressure. It appears, therefore, that tungsten filaments disintegrate rapidly in argon under the conditions which occur in a valve during operation, and it seems likely that this disintegration is due to the bombardment of the filament by the very heavy positive ions of argon.

Argon having proved impracticable helium was next tried. This gas should share with argon the advantage of not being absorbed by the hot filament, without possessing the disadvantage of causing disintegration of the filament. Experiments on oscillation ranges in helium were carried out for pressures between 4 mm. and ·042 mm. The results are shown for four different pressures in fig. 7 (Pl. VIII.) Full characteristics for helium were also drawn, and these are reproduced in fig. 8 for two different pressures. These characteristics, instead of being drawn in the usual manner, represent the results in a very convenient way suggested by B. S. Gossling. In this method anode potentials are plotted as ordinates and grid potentials as abscissæ, and each curve represents a line of constant anode (or grid) current. The curves are therefore of the nature of contours. A general discussion of these contour characteristics has been given by Gossling *.

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* Journal I. E. E. *loc. cit.*

From a study of the characteristics of helium it has been concluded that the highest pressure of helium which is practicable in valves of the design used is about 1 mm. Valve No. R2A was designed to contain helium at a pressure of about 0·6 mm. This pressure is considerably higher than would be practicable with any other gas except hydrogen, and for this reason it was thought at first that contamination by electrode gas would be less noticeable than in nitrogen. This did not prove to be the case, owing to the fact that positive ions appear in a valve containing helium when the difference of potential between the anode and the negative end of the filament amounts to about 21 volts, whereas the corresponding difference of potential required to produce positive ions in electrode gas is about 15 volts [*], so that a small admixture of electrode gas is able to cause a large alteration in the characteristics of a helium valve. This is brought out clearly in the contour characteristics, particularly in the case of the grid current contours. It will be seen from fig. 8 (Pl. VIII.) that the grid curves rise nearly vertically until a potential of about 21 volts is reached, when they suddenly bend over and become nearly horizontal. This is due to the formation of positive ions, whereby the negative space charge is partially neutralized, so that a given grid voltage is able to produce a much larger current than previously, *i. e.*, a vertical line corresponding to any particular grid potential would at this point begin to cut across the grid contours in rapid succession. Now if the helium contains a small percentage of electrode gas, these horizontal portions of the grid contours occur at about 15 volts instead of 21 volts, and, in consequence, the characteristics of the valve are very much modified.

Many attempts were made to produce satisfactory helium valves without bombarding the electrodes, the process adopted being to heat the electrodes as much as possible by radiation from the hot filament. It was found that, even when the filament was run at such a high temperature as to cause serious damage to itself, the removal of electrode gas from the anode was very incomplete, and the form of the valve characteristics was controlled mainly by the electrode gas and very little by the helium. Ultimately it was found best to subject the electrodes to heavy electron bombardment during the evacuation, just as in the case of hard valves, and then introduce the necessary quantity of helium, which had been purified by standing for some time in contact with charcoal cooled by liquid air.

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[*] Stead & Gossling, Phil. Mag. xl. p. 413 (1920).

VI. *Methods of estimating Gas Pressure in Valves.*—In the manufacture of soft valves it is very important to have a simple method of estimating the pressure of the gas before sealing off the valves from the pump. A very convenient method, which is accurate enough for the purpose, is to measure the width of the Crookes's dark space in a small vacuum-tube attached to the apparatus for the purpose. The vacuum-tube employed was provided with an aluminium disk as cathode, and a small aluminium rod in a side tube as anode. By experiment the relation between the pressure and the width of the dark space was determined for nitrogen, argon, helium, hydrogen, and carbon monoxide. The width of the cathode dark space depends on the current density through the tube [*], but a pressure indication of sufficient accuracy can be obtained if the diameter of the cathode is specified, and an induction-coil of definite size is used.

This method of estimating gas pressures is much simpler for manufacturing purposes than the McLeod gauge method, and has the advantage of not introducing mercury vapour into the apparatus. It is necessary that mercury vapour should be excluded since it ionizes at about 10·5 volts and a small trace of mercury vapour has a large influence on the characteristics. If evacuation is performed by means of a rotary oil pump and the dark space method of estimating pressures is employed, it is not necessary to introduce the complication of a liquid air-trap to keep back mercury vapour.

The dark space method is, of course, no longer available after a valve has been sealed off from the pump, and it is very useful to have another pressure test which is applicable to sealed valves. For this purpose use was made of the phenomenon known to the staff of H.M. Signal School as "backlash." When the difference of potential between the anode and the negative end of the filament becomes sufficiently great positive ions are produced by collision, and, if under these conditions the potential of the grid is made a little negative to the negative end of the filament, the grid will pick up positive ions and the grid current will be reversed. The magnitude of this backlash current depends on the anode and grid potentials and on the pressure and nature of the gas, and it forms a very simple and convenient method of estimating pressures in sealed valves. The procedure usually adopted was to make the grid have a definite negative potential (usually −2 volts) and then increase the

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[*] Aston, Proc. Roy. Soc. 1907, 1911.

anode potential until the backlash current rose to a specified value (generally 1 microampere). The anode voltage required for this purpose increases as the pressure is diminished, and the relation between these two quantities may be determined by experiment for any given gas. It is evident that the required anode voltage will be related in some way to the ionization potential of the gas, but the relation between the two is by no means a simple one. Experiments have been carried out for nitrogen, helium, argon, hydrogen, and carbon monoxide, and for grid potentials of $-1{\cdot}5$, $-2{\cdot}0$, and $-2{\cdot}5$ volts. The results for a grid potential of -2 volts are shown in fig. 9 (Pl. VIII.).

VII. *Cooling of Filaments by surrounding Gases.*—When a filament is surrounded by a gaseous atmosphere the power required to maintain it at a given temperature is greater than when the filament is *in vacuo*, owing to loss of heat by conduction and convection through the gas, as well as by radiation. It was necessary to know whether this effect would be appreciable at pressures likely to be used in valves. The method adopted was to measure the current required to maintain the filament at a given resistance, first *in vacuo* and then when surrounded by gases at various pressures. From these measurements it was possible to plot curves showing the number of watts taken by the filament at different gas pressures. Measurements of this kind were made for nitrogen and helium, and the results are shown in fig. 10 (Pl. VIII.). It was found that in nitrogen at a pressure of 1 mm of mercury the watts required were only 10 per cent. more than in a good vacuum, whilst at the pressure employed in the R2 valve (·06 mm.) the additional watts amounted to 1 per cent. The effect is thus of little importance in this latter case. The cooling due to helium is appreciably larger, amounting to about 13 per cent. at a pressure of 1 mm., and 20 per cent. at 4 mm. In hydrogen it is still larger, but no systematic observations were made. It will be observed that, both for helium and nitrogen, as the pressure increases the curves become gradually less steep up to a certain point, and then the steepness begins to increase again rather suddenly. This effect was always observed and appears to be genuine and not due to accidental causes. It seems likely that the upward turn of the curve occurs when convection in the gas begins to be appreciable.

VIII. *Objections to Nitrogen atmospheres.*—As already mentioned, nitrogen had to be abandoned as an atmosphere for soft valves because the pressure did not remain constant under working conditions. Special experiments were made to investigate this effect. Life tests carried out at Portsmouth indicated that a fairly rapid absorption of nitrogen occurred when the filament was running, and a voltage of the order of 20, or more, was applied to the anode. Very little nitrogen was left after 50 hours of such treatment. The remaining gas began to ionize at a potential of about 13 volts, and its spectrum showed lines of argon and the usual carbon monoxide bands due to electrode gas. Atmospheric nitrogen had been employed and this had apparently been absorbed, leaving a residue of argon, which would ionize at about 12·5 volts, and at the same time the electrodes had given out carbon compounds.

In order to investigate this effect more fully special experiments were carried out in the Cavendish Laboratory. A valve was attached to a "dark-space tube" (see paragraph VI. above) and was exhausted and filled with nitrogen until the dark space was 9 mm. in length. No change of pressure occurred as the result of sealing this apparatus off from the pump, nor was there any change after the valve had been allowed to stand idle for a week. The filament current was then turned on, and it was found that after 50 hours running the length of the cathode dark space was reduced to 8·5 mm. At the same time the discharge had become somewhat less pink in colour. Thus a small quantity of gas must have been liberated from the electrodes, or from the walls of the tube. A potential of 20 volts was then applied to the anode, and after 7 hours the length of the dark space was found to have increased to 11 mm. At the end of 30 hours' continuous running under these conditions the dark space was 14·5 mm. in length. At the same time the colour of the discharge had altered from pink to blue. It appeared, therefore, that nitrogen was being absorbed and that electrode gas was being given out.

To test this point further a small spectrum tube, provided with external tin-foil electrodes, was attached to another valve, and the arrangement was then exhausted and filled with nitrogen at a pressure of about 0·5 mm. Initially the spectrum consisted chiefly of nitrogen bands, but a few faint argon lines and carbon monoxide bands were also visible. The general colour of the discharge was pink. The apparatus was sealed off from the pump and allowed to stand idle for 10 days. This produced no change in the spectrum, and there was no great alteration after the filament had been run for 33 hours. As soon as a potential of 20 volts was applied to the anode the pink colour of the discharge began to alter, and after 24 hours there was no trace of pink colour left. The spectrum then showed carbon monoxide bands strongly, one or two faint nitrogen bands, and argon lines as before.

It seems clear from these experiments that under the ordinary working conditions of a valve, nitrogen is absorbed and electrode gas given out. Furthermore, the evolution of electrode gas is at least in part a purely thermal effect whilst the absorption of nitrogen is apparently electrochemical rather than chemical, since it does not take place unless there is a potential on the anode. There seems reason to believe that no absorption of nitrogen takes place unless the difference of potential between the anode and the negative end of the filament exceeds the ionization potential of nitrogen (*i. e*, about 17·5 volts) so that probably tungsten at a temperature of 2300° K does not combine with nitrogen molecules, but is capable of uniting with nitrogen presented in the form of rapidly moving positive ions. This is in agreement with the result previously obtained by Langmuir [*].

IX. *Summary*—An account is given of experiments which led up to the design of a satisfactory soft valve for Naval uses. The chief points considered are :—

(1) The effect on the valve characteristics of the position of the grid with respect to the filament, and of the closeness of the grid structure.

(2) The effect of the pressure and nature of the gas on the valve characteristics and on the production of oscillations. Nitrogen, argon, and helium were studied from this point of view, and it was found that there were serious objections to the first two gases, but that helium was very satisfactory.

(3) A method of estimating the pressure of the gas in a sealed valve is given, and the effect of nitrogen and helium in cooling the filament by conduction and convection is considered.

In conclusion the writer wishes to express his sincere thanks to Professor Sir J. J. Thomson for his unfailing kindness, and his interest in the work.

Cavendish Laboratory,
Cambridge,
August 1920.

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[*] Phys. Rev. ii. p. 450 (1913), and Jour. Amer. Chem. Soc. xxxv. p. 943 (1913).

XLII. *A Method of finding the Scalar and Vector Potentials due to the Motion of Electric Charges.*

To the Editors of the Philosophical Magazine.

GENTLEMEN,—

PROFESSOR A. ANDERSON has published in No. 236 of the Philosophical Magazine (August 1920) "A Method of finding the Scalar and Vector Potentials due to the Motion of Electric Charges." He obtains divergent results from those obtained by myself in 1898 (*Éclairage Électrique*, tome xvi. p. 5). I have no observation to present for the calculations of the beginning, but I do not agree with Professor Anderson when he states the following proposition :—

"If $\mathrm{A}=\iiint\frac{\rho\,dx'\,dy'\,dz'}{r\left[1-\frac{u_r}{c}\right]}$, then $\nabla^2\mathrm{A}-\frac{1}{c^2}\cdot\frac{\partial^2\mathrm{A}}{\partial t^2}$ will vanish at any point P where there is no electricity."

For that, it should be necessary that the elementary volume $dx'\,dy'\,dz'$ and the electric charge be carried by the same motion ; which is not the case.

In addition to this, it is easy to show that $\nabla^2\mathrm{A}-\frac{1}{c^2}\frac{\partial^2\mathrm{A}}{\partial t^2}$ will not vanish. In order to simplify the writing, I will suppose that u is constant as regards its magnitude and direction, in all points and at any time. In this case

$$\frac{1}{c^2}\frac{\partial^2\mathrm{A}}{\partial t^2}=\iiint\frac{\frac{\partial^2\rho}{\partial\theta^2}dx'\,dy'\,dz'}{c^2\left[r-\frac{1}{c}\Sigma u_x(x-x')\right]},\quad \theta=t-\frac{r}{c},$$

$$\frac{\partial^2\mathrm{A}}{\partial x^2}=\iiint\left\{\frac{\partial^2\rho}{\partial\theta^2}\frac{(x-x')^2}{c^2r^2\left[r-\frac{1}{c}\Sigma u_x(x-x')\right]}\right.$$

$$+\frac{\partial\rho}{\partial\theta}\frac{-r^2+3(x-x')^2-2\frac{r}{c}u_x(x-x')+\frac{1}{cr}\left[r^2-(x-x')^2\right]\Sigma u_x(x-x')}{cr^2\left[r-\frac{1}{c}\Sigma u_x(x-x')\right]^2}$$

$$\left.+\rho\frac{\left[-1+\frac{(x-x')^2}{r^2}\right]\left[r-\frac{1}{c}\Sigma u_x(x-x')\right]+2r\left[\frac{u_x}{c}-\frac{x-x'}{r}\right]^2}{r\left[r-\frac{1}{c}\Sigma u_x(x-x')\right]^3}\right\}dx'dy'dz'.$$

$\frac{\partial^2 A}{\partial y^2}$ and $\frac{\partial^2 A}{\partial z^2}$ will be obtained by cyclic interchange. Thus

$$\nabla^2 A - \frac{1}{c^2}\frac{\partial^2 A}{\partial t^2} = \iiint 2\rho \frac{r\frac{u^2}{c^2} - \frac{1}{c}\Sigma u_x(x-x')}{r\left[r - \frac{1}{c}\Sigma u_x(x-x')\right]^3} dx'\,dy'\,dz'$$

and does not vanish.

I am, Gentlemen,
Yours faithfully,
A. LIÉNARD.

XLIII. *The Activity of Water in Sucrose Solutions.*

By W. E. GARNER *and* IRVINE MASSON *.

JONES and LEWIS (Trans. Chem. Soc. 1920, cxvii. p. 1125), in investigating the catalysis of the inversion of cane-sugar by decinormal sulphuric acid, have obtained values for the activities of the hydrogen ion in solutions of varying sucrose content with the aid of the hydrogen electrode. These activities increase markedly with increasing concentration of sucrose; thus, "when the sucrose content is 70 per cent. the thermodynamic concentration [per litre of solution] of the hydrogen ion is 0·162 at 20° C. and 0·152 at 40° C., although the maximum actual concentration cannot exceed 0·10."

It occurred to us that this increase in the activity of the hydrogen goes *pari passu* with the abnormalities in the osmotic pressures of sucrose solutions observed by Berkeley and Hartley and by Morse and Frazer. Arrhenius (*Zeit. phys. Chem.* 1899, xxviii. p. 317) has already correlated velocities of inversion with earlier measurements of osmotic pressure. That such a connexion exists is best shown by comparing the ratios of the activity of the hydrogen ion in sucrose solution to that in pure water with the ratios of the observed osmotic pressure to the osmotic pressure calculated according to van't Hoff. The former set of ratios is denoted in the following table by $(\mathrm{H}^{\bullet})\mathrm{sucr.}/(\mathrm{H}^{\bullet})\mathrm{aq.}$, and the latter by $\mathrm{P}_{\mathrm{obs.}}/\mathrm{P}_{\mathrm{cal.}}$.

The ratios of the solubility of hydrogen gas in pure water to those in sucrose solutions are also given in the last column, as a further indication of the change in the activity of the

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* Communicated by Prof. F. G. Donnan, F.R.S.

water. It has already been pointed out by Müller (*Zeit. phys. Chem.* 1912, lxxxi. pp. 483–503 ; see also J. C. Philip, Trans. Faraday Soc. 1907, iii. pp. 140–146) that other gases are quantitatively similar to hydrogen in this respect.

Sucrose. mols./litre.	*Lewis & Jones.* $\frac{(H^\bullet)\text{sucr.}}{(H^\bullet)\text{aq.}}$		*Berkeley & Hartley.* $P_{\text{obs.}}/P_{\text{cal.}}$	*Morse & Frazer.* $P_{\text{obs.}}/P_{\text{cal.}}$		$\frac{H_2\text{ aq.}}{H_2\text{ sucr.}}$
	20°.	*40°.*	*0°.*	*20°.*	*40°.*	*15°.*
·292	1·13	1·12	1·11	1·13	1·09	1·11
·585	1·30	1·24	1·20	1·23	1·21	1·23
·877	1·49	1·56	1·36	(1·38)	(1·33)	1·39
1·169	1·75	1·82	1·56	...	...	1·61
1·460	1·97	2·18	1·80	...	...	1·87
1·755	2·32	2·60	2·11	...	...	2·17
2·047	2·70	3·04	2·49	...	...	

There is a very clear parallelism between these ratios, which is well shown when the data are plotted as curves.

The general agreement shown by these figures indicates that the causes which are operative in increasing the activity of the hydrogen ion are substantially the same as those which cause the deviations of the osmotic pressure from van't Hoff's law and the decrease in the solubility of gases in sugar solutions. These causes would appear to be at work independently of the presence of hydrogen ions.

Jones and Lewis have further shown, from their experiments on inversion velocities, that "the environmental catalytic influence of a molecule of sucrose is identical in magnitude with that exerted by one molecule of dextrose together with one molecule of lævulose." Osmotic data are available for glucose (Berkeley and Hartley) as well as for sucrose ; and assuming that a molecule of lævulose is identical in its behaviour with a molecule of glucose, we can again tabulate the ratios of observed to calculated osmotic pressures.

J. & L., Sucrose concentration.	*·292*	*·585*	*·877*	*1·169*	*1·460*	*1·755*	*2·047*
$\frac{P_{\text{obs.}}}{P_{\text{cal.}}}$; Sucrose, B. & H., 0° C.	1·11	1·20	1·36	1·56	1·80	2·11	2·49
$\frac{P_{\text{obs.}}}{P_{\text{cal.}}}$; Glucose+lævulose, B. & H., 0° C.	1·07	1·19	1·34	1·52	1·72	(2·0)	(2·3)
$\frac{(H^\bullet)\text{sucr.}}{(H^\bullet)\text{aq.}}$; J. & L. 20° C.	1·13	1·30	1·49	1·75	1·97	2·32	2·70

Similar ratios are obtained for the solubilities of oxygen in glucose solutions.

It is quite clear in the first place, on osmotic grounds alone, that the activity of the hydrogen ion should be practically unaffected by the inversion of the sucrose to glucose and lævulose ; and in the second place, that the velocity-constant of the inversion will not be influenced by the change in environment which occurs.

The increase in the activity of the hydrogen ion may be ascribed to a virtual increase in the concentration of the sulphuric acid, due to the inactivating effect of sucrose molecules upon water molecules. This effect may consist in hydration of the sucrose, and in this case there would be an actual increase in the hydrion concentration in the free water. The water which is thus fixed by the sugar (changing from $10\,H_2O$ to $5\,H_2O$ per sucrose-molecule in the foregoing series) is not available as a solvent for the hydrogen ion or for gases, nor is it osmotically active. From the data of Jones and Lewis it appears that this water is nevertheless chemically active, since the rate of inversion is proportional to the total water present and not merely to the free water—a fact which is difficult to explain on any supposition other than the above.

We communicated these results to Professor Lewis, who very kindly suggested that they might be published.

University College, London.

XLIV. *The Collisions of α-Particles with Hydrogen Nuclei.*

By C. G. Darwin, *M.A., Fellow and Lecturer of Christ's College, Cambridge* [*].

I. *Introduction and Summary.*

In a series of four papers entitled "On the Collisions of α-particles with Light Atoms" in the Philosophical Magazine of June 1919, Sir Ernest Rutherford treated of a long course of experiments he carried out on this subject. The first of these papers [†] dealt with hydrogen, and he drew several interesting conclusions, the most striking of which was that in these collisions the α-particle and the hydrogen nucleus could not both be regarded as simple point charges. He roughly explained his observations by imagining that

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[*] Communicated by the Author.

[†] Rutherford, Phil. Mag. xxxvii. p. 538 (1919).

the α-particle is a plate of radius 3×10^{-13} cm. As long as the α-particle does not approach within this distance of the hydrogen nucleus, the ordinary law of repulsion holds, but if it does a collision ensues, which sweeps the latter straight forwards. His experiments were intended to be of a preliminary nature, and so were not carried to any high degree of accuracy.

The object of the present paper was to submit these experiments to a more rigorous analysis, so as to try and discover more definite information about the structure of the nucleus. But I have been recently informed through the kindness of Sir E. Rutherford that more detailed experiments are in progress in the Cavendish Laboratory to redetermine the curves with greatly improved methods. In particular, the number of hydrogen particles projected at various angles to the primary beam of α-particles is under determination by direct observation. The results so far obtained indicate that some of his earlier curves require modification. It thus appeared desirable to omit the more speculative conclusions to which I had been led, until the new more accurate experiments are finished, and the paper only makes a study of the question of the reduction of the earlier experiments to their simplest terms and an examination of what certain assumed patterns of nucleus would give.

We first discuss what is the complete information which can be obtained from experiments of any kind whatever on the subject, and show that these must lead to a certain relation, which we call the *collision relation.* This relation is then found for Rutherford's experiments, and it is seen to be quite different from that for the collision of two point charges. The complicated reduction was carried out in considerable detail before it was realised that the new methods were possible. In view of the extraordinary difficulty of experimenting in the subject and the probability that these new methods will not always be possible, it seems desirable to retain this detail, as showing how quite complicated experiments can be reduced to simple terms. The rest of the paper is an attempt to make models of nuclei which shall give the same collision relation as the experimental. These models must all obey the inverse square law at great distances, and the attempt is of necessity limited to cases where the orbits are integrable, which makes them rather artificial. In all of them one particle is still taken as a point charge, and the complexity is attributed to the other. In view of the trend of modern physics, it is natural to suppose that the α-particle is the complex one. From the first model we conclude with some probability that the experiments cannot be accounted for, if the α-particle is equally likely to be orientated in any direction—in fact, Rutherford's suggestion of a plate is supported. Two plate-like models are then tried, and are seen to give closer resemblance to the experimental curves. For the present, however, until the latter are confirmed it would appear undesirable to draw conclusions.

2. *The Collision Relation.*

Throughout the paper we shall be dealing with the action of an α-particle in setting a hydrogen nucleus in motion. It will conduce to shortness to refer to the two simply as α and H respectively. We shall first consider what conclusions can be drawn purely inductively from any type of experiment on the collisions of α with H. Neither particle may now be assumed to be a point charge, and the law of interaction is supposed to be quite unknown. No experiments on any line that could be devised (at present, at any rate) could do more than give a statistical description of the numbers of H projected in various directions and with various velocities by a known beam of α. We shall see that all such experiments can be made to lead to a certain relation between three variables, which we shall call the *collision relation.*

In the first place, the velocity of H is determined by the angle between the initial line of motion of α and the line of final motion of H. This angle will be called the *angle of projection.* For, let E, M be the charge and mass of α, and let V be its initial velocity. Let e, m be the charge and mass of H, initially at rest, and let u be its final velocity at angle θ to the initial line of motion of α, so that θ is the angle of projection.

Then by a simple application of momentum and energy it is found that

$$u=\frac{2\mathrm{M}}{\mathrm{M}+m}\mathrm{V}\cos\theta=\frac{8}{5}\mathrm{V}\cos\theta, \quad . \quad . \quad . \quad (2{\cdot}1)$$

since $\mathrm{M}=4m$. So it is a matter of indifference whether the experiments observe u or θ. We shall use θ.

In calculating the above it has been tacitly assumed that the collision involves no emission of radiation or change in the rotational motion of either body. Unless this is so, no progress can be made with the present method, as the conservation of energy becomes inapplicable; but the absence of rotational change is made very probable by an argument based on the quantum theory, similar to one that occurs in the study of the specific heats of a gas. Let I be the moment of inertia of one of the particles, and suppose that ν is a permissible frequency for rotation about the axis of I. Then the energy of rotation is $\frac{1}{2}\mathrm{I}(2\pi\nu)^2$, and by quantum principles this must be equal to $nh\nu$, where n is zero or an integer. Solving, we have

$$\nu=\frac{nh}{2\pi^2\mathrm{I}}, \quad \ldots \ldots \quad (2\cdot2)$$

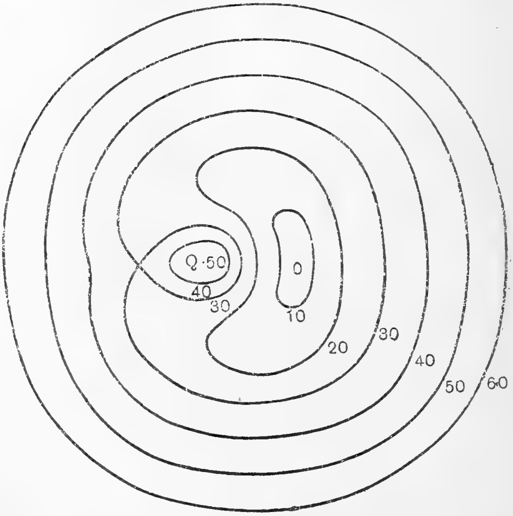
and so the possible energies are of the form

$$\frac{n^2h^2}{2\pi^2\mathrm{I}}. \quad \ldots \ldots \quad (2\cdot3)$$

We conjecture the α-particle to be not larger than a sphere of radius about 2×10^{-13} cm., while its mass is known to be $6\cdot5\times10^{-24}$ gr. This makes $\mathrm{I}=1\cdot04\times10^{-49}$ gm.cm.^{2} and $\nu=n\times3\cdot2\times10^{21}$ sec.$^{-1}$, while the energy comes to be $n^2\times2\cdot1\times10^{-5}$ erg. The energy of translation of an α-particle is $1\cdot3\times10^{-5}$ erg, and if $n=1$ these quantities are nearly equal. Thus if the collision set up a rotation in α, there would be little energy left to produce the H-particle, and it would not be detected. On the other hand, if rotational energy were converted into translational, it would force itself on our attention as an apparent breach of the conservation of energy. As this has not been observed, and as the first case could not be observed, it appears correct to neglect the possibility of changes of rotational motion. It may be remarked that the validity of our assumption might be tested by experiments which observed both velocity and direction of the H-particles simultaneously. If particles of different velocities were found going in the same direction, the experiments would have to be described by a more complicated collision relation, involving four instead of three variables.

The other quantity, which occurs in the experiments, is the number of H-particles produced for the given angle of projection θ. Now we can never control the particular types of collision that will occur. So this number is a direct measure of the probability of a collision that will produce an angle θ, and this probability can be translated into the area of a certain region in the following way. First consider a set of collisions in which both particles have given orientations, and for which α has given initial velocity and approaches from a fixed direction, so that the position of the initial line of α is the only variable. Take a plane through H perpendicular to the direction of this line, and choose some point of α, which may be called its centre. The line of approach may then be completely described by the point where the centre of α would cut the plane, if there were no deflexion. So, given this point, the orbit is determinate and with it θ, the angle of projection of H. Corresponding to every point of the plane there is a value for θ, and we can thus mark out on it curves of constant θ. We shall call the figure so made a *projection diagram*; it will be on a definite scale, of the order of 10^{-13} cm. in the actual problems we are going to consider. Fig. 1 is a hypothetical example.

Fig. 1.



HYPOTHETICAL COLLISION DIAGRAM.

Each number in the figure indicates the number of degrees of arc of the angle of projection in the corresponding collision.

Now we *define a quantity* P *as the area on the diagram for which the angle of projection is less than* θ. If α and H can have variable orientations, P is the averaged area in all the projection diagrams that may occur. Then in an experiment the number of H-particles projected at angles less than θ will be equal to the product of P by a number of known factors, the latter depending on considerations of probability. Therefore, all that experiment can do is to determine the relation between P and θ, or, allowing for experiments with different velocities of the α-particles, between P, θ, and V. This is the collision relation.

Our first task then is to find the collision relation for Rutherford's experiments. Afterwards we shall have to calculate the projection diagrams for our various models and deduce from them the corresponding theoretical collision relations. It will be well, therefore, to consider some of the characteristics that this relation may be expected to have. Let us suppose that the forces between the nuclei do not include any which can produce impulsive changes of velocity, so that the distribution of the curves in the diagram is continuous in the widest sense of the word. In spite of this continuity P may have discontinuities of its derivative. For example, in fig. 1 the P-curve would have a discontinuity of tangent at 50°. For consider how P behaves as θ diminishes through 50°. For angles just above, it is shrinking at a rate given by the rate of decrease of the outer curves, but on passing 50° a fresh area must be subtracted, that of the small patch round Q. This leads to a discontinuity of the tangent.

In the case where both α and H were regarded as point charges *, a relation was obtained from the orbits of the form

$$p=\mu \tan \theta, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (2{\cdot}4)$$

where

$$\mu=\mathrm{E}e\left(\frac{1}{\mathrm{M}}+\frac{1}{m}\right)\frac{1}{\mathrm{V}^2}. \quad . \quad . \quad . \quad . \qquad (2{\cdot}5)$$

Here p is the length of the perpendicular from H on to the initial line of motion of α. By this relation p can be uniquely determined from θ, so that the projection diagram consists of a series of circles, and unlike fig. 1 has no maxima or minima. We should therefore have $\mathrm{P}=\pi p^2$ and (2·4) would express the collision relation as it stands. Now it is more convenient to deal with lengths than with areas, so in the general case we shall define a length $\bar{p}$ by the equation $\mathrm{P}=\pi\bar{p}^2$, and shall use $\bar{p}$ instead of P. It should be remarked that even in cases where the projection diagram is entirely composed of concentric circles, so that θ can be determined by p alone, it does not follow that $\bar{p}$ is the same as p. For several values of p may lead to the same θ, whereas $\bar{p}$ from its definition in terms of P is necessarily a single-valued function of θ. The deduction of $\bar{p}$ from p is

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* Rutherford, *loc. cit.* The relation is proved with a slightly different notation in C. G. Darwin, Phil. Mag. xxvii. p. 499 (1914).

quite a simple matter, and in the discontinuities that may result, the tangent on one side is vertical. There is no need to treat of the matter further here, as examples occur in § 8 and § 9.

3. *Methods of Reducing the Experiments.*

In Rutherford's experiments* a point source of radium C was placed in an atmosphere of hydrogen. The α-particles, going in all directions, encountered here and there the nuclei of hydrogen atoms and gave them a high velocity. Some of these H-particles struck a zinc sulphide screen and produced scintillations in it. The ranges of these scintillating particles were found by interposing metal foils of varying thickness in front of the screen, so as to stop the slower ones. Our task is to deduce the collision relation from this rather complicated experimental arrangement, which was made necessary by the extraordinary difficulty of the practical problem. The process is more difficult than most reductions of experiments, because it involves the solution of what is really an integral equation, though one of a very degenerate type.

The actual observations were the ranges of the H-particles, but we can with fair confidence translate these ranges into velocities. For there is a satisfactory theory explaining the loss of velocity of α-particles in passing through matter, and it is simple to modify this so as to apply to H-particles †. According to this theory the rate of loss of velocity of an H-particle should be the same as that of an α-particle of the same speed. The theory fails to give a definite value for the range, as it only deals with the higher velocities, but we shall not be far wrong in saying that the ranges should be equal. As we shall see, the experiments we are considering give a partial confirmation of this. Since we are not going to work to any high degree of accuracy we shall take Geiger's empirical rule for the loss of velocity, different though it is from the theoretical, and apply it to the H-particles. This rule is that the remaining range of a particle at any distance along its path is proportional to the cube of its velocity. It is convenient as leading to an analytical expression, and the errors introduced, if any, will be much less than those of the actual experiments. It may, however, be remarked, that should future experiments destroy the validity of Geiger's rule, it would be quite possible to recast the following work

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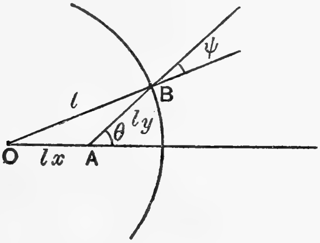
* Rutherford, *loc. cit.*

† Darwin, *loc. cit.*

with the improved relation. But before such recalculation will be worth while, it will be necessary to have much more accurate experiments to apply it to.

In the actual experiments the α-particles went in all directions and were observed in a fixed direction. This makes rather a difficult geometrical problem, and it is simpler to regard the matter as follows. Conceive that all the α-particles are projected along a single line from the centre of a sphere. If we take the total number that pass out of the sphere and multiply by the ratio of the area of the observing screen to the area of the whole sphere, we shall get the same number as in the actual arrangement. Moreover, a foil covering the sphere will correspond to the plane foil of experiment. Let l (fig. 2) be the radius of the sphere and let Q α-particles be emitted from O along OA. Consider an element at distance lx from O. Then if N be the number of atoms of hydrogen per c.c., $\mathrm{QNP}l\,dx$ will be the number of H-particles produced in the length $l\,dx$, for which the angle of projection is less than θ. Here P is the quantity defined in § 2, and it is to be regarded simply as an unknown function of θ. We have neglected the diminution of Q, due to the loss of those α-particles which make H-particles, as this is very small indeed. By (2·1) the H-particles will all have velocity greater than $\frac{8}{5}\mathrm{V}'\cos\theta$, where V' is the α-particle's velocity at A. We have to determine under what conditions they will traverse the foil so as to scintillate. If V is the initial velocity of the α-particles and r their range measured in cms. in hydrogen, then $\mathrm{V}'=\mathrm{V}\sqrt[3]{1-\frac{lx}{r}}$ by Geiger's rule, and so $\frac{8}{5}\mathrm{V}\cos\theta\sqrt[3]{1-\frac{lx}{r}}$ is the initial velocity of the H-particle. It therefore has a residual range $\left(\frac{8}{5}\right)^3\cos^3\theta\,(r-lx)$, or with sufficient accuracy

Fig. 2.



$4\cos^3\theta(r-lx)$, since theory tells us that the range is the same as for an α-particle of the same velocity. It then travels through the gas a distance $AB=ly$ and then through a metal foil. Let the stopping power of this foil be t cms., measured in hydrogen. It traverses the foil obliquely at angle ψ, and so, when through, it will have residual range $4\cos^3\theta(r-lx)-ly-t\sec\psi$, and only if this quantity is positive will the H-particle make a scintillation.

The quantities y and ψ are determinate in terms of x and θ, so that the inequality

$$4\cos^3\theta(r-lx)>ly+t\sec\psi \quad . \quad . \quad . \quad (3{\cdot}1)$$

fixes a limiting value of θ for every value of x. Under conditions where l is a considerable fraction of r, the equation may not give a solution at all for values of x near unity. This means that the α-particles, when they have got to $x=1$, have lost so much velocity that even their most favourable encounters are incapable of producing a scintillating particle. We shall not need to consider this case here, but shall find that the case where l is only a small fraction of r is the important one. In the experiments it ranged between 10 and 20 per cent., and the former belongs to the most important experiments. Neglecting l/r, we have

$$\cos^3\theta>\frac{t}{4r}\sec\psi. \quad . \quad . \quad . \quad . \quad . \quad . \quad (3{\cdot}2)$$

Substituting the value of ψ,

$$\cos^3\theta>\frac{t}{4r}\Big/\sqrt{(1-x^2\sin^2\theta)},$$

or in a form convenient for computing

$$x<\operatorname{cosec}\theta\sqrt{\left(1-\left(\frac{t}{4r}\right)^2\sec^6\theta\right)}. \quad . \quad . \quad (3{\cdot}3)$$

The terminal values are

$$\text{for } x=0 \qquad \cos\theta_0=\sqrt[3]{\frac{t}{4r}},$$

$$\text{for } x=1 \qquad \cos\theta_1=\sqrt[4]{\frac{t}{4r}}.$$

Thus $\theta_1<\theta_0$, and so fewer particles near $x=1$ give scintillations than near $x=0$. This is not because of the reduction of velocity of the α-particles, but because those thrown off at a given angle strike the stopping foil more obliquely.

If the area observed is A sq. cms., then the number seen will be

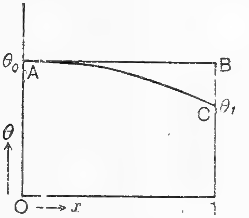
$$\nu=\nu_0\int_0^1 \mathrm{P}\,dx, \quad . \quad . \quad . \quad . \quad . \quad . \qquad (3\cdot4)$$

where
$$\nu_0=\frac{\mathrm{A}}{4\pi l^2}\mathrm{QN}l. \quad . \quad . \quad . \quad . \quad . \quad . \qquad (3\cdot5)$$

In this equation ν is a known function of $\frac{t}{4r}$, P is an unknown function of θ, while x is a known function of θ, depending on the parameter $\frac{t}{4r}$. This is an integral equation of the 'first kind' of the type known as Volterra's, with 'kernel' $\frac{dx}{d\theta}$. The present writer is not competent to discuss the general theory of such equations, but it appears that the following process gives quite a satisfactory solution. This is because the limits of integration θ_0 and θ_1 are in all cases not very far apart.

Fig. 3 illustrates an x, θ curve, with the difference between

Fig. 3.



θ_0 and θ_1 somewhat exaggerated. The tangent at θ_0 is always horizontal. Assume that P is a function of θ, that can be expanded in a Taylor series. Then in the field of integration

$$\mathrm{P}(\theta)=\mathrm{P}(\theta_0)+(\theta-\theta_0)\frac{d}{d\theta_0}\mathrm{P}(\theta_0)+\frac{1}{2}(\theta-\theta_0)^2\frac{d^2}{d\theta_0^{\,2}}\mathrm{P}(\theta_0), \quad (3\cdot6)$$

and so

$$\frac{\nu}{\nu_0}=\mathrm{P}(\theta_0)\int_{\theta_1}^{\theta_0}dx+\frac{d\mathrm{P}}{d\theta_0}\int_{\theta_1}^{\theta_0}(\theta-\theta_0)dx+\frac{1}{2}\frac{d^2\mathrm{P}}{d\theta_0^{\,2}}\int_{\theta_1}^{\theta_0}(\theta-\theta_0)^2dx.$$

The first integral is equal to unity, the second is the area of ABC taken negatively, and the third is twice the moment of this area about AB. The last two quantities are calculable by quadrature and depend only on $\frac{t}{4r}$. We denote them by

$$A_1\left(\frac{t}{4r}\right),\quad A_2\left(\frac{t}{4r}\right).$$

Then
$$\frac{\nu}{\nu_0}=P(\theta_0)-A_1\frac{dP}{d\theta_0}+\tfrac{1}{2}A_2\frac{d^2P}{d\theta_0{}^2}+\dots$$
$$=P(\theta_0-A_1)+\tfrac{1}{2}(A_2-A_1{}^2)\frac{d^2P}{d\theta_0{}^2}+\dots\quad . \qquad (3{\cdot}7)$$

The solution can then be completed by successive approximations. First, neglecting second differentials we obtain a curve for $P(\theta)$ from the experimental values of ν. Then, by means of the method of finite differences, we calculate $\frac{d^2P}{d\theta_0{}^2}$ and so get a correction term for ν. In the actual curves analysed it was found unnecessary even at the point of greatest curvature to make any correction for second differentials. This implies that no discontinuities were found. It may here be said that a slight modification of our method would be capable of dealing with any of simple type that might occur.

In obtaining the P, θ relation we neglected the loss of velocity of the α-particles in their passage through the gas. If this is to be allowed for, we have to use the accurate equation

$$\frac{\nu}{\nu_0}=\int_0 P(\theta, V)d\theta,$$

taken between the limits given by (3·1) instead of (3·2).

Assuming a Taylor expansion for P in V, we can write this as

$$\frac{\nu}{\nu_0}=\int_0\left\{P(\theta, V)-\frac{1}{3}\frac{l}{r}V\frac{\partial P}{\partial V}x\right\}dx.$$

The process of approximation could be applied to this, though it would be very cumbrous. We should require a new, more accurate set of curves like that in fig. 3, but calculated from (3·1) instead of (3·2). There would now be two parameters $\frac{t}{4r}$ and $\frac{l}{r}$, both influencing the calculations. We should then re-evaluate A_1 etc. We should also need from our previous solution to get relations between P and V, by taking together the experiments with α-particles of different velocities. In this way $\frac{\partial P}{\partial V}$ could be found, and so the second term evaluated, and applied as a correction to ν.

A good approximation will be obtained by neglecting the change in the curves of fig. 3 and also taking

$$\frac{\partial}{\partial \mathrm{V}} \mathrm{P}(\theta, \mathrm{V}) = \frac{\partial}{\partial \mathrm{V}} \mathrm{P}(\theta_0 - \mathrm{A}_1, \mathrm{V})$$

for all the values of θ in the rather short range of integration. Then

$$\int_0^1 \left\{ \mathrm{P}(\theta, \mathrm{V}) - \frac{1}{3}\frac{l}{r}\mathrm{V}\frac{\partial}{\partial \mathrm{V}}\mathrm{P}(\theta_0 - \mathrm{A}_1, \mathrm{V})x \right\} dx$$

$$= \mathrm{P}(\theta_0 - \mathrm{A}_1, \mathrm{V}) - \frac{1}{3}\frac{l}{r}\mathrm{V}\frac{\partial}{\partial \mathrm{V}}\mathrm{P}(\theta_0 - \mathrm{A}_1. \mathrm{V}) \cdot \frac{1}{2}$$

$$= \mathrm{P}\left(\theta_0 - \mathrm{A}_1, \mathrm{V}\left(1 - \frac{1}{6}\frac{l}{r}\right)\right)$$

$$= \mathrm{P}(\theta_0 - \mathrm{A}_1, \overline{\mathrm{V}}), \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3{\cdot}9)$$

where $\overline{\mathrm{V}}$ is the mean velocity of the α-particles in their passage through the gas.

Thus we see that a complete determination is possible for the collision relation between P, θ, and V. If at any time really accurate experiments should be carried out on the present principles, then it might be worth while to carry out in detail the processes sketched, but the labour would be very considerable, and it is to be hoped that it will be found possible to cast future experiments in a more tractable form. In view of the inaccuracy of the present ones I have contented myself with the simpler reduction previously described, together with the correction given in (3·9).

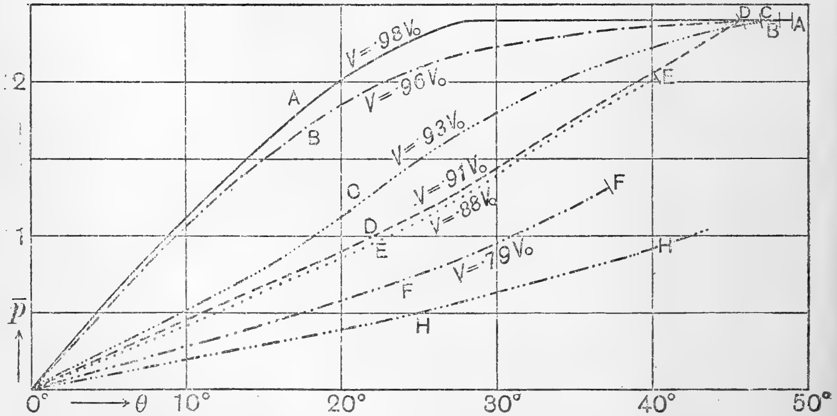
4. *The Reduction of the Experiments.*

The reduction of Rutherford's experiments then took the following course: (The pages mentioned in this section refer to the pages of his paper already cited.) Values of $t/4r$ were taken at every unit decimal between 0 and 1. The values of θ_0 were calculated for these, and also the (x, θ) curves, illustrated in fig. 3. By mechanical quadrature the areas and moments were found, so as to give A_1 and A_2. A table of the function $\theta_0 - \mathrm{A}_1$ in terms of $t/4r$ was thus drawn up, so that any value could be found by interpolation.

The experimental curves of p. 550 were then measured up. They are drawn in arbitrary units, different for each curve, and require to be brought to the same scale by means of the table on p. 553. This process was carried out by reducing the latter numbers with the help of the curves of p. 550 to a standard range of 7 cms. The absolute size is determined from the experiment described on p. 554, which deals with the flat part of curve A. With our definition of $\bar{p}$ instead of p, the calculation on p. 555 stands good, and we thus have as the absolute scale $\bar{p}=2{\cdot}4\times10^{-13}$ cm. for the value of $\bar{p}$ on this flat part.

In the curves of p. 550 the fastest H-particles at every range have very nearly four times the range of the exciting α-particles. This is the confirmation, referred to above, of the law we have assumed for the loss of velocity of H-particles. For the reduction the longest range at which scintillations were seen was taken as $4r$. The abscissa of a point on the curve is its t. Rutherford measured these quantities in their equivalents in cms. of air, not as we have done in hydrogen, but this will not affect the ratio $t/4r$. From the values of $t/4r$ we get θ_0-A_1 by means of the table. These are to be associated with values of $\bar{p}$ proportional to the square root of ν. The value of the second differential was tested at the point of greatest curvature in A, and it was found that the resulting correction was negligible.

Fig. 4.

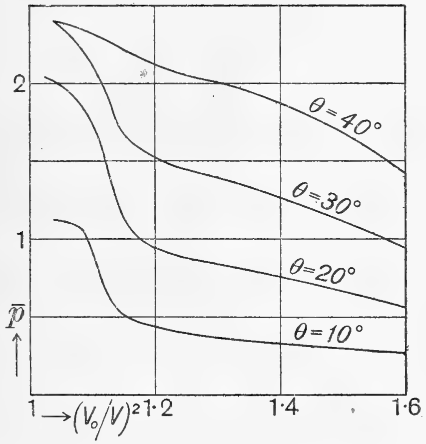


EXPERIMENTAL COLLISION RELATION ($\bar{p}$, θ).

$\bar{p}$ is measured in units of 10^{-13} cm. Each curve, as determined from experiment, ends at the point marked by a short cross line. H is the curve given by the collision of point charges, when a has the same velocity as in F.

The curves of fig. 4 are the results of this analysis. The curve H is the theoretical curve for point charges of the same velocity as F. It would appear from the uneven way in which the curves are arranged, that there are some errors in the experiments of p. 553, which connected those curves together, especially in E and possibly B. From fig. 4 we deduce fig. 5, where $\bar{p}$ is plotted against $(V_0/V)^2$ for constant values of θ. Here V_0 is 2×10^9 cm. per sec., the velocity of the fastest α-particles. The reason for taking this function of V as abscissa is, that it is the form in which V naturally occurs, when the law of interaction is the inverse square. The irregularity of the lie of the curves of fig. 4 becomes very marked in the curves of fig. 5, and to obviate this some smoothing has been applied to them.

Fig. 5.



Experimental Collision Relation ($\bar{p}$, V).

The constant θ curves have been plotted with $(V_0/V)^2$ as abscissa, where $V_0 = 2 \times 10^9$ cms. per sec.

5. *The Reduction of the α-particle to Rest.*

We now proceed to study what special laws of force can produce collision relations in any way resembling figs. 4 and 5. In doing so we are practically limited to those with integrable solutions, as the task of investigating orbits by the method of small arcs is too great to be undertaken, until some definite evidence is obtained as to the probable structure of the nuclei—especially when it is remembered that it would be necessary to treat of variations of P with both θ and V.

Before proceeding with these special cases it will be convenient to prove a simple general theorem applicable to the collision of any two bodies. It is only a slight extension of the astronomical process of reducing one of two bodies to rest. We suppose them both incapable of rotation. Then if points fixed in the bodies are chosen as their 'centres,' and if these centres are at x_1, y_1, z_1 ; x_2, y_2, z_2, the mutual potential W will be a function of $x_1 - x_2$, etc.

The equations of motion then are

$$\left.\begin{aligned} M\ddot{x}_1 &= -\frac{\partial W}{\partial x_1} \text{ etc.,} \\ m\ddot{x}_2 &= -\frac{\partial W}{\partial x_2} = \frac{\partial W}{\partial x_1} \text{ etc.} \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad (5{\cdot}1)$$

Suppose that α is approaching along the direction of the x-axis from the positive side. The momentum integrals are

$$M\dot{x}_1 + m\dot{x}_2 = -MV, \quad M\dot{y}_1 + m\dot{y}_2 = 0, \quad M\dot{z}_1 + m\dot{z}_2 = 0. \quad (5{\cdot}2)$$

Also if $\xi = x_1 - x_2$ etc. the relative motion is given by

$$\frac{Mm}{M+m}\ddot{\xi} = -\frac{\partial W}{\partial \xi} \text{ etc.} \quad . \quad . \quad . \quad . \quad (5{\cdot}3)$$

Suppose (5·3) to be solved for the relative orbit. Let the angle between the initial and final asymptotes be δ, and suppose that the y-axis is so chosen that the final motion is in the plane x, y. From the conservation of energy the final velocity is V, and so the final values of $\dot{\xi}$ etc. are

$$\left.\begin{aligned} \dot{\xi} &= \dot{x}_1 - \dot{x}_2 = V\cos\delta, \\ \dot{\eta} &= \dot{y}_1 - \dot{y}_2 = V\sin\delta, \\ \dot{\zeta} &= \dot{z}_1 - \dot{z}_2 = 0. \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad (5{\cdot}4)$$

Solving for $\dot{x}_2$ etc., we get

$$\begin{aligned} \dot{x}_2 &= -MV(1+\cos\delta)/(M+m), \\ \dot{y}_2 &= -MV\sin\delta/M+m, \\ \dot{z}_2 &= 0, \end{aligned}$$

and so the angle of projection is given by

$$\tan\theta = \dot{y}_2/\dot{x}_2 = \tan\frac{\delta}{2}.$$

So we have $\theta = \frac{\delta}{2}$ and we enunciate the result as follows.

The angle of projection of H is half the angle between the asymptotes calculated for the problem of relative motion, in which one body is constrained to remain at rest at the initial position of H, while the other, endowed with mass

$Mm/(M+m)$, is projected along the same line and with the same velocity as α in the actual problem.

It is indifferent whether it is H or α that is reduced to rest, and this shows that though it might be possible from experiments to attribute definite structures with certainty to H and α, yet the experiments could give no information as to which had which. But there is strong evidence from other sources that α is more complex than H, so in considering our models we shall reduce α to rest and project H past it. In all the cases we consider, H is a simple point charge.

6. *The Elastic Sphere.*

In our first model H is supposed to move according to the ordinary law of the inverse square, unless or until it approaches within a certain distance of the centre of α. If it gets within this distance it bounces off, as if from a hard elastic sphere. The calculation is quite straightforward, and need only be given briefly. If an auxiliary angle λ is taken, given by the relation

$$p=\mu\tan\lambda, \quad \ldots\ldots\ldots \quad (6\cdot1)$$

where, as in (2·5),

$$\mu=\mathrm{E}e\left(\frac{1}{\mathrm{M}}+\frac{1}{m}\right)\frac{1}{\mathrm{V}^2},$$

then the orbit under the inverse square law is the hyperbola

$$\frac{p}{r}\sin\lambda=\cos(\lambda-\phi)-\cos\lambda, \quad \ldots \quad (6\cdot2)$$

and so the angle between the asymptotes is 2λ. This is for the case where the apsidal distance $p\cot\frac{\lambda}{2}$ is greater than b, the radius of the sphere. If $p\cot\frac{\lambda}{2}<b$, that is if

$$\frac{\mu}{b}<\cot\lambda\tan\frac{\lambda}{2},$$

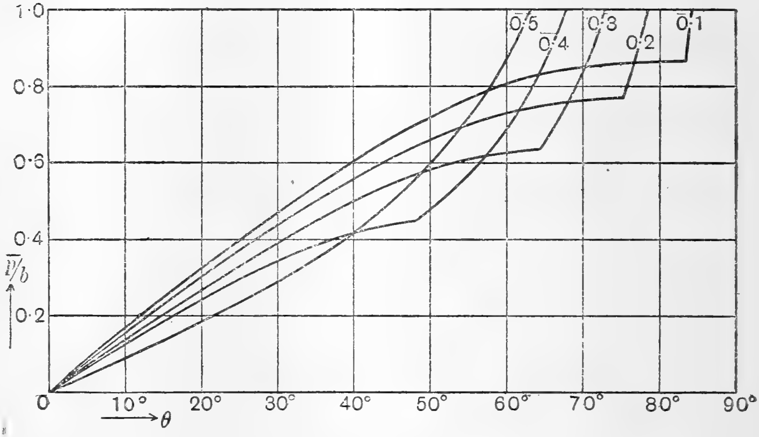
an impact occurs and it occurs when $\phi=\phi_1$, where ϕ_1 is given by

$$\frac{p}{b}\sin\lambda=\cos(\lambda-\phi_1)-\cos\lambda.$$

In this case the angle between the asymptotes is simply $2\phi_1$.

Fig. 6 shows the relations between p and θ for values of V determined by $\mu=0{\cdot}1, 0{\cdot}2, 0{\cdot}3, 0{\cdot}4, 0{\cdot}5$. If $\mu \geqslant 0{\cdot}5$ no impacts occur at all, and the curve is simply $p=\mu \tan\theta$. As p is single-valued in θ, it is the same as $\bar{p}$. The curves have little resemblance to those of fig. 4. In particular they lie too close together for the small values of θ and none of them is in the least like curve C. Note that the discontinuities of the curves are due to the discontinuity of force and not to the cause described in § 2.

Fig. 6.



Collision Relation for Elastic Sphere of radius b. The numbers on the $(\bar{p}, \theta)$ curves refer to the values of μ/b, that is, are proportional to $1/V^2$.

The elastic sphere would appear to represent, in a general way, systems of any shape, but orientated equally in all directions, and from its failure to give anything resembling the experimental curve, we conclude that the parts of α must be arranged in some way that can throw H definitely forwards. It may be that a spherical system could be devised to do so, if the operation of the inverse square law at the larger distances were abandoned, but the behaviour when V is small vetoes this possibility. Thus Rutherford's suggestion that his experiments point to a plate-like nucleus receives further support. Observe that there is nothing inherently improbable in this, as the uniform orientation may be connected with the circumstances attending the break-up of the radioactive atom.

7. *An Attempted Application of the Quantum.*

Before passing on to the next model, it will be convenient to consider the matter on quite different lines. Bohr's theory of spectra is based on the idea that of all mechanically possible types of orbit that a system can describe, only a certain number are actually permissible. Might it not be that the nuclei are both point charges, and that the peculiarities of our curves are in some way connected with this question of permissible orbits? Almost all the work on this subject has, of course, been carried out with the elliptic orbits of electrons moving round nuclei; but Epstein * has discussed the question of hyperbolic orbits—though his method of quantising them appears to the present writer to be of very doubtful validity. In his case the forces were attractive, but we can see no reason why repulsive forces should be excluded. However that may be, a very little calculation shows that there is no help to be found in the quantum theory. For any application of it, on lines at all similar to those used in other work (whether we follow Epstein the whole way or not), must lead to the condition that the angular momentum of the system is a multiple of the quantum. Thus we get the relation

$$MpV = nh.$$

If we take $V = 2 \times 10^9$ cm. per sec. for fast α-particles, we get $p = n \times 5 \times 10^{-13}$ cm. Standing by itself, this might conceivably be taken to mean that, when p was less than 5×10^{-13} cm., there could be no deflexion at all, a result something like what is observed. But the argument breaks down completely when variations of V are considered, for the lower the velocity the larger the limiting value of p becomes. So we should get the result that the behaviour at low velocities is even more abnormal than at high, directly contrary to the facts. We must therefore abandon the hope of help from the quantum theory, and explain its non-applicability by saying that that theory is a theory of radiation and that if (as may be the case here) radiation does not occur, quantum principles do not apply.

8. *The Elastic Plate.*

For our next model we take the simplest type of plate-like form that gives a soluble problem. In this α repels H with a force acting according to inverse square of the distance from its centre, but this centre is surrounded by a circular

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* P. S. Epstein, *Ann. d. Phys.* vol. l. p. 815 (1916).

plate, so that if H strikes this plate, it bounces off elastically. The plate is perpendicular to the direction of the initial motion of H. The orbit is again given by (6·2), where λ has the same meaning as before. If there is no impact, the angle between the asymptotes is therefore 2λ. An impact occurs if $r<b$ when $\phi=\frac{\pi}{2}$, that is if $\frac{p}{b}<1-\cot\lambda$ or if we substitute for λ its value in terms of p and solve, when

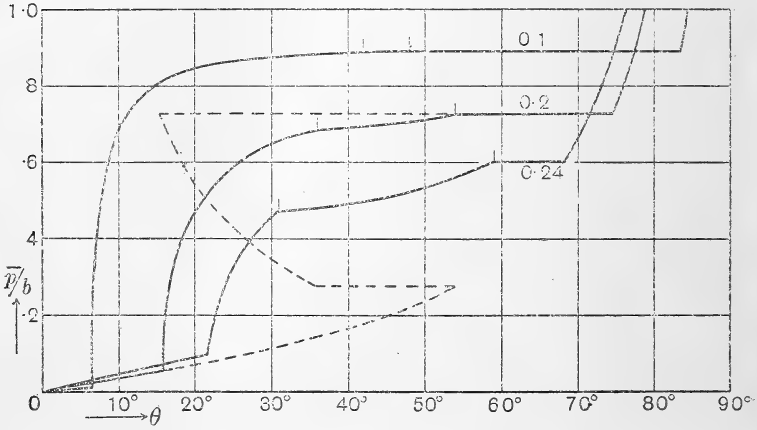
$$1-\sqrt{1-\frac{4\mu}{b}}<\frac{2p}{b}<1+\sqrt{1-\frac{4\mu}{b}}. \quad . \quad (8·1)$$

In this case it is easy to see that the second half of the orbit is exactly what it would be without the impact, if H had approached along the same line, but from the other end. So the angle between the asymptotes is $\pi-2\lambda$. The relation between p and θ is therefore

$$p=\mu\cot\theta \text{ or } p=\mu\tan\theta, \quad . \quad . \quad . \quad . \quad (8·2)$$

according as p does or does not satisfy (8·1).

Fig. 7.



Collision Relation for Elastic Plate of radius b. The numbers on the (p, θ) curves refer to the value of μ/b, that is are proportional to $1/V^2$. The points of discontinuity are marked by a cross stroke, where they are not obvious. The dotted curve, with its firm line continuations, gives the relation between p and θ for $\mu/b=0·2$.

If the incident velocity is so low that $\mu>b/4$, there are no collisions at all. If $\mu<b/4$ we encounter the complication that p is not a single-valued function of θ. The case where $\mu=0·2b$ is shown by the dotted curve in fig. 7. In these

cases it is necessary to change p into $\bar{p}$. Let p_1, p_2, p_3 be the three ordinates of p for any value of θ. Then

$$\bar{p}^2 = p_1^2 - p_2^2 + p_3^2,$$

as may be seen by making a sketch of the projection diagram. Fig. 7 shows the collision relations for the three values of V determined by $\mu/b = 0{\cdot}24$, $0{\cdot}2$, $0{\cdot}1$. The discontinuities are partly due to those of p and partly to the conversion of p into $\bar{p}$. In some ways the curves are better than those of fig. 6, but they are still very unlike the experimental. One part of the experimental curve A is however well illustrated and explained. The existence of its flat part implies a discontinuity of θ as p varies.

9. *The Bipole* *.

Of the problems of orbits that are integrable, there is only one of a more complicated type than those we have considered. This is the celebrated problem of Jacobi of two centres of force; so in order to push our investigation further we are compelled to call in its help. Now we want a body that acts something like a plate, and the field round two equal centres of force is of course quite different from this, but by considering one special set of orbits, we can deal with cases of motion through a set of equipotential surfaces not very unlike those of a disk. The comparison will be carried out in the next section. Here we shall suppose that α is split into halves at distance $2a$ apart, and that H is projected towards them along some line at right angles to the line joining these halves. This will give us a relation between p and θ. We then imagine this relation to be true for all azimuths and not merely in one plane. With this assumption we convert p into $\bar{p}$, and so obtain a set of curves for the collision relation.

The solution is found by means of the transformation

$$\left.\begin{aligned} x &= a \cosh \xi \cos \eta \\ y &= a \sinh \xi \sin \eta \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad . \quad (9{\cdot}1)$$

and following known processes the orbits due to projection

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* This term is used in the sense of two equal charges, not as is usual of two equal and opposite charges.

at right angles to the line of the bipole are given by the equation

$$\int_0^{\operatorname{sech}\xi} \frac{du}{\sqrt{(1-u^2)(1-2ku-\gamma^2u^2)}} = \int_0^{\frac{\pi}{2}-\eta} \frac{d\psi}{\sqrt{(\gamma^2-\sin^2\psi)}} . \quad (9{\cdot}2)$$

where $k=\frac{\mu}{a}=\frac{\mathrm{E}e}{a\mathrm{V}^2}\left(\frac{1}{\mathrm{M}}+\frac{1}{m}\right)$

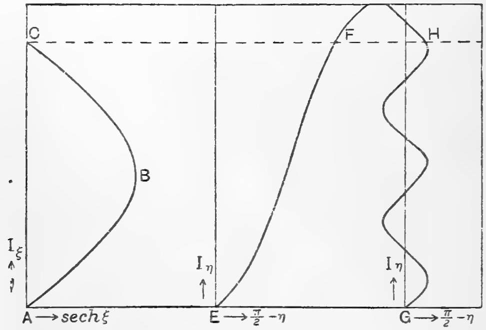
and $\gamma=\frac{p}{a}$.

The orbits are thus given by equations between two elliptic integrals. It will be convenient to consider their general character. Consider the left of (9·2), which we shall call I_ξ. As ξ diminishes from infinity sech ξ increases from zero and with it I_ξ. This process continues till sech ξ has reached the value of the least positive root of the quartic under I_ξ. These roots are

$$1, \ -1, \ \frac{\sqrt{\gamma^2+k^2}-k}{\gamma^2}, \ -\frac{\sqrt{\gamma^2+k^2}+k}{\gamma^2} . \quad . \quad (9{\cdot}3)$$

Different cases arise according as the first or third of them is the lesser. If the first, that is if $\gamma^2<1-2k$, then sech ξ can reach the value 1, that is ξ can vanish. This implies that H can go right through between the two halves of α, a state of affairs that we do not require to study, as it obviously will not give a suitable orbit. So we take $\gamma^2>1-2k$. Then sech ξ can never exceed the value given by the third root of (9·3). The point where it reaches it is a turning point and may be called the apse. After the apse sech ξ must diminish again, while I_ξ continues to increase; and when sech ξ has reached zero I_ξ will be double its value at the apse. In fig. 8 there is a sketch of the general character of I_ξ plotted against

Fig. 8.



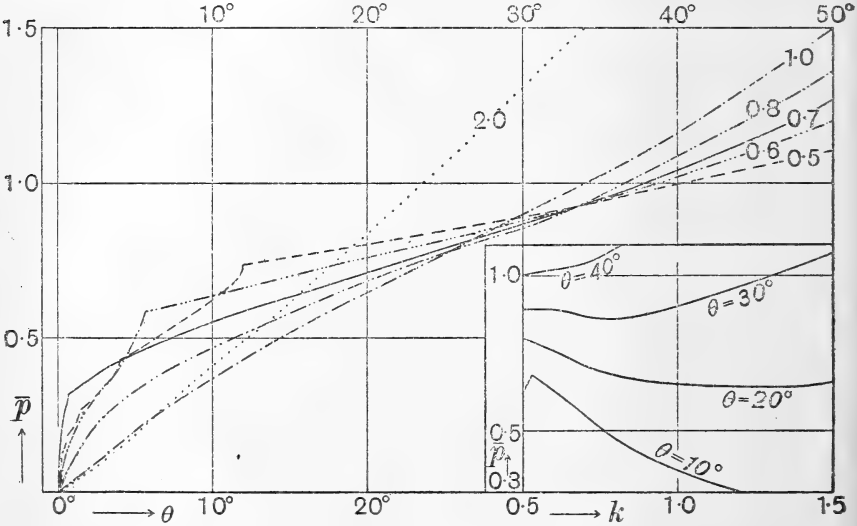
sech ξ; the exact shape will depend on the values of k and γ.

Now consider I_η, the integral on the right. Here two cases arise which exhibit an important difference. If $\gamma > 1$ (that is, if $p > a$) the integrand has no turning points, as its denominator never vanishes. If we plot I_η against $\frac{\pi}{2} - \eta$, we have a curve like EF in fig. 8. But if $\gamma < 1$ the denominator vanishes as soon as $\sin\left(\frac{\pi}{2} - \eta\right) = \gamma$, and the curve bends back like I_ξ. $\frac{\pi}{2} - \eta$ then shrinks through zero to the value given by $\sin\left(\frac{\pi}{2} - \eta\right) = -\gamma$, where there is another turning point. So the curve is a sinuous line like GH in fig. 8. The three curves are all drawn with the same horizontal base line in the diagram; k and γ will determine the size and shape of I_ξ, and γ of I_η, and also whether the latter is of the type EF or GH. To see the form of any orbit it is only necessary to select the proper k and γ curves and then draw a number of horizontal lines. These will give simultaneous values of ξ and η. In particular the line through C in fig. 8 will give the value of η on the second asymptote. As η is the ordinary vectorial angle when ξ is large, the value of θ is simply $\frac{1}{2}\left(\frac{\pi}{2} - \eta\right)$. Notice that when $\gamma < 1$, θ is almost as likely to be negative as positive.

It is unnecessary to describe in detail the rather tedious processes involved in the computation. I_ξ has to be transformed to reduce it to a standard elliptic integral of the first kind, and two cases arise, differing only analytically, according as the second or fourth root of (9·3) is the larger. In both its cases I_η needs little transformation, but the calculation is slightly complicated by the fact that the value of I_ξ to which it is equated must always be diminished by some multiple of the 'complete elliptic integral,' (which determines the number of oscillations in the orbit) before recourse can be had to the tables.

If we wish to obtain complete curves between p and θ, we are limited by the fact that γ^2 must be greater than $1 - 2k$ for all values of γ, and therefore k must not be less than 1/2. Calculations were therefore made taking $k = 0·5$, 0·6, 0·7, 0·8, 1·0, 2·0, and in each case the value of θ was computed for a succession of values of p. The curve 0·5 has an infinite number of oscillations as θ approaches zero. This is connected with the fact that if $\gamma=0$, H has initially exactly such a motion that it would come to rest midway between the halves of α. For 0·6, 0·7 there is a single oscillation, but in the higher values θ never becomes negative. After obtaining the p, θ curves we convert them into $\bar{p}, \theta$, a process only affecting those for which $k \leqslant 0{\cdot}7$. Fig. 9 shows the curves. The discontinuities in 0·5, 0·6, 0·7 are of the type with one tangent vertical. Towards 90° all the curves rise very steeply, but it has been necessary to omit these angles in order that the important part, from 0° to 50°, might be drawn on a reasonable scale for comparison with fig. 4. From the $\bar{p}, \theta$ curves with constant k, we deduce the $\bar{p}$, V with constant θ. These are shown in the inset of fig. 9, plotted with k (that is, $1/V^2$) as abscissa. This completes the description of the collision relation.

Fig. 9.



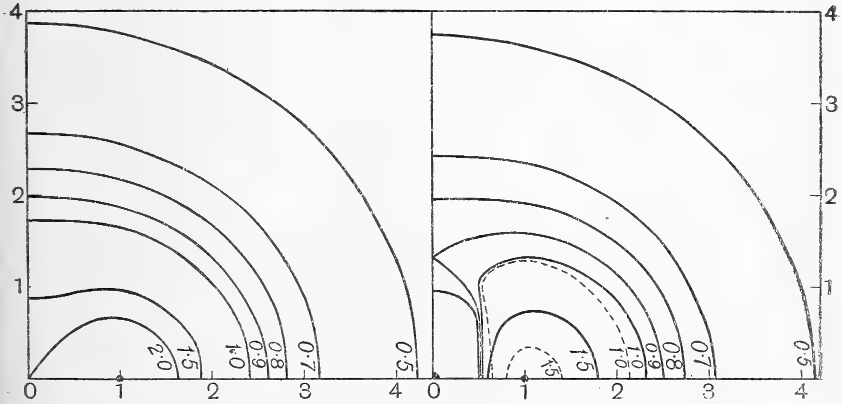
COLLISION RELATION FOR BIPOLE.

The distance between the poles is 2 units. The numbers on the $(\bar{p}, \theta)$ curves refer to the value of k, which is proportional to $1/V^2$. The inset gives the $(\bar{p}, V)$ curves in the same form as fig. 5.

10. *The Square Nucleus.*

The collision relation of § 9 was calculated on the very arbitrary assumption that we can represent the action of a plate by means of a pair of poles, twisted round so that they always lie in the plane in which the orbit is described. It is next necessary to see whether this is at all a reasonable assumption. To do so we take a model nucleus and compare its equipotential surfaces with those of the bipole. The recent evidence indicates that the nucleus of helium is made up of two electrons and four protons, to adopt the name suggested by Rutherford for the unit positive nucleus, which in its isolated state is hydrogen. It was therefore natural to choose a model composed out of these parts, and for simplicity of calculation they were supposed arranged with the protons one at each corner of a square and the two electrons together at the centre. It was not intended that this should really represent the structure of the nucleus, (and indeed there is very great difficulty in any model of this type in connexion with the theory of the relation of mass to energy,); it was merely adopted as giving a convenient form

Fig. 10*a*. Fig. 10*b*.



Equipotentials of Bipole (*a*) and Square Nucleus (*b*).

The potentials are measured with *e*/*a* as unit, where *a*, the unit of length in the figures, is the half distance between the poles of the bipole, or the radius of the square. The dotted lines are the equipotentials in a plane inclined at 45° to the others.

for examining the differences between the equipotentials of a real model and those of the bipole. Fig. 10 *a* shows the equipotentials of the latter. In fig. 10 *b* the firm lines are the equipotentials in a plane through the centre of the square containing two of the protons, and the dotted lines show those in a plane at 45° to the others; in the outer parts the two are indistinguishable. Now the inner parts of the figures are very different and some calculations of the position of the apses in the bipole orbits show that H does penetrate into these regions. So it is really only possible to put the argument in very vague form and to say that there appears to be no obvious characteristic of fig. 10, which would suggest that the bipole should have a more plate-like effect than the square nucleus.

11. *Conclusion.*

The comparison of any theoretical collision relation with experiment is a very exacting test, for it is a comparison of relations between *three* variables, and there is only one adjustable constant, the diameter of the nucleus. All the others, masses, charges, and velocity of approach are known from other work. Comparing fig. 9 with figs. 4 and 5, a rough measure of agreement was found with a nucleus of diameter $2{\cdot}7 \times 10^{-13}$ cm., but in view of the doubt that has been thrown on the experiments not much confidence can be placed in this. It is better to leave the decision as to the shape and size of the nucleus, until accurate experiments are made. One point of very great interest may, however, be noted as a possibility. Should future experiments confirm the flatness of curve A in fig. 4, it would be very strongly suggestive of the type of collision relation exhibited in fig. 7: that is to say, it would indicate that there was a discontinuity in the law of force between the nuclei.

XLV. *On the Structure of the Molecule and Chemical Combination.* *By* Sir J. J. THOMSON, *O.M., F.R.S.* [*]

FROM many points of view the structure of the molecule is even more important than that of the atom. For the structure of the molecule involves the method by which the atoms are linked together to form stable systems of different kinds, and is thus at the root of the enormously important question of chemical combination. In fact the extension of the electronic theory of matter to chemistry depends upon the solution of the problem of the structure of the molecule.

Regarding an atom as a stable arrangement of a mixture

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[*] Communicated by the Author. Many of the results in this paper were given by me in my lectures at the Royal Institution between 1914 and 1919, but owing to the pressure of other duties I have not hitherto had leisure to prepare them for publication.

of a positive charge and, except for hydrogen, a considerably larger number of electrons, we have to consider what happens when two or more such atoms are brought close together; what kind of rearrangement of the electrons takes place when the atoms unite and form a coherent system, and what are the considerations which determine the properties and stability of such a system.

I assume, as in my paper in the Phil. Mag. April 1919, that in the atom the electrons are in equilibrium under their mutual repulsions and the attractions exerted upon them by the positive charges. The repulsions between the electrons are supposed to follow the usual law and to vary inversely as the square of the distance. The force between the positive charge and the atom is supposed to be more complicated, and to vary with the distance r according to a law expressed by $\phi\left(\frac{c}{r}\right)$ where c is a length. When r is either very large or very small compared with c, $\phi\left(\frac{c}{r}\right)$ reduces to $1/r^2$, but when r is comparable with c the force is no longer of this simple type, but vanishes at certain distances, changing from attraction to repulsion or *vice versâ*.

My reasons for preferring this equilibrium arrangement to the more usual one of electrons describing orbits under forces varying rigorously as the inverse square, is that unless each electron is allowed to have a separate and isolated orbit the orbital arrangement is essentially unstable. Thus if the two electrons in a helium atom were to follow one another round the same circular orbit, the system would be quite unstable. Again, the scattering of light by hydrogen molecules is not that which would occur if the electrons in the molecule described orbits (J. J. Thomson, Phil. Mag. ser. 6, vol. xl. p. 393).

When we have to consider molecules containing perhaps dozens of electrons, the motion would be so hopelessly intricate and confused if these were all describing these large orbits that, apart from the fundamental difficulty about stability, the model would give us but little assistance in forming a mental picture of what goes on in chemical combination.

Assuming that the positive charge exerts on an electron a central force which changes from attraction to repulsion and therefore vanishes at certain distances, we proceed to consider how the electrons would be arranged round a central charge.

If there is only one electron in the atom it must be at one of the places where the force exerted by the positive charge vanishes, and in order that the equilibrium may be stable the force must become attractive when the electron is displaced from this position away from the centre and repulsive when it is displaced towards it.

If there are two electrons A and B in the atom, these must be situated so that the centre of the positive charge is midway between A and B; the distance AB is determined by the condition that the repulsion between A and B is equal to the attraction exerted by the positive charge on either of these electrons; three electrons will arrange themselves at the corners of an equilateral triangle, four at the corners of a regular tetrahedron and so on; the electrons are on the surface of a sphere concentric with the positive charge. When there are any number of electrons, the conditions for equilibrium are that the electrons should be so symmetrically placed that the force exerted on any electron by the other electrons should be along the radius, and that the magnitude of this radial force should be the same for all the electrons.

It can be shown without difficulty (J. J. Thomson, Phil. Mag. ser. 6, vol. vii. p. 237) that the radial force on an electron P due to the other electrons Q, R, S . . . is equal to $e^2 \mathrm{S}_n/4\pi r^2$, where $r = \mathrm{OP}$, O being the positive charge, e is the charge on an electron, and

$$\mathrm{S}_n = \frac{1}{\sin\frac{1}{2}\mathrm{POQ}} + \frac{1}{\sin\frac{1}{2}\mathrm{POR}} + \frac{1}{\sin\frac{1}{2}\mathrm{POS}} + \ldots$$

or if we take θ to denote one of the angles $\frac{1}{2}$POQ . . .

$$\mathrm{S}_n = \Sigma \frac{1}{\sin\theta},$$

where Σ denotes that the sum is to be taken for all the angles. Hence if we can find a distance r so that the attractive force exerted by a positive charge on a electron at this distance is equal to $e^2\mathrm{S}_n/4\pi r^2$, and if the electrons are so symmetrically arranged that S_n is the same for each electron, the electrons will be in equilibrium under the central force.

This equilibrium, however, will be unstable unless another condition is satisfied, and it is the limitation imposed by this condition that in my opinion determines the structure both of the atom and the molecule.

A simple illustration will show the stringency of this condition. Suppose that the electron P is displaced along OP from its position of equilibrium by a small distance δr, all the other electrons remaining fixed, then it can be shown that in consequence of this displacement of P the repulsion exerted by the other electrons along OP is increased by

$$\delta r \cdot \frac{e^2}{8r^3} \Sigma \left(\frac{1}{\sin^3 \theta_s} - \frac{3}{\sin \theta_s} \right).$$

If $e\phi\left(\frac{c}{r}\right)$ is the attraction exerted by the central positive charge on the electron at P, then when P is displaced the attraction is increased by

$$e\,\delta r \cdot \frac{d}{dr}\phi\left(\frac{c}{r}\right),$$

but unless this increase in the attraction is greater, or at any rate not less than that in the repulsion of the electrons, the electron when displaced will not return to its original position and the equilibrium will be unstable: hence for any symmetrical distribution of electrons to be stable

$$\frac{d}{dr}\phi(c/r) > \frac{c}{8r^3} \Sigma \left(\frac{1}{\sin^3 \theta_s} - \frac{3}{\sin \theta_s} \right). \quad . \quad . \quad . \quad (1)$$

As the number of electrons increases the angle between an electron and its nearest neighbour diminishes and the right hand side of this expression becomes greater. It becomes exceedingly large when the electrons are so numerous that θ is a small quantity, and so the equilibrium will be unstable unless $\frac{d}{dr}\phi\left(\frac{c}{r}\right)$ is also very large. Now whatever be the law of force, we may assume that at the same distance from the centre both the magnitude of the force and its rate of increase with the distance will be proportional to the magnitude of the central positive charge; this central charge is proportional to the number of electrons. We can see that if we disregard stability we can find a position of equilibrium even though the number of electrons is very large, for as we increase this number we increase the repulsive force $e^2 S_n / 4r^2$ exerted by the electrons, but the increase in S_n is comparable with the increase in the number of the electrons and therefore comparable with the increase in the central attractive force. Thus as the number of electrons increases, the attractive and repulsive forces can keep pace with each other, and it will be possible to find a position of equilibrium for a wide range in the number of electrons. When, however, we come to consider the stability of the arrangement the case is very different. We see that the right-hand side of (1) contains the term $\Sigma(1/\sin^3\theta_s)$ and this term when θ_s is small increases as $1/\theta_s^3$. If the electrons were arranged round a ring $1/\theta_s^3$ would be proportional to n^3, and if the electrons were on a sphere it would vary as $n^{\frac{3}{2}}$ where n is the number of the electrons, in either case it increases much more rapidly than n; n however measures the rate of increase of the left-hand side of (1). Hence even if for small values of n, $\frac{d}{dr}\phi\left(\frac{c}{r}\right)$ is greater than is necessary for stability by a considerable margin, as n increases the value of this quantity required for stability increases faster than the increase in the charge; hence the margin of stability must ultimately disappear and the arrangement become unstable. I have worked out the value E of the central charge required to make the equilibrium stable for symmetrical distribution of varying numbers of electrons when the force due to the central charge E is expressed by $\frac{\mathrm{E}}{r^2}\left(1-\frac{c}{r}\right)$. This law of force changes from attraction to repulsion only once, *i. e.* when $r=c$, and is thus about the simplest of its type. It will probably represent with considerable accuracy the law of force in the neighbourhood of a position of equilibrium for a much more general type of force. I hope to give the analysis in a separate paper. I will here quote two results which I have already given in lectures at the Royal Institution. The first table gives the value of the ratio of E/e to make the equilibrium of n electrons arranged at equal intervals round a circular ring stable when the motion is confined to the plane of the ring.

$n=$	1	2	3	4	5	6	7	10	12	14	16	18
E/e....	·70	·75	1·58	3·10	4·76	7·32	14·2	24·48	38·9	58	83	115

Thus to keep 10 electrons stable in a ring would require a central charge at least 25 times that on a single electron, while if there are more than 5 electrons in a ring the charge at the centre must be greater than the sum of the charges on the electrons.

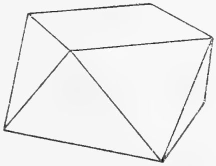
The next table gives the value of E/e when the electrons are arranged at the corners of a regular polyhedron instead of at equal intervals round a ring in one plane.

$n=$	1	2	3	4	6	8	12	20
E/e....	·70	·75	1·58	2·44	4·8	7·6	13	30

These results are independent of the value of c, the constant in the expression for the law of force.

It may be pointed out that the cube is not the stable arrangement of 8 electrons. This as Föppl (Proceedings of the International Congress of Mathematics, Cambridge 1912, vol. xi. p. 188) has shown is an arrangement when the 8 electrons are arranged in two sets of 4 in parallel planes, the electrons in each set being at the corners of a square, one square being twisted through an angle of 45° relative to the other so that the projection of the electrons on a plane parallel to either square is a regular octagon. This arrangement for 8 electrons shown in fig. 1 is analogous to that for 4 or 6, for the tetrahedron may be regarded as two sets of 2 at right angles to each other and to the line joining their middle points; the projections of the electrons on a plane at right angles to this line are at the corners of a square, while the octahedron may be regarded as two sets of 3 points, each set forming an equilateral triangle; the planes of these triangles are parallel and the triangles are twisted relatively to each other so that the projection of the electrons on a plane parallel to either triangle is a regular hexagon. Whereas all the plane faces of a cube are four-sided, the twisted polyhedron has 8 triangular faces as well as 2 four-sided ones, thus two such polyhedra could be placed so as to have either 2, 3, or 4 corners in common.

Fig. 1.



The stable arrangement for 5 electrons is when three are at the corners of an equilateral triangle with its centre at the central charge and the other two are at equal distances on opposite sides of this triangle. With 7 electrons five are at the corners of a pentagon whose plane passes through the central charge and the other two are at equal distances on opposite sides of this plane.

The point in the tables to which I wish to direct special attention is that, whether the electrons are arranged in one plane or distributed over the surface of a sphere, whenever the number of electrons exceeds a limit, which may vary with the law of force, the positive charge required to keep them in stable equilibrium will exceed the sum of the negative charges on the electrons. Thus with the law of force just assumed, if there are more than 5 electrons in the ring or more than 8 on the surface of the sphere, the positive charge at the centre will have to be greater than the charge on the electrons. I believe that it is this fact which governs the structure both of the atom and the molecule and determines the qualities which the chemists group under the term valency.

For the chemical atom is electrically neutral, the positive charge is equal to but not greater than the sum of the charges on the negative electrons, hence no arrangement of electrons is possible which requires for its stability a central charge greater than the sum of the charges on the electrons.

Very simple considerations will show the consequences which follow from this fact. I will suppose that the electrons are arranged symmetrically on the surface of a sphere whose centre is at the centre of the positive charges. This arrangement, as we have seen, is possible if there are not more than 8 electrons, and we can have atoms with from one to eight electrons on the surface of a sphere surrounding a positive charge equal to the sum of the charges on the electrons.

Now let us consider the case of an atom containing 9 electrons. The symmetrical distribution of 9 electrons over the surface of a sphere requires a central charge of more than 9 units to keep it in stable equilibrium, but when there are 9 electrons the central charge is 9 and hence is insufficient for this distribution. A new distribution will be required which will be of the following kind. The central charge 9 can hold 8 electrons in equilibrium on the surface of a concentric sphere, so that 8 of the electrons will group themselves round the central charge and there will be one over, this will go outside the shell of 8 and find a position of stable equilibrium at a greater distance from the centre. Thus the external layer of this atom will contain only one electron and in this respect will resemble the atom with one electron and unit positive charge. Now suppose we have an atom with 10 electrons, eight of these will form the inner shell and two will be left over to form the outer; thus we shall have an atom resembling as far as the outer ring is concerned the 2 electron atom with the double positive charge. If we increase the number of electrons still further we shall get outer rings with 3–4 - - - 8, but there will never be more than 8 in the outer shell. When the electrons have increased so much that 8 on the outer shell is not sufficient to accommodate them, a third shell will be formed, and when this is filled up with 8 electrons a fourth will be formed, and when this is filled up a fifth and so on. Thus, if we confine our attention to the outer layer and arrange the elements in the order of the number of electrons they contain, there will be a periodicity in the number of electrons in the outer layer; it will increase from one to eight, then drop again to one, increase again to eight, drop to one, and so on. Thus as far as properties depending on the outer layer are concerned the element will exhibit a periodicity similar to that expressed by Mendeleef's periodic law in Chemistry.

It may be pointed out, however, that the number of elements included within a period may possibly for the elements of large atomic weight be greater than eight. This might arise in the following way. Suppose that for a particular value of N a central charge N can hold n electrons in stable equilibrium, while $N+1$ can hold $n+1$. Then the atom of the element whose atomic number is N would have a layer of n electrons next the centre and other layers outside, the atom of the element next in order would have a central charge of $N+1$ units and contain $(N+1)$ electrons; but as a charge $N+1$ can hold $n+1$ electrons in stable equilibrium the innermost layer might now contain $n+1$ electrons, and thus the additional electron might be used up in the inner layer and not affect the number in the surface layer. Thus the atoms of the elements whose atomic numbers are N and $N+1$ will have different atomic weights and different central charges, while the number of electrons in the surface layer will be the same: hence, if there are any properties which depend exclusively on the number of electrons in the outer layer, these two elements will have these properties in common. If we are right in supposing that the valency of the element is a property of this kind, then the two elements will have the same valency.

When there are a great number of electrons in the atom arranged in many layers, it may require the addition of several electrons before a new electron finds its way to the outer layer, and thus there might be a considerable number of elements with different atomic weights but with very similar chemical properties. There are groups of elements such as the iron, nickel, and cobalt group, the rhodium group, and the crowd of elements known as the rare earths which fulfil this condition.

Unless the atom is electrified the total charge inside the outer layer of electrons must always be equal to the charge on those electrons. Thus, however many electrons there may be in the atom, the electrons in the outer layer cannot be under the influence of an effective charge greater than the sum of the charges on these electrons, so that with the law of force we are considering no element could have more than 8 electrons in the outside layer.

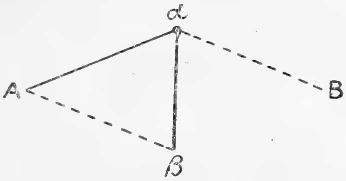
I shall suppose (as in my paper on the "Forces between Atoms and Chemical Affinity," (Phil. Mag. ser. 6, vol. xxvii. p. 757, 1914), that those atoms which have one electron in the outer layer form the first Mendeleefian group, those with two in this layer the second group, and so on. Thus the hydrogen and lithium atoms are supposed to have one electron in the outer layer, the beryllium atom 2, the boron atom 3, the carbon atom 4, the nitrogen atom 5, the oxygen atom 6, the fluorine atom 7, and the neon atom 8.

Let us now consider if any light is thrown on this view by the evidence afforded by positive rays. If eight is the maximum number of electrons which can exist in the outer layer, then the atom of neon already possesses that number and cannot accommodate another electron, and so cannot receive a negative charge. On the other hand, the atoms with a smaller number of electrons in the outer layer have, as a reference to the table 2 shows, a superfluity of stability, and so could accommodate another electron and thus acquire a negative charge ; the superfluity is however not great enough for them to accommodate two so that we should not expect to find any with a double negative charge.

In experiments with positive rays neon has never been observed with a negative charge, while negative charges are common on hydrogen, chlorine, oxygen, and carbon ; we have no information about boron and beryllium : thus far the evidence is in favour of the view. On the other hand, no negative charge has as yet been observed on the atom of nitrogen : this is remarkable, as the atoms of the elements on either side of it, carbon and oxygen, readily acquire a negative charge. It must be remembered, however, that the number of atoms with negative charges varies very much with the conditions of the discharge and the gases in the discharge-tube. To get a negative charge a neutral atom has to drag an electron from another atom or molecule, and unless an atom of nitrogen came in contact with an atom of some substance which held its electrons more loosely than the nitrogen, it might not be able to capture the electron necessary to give it the negative charge. It will be seen that all the atoms except hydrogen, lithium, sodium, potassium, could lose more than one electron and thus have double or treble positive charges : this is in accordance with the evidence afforded by the positive rays.

We have, however, to explain why an atom with 7 electrons in the outer ring, such as that of Fl or Cl, can have the same valency as an atom containing only one electron.

Fig. 2.

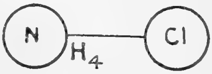


Suppose that A (fig. 2) is the centre of the 7 electron atom, B that of a 1 electron atom. Let α be an electron on the first atom, β one on the second. If these electrons place themselves somewhat as in the figure, the attractions they exert on A and B may keep these together in spite of the repulsion between the positive charges A and B, while the attractions A and B exert on α and β may keep these together in spite of the repulsion between them. The addition of β to the seven electrons already round A will raise the number in the outer layer round A to eight. Now suppose we attempted to attach another 1 electron atom B′ to A. This would introduce another electron in the layer round A, raising the number in this layer to 9. But we have seen that 9 electrons in one layer cannot be kept in stable equilibrium by a positive charge of nine units, but 9 units of positive electricity is all we have at our disposal, and two of these units are outside the layer: though they are outside they will make the arrangement of the electrons more stable than if they were absent; they will not, however, increase the stability more than if they were inside, and even in that case they could not make the arrangement stable. Thus A cannot hold a second atom of the type B, so that the compound AB_2 is impossible, while that represented by AB is saturated. Thus, if B be taken as the type of a monovalent atom, A with its seven electrons would act like a monovalent atom and would thus conform in this respect to the behaviour of Fl. The electrons in the molecule are thus arranged in a cell containing 8 electrons and surrounding the core of the fluorine atom. Next, suppose that A instead of having 7 electrons in the outer layer has only 6, and let an atom B get attached to it in the way we have supposed. This will raise the number of electrons in the layer round A to seven; it can attach another atom before the number of electrons in the outer layer is raised to 8, and the limit of stability reached. Thus, if A were a 6 electron atom it could form the unsaturated compound AB and the saturated one AB_2, in which again there is a cell of 8 electrons round the more heavily charged atom; A could not combine with more than 2 of the B atoms. Thus, A with its 6 electrons would act like a divalent atom, and in this respect could represent oxygen. We see, too, that if A contained 5 electrons it could form the unsaturated compounds AB, AB_2, and the saturated one AB_3, but not any containing more than 3 B atoms. The 5 electron atom would thus act like a trivalent atom, and could thus in this respect represent nitrogen, the electrons in the molecule forming a cell of 8 round the nitrogen atom.

We see too that since the attachment of a 1 one-electron atom like that of hydrogen requires two electrons on the layer round the atom with which it is combined, we cannot have more than four hydrogen atoms attached to a single atom; this is in accordance with experience.

We can illustrate by means of the 5 electron atom the fact that, on the view of the constitution of the atom we have assumed, an atom may possess two valencies, the sum of these two being always equal to eight. Let us take for example the nitrogen atom, then, if it enters into combination with hydrogen, we have seen that 3 hydrogen atoms will saturate it as they will bring the number of electrons in the layer round the nitrogen nucleus up to 8, the limiting number. But suppose that, instead of combining with a hydrogen atom, it combines with chlorine whose atom we assume to have 7 electrons in the outer layer. One of the electrons of the nitrogen atom may join the layer round the chlorine atom, bringing the number of electrons up to 8, the limiting number, and leaving 4 electrons in the layer round the nitrogen nucleus. These 4 electrons can link up with four atoms of hydrogen, this will bring the number of electrons in the layer round the nitrogen nucleus up to 8, the limiting number, giving the compound NH_4Cl, the arrangement of the electrons being as represented in fig, 3, where the ring denotes a cell of 8 electrons.

Fig. 3.



As the chlorine nucleus has a positive charge of 7 and is surrounded by a layer containing 8 electrons, the system $\textcircled{\text{Cl}}$

has a unit negative charge. Since there is a positive charge of 5 on the nitrogen nucleus and one of 4 on the 4 hydrogen atoms, there are 9 positive charges on the system NH_4 while it is surrounded by only 8 electrons: thus on the balance there is a single positive charge on the system $\textcircled{NH_4}$. Thus the compound might be represented by $(NH_4)_+Cl_-$, and would when electrolysed give NH_4 and Cl as ions. In this case the molecule contains 2 cells of 8, one surrounding the nitrogen core and the other that of the chlorine.

We have seen that the nitrogen atom could not take up more than three hydrogen ones, so that we could not have the compound NH_5; we could, however, have that represented by NCl_5. In this case each of the 5 electrons of the nitrogen nucleus would have gone off to complete the tale of 8 electrons round each of the 5 chlorine ions, and these cells would surround the nitrogen atom.

Those electrons on the shells round the 5 chlorine atoms which are nearest to the nitrogen nucleus will form a layer round the nitrogen atom, and we should not expect stable equilibrium if the number of electrons exceeded 8. We should get this arrangement if three of the layers round the chlorine atoms presented an edge—each then would supply two electrons to the nitrogen layer, the other two layers would each present a corner and furnish one to the layer round the nitrogen; thus two of the chlorine atoms would be more loosely attached to the nitrogen than the other three.

The compound NCl_3 would also be possible on this scheme, for an electron round the nitrogen nucleus might link up with one from the chlorine and both form part of the layer round the chlorine as well as that round the nitrogen, as the number of electrons in the layer round the nitrogen nucleus cannot be greater than eight; since there are 5 already round the nitrogen we can only get three chlorine atoms linked up in this way. Thus it is possible to have the two chlorides NCl_3, NCl_5; though NCl_5 does not seem to have been prepared, phosphorus which, like nitrogen, is a pentavalent element, is known to have two chlorides, PCl_3 and PCl_5. If, instead of an atom with five electrons in the outer layer, we had taken one like the atom of oxygen with six, it might attach to itself in the way just described six atoms of chlorine, and so be apparently hexavalent, as sulphur is in the remarkable compound SF_6 discovered by Moissan. In this way we are, as I pointed out in the paper on Chemical Combination already referred to, led to conclusions with regard to valency very similar to those advanced by Abegg and Bodländer (*Zeit. anorg. Chemie*, 1899, xx. p. 453, 1904, xxxix. p. 330), who ascribed to each element two valencies according as it was combined with a more electro-negative or more electro-positive element, the sum of these valencies always being eight. Cohen (Organic Chemistry, vol. 2. p. 3) says that the weak point of this scheme is that there are no compounds in which the alkali metals possess the valency 7. We should not expect, on the views given above, that these compounds could exist, for on this view the two valencies arise from saturation being arrived at in two different ways, one, *e.g.*, when nitrogen was acting like a triad, through the layer round the atom having acquired the maximum number of atoms consistent with stability; the other, *e. g.*, when nitrogen acts like a pentad, when all the electrons associated with the atom have been used up in binding other atoms to it. We should not expect those atoms which possess only a small number of electrons to show both kinds of valencies. For example, we should not expect to find hydrogen acting like a septavalent element, for this would mean that a layer of eight electrons was in stable equilibrium round the hydrogen nucleus which has only unit positive charge. Now a layer of eight electrons requires a very considerable positive charge to keep it in stable equilibrium, and it is improbable that a single positive charge inside, even though it were assisted by seven positive charges outside, would be able to do so.

The freedom of motion of the electrons in an atom is of importance in connexion with the attraction which the atom is likely to experience from other atoms. We can illustrate this point by considering an atom containing one electron α which will be free to move in any direction provided its distance from the centre of the atom does not vary. Suppose that the atom is placed near a positively electrified body B, then whatever may have been the initial position of the electron α, it will swing round A until it gets as near as possible to B; the attraction of B on α will then be greater than the repulsion between B and the positive nucleus A, so that the atoms will always be attracted towards B. If α had not been free to adjust itself under the force exerted by B, the atom would just as likely be repelled from B as attracted by it. We see from this that a very rigid disposition of the electrons in an atom will diminish the likelihood of its being attracted by or attracting other atoms.

It is worthy of notice that in a case like one when 8 electrons are distributed over a single layer and are on the verge of instability, the readjustment of the electrons is prevented by another cause besides that of rigid arrangement. For any readjustment of their positions would be accompanied by a diminution in the minimum distance between two electrons, but in the undisturbed state this minimum distance has almost reached the limit consistent with stability; thus any rearrangement of the electrons will tend to be unstable and to break up. Thus the system comprising the atom and its electrons will not experience the attraction to which it would be subject if the rearrangement of the electrons could be maintained.

The distance of the outer ring of electrons from the centre of the atom will vary with the valency of the atom, and can be calculated from the condition that the attraction of the electron by the central charge balances its repulsion by the other electrons in the ring.

We have assumed that the attraction between a positive charge and an electron is of the form $\frac{a}{r^2}-\frac{b}{r^3}$; we must consider the expressions for a and b when there are negative electrons as well as positive charges inside the outer ring. Since the force between electrons varies rigorously as $1/r^2$, the existence of these inside the ring will not affect the term b/r^3 in the expression for the attraction: this term will be proportional to the positive charge, and therefore to the atomic weight N. On the other hand, these electrons will affect the term a/r^2: they will make a proportional to the difference between the positive charge and the charge on the electrons inside the outer ring; but this difference in a neutral atom is equal to the charge on the electrons in the outer ring. Thus we may write the attractive force on an electron in the outer layer in the form

$$\frac{ne}{r^2}-\frac{\mathrm{N}e\,.\,\alpha}{r^3},$$

where N is the atomic weight of the element and n the number of electrons in the outer ring, α is a constant. The repulsion exerted by the other electrons in the outer layer is equal to $\mathrm{S}_n/4r^2$: where $\mathrm{S}_n=\Sigma\frac{1}{\sin\theta}$, where 2θ is the angle subtended at the centre of the atom by the line joining

2 electrons in the outer ring. Since the repulsion must balance the attraction, we have

$$\frac{n\,e}{r^2} - \frac{\mathrm{N}e\,.\,\alpha}{r^3} = \frac{\mathrm{S}_n}{4r^2}\,e$$

or

$$r = \frac{\mathrm{N}\alpha}{n - \frac{\mathrm{S}_n}{4}}$$

For hydrogen $\mathrm{N}=n=1$, $\mathrm{S}_n=0$, so that $r_{\mathrm{H}}=\alpha$;
for lithium...... $\mathrm{N}=7$, $n=1$, $\mathrm{S}_n=0$, ,, $r_{\mathrm{Li}}=7\,\alpha$;
for beryllium... $\mathrm{N}=9$, $n=2$, $\mathrm{S}_n=1$, ,, $r_{\mathrm{Be}}=5{\cdot}14\,\alpha$;
for boron ...... $\mathrm{N}=11$, $n=3$, $\mathrm{S}_n=2{\cdot}3$, ,, $r_{\mathrm{B}}=4{\cdot}52\alpha$;
for carbon ...... $\mathrm{N}=12$, $n=4$, $\mathrm{S}_n=3{\cdot}66$, ,, $r_{\mathrm{C}}=3{\cdot}88\,\alpha$;
for nitrogen ... $\mathrm{N}=14$, $n=5$, $\mathrm{S}_n=5{\cdot}2$, ,, $r_{\mathrm{N}}=3{\cdot}78\,\alpha$;
for oxygen...... $\mathrm{N}=16$, $n=6$, $\mathrm{S}_n=6{\cdot}68$, ,, $r_{\mathrm{O}}=3{\cdot}69\,\alpha$;
for fluorine ... $\mathrm{N}=19$, $n=7$, $\mathrm{S}_n=8{\cdot}08$, ,, $r_{\mathrm{F}}=3{\cdot}8\,\alpha$;
for neon......... $\mathrm{N}=20$, $n=8$, $\mathrm{S}_n=10{\cdot}1$, ,, $r_{\mathrm{Ne}}=3{\cdot}63\alpha$;
for sodium...... $\mathrm{N}=23$, $n=1$, $\mathrm{S}_n=0$, ,, $r_{\mathrm{Na}}=23\,\alpha$;

Thus, taking the elements from lithium to neon, we see that the radius of the outer layer is greatest for the lightest element and diminishes rapidly at first and then very slowly to the end of the series; when we pass from neon, the last element in this period, to sodium the first in the next, there is a great increase in the radius: this increase is again followed by a diminution which continues until we reach argon; when we pass to potassium, the first element in the next series, there is again a large increase. The radii of the layers in the same group like lithium, sodium, potassium, fluorine, chlorine, bromine, increase with their atomic weights. Thus the relation between the radii of the layers and the atomic weight is represented on this theory by a graph differing from the historic one for the atomic volumes and atomic weight given by Lothar Meyer and reproduced in almost every text-book of chemistry: here the minimum atomic volume comes in the middle of a period and not at the end. Recent investigations, however, have shown that the relation between atomic volume and atomic weight is not accurately represented by a curve of the type of that of

Lothar Meyer. Thus, Gervaise le Bas, 'Molecular Volume of Liquid Chemical Compounds,' p. 237, says :—

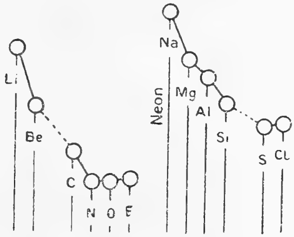
"1. There is a periodic relation between the atomic volume and the atomic weight of the elements.

2. There is a tendency for the atomic volume to diminish in each series as the atoms increase in weight. The smallest occur in group 7.

3. There is a general increase in the atomic volume of the members of each group from series one onwards, that is in the direction of increasing atomic weight."

This is in entire agreement with the results we have just found. The same thing is beautifully shown by the experiments of W. L. Bragg (Phil. Mag. xl. p. 169, 1920), which give a curve (fig. 4) for the atomic radii which for the period from lithium to neon agrees remarkably well with the figures given above.

Fig. 4.



We should expect, since the law of force we have assumed is probably only an approximation holding near a position of equilibrium, that the relative values of the radii of the elements within one period would be more reliable than those for elements in different periods as the value of α would vary from one period to another.

Work required to separate one electron from the atom.

The work required to separate an electron will depend upon whether the electron is ejected so quickly that the electrons left in the atom have not had time to alter their positions appreciably before the ejected electron has passed out of the sphere of their influence, or whether the process takes place so slowly that the other electrons subside gradually into positions of equilibrium without acquiring any kinetic energy. If W_1, W_2 represent the amounts of work for these processes respectively, it is easy to show that if the attraction is represented as on page 523 by

$$\frac{ne}{r^2}-\frac{N\alpha e}{r^3},$$

$$W_1=\frac{e^2}{2N\alpha}\left(n-\frac{S_n}{4}\right)\left(n-\frac{3}{4}\frac{S_n}{4}\right),$$

$$W_2=\frac{e^2}{2N\alpha}\left\{n\left(n-\frac{S_n}{4}\right)^2-(n-1)\left(n-\frac{S_{n-1}}{4}\right)^2\right\}.$$

W_1 is always greater than W_2. From these expressions we find the following values for W_1 for different types of atom.

Gas.	W_1.	W_2.	Gas.	W_1.	W_2.
Hydrogen..	$\frac{1}{2}\frac{e^2}{\alpha}$	$\frac{1}{2}\frac{e^2}{\alpha}$	Nitrogen...	$\frac{1}{2}\frac{e^2}{\alpha}\times\cdot55$	$\frac{1}{2}\frac{e^2}{\alpha}\times\cdot17$
Lithium ...	$\frac{1}{2}\frac{e^2}{\alpha}\times\cdot333$	$\frac{1}{2}\frac{e^2}{\alpha}\times\cdot333$	Oxygen ...	$\frac{1}{2}\frac{e^2}{\alpha}\times\cdot55$	$\frac{1}{2}\frac{e^2}{\alpha}\times\cdot4$
Beryllium..	$\frac{1}{2}\frac{e^2}{\alpha}\times\cdot54$	$\frac{1}{2}\frac{e^2}{\alpha}\times\cdot53$	Fluorine...	$\frac{1}{2}\frac{e^3}{\alpha}\times\cdot55$	$\frac{1}{2}\frac{e^2}{\alpha}\times\cdot4$
Boron	$\frac{1}{2}\frac{e^2}{\alpha}\times\cdot62$	$\frac{1}{2}\frac{e^2}{\alpha}\times\cdot5$			
Carbon ...	$\frac{1}{2}\frac{e^2}{\alpha}\times\cdot55$	$\frac{1}{2}\frac{e^2}{\alpha}\times\cdot48$	Sodium ...	$\frac{1}{2}\frac{e^2}{\alpha}\times\cdot09$	

Thus the ionizing potential is least for lithium ; it then rises sharply for beryllium, and remains nearly constant for the rest of the period ; there is a great drop in passing to sodium, and again a rapid rise, the ease with which the atoms of the alkali metals are ionized in comparison with those of other elements of the same period is very marked, and accounts for the sensitiveness of the heavier alkali metals to light of long wave-length.

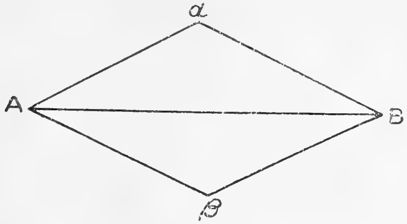
The ionizing potentials in the preceding table relate to atoms and not to molecules.

Formation of Molecules.

The molecules we consider are in the gaseous state, where each molecule is separated so far from its neighbours that it can be regarded as having an individual existence and not merely as forming a brick in a much larger structure. The term molecule when applied to the solid state is quite ambiguous without further definition : for example, from many points of view we can quite legitimately consider the whole of a large crystal as forming a single molecule. We should naturally expect that when the atoms are crowded together as in a solid, when each atom may come under the influence of a large number of neighbours, the arrangement of the electrons relatively to the atom may differ substantially from that in a gaseous molecule.

Let us now take the simplest type of molecule, that of an elementary gas when the two atoms forming the molecule are identical. For the union of two atoms each containing one electron we have the arrangement represented in fig. 5, where the positive charges AB and the electrons α β are at the corners of a parallelogram. The angle of the parallelogram will depend upon the law of force between two positive charges when separated by molecular distances. If $e^2\phi\left(\frac{c}{r}\right)$ is the attraction between unit positive charge and an electron at a distance r, $e^2\psi\left(\frac{c}{r}\right)$ the repulsion between two positive charges at a distance r, then if $A\alpha$ (fig. 5) $=r$ and the angle $A\alpha\beta=\theta$ we have for equilibrium

Fig. 5.



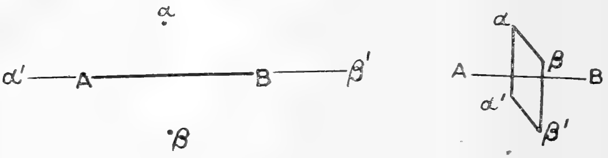
$$2e^2\phi\left(\frac{c}{r}\right)\cos\theta=\frac{e^2}{4r^2\cos^2\theta},$$

$$2e^2\phi\left(\frac{c}{r}\right)\sin\theta=e^2\psi(2r\sin\theta).$$

Thus we may regard the electrons in the atoms as a kind of hook by which one atom gets coupled up with another; this disposition of electrons may be regarded as forming what is called the "bond" by the chemists. Inasmuch as each bond of this kind requires two electrons, the symbol, whether a line or a dot, used by chemists to denote such a bond represents *two* electrons; if, as in my paper "Forces between Atoms and Chemical Affinity," *l. c.*, we use two of these symbols in place of one, the number of symbols will equal the number of electrons concerned in binding the atoms together; as was shown in that paper and will be seen in this, there is much more involved than a mere change in notation.

The union of two bi-electron atoms may be either of the type fig. 6 or 6 *a*. When only one pair of electrons is used

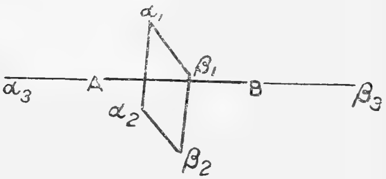
Figs. 6 & 6 *a*.



up in uniting the atoms, this type would be represented by —A—A—; or both the electrons in each atom might be used for coupling the atoms, the four electrons being at the corners of a square in a plane bisecting AA at right angles, fig. 6 *a*. This system of four electrons between the positive charges may be regarded as a double bond between the atoms. Inasmuch as the equilibrium of four electrons in one plane when the displacements are not confined to the plane requires very strong restoring forces to make it stable, we should not expect the double bond to be permanent when the positive charges which exert these forces are as small as in this case when their sum is only equal to the charges on the four electrons.

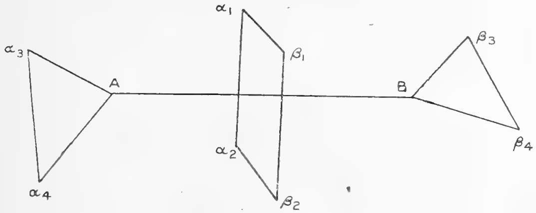
If there are two-trielectron atoms in the molecule, if all the electrons were coupling up the atoms there would be a hexagonal ring of electrons in a plane bisecting AA at right angles; as this ring requires a central positive charge greater than 6 to keep it in stable equilibrium, it would be unstable, and the more probable arrangement of electrons is

Fig. 7.



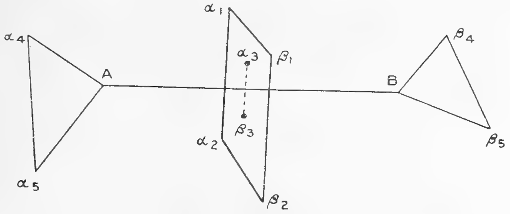
the octahedral one shown in fig. 7, and represented symbolically by —A=A—. With two four-electron atoms in the molecule we have 8 electrons to dispose of, the maximum number which can be in equilibrium on the surface of a single cell. A way in which these might be distributed with 4 electrons between the two atoms is shown in fig. 8, where there is a double bond between the atoms A, B.

Fig. 8.



When we proceed to consider the union of two five-electron atoms, new considerations come in, for in the diatomic molecule we have ten electrons, two more than can be accommodated on a single layer. A simple distribution of these 10 electrons is shown in fig. 9 where we have an outer cell,

Fig. 9.



$\alpha_1\, \alpha_2\, \alpha_4\, \alpha_5\, \beta_1\, \beta_2\, \beta_4\, \beta_5$ of 8 electrons, inside this are the two positive charges and midway between these two electrons $\alpha_3\, \beta_3$ which together with the double bond $\alpha_1\, \beta_1\, \beta_2\, \alpha_2$ help to hold the positive parts together. With two atoms each containing 6 electrons, there are 12 electrons to accommodate in a diatomic molecule. These will be sufficient to surround each positive charge with a cell containing 8 electrons, provided the cells have 4 electrons in common; these electrons coming between the two positive charges will form a double bond tending to bind them together.

When each of the atoms contains 7 electrons, we have 14 electrons at our disposal in the diatomic molecule; these are sufficient to surround each of the atoms with a cell of 8 electrons, provided the cells have two electrons in common, these two electrons forming a single bond. We might expect as the two cells have only two electrons in common, while those for the six-electron atom had four, that the cells of the seven-electron atom would *ceteris paribus* come apart more easily than those of six-electron atoms. Thus we should expect substances with a seven-electron atom like chlorine to be more easily dissociated and more energetic in their action than those like oxygen, whose atom only contains 6 electrons.

We can apply similar considerations to the union of atoms of two dissimilar elements, and we shall take as our text the molecule of carbon monoxide. This substance has excited a good deal of attention, as though from the valency point of view it is highly unsaturated, its physical properties, for example the difficulty with which it is liquefied, indicate than its molecules exercise even less than the normal attraction upon each other. We know from experiments on the Positive Rays that non-permanent molecules can be formed which violate all principles of valency. Thus, when these rays go through such a gas as $COCl_2$ we get evidence of the existence of all kinds of combinations of carbon and chlorine atoms. Where valency comes in is in connexion with the duration of the compound after it has been formed.

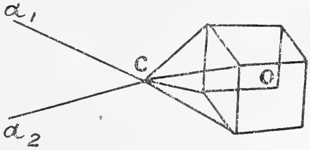
A molecule built up in conformity with valency principles is one which, as we have seen, is not likely to attract or be attracted by other molecules as much as one that is not made up on these lines. The principle of valency depends in fact upon the "survival of the unattractive"; attractive molecules die young. If we can secure this unattractiveness by other than valency conditions there is no reason why the compound should not be as permanent as the orthodox one. Now let us consider the case of CO. Here we have 10 electrons to dispose of, the same as for N_2, and we might expect that a somewhat similar arrangement to that shown in fig. 9 would produce a stable and permanent molecule. As the positive charge on the carbon atom is not the same as that on the oxygen, the cell will be distorted and will not be symmetrical about the plane through the middle point of the line joining C and O. It is interesting to notice that some of the physical properties of CO and N_2 are very similar. Thus, if a and b are the constants for these gases in Van der Waals' equation, b being what is called the atomic volume and a a constant which is connected with the magnitude of the force which one molecule exerts upon another,—then (Kaye and Laby's Tables):

	a.	b.
for CO.........	·00275	·00168
N_2.........	·00259	·00165

Thus we see that a combination of this type, not determined in any way by considerations of valency, might be as stable and as saturated as the combination of atoms in the molecule of an elementary gas. One condition for the existence of compounds of this type is that the sum of the electrons on the two atoms should be greater than 8. Thus we should not expect, what perhaps on the ordinary theory of valency we might expect, that the existence of CO implied the independent existence of CH_2, for this would have only 6 disposable electrons, and as these are insufficient to produce a completely saturated layer, they could not form a completely saturated compound.

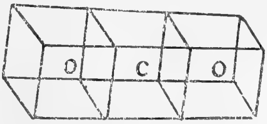
It is important to point out that we distinguish between the molecule of carbon monoxide and that of the carbonyl radicle CO. In the latter we suppose that two out of the four electrons of the carbon atom have gone to unite it with the oxygen and to make up the 8 required to form the cell round the oxygen atom, while the other two are free to join up with other electrons, so that while the molecule of CO is represented by fig. 9 that of the carbonyl radicle is represented by

Fig. 10.



The constitution of CO_2 is I think best regarded as the union of an oxygen atom with the two electrons $\alpha_3\,\alpha_4$ of the carbonyl radicle and having the configuration represented by fig. 11, where the 16 electrons are arranged in 3 cells of 8 round each of the atoms, each cell having 4 electrons in common with its next neighbour, these forming a double bond between each pair of adjacent atoms. To simplify the diagram the cells are represented as cubes.

Fig. 11.



It must be acknowledged that there are some compounds which we might expect from this point of view and of which we have no evidence: for example, the molecule NF contains 12 electrons like the oxygen molecule, and we might expect that a configuration of electrons resembling with some distortion that of O_2 might give a stable molecule with properties somewhat similar to O_2. So far as I am aware, no such compound has ever been detected or suspected. It must be remembered that the molecule N_2 which forms the type for CO is singularly inert: if a substance formed on the model of O_2 were to have more energetic chemical qualities than those of its prototype, it might enter into combination so readily as to escape observation in the free state.

We have, however, in the compounds of nitrogen and oxygen probably other examples of this type of molecule. The molecule of NO contains eleven electrons: if we take eight of these to form the outer cell we are left with three which, by taking up positions at the corners of a triangle in a plane between the atoms of nitrogen and oxygen, may help to keep these atoms together. The molecule of nitrogen monoxide is not the only form in which the combination NO occurs; besides this, there is the radicle NO, and just as we supposed the carbonyl radicle to have a different configuration from that of the molecule, so we suppose that the radicle NO has a configuration where 8 electrons form a cell of 8 round the oxygen, while there is a cell of 7 round the nitrogen, the two cells having 4 electrons in common. Thus the cell round the nitrogen is unsaturated and the combination will act like a seven-electron atom, *i. e.* be univalent.

The three electrons left over from the five nitrogen electrons after two have been used to saturate the cell round the oxygen atom might bind three hydrogen ions, and thus it is possible that under certain circumstances the radicle might act as if its valency were three.

The view we have taken of the structure of molecules is consistent with the existence of many compounds which violate the ordinary valency conditions. According to this view a molecule would be saturated, if the disposable electrons—*i. e.*, those which before combination occupied the outer layers of the atoms which make up the molecule—can be arranged in a number of connected cells, each cell containing 8 electrons. The cells must be numerous enough for the core of each atom which possesses four or more electrons to be placed inside a separate cell. The reason that each cell must have a heavily charged nucleus inside it is to ensure stability. If the cell stood alone it would require a positive charge of about seven units to ensure stability; when it forms one of a group and there are positive charges outside it, it will not require so many inside to make it stable: the assumption that four will be sufficient is an arbitrary one, which fits in with chemical facts. It is clear that there must be a lower limit, though its calculation would be long and tedious. A simple way of determining the number of disposal electrons is by the rule that it is equal to the sum of the valencies of the atoms in the molecule: we must be careful, however, when we apply this rule, to count the valency of the halogens as seven and not one, that of oxygen and sulphur as six and not two, and that of nitrogen and phosphorus as five and not three.

Perhaps the clearest way of seeing the differences between the consequences of the view of combination we are discussing and those of the ordinary theory of valency, is to consider some special cases.

Let us first take the case when there is only one cell: then, to get the eight electrons required to complete this, a four-electron atom must combine either with an atom with four electrons, or preferably with a number of less highly charged atoms, the sum of the charges on these atoms being equal to four. A five-electron atom must combine either with a three-electron atom, or singly or doubly charged atoms containing altogether three electrons; the six-electron atom must combine either with one two-electron atom or with two one-electron atoms; the seven-electron atom must combine with one one-electron atom; while the eight-electron atom could not combine at all. These results are identical with those we should arrive at from the ordinary theory of valency, provided we ascribed to the 8, 7, 6, 5, 4 electron atoms valencies 0, 1, 2, 3, 4 respectively.

Let us now, however, consider the case when we have two of these 8-electron cells connected together: these will require 12 electrons if the two cells have four electrons in common, 13 electrons if they have three in common, and 14 if they have two in common. We do not consider the case when they have only one electron in common, as it seems probable that this connexion would be too slight to keep the two cells together.

Thus if we have 12, 13, or 14 electrons at our disposal, and only two highly charged atoms, we can make up a molecule which will fulfil the conditions of saturation. Let us consider the combinations possible for a four-electron atom. We shall, for clearness, always in what follows represent 4, 5, 6, 7, 8 electron atoms by the symbols C, N, O, F, Ne respectively, while we shall represent the one-electron atom by H. Consider the combinations in which the C, N, O ... atoms form part of a molecule in which there are two cells together containing 12 electrons. The possible combinations are given in the following table :—

Two cells—quadruple connexion.

C.Ne, C.F.H, *COH_2, *CNH_3, *C_2H_4, *N_2H_2, *NOH, NF, *O_2.

When there are 13 electrons in the two cells we have the following results :—

Two cell—triple connexion.

C.Ne.H, C.F.H_2, COH_3, CNH_4, C_2H_5, N.Ne, N.F.H, NOH_2, N_2H_3, OF, O_2H.

When there are 14 electrons in the two cells we have the following results :—

Two cells—double connexion.

$CNeH_2$, *C.F.H_3, *COH_4, *CNH_5, *C_2H_6, N.NeH, *NFH_2, *NOH_3, *N_2H_4, OF.H, *O_2H_2, *F_2, ONe.

Of these only those marked with an asterisk fulfil the conditions required for the ordinary theory of valency. The theory does not assert that all these compounds can be formed: for example, we have included compounds containing the 8-electron atoms, these are the atoms of the inert elements. Owing to the very small attraction which such atoms exert on other atoms, it is not at all likely that such compounds can be prepared ; or, again, some of the compounds might break up with great rapidity into still more stable forms. All that the theory involves is that such molecules would be saturated and would be stable if they were subject only to small disturbances.

It will be noticed that practically all the compounds where the two cells have two electrons in common, *i.e.* are united along an edge, fulfil the condition of being in accordance with valency principles ; that for those where the cells have four electrons in common a considerable number are in accordance; while none of those which have three electrons in common satisfy valency conditions. We can see without much difficulty that contact along an edge of a cell is analogous to what is described on the valency theory as union by a single bond; while the fourfold connexion, *i.e.* contact along two lines, is analogous to the state known as union by a double bond.

If we use the modified notation previously described, the number of bonds between two atoms is equal to the number of electrons their two cells have in common.

There is nothing in the ordinary valency theory analogous to the cells having three electrons in common, though there is on the modification I gave of that theory in the paper on chemical combination already referred to.

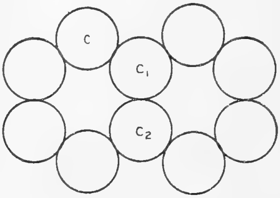
The three-electron contact has to be invoked (if we suppose the one-electron contact to be too fragile) to account for the existence of compounds like ClO_2, which are inexplicable on the theory of valency if chlorine has any odd valency. On the theory we are discussing there are 3 cells in this compound, and these have to be formed from 19 electrons: this could be done by making the cell surrounding one of the oxygen atoms have a double contact with the other oxygen cell, while one of these oxygen cells has a threefold contact with the chlorine cell.

Another example where the threefold contact might come in is the well-worn one of the benzene ring. In benzene C_6H_6 we have on our theory to make up 6 cells and we have 30 electrons with which to do it. The simplest and most symmetrical way of doing this is to have the six cells in contact round a ring with threefold contact between each two. As two opposite triangular faces of the twisted cell represented in fig. 1 are inclined to each other, this could be done without introducing much strain in the system. With this arrangement we have complete symmetry, and it is analogous to the Armstrong and Baeyer or central theory of the benzene ring. The analogue on our theory to Kekulé's conception of the constitution of the benzene ring would consist of three sets of pairs of cells, the cells in one pair having fourfold contact with each other, but only double contact with a cell in a neighbouring pair. It has often been pointed out that it is difficult to explain on this view why we do not get, when we replace two atoms of hydrogen by two atoms of chlorine, more isomers than have been observed.

Let us consider from the point of view of this theory the changes which must take place in the disposition of the cells if one of the carbon atoms takes up another atom in addition to the hydrogen atom already attached to it. Let us suppose that this is another atom of hydrogen. This atom will introduce another electron in the system which will have to be accommodated in the eight cells. To do this the contact between one or more of the cells must be altered; to find room for the new electron one of the three electron contacts must be reduced to a line contact, *i.e.* two of the cells must have only an edge in common instead of the triangular face which formed their interface before the new electron was introduced. But if one cell moves so that it opens out the contact with the cell on its left from a three- to a two-electron contact, it will also alter its contact with the cell on the right, unless this moves also, and reduce this also to a two-electron contact; hence the system will be unsaturated unless another electron is introduced to fill up the gap caused by the loss of a contact. To supply this electron another atom of hydrogen must be introduced. Hence we see that the addition of the hydrogen atoms must occur in pairs, and unless there is a movement of more than one cell throughout the chain these pairs must be on adjacent carbon atoms; if one half the ring were to move as a rigid body past the other half, the pairs would be on opposite carbon atoms.

These conclusions are borne out by the study of the additive compounds of benzene.

Fig. 12.



When there are two rings, as in naphthaline $C_{10}H_8$, there will be 10 cells and 48 electrons; this would correspond to an arrangement like that in fig. 12, where the two cells round the two central carbons have two-electron contact, while the ten contacts between the outer cells are three-electron ones.

The view that the electrons in a molecule are arranged in sets of eight forms the basis of the very interesting papers on chemical combination recently published by Professor Lewis and Mr. Langmuir, their view as to the origin of these sets differs from that given in this paper.

We shall now turn from the chemical side of the theory to the physical one, and consider how far its consequences are in accordance with our knowledge of the physical properties of atoms and molecules. As positive ray analysis is the most powerful method for the study of individual atoms and molecules, we shall consider from the point of view of the theory the evidence afforded by this method.

Let us begin by considering the positively electrified atom. Such an atom is one that has lost one or more of its electrons: but diminishing the number of the electrons round the central charge will increase the stability of those that are left; hence, if we could produce them, atoms which had lost one or more charges would be stable and might be expected to be in evidence in the positive rays. The magnitude of the positive charge cannot be greater than that produced by removing all the electrons from the outer ring; hence we could not have a positive charge of more than 8 units, which is the maximum number of electrons in the outer ring: as a matter of fact, 8 is the greatest charge yet observed on any atom in the positive rays. Again, hydrogen has never been observed with more than a single positive charge, which is in agreement with the view that its atom contains but one electron. Another way in which the theory might be tested would be to study the positive rays for the vapours of the alkali metals and the alkaline earths: these on our theory contain respectively one and two electrons in the outer ring; hence the vapour of an alkali metal ought not to be able to acquire a double charge, nor that of an alkaline earth a triple one. It is difficult to get these vapours in the positive rays in the ordinary way, but it is hoped that the application of the positive ray methods to anode rays will result in a thorough study of the properties of the atoms of metallic vapours. One result we may deduce already from our observations on the positive rays, and that is that though chlorine is regarded as a monovalent element it contains more than one electron in its outer layer, for chlorine atoms with a double positive charge occur whenever the positive rays pass through chlorine. As chlorine on our view has 7 electrons in its outer layer, it would not be surprising to find chlorine atoms with three or four positive charges.

Let us now turn to the case of positively electrified *molecules.* These with one positive charge are to be found in almost every positive ray experiment, though I am not prepared to say that every molecule can survive the removal of an electron and acquire a positive charge without dissociating into atoms.

As an example of the way in which the removal of an electron might dissociate a molecule into atoms, we may take the case of a molecule of chlorine in which the two cells are supposed to have only two electrons in common; if one of these two were removed, the two cells would be connected only by a single electron which might quite likely be insufficient to hold them together and the molecule would dissociate. This, however, would only happen if the electron removed were one of two special electrons; if any other of the 14 electrons were removed the connexion between the cells would be unimpaired and the molecule, though unsaturated, would be stable: as a matter of fact, positively electrified chlorine molecules are found in the positive rays.

It is when we consider systems with more than one positive charge that the difference between atoms and molecules becomes most apparent; for while the parabolas corresponding to doubly charged atoms are to be found on nearly every positive ray photograph, those corresponding to doubly charged molecules though not unknown are rare. If the molecules were those of elementary gases, the parabola of the molecule with the double charge would coincide with that of the atom with a single charge, and so might escape notice; this, however, would not apply to molecules of compound gases, and even in the case of elementary gases the existence of doubly charged molecules would modify the appearance of the parabola due to the singly charged molecule in a way that would lead to their detection. This very striking difference between atoms and molecules is, I think, due to the fact that it is the electrons which hold together the atoms in a molecule; they are, in fact, structural, and in general, when more than one of them is removed, the structure is weakened to such an extent that the molecule splits up and ceases to be a molecule.

The mechanism by which the double charges are produced may be one which is much more likely to produce doubly charged atoms than doubly charged molecules. If the double charge was due to the system losing one electron by one collision and a second one by a subsequent collision, we should not expect to find the marked discrepancy between the numbers of atoms and molecules with double charges. There is, however, ample evidence from the positive rays that the atoms acquire their double charges by a single operation, and not in this way. This operation may well be the breaking up of a molecule containing the atoms. Thus, for example, if from a molecule of oxygen an octet were to break away it would leave behind it an atom of oxygen and four electrons; as the oxygen atom carrries 6 positive charges, the system would have a double charge. Thus if the double charge, as seems probable, is due to the breaking up of molecules, we should expect to find the atoms rather than the molecules in possession of these charges. If one of the products of dissociation of a complex molecule were a simpler molecule, we might get this with a double charge. Thus if a molecule of CO_2, for example, were to dissociate by an octet round one of the oxygen atoms breaking away from the system, the atoms C and O with 8 electrons would be left behind ; as the positive charges on the atoms amount to 10 units, the molecule of CO formed in this way would have a double charge. As a matter of fact, CO is one of the few doubly charged molecules I have found in the positive rays.

Let us now turn to the negatively electrified atoms. These have had an extra electron added to the outer layer ; but if a cell of eight is the maximum number consistent with stability, it is clear that an atom such as one of neon or argon, which already contains 8 in the outer layer, is not in a condition to receive another electron and so cannot be negatively charged. This is borne out by observations on the positive rays, for we never find the atoms of these elements occurring with a negative charge. Again, we have supposed that in molecules, when the number of electrons is sufficient, the electrons are arranged in a series of cells of 8 ; since each cell is a saturated system, there is no place for an electron to find a resting-place and so no possibility of the molecule acquiring a negative charge. We find this borne out by the positive rays : a negatively electrified molecule is exceptional, though there are cases like those of oxygen and unsaturated hydrocarbons where the molecule can acquire a negative charge.

The molecules which can receive a negative charge are, I think, those consisting of a pair or more of cells which have more than two electrons in common. Let us take, for example, the molecule of oxygen: we have regarded this as consisting of two cells with four electrons in common ; as long as this contact is intact, neither cell can receive an electron without losing its stability by containing more than 8 electrons. Suppose, now, that the contact were to open out so that the cells had only two electrons in common ; there would now be room on each cell for an electron without the number on either cell exceeding 8, so that this molecule might receive one or even two negative charges. Compare this case with that of a molecule of chlorine, where the two cells have only two electrons in common : it cannot receive an electron while the contact is intact, and any loosening of the contact would lead to a separation of the atoms; hence we should anticipate that while a molecule of oxygen could acquire a negative charge, one of chlorine could not. A molecule in which none of the cells have more than two electrons in common is one which, if its structure were interpreted in terms of the ordinary theory of valency, would contain no double bonds, *i.e.* it would be a saturated compound; such a molecule would on this theory be incapable of receiving a negative charge, whereas if it contain double or triple bonds some of the cells would have more than two electrons in common, and by opening out this contact might be able to accommodate more electrons and thus receive a negative charge.

The negative charge on the molecule indicates that the contact or, in the usual terminology, the double bond has been loosened; so that it would seem possible that an effective way of loosening this bond, *i.e.* turning unsaturated into saturated compounds, would be to give the molecules a negative charge by exposing them to a stream of electrons. This may play a part in the Sabatier-Senderens method of reducing unsaturated compounds by passing them over finely divided metals at a high temperature. The hot metal is well known to be a source of electrons.

Let us consider from this point of view the formation of negative ions in another case—that of a gas ionized by Röntgen rays. We may suppose that the first effect of these rays is to eject electrons from the molecules, so that initially the negative ions are electrons. They will remain electrons, and so have much greater mobility than the positive ions, unless they can attach themselves to atoms or molecules. But if the gas, like neon or argon, consists of atoms with 8 electrons in the outer layer there is no room for the electron on the atom, and if it collides with the atom it will rebound and remain a free electron—the mobility of the carriers of negative electricity will be that of an electron, and will far transcend that of the positive ion. Franck and Hertz long ago called attention to the great mobility of the negative ion in argon. Next suppose that the gas is not monatomic, but that in the molecule there is only a single octet of electrons, as in N_2 and CO; in this case, again, there is no room for an electron, and we should expect the electron to remain free and have a high mobility. Franck and Hertz have shown that this is true for nitrogen. Suppose, however, that the molecule, like those of oxygen or chlorine, contains two octets with some electrons in common; then, by opening up the contacts the molecule could accommodate more electrons, so that in this case the electron could attach itself to the molecule and thereby make its mobility comparable with that of the positive ion. With oxygen, where the octets have four electrons in common, the opening of the contacts might occur without the separation of the atoms. In chlorine, however, where the octets have only two electrons in common, the opening of the contact would probably result in the disruption of the molecule into a negatively electrified chlorine atom and a neutral one, so that the electron would be attached to a chlorine atom and not to a chlorine molecule. Similar considerations will apply to more complicated molecules, and we are led to the conclusion that when the molecule of the gas contains two or more octets having electrons in common, the electron may be caught by the gas and its mobility reduced. If the molecule contains the "double bonds" of the chemists, *i.e.* if two or more octets have 4 electrons in common, the electron will attach itself to the molecule of the gas. If there are only single bonds in the molecule the molecule will be dissociated by the electron and the electron will be attached to one of the products of dissociation.

If, as in the water molecule, though there is only one octet there are a number of positively charged atoms outside it, the electron might attach itself to one of these atoms without destroying the equilibrium of the cell.

It is interesting to find that, as shown by observations on positive rays, radicles such as OH, CH_2, CH_3, which are highly unsaturated molecules, readily acquire negative charges.

The ability of an atom to receive a negative charge depends on its positive core being able to hold in stable equilibrium one more electron than is found in the outer layer of the neutral atom: thus the existence of the negative hydrogen atom shows that a single positive charge can hold two electrons in stable equilibrium; the negative carbon atom shows that a positive charge 4 can hold 5 electrons in stable equilibrium; the negative oxygen atom that 7 electrons can be held by a charge 6; and the negative chlorine atom that 8 can be held by a charge 7. The absence of a negatively electrified nitrogen atom is remarkable, since the atoms of the elements on either side of it readily acquire a negative charge. The reason may be that, though the nitrogen atom could hold a negative charge if it could get it, it is not able to snatch one from the molecules of the gas through which it is passing.

When the number of electrons in a molecule is less than eight, as it would be in such compounds as NaH, BeH_2, BH_3, the molecules might be expected, if they could be obtained in the gaseous state, to behave something like an atom having on its outer layer the same number of electrons as are present in the molecule. These molecules might therefore be expected to be able to acquire a positive charge.

It is worthy of notice that, with the exception of hydrogen and helium, no elements whose diatomic molecule contains less than 8 electrons is gaseous at ordinary temperatures. We have therefore no data by which we can compare the values of the physical constants for molecules of this type with molecules where the electrons are numerous enough to make up a series of saturated cells.

We can, however, compare the properties of a gas like nitrogen, whose molecule contains only one cell, with that of the molecule of oxygen which is made up of two cells. It is, perhaps, a little surprising that what is called in the Kinetic Theory of Gases the radius of the molecule is actually less for the two-celled oxygen molecule than for the single-celled nitrogen one. We must remember, however, that unless we regard the atoms and molecules as hard impenetrable solids, the size of the atom depends upon other than geometrical considerations; it depends essentially upon the range of the forces exerted by the electrons and positive charges in the molecule. Thus a system consisting of a uniform distribution of negative electricity over the surface of a sphere and an equal positive charge at the centre would produce no effect outside the sphere; an atom of this character would have a long free path and behave like a small atom; while an atom consisting of two equal point-charges, one positive and the other negative, would have a large stray field and might behave like a much larger atom than the spherical one, even though the distance between the point-charges were less than the radius of the sphere. The nitrogen molecule, containing 10 electrons, has two of these inside the cell; this will introduce a want of symmetry and increase the stray field, and thereby the apparent size of the molecule. But if there is not a break in the size of the molecule as we pass from nitrogen to oxygen, there is one in another property, which, as it depends essentially on the configuration of the electrons, is, I think, very suggestive. I allude to the effect of the molecule on the scattering of polarized light investigated by Lord Rayleigh (Proc. Roy. Soc. A. xcviii. p. 57). The light scattered by a single electron in a direction at right angles to the incident beam is completely polarized, and can be extinguished by a Nicol prism. The same thing is true when the light is scattered by a perfectly symmetrical body such as a sphere; if the scattering body is not perfectly symmetrical—if, for example, it is ellipsoidal instead of spherical—the scattered light is never completely polarized, and therefore cannot be completely quenched by a nicol. The ratio of the minimum to the maximum intensity of the light as seen through a nicol, which would be zero for a sphere, would increase with the ellipticity of the ellipsoid, and may be taken as an indication of the deviation of the scattering body from sphericity.

For the scattering of light by the electrons in the molecules of different gases, we should expect the arrangement of the electrons in one shell to approach nearer to the spherical symmetrical form than an arrangement in two, and that two cells would approximate to the symmetrical more closely than three; so that the ratio of the minimum to the maximum intensity of the light would increase with the number of cells. The following are Lord Rayleigh's determinations of this ratio for several gases :—

Gas.	Ratio expressed as a percentage.
Argon	·46
Hydrogen	3·83
Nitrogen	4·06
Oxygen	9·4
Carbon dioxide . .	11·7
Nitrous oxide . . .	15·4

We see how small this ratio is in the case of the symmetrical distribution of electrons round the argon atom, but the most striking feature of the table is the great jump between nitrogen and oxygen ; this is just what we should expect on our view of the constitution of these molecules, as the oxygen molecule contains two cells while the nitrogen only has one. The carbon dioxide and nitrous oxide molecules are 3-celled molecules, and we should expect them to have a higher value than the 2-celled molecule oxygen.

The measurement of this ratio promises to be a very valuable aid in determining the configuration of the molecule : it would, for example, be very interesting to know what is the effect of replacing one or more of the hydrogen atoms in CH_4 by halogens, and to compare its value for CH_4, C_2H_2, C_2H_4, and C_2H_6 ; the first and second of these are one-celled molecules, while the third and fourth have two cells.

In one sense even the octet is not fully saturated, for though it cannot receive any more electrons its own electrons may serve, as it were, as party walls against which the electrons round other atoms may fill up their gaps and become octets ; thus, for example, one of the electrons in an octet might complete the tale of 8 for a 7-electron chlorine atom. This might come about as follows :—We know from experiments on positive rays that the chlorine atom when moving through a gas seizes and retains an electron torn from one of the molecules of the gas. In the positive rays the chlorine atoms are moving at a high speed, and have therefore great energy ; this enables them to drag the electrons they grip out of the molecules. If, however, they have less than a critical amount of energy they will not be able to tear off the electron ; this will grip them, and the chlorine atom will become the prisoner of the molecule.

Any of the electrons in any of the octets might act in this way to fill the gaps in the layer of electrons round an atom. Thus any octet might act like a nucleus from which chains and side-chains of atoms ramified in every direction. As far as geometrical considerations are concerned, there is nothing to limit the number of certain kinds of atoms which could be linked together in this way. It is probable that systems built up in this way are too weak to have any but the most transient existence; but the possibility of their formation to a limited extent in certain cases ought not to be lost sight of.

XLVI. *On the Resistance of Solutions of Copper Sulphate in Glycerine.* By A. GILMOUR, *M.Sc.*, 1851 *Exhibition Scholar, Queen's University, Belfast* [*].

IN many laboratories a high resistance of very small inductance and capacity is becoming very desirable. In the physical laboratory of Queen's University, Belfast, a mixture of glycerine and copper sulphate has been used for some years for high-resistance potentiometer and other high-resistance work. During the past year variable resistances of very large value and negligible inductance and capacity were required in connexion with some work undertaken on valves [†], and, on investigation, glycerine-copper sulphate solution proved very suitable.

Though glycerine has been known for a long time as a solvent of very wide range—at least as extensive as that of water—very little work has been done on the resistance of glycerine solutions prior to that of Jones and his collaborators [‡]. They investigated the molecular conductivity of a large number of salts dissolved in glycerine and in mixtures of glycerine and water, but make no mention of glycerine-copper sulphate mixtures. In this paper the specific resistances of various mixtures of these substances are dealt with for convenience in laboratory work.

Method of Measurement.

The cell used to measure the resistance was an inverted U-tube *a* (fig. 1) into the horizontal part of which a glass tube with a bulb *b* was sealed; the latter acted as a reservoir for any extra solution drawn up by the rubber bulb *c*, by compressing and releasing which the U-tube was filled. The ends of the U-tube were encased by spirals of copper wire *e e*, which acted as electrodes and dipped well under

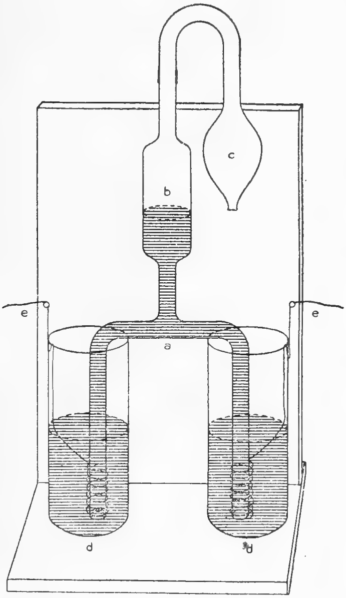
[*] Communicated by the Author.

[†] Beatty and Gilmour, Phil. Mag. Sept. 1920.

[‡] Jones and collaborators, Carnegie Inst. of Washington, No. 180, 1913.

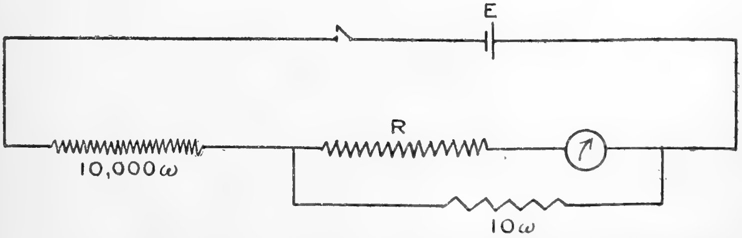
the surface of the liquid in the test-tubes *d d*. The diameter of these test-tubes was large compared with that of the tube *a*, so that any casual displacement of the ends of the tube *a* had no effect on the resistance measured. This was verified experimentally as was also the fact that the height of the liquid in *b* had no effect on the resistance.

Fig. 1.



For the higher values of the resistance the cell was connected in series with a battery and a Broca galvanometer and the resistance calculated from the deflexion and the voltage applied. For a few of the lower values the galvanometer was too sensitive. A resistance of 10 ohms was put in parallel with the unknown resistance R and the galvanometer. When a resistance of 10,000 ohms was put in series in the circuit (fig. 2) the galvanometer deflexions were

Fig. 2.



within the scale of the instrument, and the resistance R could again be calculated, knowing the deflexion and voltage applied. The sensitivity of the galvanometer was about $1\cdot3\times10^{-8}$ amp. per mm. scale-division, and was tested at the beginning and end of each set of readings to ensure that it remained constant during the experiment.

To get the specific resistance factor of the cell, the latter was filled with clean mercury and the resistance measured. The specific resistance of mercury divided by this measured resistance gives the factor of multiplication which transforms the actual resistance of any liquid in the cell into the specific resistance in ohms per c.c.

Solutions.

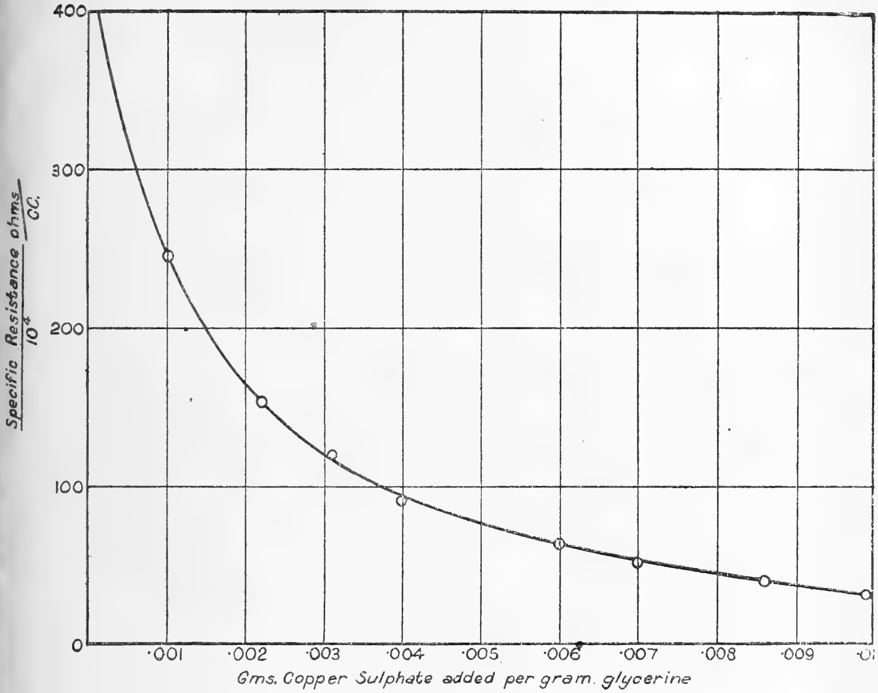
Owing to the difficulty of obtaining drainage of the glycerine from burettes and measuring-vessels, the composition of the solution is expressed in grams of copper sulphate per gram of glycerine. Blue crystallized copper sulphate $CuSO_4, 5H_2O$ was used. The required weight of it was added to a known weight of glycerine, and the whole heated slightly until the copper sulphate dissolved. The resistances were measured while the solutions were fresh.

Results.

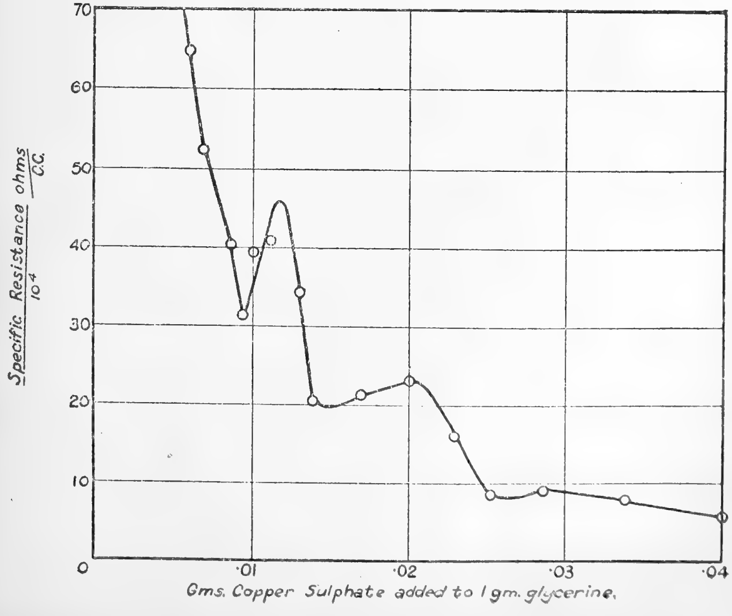
The results are given in the following table and are represented in Curves I, II, III. It will be seen that when the concentration of the copper sulphate (as measured by the scale of numbers used as abscissæ) lies between ·01 and ·03, maxima and minima occur in the curve. From the positions of the points in the vicinity of the first maximum it seems possible that subsidiary peaks are there present, but no close investigation has been made of these, as this

Gms. $CuSO_4$ added to 1 gm. glycerine.	$\frac{\text{Sp. Res.}}{10^4}$ in ohms/cc.	Gms. $CuSO_4$ added to 1 gm. glycerine.	$\frac{\text{Sp. Res.}}{10^4}$ in ohms/cc.	Gms. $CuSO_4$ added to 1 gm. glycerine.	$\frac{\text{Sp. Res.}}{10^4}$ in ohms/cc.
0	441	·0102	39·4	·0338	7·91
·0010	245	·0110	41·0	·0400	5·93
·0022	153·7	·0130	34·4	·0474	5·52
·0033	119·0	·0140	20·5	·0691	4·70
·0040	91·9	·0170	21·2	·1004	3·40
·0060	64·8	·0200	23·0	·1291	2·50
·0070	52·5	·0230	16·2	·1683	1·36
·0086	40·1	·0254	8·78	·2008	1·20
·0095	31·5	·0285	9·05	·2584	1·08

CURVE I.

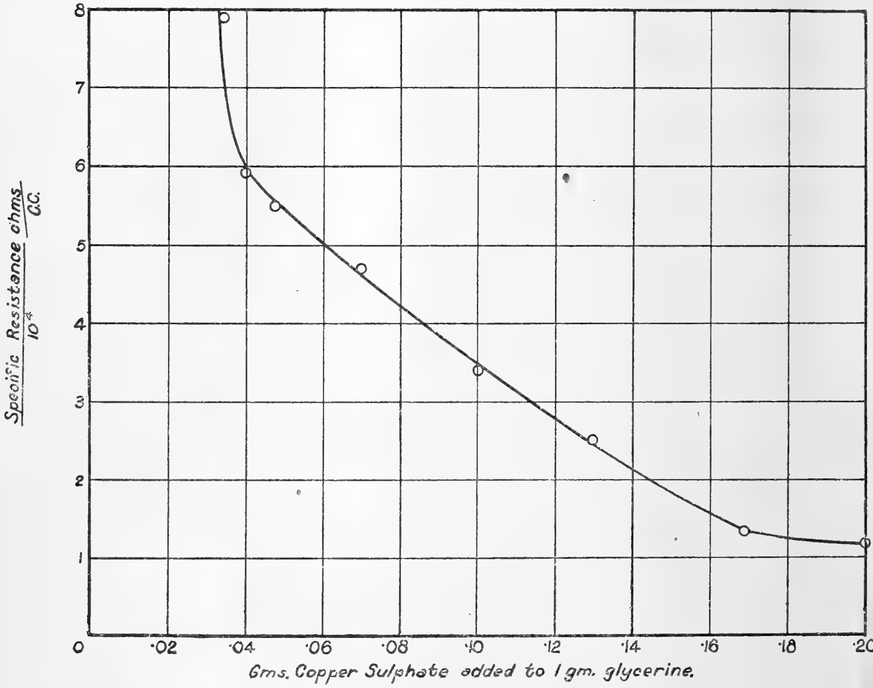


CURVE II.



portion of the curve is evidently one to be avoided by any person desiring to make up a solution to give a definite resistance. The peaks correspond, in a general way, to the discontinuities in the curves for solutions of other salts got by Jones and his collaborators *, and attributed by them to the differences in association and dissociation of the molecules when the amount of the salts and water added to the glycerine is varied.

Curve III.



Glycerine solutions are hygroscopic and their resistances, especially the higher values, vary slightly as they absorb water. Accordingly, it is desirable to seal in the electrodes when a fixed resistance is required; while in the case of variable resistances, the rod carrying the movable electrode may be made to slide in a hole in a rubber cork which forms a fairly air-tight joint. The solutions have been found to be very free from polarization, and the curves show that a very wide range of resistance is obtainable.

In conclusion, the writer wishes to thank Dr. Beatty, who suggested this research, for his continued interest and advice during its progress.

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* *Loc. cit.*

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[SIXTH SERIES.]

APRIL 1921.

XLVII. *On the Supposed Weight and Ultimate Fate of Radiation.* *By* Sir OLIVER LODGE *.

IN regions where our ignorance is great, occasional guesses are permissible. Some guesses occur in this paper: let an apology for them be understood.

If light is subject to gravity, if in any real sense light has weight, it is natural to trace the consequences of such a fact. One of these consequences would be that a sufficiently massive and concentrated body would be able to retain light and prevent its escaping. And the body need not be a single mass or sun, it might be a stellar system of exceedingly porous character so that light could penetrate freely into the interior and be subject to the combined gravitative attraction of all the constituent masses.

Given a material universe of any shape, bounded by a surface S, with an aggregate mass M distributed anyhow inside it, the average intensity of gravity, F, at the surface is given by Green's theorem as $-FS=4\pi M$. A large enough mass, not spinning too rapidly, tends to be spherical; so, if the average density of the distributed matter is ρ and the radius is R, $F=-\frac{4}{3}\pi\rho R$.

If the distribution is taken as uniform, for the sake of an example, that is with equably distributed masses and great

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* Communicated by the Author; being the substance of an Address to the Students' Math. & Phys. Soc. of the University of Birmingham.

interspaces so as to be extremely porous,—for a heterogeneous mass is more not less effective than a homogeneous one,—the average force at any point in the interior of the system at a distance r from its centre of gravity, introducing the Newtonian constant to make the specification complete, is

$$f = -\tfrac{4}{3}\pi\rho G r.$$

And the potential there is

$$V = 2\pi\rho(R^2 - \tfrac{1}{3}r^2)G.$$

The speed of anything amenable to gravitation would be a maximum near the centre of such a system and a minimum near or outside the periphery; consequently light unable to escape would accumulate near the boundary, and if liberated by an expansion or other catastrophe happening to the system, such as might occur through a gradual growth of instability, would burst forth in a blaze. Such a blaze, rapidly rising in intensity, would die down gradually during the time that the deeper seated portions of the luminous shell took to rise to the surface.

The speed which a heavy body could acquire by falling
from periphery to centre is $\sqrt{(gR)}$,
by falling from infinity to surface $\sqrt{(2gR)}$,
and by falling from infinity to centre $\sqrt{(3gR)}$,

g being the maximum gravitational intensity $\tfrac{4}{3}\pi\rho RG$, where G is the Newtonian constant whose value in c.g.s. units is 666×10^{-10} and whose dimensions are the square of a velocity divided by a linear density, or $M^{-1} L^{+3} T^{-2}$.

This means that a body able to prevent light emanating from centre from escaping altogether would have

$$\rho_1 R_1^2 = \frac{c^2}{4\pi G}.$$

To prevent light from centre from reaching the surface would need

$$\rho_3 R_3^2 = \frac{3c^2}{4\pi G}$$

while to control light from surface into an orbit, the intermediate value

$$\rho_2 R_2^2 = \frac{3c^2}{8\pi G}$$

would suffice.

The estimates are all of the same order; so, taking the intermediate value, we find that a system able to control and retain its light must have a density and size comparable to

$$\rho R^2 = \frac{27 \times 10^{20}}{25 \times 6{\cdot}66 \times 10^{-8}} = 1{\cdot}6 \times 10^{27} \text{ c.g.s.}$$

It is hardly feasible for any single mass to satisfy this condition; either the density or the size is too enormous:—

A globe as big as the earth must have a density 4×10^9 grammes per c.c.; as big as the sun a sufficient density is 4×10^5 grammes per c.c. If of the density of water its radius must be 4×10^{13} centimetres, or 600 times the linear dimension of the sun, so that the mass would be excessive—about 10^{41} grammes.

For a body of density 10^{12},—which must be the maximum possible density, as its particles would be then all jammed together,—the radius need only be 400 kilometres. This is the size of the most consolidated body. For anything smaller than that the effect would be impossible.

If a mass like that of the sun ($2{\cdot}2 \times 10^{33}$ grammes) could be concentrated into a globe about 3 kilometres in radius, such a globe would have the properties above referred to; but concentration to that extent is beyond the range of rational attention. The earth would have to be still more squeezed, into a globe 1 centimetre in diameter.

But a stellar system—say a super spiral nebula—of aggregate mass equal to 10^{16} suns, say 10^{49} grammes, might have a group radius of 300 parsecs, or 10^{21} centims., with a corresponding average density of 10^{-15} c.g.s., without much light being able to escape from it. This really does not seem an utterly impossible concentration of matter. Though it is true that the average empty space enclosing each "sun" would have a radius of only about 22 times the distance of Neptune; and the suns would subtend nearly one and a half seconds of arc as seen from each other.

Note on Units.

The constant G/c^2 is the reciprocal of a linear density, and is $666 \times 10^{-10} \div 9 \times 10^{20} = 74 \times 10^{-30}$ centim. per gramme. The quantity Gm/c^2 is accordingly of the dimensions of a length, and is commonly spoken of by relativists as a mass of so many kilometres. In this sense the sun's mass is said to be 1·5 kilometres, while the earth's mass is only half a centimetre.

Inattention to dimensions, and consequent incomplete specification, often seems to save arithmetic, but it can hardly be conducive to philosophic thinking save in exceptionally skilled hands. It is legitimate to employ $\frac{1}{3} \times 10^{-10}$ second as a unit of time, if worth while, so as to submerge the velocity of light by calling it 1,—really 1 centim. per unit. (For of course it retains its dimensions and is not merely a number—neither 1 nor any other number—though it runs the risk of being so treated, or even forgotten altogether, in spite of its importance; which is so great that it is one of the few absolute things that relativists have allowed themselves to retain.) The abolition of gravity makes the convention easy that G shall equal 1 too, and then mass becomes a length, in so far as length survives, and is also identified with energy. In fact the units to which we are accustomed get purposely muddled, without really proving any kind of identity such as is evidently hoped for by some writers.

If light has weight in any real sense it is tempting to treat it as a gas of low molecular weight and great velocity, for since its pressure is equal to the energy per c.c., $p=\frac{1}{2}\rho c^2$, it obeys Boyle's law like a gas of uniform very high temperature, the $\frac{1}{2}c^2$ taking the place of $\frac{1}{3}u^2$ appropriate to the mean square speed of molecular motions; for the 3 in gas-theory has reference to the three dimensions of space, which are not applicable to the one- or two-dimensional motion of light.

The density of solar radiation at the earth's distance can be estimated from the measured pressure as of the order 10^{-25} gramme per c.c.

At any other place, if the gravitational potential is V and the gas constant $p/\rho=k$, the density should be obtainable from

$$\rho=\mathrm{A}e^{-\frac{\mathrm{V}}{k}}$$

if it is like one of the atmospheric gases.

Now none of what we have been reckoning feels as if it were at all likely to happen. It is not likely that the speed of light escaping from say a spiral nebula and managing to reach the earth has been so much retarded *en route* that it arrives with a speed below the normal. It is unlikely that a ray travels quicker as it approaches a gravitative mass. What has been established by observation is that passing rays are deflected towards such a mass, the effect of gravitation being therefore to reduce the speed below the normal, never to increase it.

For the deflexion is as if space, or rather ether, had the refractive index $1+2\mathrm{V}/c^2$ in regions of gravitational potential V; and, if that is a right way of expressing the fact, light must travel slower as it approaches a mass, not at all as if it had weight but as if it were entering a denser medium. The retarding influence of matter is apparently felt before the substance is actually entered, as if matter could exert an optical effect by its very neighbourhood—as it undoubtedly exerts a gravitational effect.

Hence it appears erroneous to speak of light as having weight, or to treat it as a substance to which Newtonian considerations can be applied. It must be deflected optically, not mechanically. The ether is affected by the gravitational potential--the tension set up in it by a mass—so that its refractive index $\sqrt{(\mu\kappa)}$ is increased. The property corresponding to rigidity, $2\pi/\kappa$, is probably reduced by the gravitational tension or reduction of ether pressure, GM/r, caused by the neighbourhood of a mass of matter (*cf.* my paper in Phil. Mag. for February 1920, page 172).

Trapping of Light by Relativity Method.

It seems worth while to consider further the optical behaviour of a very extensive and concentrated stellar system, using the Einstein method of calculation and avoiding the conceptions of ordinary weight as applicable to light.

It is well known that a gravitational influence on a beam of light originated in the theory of relativity, which blossomed into equations summarising in striking fashion this and much other information; and it may be interesting to physicists who have not specially attended to relativity to show, in elementary fashion, how that theory would treat the subject.

A vital Relativity equation—first given I understand by Schwarzschild (see Eddington's *Report*, pp. 43–47)—is the expression for the square of a small interval between two point-events, which in ordinary geometry is merely the distance between two neighbouring points, or in polar coordinates

$$ds^2 = dr^2 + r^2 d\theta^2 + r^2 \sin^2\theta\, d\phi^2.$$

Minkowski introduced time, and Einstein introduced gravitation, so that it became

$$ds^2 = 1/\gamma\,.\,dr^2 + r^2 d\theta^2 + r^2 \sin^2\theta\, d\phi^2 - \gamma c^2 dt^2. \quad . \quad (1)$$

where time, as imaginary space, contributes to the interval, and where γ represents a numerical factor involving a potential due to neighbouring gravitational matter—say a mass m at distance r—a factor which is nearly equal to unity in ordinary cases, and is such that

$$1-\gamma = \sum \frac{2Gm}{rc^2};$$

more generally, if V is the gravitation potential—say inside a body or system of bodies—

$$\gamma = 1 - 2V/c^2.$$

Now in the absence of gravitation the speed of light must be simply c, or in other words,

$$\left(\frac{dr}{dt}\right)^2+\left(\frac{r\,d\theta}{dt}\right)^2+\left(\frac{r\sin\theta\,d\phi}{dt}\right)^2=c^2;$$

and, since $\gamma=1$ when there is no gravitation, it follows from (1) that, for light in empty space, $ds=0$.

But the essence of generalised relativity is that the interval ds, whatever it may be, is invariant and independent not only of choice of coordinate axes and of steady motion, but independent of gravitational and all other acceleration likewise; hence, for light, ds is still zero even when γ is not equal to 1.

So in general the velocity of light in a gravitational field can be reckoned from the equation

$$\frac{1}{\gamma}\left(\frac{dr}{dt}\right)^2+\left(\frac{r\,d\theta}{dt}\right)^2+\left(\frac{r\sin\theta\,d\phi}{dt}\right)^2=\gamma c^2 \quad . \quad . \quad (2)$$

For a ray travelling along a radius vector, $r\,d\theta$ &c being then zero, this gives

$$v=\frac{dr}{dt}=\gamma c;$$

while for a ray inclined at angle ϵ to a radius vector,

$$\frac{1}{\gamma}(v\cos\epsilon)^2+(v\sin\epsilon)^2=\gamma c^2,$$

or

$$\frac{v}{c}=\frac{\gamma\sec\epsilon}{\sqrt{(1+\gamma\tan^2\epsilon)}} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

The reciprocal of this expression is the refractive index in a gravitational field, which, to a first approximation, is

$$\mu=1+\frac{2\text{GM}}{rc^2}\text{ radially, and }1+\frac{\text{GM}}{rc^2}\text{ tangentially.}$$

Incidentally we can see that my elaborate experiment on the influence of matter on ether in its neighbourhood (Phil. Trans. A. 1893 and 1897), which gave a negative result when tested between a pair of whirling disks, would have failed to give a positive result even if the whirling disks had been so absurdly massive as to cause a gravitational potential comparable with that existing at the surface of the sun. For, in accordance with Fizeau's experiment, light is carried forward with the extra speed $v(1-1/\mu^2)$, which, since the peripheral speed v was usually 150 metres a second in that experiment, gives a ratio to the velocity of light

$$\frac{4\text{GM}}{\text{R}c^2}\cdot\frac{v}{c}=4\times10^{-12}.$$

The smallest ratio that could have been observed in that apparatus was the hundredth of a wave-length divided by the total effective journey of the light, say 22 metres,—a ratio which is 3×10^{-10}, or seventy-fold lacking in what would be necessary.

Similarly the Michelson-Morley experiment, even if performed on the surface of a mass as big as the sun, would be likely still to give a negative result: hence it is useless to try to repeat the experiment on some mountain elevation or in the interior of the earth.

So, by (3), it is possible for the speed of light to be zero in a region where $\gamma=0$, that is in the neighbourhood of a mass so great that

$$\frac{2M}{R}=\frac{c^2}{G};$$

and in that case light cannot altogether escape from the body. It would have a better chance of circling round and round the body, with $\epsilon=90^\circ$, because $rd\theta/dt$ is controlled by $\sqrt{\gamma}$ instead of γ; but strictly speaking it could not do this either, but would be stopped in its tracks.

Einstein's method thus makes the speed of light a minimum where ordinary gravitational considerations would make it a maximum. Instead of increasing in speed as it approaches a massive body, light lessens in speed, as if it were repelled, not attracted. But the striking discordance between the two systems is that, whereas the speed of light fully subject to gravity would depend on the distance it had travelled against a retarding force, Einstein makes it assume a velocity characteristic of the place where it is at each moment, without reference to past history.

So taking γ as equal to $1-\frac{2V}{c^2}$, and the velocity of light along a radius as γc, the velocity of light anywhere inside a stellar system such as has been considered above, pp. 549 and 551, at a distance r from its centre, is

$$v=c-\frac{4\pi\rho}{c}(R^2-\tfrac{1}{3}r^2)G.$$

An aggregate mass whose

$$\rho R^2=c^2/4\pi G=1{\cdot}1\times 10^{27} \text{ c.g.s.}$$

would therefore reduce light at its centre to relative rest.

The question has often been asked, What becomes of all the radiation poured into space by innumerable suns through incalculable ages? Is it possible that some of it is trapped, without absorption, by reservoirs of matter lurking in the depths of space, and held until they burst into new stars?

And a further more important question begins to obtrude itself:—What happens to light when, in free though modified ether, it is stopped relatively to a gravitational mass? Does it retain its energy, mainly in rotational form, tie itself into electrons, and add to the mass of the body?

To attempt an answer would involve looking into the manner in which light is retarded by a denser medium, the front being thrown back upon itself (see a detailed theory of refraction by Sir J. J. Thomson, Phil. Mag. June 1920, p. 687, and Dec. 1920, p. 715); also into the suspected tendency of a wave front to break up laterally and concentrate into *quanta*-like units akin to those whence it arose. It may suffice for the present to remark that the familiar expression for reflected amplitude, $\frac{\mu-1}{\mu+1}$, is represented in the gravitational case by $\frac{V}{V+c^2}$, or what would be written as $\frac{m}{r+m}$, or practically $\frac{m}{r}$, if, as usual, μ is written $1+\frac{2m}{r}$. Also that if the momentum conveyed per second through any area is equal to the energy per unit length even in a dense medium,—which was Prof. Poynting's assumption and is now being made the subject of careful experimentation by Dr. Barlow,—the momentum of that part which is transmitted increases, because of the longitudinal compression, the increase being accompanied by a pull. The resultant force acting on light when suddenly entering a denser medium perpendicularly comes out $2 . \frac{1-\mu}{1+\mu}$ times the incident energy-density; the transmitted energy being $\frac{4\mu}{(\mu+1)^2}$ of the original, and the transmitted momentum per second $\left(\frac{2\mu}{\mu+1}\right)^2$; which momentum, instead of remaining uniformly distributed, may become concentrated and localised in specks in the immediate neighbourhood of matter or an intense electric field. For the potential close to an electron is one third of a million volts, and the gradient is enormous. The theory of refraction above referred to shows that, inside matter, lateral elements of a wave must contribute to the formation of the new and retarded wave-front; hence lateral as well as longitudinal concentration is to be expected.

We may also observe that the density of ordinary sunshine near the earth, being $\frac{1}{2}p/c^2$, is of the order 10^{-25} gramme per c.c.; so if 10 cubic millimetres of earth sunshine, or the equivalent of 1/4600 c.mm. of solar emission, could be checked and condensed till its density was 10^{12} it might be converted into an electron of mass 10^{-27} gramme. Its original momentum, 3×10^{-17}, will presumably be communicated or restored to whatever stopped its translation; while the energy of the light, which is c^2 times its mass, will become the constitutional or vortical energy of the electron, whose intrinsic circulation must therefore be of the same order as c.

Einstein's presentation of the kinetic energy of matter is harmonious, for it takes the form $mc^2 + \frac{1}{2}mv^2$, the first term being apparently the intrinsic or constitutional energy of each ultimate particle.

On this view the interior of an enormous stellar system could be the seat of the generation of matter, on rather different lines from those suggested by Prof. Eddington in his Address to Section A at Cardiff last year. If the concentrated and controlled luminosity were locked up satisfactorily as an electron, its existence would be permanent; but is it possible that under exceptional conditions a portable collection of half-formed, either half-materialised or half-shattered, units could go about together in an unstable state, as globe-lightning, and be liable to explode back into light again?

Dr. Barlow questions whether the stoppage of light here considered need be very different from ordinary stoppage by absorption, and asks whether the β particle ejected by an impinging X or γ ray is liberated merely, or manufactured! I do not attempt to answer any of these questions at present, but I may refer to page 466 of the Phil. Mag. for May 1919 (where there is a misprint in the footnote of 10^{-12} for 10^{12}), and also to the Phil. Mag. for February 1920, pp. 172 & 173, especially to these words:—

> "A wave-front is an evanescent kind of matter—a sort of attempt of an accelerated electron to reproduce itself; the question is how such a peculiarity, when generated, can be made permanent and its violent locomotion checked. We must find out how to disturb the ether in such a way that the modification shall remain concentrated, and not instantly rush away and disperse itself with the speed of light. The electric and the magnetic components must be separated, the one kept and the other annulled."

Or, at least, they must be put out of phase. The simultaneous electric and magnetic displacements which constitute a light-wave start from a Hertz vibrator with 90° difference in phase, though the electric rapidly overtakes the magnetic, and the two get into step. Retardation should reverse the process set forth by Hertz in his *Electric Waves*, Chapter IX, (especially pp. 142, 146), and cause them to finish in a modification of their original predicament—absorption corresponding to emission,—thus reproducing the kind of etherial singularity whence they arose.

XLVIII. *The Spectrum of Hydrogen Positive Rays. By* L. Vegard, *Dr. Philos. and Professor of Physics at the University of Christiania* [*].

IN the number of the Philosophical Magazine for August 1920, Mr. G. P. Thomson has published some results of investigations with regard to the light produced by positive rays in hydrogen.

He has studied the light emission of positive rays of a different composition, and according to his interpretation of the results the positive rays themselves in the molecular form should emit the so-called second or many-lined spectrum of hydrogen. This would mean that the second spectrum should show a Döppler effect—a result which is opposed to all evidence so far as yet obtained.

In spite of a great many searches for such an effect by Stark [†], Wilsar [‡], Rau, and the author under very varied conditions, it has not been found. Under these circumstances it seems that the indirect evidence given by the experiments of Mr. Thomson is by no means conclusive, and it is my intention to direct the attention to certain facts which may give an interpretation of Thomson's experiments without assuming that the positive rays themselves emit the second hydrogen spectrum.

In a series of papers published in *Ann. der Physik* from the year 1912 to 1917 [§] I have given results of investigations with regard to the laws governing the light-emission of positive rays, and I shall briefly mention some results that have a bearing on the present question :—

(1) The second spectrum of hydrogen is part of the "unmoved" spectrum of the positive rays, and the intensity of the lines of the second spectrum increases in the same rate as the unmoved spectrum of the series lines.

(2) The ratio between the intensity of the moved and unmoved spectrum was found to vary very greatly with the velocity of the rays and with the pressure in the observation chamber.

Thus when the potential varied from 8500 to 27,000 volts, the ratio between the moved and unmoved intensity of H_β

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[*] Communicated by the Author.

[†] J. Stark, *Ann. d. Phys.* vol. xxi. p. 425 (1906).

[‡] H. Wilsar, *Ann. d. Phys.* vol. xxxix. p. 1251 (1912).

[§] L. Vegard, *Ann. d. Phys.* vol. xxxix. p. 111 (1912); vol. xl. p. 711 (1913); vol. xli. p. 625 (1913); vol. lii. p. 72 (1917).

decreased from 7·18 to 1·64, corresponding to a pressure in the observation chamber of 0·035 mm. Hg. For a given potential—20,000 volts, say,—an increase of pressure from 0·035 to 0·10 mm. Hg produced a diminution of the ratio between moved and unmoved intensity from 3·22 to 1·24.

Mr. Thomson has not separated moved and unmoved intensity, but only drawn his conclusions from the fact that under varied conditions he gets a variation of the ratio between the intensity of the line-spectrum and the second (band-) spectrum of hydrogen. But it follows, from what is already said, *that this ratio can be greatly varied as an effect of variations of pressure and velocity, and with a high pressure in the observation chamber and a large velocity of the rays the second spectrum appears on the plate very prominent relative to the line-spectrum, and this effect was found under conditions where no Döppler effect of the second hydrogen spectrum was to be observed.*

Now the hydrogen spectrum given by Thomson corresponds to discharges in tubes of different form with a different length of the dark space, and probably also differences of pressure in the observation chamber; and there is then every reason to believe that the changes he observes with regard to the ratio between the intensity of the series lines and that of the many-lined spectrum may be regarded only as an effect of velocity and pressure variations.

Mr. Thomson's interpretation of his results could only have any weight when it was proved that the effective potential of the tube and the pressure in the observation chamber were the same in both cases.

In connexion with the question regarding the origin of the moved intensity, discussed by Mr. Thomson, I should like to point out that the same problem has been treated by the author in the papers referred to.

From certain observations[*] I was able to draw the conclusion that the unmoved intensity was produced from the direct bombardment of gas molecules by the positive rays.

Further, I made experiments that showed[†] that at any rate a considerable part of the moved intensity is produced by the neutral part of the positive ray bundle, and that a complete ionization is not necessary to bring a neutral atom to emit light. I shall here briefly describe my experimental arrangement, and give results of calculations which I made some time ago, but which have not previously been published.

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[*] L. Vegard, *Ann. d. Phys.* vol. xli. p. 638 (1913).

[†] L. Vegard, *ibid.* vol. xxxix. p. 162 (1912), and vol. lii. p. 86 (1917).

I used a cylindrical tube of the form shown in the figure and of the very same type as those used in the experiments of Wilsar on the Döppler effect of hydrogen lines. The rays pass through a narrow boring in the cylindrical cathode to the observation chamber, which ends in a glass tube about 5 cm. long and 0·75 cm. diameter.

This narrow glass tube was placed between the poles of a strong electromagnet, in the way shown in the figure, and the rays could be made to pass a strong magnetic field of 11,500 gauss for a distance of 4·2 cm. The luminosity was analysed and measured at the end of the glass tube by means of a spectrograph of high light-power and a suitable dispersion for observations of the Döppler effect. The direction of the collimator axis formed an angle of about 35° with the direction of the rays, and it was thus obtained that only light from the very end of the glass tube passed into the spectrograph.

During discharge a constant current of hydrogen was maintained by continual pumping. By a simple arrangement, the direction of the current of the gas through the boring in the cathode could be reversed; and in this way the potential of the discharge-tube could be changed independently of the pressure in the observation chamber. The discharge-tube was surrounded by a thick cylinder of soft iron, to prevent the magnetism from exercising an influence on the discharge. The spectrograms were taken under the very same discharge conditions, with and without a magnetic field, and for some different pressures. When the magnetic field was put on, the positively-charged rays were driven into the wall as soon as they were formed, and the luminosity at the end was mainly produced by uncharged particles, and most of the moved intensity was emitted from rays in the neutral state.

By means of the idea of the mean free path of the positive rays introduced by W. Wien *, we can calculate the diminution of light-intensity which would be produced by the magnetic field provided that the light was entirely emitted from the neutral rays.

Let us consider a cross-section (A–B) of the bundle just before the rays enter into the magnetic field; and let us suppose that the bundle at this place has reached statistical equilibrium, so that the ratio between the number of charged and uncharged particles would remain constant provided there was no magnetic field.

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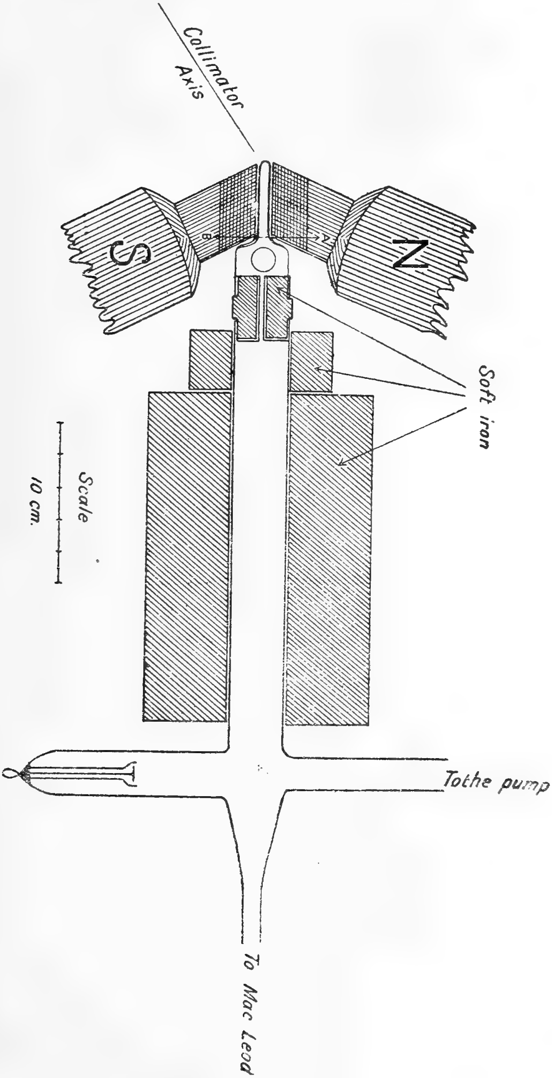
* W. Wien, *Ann. d. Phys.* vol. xxxix. p. 519 (1912).

Let the number of the positive and neutral carriers that pass this cross-section in unit time be n_1 and n_2 respectively. Now, if the "moved intensity" is due to the neutral rays



only, the moved intensity I observed without a field should be proportional to $n_2 e^{-\mu l}$, where μ is the coefficient of absorption and l the distance traversed from the cross-section to the end of the glass tube, where the intensity is measured, and the intensity I will be

$$I = k_2 e^{-\mu l} n_2.$$

When the field is put on, first of all the positive rays n_1 present at (A–B) are deflected into the glass wall, and then the positive rays which on the way l are produced from the neutral part are also brought out of the field of view of the spectrograph.

This gives for the observed moved intensity I_m with a magnetic field :

$$I_m = k_2 n_2 e^{-\frac{l}{L_2}} e^{-\mu l}.$$

Consequently :

$$\frac{I}{I_m} = l^{+\frac{l}{L_2}}.$$

L_2 is the mean distance which a neutral ray moves before it takes up a positive charge, or "the mean free path" of the neutral ray.

If we know L_2, we shall be able to calculate the ratio

$$I/I_m.$$

Now, according to Wien, "the mean free path" changes comparatively little with the velocity of the rays, but very considerably with the pressure in the observation chamber.

Wien's measurements, however, are confined to fairly low pressures, ranging from about $5 . 10^{-4}$ to $4 . 10^{-2}$ mm. Hg., while the pressure in my experiments varied between $3{\cdot}5 . 10^{-2}$ and 10^{-1} mm. Hg.

Wien gives the following values :—

For $p = 0{\cdot}0051$ mm., the quantity $L\frac{p}{p_0} = 5{\cdot}25 . 10^{-5}$ cm.

„ $p = 0{\cdot}039$ „ „ „ $L\frac{p}{p_0} = 10 . 10^{-5}$ cm.

p_0 is atmospheric pressure, and L is defined by the equation

$$\frac{1}{L} = \frac{1}{L_1} + \frac{1}{L_2}.$$

The ratio L_2/L_1 is for the same pressures 6·1 and 2·6 respectively. L is "the mean free path" of the positively charged carriers.

"The mean free path" corresponding to the lowest pressures used in my experiments can be found fairly accurately from these values. In the case of the highest pressure 0·1 mm., however, we have to extrapolate over a fairly wide range; so the value found for "the mean free path" will be somewhat uncertain.

In the table are given "the mean free paths" L_1 and L_2 as they are derived from Wien's experiments:—

Pressure.	L_2.	L_1.	$\left(\frac{I}{I_m}\right)_{\text{cal.}}$	$\left(\frac{I}{I_m}\right)_{\text{obs.}}$
0·035 mm.	7·9 cm.	2·78 cm.	1·70	1·89
0·056 ,,	5·0 ,,	2·57 ,,	2·31	2·33
0·10 ,,	3·15 ,,	2·58 ,,	3·81	2·91

The effective length of the magnetic field is put equal to that of the pole-pieces (4·2 cm.). Now we observe the light at a distance of about 0·5 cm. from the end of the narrow glass tube, and, further, some of the rays that acquire a positive charge during the passage of the last centimetre of the glass tube may come into the field of view. This will be equivalent to a diminution of the distance l.

On the other hand, the spreading of the magnetic lines will produce some increase of the effective length l. To diminish these errors, the distance l ought to be somewhat longer than in my experiments.

Comparing the calculated and observed values, we see that they show a fairly good agreement for the smaller pressures.

In the case of the highest pressure the value taken for "the mean free path" is uncertain, and when L_2 becomes small errors in the distance l will have a comparatively great effect.

At any rate, we can conclude from these numbers that the *greater part of the moved intensity is emitted from the rays in the neutral state.*

Some moved intensity may also be emitted when the positive rays get neutralized; but this quantity should be small as compared with that produced through the bombardment of the neutral rays with the gas molecules.

When we remember that the light-intensity measured by means of the spectrograph is proportional to the intensity per unit length of the ray bundle, the measured intensity should be given by an expression of the form:

$$I = k\left(\kappa_1 \frac{n_1}{L_1} + \kappa_2 \frac{n_2}{\lambda}\right) e^{-\mu l}.$$

λ is the mean distance which a neutral ray has to pass between successive collisions which excite the ray to light emission. $\frac{n_2}{\lambda}$ is then the number of such collisions which the neutral rays suffer in unit time per unit length. κ_2 measures the probability that such a collision shall result in the emission of the particular spectral line considered.

In a similar way, $\frac{n_1}{L_1}$ measures the number of times the positively charged rays get neutralized per unit length of path in unit time; and if we assume that a neutralization is always accompanied by light-emission, $\frac{n_1}{L_1}$ also measures the number of times a positively charged ray is brought to emit light through neutralization per unit length of path and in unit time.

κ_1 measures the probability that a neutralization shall result in the emission of the particular spectral line considered.

k is a constant, which depends on the apparatus and the units of the light-intensity.

If we put the magnetic field on, the luminosity produced by neutralization becomes practically zero and we get:

$$I_m = k\kappa_2 \frac{n_2}{\lambda} e^{-\frac{l}{L_2}} e^{-\mu l}$$

and

$$\frac{I}{I_m} = \frac{\kappa_1 \frac{n_1}{L_1} + \kappa_2 \frac{n_2}{\lambda}}{\kappa_2 \frac{n_2}{\lambda}} e^{+\frac{l}{L_2}}$$

$$= \left(\frac{\kappa_1}{\kappa_2} \frac{n_1}{n_2} \frac{\lambda}{L_1} + 1\right) e^{+\frac{l}{L_2}}.$$

If the ray bundle at the section (A–B) is in a state of statistical equilibrium, we have according to Wien:

$$\frac{n_1}{n_2} = \frac{L_1}{L_2}$$

and

$$\frac{I}{I_m} = \left(1 + \frac{\kappa_1}{\kappa_2} \frac{\lambda}{L_2}\right) e^{+\frac{l}{L_2}}.$$

As we saw, $(I/I_m)_{\text{obs.}}$ came out nearly equal to e^{l/L_2}, and thus $\frac{H_1}{H_2} \frac{\lambda}{L_2}$ should be a small quantity for H_α and H_β.

If the probability that a "light impact" of the neutral ray shall result in the emission of H_β, say, is the same as the probability for a H_β emission resulting from neutralization, then $\kappa_1 = \kappa_2$ and $\frac{\lambda}{L_2}$ should be a fairly small quantity.

Now if "the mean free path" λ of successive "light collisions" of the neutral ray is of the same order of magnitude as the mean free path of the gas molecules in the observation chamber, we find, as a matter of fact, that $\frac{\lambda}{L_2}$ is a small quantity.

For the pressures used we find the following values :—

Pressure.	λ.	$\frac{\lambda}{L_2}$.
0·035 mm.	0·39 cm.	0·050
0·056 ,,	0·245 ,,	0·049
0·10 ,,	0·135 ,,	0·043

Probably the mean distance between successive light-emissions will be somewhat greater than "the mean free path" of the molecules in the observation chamber, and the true values of $\frac{\lambda}{L_2}$ somewhat greater than those here given.

The question with regard to the origin of the emission of light from a bundle of positive rays has been the subject of much discussion. In a number of papers Stark has tried to prove that the moved intensity is produced by the positive ions, while Wien found that certain of his experiments were best explained by the assumption that the moved intensity came from the neutral part of the bundle.

The results of my experiments, as I have stated in my previous papers, have so far confirmed Wien's assumption, as, at any rate, the greater part of the moved intensity comes from the neutral ray. But, as we have already remarked, we ought to distinguish between the emission produced by the neutral ray on account of collisions with gas molecules and the emission due to neutralization of the positively charged carriers, and the last part will in so far be an effect of the positive rays as it will be removed when the positively charged rays are removed from the field of view.

In this connexion I should like to mention another result of my experiments with the same arrangement, which is hard to understand without assuming that the positive carriers through the neutralization process are engaged in the emission of moved intensity. *It was found that the ratio between moved and unmoved intensity was not changed by the effect of the magnetic field*, or the changes were inside the limits of possible errors.

As before, the moved intensity without a magnetic field is given by an expression of the form :

$$I_0 = (k_1 n_1 + k_2 n_2) e^{-\mu l} ;$$

and the unmoved intensity J_0 :

$$J_0 = (q_1 n_1 + q_2 n_2) e^{-\mu l} ;$$

and the ratio between the moved and unmoved intensity :

$$f_0 = \frac{I_0}{J_0} = \frac{k_1 n_1 + k_2 n_2}{q_1 n_1 + q_2 n_2}.$$

When the magnetic field is put on, moved as well as unmoved intensity is produced by the neutral rays:

$$I_m = k_2 n_2 e^{-\frac{l}{L_2}} e^{-\mu l},$$

$$J_m = q_2 n_2 e^{-\frac{l}{L_2}} e^{-\mu l},$$

$$f_m = \frac{I_m}{J_m} = \frac{k_2}{q_2}.$$

Now the experiments give as the mean of four measurements :

$$f_0/f_m = 0\cdot98, \quad \text{or approximately} = 1.$$

This gives

$$\frac{k_2}{q_2} = \frac{k_1 n_1 + k_2 n_2}{q_1 n_1 + q_2 n_2} = \frac{k_1}{q_1}.$$

Now if we assume that the positive as well as the neutral rays, by their impact with the gas molecules in the observation chamber, produce "unmoved intensity," then q_1 is different from 0, and as k_2 and q_2 are different from 0 k_1 must be different from 0.

As the positive hydrogen nucleus cannot emit the series lines, the positive hydrogen rays can only emit light at the very moment they pass into the neutral state, and $k_1 \neq 0$ means *that light is emitted as the result of the neutralization process.*

For other gases we have to reckon with the possibility that the ray may emit light *also in the positively charged state* as the result of such collisions with gas molecules, which do not result in neutralization.

Physical Institute, Christiania,
October 28, 1920.

XLIX. *The Spectrum of Hydrogen Positive Rays.*

To the Editors of the Philosophical Magazine.

Gentlemen,—

I WISH to thank you very much for allowing me to comment on Professor Vegard's paper. He ascribes the differences which I found in the spectra of the hydrogen positive rays to chance variations in pressure and in the energy of the rays. This does not explain the marked correlation which I showed to exist between the ratio of the intensities of the spectra, and the proportion between the atoms and molecules of hydrogen in the rays as determined by direct analysis. In point of fact, the difference in pressure was in the reverse direction to that required for his explanation, the pressure being lower during the experiments when the second spectrum was obtained. While there were undoubtedly variations in the potential difference used in the discharge, the resulting variation in energy of the rays was small compared with the range in energy at any instant, which, as determined from the electrostatic deflexion of the rays, was often 3 or 4 to 1.

The close connexion between the two resting spectra found by Prof. Vegard is to be expected on my theory as long as the positive rays are all of one kind—that is, as long as there is only one "moved" spectrum. They presumably correspond, the series spectrum to dissociation with ionization (partial or complete) and the second spectrum to ionization only, of the gas in the observation chamber, with the subsequent return of the electron which is the actual source of the light emitted. As long as the particles forming the rays were of the same nature, the ratio of the numbers of collisions of these two types might be expected to be the same.

As mentioned in my previous paper, the failure to find any "moved" second spectrum is a point which needs explanation; but I still think that the use of cylindrical tubes of comparatively small diameter is largely responsible, as in similar circumstances I too got a purely atomic beam of rays.

Professor Vegard suggests no alternative theory as to the nature of the carriers of the second spectrum, and the molecular theory will seem much the most probable, especially in view of the recent work of Mr. Saha *, at least until it can be shown that rays which when analysed show the presence of appreciable numbers of molecules give no "moved" second spectrum.

My experiments were not such as to throw light on the electric condition of the atoms which emit the series spectrum, and I fully accept Professor Vegard's conclusion that light is emitted both when the positively charged atom is neutralized and when the neutral atom collides with the molecules of the gas.

Yours faithfully,
G. P. THOMSON, M.A.

Corpus Christi College,
Cambridge.

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* Phil. Mag. June 1920.

L. *The Torsion of Closed and Open Tubes.*

To the Editors of the Philosophical Magazine.

GENTLEMEN,—

THE formulæ for closed and open tubes under torsion given by Dr. Prescott in the November number of the Philosophical Magazine have already been published by me in two papers: (A) "The Calculation of Torsion Stresses in Framed Structures and Thin-walled Prisms" (Brit. Assoc. Report, 1915, and 'Engineering,' October 15th, 1915), and (B) "The Torsion of Solid and Hollow Prisms and Cylinders" ('Engineering,' Nov. 24th and Dec. 1st, 1916).

Formula (32) of Dr. Prescott's paper, giving the stress in a thin tube, is stated in § 2 of paper B, and is a particular case of the theorem which forms the main subject of paper A, viz.:—If a hollow cylinder or prism, either continuous-walled or of framework, and having plane ends perpendicular to its length, be subjected to a twisting moment by couples in the planes of its ends, the total longitudinal shear is everywhere constant and equal to the twisting moment multiplied by the length of the cylinder and divided by twice the area of one of its ends.

This theorem was proved very simply from elementary considerations without using the equations of elasticity. As applied to frameworks, it was used in the calculation of the torsion stresses in the suspended span of the Quebec Bridge, and has also been applied to aeroplane fuselages.

The formula (35) for the angle of twist of a tube given by Dr. Prescott is also stated in § 2 of paper B, and is deduced there from the work stored in the tube during torsion.

The formula (56) for the angle of twist of a thin strip (called an "unclosed tube" by Dr. Prescott) is slightly more general than my formula in paper B § 6 eq. 24, which is only true when the strip is of uniform width, but equation (25) of paper B § 6 gives the extension of (24) to rolled sections.

Dr. Prescott does not give explicitly a formula for the shear stress in a thin strip in terms of the torque, but, by combining his equations (53) and (56), it follows that

$$S = \frac{3Q}{t^2 l},$$

which is the formula given by me in B § 5 and extended to structural steel sections in § 6.

This result and formula (32) have been used in the design of the spars of the tail-planes of aeroplanes which are under torsion from the king-posts carrying the control wires.

I also showed in paper B that formula (32) could be used to solve the general (St. Venant) problem of torsion by considering a solid shaft as made up of tubes of shear. This method gives the same results as the St. Venant theory, without involving the use of the equations of elasticity or of conjugate functions.

The approximate formulæ used in engineering practice for elliptical and rectangular tubes often give results which are greatly in error. This matter is considered in paper B § 7.

Yours faithfully,
C. Batho.

Dept. of Civil Engineering
and Applied Mechanics,
McGill University, Montreal,
November 30th, 1920.

To the Editors of the Philosophical Magazine.

Gentlemen,—

The results in my paper on "The Torsion of Tubes" were worked out about last March in connexion with a book I am writing. I had not then seen or heard of any of Professor Batho's work on the subject, and I thought my results were quite new. After I had decided to publish my results in the form of a paper I saw in 'Engineering' the paper (A) mentioned above, but this paper had so little in common with my own that I saw no reason for withholding publication. It was only after I had sent off the final proofs of my paper that Professor Batho's paper (B) came to my notice. If I had seen this paper earlier I should probably not have published my results in a separate paper. However, I now think that it was worth while to publish my results because our methods are so different, and especially as the correspondence in 'Engineering' shows that there were people who did not believe in Professor Batho's methods. Moreover, as he points out above, my formula for an open tube (or thin strip) applies to strips of variable thickness, whereas his applies to strips of constant thickness only.

Professor Batho's claim that he does not use the equations of elasticity cannot be upheld for his work on the thin strip, since, for this purpose, he borrows St. Venant's results for a prism of rectangular section.

J. Prescott

Jan. 7th, 1921.

LI. *The Mass of the Long-range Particles from Thorium C.* *By* Sir E. Rutherford, *F.R.S., Cavendish Professor of Physics, University of Cambridge*[*].

It is well known that thorium C disintegrates in two ways with the emission of α particles of range 8·6 cm. and 5·0 cm at 15° C. In 1914, Dr. A. B. Wood and myself showed that a small number of particles—about 1 in 10,000 of the total—were expelled with the long range of 11·3 cm. The range and number of these particles were determined by the scintillation method, and from the brightness of the scintillations it was supposed that the expelled particles were atoms of helium. In a subsequent paper the writer showed that the passage of α particles from radium C through nitrogen and oxygen led to the production of a small number of swift particles, which had a range 1·29 times that of the impinging α particles. Since the ratio of the ranges of the thorium particles, viz. 11·3 cm. and 8·6 cm., is of about the same magnitude, viz. 1·32, it was suggested that possibly the long-range particles emitted by thorium C might arise from collision of the α particles of range 8·6 cm. with the oxygen of the mica, which was used as an absorbing screen to cut off the particles of range 8·6 cm. This, however, seemed unlikely, as the number of the long-range particles from thorium C were about ten times greater than would have been expected on this hypothesis. In order to make certain of this point, Dr. A. B. Wood kindly undertook to repeat the experiments, using aluminium instead of mica as an absorbing screen. This experiment, an account of which is given in an accompanying paper, showed conclusively that the long-range particles could not be ascribed to the oxygen in the mica; and at the same time a more accurate estimate was made of the relative number of long-range particles, which were found to be 1/10,000 of the total number of α particles from thorium C

In the course of recent experiments the writer obtained some evidence that the short-range particles of the α-ray type of the range 9 cm. appearing in oxygen and nitrogen were not atoms of oxygen or nitrogen, but atoms of mass about 3. It was a matter of great interest to examine whether such atoms were liberated in radioactive changes in addition to atoms of helium of mass 4. If this proved to be the case, it would afford a more direct method of determining the mass of the new atoms with accuracy, since they

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[*] Communicated by the Author.

would be emitted in number from the radioactive source instead of from the volume of the gas bombarded by α rays.

In the original experiments of Dr. Wood and the writer, the active deposit of thorium was used as a source of radiation, and with the active material available, the γ-ray activity of the source of radiation was about 2/100 of a milligram of radium in equilibrium. In order to obtain about 20 scintillations a minute, due to the long-range particles, it was necessary in these experiments to place the mica within a few millimetres of the zinc-sulphide screen In order, however, to determine the mass of the particles by observing the amount of their deflexion in a magnetic field, it was imperative to work at a distance of at least 5 centimetres, and to employ a source at least 100 times stronger.

By the generosity of Dr. Herbert McCoy of Chicago, well known for his contributions to our knowledge of the radioactivity of thorium, I was presented with a quantity of radiothorium of γ-ray activity equal to 24 milligrams of radium. In order to obtain powerful sources of thorium C, this material, after suitable chemical treatment, was obtained in a dilute solution of small volume. I am indebted to Mr. Chadwick for his kindness in preparing this solution in a form to yield the maximum amount of thorium C. This was obtained by exposing one side of a nickel plate of area about one square centimetre in the hot solution for one hour. During the exposure the plate was kept in slow rotation by a small motor. By this method it was possible to obtain an amount of thorium C on the surface of the nickel plate equal in γ-ray activity initially to about 8 milligrams of radium. The activity of this source decayed with the time according to the period of thorium C, viz. to half value in one hour.

This source gave a sufficient number of long-range particles of range 11·3 cm. to determine their bending in a magnetic field by the scintillation method. Two different arrangements were employed. The first was similar to that described in the Bakerian Lecture (Proc. Roy. Soc. A, 1912), and illustrated in fig. 1 of that paper. The α rays from the nickel plate passed between two parallel plates, 4 cm. long and 2 cm. apart, placed in an exhausted rectangular box between the poles of a large electromagnet. The plates were distant 1·2 cm. from an opening in the end of the box, which was covered by an aluminium plate of stopping power for α particles equal to 5·4 cm. of air. To examine the deflexion of the long-range particles, additional absorbers were added to stop completely the α rays of range 8·6 cm. For this purpose the total absorption in the path of the rays was adjusted to 9·4 cm. of air. The zinc-sulphide screen was placed close to the aluminium absorbers, and the microscope so adjusted that, on exciting the magnetic field in one direction, the α particles fell over the whole surface of the screen viewed by the microscope, and by reversing the field the scintillations were confined to the lower half of the field of view. The ratio of the number of scintillations per minute, usually 4 or 5 to 1, was determined for the two fields, and this gave a measure of the amount of deflexion of the rays. This ratio was compared directly with that found for the thorium particles of range of 8·6 cm. For this purpose a much weaker source of α rays was obtained by dipping a nickel plate for a few seconds in a more dilute solution of radiothorium. The experimental arrangement was identical with that described above, except that the absorption in the path of the α rays was reduced to 5·4 cm. of air. This stopped the α particles of range 5·0 cm. from thorium C, while those of range 8·6 cm. gave bright scintillations on the screen. In all cases it was found that the long-range particles were less deflected than those of range 8·6 cm. By determining the value of the ratio for different strengths of magnetic field, the relative magnetic deflexion of the two types of rays could be directly compared. The different determinations made in this way varied between 1·08 and 1·12, with an average value of 1·10.

In order to confirm these results, the deflexions were compared by a more direct method, identical in principle with that employed previously to measure the deflexion of the swift H atoms set in motion by impact with α particles (Phil. Mag. xxxvii. p. 563 (1919)). The rays from the source placed behind a horizontal slit of width 1 mm. passed through another slit of equal width placed midway between the first slit and the zinc-sulphide screen. The distance between the source and screen was 8·3 cm. With the magnetic fields employed, the band of scintillation observed on the screen, due to α particles of range 8·6 cm. from a strong source of thorium C, was displaced 5·7 mm. by reversal of the field. The amount of deflexion of the long-range particles by reversing a current through the electromagnet of 6 amps. was directly compared with the deflexion due to the α rays of range 8·6 cm. from thorium C under similar conditions. For a current of 5 amp., the deflexion of the pencil of rays, range 8·6 cm., was ·965 of the pencil of long-range particles with a current of 6 amp., giving a field 1·12 that for 5 amp. For equal fields, the deflexion of particles of range 8·6 cm. was thus 1·08 times that for the particles of 11·3 cm. range.

If the particles of range 11·3 cm. are ordinary α particles, the relative deflexion of α particles of range 8·6 and 11·3 cm. to be expected is $\sqrt[3]{\frac{11 \cdot 3}{8 \cdot 6}} = 1 \cdot 10$, since the velocity of the α particles varies as the cube root of the range. The observed values by the two methods are 1·10 and 1·08, or a mean of 1·09. By making a number of experiments, no doubt the relative bending could be determined with more precision; but this was not thought necessary, as the agreement is sufficiently close to indicate that the long-range particles from thorium C are ordinary α particles of mass 4. It is of interest to note that if these long-range particles were atoms of mass 3 carrying two charges such as are observed to be released as a consequence of the collision of α particles with N and C atoms, the deflexion of the α particles of 8·6 cm. range should be ·90 of that of the particles of 11·3 cm. range instead of the observed value of 1·09. The data on which this calculation is based have been given in the Bakerian Lecture (*loc. cit.*).

The experiments recorded in this paper thus negative the idea that particles of mass 3 are ejected from thorium C. It should be pointed out that the agreement of the atomic weights of radium, uranium-lead, and thorium-lead with the values calculated from the emission of α particles show that no particles of mass 3 are expelled in the main series of radioactive changes of uranium and thorium. Similarly, no certain evidence has been obtained of the emission of H atoms. From numerous experiments I have made, I am inclined to believe that most of the H atoms observed from a source of radium C under normal experimental conditions must be ascribed to occluded hydrogen.

It has been generally considered that the expulsion of two distinct sets of α rays from thorium C is a proof that this product suffers a dual transformation. From analogy with the dual disintegration of radium C, it is supposed that 35 per cent. of the atoms of thorium C break up with the emission of α particles of range 5·0 cm., giving rise to thorium D, which breaks up with the expulsion of a β particle. The reverse process is considered to take place in the other branch, 65 per cent. of the atoms of thorium C first emitting β particles and giving rise to thorium C′, which is very rapidly transformed with the emission of the swift α particles of range 8·6 cm. While it is difficult to give a definite proof of this scheme of transformation, the general facts strongly support it.

The emission of α particles of range 11·3 cm. shows that the modes of disintegration are even more complicated than the above. There is no information to guide us as to the origin of these very swift particles, except that they appear to arise from thorium C and decrease in number at the same characteristic rate. The system of transformation may be similar to one of those outlined, or thorium C′ may break up in two ways with the emission of rays of ranges 8·6 and 11·3 cm.

Suppose, however, 1/10,000 of the atoms of thorium C break up directly with the emission of these very swift particles. The atomic number of the resulting product, viz. 81, is that of thallium, but the atomic weight is 208 instead of 206 observed for ordinary thallium. It is of interest to note that Merton (Proc. Roy. Soc. A, xcvi. p. 393 (1920)) found that thallium from pitchblende residues gave a longer wave-length than ordinary thallium, indicating the presence of an isotope of higher mass. Soddy* considered the question whether thallium could be an end product of the two main branches of thorium C, but found that the amount of thallium in a particular thorium mineral, Ceylon thorite, was far too small to admit of such a possibility. The amount of lead, mainly thorium lead, in this mineral was 0·4 per cent. If thallium results as the end product of the new branch, the amount of thallium should be about ·00004 per cent., or for each 100 grams of thorium-lead, 10 milligrams of the thallium isotope—supposed stable—should be obtainable. Soddy states that some thallium was found in the mineral, but in very small amount not determined, but certainly less than ·005 per cent.

It would be of interest to examine whether such a thallium isotope is present in thorium minerals in amount to be expected on the above hypothesis.

I am much indebted to Dr. Ishida and Mr. Chadwick for their assistance in counting scintillations.

Cavendish Laboratory,
Dec. 1920.

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* Soddy, 'Nature,' cii. pp. 356, 444 (1919).

LII. *Long-range Particles from Thorium Active Deposit.*

By A. B. Wood. *D.Sc.*[*]

§ 1. IN 1916 Sir Ernest Rutherford and the author published a paper[†] describing experiments which gave evidence that thorium active deposit emits a small number of high-velocity particles, of range about 11·3 cm., in addition to the main group of α-rays of ranges 5·0 cm. and 8·6 cm. At the time the experiments were made there seemed no reason to suppose that these long-range particles were other than α-particles. More recently, however, Sir E. Rutherford has shown[‡] that when α-particles are fired into a medium containing light atoms, some of these atoms are enormously accelerated by close collision of the α-particles with their nuclei, and consequently attain very high velocities and correspondingly long ranges. Such high-velocity particles behave in a manner similar to that of the α-particles themselves in that they can produce intense ionization and scintillation effects. It was calculated that oxygen atoms, for example, could be accelerated by close collision with α-particles of 8·6 cm. range, so that they attained the high velocity corresponding to a range of 11·1 cm. in air.

In the investigation of the long-range particles from thorium[§] a mica screen was employed to absorb the ordinary α-particles of ranges 5·0 and 8·6 cm., the long-range particles pentrating the mica and striking a zinc-sulphide screen. The maximum range of such particles was found to be 11·3 cm., a value differing only slightly from that deduced by Sir E. Rutherford for the range of oxygen atoms accelerated by 8·6 cm. α-particles. Since mica contains a considerable amount of oxygen, it seemed not improbable that the "long-range α-particles from thorium" might conceivably be oxygen atoms originating in the mica absorbing screen. A fraction of the long-range particles must undoubtedly have originated in this manner, since Rutherford has shown this to be so when RaC α-particles are fired through mica screens. On the other hand, the number of long-range particles observed in the case of thorium, about 1 in 10,000 of the total number of α-particles, is about 10 times the number obtained in the RaC experiments, viz. 1 in 100,000.

It was considered desirable, therefore, to re-determine the

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[*] Communicated by Professor Sir E. Rutherford, F.R.S.

[†] Rutherford & Wood, Phil. Mag. xxxi. April 1916.

[‡] Rutherford, Phil. Mag. xxxvii. June 1919.

[§] Rutherford and Wood, *loc. cit.*

number of long-range particles from thorium, and to differentiate, if possible, between long-range α-particles and high-velocity oxygen atoms.

§ 2. *Experimental Arrangements.*

The experiments carried out in 1916 gave an approximate value of 1 in 10,000 for the number of long-range particles relative to the total number of α-particles emitted by the source of thorium active deposit. Unless a very intense source be employed, therefore, it is not possible to obtain a sufficient number of long-range particles for purposes of investigation. To obtain this intense source of active deposit, the flat tip of a copper wire, 0·8 mm. diameter, was exposed as negative electrode to a strong source of radio-thorium *, until the active deposit collected was in equilibrium with the emanation. The electrodes in the exposure chamber were arranged so as to concentrate the field on the tip of the copper wire, thus ensuring a maximum amount of active deposit collected. When sufficiently active, the wire was removed and mounted, as shown in fig. 1, with the active surface 4·5 mm. distant from a zinc-sulphide screen. Usually the solid angle of the α-ray stream from the end of the wire was limited by a small hole, 1·1 mm. diameter, through a brass plate 1·8 mm. thick. The absorbing screens employed to stop the 8·6 cm. α-particles were placed over this hole, and could easily be removed or interchanged without disturbing the relative positions of the active wire and zinc-sulphide screen. The latter was carried on an extension tube, fitted to the microscope so that it always remained in focus. This zinc-sulphide screen, prepared by Mr. F. H. Glew, was of specially fine grain, the area under observation by the microscope being completely covered with small crystals. The microscope, of magnifying power 50, covered a field 2 mm. diameter. Under these conditions the maximum number of long-range particles observed was about 10 per minute. In cases where such a small number of scintillations is to be counted it is, of course, necessary to take all possible precautions against slight contamination of screens, etc., by radioactive matter. Consequently, it is essential to make "dummy" experiments, in which the active source is removed, as a matter of routine.

To ensure that the relative positions of the microscope, ZnS screen, absorbing screen, and active source always remained the same through an experiment, they were all

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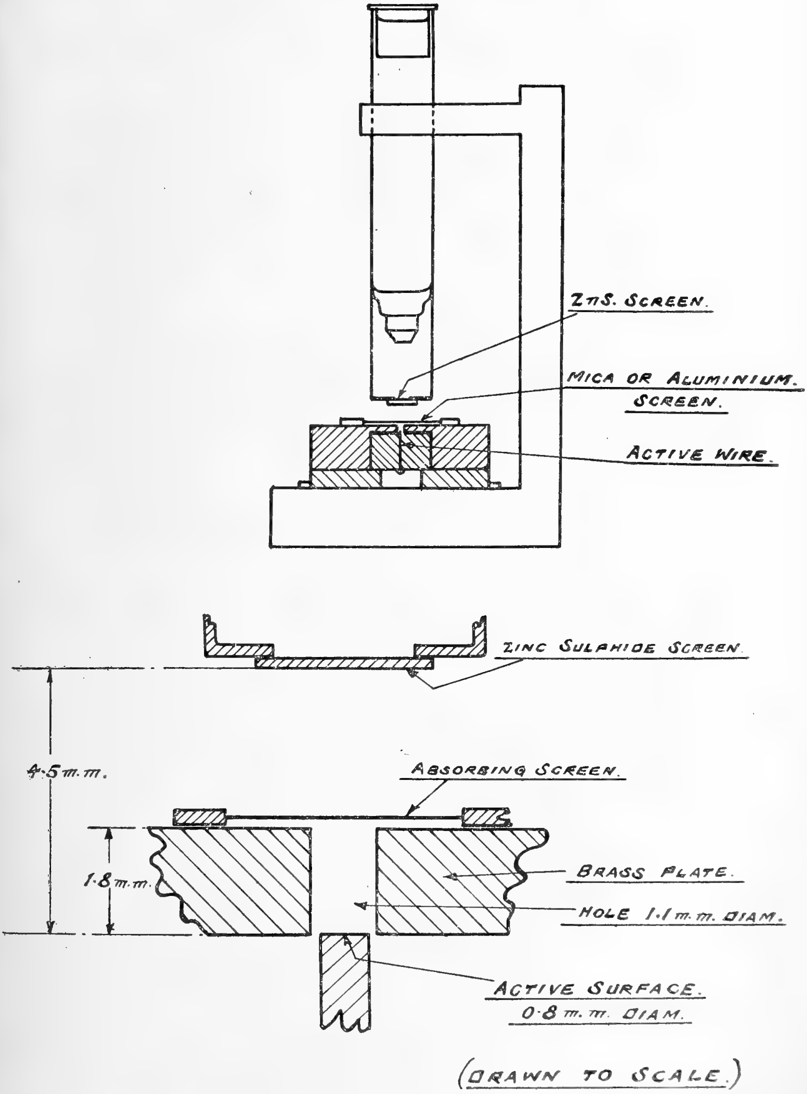
* Very kindly lent for these experiments by Prof. L. R. Wilberforce, of Liverpool University.

mounted on a common supporting base in such a way that the axis of the microscope passed through the centre of the active surface and the hole in the brass plate (see fig. 1).

Fig. 1.



The air-equivalents of the absorbing screens used in the investigation were determined first of all by weighing measured areas and deducing the air-values from data supplied in a paper by Marsden and Richardson*. The values thus deduced were afterwards carefully checked by direct observation. When mica was used as absorber of the 8·6 cm. α-particles, two thin layers, each of air-equivalent 4·3 cm., were superposed, the total "effective" air-equivalent from active source to ZnS screen in that case being 8·6 cm. + ·45 cm. or 9·05 cm. Thus any particle originating in the active material and striking the zinc-sulphide screen must have a range greater than 9 cm. In certain experiments aluminium screens were used to stop the 8·6 cm. α-particles. For this purpose four layers, each of 2·0 cm. air-equivalent, and one layer of 0·5 cm. air-equivalent, were superposed, the total effective air-value from active source to ZnS screen being in this case 8·95 cm.

The methods of counting scintillations and experimental precautions necessary have frequently been described elsewhere, and need no further mention here.

§ 3. *Relative number of Long-range α-particles and Oxygen atoms.*

In order to discriminate between the scintillations produced by α-particles and oxygen atoms, two methods have been employed:—

(*a*) By varying the position of a mica absorbing screen between the active source and the zinc-sulphide screen.

(*b*) By comparing effects with mica and aluminium absorbing screens.

(*a*) In this method the distance between the active source and the zinc-sulphide screen is kept constant, whilst the mica absorbing screen (of air-equivalent 8·6 cm.) is placed (1) near to the active source, (2) near to the zinc-sulphide screen, the number of "long-range" scintillations being carefully determined for each position of the mica (see fig. 2). In this comparison the brass aperture (fig. 1) restricting the divergence of the X-ray beam incident on the mica is removed. It will readily be seen that if all the long-range particles come directly from the active deposit, no appreciable difference in the number of scintillations counted should be observed as the mica is moved from position (1) to (2). If, however, the long-range particles all originate in the mica, it can be shown that the number of scintillations observed in case (1) when the mica is close to the

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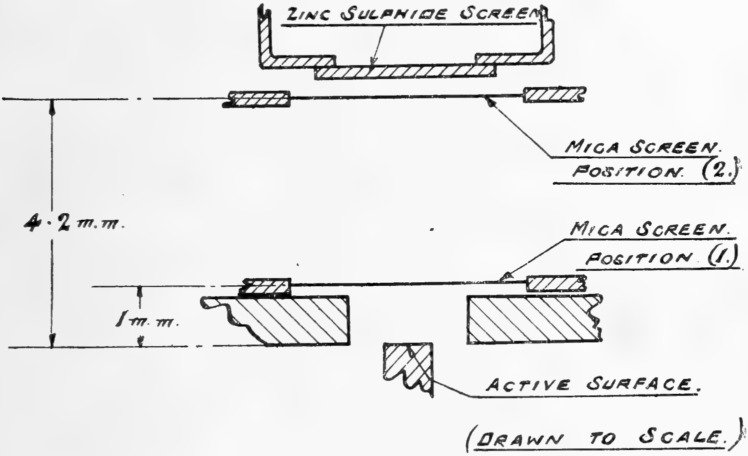
* Marsden & Richardson, Phil. Mag. Jan. 1913.

active source, should be about 1·5 times the number observed in position (2) when the mica is placed near the zinc-sulphide screen and further away from the active source.

Fig. 2.



In a particular experiment of this nature the total distance from the active deposit to the zinc-sulphide screen was 5 mm., the distance from the mica to the active source in (1) being 1 mm., and in (2) 4·2 mm. Within the limits of possible error in experiment, the number of scintillations observed was the same in each case. Repetitions of this experiment always gave the same result. Whilst these observations suggest that the majority of the long-range particles come from the active source, they do not entirely preclude the possibility of a small fraction originating in the mica. Before discussing this result, therefore, it appears desirable to give the data obtained by the second method of long-range particle analysis.

(*b*) In the second method a direct comparison is made between the number of long-range scintillations observed when screens of (1) mica and (2) aluminium are used to stop the 8 6 cm. α-particles. A comparison here is made between the effects observed when the absorbing screen contains oxygen, as in mica, and when it is free from oxygen (except as a slight impurity perhaps), as in aluminium.

If the long-range particles consist entirely of oxygen atoms from the mica, it is evident that the number of scintillations observed through mica will be far greater than the number through aluminium. Alternatively, if all the long-range particles originate in the active source, the number of scintillations observed through aluminium will be exactly the same as the number through mica—always assuming, of course, that the aluminium and mica screens have the same air-equivalents. If, however, both oxygen atoms and long-range α-particles are present, the result will be a compromise between these two extremes, *i. e.* the number through aluminium will be somewhat less than the number through mica. An experiment of this nature should therefore provide a crucial test of the origin of the long-range particles.

It has already been mentioned that the mica screen consisted of two layers, each of air-equivalent 4·3 cm.—total 8·6 cm., whilst the aluminium screen was made up of four layers of 2·0 cm. air-equivalent and one layer of 0·5 cm. air-equivalent—total 8·5 cm. It will then be seen that the "comparison" screens were, as nearly as practicable, of equal stopping power, the aluminium screen, if at all, having a slightly less air-equivalent than that of the mica. In carrying out an experiment, a careful comparison was made of the number of scintillations observed when the 8·6 cm. α-particles were absorbed in each of these screens, the relative positions of the active source and zinc-sulphide screen being fixed during an experiment whilst the absorbing screens were interchanged. As a result of such comparisons it was observed that the number of long-range scintillations was almost the same in the two cases. A slight excess, in favour of the mica, was however always obtained, this excess amounting only to a small fraction—less than 10 per cent. of the total. Thus it was found in one comparative test that the number of scintillations observed through the mica screen averaged 10·0 per minute, whilst with the aluminium screen the average was 9·4 per minute. On another occasion the number through the mica screen averaged 6·0 per minute, whilst aluminium gave 5·5 per minute. The differences are small, and fall very near to the limits of error of experiment. They are significant, however, in suggesting, what otherwise seems probable, that a small fraction, 7 or 8 per cent., of the long-range particles comes from the mica, the greater proportion, over 90 per cent., coming directly from the active deposit.

The two methods of analysis lead therefore to the same result, viz. that the greater proportion of the long-range particles originates in the active deposit. It seems probable, on general grounds, that these particles are α-rays, but this point can only be settled conclusively by some form of magnetic deflexion experiment, in which the value of e/m, and consequently the mass of the particles, is determined *.

§ 4. *Number of Long-range Particles.*

In 1916 a determination was made, by two different methods, of the ratio of the number of long-range α-particles relative to the number of ordinary α-particles. The value then obtained, 1 in 10,000, was based on the results of a few observations only, and may consequently be not entirely reliable. Since the publication of Sir Ernest Rutherford's paper on "Collision of α-particles with Light Atoms"†, it became of considerable importance to re-determine this ratio more carefully.

The first method employed in the earlier determination of the ratio was briefly as follows. The number of long-range particles per minute was measured with the source fixed at a known distance from the zinc-sulphide screen. After the active deposit had decayed *in situ* for 32 hours, the mica absorbing screen was removed and the number of α-particles from thorium C measured at distances from 6 to 7 cm. so as to include only the 8·6 cm. α-particles from thorium C. Reducing both sets of observations to the same distance from the active source and correcting for the decay in 32 hours, a value of the ratio could at once be deduced. In the second method the source and zinc-sulphide screen remained fixed at the same distance apart (about 5 mm.) throughout the measurements, a comparison being made between the number of long-range particles penetrating the mica screen initially and the total number of α-particles (5·0 and 8·6 cm.) from thorium C six days afterwards (the mica screen being removed). Assuming the decay period of thorium active deposit to be 10·6 hours, the number of ordinary α-particles at the commencement of the experiment could easily be obtained and the required ratio evaluated.

Recent experiments have proved the latter to be the more reliable method, the results now given being based entirely

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* The magnetic deflexion of the long-range particles has since been accomplished by Sir Ernest Rutherford, who has proved conclusively that the long-range particles are α-rays.

† Phil. Mag. June 1919.

on such observations. The following example serves to illustrate the method of determining the ratio :—

(a) *Long-range particles* (through mica screen of air-equivalent 8·6 cm.).

Number of scintillations on Nov. 24, 1919, at 8 P.M.	8·3 per minute (mean of 15 observations each of 1 minute duration).

(b) *Ordinary α-particles* (mica screen removed, source and zinc-sulphide screen undisturbed).

Number of scintillations on Nov. 30, 1919, at 8 P.M. (144 hrs. after (a)).	10·2 per minute (mean of 15 observations each of 1 minute duration).

Let N be the total number of ordinary α-particles emitted per minute from the source to the zinc-sulphide screen on Nov. 24 at 8 P.M., and assuming the decay constant λ for thorium active deposit to be 0·064 hr.$^{-1}$, we have

$$\frac{10\cdot2}{\mathrm{N}}=e^{-\cdot064\times144},$$

whence N=102,000 per minute at 8 P.M., Nov. 24, 1919.

(c) *Ratio.*

$$\frac{\text{Number of long-range particles}}{\text{Number of ordinary }\alpha\text{-particles}}=\frac{8\cdot3}{102{,}000}=\frac{1}{12{,}300}.$$

The most probable value of the ratio, obtained as a result of all determinations similar to this, is 1 in 11,000, a value only slightly lower than that given in 1916, viz. 1 in 10,000.

The ratio 1 in 11,000 was obtained with a mica absorbing screen. It has been mentioned, however, in paragraph § 3, that the value observed when an aluminium screen was used was about 7 or 8 per cent. lower than this, viz. 1 in 12,000. If the difference between the two values is real, the result implies that the proportion of "oxygen atoms" from the mica to the total number of ordinary α-particles is in the ratio of 1 to 100,000 approx., a value of the same order as that obtained by Sir Ernest Rutherford when RaC α-particles were fired into the mica.

Further experiments should be made, however, with a more active source before this point can be definitely established.

§ 5. *Particles of Range greater than* 11·3 *cm. Hydrogen atoms.*

When RaC α-particles (7·0 cm. range) were absorbed in mica, Sir E. Rutherford showed that, in addition to the oxygen atoms of range 9 cm., a number of hydrogen atoms of range 28 cm. were always observed[*]. It might be inferred, therefore, that hydrogen particles would be produced in a similar manner by thorium C α-particles.

A careful examination has consequently been made to detect particles of ranges greater than 11·3 cm. For this purpose the source of thorium active deposit was covered with the 8·6 cm. mica screen as before, and with an additional 3·6 cm. mica layer which, with 0·5 cm. of air, makes a total of 12·7 cm. as the air-equivalent between active source and zinc-sulphide screen. In one series of observations only 2 scintillations were observed in 20 "counts," each of 1 minute duration—*i. e.*, the average number was 0·1 per minute, whereas the number of particles whose range exceeded 9 cm. was 8 per minute in the same experiment. On this basis, it appears that the number of particles of range greater than 12·7 cm. probably does not exceed 1 per cent. of the number of range greater than 9 cm., *i. e.* not more than 1 in 10^6 of the total number of α-particles emitted from thorium C. It is hardly necessary to explain that this value is only very approximate, depending, as it does, on such slender evidence.

§ 6. *Summary of Results.*

The results obtained in 1916 relating to the long-range particles from thorium active deposit have been confirmed. A re-determination of the total number of long-range particles (ranges exceeding 8·6 cm.) to the total number of ordinary α-particles (ranges 5·0 and 8·6 cm.) gives a value 1 in 11,000 as compared with 1 in 10,000 obtained in 1916.

It has been shown that at least 90 per cent. of these particles originate in the active deposit, whilst the remainder are probably produced by intimate collision of α-particles with the oxygen atoms contained in the mica absorbing screen which was employed to stop the α-particles of 8·6 cm. range. Slight evidence has been obtained of the existence of high velocity particles, probably hydrogen atoms, of ranges exceeding 11·3 cm. The proportion of these relative to the total number of long-range particles is probably not greater than 1 in 100. Thus we have for thorium active deposit :—

(1) Number of long-range particles (probably α-rays) relative to the total number of α-particles. } 1 in 10^4 approx.

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[*] Rutherford, Phil. Mag. June 1919.

(2) Relative number of high-velocity oxygen atoms produced by collision of α-particles with oxygen in mica.	1 in 10^5 approx.
(3) Relative number of particles of range exceeding 11·3 cm. (probably hydrogen atoms).	1 in 10^6 approx.

Items (2) and (3) are only rough estimates based on rather slender evidence.

In conclusion, I should like to express my warmest thanks to Sir Ernest Rutherford for many helpful suggestions during the progress of this research. Also to Prof. L. R. Wilberforce for his kindness in lending me the source of radio-thorium and the microscope used in the experiments.

LIII. *On a Method of Analysis suitable for the Differential Equations of Mathematical Physics.* *By* W. L. Cowley, *A.R.C.Sc., D.I.C., and* H. Levy, *M.A., D.Sc.*[*]

Part I.

ATTEMPTS to obtain solutions of the differential equations of mathematical physics applicable to problems of practical importance are usually handicapped by the fact that the differential equations arising cannot be solved in terms of the simpler functions without the introduction of restrictions tending to invalidate or limit seriously the application. To the practical calculator, however, the derivation of a complete solution expressible in this form is of merely secondary importance, and does not in fact frequently weigh with him at all. Since the instruments at his disposal can at best only attain a certain degree of accuracy, an equally approximate solution, otherwise unrestricted, of the equations is sufficient. But when the investigation does not merely deal with the evaluation of some quantity for a definite problem, but with the selection of a particular member of a class satisfying certain requirements, particular forms of solution, especially particular forms of expansion, are most convenient for the analysis.

On general considerations the practical requirements that must be satisfied by such solutions are easily outlined. The evaluation of the separate terms in the solution should be suitable for application in a drawing office; that is to say, they should depend purely on graphical or arithmetical

[*] Communicated by the Authors.

methods demanding the use of a slide rule, a multiplying machine, or planimeter. Graphical processes depending on differentiation ought to be avoided, as experience shows that without a considerable degree of labour inaccuracy is unavoidable. In the second place, if the expansion is in series the convergence should be as rapid as possible and the rate of convergence should be more or less evident at each stage. Of the common methods of expansion, the Fourier's series type is perhaps the most suitable so far as these considerations are concerned. An ordinary power series, on the other hand, may or may not be rapidly convergent over the whole range according to the nature of the problem considered. The crux of the matter is reached from a third consideration, which requires that the form of expansion should be sufficiently expressive to allow of further analysis of the solution to determine the properties of, say, the class of problem under consideration and the particular member desired. Consider, for example, the problem where a series of struts whose law of cross-section is given for the whole class, say $I=I_0R$, where I and I_0 are the moments of inertia at any section x and at a standard section respectively, and R is a non-dimensional function of x specifying the law of variation in I along the length.

Let F be an end-thrust acting longitudinally along the strut and l the length of the strut. Let the problem be, to determine the particular member (that is to say, the value of l) of this class which when under given eccentricity will give a deflexion α at the middle, the strut being simply supported at the ends. The differential equation for the flexure of the strut is

$$\mathrm{EI}\frac{d^2y}{dx^2}+\mathrm{F}y=0$$

when the origin is taken at one end.

This of course holds for all values of EI_0, F, and l, and therefore represents apparently at first sight a four-fold infinity of problems; but it is easily shown that this is not the case, but rather that the passage from one member of the class to another is brought about by the variation in value of one expression involving all these quantities, so that in reality there will exist merely a one-fold infinity of members in the class. Writing $x=lx'$, $y=ly'$ and inserting this in the differential equation we find, dropping dashes,

$$\frac{d^2y}{dx^2}+\mathrm{C}\frac{y}{\mathrm{R}}=0,$$

where $\mathrm{C}=\frac{\mathrm{F}l^2}{\mathrm{EI}_0}$ and R is a function of x.

For a given form of R and definite end conditions the nature of the solution of this equation depends only upon the value adopted for C, and all the members of the class are included among the one-fold values of this class variable. In fact, if the differential equation representing any physical problem is thrown into the non-dimensional form, one or more class variables, themselves non-dimensional, will be derived in the manner shown above. The differential equation of the steady two-dimensional flow of a viscous fluid, for example, when thrown into the non-dimensional form

$$\nabla^2\zeta = \frac{Vl}{\nu}\left[u\frac{\partial\zeta}{\partial x} + v\frac{\partial\zeta}{\partial y}\right],$$

where ζ represents the vorticity, V the speed of the moving body, l its length, and ν the coefficient of viscosity, indicates that the properties of the motion will centre round a consideration of the modifications in the solution as the non-dimensional quantity Vl/ν varies.

The general solution of differential equations involving such a class variable satisfying certain boundary conditions which may themselves involve the class variable may be written in the form

$$f(x, y, C) = 0. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

The ordinary power series solution presents (1) in the form

$$a_0 + a_1x + b_1y + a_2x^2 + b_2y^2 + c_2xy + \text{etc.} = 0, \quad . \quad (2)$$

where of course the quantities a, b, etc. are functions of the class variable C. In this form each of these coefficients may be more or less complicated expressions of C, and the whole expression is not in a form suitable for investigating the relative properties of the members of the class, although it may be convenient for other purposes.

There is, however, another method of presentation of (1) more suitable from the present point of view, viz.:

$$X_0 + X_1c + X_2c^2 + \text{etc.} = y, \quad . \quad . \quad . \quad . \quad (3)$$

where X_0, X_1, etc. are now functions of x. The investigation of the properties of the class and selection of a particular member satisfying certain optimum conditions can then be easily derived from this expansion. Whether or not the solution can be thrown into this form in general is a matter which cannot be discussed here, but will clearly depend upon the form of the differential equation and the boundary conditions. Each case will require to be treated on its merits.

Since the above expression for y may be assumed to remain true for an infinite number of values of the class variable, the boundary relations for each of the functions X_0, X_1, etc. can be expressed in a particularly simple form. Suppose, for example, the boundary condition be that at a certain point, say $x=\alpha$,

$$y = \alpha_0 + \alpha_1 C + \alpha_2 C^2 + \text{etc.};$$

then the boundary conditions for the functions X_0 etc. are, at $x=\alpha$,

$$X_0 = \alpha_0,\ X_1 = \alpha_1,\ \text{etc.}$$

In the same way, if the differential equation be one involving more than one independent variable, a similar series of boundary conditions for X_0, X_1, etc., now functions of more than one independent variable, are easily derived.

In those cases where the equations are linear and involve only one independent variable there is another method of presentation which bears a close resemblance to that given above. Taking an equation of the fourth order for example, the general solution may be written in the form

$$y = \mathrm{A}f_1(x) + \mathrm{B}f_2(x) + \mathrm{C}f_3(x) + \mathrm{D}f_4(x), \quad . \quad . \quad (4)$$

where A, B, C, and D are four quantities to be determined by the boundary conditions. Instead of, as in the previous method, supposing y expanded in a power series of the class variable, we may imagine each of the functions f_1, f_2, f_3, f_4 so expanded. If these expansions can be determined, the boundary conditions when inserted will provide similar expansions for A, B, C, and D. For certain cases, particularly with linear equations of this type, this latter form of presentation is frequently the simplest. The first step in the analysis is to determine the form of the coefficients in the expansion. Unless it can be proved that a convergent expansion of this type is always possible, it will be necessary to check the result *à posteriori*. For illustrating these points the classical engineering problem of determining the whirling speed of a shaft whose cross-section varies along its length will be treated in detail. It will be seen that all the conditions of rapidity of convergency and ease of calculation are most satisfactorily fulfilled. Generally the method that will be adopted will be to assume an expansion for the dependent variable as a power series in the class variable, to insert this in the differential equation, and equate the coefficients of the various powers of the class variable to zero. A series of simpler differential equations will then be obtained for each of the coefficients, with simplified boundary conditions. Consider the following example. The equation whose solution is required at whirling is, expressed in non-dimensional form,

$$\frac{d^2}{dx^2}\left(\mathrm{R}\frac{d^2y}{dx^2}\right)-\theta^4\mu y=0, \quad . \quad . \quad . \quad . \quad (5)$$

where $\mathrm{R}=\mathrm{I}/\mathrm{I}_0$ = the ratio of the moments of inertia at any point x to that at any standard section ;

y = deflexion of the shaft axis at any point x.

This quantity is assumed so small that it is legitimate to write $\frac{1}{l}\frac{d^2y}{dx^2}$ for the curvature where l is the length of the shaft.

$$\theta^4=\frac{\omega_0}{g}\frac{q^2l^4}{\mathrm{EI}_0}, \quad \mu=\frac{\omega}{\omega_0}, \quad . \quad . \quad . \quad . \quad (6)$$

where E = elastic constant, $\frac{\omega}{g}$ = mass per unit length, and $2\pi/q$ = period of rotation of shaft. The boundary conditions that we may suppose imposed on the shaft are given as deflexions and bending moments at the extremities of the shaft.

For the sake of definiteness, let these conditions be

$$\left.\begin{array}{lll} x=0 & y=0, & \text{Bending moment} = \mathrm{M}_0\mathrm{EI}_0, \\ x=1 & y=0, & \quad ,, \qquad\quad ,, \qquad\; = 0. \end{array}\right\} \quad . \quad . \quad (7)$$

The class variable in this case is clearly θ^4, while R and μ are given non-dimensional functions of x. It will for the moment be assumed that y can be expanded in ascending powers of θ^4 in the form

$$y=y_0+y_1\theta^4+y_2\theta^8+y_3\theta^{12}. \quad . \quad . \quad . \quad . \quad (8)$$

Inserting this in equation (5) and equating coefficients of powers of θ to zero,

$$\begin{array}{l:l:l:} \frac{d^2}{dx^2}\left(\mathrm{R}\frac{d^2y_0}{dx^2}\right) & -\theta^4\mu y_0 & \\ & +\theta^4\frac{d^2}{dx^2}\left(\mathrm{R}\frac{d^2y_1}{dx^2}\right) & -\theta^8\mu y_1 \\ & & +\theta^8\frac{d^2}{dx^2}\left(\mathrm{R}\frac{d^2y_1}{dx^2}\right) \end{array}$$

$$\ldots\ldots=0.$$

Hence

$$\frac{d^2}{dx^2}\left(\mathrm{R}\frac{d^2y_0}{dx^2}\right) = 0, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (9_0)$$

$$\frac{d^2}{dx^2}\left(\mathrm{R}\frac{d^2y_n}{dx^2}\right) = \mu y_{n-1}. \quad . \quad . \quad . \quad . \quad . \qquad (9_n)$$

From (9_0) it follows that

$$y_0 = \mathrm{A}\int_0^x dx\int_0^x \frac{x\,dx}{\mathrm{R}} + \mathrm{B}\int_0^x dx\int_0^x \frac{dx}{\mathrm{R}} + \mathrm{C}x + \mathrm{D}\,; \quad . \quad . \qquad (10)$$

therefore

$$\begin{aligned} y_1 = &\int_0^x dx\int_0^x \frac{dx}{\mathrm{R}}\int_0^x dx\int_0^x dx\,\mu y_0 \\ = \mathrm{A}&\int_0^x dx\int_0^x \frac{dx}{\mathrm{R}}\int_0^x dx\int_0^x dx\,\mu\int_0^x dx\int_0^x \frac{x\,dx}{\mathrm{R}} \\ +\mathrm{B}&\int_0^x dx\int_0^x \frac{dx}{\mathrm{R}}\int_0^x dx\int_0^x \mu\,dx\int_0^x dx\int_0^x \frac{dx}{\mathrm{R}} \\ +\mathrm{C}&\int_0^x dx\int_0^x \frac{dx}{\mathrm{R}}\int_0^x dx\int_0^x dx\,\mu x + \mathrm{D}\int_0^x dx\int_0^x \frac{dx}{\mathrm{R}}\int_0^x \mu\,dx \end{aligned}$$

etc. . . . . (11)

Inserting this expression for $y_0\, y_1 \ldots$ in (8), y takes the form

$$y = \mathrm{A}f_1(x) + \mathrm{B}f_2(x) + \mathrm{C}f_3(x) + \mathrm{D}f_4(x), \quad . \quad . \qquad (12)$$

where

$$f_1(x) = \int_0^x dx\int_0^x \frac{x\,dx}{\mathrm{R}} + \theta^4\int_0^x dx\int_0^x \frac{dx}{\mathrm{R}}\int_0^x dx\int_0^x \mu\,dx\int_0^x dx\int_0^x \frac{x\,dx}{\mathrm{R}} + \ldots \qquad (13)$$

$$f_2(x) = \int_0^x dx\int_0^x \frac{dx}{\mathrm{R}} + \theta^4\int_0^x dx\int_0^x \frac{dx}{\mathrm{R}}\int_0^x dx\int_0^x \mu\,dx\int_0^x dx\int_0^x \frac{dx}{\mathrm{R}} + \ldots \qquad (14)$$

$$f_3(x) = x + \theta^4\int_0^x dx\int_0^x \frac{dx}{\mathrm{R}}\int_0^x dx\int_0^x \mu x\,dx + \ldots \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (15)$$

$$f_4(x) = 1 + \theta^4\int_0^x dx\int_0^x \frac{dx}{\mathrm{R}}\int_0^x dx\int_0^x \mu\,dx + \ldots \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (16)$$

The forms which these integrals adopt, although apparently involved, are in reality very simple. It should be remarked in the first place that the law of formation is clearly bound up with the relation

$$y_n = \int_0^x dx \int_0^x \frac{dx}{R} \int_0^x dx \int_0^x dx\,\mu y_{n-1}.$$

The four arbitrary constants A, B, C, D are as yet undetermined, but will shortly be found from the boundary conditions.

Inserting the boundary conditions (7) in (12), the four following equations for A, B, C, D are easily derived:

$$\left.\begin{aligned} 0 &= Af_1(0)+Bf_2(0)+Cf_3(0)+Df_4(0) \\ 0 &= Af_1(1)+Bf_2(1)+Cf_3(1)+Df_4(1) \\ M_0 &= Af_1''(0)+Bf_2''(0)+Cf_3''(0)+Df_4''(0) \\ 0 &= Af_1''(1)+Bf_2''(1)+Cf_3''(1)+Df_4''(1) \end{aligned}\right\} . \qquad (17)$$

Referring now to the expressions for these functions, it is to be noted that

$$f_1(0) = f_2(0) = f_3(0) = 0, \qquad f_4(0) = 1,$$

$$f_1''(0) = 0, \quad f_2''(0) = 1/R_0, \quad f_3''(0) = 0, \quad f_4''(0) = 0.$$

Hence it follows that $B = M_0R_0$ and $D = 0$, where A and C are likewise easily determined. The interest of this problem does not, however, lie, from the engineer's point of view, in determining the deflected position of the shaft under the applied bending moment EI_0M_0 at the end, for any given period of rotation $2\pi/q$, but rather in finding those particular values of q for which the shaft whirls. Since the analysis is based on the assumption that the deflexions are small, whirling would involve a violation of the basic assumptions, and the deflexion would appear from the equations to be infinite. The condition that this should be so, and therefore that whirling should occur, is easily derived from (17), viz.

$$\begin{vmatrix} f_1(0) & f_2(0) & f_3(0) & f_4(0) \\ f_1(1) & f_2(1) & f_3(1) & f_4(1) \\ f_1''(0) & f_2''(0) & f_3''(0) & f_4''(0) \\ f_1''(1) & f_2''(1) & f_3''(1) & f_4''(1) \end{vmatrix} = 0, \qquad (18)$$

which, when the particular values for the functions are inserted, becomes

$$f_1(1)f_3''(1) = f_1''(1)f_3(1), \quad . \quad . \quad . \quad . \quad (19)$$

where

$$f_1(1)=\int_0^1 dx\int_0^x \frac{x\,dx}{R}+\theta^4\int_0^1 dx\int_0^x \frac{dx}{R}\int_0^x dx\int_0^x \mu\,dx\int_0^x dx\int_0^x \frac{x\,dx}{R}+\dots \quad . \quad . \quad . \qquad (20)$$

$$R_1 f_1''(1)=1+\theta^4\int_0^1 dx\int_0^x \mu\,dx\int_0^x dx\int_0^x \frac{x\,dx}{R}+\dots \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (21)$$

$$f_3(1)=1+\theta^4\int_0^1 dx\int_0^x \frac{dx}{R}\int_0^x dx\int_0^x \mu x\,dx+\dots \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (22)$$

$$R_1 f_3''(1)=\theta^4\int_0^1 dx\int_0^x \mu x\,dx+\dots \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (23)$$

For purposes of evaluation of the coefficients of the powers of θ^4 in each of the above, it is to be noted that the first term on the right of (20) will be automatically evaluated in determining the second term on the right of (21), which again will be evaluated in calculating the second term on the right of (20), and so on. The labour, therefore, in determining the coefficients in the above four series is really only half what it would appear to be at first sight. The geometrical outline of the shaft, and the loading, are of course introduced through the functions R and μ, and since all the integrals can be rapidly evaluated by means of a planimeter, or otherwise, as will shortly be seen, the most general case will be almost as simple as any other by this method. As an illustration, however, an example will be taken which is capable of solution by another method in order that a comparison may be made.

Example.—Consider a homogeneous circular shaft simply supported at the ends and built of three portions, the middle portion being of diameter d and length $l/2$, and the two ends adjoining the supports of diameter $d/4$ and length $l/4$. It is required to find the first whirling speed of this shaft.

The mid-section or the section of reference has a diameter d, and w_0 and I_0 can easily be evaluated from this dimension. μ will be 1/16 over the two end portions and unity over the middle piece, while R will be 1/256 and unity respectively. The series $f_1(x)$ and $f_1''(x)$ can be worked out simultaneously as well as the two series $f_3(x)$ and $f_3''(x)$, as can be seen from the expansions. The steps in the calculation are evident. For example, in evaluating $f_1(x)$ and $f_1''(x)$, x/R is first obtained for various values of x. Each figure is then divided by the maximum occurring in the column for convenience in plotting, a method that may be used throughout whenever a curve is to be integrated. From the graph the integral curve, $\int_0^x \frac{x\,dx}{R}$ can easily be obtained, the mean-ordinate method being most convenient. Thus at 0·05 the integral will be 0·05 times the average height of the curve between 0 and 0·05, at 0·1 it will be the area up to 0·05 plus 0·05 times the average ordinate between 0·05 and 0·1, and so on. These integrations can rapidly be performed, and if it is necessary for the sake of accuracy to integrate thoroughly a portion of the curve where the ordinates are small, this portion can be plotted to a larger scale and allowance made accordingly.

The actual details in the calculation will be omitted, but experience shows that the various steps in the process can easily be carried through accurately by anyone not even conversant with the calculus. We then find

$$
\begin{aligned}
0 &= f_1(1)\,f_3''(1) - f_3(1)\,f_1''(1) \\
&= \theta^4(13{\cdot}28 + 2{\cdot}84\,\theta^4 + 0{\cdot}00127\,\theta^8)(0{\cdot}1174 + 2{\cdot}4\times 10^{-4}\,\theta^4 \\
&\qquad\qquad + 5{\cdot}54\times 10^{-8}\theta^8) \\
&\quad - (1 + 0{\cdot}585\,\theta^4 + 1{\cdot}033\times 10^{-3}\,\theta^8 + 2{\cdot}656\times 10^{-7}\times 10^{12}) \\
&\quad \times (1 + 0{\cdot}601\,\theta^4 + 2{\cdot}36\times 10^{-4}\,\theta^8 + 5{\cdot}1 + 10^{-8}\,\theta^{12}),
\end{aligned}
$$

i. e. $\quad 1 - 0{\cdot}375\,\theta^4 + 0{\cdot}015\,\theta^8 - 0{\cdot}000072\,\theta^{12}\ldots = 0.$

This gives

$$\theta^4 = \frac{w_0 q^2 l^4}{g\mathrm{EI}_0} = 3,$$

from which q is at once derived, and it will be seen that the term in θ^{12} may be neglected, the series being so rapidly convergent.

As a check, the whirling speed in the present instance can be found by an alternative method applicable when the shaft is composed only of portions of uniform section. The method consists in supposing the shaft simply supported at the positions where the section suddenly changes, as well as at the ends, and then utilizing the conditions that the reactions at these intermediate supports are zero. The portions between two neighbouring supports are uniform, and therefore the functions f_1, f_2, etc. are directly integrable algebraically*. This method gives

$$\theta^4 = 2{\cdot}91$$

$$\text{as against}\quad \theta^4 = 3 \quad \text{by the previous method.}$$

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* Advisory Committee for Aeronautics Report, Q & M. 690.

It has become apparent that for all practical purposes the series used in the foregoing analysis are rapidly convergent. In point of fact the rapidity of convergence can easily be investigated, for each one of the functions $f_1(x) \ldots f_4(x)$ is absolutely convergent.

Let μ_1 and R_1 be the greatest and least values of μ and R over the range of x considered ; then, for example,

$$|f_4(x)| < 1+\frac{\theta^4\mu_1}{R_1}\int_0^x dx\int_0^x dx\int_0^x dx\int_0^x dx + \ldots ;$$

$$i.\ e., \qquad < 1+\frac{\theta^4x^4}{4\,!}\,\frac{\mu_1}{R_1}+\frac{\theta^8x^8}{8\,!}\,\frac{{\mu_1}^2}{{R_1}^2}\ldots .$$

Hence

$$\frac{n\text{th term}}{n-1\text{th term}}=\frac{\theta^4x^4\mu_1}{R_1}\times\frac{1}{(4n-4)(4n-5)(4n-6)(4n-7)},$$

showing that the series converges with extreme rapidity.

PART II.

§ 1. *General.*—The equations of the flow of an incompressible viscous fluid stand almost unique in mathematical physics in virtue of the fact that no single unrestricted and complete solution has yet been obtained in any problem. Failing a complete mathematical breakdown of the equations, attempts have been made to derive information regarding fluid-motion phenomena by the elimination on the one hand of one of the most vital characteristics of the fluid, viscosity, and on the other by such limitations on the motions as would involve no effect due to the inertia of the fluid. From the physical standpoint, however, based on the accumulated experimental experience of these questions acquired in hydro- and aerodynamical research, one is inevitably driven to the conclusion that both these factors are of prime importance and must play a prominent part in any explanation of the characteristics of fluid motion.

Certain outstanding experimental facts found by previous workers may here be mentioned, as they bear critically on the problem at issue. Osborne Reynolds [*] discovered the existence of a critical state of flow originating during flow of water in pipes above a certain value of vl/ν, where v is the mean velocity in the pipe, l the diameter, and ν the viscosity of water. Further investigations on this point by

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[*] Phil. Trans. 1883, p. 935.

Stanton and Pannell[*] have borne out the fact that the motion of air in pipes obeyed the same law and likewise exhibited a corresponding critical state for the same value of Reynold's number. These two researches and the vast additional amount of aeronautical evidence accumulated during the past few years indicate that the whole process of the motion in any problem centres round the value of the non-dimensional expression vl/ν. This vital fact will be utilized as suggesting a means for analysing the equation. With regard to the state of affairs existing at the surface of a body in a fluid, it has long been recognized that on general grounds no relative motion may be expected to exist, and much confirmatory evidence for this view has been found for slow motions of fluids. For the present it will be assumed that the fluid with which we deal has density (ρ), viscosity (ν), and is incompressible; and these are the only properties of the fluid that affect the question apart from the "no slip" condition at the boundary.

§ 2. Under these circumstances the accepted equations of motion restricting ourselves to two dimensions:

$$X-\frac{1}{\rho}\frac{\partial p}{\partial x}=\frac{\partial u}{\partial t}+u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}-\nu\nabla^2 u, \quad . \quad . \quad . \quad (1)$$

$$Y-\frac{1}{\rho}\frac{\partial p}{\partial y}=\frac{\partial v}{\partial t}+u\frac{\partial v}{\partial x}+v\frac{\partial v}{\partial y}-\nu\nabla^2 v, \quad . \quad . \quad . \quad (2)$$

along with the equation of continuity,

$$\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}=0, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

where the usual notation is employed.

The last equation admits of the introduction of the stream-function ψ without loss of generality, so that the fundamental equations may be re-written:

$$X-\frac{1}{\rho}\frac{\partial p}{\partial x}=-\frac{\partial}{\partial t}\left(\frac{\partial\psi}{\partial y}\right)+\frac{\partial\psi}{\partial y}\frac{\partial}{\partial x}\left[\frac{\partial\psi}{\partial y}\right]-\frac{\partial\psi}{\partial x}\frac{\partial^2\psi}{\partial y^2}+\nu\frac{\partial}{\partial y}\nabla^2\psi, \quad . \quad . \quad (4)$$

$$Y-\frac{1}{\rho}\frac{\partial p}{\partial y}=\frac{\partial}{\partial y}\left(\frac{\partial\psi}{\partial x}\right)-\frac{\partial\psi}{\partial y}\frac{\partial^2\psi}{\partial x^2}+\frac{\partial\psi}{\partial x}\frac{\partial}{\partial y}\left(\frac{\partial\psi}{\partial x}\right)-\nu\frac{\partial}{\partial x}\nabla^2\psi. \quad . \quad . \quad (5)$$

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[*] Phil. Trans. A, vol. 214.

The usual process is to eliminate the pressure terms from these equations, giving an equation in ψ alone in the form

$$\frac{\partial Y}{\partial x}-\frac{\partial X}{\partial y}=\frac{\partial}{\partial t}\nabla^2\psi-\frac{\partial\psi}{\partial y}\frac{\partial}{\partial x}\nabla^2\psi+\frac{\partial}{\partial x}\frac{\partial}{\partial y}\nabla^2\psi-\nu\nabla^4\psi. \quad (6)$$

This equation, derived by differentiating (4) and (5), is one order higher than each of the original, and therefore an equivalent system to (4) and (5) must associate with (6), an equation of one order lower, otherwise extraneous problems may be introduced not contemplated in the discussion. These extraneous cases may be excluded by the following process. Let A and B be two points situated in the fluid, the extremities of any given curve ; then from equations (4) and (5),

$$\int_B^A (Xdx+Ydy)-\frac{1}{\rho}(p_A-p_B)=\int_B^A\frac{\partial}{\partial t}(udx+vdy)$$
$$+\int_B^A u\left\{\frac{\partial u}{\partial x}dx+\frac{\partial v}{\partial x}dy\right\}+\int_B^A v\left\{\frac{\partial u}{\partial y}dx+\frac{\partial v}{\partial y}dy\right\}$$
$$-v\int_B^A(\nabla^2 u\,dx+\nabla^2 v\,dy). \quad . \quad . \quad . \quad . \quad . \quad . \quad (7)$$

Equations (6) and (7) are the equivalent of (4) and (5). Equation (7) is of course an expression giving the pressure difference between two points A and B, and as such need not be considered in relation to (6) except in so far as it must be used for the interpretation of any externally applied conditions. It will then furnish information with which the solution of (6) must be consistent. For example, if points A and B are maintained through some external agency at a constant difference in pressure P, then equation (7) will furnish a condition which must be satisfied by the solution of (6) along every curve, lying wholly in the fluid, that can be drawn between A and B. The necessary and sufficient condition that the external forces can only be of a conservative nature is that $\int_B^A (Xdx+Ydy)$ should be independent of the path. It follows in that case that the path from A to B may be arbitrarily chosen, and one may be selected in any convenient manner, the integration along all other paths reconcilable to this being the same. In general it will be found most convenient to select as a portion of the path the contour of the boundary along which the velocities are specified.

Consider a contour completely enclosing one of the boundaries; then the terms on the left-hand side vanish, and the equation (7) becomes

$$\int_c \left(\frac{\mathrm{D}u}{\mathrm{D}t}dx+\frac{\mathrm{D}v}{\mathrm{D}t}dy\right)=\nu\int_c(\nabla^2 u\,dx+\nabla^2 v\,dy)$$

$$=2\nu\int\frac{\partial\zeta}{\partial n}ds. \quad . \quad . \quad . \quad . \quad . \quad (7)$$

Any such contour is reconcilable with a contour round the actual outline of the boundary. Equation (7) has three particular forms, when no external forces exist, of interest to the present problem. Round a body at rest, where u, v, $\frac{\partial u}{\partial t}$, etc. are zero, it takes the form

$$\int_{\text{body}} 2\nu\frac{\partial\zeta}{\partial n}ds=0. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (7\,a)$$

Round a body moving with constant velocity U in the direction x,

$$\int_{\text{body}} 2\nu\frac{\partial\zeta}{\partial n}ds=\int_{\text{body}}\frac{\partial}{\partial n}\left(\mathrm{U}v-\frac{\partial\zeta}{\partial n}\right)ds. \quad . \quad . \quad (7\,b)$$

Along a boundary at rest, such as a wall of a channel where the fluid is under a given pressure-head P between the ends maintaining the flow, it becomes

$$2\nu\int_{\text{boundary}}\frac{\partial\zeta}{\partial n}ds=\frac{\mathrm{P}}{\rho}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (7\,c)$$

Before a solution of equation (6) can be accepted, therefore, where the expression for ψ satisfies the appropriate boundary condition of the problem, it is necessary to satisfy the various forms of (7) where the integral is taken round each boundary of the fluid. The boundary conditions that are to be inserted are of course those that involve the statement that there is no slip.

§ 3. *Steady motion in two dimensions.*—Consider the case where the boundaries of the fluid are in steady motion, and where in addition it is assumed that the fluid everywhere moves steadily. As far as experimental evidence shows, these two assumptions would appear to be entirely independent of each other; for in all known cases above a certain value of vl/ν a steadily moving body appears to give rise to a periodic eddying in its wake. Below this critical value, however, these two assumptions appear to be perfectly consistent. If steady motion of the fluid is not possible above this critical value, an indication of this would be expected to be given from the form of the solution in the steady case.

It will be convenient to throw the equations of motion into the non-dimensional form by the following substitutions :—

$$x = \mathrm{L}x',$$
$$y = \mathrm{L}y',$$

where L is the length of any particular part of the moving body, and x' and y' are now merely variable numbers ;

$$u = \mathrm{U}u',$$
$$v = \mathrm{U}v',$$

where U is the steady velocity of one of the boundaries, and u' and v' also variable numbers. Under these circumstances

$$\psi = \mathrm{UL}\psi'.$$

Inserting these in the equations of motion and in the boundary conditions, we find, omitting dashes,

$$\nabla^4\psi + \mathrm{C}\left\{\frac{\partial\psi}{\partial y}\frac{\partial}{\partial x}\nabla^2\psi - \frac{\partial\psi}{\partial x}\frac{\partial}{\partial y}\nabla^2\psi\right\} = 0, \quad . \quad . \quad (8)$$

where $\mathrm{C} = \mathrm{UL}/\nu$.

In the case where the body is in motion with uniform velocity in the direction of x there are in addition

$$\left.\begin{array}{l} \dfrac{\partial\psi}{\partial x} = \ \ 0 \\ \dfrac{\partial\psi}{\partial y} = -1 \end{array}\right\} \text{round the moving body ;} \quad . \quad (10)$$

and when the boundary is at rest

$$\left.\begin{array}{l} \dfrac{\partial\psi}{\partial x} = 0 \\ \dfrac{\partial\psi}{\partial y} = 0 \end{array}\right\} \text{round body at rest.} \quad . \quad . \quad . \quad (11)$$

The integral condition in these two cases becomes

$$2\int_{\text{body}} \frac{\partial\zeta}{\partial n}\,ds = \mathrm{C}\int_{\text{body}} \frac{\partial v}{\partial n}\,ds \ \text{ round the moving body,} \qquad (12)$$

$$2\int_{\text{body}} \frac{\partial\zeta}{\partial n}\,ds = 0 \ \text{ round body at rest.} \quad . \quad . \quad . \quad . \quad . \quad . \quad (13)$$

It is in this non-dimensional form that the equations may be more conveniently analysed. Reverting to the experimentally known facts that the value of UL/ν determines for any particular problem the nature of the flow, we see that this finds its counterpart in the presence of that quantity as a variable parameter C in the differential equation. It is clear, in fact, that mathematically as well as physically the whole problem may be made to centre round this parameter; and since we are more directly concerned in the modification in the state of the flow over the whole field as it were as C varies, rather than with a comparison with the state of flow at one point in the x, y plane with that of another, the most natural form of solution that is suggested would be a solution as a series in powers of C. Whether or not such an expansion is possible as a convergent series is a matter which will be entered into shortly, but for the moment it is proposed to assume that the expression for ψ the stream-function may be written in the form

$$\psi = \psi_0 + \psi_1 C + \psi_2 C^2 + \ldots \quad . \quad . \quad . \quad (14)$$

[It may here be remarked that if the problem of the motion of a body had been solved for a given value of C, the solution for the same body moving backwards may be derived by writing $-C$ for C in this expression.] Inserting this in equations (8), (10), (12), etc., and equating the coefficients of each power of C to zero, since the equations hold for all values of that variable, the following system is obtained :—

$$\left.\begin{array}{ll}
\nabla^4\psi_0 = 0, & \quad . \quad . \quad . \quad . \quad . \quad (8a)\\
\dfrac{\partial\psi_0}{\partial x} = 0, \text{ and } \dfrac{\partial\psi_0}{\partial y} = -1 & \quad . \quad . \quad . \quad . \quad . \quad (10a)\\
\text{round the moving body} & \\
\displaystyle\int_{\text{body}} ds \frac{\partial}{\partial n}(\nabla^2\psi_0) = 0, & \quad . \quad . \quad . \quad . \quad . \quad (12a)
\end{array}\right\} \text{Coefficient of } c^0.$$

$$\left.\begin{array}{ll}
\nabla^4\psi_1 = \dfrac{\partial\psi_0}{\partial y}\dfrac{\partial}{\partial x}\nabla^2\psi_0 - \dfrac{\partial\psi_0}{\partial x}\dfrac{\partial}{\partial y}\nabla^2\psi_0, & \quad . \quad . \quad . \quad (8b)\\
\dfrac{\partial\psi_1}{\partial x} = 0, \text{ and } \dfrac{\partial\psi_1}{\partial y} = 0 & \quad . \quad . \quad . \quad . \quad . \quad (10b)\\
\text{round the boundaries} & \\
\displaystyle\int_{\text{body}} ds \frac{\partial}{\partial n}\nabla^2\psi_1 = \int_{\text{body}} ds \frac{\partial}{\partial n}\frac{\partial\psi_0}{\partial n}, & \quad . \quad . \quad . \quad . \quad . \quad (12b)
\end{array}\right\} \text{Coefficient of } c'.$$

etc., etc.

In the above the expanded expression for ψ has been inserted not merely in the differential equation, but also in the integral and boundary conditions, and in all cases the coefficients of powers of C equated to zero. It will be seen that there are thus sufficient equations and boundary conditions for each of the functions ψ_0, ψ_1, etc. to determine them uniquely. It is not proposed here to prove the uniqueness of these expressions, but rather to point to an analogous series of physical problems with which each of these equations correspond.

§ 4. *Parallelism in the theory of elastic plates. Flexure of a flat plate.*—If a flat plate of flexural rigidity EI per unit width, Poisson's ratio σ be loaded laterally with an intensity Z per unit area, the plate being supported in any given manner along the edges, then if ψ be the deflexion at any point measured relative to a given horizontal surface, the equation determining ψ is

$$D\nabla^4\psi = Z, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (15)$$

where $D = EI/(1-\sigma^2)$ and $I = 2/3h^3$, h = thickness of plate, while the shearing force N per unit length of any curve drawn upon the plate is in the direction perpendicular to the plate;

$$N = -D\frac{\partial}{\partial n}\nabla^2\psi, \quad . \quad . \quad . \quad . \quad . \quad (16)$$

when n represents the normal to the curve.

A complete parallelism may now be established between the problem of the flexure of a flat plate under lateral loading and each of the problems involved in the determination of the functions $\psi_0, \psi_1, \ldots$, the boundaries of the flat plate being the same as those of the fluid.

For the evaluation of the function ψ in that case the following equations and boundary conditions were required to be satisfied:—

$$\nabla^4\psi_n = f(\psi_0, \psi_1 \ldots \psi_{n-1})\,; \quad n \neq 0,$$

$$\frac{\partial\psi_n}{\partial x} = \frac{\partial\psi_n}{\partial y} = 0 \text{ along every boundary.}$$

$$\int_{\text{body}} \frac{\partial}{\partial n}(\nabla^2\psi_n)\,ds = \int_{\text{body}} ds\frac{\partial}{\partial n}\left(\frac{\partial\psi_{n-1}}{\partial x}\right).$$

For the evaluation of ψ_0,

$$\nabla^4\psi_0 = 0,$$

$$\frac{\partial\psi_0}{\partial y} = -1, \quad \frac{\partial\psi_0}{\partial x} = 0 \text{ along the moving boundary,}$$

and

$$\int_{\text{body}} \frac{\partial}{\partial n}(\nabla^2\psi_0)\,ds = 0 \text{ round the body.}$$

If now the value of ψ for the deflexion of the flat plate be identified with the stream-function component ψ_0, it is a simple matter to interpret the boundary conditions in terms of those for the plate.

Consider first the determination of ψ_0. The fundamental equation

$$\nabla^4\psi_0 = 0$$

indicates that it may be taken to represent the deflexion of an unloaded flat plate.

Along the curve representing the moving boundary, however, the two component slopes are given—viz., $\frac{\partial\psi_0}{\partial x} = 0$, $\frac{\partial\psi_0}{\partial y} = -1$; and this curve must therefore be clamped to have these slopes at each point. Although this determines the relative elevation of each point of this boundary, it does not fix the elevation of the whole curve with reference to, say, that of the other boundary. This is immediately derived from the interpretation of the remaining condition,

$$\int_c \frac{\partial}{\partial n}(\nabla^2\psi_0)\,ds = 0.$$

From equation (16) this implies that

$$\int_c \mathrm{N}\,ds = 0\,;$$

that is to say, the total shearing force round the curve corresponding to the fixed boundary must be zero. Consequently the absolute elevation of the curve corresponding to the moving boundary must be so adjusted that no resultant shearing force is brought to bear on the curve corresponding to the fixed boundary. The conditions can be realized with comparative simplicity in practice; but for the present it is sufficient to note that under the circumstances described a measurement of the elevations of each point on the plate will determine the function ψ_0 directly.

By inserting this value for ψ_0 in the expression on the right-hand side of the equation (8 *b*) an expression is derived which, when interpreted in the light of the parallelism explained above, determines the lateral loading which must be imposed on the same plate as before. The boundary conditions can be interpreted with equal ease and simplicity, and by a direct measurement of the deflexions ψ_1 is at once derived. A repetition of this process leads successively to the evaluation of the functions ψ_2, ψ_3, etc.

§ 5. The parallelism with the flat plate demands that if the solution is to be obtained by an experiment of this nature, the lateral loading will vary in magnitude and in sign at different parts of the plate, a condition difficult to realize in practice. By a process of graphical integration, however, the problem may be reduced to that of an unloaded plate with modified boundary conditions.

Consider, for example, the equation

$$\nabla^4 Z = f(x, y) ;$$

then a particular integral is

$$Z_1 = \iint dx_2 dy_2 \Big[\log r_2 \iint \log r_1 f(x_1 y_1) dx_1 dy_1\Big].$$

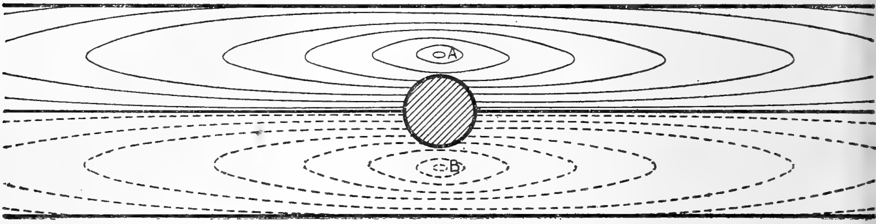
This equation may be evaluated graphically, or, if convenient, analytically, and by the ordinary transformation the equations whose solution is required can be reduced to

$$\nabla^4 Z = 0.$$

This reduces the problem once more to an unloaded plate, the method of fixing at the boundaries being determined quite simply from the original conditions and the derivatives of Z_1.

§ 6. Without attempting to enter into too great detail or to evaluate the results in a given case, the nature of the disposition of the stream-lines can be seen on general grounds in any particular problem. Take, as illustration, the problem of the motion of a cylinder about its direction of motion moving at uniform speed down the centre line of a channel. To determine the first term in the expansion outlined above, we must consider an unloaded flat plate bounded externally by the parallel walls of the channel and internally by the section of the cylinder. Along the walls the velocity of the fluid is zero, and consequently the plate must be clamped horizontally along these boundaries. Round the inner boundary the slope at each point with respect to x must be zero, where x is along the channel, while with respect to y it is unity. At the same time there must be no total reaction on each boundary in consequence of the integral condition. This implies that the cylinder must be placed under the influence of a pure couple about the axis of x of such magnitude as to produce unit slope. Any slope, provided it is small, may be taken as standard. This does not imply that the velocity of the body is small, but rather that the analogy of the flat plate will only be valid when the deflexions of the latter are small. The stream-lines corresponding with the first term in the expansion can now be found by plotting the contours of the plate. On general grounds it is clear that these will be of the nature shown in fig. 1, where points A and B, the positions of maximum and minimum deflexion, are stationary points at the instant relative to the walls of the channel. If the body be considered at rest and the planes be supposed in motion past it, the problem can be reproduced by tilting the whole system in the previous case until the section of the cylinder is once more in a horizontal plane. If the cylinder is symmetrical about a plane perpendicular to the axis of the channel, for example, a circular or elliptical cylinder, the system of stream-lines obtained above would be symmetrical about the lengthwise and crosswise plane. A consideration of the next term in the expansion, however, indicates how asymmetry about the crosswise plane arises. This is due to the fact that the expression on the right-hand side of (8 b) is of the same sign in opposite quadrants. The result of this is, that whereas the loading still maintains symmetry about the x axis, asymmetry occurs about the y axis. The stationary points A and B are thus, as can be seen by a simple inspection, similar to that already indicated, moved further back towards the rear of the body, the extent depending upon the value of VL/ν with which ψ_1 is associated. It is evident that the exact expressions for the various functions could be obtained by experiment, and until that is carried through, no general conclusions can be drawn, but the general nature of the stream-lines is clearly indicated in the foregoing discussion. It is hoped in a future paper to indicate how the system of equations (8 b), (8 c), etc. can be solved graphically.

Fig. 1.



§ 7. *Simple cases for verification.*—As a check on the analysis the case of a cylinder rotating in a viscous fluid inside a concentric cylinder is here worked out. It should be noticed that in this case the body is not moving forward with constant speed but rotating with constant angular velocity; but the necessary modifications and boundary conditions can easily be obtained. The result can then be compared with that obtained by the ordinary methods.

Consider the case of a circular cylinder of radius a spinning with uniform angular velocity ω, and coaxial with another cylinder of radius b, the region between being filled with a viscous fluid.

The boundary conditions are then, writing $C=\frac{a^2\omega}{\nu}$, and using non-dimensional quantities,

$$\text{Along } r=1 \begin{cases} \psi_0 = \psi_1 = \ldots = 0, \\ \frac{\partial\psi_0}{\partial r}=1, \ \frac{\partial\psi_1}{\partial r} = \ldots = 0\,; \end{cases}$$

$$\text{along } r=\frac{b}{a} \begin{cases} \frac{\partial\psi_0}{\partial r}=\frac{\partial\psi_1}{\partial r} = \ldots = 0, \\ \int_c ds\frac{\partial}{\partial n}(\nabla^2\psi_0) = \text{etc.} = 0. \end{cases}$$

The equations for ψ_0 etc. are :

$$\nabla^4\psi_0 = 0.$$

$$\therefore \quad \frac{1}{r}\frac{\partial}{\partial r}\left[r\frac{\partial}{\partial r}\left\{\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial\psi_0}{\partial r}\right)\right\}\right]=0, \quad . \quad . \quad (17)$$

and the equations for ψ_1 etc. all reduce to the same, the right side vanishing in virtue of the fact that

$$\frac{\partial\psi}{\partial y}\frac{\partial}{\partial x}(\nabla^2\psi)-\frac{\partial\psi}{\partial x}\frac{\partial}{\partial y}(\nabla^2\psi)$$

$$=\frac{xy}{r^2}\left[\frac{\partial\psi}{\partial r}\frac{\partial}{\partial r}(\nabla^2\psi)-\frac{\partial\psi}{\partial r}\frac{\partial}{\partial r}(\nabla^2\psi)\right]=0.$$

Hence the expansion in C consists of the terms in ψ_0 only.

The solution of (17) is

$$\psi_0 = Ar^2\log r + Br^2 + C\log r + D, \quad . \quad . \quad (18)$$

where the four constants are to be derived from $\psi_0=0$,

$$\frac{\partial\psi_0}{\partial r} = 1 \text{ at } r=1,$$

and

$$\frac{\partial\psi_0}{\partial r}=0, \ \int_c ds\frac{\partial}{\partial n}(\nabla^2\psi_0) = 0 \text{ at } r=b/a.$$

Now, round $r=b/a$

$$\int_c ds \frac{\partial}{\partial n}(\nabla^2\psi_0) = \int_0^{2\pi} \frac{2\pi b\, d\theta}{a} \left[\frac{\partial}{\partial r}(\nabla^2\psi_0)\right]_{r=b/a};$$

$$\therefore \quad \left[\frac{\partial}{\partial r}(\nabla^2\psi_0)\right]_{r=b/a} = 0.$$

Inserting these four conditions in the expression (18), for ψ_0 we derive finally

$$\psi_0 = \frac{1}{2(a^2-b^2/a^2)}\left[r^2 - \frac{b^2}{a^2}\log r^2 + 1\right].$$

Hence the angular velocity at any point is

$$\Omega = \frac{1}{r}\frac{\partial\psi_0}{\partial r} = \frac{a^2\omega}{a^2-b^2}\left(1-\frac{b^2}{R^2}\right),$$

where R is the actual distance from the centre to the point (*cf.* Lamb's 'Hydrodynamics,' § 333).

Justification of expanding in powers of C *near* C=0.

The foregoing analysis is based fundamentally on the assumption that a convergent expansion for ψ in ascending integral powers of C valid over at least a finite range from $C=0$ is possible.

It has been customary in dealing with problems of fluid motion to impose tentatively on the equations the restriction that the inertia terms are negligible for slow motions. In effect this is equivalent to a neglect of the second group of terms in (8), the assumption being that if a solution of $\nabla^4\psi=0$ can be found, the inertia terms $\frac{\partial\psi}{\partial x}\frac{\partial}{\partial y}\nabla^2\psi$ etc. will at most be of the same order as $\nabla^4\psi$; and since C is small in comparison with unity, that group of terms may be neglected. That this is justifiable *à posteriori* is clear when we remember that, on the analogy of the flat plate, the equation

$$\nabla^4\psi_0 = 0$$

will represent the deflexion of an unloaded flat plate where the boundaries are compelled to satisfy certain conditions as regard slope etc. In order that the analogy may be valid, these slopes, although reckoned as of the order of standard slopes, are small. It follows that all the derivatives of ψ_0 which occur in the equation are approximately of the same order, and therefore those involving the factor C in the equation may be neglected in comparison with $\nabla^4\psi$. On this physical argument it seems justifiable to assume that there is a solution for the differential equation in the neighbourhood of $C=0$, approximately given by

$$\psi = \psi_0,$$

where

$$\nabla^4\psi_0 = 0.$$

Let the solution therefore be written

$$\psi = \psi_0+\chi$$

for C small, and let us assume that χ is also small. Inserting this in the differential equation and boundary conditions, we find

$$\nabla^4\chi = C\left\{\frac{\partial\psi_0}{\partial x}\frac{\partial}{\partial y}\nabla^2\psi_0-\frac{\partial\psi_0}{\partial y}\frac{\partial}{\partial x}\nabla^2\psi_0\right\}$$
$$+C\left\{\frac{\partial\psi_0}{\partial x}\frac{\partial}{\partial y}\nabla^2\chi \text{ etc.}\right\}+C\left\{\frac{\partial\chi}{\partial x}\frac{\partial}{\partial y}\nabla^2\chi \text{ etc.}\right\}. \quad (20)$$

Assuming χ and its derivatives are small when C is small, an assumption that will be checked *à posteriori*, we must therefore write to a first approximation

$$\nabla^4\chi = C\left\{\frac{\partial\psi_0}{\partial x}\frac{\partial}{\partial y}\nabla^2\psi_0-\frac{\partial\psi_0}{\partial y}\frac{\partial}{\partial x}\nabla^2\psi_0\right\},$$

where χ must satisfy the boundary condition

$$\frac{\partial\chi}{\partial x}=\frac{\partial\chi}{\partial y}=0$$

round all the boundary, and the corresponding integral conditions. The quantity χ therefore represents once more the deflexion of a flat plate loaded with a definite finite distribution, but with density proportional to C. It therefore follows that we must write $\chi=C\psi_1$, where ψ is finite and independent of C and ψ_1 satisfies the differential equations already used in determining ψ_1 in the previously assumed expansion. By referring back to equation (20) it is now evident that we were justified in neglecting the remaining terms on the right-hand side, for all the derivatives of χ from the flat-plate analogy are of the order C. This step-by-step process may be continued along the same lines where we find that on seeking to determine the finite function, which is a solution of the differential equation for small values of C, the power series previously assumed is derived, ψ_0, ψ_1, ψ_2, etc. being determined from the same system of equations as before. In point of fact, the expression for ψ so found is a Taylor's series regarding C as the variable.

Theorem.

There is no expression, when Ul/ν is small, for the stream function of the viscous steady-motion equations in two dimensions, expressible in a finite number of terms in powers of Ul/ν other than those explicitly independent of Ul/ν. For if so, let

$$\psi = \Sigma\,\psi_n C^n,$$

and insert this as before in the differential equation. Equating the coefficient of C^{2n+1} in the equation to zero, we find for ψ_n

$$\frac{\partial\psi_n}{\partial x}\frac{\partial}{\partial y}\nabla^2\psi_n - \frac{\partial\psi_n}{\partial y}\frac{\partial}{\partial x}\nabla^2\psi_n = 0, \quad . \quad . \quad (21)$$

with the conditions that

$$\frac{\partial\psi_n}{\partial x} \text{ and } \frac{\partial\psi_n}{\partial y} \text{ and } \int_{\text{body}} ds\,\frac{\partial}{\partial n}\left(\frac{\partial\psi}{\partial x}\right)$$

are zero round the boundaries.

Equation (21) is clearly the limiting case of our fundamental equation (8) when ν tends to zero. There is, however, this difference from the common conception of the perfect fluid—that here the no-slip condition is maintained by the presence of some distribution of vorticity. In fact, (21) may be written

$$\frac{\partial\psi_n}{\partial y}\Big/\frac{\partial\psi_n}{\partial x} = \frac{\partial}{\partial y}\nabla^2\psi_n\Big/\frac{\partial}{\partial x}\nabla^2\psi_n,$$

which says that the lines of constant values of $\nabla^2\psi_n$ are coincident with those of constant ψ_n; *i.e.*,

$$\nabla^2\psi_n = f(\psi_n),$$

where f is a function to be determined from the boundary conditions. We may regard equation (21) as the limiting case of the flow of a fluid of exceedingly small viscosity; but since there are no externally applied forces or pressures causing motion of the fluid, and no motion of any boundaries, there cannot possibly be flow of any nature. Under these circumstances ψ_n will be constant, and may accordingly be ignored. This clearly applies to all values of n down to $n=1$ when the boundary conditions are now no longer zero.

It may be remarked in passing that the solutions of problems in steady motion independent of Ul/ν must therefore be such as to satisfy the two equations :

$$\nabla^4\psi = 0$$

$$\text{and} \qquad \nabla^2\psi = f(\psi),$$

where f is arbitrary, along with the appropriate boundary conditions. Such problems, of course, give a system of stream-lines in virtue of $\nabla^4\psi$, which are not changed when the direction of motion of the body is reversed. The only known solutions of this type so far obtained are those of the steady motion of a fluid between the walls of a channel and the steady relative rotation of two concentric cylinders.

It may be remarked, in conclusion, that certain classes of problem, although violating the ordinary condition of steadiness, viz. $\frac{\partial u}{\partial t}$ and $\frac{\partial v}{\partial t} = 0$ everywhere, can still be reduced to that of a steady-motion case by the superposition of a uniform linear velocity or angular velocity upon the axes of reference. For example, if every point of a cylinder of any given shape describe a circular path inside and concentric with a given circular cylinder, the motion may be reduced to that of a steady case.

LIV. *Wave Propagation over Parallel Wires: The Proximity Effect.* By JOHN R. CARSON, *Department of Development and Research, American Telephone and Telegraph Company**.

I. *Introduction.*

THE importance of the problem dealt with in the present paper—wave propagation along a conducting system composed of two similar and equal parallel wires—has been emphasized by modern developments in telephonic transmission such as the carrier wave system of the American Telephone and Telegraph Company, and the utilization of loaded cable circuits in which the wires are in very close juxtaposition. For such systems, where the frequencies employed are relatively high and the wires very close together, considerable theoretical work has been found necessary to reduce the solution to a form available for immediate engineering use, in spite of the previous

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* Communicated by the Author.

valuable researches of such mathematicians as Mie[*] and Nicholson[†].

In the present paper the analysis of the problem starts with Maxwell's equations, but one simplifying assumption is introduced *ab initio*—namely, that the exponential propagation factor is a small quantity. The approximations involved in this assumption are fully justified in all physical systems which could actually be employed for the transmission of electrical energy; so that from a practical standpoint the restriction thus imposed on the generality of the solution is purely formal. By aid of this simplifying assumption the determination of the current distribution in the wires is essentially reduced to a two-dimensional problem, which is solvable from the boundary conditions satisfied by the *tangential magnetic force* and the *normal magnetic induction* at the surfaces of the wires. With the current distribution in the wires thus determined, the exponential propagation factor γ is solvable by applying the law $\operatorname{curl} \mathrm{E} = -\mu d/dt\, \mathrm{H}$ to an appropriate surface bounded by a contour which includes line elements in the surfaces of the wires. By this means it is shown that the propagation factor satisfies an equation of the form

$$\gamma^2/ip\mathrm{K} = 2\mathrm{Z} + ip\mathrm{L},$$

where K is the electrostatic capacity between wires, Z the "impedance" of the wire per unit length, and L the inductance corresponding to the magnetic flux between the wires. This equation is of exactly the same form as that derivable from the *telegraph equation*, but differs therefrom in that Z and L are both functions of the frequency $p/2\pi$ and the parameter k (ratio radius of wire to interaxial separation between wires). As formulated in the present paper, the actual calculation of Z and L involves only the computation of Bessel functions.

The method of solution sketched above and worked out

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[*] G. Mie, *Annalen der Physik,* vol. ii. pp. 201–249 (1900). In this paper the problem is attacked in a fundamental manner. The results arrived at are, however, limited to a restricted range of frequencies and the parameter k (ratio radius of wire to interaxial separation). Furthermore, Mie's method of attack does not admit of extension to other types of transmission systems in which the surfaces of the conductors are generated by lines parallel to the axis of transmission.

[†] J. W. Nicholson, Phil. Mag. vol. xvii. p. 255 (1909), and vol. xviii. p. 417 (1909). In these papers formulas are derived for the resistance and reactance of parallel wires which are valid for a very wide range of frequencies, but which are applicable only when the ratio of the radius of the wire to the interaxial separation between wires is a relatively small quantity.

in the following section of this paper has one substantial advantage which gives it an interest extending beyond the specific problem : it is quite generally applicable to problems in wave propagation along conducting systems in which the surfaces of the conductors are generated by lines parallel to the axis of propagation. For example, it has been successfully applied by the writer to the problem of wave propagation along a wire parallel to the plane surface of a semi-infinite solid of finite conductivity ; the corresponding practical problem is, of course, transmission over a ground return circuit. Again, it has been applied to quantitatively investigate the effect of a concentric ring of iron armour wires on submarine cable transmission.

From an engineering standpoint the most important effect in parallel wire transmission is the dissipation of energy in non-magnetic wires. Consequently formulas for the alternating current resistance of the wire have been worked out in detail, and the functions involved have been computed and graphed, the data being collected in section III of this paper. The a.c. resistance of the wire is expressed in the form

$$\mathrm{R} = \mathrm{CR}_0,$$

where R_0 is the a.c. resistance of the wire *when the return conductor is concentric* (and is therefore calculable from well-known formulæ and tables), and C is a correction factor which formulates the modifying effect of the current in the adjacent wire. This is termed the *proximity effect correction factor*, following a usage suggested by Kennelly *. The correction factor C approaches an upper limit C_m, which is a function of the parameter k only (ratio of radius to inter-axial separation between wires), which it approaches in accordance with an asymptotic formula derived from the asymptotic expansion of the Bessel functions involved. By aid of the data of section III the calculation of C is reduced to a very simple matter.

II. *Mathematical Analysis and Derivation of Formulæ.*

The conducting system under consideration, as stated, consists of two long similar and equal parallel wires of circular cross-section, in which equal and opposite currents are flowing. The radius of the wire is denoted by a, its conductivity and permeability by λ and μ respectively, and

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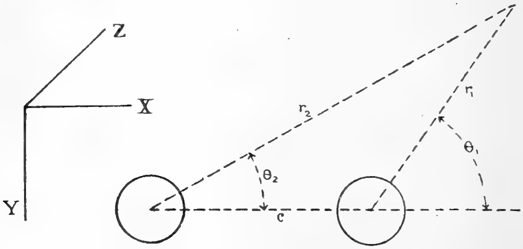
* Kennelly, Laws, and Pierce, Proceedings A.I.E.E. 1915, pp. 1749–1813.

the interaxial separation between wires by c. The co-ordinates of any point in the system with respect to the axis of one wire are denoted by r_1, θ_1, and the co-ordinates of the same point with respect to the axis of the second or return wire by r_2, θ_2, as shown in the sketch herewith.



Before proceeding with the analysis of the specific problem, a very brief discussion of the fundamental field equations will be given, in order to indicate the significance of certain important simplifying assumptions employed in the subsequent analysis and the restrictions thus imposed on the generality of the solution. It may be remarked that these simplifying assumptions are quite generally applicable to problems in wave propagation where the surfaces of the conductors are generated by lines parallel to the axis of propagation. The discussion starts with Maxwell's equations in a continuous medium:

$$\left.\begin{aligned}\text{curl E} &= -\mu ip\,\text{H},\\ \text{curl H} &= (4\pi\lambda+\text{K}ip)\,\text{E},\\ \text{div. E} &= 0,\\ \text{div. H} &= 0.\end{aligned}\right\} \quad . \quad . \quad \text{(I.)}$$

In these equations E and H denote the electric and magnetic forces, while λ, μ, and K are the conductivity, permeability, and specific inductive capacity of the medium. It is assumed throughout the following that elm. c.g.s. units are employed. The axis of propagation will be taken as the axis of Z, and it will be assumed that the electric and magnetic forces vary as $\exp(ipt-\gamma z)$; consequently the frequency is $p/2\pi$, γ is the propagation factor, and the operators d/dt and $\partial/\partial z$ are replaceable by ip and $-\gamma$ respectively. All six vector components (E_{xyz}, H_{xyz}) satisfy the wave equation

$$(\partial^2/\partial x^2+\partial^2/\partial y^2)\phi = -(m^2+\gamma^2)\phi, \quad . \quad . \quad \text{(II.)}$$

where

$$m^2 = -\left(4\pi\lambda\mu ip-(p/v)^2\right) \quad \text{and} \quad v = 1/\sqrt{\text{K}\mu}.$$

It will be found convenient to write the field equations in the form:

$$(m^2-\gamma^2)\mathrm{H}_x = -\frac{m^2}{\mu ip}\frac{\partial}{\partial y}\mathrm{E}_z+\gamma\frac{\partial}{\partial x}\mathrm{H}_z, \quad . \quad . \quad . \quad (1)$$

$$(m^2-\gamma^2)\mathrm{H}_y = \frac{m^2}{\mu ip}\frac{\partial}{\partial x}\mathrm{E}_z+\gamma\frac{\partial}{\partial y}\mathrm{H}_z, \quad . \quad . \quad . \quad (2)$$

$$(m^2-\gamma^2)\mathrm{E}_x = \gamma\frac{\partial}{\partial x}\mathrm{E}_z+\mu ip\frac{\partial}{\partial y}\mathrm{H}_z, \quad . \quad . \quad . \quad (3)$$

$$(m^2-\gamma^2)\mathrm{E}_y = \gamma\frac{\partial}{\partial y}\mathrm{E}_z+\mu ip\frac{\partial}{\partial x}\mathrm{H}_z. \quad . \quad . \quad . \quad (4)$$

From equations (3) and (4),

$$-\mu ip\,\mathrm{H}_z = \frac{\partial}{\partial x}\mathrm{E}_y-\frac{\partial}{\partial y}\mathrm{E}_x. \quad . \quad . \quad . \quad . \quad . \quad (5)$$

We now introduce the assumption, essential to the subsequent analysis, that γ and p/v are very small quantities of comparable orders of magnitude. That is to say, they are very small compared with unity and also compared with the value of m *in the conductors.* The justification for these assumptions and their immediate corollaries, introduced *ab initio*, resides in the fact that the solution obtained by their aid actually satisfies the necessary conditions in transmission systems of ordinary dimensions, even if the frequency exceeds a million cycles per second.

From equations (3), (4), and (5) it follows that the electric force in the plane normal to the axis of propagation is of the order of magnitude of $\gamma/(m^2-\gamma^2)$ compared with the axial component E_z. In the conductors this is a very small quantity of the order of magnitude of $\gamma/4\pi\lambda\mu p$, while in the dielectric it is a large quantity of the order of magnitude of $1/\gamma$. Consequently *in the conductors* the electric force in the plane normal to the axis of propagation will be ignored in comparison with E_z; in the dielectric, however, the former is large compared with the latter. By corresponding considerations the axial magnetic force H_z is very small compared with the magnetic force in the plane XY, both in the conductors and in the dielectric.

As a consequence of the foregoing, the magnetic force *in the conductors* is derivable from

$$\mu ip\,\mathrm{H}_x = -\frac{\partial}{\partial y}\mathrm{E}_z, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (6)$$

$$\mu ip\,\mathrm{H}_y = \frac{\partial}{\partial x}\mathrm{E}_z, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (7)$$

which replace (1) and (2).

We are now prepared to take up the analysis of the problem of wave propagation along parallel wires; in the course of this analysis the significance and utility of the simplifying assumptions will become more apparent.

From the general solution of the wave equation in polar co-ordinates and the special conditions of symmetry which obtain, the axial electric force in wire ♯1 is given by the Fourier-Bessel expansion

$$E_z = \sum_0^\infty A_n J_n(\rho_1) \cos n\theta_1, \quad . \quad . \quad . \quad . \quad . \quad (8)$$

and in wire ♯2 by

$$E_z = -\sum_0^\infty (-1)^n A_n J_n(\rho_2) \cos n\theta_2, \quad . \quad . \quad . \quad (9)$$

where

$$\rho_1 = ir_1\sqrt{4\pi\lambda\mu ip}, \quad \rho_2 = ir_2\sqrt{4\pi\lambda\mu ip}.$$

In these equations $J_n(\rho)$ is the Bessel function of order n and argument ρ, and the coefficients $A_0 \ldots A_n$ are to be determined from the boundary conditions at the surfaces of the wires. In either wire the magnetic force is then derivable from

$$\left.\begin{aligned} \mu ip\, H_\theta &= \frac{\partial}{\partial r} E_z, \\ \mu ip\, H_r &= -\frac{1}{r}\frac{\partial}{\partial \theta} E_z, \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad (10)$$

where r, θ denote either r_1, θ_1 or r_2, θ_2 according as wire ♯1 or wire ♯2 is under consideration. From the symmetry of the system, however, the satisfaction of the boundary conditions imposed at the surface of one wire insures their satisfaction at the surface of the other.

In the dielectric the electric and magnetic forces are expressible as Fourier-Bessel expansions, the Bessel functions, however, being of the "external" or second kind. In accordance with the assumption, however, that γ is a small quantity of the order of magnitude of p/v, it follows that so long as pc/v (where c is the separation between wires) is a small quantity compared with unity, the Bessel functions in the neighbourhood of the wires may be replaced by the limiting forms which they assume for vanishingly small arguments. In particular, the magnetic forces H_x and H_y in the neighbourhood of the wires are expressible as

$$\mathrm{H}_x = \sum_{n=1}^{\infty} \mathrm{B}_n \left\{ \frac{\sin n\theta_1}{r_1{}^n} + (-1)^n \frac{\sin n\theta_2}{r_2{}^n} \right\},$$

$$\mathrm{H}_y = \sum_{n=1}^{\infty} \mathrm{C}_n \left\{ \frac{\cos n\theta_1}{r_1{}^n} + (-1)^n \frac{\cos n\theta_2}{r_2{}^n} \right\}.$$

The magnetic force in the dielectric is thus expressed in terms of two symmetrical waves centred on the axes of the two wires respectively.

From the equation div. H=0 it follows that

$$\frac{\partial}{\partial x}\mathrm{H}_x + \frac{\partial}{\partial y}\mathrm{H}_y$$

differs from zero only by $\gamma \mathrm{H}_z$, which is a very small quantity since both γ and H_z are small. With very small error we may therefore write

$$\frac{\partial}{\partial x}\mathrm{H}_x + \frac{\partial}{\partial y}\mathrm{H}_y = 0,$$

which determines the relation between the B and C coefficients of (11) and gives

$$\mathrm{H}_x = \sum_{n=1}^{\infty} \mathrm{B}_n \left\{ \frac{\sin n\theta_1}{r_1{}^n} + (-1)^n \frac{\sin n\theta_2}{r_2{}^n} \right\}, \quad . \quad . \quad (12)$$

$$\mathrm{H}_y = -\sum_{n=1}^{\infty} \mathrm{B}_n \left\{ \frac{\cos n\theta_1}{r_1{}^n} + (-1)^n \frac{\cos n\theta_2}{r_2{}^n} \right\}. \quad . \quad . \quad (13)$$

In the dielectric the electric forces satisfy the wave equation II, and are therefore expressible as two Fourier-Bessel expansions oriented on the axes of the two wires. In accordance with our assumption, however, that γ and p/v are very small quantities, the Bessel functions are replaceable in the neighbourhood of the wires by the limiting forms which they assume for vanishingly small arguments. The same result is arrived at if we take E_x and E_y as satisfying the equations

$$(\partial^2/\partial x^2 + \partial^2/\partial y^2)\mathrm{E}_x = 0,$$
$$(\partial^2/\partial x^2 + \partial^2/\partial y^2)\mathrm{E}_y = 0.$$

Furthermore, from the relative magnitudes of E_z and the electric force in the plane XY in the dielectric, the equation div. E=0 may with very slight error be replaced by

$$\frac{\partial}{\partial x}\mathrm{E}_x + \frac{\partial}{\partial y}\mathrm{E}_y = 0.$$

These equations are satisfied if we introduce a function V which satisfies the equation

$$(\partial^2/\partial x^2+\partial^2/\partial y^2)\mathrm{V} = 0$$

and then derive E_x and E_y from V in accordance with

$$\mathrm{E}_x = -\frac{\partial}{\partial x}\mathrm{V},$$

$$\mathrm{E}_y = -\frac{\partial}{\partial y}\mathrm{V}.$$

Now, at the surfaces of the wires the tangential electric force in the plane XY, which is continuous, is very small compared with the normal component. Consequently very small error is introduced if in determining V it is taken as constant over the circumferences of the wires in the plane XY. It follows at once that

$$\mathrm{V} = \mathrm{V}_0 e^{(ipt-\gamma z)}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (14)$$

where V_0 *is the electrostatic potential and the surfaces of the wires are equipotential surfaces.* The determination of E_x and E_y in the dielectric is therefore reduced to a two-dimensional electrostatic problem, in which the surfaces of the two wires are equipotential surfaces.

The solution of our problem—namely, the determination of γ and the coefficients $\mathrm{A}_0 \ldots \mathrm{A}_n$ and $\mathrm{B}_1 \ldots \mathrm{B}_n$ of equations (8) and (12)—is obtained by formulating and satisfying the boundary conditions which obtain at the surfaces of the wires. These are that the tangential electric and magnetic forces are continuous. The current distribution in the wire, which carries with it its alternating current resistance, is, however, determinable by a less general statement of the boundary conditions; namely, that *the tangential magnetic forces and the normal magnetic induction are continuous.* With the current distribution in the wire determined, the propagation factor γ is determined without difficulty, as is shown subsequently.

Before proceeding with the determination of the coefficients $\mathrm{A}_0 \ldots \mathrm{A}_n$ of equation (8), the alternating current resistance of the wire will be formulated. Let the value of ρ_1 at the surface of wire #1 be denoted by

$$\xi = bi\sqrt{i} = ia\sqrt{4\pi\lambda\mu ip},$$

and let

$$\mathrm{A}_n = h_n\mathrm{A}_0 \quad (n = 1, 2, 3 \ldots).$$

Then, omitting the subscript in θ_1, the axial electric force at the surface of wire ♯1 is

$$A_0(J_0(\xi)+h_1J_1(\xi)\cos\theta+h_2J_2(\xi)\cos 2\theta+\ldots), \quad (15)$$

and the value of the tangential magnetic force H_θ at the surface of the wire is by (10):

$$\frac{\xi A_0}{a\mu ip}(J_0'(\xi)+h_1J_1'(\xi)\cos\theta+h_2J_2'(\xi)\cos 2\theta+\ldots). \quad (16)$$

Since 4π times the total current I flowing in the wire is equal to the line integral of the magnetic force H_θ around the circumference of the wire, it follows at once that

$$\frac{\xi J_0'(\xi)}{\mu ip} A_0 = 2I, \quad \ldots\ldots \quad (17)$$

which determines the fundamental coefficient A_0 in terms of the current in the wire.

The resistance R of the wire per unit length is conveniently defined as the mean dissipation per unit length, divided by the mean square current. The dissipation W in the wire is very conveniently and simply formulated by Poynting's theory of the energy-flow in the electromagnetic field, which, applied to the present problem, gives

$$W=\frac{a}{4\pi}\int_0^{2\pi}E_zH_\theta d\theta, \quad \ldots\ldots \quad (18)$$

where E_z and H_θ are the values at the surface of the wire, as given by (15) and (16). If these series are substituted for E_z and H_θ in (18) and the value of A_0 is taken from (17), and if the resulting expressions are realized, it follows without difficulty that

$$R=R_0\left\{1+1/2\sum_{n=1}^{\infty}|h_n|^2\frac{u_nv_n'-u_n'v_n}{u_0v_0'-u_0'v_0}\right\}, \quad \ldots \quad (19)$$

where

$$u_n+iv_n=J_n(\xi)=J_n(bi\sqrt{i}),$$

$$u_n'+iv_n'=\frac{d}{db}J_n(bi\sqrt{i}).$$

R_0 denotes the a.c. resistance of the wire when the coefficients $h_1 \ldots h_n$ are all zero; that is, R_0 *is the resistance of the wire where the return wire is concentric*, which is calculable from well-known formulæ and tables. The functions u_n and v_n, it will be observed, correspond precisely with the well-known ber and bei functions, which are similarly derived from the Bessel function of zero order and complex argument $bi\sqrt{i}$. From (19) the *proximity effect correction factor* C is given by

$$C = 1 + 1/2 \sum_{n=1}^{\infty} | h_n |^2 \frac{u_n v_n' - u_n' v_n}{u_0 v_0' - u_0' v_0}. \quad \ldots \quad (20)$$

By aid of formulæ (19) and (20) the a.c. resistance of the wire is calculable, once the coefficients $h_1 \ldots h_n$ or $A_1 \ldots A_n$ are determined; to this determination we now proceed.

As stated and discussed above, the harmonic coefficients are determined by the continuity of the tangential magnetic force and the normal magnetic induction at the surfaces of the wires; that is, by the continuity of $H_{\theta 1}$ and μH_{r1} at $r_1 = a$, and of $H_{\theta 2}$ and μH_{r2} at $r_2 = a$. From considerations of symmetry, however, these boundary conditions need be formulated at the surface of one wire only, and their satisfaction at the surface of either wire insures their satisfaction at the surface of the other. To formulate these conditions at the surface of wire #1, we require that the tangential and normal components of the magnetic force at the surface of this wire be expressed in terms of r_1 and θ_1 only, whereas H_x and H_y of formulæ (12) and (13) are expressed in terms of both r_1, θ_1 and r_2, θ_2. As a preliminary, we therefore require the expansion of H_x and H_y, as given by equations (12) and (13) in terms of r_1 and θ_1 alone. This is effected by the following transformations:—

$$\frac{\cos s\theta_2}{r_2^s} = \frac{1}{c^s} \left\{ \begin{array}{l} 1 - \frac{s}{1\,!}(r_1/c) \cos \theta_1 + \frac{(s)(s+1)}{2\,!}(r_1/c)^2 \cos 2\theta_1 \\ \quad - \frac{(s)(s+1)(s+2)}{3\,!}(r_1/c)^3 \cos 3\theta_1 \ldots, \end{array} \right. \quad (21)$$

$$\frac{\sin s\theta_2}{r_2^s} = \frac{1}{c^s} \left\{ \begin{array}{l} \frac{s}{1\,!}(r_1/c) \sin \theta_1 - \frac{(s)(s+1)}{2\,!}(r_1/c)^2 \sin 2\theta_1 \\ + \frac{(s)(s+1)(s+2)}{3\,!}(r_1/c)^3 \sin 3\theta_1 \ldots \ldots \end{array} \right. \quad (21a)$$

(It may be remarked in passing that these transformations may be very advantageously employed in calculating the capacity coefficients of a system of parallel cylinders.)

If these transformations are substituted in (12) and (13), H_x and H_y in the dielectric are expressed entirely in terms of r_1 and θ_1, or, omitting subscripts, in terms of r and θ. If we now employ the relations

$$H_r = H_x \cos \theta + H_y \sin \theta,$$
$$H_\theta = H_y \cos \theta - H_x \sin \theta,$$

we get, after rearrangement and simplification, the following infinite series for the tangential and normal components of the magnetic force in the dielectric at the surface of wire ♯1 :—

$$
\begin{aligned}
H_\theta = {} & -B_1/a - \cos\theta\,(B_2/a^2 - \Sigma_0) \\
& - \cos 2\theta \left(B_3/a^3 + \frac{1}{1\,!}\,(a/c)\,\Sigma_1 \right) \\
& - \cos 3\theta \left(B_4/a^4 - \frac{1}{2\,!}\,(a/c)^2\,\Sigma_2 \right) \\
& - \ldots \qquad (22)
\end{aligned}
$$

$$
\begin{aligned}
H_r = {} & \sin\theta\,(B_2/a^2 + \Sigma_0) \\
& + \sin 2\theta \left(B_3/a^3 - \frac{1}{1\,!}\,(a/c)\,\Sigma_1 \right) \\
& + \sin 3\theta \left(B_4/a^4 + \frac{1}{2\,!}\,(a/c)^2\,\Sigma_2 \right). \qquad (23)
\end{aligned}
$$

In these expressions the Σ's denote the following infinite series :—

$$
\left.
\begin{aligned}
\Sigma_0 &= B_1/c - B_2/c^2 + B_3/c^3 - \ldots, \\
\Sigma_1 &= B_1/c - 2B_2/c^2 + 3B_3/c^3 - 4B_4/c^4 - \ldots, \\
\Sigma_2 &= 1\,.\,2\,.\,B_1/c - 2\,.\,3\,.\,B_2/c^2 + 3\,.\,4\,.\,B_3/c^3 - \ldots, \\
& \ldots\ldots\ldots\ldots \\
\Sigma_n &= n\,!\,B_1/c - \frac{(n+1)\,!}{1\,!}\,B_2/c^2 + \frac{(n+2)\,!}{2\,!}\,B_3/c^3 - \ldots.
\end{aligned}
\right\} \qquad (24)
$$

From (10), (15), and (17) the tangential and normal components of the magnetic force at the surface of the wire are, in terms of the internal solution and the current I in the wire :

$$
H_\theta = (2I/a)\left(1 + \frac{J_1'}{J_0'}\,h_1 \cos\theta + \frac{J_2'}{J_0'}\,h_2 \cos 2\theta + \ldots\right), \qquad (25)
$$

$$
H_r = (2I/a)(1/\xi J_0')(J_1 h_1 \sin\theta + 2J_2 h_2 \sin 2\theta + \ldots), \qquad (26)
$$

where the argument of the Bessel functions $J_0 \ldots J_n$ and $J_0' \ldots J_n'$ is $\xi = ai\sqrt{4\pi\lambda\mu i p}$.

The boundary condition of the continuity of H_θ at the surface of the wire gives by direct equation of corresponding terms of (22) and (25):

$$\left.\begin{aligned} 2\mathrm{I}/a &= -\mathrm{B}_1/a, \\ \mathrm{B}_1 &= -2\mathrm{I}, \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad . \quad (27)$$

and

$$(2\mathrm{I}/a)\frac{\mathrm{J}_n'}{\mathrm{J}_0'}h_n = -\mathrm{B}_{n+1}/a^{n+1} + \frac{(-1)^{n-1}}{(n-1)\,!}(a/c)^{n-1}\Sigma_{n-1}, \qquad (28)$$

$$n = 1, 2, 3, \ldots .$$

Similarly, the boundary condition of the continuity of the normal magnetic induction applied to (23) and (26) gives:

$$(2\mathrm{I}/a)\frac{n\mu\mathrm{J}_n}{\xi\mathrm{J}_0'}h_n = \mathrm{B}_{n+1}/a^{n+1} + \frac{(-1)^{n-1}}{(n-1)\,!}(a/c)^{n-1}\Sigma_{n-1}, \qquad (29)$$

$$n = 1, 2, 3 \ldots .$$

From (28) and (29):

$$\mathrm{B}_{n+1} = -a^n\mathrm{I}h_n\frac{\xi\mathrm{J}_n' - n\mu\mathrm{J}_n}{\xi\mathrm{J}_0'} \quad . \quad . \quad . \quad . \quad (30)$$

and

$$(\mathrm{I}/a)\frac{\xi\mathrm{J}_n' + n\mu\mathrm{J}_n}{\xi\mathrm{J}_0'}h_n = \frac{(-1)^{n-1}}{(n-1)\,!}(a/c)^{n-1}\Sigma_{n-1}. \quad . \quad (31)$$

It is now convenient to introduce the following notation:

$$\left.\begin{aligned} \sigma_n &= (\xi\mathrm{J}_n' - n\mu\mathrm{J}_n)/\xi\mathrm{J}_0', \\ \rho_n &= (\xi\mathrm{J}_n' - n\mu\mathrm{J}_n)/(\xi\mathrm{J}_n' + n\mu\mathrm{J}_n), \\ q_n &= \sigma_n h_n, \\ a/c &= k. \end{aligned}\right\} \quad . \quad . \quad (32)$$

In terms of this notation it follows from (30) that

$$\mathrm{B}_{n+1} = -a^n q_n \mathrm{I}. \quad . \quad . \quad . \quad . \quad . \quad . \quad (33)$$

If the B coefficients in the Σ functions as defined by (24) are replaced by their values as given by (27) and (33), it is easy to show that equations (31) may be written as

$$q_n = (-1)^n 2\rho_n k^n - \frac{(-1)^n}{(n-1)\,!}\rho_n k^n\left(\frac{n\,!}{1\,!}kq_1 - \frac{(n+1)\,!}{2\,!}k^2 q_2 + \ldots\right),$$

which may conveniently be written as

$$q_n = (-1)^n 2\rho_n k^n - \frac{(-1)^n}{(n-1)\,!}\rho_n k^n \Sigma_n(q), \quad . \quad . \quad (34)$$

$$n = 1, 2, 3 \ldots .$$

Equations (34) constitute an infinite system of equations in the infinitely many variables $q_1 \ldots q_n$, and on their solution depends the determination of the harmonic coefficients $h_1 \ldots h_n$.

The solution of (34) is to be obtained by some process of successive approximation. For example, a formal solution is gotten by taking $q_1 \ldots q_n$ as the limit of the sequences:

$$\begin{array}{l} q_1^{(0)}, q_1^{(1)}, q_1^{(2)}, q_1^{(3)}, \ldots q_1^{(s)} \ldots \\ \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \\ q_n^{(0)}, q_n^{(1)}, q_n^{(2)}, q_n^{(3)}, \ldots q_n^{(s)}, \end{array}$$

where the successive terms of the sequences are defined by the relations:

$$q_n^{(0)} = (-1)^n 2\rho_n k^n, \quad n = 1, 2, 3 \ldots$$

and
$$q_n^{(s+1)} = (-1)^n 2\rho_n k^n - \frac{(-1)^n}{(n-1)!} \rho_n k^n \Sigma_n (q^{(s)}).$$

The method of solution results in a convergent sequence provided the parameter k is less than its limiting value 1/2, and for values of k likely to be encountered in practice a very rapidly convergent sequence.

Another method of successive approximations which may often be advantageously employed may be termed the method of successive ignorations. This consists in first ignoring all the variables except q_1 and determining its first approximate value $-2\rho_1 k$ from the first equation of the system. A second and higher approximation is then gotten by retaining q_1 and q_2 and evaluating them from the first two equations. A third and still higher approximation results from retaining q_1, q_2, and q_3 and solving for them from the first three equations. This process is to be continued until the convergence of the sequence is evident. This latter method of solution likewise results in a convergent sequence, and works very well in practice unless the parameter is too close to its limiting value.

While some such process of approximation is to be employed in the general case, and indeed has been successfully applied by the writer to several similar problems, a simpler method of solution fortunately suggests itself in the special case of greatest practical importance—namely, when the wires are composed of non-magnetic metals and, in consequence, the permeability μ is equal to unity. The resulting formulæ have the added advantage of being asymptotic in character, and consequently give the values of $q_1 \ldots q_n$ with increasing precision in the practically important range of values. It should be remarked that the formulæ now to be derived constitute asymptotic solutions also when μ is greater than unity; they cannot, however, be safely applied when the permeability is large unless the frequency is very high.

Restricting attention, therefore, to the case where $\mu=1$, we observe that the functions ρ_n and σ_n of (32) may be written as

$$\sigma_n = \mathrm{J}_{n+1}/\mathrm{J}_1,$$
$$\rho_n = -\mathrm{J}_{n+1}/\mathrm{J}_{n-1}.$$

These identities follow from the definitions of equations (32) and well-known properties of Bessel functions. We know also that when the argument ξ is large compared with the order n, the function ρ_n becomes closely equal to its limiting value unity. We are therefore led to consider the auxiliary system of equations in the auxiliary variables $p_1 \ldots p_n$, which is obtained from (34) by replacing the functions $\rho_1 \ldots \rho_n$ therein by their common limit unity. That is, we define the auxiliary variables $p_1 \ldots p_n$ by the following system of equations:—

$$p_n = (-1)^n 2k^n - \frac{(-1)^n}{(n-1)!} k^n \Sigma_n(p), \quad . \quad . \quad . \quad (35)$$
$$(n = 1, 2, 3, \ldots).$$

Now, since the functions $\rho_1 \ldots \rho_n$ approach the common limit unity as the argument ξ approaches infinity, it is evident that $p_1 \ldots p_n$ are simply the limiting values assumed by the variables $q_1 \ldots q_n$ when the wire is of infinite conductivity. From the known surface distribution of magnetic force in this case the following solution of equations (35) at once suggests itself, and may be readily verified; the variables $p_1 \ldots p_n$ are simply the Fourier coefficients of the expansion

$$\frac{1}{1+2k\cos\theta} = \mathrm{K}(1+p_1\cos\theta+p_2\cos 2\theta+\ldots). \qquad (36)$$

From (36) it is easy to show that

$$\left.\begin{aligned} p_1 &= -2ks, \\ p_n &= (-1)^n 2k^n s^n, \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad . \quad (37)$$

where s is the series ratio

$$s = \frac{1+\frac{3}{1\,!}k^2+\frac{4\,.\,5}{2\,!}k^4+\frac{6\,.\,7\,.\,8}{3\,!}k^6+\dots}{1+\frac{2}{1\,!}k^2+\frac{\text{[illegible]}}{2\,!}k^4+\frac{5\,.\,6\,.\,7}{3\,!}k^6+\dots}$$

$$=2\frac{1-\sqrt{1-(2k)^2}}{(2k)^2}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (38)$$

Having thus solved equations (35) for the variables $p_1 \dots p_n$, it is easy to show from (34) and (35) that if we write

$$q_n = p_n + d_n,$$

the variables $d_1 \dots d_n$ satisfy the system of equations:

$$d_n = (\rho_n - 1)p_n - \frac{(-1)^n}{(n-1)\,!}\rho_n k^n \Sigma_n(d). \quad . \quad . \quad . \quad (39)$$

The system of equations (39) in $d_1 \dots d_n$ admit of solution by successive approximations, as discussed in connexion with the solution of the corresponding system of equations (34) in $q_1 \dots q_n$. For the important case of non-magnetic conductors, however, a very close approximation to the exact solution is obtained by replacing (39) by the approximations:

$$d_n = (\rho_n - 1)p_n + (-1)^n n\rho_n k^{n+1} d_1. \quad . \quad . \quad (40)$$

This gives to the same order of approximation

$$q_n = \rho_n p_n + (-1)^n n\rho_n (1-\rho_1) k^{n+1} p_1. \quad . \quad . \quad (41)$$

Since by (32) $h_n = q_n/\sigma_n$, this gives for non-magnetic conductors

$$h_n = p_n \frac{\mathrm{J}_1}{\mathrm{J}_{n-1}}\left(1 - n(1-\rho_1)k^2/s^{n-1}\right)$$

$$= p_n \frac{\mathrm{J}_1}{\mathrm{J}_{n-1}}\left(1 - 2n\frac{k^2}{s^{n-1}}\frac{\mathrm{J}_1}{\xi \mathrm{J}_0}\right). \quad . \quad . \quad . \quad . \quad (42)$$

We are now in a position to formulate the proximity effect correction factor C of equation (20), which involves the harmonic coefficient $h_1 \dots h_n$. From (42) to the same order of approximation as (40)

$$|h_n|^2 = p_n^2 \frac{u_1^2+v_1^2}{u_{n-1}^2+v_{n-1}^2}(1+2ngk^2/s^{n-1}),$$

where g denotes the function

$$g = \frac{\sqrt{2}}{b}\frac{u_1(u_0+v_0)-v_1(u_0-v_0)}{u_0^2+v_0^2}. \quad . \quad . \quad . \quad . \quad (43)$$

It will be remembered, of course, that

$$b = a\sqrt{4\pi\lambda\mu p}$$

and $$J_n(i\sqrt{i}\,b) = u_n + iv_n.$$

If the foregoing is substituted in (20), some easy simplifications give

$$C = 1 + \frac{2}{aR_0}\sqrt{\frac{\mu p}{\pi\lambda}}\,(S_1 - 2gk^2S_2), \quad . \quad . \quad . \quad (44)$$

where

$$S_1 = \sum_{n=1}^{\infty} w_n k^{2n} s^{2n}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (45)$$

$$S_2 = \sum_{n=1}^{\infty} n w_n k^{2n} s^{2n} s^{n+1}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (46)$$

$$R_0 = \frac{1}{a}\sqrt{\frac{\mu p}{\pi\lambda}}\,\frac{u_0 v_0{}' - u_0\, v_0}{u_1{}^2 + v_1{}^2}, \quad . \quad . \quad . \quad . \quad . \quad . \quad (47)$$

$= \textit{resistance of wire with concentric return},$

and w_n is the auxiliary function

$$w_n = \frac{u_n v_n{}' - u_n{}' v_n}{u_{n-1}^{\,2} - v_{n-1}^{\,2}}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (48)$$

Equation (44) is the formula for the correction factor C for non-magnetic conductors, the evaluation of which is discussed in section III. For the purposes of numerical calculation, the following asymptotic expressions are useful. For values of the argument b equal to or greater than 5,

$$g \sim \sqrt{2}/b,$$

$$\frac{1}{aR_0}\sqrt{\frac{\mu p}{\pi\lambda}} \sim \sqrt{2} - 1/b. \quad . \quad . \quad . \quad . \quad . \quad (49)$$

Consequently for $b \geq 5$,

$$C \doteq 1 + 2(\sqrt{2} - 1/b)\left(S_1 - \frac{2\sqrt{2}}{b}k^2S_2\right). \quad . \quad . \quad (50)$$

From the asymptotic expansions of J_n it is easy to show that for $b > n^2$

$$w_n \sim 1/\sqrt{2} - (2n-1)/2b. \quad . \quad . \quad . \quad . \quad . \quad (51)$$

If this is substituted for w_n in S_1 and S_2 of equation (44), some easy simplifications give

$$C \sim C_m(1-A/b), \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (52)$$

where

$$C_m = \frac{1+k^2s^2}{1-k^2s^2}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (53)$$

and

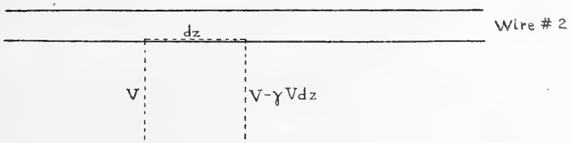
$$A = 2\sqrt{2}\frac{k^2s^2}{1-k^4s^4}\left[1+2k^2\left(\frac{1-k^2s^2}{1-k^2s}\right)^2\right]. \quad . \quad (54)$$

The limiting value of the correction factor is therefore C_m, and this is a function only of the parameter $k=a/c$. The asymptotic formula (54) can be used under the following conditions and in accordance with the following rule:—If the series $\sum k^{2n}s^{2n}$ converges to a required order of approximation in a finite number of terms n, then the correction factor C may be calculated from (53) provided the argument b is such that

$$b > n^2 > 5.$$

The correction-factor formulæ need not be further considered here, as they are fully discussed in section III. We shall therefore now proceed to complete the solution of the problem by formulating the propagation factor γ. In this discussion it will be assumed that $h_1 \ldots h_n$ and $q_1 \ldots q_n$ have been evaluated in accordance with the methods fully discussed above. For non-magnetic wires they may be calculated from (41) and (42), while in other cases any of the methods of successive approximations discussed above may be applied to equations (34).

The propagation factor γ is determined by applying the law curl $E = -\mu ip H$ to any appropriate surface, the contour of which includes a line segment dz in the surface of each wire and two lines in the dielectric joining their corresponding ends. The most convenient surface to take is a plane surface in the plane of the axes bounded by the



elements dz in the inner or adjacent surfaces of the wires and the straight lines in the dielectric joining the corresponding ends of the elements dz, as shown in the cross-section sketch herewith.

To apply the law $\operatorname{curl} \mathrm{E} = -\mu i p \mathrm{H}$ to this surface, it is only necessary to calculate the magnetic flux through the surface and the line integral of the electric force around the contour. The contribution to the line integral from the elements dz is, by (15) and (17),

$$dz \, \frac{2\mu i p \mathrm{J}_0}{\xi \mathrm{J}_0{}'} (2\mathrm{I}) \left(1 - h_1 \frac{\mathrm{J}_1}{\mathrm{J}_0} + h_2 \frac{\mathrm{J}_2}{\mathrm{J}_0} - \ldots \right),$$

which may be written as

$$2\mathrm{IZ} dz = 2\mathrm{Z}_0 \mathrm{I} dz \left(1 - h_1 \frac{\mathrm{J}_1}{\mathrm{J}_0} + h_2 \frac{\mathrm{J}_2}{\mathrm{J}_0} - \ldots \right), \quad . \quad . \quad . \quad (55)$$

where $\mathrm{Z}_0 = 2\mu i p \mathrm{J}_0 / \xi \mathrm{J}_0{}'$ and the argument of the Bessel functions is $\xi = ia\sqrt{4\pi\lambda\mu i p}$. Z_0 is the "internal" or "self-impedance" of the wire when the return wire is either concentric or at such a distance as to make the proximity effect negligible.

To calculate the contribution to the contour integral of the lines in the dielectric joining the corresponding ends of the segments dz, it will be recalled that the electric force in the dielectric in the plane XY is derivable as the gradient of a scalar or electrostatic potential, as given in equation (14). Consequently the contributions of these lines are simply

$$\left(\mathrm{V}_0 + \frac{\partial}{\partial z} \mathrm{V}_0 - \mathrm{V}_0 \right) dz = -\gamma \mathrm{V}_0 dz,$$

where V_0 is the *electrostatic potential between the two wires.* If K denote the electrostatic capacity between the two wires, then

$$\mathrm{V}_0 = \frac{\gamma}{ip\mathrm{K}} \mathrm{I} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (56)$$

and

$$-\gamma \mathrm{V}_0 dz = -\frac{\gamma^2}{ip\mathrm{K}} \mathrm{I} dz,$$

and the total line integral of electric force around the contour is

$$\left(2\mathrm{Z} - \frac{\gamma^2}{ip\mathrm{K}} \right) \mathrm{I} dz. \quad . \quad . \quad . \quad . \quad . \quad . \quad (57)$$

The calculation of the electrostatic capacity K involves merely the solution of the two-dimensional potential problem in which the surfaces of the wires are equipotential.

We have now to calculate the magnetic flux through the surface; it is

$$\Phi = -\int_a^{c-a} \mathrm{H}_y dr,$$

which by reference to equation (13) becomes

$$-2dz\int_a^{c-a} dy(\mathrm{B}_1/y - \mathrm{B}_2/y^2 + \mathrm{B}_3/y^3 - \ldots)$$

$$= -2dz\left[\mathrm{B}_1 \log\left(\frac{c-a}{a}\right) - \mathrm{B}_2\left(\frac{1}{a} - \frac{1}{c-a}\right) + \tfrac{1}{2}\mathrm{B}_3\left(\frac{1}{a^2} - \frac{1}{(c-a)^2}\right) - \ldots\right].$$

From (27) and (33) this reduces to

$$\Phi = 2\mathrm{I}dz\left\{2\log\left(\frac{c-a}{a}\right) - q_1\left[1 - \frac{a}{c-a}\right] + \tfrac{1}{2}q_2\left[1 - \left(\frac{a}{c-a}\right)^2\right] + \ldots\right\} \quad . \quad . \quad . \quad . \quad . \quad . \quad (58)$$

The law curl $\mathrm{E} = -\mu ip\mathrm{H}$ now gives at once

$$\frac{\gamma^2}{ip\mathrm{K}} = 2\mathrm{Z} + ip\mathrm{L}, \quad . \quad . \quad . \quad . \quad . \quad . \quad (59)$$

where

$$\mathrm{Z} = \mathrm{Z}_0(1 - h_1\mathrm{J}_1/\mathrm{J}_0 + h_2\mathrm{J}_2/\mathrm{J}_0 - \ldots), \quad . \quad . \quad . \quad . \quad . \quad . \quad (60)$$

$$\mathrm{L} = \mathrm{L}_0\left(1 - 2q_1\frac{1 - \frac{k}{1-k}}{\mathrm{L}_0} + q_2\frac{1 - \left(\frac{k}{1-k}\right)^2}{\mathrm{L}_0} + \ldots\right), \quad . \quad (61)$$

and

$$\mathrm{Z}_0 = 2\mu ip\mathrm{J}_0/\xi\mathrm{J}_0', \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (62)$$

$$\mathrm{L}_0 = 4\log\left(\frac{1-k}{k}\right). \quad . \quad . \quad . \quad . \quad . \quad . \quad (63)$$

Z may therefore be regarded as the impedance of the wire, and L the inductance corresponding to the magnetic flux between the wires; Z_0 and L_0 are their limiting values when the parameter k is vanishingly small—that is, when the proximity effect is negligibly small. While it is convenient from this standpoint to regard L as the inductance per unit length of the circuit, it must be carefully borne in mind that both Z and L are complex. Consequently the true resistance R* and reactance X *of the circuit* are defined by the relation

$$R+iX = 2Z+ipL. \quad . \quad . \quad . \quad . \quad . \quad . \quad (64)$$

Having calculated $q_1 \ldots q_n$ and $h_1 \ldots h_n$ in accordance with methods discussed above, it is a straightforward process to calculate Z, L, R, and X from (60), ... (64), the only operations involved being the evaluation of the Bessel functions appearing in the formulas. For very low frequencies, Z and L approach the limit Z_0 and L_0 respectively. On the other hand, when the frequency is sufficiently high they approach upper limits corresponding to

$$q_n \sim p_n = (-1)^n 2k^n s^n$$

$$h_n J_n / J_0 \sim 2k^n s^n \quad \text{when } n \text{ is even.}$$

Consequently,

$$(1+h_2 J_2/J_0 + h_4 J_4/J_0 + \ldots) \sim \frac{1+k^2 s^2}{1-k^2 s^2} = C_m,$$

where C_m is the upper limit of the correction factor C. It may also be shown that

$$2Z+ipL \sim 2C_m Z_0 + 4ip \log\left(\frac{1}{ks}\right).$$

The limiting values of Z and L correspond to the surface distribution of currents which would exist if the wires were of infinite conductivity.

The calculation of Z and L from the foregoing formulas and tables of Bessel functions is not a difficult matter. The writer, however, hopes when time permits to prepare numerical tables and the theoretical data for the computation of Z and L, similar to those given in section III for the correction factor C. The latter function is, however, of much the greater engineering importance.

III. FORMULÆ FOR CORRECTION FACTOR C FOR NON-MAGNETIC CONDUCTORS.

List of Symbols.

a = radius of wire in cm.
c = interaxial separation between wires in cm.
k = ratio a/c.
λ = conductivity of wire in elm. C.G.S. units.
μ = permeability " " "
p = 2π times frequency in cycles per second.
$i = \sqrt{-1}$.
$b = a\sqrt{4\pi\lambda\mu p}$.

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* The circuit resistance is, of course, twice that of the wire.

$J_n(bi\sqrt{i}) = u_n + iv_n$.
= Bessel function of order n and argument $bi\sqrt{i}$.
R = resistance of wire per unit length.
R_0 = ,, ,, ,, ,, *with concentric return.*
C = Proximity Effect Correction Factor

$$R = CR_0. \qquad \text{(I.)}$$

The auxiliary functions involved are :

$$R_0 = \frac{1}{a}\sqrt{\frac{p}{\pi\lambda}}\,\frac{u_0v_0' - u_0'v_0}{u_1^2+v_1^2}, \qquad (1)$$

$$g = \frac{\sqrt{2}}{b}\,\frac{u_1(u_0+v_0)-v_1(u_0-v_0)}{u_0^2+v_0^2}, \qquad (2)$$

$$w_n = \frac{u_nv_n' - u_n'v_n}{u_{n-1}^2+v_{n-1}^2}, \qquad (3)$$

$$s = 2\,\frac{1-\sqrt{1-(2k)^2}}{(2k)^2}. \qquad (4)$$

The formula for the correction factor C is then

$$C = 1 + \frac{2}{aR_0}\sqrt{\frac{p}{\pi\lambda}}\,(S_1 - 2gk^2S_2), \qquad \text{(II.)}$$

where

$$S_1 = \sum_{n=1}^{\infty} w_nk^{2n}s^{2n}, \qquad (5)$$

$$S_2 = \sum_{n=1}^{\infty} nw_nk^{2n}s^{n+1}. \qquad (6)$$

For values of the argument $b \geq 5$,

$$C \doteq 1 + 2(\sqrt{2} - 1/b)\left(S_1 - \frac{2\sqrt{2}}{b}k^2S_2\right). \qquad \text{(III.)}$$

For larger values of the argument b, the correction factor C approaches an upper limit

$$C_m = \frac{1+k^2s^2}{1-k^2s^2} \qquad \text{(IV.)}$$

in accordance with the asymptotic formula

$$C \sim C_m(1 - A/b), \qquad \text{(V.)}$$

where

$$A = 2\sqrt{2}\,\frac{k^2s^2}{1-k^4s^4}\left[1 + 2k^2\left(\frac{1-k^2s^2}{1-k^2s}\right)^2\right]. \qquad (7)$$

Fig. 1.

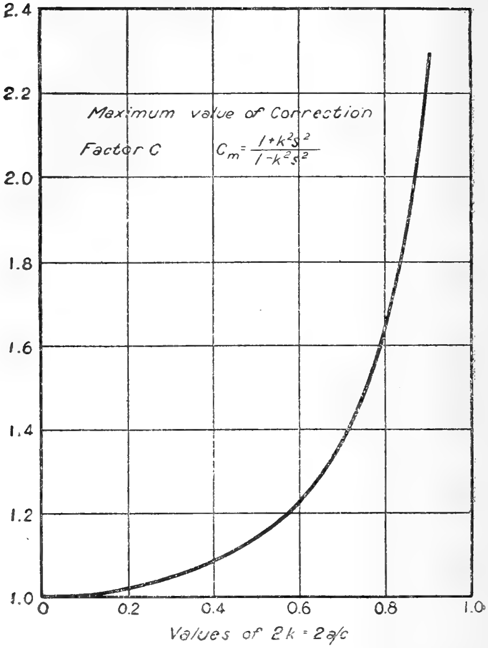
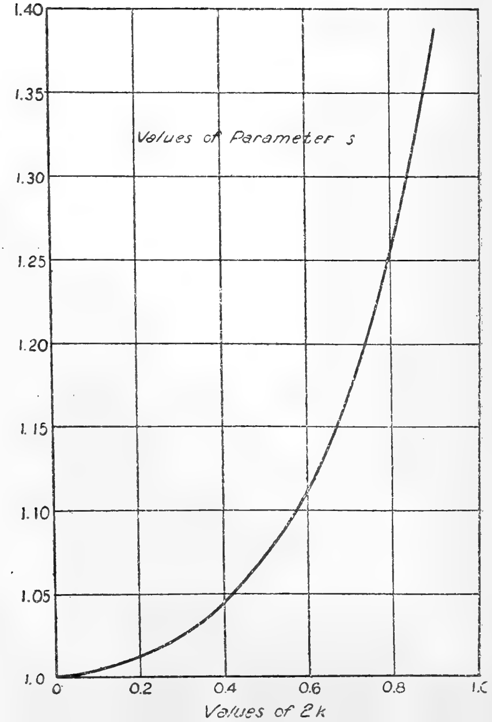

Maximum value of Correction
Factor C
$C_m = \frac{1+k^2s^2}{1-k^2s^2}$
2.4
2.2
2.0
1.8
1.6
1.4
1.2
1.0
0
0.2
0.4
0.6
0.8
1.0
Values of 2k = 2a/c

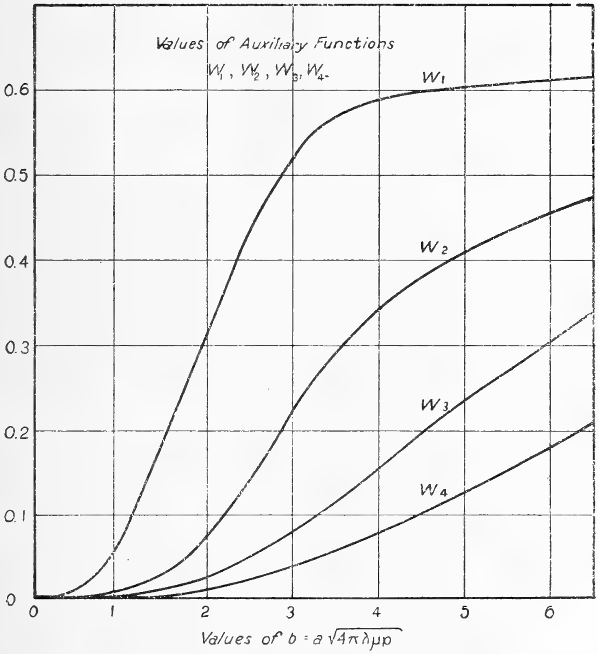
Fig. 2.


Values of Parameter s
1.40
1.35
1.30
1.25
1.20
1.15
1.10
1.05
1.0
0
0.2
0.4
0.6
0.8
1.0
Values of 2k

The correction factor C may be calculated from the asymptotic formula V instead of II or III under the following condition and in accordance with the following rule:—If the series $1+k^2s^2+k^4s^4+\ldots$ converges to a required order of approximation in a finite number of terms n, then formula V may be employed, provided that $b \geq n^2 \geq 5$.

The auxiliary functions involved in the foregoing formulæ have been computed and are plotted in the accompanying curves, the accuracy of which is believed to be sufficient for all engineering purposes. An example of their use will now be given in calculating the correction factor C for the following representative case: $b=5$ and $2k=2a/c=0{\cdot}75$.

Fig. 3.



From the curves of fig. 3,

$$w_1 = 0{\cdot}605$$
$$w_2 = 0{\cdot}410$$
$$w_3 = 0{\cdot}235$$
$$w_4 = 0{\cdot}130.$$

Fig. 4.

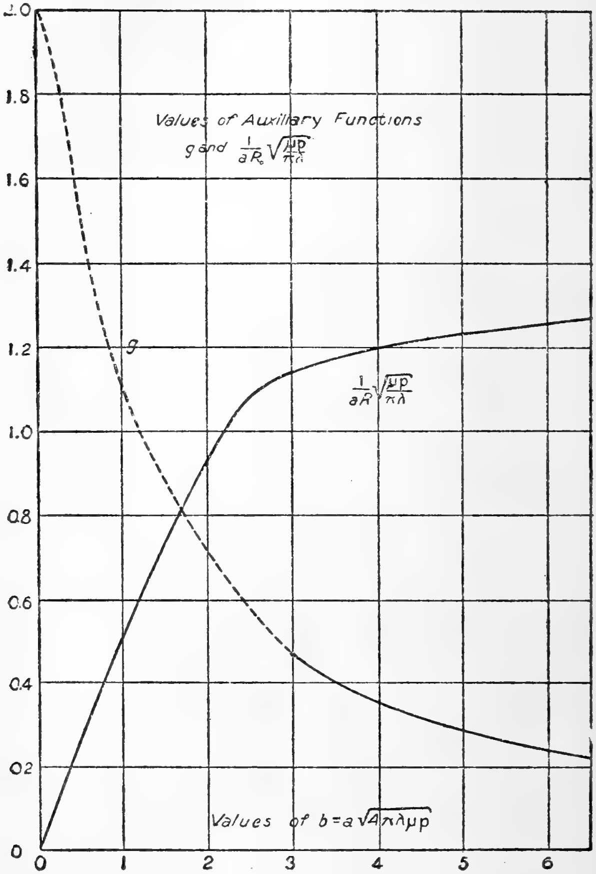
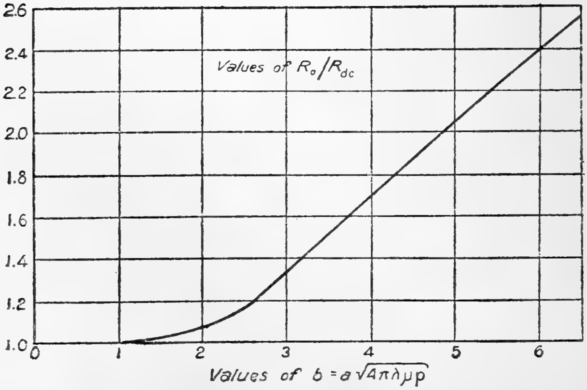

Values of Auxiliary Functions
g and $\frac{1}{aR_0}\sqrt{\frac{\mu p}{\pi\lambda}}$
g
$\frac{1}{aR}\sqrt{\frac{\mu p}{\pi\lambda}}$
Values of $b=a\sqrt{4\pi\lambda\mu p}$
2.0
1.8
1.6
1.4
1.2
1.0
0.8
0.6
0.4
0.2
0
0
1
2
3
4
5
6

Fig. 5.


Values of R_0/R_{dc}
Values of $b=a\sqrt{4\pi\lambda\mu p}$
2.6
2.4
2.2
2.0
1.8
1.6
1.4
1.2
1.0
0
1
2
3
4
5
6

From the curves * of fig. 4,

$$g = 0{\cdot}285 \quad \text{and} \quad \frac{1}{a\mathrm{R}_0}\sqrt{\frac{p}{\pi\lambda}} = 1{\cdot}225.$$

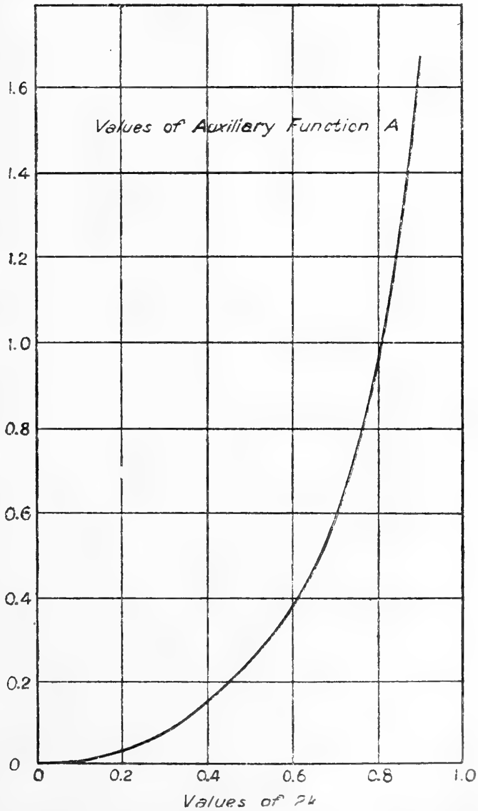
From the curve of fig. 2,

$$s = 1{\cdot}20.$$

Consequently,

$$\begin{array}{rcl@{\qquad\qquad}rcl}
w_1k^2s^2 & = & \cdot1225 & w_1k^2s^2 & = & \cdot1225 \\
w_2k^4s^4 & = & \cdot01681 & 2w_2k^4s^3 & = & \cdot02801 \\
w_1k^6s^6 & = & \cdot001952 & 3w_3k^6s^4 & = & \cdot004065 \\
w_4k^8s^8 & = & \cdot0002187 & 4w_4k^8s^5 & = & \cdot0005058 \\
\hline
\mathrm{S}_1 & = & \cdot1415 & \mathrm{S}_2 & = & \cdot1551
\end{array}$$

Fig. 6.



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* The first four orders of w_n are sufficient for practical purposes Higher orders can be calculated from the Bessel function recurrence formulæ.

Substitution in formula II gives

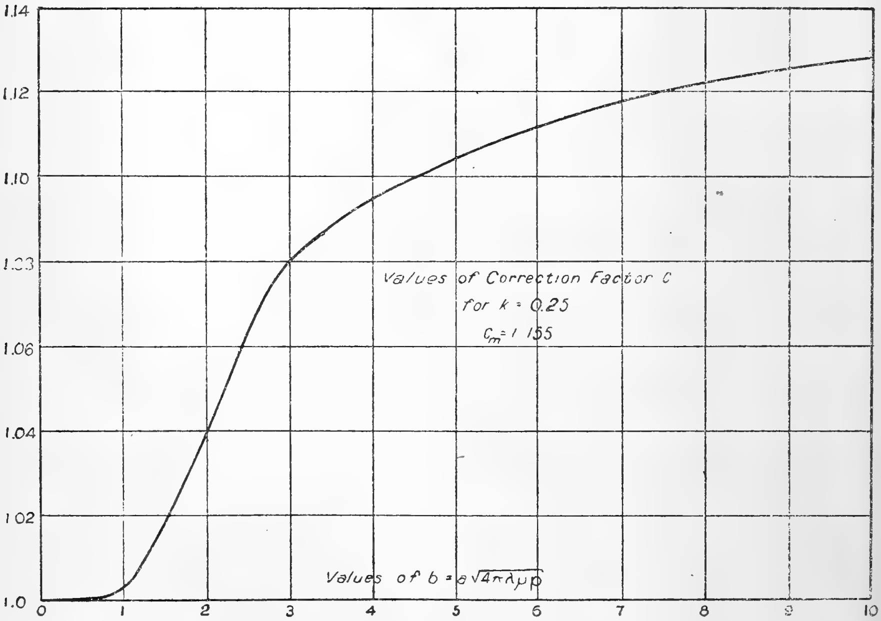
$$C = 1+2{\cdot}45\,({\cdot}1415-{\cdot}01223)$$
$$= 1{\cdot}317.$$

From the curve of fig. 1 the maximum value C_m of the correction factor for this case is 1·51, and from the curve of fig. 6 the value of the auxiliary function A is 0·76. The asymptotic formula V therefore gives:

$$C \sim 1{\cdot}51\,(1-0{\cdot}76/5) = 1{\cdot}28.$$

As would be expected from the magnitudes involved, the asymptotic formula therefore gives a result which is in error by a small amount.

Fig. 7.



To give an idea of the variation of the correction factor C with frequency, it has been computed for $k=0{\cdot}25$ and plotted in the curve of fig. 7 as a function of the argument b. This case is of practical interest, since the corresponding conductor spacing is that of telephone cable circuits. For this ratio of a/c the value of $C_m=1{\cdot}155$; this limiting value is however approached quite slowly, as is evident from an inspection of fig. 7.

To facilitate the calculation of R_0, the ratio R_0/R_{dc} has been plotted in fig. 5. R_d denotes, of course, the d.c. resistance of the wire. For values of the argument $b \geq 5$,

$$R_0/R_{dc} \sim b/(2\sqrt{2}-2/b),$$

which makes the calculation very simple.

April 17th, 1920.

LV. *The Separation of Miscible Liquids by Distillation.*

By A. F. DUFTON, *B.A., Frecheville Research Fellow, Royal School of Mines* *.

1. IN the search for a perfect apparatus for the separation of mixtures by distillation the greatest advances have been made in industrial practice. M. Sorel, one of the leading French authorities upon the distillation and the rectification of alcohol, in reviewing the principles underlying the construction of stills, writes :—

"La plus grande partie des données dont nous avons besoin peuvent être determinées dans la laboratoire du physicien. Malheureusement bien peu de savants s'en sont occupés, soit que le sujet leur parût peu important, soit que l'impossibilité jusqu'ici reconnue d'arriver à des lois mathématiques les ait rebutés. Il faut donc que le constructeur se transforme en expérimentateur. . ."

"C'est l'aveu franc et net," to quote M. Chenard †, "d'un empirisme certain."

The extent to which laboratory practice has been outstripped may be seen by comparing the still designed by Coffey ‡, of Dublin, in 1832, or Derosne's still, which give in continuous distillation on the large scale the strongest spirit which can be obtained, with the various laboratory still-heads examined by Dr. Young § in 1899. The only continuous laboratory still appears to be an experimental one devised by Lord Rayleigh ||, which consisted of a long length (12 metres) of copper tubing 15 mm. in diameter, and a similar one described by Carveth ¶.

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* Communicated by Sir E. Rutherford, F.R.S.

† Chenard, *Bulletin de l'Association des Chimistes de Sucrerie et de Distillerie de France*, 1915.

‡ 'Chemistry as applied to the Arts and Manufactures.' Vol. I. Alcohol.

§ Young, Journ. Chem. Soc. 1892, p. 679.

|| Rayleigh, Phil. Mag. (4) 1902, p. 536.

¶ Carveth, J. Phys. Chem. vi. p. 253 (1902).

The invention in 1918 by Dr. S. F. Dufton [*] of a laboratory still-head of small working volume and not inconvenient length, giving perfect separation of simple binary mixtures, *e. g.* benzene and toluene, renders possible the examination in the laboratory of the rationale of separation by distillation.

This examination is the object of the present research. The problems presented by close isomers and by the recently discovered isotopes render desirable every possible refinement in methods of separation, and a knowledge of the physical processess taking place in a still-head should render possible both the application of distillation to more difficult separations and the elimination of the empirical element in design.

To minimise the empirical element in the design of still-heads by determining more clearly the physical processes involved, calculation has been made in the following pages of the requisite flow of liquid and of vapour at each point in a theoretically perfect column, and indication has been given of the corresponding supply or withdrawal of heat involved. A quantity, termed the THERMAL EFFICIENCY, has been defined to afford a comparison of the performance of columns yielding a pure distillate and a measure of the approach towards theoretical perfection.

2. F. D. Brown [†] has pointed out that when a mixture of two liquids which mingle in all proportions is heated to ebullition, it evolves a mixed vapour containing a certain proportion of each of the two substances, the liquid mixture and the gaseous mixture being mutually related and existing together in a state of equilibrium.

It is upon this equilibrium between liquid mixture and gaseous mixture of different composition that separation by distillation depends. Ideal discontinuous distillation consists in the evaporation of liquid, in the removal of the vapour produced and in the return to the still, in liquid of the same composition as that in the still, of the whole of the less volatile constituent. The evaporation of 2·32 grams from a large mass of a mixture of equal masses of benzene and toluene, for example, yields as vapour 0·66 gram of toluene and 1·66 gram of benzene, and the return of the 0·66 gram of toluene with 0·66 gram of benzene as liquid involves the separation of 1·00 gram of benzene.

The amount of evaporation necessary and the corresponding quantity of heat required may be calculated mathematically. If in a system of two substances A and B, which mingle in

[*] S. F. Dufton, J. Soc. Chem. Industry, 1919, p. 45.

[†] Brown, Journ. Chem. Soc. 1881, p. 531.

all proportions, V_a and V_b be the masses in the vapour at any moment and L_a and L_b the masses in the liquid in contact with and in equilibrium with the vapour, and if E be a mass evaporated and C a mass condensed, the criteria for equilibrium are

$$\frac{\dfrac{\partial L_a}{\partial E}}{\dfrac{\partial L_b}{\partial E}} = \frac{V_a}{V_b}$$

and

$$\frac{\dfrac{\partial L_a}{\partial C}}{\dfrac{\partial L_b}{\partial C}} = \frac{L_a}{L_b}.$$

If the mass L_b be conserved, as is the case in discontinuous distillation,

$$\begin{aligned}\frac{\partial L_b}{\partial E} . dE + \frac{\partial L_b}{\partial C} . dC &= dL_b \\ &= 0.\end{aligned}$$

These three equations show that

$$\begin{aligned} dE &= -\left\{\frac{\partial L_a}{\partial E} . dE + \frac{\partial L_b}{\partial E} . dE\right\} \\ &= -\left\{1 + \frac{1}{\alpha(k-1)}\right\}\left\{\frac{\partial L_a}{\partial E} . dE + \frac{\partial L_a}{\partial C} . dC\right\} \\ &= -\left\{1 + \frac{1}{\alpha(k-1)}\right\} dL_a, \qquad \qquad (1)\end{aligned}$$

where $\dfrac{L_a}{L_a + L_b} = \alpha$, the proportion of liquid which is A,

and where $k = \dfrac{V_a . L_b}{V_b . L_a}$.

Equation (1) signifies that the separation of unit mass of A involves the evaporation of a mass $\left\{1 + \dfrac{1}{\alpha(k-1)}\right\}$ of the liquid. If Q be its latent heat of evaporation, the quantity of heat required is

$$Q\left\{1 + \frac{1}{\alpha(k-1)}\right\}. \qquad \qquad (2)$$

The Thermal Efficiency of a still-head for such a binary mixture may be defined as the ratio of the mass actually separated by this amount of heat to unit mass, or, equivalently, as the ratio of the theoretical quantity of heat required for a separation to the amount actually expended. It is expressed conveniently as a percentage.

The value of k—that is, the relative composition of vapour and liquid phases in equilibrium—must be determined experimentally. Attempts have been made to express k in terms of the physical constants of the components, and some success has been attained; thus, for alcohol and water, a mixture of considerable industrial interest, Duclaux * has established the relation

$$\frac{A}{E}=p\frac{a}{a+e},$$

where p is a constant,

and where

a represents alcohol per cent. by vol. in the liquid,
e ,, water ,, ,, ,,
A ,, alcohol ,, ,, condensed vapour,
E ,, water ,, ,, condensed vapour.

For substances which are chemically closely related to each other a simpler relation exists, viz.:

$$k=\frac{P_1}{P_2},$$

where P_1 and P_2 are the vapour-pressures of the pure components at the temperature considered †.

Fig. 1 shows for mixtures of benzene and toluene the composition of liquid ‡ and vapour phases in equilibrium at atmospheric pressure, the composition of the vapour being calculated from that of the liquid by means of this relation.

As in the design of an efficient column a knowledge of the flow of vapour and of liquid is desirable, the minimum flow in an ideal column must be calculated. If in the column

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* Duclaux, *Ann. de Phys. et Chim.* 1878.

† Brown, Journ. Chem. Soc. 1881, p. 304. Zawidski, *Zeit. f. Phys. Chem.* xxv. p. 129 (1900); *cf.* also Young, 'Stoichiometry,' p. 256 (1918). Ostwald, *Lehrbuch d. Allgem. Chem.* (2 Aufl.), iii. p. 613.

‡ Spielman & Wheeler, Tables of Chem. and Phys. Constants, H.M.S.O. 1919.

the net flow be $\frac{dA}{dt}$ of substance A, the net flow of substance B being zero, the flow past any point must satisfy the equations

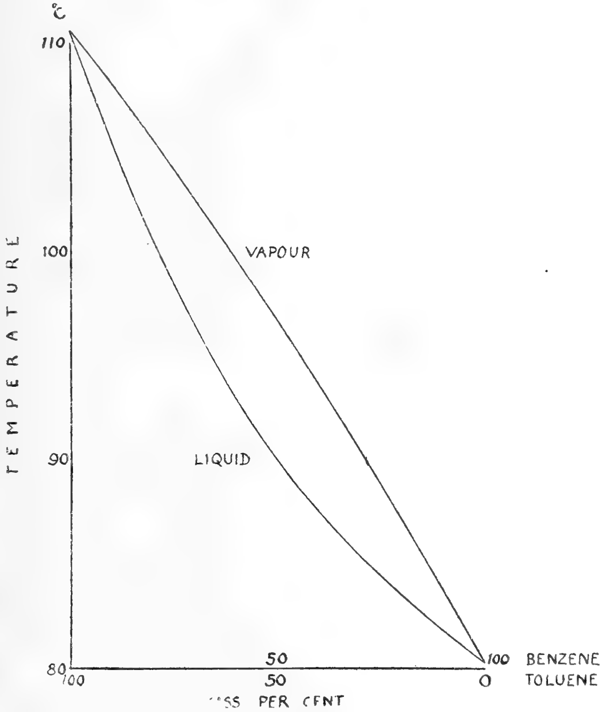
$$dV_a + dL_a = dA$$

and $$dV_b + dL_b = dB = 0\ ;$$

Fig. 1.

Mixture of Benzene and Toluene.

The Composition of Phases in Equilibrium at Atmospheric Pressure for different Temperatures.



moreover, since the flow of each component in the liquid or vapour is proportional to the amount present in that phase,

$$\frac{dL_a}{dL_b} = \frac{L_a}{L_b},$$

and $$\frac{dV_a}{dV_b} = \frac{V_a}{V_b}.$$

From these four equations it follows that

$$\begin{aligned}\frac{dA}{dL_a}&=1+\frac{dV_a}{dL_a}\\&=1+\frac{V_a}{V_b}\cdot\frac{L_b}{L_a}\cdot\frac{dV_b}{dL_b}\\&=1-\frac{V_a\,.\,L_b}{V_b\,.\,L_a}\\&=1-k,\end{aligned}$$

if there be equilibrium between liquid and vapour:

that is,
$$\begin{aligned}\frac{dL}{dA}&=\frac{1}{\alpha}\frac{dL_a}{dA}\\&=-\frac{1}{\alpha(k-1)}, \qquad (3)\end{aligned}$$

where $\frac{dL}{dt}$ is the total flow of liquid at the point

and
$$\frac{dV}{dA}=1+\frac{1}{\alpha(k-1)}, \qquad (4)$$

where $\frac{dV}{dt}$ is the total flow of vapour.

The upper portion of Table I. shows the necessary flow for the separation of one gram of benzene from a mass of a 50-per-cent. mixture of benzene and toluene.

Table I.

Temp. ° C.	C_6H_6 per cent. in liq.	Grams Vapour passing up $C_6H_6+C_7H_8$=total.	Grams Liquid flowing down $C_6H_6+C_7H_8$=total.
80·2	100	1·64+0·00=1·64	0·64+0·00=0·64
81·8	90	1·65+0·07=1·72	0·65+0·07=0·72
83·5	80	1·65+0·16=1·81	0·65+0·16=0·81
85·4	70	1·66+0·28=1·94	0·66+0·28=0·94
87·6	60	1·66+0·44=2·10	0·66+0·44=1·10
90·0	50	1·66+0·66=2·32	0·66+0·66=1·32
93·0	40	1·67+1·01=2·68	0·67+1·01=1·68
96·4	30	1·68+1·57=3·25	0·68+1·57=2·25
100·4	20	1·70+2·80=4·50	0·70+2·80=3·50
105·0	10	1·72+6·43=8·15	0·72+6·43=7·15
110·6	0	1·75+ ∞ = ∞	0·75+ ∞ = ∞

It will be seen that condensation of toluene takes place throughout the length of the column. The lower portion of the table shows that an increased amount of evaporation and condensation must take place as the mixture becomes poorer in benzene. It is of interest to observe that in the removal of a trace of toluene from "pure" benzene by distillation 1·64 gram must be evaporated for each gram collected.

The heat which must be lost in the column during the distillation of one gram of benzene from a mixture may be calculated from Table I. At the top of the column 0·64 gram of benzene must be condensed at its boiling-point. This involves a loss of 61 calories. It is convenient to divide the remainder of the column into sections, each corresponding to a temperature difference of two degrees, and to evaluate the loss in each. For these sections the loss of heat is shown in Table II.

TABLE II.

Section. °C.	Grams of Vapour condensed.	Calories lost (approx.).
80– 82	0·10	9
82– 84	0·12	10
84– 86	0·14	12
86– 88	0·16	14
88– 90	0·19	17
90– 92	0·23	20
92– 94	0·29	25
94– 96	0·37	32
96– 98	0·51	44
98–100	0·78	86
100–102	1·18	101
102–104	1·60	136
. . .	. . .	. . .
108–110·6	∞	∞

It is important to observe that for perfect thermal efficiency the liquid and vapour need not be in equilibrium at any point other than the top and bottom of the column.

The particular case in which the whole of the necessary loss of heat is effected at the top of the column is of interest. It gives the maximum flow for perfect efficiency. This flow for the distillation of one gram of benzene from a 50-per-cent. mixture of benzene and toluene is shown in Table III. The loss of heat is 117·5 calories, *i.e.* the 208 supplied for necessary evaporation less 93·5, the heat of evaporation at 90° C. of the benzene separated, plus 3 calories, the heat given up by this benzene in cooling to 80°·2 C. The loss of this 117·5 calories means the return down the column as liquid of 1·24 gram of benzene.

TABLE III.

C_6H_6 per cent. in liq.	Grams Vapour passing up $C_6H_6+C_7H_8$=total.	Temp. of Vapour.	Temp. of Liquid.	Grams Liquid flowing down $C_6H_6+C_7H_8$=total.
		°	°	
100	2·24+0·00=2·24	80·2	80·2	1·24+0·00=1·24
74·3	1·95+0·33=2·28	86·4	84·5	0·95+0·33=1·28
50	1·66+0·66=2·32	90·0	90·0	0·66+0·66=1·32

It will be seen that in the middle of the column the vapour is considerably hotter than the liquid. With benzene and toluene, owing to the approximate equality of the latent heats, this method of working gives a practically uniform flow throughout the column.

In the determination of the thermal efficiency of a column used to separate benzene from a mixture of benzene and toluene, calculation must be made of the quantity of heat which would be required if a perfect column were employed. The quantities necessary for different mixtures are given in Table IV. and plotted in fig. 2.

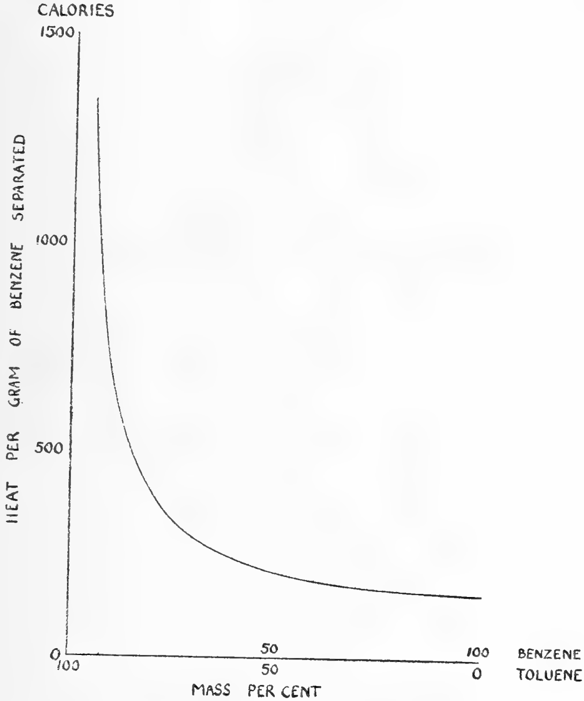
TABLE IV.

Benzene in mixture 100 α per cent.	Liquid evaporated: grams $\left\{1+\frac{1}{\alpha(k-1)}\right\}$.	Latent heat calculated from components.	Calories required to separate 1 gm. of benzene.
0	∞	84·0	∞
5	15·6	84·5	1320
10	8·15	85·1	698
20	4·5	86·3	388
30	3·29	87·4	288
40	2·68	88·6	237
50	2·32	89·7	208
60	2·10	90·9	191
70	1·94	92·0	179
80	1·81	93·2	169
90	1·72	94·3	162
100	1·64	95·5	156

3. In the experimental investigation of the thermal efficiency which can be obtained in the laboratory, a mixture of 40 c.c. of "crystallized" benzene with 10 c.c. of reputed pure toluene was boiled in a 100-c.c. vacuum flask as still. For the first experiments a Dufton column* was constructed in a glass tube, 105 cm. in length and 0·46-cm. bore. Heat was supplied electrically by means of a platinoid spiral immersed in the liquid and the rate of heating was determined by measurement of the current and the potential difference, calibration being effected by heating a known mass of water in the flask. The benzene condensed was collected in a burette and observations of its volume were made at five-minute intervals. A thermometer placed in the top of the column showed the temperature of the vapour and indicated the

Fig. 2.

The Heat required to distil Benzene from mixtures of Benzene and Toluene.



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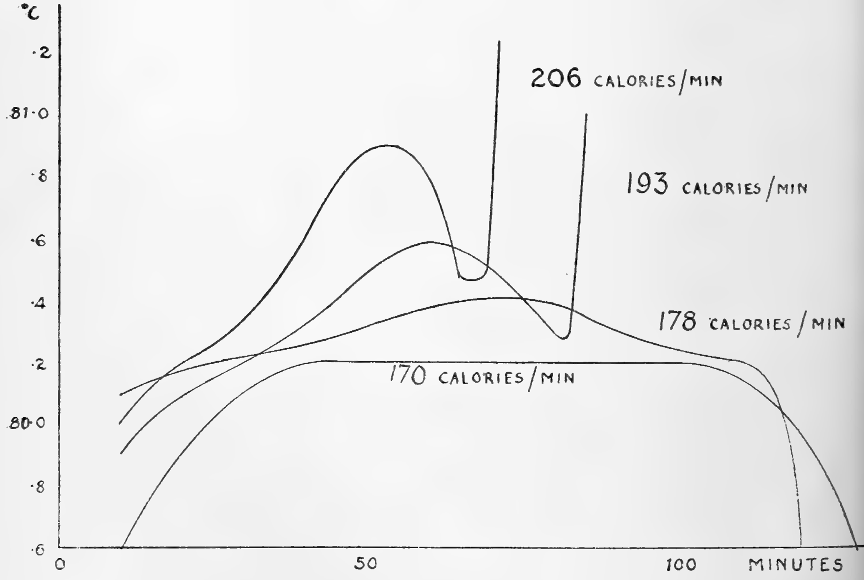
* S. F. Dufton, *loc. cit.*

purity of the distillate, a rise of one degree above the boiling-point of benzene corresponding to 2·78 per cent. of toluene. The thermometer was calibrated in position in the apparatus by the distillation of pure benzene, the boiling-point, corrected for atmospheric-pressure variation, being taken as 80°·2 C.

After some preliminary experiments, a series of four was made in which the only variation was in the rate of heating. In the first, with a heat supply of 170 calories per minute, pure benzene was separated, and when 39·5 c.c. out of the 40·0 c.c. of benzene had been collected, the thermometer fell and distillation ceased. At 178 calories per minute, in the next experiment, a trace (0·5 per cent.) of toluene passed over during part of the run. Distillation ceased when 39·3 c.c. had been collected. With a heat supply of 193 calories per minute there was a little more toluene in the distillate, and at 206 calories per minute still more.

Fig. 3.

Effect of Excessive Heat Supply on Purity of Distillate.



In these two experiments, after 40 c.c. had been distilled the temperature rose suddenly and toluene began to pass over. The temperature-time curves for the four experiments are plotted in fig. 3. The experiments show that for a given column there is an upper limit to the rate of heating, above which the distillate is not pure.

For the first of the experiments just described Table V. shows the volume of benzene collected, the percentage of benzene in the mixture remaining in the apparatus, the volume of distillate collected per minute and the thermal efficiency. The slow rise of temperature and the gradual increase in the rate of distillation and in the thermal efficiency during the first 50 minutes are attributed to the expenditure of heat in warming the column and the thermometer. When a steady state was attained, the thermal efficiency was sensibly constant at 44 per cent., falling slightly as the mixture became poor in benzene.

Table V.

Time: minutes.	c.c. collected.	c.c. per min.	Mixture per cent. of benzene.	Temp. °C.	Thermal efficiency per cent.
0	0·0	...	80	68·8	...
10	1·6	0·28	79	79·6	25
20	5·2	0·39	78	79·9	34
30	9·3	0·43	76	80·1	38
40	13·8	0·46	72	80·2	42
50	18·5	0·45	68	80·2	42
60	23·0	0·45	63	80·2	44
70	27·3	0·43	56	80·2	44
80	31·4	0·38	46	80·2	43
90	34·8	0·29	34	80·2	39
100	37·3	0·195	21	80·2	38
110	38·6	0·085	...	80·1	...
120	39·1	0·035	...	80·0	...
130	39·35	0·02	...	79·2	...
140	39·5	0·01	...	77·3	...

The composition of the mixture given in column 4 is obtained from the original composition by deducting the amount of benzene collected. When the composition falls to 20 per cent. of benzene only 2·5 c.c. of benzene remain, and the calculated percentage ceases to be trustworthy.

Experiments made with slower rates of heating showed that the greatest thermal efficiency was obtained when distillation was at the maximum rate to yield pure benzene. At 170 calories per minute the thermal efficiency was 44 per cent., with a heat supply of 146 calories per minute it was 35 per cent., and at 128 calories per minute it was only 5 per cent.

In the experiments described above, the column was lagged with cotton-wool to reduce the loss of heat, so that the liquid condensed was not sufficient to "flood" the column and to obstruct the upward flow of vapour. Experiments were made to determine the effect of further reduction in the amount of heat lost. The amount of lagging was increased until the maximum rate of heating to yield pure benzene was 151 calories per minute. In an experiment with a heat supply of 158 calories per minute the distillate contained a trace of toluene. For 151 calories per minute the thermal efficiency was 38 per cent. With a heat supply of 143 calories per minute it was 25 per cent. These experiments show that the thermal efficiency is reduced if loss of heat in the column be unduly prevented.

A shorter column, 25 cm. in length, was constructed with a bore of 0·3 cm. at the top and 0·7 cm. at the bottom, the wire for the spiral being the same throughout. With this column it was necessary to reduce the rate of heating as the proportion of benzene in the mixture diminished. Table VI. shows the results obtained in an experiment with this column. The thermal efficiency for a mixture containing 75 per cent. of benzene was over 70 per cent., and for mixtures containing more than 50 per cent. of benzene the column was more efficient than the 105 cm. column.

Table VI.

Time minutes.	c.c. collected.	c.c. per min.	Mixture per cent. of benzene.	Temp. ° C.	Calories per min.	Thermal efficiency per cent.
0	0·0	...	80	...	105·5	...
10	4·6	0·54	78	79·7	105·5	78
20	9·9	0·44	75	80·2	93·5	72
30	13·7	0·375	72	80·2	88·5	66
40	17·25	0·335	70	80·1	84·5	62
50	20·5	0·32	66	80·2	84·5	61
60	23·6	0·29	62	80·25	84·5	57
70	26·25	0·235	58	80·2	77·5	52
80	28·4	0·195	54	80·2	75	47
90	30·2	0·185	50	80·2	73	46
100	31·8	0·145	45	80·2	69	41
110	33·05	0·105	41	80·2	67·5	32
120	33·9	0·06	38	80·3	63	20

For mixtures containing less than 40 per cent. of benzene this column did not yield a pure distillate. For such mixtures a longer column must be employed.

4. The exposition which has been given of the principles underlying the art of discontinuous distillation and the confirmatory experiments which have been described are an essential preliminary to the investigation of continuous distillation.

In the continuous distillation of a binary mixture both components are constantly removed and the liquid does not become increasingly difficult of separation by reason of the accumulation of one component. Table I. shows the minimum flow of vapour and of liquid during ideal discontinuous distillation. Table VII. shows in similar manner the minimum flow in an ideal column during the continuous distillation of a mixture of equal masses of benzene and toluene. The upper portion of the column behaves as in discontinuous distillation, condensation taking place throughout its length, but in the lower portion evaporation of benzene occurs and the requisite quantity of heat to produce this evaporation must be supplied. This heat may be supplied throughout the length of the lower portion, as in the still devised by Lord Rayleigh, or, if minimum flow be not essential, by supplying a quantity of vapour of the less volatile constituent at the bottom of the column, as in the Coffey still.

TABLE VII.

Temp. ° C.	C_6H_6 per cent. in liq.	Grams Vapour passing up $C_6H_6+C_7H_8=$total.	Grams Liquid flowing down $C_6H_6+C_7H_8=$total.
Top of column.			
80·2	103	1·64+0·00=1·64	0·64+0 00=0·64
81·8	90	1·65+0·07=1·72	0·65+0·07=0·72
83·5	80	1·65+0·16=1·81	0·65+0·16=0·81
85·4	70	1·66+0·28=1·94	0·66+0·28=0·94
87·6	60	1·66+0·44=2·10	0·66+0·44=1·10
90·0	50	1·66+0·66=2·32	0·66+0·66=1·32
		2 grams liquid supplied.	1·66+1·66=3·32
93·0	40	1·12+0·68=1·80	1·12+1·68=2·80
96·4	30	0·73+0·70=1·43	0·73+1·70=2·43
100·4	20	0·43+0·71=1·14	0·43+1·71=2·14
105·0	10	0·19+0·72=0·91	0·19+1·72=1·91
110·6	0	0·00+0 75=0·75	0·00+1·75=1·75

The total evaporation necessary in the separation of a mass of the more volatile constituent is the same as that required in the discontinuous separation of the same mass from a large quantity of the mixture.

The heat required for the separation of one gram is

$$Q\left\{1+\frac{1}{\alpha(k-1)}\right\}. \quad . \quad . \quad . \quad . \quad . \quad (5)$$

The separations afforded by continuous and by discontinuous distillation may be compared. From equation (2), assuming k to be constant and equal to 2·5 for mixtures of benzene and toluene and Q to be constant and equal to 90 calories, the heat required to separate by discontinuous distillation p grams of benzene from a mixture of one gram of benzene with one gram of toluene, leaving a residue of $(1-p)$ gram of benzene and one gram of toluene, is

$$Q\int_0^p\left\{1+\frac{2-x}{(1-x)(k-1)}\right\}dx$$

$$=\frac{Q}{k-1}\left\{p\,.\,k-\log_e(1-p)\right\}. \quad . \quad . \quad . \quad . \quad (6)$$

Equation (5) shows that 21,000 calories are theoretically capable of separating 100 grams of benzene from 100 grams of toluene in continuous distillation. The same quantity of heat in discontinuous distillation will only separate from this mixture 78 grams of benzene, leaving 22 grams of benzene with the 100 grams of toluene. In order to separate 99 grams of benzene and leave 1 gram with the toluene, twice this amount of heat is required.

This comparison indicates that continuous distillation is much more economical than discontinuous. In continuous working, moreover, a column can be employed at its maximum efficiency throughout, while the length must be variable if the highest efficiency in discontinuous distillation is to be obtained.

The experimental portion of this investigation was carried out in Cambridge at the Cavendish Laboratory and one of the still-heads used was kindly lent by Dr. S. F. Dufton, to whom I am much indebted for encouragement and advice.

LVI. *Motion and Hyperdimensions.*

By F. TAVANI *.

THE object of this note is (i.) to establish a general relation showing how hyperdimensions are brought into evidence by carrying out an analytical operation which is susceptible of dynamical interpretation ; (ii.) to derive from the said relation, for the particular case of three dimensions, an important characteristic which can be extended to spaces of any dimension.

Let us consider the two complex quantities

$$a=(a_1\,a_2\ldots a_n) \quad \text{and} \quad b=(b_1\,b_2\ldots b_n).$$

After having extended to them the method of representation through the coordinates, we can assume as definition of the scalar or internal product between them

$$a/b=\sum_{n=1}^{n}(a_n b_n),$$

in which the first member represents the scalar product; according to Grassman's notation, the equation

$$\sum_{n=1}^{n}(a_n b_n)=0$$

expresses that the complex quantities a, b are perpendicular to one another.

Let us consider a sequence, $a, b, c, d \ldots n$ of n complex quantities, of n coordinates satisfying the following equations :

$$\left.\begin{array}{lll} b/a=0, & & \\ c/a=0, & c/b=0, & \\ d/a=0, & d/b=0, & d/c=0, \\ \ldots & \ldots & \ldots \end{array}\right\} \quad \ldots \quad (\text{i.})$$

from which we obtain by derivation

$$\left.\begin{array}{lll} b'/a=-b/a', & & \\ c'/a=-c/a', & c'/b=-c/b', & \\ d'/a=-d/a', & d'/b=-d/b', & d'/c=-d/c', \\ \ldots & \ldots & \ldots \end{array}\right\} \quad . \quad (\text{ii.})$$

the derivation being with respect to a variable t, of which the said quantities are assumed to be functions.

The geometrical meaning of the equations (i.) and (ii.) is an extension of that which they obviously have when the

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* Communicated by the Author.

complex quantities are complex of second order or vectors—that is to say, the equations of group (i.) say that each complex is perpendicular to all those which precede it in the given sequence; while the first equation of group (ii.) expresses the projection of b' upon a, the second equations express the projections of c' upon a and b, the third equations give the projections of d' upon a, b, c, and so on.

The quantities $a, b, c, d \ldots$ may be assumed further to verify the conditions: $a^2=1$, $b^2=1$, $c^2=1 \ldots\ldots$

and

$$\left.\begin{array}{llll} b=\dfrac{a'}{\text{mod.}\, a'}, & b=\dfrac{c'}{\text{mod.}\, c'}, & & \\ c=\dfrac{a'}{\text{mod.}\, a'}, & c=\dfrac{b'}{\text{mod.}\, b'}, & c=\dfrac{d'}{\text{mod.}\, d'}, & \\ d=\dfrac{a'}{\text{mod.}\, a'}, & d=\dfrac{b'}{\text{mod.}\, b'}, & d=\dfrac{c'}{\text{mod.}\, c'}, & d=\dfrac{e'}{\text{mod.}\, e'}, \\ \ldots\ldots & & & \end{array}\right\} \quad . \text{(iii.)}$$

which are obviously equivalent to those of the following group:

$$\left.\begin{array}{lll} a'=b\,\text{mod.}\, a', & c'=b\,\text{mod.}\, c', & \\ a'=c\,\text{mod.}\, a', & b'=c\,\text{mod.}\, b', & d'=c\,\text{mod.}\, d', \\ \ldots\ldots & & \end{array}\right\} \quad . \text{(iii.}_1\text{)}$$

Let P be a point moving in a space of n dimensions as a function of a variable t (time), so that

$$\text{mod.}\,P'=v \text{ (speed of P)}$$

and

$a=\dfrac{P'}{v}$, an equation analogous to those of the group (iii.).

Let also $\pi_1\, \pi_2\, \pi_3 \ldots \pi_n$ indicate points, such that the complex (π_1-o) has the direction of P' and passes through the fixed point o. In the case of complex of second order, and P moving in ordinary space, π_1 describes, while P moves, what is called the spherical indicatrix of the tangents to the path of P. We assume the point o to be such that $\pi_1=o+a$, and in a similar way

$$\pi_2=o+b, \quad \pi_3=o+c, \quad \pi_4=o+d. \quad . \quad . \quad \text{(iv.)}$$

In the case of complex of second order, vectors, and P

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* The meaning of these equations is obvious: they express that the complexes are of modulus $=1$, and each perpendicular to its derivate.

moving in ordinary space, π_3 describes, while P moves, the second spherical indicatrix.

From (iv.) we obtain by derivation

$$\text{mod.}\,\pi_1'=\text{mod.}\,a',\ \text{mod.}\,\pi_2'=\text{mod.}\,b',\ \text{mod.}\,\pi_3'=\text{mod.}\,c'\ldots\ (\text{iv.}_1)$$

Let us now introduce the notion of curvature ρ_1 through the relation

$$\rho_1=\frac{ds}{d\sigma_1},$$

in which ds is the portion of the path described by P, and $d\sigma_1$ the portion of the path described by π_1, while P describes ds.

Then we have

$$\rho_1=\frac{ds}{d\sigma_1}=\frac{\left(\dfrac{ds}{dt}\right)}{\left(\dfrac{d\sigma_1}{dt}\right)}=\frac{v}{\text{mod.}\,\pi_1'}=\frac{v}{\text{mod.}\,a'}.$$

Therefore $\text{mod.}\,a'=\dfrac{v}{\rho_1}$, and with a similar reasoning we obtain

$$\text{mod.}\,b'=\frac{v}{\rho_2},\quad \text{mod.}\,c'=\frac{v}{\rho_3},\quad \text{mod.}\,d'=\frac{v}{\rho_4}\ldots,\quad .\quad (\text{v.})$$

where ρ_2, ρ_3, $\rho_4\ldots$ have a meaning similar to that of ρ_1 established in a similar way by extending the same reasoning to π_2, π_3, $\pi_4\ldots$.

It is easy now to find the expressions of the projections of b', c', $d'\ldots$ upon the axis a, c, a, b, d, a, b, c, e, ... taken as systems of reference; by replacing in the second members of the equations (ii.), the values of a', b', c', $d'\ldots$ as given by (iii.$_1$), and making use of the equations (v.), we obtain:

$$\left.\begin{array}{l}(1)\left\{\begin{array}{l} b'/a=-b/a'=-b/b\,\text{mod.}\,a'=-\dfrac{v}{\rho_1}\\[2ex] b'/c=-b/c'=-b/b\,\text{mod.}\,c'=-\dfrac{v}{\rho_3}\end{array}\right.\\[4ex] (2)\left\{\begin{array}{l} c'/a=-c/a'=-c/c\,\text{mod.}\,a'=-\dfrac{v}{\rho_1}\\[2ex] c'/b=-c/b'=-c/c\,\text{mod.}\,b'=-\dfrac{v}{\rho_2}\\[2ex] c'/d=-c/d'=-c/c\,\text{mod.}\,d'=-\dfrac{v}{\rho_4}\\ \ldots\ldots\ldots\ldots\ldots\ldots\end{array}\right.\end{array}\right\}\ .\ .\quad (\text{vi.})$$

Therefore the expression of b' through a, c, taken as a system of reference, is

$$\left.\begin{array}{ll} (1) & b' = -\frac{v}{\rho_1} a - \frac{v}{\rho_3} c \text{ [*]} \\ (2) & c' = -\frac{v}{\rho_1} a - \frac{v}{\rho_2} b - \frac{v}{\rho_4} d \\ \ldots & \ldots\ldots\ldots\ldots\ldots \end{array}\right\}, \quad \ldots \quad \text{(vii.)}$$

and in general

$$V_p' = \sum_{m=1}^{m=p+1} \left(-\frac{v}{\rho_m} V_m\right) + \frac{v}{\rho_p} V_p \quad \ldots \quad \text{(viii.)}$$

valid from $p=2$ upwards,

where V denotes the quantity of the general type defined for $a, b, c, d \ldots$.

Let us make an important application of the relation (viii.).

From $a = \frac{P'}{v}$ we obtain $P' = va$, then by derivating $P' = va$ twice with respect to t, and replacing in the second member to a' its value,

$$a' = b \text{ mod. } a' = \frac{v}{\rho_1} b,$$

and to b' its value given by vii. (1), we arrive at

$$P''' = \left(v'' - \frac{v^3}{\rho_1^2}\right) a + \left(\frac{3vv'}{\rho_1} - \frac{v^2}{\rho_1^2} \rho'\right) b - \frac{v^3}{\rho_1 \rho_3} c \text{ [†]} \quad \ldots \quad \text{(ix.)}$$

We can repeat the derivation of (ix.), and replace to a', b', c' their values given by (viii.), so that in the second member appear only terms consisting of the quantities a, b, c, d, with coefficients which are elementary functions of v and the curvatures $\rho_1 \rho_2 \rho_3, \ldots$. If we indicate by $\phi(v. \rho)$ such coefficients, we can write in general

$$\frac{d^n P}{(dt)^n} = \sum_{n=1}^{n} \phi_n(v, p) V_n, \quad \ldots\ldots \quad \text{(x.)}$$

where $V_1 = a$, $V_2 = b$, $V_3 = c \ldots\ldots$

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[*] This equation is given by Prof. Peano in his '*Analisi infinitesimale*,' vol. ii. § 325, by a method which in this paper I have followed and generalized in order to obtain the relation (viii.).

[†] This relation is also given by Prof. Peano, *loc. cit.*

Thus we have established the relations (viii.) and (x.) which are the generalized forms of vii. (1) and (ix.), the latter being of particular use for the case of three dimensions, viz. of the physical space. We are going to make an application of it for the study of the motion of a system of reference in a space of three dimensions, a similar study being susceptible of extension to spaces of any dimension.

Let us consider the equation

$$\frac{d^3\mathrm{P}}{dt^3}=\left(v''-\frac{v^3}{\rho_1{}^2}\right)a+\left(\frac{3vv'}{\rho_1}-\frac{v^2}{\rho_1{}^2}\rho_1{}'\right)b-\frac{v^3}{\rho_1\rho_3}c.$$

If we assume a, b, c to be three vectors normal to one another so as to form a system of reference with the origin P, then the analytical expression of P as function of time, $\phi(t)$, represents the law of motion of the system, and $\phi'(t)$ and $\phi''(t)$ the velocity and the acceleration of the motion. Extending the meaning of ϕ', ϕ'', we may consider ϕ''', ϕ''''..... as the hyperaccelerations of the system of order three, four ..., the reality of these quantities depending on the law of the motion itself.

We can now express the meaning of the above equation through the following proposition :—

"If a system of three vectors normal to one another, of origin P, is in motion with a given law $\mathrm{P}=\phi(t)$, so that the axis of X coincides with the tangent to the path of P, the hyperaccelerations of the system represented by $\frac{d^n\mathrm{P}}{dt^n}(n=3, 4, 5,\ldots n)$ are analytically expressed with respect to a system of axes perpendicular to one another, provided that the dimensions of the orthogonal system of reference are taken in number equal to the number of the order of the acceleration." The meaning of this proposition can also be expressed by saying that "the virtual displacements due to the hyperacceleration of an orthogonal system, moving with its axis of X in the direction of the tangent to the path described by the origin, are analytically expressed through a system of reference, of a number of dimensions equal to the order of hyperacceleration."

LVII. *Force-Transformation, Proper Time, and Fresnel's Coefficient.* *By* Prof. FREDERICK SLATE [*].

FOR electronic conditions, Newtonian dynamics and relativity based on a Lorentz transformation are reducible to parallelism as mathematical schemes. The former introduces variable effective inertia where the latter treats inertia essentially as constant [†]. Moreover, relativity's method here can be assigned to widely inclusive grounds. Any attempt at a detailed physics by resolving further the data of energetics must countenance some flexible factoring of energy and of energy-flux in its tentative dynamics. Lagrange's equations are known to admit such alternatives; and no cogent reason exists for bounding the range of that proper freedom by their algebraic type [‡]. As an implication of the present analysis, we achieve a broader outlook, the gain of whose perspective is worth seeking.

Let an energy-transfer (W) of calculable amount be associated with a working speed (v) at the close of an interval ($0, v$), the frame being one among a "legitimate group." An interval like (u, v) is covered by a difference. Then it is mathematically permissible to express (W) variously as a doubled kinetic energy (2E), and to prepare thus for corresponding mechanical analogues. Accordingly, write a series of equivalents; not exhaustive, but meant to exemplify useful forms:

$$\mathrm{W} \equiv 2\mathrm{E} \equiv (m_1)(z\boldsymbol{v})^2 = (m_1 z)(zv^2) = (m_1 z^2)(v^2)$$

$$\equiv (\mu')(c^2) \equiv \left(\mu \frac{vz}{c}\right)(c^2). \quad . \quad . \quad (1)$$

The first parentheses in each factoring separate an assumed inertia-coefficient from a squared velocity; auxiliary velocities ($v\sqrt{z}$, vz, c, etc.) are then one inherent feature of any such series. The last two members employ a reduction to standard (or terminal) velocity [§]; (m_1) is a constant;

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[*] Communicated by the Author.

[†] Slate, Phil. Mag. vol. xxxix. p. 433; vol. xl. p. 31; vol. xli. p. 96. These papers are cited as (I.), (II.), (III.).

[‡] Generalized velocity and momentum are defined, and have been used practically, to realize this possibility. Incidentally, Abraham's early success in extending Lagrange's equations to the electron may find partial explanation here (*Theorie der Elektrizität,* vol. ii. p. 177 (1908)).

[§] Introduced at eq. (9) of (III.). Notice also eq. (10, 11, 12) and the application in eq. (20, 21).

(z) must be in general a variable ratio. Whereas no particular factoring can modify the total energy-flux (dW/dt), certain coördinating conventions will reserve some latitude about details under each adoption of factors. A momentum (Q) involves inertia (m), velocity (v'), and tangential force (T), while (E) is invariant* :

$$2E \equiv Qv' ; \quad Q \equiv mv' ; \quad T \equiv \frac{d}{dt}(mv'). \quad . \quad . \quad (2)$$

Regarding (E) as a function of (m, v') ; and defining

$$v_1 \equiv zv ; \quad Q_1 \equiv m_1 v_1 ; \quad Q_2 \equiv \mu' c \equiv \mu v_1 ; \quad . \quad . \quad (3)$$

the indispensable connexions among the group derived from equation (1) can be symbolized by

$$\frac{dW}{dt} \equiv 2\frac{dE}{dt} = 2v_1 T_1 = cT_2 = 2\left(v'T - \frac{\partial E}{\partial m}\frac{dm}{dt}\right)$$
$$= v'T + \frac{\partial E}{\partial v'}\frac{dv'}{dt}. \quad . \quad . \quad (4)$$

The definite (constant) inertia (m_1) is uniquely linked with (T_1), for which alone the partial $(\partial E/\partial m)$ vanishes, and the principle of *vis viva* remains valid in the sense belonging to rigid dynamics. The force (T_2) is unique otherwise; its partner being (c), the complementary partial $(\partial E/\partial v')$ is suppressed. There is a second unpartitioned absorption of energy, also into kinetic form, but with an accompanying variable inertia (μ'). This may be viewed as another sense of the *vis viva* principle. The recurrence of $(c^2 d\mu'/dt)$, with differing plausible values for (μ'), has made itself noticeable throughout previous developments regarding electronic energy †.

The two abbreviated forms in equation (4), together with the two general members, yield the following relations among others :

$$v'T \equiv v'\left(m\frac{dv'}{dt} + v'\frac{dm}{dt}\right) = v_1 T_1 + \tfrac{1}{2}v'^2\frac{dm}{dt} = cT_2 - mv'\frac{dv'}{dt}. \quad (5)$$

This amounts to establishing a transition between two activities (energy-fluxes), derived in turn from a variable inertia and from one that is constant. Either value of the

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* *Cf.* (III.), p. 102 ; and eq. (12, 18).

† See (III.) *passim* ; it is plainly one goal of relativity's combinations. An important effort to construct a physical meaning for this expression is added by Sir J. J. Thomson, Phil. Mag. June 1920, p. 679.

activity may be favoured by physical evidence; but on whichever one the preference thus falls, equation (5) or some simple equivalent shows how to calculate it in terms of the other, with due aid from a correcting partial derivative.

Presented in forms like

$$0 = m_1v_1^2 - \mu'c^2 = (m_1v_1 + ic\mu')(v_1 + ic)$$
$$= z(m_1v + i\mu v)(v_1 + ic) \quad \text{[Real terms]}, \qquad (6)$$

equation (1) sets in relief, first the idea of conservation (equal gain and loss at a transfer), and secondly the needful pairing of each alternative momentum with its own velocity-factor, when the complex product is expanded. All these aspects of the above more comprehensive situation embrace essentials of that correlation between Newtonian and "non-Newtonian" dynamics upon which this discussion turns. Beside the frequent appearance of (μ', μ) just referred to, keeping the last members of equations (1, 5) somewhat at the front, mathematical prominence is assured to them through the Lagrange function and its derivative*. But this must not exclude the third member of equation (5). Not only was its type put forward earlier as a central necessity of general statement†, but it happens to offer also for the present phase rather direct contact with the systematic use of "proper time" and of "local time" peculiar to relativity. This distinction is largely superfluous for our Newtonian plan, since without according it a place, the main dynamical relations resting upon it have been reproduced. That composite scheme of time-variables must be truly secondary, if it be indeed carried into the fundamental equations through a constancy of inertia made primary. Yet the newer doctrine takes so seriously what centres upon an entire parity of time and coördinate, that more adequate review of these points is in place, for which the activity (v_1T_1) opens the way. A step or two in broader terms can be added, before limiting ourselves by the Lorentz electron's assumptions.

Define now

$$m \equiv m_1z \equiv m', \quad v' \equiv v\sqrt{z};$$

then

$$Q' \equiv m'v\sqrt{z} = m_1vz^{3/2}.$$

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* The conception of kinetic potential has this consequence. Some special coincidences have shown themselves already: (III.), p. 104; (II.), eq. (47).

† In (I.), eq. (10); (II.), eq. (3).

With these values, equation (5) gives

$$\frac{v'\mathrm{T}'}{z} - v^2\frac{dm'}{dt} = v\mathrm{T}_1 - \tfrac{1}{2}v^2\frac{dm'}{dt} = \frac{1}{z}\frac{d\mathrm{E}}{dt} - \frac{\mathrm{E}}{z^2}\frac{dz}{dt} = \frac{d}{dt}\left(\frac{\mathrm{E}}{z}\right). \quad (7)$$

From two factorings of (Q_1), an important duplicate expression of T_1 follows:

$$\mathrm{T}_1 \equiv \frac{d}{dt}(m_1 z v) = m_1\frac{dv_1}{dt} = m'\frac{dv}{dt} + v\frac{dm'}{dt}. \quad . \quad . \quad (8)$$

Since an activity of importance for the electron grows out of $(v\mathrm{T}_1)$, which equation (7) connects with an energy-transfer (W/z), the latter quantity will presently claim attention. Next define $m \equiv m_1 z^2 \equiv m''$; with $v' \equiv v$, and $\mathrm{Q}'' \equiv m''v$. These lead at once to

$$2v_1\mathrm{T}_1 - v\mathrm{T}'' - m_1 z^2 v\frac{dv}{dt} = 0 = z_1\left[2v_1\mathrm{T}_1 - v\mathrm{T}'' - m_1 z^2 v\frac{dv}{dt}\right], \quad (9)$$

the factor (z_1) being arbitrary. Finally it is evident that

$$2z\left[v\mathrm{T}_1 - c^2\frac{dm'}{dt}\right] = c^2\left[\frac{d\mu'}{dt} - \frac{dm''}{dt}\right]. \quad . \quad . \quad . \quad (10)$$

The supposition is continued, that C.G.S. units of length and time (fluxion time) fix values for all observed velocities; either (v_o) in the standard frame (F), or (v_o') in any frame (U). Therefore $(dt = dt_o')$ in the defining ratios

$$v_o \equiv \frac{dx_o}{dt_o}; \quad v_o' \equiv \frac{dx_o'}{dt_o'}; \quad . \quad . \quad . \quad . \quad (11)$$

the accents add only a helpful indication of the "observing-frame." This will not bar auxiliary time-differentials, for convenient expression of auxiliary velocities, originally determined by C.G.S. units. Thus, if in relation to (F, U)

$$\left.\begin{aligned} d\tau &\equiv \frac{dt_o}{z}; \quad \frac{dx_o}{d\tau} = v_1, \; \frac{d}{d\tau}\left(\frac{dx_o}{d\tau}\right) = \frac{dv_1}{d\tau} = z\frac{dv_1}{dt_o}; \\ d\tau' &\equiv \frac{dt_o'}{z'}; \quad \frac{dx_o'}{d\tau'} = z'v_o' \equiv v_1', \; \frac{d}{d\tau'}\left(\frac{dx_o'}{d\tau'}\right) = \frac{dv_1'}{d\tau'} = z'\frac{dv_1'}{dt_o'}. \end{aligned}\right\} \quad (12)$$

Consequently, multiples of previous forces and activities may be drawn upon for the algebra, from parallel series like

$$\left.\begin{aligned} m_1\frac{d^2x_o}{d\tau^2} &= z\mathrm{T}_1, \quad \frac{dx_o}{d\tau}\left(m_1\frac{d^2x_o}{d\tau^2}\right) = z\frac{d}{dt_o}\left(\frac{m_1 v_1^2}{2}\right) = z\frac{d\mathrm{E}}{dt_o} = \frac{d\mathrm{E}}{d\tau}; \\ m_1\frac{d^2x_o'}{d\tau'^2} &= z'\mathrm{T}_1', \quad \frac{dx_o'}{d\tau}\left(m_1\frac{d^2x_o'}{d\tau'^2}\right) = z'\frac{d}{dt_o'}\left(\frac{m_1 v_1'^2}{2}\right) = z'\frac{d\mathrm{E}'}{dt_o'} = \frac{d\mathrm{E}'}{d\tau'}. \end{aligned}\right\} \quad (13)$$

The special auxiliaries (v_c', v_c), which in our rendering replace the Einstein velocities, can be brought formally closer to the latter under a similarly expanded notation. Instead of accepting them initially as "distorted" (but C.G.S.) velocities *, throw each of their defining ratios into terms of its own "distorted" length and time :

$$\left.\begin{aligned} v_c' &\equiv \frac{v_o - u}{1 - \frac{uv_o}{c^2}} \equiv \frac{dx'}{dt_c'}; \quad v_c \equiv \frac{v_o' + u}{1 + \frac{uv_o'}{c^2}} \equiv \frac{dx_c}{dt_c}; \\ &\text{then} \\ \frac{dx_c}{dt_o'} &= \gamma(u)(v_o' + u)\ ; \quad \frac{dx_c'}{dt_o} = \gamma(u)(v_o - u). \end{aligned}\right\} \quad (14)$$

In effect, details are here borrowed from relativity ; but again another conception of them is attached to the notation :

$$x_c' \equiv \gamma(u)(x_o - ut_o)\ ; \quad t' \equiv \gamma(u)\left(t_o - \frac{u}{c^2}x_o\right);$$

$$x_c \equiv \gamma(u)(x_o' + ut_o')\ ; \quad t_c \equiv \gamma(u)\left(t_o' + \frac{u}{c^2}x_o'\right). \quad . \quad (15)$$

Let now the more general (z, z') be particularized provisionally as $(\gamma(v_o), \gamma(v_o'))$. It is directly provable that

$$d\tau \equiv \frac{dt_o}{\gamma(v_o)} = \frac{dt_c'}{\gamma(v_c')}; \quad d\tau' \equiv \frac{dt_o'}{\gamma(v_o')} = \frac{dt_c}{\gamma(v_c)};$$

$$d\tau = d\tau' \quad [v_o = v_o']\ ; \quad . \quad (16)$$

which is our version of "proper-time invariance" for frames (U). At this point, as at others, a group of four quantities doubles relativity's pair, and a pair ($d\tau$, $d\tau'$) distinguishes where relativity does not. Both duplications are referable to combining the two aspects of "simultaneous": it is evident that, whenever observed (v_o', v_o) coincide with specified series (v_c', v_c), the values calculated for the other frame will coincide also with the (v_o, v_o') corresponding to such (v_c', v_c).

Differentiate equations (14), noting equations (16) and previously demonstrated equalities † ; which gives

$$\left.\begin{aligned} \gamma^2(v_o)\frac{dv_o}{d\tau} &= \gamma^3(v_o)\frac{dv_o}{dt_o} = \gamma^2(v_c')\frac{dv_c'}{d\tau} = \gamma^3(v_c')\frac{dv_c'}{dt_c'} = \frac{T_o}{m_0}; \\ \gamma^2(v_o')\frac{dv_o'}{d\tau'} &= \gamma^3(v_o')\frac{dv_o'}{dt_o'} = \gamma^2(v_c)\frac{dv_c}{d\tau'} = \gamma^3(v_c)\frac{dv_c}{dt_c} = \frac{T_o'}{m_0\gamma(u)}. \end{aligned}\right\} \quad (17)$$

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* See (I.), pp. 436, 438 ; (II.), pp. 40, 41.

† In (II.), eq. (19, 25). Consider eq. (21, 24) there and eq. (17) here for features of likeness and difference. A "reduction-factor" and a "distorted time" are equivalent in operation.

The results are patterned after a "force-invariance for frames (U)." It is relevant to completed comparison, that Newtonian dynamics and relativity control these similar resources. Nevertheless, no such steps of algebraic manipulation, subject mainly to notational consistency, can acquire physical standing in either plan, except by the one test.

Under the supposition $z \equiv \gamma(v_o)$, we approach the assumptions for the Lorentz electron and the frame (F); but retain as before a constant inertia (m_1), whose ratio to the standard (m_0) can be assigned later. Resume from previous notation and values :

$$\mathrm{T}_1 = m_1\gamma^3(v_o)\frac{dv_o}{dt_o}\,;\ \mathrm{T}_o \equiv \frac{m_0}{m_1}\mathrm{T}_1\,;\ m' \equiv m_1\gamma(v_o)\ ;\ v\mathrm{T}_1 = c^2\frac{dm'}{dt_o}\,^*. \quad (18)$$

Add, from equations (12, 9) :

$$v_1\mathrm{T}_1 = \frac{d\mathrm{E}}{dt_o} = c^2\frac{dm'}{d\tau}\,;\quad \frac{1}{\gamma(v_o)}\frac{d\mathrm{E}}{dt_o} = \frac{1}{\gamma^2(v_o)}\frac{d\mathrm{E}}{d\tau}\,;\quad c^2\frac{d\mu'}{dt_o} - c^2\frac{dm''}{dt_o} = 0. \quad (19)$$

Whatever condition reduces to zero the first parenthesis in equation (10) makes the difference $(m'' - \mu')$ constant. This requires

$$m_1 z^2 = (m_1 z^2)\frac{{v_o}^2}{c^2} + c_1\,;\quad c_1 = m_1 z^2\left(1 - \frac{{v_o}^2}{c^2}\right) = m_1$$

$$[z^2 \equiv \gamma^2(v_o)]. \quad (20)$$

Further,

$$\mu' = m_1(\gamma^2(v_o) - 1) = \frac{m_1{v_o}^2}{c^2 - {v_o}^2}\,;\quad \mu \equiv \mu'\frac{c}{v_1} = \frac{m_1 v_o}{\sqrt{c^2 - {v_o}^2}}\,;$$

$$\mu c = m_1\gamma(v_o)v_o. \quad (21)$$

The "transverse mass" and the momentum of relativity's Lorentz electron emerge thus simply from general expressions through an assumption regarding activity. Remark that $(1 - {v_o}^2/c^2)$ is an intrinsic factor in the resistance problem, and in equation (20); its equality with $(1/z^2)$ is introduced by a condition open to revision.

To class "multiple forces," like $(z\mathrm{T}_1,\ z'\mathrm{T}_1')$ in equations (13), as being of "Minkowski" type suggests bringing the original $(\mathrm{T}_1,\ \mathrm{T}_1')$ under a "Newton" type ; indeed the latter are part of normal Newtonian dynamics. The work and the impulse of the two force-classes offer several points

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* The general bearing of the "resistance problem" and its terminal velocity is to be kept in mind : (II.), p. 35 ; (III.) *passim*, but especially eq. (6, 22). Distinguish two uses of the symbol (v_1).

to work out. Consider first in (F) the force (K_1) and its activity (A_1):

$$K_1 \equiv m_1 \frac{d^2 x_o}{d\tau^2} = \gamma(v_o) T_1 ; \quad A_1 \equiv \frac{dx_o}{d\tau} K_1 = \gamma(v_o) v_1 T_1$$
$$= \gamma^2(v_o) c^2 \frac{dm'}{dt_o}. \quad (22)$$

Keeping equations (1, 4) in view, determine the work (W_1) for an interval (u, v_o):

$$\begin{aligned} W_1 \equiv 2\int_u^{v_o} K_1 \frac{dx_o}{d\tau} d\tau &= m_1 c^2 \left(\frac{c^2}{c^2 - v_o^2} - \frac{c^2}{c^2 - u^2} \right) \\ &= m_1 c^2 \left(\frac{v_o^2}{c^2 - v_o^2} - \frac{u^2}{c^2 - u^2} \right) \\ &= c^2(\mu' - \mu_0') = W - W_0 \; [z \equiv \gamma(v_o)]. \end{aligned} \quad . \; . \; . \; . \; (23)$$

Therefore

$$m_1 c^2 \frac{v_o^2}{c^2 - v_o^2} - m_1 c^2 \frac{c^2}{c^2 - v_o^2} = m_1 c^2 \frac{u^2 - c^2}{c^2 - u^2} = -m_1 c^2 ; \quad (24)$$

which might be rated a trivial identity, did not our purpose connect it profitably with equations (6, 18)*. Denote by (R') the product of the real terms in equation (6), by (R'') the product of the imaginary terms, and by (R_1) their sum Then for present values $(R' + R'')$ is seen to repeat the first member of equation (24). Hence $dR_1/dt_o = 0$, and referring to equation (7) also,

$$\left. \begin{aligned} \frac{d}{dt_o}\left(\frac{R'}{\gamma(v_o)}\right) + \frac{d}{dt_o}\left(\frac{R''}{\gamma(v_o)}\right) &= \frac{d}{dt_o}\left(\frac{R_1}{\gamma(v_o)}\right) ; \\ \frac{d}{dt_o}\left(\frac{W}{\gamma(v_o)}\right) - c^2 \frac{dm'}{dt_o} \qquad &= m_1 \gamma(v_o) v_o \frac{dv_o}{dt_o}. \end{aligned} \right\} \quad . \; . \; (25)$$

Without elaborating every detail, this outline goes far enough to be convincing: it retraces essentially the "step up and step down" with the factor $(\gamma(v_o))$ which reaches Newtonian activity in relativity's procedure—attainable brevity or directness is not for the moment an issue. The repetition with quantities belonging to a frame (U) is so nearly routine that it is omitted. On any line of analysis, the decision lies in the resistance problem, whether at transfer a reduction factor shall be applied to values first written for (F) or for (U). The turning-point is located in

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* Moreover, this "invariant function" is plainly an offshoot from the fundamentals of the resistance problem.

the answer to the question: Which relative speed (v) is physically responsible for the resistance (m_1kv^2)?

The manifold bearings of the force (T_1) justify adding a word about its impulse; and transitions between frames (F, U) where it occurs. As in equation (23), take the interval (u, v_o) in (F), important in corresponding to the interval (0, v_o-u) or (0, v_c') in (U). Adjust a velocity (v_u) in (U) to meet the condition making momentum-changes permanently equal; as (v_c') affected an energy-change:

$$m_1v_u \equiv m_1(\gamma(v_o)v_o-\gamma(u)u). \quad . \quad . \quad . \quad . \quad (26)$$

Then the equal time-rates entail

$$m_1\frac{dv_u}{dt_o} = m_1\frac{d}{dt_o}(\gamma(v_o)v_o) \equiv m_1\frac{dv_1}{dt_o} \equiv T_1; \quad . \quad . \quad (27)$$

the tangential accelerations and their forces thus measured in (F, U) are equal; and the activities as well,

$$v_uT_1 = T_1(\gamma(v_o)v_o-\gamma(u)u). \quad . \quad . \quad . \quad . \quad (28)$$

Consequently,

$$\left.\begin{aligned} T_1(v_1-v_u) &= T_1\gamma(u)u = \gamma(u)T_1[v_o-(v_o-u)]; \\ T_1v_u-T_1\gamma(u)(v_o-u) &= T_1v_o(\gamma(v_o)-\gamma(u)). \end{aligned}\right\} \quad (29)$$

Understanding that (m_0) in (T_a') is replaced by (m_1) for this occasion, the work-relation appears*:

$$\int_0^u T_1v_u\,dt_o-\gamma(u)\int_0^{v_c'} T_a'v_c'\,dt_o = \int_u^{v_o} T_1v_o(\gamma(v_o)-\gamma(u))\,dt_o$$

$$= \frac{c^2}{2}\int_u^{v_o}\frac{d\mu'}{dt_o}\,dt_o-c^2\gamma(u)\int_u^{v_o}\frac{dm'}{dt_o}\,dt_o. \quad (30)$$

This result does something to enlarge command of interdependence between (F, U). But its better service, perhaps, is to enforce again two dynamical ideas that pervade these investigations: first, that the attacks through energy and through momentum, though reconcilable, are not entirely congruent; and secondly, that the treatment of variable inertia breaks away from what suffices for constant inertia. Each term in the first member builds upon its own equal acceleration (dv_1/dt_o, dv_o/dt_o) in (F, U). This is not on the surface true of the second member, since (T_1v_1) belongs to the third member of equation (8), and (T_1v_o) to its fourth member†. The above expansion repeats for an observation-

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* Relying on (I.), pp. 436, 438; or on (II.), p. 44.

† *Cf.* the earlier comment; (II.), pp. 38, 39.

frame (U); with its distinctive notation, and introducing for (F) a companion to (v_c) similar to (v_u) in (U).

Some mention has been made already of a "Fresnel coefficient" (κ), and of a meaning for it as "inertia-drag," when associated with the frame (F). There proves to be, however, a pair of such coefficients (κ, κ'), symmetrically related to (F, U), like ($d\tau$, $d\tau'$) of equations (12). Something remains to say about this pair, connected with our Newtonian forces (T_o, T_a, T_o', T_a'), including now under those symbols values for either the more general (m_1) or the more particular (m_0), as the context may indicate. We can quote for observation-frame (F) *:

$$\left.\begin{aligned}\kappa &\equiv \frac{v_o - v_c'}{u} = \frac{c^2 - v_o^2}{c^2 - uv_o} = \frac{c^2 - v_c'^2}{c^2 + uv_c'} = 1 - \frac{v_o v_c'}{c^2}\,;\\ \kappa T_a' &= m_1\gamma(v_o)\frac{dv_o}{dt_o};\quad \kappa T_o = m_1\gamma(u)\gamma(v_c')\frac{dv_c'}{dt_c}.\end{aligned}\right\}\quad(31)$$

Basing frankly on symmetry for the defining ratio (but succeeding members are demonstrable), write then for an observation-frame (U), noting ($u' = -u$):

$$\kappa' \equiv \frac{v_o' - v_c}{u'} = \frac{v_c - v_o'}{u} = \frac{c^2 - v_o'^2}{c^2 + uv_o'} = \frac{c^2 - v_c^2}{c^2 - uv_c} = 1 - \frac{v_o' v_c}{c^2}.\quad(32)$$

In the special view of relativity, (κ, κ') become equal. Or here visibly through the coincidences that are mentioned below equations (16). But our Newtonian scheme, in the several instances enumerated, gives enlarged reciprocity to frames (F, U); *independent* phenomena originating (we may say) in either are convertible into terms of the other. The contrast with treating the *same* phenomena indifferently in all frames of the group is certainly not to Newton's disadvantage.

Under the definition of (κ'), the proofs are direct that

$$\left.\begin{aligned}\kappa'\left(\frac{T_o'}{\gamma(u)}\right) &= m_1\gamma(u)\gamma(v_c)\frac{dv_c}{dt_o}\,;\quad \kappa'\left(\frac{T_a}{\gamma(u)}\right) = m_1\gamma(v_o')\frac{dv_o'}{dt_o'}\,;\\ \frac{T_a}{\gamma(u)} &= m_1\gamma(v_o')\frac{dv_o'}{dt_o'} + (v_o' + u)\frac{d}{dt_o'}(m_1\gamma(v_o')).\end{aligned}\right\}\quad(33)$$

These round out the symmetry because, allowing always

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* From (II.), pp. 38, 43. Eq. (5, 13, 24, 26) there define (T_o, T_a, T_o', T_a').

for the weighting factor $(\gamma(u))$, and remembering $(dt_o = dt_o')$, the last equality stands in exact parallel with

$$\mathrm{T}_a' \equiv m_1\gamma(v_o)\frac{dv_o}{dt_o} + (v_o - u)\frac{d}{dt_o}(m_1\gamma(v_o))\,^*. \quad . \quad (34)$$

Next use (κ, κ') with those multiples $(\mathrm{K_F}, \mathrm{K_F}', \mathrm{K}_u', \mathrm{K}_u)$ of $(\mathrm{T}_o, \mathrm{T}_a, \mathrm{T}_o', \mathrm{T}_a')$ which have been subjected to a "Minkowski transformation"; they are all of the same type as (T_o) itself†. The combinations disclose another phase of symmetry:

$$\left.\begin{aligned} \kappa\mathrm{K_F} &= \gamma(u)m_1\gamma(v_c')\frac{dv_c'}{dt_o}; \quad & \kappa'\mathrm{K_F}' &= \gamma(u)m_1\gamma(v_o')\frac{dv_o'}{dt_o'}; \\ \kappa'\mathrm{K}_u' &= \gamma(u)m_1\gamma(v_c)\frac{dv_c}{dt_o}; \quad & \kappa\mathrm{K}_u &= \gamma(u)m_1\gamma(v_o)\frac{dv_o}{dt_o}. \end{aligned}\right\} \quad (35)$$

Since $\gamma(u) = \gamma(u')$, the common factor $\gamma(u)$ signalizes again the fully reciprocal relations of (F, U) to transfer back and forth. The effect otherwise of (κ, κ') is to cancel from each force's companion the term depending on variable inertia ‡. Put together equations (17, 33, 34), bringing out once more the vital divergence from relativity through employing a time-variable uniformly in all frames. Whatever the influence in other directions of such comparisons, they will not prejudice judgment to undervalue the mental vision which detected this fitness of local (or distorted) time to restore an invariance in magnitude, as well as preserve a mathematical type.

By drawing upon the sequence of equations (17), the original equation (9) can be employed to recalculate for transfers among frames (U). That repetition passed over as nearly self-evident, new conclusions remain to extract from the comment attached to equation (5), that coördinates one activity $(v_1\,\mathrm{T}_1)$ with an adjustable set of activities $(v'\,\mathrm{T})$. To grasp its full scope, and to weigh habitually all its alternatives, may tend to clear perplexing or obscure questions, when the energy is assigned in advance, to which that equation shall be accommodated. Exhaustive inquiry

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* Laid down by way of fundamental premise in (II.), eq. (12).

† See (II.), p. 43. They will not be confused with "Minkowski forces," like $(z\mathrm{T}_1, z'\mathrm{T}_1')$ in eq. (13) above.

‡ Returning closely to the assumption of (II.), p. 37. This point gains in importance, when the intimate bearing of Fresnel's coefficient upon the physical action is conceded.

into the combination

$$v'\mathrm{T} = (zv)\mathrm{T}_1 + \tfrac{1}{2}v'^2\frac{dm}{dt} \quad . \quad . \quad . \quad . \quad (36)$$

should be generally fruitful, under suggestion from the successful analysis for Abraham's electron *. Bringing in electromagnetic activity $(d\mathrm{M}/dt_o)$ at the value which mainly controls the reasoning here,

$$\frac{d\mathrm{M}}{dt_o} = v_o\left[\mathrm{T}_1 + \frac{m'}{4}\frac{dv_o}{dt_o}\right], \quad . \quad . \quad . \quad . \quad (36\,a)$$

its divergence from $(v_o\mathrm{T}_1)$ is not such a discrepancy logically, as leaning on $(v_1\mathrm{T}_1)$ exclusively would imply. Nor is its removal at once compulsory, by inventing some release of the electron's internal energy, for example. But concession must be made to two possibilities: first that (z) is not exactly equal to $(\gamma(v_o))$; and, secondly, that the total energy-flux is greater than $(v_1\mathrm{T}_1)$, on account of imperfect conversion into electromagnetic form. We can illustrate both points by returning to equation (9), put into these terms :

$$\begin{aligned} v_o\left(\frac{\mathrm{T}''}{\gamma(v_o)}\right) &= 2v_o\mathrm{T}_1 - m'v_o\frac{dv_o}{dt_o} = v_o\mathrm{T}_1 + v_o^2\frac{dm'}{dt_o} \\ &= v_o\mathrm{T}_1\left(1+\frac{v_o^2}{c^2}\right). \quad . \quad (37) \end{aligned}$$

If (m_1) and the last parenthesis be associated, the larger energy-flux denoted by the first member would appear as due to a graded increase of effective inertia. The idea read into Fresnel's coefficient is revived. Proceed next to revised magnitudes of (μ', μ), determined by an imperfect conversion that leaves $(m_1v_o^2/2)$ of (literal) kinetic energy :

$$\left.\begin{aligned} &\mu_1'c^2 \equiv m_1v_1^2 - \frac{m_1v_o^2}{2}; \;\; \mu_1' = \frac{\mu'}{2}\left(1+\frac{v_o^2}{c^2}\right) = \frac{v_o^2}{c^2}\left(\frac{m''+\mu'}{2}\right); \\ &\quad \mu_1 \equiv \frac{c}{v_1}\,\mu_1' = \frac{\mu}{2}\left(1+\frac{v_o^2}{c^2}\right). \end{aligned}\right\} \quad (38)$$

Hence

$$2\gamma(v_o)\mu_1'\frac{c^2}{v_o^2}\frac{dv_o}{dt_o} = \mathrm{T}_1\left(1+\frac{v_o^2}{c^2}\right) = \frac{\mathrm{T}''}{\gamma(v_o)} \; ; \quad . \quad . \quad (39)$$

and the set $(\mathrm{T}'', \mu_1', \mu_1)$ grow out of $(\mathrm{T}_1, \mu', \mu)$ by introducing the same essential factor, which equations (38) make

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[*] In (III.), eq. (19) to (24). We are continuing the substitution of (m_1) for (m_0).

characteristic of that imperfect conversion. This becomes significant for comparison with previous results *. Continuing along the same line, we find for the assumed time-rate of electromagnetic energy,

$$\begin{aligned} v_o\left(\mathrm{T}_1+\frac{m'}{4}\frac{dv_o}{dt_o}\right) &= \left(\frac{\mathrm{T}''}{2\gamma(v_o)}+\tfrac{3}{4}m'\frac{dv_o}{dt_o}\right)v_o \\ &= \left[\gamma(v_o)\mu_1'\frac{c^2}{{v_o}^2}+\tfrac{3}{4}m'\right]v_o\frac{dv_o}{dt_o}, \qquad (40) \end{aligned}$$

where the brackets again set off an effective inertia. The facts can be summarized into saying that a composite operator (scale-factor) must be applied; each element in it to the proper quota of (m_1). Equation (9) fixes, for the general ratio (z) of equation (1), a mid-way point between the activities (energy-fluxes) denoted by $(2v_o\mathrm{T}_1, \mathrm{T}''v_o/\gamma(v_o))$. But specializing (z) into $(\gamma(v_o))$ throws the assigned electromagnetic activity unsymmetrically into that interval. This outcome conforms reasonably with building on the basis afforded by the last member of equation (7). Provided that

$$\left.\begin{aligned} \frac{\mu_2'c^2}{z} \equiv \frac{2\mathrm{E}}{z}-\frac{n\mathrm{E}}{z^2}; \quad \frac{d}{dt}\left[\frac{\mu_2'c^2}{z}+\frac{n\mathrm{E}}{z^2}\right] &= \frac{2}{z}\frac{d\mathrm{E}}{dt}-\frac{2\mathrm{E}}{z^2}\frac{dz}{dt} \\ &= 2v\mathrm{T}_1-v^2\frac{dm'}{dt} = v\mathrm{T}_1+m'v\frac{dv}{dt}; \\ \frac{d}{dt}\left[\frac{\mu_2'c^2}{z}+\frac{n\mathrm{E}}{z^2}\right]-\tfrac{3}{4}m'v\frac{dv}{dt} & \\ &= v\left(\mathrm{T}_1+\frac{m'}{4}\frac{dv}{dt}\right). \end{aligned}\right\} \qquad (41)$$

The preceding treatment of electromagnetic activity is in the first instance empirical, let it be granted. For one thing, it adheres to an activity-value whose exact validity is perhaps not yet beyond question. Supposing, however, this datum to remain unshaken under renewed critical examination, the foregoing "cut-and-try" result can be rationalized, at least partially, by comparing it with the routine in terms of (μ', μ) and of (μ_1', μ_1). Let us lay out the main steps of that analysis, by way of conclusion. Consider first the activities *for the speed* (v_1), based on equations (1, 38),

$$v_1\frac{d}{dt}(c\mu) = c^2\frac{d\mu'}{dt}-c\mu\frac{dv_1}{dt}; \quad v_1\frac{d}{dt}(c\mu_1) = c^2\frac{d\mu_1'}{dt}-c\mu_1\frac{dv_1}{dt}. \qquad (42)$$

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* Particularly (III.), eq. 29; and the routine of eq. (11, 12) there.

For present conditions, on adding in the second case the "unconverted activity," $(2v_1T_1)$ appears repartitioned thus :

$$\left.\begin{aligned} A &\equiv v_1\frac{d}{dt}(c\mu) = 2v_1T_1 - v_1T_1\,; \\ A_1 &\equiv v_1\frac{d}{dt}(c\mu_1) + m_1v\frac{dv}{dt} = 2v_1T_1 - \tfrac{1}{2}vT''. \end{aligned}\right\} \quad (43)$$

Hence

$$\left.\begin{aligned} \frac{A-A_1}{2} &= -\frac{m''}{4}v\frac{dv}{dt}\,; \\ \frac{A+A_1}{2} &= v_1T_1 + \frac{m''}{4}v\frac{dv}{dt} = \frac{vT''}{2} + \tfrac{3}{4}m''v\frac{dv}{dt}. \end{aligned}\right\} \quad (44)$$

This agrees with one particular (empirical) rearrangement of equation (9), after reducing the working-speed to (v); exploiting the flexibility of a zero-difference :

$$\left[2vT_1 - \frac{v}{2}\frac{T''}{\gamma(v)} - \frac{m'}{4}v\frac{dv}{dt}\right] - \left[\frac{v}{2}\frac{T''}{\gamma(v)} + \tfrac{3}{4}m'v\frac{dv}{dt}\right] = 0. \quad (45)$$

And the magnitude of the last item coincides with the time-rate of the corresponding electromagnetic potential. The quoted value of electromagnetic energy-flux rejects, we might say, both extreme suppositions, of (A, A_1); it is determined symmetrically between them. How far does this support the assertion of *mass* in the Lorentz electron ?

Proper candour can admit this whole system of equations to be finally inconclusive, and yet hold to their present usefulness. So long as aggregates only are accessible, the search for their physical constituents will grope more or less blindly. The close of that period will be hastened by first enlarging the list of possibilities, and at last weighing them impartially. The simple thoughts of this paper do no more than exemplify a method, it is true, without exhibiting it in formal terms or delimiting it. But it can scarcely be doubtful that such a widening of Lagrange's plan is of good promise for the further discussion of energy-fluxes in terms of mechanics.

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LVIII. *Some Problems relating to Rotating Fluid in the Atmosphere. By* GEORGE GREEN, *D.Sc., Lecturer in Natural Philosophy in the University of Glasgow* *.

ACCORDING to the modern view regarding the constitution of cyclones and anticyclones the characteristic movements of the air are in planes parallel to the earth's surface. Upward currents may occur in certain parts of the system, but they do not form the essential feature of the motion. This view may, or may not, prove to be correct. It is therefore a problem of some interest to determine under what conditions horizontal motions of the type generally associated with the normal cyclone and anticyclone are possible in the atmosphere. Certain aspects of the problem of the travelling cyclone have already been dealt with by Dr. Jeffreys †, and by the late Lord Rayleigh ‡, and by Sir Napier Shaw §.

The present paper deals with the same problem in a different way. Its purpose is to show that certain motions of the type generally associated with the normal cyclone are consistent with the hydrodynamical equations of motion and with the condition of continuity of the fluid, and are therefore possible motions of the atmosphere. The difficulty pointed out by the late Lord Rayleigh regarding the boundary within which motion takes place, is not touched upon in this paper.

Let us assume that the air near any point O at the earth's surface is in uniform rotation relative to the earth about a vertical axis OZ, drawn through O upwards. This point O may be at rest or in motion relative to the earth while the fluid layer in contact with the earth rotates uniformly about O as a centre of rotation. We can now refer the motion of each particle of fluid to three rectangular axes defined below drawn through the centre of the earth O′.

O′X is drawn parallel to a horizontal line drawn due East from point O.

O′Y is drawn parallel to a horizontal line drawn due North from point O.

O′Z is drawn parallel to a vertical line drawn through point O.

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* Communicated by the Author.

† Phil. Mag. January 1919, p. 1.

‡ Phil. Mag. September 1919, p. 420.

§ Geophysical Memoirs of the Meteorological Office.

The coordinates of any point referred to point O at each instant we shall denote by (x, y, z) and the coordinates of the same point referred to the three parallel axes through O′ are then $(x, y, z+\mathrm{R})$, where R denotes the radius of the earth approximately. For our present purpose we may treat the earth as a perfect sphere. The three axes which we have chosen through O′ are at each instant rotating axes. We shall accordingly represent by

ω_x:—the angular velocity of rotation of axes O′Y and O′Z about axis O′X,

with ω_y and ω_z as corresponding quantities for the other axes respectively. The corresponding rates of change of these quantities are then denoted by $\dot{\omega}_x$, $\dot{\omega}_y$, $\dot{\omega}_z$ respectively, and the components of the velocity of the fluid relative to the moving axes specified above, at each instant of time t, are denoted by u, v, w respectively.

Any possible motion of the fluid must now satisfy the following system of equations :—

(a) The equations of motion :

$$\left.\begin{aligned}
\frac{du}{dt}-2\omega_z v+2\omega_y w-\dot{\omega}_z y+\dot{\omega}_y(z+\mathrm{R})-(\omega_y{}^2+\omega_z{}^2)x&=-\frac{1}{\rho}\frac{\partial p}{\partial x}\\
\frac{dv}{dt}-2\omega_x w+2\omega_z u-\dot{\omega}_x(z+\mathrm{R})+\dot{\omega}_z x-(\omega_z{}^2+\omega_x{}^2)y&=-\frac{1}{\rho}\frac{\partial p}{\partial y}\\
\frac{dw}{dt}-2\omega_y u+2\omega_x v-\dot{\omega}_y x+\dot{\omega}_x y-(\omega_x{}^2+\omega_y{}^2)&=-g-\frac{1}{\rho}\frac{\partial p}{\partial z}
\end{aligned}\right\}\text{(A)}$$

where $$\frac{d}{dt}=\frac{\partial}{\partial t}+u\frac{\partial}{\partial x}+v\frac{\partial}{\partial y}+w\frac{\partial}{\partial z}.$$

(b) The equation of continuity of the fluid :

$$\frac{d\rho}{dt}+\rho\left(\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}+\frac{\partial w}{\partial z}\right)=0. \quad . \quad . \quad . \quad \text{(B)}$$

(c) The equation determining the physical nature of the fluid :—

$p=k\rho$ for an isothermal atmosphere,

$p=k\rho^\gamma$ for an atmosphere in convective equilibrium, where γ denotes the ratio of the specific heat of air pressure constant to the specific heat volume constant.

(*d*) The boundary equations;—which we may take to be those represented by the conditions,

$$w=0,\quad \frac{dw}{dt}=0,\quad \text{at } z=0 \text{ and at } z=h,$$

h being the upper limit of the troposphere which we assume to be a fluid of finite depth.

For convenience we have taken $p=k\rho$ throughout; the results obtained in the paper can, however, be easily modified to suit other physical conditions than that of an isothermal atmosphere.

These equations are simplified to some extent by the assumption which we now make that the motion of the fluid in each layer parallel to the ground at point O is entirely horizontal. This makes w and $\frac{dw}{dt}$ each zero. Moreover, in the special cases of motion to be considered in the present paper the terms $\dot{\omega}_x$ and $\dot{\omega}_y$ are also zero. The values to be assigned to the terms ω_x, ω_y, ω_z depend in part on Ω the rotational velocity of the earth about its axis, on ϕ, the latitude of the point O, and on the motion of the point O relative to the earth. If we denote by (U, V) the components of the velocity of point O in the directions due East and due North respectively, we have

$$\omega_x=-\dot{\phi}=-\frac{\mathrm{V}}{\mathrm{R}};\quad \omega_y=\Omega\cos\phi+\frac{\mathrm{U}}{\mathrm{R}};\qquad \omega_z=\Omega\sin\phi.$$

$$\dot{\omega}_x=-\frac{\dot{\mathrm{V}}}{\mathrm{R}};\quad \dot{\omega}_y=-\Omega\sin\phi\,.\,\dot{\phi}+\frac{\dot{\mathrm{U}}}{\mathrm{R}};\quad \dot{\omega}_z=\Omega\cos\phi\,.\,\dot{\phi}.$$

To obtain the values of $(\omega_x^2+\omega_y^2)z$, $(\omega_y^2+\omega_z^2)x$, $(\omega_z^2+\omega_x^2)y$ to be used in our equations we omit all terms containing Ω^2 which appear in ω_y^2 and ω_z^2. These terms are already allowed for in treating gravity as a force uniform in direction over the whole field around point O. That is, the terms referred to are compensated for in the variation in the direction of gravity around point O, and do not affect the motions now being considered.

Case I.—The first motion of the atmosphere to which we shall apply the above equations is that corresponding to a uniform rotation of the atmospheric layer in contact with the earth about a vertical axis through a point O which is at rest relative to the earth's surface. We assume that each particle of fluid in any of the upper layers describes a circle with uniform angular velocity ω about a centre in its own plane. This type of motion is represented by

$$u=-\omega(y-\beta z)\ ;\ v=\omega x\ ;\qquad w=0.$$
$$\omega_x=0\ ;\qquad \omega_y=\Omega\cos\phi\ ;\ \omega_z=\Omega\sin\phi.$$

The equations (A) and (B) of page 666 then take the form

$$\left.\begin{aligned} -(\omega^2+2\omega\Omega\sin\phi)x &= -k\frac{\partial\log\rho}{\partial x}\\ -(\omega^2+2\omega\Omega\sin\phi)(y-\beta z) &= -k\frac{\partial}{\partial y}\log\rho\\ +2\omega\Omega\cos\phi(y-\beta z) &= -g-k\frac{\partial}{\partial z}\log\rho \end{aligned}\right\}\quad (1)$$

$$\left(\frac{\partial}{\partial t}+u\frac{\partial}{\partial x}+v\frac{\partial}{\partial y}\right)\log\rho=0.\quad . \quad . \quad . \quad . \quad (2)$$

The integration of the first three equations gives

$$k\log\rho=\tfrac{1}{2}(\omega^2+2\omega\Omega\sin\phi)\{x^2+(y-\beta z)^2\}-gz+C,\quad (3)$$

as the general equation determining the density and pressure at any point in the neighbourhood of O. The constant C represents the value of $k\log\rho$ at point O itself. In order that the above integral may satisfy the third equation of motion, β must be chosen according to the equation

$$\beta=\frac{2\Omega\cos\phi}{\omega+2\Omega\sin\phi}.\quad . \quad . \quad . \quad . \quad . \quad . \quad (4)$$

The continuity equation is then fulfilled also; and the equation (3) corresponds to conditions of pressure and of motion which are possible in the atmosphere.

When ω is taken positive in the same direction as the earth's rotation, the atmospheric motion described above corresponds with that obtaining in the stationary cyclone. When ω is taken negative the motion corresponds with that obtaining in the stationary anticyclone. In each plane parallel to the surface of the earth the air is in motion about a definite centre determined by the height of the plane above the earth's surface at O. Referred to the point O, the centre of isobars drawn on the earth's surface, the line of centres of rotation lies in the meridian plane ZOY and is inclined to the vertical line drawn through O towards the North in the Northern hemisphere at an angle i given by

$$\tan i=\beta=\frac{2\Omega\cos\phi}{\omega+2\Omega\sin\phi}.\quad . \quad . \quad . \quad . \quad (5)$$

We may regard this line of centres as an axis of the cyclone or anticyclone. In a cyclone, with ω small, the axis is very nearly parallel to the axis of rotation of the earth. Corresponding to larger and larger values of ω positive, the axis tends more and more towards the vertical line drawn from O the centre of isobars. In anticyclonic motion on the other hand, with ω small, the axis is very nearly parallel to the axis of rotation of the earth; and with increasing values of negative ω the axis tends towards the horizontal line drawn due north from O the centre of isobars. The greatest admissible value of ω negative, according to the above equation, is clearly given by

$$\omega_g = 2\Omega \sin \phi. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (6)$$

Thus for an anticyclone the maximum angular velocity at the equator is zero and at the pole it is 2Ω. This result would imply that no anticyclonic motion could be observed at the equator; at least no motion of the type under consideration in which the motion of each particle is in a horizontal plane. The absence of permanent or semi-permanent centres of high pressure from the equatorial belt of the earth's surface is in agreement with the theoretical result obtained above, and may be regarded as some confirmation of the idea that in the larger cyclonic and anticyclonic movements horizontal motions predominate. The assumption that the motion is horizontal cannot of course be considered as one likely to be strictly fulfilled in the equatorial region.

The relation of the upper winds to ground winds at any point of the earth's surface within the area covered by a cyclone or anticyclone can be readily determined for a stationary cyclone or anticyclone from the equations already given. The inclination of the axis of rotation could also readily be determined from observations of the ground wind and upper winds at one or two points within the area. We may take for example two points of observation—A due north of O and B due south of O. At B the upper winds continuously increase in velocity with height above the surface. At A the velocity of wind decreases with height until the axis is reached where the wind velocity is zero. Further increase in height is accompanied by increasing wind velocity in the opposite direction to that of the wind at the ground. If our points of observation do not lie due north or due south from O, upper wind velocities change in direction and amount with height in a manner depending on the position of the point of observation relative to O. The change of wind direction with height increases in some cases to nearly 180 degrees.

Case II.—The next question which arises is this: What is the smallest modification of the conditions of motion described in Case I. which will correspond with a uniform motion of O, the centre of isobars at the earth's surface, relatively to the earth, without alteration of the motion of the fluid relative to O? Let us suppose first that O has a steady velocity U in a due east direction, other conditions of motion being as in Case I. This type of motion is represented by

$$u=-\omega(y-\beta z)\ ; \quad v=\omega x\ ; \qquad w=0.$$

$$\omega_x=0: \qquad \omega_y=\Omega\cos\phi+\frac{\mathrm{U}}{\mathrm{R}}\ ; \quad \omega_z=\Omega\sin\phi.$$

$$\dot{\omega}_x=0\ ; \qquad \dot{\omega}_y=0\ ; \qquad \dot{\omega}_z=0.$$

Since U is to be regarded as a small velocity, we may neglect terms of the order $\left(\frac{\mathrm{U}}{\mathrm{R}}\right)^2$ or $\Omega\,.\,\frac{\mathrm{U}}{\mathrm{R}}$ in the equations of p. 666 which then reduce to

$$\left.\begin{aligned} -(\omega^2+2\omega\Omega\sin\phi)x&=-k\frac{\partial}{\partial x}\log\rho\\ -(\omega^2+2\omega\Omega\sin\phi)(y-\beta z)&=-k\frac{\partial}{\partial y}\log\rho\\ +2\omega\left(\Omega\cos\phi+\frac{\mathrm{U}}{\mathrm{R}}\right)(y-\beta z)&=-g-k\frac{\partial}{\partial z}\log\rho \end{aligned}\right\} \quad (7)$$

$$\left(\frac{\partial}{\partial t}+u\frac{\partial}{\partial x}+v\frac{\partial}{\partial y}\right)\log\rho=0. \quad . \quad . \quad . \quad . \quad (8)$$

The integration of these equations again leads to

$$k\log\rho=\tfrac{1}{2}(\omega^2+2\omega\Omega\sin\phi)\{x^2+(y-\beta z)^2\}-gz+\mathrm{C}. \quad (9)$$

as in Case I., but the constant β has now the value given by

$$\beta=\frac{2\left(\Omega\cos\phi+\frac{\mathrm{U}}{\mathrm{R}}\right)}{\omega+2\Omega\sin\phi} \quad . \quad . \quad . \quad . \quad . \quad . \quad (10)$$

It will be seen from this relation that the essential difference between this motion and that considered in Case I.

is that the axis of the cyclone or anticyclone is now inclined to the vertical line through O at a greater angle than before, for the same value of ω (positive) in each case. The axis still remains in a meridian plane through O at each instant. When the inclination of the axis to the vertical line OZ exceeds the value given in equation (5), the centre of isobars moves towards the East; when the inclination falls short of the value given in equation (5), the centre of isobars moves towards the West. A westward velocity equal to the velocity of a point on the earth's surface brings the centre of isobars to rest in space, and the axis of rotation is then vertical. (See also Case IV.)

Case III.—We proceed now to consider what modification of the conditions of motion described in Case I. would correspond with a motion of the centre of isobars towards the North as well as towards the East. If we take U, V as the velocity components of O east and north respectively, and if we assume that the motion of the fluid relative to the point O is of the same type as that of the previous cases of motion, we can represent the conditions of motion by

$$u = -\omega(y - \beta z)\,; \qquad v = \omega(x - \alpha z)\,; \qquad w = 0.$$

$$\dot{\omega}_x = -\frac{\mathrm{V}}{\mathrm{R}} = -\dot{\phi}\,; \qquad \omega_y = \Omega\cos\phi + \frac{\mathrm{U}}{\mathrm{R}}\,; \qquad \omega_z = \Omega\sin\phi.$$

$$\dot{\omega}_x = 0 = -\ddot{\phi}\,: \qquad \dot{\omega}_y = 0\,; \qquad \dot{\omega}_z = \Omega\cos\phi\,.\,\dot{\phi}.$$

Owing to the existence of the component of velocity V towards the north the angular velocity of the axes OX and OY about OZ is continually undergoing change. If we regard the total angular velocity of a particle of fluid about OZ as constant, while the relative angular velocity ω changes, this gives the conditions

$$\omega + \Omega\sin\phi = \text{constant} \quad . \quad . \quad . \quad . \quad (11)$$

and $$\dot{\omega} + \Omega\cos\phi\,.\,\dot{\phi} = 0\,. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (12)$$

As the immediate intention is to determine the effect of a small modification of the conditions of stationary cyclonic motion, we may treat terms of the order, Ω^2, $\frac{\Omega\mathrm{U}}{\mathrm{R}}$, $\Omega\frac{\mathrm{V}}{\mathrm{R}}$, $\frac{\mathrm{U}^2}{\mathrm{R}^2}$, $\frac{\mathrm{V}^2}{\mathrm{R}^2}$ as negligible. To the order of approximation stated, the equations to be fulfilled then take the form

$$\left.\begin{array}{l} -(\omega^2+2\omega\Omega\sin\phi)(x-\alpha z)=-k\dfrac{\partial}{\partial x}\log\rho \\ -(\omega^2+2\omega\Omega\sin\phi)(y-\beta z)=-k\dfrac{\partial}{\partial y}\,.\log\rho \\ 2\left(\Omega\cos\phi+\dfrac{\mathrm{U}}{\mathrm{R}}\right)\omega(y-\beta z)-2\omega\dot{\phi}(x-\alpha z)=-g-k\dfrac{\partial}{\partial z}\,.\log\rho \end{array}\right\} . \quad (13)$$

$$\left(\frac{\partial}{\partial t}+u\frac{\partial}{\partial x}+v\frac{\partial}{\partial y}\right)\log\rho=0 \quad . \quad . \quad . \quad . \quad . \quad (14)$$

An equation similar to 3 and 9 can readily be obtained by integration, namely :—

$$k\log\rho=\tfrac{1}{2}(\omega^2+2\omega\Omega\sin\phi)\{(z-\alpha z)^2+(y-\beta z)^2\}-gz+\mathrm{C}. \quad (15)$$

where C is again the value of $k\log\rho$ at each instant at O, the centre of isobars at the surface of the earth. In order that the third equation (13) above may be fulfilled, α and β must be chosen so that

$$\alpha=-\frac{2\dot{\phi}}{\omega+2\Omega\sin\phi} \quad . \quad . \quad . \quad . \quad . \quad . \quad (16)$$

$$\beta=\frac{2\left(\Omega\cos\phi+\dfrac{\mathrm{U}}{\mathrm{R}}\right)}{\omega+2\Omega\sin\phi} \quad . \quad . \quad . \quad . \quad . \quad . \quad (17)$$

The values of α and β so chosen are not constants but functions of the time. Their variations with respect to time, however, are of the order of magnitude of the terms which we have agreed to neglect. The continuity equation is also in this case not exactly fulfilled, though its fulfilment is secured to the desired order of approximation.

From the equations which we have obtained for this case it appears that the axis of rotation does not remain in the meridian plane when the centre of isobars has a motion towards the North or South. The inclination of the axis to the meridian plane through O is towards the west side when the motion of O is towards the north, and towards the east side when the motion of O is towards the south, in the Northern hemisphere. According to equations (16) and (17) above, a knowledge of the angle of inclination of the axis of the rotating fluid is all that is necessary to enable us to determine the rate at which the system moves to the North or East.

In view of the approximate nature of the solutions contained in Cases II. and III. it cannot be said that the motions discussed are possible steady motions in the atmosphere. If a motion of the type indicated were established it would be subject to gradual modification owing to the cumulative influence of the smaller terms which have been omitted in the discussion given above. The chief interest to us lies in the fact that these motions retain all the characteristics of the motion of the air in the stationary cyclone or anticyclone, while indicating the nature of the slight disturbances which would produce motion of the system as a whole.

Case IV.—It is of interest to find also a steady motion which would correspond with the moving cyclone or anticyclone and which would be free from the limitations as to the smallness of U and V which have been imposed in Cases II. and III. This can be done in the case where the centre of isobars at the earth's surface has a uniform motion in a direction due East or due West. Let us consider the conditions of motion represented by

$$\begin{array}{lll} u=-\omega(y-\beta z)\,; & v=\omega' x\,; & w=0. \\ \omega_x=0\,; & \omega_y=\Omega\cos\phi+\dfrac{\mathrm{U}}{\mathrm{R}}; & \omega_z=\Omega\sin\phi. \\ \dot{\omega}_x=0\,; & \dot{\omega}_y=0\,; & \dot{\omega}_z=0. \end{array}$$

These conditions of motion must satisfy the system of equations

$$\left.\begin{aligned} -(\omega\omega'+2\omega'\Omega\sin\phi)x-\frac{\mathrm{U}}{\mathrm{R}}\left(2\Omega\cos\phi+\frac{\mathrm{U}}{\mathrm{R}}\right)x \\ =-k\frac{\partial}{\partial x}\log\rho \\ -(\omega\omega'+2\omega\Omega\sin\phi)(y-\beta z)=-k\frac{\partial}{\partial y}\log\rho \\ 2\omega\left(\Omega\cos\phi+\frac{\mathrm{U}}{\mathrm{R}}\right)(y-\beta z)-\frac{\mathrm{U}}{\mathrm{R}}\left(2\Omega\cos\phi+\frac{\mathrm{U}}{\mathrm{R}}\right)z \\ =-g-k\frac{\partial}{\partial z}\log\rho \end{aligned}\right\}. \quad (18)$$

$$\left(\frac{\partial}{\partial t}+u\frac{\partial}{\partial x}+v\frac{\partial}{\partial y}\right)\log\rho=0. \quad . \quad . \quad . \quad (19)$$

The equation derivable from these to represent the distribution of pressure or density throughout the fluid consistent with the above motion is

$$k\log\rho=\tfrac{1}{2}\left(\omega\omega'+2\omega'\Omega\sin\phi+2\frac{\mathrm{U}}{\mathrm{R}}.\,\Omega\cos\phi+\frac{\mathrm{U}^2}{\mathrm{R}^2}\right)x^2$$
$$+\tfrac{1}{2}(\omega\omega'+2\omega\Omega\sin\phi)(y-\beta z)^2+\tfrac{1}{2}\frac{\mathrm{U}}{\mathrm{R}}\left(2\Omega\cos\phi+\frac{\mathrm{U}}{\mathrm{R}}\right)z^2$$
$$-gz+\mathrm{C},\quad\ldots\ldots\quad(20)$$

where C is the value of $k\log\rho$ at point O, the centre of isobars drawn at the earth's surface.

It is easy to show that the required conditions are fulfilled provided

$$\omega'^2=\omega'\omega+\frac{\mathrm{U}}{\mathrm{R}}\left(2\,\Omega\cos\phi+\frac{\mathrm{U}}{\mathrm{R}}\right),\quad\ldots\quad(21)$$

and $$\beta=\frac{2\left(\Omega\cos\phi+\frac{\mathrm{U}}{\mathrm{R}}\right)}{\omega'+2\Omega\sin\phi}.\quad\ldots\ldots\quad(22)$$

From these equations we learn that the relative path of each fluid particle about the centre of rotation in its own plane is not a circle but an ellipse whose axes are due east and due north respectively, these axes being proportional to $1/\sqrt{\omega'}$ and $1/\sqrt{\omega}$ respectively. We fall back again upon our former case of a circular path if

$$\frac{\mathrm{U}}{\mathrm{R}}=-2\Omega\cos\phi;\quad\ldots\ldots\quad(23)$$

that is, if the centre of isobars moves relative to the earth uniformly west at a speed equal to twice the speed of a point on the earth's surface coinciding with it at any instant. It can readily be verified from equation (20) above that the isobars are ellipses concentric and coaxal with the ellipses representing the paths of the fluid. Hence isobars are also lines of flow in all the cases considered.

If the motion of the air in cyclones and anticyclones is, as we have assumed, mainly horizontal, it may be possible to obtain from actual observations some confirmation of the conditions of motion and corresponding pressure distributions indicated by theory in the above cases. As we have already indicated for the case of a stationary cyclone or anticyclone, the inclination of the axis, or line of centres of rotation in the upper layers of the atmosphere, could be determined by observations of the wind velocity at various heights above the ground taken at one or two points within the area covered by the rotating fluid. The variation of pressure with height above the ground indicated in the various cases may also be capable of observation, although this would no doubt prove a much more difficult matter. The solutions obtained above deal only with a very special atmospheric motion in which the angular velocity of the rotating fluid is uniform throughout the whole mass of fluid and horizontal temperature gradients are neglected. This may also prove to be not in accordance with actual observation. Other cases of interest would be those in which the angular velocity of rotation varies with height above the earth's surface and with distance from the centre of rotation. Perhaps the most interesting point established in connexion with the cases considered above is the very great importance of the angular motion of the earth about its axis in relation to cyclone and anticyclone motion. In this connexion it would be of interest to determine the effect of small vertical movements of the air in tending to bring the axis of the cyclone more towards the vertical; and this would probably lead to results more in accordance with actual observations than those which we have obtained above. I hope, however, to continue this investigation in a later paper.

LIX. *Precision-measurements in the X-Ray Spectra. Part IV.—K-Series, the Elements Cu—Na.* By Elis Hjalmar [*].

In a former part of this paper [†] Prof. M. Siegbahn has given an account of a precision-measurement of the $K\alpha_1$-line in the region Cu—Cl. I have continued these investigations

(1°) on the $K\beta_1$-line for the same elements; and

(2°) on the $K\alpha_1$- and $K\beta_1$-lines in the following domain, S—Na.

The first of these examinations was made with calcite as analysing crystal, the latter with gypsum.

The apparatus used, the new vacuum spectrograph with a metallic X-ray tube, was described by Prof. Siegbahn [†] and specially by Dr. W. Stenström [‡]. The slit between the tube and the spectrograph was for the measurements in the region

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[*] Communicated by the Author.

[†] Phil. Mag. xxxvii. June 1919.

[‡] W. Stenström, "Experimentelle Untersuchungen der Röntgenspektra," Diss., Lund, 1919.

Cu—Cl ($\lambda < 5{\cdot}3$ Å.U.) covered with a thin foil of aluminium, thickness 7 μ; in the following researches on S—Na ($5{\cdot}3 < \lambda < 12$ Å.U.) with a very thin goldbeaters' skin, which was coloured red with erythrosin (conc. alcoholic solution of tetraiodfluorescin) in order to keep out the luminiscence light from the anode. This foil, however, may be recommended, even for some shorter wave-lengths.

The form in which the elements were investigated is to be seen from Table I. For Al and Mg respectively two and three different materials were used. No influence of the chemical compound on the wave-lengths or the intensity of the lines could be observed in these cases.

TABLE I.

Element.	Material.	Way of application.
Cu ...	Pure metallic sheet.	Soldered on anode.
Ni ...		
Co ...		
Va ...	Acid, powder.	Rubbed on Ag-anode.
Ti......		
Sc......	Sulphate ,,	,, ,,
Ca ...		
K......	KCl, ,,	,, ,,
Cl......		
Ba ...	$BaSO_4$,,	,, Cu-anode.
Ag ...	Metallic sheet.	Soldered on ,,
Mo ...	Metallic powder.	Rubbed on ,,
S	Cu_2S+Na_2S, powder.	,, ,,
P	P_2O_5, powder.	,, ,,
Si	Pure Si-powder.	,, ,,
Al.. ...	$AlCl_3$, powder.	,, ,,
	Pure met. sheet.	Soldered on ,,
Mg ...	$MgCl_2$, powder.	Rubbed on ,,
	MgO, ,,	,, ,,
	Pure met. band.	Soldered on ,,
Na ...	Na_2SO_4, powder.	Rubbed on ,,

The distance between the lines is measured on a comparator. As the lines are curved to each other, the distance is measured as max. or min. distance with an accuracy of about 0·007 mm. The distance between the turning axis and plate is found to be 126·14 mm. The inevitable experimental errors will cause a total deviation in λ less than 0·01 per cent. of λ. The value of the lattice constant of calcite has been determined in this laboratory* with a very great precision to be

$$\log 2d = 3{\cdot}7823347 - 11,$$

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* M. Siegbahn, *loc. cit.*

that of rock-salt is taken as $\log 2d = 3{\cdot}7503541 - 11$ at 18° C. In this work the unit of wave-lengths is 10^{-11} cm., according to the proposal by Prof. Siegbahn [*], and is here indicated by X.U. The results of the examinations with calcite are given in the following table, in which

λ = wave-length (mean value).
ν = frequence.
R = Rydberg's constant ($\log R = 5{\cdot}0403650$).

TABLE II.

Element.	Line.	λ X.U.	$\frac{\nu}{R}$
29 Cu	β_1	1388·87	656·10
	β_2	1382	659·4
28 Ni	β_1	1496·69	608·84
	β_2	1484·03	614·03
	β'	1498·11	608·25
27 Co	β_1	1617·15	563·49
	β_2	1606	567·3
	β'	visible, not measured.	
23 Va	β_1	2279·68	399·73
	β_2	2265·37	402·25
	β'	2285·26	398·74
22 Ti	β_1	2508·74	363·22
	β_2	2493·67	365·42
	β'	2515·06	362·31
21 Sc	β_1	2773·66	328·52
	β_2	2755	330·7
20 Ca	β_1	3082·97	295·57
	β_2	3067·40	297·07
	β''	3079·57	295·90
19 K	β_1	3446·38	264·40
	β''	3442·70	264·68
17 Cl	β_1	4394·50	207·36
	β''	4391	207·5

For the faint lines β', β_2, and β'' the accuracy is commonly less, because they are measured relatively to the β_1-line. From his theory of X-ray spectra, A. Sommerfeld has concluded that the β_1-line must have a component [†]. Simultaneously the line β' was observed in this laboratory in some photograms [‡]. I could follow this line right to Ti. At Sc there was no component, but at the next element Ca the other component β'' appeared, and was also discovered at the following K and Cl. The line β_2 could be followed only

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[*] M. Siegbahn, *Arkiv f. mat., astr. & fys.* Bd. xiv. 1919.
[†] A. Sommerfeld, *Sitz. Ber. d. Bayr. Akad. d. Wiss.* 1918.
[‡] M. Siegbahn, *loc. cit.*

up to and including Ca. At K it was not to be found, though the exposure was extended a long time.

Before I could advance in the next domain S—Na, it was necessary to get a precision-measured value of the lattice constant of gypsum. The oldest value fixed by E. Friman* was not satisfactory, so W. Stenström † set about a new investigation. His working may here be a little recapitulated, because there is a very interesting fact in it. From six spectrograms of Cu β_1 in the first three orders, he calculated a mean value of the lattice constant. But when using this value *on another line with longer wave-length,* he found that Bragg's formula $n\lambda = 2d \sin\phi$ (n is the order of the spectrum) *gave decreasing* λ *for increasing* n. The deviations were much too great to be explained through experimental errors. Stenström is of the opinion that these deviations are dependent on refraction of the rays in the crystal. However this may be, the matter is of such an importance that it is here scrutinized anew. Another circumstance is here considered. The temperature coefficient k is for gypsum rather high, $k=0{\cdot}000025$. It is easily calculated that this fact will sometimes cause a deviation in λ, which is of the same magnitude as the experimental errors. But corrections due to the temperature are easily carried out. The temperature has here been exactly determined in the spectrograph. 18°·0 C. is taken as standard temperature.

At first the lattice constant was determined from Cu β_1 in three orders on 16 plates. The mean values are given in Table III.

TABLE III.

Cu $\beta_1 \lambda = 1388{\cdot}87$ X.U.

n.	ϕ_{18}.	log $2d$.
1	5° 15′ 34″·5	1·18044−8
2	10 33 39 ·8	57
3	15 57 12	68

ϕ_{18} is the corrected angle of reflexion. The total mean value becomes

$$\log 2d = 1{\cdot}18056 - 8,$$

and is here used in Tables IV.–VI.

The deviations from Bragg's formula can now be examined.

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* E. Friman, "Untersuchungen über die Hochfrequendspektra der Elem.," Diss., Lund, 1916.

† W. Stenström, *loc. cit.*

Besides the β_1-lines in Table II., the following are employed :—

Ba Lβ_1 $\lambda=2562{\cdot}35$ X.U.
K Kα_1 $\lambda=3733{\cdot}86$
Ag Lβ_1 $\lambda=3926{\cdot}56$
Mo Lβ_1 $\lambda=5166{\cdot}44$

The results of the measurements are given in Tables IV.–VI. Here λ_g is the value of λ obtained with gypsum and calculated with Bragg's formula ; $\partial\lambda$, the difference between the value and the wave-length λ_c, obtained with calcite. The last column contains $\partial\lambda$ in per cents. of λ_c.

TABLE IV.

First order.

Line.	ϕ_{18}.	λ_g X.U.	$\partial\lambda$.	$\frac{100\,\delta\lambda}{\lambda}$.
Va β_1 ……	8° 39′ 8″·3	2279·90	+0·22	0·009
Ti β_1 ……	9 31 43 ·2	2508·82	+0·08	0·003
Ba β_1 ……	9 44 7 ·1	2562·71	+0·36	0·014
Sc β_1 ……	10 32 49 ·0	2774·00	+0·34	0·012
Ca β_1 ……	11 44 10 ·2	3082·71	−0·26	0·008
K β_1 ……	13 8 37 ·1	3446·08	−0·30	0·009
Ag β_1 ……	15 0 56 ·3	3926·45	−0·11	0·003
Cl β_1 ……	16 51 13 ·0	4393·90	−0·60	0·014
Mo β_1 ……	19 55 48 ·0	5166·00	−0·44	0·009

TABLE V.

Second order.

Line.	ϕ_{18}.	λ_g X.U.	$\partial\lambda$.	$\frac{100\,\delta\lambda}{\lambda}$.
Va β_1 ……	17° 29′ 57″·8	2278·53	−1·15	0·050
Ti β_1 ……	19 19 21 ·6	2507·33	−1·41	0·056
Ba β_1 ……	19 45 14 ·9	2561·12	−1·23	0·048
Ca β_1 ……	23 59 31 ·0	3081·14	−1·83	0·059
K β_1 ……	27 2 15 ·2	3444·62	−1·76	0·051
Ag β_1 ……	31 11 16 ·9	3924·00	−2·56	0·065
Mo β_1 ……	42 57 11 ·1	5163·44	−3·00	0·058

TABLE VI.

Third order.

Line.	ϕ_{18}.	λ_g X.U.	$\partial\lambda$.	$\frac{100\,\delta\lambda}{\lambda}$.
Va β_1 ……	26° 48′ 28″·1	2278·31	−1·37	0·060
Ba β_1 ……	30 27 15 ·0	2560·47	−1·98	0·077
Ca β_1 ……	37 33 47 ·2	3079·71	−3·26	0·106
K α_1 ……	47 36 2 ·6	3730·42	−3·44	0·092

The deviations in Table IV. are very small. Most of them lie within the limits of the experimental errors. The lattice constant may accordingly be more exactly determined from these measurements than from Table III. The results of the calculations about this are given in Table VII.

TABLE VII.

Line.	ϕ_{18}.	λ_cX.U.	log 2*d*.
Va β_1	8° 39′ 8″·3	2279·68	1·1805169 − 8
Ti β_1	9 31 43 ·2	2508·74	5500
Ba β_1	9 44 7 ·1	2562·35	5031
Sc β_1	10 32 49 ·0	2773·66	5047
Ca β_1	11 44 10 ·2	3082·97	6069
K β_1	13 8 37 ·1	3446·38	5861
Ag β_1	15 0 56 ·3	3926·56	5739
Cl β_1	16 51 13 ·0	4394·50	6157
Mo β_1	19 55 48 ·0	5166·44	6003

Thus, as a definite value, we get

$$\log 2d = 1·1805620 - 8(\pm 0·0000149).$$

On looking at Table IV. one will be at once astonished at the change in the sign of $\delta\lambda$ at Ca. But from what has been said above it is evident that the importance of this must not be over-rated. It will seem as if the measurements above Ca ($\lambda > 3·1$ Å.U.) will give too small values of the wave-lengths, but as the deviations are very small ($< 0·01$ per cent. of λ), no corrections may be made. In the second and third orders the deviations are larger, exactly as it was established by Stenström.

The results of the investigations in the domain S—Na are collected in Tables VIII.–XIII. Here I have discovered*

TABLE VIII.

S.

Line.	λ_g X.U.	$\frac{\nu}{R}$.
α_1	5360·66	169·99
$\alpha_1{}'$	visible, not measurable.	
α_3	5328·33	171·02
α_4	5321·75	171·23
$(\alpha_5\alpha_6)$...	5277	172·68
β_3	5047	180·55
β_1	5019·13	181·55

TABLE IX.

P.

Line.	λ_g X.U.	$\frac{\nu}{R}$.
α_1	6141·71	148·37
$\alpha_1{}'$	visible, not measurable.	
α_3	6102·19	149·33
α_4	6095·00	149·51
β_3	5820·40	156·56
β'	visible, not measurable.	
β_1	5785·13	157·51

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* In a photogram of Ca they could also be observed.

TABLE X.

Si.

Line.	λ_g X.U.	$\frac{\nu}{R}$.
α_1	7109·17	128·18
α_1'	7083	128·65
α_3	7063·82	129·02
α_4	7053·72	129·18
α_5	7014	129·92
α_6	7003	130·12
β_3	6793	134·15
β'	6744·20	135·11
β_1	6739·33	135·21

TABLE XI.

Al.

Line.	λ_g X.U.	$\frac{\nu}{R}$.
α_1	8319·40	109·53
α_1'	8285·60	109·98
α_3	8264·60	110·26
α_4	8253·00	110·41
α_5	8205·80	111·05
α_6	8189·20	111·27
β_3	8025	113·55
β_1'	visible, not measurable.	
β_1	7940·50	114·76

TABLE XII.

Mg.

Line.	λ_g X.U.	$\frac{\nu}{R}$.
α_1	9867·75	92·34
α_1'	9826·(5)	92·73
α_3	9799·40	92·99
α_4	9786·20	93·11
α_5	9730	93·65
α_6	9712	93·83
β_3	9647	94·46
β'	visible, not measurable.	
β_1	9534·50	95·57

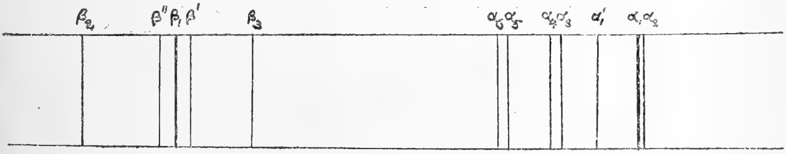
TABLE XIII.

Na.

Line.	λ_g X.U.	$\frac{\nu}{R}$.
α_1	11883·6	76·68
α_1'	11835	76·99
α_3	11802·4	77·21
α_4	11781·4	77·34
α_5	visible, not measurable.	
α_6	,, ,,	,,
β_1	11591	78·62

some new faint lines (α_1', α_5, α_6, β', and β_3). The mutual position of all the lines can be seen from the figure. The lines α_1 and α_2 appear here unseparated as one line, in the tables indicated by α_1. The line β_3, however, is to be looked upon as the continuation of β' in the region Ni—Ti.

Diagram of a K-Spectrum.



Judging from this investigation, a complete K-spectrum would have the appearance which is shown in the figure. It is, however, to be noticed that with no element do all the lines occur.

Physical Laboratory,
University of Lund,
June 1920.

LX. *Notices respecting New Books.*

Où en est la Météorologie? By ALPHONSE BERGET. 8vo. Pp. 303.
Gauthier Villars. Price not indicated.

THIS work is one of a series intended to give general accounts of the present positions of various scientific subjects. As a summary of our present knowledge of meteorology it is excellent, and often remarkably up to date, even including results published in the early part of 1920. The chapters on the constitution of the atmosphere and its optical and electrical properties are particularly good. Full treatment of particular points cannot naturally be given in a small work covering so much ground as this, and some important matters are omitted; but it is a notable achievement, in so diffuse a subject, to have made the striking omissions as few as they are. The inclusion of references to original papers and an index would have increased the value of the book considerably.

Dr. Berget is distinctly hostile to Germany in general and German science in particular, and expresses his views somewhat freely, and often soundly. But one may disagree with his objection on p. 155 to the complexity of Koppen's 11-compartment classification of climates, when he himself requires 6 classes to deal with France alone. He also disapproves of millibars and absolute temperature. It may be noted that on p. 84, "par centimètre carré" should be "par mètre carré"; on p. 90 the pattern of wind-screen shown is not that used in Britain; on p. 179 there is no indication of the unit of velocity; and that on pp. 275–6 "weather" is three times misprinted. The section on the temperature of the stratosphere makes no mention of Gold or Humphreys; that on the lunar tide in the atmosphere, none of Chapman; that on the variations of wind, none of G. I. Taylor; those on the structure of cyclones and methods of forecasting, none of Bjerknes; and if there is any mention in the book of Hann, Shaw, or Dines, it is not easily found. The account of the general circulation is based on the conventional theory of the geography books, but bad as this theory is, it presumably had to be included in a work on the present position of meteorology, as it is still the only one available. The blame for this state of affairs rests on meteorologists in general, and not on Dr. Berget in particular.

But in spite of these defects, most meteorologists will find in this book much that is new and interesting to them, and the definite mistakes are very few.

H. J.

LXI. *Proceedings of Learned Societies.*

GEOLOGICAL SOCIETY.

[Continued from p. 160.]

March 10th, 1920.—Mr. R. D. Oldham, F.R.S., President, in the Chair.

THE following communication was read :—

'The Lower Palæozoic Rocks of the Arthog-Dolgelley District (Merionethshire).' By Prof. Arthur Hubert Cox, M.Sc., Ph.D., F.G.S., and Alfred Kingsley Wells, B.Sc., F.G.S.

This paper gives an account of the geology of the country between the Cader Idris range and the Mawddach Estuary. The stratigraphical succession is as follows :—

?LLANDEILO	Lower Basic Volcanic Series.	
LOWER LLANVIRN	Cefn-Hir Ashes.	
	Crogenen Slates.	
	Bryn Brith Beds.	
	Moelyn Slates.	
	'China-Stone' Ashes.	Lower Acid or Mynydd Cader Volcanic Series.
LOWER LLANVIRN and ARENIG	Pont Kings Slates.	
ARENIG	Lower Ashes.	
	Basement Series.	
	?Unconformity.	
UPPER CAMBRIAN	Tremadoc Slates.	vi. Upper Pencil-Slates. v. Upper *Dictyonema* Band. iv. *Asaphellus* Beds. iii. Lower Pencil-Slates. ii. Lower *Dictyonema* Band. i. *Niobe* Beds.
	Dolgelley Beds.	
	Ffestiniog Beds.	
	Maentwrog Beds.	[Not seen south of the Mawddach Estuary.]

The Upper Cambrian beds are similar, both lithologically and faunally, to the corresponding beds on the north-east and north-west of the Harlech Dome. Discontinuity between the Cambrian and Ordovician Systems is shown by the marked contrast in lithology between the uppermost beds of the Tremadoc Slates and the succeeding arenaceous Basement Series of the Ordovician System. The Ordovician succession includes in its lower portion two distinct volcanic series ; the lower one immediately succeeds the Basement Series, and consists mainly of rhyolitic ashes, with occasional rhyolite-flows which thin out westwards. Coincidently with the thinning of the volcanic rocks, slate-bands with an Arenig and Lower Llanvirn fauna appear interbedded with the rhyolitic rocks. The higher volcanic series is separated from the lower by the main mass of the *Bifidus* Beds, and includes a great thickness of 'andesitic' ashes and agglomerates, together with abundant spilite-flows often occurring as pillow-lavas.

Intrusive igneous rocks in the form of transgressive sills and laccolitic bodies occur at all horizons in the succession. They are of two main types: (i) diabases of normal Welsh types, and (ii) granophyres (eurite type). Basic intrusions are the more numerous, but the acid rocks which build one large sill repeated by strike-faulting, have the greater bulk. Progressive deep-seated differentiation of the granophyric magma has given rise to a series of small intrusions near the main sill. The rocks in these minor intrusions show all transitions from eurites, through markfieldites, to quartz-diabase, but all agree in showing granophyric structures, and they are usually arranged in an upward order of decreasing density. The main sill is the youngest intrusion in the district, and is of uniform composition throughout, except that differentiation in place has resulted in the production of narrow basic selvages at both the upper and the lower margins. Petrographical and structural considerations unite in justifying the assumption of a pre-Upper Bala date for all the intrusions.

Faulting and folding were already in progress before the intrusion of the hypabyssal rocks. The faults and the igneous rocks, both bedded and intrusive, decrease in importance westwards coincidently with a change of strike. The early movement acted along the same lines as the later, more powerful 'Caledonian' movements.

April 21st.—Mr. R. D. Oldham, F.R.S., President,
in the Chair.

The following communication was read:—

'The Devonian of Ferques (Lower Boulonnais).' By John William Dudley Robinson, B.Sc., F.G.S.

In the Lower Boulonnais, between Calais and Boulogne, lies a small tract of Devonian rocks. They form a link between the Devonian beds in Belgium, France, and Germany, and those of England geographically, and also geologically, since they appear to have been formed in a narrow strait which joined the open seas extending towards the Atlantic and over Germany and Russia.

They have been regarded as undisturbed beds lying on a Silurian land-surface, covered conformably by Carboniferous Limestone, but separated from it by a gap due to cessation of deposition. Detailed mapping shows that the Carboniferous and Devonian are separated by a thrust-fault and that the Devonian has been riven into blocks by dip-faults or 'tears' caused by the stresses set up by the thrust-faulting. These blocks are seen to have been pushed farther and farther to the north as one proceeds westwards. The failure to recognize this structure has caused beds to be inserted in the succession which are, in fact, merely parts of other bands. The palæontology, therefore, must be reworked for correct zoning.

The structure thus shown indicates that the Devonian coast-line may well have taken a northward trend under the Straits of Dover and passed north of the Kent Coalfield and the Thames.

The more complicated structure of the Devonian thus revealed indicates a corresponding increase of complexity in the Carboniferous strata, which may be of importance in unravelling the structure of the Kent and Boulonnais Coalfields.

The Devonian beds of the area described are practically unaltered and fossiliferous in many parts. Some of the types of rock described have previously escaped notice.

THE
LONDON, EDINBURGH, AND DUBLIN
PHILOSOPHICAL MAGAZINE
AND
JOURNAL OF SCIENCE.

[SIXTH SERIES.]

MAY 1921.

LXII. *The Disappearance of Gas in the Electric Discharge.*—II. *By* The Research Staff of the General Electric Company, London *. (*Work conducted by* N. R. Campbell.)

Observations on Different Gases.

In the first portion of this paper (Phil. Mag. xl. p. 585, Nov. 1920) there were discussed (1) the electrical conditions in which gases disappear from a discharge-tube with a hot cathode, (2) the chemical actions involved in the discharge, especially those connected with the disappearance of carbon monoxide. We shall now consider the disappearance of other gases and the destination of the gas that disappears. It has been found that some of the minor conclusions under (2) of the previous paper are erroneous; consequently, carbon monoxide will be included again among the gases to be discussed.

12. *Hydrogen.*—Langmuir, in a very complete investigation †, has shown that hydrogen will disappear from a vessel containing a hot tungsten filament, even when no discharge passes through the gas. The disappearance takes place even when the temperature of the tungsten is as low as

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* Communicated by the Director.

† I. Langmuir, Journ. Amer. Chem. Soc. xxxiv. p. 1310 (1912).

1300° K. and the greatest potential difference in the vessel as small as 10 volts; in these circumstances there is no glow discharge. No other gas is known to disappear in the absence of the discharge, unless, like oxygen and chlorine, it can react chemically with the tungsten to form well-known compounds, or unless the temperature of the tungsten is so high that there is rapid volatilization of the metal. Consequently, the phenomenon differs from all those which form the main object of this investigation. But since hydrogen is one of the gases the disappearance of which is of most importance, and since the form of discharge discussed here involves the presence of an incandescent cathode which would cause the gas to disappear even if there were no discharge, the matter requires attention.

The experiments made on the disappearance of hydrogen in the absence of a glow discharge confirmed completely those of Langmuir. If the gas has access to glass cooled in liquid air, some of the gas is liberated again when the cooled glass is warmed to room temperature (the filament being now cold), and is not re-absorbed when the glass is cooled again. In a few experiments all the gas that had disappeared was liberated on warming the cooled glass; in others only a part of it reappeared. When only part reappeared, a further portion could be liberated by heating to about 600° K. that part of the glass which had been at room temperature during the disappearance. It was always impossible to recover all the hydrogen by this method; possibly the residue was adhering to portions of the glass which, for mechanical reasons, could not be heated, such as the connecting-tubes and the taps or cut-offs which were necessary to isolate the part of the apparatus in which the disappearance occurred; but the general results of the work here described indicated that this is not the only explanation. Increasing the temperature of the oven from 600° K. to 730° K. (when the glass began to soften) did not increase the quantity restored. By baking the vessel before warming up the cooled glass, then warming up the cooled glass, and, lastly, baking again, it could be shown that some of the gas liberated from the cooled glass recondensed on the glass at room temperature and could be liberated thence by baking.

If the gas that disappeared was pure hydrogen, the gas restored was also pure hydrogen. This important result could be established with great certainty by measurements of the glow potential. It was pointed out in § 6 that the glow potential of hydrogen differs so greatly from that of any other gas, and is so sensitive to impurities, that there is no possibility of a mistake.

It is clear then that, in accordance with the conclusion of Langmuir, the hydrogen that disappears adheres to the glass, especially to the cooler parts, and that it can be liberated therefrom, at least in part, by rise of temperature. In the first few moments after liberation it is still in a condition to adhere to the glass, but it generally loses that power and reverts to the normal condition of hydrogen. There appears to be a definite quantity of hydrogen that can adhere to a definite area of glass at a definite temperature ; the rate of disappearance of the gas, while the filament is hot, falls continuously and approaches zero asymptotically. This quantity increases as the temperature of the glass decreases, but it also varies with the exact condition of the glass. Sometimes no gas will adhere to the glass until it is well below room temperatures ; sometimes at room temperatures as much will adhere as corresponds to a monomolecular layer on the glass, and perhaps even more. At the temperature of liquid air much more than a molecular layer will adhere ; but even here the amount that will adhere is definitely limited, and a state can be reached in which no more hydrogen can be caused to disappear.

According to Langmuir, the abnormal hydrogen, produced by the contact with hot tungsten, which will adhere to glass, is made up of monatomic molecules. This conclusion is based partly on the abnormal thermal conductivity of the hydrogen, partly on its abnormal chemical activity. Langmuir found that the active hydrogen, liberated by warming the cooled glass, would combine with oxygen in the cold. We have not been able to confirm certainly this abnormal activity, but we do not pretend that our experiments are conclusive. It appears to us that any theory of the disappearance must be framed in the light of the observations on other gases that will be described below.

13. *Disappearance of hydrogen in the discharge.*—If the temperature of the filament and the potential between cathode and anode are raised so that a discharge passes through the hydrogen, the phenomena are somewhat different. First, the gas does not remain pure hydrogen. It becomes contaminated with a substance which has a lower glow potential and is condensible in liquid air. This substance is almost certainly water. Its identity is judged partly by the temperature at which it will condense, partly

by its action upon sodium, which results in the evolution of hydrogen. It is not definitely known whether the water is produced by the oxidation of the hydrogen * or evolved as water from the glass walls (see § 17). Second, the quantity that can be made to disappear is less than that which will disappear if there is no discharge. If the walls are saturated without the discharge, and the discharge is then started, hydrogen is evolved and not absorbed. However, in order that this effect may be observed, the hydrogen must be perfectly pure; if traces of impurity are present, the discharge may cause the disappearance of more hydrogen. Third, until the limit of absorption is reached, disappearance is more rapid with the discharge than without it.

As before, part of the hydrogen can be liberated again by rise of temperature of the walls, whether they have been cooled in liquid air or have been at room temperature. In some cases all the hydrogen absorbed during the discharge has been liberated in this manner. Any loss may be explained by the oxidation of the gas to water, and this explanation receives some support from experiments in which the original pressure was restored by allowing the gas access to sodium or heated magnesium. But since it was not possible in all cases so to restore the original pressure, possibilities indicated by later experiments must be taken into account; the hydrogen may adhere to the walls in a way which cannot be reversed by heating to the softening point of the glass.

14. *The disappearance of carbon monoxide.*—In the first paper (§ 10) it was said that, in the absence of a cooled tube to remove the carbon dioxide formed, the absorption of this gas reached a limit when a pressure of 0·009 mm. was reached, although the discharge still continued. It has been found since that this observation was vitiated by the presence of a trace of mercury vapour (*cf.* § 7). If mercury vapour is completely removed †, the disappearance continues until the discharge stops, owing to the rise of the glow potential—at any rate, if a potential greater than 300 volts is not employed. If fresh CO is admitted, it disappears again under the discharge at a rate apparently unchanged; a limit to the

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* If there are reducible substances present, *e.g.* copper oxide on the wires, they are certainly reduced by the discharge in hydrogen. But the lamps used here were thought to be free from such substances.

† The belief, apparently current, that mercury can be completely removed by a trap cooled in solid CO_2 is erroneous (*cf.* M. Knudsen, *Ann. d. Phys.* l. p. 472 (1916)). The glow potential can certainly detect the vapour of mercury at that temperature. On the other hand, no test has been found which will show the vapour at the temperature of liquid air.

absorption has not been found, although a quantity has been absorbed equal to at least five times that representing a monocular layer on the walls.

If the vessel is now baked (the temperature of baking, so long as it is over 600° K. is apparently immaterial), the greater part of the gas is restored. Most of it is restored as CO_2, and not as CO. The amount of CO liberated from the walls was never found to be greater than $\frac{1}{3}$ a monomolecular layer. But if it is supposed, as indicated in § 10, that the CO_2 is formed from the CO by abstraction of carbon, so that a given volume of CO_2 represents double that volume of CO, then at least 90 per cent. of the gas can be restored by baking. It was not investigated whether there was also water vapour in the restored gas, but it now appears probable there was. Water vapour seems always to be produced when a discharge is passed in a vessel made of glass (lead-soda) used in these experiments.

15. *The disappearance of nitrogen.*—Nitrogen disappears from the vessel under the action of the discharge, exactly as does carbon monoxide, in apparently unlimited quantities. But at the same time the filament wastes rapidly and the walls become covered with a black deposit. Some blackening always occured after prolonged operations in hydrogen and CO, but it was altogether of a lesser order. In nitrogen, absorption of gas and blackening of the glass (measured by the wastage of the filament) are roughly proportional.

When the walls are heated up to the softening point of the glass, only a very small proportion of the gas absorbed is evolved; and much of this gas is not nitrogen. The nitrogen cannot be liberated in any quantity by such baking. (Again no inquiries for water vapour were made.) This fact was taken at first to indicate the formation of Langmuir's nitride WN_2 by reaction between the tungsten and the gas; indeed, Langmuir asserts that the compound is formed in the discharge. On the other hand, he insists that the compound is brown; while the deposit on the walls was grey or black, and indistinguishable in colour from that formed in other gases. An alternative (or, perhaps, additional) explanation of the disappearance of nitrogen will be indicated later; it is that the liberation of the nitrogen from the walls is prevented by the deposition over it of tungsten spluttered from the cathode.

16. *The disappearance of argon.*—The gas employed always contained about 5 per cent. of nitrogen, but was free from oxygen.

Under the discharge the gas disappeared like other gases, but at a very much slower rate. The rate of disappearance increased very suddenly when the pressure fell below ·005 mm., but even at these low pressures was not more than one-fifth of the rate at which nitrogen would disappear. (In nitrogen there was a similar rapid increase in the rate of disappearance at about ·012 mm.) At the same time, rapid blackening of the bulb occurred. In both nitrogen and argon there is the intimate connexion between cathode spluttering and absorption of gas which Vegard* has noted in the discharge without an incandescent cathode.

An interesting fact may be mentioned, though it has probably no direct bearing on the matter under discussion. The blackening which occured with argon was not, like that with nitrogen, distributed almost uniformly over the lamp. Round the central portion of the lamp where the electrons projected from the cathode would be expected to hit the walls, there was absolutely clear glass with no sign of blackening; on this clear glass were the sharp shadows of the anode wire and the wire supporting the filament; and elsewhere the division between the blackened portions and the clear glass was absolutely sharp. The black material is not deposited on the portion of the walls struck by electrons projected at right angles to the cathode surface. Traces of a similarly unequal blackening were seen in nitrogen, but they were only traces of a distribution that is perfectly definite with argon.

When the lamp in which argon had disappeared was heated, more gas was evolved (non-condensible in liquid air) than had been absorbed. It appeared probable that all the absorbed argon was contained in the evolved gas, together with hydrogen, doubtless derived from the glass, although, as always, the glass had been baked until no more gas was evolved before the experiment started.

17. *The discharge in mercury vapour.*—If the vessel is baked just below the softening point of the glass until no more gas is evolved, and completely exhausted of all gases other than mercury vapour, and if the discharge is then passed through this vapour in the cool lamp, the pressure in the lamp increases rapidly. The gas evolved is hydrogen, perhaps mixed with a little water. As long as there is liquid mercury present from which a continual supply of vapour can be drawn, the evolution continues without

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* L. Vegard, *Ann. d. Phys.* l. p. 769 (1916).

apparent diminution while the discharge lasts. In 48 hours enough gas to fill the lamp to a pressure of 1 cm. has been evolved without any signs of fatigue, so long as the gas formed is continually pumped away. Meanwhile the filament wastes rapidly, and the experiment is brought to a close by its breakage. Other experiments conducted in this laboratory have shown that it is probable that the gas is evolved from the walls in the first instance as water, which is reduced to hydrogen by the tungsten filament. If the supply of mercury vapour is not unlimited, the action seems to come to an end after many hours; the mercury has disappeared. It is not known what becomes of the mercury, and indeed it is not absolutely certain that it disappears. The process is interesting as showing definitely another action which must be taken into account in considering the disappearance of gases under the discharge. The discharge here evolves gas from the walls. It has been already noted that this action occurs also in argon, and probably in hydrogen ; it may also occur in nitrogen and CO. And it should be noted that the gas thus liberated cannot be liberated by mere heating of the walls to their softening point; gas can be attached to the walls in some such way that it can be liberated by the discharge, but not by heating. Of course, the attachment may consist of chemical combination; it is possible that glass contains hydrogen chemically combined, probably as water. But it should be observed that the hydrogen liberated, if piled up on the glass, would form a layer at least 25 molecules thick ; some of it must therefore have come from a layer at least 25 molecules deep. Since the potential driving the discharge in these experiments was often as low as 50 volts, it is hardly to be expected that the electrons or ions could penetrate so far into the glass simply in virtue of the energy which they receive from the discharge. It seems easier to believe that a layer on the surface, subject to the action of these particles, is constantly renewed by diffusion from within.

18. *Conclusions.*—The following conclusions are based on these experiments, though some of the evidence for them is derived from the next section :—

(1) All gases can be made to adhere to glass by the discharge in such a way that part, at least, can be restored by heating the glass.

(2) The amount of gas that can be made so to adhere depends on the nature of the gas and on the state of the glass.

(3) The adhesion is not due primarily to chemical reaction, although such reaction (as in the conversion of CO to CO_2) may aid adhesion by converting the gas into another which adheres more readily.

(4) The discharge can also liberate gas from the walls, doubtless by bombardment of the charged particles, and some of the gas so liberated cannot be liberated by heating the glass to the softening point.

(5) The limit in the disappearance of the gas is reached, when the rate at which gas is caused to adhere to the glass by the discharge becomes equal to the rate at which it is liberated by the bombardment.

THE ACTION OF PHOSPHORUS.

The main object of the research was to elucidate the well-known action of phosphorus in promoting the disappearance of gas. The experiments so far described were undertaken in the hope that they would throw light on that action. We must now proceed to discuss what happens when phosphorus is present in the discharge vessel.

19. *The physical properties of phosphorus.*—It is well known that phosphorus exists in at least two forms at room temperatures—the white and the red; the red is stable, the white metastable. The transition point is usually given as 520° K., and heating to just below this temperature is necessary to convert the white rapidly into red. On the other hand, the conversion of red into white does not proceed rapidly at tempertures below 720° K., which is just above the softening point of the glass used. It was not possible to convert red into white rapidly without passing the softening point of the glass; but a slow conversion doubtless takes place at lower temperatures.

At room temperature the red phosphorus has a vapour-pressure too low to be determined by any of the ordinary methods. In the course of the experiments a rough determination of the vapour-pressure of white phosphorus at 290° K was made by a modification of Victor Meyer's method. For this purpose an evacuated vessel was filled with the vapour, and then placed in communication for a few seconds through a narrow tap with a vessel containing a neutral gas at a known greater pressure. If it is assumed that no phosphorus diffuses out against the entering stream of gas, the vapour-pressure can be calculated from the amount of that gas entering. It was found that at 290° K. the vapour-pressure was 0·014 mm.

No previous measurements at such low temperatures could be found, but the vapour-pressure has been determined recently in the chemical department of this laboratory by Walker's method of drawing a neutral gas over the phosphorus and determining the weight of the material subsequently frozen out of the stream. The results will be published elsewhere, but they were sufficiently concordant with the value just given, if the molecule of the vapour is taken (in accordance with the accepted view) to be P_4.

20. *The discharge in phosphorus vapour.*—A side tube containing white phosphorus was attached to the lamp (fig. 1) and separated from it by a tap A *. The lamp was also separated by a tap B from the remainder of the apparatus. By exhausting with both taps open, closing B and then A, the lamp could be filled with phosphorus vapour at the pressure corresponding to room temperature; or, by cooling the side tube, at the pressure corresponding to any lower temperature (T).

There was no evidence that heating a tungsten filament in the vapour to 2500° K. produced any change either in the filament or in the vapour. But if the potential of the anode was gradually raised, a definite glow potential, similar in all respects to that discussed in § 5, was reached at 46·5 volts ($T=290$). At $T=273$, V_g was 58 volts. Thus the glow potential, as in other gases, is greater at the lower pressure. The glow had a characteristic blue colour, but observations with a pocket spectroscope did not disclose any recognizable lines other than those of CO, which were doubtless derived from the tap grease.

The glow, and the increased current which accompanies it, only lasts for a second or two. It very soon ceases, and is not renewed unless the potential is raised further. The phosphorus vapour disappears in the discharge, like most other gases, but much more rapidly than any of the other gases investigated. And it is easy to determine what has happened to the vapour that has disappeared (or at least to a great part of it). At the moment that the glow discharge passes, the walls of the lamp assume a faint yellow tint, precisely that of the "yellow bulb," which is so familiar to lamp makers. If further charges of vapour are admitted and the process repeated, the vapour disappears as before, and the coloration of the wall gradually deepens to brown.

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* Of course, taps always mean the presence of grease-vapour; and it is impossible to be *sure* that this vapour has no effect on the result. But all the evidence that could be obtained by varying the conditions in all possible ways convinced us that the grease vapour had no effect whatever on the experiments about to be described.

The yellow substance does not disappear from the walls if the lamp is exhausted for many hours to a very high vacuum through a liquid-air trap. If oxygen is admitted to the vessel, it is not absorbed at room temperature. The yellow substance is not therefore, as seems to have sometimes been believed, white or "yellow" phosphorus. It is almost certainly *red* phosphorus or some modification with similar properties. If the lamp is baked to 650° K. or more, the yellow coloration disappears gradually; some of it merely distils to the cooler connecting tubes, but some collects in the liquid-air trap as white phosphorus, which will absorb oxygen at room temperatures. If a little oxygen is admitted during baking, it is absorbed when the temperature reaches 650° K. These observations are completely accordant with the view that the yellow substance is red phosphorus, deposited on the walls in an extremely fine film. The discharge converts the white vapour into the red solid; the change is one of those chemical actions (see § 11) we should expect to be brought about by the discharge.

Accordingly, the disappearance of phosphorus vapour under the discharge does not seem to differ essentially from that of the other gases that have been noticed. In all cases the gas that has disappeared is deposited on the walls; in phosphorus, as in CO, this deposition is aided by a chemical change in the gas which makes it adhere more readily to the walls. Nevertheless, these are important peculiarities about phosphorus. The first is that the product of the chemical change is a stable solid body which has no appreciable vapour-pressure even when it is not adhering to glass*; no evidence of the formation of such stable solids has been found in the other gases.

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* An attempt was made to determine the vapour-pressure of red phosphorus by the method just mentioned; but the only result that could be obtained was that the vapour-pressure at room temperature is less, and probably much less, than 0·001 mm. The glow potential of red phosphorus vapour, whether introduced as a powder or deposited on the walls by the discharge through white phosphorus vapour, is certainly greater than 600 volts—a result which indicates again that the vapour-pressure must be very small.

In the earlier work some very puzzling indications were obtained of a residual vapour-pressure of the substance deposited on the walls amounting to 0·0009 mm., giving a glow potential as low as 32·5 volts. It is now believed that this vapour was that of oxides (or possibly other compounds) of phosphorus, formed by the reaction of the phosphorus vapour with carbon monoxide under the influence of the discharge. But as it has been found so far impossible to reproduce these early experiments, the matter has not been cleared up satisfactorily. However, no doubt is now entertained that the vapour-pressure of red phosphorus is inappreciable for these experiments.

The second concerns the values of the glow potential. If the molecular weight of phosphorus vapour is 124, the glow-potential curve corresponding to fig. 4 should lie between those of argon and mercury (*cf.* § 6). Only one point on the curve is definitely known ($p=0{\cdot}014$ mm., $V_g=46{\cdot}5$), but this point (and probably that for T=273 K.) lies well above the argon curve and close to that for CO. This discrepancy suggests interesting considerations which will be raised when the general theory of the glow potential is discussed in a later communication; but it may be pointed out that an explanation would be obtained if it could be established that the ionization of the phosphorus molecule splits it into its constituent atoms, so that the weight of the phosphorus ion is 31. At present, however, this suggestion is merely speculative*.

The third peculiarity lies in the great difference between V_g and V_g', the rising and falling glow potentials. It is impossible to state the difference quantitatively without measuring fully the glow-potential curves; such measurement is difficult, because it involves a rapid method of determining the pressure of the phosphorus vapour present in the vessel. But an indication is obtained by observing by what amounts the potential must be raised, in order to start the discharge once more when it has ceased, owing to the disappearance of the vapour (*i.e.* by tracing out the line NABCDE in fig. 3); this amount is the difference between V_g and V_g' at the same pressure. If in argon (fig. 3) we start with a glow potential of 38 volts, the successive values of V requisite to restart the discharge when it has stopped will be 38, 54, 76, 105, 150, ... In phosphorus vapour, the successive values are approximately 46·5, 78, 350. They are difficult to determine because, at the lower pressures, the glow is so transient that it may be missed; moreover, unavoidable impurities become important; but there is no doubt that, after the glow has been started at a potential less than 100 volts, the restarting of the glow after it has stopped needs a potential very much higher than in similar conditions in other gases. No explanation is attempted at present of this difference; but it will appear that the fact is of great practical importance.

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* *Note added* Feb. 9, 1921.—It has been discovered that these conclusions are doubtful. The vapour-pressure of white phosphorus in the presence of hot tungsten is considerably lower than its normal vapour-pressure. The hot tungsten converts some, but not all, of the vapour into the red modification; equilibrium is reached when the concentration of the vapour present is less than the normal, but still finite. The high glow potentials are doubtless due to this lowering of the equilibrium vapour-pressure.

As in other gases, no evidence could be obtained that a discharge through phosphorus vapour under a potential less than the glow potential would cause any disappearance of the vapour, even if the (i, V) characteristic indicated considerable ionization.

21. *The disappearance of gases in the presence of phosphorus.*—For this investigation the lamp was filled with the gas in question to a known pressure; tap B was then closed, A opened, and sufficient time allowed for the phosphorus vapour to attain by diffusion its equilibrium vapour-pressure throughout the lamp. The filament was then heated, and the potential V raised till the glow discharge occurred. The glow potential was never higher than that of the phosphorus vapour; if there was much gas present it was less. The presence of phosphorus did not seem to alter considerably the current passing in the glow discharge; since this current represents the saturated thermionic current, the thermionic emission was unaltered.

The gas disappeared initially in the discharge much more rapidly than if phosphorus were not present, except possibly if the gas were argon. If the gas were hydrogen, the gas would not disappear completely unless the initial pressure of the gas were below a certain limit (about 0·04 mm. in the lamp of fig. 1). If it were CO or nitrogen, it would disappear however great the initial pressure[*]; in nitrogen the disappearance was accompanied as before by a rapid blackening of the walls and wastage of the filament. In argon, so far as could be ascertained, the presence of the phosphorus made little difference, but the disappearance of this gas is so slow and irregular that it is difficult to decide anything definitely about it. Mercury could not be examined, because the metal acts directly with the vapour, and one or other disappears completely without any discharge at all.

If, instead of a definite charge of phosphorus vapour being mixed with the gas initially, the tap A were left open so that a continual supply of the vapour was provided, then the disappearance continued rapidly with hydrogen, nitrogen, and CO as long as the discharge lasted and fresh gas was supplied; now the amount of hydrogen that could be made to disappear was, like that of the other two gases, unlimited.

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[*] This statement is not strictly true. If the initial pressure is too high, greater than 0·15 mm., the disappearance of gas begins almost indefinitely slowly. But if, while the pressure of the gas is always kept below (say) 0·05 mm., fresh gas is admitted as the disappearance proceeds, the amount of CO and nitrogen that can be made to disappear seems almost unlimited.

These statements apply to the total quantity of gas which can be made to disappear. What is far more important for some purposes is the lowest pressure that can be reached, even if the amount of gas initially present is very small. In § 9 it was pointed out that the lowest pressure that can be reached in the absence of phosphorus is that at which the falling glow potential becomes equal to the potential which can be applied between the electrodes. This pressure depends largely on the form of the electrodes, and is in general lower the more uniform the field between them. In the lamp of fig. 1, with an applied potential of 150 volts, the limits were for H_2 0·01 mm., for N_2 and CO 0·0012. (To reach this limit in hydrogen it is necessary to remove the water formed.) On the other hand, in the presence of phosphorus, with the same potential, the discharge and the absorption of gas do not cease until the pressure has fallen to 0·0002 mm., and when the discharge ceases at this pressure it is not started again by raising the potential to 300 volts.

An explanation of this effect of phosphorus in lowering the limit of pressure that can be reached may be based on the facts narrated already. The absorption of gas ceases when the glow discharge ceases; and the glow discharge ceases when the falling glow potential becomes equal to the applied potential. The admixture of phosphorus vapour with the gas increases the pressure corresponding to any given partial pressure of the gas, and thus decreases the glow potential corresponding to that partial pressure. It therefore enables the discharge and the absorption of gas to continue when the partial pressure has fallen so low that, if the gas were present alone and its partial pressure were the total pressure, the potential applied could no longer be sufficient to maintain the discharge. When the partial pressure of the gas has fallen sufficiently, the phosphorus vapour, and not the gas, begins to disappear; and the disappearance of this vapour proceeds until the pressure of the phosphorus has fallen so low that the discharge can proceed no longer. And when the discharge ceases because the falling glow potential has become equal to the applied potential, it is not started again except by a very great increase of potential, owing to the wide difference between the falling and rising glow potentials in this vapour (*cf.* § 9).

But why does the gas disappear before the phosphorus vapour (or before the greater part of it), as this explanation demands? Since pure phosphorus vapour disappears much more rapidly than pure gas, it might be expected that the order of disappearance would be reversed. It is not because the current is carried by the gas rather than the phosphorus vapour, for there is evidence that, even in the presence of gas, part of the discharge is carried by the phosphorus. Such evidence cannot be based on spectroscopic observations, for the spectrum of phosphorus is not readily discernible; but it is obtained from the observation that, if sufficient phosphorus vapour is present, a "yellow bulb" is formed before all the gas has disappeared. The reason for the persistence of the phosphorus is, we believe, to be found in the reversibility under the discharge of the reaction which leads to the conversion of the white phosphorus vapour into the red solid. Even when nothing but phosphorus is present, the reaction is probably reversible, in accordance with our general principles. But the equilibrium lies so far on the side of red phosphorus that the residual pressure of white is barely detectable*. But when the red phosphorus on the walls is bombarded, not only by phosphorus molecules which may be themselves converted into solid, but also by gas molecules which cannot be so converted, the equilibrium is pushed towards the vapour phase. So long as there is gas present in considerable quantity, the conversion of white into red is never complete; there is always enough white phosphorus re-evaporating to maintain the discharge; and it is only when the gas has been greatly reduced in quantity that the equilibrium moves once more towards the solid phase, and a complete disappearance of all gaseous molecules is obtained.

IS THE ACTION OF PHOSPHORUS CHEMICAL?

22. We have no doubt ourselves that this explanation of the lower limit of pressure attainable in the presence of phosphorus vapour is right in essentials; such uncertainty as affects its detail will probably be removed when the exact mechanism by which the glow discharge causes the adhesion of gas to the walls is better known. But it appears to be believed very generally that the action of phosphorus in aiding absorption by the discharge is chemical, and arises from the formation of definite chemical compounds of phosphorus with the gas; such a theory seems to pervade all the

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* But it *is* detectable by that very sensitive test, the glow when air or oxygen is admitted. If air is admitted to a bulb in which phosphorus vapour has been absorbed by the discharge, whether with or without gas, the flash can always be seen. On the other hand, it is *not* seen if powdered red phosphorus is placed in the bulb and freed from white by long-continued exhaustion in contact with the vessel cooled in liquid air.

literature of the subject that we have discovered. It is therefore necessary to consider that theory.

It is doubtless based on the belief that a greater amount of gas can be absorbed in the presence of phosphorus than in its absence. This belief is doubtless well-founded, but it must be insisted strongly that the facts described so far afford no warrant for it. The important action which we have just discussed depends only on a reduction of the lower limit of pressure attainable and on a consequent increase of the potential that can be applied without a discharge passing. The attainment of this lower limit is not necessarily associated with the absorption of more gas, for the limit is set, not by the cessation of absorption if the discharge continues, but by the cessation of the discharge.

So far, then, there is no evidence for the chemical theory. We will now proceed to give the evidence that has been obtained against it.

This evidence may be summarized in three statements :—

(1) If the gas that has disappeared is restored, it is found to be in the same chemical state as it would have been if it has disappeared in the absence of phosphorus.

(2) There is no simple relation between the quantity of gas that can be made to disappear and the quantity of phosphorus necessary for its disappearance. There is nothing approaching to a "law of constant proportions".

(3) The amount of gas that will disappear depends very greatly on the surface condition of the walls of the discharge vessel.

These three statements will be expanded in order.

23. *Restoration of absorbed gas.*—(1) When the gas has disappeared in the presence of phosphorus, part of it at least can usually be restored by baking the vessel as described in § 12. If sufficient phosphorus has been used to give a "yellow bulb," the evolution of gas appears to accompany closely the disappearance of the yellow colour. If the gas that has disappeared is CO, the gas can be restored almost completely as a mixture of CO and CO_2, the proportion of the latter being the greater as the total amount increases. That is exactly the result obtained without phosphorus. If it is argon, the gas restored is argon with some hydrogen—as before. If it is hydrogen, the gas restored is hydrogen (partly in the "active" condition) together with water vapour. Sometimes the whole of the gas can be restored; sometimes only part; the proportion restored varies considerably from experiment to experiment in much the same way

and within much the same limits as if there were no phosphorus present. The question arises whether any phosphine is restored. Langmuir found that his active hydrogen would react with phosphorus to form phosphine, and, in accordance with the principle of § 11 that any chemical compound will be formed to some extent that can be formed, some phosphine is to be expected. It is very difficult to detect in the presence of hydrogen, phosphorus vapour, and water; the only method seems to be to condense as much as possible of the gases restored (*i. e.* P, H_2O, PH_3) in liquid air, and to note the increase of pressure as the temperature of the cooled tube is allowed to rise. Phosphine boils at 153° K. Some observations made in this manner seemed to indicate the presence of a small amount of gas which volatilized between 120° and 180° K.; since water does not volatilize appreciably till 200° K., and phosphorus until still higher temperatures, the gas may have been phosphine. But its amount was variable and never more than 25 per cent. of the restored hydrogen; accordingly, even if phosphine is formed, its formation cannot be the main cause of the increased amount of gas that will disappear. Moreover, since phosphine is a gas, its formation will only result directly in the reduction of the volume of hydrogen by one-third. But since it is possible that umlimited quantities of this gas, as of CO, can be absorbed in the discharge, its formation might aid in the reduction of the hydrogen.

If the gas that has disappeared is nitrogen, then no gas is restored by baking the vessel [*]. But in this case the yellow layer does not disappear on baking. This fact, taken in conjunction with the marked blackening of the walls, appears to us to suggest that the nitrogen and the phosphorus are not liberated on baking, simply because they are held to the walls by a layer of spluttered tungsten deposited over them. When the general nature of the results is considered, this appears to us a more plausible view than that the nitrogen disappears by the formation of a nitride. The loss in weight of the filament was always decidedly less than that indicated by Langmuir's formula

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[*] More accurately, by baking it for 5 minutes, the time sufficient to restore all the CO and, in that case, remove all the yellow. If the vessel is baked for some hours, the yellow disappears and possibly gas is restored; but since it seems impossible by previous baking to exhaust entirely the gas from the walls, the gas coming off may not be the absorbed nitrogen. The previous baking in all cases had been so prolonged that the pressure would not rise by more than 0·0001 mm. during 5 minutes' subsequent baking, unless gas had been absorbed in the interval.

WN_2; it approached more nearly that corresponding to WN_3. But once more, according to § 11, it is probable that some nitride is formed.

24. *The absence of "constant proportions."*—A very large number of observations have been made with the object of determining whether the amount of gas that disappears under the influence of phosphorus is proportional to the amount of phosphorus present. It is clear that, of the gases mentioned, hydrogen alone is suitable for the investigation, because it alone shows a definite limit to the disappearance. The following experiments appear conclusive :—

(*a*) If the law of constant proportions were true, the pressure of hydrogen which would disappear when mixed with the vapour of phosphorus at a definite temperature should be independent of the size or shape of the vessel. On the contrary, it was found to depend considerably, and in a very complex manner, on both the size and shape.

(*b*) In a lamp of a certain type the pressure of hydrogen which would disappear when mixed with one charge of phosphorus vapour was 0·039 mm. If hydrogen was filled to a greater pressure than this, together with a charge of phosphorus, and the discharge passed, the pressure could be reduced by about 0·039 mm. If, now, a second charge of phosphorus was mixed with the hydrogen left, the amount that could be made to disappear was much less than 0·039 mm.; and if further charges of phosphorus were added, the decrease of pressure due to each charge appeared to decrease still further. The subsequent charges do not cause the absorption of as much hydrogen as the first.

(*c*) The same experiment can be repeated in rather a different form. If, instead of mixing the phosphorus vapour with the hydrogen, we first introduce a charge of the vapour, pass the discharge through this vapour, depositing it on the walls as a yellow film, and then introduce hydrogen; the amount of hydrogen that can be made to disappear is the same as that which would disappear if the phosphorus had been in the form of vapour, *i. e.* 0·039 mm. If, however, we introduce initially and deposit on the walls, not one charge of the vapour, but many charges, the amount of hydrogen subsequently introduced that will disappear does not seem to change. When the phosphorus is introduced in this manner, the amount of hydrogen that can be made to disappear seems independent of the amount of phosphorus present, so long as it is above a certain limit.

(*d*) If not pure hydrogen, but hydrogen mixed with CO, is admitted, the amount of hydrogen that can be absorbed is almost or quite independent of the amount of CO that is absorbed at the same time. Thus, in one experiment, the limit for pure hydrogen was 0·042 mm.; if 0·04 mm. of hydrogen together with 0·06 mm. of CO, making a total pressure of 0·10, all this mixture was absorbed. This appears to be the strongest proof that the action is not chemical, at least in the case of CO.

On the other hand, direct experiments showed that the amount of hydrogen that can be absorbed in the discharge does increase in some measure and in some conditions with the amount of phosphorus vapour mixed with it. It was difficult to obtain quantitative results, but such measurements as could be made did not indicate proportionality between phosphorus added and hydrogen absorbed. Some of these observations are noted below.

25. *The influence of the walls of the vessel.*—These experiments prove conclusively that the disappearance of the gas is not due simply to chemical combination with the phosphorus. The next experiments show that the state of the walls of the vessel influences profoundly the disappearance of the gas.

(*a*) The most striking experiment was made for the first time by accident. If the lamp is filled with phosphorus vapour, and then air admitted, the phosphorus is oxidized and deposited on the walls, and alters somewhat their lustre. If the lamp is now exhausted, and either hydrogen or CO admitted (the other gases were not investigated), then none of the gas can be made to disappear by the discharge, whether or no phosphorus is present. If phosphorus is present, it is deposited on the walls as usual as a yellow film, but no gas disappears. The film of phosphorus oxides on the walls entirely prevents the absorption of gas.

(*b*) Other experiments directed to changing the state of the walls so that the amount of gas absorbed should be altered were not definitely successful, although indications which it is hoped to discuss in a later communication have been obtained. But it could be proved that this state affected the condition of the gas after disappearance. For this purpose hydrogen was introduced into the lamp at various pressures, and the discharge passed with the tap A open, so that an unlimited supply of phosphorus vapour was available. In those circumstances any quantity could be made to disappear. The lamp was now baked in conditions known to liberate in a few minutes all the gas that could be liberated by baking at all (unless it is prolonged for many hours), and the pressure of the restored gas measured. Since the gas was in contact with liquid air, only the restored hydrogen, and not water or phosphine, was measured in this way. It was found that, in any given state of the vessel, the ratio of hydrogen re-evolved to that absorbed was remarkably constant and independent of the amount absorbed ; in the lamps used the normal ratio was 0·6. This ratio could be repeated many times by absorbing and liberating hydrogen in the same vessel, but if the experiment was sufficiently prolonged, the ratio would at some stage always fall suddenly to a lower value, and again remain constant at that value. Later it would fall again until usually a ratio as low as $\frac{1}{3}$ could be reached. It was noticed that any signs of blackening of the walls were immediately accompanied by a fall in the ratio, *i. e.* by the re-evolution of less gas on baking.

An attempt was made to determine whether the hydrogen which did not reappear on baking could be accounted for as water and phosphine (and " active hydrogen ") condensed in liquid air. In a few experiments, which were always the first experiment made on a new lamp, it seemed that, by allowing the products of baking to act upon sodium, practically all the hydrogen could be restored. But when the ratio of re-evolved to absorbed hydrogen fell, there was no rise in the quantity that could be restored by action on sodium. The fall is not due to the production of a larger proportion of water ; the hydrogen disappears in a way that cannot be reversed by baking.

26. As against all these facts tending to show that chemical action is not the cause of the absorption of hydrogen in the presence of phosphorus, we can only set one which supports that view directly. If the lamp in fig. 1 is filled with phosphorus vapour without other gas, the discharge causes the walls to become yellow. If an amount of gas is present which is near the limit that can be absorbed, then the yellow coloration does not appear. Further, if the walls are first made yellow by the discharge in pure phosphorus and then gas admitted and absorbed, the yellow colour vanishes. These facts might indicate that a colourless compound of phosphorus and the gas is formed. But the general nature of our results leads us rather to believe that the admixture of gas with the very thin layer of phosphorus on the walls * changes its optical properties.

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* How thick is the thinnest layer which will give a yellow colour, it is impossible to say without a better knowledge of the cross-section of a phosphorus molecule. It may be only one molecule thick, but it may be 2 or 3 thick—the latter alternative fits in best with the views suggested in the following section.

27. *How does phosphorus increase absorption?*—But if the action of phosphorus is not chemical, why is the amount of gas that can be absorbed with phosphorus greater than that which can be absorbed without? For it is undoubtedly greater—at least in hydrogen. If the supply of phosphorus vapour is unlimited, then there is no limit to the amount of hydrogen that can be absorbed. We suggest that the answer is this. Under the discharge a limit is set to the absorption of hydrogen, because absorption is balanced by evolution of gas already absorbed in the walls by the bombardment of the changed particles. When phosphorus is mixed with the hydrogen, it is deposited on the walls as a stable solid; this solid shields from bombardment the absorbed hydrogen and prevents it from being detached. If sufficient phosphorus is present to form a solid layer over the hydrogen absorbed on the glass, this layer provides a new surface on which gas may once more be absorbed. If the supply is unlimited, an unlimited number of layers can be found, consisting alternately of hydrogen and phosphorus, and there is no limit to the amount of gas that can be made to disappear. (Of course, the layers of gas and phosphorus are not likely to be completely distinct.)

On this view, if conditions can be obtained in which absorption of hydrogen takes place without evolution, then the amount absorbed should not be increased by phosphorus unless it is present in such quantity that it can form a layer covering the hydrogen completely and providing a new surface for absorption. For in such conditions the limit to absorption will be fixed by the amount of gas which can adhere, all at one time, to the surface of the walls. Now, such conditions can be obtained in hydrogen, for the gas can be made to adhere to the walls in the presence of hot tungsten without any discharge at all. It is strong support for our view that we can discover no evidence whatever that more hydrogen can be absorbed by the discharge in the presence of phosphorus than can be absorbed by Langmuir's method without any phosphorus at all—*unless* so much phosphorus is supplied that a distinct yellow coloration appears on the walls. In order that phosphorus may increase absorption above the maximum quantity that can be absorbed in its absence, so much must be used that there is definite evidence of the formation of a complete layer of phosphorus on the glass. It is difficult experimentally to establish that proposition with perfect certainty, but we have observed nothing that cannot be reconciled with it.

On the other hand, it has been noted that the amount of hydrogen that can be absorbed when the glow-discharge passes is definitely less than that absorbed without the discharge. This difference is due to evolution by bombardment. The difference is certainly decreased by the addition of phosphorus vapour, even when there is not enough to form a yellow bulb; but there is no evidence that, while the bulb does not turn yellow, the sign of the difference can be reversed by phosphorus and more absorbed with than without it. The marked increase in the absorption of hydrogen which is obtained with an unlimited supply of phosphorus vapour is invariably associated with the formation of a "yellow" bulb, the depth of the coloration of which increases with the amount of hydrogen absorbed.

Summary.

The paper is a continuation of that in Phil. Mag. xl. p. 585, Nov. 1920. It discusses the disappearance under the electric discharge of hydrogen, carbon monoxide, nitrogen, argon, both in the absence and the presence of phosphorus vapour.

§ 12. In the presence of incandescent tungsten, *hydrogen* disappears without the passage of the discharge, as discovered by Langmuir. The gas disappearing adheres without change to the surface of the glass, especially to the cooler parts of it.

§ 13. When the discharge passes, the rate of disappearance of *hydrogen* is not greatly altered; but the phenomenon is complicated by the simultaneous liberation of absorbed gas by bombardment and by the appearance of water vapour. The gas that has disappeared is again adhering to the glass walls either as hydrogen or as water.

§ 14. In the discharge, carbon monoxide is converted into dioxide (see § 10); the dioxide, as well as the original monoxide, adheres to the glass walls and can be liberated thence by heating. The quantity of CO_2 which will adhere to the glass is very much greater than that of hydrogen which will so adhere.

§ 15. Nitrogen disappears under the discharge, but cannot be liberated again by baking the evacuated vessel. At the same time the filament wastes and the walls are blackened. It is probable that Langmuir's nitride WN_2 is formed, but it

is thought that part of the gas is held to the walls in its original form by a covering layer of tungsten.

§ 16. Argon disappears like nitrogen with much blackening of the bulb; but the gas can be restored by baking together with hydrogen produced by the bombardment of the glass.

§ 17. Mercury has not been proved to disappear. The discharge through the vapour liberates large quantities of gas from the glass, even if they have been previously baked in a high vacuum.

§ 18. The conclusions based on this part of the work are given on p. 691.

§ 19. Some of the relevant properties of phosphorus vapour are stated.

§ 20. Phosphorus vapour disappears rapidly in the discharge, being converted into red phosphorus which is deposited on the walls. The glow potential of phosphorus vapour is anomalous.

§ 21. Hydrogen, carbon monoxide, and nitrogen when mixed with phosphorus vapour disappear in the discharge together with the phosphorus. A lower final pressure of gas can be reached with a given applied potential in the presence of phosphorus than in its absence. This fact is correlated with the abnormal glow potential of phosphorus vapour.

§ 22. It is asked whether there is any evidence for the prevalent view that the action of phosphorus in removing gases in the discharge is chemical. Evidence against this view is adduced thus:—

§ 23. The state of these gases when they have disappeared in the presence of phosphorus seems not to differ from the corresponding state in the absence of phosphorus.

§ 24. There is no indication of a "law of constant proportions."

§ 25. The state of the glass walls modifies profoundly the absorption of gas.

§ 27. An alternative theory of the effect of phosphorus is proposed. It is suggested that the deposited red phosphorus covers the deposited gas and prevents it from being liberated again by bombardment. At the same time it provides a new surface on which gas can be absorbed.

LXIII. *The Physical Significance of the Least Common Multiple.* *By* NORMAN CAMPBELL, *Sc.D., and* E. C. C. BALY, *C.B.E., M.Sc., F.R.S.* *

NOTE.—This paper is the result of a long private discussion between its authors. They have resolved some of their differences and misunderstandings, but since a residuum of disagreement remains they think it well to express their views in the form of a criticism and a reply.

I. (By N. R. C.)

1. IN interpreting his work on absorption spectra Prof. Baly (*e.g.* Phil. Mag. xl. p. 1, 1920) has based some of his conclusions on the observation that one of the frequencies (or wave-numbers) characteristic of the spectrum is the least common multiple of other characteristic frequencies. So far as I can discover, this is the first time that physical significance has been attributed to the L.C.M. of measured magnitudes; the occasion seems appropriate to examine what that significance may be.

2. The conception of an L.C.M. is usually applied only to integers; Prof. Baly applies it to fractions, still meaning, of course, by the L.C.M. the least common *integral* multiple. No objection can be taken to this extension in general; any fractions, so long as they are commensurable, have an L.C.M. which is given by

$$\text{L.C.M.} = \frac{\text{L.C.M. of numerators of the fractions}}{\text{H.C.F. of denominators of the fractions}}, \qquad (1)$$

where H.C.F. means highest common integral factor. But it may be observed at once that the conception is applicable only if the fractions are commensurable; there cannot be any L.C.M. of the length of the side of a square and the length of its diagonal. In speaking of an L.C.M., Prof. Baly is assuming that the real frequencies characteristic of an absorption spectrum are commensurable. Though no arguments can be raised against this assumption, I believe that none can be raised for it; there can be no arguments until relations are found for absorption spectra such as are known for the "hydrogen-like" emission spectra. The frequencies of the Balmer series are doubtless commensurable; but it is still doubtful whether the frequencies of the similar spectra in other elements are so. However, Prof. Baly might well claim that the success of his interpretation

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* Communicated by the Authors.

justified his assumption, and therefore no further objection on this score can be taken.

3. The point to which I wish to draw attention is that the L.C.M. given by (1) is determinate only if the fractions are known with perfect exactitude ; it is quite indeterminate if there is any experimental error whatever. In this the L.C.M. differs from the functions which are most frequently used in physics. Thus, if we are concerned with the product of two magnitudes A and B, an uncertainty of (say) 1 per cent. in their values produces an uncertainty of only about 1 per cent. in their product. But the same uncertainty, or any uncertainty whatsoever, produces an infinite uncertainty in their L.C.M. If we do not know the values exactly, then there is an infinite number of values covering an infinite range, any of which may be the L.C.M.

For there is an infinite number of ratios p/q (where p and q are prime to each other) which differ by less than any assigned amount from each other ; whatever our experimental accuracy, there is an infinite number of pairs p, q the ratio of which can be used with equal right to represent the measured value. Let these ratios for one of the measured values be $\frac{p_1}{q_1}, \frac{p_2}{q_2} \ldots$; for the other $\frac{r_1}{s_1}, \frac{r_2}{s_2} \ldots$. Then, by (1), the L.C.M. of $\frac{p_1}{q_1}$ and $\frac{r_1}{s_1}$ will not be equal, or even approximately equal in general, to the L.C.M. of $\frac{p_2}{q_2}$ and $\frac{r_2}{s_2}$. If (p_1, q_1, r_1, s_1) are all prime to each other then the L.C.M. is p_1r_1, and will differ enormously according to the approximation adopted. Thus, if we cannot measure a magnitude to 1 in 1000, we cannot have any reason to adopt the approximation 22/7 rather than the approximation 355/113 ; but if we take the L.C.M. of this magnitude with (say) 31/10, the two results we shall get will be 682 and 11005, which are in the ratio of 22 to 355. Their ratio is not that of any small integers ; they are magnitudes as completely different as any magnitudes can be.

4. This is so obvious that, if we had been in the habit of expressing fractions in the vulgar form, the indeterminateness of the L.C.M. of any fractions of which the value is not known with complete mathematical accuracy would have been immediately apparent. It is only if we use, as we always do, decimal notation (a term which will be employed to denote also similar notation based on a radix other than 10) that there is any appearance of determinateness. For when we adopt such notation, we fix the denominator of all

our fractions and therefore fix also the numerator and the L.C.M. Then (1) leads immediately to the following rule, which is that actually employed by Prof. Baly :—Express the values with the same number of decimal places; remove the decimal point; take the product of the resulting integers and replace the decimal place in the product. Thus the L.C.M. of 7·7 and 5·6 is 431·2. It is to be observed that we must take the product and not the L.C.M. of the integers. For the number of decimal places expressed are those which represent amounts greater than the experimental error; we do not know what the remaining places are. If we took the L.C.M. of 77 and 56 (viz. 616) in place of the product 4312, we should be assuming that the unexpressed places were all zeros, and for such an assumption there is not, of course, the slightest justification.

This rule, which is imposed on us by decimal notation. implies the choice of one out of the infinite number of alternatives for the ratio p/q which is to represent the value. Moreover, the choice is determined wholly by the radix of the notation and the physical unit. But the L.C.M. is determined by the choice; and the L.C.M. arrived at is thus a quite arbitrary selection from an infinite number of possible alternatives.

5. In place of a general discussion it will perhaps be better to take a single example. Suppose that the real value of the two magnitudes is 7·69438 . . . and 5·62936 . . . inches, and that the possible experimental error is not less than 1 per cent. Then we shall choose the numbers already given, 7·7 and 5·6, and find 431·2 as the L.C.M. But if we had been measuring in centimetres and had never heard of inches (obviously a permissible supposition) the real values would have been 19·5438 . . . and 14·2986 . . . ; remembering our 1 per cent. error, we shall choose 19·5 and 14·3, of which the L.C.M. is 2788·5 cm. or 1976·6 inches—a result quite different from the value 431·2 at which we arrived before. (It is, of course, 10/2·54 times as great.) Accordingly a change of unit leads, by the rule to which we are forced, to the identification of a perfectly different length to represent the L.C.M. of the original lengths.

Now let us change our radix to 3, and express number in the scale of 3 in italics. Then 7·69438 . . . ≡ *21·20020202* . . . and 5·62936 . . . ≡ *12·12122221* . . . . With an accuracy of 1 in 243 we shall choose *21·200* and *12·121*. Of these the L.C.M. is *1120212·200* ≡ 1157·7—again quite a different result from our original 431·2.

Indeed it is not difficult to see that by an appropriate choice of unit or of radix, and employing always the rule which Prof. Baly employs, it is possible to make out any number whatsoever to be the L.C.M. of any other numbers whatsoever. And this indefiniteness, it must be insisted, does not result from Prof. Baly's use of the wrong rule; so long as there can be a right rule to determine something that does not exist, it is the right rule. The plain fact of the matter is that, if the numbers of which the L.C.M. is to be taken are not known with perfect and complete mathematical accuracy, there is no such thing as the L.C.M. His rule merely adopts one out of an infinite number of alternatives; and the alternative which it adopts depends wholly on the unit of measurement and the scale of notation.

6. Are we then to conclude that the numerical agreements which he finds are pure coincidences? The answer to that question I leave to those familiar with spectroscopic work, for they alone can determine whether the accuracy of the measurements is such as to make the probability of the necessary coincidence sufficiently great for that explanation to be plausible. But it seems that if the answer is negative, there is only one conclusion to be drawn.

If the L.C.M. is to be significant the values must be known with complete accuracy. There is only one kind of magnitude which has no experimental error and can be known with complete accuracy; that magnitude is the number of something, which is necessarily an integer *. If we were measuring electric charges and found values $9{\cdot}6\times10^{-10}$, $14{\cdot}3\times10^{-10}$, $28{\cdot}6\times10^{-10}$ e.s.c.g.s. units, we should doubtless conclude that we were measuring 2, 3, 6 electronic charges, and might justifiably attribute significance to the fact that the third number is the L.C.M. of the first two. And the conclusion would not be invalid because there was experimental error; the error arises in determining what is the charge of which the measured charges are integral multiples; we know without any error at all that the charges are integral, and not fractional, multiples of that charge, and since our measurements are accurate enough to distinguish between successive integers, we know exactly what multiples they are of the unknown charge. We know that the real charges are px, qx, rx where p, q, r are integers; and we can draw the conclusion that qx is the L.C.M. of px and rx without knowing x. The conclusion, moreover, would be independent of any change of unit (which would merely change x) or of change of radix.

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* See 'Physics: The Elements,' Chap. xvi.

This represents, in my opinion, the only case in which the L.C.M. of magnitudes affected by experimental error can have an L.C.M. which is physically significant. Its characteristic is that all the magnitudes are integral multiples of some "natural unit" of which fractional values cannot occur. If Prof. Baly's conclusions cannot be explained as numerical coincidences, they must prove that there is an indivisible unit of frequency in the absorption spectra which he considers. But since there is, I believe, no general belief at present in such a unit, it will be well to point out that the measurements must fulfil certain conditions to be consistent with that hypothesis. If they do not fulfil them, then, so far as I can see, the agreements he finds must be numerical coincidences, however small the apparent probability of such a coincidence may be.

If measured values a, b, c are to be integral multiples of a natural unit and, at the same time, the statement is to be significant that c is the L.C.M. of a and b, then it must be possible to find an x such that, within experimental accuracy, $(a, b, c)=(p, q, r) \cdot x$, where p, q, r are integers and r is the L.C.M. of p, q.

This condition raises some difficulties when, as in Prof. Baly's examples, the values obey his rule, being expressed in units such that they are not integral. For consider $a=7{\cdot}7$, $b=5{\cdot}6$, $c=431{\cdot}2$. The only possible value of x is near to $0{\cdot}1$; but if $x=0{\cdot}1$, the numbers of natural units in each of the measured magnitudes is 77, 56, 4312. And since the only way of proving the existence of a natural unit with any satisfaction is by showing that measured values are in the ratios of *small* integers, the mere fact that all Prof. Baly's magnitudes must be represented, in terms of natural units, by integers that are by no means small, makes it difficult for him to establish from these experiments (I say nothing of others) the only proposition that will make his results significant. But once more, the only alternatives are (1) that his agreements are mere coincidences, (2) that all experimental physics is founded on a delusion and scales of notation have physical importance.

II. (By E. C. C. B.)

Whilst minor objections might be raised to some of Dr. Campbell's statements the main criticism he makes against the least common multiple is obviously sound. I naturally am in complete agreement with him in his conclusion that the least common multiple of two different energy quanta can have no physical significance unless these quanta are integral multiples of a fundamental unit. Dr. Campbell speaks throughout of frequencies and the least common multiple of these frequencies, and in conclusion he states his opinion that there is no general belief in a fundamental unit of frequency. I would point out that my theory is based on the assumption that each elementary atom is characterized not by a frequency but by its energy quantum, the frequency being due to the time factor in the process involving the absorption or emission of one quantum, namely the shift of an electron from one stationary orbit to another. To my mind the difficulty in postulating a fundamental unit of energy is not so great as it might be in postulating a natural indivisible unit of frequency, especially when it is remembered that this fundamental unit is *ex hypothesi* associated with matter.

In my papers I made the assumption that two atoms in combining together contribute each an equal share towards the total energy loss and therefore emit an equal amount of energy as whole numbers of their characteristic quanta. This very simple hypothesis naturally means that each atom evolves an amount of energy which is the L.C.M. of the quanta characteristic of the two atoms, and in putting it forward I was tacitly assuming that each elementary quantum is an integral multiple of some fundamental unit. I am very indebted to Dr. Campbell for pointing this out.

The existence of this fundamental unit of energy associated with matter does not seem to me to be difficult of acceptance. Very possibly the fundamental unit is the elementary quantum characteristic of the hydrogen atom, the quanta characteristic of other atoms being integral multiples of this. The very interesting possibility that every atomic nucleus is built up of hydrogen nuclei certainly does not increase the difficulty of belief in this, and in addition I may refer to the statement in my paper that simple arithmetical relations do seem to exist between the elementary quanta I have been able to calculate.

Apart from the question of the correctness of any particular theory, the non-existence of a fundamental unit of energy would seem to lead to utterly chaotic and uncoordinated relations between the frequencies and energy contents of substances. It would I think stultify all hopes of quantitatively coordinating energy and chemical reaction, and as is well-known many most promising results have been obtained by Bodenstein, W. C. McC. Lewis, and others in this direction.

Dr. Campbell says that all the numerical agreements I have obtained must be coincidences if the fundamental unit does not exist, but after all the multiplication of coincidences must sooner or later engender a belief in their reality. The "coincidences" I have obtained are very numerous indeed. The calculation of all the 600 individual lines in the absorption band of sulphur dioxide from the infra-red absorption spectrum of hydrogen sulphide, the calculation of the lines of water from sulphur dioxide are only two isolated cases. I laid more stress upon these because of their very great accuracy. There are many others equally striking *, and I do not think I am unduly biassed in saying that it surely is easier to believe in the existence of a fundamental unit of energy associated with matter than to disbelieve in the reality of these results.

There have also been obtained in these laboratories some results which very strongly support the L.C.M. principle and hence the existence of the fundamental unit of energy. In my paper on molecular phases † I suggested that the observed deviations from Einstein's law of the photochemical equivalent are due to the re-absorption by the surrounding reactant molecules of the energy radiated during the reaction. This will clearly result in more than one molecule reacting for every quantum of energy absorbed. In any photochemical reaction it is obvious that the energy evolved must be radiated by the products of the reaction and at frequencies characteristic of them, since if the reactant molecules radiate the energy they have absorbed they will no longer be in a reactive condition. It must be emphasised that, if the energy radiated by the products is absorbed by the reactant molecules, the quanta characteristic of the former must be *exact* multiples of the latter, which is the fundamental basis of my theory and indeed is the point at issue between Dr. Campbell and myself. If, therefore, it can be proved that in any reaction the energy radiated by the resultant molecules is absorbed by the reactant molecules, I venture to think that the integral relations between the quanta characteristic of the two will be very strongly supported, if not absolutely proved.

In the paper referred to it was pointed out that the re-absorption by the reactant molecules of the energy radiated by the resultant molecules will obviously depend upon two factors. In the case of a photochemical reaction it will depend firstly on the concentration of the reactant molecules,

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* Astrophys. Journ. xlii. p. 4 (1915).

† Phil. Mag. xl. p. 15 (1920).

and secondly on the density of the radiated energy and therefore on the intensity of the activating light. In other words, the number of molecules reacting above and beyond that demanded by Einstein's law will vary, firstly with the concentration of the reactant molecules when the intensity of the activating light is kept constant, and secondly with the intensity of the activating light when the concentration of the reactant molecules is kept constant.

The fact that the deviation from Einstein's law varies with the concentration of the reactant molecules has already been proved, and this was pointed out in my paper. Further, Bodenstein [*] has found that the amount of hydrogen chloride formed from a mixture of hydrogen and chlorine in unit time under constant illumination varies as the square of the concentration of the chlorine. Clearly, therefore, the first condition as regards the re-absorption is established.

The second condition, namely the variation in the intensity of the light with a constant concentration of the reactant molecules, has been investigated by Mr. W. Barker and myself. The photochemical union of hydrogen and chlorine under constant pressure in the presence of water has been studied, the concentration of the chlorine and hydrogen being kept constant by the dissolution of the hydrogen chloride in water as fast as it is formed. The detailed results will be published elsewhere, but a brief statement as to their nature may now be made. In the first place, it is obvious that, if re-absorption by the chlorine of the energy radiated by the hydrogen chloride takes place, the amount of hydrogen chloride formed in unit time with constant intensity of the activating light will at first conform with Einstein's law and then steadily increase up to a constant maximum. This increase in the rate of the reaction up to a constant maximum was first noted by Bunsen and Roscoe [†] and now has been amply confirmed by Mr. Barker and myself.

In the second place, we have found that these maximum rates are not proportional to the intensity of the incident light but that they increase far more rapidly than the intensity. In other words, the deviation from Einstein's law increases with the intensity of the incident light exactly as foretold from the theory. It may also be pointed out that the velocity curves indicate that at a finite intensity the number of molecules reacting for every quantum absorbed

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[*] *Zeit. Phys. Chem.* lxxxv. p. 297 (1915).

[†] Pogg. *Ann.* c. p. 481 (1855).

becomes infinite. With this critical intensity the reaction passes as an explosion wave through the mixed gases.

Reference may also be made to Slade and Higson's results * with a photographic plate, which show that the amount of silver obtained with a given exposure is not proportional to the intensity of the light but increases at a greater rate than the intensity.

All the phenomena foretold from my theory have therefore been experimentally proved. I lay great stress on the fact that this re-absorption by the reactant molecules of the energy radiated by the resultant molecules is only possible if the quanta characteristic of the former are exact integral multiples of those characteristic of the latter. Moreover, these observations lie outside the purview of Dr. Campbell's criticism and the question of coincidence does not arise.

These results would seem to have considerable importance apart from the present discussion, for they clearly establish the possibility of a new type of photocatalysis. A reaction, for instance, which requires light of extremely short wave-length (Schumann region) should be induced by light of a longer wave-length in the presence of a suitable catalyst. The criterion of this catalyst will be that it contains the same atoms as the reactant molecules. This substance when absorbing light rays of its own characteristic frequency will radiate this energy in the infra-red at frequencies which are exactly equal to those of the reactant molecules with the result that, provided the radiation density is sufficient, these molecules will become reactivated and will react. Experiments on these lines were commenced some time ago in these laboratories and are still in progress.

During the writing of this note an interesting paper has been published by Daniels and Johnston † in which is described a typical instance of this very phenomenon. These authors have proved that light of wave-length 400–460 $\mu\mu$ has no action on pure nitrogen pentoxide, but that this substance is decomposed by light of this wave-length in the presence of nitrogen dioxide which is known to absorb these rays. They put forward the same explanation as given above, but state that the energy radiated by the dioxide can be absorbed by the pentoxide if the absorption bands of the two substances in the infra-red overlap. This process is impossible on the quantum theory unless the infra-red quanta characteristic of the two are exactly equal, that is to say are integral multiples of a fundamental unit.

* Proc. Roy. Soc. xcviii. p. 154 (1920).

† J. Amer. Chem. Soc. xliii. p. 73 (1921).

In conclusion it may be said that the evidence for the exact integral relationship between the quanta characteristic of atoms and the quanta characteristic of the molecules formed by the combination of these atoms is well-nigh overwhelming. The hesitation to accept the existence of a fundamental atomic quantum of energy, possibly that associated with the atom of hydrogen, must surely give way before the experimental evidence now adduced. Whilst admitting that my previous results may by some freak of nature have been coincidences in spite of their number, I feel that the more recent work gives extraordinary support to my original hypothesis of the least common multiple principle.

LXIV. *A Null-Deflexion Constant Current Type of Hot-Wire Anemometer, for use in the Determination of Slow Rates of Flow of Gases, together with an Investigation of the Effect of the Free Convection Current upon such Determinations.*

By J. S. G. THOMAS, *D.Sc.(Lond.), B.Sc. (Wales), A.R.C.S., A.I.C., Senior Physicist, South Metropolitan Gas Company, London* [*].

Introduction.

IT was pointed out in a recent communication [†], that the laws governing the convection of heat from fine heated wires are such as to indicate the hot-wire anemometer as pre-eminently the type of instrument to be employed in the investigation of *slow* rates of flow of gases. Employing the Morris type of hot-wire anemometer in such investigations, difficulties are encountered owing to the existence of the free convection current arising from the heated wire. In recent papers [‡], the author has discussed a type of directional hot-wire anemometer, in which these difficulties are largely obviated, and by the use of which the range of application of the hot-wire anemometer may be extended to the investigation of very slow-moving streams of gas. The sole uncompensated effect arising from the existence of the free convection current in the directional type of instrument referred to, is due to the difference in the magnitudes of the

[*] Communicated by the Author.

[†] Phil. Mag. vol. xli. p. 240 (1921).

[‡] Phil. Mag. vol. xxxix. pp. 525–527 (1920); vol. xl. pp. 640–655 (1920); Proc. Phys. Soc. vol. xxxii. Part 3, pp. 196–207 (1920).

respective free convection currents arising from the two exposed wires owing to their small difference of temperature when exposed to the convective effect of an impressed stream of gas.

In the type of hot-wire anemometer investigated in detail by King [*], the heat-loss from the wire due to forced convection was ascertained by adjusting a measured current through the wire so as to bring its resistance to a value corresponding to a predetermined temperature. A Kelvin double bridge was employed, and the arm opposite the sensitive exposed platinum arm was of manganin of negligible temperature coefficient. In the Morris type of instrument, the arm of the bridge opposite to the exposed arm is constituted of a wire similar in all respects to the latter, but shielded by means of a surrounding tube from the cooling effect of any impressed gas stream. It is clear that, with this latter arrangement, the balanced condition of the bridge is, except in so far as a difference exists in the respective free convection currents from the two wires owing to their different respective dispositions to their immediate surroundings, independent of the actual temperature of the fluid medium in their neighbourhood, so long as this temperature is the same in each case. It has, however, been shown by the author [†], that the balanced condition of the bridge is, owing to the difference in the cooling effects experienced by the exposed and shielded wires, due to their respective free convection currents, dependent to some extent upon the heating current employed in the bridge. It appeared, therefore, desirable to investigate the possibility of constructing a null-deflexion type of hot-wire anemometer in which the bridge current was maintained constant, the deflexion produced by an impressed gas stream being annulled otherwise than by increasing the heating current through the arm of the bridge exposed to the stream. The present paper details some of the results obtained in the course of such an investigation.

In a previous paper [‡], attention has been directed to the fact that the resistance of a fine heated wire varies considerably when the wire, through which a constant current passes, is inclined at various inclinations to the horizontal.

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[*] Phil. Trans. A. 520, vol. ccxiv. p. 385 (1914).

[†] Phil. Mag. vol. xxxix. pp. 511, 528, fig. 16 (1920).

[‡] Proc. Phys. Soc. vol. xxxii. part 5, pp. 291–314 (1920). Also Phil. Mag. vol. xl. *loc. cit.*

With the wire horizontally disposed, the free convection current passes immediately away from the wire, whereas with the wire set vertically, the wire is laved by its free convection current whereby the heat-loss, owing to this current, is materially reduced. With variations in the inclination of the wire between the two positions specified, it is clear that the cooling the wire experiences also varies. It follows, therefore, that up to a limit of the velocity of the impressed stream of gas, possibly dependent upon the value of the heating current employed, the loss of heat from a horizontally disposed wire exposed to the cooling action of an impressed stream may be compensated for, maintaining the heating current in the wire constant, by rotation of the wire from its original horizontal position, whereby the heat-loss due to free convection is reduced so as to compensate for the thermal loss due to the impressed stream. The following experimental arrangement was employed :—

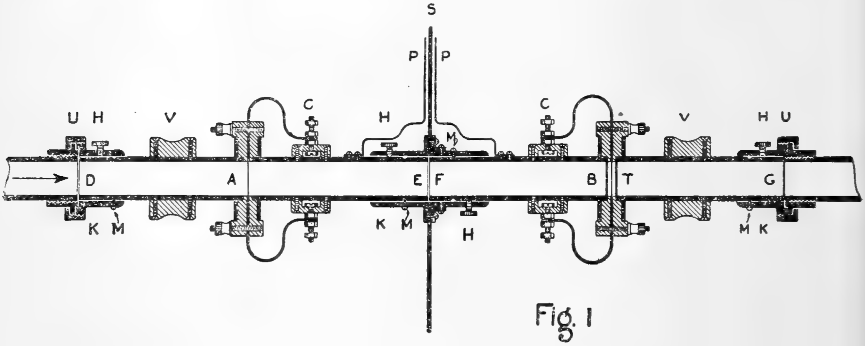

Fig. I

Apparatus.

Fig. 1 shows the anemometer-tube employed which was inserted in the flow tube of equal bore by means of the spigot unions S, S. A represents the exposed platinum wire, and B the shielded wire, cut from the same specimen as A, and surrounded by the shielding tube T. The ends of the respective wires are connected by means of short lengths of thick copper wires to the screw terminals C, C. The mode of insertion of the wires A and B in the anemometer-tube is similar to that described in previous papers. The anemometer-tube was made in two separate sections, DE and

FG, the ends of which were carefully turned so as to be accurately at right angles to the axis of the tube. The ends of these sections were likewise carefully turned so as to afford a good sliding fit into the sockets K, K, K shown, and the tubes could be separately rotated axially. No difference in the anemometer readings corresponding to a given flow could be detected whether the outer portions of these sockets were sealed with wax or not. The small screws M, the ends of which moved in triangular grooves cut at right angles to the axis of the tube as shown, served as guides during rotation of the tubes. After rotation, the positions of the tubes were secured by means of the screws H. (Although the diagram shows only a single screw M and a single screw H at each of the sockets, it may be mentioned that actually in every case three screws disposed radially at angles of 120° were used.) The inclination of either wire A or B to its initial horizontal position was read by means of the pointers P, P which were attached to the respective sections of the anemometer-tube as shown, and which moved over the circular plate S of 8 inches diameter similarly affixed to the central socket-tube K, and divided into degrees. V, V were ebonite blocks affixed to the anemometer-tube in order to facilitate the rotation of the respective sections of the tube. The portions of the anemometer-tube in the neighbourhood of the wires A and B for a distance of 3 inches on either side of the wires were wrapped round with soft felt and the wires were inserted in a Wheatstone bridge in the manner detailed in the papers previously referred to. The constant ratio arm was throughout adjusted to 1000 ohms. The bridge current was adjusted to any desired value and maintained constant by means of a rheostat. The drop of potential across either wire was ascertained by means of a Weston voltmeter of resistance about 200 ohms, the indications of which were correct to within $\frac{1}{4}$ per cent. The resistances of the wires were deduced therefrom, using the value of the current employed in the bridge. The temperatures of the wires in any case were ascertained therefrom, employing the values of their respective resistances at atmospheric temperature, determined by means of a Callendar and Griffiths bridge, and the value of the temperature coefficient of the portion of wire from which they were cut. Precautions were taken to age the wires before measurements were made.

Experimental Results and Discussion.

The following particulars refer to anemometer-tube R 3, used in the present investigation :—

Ratio arm.	1000 ohms.
Internal Diameter of Flow Tube. ...	2·039 cm.
External Diameter of Flow Tube. ...	2·238 cm.
Internal Diameter of Shielding Tube.	0·242 cm.
External Diameter of Shielding Tube.	0·419 cm.
Mean Diameter of unprotected and protected wire.	0·101 mm.
Temperature coefficient of wire.	0·003588.
Ro. of exposed wire (C=0·02 amp.)	0·2337 amp.
Ro. of shielded wire (C=0·02 amp.)	0·2445 amp.

The whole of the flow system, including the anemometer-tube, was tested for leakage by closing the outlet and establishing a pressure of 10·5 inches of water within the tube. At this pressure the leak was ascertained to be 0·06 cubic feet per hour. As the pressure in the flow tube during a series of calibrations never exceeded 0·1 inch of water, the leakage in the system is obviously extremely small and negligible.

A series of determinations was made of the variation occurring in the values of the resistances of the respective wires, heated by various currents (1·5 to 0·9 amp.) with variations in their inclinations to the horizontal. No detectable variation with inclination occurred in the resistance of the shielded wire, for all values of the inclination from the horizontal to the vertical position. Subsequent investigation showed that any such variation as occurred was certainly less than 0·2 per cent. of the initial resistance of the wire. The enclosure shielding this wire appears, therefore, to have been of such dimensions that no appreciable convection current from the wire employed is set up therein when the heating current employed has any value up to 1·5 amp. The determination of the dimensions of a chamber wherein the free convection effect experienced by a fine heated wire is negligible is of importance in connexion with the design of the katharometer *, and the method of rotation of the chamber as described affords a ready means of ascertaining the degree of elimination of such free convection current in any given case. In the case of the exposed wire, considerable alteration of resistance

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* Daynes, Proc. Roy. Soc. A. vol. xcvii. p. 276 (1920).

accompanied its rotation through successive angles from the horizontal to the vertical position. The magnitude of these variations when heating currents of from 0·9 to 1·5 amp. were employed will be seen from fig. 2, wherein are given the results for successive angles of inclination to the horizontal increasing by 10°, readings in each case being taken with the pointer indicating the same inclination to the right and left of its initial vertical position. The initial horizontal position of the exposed wire can be very accurately

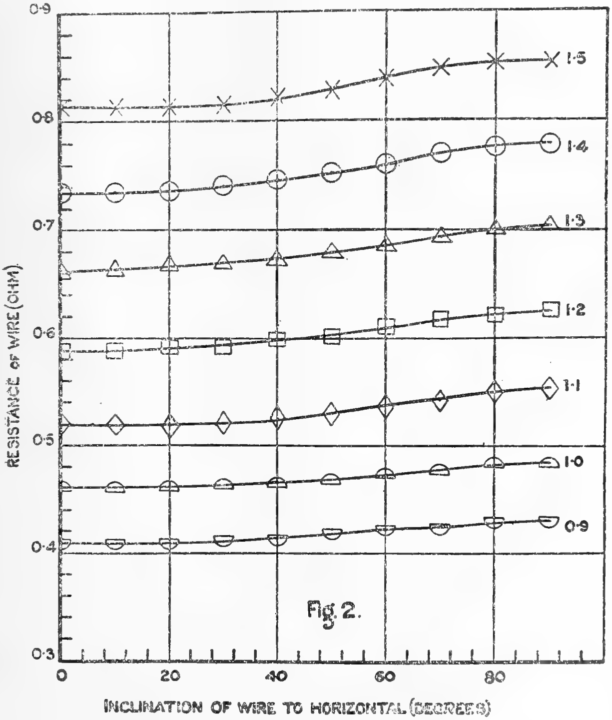

Fig. 2.

determined by employing the bridge as previously described. It is clear that if the bridge is balanced with the wire either horizontal or vertical, equal galvanometer deflexions are obtained on rotation of the exposed wire through the same angle in a clockwise or counter-clockwise direction. The temperature of the surroundings varied from 16° C. to 20°·8 C. while the results shown in fig. 2 were being obtained. For the purposes of comparison it appeared desirable to reduce the experimental results to a uniform basis in which the temperature of the surroundings of the enclosure containing the wire was 0° C. An investigation of the dependence of the temperature of a fine platinum wire heated by a constant current within an enclosure, upon the surrounding temperature, showed that, in the present case, no appreciable error would be introduced by assuming that an increase in the temperature of the medium surrounding the enclosure containing the wire, is accompanied by an increase in the temperature of the heated wire equal to such increase of temperature. The results plotted in fig. 2 have thus been deduced from the experimental results, assuming that the enclosure is surrounded by a medium at 0° C. The values of the resistance and temperature of the wire in the horizontal and vertical positions when heated by various currents, the surroundings being at 0° C., are set out in Table I. herewith :—

TABLE I.

Heating Current (amp.).	Resistance of Wire (ohm).		Mean temperature of Wire (° C.).		Ratio $\frac{R_v}{R_h}$.	$\theta_v-\theta_h$ (° C.).
	Wire Horizontal. R_h.	Wire Vertical. R_v.	Wire Horizontal. θ_h.	Wire Vertical. θ_v.		
0·9	0·4189	0·4389	225	250	1·048	25
1·0	0·4680	0·4950	288	322	1·058	34
1·1	0·5273	0·5637	364	414	1·069	50
1·2	0·5958	0·6350	456	510	1·066	54
1·3	0·6692	0·7154	557	622	1·069	65
1·4	0·7428	0·7936	663	738	1·068	75
1·5	0·8260	0·8680	786	852	1·050	66

It will be seen that on rotation of the wire from the horizontal to the vertical position, an increase of from 5 to 6 per cent. occurs in the resistance of the wire. The accompanying increase in the temperature of the wire is shown in the 7th column. The temperatures θ_h and θ_v are probably correct to within 0·5 per cent.

Employing the unshielded wire in the manner already described, a series of determinations was made of the angular rotation of the wire about the axis of the flow tube necessary in order to maintain a balanced condition of the bridge when the wire was subjected to the cooling action of a current of air moving with a determinable mean velocity in the tube, the bridge current being maintained constant meanwhile. The air stream was derived from a 5 cubic feet gas holder, and was controlled and its velocity determined in the manner detailed in previous papers. The results obtained employing values of the bridge current ranging from 0·9 amp. to 1·5 amp. are shown in fig. 3. The main features of the calibration curves are briefly as follows:—With gradual increase of the impressed stream from zero, the initial comparatively large rotation necessary to restore balance of the bridge is succeeded by a region of velocities for which the necessary rotation increases but slowly with increase in the magnitude of the impressed velocity. With subsequent increase of the impressed velocity the necessary rotation increases extremely rapidly. These characteristics of the calibration curves are readily interpreted by reference to the inclination-resistance curves of fig. 2. It is seen that with continuous increase of the inclination of the wire to the horizontal, the resistance of the wire initially increases extremely slowly. With further rotation, the resistance increases comparatively rapidly until inclinations approaching the vertical are reached, when the rate of variation of resistance with inclination again becomes very small. The velocity-inclination curves (fig. 3) for inclinations of the

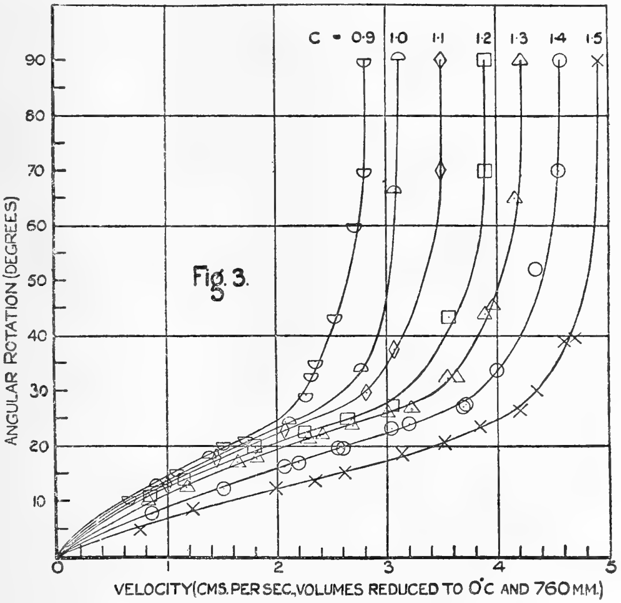

Fig. 3.

wire approaching verticality are seen to be practically perpendicular to the axis of velocities. These extremely steep portions of the calibration curves represent a condition of affairs where the velocity of the impressed stream has attained such a value that the temperature of the resultant convection current from the wire differs very little from atmospheric, so that very little temperature change occurs in the wire on rotation. Moreover the fact that the convection current from the wire is of approximately uniform temperature over a region considerably wider than the diameter of the wire *, is a factor likewise operative in accentuating the steepness of the calibration curves in this region compared with their smaller inclination in the region of the origin.

The curves in fig. 3. show that, employing a current of 1·5 amp., the type of anemometer described may be usefully employed for the determination of velocities up to about 4·5 cm. per sec. The range of application diminishes with decrease in the bridge current employed. For a current of 0·9 amp. the maximum velocity measurable with accuracy is about 2 cm. per sec.

Determination of the effective velocity of the free convection current arising from the wire when inclined to the horizontal.

As already remarked, the cooling effect, due to the free convection current, experienced by the heated unshielded wire in a position inclined to the horizontal is less than when the wire is horizontal. Such diminished cooling effect arises from the greater thermal shielding influence afforded by the free convection current in the case of the inclined wire, and may for purposes of calculation be ascribed to a diminution in the velocity of the free convection current. The approximate values of such effective velocities under the conditions of the present experiments can be readily determined from the results represented in fig. 3. In the case of any one of the curves shown therein, the total heat-loss from the wire is the same for any point on the curve, as the points represent a balanced condition of the bridge. If, then, v_0 = effective velocity of the free convection current with the wire horizontal, v_α = the effective velocity of the same when the wire is inclined at an angle α to the

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* Proc. Phys. Soc. vol. xxxii. Part 5, p. 301 (1920).

horizontal, and if, moreover, V_α=velocity of the impressed stream corresponding to the rotation α, we evidently have, since the total cooling effect experienced by the wire is the same in all cases,

$$v_\alpha=\sqrt{v_0^2-V_\alpha^2},$$

assuming that the cooling effect due to the walls of the tube etc. is the same in all cases. This assumption would be most strictly justified in the case of experiments carried out in a channel of large dimensions compared with the length of the heated wire. Values of v_0 appropriate to various temperatures of the heated wires in the present case were determined in the manner described in a previous paper[*]. The present results are not strictly comparable with the results obtained in the previous paper owing to the differences in the mounting of the wire, the diameter and lagging of the tube, also the difference in the diameter and temperature coefficient of the wires in the two cases. In each case, the velocity of the free convection stream, assuming the stream to be at the temperature of the wire, was found to be linearly related to the temperature of the wire, the relation being of the form $V=\kappa(\theta-20)$ where V is the velocity of the free convection current, and $\theta°$ C. the temperature of the wire. While the determinations were being made, the mean atmospheric temperature was 20° C. Evidence has previously been given[†] that the free convection stream is not raised to the temperature of the wire owing to the existence of a stagnant gas film surrounding the wire. The effective velocity of the free convection stream, assuming its temperature to be T, the wire being horizontal and at temperature θ, is, in the present case, given by the relation

$$V=0{\cdot}026(\theta-20)\left(\frac{T+273}{\theta+273}\right).$$

For purposes of comparison with results previously given in various calibration curves, in which velocities have been recorded, assuming the impressed air stream to be at 0° C., the values of the effective velocity of the free convection current for various inclinations of the wire have been calculated as already explained, taking the temperature of the stream as

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[*] Phil. Mag. vol. xxxix. pp. 518–523 (1920).

[†] *Ibid.* pp. 531–534. See also Langmuir, Proc. Amer. Inst. Elec. Eng. xxxi. pp. 1011–1022. Trans. Amer. Electrochem. Soc. xxiii. p. 293 (1913).

0° C. The results obtained are set out in Table II. herewith :—

TABLE II.

Diameter of Wire 0·101 mm.

Current (amp.).	Temperature of Wire (° C.)	Effective Velocity of Free Convection Current when Wire is inclined to Horizontal at Angle Specified (cm. per sec.). Temperature of Stream taken as 0° C.					
		0°	10°	20°	30°	40°	90°
0·9	212	2·8	2·7	2·4	1·7	1·3	0
1·0	279	3·3	3·2	2·9	2·0	1·5	1·0
1·1	354	3·8	3·7	3·4	2·6	2·1	1·5
1·2	445	4·2	4·1	3·8	2·8	2·1	1·5
1·3	548	4·6	4·5	4·1	3·0	2·5	1·8
1·4	651	4·9	4·7	4·1	3·1	2·6	1·8
1·5	766	5·1	4·8	3·7	2·6	2·1	1·4

It is clear that the numbers in the above table are only of significance for comparative purposes, as the values of v_0, given in the third column, were obtained by employing the tube in a vertical position, whereas the values of the respective velocities given in the remaining columns are deduced therefrom, employing results obtained with the axis of the flow tube horizontal. The various results would be strictly comparable only in the case where the experiments were carried out in a channel whose dimensions were large compared with the length of the wire employed. Further, since $dv_\alpha = \frac{v_0}{v_\alpha} dv_0$, it is clear that the effect of an error in v_0 upon the value of v_a deduced therefrom will be greater, the greater the inclination of the wire to the horizontal. Thus, an error of 0·1 cm. per sec. in the determination of v_0 corresponding to a heating current of 1·5 amp. introduces an error of about 0·4 cm. per sec. in the deduced value of v_a at 90°. For this reason, therefore, no great accuracy can be attributed to the absolute values of the velocities, relating more particularly to large inclinations of the wire to the horizontal, contained in Table II. The curves yielded by the results in Table II., plotting velocities as ordinates against the respective inclinations as abscissæ, resemble that obtained by

King * for the relation of the convection constant β_0 to the inclination of the wire. The results clearly indicate that very considerable reduction in the magnitude of the effective velocity of the free convection current occurs when the wire is rotated from the horizontal to the vertical position. Such considerable reduction is to be anticipated from a comparison of the results contained in a previous paper †, showing the variation of the resistance of a heated wire, when subjected to an impressed downwardly-directed current of air, and the variation of the resistance of the wire when rotated from the horizontal to the vertical position given in Table I. The latter variation is of the same order of magnitude as that in the previous work corresponding to complete elimination of the free convection effect in the case of a horizontal wire. The effective velocity of the free convection current corresponding to any inclination of the wire could be very accurately determined by experiments along these lines carried out in a wind channel.

It has previously been pointed out ‡ that an inversion in the respective sensitivities of an anemometer of the Morris type with horizontal wire, employing two different values of the heating current, occurs as the velocity of the impressed stream is gradually increased from zero, and that the velocity of the impressed stream corresponding to this point of inversion is larger, the larger the free convection current corresponding to the greater of the two heating currents concerned. A similar effect due to a decrease in the magnitude of the effective velocity of the free convection current, when the wire is rotated from the horizontal to the vertical position, is shown in fig. 4, wherein are given the forms of the calibration curves obtained employing currents of 1·5 and 1·0 amp. in the anemometer bridge, the bridge in each case being balanced in the absence of an impressed flow of air with the unshielded wire (1) horizontal and (2) vertical, corresponding to the respective inclinations at which the wire was subsequently used. The galvanometer shunt was throughout equal to 27 ohms. It is seen that the vertical disposition of the wire affords the greater sensitivity (*cf.* curves A and C, or B and D), and that whereas in the case of the horizontal wire, the larger heating current affords the greater sensitivity only for velocities greater

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* Phil. Trans. A. *loc. cit.* p. 425.

† Phil. Mag. vol. xxxix. pl. xii. fig. 13 (1920).

‡ Phil. Mag. vol. xxxix. p. 515 (1920).

than 5·6 cm. (point P in diagram), with the vertical disposition of the wire the inversion in the respective sensitivities corresponds to an impressed velocity of about 1·9 cm. per sec. These velocities are of the same order of magnitude as those given in Table II. for the respective effective velocities of the free convection current for the horizontal and vertical disposition of the wire when heated by a current of 1·5 amp.

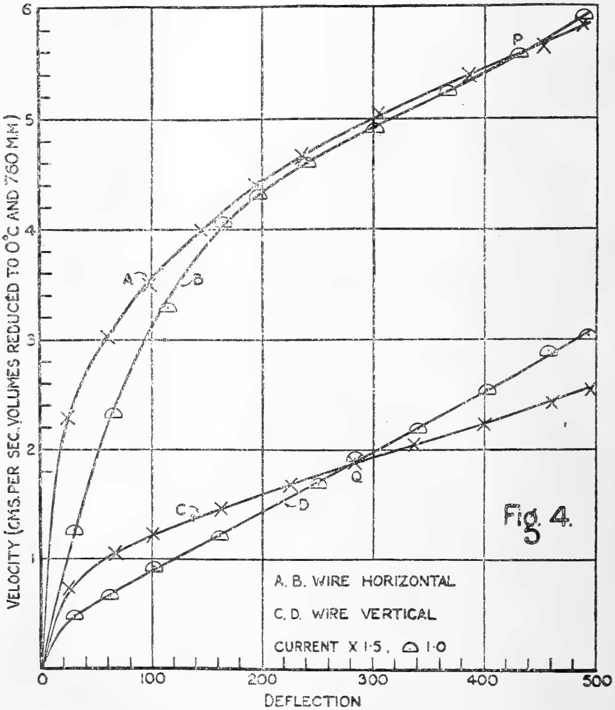

Fig. 4.

The alteration in the value of the effective velocity of the free convection current occurring on rotation of the wire was utilized in the following manner to determine the magnitude of the velocity of the impressed stream of air for which the effect of the free convection current became negligible in the case of a stream of air flowing horizontally. This matter has been considered theoretically by King *. In the present

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* Phil. Trans. A. *loc. cit.* p. 426.

instance the bridge was balanced *in the absence of an impressed flow with the wire disposed horizontally*, and thereafter calibration curves were determined with the wire disposed either *horizontally or vertically*, readings corresponding to the respective horizontal and vertical dispositions being taken alternately for the same value of the impressed velocity of the air stream. Similar determinations were then made with the bridge balanced in the absence of an impressed flow with the wire disposed *vertically*. The temperature of the wire corresponding to the various flows

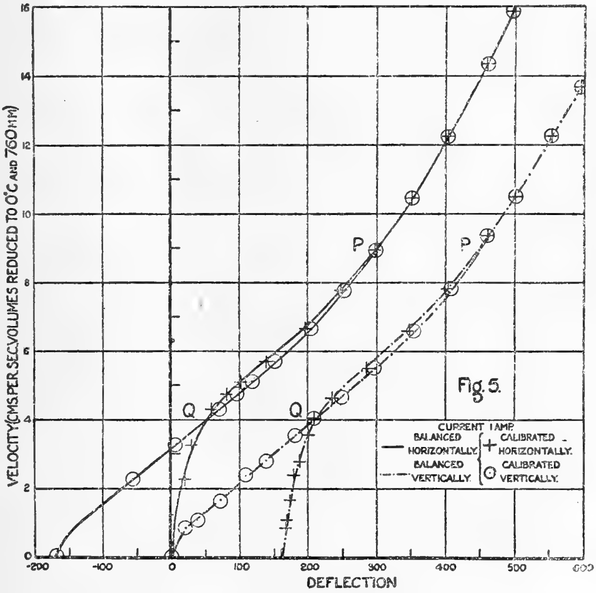

Fig. 5.

was determined in the manner previously described. Calibration curves were determined employing values of the heating current ranging from 0·9 to 1·5 amp. They all showed the characteristic features of the curves drawn in fig. 5, which gives the results obtained when a heating current of 1·000 amp. was used in the bridge. A difference of 1 scale division in the respective deflexions when the wire was disposed vertically or horizontally could be detected with ease. With the exception of the velocity corresponding to the point Q, the points P obviously correspond to a minimum value of the impressed velocity for which the deflexion is independent of the inclination of the wire, and consequently, in accordance with what has preceded, of the magnitude of the velocity of the free convection current. Velocities between those represented by the points P and Q can be determined from the calibration curve, irrespective of the inclination of the wire, and hence of the free convection current, with a possible percentage error not exceeding 6 per cent. The results obtained, using the respective heating currents mentioned, are summarized in Table. III. The respective temperatures of the wire in the horizontal and vertical positions in the absence of flow are contained in Table I. (fourth and fifth columns respectively).

Table III.

Heating Current (amp.).	Temperature of Wire corresponding to point P (fig. 5), for which the deflexion is independent of inclination of wire. θ_p (° C.).	Impressed velocity corresponding to P. (cm. per sec.).	Temperature of wire corresponding to point Q (fig. 5). θ_q (° C.).	Impressed velocity corresponding to Q. (cm. per sec.).
0 9	179	7	215	3·7
1·0	233	9	278	4·0
1·1	281	11	352	4·5
1·2	331	13	439	5·0
1·3	390	15·5	538	5·6
1·4	470	18·5	633	6·3
1·5	547	22·0	757	6·8

The results obtained in this table and in columns 4 and 5 of Table I. indicate that the velocity V_p corresponding to the point P, for which the deflexion is independent of the inclination of the wire, is related to the corresponding temperatures θ_p, θ_h, and θ_v of the wire by the empirical relations,

$$V_p = 0{\cdot}040\,\theta_p = 0{\cdot}029\,\theta_h = 0{\cdot}026\,\theta_v.$$

The corresponding empirical relation in the case of the points represented by Q is

$$V_q = 0{\cdot}0060\,\theta_q + 2{\cdot}4 = 0{\cdot}0057\,\theta_h + 2{\cdot}4$$
$$= 0{\cdot}0052\,\theta_v + 2{\cdot}4.$$

Finally, in fig. 6 are given the forms of the calibration curves obtained employing a heating current of 1·3 amp. in the bridge for various inclinations (specified in the diagram) of the wire to the horizontal. The bridge was in every case initially balanced in the absence of an impresed flow and the galvanometer shunt adjusted throughout to 4 ohms. The diminished influence of the free convection current upon the form of the calibration curve as the wire is more inclined to the horizontal is clearly brought out by the curves, the initial steep portion of the curves becoming continually less pronounced as the inclination of the wire to the horizontal is increased. Excluding these initial portions, the curves are seen to be practically parallel to one another.

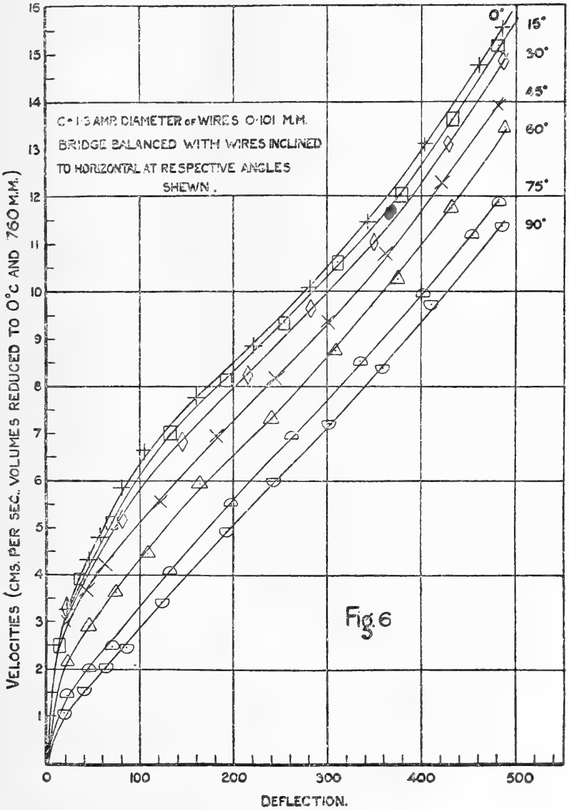

Fig. 6

The research detailed herein was carried out in the Physical Laboratory of the South Metropolitan Gas Co. Mr. W. H. B. Hall assisted in the experimental part of the work.

The author desires to express to Dr. Charles Carpenter, C.B.E., M.I.C.E , his sincerest thanks for his unfailing readiness to provide all facilities for carrying out the work, and for his inspiring interest in the investigation.

Summary.

A form of hot-wire anemometer is described in which the heating current is maintained constant, and the cooling effect experienced by the exposed wire due to a small impressed velocity of the gas stream, compensated by alteration of the inclination of the wire from its initial horizontal position. The limits of application of such a device are shown by a series of calibration curves, in which the necessary alteration is plotted against the value of the impressed velocity, for values of the heating current ranging from 0·9 to 1·5 amp. The results are employed to deduce the value of the effective free convection current corresponding to any given inclination of the wire when heated by the respective electric currents. The alteration in the effective velocity of the free convection current accompanying alteration in the inclination of the wire is utilized to determine the inferior limit of the value of the impressed velocity of the stream for which the effect of the free convection current may be safely neglected. Empirical linear formulæ are obtained relating such limiting velocities to the temperature of the wire. The effect of the free convection current upon the form of the calibration curves is illustrated by reference to the forms of the curves obtained employing two values of the heating current for both the horizontal and vertical dispositions of the wire, and for various inclinations of the wire employing a constant heating current.

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17 Jan., 1921.

LXV. *Scattering and Absorption of Hard X-Rays in the Lightest Elements.*

By TYCHO E:SON AURÉN, *Dr. phil.*[*]

BY the aid of a compensation method in adjusting the thickness of a water layer so that its absorption for a pencil of X-rays is equal to that of a liquid layer of the thickness of 1 cm. or of a sheet of metal of known thickness, I have made determinations of the relative atomic absorption coefficients of a number of elements, and I have given an account of the results of these researches in a previous paper[†]. Barkla and other authors have shown that, for many elements and at different wave-lengths, the relation of the absorption coefficients is independent of the wave-length. In calculating the coefficients, I have assumed that such was the case for all elements. As the influence of the effect of scattering in the lightest elements is so great compared with true absorption that it cannot be neglected, it was supposed that the deviations from the constant relation of the absorption coefficients in different wave-lengths was entirely due to this effect. By means of the determination of the absorption in H, which may be considered, with no appreciable error, to be exclusively dependent on scattering, a measure of the scattering effect caused by one electron was obtained, and aided by this measure I then tried to state the number of electrons, *i. e.* outer electrons, which in different elements produce scattering.

As far as I know, there has not yet been performed any investigation into the question whether the relation of the values of absorption coefficients of the lightest elements corrected for scattering is constant, which lacuna has certainly been left owing to the difficulty there is in determining accurately the extent of the scattering effect. My estimation just mentioned of the number of the outer electrons being essentially dependent on the determination of absorption in the lightest elements, I have found it necessary, in recent experiments, to try to determine more accurate values of the magnitude of absorption in these elements. Particularly it has appeared of importance to me to bring about more accurate measures regarding absorption of hydrogen. The method of research has been identical with that used earlier, with the exception of using a Coolidge bulb and considerably shorter wave-lengths than before.

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[*] From the *Meddelanden från K. Vetenskapsakademiens Nobelinstitut*, Bd. iv. No. 5. Communicated by the Author.

[†] *Medd. fr. K. Vet.-Akad. Nobelinstitut*, Bd. iv. No. 3 (1919), and Phil. Mag. xxxvii. p. 165 (1919).

The tension, applied at the bulb from a high-tension transformer, could be read off on a kilowattmeter attached to the primary circuit of the transformer, and could also be accurately regulated. In the course of my experiments tension has been continuously observed, and a continuous regulation has been made. The variations in the tension have only exceptionally reached 0·5 kv. The composition of radiation has been varied not only by variation of tension, but also by aluminium filters of the thicknesses of 0·5–3·6 cm. For each determination, as a rule, five different measurements have been made, which have very nearly agreed with one another, and have seldom diverged more than 0·5 per cent. from the mean. Repeated determinations of the absorption of the same substance have shown that the agreement of the values found is always remarkably close.

In Table I. there are noted down the tensions used in the different compositions of radiation, the thickness of the respective aluminium filters, the values observed in these radiations for the atomic absorption coefficient of Cu relative to water ($\kappa_{\mathrm{Cu/H_2O}}$), the corresponding values of the mass absorption coefficient of Cu $\left(\frac{\mu}{\rho}\mathrm{Cu}\right)$ and the effective wavelength (λ_e) in Ångström units, calculated by the aid of the table of Barkla and White*.

Table I.

Notation.	Approximate tension in kv.	The thickness of aluminium filter in cm.	$\kappa_{\mathrm{Cu/H_2O}}$.	$\frac{\mu}{\rho}$ Cu.	$\lambda_e \,.\, 10^8$ cm.
I.	70	0·5	65·4	6·30	0·359
II.	75	1·0	49·4	3·64	0·302
III.	80	1·5	40·3	2·62	0·264
IV.	85	2·0	33·8	2·02	0·237
V.	95	2·5	28·6	1·60	0·215
VI.	100	3·6	23·4	1·28	0·194

In the determination of the atomic absorption coefficient of H relative to water in my experiments published previously, I have started from the point of view that the law of additivity is strictly valid, and, consequently, that this

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* Barkla and White, Phil. Mag. xxxiv. p. 272 (1917).

coefficient can be calculated, if the corresponding coefficient of oxygen relative to water is known. As for the validity of the law of additivity, I have not, in a considerable number of my experiments in which widely different substances have been investigated, found anything in any case indicating that the law should not be strictly in force. On the other hand, it must be said that the determination of κ_{O/H_2O} is liable to an error of great importance in the calculation of κ_{H/H_2O}. This error is owing to the fact that this coefficient could only have been calculated from the difference of the molecular absorption coefficients of substances, which, in regard to their chemical composition, differ from each other by one atom of oxygen, because the experimental errors of both coefficients appear in this difference. Since the value of κ_{H/H_2O} is calculated by the subtraction of the coefficient of κ_{O/H_2O} from κ_{H_2O/H_2O} or 1, its value will be liable to error, because of the inaccuracy of κ_{O/H_2O}, and also because κ_{H/H_2O} is in itself a small quantity compared to κ_{O/H_2O}. However, I have previously shown that absorption can be determined with great accuracy in organic compounds, and it may, therefore, be of more advantage to calculate the absorption of hydrogen directly from these determinations. So I have made a new series of absorption determinations in various organic substances at the effective wave-lengths recorded in Table I. In these researches, only pure preparations free from water have been used. In Table II. are recorded absorption coefficients (κ_{A/H_2O}),calculated by the formula *

$$\kappa_{A/H_2O}=\frac{55{\cdot}5d}{m}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

where d stands for thickness in cm. of a water layer of an absorption power equal to a layer 1 cm. thick of the substance to be examined, and m means the number of gram-molecules per litre of the same substance. It appears from Table II. that a variation of wave-length has very little influence on the determination of κ_{A/H_2O}. The circumstance, therefore, that radiation has not been quite homogeneous cannot have been of any essential significance in these measurements. Certainly the fact that the compositions of both ray pencils after passing through the liquid layers have somewhat differed, cannot have exercised any noteworthy influence on the results. Many experiments have shown the determination of κ_{a/H_2O} for a certain metal to be independent of the composition of the solution, and a very close agreement has been obtained

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* *Loc. cit.* p. 170.

when examining a water solution of one salt of the metal in question and from the direct examination of the solid metal.

In the following table are noted the observed values of κ_{A/H_2O} obtained, when examining a number of organic substances. In each column is shown the observed value for the effective wave-length stated in Table I., and beneath the value calculated by means of the values for κ_{H/H_2O}, κ_{C/H_2O}, and κ_{O/H_2O} derived from the observations (Table III.). Moreover the table includes the observed values for κ_{A/H_2O} for N_2H_4 in waterfree solution (produced out of hydrazine hydrate from Kahlbaum).

Table II.—Molecular absorption coefficients in relation to molecular absorption coefficients of water.

Substance.		I.	II.	III.	IV.	V.	VI.
Methyl formate $C_2H_4O_2$ ...	obs.	3·01	3·06	3·08	3·12	3·14	3·14
	cal.	3·03	3·06	3·09	3·12	3·14	3·15
Paraldehyde $C_6H_{12}O_3$	obs.	6·42	6·67	6·70	6·83	6·86	7·04
	cal.	6·46	6·68	6·81	6·92	7·01	7·04
Propionic acid $C_3H_6O_2$......	obs.	3·64	3·76	3·82	3·86	3·90	3·92
	cal.	3·67	3·76	3·82	3·87	3·91	3·92
Allyl alcohol C_3H_6O	obs.	2·76	2·90	2·98	3·09	3·15	3·16
	cal.	2·79	2·92	2·99	3·05	3·10	3·12
Dimethyl carbonate $C_3H_6O_3$	obs.	4·52	4·61	4·67	4·67	4·73	4·74
	cal.	4·54	4·60	4·64	4·68	4·72	4·73
Ethyl acetate $C_4H_8O_2$	obs.	4·25	4·41	4·51	4·60	4·64	4·66
	cal.	4·31	4·45	4·54	4·61	4·67	4·69
Methylethyl keton C_4H_8O...	obs.	3·44	3·60	3·71	3·78	3·84	3·87
	cal.	3·43	3·61	3·72	3·80	3·86	3·89
Benzyl alcohol C_7H_8O	obs.	4·95	5·24	5·42	5·52	5·57	5·64
	cal.	4·95	5·21	5·36	5·48	5·58	5·61
Toluene C_7H_8	obs.	4·10	4·40	4·60	4·72	4·83	4·87
	cal.	4·07	4·37	4·53	4·66	4·77	4·81
Octyl alcohol $C_8H_{18}O$	obs.	6·13	6·62	6·84	6·96	7·07	7·17
	cal.	6·13	6·55	6·78	6·97	7·11	7·17
Octane C_8H_{18}..................	obs.	5·27	5·76	6·03	6·12	6·29	6·38
	cal.	5·26	5·71	5·96	6·15	6·30	6·36
Benzene C_6H_6	obs.	3·45	3·75	3·84	3·94	4·00	4·04
	cal.	3·43	3·68	3·81	3·92	4·01	4·04
Hydrazine N_2H_4	obs.	1·598	1·675	1·700	1·738	1·752	1·762

When the molecular absorption coefficient for H_2O is taken equal to one, we obtain, from the above organic compounds, values of absorption coefficients of C and H in various compounds in different proportions. If these values are treated by the least square method, we obtain the values, recorded in Table III., of the absorption coefficients of C and H in relation to water (κ_{H/H_2O}, κ_{C/H_2O}), and by means of the value κ_{H/H_2O}, we obtain the corresponding value of O (κ_{O/H_2O}), when, as stated above, the molecular absorption coefficient of H_2O is taken equal to one. It is from the observed values of κ_{A/H_2O} for N_2H_4 that the relative atomic absorption coefficient of N (κ_{N/H_2O}) has then been calculated.

TABLE III.—The atomic absorption coefficients of H, C, N, and O in relation to the molecular absorption coefficient of H_2O.

	I.	II.	III.	IV.	V.	VI.
κ_{H/H_2O}	0·068	0·081	0·088	0·093	0·096	0·098
κ_{C/H_2O}	0·504	0·532	0·547	0·560	0·572	0·575
κ_{N/H_2O}	0·663	0·675	0·674	0·683	0·684	0·685
κ_{O/H_2O}	0·874	0·838	0·824	0·814	0·808	0·804

By the method used before, *i. e.* by the aid of the difference between the molecular absorption coefficients of pairs of compounds, which in chemical respect differ only by one or more atoms of oxygen, the following mean values for κ_{O/H_2O} will be obtained :—

	I.	II.	III.	IV.	V.	VI.
κ_{O/H_2O}...	0·86	0·85	0·83	0·81	0·79	0·78

These values agree very closely with those recorded in Table III., yet they must from the reason stated above be considered less certain than those last mentioned. If not only κ_{H/H_2O} and κ_{C/H_2O} but κ_{O/H_2O} by the aid of the least square method is directly calculated from the values of κ_{A/H_2O} of the examined compounds of H, C, and O (Table II.), values are obtained, which inconsiderably differ from those given in Table III. The probable errors of the coefficients found calculated by the said method reach, for κ_{H/H_2O} a mean of 4 per cent., and for κ_{C/H_2O} and κ_{O/H_2O} 1 per cent.

The atomic absorption coefficient of a certain element can be expressed by the formula

$$\kappa_a = k\,\lambda^b + \sigma_a, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

where b is a constant, which has closely the same value for all elements ; k_a and σ_a are constants, which are characteristic for the element in question. The latter term (σ_a) due to the scattering effect is indicated as the atomic scattering-coefficient. The atomic absorption coefficient of a certain element (a_1) in relation to another element (a_2) will then be expressed thus:

$$\kappa_{a_1/a_2} = \frac{k_{a_1}\lambda^b + \sigma_{a_1}}{k_{a_2}\lambda^b + \sigma_{a_2}} . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

At a decreasing wave-length, the term $k_a\lambda^b$ will become still smaller and smaller in relation to σ_a, and, in the lightest elements at very short wave-lengths, will be of no significance as compared with the second term of equation (2). The limit of κ_{a_1/a_2} at a decreasing wave-length must therefore be:

$$\kappa_{a_1/a_2} = \frac{\sigma_{a_1}}{\sigma_{a_2}} . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (4)$$

From the experiments, we see also that κ_{a/H_2O} for H, C, N, and O at a decreasing wave-length seems to approach fixed limits. These limits (see column VI., Table III.) stand very nearly to each other in the ratio of the numbers 1 : 6 : 7 : 8, and these numbers are the same as the atomic numbers of the respective elements. This issue is in good agreement with the theory advocated by J. J. Thomson, according to which the scattering effect is proportional to the number of electrons included in the atom. This gives a support to the current notion of the structure of atoms, which is that the number of electrons corresponds to the atomic number. The supposition I have earlier tentatively suggested that the scattering effect is produced solely by the outer electrons cannot, therefore, be maintained, at least not for the range of wave-lengths made use of here. It then follows that the mass scattering coefficient, except in some of the lightest elements, must be notably higher than I previously thought to be the case. In accordance with what Barkla and Dunlop * have shown, we see that the relation between the mass scattering coefficients for Cu and Al at short wave-lengths is fairly equal to one, whence it follows that the relation between the scattering coefficients of these elements also becomes very nearly the

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* Barkla and Dunlop, Phil. Mag. xxxi. p. 229 (1916).

same as the relation between the atomic numbers. Concerning all elements as far as Cu inclusively, and within the range of wave-lengths I have investigated, it is in all probability true that the scattering effect is proportional to the atomic number, and it is likely to be true also as regards several elements immediately following Cu. On the contrary, it is not true concerning the heaviest elements as it appears from the determinations, made by the above mentioned writers, as to the relation between the mass scattering coefficients of Pb and Al.

In the case of hard X-rays, it can be said with certainty that the true absorption ($\kappa_a \lambda^b$) due to H may be regarded as infinitely small as compared with the scattering effect. Since, in conformity to the current notion, there is only one electron combined with the nucleus of an atom of hydrogen, then the absorption observed in hydrogen must be equal to the scattering effect due to one electron. The mass scattering coefficient per electron can easily be calculated, when we know the mass absorption coefficient of Cu $\left(\frac{\mu}{\rho}\mathrm{Cu}\right)$ for the effective wave-lengths used in the experiment. If the values of $\kappa_{\mathrm{Cu/H_2O}}$ recorded in Table I. be corrected by deducting 29 $\kappa_{\mathrm{H/H_2O}}$ from them, we obtain the values 63·4, 47·0, 37·7, 31·0, 25·8, 20·6 for the respective wave-lengths. By means of these values we calculate from the values of $\kappa_{\mathrm{H/H_2O}}$ the scattering coefficient per electron in relation to the true atomic absorption coefficient of Cu (q). In Table IV. we again find the values of q corresponding to values of $\frac{\mu}{\rho}\mathrm{Cu}$.

TABLE IV.

	I.	II.	III.	IV.	V.	VI.	Mean.
q	0·00107	0·00172	0·00233	0·00300	0·00372	0·00476	—
$\frac{\mu}{\rho}$Cu ...	6·30	3·64	2·62	2·02	1·60	1·28	—
$q\frac{\mu}{\rho}$ Cu ...	[0·00674]	0·00626	0·00610	0·00606	0·00595	0·00610	0·00609

The radiation of the greatest wave-length (I.) is probably less homogeneous than the other radiations, which had to pass through thicker filters. With the exception of the values of $q\frac{\mu}{\rho}\mathrm{Cu}$ for effective wave-length (I.), it appears from the table that q is inversely proportional to $\frac{\mu}{\rho}\mathrm{Cu}$, but it may be that this relation does not hold good for other ranges of wave-lengths and is accidental in the present case. However, in these experiments, the relations may be used to find the mass scattering coefficient of a certain element. Taking M_{Cu} and M_a to denote the atomic weights of Cu and the element (a) at issue and Z_a the atomic number of the element, then the following formula for the mass coefficient $\left(\frac{s}{\rho}\right)$ holds good:

$$\frac{s}{\rho}a = \frac{M_{Cu} \cdot Z_a \cdot q}{M_a\,(1+29\,q)} \cdot \frac{\mu}{\rho}\text{Cu}. \quad . \quad . \quad . \quad . \quad (5)$$

As $M_{Cu}=63{\cdot}6$ and $q\frac{\mu}{\rho}\text{Cu}=0{\cdot}00609$ (Table IV.), $\frac{s}{\rho}$ at a greater wave-length, when $29\,q$ may be neglected, can be expressed thus:

$$\frac{s}{\rho}a = \frac{Z_a}{M_a} \cdot 0{\cdot}387. \quad . \quad . \quad . \quad . \quad . \quad . \quad (6)$$

As for H. the value 0·387 is obtained, which is somewhat lower than what I found before (0·445), owing to the fact that absorption of hydrogen could not, with sufficient accuracy, be ascertained. The value I have found now, however, agrees very closely with what has been calculated according to Thomson's theory and what Barkla has experimentally found. For Al formula (6) gives the value 0·187 instead of the known value 0·2. We infer from formula (5), however, that $\frac{s}{\rho}$ decreases with the wave-length, because q is continually increasing (*cfr.* Table IV.). At $\lambda = 0{\cdot}166$ Å., $\frac{s}{\rho}$ therefore attains the value of 0·157, which is in good conformity with the fact that $\frac{s}{\rho}$ cannot have the same value at all wave-lengths. Already at $\lambda=0{\cdot}16$ Å, the mass absorption coefficient of Al is 0·18, and the mass scattering coefficient, being part of the whole absorption coefficient, cannot be greater than the whole. Richtmeyer and Grant * have recently published the results of experiments of absorption in about the same wave-length range and, as an

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* Phys. Rev. xv. p. 547 (1920).

average for $\frac{s}{\rho}$ of Al, have found 0·15, which is in close agreement with the value above.

According to the atom model of Rutherford-Bohr the electrons are imaged as discrete particles, which are moving in rings around the nucleus. In the model, recently worked out by Lewis and Langmuir, the electrons, however, are thought to form coaxial rings of electricity, "ring-electrons." For the purpose of obtaining material to decide between the two assumptions, Schott* has worked out the theory of scattering on the basis of the first theory. As a matter of fact the paper gives the results for both assumptions, for the ring-electron is nearly a limiting case of discrete electrons. Yet the paper shows that the mass scattering coefficient of hydrogen $\left(\frac{s}{\rho}\mathrm{H}\right)$ reaches the value of 0·398, and is quite independent of the wave-length. As is shown in formula (6), it follows from my experiments that $\frac{s}{\rho}\mathrm{H}$ for greater wave-lengths has a value 0·387, which thus is in good agreement with the said value and the value earlier calculated by Thomson. The mass scattering coefficient nevertheless is not independent of the wave-length but is decreasing with increasing wave-length (formula 5), which does not agree with the said result of the investigation of Schott.

In two papers on "The Size and Shape of the Electron"†, Compton, on the basis of the atom model of Lewis and Langmuir, has made a theoretical investigation for finding how the scattering effect is dependent on the wave-length. If the electron is thus taken as a flexible ring of electricity, Compton finds that the scattering coefficient will be expressed by the formula:

$$\sigma=\sigma_0\left(1-29{\cdot}61\left(\frac{a}{\lambda}\right)^2+524{\cdot}2\left(\frac{a}{\lambda}\right)^4-5398\left(\frac{a}{\lambda}\right)^6+\ldots\right) \quad (7)$$

where σ_0 stands for the value of scattering coefficient at great wave-lengths, and a means the radius of the electron as calculated by Compton and found to be $(1{\cdot}85\pm0{\cdot}05).10^{-10}$ cm. In the following table, for the effective wave-lengths (λ_e) used by me, the values for $\frac{\sigma}{\sigma_0}$ derived from my experiments $(\sigma_0=0{\cdot}387)$ have been combined with those calculated by

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* Schott, Proc. Roy. Soc. xcvi. p. 410 (1920).

† Compton, Phys. Rev. xiv. p. 20 (1919).

formula (7) and by the empirical formula

$$\frac{\sigma}{\sigma_0}=\left(1-\frac{0\cdot00445}{\lambda^2}\right), \quad . \quad . \quad . \quad . \quad . \quad . \quad (8)$$

where λ is expressed in Ångström's units.

Table V.

Notation.	λ_e.	$\frac{\sigma}{\sigma_0}$ observed.	$\frac{\sigma}{\sigma_0}$ calculated by form. 7;	by form. 8.
I.	0·359	0·969	0·925	0·966
II.	0·302	0·953	0·896	0·951
III.	0·264	0·935	0·867	0·936
IV.	0·237	0·920	0·838	0·921
V.	0·215	0·902	0·807	0·904
VI.	0·194	0·879	0·770	0·882

It will be seen from the table that the empirical formula (8) agrees very closely with the experimental values of $\frac{\sigma}{\sigma_0}$, whereas formula (7) gives values which decrease with decreasing wave-lengths more rapidly than is shown by the experiments. By giving the constants in formula (7) another value, a closer agreement with the experimental results may, of course, be easily obtained. Without discussing the theoretical assumptions which form the basis for formula (7), I may state that it is evident that the relation between the scattering coefficient and the wave-length can be expressed by a function of that form.

In order to be able to appreciate the magnitude of the true absorption ($k_a\lambda^b$), the values found of κ_{a/H_2O} for C, N, and O have been corrected for scattering by deducting from these values the products of 6, 7, and 8 times κ_{H/H_2O} respectively (Table VI.). The remainders (Δ) thus found give a measure of the true absorption at the wave-lengths used.

It appears from the table that at the greater wave-lengths, (I., II., & III.), the atomic absorption coefficients of C and N are in very close proximity to, respectively, 6 and 7 times κ_{H/H_2O}. In the case of the lightest elements, as far as N inclusively, it seems that absorption can chiefly be considered as a scattering effect, whereas true absorption seems to be of very little account. The difference between κ_{O/H_2O} and

TABLE VI.

	I.	II.	III.	IV.	V.	VI.
κ_{C/H_2O} …………	0·504	0·532	0·547	0·560	0·572	0·575
$6\kappa_{H/H_2O}$ ………	0·408	0·486	0·528	0·558	0·576	0·588
Δ ……………	0·096	0·046	0·019	0·002	−0·004	−0·013
κ_{N/H_2O} …………	0·663	0·675	0·674	0·683	0·684	0·685
$7\kappa_{H/H_2O}$ …………	0·476	0·567	0·616	0·651	0·672	0·686
Δ ……………	0·187	0·108	0·058	0·032	0·012	−0·001
κ_{O/H_2O} …………	0·874	0·838	0·824	0·814	0·808	0·804
$8\kappa_{H/H_2O}$ …………	0·544	0·648	0·704	0·744	0·768	0·784
Δ ……………	0·330	0·190	0·120	0·070	0·040	0·020

$8\ \kappa_{H/H_2O}$ is considerably greater, which shows that true absorption is proportionally greater in O than in lighter elements. If we compare the values of κ_{a/H_2O} for the three elements with one another, we find that while the values for C and N are continually increasing towards a maximum, they are, on the contrary, decreasing for O and approaching to a minimum. Indeed, according to formula 3 this must be the case, if the term $k_a\lambda^b$ is proportionally much greater in the case of O than for the other two elements. As I have previously found, it also results from these experiments that true absorption is increasing much more rapidly at O than with the preceding elements. I have assumed, as a possible explanation of this increase of absorption, the suggestion that there is a re-grouping taking place in the electrons, in so far as four electrons are concentrating nearer to the nucleus and are forming an inner region, whereas the remaining four electrons form a group of outer electrons (valence electrons). The suggestion that the number of the outer electrons is likely to be four at O is supported by the fact that of all chemical compounds hitherto known of which this element is a constituent, oxygen does not appear to be capable of a greater number of valences than four.

Nobel Institution for Physical Chemistry.
Stockholm, February 1920.

LXVI. *On the Correction for Shear of the Differential Equation for Transverse Vibrations of Prismatic Bars.* *By* Prof. S. P. TIMOSHENKO *.

IN studying the transverse vibrations of prismatic bars, we usually start from the differential equation

$$\mathrm{EI}\frac{\partial^4 y}{\partial x^4}+\frac{\rho\Omega}{g}\frac{\partial^2 y}{\partial t^2}=0, \quad . \quad . \quad . \quad . \quad . \quad (1)$$

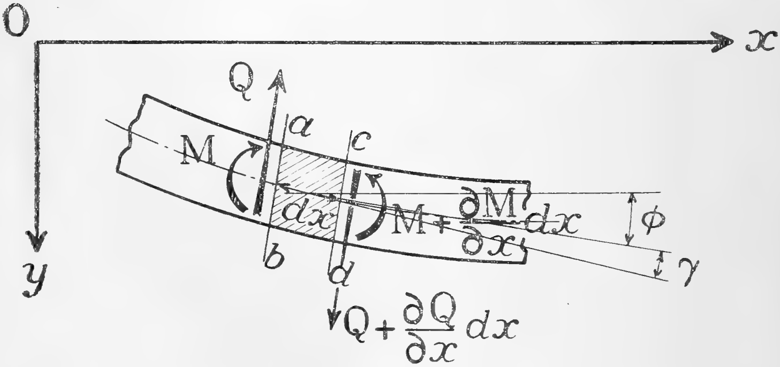
in which EI denotes the flexural rigidity of the bar,
Ω the area of the cross-section,
and $\frac{\rho}{g}$ the density of the material.

When the "rotatory inertia" is taken into consideration, the equation takes the form

$$\mathrm{EI}\frac{\partial^4 y}{\partial x^4}-\frac{\mathrm{I}\rho}{g}\frac{\partial^4 y}{\partial x^2 \partial t^2}+\frac{\rho\Omega}{g}\frac{\partial^2 y}{\partial t^2}=0. \quad . \quad . \quad . \quad (2)$$

I now propose to show how the effect of the shear may be taken into account in investigating transverse vibrations, and I shall deduce the general equation of vibration, from which equations (1) and (2) may be obtained as special cases.

Fig. 1.



Let $a\ b\ c\ d$ (fig. 1) be an element bounded by two adjacent cross-sections of a prismatic bar. M and Q denote respectively the bending moment and the shearing force.

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* Communicated by Mr. R. V. Southwell, M.A. Translated from the Russian by Prof. M. G. Yatsevitch.

The position of the element during vibration will be determined by the displacement of its centre of gravity and by the angular rotation ϕ in the (x, y) plane: the axis Ox may be taken as coinciding with the initial position of the axis of the bar.

The angle at which the tangent to the curve into which the axis of the bar is bent (the curve of deflexion) is inclined to the axis Ox will differ from the angle ϕ by the angle of shear γ. Hence, for very small deflexions, we may write

$$\frac{\partial y}{\partial x}=\phi+\gamma. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

For determining M and Q we have the familiar expressions

$$\mathrm{M}=-\mathrm{EI}\frac{\partial\phi}{\partial x}, \quad \mathrm{Q}=\lambda\mathrm{C}\Omega\gamma=\lambda\mathrm{C}\Omega\left(\frac{\partial y}{\partial x}-\phi\right), \quad . \quad (4)$$

where C denotes the modulus of rigidity, for the material of the bar, and λ is a constant which depends upon the shape of the cross-section.

The equations of motion will now be:—

for the rotation—

$$-\frac{\partial\mathrm{M}}{\partial x}dx+\mathrm{Q}dx=\frac{\rho\mathrm{I}}{g}\frac{\partial^2\phi}{\partial t^2}dx,$$

or
$$\mathrm{EI}\frac{\partial^2\phi}{\partial x^2}+\lambda\mathrm{C}\Omega\left(\frac{\partial y}{\partial x}-\phi\right)-\frac{\rho\mathrm{I}}{g}\frac{\partial^2\phi}{\partial t^2}=0, \quad . \quad . \quad . \quad (5)$$

if we substitute from equations (4);

for translation in the direction of Oy—

$$\frac{\partial\mathrm{Q}}{\partial x}dx=\frac{\rho\Omega}{g}\frac{\partial^2 y}{\partial t^2}dx,$$

or
$$\frac{\rho\Omega}{g}\frac{\partial^2 y}{\partial t^2}-\lambda\mathrm{C}\Omega\left(\frac{\partial^2 y}{\partial x^2}-\frac{\partial\phi}{\partial x}\right)=0. \quad . \quad . \quad . \quad . \quad (6)$$

Eliminating ϕ from (5) and (6), we obtain the required equation in the form

$$\mathrm{EI}\frac{\partial^4 y}{\partial x^4}+\frac{\rho\Omega}{g}\frac{\partial^2 y}{\partial t^2}-\frac{\rho\mathrm{I}}{g}\left(1+\frac{\mathrm{E}}{\lambda\mathrm{C}}\right)\frac{\partial^4 y}{\partial x^2\partial t^2}+\frac{\rho^2\mathrm{I}}{g^2\lambda\mathrm{C}}\frac{\partial^4 y}{\partial t^4}=0. \quad (7)$$

Introducing the notation

$$\frac{\mathrm{EI}g}{\rho\Omega}=a^2, \quad \frac{\mathrm{I}}{\Omega}=k^2,$$

we may write equation (7) in the form

$$a^2\frac{\partial^4 y}{\partial x^4}+\frac{\partial^2 y}{\partial t^2}-k^2\left(1+\frac{\mathrm{E}}{\lambda\mathrm{C}}\right)\frac{\partial^4 y}{\partial x^2\partial t^2}+\frac{k^2\rho}{g\lambda\mathrm{C}}\frac{\partial^4 y}{\partial t^4}=0. \quad (8)$$

In order to estimate the influence of the shear upon the frequency of the vibrations, let us consider the case of a prismatic bar with supported ends. The type of the vibrations may be assumed to be given by

$$y = \mathrm{Y} \sin \frac{m\pi x}{l} \cos p_m t, \quad . \quad . \quad . \quad . \quad . \quad (9)$$

where l represents the length of the bar, and p_m is the required frequency. By substitution from (9) in equation (8), we obtain the following equation for the frequency:

$$\alpha^2 \frac{m^4\pi^4}{l^4} - p_m{}^2 - \frac{m^2\pi^2k^2}{l^2}\left(1+\frac{\mathrm{E}}{\lambda\mathrm{C}}\right)p_m{}^2 + \frac{k^2\rho}{g\lambda\mathrm{C}}p_m{}^4 = 0. \quad (10)$$

If only the first two terms on the left side of this equation are retained (this will correspond to the equation (1)), we have

$$p_m = \alpha \frac{m^2\pi^2}{l^2} = \frac{\alpha\pi^2}{\mathrm{L}^2}, \quad . \quad . \quad . \quad . \quad . \quad . \quad (11)$$

where $\mathrm{L} = \frac{l}{m}$ represents the length of a wave.

By retaining the first three terms of equation (10) (*i. e.* by neglecting the terms which involve λ), we find

$$p_m = \frac{\alpha\pi^2}{\mathrm{L}^2}\left(1 - \frac{1}{2}\frac{\pi^2k^2}{\mathrm{L}^2}\right) \quad . \quad . \quad . \quad . \quad . \quad . \quad (12)$$

approximately: this result corresponds to equation (2), where the rotatory inertia is taken into consideration.

By using the complete equation (10), and neglecting small quantities of the second order, we find

$$p_m = \frac{\alpha\pi^2}{\mathrm{L}^2}\left[1 - \frac{1}{2}\frac{\pi^2k^2}{\mathrm{L}^2}\left(1+\frac{\mathrm{E}}{\lambda\mathrm{C}}\right)\right] \quad . \quad . \quad . \quad (13)$$

approximately.

Assuming the values

$$\lambda = \tfrac{2}{3}, \qquad \mathrm{E} = \tfrac{8}{3}\mathrm{C},$$

we have

$$\frac{\mathrm{E}}{\lambda\mathrm{C}} = 4,$$

and hence we see that the correction for shear is four times greater than the correction for rotatory inertia. The value of the correction of course increases with a decrease in the wave-length L, *i. e.*, with an increase in m.

Yougoslavia, Videm.
Summer 1920.

LXVII. *On the Einstein Spectral Line Effect.* *By* H. J. PRIESTLEY, *Professor of Mathematics, University of Queensland* *.

THE prediction of the Einstein displacement of the spectral lines is based on two assumptions, namely :

(*a*) the atom behaves as a natural clock, giving a value of ds which is the same for each vibration ;

(*b*) the time period dt of a vibration is transmitted by the radiation from the source to the observer.

If we assume, as an alternative to (*b*), that the Einstein interval ds is transmitted by the radiation, the effect will arise on the transference, not of the source, but of the observer to a different gravitational field. The object of the present paper is to discuss the reasonableness of rejecting (*b*) and adopting the alternative.

The usual method of discussing the deflexion of a beam of light by a gravitating field, by applying the principle of Least Time to determine the course of a ray propagated in three-dimensional space with velocity $1-2mr^{-1}$, leads one towards the adoption of (*b*). The use of the principle of Least Time appears to imply an underlying constancy of period. Consequently it is advisable to investigate the deflexion by a method which makes no appeal to Pre-Relativity Physics.

The equations obtained from

$$\delta\int ds=0,$$

$$ds^2=-\gamma^{-1}\,dr^2-r^2\,d\phi^2+\gamma\,dt^2,$$

are

$$\left(\frac{dr}{ds}\right)^2+r^2\left(\frac{d\phi}{ds}\right)^2=k^2-1+\frac{2m}{r}+2m\frac{h^2}{r^3}, \quad . \quad . \quad . \quad (1)$$

$$r^2\frac{d\phi}{ds}=h, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

$$\gamma\frac{dt}{ds}=k, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

where h and k are constant.

Elimination of s from (1) and (2) by means of (3) leads to

$$\left(\frac{dr}{dt}\right)^2+r^2\left(\frac{d\phi}{dt}\right)^2=\gamma^2\left[1-k^{-2}+2mk^{-2}/r+2mh^2k^{-2}/r^3\right], \quad (4)$$

$$r^2\frac{d\phi}{dt}=\gamma h/k. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (5)$$

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* Communicated by the Author.

If the last term in (4) be transformed by means of (5), the equation (4) becomes

$$\left(\frac{dr}{dt}\right)^2+\gamma r^2\left(\frac{d\phi}{dt}\right)^2=\gamma^2[1-k^{-2}\gamma^{-1}]. \quad . \quad . \quad (6)$$

Further, by the Principle of Equivalence,

$$\left(\frac{dr}{dt}\right)^2+\gamma r^2\left(\frac{d\phi}{dt}\right)^2=\gamma^2 \quad . \quad . \quad . \quad . \quad . \quad (7)$$

along the ray.

It follows from (6) and (7) that the constant k must be infinite for a light pulse. In that case h is also infinite and (5) can be written

$$r^2\frac{d\phi}{dt}=a\gamma, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (8)$$

where a is the ratio of the two infinite constants.

Elimination of t from (7) and (8) leads to

$$\frac{du^2}{d\phi}+\gamma u^2=a^{-2},$$

where $u=r^{-1}$,

and, on differentiation, this equation becomes

$$\frac{d^2u}{d\phi^2}+u=3mu^2. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (9)$$

Since $m=1{\cdot}47$ km. and u cannot be greater than $(697000)^{-1}$, the term on the right-hand side of (9) is small compared with u. Hence (9) can be solved by approximate methods.

If $u=A\cos\phi$ is a first approximation, the second is given by

$$u=A\cos\phi+\tfrac{3}{2}mA^2[1-\tfrac{1}{3}\cos 2\phi]. \quad . \quad . \quad (10)$$

The directions of the asymptotes are given by $u=0$, or

$$2mA^2\cos^2\phi-2A\cos\phi-4mA^2=0.$$

If second powers of mA be neglected, the solution of this equation is

$$\cos\phi=-2mA,$$

$$\phi=\pm\left[\frac{\pi}{2}+2mA\right].$$

Hence the angle between the asymptotes of the light path is approximately $4mA$. It follows from (10) that $A=R^{-1}$, where R is the distance of perihelion.

The deflexion of the ray is therefore $4m/R$, the value verified by the Solar Eclipse Expeditions.

Since the deflexion can be thus established without any appeal to the Principle of Least Time or any other ideas of pre-relativity Physics, there is no *à priori* reason for adopting assumption (*b*) above.

The following argument suggests that the proposed alternative is more in accordance with the ideas of Relativity.

Consider two light pulses leaving A at times t_A, t_A+dt_A and arriving at B at times t_B and t_B+dt_B.

Since, by the Principle of Equivalence, $ds=0$ along the world line of each pulse, it seems that the Einstein interval ds_A between the two departures from A is equal to the interval ds_B between the two arrivals at B. That is, the interval ds is transmitted by the radiation.

University of Queensland,
Brisbane.

LXVIII. *The Degradation of Gamma-Ray Energy.* By Arthur H. Compton, *Ph.D., Wayman Crow Professor of Physics, Washington University*[*].

It has long been known that when matter is traversed by gamma rays, it becomes a source of secondary[†] gamma radiation. The relation between the primary and the secondary gamma rays, however, has not been definitely established. Although the secondary radiation is very appreciably less penetrating than the primary rays, it has usually been considered to be due principally to true scattering[‡]. It is the purpose of the present paper to investigate the nature and the general characteristics of secondary gamma rays, and to study the mechanism whereby

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[*] Communicated by Prof. Sir E. Rutherford, F.R.S.

[†] In this paper the term "secondary" gamma radiation is used to denote any radiation of the gamma type excited either directly or indirectly by the passage through matter of primary gamma rays. By "scattered" radiation is meant the radiation emitted by the electrons in matter (that due to the positive nuclei is theoretically negligible in comparison) due to the accelerations to which they are directly subjected by the primary rays. The term "fluorescent" radiation signifies as usual radiation of the energy absorbed from the primary beam and stored temporarily in the kinetic and potential energies of the electrons. Its frequency therefore depends jointly upon the frequency of the primary rays and the nature of the radiator.

[‡] *Cf. e. g.* E. Rutherford, 'Radioactive Substances, etc.,' p. 282. J. A. Gray, Phil. Mag. xxvi. p. 611 (1913). D. C. H. Florance, Phil. Mag. xxvii. p. 225 (1914). K. W. F. Kohlrausch, *Phys. Zeitschr.* xxi. p. 193 (1920).

comparatively soft secondary radiation is excited by relatively hard primary radiation.

From theoretical considerations, both scattered and fluorescent radiation should undoubtedly be present in secondary gamma rays. According to J. J. Thomson's well-known theory [*], when the wave-length is so short that there is no appreciable co-operation in the scattering by the different electrons, the mass scattering coefficient should be about 0·2 for all elements and all wave-lengths (if the number of electrons per atom effective in the scattering is equal to the atomic number, as seems to be the case for hard X-rays). The magnitude of the scattering to be expected is considerably reduced if the wave-length approaches the size of the electron, and may, indeed, become very small if the ratio of the wave-length to the diameter of the electron approaches unity [†]. There is, however, on the basis of the classical electrodynamics, no means of eliminating completely the scattered radiation.

Fluorescent radiations of a comparatively soft type, presumably the characteristic K radiations, have been detected in the secondary gamma rays from elements of high atomic weights [‡]. But, in addition to this, there should be excited in all elements a harder fluorescent radiation by the impact of the high-speed beta particles liberated by the primary rays. The number of such electrons expelled by gamma rays is known to be much the same per unit mass for all elements, and it has been shown experimentally [§] that gamma rays in not greatly different amounts per unit mass are excited when beta rays fall upon different substances. Thus one would expect to find in the secondary gamma rays an appreciable amount of fluorescent radiation, which, like the scattered radiation, does not differ greatly according to the element used as radiator.

The usual method of distinguishing between scattered and fluorescent radiation is by comparing the absorption coefficients of the primary and secondary radiations. It is assumed that the scattered rays are of the same hardness as the primary rays, whereas all known high-frequency fluorescent radiations are of a less penetrating type. Gray [‡] and Florance [§], however, have rejected this criterion, for although they find that the secondary radiation excited by hard gamma rays is of a distinctly softer type than the

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[*] J. J. Thomson, 'Conduction of Electricity through Gases,' 2nd ed. p. 325.

[†] A. H. Compton, Phys. Rev. xiv. p. 23 (1919).

[‡] J. A. Gray, *loc. cit.*

[§] D. C. H. Florance, *loc. cit.*

primary radiation, they conclude that the primary rays are truly scattered, but in the process of scattering are so modified as to become less penetrating. It is therefore important to determine under what circumstances, if any, the hardness of the scattered rays may differ from that of the primary rays.

If the scattering is due to electrons of negligible dimensions which are separated far enough to act independently of each other, there is no question but that the scattered ray will be exactly similar to the primary ray in every respect except intensity ; for since the accelerations to which each electron is subject are strictly proportional to the electric intensity of the primary wave which traverses it, and since the electric intensity of the scattered ray (at a great distance) due to each electron is proportional to its acceleration, the electric vector of the scattered wave is strictly proportional to the electric vector of the primary wave. Thus the frequency, the wave-form, the damping, etc., will be the same in both beams. Radiations scattered by such electrons should, therefore, be identical in character with the primary waves.

Whatever type of scattering unit be assumed, it is also clear that, if the primary wave is perfectly homogeneous—*i. e.*, if it is an indefinitely long train of simple harmonic waves of constant frequency,—the scattered waves must also be homogeneous and of the same frequency. If, however, the scattering unit—whether a group of electrons or the individual electron—is of dimensions comparable with the wave-length of the incident radiation, theory demands that the scattering, especially at large angles, shall be less for short than for longer waves. This prediction is confirmed by measurements of the scattering of X-rays and gamma rays over a wide range of frequencies. If the primary beam consists of very short, highly-damped pulses, or of waves of some irregular form, it may, of course, be considered as the Fourier integral of a large number of long trains of waves of different wave-lengths. Thus, unless the primary beam consists of long trains of monochromatic waves, the scattered radiation will, in general, be softer than the primary rays, and the hardness will be greater at small than at large angles with the incident beam. This corresponds qualitatively with the properties of the secondary gamma rays *.

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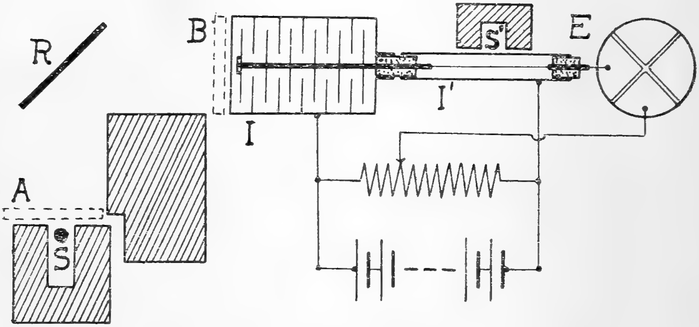
* This explanation of the difference in hardness, as well as an explanation of the distribution of the intensity of the scattered gamma radiation, has been discussed in detail, for the special case of scattering by a ring electron of comparatively large size, by the writer (Phys. Rev. xx. p. 30 (1919)).

While it is possible to account in this manner for the difference in penetrating power of the primary and secondary radiation if a sufficiently heterogeneous primary beam is postulated, it is clear that, as a result of scattering, there can be no transformation of radiation of one frequency into radiation of another frequency. That is, the scattered rays can be no softer than the softest components of the primary rays, and removal by filtering of the softer components of the primary radiation must harden also the secondary beam.

An experimental method of determining the relative amount of scattering and fluorescence has been applied to a study of the secondary radiation excited by the hard gamma rays from radium C. In figure 1 is shown diagrammatically the arrangement of the experiment. A source of hard gamma rays S excites secondary radiation in a block R, and the intensity of this radiation is measured by an ionization chamber I, which is screened by heavy lead blocks from the direct beam of gamma rays, and which in the final experiments is surrounded on four sides by about 4 cm. of lead to keep out secondary radiation from the walls of the room. The ionization current when the radiator R is removed, is approximately balanced by the ionization current produced by an adjustable source of gamma rays S in a second chamber I′. The intensity of the secondary radiation is then measured by the difference in the readings of the electrometer E when the radiator R is in place and when removed.

Fig. 1.



The test for the presence of fluorescent radiation was made by comparing the intensity of the secondary radiation when an absorption screen was placed alternately in position A, in front of the source of gamma rays, and position B, in front of the ionization chamber. Supplementary tests showed that when the primary rays entered directly into the ionization chamber, the absorption was nearly the same whether the lead screen was at A or at B[*]. Let us suppose that the primary beam consists of any number of components of different wave-lengths $\lambda_1, \lambda_2, \ldots$, that $I_1, I_2, \ldots$ are the intensities of these components in the primary beam, $c_1, c_2, \ldots$ the fractions of each component scattered into the ionization chamber, $k_1, k_2, \ldots$ the fractions of the respective energies transmitted through the absorption screen when placed at A, and $k_1', k_2', \ldots$ the corresponding fractions when placed at B. Then it is clear that, when the absorption screen is placed at A, the intensity of the beam scattered into the ionization chamber is

$$I = c_1 k_1 I_1 + c_2 k_2 I_2 + \ldots = \Sigma c_s k_s I_s,$$

and, when placed at B, the intensity is

$$I' = c_1 k_1' I_1 + c_2 k_2' I_2 + \ldots = \Sigma c_s k_s' I_s.$$

But for all wave-lengths, k_s is very nearly equal to k_s'. Hence I is nearly equal to I′: that is, for truly scattered radiation, the observed intensity of the secondary radiation should be approximately the same whether the absorbing plate is in the position of A or B.

If, on the other hand, the primary radiation excites in the radiator R a fluorescent radiation which is more readily absorbed than the primary rays, the observed intensity of the secondary radiation will be less when the absorption screen is in the position B; for if k_p is the fraction of the primary radiation transmitted through the absorption screen, while k_s is the corresponding transmission factor for the fluorescent radiation, the ratio I'/I of the intensity of the fluorescent radiation when the screen is at B to that when the screen is at A is obviously k_s/k_p. Thus the effect of any fluorescent radiation will be to make the fraction I'/I less than unity.

If all the secondary radiation is of the fluorescent type, the ratio k_s/k_p, and hence also of I'/I, should become indefinitely small as the thickness of the absorption screen is

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[*] The supplementary experiment referred to showed that for the gamma rays from radium C filtered through 2 mm. of lead, and using an absorption screen of 1 cm. of lead, the value of k was 0·57 and of k' was 0·52. The difference is doubtless due to the difference in the amount of secondary radiation reaching the ionization chamber in the two cases. This difference will be relatively less important for softer radiation, but will be relatively somewhat more prominent for greater thicknesses of the screen.

increased. For wholly scattered radiation, as we have just seen, the value of this ratio should remain approximately unity for all thicknesses of the absorption screen. If the secondary radiation is a mixture of the two types, it will be seen that the ratio I′/I should approach, for large thicknesses of the absorption screen, the constant value

$$I'/I = c_s/(c_s + c_t), \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

where c_s is the fraction of the primary beam scattered into the ionization chamber with no absorption screen at B, and c_t is the corresponding fraction for the fluorescent radiation. Thus by measuring the ratio of the intensity of the secondary radiation when suitable absorption screens are placed alternately in front of the ionization chamber and the source, it is possible to determine the relative magnitude of the scattered and the fluorescent radiation.

The results of measurements of this ratio at three different angles with the primary beam are shown in Table I.

TABLE I.

Thickness of lead screen.	Ratios I′/I for iron radiator. 45°.	90°.	135°.
0	1	1	1
0·15 cm.	...	...	0·45
0·5	...	0·30	−0·02
1·0	0·52	0·13	−0·02
2·0	0·39	0·02	...
3·0	0·26	...	...
4·1	0·20	...	...

At each angle the measurements were continued until the intensity was too low for accurate determinations of I′/I. The probable error of the final measurements of the ratio at 135° was about 0·02, at 90° about 0 03, and at 45° about 0·04. On the basis of the above discussion, we may therefore conclude that for gamma rays which have traversed several centimetres of lead the secondary radiation at angles greater than 90° is, except for the small probable error, all of the fluorescent type.

At 45° it appears that the value of the ratio I′/I is approaching a constant value for large thicknesses of the absorption screen. The limiting value of this ratio would seem to be of the order of 5 or 10 per cent., which, according to expression (1), would represent approximately the fraction of the secondary radiation at this angle which is due to true scattering. It is clear, in view of the magnitude of the probable error, that such extrapolation for large thicknesses of the screen is precarious, and the evidence for any true scattering cannot be considered conclusive. Probably at least 90 per cent. of the secondary radiation at this angle is of the fluorescent type.

Though we thus find that there is very little of the penetrating primary gamma radiation present in the secondary rays, it is not impossible that some of the softer components of the unfiltered primary beam may be appreciably scattered, but be so strongly absorbed that they are not detected through the lead screens employed. An upper limit to the amount of such soft scattered radiation that may be present can be assigned in the following manner. If at any specified angle a certain screen suffices to make the ratio I′/I sensibly zero, it is clear that with this screen in position A there is no appreciable scattered radiation entering the ionization chamber, and practically all of the secondary radiation is fluorescent. The absorption coefficient for the primary rays which excite the fluorescent radiation may then be determined by measuring the decrease in ionization when additional screens are placed at A. Assuming that this absorption coefficient remains constant for all thicknesses of the absorption screen, which experiments on the primary beam show is very nearly the case, the intensity of the fluorescent radiation when no absorption screen is employed is

$$J_0 = J \, . \, e^{\mu x},$$

where J is the observed intensity for a screen of thickness x, and μ is the linear absorption coefficient. Of the total secondary radiation I_0 observed when no screen is employed, the fraction J_0/I_0 at least consists of fluorescent radiation. Of the remainder, $(I_0 - J_0)/I_0$, a part may be fluorescent, since the true value of μ is presumably greater for small thicknesses of the absorption screen, and the rest will represent the truly scattered radiation. Thus the fraction $(I_0 - J_0)/I_0$ is an upper limit to the amount of scattered radiation which may be present in the secondary gamma rays when no absorption screens are employed.

Since theoretical considerations would lead one to expect the scattering to be greater at the smaller angles, an experimental determination of the value of this fraction was made for the secondary radiation at 45°. In this experiment the window of the ionization chamber consisted of 0·15 cm. of lead, and the same thickness of lead surrounded the source of gamma rays. This was necessary in order to cut out the beta rays. The results are shown in the following table :—

TABLE II.

x.	μ.	J/I_0.	J_0/I_0.	$(I_0-J_0)/I_0$.
3·1 cm.	0·57	0·17	0·98	0·02

The values in the fourth and fifth columns are calculated from the experimental data in the first three columns. After the primary gamma rays have passed through 3·1 cm. of lead, we have seen that probably not more than 10 per cent. of the secondary radiation is of the scattered type, but the intensity is so weak that the probable error of these measurements is necessarily rather large. It may be concluded, however, that at 45° probably not as much as 15 per cent. of the whole secondary radiation consists of scattered primary rays.

At the larger angles the results given in Table I. show that if there is any appreciable scattered radiation it must be of a very soft type, and since it is unable to penetrate a centimetre of lead, it cannot be identified with the hard gamma rays from radium C. It will be seen from this table also that the absorption coefficient of the fluorescent radiation is rapidly approaching that of the primary rays at the smaller angles. This fact, together with geometric difficulties which prevent securing intense secondary radiation at the smaller angles, makes very difficult any effort to separate the scattered and fluorescent radiation at angles much smaller than 45°. The question of the presence of scattered radiation at the smaller angles will be discussed in another paper on the basis of some experiments of a different type *. In this paper some positive evidence will be presented for the existence of true scattering at angles less than 15°.

In order to find out how far the actual scattering falls short of that to be expected from theoretical reasoning, a measurement was made of the relative intensity of the primary and secondary radiation at several angles. If the dimensions of the electron are negligible compared with the wave-length of the gamma rays, and if the electrons all scatter independently of each other, the usual theory † gives for the intensity of the scattered radiation

$$\frac{I_\theta}{I}=\frac{Ne^4(1+\cos^2\theta)}{2m^2l^2C^4}, \quad . \quad . \quad . \quad . \quad . \quad . \qquad (2)$$

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* *Infrà*, p. 770.

† J. J. Thomson, *loc. cit.*

where I is the intensity of the primary beam at the ionization chamber when the radiator is replaced by the source of gamma rays, N is the number of electrons which are effective in scattering, e and m are the charge and mass respectively of the electron, θ is the angle with the primary beam at which the scattered beam is observed, l is the distance of the radiator from the source of gamma rays, and C is the velocity of light. Taking the number of electrons per atom as equal to the atomic number, and using the experimental values 10·3 cm. for l and 234 g. for the mass of the iron radiator, this expression gives for the ratio I_0/I at 90° the value 0·023. The experimental value of this ratio was 0·0017. But of this we have seen that less than 3 per cent. probably represents true scattering. The value of the ratio I_s/I, where I_s is the observed true scattering, is therefore less than 0·00005, only 2 per cent. of that required by theory. Similar results for the scattering at 45° and 135° are given in Table III. It will be seen from this table that at large angles, if there is any true scattering, it is probably less than a thousandth part of the amount predicted on the basis of the usual electron theory.

TABLE III.

Angle.	I_θ/I observed.	I_s/I observed.	I_s/I calculated.	$\frac{I_s \text{ obs.}}{I_s \text{ calc.}}$
45°	0·015	c. 0·001	0·035	c. 0·03
90°	0·0017	$<$0·00005	0·023	$<$0·002
135°	0·0008	$<$0·00002	0·035	$<$0·0005

It is not impossible to account for this very low value of the scattering on the basis of the classical electrodynamics, if suitable assumptions are made with regard to the wave-length of the primary gamma rays and the properties of the electron. Thus, for example, the writer has shown elsewhere[*] that if the electron is a rigid sphere which is not subject to rotational displacements by the primary beam, the scattering at all angles becomes negligible when the ratio of the wave-length to the radius of the electron is less than about 2·4. Certain other types of electron give a similar result for different values of this ratio. If this explanation is the correct one, the wave-length of these gamma rays must be considerably shorter than that of the hardest X-rays which have yet been studied, since for these rays the scattering, though somewhat smaller than that predicted by the usual theory, is apparently of the proper order of

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[*] A. H. Compton, Phys. Rev. xx. p. 25 (1919).

magnitude *. It seems premature to attempt any detailed explanation of the failure of the usual electron theory until more definite information is available with regard to the wave-length of the hard gamma rays.

The Characteristics of the Fluorescent Radiation.

Let us now consider the properties of the fluorescent radiation excited by the hard gamma rays. The observed relative intensities of the secondary radiation from aluminium, iron, and lead at different angles with the primary beam, when gamma rays from radium C filtered through a centimetre of lead are employed, are shown in Table IV.

Table IV.

Secondary radiator.	Angle							
	30°.	45°.	60°.	75°.	90°.	120°.	135°.	150°.
Al......	...	6·2	4·0	1·7	(1·0)	0·7	0·4	0·3
Fe......	10	7·6	4·6	2·0	(1·0)	0·6	0·5	0·4
Pb......	11	6·8	4·3	2·2	(1·0)	0·8	...	0·7

In comparing the intensity of the scattered beam at any two angles, θ_1 and θ_2, the effect of absorption was eliminated as completely as possible by the well-known method of placing the radiating plate with its normal at an angle $(\theta_1+\theta_2)/2$ with the primary beam. In this case the absorption is the same at the two angles at which the secondary radiation is compared. Since the window of the ionization chamber consisted of 0·15 cm. of lead, any soft fluorescent radiation was strongly absorbed.

Data similar to those given in this table have been published by Florance † and Kohlrausch ‡, except that in the present case the effect of absorption by the radiator has been largely eliminated, and the primary beam was rendered homogeneous by filtering through a suitable lead screen §.

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* *C. f. e. g.* Hull & Rice, Phys. Rev. viii. p. 326 (1916). Barkla & White, Phil. Mag. xxxiv. p. 277 (1917).

† D. C. H. Florance, Phil. Mag. xx. p. 921 (1910).

‡ K. W. F. Kohlrausch, *loc. cit.*

§ It should be noted that on account of the differing hardness of the secondary radiation at different angles, the relative intensity observed at a given angle depends upon the fraction of the radiation absorbed by the ionization chamber. The ionization chamber used in the present experiments was so designed that it absorbed a large part of even the hard primary rays. This probably accounts for the fact that the writer's experiments show relatively more intense radiation at the smaller angles where the secondary rays are hard, than do the experiments of Florance and Kohlrausch, whose ionization chambers presumably absorbed only a small fraction of the incident radiation.

In common with these experimenters, it is found that the secondary radiation, which in the writer's work consisted almost wholly of fluorescent radiation, is very much more intense at small angles than at large angles with the incident gamma rays.

The relative amount of fluorescent radiation excited in different substances per unit mass is shown in Table V.

TABLE V.

Angle.	Relative fluorescence per unit mass:				
	Paraffin.	Al.	Fe.	Sn.	Pb.
135°	1·12	1·04	(1·00)	0·78	0·74
45°	1·7	0·9	(1·0)	0·8	0·9

The readings at 135° were taken for primary rays filtered through 0·5 cm. of lead, and those at 45° were with a 4·1 cm. lead filter. Thus it was made certain that practically all of the secondary radiation was of the fluorescent type. Sufficiently thin plates of the radiating materials were employed that the necessary corrections for the absorption of the primary and secondary radiation in the radiator were not large. The values here given therefore represent the amount of the fluorescent radiation excited in unit mass of the different radiators, which penetrates the 0·15 cm. lead window of the ionization chamber. It will be seen that the values do not differ greatly over a wide range of atomic weights.

The constancy is even more marked when the fluorescence per electron is calculated by multiplying each of the above values by the ratio (atomic weight)/(atomic number), as is done in the following table. At the angle 135° the constancy of these values is somewhat accidental, since, as we shall see, the absorption coefficient of the fluorescent radiation at this angle is considerably greater for the radiation from the light than for that from the heavy elements. At 45°, however, the hardness of the fluorescent radiation is practically the same for the different radiators, so the constancy of the values at this angle is of real significance. The result expressed by this table is confirmed by the more quantitative experiments of Ishino *, who found that the magnitude of

TABLE VI.

Angle.	Relative fluorescence per electron:				
	Paraffin.	Al.	Fe.	Sn.	Pb.
135°	0·91	1·01	(1·00)	0·87	0·87
45°	1·4	0·9	(1·0)	0·9	1·1

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* M. Ishino, *loc. cit.*

the total secondary radiation (which includes any scattered rays that may be present) per atom is more nearly proportional to the atomic number than to the atomic weight. The amount of the fluorescent radiation excited is therefore approximately proportional to the number of electrons traversed by the primary gamma rays.

It will be seen on examining Table I. that the fluorescent radiation at small angles with the primary beam is considerably harder than that at right angles. This matter was examined in greater detail for a number of different elements, with the results shown in Table VII. Care was taken in these experiments also to eliminate any possible soft scattered radiation by interposing suitable absorption screens between the source of gamma rays and the radiating material. As a result, the intensity of the secondary radiation was so low that the values of the absorption coefficients obtained can be considered only approximate. The data suffice to show, however, that while the radiation from all substances is harder at small angles, the difference is less for elements of high atomic weight, so that whereas at large angles the radiation from the heavier elements is considerably more penetrating, at 45° the hardness differs but little from element to element*.

Table VII.

Angle.	Mass absorption coefficients in Lead of the fluorescent radiation excited in different materials by hard gamma rays:				
	Paraffin.	Al.	Fe.	Sn.	Pb.
45°	0·10	0·10	0·11	0·09	0·05 (?)
90°	...	...	0·21	...	...
135°	0·78	0·50	0·50	0·32	0·15

Interesting information is obtained on examining the absorption coefficients of this penetrating fluorescent radiation in various materials. This was done for the secondary radiation from iron at 135° after the primary gamma rays had been filtered through 0·5 cm. of lead, with the results shown in the following table. With a similar geometric

Table VIII.

Mass absorption coefficients in different elements of the fluorescemt radiation at 135° excited in iron by hard gamma rays from radium C:

Pb.	Sn.	Fe.	Al.
0·50	0·18	0·08	0·07

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* The constancy observed for different elements at 45° is confirmed by the measurements of Florance, Phil. Mag. xx. p. 935 (1910).

arrangement, the mass absorption coefficient of the primary rays in lead was 0·062, which presumably means that this fluorescent radiation is of very appreciably longer wave-length than the primary gamma rays. On the other hand, the experiments of Hull and Rice [*] show that X-rays of wave-length $0{\cdot}122 \times 10^{-8}$ cm. have a mass absorption coefficient in lead of about 3·0, which indicates that even the softest part of this fluorescent radiation is of shorter wave-length than the critical wave-length $0{\cdot}147 \times 10^{-8}$ cm. required to excite the characteristic K radiation in lead. This conclusion is confirmed by the fact that the mass absorption of this fluorescent radiation is greater in lead than in tin, which is the reverse of the case for wave-lengths between the K radiation from lead and the radiation from tin. There can thus be no question but that the fluorescent rays under examination are of a distinctly harder type than the characteristic K radiation from even the heaviest elements.

In an experiment with a Coolidge tube operated by an induction coil at a maximum potential of 196,000 volts, Rutherford has obtained X-rays whose mass absorption coefficient in lead is as low as 0·75 [†]. This is practically the same as the value observed for the fluorescent gamma radiation from paraffin at 135° (Table VII.). According to the quantum relation, $h\nu = eV$, the wave-length in Rutherford's experiment must have been greater than 0·063 Å.U. The wave-length of the softest part of this penetrating fluorescent radiation must therefore lie between 0·06 and 0·12 Å.U.

It is interesting to note that these secondary gamma rays bridge the gap which has existed between the hardest X-rays and the very penetrating gamma rays; for as we have just seen, the softest part of this secondary radiation falls within the wave-length of the hardest X-rays, while Table VII. shows that at small angles it is nearly as penetrating as the hard gamma rays from radium C.

The Origin of the Fluorescent Radiation.

Although the secondary gamma radiation under examination seems, without doubt, to be fluorescent in nature, it differs in several important respects from the characteristic fluorescent K and L radiations excited in matter when traversed by hard X-rays. In the first place, whereas these characteristic radiations differ greatly in hardness from

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[*] Hull & Rice, *loc. cit.*

[†] E. Rutherford, Phil. Mag. xxxiv. p. 153 (1917).

element to element, the secondary gamma rays, especially at small angles with the incident beam, are of nearly the same hardness over a wide range of atomic numbers. And in the second place, while the characteristic radiations are found to be distributed uniformly with regard to intensity and quality at all angles with the primary beam, the fluorescent gamma rays show marked asymmetry in both quantity and quality in the forward and reverse directions. There is therefore good reason to suppose that the oscillators which give rise to this fluorescent radiation are radically different in character from those which are responsible for the K, L, and M characteristic radiations *.

An explanation of the origin of the fluorescent radiation which appears to be satisfactory is that the high-speed secondary beta particles liberated in the radiator by the primary gamma rays excite the secondary gamma rays as they traverse the matter of the radiator. On this view the fluorescent gamma rays should be identical in character with the so-called "white" radiation excited in the target of an X-ray tube by the impact of the cathode particles. Experiments have shown that when cathode rays or beta rays strike a target which is so thin that the particles are not greatly scattered and in which no considerable amount of characteristic radiation is excited, the X-rays emitted are more intense and harder in the general direction of the cathode ray beam than in the reverse direction †. This asymmetry is of the same kind as that observed for the secondary gamma rays, and though not so marked, is found to increase with the speed of the impinging electrons. For speeds comparable with those of fast beta rays the asymmetry may well become as great as that observed in the present experiments. But it is also known that the beta rays liberated by gamma rays are much more intense in the direction of the gamma ray beam than in the reverse

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* The idea suggested itself that the secondary radiation which was being studied was a fluorescent radiation excited in the lead screens which surrounded the source of gamma rays, this fluorescent radiation being in turn scattered by the radiator into the ionization chamber. It is obvious that such a radiation would not be eliminated by placing additional lead screens over the source, while the ionization would be considerably reduced by placing screens over the ionization chamber. Considerations of the energy involved and of the characteristics of the secondary radiation rendered this suggestion improbable, but the possibility was definitely eliminated by removing all the lead screens and replacing them with iron. The phenomenon in this case was identical with that when lead screens were employed.

† G. W. C. Kaye, Proc. Camb. Phil. Soc. xv. p. 269 (1909). J. A. Gray, *loc. cit.*

direction. Indeed, "the results indicate that the beta particles initially escape in the direction of the gamma rays, and with the same speed for all kinds of matter"*. Thus the hypothesis that the fluorescent gamma radiation is due to the impact of the secondary beta particles accounts qualitatively for the observed asymmetry in the hardness and intensity of the secondary gamma rays.

With regard to the relative intensity of the fluorescence excited in different materials, attention may be called to the fact that the number of the beta particles excited by hard gamma rays is approximately the same per electron in different elements †. There is a somewhat larger number produced in the very heavy elements such as mercury and lead, which appears to be connected with the excitation of the characteristic K radiation in these elements. But since such radiation is too soft to have an appreciable effect in the present investigation, it seems probable that the number of *effective* beta particles excited in these elements does not differ greatly from that for the lighter elements. If, therefore, the simple assumption is made that the amount of secondary gamma rays excited depends only upon the number of electrons traversed by the secondary beta particles, our hypothesis gives a satisfactory account of the fact that the amount of secondary gamma radiation per electron is practically the same for all elements.

The greater scattering of beta particles by elements of high atomic weight means that in these elements a relatively larger number of the beta rays move in a direction opposed to the primary beam. For this reason we should expect, as Table VII. shows is actually the case, that at large angles with the incident beam the fluorescent radiation from the heavier elements will be more penetrating than that from the light ones.

An estimate of the relative energy in the secondary rays can be obtained by integrating over the surface of a sphere the observed relative intensity of the scattered beam at various angles. A rough summation of this kind, using the data of Tables III. and IV. and extrapolating by the help of Kohlrausch's data for the very small angles, shows that the ratio of the energy (as measured by the ionization) of the secondary radiation from iron to the total energy absorbed from the primary beam is about 0·69. This result is in good agreement with Ishino's estimate that the "scattering" by iron accounts for 62 per cent. of the total absorption. A

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* E. Rutherford, 'Radioactive Substances, etc.,' p. 276.

† Eve, Phil. Mag. xviii. p. 275 (1909).

correction must be applied to this value to make allowance for the fact that while the greater part of the secondary radiation which enters the ionization is absorbed, in the present experiment only about half of the primary beam was thus absorbed. Taking this correction factor to be about 0·7, we find that approximately 50 per cent. of the absorbed primary rays is transformed into radiation of sufficiently high frequency to penetrate 0·15 cm. of lead*. The efficiency of transformation of the energy is therefore of a much higher order than that observed in an X-ray tube operating at usual potentials, in which case not as much as 1 per cent. of the energy of the cathode rays appears as X-rays. It is possible that this difference is to be accounted for by an excitation of gamma rays when the secondary beta particles are liberated in addition to that produced when they collide with other electrons.

A question of great theoretical importance is—What kind of oscillator can give rise to radiation which not only is more intense in one direction than in another, but also differs in wave-length in different directions? Since the secondary radiation differs in frequency from the primary rays, it would seem impossible to invoke any interference between the radiation from the different oscillators to account for this phenomenon. Such an explanation is rendered the more difficult by the fact that to explain the different hardness of the rays in different directions, oscillators of different frequencies would have to be present, between which there could be no fixed phase relations. An obvious means of accounting for the observed phenomenon is to suppose that the radiator which gives rise to the secondary rays is moving at high speed in the direction of the primary beam. In this case, both the intensity and the frequency of the fluorescent radiation will be greater in the forward than in the reverse direction, as is demanded by the experiments.

A rigid calculation of the relative intensity of the fluorescent radiation at different angles, according to this hypothesis, is not at present possible, because the scattering of the beta particles results in an irregular distribution of their velocities. It will nevertheless be instructive to consider the relative energy radiated in different directions by an oscillator moving at a speed comparable with that of light. It can be

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* The estimate here made of the efficiency of transformation is, of course, based upon the assumption that unit energy of one frequency produces the same total number of ions as unit energy of another frequency. Though this assumption has not been tested over the range of frequencies here considered, it does not appear probable that any error thus introduced can change the order of magnitude of the result.

shown that for an electron whose acceleration is unpolarized relative to the observer[*], and which is travelling at a velocity βC, the mean square of the electric vector at a great distance r is[†]

$$E^2=\frac{e^2\overline{\Gamma}^2}{16\pi^2C^4r^2}\times\frac{(1-\beta\cos\theta)^2-\frac{1}{3}(1-\beta^2)\sin^2\theta+\frac{1}{3}\cos\theta(2\beta-\cos\theta-\beta^2\cos\theta)}{(1-\beta\cos\theta)^6}, \quad (3)$$

where $\overline{\Gamma}^2$ is the mean square of the acceleration relative to the observer at the moment the pulse under observation left the electron, e is the charge of the oscillator, C is the velocity of light, and θ is the angle between the direction of motion of the particle and the observed beam. Thus the ratio of the intensity of the radiation at an angle θ_1 to that at an angle θ_2 is given by the expression

$$R=\frac{I_1}{I_2}=\left\{\frac{1-\beta\cos\theta_2}{1-\beta\cos\theta_1}\right\}^6\frac{(1-\beta\cos\theta_1)^2-\frac{1}{3}(1-\beta^2)\sin^2\theta_1+\frac{1}{3}\cos\theta_1(2\beta-\cos\theta_1-\beta^2\cos\theta_1)}{(1-\beta\cos\theta_2)^2-\frac{1}{3}(1-\beta^2)\sin^2\theta_2+\frac{1}{3}\cos\theta_2(2\beta-\cos\theta_2-\beta^2\cos\theta_2)}. \quad (4)$$

Assuming that all the radiating particles are moving in the same direction, the ratio of the intensity of the fluorescent radiation at the angle 45° to that at 135° has been calculated from this expression, with the results shown in figure 2. It will be seen that it is possible on this view to account for any reasonable degree of asymmetry of the secondary radiation. In the case of paraffin, in which the least scattering of the beta particles occurs, the observed ratio of the intensities at 45° and 135° was about 20. In addition to the effect of the scattering of the secondary beta rays, experimental errors arise because much of the soft radiation at 135° is absorbed before it enters the ionization chamber, while a considerable part of the hard radiation at 45° traverses the ionization chamber without being absorbed. The rapid increase of R with β, however, makes it reasonably certain, on the present view, that the average speed of the oscillators which emit the secondary gamma radiation does not differ greatly from half the speed of light.

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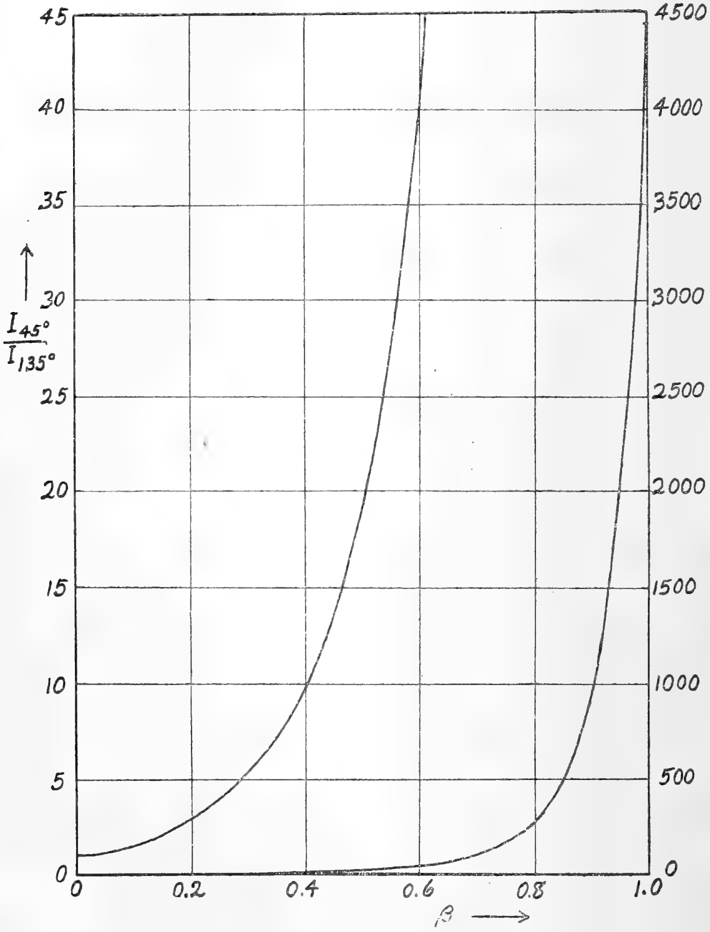
[*] Of course such an oscillator will not be unpolarized relative to an observer moving with it. A slight polarization will not, however, make any great difference in the value of the ratio (4).

[†] The values of the three components of the electric vector from which this expression is derived may be found, *e.g.* in O. W. Richardson's 'Electron Theory,' p. 256.

Since the speed of even the swiftest alpha particles is only about one-tenth that of light, it is clear that the radiating particles cannot have mass comparable with atoms, but must be individual electrons. We are thus led to the idea that it is the vibrations of the secondary beta particles themselves which give rise to the fluorescent gamma rays*.

Fig. 2.



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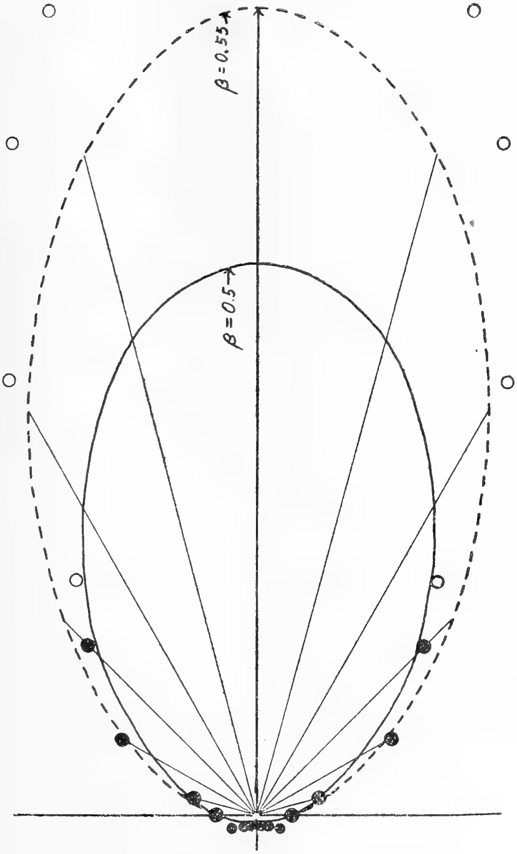
* The corresponding hypothesis that the "white radiation" from an X-ray tube is due to vibrations of the cathode particles has been suggested on the basis of similar considerations by D. L. Webster, Phys. Rev. xiii. pp. 303–305 (1919).

Substituting the value $\beta=0{\cdot}5$ in equation (4), we can calculate the relative scattering to be expected at various angles. The result is shown in the solid curve of figure 3.

Fig. 3.



Curves showing intensity at different angles with motion of oscillator whose velocity is βC.
Solid circles writer's, open circles Kohlrausch's values of relative intensity of gamma rays.

The solid circles in this figure represent the writer's observations on aluminium, as given in Table IV. The open circles at the small angles show the results of Kohlrausch referred to the writer's value at 45°. It is not impossible that the large amount of secondary radiation at these small angles is due in part to the presence of some true scattering. However this may be, the generally satisfactory form of the theoretical curve suggests that we are working along the right line.

If this view of the origin of the fluorescent radiation is the correct one, we are supplied with a means of estimating roughly the wave-length of the primary gamma rays. It has been shown above that the wave-length of the softest part of the fluorescent radiation lies between 0·06 and 0·12 Å.U., and probably nearer the former. But according to the Doppler principle, if the oscillators producing the fluorescent radiation are moving in the direction of the primary rays, the ratio of the wave-length at an angle θ_1 to that at an angle θ_2 is

$$\frac{\lambda_1}{\lambda_2}=\frac{1-\beta\cos\theta_1}{1-\beta\cos\theta_2} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (5)$$

Thus, if we take λ_2 to be about 0·08 Å.U. at $\theta_2=135°$, and the value of β to be 0·5, the wave-length of the penetrating fluorescent radiation at 45° is about 0·04 Å.U. This result does not vary greatly with different values of β; but the extreme hardness of the fluorescent radiation at 45° indicates that it is more nearly similar to the primary rays than to the soft secondary radiation which appears at the larger angles. Thus we shall probably not be far wrong in assigning a value 0·02 to 0·03 Å.U. as the wave-length of the most effective part of the hard gamma rays from radium C. This result is not in disaccord with the calculations of Rutherford * based upon the quantum hypothesis.

Summary.

The principal experimental results of this investigation may be summarized as follows :—

By far the greater part of the secondary gamma radiation from matter traversed by the hard gamma rays from radium C is fluorescent in nature. If any truly scattered radiation is present, at 45° it probably amounts to less than 15 per cent., and for angles greater than 90° to less than 3 per cent. of the secondary rays.

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* E. Rutherford, Phil. Mag. xxxiv. p. 153 (1917). According to the quantum relation, an electron must have a velocity $\beta=0·8$ in order to excite radiation of wave-length 0·04 Å.U. On the present view, therefore, the radiating beta particle must already have lost a large part of its energy of translation.

At large angles with the primary beam the scattered energy is probably less than 0·001 of that required by the usual electron theory.

The secondary fluorescent radiation is found, in accord with observations by others on the whole secondary radiation, to be harder and more intense at small angles with the incident beam than at large angles, and tables are given showing the manner of this variation.

While at large angles the radiation from heavy elements is somewhat more penetrating than that from the light elements, at small angles both the hardness and the intensity of the fluorescent radiation are approximately the same from elements covering a wide range of atomic numbers.

A study of the absorption coefficients of this radiation in various elements shows that the softest parts of it, though of shorter wave-length than the K radiation from lead, are not harder than the most penetrating X-rays. The hardest parts approach in penetrating power the primary gamma rays from radium C.

It is pointed out that the very small scattering observed is not incompatible with the classical electrodynamics, if the wave-length of the gamma rays and the diameter of the electron are of the same order of magnitude.

A satisfactory qualitative explanation of the observed fluorescent radiation is found in the gamma rays produced by the impact of the secondary beta particles liberated in the radiator by the primary gamma rays.

The observed asymmetry in the intensity and hardness may be accounted for if the oscillators which give rise to the fluorescent radiation are electrons moving in the direction of the primary beam with about half the speed of light.

The wave-length of the softest part of the observed fluorescent radiation is shown to lie between 0·06 and 0·12 Å.U., probably nearer the former value, while the wave-length of the hardest part is probably about half as great. By a comparison of absorption coefficients, the effective wave-length of the hard gamma rays from radium is estimated as about 2 or 3×10^{-10} cm.

The writer performed these experiments at the Cavendish Laboratory as National Research Fellow in Physics. He desires to express his appreciation of the interest which Professor Rutherford has shown in the work.

Washington University,
St. Louis,
September 24th, 1920.

LXIX. *The Wave-Length of Hard Gamma Rays.* By

ARTHUR H. COMPTON, *Ph.D., Physics Laboratory, Washington University* *.

THE only recorded attempt to measure directly the wave-length of hard gamma rays is apparently that of Rutherford and Andrade †, using the method of reflexion from a crystal of rock-salt. In these experiments spectrum lines were observed at angles as small as about 44 minutes, corresponding to a wave-length of about 0·07 Å.U. It was thought that this line, as well as one of wave-length 0·10 Å.U., could be detected through a 6-millimetre screen of lead, which would make it appear that these lines represent the hard gamma rays from radium C. Professor Rutherford informs me, however, that the appearance of these lines through the lead screen was doubtful. His more recent measurements of the absorption of X-rays of very high frequency ‡ have indicated rather that radiation, whose wave-length is about 0·08 Å.U., has an absorption coefficient in lead that is very much greater than that of the hard gamma rays from radium. Thus, while the crystal reflexion measurements show that radium gives off gamma rays of wave-lengths 0·07 Å.U. and longer, the very penetrating radiation which it emits probably has a much shorter wave-length.

Various lines of theoretical reasoning suggest that there are in hard gamma rays components ranging in wave-length from 0·01 to 0·04 Å.U. Rutherford has pointed out § that radium C gives off beta rays with an energy corresponding to a fall through from 5 to 20×10^5 volts. According to the quantum relation, $h\nu = eV$, the limiting wave-length produced by the slower of these electrons would be about 0·03 Å.U., while that due to the fastest ones would be as short as 0·007 Å.U. In the second place, using an absorption formula which is satisfactory for hard X-rays of known wave-length, it is found || by extrapolation that the absorption coefficient of hard gamma rays corresponds to a wave-length of about 0·04 Å.U. And finally, knowing approximately the

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* Communicated by Prof. Sir E. Rutherford.

† Rutherford and Andrade, Phil. Mag. xxviii. p. 263 (1914).

‡ E. Rutherford, Phil. Mag. xxxiv. p. 153 (1917).

§ *Ibid.*

|| A. H. Compton, Washington University Studies, Scientific Series, Jan. 1921.

wave-length of the "incident" secondary gamma radiation, and calculating from this the wave-length of the "emergent" secondary radiation on the hypothesis that the difference in wave-length is a Doppler effect due to motion of the particles emitting the secondary radiation, the wave-length of the primary gamma rays can be estimated, since the absorption coefficient of the primary and the "emergent" secondary radiation is nearly the same. This method leads to a value of between 0·02 and 0·03 Å.U. for the effective wave-length of the hard gamma rays from radium *.

In the present paper a new method of measuring the wave-length of high frequency radiation will be proposed, and the method will be applied to the determination of the wave-length of gamma rays. Instead of studying the spectrum lines reflected by a grating composed of regularly arranged atoms in a crystal, this method consists in observing the diffraction pattern due to the individual atoms. To consider an optical analogy, if the reflexion of X-rays from a crystal is compared with the spectrum from a ruled grating, the method of atomic diffraction corresponds to a study of the diffraction pattern due to a large number of parallel lines ruled at random distances. The distance between the different order lines in the spectrum is determined by the grating space between the lines ruled on the grating, while the distance between the bands of the diffraction pattern is determined by the breadth of the individual lines. The advantage of the method as applied to gamma-ray measurements lies in the fact that the effective diameter of the atom is much smaller than the distance between two atoms in a crystal, so that the effective width of the diffraction band is much greater than the distance between two spectrum lines. Thus, whereas the spectrum of hard gamma rays from a crystal grating would have to be studied at angles less than 1/2 degree, atomic diffraction measurements may be made at angles in the neighbourhood of 10 degrees. In order to use the method quantitatively, it is of course necessary to know the effective diameter of the atom. This may be determined, in a manner that will be described below, by measurements with X-rays of known wave-length.

Debye has shown that if an atom is composed of N electrons, and if at any instant the distance between the *m*th and the *n*th electron is s_{mn}, the probable intensity of the X-rays

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* A. H. Compton, Phil. Mag. *suprà*, p. 749.

scattered at an angle θ with the primary beam whose intensity is I is *,

$$I_1 \sum_1^N \sum_1^N \frac{\sin\left[\frac{4\pi}{\lambda} s_{mn} \sin\frac{\theta}{2}\right]}{\left[\frac{4\pi}{\lambda} s_{mn} \sin\frac{\theta}{2}\right]},$$

where λ is the wave-length and I_1 is the intensity of the rays scattered by a single electron. This expression supposes that the forces holding the electrons in position are negligible in comparison with the forces due to the traversing radiation—an hypothesis supported by experiments on the scattering of X-rays. If $p_{mn} . ds$ is the probability that the distance s_{mn} will lie between s and $s+ds$, the average value of the intensity for all possible arrangements of the electrons in the atoms is,

$$I_\theta = I_1 \sum_1^N \sum_1^N \int_0^\infty \frac{\sin\left[\frac{\sin\theta/2}{\lambda} . 4\pi\, s_{mn}\right]}{\frac{\sin\theta/2}{\lambda} . 4\pi\, s_{mn}} p_{mn}\, ds_{mn},$$

or

$$\frac{I_\theta}{I_1} = F\,(N, p, \sin\theta/2/\lambda).$$

Since for any particular atom the quantities N and p remain constant, for an atom of atomic number N this ratio may be written,

$$\frac{I_\theta}{NI_1} = \psi_N (\sin\frac{\theta}{2}/\lambda).$$

If the quantity $\sin(\theta/2)/\lambda$ is sufficiently large, it will be seen that co-operation in the scattering by different electrons will be almost wholly a matter of chance, and the "excess scattering" function ψ will become practically unity. On the other hand, for very small values of this quantity co-operation between the electrons will be almost complete and the value of the function ψ will approach N. For intermediate values of $\sin(\theta/2)/\lambda$ the function will have a different value for every atom, since for no two atoms will the probabilities p_{mn} be identical. The experimental values of $\psi = I_\theta/I_1 . N$ for different materials, as measured by Barkla and Dunlop †, together with the values calculated for certain

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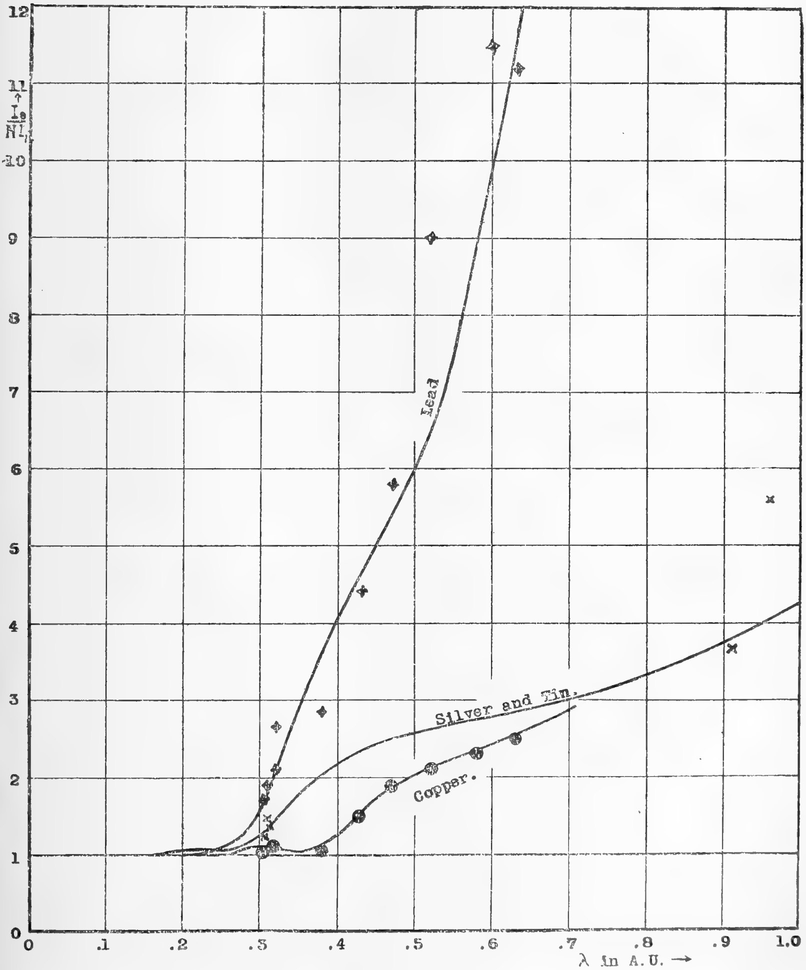
* P. Debye, *Ann. d. Phys.* xlvi. p. 809 (1915).

† Barkla and Dunlop, Phil. Mag. xxxvii. p. 222 (1916).

arbitrary arrangements of the electrons in the atoms of the different elements *, are shown in fig. 1. The measurements for the different wave-lengths were all made at an angle $\theta = 90°$.

Fig. 1.



Let us suppose that according to these experiments the ratio of the value of ψ for lead to its value for copper is R when $\sin(\theta/2)/\lambda = c$. Then, if for some unknown wave-length λ' the value of this ratio becomes R at an angle θ', it

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* *Cf.* A. H. Compton, Phil. Mag. (soon to be published).

is clear that $\sin(\theta'/2)/\lambda' = c = \sin(\theta/2)/\lambda$, whence

$$\lambda' = \frac{\sin\theta'/2}{\sin\theta/2}\lambda. \quad \ldots \ldots \quad (2)$$

Thus, if it is possible to find an angle at which gamma rays scattered from lead and copper are in the same ratio as the X-rays scattered at 90° in Barkla and Dunlop's experiments, we have the data necessary to calculate the effective wave-length of the gamma rays.

Scattering of Hard Gamma Rays at Small Angles.—An element of uncertainty is introduced into the application of this method of determining the wave-length of gamma rays by the fact that recent measurements have shown[*] that only a very small part, if any, of the secondary gamma rays observed at large angles with the primary beam is truly scattered radiation. At an angle of 45° these measurements indicated that perhaps 5 or 10 per cent. of the secondary radiation consisted of scattered primary rays, though the absorption coefficient of the fluorescent secondary radiation was so nearly the same as that of the primary rays that it was not possible to establish with certainty the existence of any scattered rays. With the hope of placing the present wave-length experiments on a more certain footing, careful examination of the character of the secondary radiation at 22°·5 was made, using, with some refinements, the same general method as that employed in the earlier experiments. It was found that, though for small thicknesses of the absorption screen the absorption coefficient of the secondary radiation differed by only about 8 per cent. from that of the primary beam, even after traversing 5·6 cm. of lead the two absorption coefficients were still measurably different. Thus at least 50 per cent. of the radiation at this angle is certainly fluorescent in nature, but the fluorescent radiation is absorbed at so nearly the same rate as the primary rays that it was impossible to decide what part of the remaining 50 per cent. was fluorescent and what part might be scattered radiation. The experiments are, however, consistent with the view that for angles smaller than 30° with the primary beam a considerable portion of the secondary gamma radiation consists of truly scattered radiation.

Several investigators have found that at angles greater than 30° the ratio of the secondary radiation from one element

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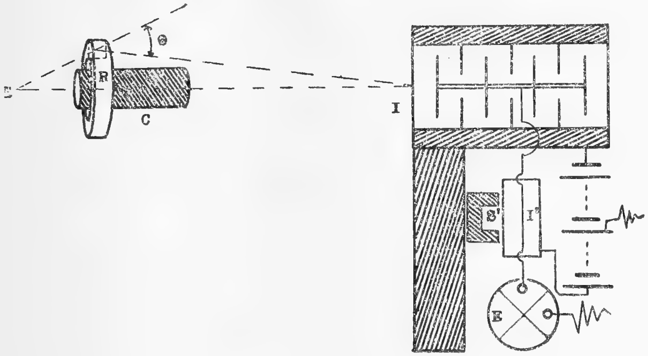
[*] A. H. Compton, *suprà*, p. 749.

to that from another is practically independent of the angle *. An examination was therefore made of the relative intensity of the secondary radiation from copper and lead at angles less than 30°. The experimental arrangement is shown in fig. 2. A strong source of hard gamma rays S (usually about 100 millicuries) is placed at a point on the axis of the ionization chamber I, and a lead cylinder C is placed between to cut off the primary gamma rays. The sample of copper or lead under examination is in the form of a ring R supported coaxially on the lead cylinder by a piece of cardboard. It is clear that all parts of the ring will scatter gamma rays into the ionization chamber at approximately the same angle. In order to secure the greatest possible intensity, the dimensions of the apparatus were so adjusted that the maximum differences in the angle θ made with the primary beam by the secondary rays entering the ionization chamber were between $\theta/2$ and $3\theta/2$. By this arrangement ionization due to the secondary radiation from the ring R can be obtained which is very considerably greater than that due to stray rays from the walls of the room, etc. It was found convenient, however, to balance the stray radiation against ionization produced by a small source S' of gamma rays in a second chamber I'. The secondary radiation was then measured by the difference in the readings of the electrometer when the radiator R was in place and when removed.

Fig. 2.



The results of the experiments are shown in the following table. The values of the observed ratios of the intensity of

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* *Cf.*, *e. g.*, M. Ishino, Phil. Mag. xxxiii. p. 140 (1914); K. W. F. Kohlrausch, *Phys. Zeitschr.* xxi. p. 193 (1920); A. H. Compton, *suprà*, p. 749.

the secondary rays from lead to that from copper, shown in the second column, are the averages of large numbers of readings :—

Table I.

Angle.	Observed Ratio. $\frac{\text{Intensity for Pb.}}{\text{Intensity for Cu.}}$	Relative Intensity per electron.
30°	·605±·012	·86±·02
20°	·601±·009	·85±·01
15°	·638±·008	·89±·01
10°	·677±·023	·95±·03

In the third column the observed intensities are corrected for the difference in absorption of the rays in traversing the lead and copper rings, and the ratio of the relative intensity per electron is calculated, taking the number of electrons per atom as equal to the atomic number. It will be seen that while at all angles at which measurements were made the value of this ratio is slightly less than the theoretical value unity, the ratio shows a tendency to increase at the smaller angles. It is unfortunate that at still smaller angles the energy of the secondary radiation was so low that no satisfactory measurements could be made.

It may be mentioned that Kohlrausch * has recently made measurements similar to these at angles as low as 10°, and that his measurements do not show this tendency for the intensity of the secondary rays from lead to increase at the small angles more rapidly than that from the lighter elements. While Kohlrausch used several times as strong a source of gamma rays as that employed in these experiments, his apparatus was not specially adapted to taking measurements at small angles, and it would appear that his probable error at these angles was greater than that of the present measurements. In view of the consistency of the experiments here recorded, it appears probable that the observed increase in the ratio at 10° is not the result of chance.

The Wave-Length of the Gamma Rays.—A comparison of the values given in Table I. with the data in fig. 1 shows that the present experiments have not been carried to angles sufficiently small to give values in I_{Pb}/I_{Cu} overlapping Barkla

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* K. W. F. Kohlrausch, *loc. cit.*

and Dunlop's experimental values for the scattering of X-rays at 90°. Since their measurements included wave-lengths as short as 0 3 Å.U., we may conclude from equation (2) that the effective wave-length of the gamma rays here employed is less than

$$\frac{\sin 5^\circ}{\sin 45^\circ} \times 0{\cdot}3 = 0{\cdot}037 \text{ Å.U.}$$

If Barkla and Dunlop's results are extrapolated according to the theoretical curve for lead shown in fig. 1, it will be seen that the ratio I_{Pb}/I_{Cu} at 90° begins to increase appreciably for wave-lengths in the neighbourhood of ·2 to ·25 Å.U. If from Table I. we take 10° as the angle at which the ratio begins to increase for gamma rays, the effective wave-length of these waves is by equation (2) between 0·025 and 0·030 Å.U. According to the present experiment this may therefore be taken as the approximate wave-length of hard gamma rays which have traversed about 8 mm. of lead.

The three principal elements of uncertainty which enter into this calculation are: (1) the wave-length 0·2 Å.U. at 90° is an extrapolated value, (2) a possible error in the experiments, and (3) the lack of positive evidence that the radiation measured in these experiments contains an appreciable fraction of truly scattered rays. None of these difficulties seem sufficiently serious to render improbable the correctness of the result as to order of magnitude. Indeed this method of estimating the wave-length of hard gamma rays is perhaps the most direct one that has been employed, and in as far as its results are in agreement with the predicted values considered at the beginning of the paper, they may be taken as a support of the theoretical bases of these predictions.

The writer performed this experiment at Cavendish Laboratory as National Research Fellow. He desires to thank Professor Sir E. Rutherford for the free use of the laboratory facilities and for valuable suggestions with regard to the experimental procedure.

Washington University,
Saint Louis, U.S.A.
December 1, 1920.

LXX. *Dissociation of Hydrogen and Nitrogen by Electron Impacts.* *By* A. LL. HUGHES, *D.Sc., Research Professor of Physics, Queen's University, Kingston, Canada**.

HYDROGEN.

LANGMUIR † found that hydrogen could be dissociated by contact with a tungsten (or other metallic) wire when its temperature was raised above 1300° K. He calculated from his experiments that the work necessary to dissociate a gram molecule of hydrogen was 84,000 calories. This may be expressed in terms of the energy necessary to dissociate a single molecule. It is the energy which an electron would acquire in falling through a potential difference of 3·6 volts.

It occurred to the writer that possibly the molecules could be dissociated by direct impact of electrons possessing energy in excess of that corresponding to 3·6 volts. The experiment failed to show any appreciable dissociation by electrons whose energy corresponded to 3·6 volts. It, however, led to an investigation of the disappearance of hydrogen when subjected to bombardment by electrons possessing higher energies. A number of investigations have from time to time been carried out on the reduction in pressure when an electric discharge is passed through a gas. In some respects, the central idea of the present investigation is new. The experimental conditions are simplified so as to give as direct information as possible as to the ratio of the number of molecules disappearing from the gas in terms of the number of collisions between electrons and molecules, for different values of the electron velocities. A stream of electrons is passed through hydrogen and the progressive decrease in pressure noted. It is assumed, for reasons discussed later, that the hydrogen which disappears, does so, because it is dissociated into atoms which condense on the walls when they strike them.

Apparatus.—The final form of apparatus used is shown in fig. 1. The earlier experiments showed the necessity for trapping the atomic hydrogen as completely as possible. This was secured (as in some of Langmuir's experiments) by keeping the experimental tube in liquid air throughout a set of observations. This called for a source of electrons

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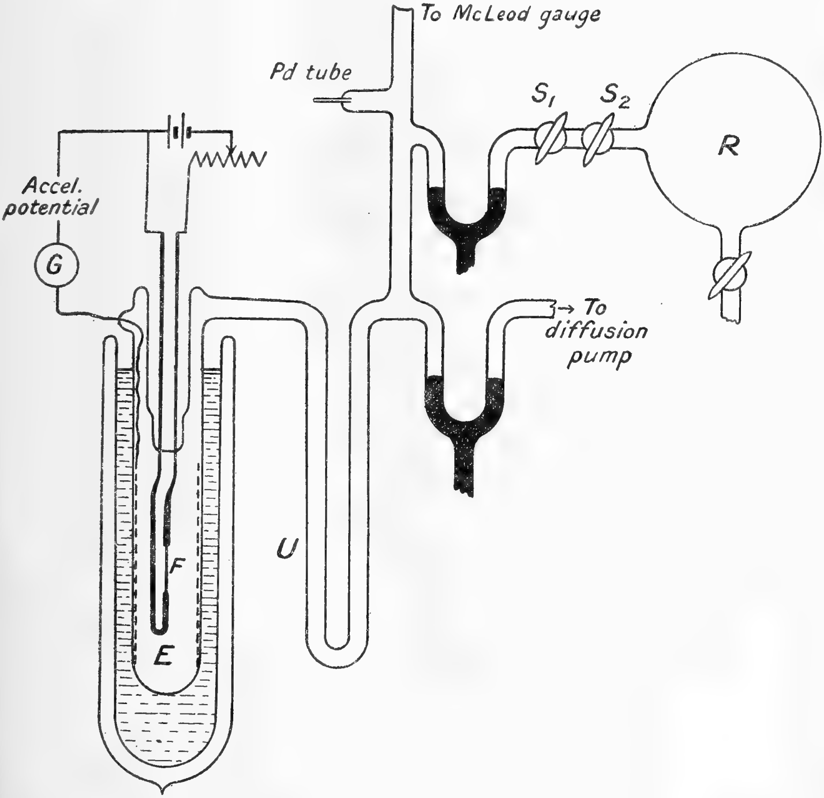
* Communicated by the Author.

† Langmuir, Journ. Amer. Chem. Soc. xxxvii. p. 451 (1915).

which would give out as little heat as possible in order to conserve liquid air. The experimental tube E was a glass tube 2 cm. in diameter internally. The source of electrons was a platinum strip (F) 1 cm. long and 1 mm. wide, coated with a mixture of BaO and SrO. The anode was a nickel gauze (40 mesh to the inch) which fitted snugly inside the glass tube. The advantage of this type of anode was that there were no large areas of glass surface to become charged up and give erratic results, as so often occurs when electrons and ions have an opportunity to lodge on such areas. The number of electrons available was controlled by a rheostat, while the energy of the electrons was determined by the accelerating potential applied between the gauze and the filament. The experimental tube was connected to a mercury trap and a diffusion pump through a U-tube. The

Fig. 1.



pressure was measured by a McLeod gauge. Hydrogen was admitted through a palladium tube. The apparatus was thoroughly outgassed by prolonged pumping and heating to 400° C. When systematic observations were carried out, the apparatus was subjected to an outgassing for 30 minutes at 400° C. between each run, the filament being heated to a white heat, and the nickel gauze being bombarded by electrons. The runs tabulated below were obtained after several weeks' preliminary testing, during which there had been many short heat treatments with the pump in action, and during which no gas had been admitted except hydrogen up to pressures of from ·001 to ·1 mm. It is extremely unlikely therefore that there could be any impurity in the hydrogen. To keep mercury out of the experimental tube E, it was customary, after heating both E and U, to surround the U-tube with liquid air and to continue the heating and pumping for some minutes to ensure that no appreciable amount of mercury was left in E.

Method of Experiment.—Before admitting the hydrogen, a test was always made (1) on the amount of gas given out when the filament was heated at the temperature to be used in experiment on the hydrogen, and (2) on the amount of gas given out when electrons were driven across the tube. (1) was in general not measurable, and (2) was usually very small compared with the pressure changes observed when the hydrogen was in the apparatus. Then hydrogen was admitted to approximately the pressure desired. A test was now made for the amount of hydrogen cleaned up by the hot filament—the purely thermal effect discovered by Langmuir. It was found that, at any temperature above a dull red heat, there was a very appreciable clean up. The experiments on the clean up due to the electron stream had therefore to be carried out with a lower filament temperature. Fortunately, the filament gave out an ample supply of electrons when heated to a temperature at which it was barely visible in the dark. The electrons were driven across by the accelerating potential chosen, the electron stream being held constant, if necessary, by slight adjustment of the heating current. Pressures were read every four minutes. It was impracticable to take gauge readings oftener than this. At the end of 40 minutes the electron current was stopped and the constancy of the pressure with the filament still hot, but without an electron current, was checked. The residual hydrogen was pumped out to a pressure less than ·00001 mm. Then the evolution of gas due to electron bombardment of the gauze was measured, using the same accelerating potential as before. This was appreciable, especially with the higher accelerating potentials and after large amounts of hydrogen had been cleaned up.

The liquid air had to be replenished from time to time, to keep its surface at the same level around the tube E.

Results.

The clean up with different initial pressures is shown in Table I. The accelerating potential was 71 volts in every run.

TABLE I.

Electron current=650 microamps.
Accelerating potential=71 volts.
Pressures taken at intervals of 4 minutes.

A	B	C	D
5080×10^{-5} mm.	2160×10^{-5} mm.	942×10^{-5} mm.	200×10^{-5} mm.
4380	1602	642	122
4060	1262	482	77
3880	1070	398	52·2
3550	926	338	35·6
3380	800	292	27·4
3200	730	252	19·2
3060	656	230	16·8
2890	592	204	12·4
2740	546	184	9·6
2600	512	164	8·8

The absolute amount of hydrogen cleaned up decreases as the initial pressure is made smaller, but the ratio of the initial pressure to the final pressure increases.

Table II. gives the results obtained on varying the accelerating potential, but keeping the other conditions constant. It was impossible to do more than to get the initial pressure approximately constant.

A slight but definite clean up was observed at 13·3 volts, a pressure decrease from 432×10^{-5} mm. to 408×10^{-5} mm. being observed in 60 minutes with an electron current of 185 microamperes. (This will be referred to as run L.) No evidence of any clean up at all could be obtained when the electrons were accelerated by 8·9 volts. A careful test was made over a period of 90 minutes, and the very slight

TABLE II.

Electron current 650 microamperes.
Initial pressures approximately constant.
Unit of pressure 10^{-5} mm.
Pressure taken at intervals of 4 minutes.

Pressure.

E	F	G	H	I	J	K
291 volts.	141 volts.	71 volts.	43 volts.	28·3 volts.	23·7 volts.	17·9 volts.
918	970	942	970	900	892	884
656	636	642	730	724	738	796
532	480	482	586	612	662	744
462	398	398	512	552	592	708
398	344	338	450	508	572	692
376	294	292	404	456	506	676
344	264	252	364	414	480	662
328	248	230	328	392	450	656
298	224	204	304	360	416	648
288	216	184	278	338	398	642
270	200	164	252	318	380	—

clean up obtained was no bigger than the effect (a small thermal clean up) obtained during the next 90 minutes without an electron current.

Proportionality between electron currents and clean up.—Run G was repeated with the electron current reduced fivefold. The interval between the pressure readings was increased five fold. The readings obtained were practically identical with those of run G, showing that the clean up, over this range of five-fold decrease in the electron current, is proportional to the electron current.

The Ratio $\dfrac{\textit{Number of Hydrogen Molecules cleaned up}}{\textit{Number of Collisions between Electrons and Molecules}}$.

Let

V = total volume of apparatus (= 302 c.c.),
V_1 = part of volume at room temp. (= 267 c.c. approx.),
V_2 = ,, ,, liquid air temp. (= 35 c.c. approx.),
n_1 = no. of molecules per c.c. in V_1,
n_2 = ,, ,, ,, V_2,
n' = total number of molecules ;

then $n' = n_1 V_1 + n_2 V_2 = n_1 \left(V_1 + \frac{n_2}{n_1} V_2 \right)$.

N = number of electrons emitted per unit time,
λ = mean free path of electron (in V_2) ;

then $N\left(1 - e^{-\frac{x}{\lambda}}\right) dt$ = number of electrons which collide with molecules between the filament and gauze in time dt. ($x = 1$ cm.)

Let b = fraction of collisions resulting in disappearance of the molecule from the gas, and
dn' = decrease in the total number of molecules in dt.

Then $dn' = bN(1-e^{-\frac{1}{\lambda}})dt$,

$$dn_1\left(V_1+\frac{n_2}{n_1}V_2\right)=bN(1-e^{-\frac{1}{\lambda}})dt,$$

n_1 the number of molecules per c.c. in V_1 is proportional to the pressure p. Hence $n_1=ap$ ($a=3{\cdot}55\times10^{11}$ at 10^{-5} mm. pressure at 0° C.).

$$a\,dp\left(V_1+\frac{n_2}{n_1}V_2\right)=bN(1-e^{-\frac{1}{\lambda}})\,dt.$$

Hence

$$b=\frac{a\left(V_1+\frac{n_2}{n_1}V_2\right)}{N(1-e^{-\frac{1}{\lambda}})}\frac{dp}{dt}.$$

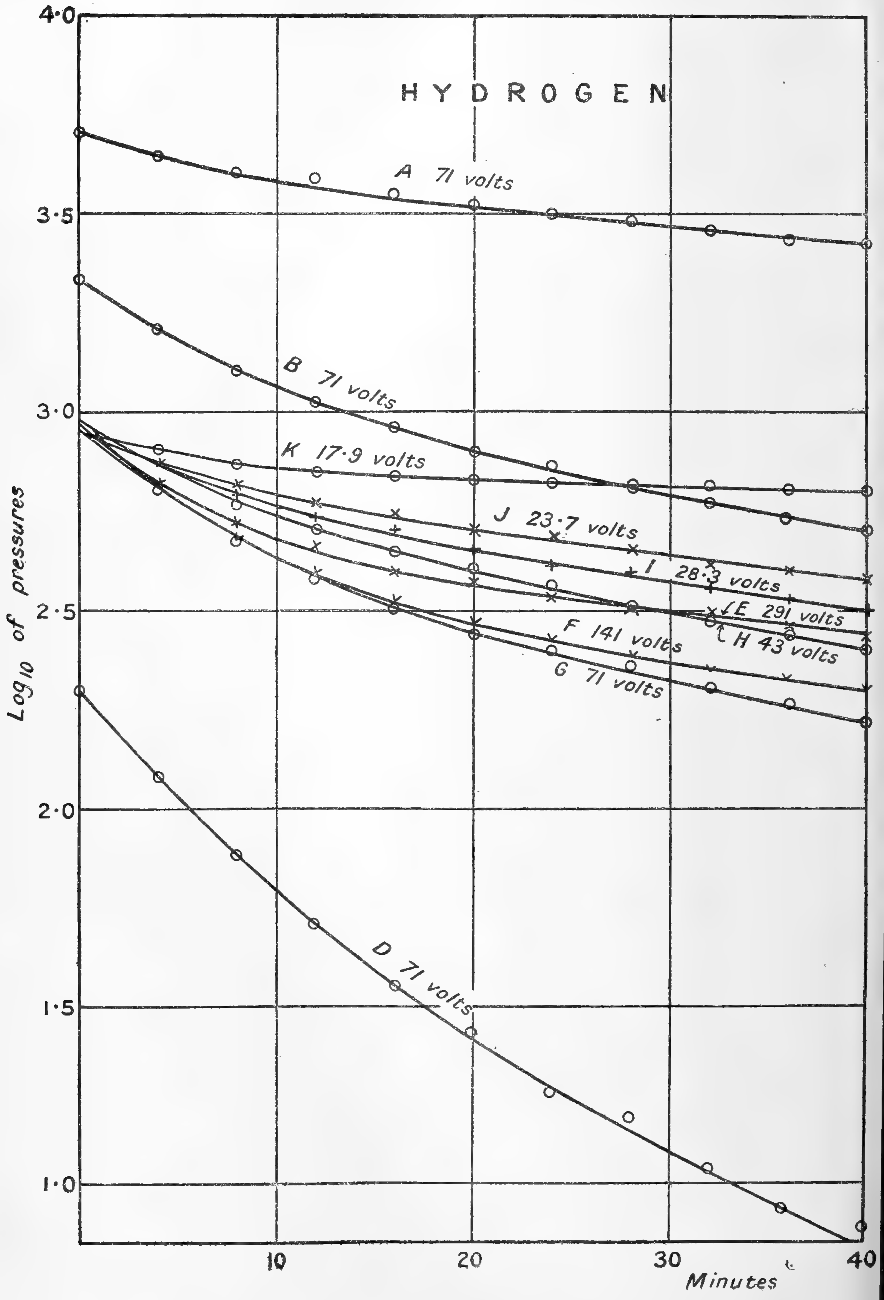
Numerical values can be substituted for all the symbols on the right-hand side and so b can be calculated. Before doing so, it is well to consider some possible implications of the equation. If the mean free path, λ, of an electron is considerably larger than the path (1 cm.) from the filament to the gauze, then the factor $(1-e^{-\frac{1}{\lambda}})$ may be written $\frac{1}{\lambda}$, which in turn is proportional to the pressure p. Hence the equation becomes

$$b=\frac{1}{p}\frac{dp}{dt}\times\text{const.}$$

$$=\frac{d(\log p)}{dt}\times\text{const.}$$

Now if b were a constant, *i. e.* if the number of molecules disappearing always bore a constant ratio to the number of collisions, we should have $\log p$ a linear function of t. The results given in Tables I. and II. are shown in fig. 2 in which the ordinates are $\log_{10} p$. In no case does $\log_{10} p$ appear to be a straight line, the curves all show a decreasing rate of disappearance. This can be explained on the assumption that the surface takes up the gas which has disappeared and that as the area free to take up gas diminishes, the rate of clean up must diminish. Thus superposable curves were always obtained when starting with the same initial pressure and using the electrons of the same energy, provided that

Fig. 2.


HYDROGEN
A 71 volts
B 71 volts
K 17·9 volts
J 23·7 volts
I 28·3 volts
E 291 volts
H 43 volts
F 141 volts
G 71 volts
D 71 volts
4·0
3·5
3·0
2·5
2·0
1·5
1·0
10
20
30
40
Minutes
Log10 of pressures

each time there had been a thorough outgassing. If the outgassing were omitted, other conditions being the same, the rate of clean up was diminished considerably. For the two lower pressures (the columns C and D, Table I.) the log curve corresponding to the lower pressure of the two has less curvature, as might be expected from the fact that the amount of gas which has disappeared and which retards the clean up is always less in case D than in case C. (C and G are identical.)

To calculate "b" for each run, it is clear that we must take the value of dp/dt when $t=0$, for this is the time when "b" is least affected by the supposed inability of the surface to take up all the dissociated gas reaching it. It was found convenient to deduce dp/dt at $t=0$ from the initial slope of the corresponding log curve by means of the relation $\frac{dp}{dt}=2\cdot30\frac{p\,d\log p}{dt}$. The pressure unit was 10^{-5} mm., the time unit was 1 minute. N the number of electrons emitted in one minute was found to be $2\cdot44\times10^{17}$ (from the electron current of 650 microamps.).

There remain to be calculated the free path λ for the *electron*, and n_2/n_1 for the ratio of the densities of the gas in the parts of the apparatus at liquid air temperature and at room temperature respectively. When the mean free path of the *molecule* is considerably less than the distance apart of the walls of the apparatus the ordinary gas laws hold, and $n_2/n_1=\rho_2/\rho_1=T_1/T_2$ where T_1 and T_2 are the absolute temperatures. However, as Knudsen * has found, when the mean free path of the molecule is considerably less than the diameter of the tubes, another set of gas laws is applicable, from which we get $n_2/n_1=\rho_2/\rho_1=\sqrt{T_1/T_2}$. We shall refer to these two sets of laws as the high pressure, "H.P.", and the low pressure, "L.P.", laws respectively. The critical distance in this apparatus is about 3 mm., this being the distance between the inner tube and outer tube at the level of the surface of the liquid air, for this is the place where the temperature transition occurs. (The hydrogen molecule has a mean free path of 3 mm. at 290×10^{-5} mm. pressure at 0° C.)

There would have been less ambiguity if the experiments could have been carried out at pressures much below the critical values, where the "low pressure" laws apply accurately. This, unfortunately, would have meant restricting the range of observations to within a very few mm. at the

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* Knudsen, *Ann. der Phys.* xxxi. p. 205 (1910).

top of the McLeod gauge. It was decided, therefore, to start at higher pressures in spite of the fact that sometimes neither set of gas laws would apply accurately:

$$\frac{n_2}{n_1} = \frac{293}{90} = 3{\cdot}25 \quad \text{if "high pressure" gas laws apply.}$$

$$= \sqrt{\frac{293}{90}} = 1{\cdot}80 \quad \text{if "low pressure" gas laws apply.}$$

$\lambda =$ mean free path of an *electron* in the experimental tube at 90° K. This was taken as $4\sqrt{2} \times$ m.f.p. of a *molecule*. There will be two values according as to whether the "high pressure" or "low pressure" laws apply. Jeans gives $11{\cdot}6 \times 10^{-6}$ cm. as the m.f.p. of the hydrogen molecule at 760 mm. pressure and 0° C.

The values of "*b*" calculated for runs A, B, C, and D (Table I.) are given in Table III. For the initial pressures A and B, only the "H.P." values are admissible.

Table III.

Values of "*b*" $= \dfrac{\text{Number of molecules disappearing.}}{\text{Number of collisions between electrons and molecules.}}$

Accelerating Potential = 71 volts.

		A	B	C	D
Initial Pressure =		5080 $\times 10^{-5}$ mm.	2160 $\times 10^{-5}$ mm.	942 $\times 10^{-5}$ mm.	200 $\times 10^{-5}$ mm.
m.f.p. of H_2 molecule	"H.P." law....	·053 cm.	·124 cm.	·286 cm.	1·30 cm.
	"L.P." law....	—	—	·520 cm.	2·45 cm.
"*b*"	"H.P." law............	·129	·136	·122	(·115)
	"L.P." law............	—	—	·168	·178

The initial pressure in C is in the region where neither law is strictly applicable. The real value of "*b*" will, therefore be between ·122 and ·168 for C. For the pressures used in D, "L.P." laws are evidently to be used and the "L.P." value of "*b*" should be taken.

It is seen that "*b*," which measures the clean up per collision, increases progressively as the initial pressure decreases. At the lowest pressure, D, about 1 out of every 6 collisions results in the disappearance of a molecule. Here the conditions are the simplest, the electrons practically never make two collisions, and the atoms resulting from a collision with an electron have a good chance of reaching the walls and condensing there.

In Table IV. the values of "b" are given for approximately the same initial pressure ($\triangleq 950 \times 10^{-5}$ mm.), but for different accelerating potentials. Runs E, F, ... K (Table II.) were used. (The initial pressure in run L, which is included, was only 430×10^{-5} mm.).

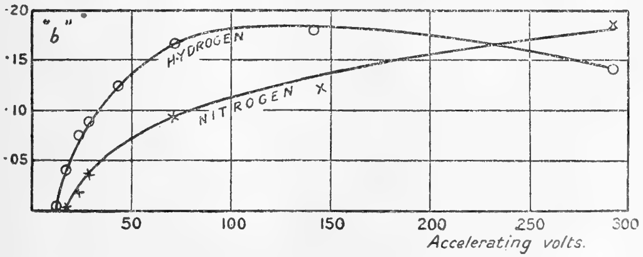
Table IV.

Values of "b."

		E	F	G	H	I	J	K	L
		291 volts.	141 volts.	71 volts.	43 volts.	28·3 volts.	23·7 volts.	17·9 volts.	13·3 volts.
"b"	"H.P." law.	·100	·133	·122	·088	·064	·056	·029	·0031
	"L.P." law.	·141	·180	·168	·122	·088	·077	·041	·0045

For 8·9 volts, "b" could not be measured, it was certainly less than ·0005. The values for "b" are shown in fig. 3. It will be seen that "b" rises rapidly between 13 and about 40 volts, and does not change much after about 60 volts. (An earlier experiment showed that "b" was much the same for 600 volts as for 70 volts; these later and more accurate experiments indicate a drop between 140 and 290 volts. It is proposed to investigate this further.)

Fig. 3.



Control Experiments with Helium.—Helium was admitted to a pressure of 950×10^{-5} mm., and a run was taken with 71 volts accelerating potential. The clean up was less than one two-hundredth part of that in the corresponding hydrogen run (G). The helium was purified for a short time over charcoal in liquid air. The slight clean up was no doubt due to impurities, probably hydrogen, as the apparatus was not outgassed for more than half an hour, nor were excessive precautions taken to secure pure helium.

DISCUSSION.

It is necessary to give reasons for believing that the disappearance of the gas is due to dissociation of hydrogen into atoms which condense on the walls.

Langmuir* found that pure hydrogen at a low pressure in a bulb containing an incandescent filament disappeared. On carrying out the experiment with the bulb, or a side tube, immersed in liquid air, a peculiar effect was noticed. After the pressure decrease had been in progress for some time, the liquid air was removed (the filament heating current being cut off) and the pressure rose. On replacing the liquid air, the pressure fell a little, but *not* to its original value. There was, therefore, as Langmuir terms it, a *non-recondensible* gas. Langmuir accounted for this on the supposition that hydrogen molecules on impact with the very hot filament were dissociated into atoms, and that these atoms, if they had a clear run to the walls, would condense on them, the effect being the more marked at low temperatures. On removing the liquid air, some of the atoms would come off and re-combine with other atoms to form *molecules* which could not be condensed. Thus Langmuir gives a natural explanation for the "non-recondensible" gas. The more atoms already on the surface, the greater the chance of an atom impinging on the surface, striking one of them and recombining to form a molecule, which leaves the surface and so reduces the apparent rate of dissociation of the gas. From experiments on the transfer of heat through hydrogen from an incandescent filament, Langmuir calculated the amount of hydrogen dissociated. He states that, under the most favourable conditions, the observed decrease in pressure corresponds to only 1/7 of the amount actually dissociated, and may often be much less.

Non-recondensible gas was obtained in these experiments. The following is an illustration from one set of observations. Hydrogen was admitted to a pressure of 548×10^{-5} mm. and cleaned up by electron impact to a pressure of 100×10^{-5} mm. The liquid air was then removed, the pressure rose, the liquid air was replaced and the pressure fell to 263×10^{-5} mm. Hence the amount of non-recondensible gas was proportional to $(263-100) \times 10^{-5}$ mm., and the amount of hydrogen which originally disappeared was proportional to $(548-100) \times 10^{-5}$ mm. Thus the amount of non-recondensible gas formed 37 per cent. of the amount of hydrogen cleaned up. Then, *without outgassing*, more hydrogen was admitted until

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* Langmuir, Journ. Amer. Chem. Soc. xxxvii. p. 451 (1915).

the pressure was again about 550×10^{-5} mm. On repeating the experiment the yield of non-recondensible gas was 54 per cent. The procedure was carried through twice again, the yields rising to 67 per cent. and 81 per cent. Thus, taking the presence of non-recondensible gas as a criterion of dissociation of hydrogen, there is strong evidence in favour of the view that hydrogen can be dissociated by electron impacts. The progressive rise in the yield corresponds to the decreasing amount of surface available for the atomic hydrogen.

Let us consider some alternative explanations. There may be chemical action between some gaseous impurity (which must obviously be gaseous at 90° K) and the hydrogen when ionized by electrons. It is extremely unlikely that the impurity would be present in sufficient quantity to account for a clean up of as much as four-fifths of the hydrogen time after time. It may be urged that hydrogen when ionized combines with the nickel, or something on it or on the glass. But in neither of these alternatives is there any reason to suppose that the compound would show the exceedingly characteristic non-recondensible effect. We may now take up the behaviour of ionized hydrogen, which in some way may show the non-recondensible effect. Take first the positively charged hydrogen molecule. Such a charged molecule would be driven to the filament, and to account for the clean up found in many runs, a layer of hydrogen 100 molecules deep would be formed on the filament. Again, a clean up of this type would be unaffected by removing the liquid air if the filament were kept hot. Such was not the case. Take now the case of negatively charged hydrogen molecules. These would be driven to the gauze. If they stuck there by virtue of their charge, a very small clean up would lead to surface charge sufficient in amount to annul the field and reduce the electron current almost to zero, which is not the case. If, on the other hand, they gave up their charge, there is no reason to suppose that they would stay on the surface any more than any other uncharged molecule. Another objection is that electrons with energies less than 10 volts would be as likely to form a negative ion by uniting with a molecule as electrons with greater energies. The clean up, however, below 13 volts is very small. Still another reason against this view is that various investigations have shown that the negative ion in hydrogen is an electron, proving that there is no tendency for a hydrogen molecule to unite with an electron.

Franck, Knipping, and Krüger [*] have recently published an account of the ionizing potentials of hydrogen, and they suggest that there are two types of ionization. The first type is ionization, without dissociation, when the electrons have energies in excess of 11 volts. The second type of ionization is accompanied by dissociation. Since 3·6 volts measures the energies required for dissociation and 13·5 volts measures the theoretical energy required for ionizing one atom, there should be an ionizing potential at $3·6+13·5=17·1$ volts, corresponding to dissociation and ionization of one of the atoms. They verified this experimentally. Similarly they found a radiating potential at 13·6 volts corresponding to dissociation combined with radiation from one of the atoms ($3·6+10·2=13·8$ volts). They also verified the existence of an ionizing potential at $3·6+2\times13·5=30·6$ volts, corresponding to dissociation and ionization of *both* atoms. On this theory, then, one would expect dissociation to accompany these types of ionization of, and radiation from, hydrogen. On account of the first type of ionization (11 volts), it is possible that all ionizing collisions are not all dissociating collisions.

Wendt [†] has found that if hydrogen is driven through a tube across which an electric discharge is passing, the hydrogen afterwards contains a small amount of triatomic hydrogen H_3. In his experiments, the pressure was high enough to insure that the mean free path of a molecule was considerably less than the diameter of the tube, so it is natural to infer that the first step in the formation of H_3 is the dissociation of H_2 by electron impacts into atoms H, which in turn unite with molecular hydrogen at the next collision to form H_3. It is difficult to see any way of accounting for H_3 without the intermediate step of the formation of H. Dempster [‡] has found evidence for the view that the formation of H is a necessary step in the formation of H_3. The hydrogen spectrum contains several series (*e.g.*, the Balmer series and the Lyman series), which are, according to Bohr's theory, characteristic of the atom. It thus appears that there are good grounds for believing that atomic hydrogen is formed in the electric discharge, and probably as a result of direct impact of electrons on molecules.

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[*] Franck, Knipping, and Krüger, *Verh. d. Deutsch. Phys. Ges.* xxi. p. 728 (1919).

[†] Wendt, Nat. Acad. Sci. Proc. v. p. 518 (1919).

[‡] Dempster, Phil. Mag. xxxi. p. 438 (1916).

It appears then that the simplest explanation of the disappearance of the hydrogen in this investigation is that it is due to the dissociation of the hydrogen into atomic hydrogen which condenses on the walls, especially at low temperatures. The experimental value "*b*" for the yield of atomic hydrogen per collision is probably too low for several reasons.

(*a*) Some atoms may collide with other atoms forming ordinary molecular hydrogen which does not condense.

(*b*) Some atoms may collide with hydrogen molecules forming H_3 which may not condense as readily.

(*c*) An atom may hit a spot on the cold surface already occupied by another atom. A hydrogen molecule will be formed and will leave the surface. Hence this dissociation does not contribute to the pressure decrease.

(*d*) If, after an appreciable clean up of hydrogen, the gas is completely pumped out, and electrons are accelerated so as to bombard the gauze and glass, it is found that there is a considerable evolution of hydrogen, much more than after a thorough outgassing. This effect no doubt goes on all the time during a clean up, and tends to reduce the rate of clean up, especially towards the end of a run. (This evolution due to bombardment increases, for a given electron current, with the accelerating potential, but not so quickly. Calculation shows that the total heating effect at the gauze of such a bombardment is absolutely negligible.) These effects are more clearly marked at the higher pressures, and no doubt influence somewhat even the *initial* values of the clean up.

(*e*) It may be that not every atom which strikes a cold surface condenses. Langmuir states that the measurements of the clean up in a cold bulb in which hydrogen is being dissociated thermally by an incandescent filament, gives under the most favourable conditions a value only 1/7 of the amount of dissociation actually occurring as deduced from the measurements of heat transfer through the gas. This factor is no doubt determined largely by the dimensions and shape of the bulb and the conditions of the experiment, and it would not be justifiable to multiply our experimental determination of "*b*" by 7 to get the real amount dissociated. Nevertheless it is significant that the highest value of "*b*," viz. ·18 (Tables III. and IV.), is roughly 1/6 of the maximum possible yield, *i.e.*, one dissociation per collision.

The hydrogen curve in fig. 3 shows the yield for different accelerating potentials. It attains a value not far from its maximum at about 70–100 volts. As the electrons under our experimental conditions may collide with any velocity between zero and that corresponding to the applied accelerating potential, the curve which would be obtained if the electrons always collided with the energy corresponding to the applied potential would attain its maximum at a lower value, possibly somewhere between 40 and 80 volts. (One disadvantage of working at higher pressures, *e. g.* A in Table III., is that the electrons are forced to make most of their collisions close to the filament, so that their energies on collision are less than would be inferred from the applied field.)

On the basis of Franck, Knipping, and Krüger's view of ionization, there are good grounds for believing that ionization and dissociation of hydrogen are bound up with one another. So far as this writer knows, there are no results giving the efficiency of ionization per collision as a function of the energy of the electrons, *i. e.* the fraction of collisions between electrons and molecules resulting in ionizations. Johnson * has made some experiments on the total ionization produced in a gas by electrons. He found that the accelerating potential had to be raised to about 50 volts before he obtained one ion per electron passing through the gas. This is not quite the same thing as saying that a collision between a hydrogen molecule and an electron possessing energy corresponding to 50 volts always results in ionization, but, for want of more direct information, we may tentatively assume that an electron of such energy will give ionization at practically every impact. If, at the higher voltages, ionization of the 17 volt type predominates, *i. e.* dissociation of the molecule combined with ionization of one atom, then each collision would result in the dissociation of the molecule and the production of one ion. This is what our investigation suggests, if we make use of Langmuir's result, that the rate of clean up measured, under the most favourable conditions, only one seventh of the amount of dissociation really taking place. (It is admitted that the application of Langmuir's numerical result to this investigation is open to question, but it seems certain that the clean-up measurements underestimate the amount of dissociation.) The view that the efficiency of dissociation is unity, *i. e.* that every collision between a molecule and an electron will result in dissociation when the electron has enough energy, has an attractive simplicity.

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* Johnson, Phys. Rev. x. p. 609 (1917).

No measurable amount of dissociation could be obtained when electrons of energy 8·9 volts were used. One might expect some dissociation at, or above, 3·6 volts, since this corresponds to the measured work of dissociating the hydrogen molecule. Either no dissocation at all occurs, or else it is produced in so small an amount as not to be detected in these experiments. The latter would demand very special circumstances, either in the momentary state of the hydrogen molecule, or in its orientation at impact, to give dissocation at these low values. The alternative view is that dissociation cannot occur until radiation of the proper type (the 13·5 volt effect, corresponding to dissociation and radiation from one atom) or ionization (the 17·0 volt effect corresponding to dissociation and ionization of one atom) takes place, *i. e.* dissociation by electron impacts must accompany some other effect.

The "clean up" of hydrogen by say an electric discharge in the presence of sodium or potassium, is probably due to combination between the metal and the atomic hydrogen produced. It is possible that lining the experimental tube with sodium would show a more rapid clean up than in the experiment if the chemical forces were stronger than the forces controlling temperature condensation.

NITROGEN.

Preparation.—Nitrogen was prepared by warming a solution of ammonium chloride and sodium nitrite. Ten litres were collected over water. This was passed slowly through calcium chloride, soda lime, red-hot copper turnings, and phosphorus pentoxide, into a reservoir R of one litre capacity permanently attached to the apparatus (fig. 1). Several litres of the nitrogen were used to wash out the apparatus before finally filling the reservoir. Small quantities of nitrogen could be admitted into the apparatus by manipulating the stopcocks S_1 and S_2.

Method of Experiment.—Experiments were conducted in the same way as those on hydrogen.

Results.

The clean up of nitrogen for different accelerating potentials is shown in Table V. The experiments on nitrogen were not as extensive as those on hydrogen. No experiments have, up to the present, been carried out on the effect of varying the initial pressure.

TABLE V.

Electron Current 650 microamps.
Initial Pressures approx. const.
Unit of pressure $= 10^{-5}$ mm.

Time interval (mins.).	Exp. tube E in liquid air.						Exp. tube at 20° C.	
	M	N	O	P	Q	R	S	T
	291 volts.	145 volts.	72 volts.	29 volts.	23·8 volts.	17·8 volts.	72 volts.	72 volts.
	596	566	520	520	552	650	900	890
3	364	376	376	456	520	650	662*	724
3	288	304	294	416	494	630	580	642
3	244	264	264	392	464	630	532	578
3	212	224	234	382	444	612	480	532
3	181	204	208	348	420	612	444	506
5	144	176	192	316	404	600	392	450
5	118	162	166	282	388	586	354	404
5	101	141	144	274	370	580	318	376
5	94	122	132	260	348	580	392	344
5	88	112	128	255	344	572	260*	328

* The first and last time intervals in S were 4 mins. The other intervals were as shown in the first column.

Runs S and T were taken under different conditions from the rest. In run S the whole apparatus was at room temperature so that mercury vapour had access to the experimental tube. In run T the U-tube was surrounded by liquid air to freeze out mercury from the experimental tube. The initial pressures were chosen so as to give the electrons much the same mean free path as in the runs M, N, ... R.

The Ratio $\dfrac{\textit{Number of Nitrogen Molecules cleaned up}}{\textit{Number of Collisions between Electrons and Molecules}}$.

The value of "*b*" given in Table VI. was calculated for nitrogen in exactly the same way as for hydrogen. The calculations of various mean free paths were based on the value given by Jeans in his 'Dynamical Theory of Gases,' viz. $5·7 \times 10^{-6}$ cm. at 760 mm. pressure and 0° C.

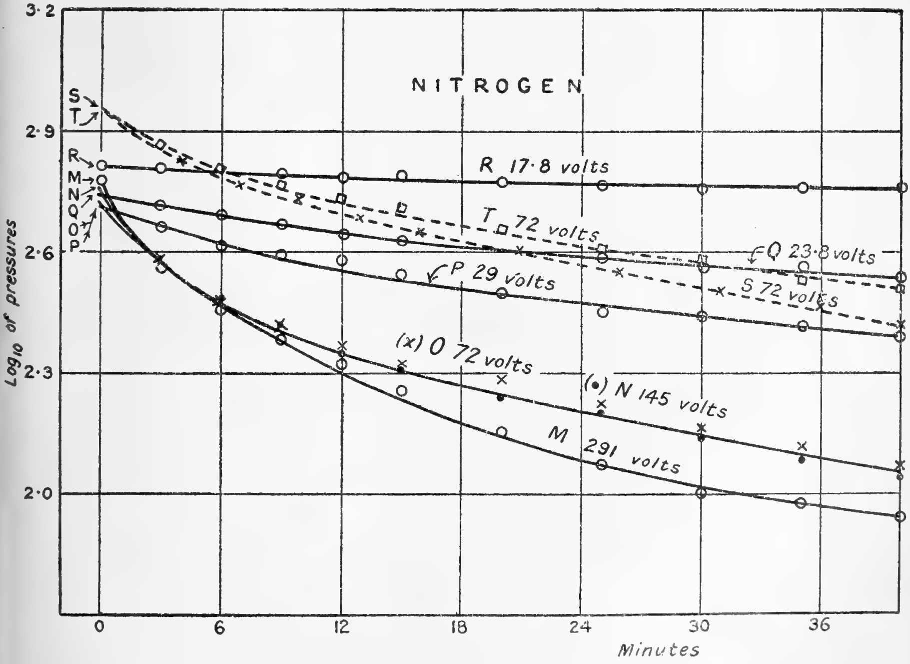
It was rather surprising that the clean up without liquid air around the experimental tube (runs S and T) were as great as they were.

TABLE VI.

Values of "*b*" for nitrogen.

	M	N	O	P	Q	R	S	T
	291 volts.	145 volts.	72 volts.	29 volts.	23 8 volts.	17·8 volts.	72 volts.	72 volts.
"*b*" "H.P." Law	·141	·095	·070	·028	·0135	·0019	·11	·091
"*b*" "L.P." Law	·186	·123	·094	·038	·0185	·0025		
m.f.p. for molecule. "H.P." Law ..	·24 cm.						·48 cm.	
"L.P." Law ...	·44 cm.							

Fig. 4.



The values for "*b*" for runs M, N, ... R are plotted in fig. 4.

Discussion.

Strutt[*] found that nitrogen from a discharge-tube had active properties, and concluded that it was atomic nitrogen rather than a more complex type of molecule. He mentions that no reduction in pressure was observed when the discharge-tube was immersed in liquid air during the passage of the current. Wendt and Grubb[†] have shown that N_3 exists in a stream of nitrogen drawn through a discharge-tube. (As in Wendt's experiments on hydrogen, the gas was at a higher pressure than in these experiments, a condition which favours collisions of atomic nitrogen with molecular nitrogen to form N_3.) No temperature dissociation of nitrogen such as Langmuir found for hydrogen has been observed. Hence we have nothing to give us an idea of how much the clean-up method underestimates that amount of dissociation of nitrogen (should this be the explanation for the clean up).

The strongest evidence for the view that nitrogen is dissociated in these experiments is that the "non-recondensible" effect was obtained. Thus, after a thorough outgassing, the ratio of the non-recondensible gas to the amount cleaned up was 16 per cent.; after admitting more nitrogen without outgassing, and repeating, the ratio rose to 31 per cent.; and on repeating the procedure for a third time the ratio amounted to 67 per cent. Less nitrogen was recovered as a rule than in the corresponding experiments on hydrogen. In one experiment, with both the tubes surrounded by liquid air during the clean up, 25 cu. mm. (referred to atmos. press. and temp.) were "lost" after a considerable clean up, on warming up to room temperature. On warming the tubes to 180° C. for 20 minutes, 7·1 cu. mm. were recovered (nearly all of which came off in the first five minutes). On warming to 265° C. for 20 minutes, 2·9 cu. mm. came off, and on further warming for 20 minutes to 390° C., 6·1 cu. mm. came off. In both the latter cases, as in the first, the evolution was about complete in the first five minutes. Thus, in all, 17·1 cu. mm. were recovered, out of the 25 cu. mm. which had disappeared. These results show that the nitrogen disappearing during a clean up clings rather tenaciously to the walls of the apparatus even at fairly high temperatures.

Attention is called to the large values of "*b*" for runs S and T, where there was no liquid air around the experimental tube to accelerate the clean up, as was thought

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[*] Strutt, Proc. Roy. Soc. lxxxv. p. 219 (1911).

[†] Wendt and Grubb, Science, lii. p. 159 (1920).

would happen from analogy with hydrogen. When mercury vapour is prevented from reaching the experimental tube by immersing the U-tube in liquid air, a slightly smaller value for "*b*" is obtained (compare runs S and T). Possibly mercury vapour in the experimental tube helps to "fix" the atomic nitrogen as it is formed.

It will be noticed (fig. 3) that the value of "*b*" for nitrogen is considerably smaller than for hydrogen, as long as accelerating potentials below 200 volts are considered. At higher potentials the nitrogen curve crosses the hydrogen curve.

It is proposed to extend these investigations further. The evidence so far obtained points to the desirability of working at very low pressures, when the simplest conditions are obtained. Possibly a Knudsen gauge or an ionization gauge will be more satisfactory than a McLeod gauge.

A peculiar clean-up effect.—The first experiments on nitrogen gave very erratic results. These were traced down to an effect which was obtained with nitrogen, but not with hydrogen. If the filament in E were kept warm (*i.e.*, at any temperature above a barely visible red heat) and the tube E surrounded by liquid air, the pressure remained absolutely steady provided that there was no electron current. If the filament heating current were cut off, the pressure fell slowly. In one experiment, the pressure fell from 32×10^{-5} mm. to 4×10^{-5} mm. in 120 minutes. On heating the filament, the initial pressure was regained in so short a space of time as 30–40 seconds (the liquid air was being around the tube all the time). This cycle of events could be repeated indefinitely, and the initial pressure could always be regained by a short heating. A saturation effect was shown in this type of clean up. Thus, starting with a pressure of 4060×10^{-5} mm. with liquid air around the tube and the filament at a dull red heat, it fell to 3120×10^{-5} mm. in five minutes after cutting off the heating current. The fall in the next fifteen minutes was only to 3040×10^{-5} mm. On heating the filament for less than a minute, the pressure rose to the initial value 4060×10^{-5} mm. The change of pressure due to cooling the tubes by liquid air was only that corresponding to the temperature change. Hence this effect is located at the filament. The only substances in this experiment that can account for this type of clean up are platinum, BaO, and SrO, and possibly the copper leads to the platinum. The effect calls for further investigation.

Summary.

The disappearance of hydrogen and nitrogen at low pressures when an electron stream is passed through them has been investigated. For hydrogen, no disappearance was obtained unless the electrons had energy above 13 volts. The rate of disappearance rose rapidly as the energy of the electron was increased to about 70 volts, after which no rapid change was noted (the rate appeared to diminish somewhat when the energy of the electrons was raised from 150 to 300 volts). For nitrogen, the rate of disappearance was at first much less than for hydrogen, but when the energy of the electrons was raised sufficiently (roughly 200 volts) the rate of disappearance of the nitrogen exceeded that for hydrogen.

Reasons are given for believing that this disappearance is due to the splitting of the molecules into atoms when electrons collide with the molecules, and that these atoms condense on the adjacent surfaces particularly if they are cold. The maximum rate of disappearance of hydrogen occurred when electrons of energy corresponding to 140 volts were used. About one molecule disappeared for every six collisions. Langmuir's work on the thermal dissociation of hydrogen showed that the clean up, even under the most favourable circumstances, accounted for only one-seventh of the amount of dissociation. Hence it is possible that when the electrons have the right amount of energy a dissociation may occur almost at every collision.

A peculiar clean up, apparently due to absorption of nitrogen by a platinum filament covered with BaO and SrO when cooled to the temperature of liquid air, was noticed.

The author wishes to express his thanks to Mr. A. E. Harkness for his assistance in taking some of the observations.

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December, 1920.

LXXI. *Note on the Abnormality of Strong Electrolytes.* *By* DAVID LEONARD CHAPMAN *and* HERBERT JOHN GEORGE *.

IN a series of papers recently published in the 'Journal of the Chemical Society' (Trans. Chem. Soc. cxiii. pp. 449, 627, 707, and 790 (1918)) J. C. Ghosh has endeavoured to explain the anomalous behaviour of strong electrolytes. From his theory the electrical conductivities and the osmotic pressures of solutions of salts can be deduced, and the values thereby calculated for the ranges of concentration examined are in good agreement with the experimental results. This theory has attracted much attention, and on the whole has been favourably received, although also it has been subjected to a searching adverse criticism from several points of view by J. R. Partington (Trans. Far. Soc. xv. p. 98 (1919)).

For a deductive theory it embodies an unusually large number of assumptions. Thus the author postulates that

(1) the dissolved salt is completely ionized ;
(2) the mean disposition of the charged ions is regular, and similar to the arrangement of the atoms when they assume a crystalline structure ;
(3) the component ions of a salt molecule form a completely saturated electrical doublet, and the work necessary for separating the component ions of a molecule is the electrical work, A, done in moving the ions constituting a doublet from their fixed mean distance in the solution to an infinite distance apart ;
(4) the free ions (conducting ions) are those whose kinetic energy is greater than the work to be done in separating them to an infinite distance.

In addition to these the author makes other implicit and stated assumptions which it is not necessary to specify here.

Our present purpose is merely to indicate what appears to us to be a false deduction from the postulate (4) quoted above, and to show that, if the proper correction is made, the theory in its present form fails to account for the facts. It is sufficient to consider the simplest case to which the theory has been applied—namely, that of a uni-univalent salt such as potassium chloride. According to Ghosh, it follows, in this case, from postulate (4) that the kinetic energy of a free ion must exceed $\frac{A}{2}$, *i. e.*, half the work

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* Communicated by the Authors.

required to separate the ion from its partner; and he states that the number of ions which satisfy this condition is given by the expression

$$Ne^{-\frac{A}{2kt}}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (A)$$

in which N is the total number of ions, k the gas constant for a single molecule, and t the absolute temperature. The latter statement is we believe fallacious; for, according to a well-known result in the kinetic theory of gases, the number of molecules which have a velocity in excess of c_0 is

$$4\pi N\left(\frac{m}{2\pi kt}\right)^{\frac{3}{2}}\int_{c_0}^{\infty} e^{-\frac{mc^2}{2kt}} \,.\, c^2 \,.\, dc,$$

m being the mass of a molecule and c its velocity.

On changing the variable from c to $x=\sqrt{\frac{mc^2}{2kt}}$ this number becomes

$$\frac{4N}{\pi^{\frac{1}{2}}}\int_{x_0}^{\infty} x^2e^{-x^2}dx,$$

which, after integration by parts, assumes the form

$$\frac{2N}{\pi^{\frac{1}{2}}}x_0e^{-x_0^2}+\frac{2N}{\pi^{\frac{1}{2}}}\int_{x_0}^{\infty} e^{-x^2}dx.$$

In terms of the probability integral defined by

$$\text{erf } x=\frac{2}{\pi^{\frac{1}{2}}}\int_{x}^{\infty} e^{-x^2}dx$$

this number becomes

$$N\left(\frac{2}{\pi^{\frac{1}{2}}}x_0e^{-x_0^2}+\text{erf } x_0\right). \quad . \quad . \quad . \quad . \quad (B)$$

(*Vide* Jeans, 'The Dynamical Theory of Gases,' 2nd ed. pp. 34, 35.)

However, $\frac{A}{2kt}$ in expression (A) is equal to x_0^2, and therefore the number of free ions as estimated by Ghosh is

$$Ne^{-x_0^2}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (C)$$

Now, the number of free ions divided by the total number of ions is $\frac{\mu_v}{\mu_\infty}$, μ_v being the molecular conductivity at dilution v, and μ_∞ the molecular conductivity at infinite dilution. Therefore, according to Ghosh, the value of this quotient is $e^{-x_0^2}$, whereas, according to (B), it should be $\frac{2}{\pi^{\frac{1}{2}}} x_0 e^{-x_0^2} + \operatorname{erf} x_0$.

The values of e^{-x^2} and $1 - \operatorname{erf} x$ are tabulated in the appendix to Jeans's 'Treatise on the Dynamical Theory of Gases.' With the aid of this table we have calculated and compared below the two sets of values of $\frac{\mu_v}{\mu_\infty}$.

x.	$\frac{\mu_v}{\mu_\infty}$ from (C).	$\frac{\mu_v}{\mu_\infty}$ from (B).
0·2	0·96080	0·99413
0·3	0·91393	0·98075
0·4	0·85214	0·95623
0·5	0·77880	0·91889

A comparison of the two columns of figures is sufficient to show that, as the values of $\frac{\mu_v}{\mu_\infty}$ calculated from the expression (C) are in good agreement with the experimental numbers, the values of the same quotient calculated with the aid of expression (B) must show a considerable discrepancy. Accordingly, Ghosh's theory in its present form is not in agreement with the facts if the number of ions whose kinetic energy exceeds a specified value is correctly given by the commonly accepted formula.

Jesus College, Oxford.

LXXII. *On the Discontinuous Flow of Liquid past a Wedge.* *By* W. B. MORTON, *M.A., Queen's University, Belfast* [*].

IN the well-known case of two-dimensional motion solved by Bobyleff, a wedge is set in an infinitely extended stream in such a way that a stream-line divides at the apex of the wedge and runs along the two sides into the two surfaces of discontinuity which extend to infinity. This requires, for a wedge of given angle set in a given manner relative to the stream, that the breadths of the two sides should be in a definite ratio. The question arises as to the character of the motion when this ratio is departed from.

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[*] Communicated by the Author.

A survey of the conditions of the problem may be obtained as follows. Start with a single plane lamina set at angle α to the direction of the stream at infinity. The stream divides at a point on the upstream side of the centre. Now build out a second plane from the upstream edge of the first, at angle β with the stream, until it reaches the free surface of the liquid. This happens when the ratio of the second breadth to the first is less than the Bobyleff value. Up to this point, of course, the former state of motion persists, the second face of the wedge lying entirely in the region of dead water behind the first. But now a new kind of motion sets in: the stream-line, which passes round the angle of the wedge, is interrupted where it runs along the second plane near its edge. Beginning at the corner, it first sweeps round a pocket of dead water enclosed between it and the second plane, it then becomes tangent to the plane and runs along it to its edge and thence to infinity. There is evidently a point of inflexion before the stream-line joins the plane. As the second plane is extended, the point on the first plane, where the stream divides, moves towards the corner, until we reach Bobyleff's case. Still further extension gives the reverse change in the character of the motion, the rôles of the planes being interchanged until we get to the one-plane case round the second plane, with the first plane lying entirely in the dead water.

It is proposed in the present note to discuss the general or transitional case. The treatment is quite straightforward on the well-known method of conformal representation, but it derives some interest from the fact that there are two different constant values of the velocity on the free portions of the stream-line, one along the infinite branches and another, smaller than this, round the pocket of dead water.

The two critical breadth-ratios can be found from the two known solutions, Kirchhoff's and Bobyleff's. The expressions obtained are, in general, complicated, but become manageable for the special case of a right-angled wedge $\left(\beta=\frac{\pi}{2}-\alpha\right)$. Taking the breadth of the "first plane" as unity, the breadth of the second when it just touches the free surface is

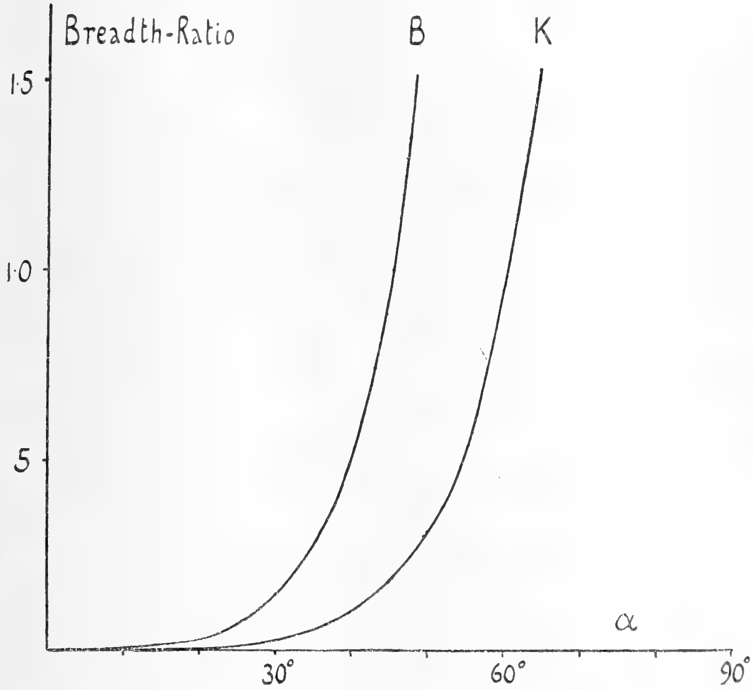
$$\frac{\sin\alpha}{\pi\sin\alpha+4}\left\{\frac{\sin\alpha\sin\theta(1-\cos\alpha\cos\theta)}{(\cos\alpha-\cos\theta)^2}-\log\frac{\sin\frac{1}{2}(\theta+\alpha)}{\sin\frac{1}{2}(\theta-\alpha)}\right\}, \quad . \quad . \quad . \quad . \quad (1)$$

where α is the inclination of the first plane to the stream and θ is given by $\cos\theta=\cos\alpha/(2+\cos\alpha)$. And the value when the stream divides at the corner is

$$\int_0^{\alpha} \sin^{\frac{3}{2}}(\alpha+\phi) \, . \, \sin^{\frac{1}{2}}(\alpha-\phi) \, . \, \sin 2\phi \, d\phi \Big/ \int_{\alpha}^{\frac{\pi}{2}} \sin^{\frac{3}{2}}(\phi+\alpha) \times \sin^{\frac{1}{2}}(\phi-\alpha) \, . \, \sin 2\phi \, d\phi. \quad . \quad . \quad . \quad (2)$$

These values are plotted in fig. 1, in the curves marked

Fig. 1.



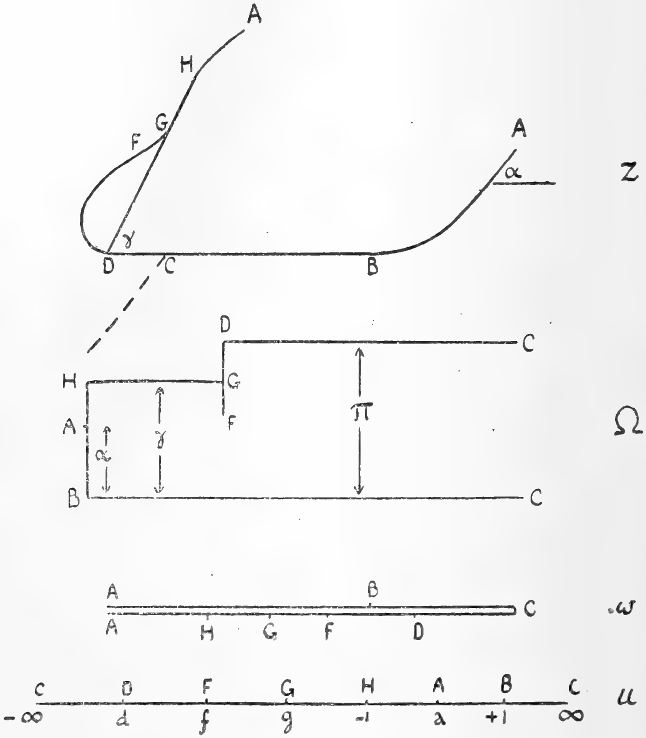
K and B respectively. It is obviously sufficient to let α vary from 0 to $\frac{\pi}{4}$.

In the general case let γ be the angle between the sides of the wedge. The diagrams on fig. 2 give the boundaries in the planes of

$$z = x + iy,$$
$$\omega = \phi + i\psi,$$
$$\Omega = \log\left(-Q\frac{dz}{d\omega}\right) = \log\frac{Q}{q} + i\theta,$$

and the arrangement of the corresponding points along the real axis of the auxiliary variable u. Here Q is the velocity along the infinitely extended free surfaces, introduced explicitly to keep the dimensions right, as Greenhill does in his "Report." The diagrams are not drawn to scale. The

Fig. 2.



values -1, $+1$, ∞ are chosen for the points H, B, C, leaving A, D, F, G to be fixed by the conditions of the problem. Indicate the corresponding values of u by small letters, then d, f, g are negative and a of doubtful sign, and the transformations are given by

$$\frac{d\omega}{du} = (u - a)^{-3}, \quad . \quad . \quad . \quad . \quad . \quad . \qquad (3)$$

$$\frac{d\Omega}{du} = (u - f)(u^2 - 1)^{-\frac{1}{2}}(u - d)^{-\frac{1}{2}}(u - g)^{-\frac{1}{2}}. \quad . \quad . \qquad (4)$$

It will be seen that the analysis applies to a more general case than that with which we started, for there is in it nothing to imply that the second plane passes through D, the upstream edge of the first. What we really have is the case in which the left-hand free stream-line is caught for a portion of its length by a second plane set anyhow with respect to the first. The four quantities to be disposed of, a, d, f, g, are determined by the angle γ between the planes, the angle of impact α on the first plane and the two coordinates which fix H, the outer edge of the second plane, with respect to DB. If arbitrary values are assigned to dg, f is determined by γ and then a by α by the relations :

$$\int_{-1}^{1}(u-f)(1-u^2)^{-\frac{1}{2}}(u-d)^{-\frac{1}{2}}(u-g)^{-\frac{1}{2}}\,du=\gamma\,. \quad . \quad (5)$$

and

$$\int_{a}^{1}(u-f)(1-u^2)^{-\frac{1}{2}}(u-d)^{-\frac{1}{2}}(u-g)^{-\frac{1}{2}}\,du=\alpha. \quad . \quad . \quad (6)$$

In other words, we can ensure that the planes shall make an assigned angle with each other, and that the stream shall meet them in an assigned direction, but the breadth and position of the second plane relative to the first can only be found by trial. Thus, to obtain solutions for a wedge, we could keep, say d, fixed, and work out the configuration of the z-plane for different values of g, until we found the value which made the second plane pass through the edge of the first, repeating the process with other d's until a range of wedge-cases, with varying-breadth ratio, could be obtained. This lengthy procedure is considerably shortened when the wedge is right-angled, for then, taking the origin of z at D, the required condition is the vanishing of x at the point G.

Reverting to the general case, let Q′ be the velocity over the free stream-line bounding the pocket DFG. Then

$$\log\frac{Q}{Q'}=\text{HG on the }\Omega\text{ diagram}$$

$$=\int_{g}^{-1}(u-f)(u^2-1)^{-\frac{1}{2}}(u-d)^{-\frac{1}{2}}(u-g)^{-\frac{1}{2}}du. \quad . \quad . \quad (7)$$

Along DFG the real part of Ω is $\log\frac{Q}{Q'}$, its imaginary part is $i\chi$, say, where

$$\chi=\pi-\int_{d}^{u}(u-f)(u^2-1)^{-\frac{1}{2}}(u-d)^{-\frac{1}{2}}(g-u)^{-\frac{1}{2}}du, \quad . \quad (8)$$

χ is minimum for $u=f$ corresponding to the inflexion F and then rises to γ at $u=g$.

Taking the origin at D we have along DFG

$$z=\mathrm{Q}^{-1}\int_d^u \exp\Omega\,.\,(a-u)^{-3}\,du=\mathrm{Q}'^{-1}\int_d^u \exp(i\chi)(a-u)^{-3}\,du \quad . \quad . \quad . \quad . \quad . \quad . \quad (9)$$

If the real part of z vanishes at G (when the planes form a right-angled wedge), then

$$\int_d^g \cos\chi\,.\,(a-u)^{-3}\,du=0. \quad . \quad . \quad . \quad . \quad . \quad (10)$$

The Ω-integrals can be expressed by elliptic functions, but the formulæ are clumsy and inconvenient for purposes of numerical calculation. For the z-integrals one is obliged to have recourse to mechanical or arithmetical quadrature. I have carried out the calculations * for a rectangular wedge whose faces make angles of 45° with the stream. For this purpose the integrals were first transformed so as to cover the range from 0 to $\frac{1}{2}\pi$ of an angular variable.

Putting $u=-\cos 2\theta$ in (5) the equation giving f becomes

$$-f\,.\int_0^{\frac{\pi}{2}}(-d-\cos 2\theta)^{-\frac{1}{2}}(-g-\cos 2\theta)^{-\frac{1}{2}}\,d\theta$$

$$=\frac{\pi}{4}+\int_0^{\frac{\pi}{2}}\cos 2\theta(-d-\cos 2\theta)^{-\frac{1}{2}}(-g-\cos 2\theta)^{-\frac{1}{2}}d\theta\,. \quad (11)$$

($-d$, $-g$, are positive quantities greater than unity).

The two integrands having been tabulated for $2\frac{1}{2}°$ intervals the integrals were calculated approximately by Simson's rule and f found. It is to be noticed that the values assumed for d and g must be such that the resulting value of f lies between them. To get a rough idea of the restriction thus imposed on d, g, we may reason as follows. We get an approximation to the integral on the left of (11) by giving the integrand the constant value $(dg)^{-\frac{1}{2}}$, corresponding to the half-way value of θ: this gives $\pi/2\sqrt{dg}$. The integral on the right has a small negative value, say $-\mathrm{A}\pi$. Thus we get as an approximate value

$$-f=\sqrt{dg}\left(\tfrac{1}{2}+\mathrm{A}\right),$$

and this has to lie between $-g$ and $-d$. It follows that

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* I am much indebted to Miss L. Beck, B.Sc., for help in carrying out the computations and integrations.

$-d$ is something like four times $-g$ and that $-f$ lies close to $-g$. This means that the inflexion on the stream-line is near the point G, which is physically obvious.

Having obtained f, and using the values already tabulated, it is easy to get a table of the integrand which is equivalent to that on the left hand of equation (5) or (6). Since $\alpha=\frac{\pi}{4}$ we have $\int_a^1=\frac{1}{2}\int_{-1}^1$, so we have to find the value of θ at which the transformed integral is bisected, say θ_0, then $a=-\cos 2\theta_0$.

The integral in (8) is now transformed by the substitution $u=\frac{1}{2}(d+g)+\frac{1}{2}(d-g)\cos 2\theta$ and the value of $(\pi-\chi)$ found by step-by-step integration. Finally, the integrand in (10) transformed to θ is tabulated and the integral found by Simson's rule. Having done this for an assumed value of d with different values of g, it was possible to find by interpolation the g which makes the second plane pass through the edge of the first. Then the other integrals which specify the details of the motion can be evaluated for the special values of the constants d, g, f, a. The integrations were carried out partly arithmetically and partly by use of a Coradi integraph.

The results are shown on fig. 3. The abscissa is the ratio of the breadth of the second plane to that of the first. The curves begin at the value ·18, which is the ordinate of the curve K on fig. 1 for $\alpha=45°$. This gives the breadth of the second plane, built out at right angles to the first, when it just touches the stream. The corresponding values of the constants are easily found to be

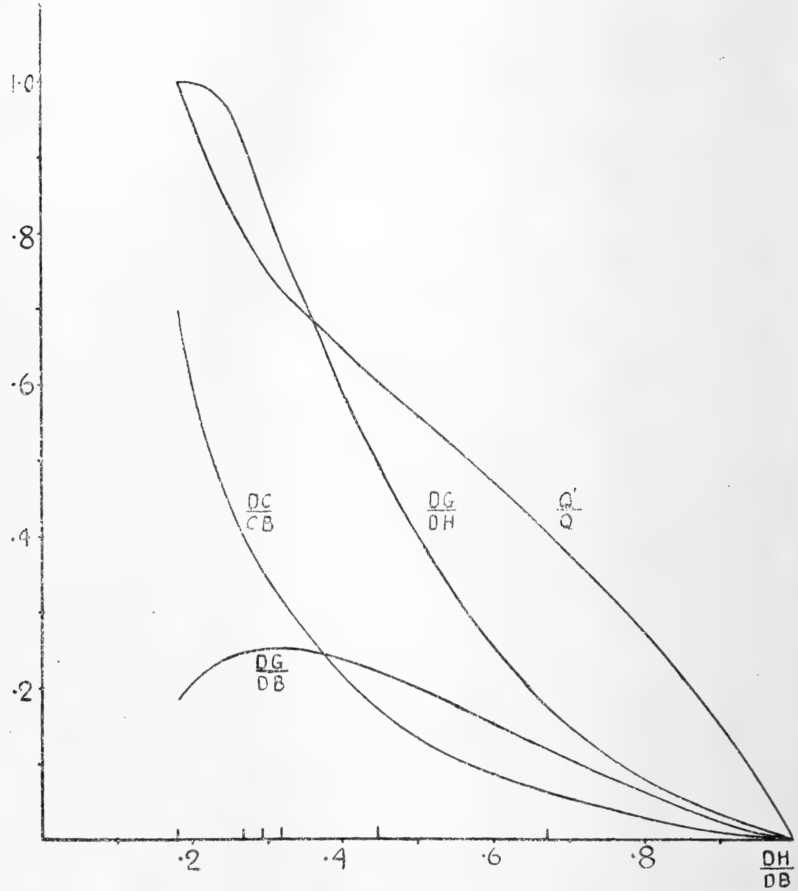
$$d=-(3+\sqrt{2}), \qquad g=f=-1, \qquad a=\tfrac{1}{2}(\sqrt{2}-1).$$

The other cases actually calculated are indicated by the positions of the marks on the upper side of the horizontal axis. They correspond to the values

$-d=$	$-g=$
6	1·049
6·5	1·099
7	1·156
10	1·558
20	3·023

The graph DG/DH shows the shrinking of the pocket of dead water to nothing as the Bobyleff case is approached, when the breadths of the planes are equal. The actual

Fig. 3.



length of the pocket passes through a maximum, as indicated on the graph marked DG/DB. The approach of the point of division of the stream to the corner is shown by DC/CB, which is plotted on a tenfold scale. Q′/Q is the ratio of the velocity along the boundary of the pocket to that along the ultimate free stream-lines.

LXXIII. *The Motion of a Simple Pendulum after the String has become Slack.* By ARTHUR TABER JONES, *Ph.D., Associate Professor of Physics at Smith College, U.S.A.* *

INTRODUCTION.—Under the above title Professor W. B. Morton recently published † a very interesting note. The pendulum is supposed to swing in a plane, the string to be inextensible, and the velocity of the bob to be sufficient to carry it higher than the point of support, but not sufficient to bring it to the top of its circular path. At some point, higher than the centre of the circle, the tension of the string vanishes and the path of the bob becomes a parabola. At the point where this parabola intersects the circle, the string tightens again and jerks the bob out of its parabolic path. In ideal cases this jerk may be thought of as perfectly elastic or as perfectly inelastic. If it is perfectly elastic it reverses the radial component of the bob's velocity, and if it is perfectly inelastic it destroys this radial component. In the case which Professor Morton has discussed the jerk is treated as perfectly inelastic. Throughout the present note the jerk is treated as perfectly elastic ‡. With a real string the jerk is far from being perfectly elastic or perfectly inelastic, so that the case of a real pendulum is an intermediate one.

The general case.—If the jerk is perfectly elastic, each path consists, in general, of a circular arc followed by an infinite series of parabolic arcs. For the first parabola Professor Morton points out that the level of no velocity is given by

$$l=\tfrac{3}{2}r\cos\alpha, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

where l stands for the distance from the centre of the circle up to the level of no velocity, r for the radius of the circle, and α for the angle which the vertical diameter of the circle makes with the radius to the point where the string first slackens. On the present hypothesis no energy is lost in the jerks, so that (1) gives the level of no velocity throughout an entire path—that is, for every one of an infinite series of parabolas.

If we take the origin at the centre of the circle, and

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* Communicated by the Author.

† Phil. Mag. (6) xxxvii. p. 280 (1919). I have checked Professor Morton's results and obtained the same expressions that he has. There is a misprint at the bottom of p. 282, where the equations should read: $f(\alpha_2)=\frac{3}{2}\cos\alpha_1$, $f(\alpha_3)=\frac{3}{2}\cos\alpha_2$.

‡ This problem may, of course, just as well be thought of as having to do with the motion of a particle which slides and bounds in a single plane inside of a frictionless spherical cavity, the impacts being perfectly elastic.

if the motion of the bob on the first parabola is toward the right, then the first parabolic arc starts at the point $(-r\sin\alpha, +r\cos\alpha)$, and ends, as Professor Morton shows, at the point $(+r\sin 3\alpha, +r\cos 3\alpha)$. If we take the origin at the initial point of the sth parabolic arc, the equation of the sth parabola may be written

$$\frac{y}{r}=\frac{x}{r}\tan\alpha_s-\frac{r}{4h_s\cos^2\alpha_s}\left(\frac{x}{r}\right)^2, \quad . \quad . \quad . \quad (2)$$

where α_s stands for the angle of elevation at the initial point of the parabolic arc, and h_s for the distance from this initial point up to the level of no velocity. For $s=1$ we have $\alpha_1=\alpha$ and $h_1/r=\frac{1}{2}\cos\alpha$. For $s=2$ we find

$$\tan\alpha_2=-\frac{64\cos^6\alpha-112\cos^4\alpha+48\cos^2\alpha-3}{64\cos^6\alpha-144\cos^4\alpha+96\cos^2\alpha-17}\,.\tan\alpha \quad . \quad (3)$$

and

$$\frac{h_2}{r}=\tfrac{1}{2}\cos\alpha\left[9-8\cos^2\alpha\right].$$

For $s>2$ the expressions for α_s, and probably for h_s, become very complicated, and I have not attempted to use them.

To find the path which corresponds to any given α, I proceed as follows. After using (1), (2), and (3) to find the equations of the first and second parabolas, I determine the co-ordinates of the end of the second parabolic arc by a process of successive approximation. Knowing these co-ordinates and the equation of the second parabola, I find the angle of incidence of the second parabola upon the circle, take the angle of reflexion as equal to the angle of incidence, and so find α_3. I can then use (1) and (2) to find the equation of the third parabola. By this method the path which corresponds to any given α may be found to any desired degree of accuracy and for as many parabolas of the series as may be desired. The general manner in which the path changes with changing α may be traced to the beginning of the fourth parabola from the curves in fig. 1.

Special cases.—In the cases which are especially interesting the paths repeat, and so form figures that are described periodically and may be thought of as somewhat analogous to the well-known Lissajous curves. These cases fall into three classes:—

(1) Cases in which the end of some one of the parabolic arcs is tangent to the circle,

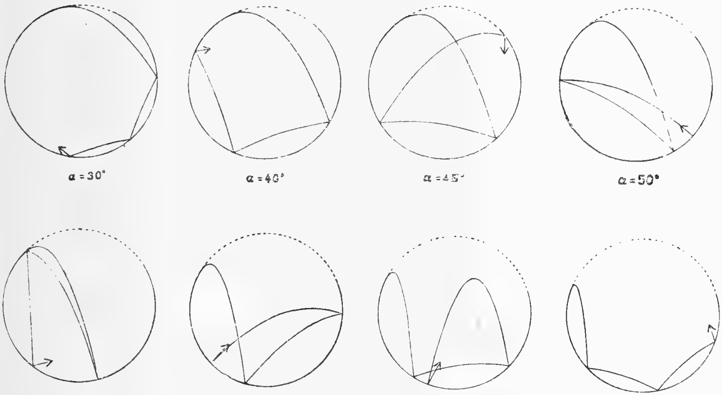
(2) Cases in which the end of some one of the parabolic arcs is perpendicular to the circle,

(3) Cases in which some one of the parabolic arcs degenerates into a vertical line.

From the first class are to be excluded those cases in which the arc in question is identical with the first parabolic arc. Otherwise this class would include all the paths which repeat. Cases of the third class occur only when the bob travels upward on the vertical part of its path, and reverses its motion before reaching the upper half of the circle. This can occur only when the level of no velocity lies below the top of the circle—that is, for $\alpha > 48° \ 11'$. The case $\alpha = 90°$ may be thought of as a limiting case of this third class. In all the cases which are periodic the paths are symmetrical with respect to the vertical diameter of the circle.

In order to find the values of α for the various special cases, a study is first made of a series of paths like those shown in fig. 1. From this study some idea is gained as to the neighbourhood in which certain cases are to be sought. A calculation of several paths in one of these neighbourhoods gives data from which the value of α for the case in question may be found to any desired degree of accuracy. I have aimed to have my values for α correct to the nearest minute.

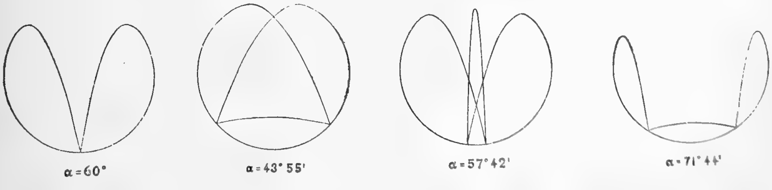
Fig. 1.



$\alpha = 30°$ $\alpha = 40°$ $\alpha = 45°$ $\alpha = 50°$

$\alpha = 55°$ $\alpha = 65°$ $\alpha = 70°$ $\alpha = 75°$

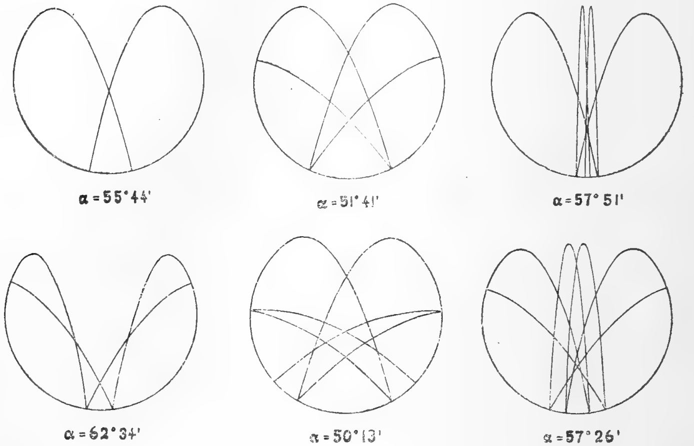
Fig. 2.



$\alpha = 60°$ $\alpha = 43° \ 55'$ $\alpha = 57° \ 42'$ $\alpha = 71° \ 44'$

I think I have found all of the periodic paths which do not involve parabolas beyond the third. In the first class there are the four cases shown in fig. 2. In one of these cases the end of the second parabolic arc is tangent to the circle, and in three of them the end of the third parabolic arc is tangent to the circle. In the second class there are the eleven cases shown in fig. 3. In one of these cases the end of the first parabolic arc is perpendicular to the circle, in three the end of the second parabolic arc is perpendicular to the circle, and in seven the end of the third parabolic arc is perpendicular to the circle. In two of these cases the complete figure has not been drawn. In these two cases it

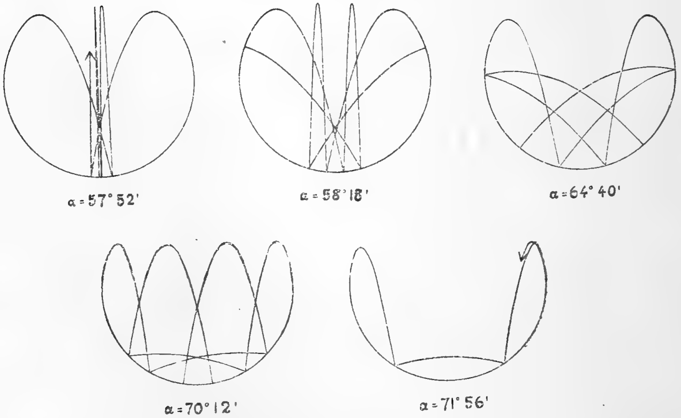
Fig. 3 [First Part].



$\alpha = 55°44'$ $\alpha = 51°41'$ $\alpha = 57°51'$

$\alpha = 62°34'$ $\alpha = 50°13'$ $\alpha = 57°26'$

Fig. 3 [Completed].

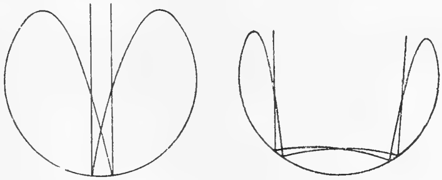


$\alpha = 57°52'$ $\alpha = 58°18'$ $\alpha = 64°40'$

$\alpha = 70°12'$ $\alpha = 71°56'$

appeared that it would be easier to imagine the rest of the figure, by remembering that it is symmetrical, than it would be to follow the complete figure. In the third class there are three cases—one in which the second parabola degenerates into a straight line, and two in which the third parabola degenerates into a straight line. Two of these three cases are shown in fig. 4. As to the third case, it can be seen from the paths shown in fig. 3 that, when the bob is on the third parabolic arc, it is moving toward one side of the figure for $\alpha = 57° 51'$, and toward the other side of the figure for $\alpha = 57° 52'$. One of the desired cases must therefore exist for a value of α which lies between these limits.

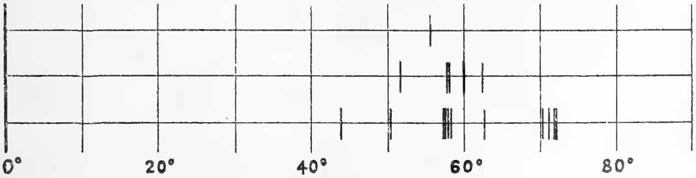
Fig. 4.



$\alpha = 58°3'$ $\alpha = 71°9'$

It is very noticeable that the values of α for which these periodic cases occur are not at all uniformly distributed throughout the possible ninety degrees. Their distribution is indicated in fig. 5. On the three horizontal lines are shown the values of α for the cases in which there are involved (1) only the first parabolic arc, (2) only the first two parabolic arcs, (3) only the first three parabolic arcs. It will be seen that, with only one exception, all of the cases here considered are included between the limits of $\alpha = 50°$ and $\alpha = 72°$.

Fig. 5.



Northampton, Mass.,
December 23, 1920.

LXXIV. *The Spectrum of Helium in the Extreme Ultra-Violet. By* Hugo Fricke *and* Theodore Lyman [*].

THE study of the Spectrum of Helium in the Schumann region, which was published some five years ago, yielded results which were difficult to interpret [†]; and the conclusions drawn from a more recent attack on the same subject [‡], though interesting, were by no means final. Both investigations suffered from the same defect, for to obtain any lines at all when helium was used in the vacuum tube it was necessary to employ a strong disruptive discharge, an electrical condition sure to introduce impurities by its action on the walls of the tube and upon the electrodes.

The upshot of the whole matter was that, though the region betwen 1700 and 600 Å.U. contained a considerable number of lines, only those at 1640 and 1215 could be attributed to helium [§]. It has been pointed out that these wave-lengths fit the relation $\nu=4\text{N}\left(\frac{1}{2^2}-\frac{1}{m^2}\right)$, and therefore probably form members of the enhanced spectrum series. Even now, however, the origin of these lines cannot be regarded as perfectly certain.

It has become increasingly evident during the progress of the present investigation that the arrangement proposed by Hicks [||] by which a considerable number of lines, many of them certainly due to impurities, were made to fit the formula for the enhanced spectrum, is without justification [¶].

The immediate cause of the renewal of the attack on the problem was the discovery by several investigators of a resonance potential in helium corresponding to a wave-length of about 600 Å.U. [**]

The improvements introduced were two in number: first, by the employment of a vacuum spectroscope with a grating of but 20 cm. radius the gas absorption was greatly reduced;

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[*] Communicated by the Authors.

[†] T. Lyman, Astrophysical Journal, xliii. No. 2, p. 89 (1916).

[‡] T. Lyman, Science, xlv. p. 187, Feb. 1917.

[§] T. Lyman, Nature, civ. p. 314 (1919).

[||] Hicks, Nature, civ. p. 393 (1919).

[¶] T. Lyman, Nature, civ. p. 565 (1920).

[**] F. Horton & A. C. Davies, Proc. Roy. Soc. xcv. p. 408 (1919). F. Horton & D. Bailey, Phil. Mag. xl. p. 440 (1920). J. Franck & P. Knipping, *Phys. Zeitsch.* xx. p. 481 (1919); *Zeitsch. f. Physik*, i. p. 320 (1920). K. T. Compton, Phil. Mag. xl. p. 553 (1920).

and second, by the use of a continuous current in the discharge-tube the chance of impurities was minimized.

The result of these improvements was the discovery of a fairly strong line at 585 Å.U. The existence of this radiation was confirmed by an observation with the vacuum grating spectroscope of 97 cm. radius, long used in this laboratory.

It is interesting to note that the resonance potential corresponding to 585, 21·2 volts, agrees rather with the results of Franck and Knipping than with those of other investigators. The significance of this fact will be considered presently.

It may be concluded that, apart from the two lines already mentioned as probably belonging to the enhanced spectrum, but whose origin is even now somewhat doubtful, only one line in the extreme ultra-violet can be ascribed to helium with any certainty at present.

The description of the experimental arrangements and a detailed discussion of results follows :—

The vacuum spectroscope containing the grating of 20 cm. radius was so arranged that its joints could be closed with Khotinski cement. This method of sealing together with the small volume facilitated the production of an excellent vacuum, and assisted in preserving the purity of the gas under examination. The discharge-tube was of quartz, the cathode being of aluminium and cylindrical in form, about 2·4 cm. long with a hole 5 mm. in diameter ; the anode was of tungsten. The cathode was always placed at the end of the tube near the slit. The helium was purified by charcoal and liquid air, the spectroscope being protected from mercury vapour by U tubes refrigerated in the usual manner. The gas when originally prepared was free from impurities. However, as it was impossible to heat the whole apparatus, the helium not infrequently showed the presence of traces of hydrogen and the oxides of carbon when examined after it had been admitted to the spectroscope.

The gas was usually at a pressure of eight tenths of a millimetre, the current being between twenty and forty milliamperes.

The dispersion of the grating was 84·6 Å.U. to the mm., the region contained in the length of the photographic plate extending from the slit image to the neigbourhood of 1200 Å.U.

Under these conditions the strongest line on all our plates occurs at 585 Å.U.; its intensity bears a constant ratio to that of the direct image of the slit. The other lines which are usually present at 686, 860, 972, 992, 1026, and 1176 show varying relative intensities ; none of them occur on all our plates. Their behaviour indicates that they are due to impurities. This conclusion is confirmed by the fact that we have found lines at 686 Å.U., 860, and 1176 when we employed a high-potential vacuum spark between carbon terminals with our small spectroscope. Millikan * has obtained radiation from the same source at nearly identical wave-lengths, namely 687·3, 858·6, and 1175·6, using a grating of greater dispersion than ours. The line 992 has already been attributed to an unknown impurity † ; 972 and 1026 are due to hydrogen †.

Hicks ‡ has suggested that as the lines 972, 992, 1026, and 1086 fit the formulæ for the enhanced spectrum of helium, they must belong to that gas. The fact that the line 1086 is not found on our plates is sufficient in itself to disprove this idea.

When we undertook to confirm our observations by means of the large spectroscope, it was only after several trials and with a very long exposure that we were able to obtain a faint but unmistakable record of λ 585. It was accompanied by fairly strong lines at 1216, 1200, 1086, and 1085, all of which have been observed before with the same instrument and are known to belong to hydrogen or to some other impurity. In the visible spectrum the discharge showed traces of hydrogen and carbon in about the same intensities as when the small spectroscope was used. In addition, nitrogen bands appeared very faintly at the end of the exposure. This impurity, not present with the smaller grating, may account for the appearance of the lines at 1200, 1086, and 1085. The region covered by the photographic plate extended from 500 to 1300 Å.U.

Owing to its faintness and to the fact that it was separated from lines of reference by a considerable distance, an exact measurement of the wave-length of this resonance line is impossible at present; however, the value of 585 ± 2 may be regarded with confidence.

The feeble character of λ 585 may be attributed to the long gas path of the large spectroscope whose grating has a radius of curvature of 97 cm. as against the radius of 20 cm. of our small instrument, if we are willing to admit a certain amount of selective absorption by helium in the neighbourhood of the line in question. This hypothesis of selective absorption is made necessary by the fact that a disruptive discharge

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* R. A. Millikan, Astrophysical Journal, lii. p. 47 (1920).

† T. Lyman, Astrophysical Journal, xliii. No. 2, p. 89 (1916).

‡ Hicks, Nature, civ. p. 393 (1919).

in helium examined with the large spectroscope yields strong impurity lines of shorter wave-length than λ 585.

It may appear curious that this resonance line was not discovered in previous researches when a disruptive discharge was employed. This may be explained in two ways: first, on the ground that the impurities known to be liberated by the disruptive discharge carry the current to the exclusion of the helium; second, by reference to the work of Compton, Lilly, and Olmstead* on the minimum arcing voltages, in which they have shown that when current density is high it is possible to obtain an arc at about 8 volts with the emission of the ordinary helium spectrum, while the resonance potential line requires upward of twenty volts. It would seem, therefore, that with high current density it might be quite reasonable to expect a bright helium spectrum in the visible spectrum without the resonance line in the extreme ultra-violet.

In conclusion, it is interesting to note that, if we follow the speculations of Bohr, Sommerfeld †, and others ‡, two resonance potential lines in helium might be expected corresponding to the two principal series of this element. The wave numbers of the limits of these two principal series are respectively 32031 and 38453, these figures corresponding to potentials of 4·0 and 4·8 volts. Taking the ionization potential of helium as 25·2 volts and subtracting from it the values 4·0 and 4·8, we obtain the values 21·2 and 20·4 as the two values of the resonance potentials in helium. The line 585 corresponds to the value 21·2: that we have only this line in the spectrum and not the line corresponding to 20·4 volts appears to agree with the work of Franck and Knipping.

Jefferson Physical Laboratory,
Harvard University,
December 1920.

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* K. T. Compton, E. T. Lilly, & P. S. Olmstead, Physical Review, xvi. p. 282 (1920). Compare T. C. Hebb, Phys. Rev. xvi. p. 375 (1920).

† Sommerfeld, *Atombau u. Spektrallinien*, 2nd Ed. p. 287.

‡ J. Franck & P. Knipping, *loc. cit.*

LXXV. *Note on the possibility of separating Mercury into its Isotopic Forms by Centrifuging. By* J. H. J. Poole, *M.A.*[*]

It appears certain from the recent work of Dr. F. W. Aston on the mass spectra of the elements that mercury is really a mixture of several isotopes of varying atomic weight. Dr. Aston in a letter to 'Nature' (Dec. 9, 1920) states that his most recent results have shown that it consists of at least six isotopes with atomic weights of 197, 198, 199, 200, 202, and 204. The exact resolution of the four isotopes between 197 and 200 has not yet been definitely determined, but the existence of some such isotopes and also the two of atomic weight 202 and 204 may be taken as definitely established. The maximum difference between the isotopes of mercury would amount to 7 units, but as we are ignorant as to the proportions of the various isotopes present it is impossible to deal with the question of their separation when centrifuged in a rigid manner. I propose to assume that the mercury can be treated as a nearly equal mixture of two isotopes only which differ in atomic weights by 4 units. This seems to be a fairly justifiable assumption to make, as the mean atomic weight of the two heaviest isotopes whose exact atomic weight is most accurately known, exceeds that of ordinary mercury by over 2 units.

As regards the equilibrium state of a mixture of two liquid isotopes in either a gravitational or centrifugal field of force, the case is not very clear. Drs. Lindemann and Aston have dealt with the subject as regards gaseous isotopes in a paper in the Philosophical Magazine for May 1919. Their application of the results obtained to the case of two liquids is, however, not so plain. The following discussion will perhaps throw some light on the problem.

Let us assume that the two isotopes differ only in mass, *i.e.*, the molecular volumes and all other properties are the same for the two. Further, let us assume that mercury is incompressible, so that the total number of molecules per c.c. is constant. Consider first the case of a column of mercury in an ordinary gravitational field. It is obvious from symmetry that if equal volumes of both isotopes are present on the whole, then at the central section of the column the number of molecules per c.c. of each isotope will be the same. Let us accordingly take this point as our origin and consider the equilibrium of a layer of mercury at a depth x below this.

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[*] Communicated by the Author.

Let n_1=number of molecules per c.c. of isotope A at this depth,

n_2=number of molecules per c.c. of isotope B at this depth,

m_1=mass of molecules A, m_2=mass of molecules B.

Then, since we have assumed that mercury is incompressible,

$$n_1+n_2=\text{a constant},$$

$$\frac{dn_1}{dx}+\frac{dn_2}{dx}=0.$$

Also the density in the layer $=n_1m_1+n_2m_2$. Consider an A molecule. It is acted on by a vertical force downwards due to its weight of m_1g, and a vertical force upwards due to flotation of $\frac{1}{n_1+n_2}(n_1m_1+n_2m_2)g$ since the volume of a molecule $=\frac{1}{n_1+n_2}$.

Hence there will be a net downward force on an A molecule of

$$\left[m_1-\frac{1}{n_1+n_2}(n_1m_1+n_2m_2)\right]g$$

$$=\frac{n_2(m_1-m_2)g}{n_1+n_2}.$$

If S is the area of the column, the total force on all the A molecules in a layer of thickness dx due to gravitational effects

$$=\frac{n_1n_2}{n_1+n_2}(m_1-m_2)\,.\,\mathrm{S}\,.\,dx.$$

Now we have assumed that the mercury is in equilibrium hence there must be a force acting which will counterbalance this effect. Since we have also assumed that the two isotopes A and B are exactly alike except as regards mass, and that the total number of molecules per c.c. is constant, it is plain that the attractive force on a molecule due to the surrounding molecules will on the average be zero, and no re-distribution of the two classes of molecules will affect this result.

It would thus appear that the only force which could counterbalance the tendency of the heavier molecules to move downwards would be due to the increase of the osmotic pressure of the latter owing to an increase in the number of them present per c.c. This really amounts to the fact that the number of A molecules which diffuse out of the layer downwards owing to the gravitational effect must be equal to the number which diffuse into the layer from below owing to the increase of concentration downwards, so that the total number of A molecules in the layer remains constant, *i.e.*, the mercury is in a state of statistical equilibrium.

Let W = average molecular energy.

Then the osmotic pressure $=\frac{2}{3}\mathrm{W}n$ where n = number of molecules per cc.

Hence we have at once :

$$\tfrac{2}{3}\mathrm{W}\,.\,dn_1=\frac{n_1n_2}{n_1+n_2}(m_1-m_2)g\,.\,dx$$

$$\frac{dn_1}{n_1}+\frac{dn_1}{n_2}=\frac{3}{2}\frac{(m_1-m_2)g\,.\,dx}{\mathrm{W}}\,.$$

But
$$\frac{dn_1}{dx}=-\frac{dn_2}{dx},$$

$$\therefore\quad\int\frac{dn_1}{n_1}-\int\frac{dn_2}{n_2}=\frac{3}{2}\cdot\frac{(m_1-m_2)g}{\mathrm{W}}\cdot\int dx,$$

$$\therefore\quad\log\frac{n_1}{n_2}=\frac{3}{2}\frac{(m_1-m_2)gx}{\mathrm{W}}+\mathrm{C}.$$

Now, since we have chosen our origin so that $n_1=n_2$ when $x=0$. C must be zero.

Hence

$$\log\frac{n_1}{n_2}=\frac{3}{2}\cdot\frac{(m_1-m_2)gx}{\mathrm{W}}\,.$$

But
$$\mathrm{W}=\frac{3}{2}\frac{\mathrm{R}\theta}{\mathrm{N}},$$

where R is the gas constant, θ the absolute temperature, and N the number of molecules per gram-molecule.

$$\therefore\quad\log\frac{n_1}{n_2}=\frac{\mathrm{N}(m_1-m_2)gx}{\mathrm{R}\theta}\,.$$

It is probable that the mercury molecule is monatomic, as all metals appear to be so.

Hence $\mathrm{N}m_1$ = one gram-atom of isotope A

and $\mathrm{N}(m_1-m_2)=\mathrm{M}_1-\mathrm{M}_2$ grm.,

where M_1 and M_2 are the atomic weights of the two isotopes.

Thus finally we have

$$\log\frac{n_1}{n_2}=\frac{(\mathrm{M}_1-\mathrm{M}_2)gx}{\mathrm{R}\theta}.$$

If we are dealing with a centrifugal field of force, the equation obviously is

$$\frac{dn_1}{n_1}-\frac{dn_2}{n_2}=\frac{3}{2}\cdot\frac{m_1-m_2}{\mathrm{W}}\cdot\omega^2 r\,.\,dr,$$

which gives
$$\log\frac{n_1}{n_2}=\frac{\mathrm{M}_1-\mathrm{M}_2}{2\mathrm{R}\theta}\cdot\omega^2 r^2+\mathrm{C}.$$

In this case we cannot eliminate C so easily as we do not know $\frac{n_1}{n_2}$ for any given value of r.

As, however, we are not very certain as to the exact conditions, I propose to deal with the question as though the centrifuge tube was in an uniform gravitational field of magnitude equal to the centrifugal field at the central section of the tube. This is of course only an approximation, but it will be probably good enough to enable us to obtain some idea of the magnitude of the result to be expected.

The particulars of the centrifuge used in our experiments on the effect of centrifuging liquid lead (see Phil. Mag. March 1920) are approximately as follows :—

Length of centrifuge tube .... $=6$ cm.
Distance of inner end of tube from centre of rotation .... $=4$ cm.
Number of revolutions per sec. $=150$
Hence acceleration at centre of tube $=\omega^2 r$
$=(150\times 2\pi)^2\times 7$,

and at outer end of tube,

$$\log\frac{n_1}{n_2}=\frac{\mathrm{M}_1-\mathrm{M}_2}{\mathrm{R}\theta}\cdot gx,$$
$$=\frac{4\times(150\times 2\pi)^2\times 7\times 3}{83{\cdot}15\times 300\times 10^6}.$$

Since $\theta=$ air temperature $=300°$ Abs. (approx.),
$\mathrm{R}=83{\cdot}15\times 10^6$, $g=\omega^2 r$,
$\mathrm{M}_1-\mathrm{M}_2=4$ $=(150\times 2\pi)^2\times 7$,

and x is to be measured from centre of tube, and is therefore $=3$ cm.

This gives finally that $\frac{n_1}{n_2}=1{\cdot}003$.

We have now to consider what effect this small change in the concentration of the two isotopes would have on the mean density.

Let $k=\frac{n_1}{n_2}$, d_1=density of isotope A,

d_2= " " B.

Then $\frac{d_1}{d_2}=\frac{M_1}{M_2}$ $=1{\cdot}02$ very nearly.

And the density of the mixture

$$=\frac{1}{1+k}(kd_1+d_2)$$

$$=\frac{d_2}{1+k}(k1{\cdot}02+1).$$

Putting $k=1{\cdot}003$ and $k=1$ respectively, we find that the difference of the densities at the outer end of the tube and at the centre equals $\cdot000015d_2$. The difference between the densities at the outer and inner ends will of course be double this, but still it only amounts to one part in 30,000. It would probably be extremely difficult to detect this difference with any certainty, and thus experiments with this centrifuge would not lead to any really satisfactory results. In this connexion it is of interest to note that J. N. Brönsted and G. Hevesy (Nature, 106. p. 144, 30 Sept. 1920) claim to have effected a certain separation in mercury of about this magnitude by fractional distillation at low pressures. They state that their density determinations can be trusted as accurate to 1 in a million, but it would be difficult to deal with so large a quantity of mercury, as they did, in centrifuging experiments. However, it would certainly seem quite feasible to construct a centrifuge in the form of a hollow disk which could be run up to at least 60,000 revs. per minute; and such a centrifuge, if its other dimensions were approximately the same, would give a difference of density of about 1½ parts in 1000, which should be quite easily detectable. The amount of separation might further be increased by successive experiments, but the results obtained would apparently hardly justify the expense entailed in constructing the special centrifuge.

Iveagh Geological Laboratory.
January, 1921.

NOTE.—It might be possible to separate liquid neon by this method, if a centrifuge could be run at a sufficiently low temperature. The case of neon is rather more hopeful as there is a difference in density of about 10 per cent. between the two isotopes, and working at such a low temperature would, of course, increase the separation effect very largely.

LXXVI. *The Gravitational Field of a Particle on Einstein's Theory.*

By F. W. Hill, *M.A., late Fellow of St. John's College, Cambridge, and* G. B. Jeffery, *M.A., D.Sc., Fellow of University College, London* *.

THE solution of Einstein's contracted tensor equation $G_{\mu\nu}=0$ for a single attracting point mass may be expressed by means of the line element

$$ds^2 = -\epsilon^{\lambda} dr^2 - e^{\mu}(r^2 d\theta^2 + r^2 \sin^2\theta\, d\phi^2) + e^{\nu} dt^2, \quad . \quad (1)$$

where r, θ, ϕ are polar co-ordinates and λ, μ. ν are functions of r only. On substitution into the equations $G_{\mu\nu}=0$, it is found that the resulting equations are insufficient to determine λ, μ, ν. One relation between them must be laid down, and this corresponds to the way in which the radius vector r is measured. Perhaps the best known form of the solution is that for which $\mu=0$, in which case we have †

$$ds^2 = -\gamma^{-1} dr^2 - r^2 d\theta^2 - r^2 \sin^2\theta\, d\phi^2 + \gamma dt^2, \quad . \quad (2)$$

where $\gamma = 1 - 2m/r$ and m is the mass of the particle.

De Sitter ‡ gives *approximate* solutions for which $\lambda=\mu$ and $\lambda+2\mu+\nu=0$.

Difficulty is sometimes felt in applications of the solutions for which λ and μ are different, and this is often met by writing $r+m$ for r in (2) by which, neglecting squares of m/r, we have

$$ds^2 = -\gamma^{-1}(dr^2 + r^2 d\theta^2 + r^2 \sin^2\theta\, d\phi^2) + \gamma dt^2, \quad . \quad (3)$$

γ having the same meaning as before.

The purpose of this paper is to show that there is an *exact* solution for which $\lambda=\mu$.

In the general theory with co-ordinates x_1, x_2, x_3, x_4, we have

$$ds^2 = g_{\rho\sigma} dx_\rho dx_\sigma,$$

where the occurrence of the same suffix twice in any term indicates that that term is to be summed for values 1, 2, 3, 4 of that suffix. The sixteen quantities $g_{\rho\sigma}$ form a symmetrical covariant tensor whose determinant is denoted by g. The contravariant tensor $g^{\rho\sigma}$ is defined to be the

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* Communicated by the Authors.

† *Cf.* Eddington, 'Report on the Relativity Theory of Gravitation,' p. 46.

‡ 'Monthly Notices, Royal Astronomical Society,' lxxvi. p. 699 (1916).

minor of $g_{\rho\sigma}$ divided by g, and we introduce the Christoffel three index symbols:

$$\{\rho\sigma,\ \tau\} = \tfrac{1}{2}g^{\tau\alpha}\left(\frac{\partial g_{\rho\alpha}}{\partial x_\sigma}+\frac{\partial g_{\sigma\alpha}}{\partial x_\rho}-\frac{\partial g_{\rho\sigma}}{\partial x_\alpha}\right).$$

Einstein's equations of gravitation in free space are then

$$\begin{aligned} G_{\sigma\tau} \equiv -\frac{\partial}{\partial x_\alpha}\{\sigma\tau,\ \alpha\} + \{\sigma\alpha,\ \beta\}\{\tau\beta,\ \alpha\} \\ + \frac{\partial^2}{\partial x_\sigma \partial x_\tau}\log\sqrt{-g} - \{\sigma\tau,\ \alpha\}\frac{\partial}{\partial x_\alpha}\log\sqrt{-g} = 0. \quad (4) \end{aligned}$$

In our case, we have from (1)

$$g_{11} = -e^\lambda, \quad g_{22} = -r^2 e^\mu, \quad g_{33} = -r^2\sin^2\theta\, e^\mu, \quad g_{44} = e^\nu,$$

$$g \;=\; -r^4\sin^2\theta\, e^{\lambda+2\mu+\nu},$$

$$g^{\rho\sigma} = g_{\rho\sigma} = 0 \text{ if } \rho \neq \sigma,$$

and $\quad g^{\rho\rho} = 1/g_{\rho\rho}.$

It is then found that of the 40 Christoffel symbols only 9 survive, and their values are easily calculated. Substituting these into (4), it is found that there are only three equations between λ, μ, ν. As given by de Sitter *, these are :

$$\mu'' + \tfrac{1}{2}\nu'' + \frac{2}{r}\mu' - \frac{1}{r}\lambda' + \tfrac{1}{2}\mu'^2 - \tfrac{1}{2}\lambda'\mu' - \tfrac{1}{4}\lambda'\nu' + \tfrac{1}{4}\nu'^2 = 0,$$

$$\begin{aligned} -1 + e^{\mu-\lambda}[1 + 2r\mu' + \tfrac{1}{2}r^2\mu'' + \tfrac{1}{2}r(\nu'-\lambda') \\ + \tfrac{1}{2}r^2\mu'(\mu' + \tfrac{1}{2}\nu' - \tfrac{1}{2}\lambda')] = 0, \end{aligned}$$

$$\tfrac{1}{2}\nu'' + \frac{1}{r}\nu' + \tfrac{1}{2}\nu'(\mu' + \tfrac{1}{2}\nu' - \tfrac{1}{2}\lambda') = 0.$$

Putting $\lambda = \mu$, we have

$$\lambda'' + \tfrac{1}{2}\nu'' + \frac{1}{r}\lambda' \qquad - \tfrac{1}{4}\lambda'\nu' + \tfrac{1}{4}\nu'^2 = 0, \quad (5)$$

$$\lambda'' \qquad + \frac{3}{r}\lambda' + \frac{1}{r}\nu' + \tfrac{1}{2}\lambda'^2 + \tfrac{1}{2}\lambda'\nu' \qquad = 0, \quad (6)$$

$$-\tfrac{1}{2}\nu'' \qquad - \frac{1}{r}\nu' \qquad - \tfrac{1}{4}\lambda'\nu' - \tfrac{1}{4}\nu'^2 = 0. \quad (7)$$

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* 'Monthly Notices, Royal Astronomical Society,' lxxvi. p. 712 (1916).

Adding these three equations, we have the following equation in λ only:

$$2\lambda''+\frac{4}{r}\lambda'+\tfrac{1}{2}\lambda'^2=0,$$

which on integration gives

$$e^{\lambda}=\left(1+\frac{m}{2r}\right)^4, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (8)$$

where m is a constant of integration and a second constant of integration has been chosen so that $\lambda \rightarrow 0$ as $r \rightarrow \infty$.

Substituting in (6), we obtain an equation in ν which readily integrates to give

$$e^{\nu}=\left(\frac{1-\dfrac{m}{2r}}{1+\dfrac{m}{2r}}\right)^2. \quad . \quad . \quad . \quad . \quad . \quad . \quad (9)$$

It is then necessary to show that (8) and (9) satisfy either (5) or (6), and this presents no difficulty.

The line element may therefore be written without approximation

$$ds^2=-\left(1+\frac{m}{2r}\right)^4(dr^2+r^2d\theta^2+r^2\sin^2\theta\, d\phi^2)$$
$$+\left(\frac{1-\dfrac{m}{2r}}{1+\dfrac{m}{2r}}\right)^2 dt^2. \qquad (10)$$

The constant m is most readily identified with the mass by considering the approximation of (10) when m/r is small and comparing it with (2). In fact, (10) may be obtained from (2) by means of the transformation

$$r=r_1\left(1+\frac{m}{2r_1}\right)^2, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (11)$$

and then dropping the suffix in r_1.

The advantage of the form (10) lies in the facility with which it can be transformed from one set of co-ordinates to another. For example, in Cartesian co-ordinates it becomes

$$ds^2 = -\left(1+\frac{m}{2r}\right)^4 (dx^2+dy^2+dz^2) + \left(\frac{1-\frac{m}{2r}}{1+\frac{m}{2r}}\right)^2 dt^2.$$

It cannot be too clearly recognized that the difference between the two forms of the line element given in (2) and (10) depends only upon the way in which we agree to measure the radial distance r. The form (10) can therefore give no physical result which is independent of radial measurements which is not already implied in (2). For example, the equation of the orbit of a particle is obtained from

$$\frac{d^2x_\alpha}{ds^2} + \{\mu\nu, \alpha\}\frac{dx_\mu}{ds}\frac{dx_\nu}{ds} = 0. \quad . \quad . \quad . \quad . \quad . \quad (12)$$

For a particle moving in the plane $\theta = \frac{1}{2}\pi$, the form (10) gives

$$\frac{d^2u}{d\phi^2} + u = \frac{m}{h^2} + \frac{6m^2u}{h^2}; \quad . \quad . \quad . \quad . \quad . \quad . \quad (13)$$

while (2) gives the more usual form

$$\frac{d^2u}{d\phi^2} + u = \frac{m}{h^2} + 3mu^2, \quad . \quad . \quad . \quad . \quad . \quad . \quad (14)$$

where, as usual, $u \equiv 1/r$.

These two differential equations have different solutions; but if we investigate a fact which does not depend upon radial measurement, the advance of the apse line per revolution, (13) and (14) agree in giving $3m^2/h^2$.

LXXVII. *On Systems with "Propagated Coupling."*

To the Editors of the Philosophical Magazine.

GENTLEMEN,—

PROFESSOR A. W. PORTER, F.R.S., and Mr. R. E. GIBBS, B.Sc., have published in No. 243 of the Philosophical Magazine a paper "On Systems with Propagated Coupling" (March 1921, p. 432). In the first part of this communication some interesting experiments are described, but in the second part (p. 434), in the paragraph headed "(ii.) Simple dynamical case illustrating maintenance of vibrations," the authors give what comes to a theoretical treatment of what happens when an ordinary telephone receiver is connected to a battery. Professor Porter and Mr. Gibbs arrive at the unexpected conclusion that this telephone receiver will produce a continuous sound.

This obviously cannot be the case, and the erroneous result arrived at is due to an unfortunate oversight of a minus sign. In fact the second equation of p. 435 must be

$$-\alpha C+\left(m\frac{d^2}{dt^2}+r\frac{d}{dt}+K\right)y=0$$

instead of

$$+\alpha C+\left(m\frac{d^2}{dt^2}+r\frac{d}{dt}+K\right)y=0$$

if the first equation is maintained.

This error, which is repeated a few times (pp. 437, 438, 440), just makes the maintenance of free undamped vibrations, which the examples are intended to illustrate, *impossible* in all cases treated, none of the differential equations given having purely imaginary roots.

In order to obtain spontaneous oscillations a variable resistance must be present in the electrodynamical system, as was the case in the authors' experiments, where a telephone receiver was coupled acoustically to a microphone transmitter, and *not* to another telephone receiver.

I am, Gentlemen,
Yours truly,
BALTH. VAN DER POL, Jun.

Physical Laboratory,
Teyler's Institute, Haarlem (Holland).

To the Editors of the Philosophical Magazine.

GENTLEMEN,—

WE must thank Dr. van der Pol for pointing out the unfortunate slip in our paper in connexion with the sign of α.

We do not agree with him, however, in regard to the necessity for assuming a variability in the resistance—though this variability was certainly present either in the microphone or in the valve set. It may be pointed out that the experiment works equally well with Brown magnetophones. The effect of the propagation of the mutual action is to create a phase-difference between y and C to an amount depending upon the separation of the reacting parts; and this, in certain positions, is equivalent to a reversal in the sign of α, which thus becomes of the same sign as given in the paper. Since the position for which this will occur is different from what we supposed, it must be admitted that the detailed algebra requires to be restated.

Alfred W. Porter.
Reginald E. Gibbs.

March 15th, 1921.

LXXVIII. *Proceedings of Learned Societies.*

GEOLOGICAL SOCIETY.

[Continued from p. 684.]

May 5th, 1920.—Mr. G. W. Lamplugh, F.R.S., Vice-President, in the Chair.

THE following communication was read:—

'A Natural "Eolith" Factory beneath the Thanet Sand.' By Samuel Hazzledine Warren, F.G.S.

The paper describes a section in the Bullhead Bed at Grays, where the conditions have been favourable for the chipping of the flints by subsoil pressure. There is evidence of extensive solution of the Chalk beneath the Tertiary deposits, and the differential movements thus brought about have occasioned much slickensiding, and remarkable effects in the chipping of the flints.

In the author's opinion the section affords the most complete and conclusive evidence hitherto obtained in support of the theory of the origin of the supposed Eolithic implements by purely natural agencies. There are not only the simpler Kentish types, such as notches, bowscrapers, and the like, but also the larger and more advanced forms of rostro-carinates which are characteristic of the sub-Crag detritus-bed. Careful digging enables the pressure-points of one stone against another and the resultant chipping effects to be studied in detail; and in many instances the flakes removed can be recovered and replaced.

A few examples are more than merely Eolithic in character. If such exceptional examples were removed from their associates, and also from the evidences of the geological forces to which they have been exposed, no investigator could be blamed for accepting them without question as of Mousterian workmanship. Individual specimens may often deceive: in order to distinguish a geological deposit of chipped flints from the débris of a prehistoric chipping-floor, it is necessary to base one's judgment upon fairly representative groups, and also to take into consideration the circumstances in which they have been discovered.

THE
LONDON, EDINBURGH, AND DUBLIN
PHILOSOPHICAL MAGAZINE
AND
JOURNAL OF SCIENCE.

[SIXTH SERIES.]

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LXXIX. *The Dielectric Constants of Electrolytic Solutions.*

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I.

THE values given by various observers for the change in the dielectric constant produced by dissolving electrolytes in water show considerable divergence. To take the case of copper sulphate—Drude[16, 17], Coolidge[13], and Palmer[47] were unable to distinguish any difference between the dielectric constant of pure water and of solutions of this salt, whereas Smale[52] asserts that a 0·05 normal solution has a dielectric constant 15·5 per cent. greater than that of water. Now, according to the Nernst-Thomson[56] rule, the dielectric constant of a solvent is intimately connected with the degree of ionization of an electrolytic solute; and it is therefore of importance that we should have a knowledge of the effect of the solute on the dielectric constant of the solution. It seems to be pretty generally assumed that this effect is to raise the dielectric constant—a view which is supported by Smale's experiments with aqueous solutions and by Walden's[60] experiments with non-aqueous solutions. The observations of Drude[14-21], Coolidge[13], Eichenwald[25], and Palmer[47] are

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* Communicated by the Author.

generally overlooked*. This may be because these observers have usually been somewhat guarded in their statements. Drude[17], for example, says: "Man würde daher auf eine geringe Abnahme der Dielectricitätsconstante mit wachsendem Salzgehalt zu Schliessen haben. Wenn auch diese Schluss wegen der Grösse der Beobachtungsfehler nicht mit voller Sicherheit vorläufig zu ziehen sein mag, so geht doch aus der Beobachtungen zweifellos hervor, dass die Dielectricitätsconstante des Wassers durch Auflösung eines Electrolyten selbst bis zu der Leitfahigkeit $K = 11 . 10^{-7}$ jedensfalls nicht vergrössert wird."

Practically all methods by which one can determine the capacity of a condenser whose plates are separated by a conducting medium involve the use of alternating currents; these may have a comparatively low frequency (2C0 to 300 radians per second, as in the experiments of Franke[28], Heerwagen[31,32], and Rosa[49]) or a considerably higher frequency (10^6 to 10^{10}, as in the experiments of Niven[45], Harrington[30], Nernst[41–44], Marx[38,39], Cole[4], Drude[14–23], Colley[11,12], Lampa[35], and many others). With moderate frequencies (not exceeding 10^5 or 10^6) it is possible to use forms of Wheatstone's bridge (see 1, 26, 27, 30, 33, 41–44, 48, 54, and 59). Silow's method of charging two electrometers simultaneously by the same source of E.M.F. and comparing their deflexions when each is filled with a different dielectric, has been applied over a wide range of frequencies (see 5–8, 28, 31, 32, 46, 47, 49, 51, 52, 55, 57).

It is not proposed to discuss either of these methods here, but it may be pointed out that the results obtained from them are only trustworthy when proper corrections are made for the current passing through the condenser by conduction.

Imperfect insulators under the influence of an alternating field may behave in one of two ways: (*a*) they may convey a current in accordance with Ohm's law and dissipate the power of the current at a rate proportional to i^2r, or (*b*) they may cause an absorption of electrical power which is out of all proportion to the conductance they display to a constant field or when only slow oscillations are used. The second form of absorption has been studied

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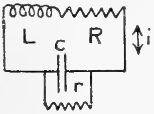
* Lecher's observation[37] that a condenser had the same effect on Ruhmkorf oscillations when it was filled with water as when it was filled with 10 per cent. sulphuric acid has been quoted in favour of the view that both these liquids have the same dielectric constant (Palmer[46]). It really shows that both behaved as conductors.

by Cohn and Arons[5, 6] and by Drude[17], and is known as anomalous absorption or dielectric polarization. It has been shown by many observers (see 2, 3, 14–23, 25, 13, 44, 61), and confirmed by the present investigation, that aqueous solutions show only the first type of absorption (that due to their conductivity) so long as the frequency used is not too high. The experiments of Drude[22], Colley[11, 12], and Rukop[50] on water, and of Wildermuth[61] on solutions, indicate that at very high frequencies anomalous absorption may become noticeable.

In most cases in which high frequency has been hitherto employed, the current was in the form of highly-damped trains of waves. It is proposed here to confine attention to continuous waves of pure sine form; the treatment of such cases is comparatively simple, and the general conclusions are similar to those obtaining for waves of more complex types.

In a circuit consisting of an inductance L, a resistance R, and a capacity C, whose dielectric has a resistance r, the relation between the amplitude of the applied or induced E.M.F. (E) and the amplitude of the current in the resistance R is given by the equation

Fig. 1.



$$\left(\frac{E}{i}\right)^2 = Z^2 = \left(R + \frac{r}{1+p^2C^2r^2}\right)^2 + \left(pL - \frac{1}{pC + p^{-1}C^{-1}r^{-2}}\right)^2, \quad . \quad . \quad . \quad (1)$$

where p = the frequency.

If such a circuit is adjusted so that i is a maximum, then the conclusions drawn from the experiment may depend on the method of adjustment employed.

A. *Tuning by adjustment of* L.

This is essentially Drude's method[14–23]. Z will clearly be a minimum when $(pL)^{-1} = pC + p^{-1}C^{-1}r^{-2}$. Now, if C_1 is the capacity of a condenser employing a good insulator and which gives a maximum current with the same value of L as does C, then we may call C_1 the apparent value of C; and it follows that $C_1 = C + p^{-2}r^{-2}C^{-1}$, or C_1 is greater than the true value of C, and that such experiments will give too high a value for the dielectric constant of the substance under investigation unless proper corrections are applied.

Comparison of the charges imparted to two condensers by an alternating E.M.F. will require correction depending on the inductance and resistance of the leads between the condensers. The most complete series of observations of this kind is that carried out by Smale[52], and his results for the ratio of the dielectric constants of certain solutions to that of water are given in Table I.

TABLE I.

Ratio of D.C. for Solutions to that of Water (Smale).

Normality ……	0·001	0·002	0·005	0·008	0·010	0·020	0·030	0·050
Solute HCl ……	0·99	1·033	1·064	1·090	1·126			
KCl ……	1·013	1·018	1·034	1·070	1·113	…	1·160	
$CuSO_4$ …	…	1·012	1·017	1·050	1·086	1·128	…	1·155

The apparent rise in the D.C. due to dissolving an electrolyte in water is most marked in the case of the best conductor (HCl), and increases as the strength of the solution increases. It will be seen from what follows that the true D.C.'s for these solutions are probably less than that of water.

B. *Tuning by altering the size or the distance between the plates of the condenser.*

As the capacity is increased, the resistance will diminish in such a way that the product Cr remains constant. The condition that Z in equation (1) may be a minimum is then $p^2CL = 1 + R/r$, or, using the same notation as before, $C = C_1(1 + R/r)$, and the uncorrected values of the D.C. of a solution obtained in this way will be lower than the true values. The method was used by Marx[39], but was only applied to water of low conductivity—*i. e.*, the ratio R/r was negligibly small.

C. *The electrolytic condenser is connected in parallel with a variable air condenser, and tuning obtained by varying the latter. The electrolytic condenser is then removed and the air condenser readjusted.*

This was the method used in the present work. The essential features of the circuit are shown in fig. 2. Oscillations were induced by a neighbouring circuit emitting continuous waves, and the capacity C was varied until the current in L was a maximum.

Fig. 2.

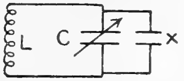
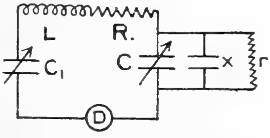


Fig. 3.



A more complete scheme of the circuit is shown in fig. 3, in which D represents a current detector and r the leakage resistance across x.

The impedance Z of such a circuit is given by

$$Z^2=\left(R+\frac{r}{1+p^2r^2(C+x)^2}\right)^2$$

$$+\left(pL-\frac{1}{pC_1}-\frac{1}{pC+px+p^{-1}r^{-2}(C+x)^{-1}}\right)^2. \quad (1\,a)$$

And on removal of x and r this becomes

$$Z^2=R^2+\left(pL-\frac{1}{pC_1}-\frac{1}{pC}\right)^2. \quad . \quad . \quad . \quad . \quad (1\,b)$$

If i and I are the maximum currents obtainable in these two cases, and C_2 and C_0 the values of C which give these currents, then

$$\frac{C_0}{C_2+x}=1-\left(\frac{I^2}{i^2}-1\right)\Big/K^2=\frac{1}{K^2}\cdot\frac{r}{R}\left(\frac{I}{i}-1\right)=\frac{r/R}{r/R+I/i+1}, \quad . \quad . \quad . \quad (2)$$

where $$K^2=\left(L-\frac{1}{p^2C_1}\right)\Big/C_0R^2.$$

Whence

$$\frac{I}{i}=\left(K^2+\left(1+\frac{r}{2R}\right)^2\right)^{\frac{1}{2}}-\frac{r}{2R} \quad . \quad . \quad . \quad . \quad (3)$$

and

$$C_2+x-C_0=\frac{C_0R\left[1-r/2R+(K^2+(1+r/2R)^2)^{\frac{1}{2}}\right]}{r}. \quad (3\,a)$$

If the insulation of the dielectric is perfect, then $r=\infty$ and $C_2+x-C_0=0$: the right-hand side of equation (3 a) is therefore a correction term, and it is desirable to reduce its value to a minimum. To do this, both R and K must be made as small as possible; this will give large values of i (see equation 3), and so facilitate accurate adjustment of C and cause the correction term to approach $2C_0R/r$.

This is illustrated by some calculated values in Tables II. and III.

TABLE II.

Calculated values of I/*i*.

K^2.	$r/R=1$.	10.	100.	1000.
1 ……	1·30	1·08	1·01	1·002
2 ……	1·56	1·16	1·02	1·005
10 ……	3·00	1·78	1·10	1·011
100 ……	9·6	6·62	1·97	1·101

TABLE III.

Values of C_2+x-C_0 corresponding to Table II.

K^2.	$r/R=1$.	10.	100.	1000.
1 ……	2·30	0·21	0·02	0·002
2 ……	2·56	0·22	0·02	0·002
10 ……	4·0	0·28	0·02	0·002
100 ……	10·6	0·77	0·03	0·002

To make r/R large, R must be small; but this will increase $K^2\left(=\left(L-\frac{1}{p^2C_1}\right)\Big/C_0R^2\right)$ unless $L-p^{-2}C_1^{-1}$ is small and C_0 is large.

This illustrates the fallacy of the general assumption that exceedingly high frequencies are necessary for all methods of determining the dielectric constant of an electrolyte. The assumption is correct when the method referred to in section I. A is used. The results given by methods B and C appear at first sight to be independent of frequency; but it must be borne in mind that R is not independent of p. As a first approximation, it is generally stated that $Rp^{-\frac{1}{2}}$ is constant for a straight wire. Some experiments made by the author seem to show that for circuits and frequencies such as were used in these experiments, Rp^{-1} is more nearly constant. The circuit is most free from errors due to leakage when it is symmetrical, *i. e.* when $C_1=C_0$. In this case, K becomes equal to $pL/2R$, and is practically constant so long as L is constant.

II. ESTIMATION OF THE FACTOR K.

The electrolytic condenser is removed, and the current estimated for various values of C; if I_0 is the maximum current when $C=C_0$ and I is the current when $C=C$, then equation (1 *b*) leads to

$$K\left(1-\frac{C_0}{C}\right)=\left(\frac{I_0^2}{I^2}-1\right)^{\frac{1}{2}}. \quad . \quad . \quad . \quad . \quad . \quad . \quad (1\,c)$$

This assumes that the mutual inductance between the transmitter and receiver is so small that variations in the current in the receiver do not appreciably affect the transmitter current. If this condition is fulfilled, the graph obtained by plotting C against $C(I_0^2/I^2-1)^{\frac{1}{2}}$ is a straight line passing through the point $(C_0, 0)$; the slope of this line $=K$. Fig. 4 shows how these lines are affected by varying the distance between the transmitter and receiver.

III. TESTS OF THE FORMULÆ DEVELOPED IN THE PRECEDING PARAGRAPHS.

The circuits used in these tests (see fig. 3) had a comparatively large resistance, and consisted of a receiving inductance, two variable condensers in series, and a thermal converter attached to a millivoltmeter. The capacity of one condenser (C_1) was kept constant and approximately equal to C_0 during each set of readings. This symmetrical form of circuit has the advantage that the voltage variation at the centre of the thermal converter is a minimum, and there is very little leakage of high-frequency current to the galvanometer leads.

The plates of the condenser x were two brass disks, the distance between which could be varied by means of a screw. In the following tables $n=$ the number of threads by which the threads were separated.

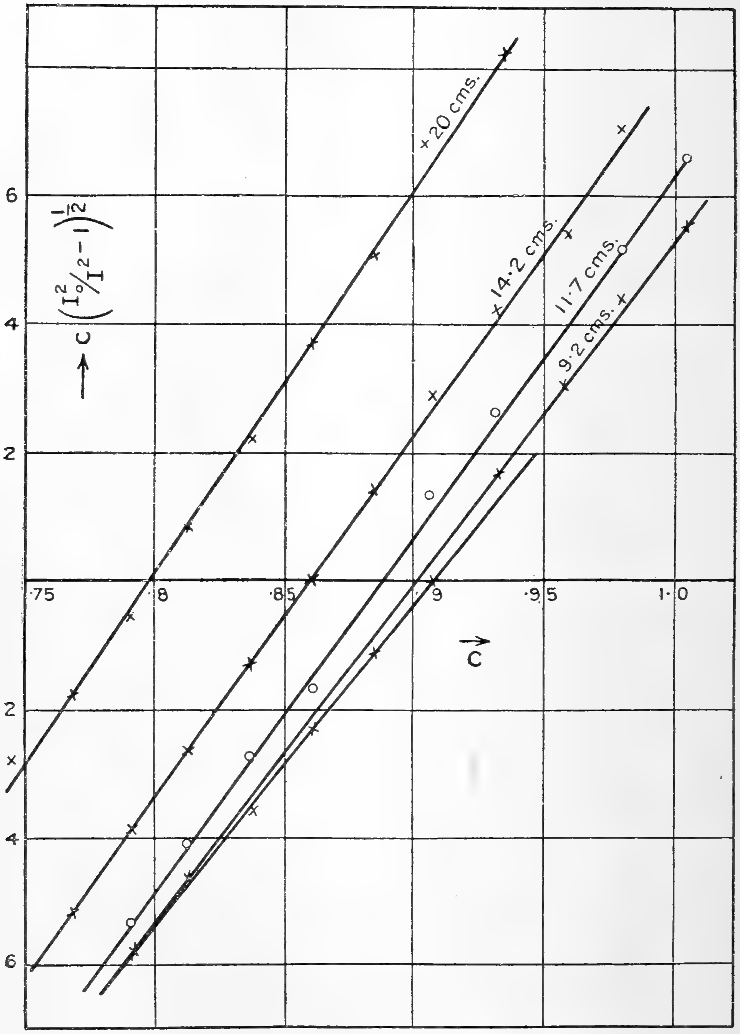
(a) *Tests with distilled water.*

Water of various degrees of purity was used. The wave-length in air was about 55 metres. In every case $K^2(I^2/i^2-1)$ was calculated (see equation 2) and found to be negligibly small.

TABLE IV.—Distilled Water.

n.	x in millimicrofarads.		
2	0·3405	......	0·3585
3	0·245	0·2485	0·238
4	0·202	0·192	0·1865
5	0·152	0·156	0·1495
6	0·127	0·131	0·1275
8	0·098	0·100	0·1015

Fig. 4.



(b) *Tests with dilute solutions of potassium chloride.*

Two different wave-lengths were used, and the circuit was varied in such a way as to change both L and R.

Table V. (*a*).—Details of Circuits.

Reference	(i.)	(ii.)	(iii.)	(iv.)
λ (metres).........	50	50	77	77
K	4·43	3·33	3·14	5·60
R (approx.)	14·7	14·6	15·3	8·2

TABLE V. (*b*).

Capacities of Conductors filled with KCl Solutions.

	I/i.				C_0-C.			
Reference	(i.)	(ii.)	(iii.)	(iv.)	(i.)	(ii.)	(iii.)	(iv.)
$n=2$	......	1·825	1·74	2·215	......	·217	·184	·202
3	1·82	1·565	1·515	1·87	·1995	·1535	·118	·155
4	1·62	1·42	1·395	1·70	·150	·131	·096	·1195
5	1·53	1·345	1·32	1·58	·121	·1015	·0815	·103
6	1·45	1·30	1·275	1·49	·101	·0895	·069	·089
8	1·36	1·235	1·21	1·38	·078	·0735	·049	·072

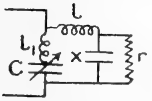
	x.			
Reference.........	(i.)	(ii.)	(iii.)	(iv.)
$n=2$	......	·3635	·413	·343
3	·2545	·236	·252	·235
4	·184	·1865	·180	·179
5	·149	·142	·148	·149
6	·124	·1235	·125	·125
8	·0955	·0985	·090	·098

The correction $C_0(I^2/i^2-1)/(K^2+1-I^2/i^2)$ in the case where the distance between the plates was three threads of the screw varied from 0·055 to 0·134, and it will be seen that concordant values of x were obtained.

IV. CORRECTION FOR INDUCTANCE OF LEADS.

A condenser of capacity C in series with an inductance l is equivalent to a condenser of capacity $C/(1-p^2lC)$. With high frequencies the term p^2lC may be appreciable even when l is small. The values given in the above tables under

Fig. 5.



the heading x will not therefore be true values of the capacity of the condensers; to obtain these, a correction must be applied for the induction of the leads joining x and C.

Let l_1 = inductance in series with C
and l = ,, ,, ,, x.

Equation (1 *a*) must then be modified, and becomes

$$Z^2 = \left(R + \frac{\alpha}{p(\alpha^2+\beta^2)}\right)^2 + \left(pL - \frac{\beta}{p(\alpha^2+\beta^2)}\right)^2, \quad . \quad (4)$$

where

$$p\alpha = \frac{r}{r^2(1-p^2lx)^2+p^2l^2}$$

and

$$\beta = \frac{C}{1-p^2l_1C} + \frac{xr^2(1-p^2lx)-l}{r^2(1-p^2lx)^2+p^2l^2}. \quad . \quad . \quad . \quad . \quad (4\,a)$$

If tuning is obtained by varying C, then α is invariant, and the condition for maximum current is

$$\left.\begin{aligned} \alpha &= \frac{KC_0'(I_0/I-1)}{K^2-I_0^2/I^2+1} = (I_0/I-1)\frac{\beta_0}{K} \\ \text{and} \qquad \beta_0 &= \frac{K^2C_0'}{K^2-I_0^2/I^2+1}, \end{aligned}\right\} \quad . \quad . \quad . \quad (5)$$

where $C_0' = C_0/(1-p^2l_1C_0)$.

The values of l and l_1 can be found by using an electrolyte with a high value of r (*e.g.* pure water). Equations (4 *a*) and (5) then become

$$\frac{C_0}{1-p^2l_1C_0} = \frac{C}{1-p^2l_1C} + \frac{x}{1-p^2lx}$$

or

$$x = DC_a = \frac{C_0-C}{1+p^2C_0(l-l_1)-p^2C(l+l_1)+p^4l_1^2CC_0}$$

where D = the dielectric constant of water and C_a = the value of x when air is used as dielectric.

The term $p^4l_1^2CC_0$ is negligible, and therefore $(C_0-C)/C_a$ will be a linear function of C.

For conducting dielectrics,

$$\left.\begin{aligned} x &= \frac{p^2l\alpha^2+\Delta(1+p^2l\Delta)}{p^4l^2\alpha^2+(1+p^2l\Delta)^2} \\ \text{and} \qquad pr &= p^4l^2\alpha + \frac{(1+p^2l\Delta)^2}{\alpha}, \\ \text{where} \qquad \Delta &= \beta - \frac{C}{1-p^2l_1C}. \end{aligned}\right\} \quad . \quad . \quad . \quad (6)$$

In such a case p^2l can be evaluated by utilizing the fact that the product of the resistance and capacity of a condenser containing a given dielectric is independent of the form of the condenser, and hence

$$p^2l\left(\alpha+\frac{\Delta^2}{\alpha}\right)+\frac{\Delta}{\alpha}=\text{constant.} \quad . \quad . \quad . \quad . \quad (7)$$

The values already quoted were obtained at a time when the importance of this correction for the leads was not realized. They are consistent with equations (5) and (6), but as sufficient care had not been taken to keep l constant, it was not thought worth while to attempt to apply the corrections for l.

The above equations may also be used to deduce the resistivity of an electrolytic dielectric : if Λ is the molecular conductivity of a solution containing n gram mols. per c.c., then its specific resistance is $1/\Lambda n$, and if the solution is put into a condenser, then the resistance of this will be $(\Lambda n 3600 \pi C_a)^{-1}$ ohm, where C_a is the capacity of the condenser in millimicrofarads when filled with air.

Hence

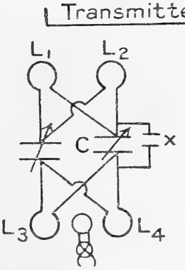
$$\Lambda=\frac{p}{3600\pi n C_a(p^4l^2\alpha+(1+p^2l\Delta)^2/\alpha)} . \quad . \quad . \quad (8)$$

When this equation is applied to the observations of Table V., Λ for 0·000755 normal KCl solution is calculated to be 105 (Kohlrausch gives 129 at 18° C.).

VI. Experiments with Improved Form of Apparatus.

The final form of the apparatus is shown diagrammatically

Fig. 6.



in fig. 6. This was designed so as to reduce L and R to a minimum ; two rectangles of brass strip connected in parallel were used as receiving inductances (L_1, L_2). These were connected to two adjustable condensers, the other plates of which were connected to two circles of brass strip (L_3, L_4) also in parallel with one another. Between these two lay two turns of thick copper wire connected to a thermal converter and a millivoltmeter.

It was found that with this arrangement the resistance for high-frequency currents in the principal receiving circuit was about 1/5 ohm. It is probable that this is not, properly speaking, the real "resistance," but a function of this, of the mutual inductance between the two circuits and various leakage effects. All of these will vary when the frequency is changed. The various values of K found at various times were used to calculate R, and the results are shown in Table VI.

TABLE VI.

λ (metres)	33·2	35·5	39·2	45·5	45·5	45·5	45·5
R (ohms)	·306	·269	·346	·211	·210	·216	·206
λ (metres)	45·5	45·5	45·7	45·7	46·2	47·3	
R (ohms)	·209	·190	·220	·200	·192	·230	

These agree fairly well with the empirical formula $R\lambda = 9{\cdot}68$. Experiments with other circuits over a wider range of wave-lengths showed that this formula is justifiable.

The condenser used in these experiments consisted of two coaxial cylinders; the inner one could be withdrawn from inside the outer by means of a screw. Mr. E. W. B. Gill kindly calibrated this condenser for me with air between the plates. It was connected to C by two stiff parallel wires, in order that the inductance of the leads might be constant. Experiments with water and with glycerol in the condenser enabled a series of values of p^2l and p^2l_1 to be found; these were in satisfactory agreement with $l = 343$ cm. and $l_1 = 67$ cm., assuming the dielectric constant of water to be constant over the range of frequencies employed. This assumption is justified by series of experiments to be described later (see Section VIII.).

Table VII. gives the results obtained for water at various wave-lengths (λ) and various temperatures. These have all been reduced to 18° C., using Coolidge's value for the temperature coefficient *.

Table VII.—Dielectric Constant of Water.

λ.	t °C.	D_t.	D_{18}.
33·2 …………	……	80·3	
34·2 …………	14	84·25	82·85
34·4 …………	……	82·05	
34·8 * …………	17·1	80·2	79·9
35·9 …………	18·2	82·2	82·25
41 …………	19·1	81·05	81·45
41·1 * …………	16·8	78·2	77·8
44·9 * …………	17	82·5	82·15
45·3 …………	18·2	81·2	81·25
45·5 …………	11·8	84·9	82·7
45·5 …………	14	84·0	82·6
45·5 …………	17·6	81·0	79·85
46·2 …………	12·2	80·95	78·95
46·2 …………	……	76·7	
46·6 …………	19·1	78·0	78·4
61·5 …………	18·6	81·4	81·6
61·6 * …………	17·4	81·25	81·05
70 …………	19·8	82·05	82·7
80·5 * …………	17·2	80·7	80·4
80·5 …………	16·6	80·7	80·2
103 …………	17·6	82·8	82·65
		Mean ……………	81·05

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* The temperature coefficients found by various observers vary between −0·28 per °C. and −0·495. Coolidge's value is in fair agreement with that found by Marx and Heerwagen.

Observer.	K_{18}.	$-\frac{\Delta D}{\Delta t}$.
Rosa [49] ………………	77·8	0·348
Cohn [8] ………………	74·7	0·333
Heerwagen [32] ………	80·52	0·362
Franke [28] ……………	81·6	0·504
Thwing [53] ……………	74·27	0·473
Drude [17] ……………	81·4	0·368
Ratz [48] ………………	80·6	0·39
Abegg [1] ………………	80·42	0·425
Coolidge [13] …………	81·25	0·352
Palmer [47] ……………	……	0·358 at 60 ∼
		0·507 at 10^6 ∼
		0·570 at 3 . 10^6 ∼
Nernst [41] ……………	80·0	0·28
Marx [37] ………………	82·3	0·362
Niven [45] ………………	79·4	0·632

Some experiments were also made in which the leads were cut down to a minimum; this reduced l to 285 cms. The results are indicated by an asterisk.

TABLE VIII.—Dielectric Constant of Glycerol.

(All these observations were made at 17°–18° C., except that for $\lambda = 45 \cdot 5$, which was at 14° C.).

λ.	D.
152	52·85
123	49·85
103	53·05
98	50·9
91·2	51·95
83·3	52·15
74	52·15
47·3	49·5
45·5	(53·05)
42·6	49·05
35·5	51·0
33·2	50·05
Mean	51·15

In the case of water the most careful determination seems to be that of Turner[59], viz. 81·12. The somewhat varied values by other observers found for glycerol indicate that D for this liquid is dependent on the frequency; at the same time it must be borne in mind, when comparing the results of different observers, that it is by no means easy to obtain glycerol free from water or to prevent its contamination by water. The various values found in the literature for D and for the square of the refractive index (n^2) are collected below:—

Observer.	Metres.	D.	n^2.
Thwing[58]	10	56·2	
Drude[18, 19, 20, 21]	2	...	39·1
	0·75	...	25·4
	0·73	19·1	
	,,	16·5	
v. Lang[36]	0·085	...	14·1
Eckert[24]	0·057	...	14·4
Merczyng[40]	0·045	...	16·8
Lampa[35]	0·008	...	3·4
	0·006	...	3·1
	0·004	...	2·62

VII. DIELECTRIC CONSTANTS OF SOLUTIONS.

Four substances were chosen for investigation:—

Sugar: a typical non-electrolyte.
Potassium chloride: a uni-uni-valent electrolyte.
Copper sulphate: a bi-bi-valent electrolyte.
Tetraethylammonium napthalene-β-sulphonate: a uni-uni-valent electrolyte with comparatively large ions.

(a) *Sugar Solutions.*—Three concentrations were used; the traces of saline impurity in commercial sugar gave the solutions an appreciable conductivity (the 20-per-cent. solution had a conductivity comparable with that of 0·001 normal KCl solution).

TABLE IX.—Dielectric Constants of Sugar Solutions.

Wave-length in air = 45·5 metres. K = 38·4.

Percentage of sugar	6·84	11·92	20·6
Temperature (° C.)	13	14	16
D for solution	81·2	69·8	(79 c.)
D for water	82·8	82·45	81·75
Ratio	0·985	0·85	

The last horizontal row gives the ratio of D for the solutions to that of water at the same temperature. Interpolation of Harrington's results[30] gives 0·975 and 0·96 for the corresponding ratios; Drude[17], using waves of 2 metres length in air, found for the square of the refractive index of a 40-per-cent. solution the value 67·5. He noted "strong absorption" equivalent to that of a solution having conductivity 3×10^{-7}, though the actual conductivity, as tested by the ordinary methods, was only 6×10^{-10}.

(b) *Potassium chloride.*—In the case of the more concentrated solutions the maximum current was so small that accuracy in adjusting the condenser was very poor. The necessary corrections are large and depend on a number of observations, and hence the values obtained for D are untrustworthy and are not quoted. It will be seen that very fair agreement was obtained between values of the molecular conductivity (Λ) and those obtained by Kohlrausch*.

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* As the apparatus was designed with a view to making the effects due to conductivity as small as possible, the accuracy with which Λ may be determined is also small.

TABLE X.—Dielectric Constants of KCl Solutions.

λ (metres)	34·2	45·5	45·5	45·5	45·5	45·5
K	22·9	38·4	45·9	57·5	57·5	45·9
Temp. (° C.)	15·4	15·4	10·6	16·8	16·0	11·4
Normality	·000755	·000755	·00151	·00755	·0151	·0151
D for solution	80·25	75·0	77·7	66·25		
D for water	81·95	81·95	83·65	81·45		
Ratio	0·98	0·915	0·93	0·815		
Λ	147	121·5	101	79·5	72·7	91·5
Λ (Kohlrausch)	121·6	121·6	106·1	120·4	116·35	105·2

(c) *Copper-sulphate Solutions.*

λ was in all cases 45·5 metres and K=57·5.

TABLE XI.

Temperature (° C.)	15·1	13·2	14·8	14·8
Normality	·00114	·00228	·00456	·0228
D for solution	75·2	78·2	73·9	
D for water	81·08	82·75	82·15	
Ratio	0·93	0·945	0·90	
Λ	89·8	73·7	67·0	41·7
Λ (Kohlrausch)	91·5	84·5	76·5	55·35

(d) *Tetraethylammonium naphthalene-β-sulphonate Solutions.*

λ and K as for copper-sulphate solutions.

TABLE XII.

Temperature (° C.)	16·6	14	14·2
Normality	·002	·005	·010
D for solution	76·3	69·0	
D for water	81·55	82·45	
Ratio	0·935	0·85	
Λ	38·8	32·5	32·8

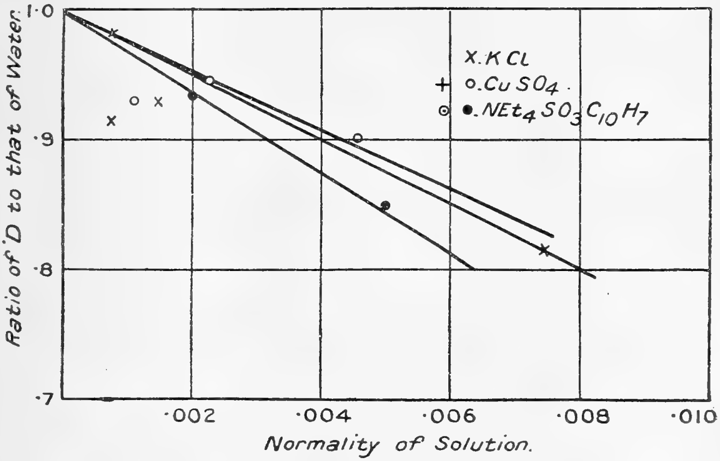
The agreement between the values obtained for the molecular conductivity and those obtained by Kohlrausch at smaller frequencies indicates that frequency has very little influence on this quantity. In the case of the tetraethylammonium salt no data are available for comparison, but data for similar acids and bases indicates that the conductivity at infinite dilution should lie between 30 and 40.

It has already been pointed out that the agreement above referred to is an indication of the absence of anomalous absorption or dielectric polarization.

Some attempts were made to calculate the correction factors in difficult cases by using Kohlrausch's data, but uncertainty as to which of the observations was to be considered faulty rendered the attempt fruitless.

The ratios of the dielectric constants of the solutions investigated to that of water have been plotted against concentration (in grm. equiv. per li.) in fig. 7. The relationship appears to be linear and practically independent of the nature of the salt.

Fig. 7.

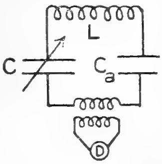


VIII.

The Dielectric Constant of Water is independent of the Frequency between Wave-lengths of 17 *and* 52 *metres.*

A receiving circuit was made up, consisting of (i.) an inductance L, which could be varied to suit the frequency employed but which was kept constant during each series of observations, (ii.) a variable condenser C, (iii.) a small inductance coupled with the detector D, and (iv.) a fixed air condenser C_a. The latter could be replaced by one having the same external dimensions, but with a greater distance between its plates. When this was filled with

Fig. 8.



water its capacity (x) was not very different from that of C_a. By noting the value of C, which brought the circuit into tune with wave of known frequency, first with C_a and then with x in the circuit, the relative values of C_a and x could be calculated. The results are shown in Table XIII.

TABLE XIII.—Capacity of Water Condenser (x), taking Air Condenser as 0·465.

Inductance.	L_0.	L_1.	L_2.	L_3.	L_4.	L_5.	L_6.	Mean.
λ = 17·15	...	...	...	...	...	·5637	·537	·550
19·2	...	...	·5602	·5540	·517	·5640	·5688	·553
19·85	...	...	...	·5750	...	·5687	·5968	·580
25·7	...	...	·5818	·5526	...	...	...	·567
31·7	...	·5965	·5697	·5767	...	...	...	·581
33·4	...	·5504	·5783	...	...	...	...	·564
35·45	...	...	·6042					
41·8	...	·5592	...	...	...	...	...	·559
47·4	·573	...	...	...	...	...	...	·573
52·3	·566	...	...	...	...	...	...	·566
Mean ...	·570	·569	·579	·5645	...	·5655	·5675	·568

The only corrections applied to the above results is a small one due to the conductivity of the water; the variations from the general mean are only such as may be expected from the probable experimental errors of the method, and show no trace of any connexion with frequency. Marx[39] has shown that D for water is constant between λ=11·1 and λ=17·8 metres.

IX. SUMMARY.

The methods in use for finding the dielectric constant of water and of solutions are criticised, and a method is described by which the necessary corrections for conductivity may be applied.

The electrolytes investigated appear to lower the dielectric constant of water, and are in this respect analogous to the majority of non-electrolytes.

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LXXX. *The Moduli of Rigidity for Spruce.*

By H. Carrington, *B.Sc., M.Sc.Tech., A.M.I.Mech.E.*[*]

[Plate IX.]

THE investigation described below was conducted during the war in the College of Technology, Manchester, for the Air Board and the Royal Aircraft Establishment. Aeroplane designers found themselves at some disadvantage, owing to the lack of information about the elastic properties of timber, and so far as the writer is aware, no published records of values of the moduli of rigidity, determined on the assumption that the wood had three planes of elastic symmetry, were then available. Torsion experiments on prisms of wood were conducted by Gerstner[†] (1833) and Pacinotti & Peri[‡] (1845), who used the old Coulomb theory that plane sections remained plane during strain to calculate the values. A short table of values of Young's Modulus along the grain, Poisson's Ratio, and the Modulus of Rigidity for beech and pine, based on the experimental results of Chevandier & Wertheim[§] (1848) and of Mallock[||]

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[*] Communicated by the Author.
[†] 'The Theory of Elasticity,' Todhunter & Pearson, vol. i. art. 810.
[‡] *Ibid.* vol. i. arts. 1250–1252.
[§] *Ibid.* vol. i. arts. 1312–1314.
[||] Proc. Roy. Soc. Lond. vol. xxix. pp. 157–161.

(1879), and assuming the partial ellipsoidal elasticity of St. Venant to hold, is given in a paper on "The Torsion resulting from Flexure in Prisms with Cross-sections of Uni-axial Symmetry only," by A. W. Young, M.A., Ethel M. Elderton, and Karl Pearson, F.R.S. (Cambridge University Press, 1918). The shortness and incompleteness of the table—for neither the humidity nor the density of the wood is given, both of which have an important bearing on the values of the elastic coefficients—and the fact that the authors had to refer back prior to 1880, indicates further the lack of information about the elastic properties of timber.

The values of the moduli were determined from torsion experiments on prisms of spruce of rectangular cross-section. Most of the prisms were about 12 inches long, and the length of the longest side of any cross-section did not exceed $1\frac{1}{4}$ inches. A photograph of the apparatus used is shown in Pl. IX. fig. 1.

The two shafts could revolve freely in ball-bearings, and each shaft was fitted with a pulley, one of which was fixed when an experiment was being performed. The torque was transmitted to the other pulley and thence to the specimen by weights placed in a scale-pan, which was suspended from the periphery of the pulley by a fine wire. The effective radius of the couple was 2 inches. The shafts were provided with jaws, into which the test pieces were fixed by wooden wedges, and alignment of the axis of the test pieces with the axes of the shafts was effected by a scribing-block. One of the shafts had a slight end play, so that longitudinal tension was avoided. It was found possible to obtain continuous stress-strain curves with increments of load of $\frac{1}{20}$ lb. in the scale-pan, equivalent to increments of torque of $\frac{1}{10}$ lb. inch.

An optical method was used to measure the angles of twist. Two frames, each supporting a small mirror, were fixed to the test piece by pointed set screws, so that the axis of each of the pair of screws in a frame was in a line intersecting the axis of the test piece at right angles (see Pl. IX. fig. 1). When the frames were fixed in position, the distance between the axes of each pair of screws was that over which the angles of twist were measured. This distance was usually about 2 inches. The magnitudes of the angles were obtained by noting the apparent motion of the cross-wires of a pair of telescopes along scales placed opposite to the mirrors and reflected down the telescopes. Each scale was about 50 inches from the corresponding mirror, and was placed so that the line of sight between the mirror and the corresponding scale moved in a plane perpendicular to the axis of the test piece.

When conducting an experiment the load in the scale-pan was increased by equal amounts, and the scales were read after each increase. The scale readings were then plotted against the torque, and resulted in a curve whose initial portion was straight. The slope of this straight portion was involved in the calculation of the moduli. Although the main object of the experiments was to obtain values of the moduli, it was considered worth while to determine values of the elastic limits and moduli of rupture when possible. These were calculated from the values of twist and torque where the curves ceased to be straight and from the maximum torque. For most of the experiments plane scales were used, and the error introduced was negligible. In other cases, where the angle of twist at the elastic limit was considerable, curved scales were employed.

The prisms were obtained from balks representing four different trees. In every case the specimens were cut from portions over 12 inches from the pith, and since the lengths of annual layers in the circumferential direction did not exceed $1\frac{1}{4}$ ins. corresponding with any cross-section, or about 2 ins. corresponding with the tested length of any prism, the layers were sensibly plane surfaces. The specimens were cut so that the annual layers were parallel, either to opposite sides or ends, and the values of the moduli were calculated on the assumption that the specimens had three planes of elastic symmetry.

The direction of the grain will be noted by ZOZ, the direction perpendicular to this and normal to the annual layers by XOX, and the direction perpendicular to the other two, which is thus tangential to the annual layers, by YOY. When a specimen with its length in the direction ZOZ is twisted about its longitudinal axis, two values of the moduli are involved, *i.e.* μyz and μzx (μyz denotes the modulus of rigidity corresponding with shear strain along planes OZX and OXY, *i.e.* along the direction YOZ).

The moduli were calculated from the slopes of the stress-strain curves and the dimensions of the test pieces. Two methods were used, and in order to explain them, consider a prism of rectangular cross-section twisted about the longitudinal axis, and let the two values of the moduli involved be denoted by μ_1 and μ_2. Also let M=twisting-moment corresponding with the angle of twist per unit length τ,

$2b$ and $2c$=lengths of the sides of the cross-section, and let

$$\frac{b}{c}\sqrt{\frac{\mu_2}{\mu_1}}>1. \quad \text{Then } \mathrm{M}=\mu_1\tau bc^3\beta, \text{ where}$$

$$\beta=\left\{\frac{16}{3}-\left(\frac{4}{\pi}\right)^5\frac{c}{b}\sum_{n=1}^{n=\infty}\frac{1}{(2n-1)^5}\tanh\frac{(2n-1)\pi}{2}\frac{b}{c}\sqrt{\frac{\mu_2}{\mu_1}}\right\}^{*}.$$

$$\text{When}\quad \frac{b}{c}\sqrt{\frac{\mu_2}{\mu_1}}>3,\quad \beta=\frac{1-0{\cdot}630\frac{c}{b}\sqrt{\frac{\mu_1}{\mu_2}}}{0{\cdot}1875}^{\dagger}$$

with an error of less than 0·1 per cent, and therefore

$$\mu_1=\frac{0{\cdot}1875}{bc^3\left(1-0{\cdot}630\frac{c}{b}\sqrt{\frac{\mu_1}{\mu_2}}\right)}\left(\frac{\mathrm{M}}{\tau}\right) \quad . \quad . \quad . \quad (1)$$

Thus the calculation of μ_1 by this equation required a knowledge of μ_2.

This method could have been used for all the experiments, but in some cases it would have necessitated excessively thin test pieces. This will be evident from a consideration of formula (1); for if $\sqrt{\frac{\mu_1}{\mu_2}}$ is greater than, or about unity, and since it is only approximately known, then $\frac{c}{b}$ must be small in order that the error in the value of

$$\left(1-0{\cdot}630\frac{c}{b}\sqrt{\frac{\mu_1}{\mu_2}}\right)$$

shall be negligible. Excessively thin pieces were undesirable because, for instance, of the impossibility of measuring the thickness with sufficient accuracy, owing to the yielding nature of the material and the difficulty in getting the surfaces planed parallel. Also in cases where the thickness of the pieces was in the direction of the width of the annual rings, thin pieces were in most cases specially to be avoided, because the thickness would have been comparable with the width of the rings, and hence the proportion of autumn to spring wood may have varied from piece to piece.

In the case of the second method, two experiments had to be performed on each test piece. After the first experiment the piece was cut so that the ratio of the sides was altered.

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[*] Todhunter & Pearson's 'Elasticity,' vol. ii. arts. 27 & 47.

[†] *Ibid.* art. 47, and 'Experimental Elasticity,' Searle, art. 40.

With the above notation let the suffixes 1 and 2 refer respectively to the first and second experiments, and let

$$\frac{b_2}{c_2}\sqrt{\frac{\mu_2}{\mu_1}} > \frac{b_1}{c_1}\sqrt{\frac{\mu_2}{\mu_1}} > 1.$$

Then $M_1 = \mu_1\tau_1 b_1 c_1{}^3 \beta_1$ and $M_2 = \mu_1 \tau_2 b_2 c_2{}^3 \beta_2$.

Therefore
$$\frac{\beta_2}{\beta_1} = \frac{\tau_1 b_1 c_1{}^3 M_2}{\tau_2 b_2 c_2{}^3 M_1} > 1 \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

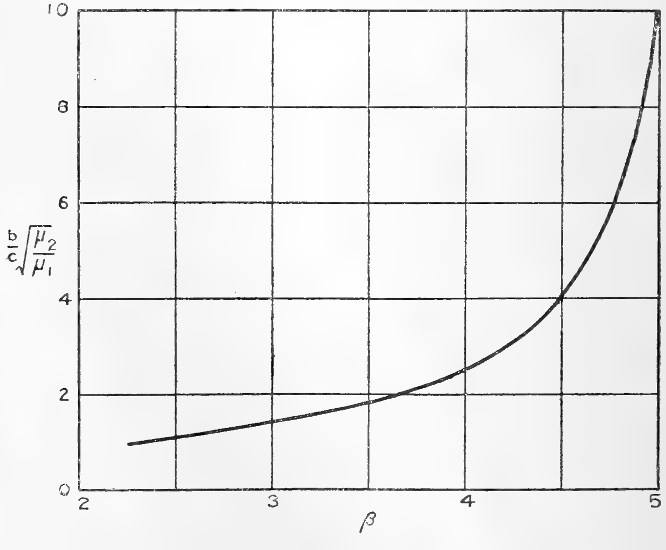
and
$$\frac{\dfrac{b_2}{c_2}\sqrt{\dfrac{\mu_2}{\mu_1}}}{\dfrac{b_1}{c_1}\sqrt{\dfrac{\mu_2}{\mu_1}}} = \frac{b_2 c_1}{b_1 c_2} > 1. \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

Values of β corresponding with different values of $\frac{b}{c}\sqrt{\frac{\mu_2}{\mu_1}}$ were calculated by St. Venant [*], and in fig. 2 a curve is drawn from the values. Since the ratios (2) and (3) were known from the experiments and measurements of the

Fig. 2.

Relation between β & $\frac{b}{c}\sqrt{\frac{\mu_2}{\mu_1}}$.

(See Todhunter & Pearson's 'History of Elasticity,' vol. ii. art. 47.)



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[*] Todhunter & Pearson's 'Elasticity,' vol. ii. art. 47.

test pieces, the values of β_1, β_2, $\frac{b_1}{c_1}\sqrt{\frac{\mu_2}{\mu_1}}$ and $\frac{b_2}{c_2}\sqrt{\frac{\mu_2}{\mu_1}}$ could be obtained by trial from the curve, and hence the values of μ_1 and μ_2. The writer is not aware of any previous record of the second method, but by making the ratios of the lengths of the sides of the cross-sections such that the ratios of $\frac{\beta_2}{\beta_1}$ were equal to 1·4 or 1·5, and avoiding the steep part of the curve corresponding with values of β_2 greater than about 4·6, it was found capable of the accuracy aimed at throughout the experiments, *i. e.* an error of less than 1 per cent.

Six types of experiments were performed :—

(1) Torsion about axis XOX, breadth of cross-section in direction YOY, thickness in direction ZOZ.
(2) Torsion about axis XOX, breadth of cross-section in direction ZOZ, thickness in direction YOY.
(3) Torsion about axis YOY, breadth of cross-section in direction ZOZ, thickness in direction XOX.
(4) Torsion about axis YOY, breadth of cross-section in direction XOX, thickness in direction ZOZ.
(5) Torsion about axis ZOZ, breadth of cross-section in direction XOX, thickness in direction YOY.
(6) Torsion about axis ZOZ, breadth of cross-section in direction YOY, thickness in direction XOX.

From experiments type (1), values of μxy were obtained.

From experiments type (2), values of μzx and μxy were obtained.

From experiments type (3), values of μyz and μxy were obtained.

From experiments type (4), values of μxy were obtained.

From experiments type (5), values of μyz and μzx were obtained.

From experiments type (6), values of μyz and μzx were obtained.

For most of the experiments of types (1) and (4) equation (1) was used. Since the values of $\sqrt{\frac{\mu xy}{\mu yz}}$ and $\sqrt{\frac{\mu xy}{\mu zx}}$ were somewhere about 0·2, it was not necessary to make the pieces excessively thin in order to calculate μxy with the required accuracy, and in many cases the cross-sections were made square. For all the experiments of the other types, with a few exceptions, equations (2) and (3), in conjunction with the graph in fig. 2, were employed.

In addition to the moduli, the following properties were also measured for each test piece. The density when tested, density when dry, humidity, shrinkages in the directions XOX and YOY, and the number of annual rings per inch. In order to obtain the density when dry, the humidity and shrinkages, each test piece was dried in an electrically-heated stove at 104° C. for from 24 to 48 hours immediately after testing. From the measurements and weights of the pieces before and after drying, the properties were calculated, obvious precautions being necessary because of convection currents and the rapidity with which the pieces absorbed moisture. The number of annual rings per inch was deduced from measurements of the thicknesses of the rings by a finely divided scale.

The values of the elastic limits and moduli of rupture were calculated by expressions of the type

$$Z_x = \mu zx \,.\, c \,.\, \tau \,.\, \gamma, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (4)$$

where $\frac{b}{c}\sqrt{\frac{\mu yz}{\mu zx}} > 1$ and Z_x is the stress in the middle of the side $2b$ corresponding with the angle of twist per unit length τ. Values of γ corresponding with different values of $\frac{b}{c}\sqrt{\frac{\mu yz}{\mu zx}} > 1$ were calculated by St. Venant, and are reproduced in Todhunter & Pearson's 'Elasticity.' In calculating values of the elastic limit, the twist per unit length to be used in equation (4) was obtained direct from the torque-twist curves. In calculating values of the moduli of rupture, it was assumed that the material remained elastic right up to the maximum torque, and the twist was taken as proportional to the twist at the elastic limit.

The results of the experiments are given in Tables I., II., and III. The balks are distinguished by the letters A, B, C, D, and about 10 values of each constant were obtained for each balk.

The means of the values are tabulated and also the means of the other properties. In order to give some idea of the variations in the properties across each balk, the probable variations of the values from the means were also tabulated. These were calculated on the assumption that the values obeyed the laws of probability and Peter's formula * was used.

* 'Theory of Errors and Least Squares,' Weld, New York (1916).

TABLE I.

Balk.		A. Mean of 10 experiments.	A. Probable variation from mean per cent.	B. Mean of 10 experiments.	B. Probable variation from mean per cent.	C. Mean of 11 experiments.	C. Probable variation from mean per cent.	D. Mean of 11 experiments.	D. Probable variation from mean per cent.
Density when tested (lb./cu. ft.)		24·8	2·40	33·3	4·0	27·6	4·0	25·7	1·55
Density dry (lb./cu. ft.)		23·4	2·95	31·5	3·6	26·4	3·4	24·2	1·60
No. of annual rings per in.		17·0	33·1	26·1	23·5	10·4	24·5	13·3	19·4
Moisture per cent		11·8	6·3	13·0	6·2	11·2	11·5	12·7	2·10
Shrinkage (per cent. on dimensions of dry test piece)	XOX	1·94	10·4	3·04	7·0	1·91	12·0	1·92	4·9
	YOY	3·29	7·7	3·84	4·9	3·26	6·4	3·79	5·7
Mod. of Rigidity (μyz) (10^4 lb./sq. in.)		9·78	3·8	12·4	7·7	9·58	8·8	9·16	7·4
Elastic Limit (yz) (lb./sq. in.)		14·30	5·1	1330	5·4	1140	9·9	995	5·4
Mod. of Rupture (yz) (lb./sq. in.)		2660	6·2	2950	6·5	2600	6·6	2390	3·5
$\frac{\text{Mod. of Rupture}}{\text{Elastic Limit}}$		1·93	8·9	2·22	9·2	2·35	14·6	2·43	5·0

TABLE II.

Balk.	A. Mean of 11 experiments.	A. Probable variation from mean per cent.	B. Mean of 13 experiments.	B. Probable variation from mean per cent.	C. Mean of 10 experiments.	C. Probable variation from mean per cent.	D. Mean of 11 experiments.	D. Probable variation from mean per cent.
Density when tested (lb./cu. ft.)	24·5	2·45	32·4	3·9	27·5	3·2	25·7	1·35
Density dry (lb./cu. ft.)	23·2	2·65	31·7	3·0	26·0	3·2	24·1	1·40
No. of annual rings per in.	18·3	30·7	27·4	19·7	9·05	17·4	13·3	17·1
Moisture per cent.	11·3	7·7	13·0	4·9	10·8	7·2	12·7	1·95
Shrinkage (per cent. on dimensions of dry test piece) XOX	1·84	8·3	2·94	7·0	1·80	7·4	1·93	4·4
Shrinkage (per cent. on dimensions of dry test piece) YOY	3·17	7·7	3·79	5·7	3·12	3·7	3·88	5·2
Mod. of Rigidity (μzx) (10^4 lb./sq. in.)	7·51	3·7	9·05	10·0	11·4	10·0	7·98	6·9
Elastic Limit (zx) (lb./sq. in.)	1160	3·2	990	6·9	1240	6·9	985	4·7
Mod. of Rupture (zx) (lb./sq. in.)	2310	3·7	2430	1·30	2950	2·5	2620	1·65
$\frac{\text{Mod. of Rupture}}{\text{Elastic Limit}}$	1·97	2·0	2·40	4·7	2·37	3·7	2·63	6·1

Table III.

Balk.		A. Mean of 11 experiments.	A. Probable variation from mean per cent.	B. Mean of 11 experiments.	B. Probable variation from mean per cent.	C. Mean of 13 experiments.	C. Probable variation from mean per cent.	D. Mean of 14 experiments.	D. Probable variation from mean per cent.
Density when tested (lb./cu. ft.)		24·9	2·40	32·8	2·75	28·5	2·70	25·8	1·40
Density dry (lb./cu. ft.)		23·6	2·20	30·4	2·70	26·8	2·60	24·4	1 20
No. of annual rings per in.		18·0	14·3	18·7	15·5	9·58	20·0	13·3	13·1
Moisture per cent.		11·0	6·9	11·4	4·3	12·2	7·5	12·1	2·30
Shrinkage (per cent. on dimensions of dry test-piece)	XOX	1·76	9·4	2·65	7·0	2·02	5·7	1·78	6·0
	YOY	3·21	10·0	3·42	5·8	3·44	5·9	3·83	6·4
Mod. of Rigidity (μxy) (10^3 lb./sq. in.)		3·77	12·0	5·09	8·7	5·32	15·0	3·74	8·0
Elastic Limit (xy) (lb./sq. in.)		4·66	21·5	99·2	14·4	104	9·3	38·9	4·8
Mod. of Rupture (xy) (lb./sq. in.)		317	16·3	393	19·0	483	15·0	256	10·0
$\frac{\text{Mod. of Rupture}}{\text{Elastic Limit}}$		6·87	12·5	3·99	11·4	4·47	10·8	6·50	8·7

It should be noted that where two experiments were performed on one test piece yielding a value for two of the moduli of rigidity, only one value of the elastic limit and modulus of rupture could be obtained. Thus the means of the elastic limits and moduli of rupture usually corresponded with a smaller number of results than the number of experiments given at the heads of the columns.

For each experiment the torque was increased at half-minute or three-quarter-minute intervals, and the amounts of the increases were such as to correspond with from eight to twenty readings up to the elastic limit, so that as a rule the elastic limit was reached in about 10 minutes or less.

From a consideration of the tables, it will appear that the mean values of the percentages of moisture differ little from 12 per cent. This sensibly constant humidity was obtained by keeping the prepared test pieces in the laboratory away from the sun, for about one month before experimenting upon them.

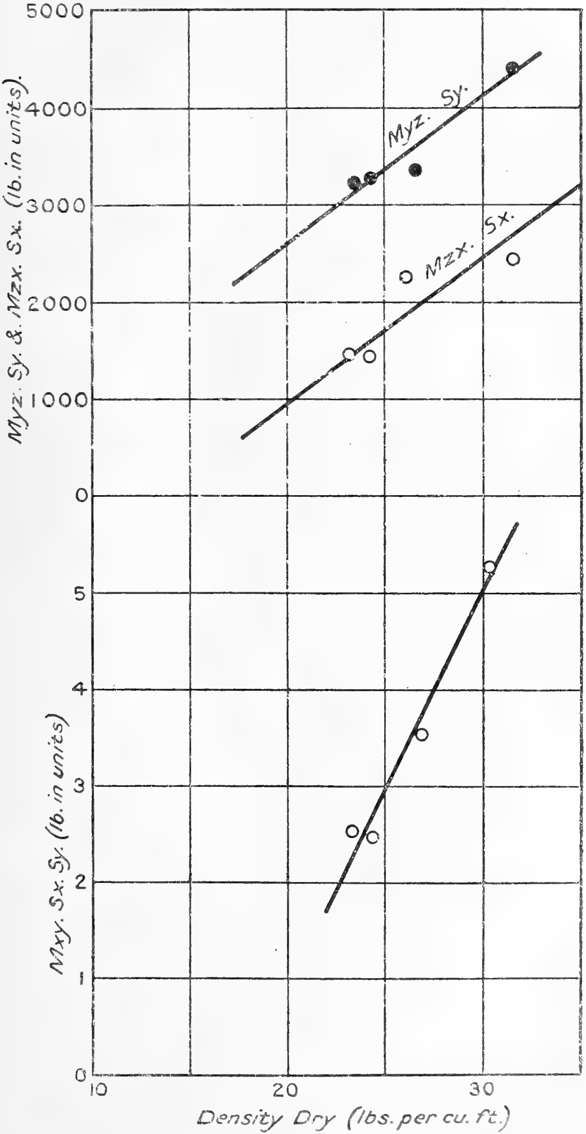
Photomicrographs of normal, tangential, and axial sections magnified about 40 diameters are shown respectively in Pl. IX. figs. 3, 4, and 5 for balk A, these being also typical of similar sections for the other balks. All the baulks were straight-grained and free from knots and shakes, and represented the good average spruce which was available for aeroplane manufacture early in the war.

Relations between the Moduli and the Physical Properties.

At the commencement of the research only balk A was experimented upon. It was then found that values of a modulus obtained from pieces which had been nearly adjacent to each other in the balk often varied considerably. This suggested that the variations might be connected with the shrinkages, which were accordingly measured. No relations, however, between the values of a particular modulus and the shrinkages or any of the measured physical properties could be found. From an examination of sections of the pieces under the microscope, differences in structure often appeared, and it seemed likely that investigation along these lines might reveal reasons for the variations. Such refinement being impracticable, it was decided to consider the variations as characteristic of the structural nature of the material, and to experiment upon three other balks, with a view to the possibility of finding relations between the means of the moduli and physical properties.

An examination of the means from Tables I., II., and III. resulted in the curves shown in fig. 6.

Fig. 6.



Here Sx and Sy denote respectively the shrinkages in the directions XOX and YOY per unit length of dry timber, and the products $\mu yz Sy$, $\mu zx Sx$, and $\mu xy SxSy$ are plotted against the dry densities. It will be noted that the points involving μxy lie sensibly on a straight line, whilst those involving μyz and μzx lie approximately on two parallel straight lines.

Relations similar to the above have been obtained for each of three values of Young's Modulus and six values of Poisson's Ratio for spruce, which the writer hopes shortly to publish, and these were found to agree with the symmetrical ratios of a body having three planes of elastic symmetry *. It may be noted that the number of annual rings per inch is not involved in any of the relations.

The writer wishes to thank the late Principal, J. C. M Garnett, M.A., for providing facilities for conducting the research, and also Professor G. Stoney, F.R.S., for many helpful suggestions. He is further much indebted to Mr. A. L. McAulay, B.Sc.Tas., for valuable assistance with the experiments.

LXXXI. *On the Colours of Mixed Plates.*—Part II. *By* C. V. Raman, *M.A., Palit Professor of Physics, and* Bhabonath Banerji, *M.Sc., Assistant Professor of Physics, University of Calcutta* †.

1. *Inadequacy of the Elementary Diffraction Theory.*

THE optical phenomena exhibited by mixed plates have been described in detail, with illustrations, in the first part of the paper. We now proceed to consider their explanation. It is obvious that the phenomena must be classed as laminar diffraction effects. But, as has already been remarked in the introduction to the first part of the paper, many of the features observed differ from what one might expect on the usual elementary theory of diffraction phenomena. Conspicuous amongst these is the special character of the diffraction-halo seen surrounding a distant light-source viewed through a mixed plate of uniform thickness. Many of the observed features of this halo are not explicable on the elementary diffraction theory: namely, the succession of dark and bright rings of widths rapidly increasing from the centre outwards, the perfect blackness of

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* Love, 'The Mathematical Theory of Elasticity,' Art. 73.

† Communicated by the Authors.

the dark rings in the outer part of the halo, and the obviously composite structure of the inner part of the halo. In seeking for an explanation of these effects, a clue is furnished by the observation already recorded, that when a mixed plate is observed by the light diffracted by it, *the whole surface of the film does not appear luminous, but only the laminar boundaries or lines of separation of the two media forming the film.* The optical effects of mixed plates are thus, in fact, the optical effects due to the scattering or radiation of light from laminar diffracting boundaries. It is necessary to study the manner in which an individual laminar boundary scatters or diffracts light incident on it; to compare this with the indications of theory, and from the observed effects to infer the aggregate result of the scattering by a large number of such boundaries irregularly situated on the film. These points we now proceed to discuss.

2. *Examination of Mixed Plates by the Method of the Foucault Test.*

The most convenient way of examining the scattering of light by a laminar boundary in directions nearly *coincident with that of the incident waves* is by the method of observation known as the Foucault knife-edge test or the Töpler Schlieren method. The theory of this method was developed by the late Lord Rayleigh on the basis of the usual elementary treatment of diffraction phenomena *, and it was shown by him that a discontinuous laminar boundary should appear as a luminous line when examined by the method of the Foucault test. A beautiful illustration of Rayleigh's theory is furnished on examining a clear piece of mica by the Foucault test, when it will be found that the striæ or boundaries in the mica between regions having slightly different thicknesses shine out as vividly-coloured lines of light in a dark field †. To study the phenomena of mixed plates by a similar method, the following arrangement is suitable. Light from a small circular aperture illuminated by an incandescent filament lamp falls upon a good achromatic lens, and is brought to a focus at a distance from it. Two

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* "On Methods for Detecting Small Optical Retardations and the Theory of the Foucault Test." Phil. Mag. Feb. 1917.

† "On the Colours of the Striæ in Mica," C. V. Raman and P. N. Ghosh. 'Nature,' October 1918. See also P. N. Ghosh, Proc. Roy. Soc. A. Series, vol. xcvi. p. 257 (1919).

pieces of good plate-glass, pressed together with a drop or two of water mixed with air between them, are placed in front of the achromatic lens. The film thus enclosed between the glasses forms the "mixed plate," which is observed through a telescope placed with its objective just behind the focus of the achromatic lens. The appearance of the film, as seen with this arrangement, depends on the form of the aperture or stop regulating the admission of light from the focal plane of the lens into the object-glass of the observing-telescope:—(A) With an aperture placed centrally in the focal plane so as to admit the light coming to the geometrical focus but cutting off the diffracted light, the water-air boundaries in the mixed plate appear as coloured lines in a bright white field. (B) When a central stop is placed symmetrically so as to cut off the light coming geometrically to a focus, and an annular aperture surrounding the stop admits only diffracted light into the observing telescope, the whole field appears dark, except the water-air boundaries, which are seen apparently doubled, shining out as two brightly-coloured lines of light running parallel to each other and separated by a fine perfectly dark line coinciding with the exact outline of the boundary. The colours seen depend only on the thickness of the film, and are independent of the size or shape of the boundaries. They are complementary to those seen in case A. A magnified photograph of the film under these conditions, showing the apparent doubling of the boundaries, was reproduced in fig. 4 of the Plate accompanying the first part of the paper. (C) When, instead of a symmetrical annular aperture, we have only a small aperture placed eccentrically in the focal plane admitting diffracted light into the observing telescope, then the full outlines of the water-air boundaries are not seen, but only two small portions of each closed boundary, such that the normals to the boundary at the two points visible are parallel to the radius vector joining the focus with the aperture placed in the focal plane. With such an excentrically-placed aperture the phenomenon of the doubling of the boundaries noticed under (B) does not occur, and we merely get a single luminous coloured line in a dark field running along the portions of the boundary visible.

The phenomena described above are closely analogous to those exhibited by the striæ in mica *. The observations show clearly that the laminar boundaries in a mixed plate

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* P. N. Ghosh, *loc. cit.*

act as centres or sources of diffracted radiation. Each element of a laminar boundary may be regarded as sending out *two* streams of radiation—one on the more retarded, and one on the less retarded side of the wave-front. In directions nearly coincident with that of regular transmission of the incident waves, these two streams are of practically equal intensity and of opposite phases. In such directions the phenomena observed are in agreement with the indications of Lord Rayleigh's theory, according to which the colour of the laminar boundary, as seen in the Foucault test, should be complementary to the colour of the central fringe in the laminar diffraction pattern produced by it. For very small angles of diffraction, therefore, the elementary diffraction theory gives results which are substantially valid. As we shall see presently, this ceases to be true when we consider larger angles of diffraction.

3. *The Unsymmetrical Scattering of Light by Laminar Boundaries: Normal Incidence.*

Very simple observation suffices to show that the scattering of light through larger angles by laminar boundaries exhibits features not indicated by the elementary theory. For this purpose, a thin film of liquid mixed with air enclosed between two glass plates is placed normally in the track of a strong pencil of light from a lantern, and viewed obliquely by the eye with or without the aid of a magnifier. It will be noticed at once that the edges of separation of liquid and air diffract light in a strikingly unsymmetrical manner. Any given edge can easily be observed diffracting light at all angles up to 90° when viewed on the side passing through the liquid; but viewed on the side passing through air, it can hardly be seen at all except in directions making less than about 10° or 15° with the direction of the incident beam, so small is the intensity of the diffracted light in this region. If a closed curved boundary enclosing air be viewed at a slight obliquity to the direction of the incident light, the two limited portions of the boundary visible appear differently coloured, one being much fainter than the other. The fainter portion which is seen through air vanishes altogether when viewed at greater obliquity, while the part of the boundary seen through the liquid, *i. e.* on the more retarded side of the wave-front, remains visible throughout, its colour changing periodically and becoming richer as the obliquity of observation is increased, and finally appearing achromatic when viewed in a direction nearly parallel to the plate. An edge too thick to show colour when observed in a direction nearly normal to the plate, will appear vividly coloured when viewed obliquely on one side of this direction, and be practically invisible from the other side. It should also be remarked that in oblique directions the portions of any one curved boundary that can be seen at a time become greatly reduced—in fact, nearly contract to single points. The normal to the boundary at these points lies in the plane of observation. Each element of a curved boundary is therefore effective in scattering light principally in a plane normal to its own direction *.

For a closer examination of the manner in which the colour of the light scattered by a laminar boundary varies with the direction of observation, the mixed plate may be placed on the table of a spectrometer, and viewed through a low-power microscope which replaces the telescope ordinarily used in the instrument. The laminar boundaries under observation should be illuminated by a somewhat narrow pencil normally incident on the plate, and in order to screen this from entering directly into the field of view of the microscope, a wire may be placed immediately in front of the objective. On turning the microscope about the axis of the spectrometer, the phenomena described in the preceding paragraph may be readily observed and studied. Viewed nearly in the direction of the incident light, the laminar boundaries appear of a uniform colour depending on the thickness of the film. On turning the microscope aside to a slightly oblique direction, each of the boundaries seen changes colour, but differently in its two parts which are seen respectively through the more and less refrangible media. For instance, a closed boundary which, seen in a nearly normal direction, appears throughout golden yellow, viewed at a slightly greater obliquity, appears red on the portion seen through the liquid and greenish blue on the portion seen through air, the latter appearing much fainter. The colours and intensities of the two parts of each boundary are interchanged when the microscope is turned over to the other side. Viewed at still greater obliquities, further fluctuations of colour occur, the sequence of these variations being quite asymmetric with respect to the two sides of the direction of the regularly transmitted pencil. A clear idea

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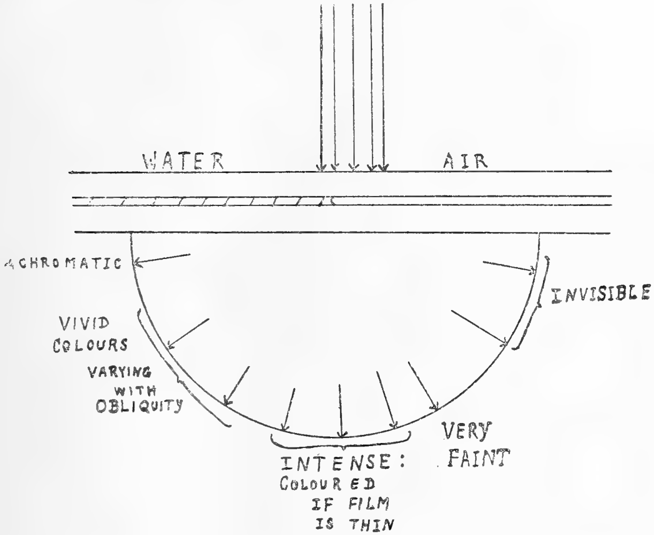
* This is generally true of all curved diffracting boundaries on which light is normally incident. See Phil. Mag. Jan. 1919, p. 127, and Sept. 1919, p. 219.

of these phenomena will be obtained from the diagram given below.

Fig. 1.



The fluctuations of colour with obliquity are most striking in the region on the left of the diagram where they commence, and are continued over to the right, where they are much less vivid, apart from the greatly decreased intensity of the diffracted light on this side. If, instead of white light, a pencil of monochromatic light be used to illuminate a laminar boundary, the light scattered by it shows a similar asymmetry, the intensity fluctuating as the obliquity is varied, and vanishing in a series of directions which are closer together or wider apart according to the thickness of the film.

4. *Explanation of the Diffraction-Haloes.*

We are now in a position, on the basis of the observations described in the preceding section, to form a general idea of the manner in which the diffraction-haloes due to a mixed plate arise. To begin with, it may be assumed that the mixed plate is of uniform thickness, and that the light is incident normally upon it. Consider the light diffracted by the film in a direction making an angle θ with the incident pencil of rays. To the aggregate scattering in this direction the laminar boundaries in the film all contribute, the effective elements of each boundary being those normal to the plane containing the incident and scattered rays. Since, as we have seen, each boundary scatters light in an unsymmetrical manner with respect to the direction of the transmitted pencil, we may divide all the effective elements into two groups, namely those which scatter light in the given direction respectively through the more and less refrangible media in the film. In summing up the effects of the elements in each group in the given direction θ, the relative phases of the scattered rays which depend on the positions of the scattering elements have to be taken into account; and as the boundaries are irregularly distributed on the film, the phases of the elements in each group may be assumed to vary arbitrarily. Further, as the dimension of each boundary—that is, the diameter of each air-bubble in the film—varies between wide limits, there is also no fixed phase-relation between the corresponding elements in the two groups. Thus, the intensity of the scattered light in the direction θ is *statistically* equal to the sum of the intensities due to the two groups separately, and the intensity due to each group is, similarly, the sum of the intensities due to its discrete elements. But for individual values of θ there may be large deviations from the statistical average, and this gives rise to the granular structure of the halo in monochromatic light, and its radial fibrous structure in white light [*]. We have seen that the colour and intensity of the scattering by each elementary boundary varies with θ in a manner depending on the thickness of the film, and since this is constant, the *average* aggregate effect of each group varies with θ in the same way as for individual elements. Considering together all the possible directions of the scattered pencil, we see that each group will give rise to a diffraction-halo with circular rings surrounding the direction of the source. The diffraction-halo due to one group extends from $\theta=0°$ up to $\theta=90°$, while that due to the other group is of sensible intensity only for small values of θ. Near the centre, therefore, the two sets of rings due to both groups of elements are superposed and the halo is composite; while in the outer part, only those elements which diffract light through the more refrangible medium have a sensible effect, and the halo is therefore simple. It is thus seen that

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[*] Compare De Haas, "On the Scattering of Light by Small Particles." Proc. Roy. Soc. of Amsterdam, 1918 p. 1278; also Lord Rayleigh, Phil. Mag. Dec. 1918.

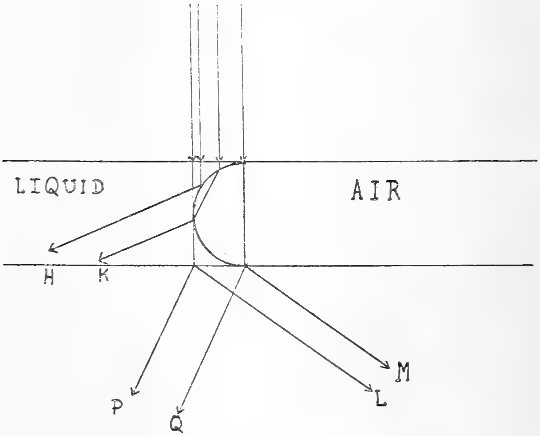
the summation of the effects of the individual boundaries in the film leads to results in close agreement with the features of the halo already described.

The foregoing treatment also enables us at once to explain the increased intensity of the halo in certain directions in the case of films containing elongated boundaries. In so far as relates to the general configuration of the halo, the arguments of the preceding paragraph apply *mutatis mutandis* also in the case of such films, and it is clear why the distortion of the boundaries leaves the circular form and positions of the rings in the diffraction-halo unaffected; the only difference is as regards the relative intensity of the halo along different radii, which depends on the aggregate length of the scattering elements effective along the respective directions. If we divide up each boundary in the film into n parts, such that the successive normals at the points of division make angles of $2\pi/n$ with each other, the aggregate scattering effect of each of the n groups of parallel elements in the film in the planes respectively normal to them would be the same, provided the *average* length of an element in each of the n groups were the same. The latter condition is satisfied, provided the boundaries in the film show no bias towards elongation in any particular direction. But if they do show such bias, the average length of an element is greatest in respect of the groups running parallel to the general direction of elongation. and least in the groups running transverse to such direction. From this, it follows that the intensity of the halo should be greatest in the plane perpendicular to the direction of elongation, and least in the plane parallel to it. This is exactly what is observed. The intensities should, in fact, be quantitatively proportional to the square of the average length of an element in each group.

5. *Mathematical Theory: Normal Incidence.*

We have now to consider the explanation of the unsymmetrical scattering by the laminar boundaries in a mixed plate, and to express the results in quantitative form. No *rigorous* treatment of the problem of diffraction by plane transparent laminæ bounded by edges appears as yet to have been put forward. In practice, the precise shape of the diffracting-edge should obviously have a considerable influence in determining the manner in which it scatters light in directions much removed from that of regular propagation of the incident waves. For instance, the striæ in mica are often found, under the microscope, to possess an echelon-like structure *. The striæ diffract light asymmetrically through large angles, but the effects observed with different striæ differ in a manner which suggests that the results are influenced by the structure of the laminar edge as well as by its total thickness. In the case of mixed plates, owing to the action of surface tension, the laminar edges are not perpendicular to the surface of the film, but have the form of a meniscus (fig. 2). We shall assume that the angle of contact of the liquid with the plate is zero, and that the meniscus is of semicircular form. The diagram is assumed to be drawn perpendicular to the plane of the film and also to the element of the scattering boundary under consideration.

Fig. 2.



It is obvious, especially in view of the curvature of the surface of separation, that the incident light would be scattered very differently towards the two sides of the boundary between the two media. On either side, scattered disturbances would emerge which have traversed different paths partly through one medium and partly through the other, and the problem is to find their relative intensities and the path-differences under which they interfere. As regards the light scattered towards the side of the more refrangible medium, it is convenient to assume tentatively that the paths traversed are those given by the laws of

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* P. N. Ghosh, "On some Phenomena of Laminar Diffraction observed with Mica." Proc. Ind. Assoc. for the Cultivation of Science Calcutta, vol. vi. pt. i. (1920).

geometrical optics. Part of the light incident on the curved meniscus would be reflected within the liquid, and if the angle of incidence were greater than the critical angle, the reflexion would be total. Part would also be refracted into the rarer medium, and after a second refraction emerge again into the liquid. If these disturbances finally travel in parallel directions, such as those shown by H and K in the plane of the diagram, the path-difference between them may be readily evaluated and shown to be

$$t(1-\mu\sin i)(\mu\cos i-\sqrt{1-\mu^2\sin^2 i})+\delta. \quad . \quad . \quad (1)$$

In this formula, t is the thickness of the film, μ the refractive index of the liquid, and i the angle of incidence on the meniscus of the light which is twice refracted. δ is the correction necessary on account of the change of phase in total reflexion. If i be nearly equal to zero, both pencils emerge nearly parallel to the direction of the incident rays, δ has the value which obtains at nearly grazing incidence, and the expression for the path-difference reduces to

$$(\mu-1)t-\lambda/2. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

The light scattered through small angles thus interferes under a difference of path which is the same as that of the regularly-transmitted pencils less half a wave-length. This agrees with what we should expect on the simple diffraction theory, the scattered light being of a colour complementary to that due to the interference of the regularly-transmitted pencils. For larger angles of scattering, the difference of path between the interfering pencils given by (1) steadily falls off in magnitude, and finally becomes zero when the angle of incidence on the meniscus is just equal to the critical angle, as the pencils then become coincident and δ is also equal to zero. We should thus expect to observe a series of maxima and minima of intensity in the scattered light in different directions, which is exactly what is found in experiment. The deviation of the interfering pencils within the liquid film is given by $2(r-i)$ where $\mu\sin i=\sin r$, and when this is equal to $(\pi-2\alpha)$ where α is the critical angle, the path-difference vanishes, and we should expect the scattered light to be achromatic. The angle of scattering θ on emergence from the film is given by the relation $\sin\theta=\mu\sin 2(r-i)$. It is worthy of note that, as the correction δ for the change of phase in total reflexion depends on the plane of polarization of the incident light, the positions of the maxima and minima in the scattered light should be slightly different for light polarized in and at right angles to the plane of incidence, the difference being greatest when the path-difference given by (1) is rather small. This point will be noticed again hereafter.

We should, of course, also consider disturbances such as those indicated by P and Q in the diagram, which have traversed paths lying wholly in one medium or the other and are then diffracted in oblique directions from the edges of the regularly-transmitted wave-fronts. If θ be the angle of diffraction on emergence from the film, the path-difference under which such disturbances interfere is

$$(\mu-1)t-t/2 \,.\, \sin\theta-\lambda/2, \quad . \quad . \quad . \quad . \quad (3)$$

the deduction of $\lambda/2$ being made on account of the phase-reversal of the diffracted ray. When the angle of scattering is not large, the path-differences given by equations (3) and (1) are identical, as may be readily shown on expanding (1) and neglecting the second and higher powers of i. Indeed, in this case it follows from the well-known principle of minimum or stationary path, that the actual course followed within the film by either of the interfering disturbances is a matter of indifference, provided the deviations from the geometrical path are not large. On the other hand, when the angle of scattering is large, the intensity of the light scattered from the curved interface between the two media towards the more refrangible medium would be far larger than that diffracted from the edges of the wave-fronts. Hence, both for small and large angles of scattering we would be justified in regarding the expression (1) for the path-difference as substantially valid.

Passing on now to consider the light scattered towards the less refrangible medium, it is clear in this case that no sensible portion of it is contributed by the curved interface between the media. The scattered light which emerges consists entirely of disturbances (such as those indicated by L and M in the diagram) diffracted from the edges of the wave-fronts. These interfere under a path-difference

$$(\mu-1)t+t/2 \,.\, \sin\theta+\lambda/2, \quad . \quad . \quad . \quad . \quad (4)$$

which now *increases* with the increasing obliquity of the diffracted light; in directions nearly normal to the film the scattered light is, as before, of a colour complementary to that due to the interference of the regularly-transmitted pencils, and its intensity is equal or comparable with that of the light similarly diffracted towards the side of the more refrangible medium. At larger angles of diffraction, however, the intensity falls off with great rapidity, and is far less than on the side of the more refrangible medium where the light scattered from the curved interface plays an important part.

We have not, so far, discussed the relative intensity of the interfering pencils scattered in any given direction. The mathematical treatment of this question may be deferred till a later stage. For the present, it may suffice to remark that experimental observation as already detailed shows the interferences to be remarkably perfect, and hence the interfering pencils must be of comparable or equal intensity throughout the region in which we have maxima and minima in the scattered light. There is no difficulty in understanding, at least in a general way, why this is the case. In directions nearly normal to the film, light is diffracted chiefly from the wave-fronts regularly transmitted through the film, and the contributions to the scattered radiation from the part of the wave-front lying on either side of each boundary should obviously be equal. In more oblique directions the scattering occurs chiefly at the curved interface between the two media, and a calculation on the principles of geometrical optics shows the intensities of the pencils emerging respectively after two refractions and after total reflexion to be comparable throughout, the intensity of the former being at first greater, then equal and finally less than that of the latter, as the angle of scattering increases. Thus there is reason to expect that throughout the range in which the scattered light can be observed, the interferences should be strongly marked.

6. *Concluding Remarks.*

Numerical computation of the position of the dark and bright rings in the halo from formula (1) of the preceding section gives results in general agreement with experiment. The width of the successive rings increases rapidly as we proceed outwards from the centre of the halo. For instance, in the particular case of a film for which the path-difference for normal transmission through the two media is five wave-lengths, the angular radii of the five dark rings in the halo as given by the formula are $\theta=0^\circ$, $\theta=9^\circ\ 20'$, $\theta=20^\circ\ 27'$, $\theta=37^\circ\ 12'$, and $\theta=73^\circ$, respectively. These quantities are of the same order as those actually observed.

A detailed quantitative comparison between experiment and theory, together with a discussion of the theory of the diffraction-haloes due to obliquely-held plates and of the effects observed with non-uniform plates, as also of the special phenomena observed with dry films of albumen, will be given in the concluding instalments of the paper.

Calcutta, India,
Dated the 14th of October, 1920.

LXXXII. *Some Dimensions of the Atom.*

By L. St. C. BROUGHALL [*].

IN the majority of the theories so far advanced on the subject of the structure of the atom, little has been said about the actual dimensions.

The diameter of the molecule varies approximately between 1 and 10×10^{-8} cm., the value for hydrogen being somewhere in the region of 2×10^{-8} cm. Now in the case of the inert gases we conclude, on the evidence afforded by the ratio of the specific heat at constant pressure to the specific heat at constant volume, that the molecules are monatomic. Therefore, any investigation on the size of the molecule is also, for these gases, an investigation on atomic dimensions.

There are several methods which have been applied to the measurement of σ, the diameter of the molecule of a gas, and the experimental values for the inert gas have been found particularly in two ways, namely—(i.) the value deduced from "b" in Van der Waals' equation; (ii.) the value obtained from the viscosity of the gas.

Now in the first case we have the equation $\left(p+\frac{a^2}{\nu}\right)(\nu-b)=R\theta$, where "$b$" is a numerical constant required to correct the simple gas law for the volume of the molecules present. In order to obtain the diameter we use the equation $\sigma=\left(\frac{3b}{2\pi N}\right)^{\frac{1}{3}}$ where $N=2{\cdot}75\times10^{19}$.

In the second case σ can be calculated from the equation $\sigma=\left({\cdot}0912\frac{\rho G}{N\eta}\right)^{\frac{1}{2}}$ where ρ = density of gas in gms./c.c. at S.T.P., G = velocity of molecule in cms./sec. at 0° C., η = viscosity, N = number of molecules per c.c. at 0° C. and 76 cm. pressure $=2{\cdot}75\times10^{19}$.

Using the above formulæ we find that the values for the inert gases are as follows:—

Gas.	Molecular diameter deduced from η.	b.
Helium	$2{\cdot}02\times10^{-8}$ cm.	$2{\cdot}30\times10^{-8}$ cm.
Neon	2·37 ,, ,,	......
Argon	3·41 ,, ,,	2·86 ,, ,,
Krypton ...	3·89 ,, ,,	3·14 ,, ,,
Xenon	4·58 ,, ,,	3·42 ,, ,,

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[*] Communicated by the Author.

Now, owing to the differences between the values obtained by the two methods, it is apparent that the diameter can only be found to one significant figure, and in some cases even this accuracy cannot be obtained. On the other hand, we would expect that the values deduced from one source would show any relation existing between the respective molecules.

Now both sets of values show an increase in σ with increase in the atomic number of the element, presuming it is a monatomic gas; but the values obtained from "b" of Van der Waals' equation show a much more remarkable fact, namely, that the increase in moving from one inert gas to the succeeding one is a constant. Thus:

$$\sigma_{\mathrm{Xe}}-\sigma_{\mathrm{Kr}}=\cdot 28\times 10^{-8}\ \mathrm{cm.};\ \sigma_{\mathrm{Kr}}-\sigma_{\mathrm{A}}=\cdot 28\times 10^{-8}\ \mathrm{cm.};$$
$$\sigma_{\mathrm{A}}-\sigma_{\mathrm{He}}=\cdot 56=\cdot 28\times 2\times 10^{-8}\ \mathrm{cm.}$$

Unluckily I have not got the value for Neon, but it is almost certain that its diameter is $2\cdot 58\times 10^{-8}$ cm.

These figures become much more interesting when compared with Langmuir's "Theory of Atomic Structure," published in the Journal of the American Chemical Society for 1919.

Now, it was found by H. J. Moseley in 1913 that the number of electrons in the atom of an element is equal to the atomic number of that element. Thus the atomic number (N) for Helium = 2; Neon = 10; Argon = 18; Krypton = 36; Xenon = 54; Niton = 86.

These figures were shown in Langmuir's paper to obey a very simple law, namely, that $N=2(1^2+2^2+2^2+3^2+3^2+4^2)$ for the inert gases. Thus N for Helium = $2(1^2)$. N for Neon = $2(1^2+2^2)$, etc.

On this, and in order to explain the valency of the elements and their magnetic properties, Langmuir advanced the explanation that each of the terms in the above equation represents a complete shell of electrons, but the Neon–Argon shells are formed into one shell, as also the Krypton–Xenon shells.

The respective shells are referred to as I, II *a*, II *b*, III *a*, III *b*, IV. Now he further states that the distance between the first shell and shell II *b* is equal to the distance between shells II *b* and III *b*. Now the above figures prove that this is the case, and it follows that each electron has the same free space at its disposal irrespective of the shell it belongs to, as the number of electrons increases in the same proportion as the space available; but owing to the existence of the shells II *b* and III *b*, one has to presume that two electrons can be squeezed into the space for one without making the atom unstable. This is the case of the Argon and Xenon atoms.

In the case of the inert gases the outer shell is always complete and there is no tendency to acquire other electrons; their valency is therefore zero. All the other elements have the outer shell incomplete, and the valency is given by the number of electrons which must be gained or lost in order to produce the stable structure peculiar to the inert gases.

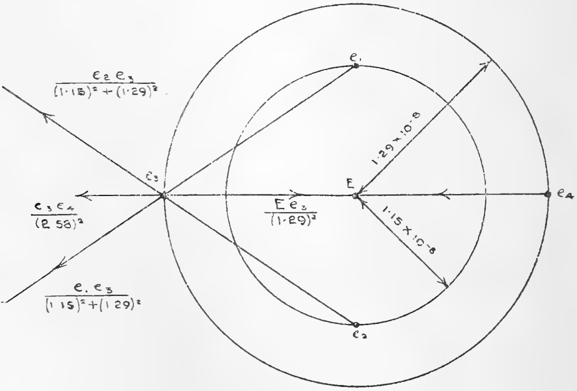
The Boron atom with an atomic number of 5 consists of a positive nucleus; then there are two electrons corresponding to the Helium atom, and finally a ring of three electrons in the shell corresponding to the Neon shell.

The diameter of the atom should therefore be the same as that of the Neon atom, namely $2{\cdot}58 \times 10^{-8}$ cm. These figures therefore give considerable support to this interesting theory; but it must be remembered that only when the atomic diameter is calculated from "*b*" of Van der Waals' equation is this constant increase in diameter to be found.

Now, knowing the ratio of the diameter of one shell to that of the next, one can obtain some further atomic values.

The element Beryllium has an atomic number "4," therefore the electrons are four in number, and so the atom can be represented by a plain figure as the electrons are distributed between two shells.

Fig. 1.



BERYLLIUM

Now all the electrons have the same charge, which we will call "*e*," and we will further call the respective electrons e_1, e_2, e_3, and e_4. Further, let us give the positive nucleus a charge "E." We can now, presuming the inverse square law holds good when the dimensions are so small, find the force acting on any one of the electrons, and as the atom is stable, so the positive charge must balance the negative charges, thus we can find "E" in terms of "e." Let us take the electron e_3.

The diameter of the Helium shell is $2{\cdot}30 \times 10^{-8}$ cm. The diameter of the second shell equals the diameter of the Neon shell $= 2{\cdot}58 \times 10^{-8}$ cm.

Attractive force acting on e_3 towards the centre $= \frac{\mathrm{E}e}{(1{\cdot}29)^2}$;

Repulsive force along e_3e_1 = repulsive force along

$$e_3e_2 = \frac{e^2}{(1{\cdot}15)^2 + (1{\cdot}29)^2}.$$

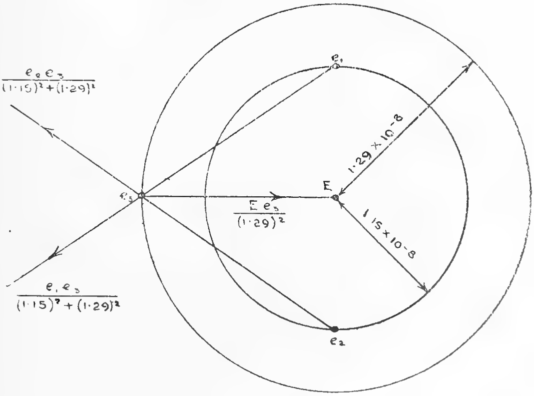
Now, the component of this force acting away from the centre is given by $\frac{e^2}{2{\cdot}99} \sin 48^\circ\ 18'$ = the same component for force acting along e_3e_2.

The force acting between e_3e_4 is $\frac{e^2}{(2{\cdot}58)^2}$, therefore it follows that

$$\frac{\mathrm{E}e}{(1{\cdot}29)^2} = \frac{2e^2 \times {\cdot}747}{2{\cdot}99} + \frac{e^2}{(2{\cdot}58)^2},$$

whence $\mathrm{E} = 1{\cdot}08e$.

Fig. 2.



LITHIUM

For Lithium with an atomic number "3," we find that $\mathrm{E} = {\cdot}83e$.

Helium, as previously stated, has two electrons arranged on opposite sides of the positive nucleus, and we can obviously represent the atom by a plane diagram.

$$\text{Attractive force on } e_1 = \frac{\mathrm{E}e}{(1\cdot15)^2};$$

$$\text{Repulsive force on } e_1 = \frac{e^2}{(2\cdot30)^2};$$

therefore $$\frac{\mathrm{E}e}{(1\cdot15)^2} = \frac{e^2}{(2\cdot30)^2} \text{ and so } \mathrm{E} = \cdot25e.$$

These values for E are calculated on the presumption that (*a*) the electrons are not in motion round the positive nucleus, and so have no acceleration towards the centre, and (*b*) only electrostatic forces are present. It may be mentioned here, however, that in the case of atoms where two electrons are squeezed into the space nominally provided for one, then there is a second force which comes into play, and this exactly balances the repulsive electrostatic force between the electrons. It is therefore an attractive force of some nature.

If we do not presume this, then when we meet with the first case of two electrons above one another there will be a very large increase in the value of E, which although possible is not probable. We can, however, find the distance to which these two forces annul each other.

Let us take a completed Neon atom, then upon adding eight more electrons we shall obtain an Argon atom, of course the positive nucleus must alter at the same time. All the shells I, II, and II *b* are now completed, and in the second shell we have crowded two electrons into the space for one.

If one above the other, it seems feasible that the distance between them should be the same as the distance between the radii of the shells II *a* and II $b = \cdot14 \times 10^{-8}$ cm. Consequently at this distance the two forces balance; but this attractive force does not seem to be noticeable between electrons which are a greater distance apart. Therefore it probably varies as a higher power of the distance than the 2nd.

Owing to the complexity of the elements with atomic numbers above 10, it becomes very difficult to calculate the dimensions of E. From the value of *b* in Van der Waals' equation we obtain no information respecting the actual arrangement of the electrons in the atom. Langmuir's positions for them have been used throughout; the figures only showing that when moving from one inert gas to the next the radius of the atomic sphere increases by a constant quantity.

LXXXIII. *The Coalescence of Liquid Spheres—Molecular Diameters.* By Wilson Taylor, *Physics Laboratory, University of Toronto, Canada* [*].

MOLECULAR phenomena are commonly considered as being caused by some kind of attraction which molecules have for each other when they are near together. This idea has been the basis for the explanation of such phenomena as the liquefaction of gases where the molecules near to each other are attracted so powerfully that they remain permanently in an aggregation. The idea was probably first suggested by Newton, and Laplace pointed out that the phenomena of surface-tension would result from the necessity for this attraction being unequally distributed among the molecules near the surface of the liquid. But he had to assume that this attraction ceased to be effective when the distance between the molecules was beyond a certain range. This has also been made the basis for the explanation of adhesion and solidification. Maxwell carried out Laplace's idea further, using it as a mathematical basis for the properties of the film in a soap-bubble. But he found difficulty in accounting for the formation of the black spots and the sudden changes in the tension of liquid films noticed by Rüker and Reinold in 1886. In recent years, however, the constitution of the soap and other films has been studied in great detail, and there has been expressed by various investigators some doubt as to the validity of the explanations which are based on the attraction of molecules as masses. From direct observation the soap-bubble appears to be composed of a number of strata whose multiples differ in thickness by steps which have been estimated to be from 1 to 10 $\mu\mu$ in height. The difficulty of explaining this by the attraction of molecules, or by any other hypothesis, has been so great that Perrin, who has made an exhaustive study of the bubble (*Annales de Physique*, Sept. 1918), speaks of it as an enigma.

This difficulty led the writer to study the action of two spheres of the same liquid when near to each other and when they came into contact; and the following may be considered as an attempt to consider an alternative basis for the explanation of molecular phenomena.

He was not able to find any direct evidence that such spheres attracted each other, no matter how small they were

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[*] Communicated by the Author.

down to microscopic dimensions, nor how closely they approached each other so long as they did not touch. But when contact was made a powerful force instantly came into play that caused the two masses to coalesce and become one sphere. This was found to be the case with all liquids, independently of the relative sizes of the spheres, and at all temperatures of the liquids. With mercury the action was instantaneous and very powerful. Sometimes, however, two or more mercury spheres would apparently lie in contact without coalescing, but on examination with the microscope there was always found some foreign substance which had accumulated on their surfaces that prevented their coming into contact. The question arose: Is it not possible that the same enveloping force which causes visible spheres to coalesce also causes the free molecules of a gas to aggregate themselves into a liquid?

Now, in any self-contained system of material masses any change in energy is from potential energy, either directly or through kinetic energy of large masses, into kinetic energy of the molecules of the system, that is, into heat. For to produce any change the forces of the system must become statically unbalanced, so that the resultant force does work against the inertia of large and small masses causing them to acquire kinetic energy.

It has long been observed that in many respects surface-tension appears to act as though it were something of the nature of an elastic envelope about a liquid mass trying to compress the mass into a smaller volume by contracting the area of the envelope. This tendency to change the volume, however, is due solely to the curvature of the enveloping surface, which tends to cause compression on the concave side and expansion of volume on the convex side. If there is no curvature there is no tendency either way to alter the volume. The fundamental nature of the tension, then, is to tend to decrease the area of the surface.

Let us consider, then, a system of three spheres of water whose diameters are ·3, ·4, and ·5 cm. The potential surface energy of the system is $\mathrm{T} \times \pi\{(\cdot 3)^2 + (\cdot 4)^2 + (\cdot 5)^2\}$, or $\cdot 50\,\pi\mathrm{T}$ ergs. If these spheres be brought into contact, they will coalesce into one sphere whose diameter is ·6 cm., and whose potential surface energy is $\mathrm{T} \times \pi(\cdot 6)^2$, or $\cdot 36\pi\mathrm{T}$ ergs. The potential surface energy which has been transformed into heat by coalescence is, therefore, $\cdot 14\pi\mathrm{T}$ ergs. If we take, according to the best available data, $\mathrm{T} = 73{\cdot}3$ dynes per cm. at 15° C., this energy amounts to $7{\cdot}71 \times 10^{-7}$ calorie, where $\mathrm{J} = 4{\cdot}184 \times 10^7$.

In the same way, if a large number of liquid spheres coalesce into a given mass, the heat produced would be considerable.

Let n = the number of these liquid spheres in a gram molecule,
d = the diameter of each sphere,
D = the diameter of the liquid gram molecule sphere,
m = the molecular weight.
ρ_θ = the density of the liquid at θ°A,
T_θ = the tension of the envelope about the liquid spheres at θ°A,
and L = the latent heat in calories of a gram mass of the gas at the boiling-point.

Equating the masses before and after coalescence we have

$$n \cdot \frac{\pi}{6} \cdot d^3 = \frac{\pi}{6} \cdot \mathrm{D}^3,$$

from which $$d = \mathrm{D} \cdot n^{-\frac{1}{3}}.$$

Also, the potential surface energy has been reduced by coalescence from $\mathrm{T} \cdot n\pi d^2$ to $\mathrm{T} \cdot \pi \mathrm{D}^2$. Therefore the amount which has been transformed into heat is $\mathrm{T} \cdot \pi(nd^2 - \mathrm{D}^2)$, which by the relation above may be written $\mathrm{T} \cdot \pi \mathrm{D}^2(n^{\frac{1}{3}} - 1)$. Now, since n is very large in comparison with unity, $n^{\frac{1}{3}} - 1$ differs but little from $n^{\frac{1}{3}}$, and thus this amount is $\mathrm{T} \cdot \pi \mathrm{D}^2 n^{\frac{1}{3}}$. Further, $\frac{\pi}{6} \cdot \mathrm{D}^3 = \frac{m}{\rho}$, and we finally see that the potential surface energy of the n liquid spheres which is converted into heat by their coalescence is $\left(\frac{36m^2\pi}{\rho^2}\right)^{\frac{1}{3}} \cdot \mathrm{T} \cdot n^{\frac{1}{3}}$, a quantity which varies as the cube root of the number of spheres obeying the law of coalescence, provided that T remains constant. This will be the case if the temperature remains constant, that is, if the heat is transferred to other masses such as the surrounding air and adjacent bodies.

Now, when a gram mass of a substance condenses from a gas into a liquid at the same temperature, there is produced L calories of heat energy which is known as the latent heat of condensation. Equating, therefore, these two amounts of energy, we have

$$\mathrm{T} \cdot \left(\frac{36\pi m^2}{\rho^2}\right)^{\frac{1}{3}} \cdot n^{\frac{1}{3}} = m\mathrm{LJ},$$

from which $$n = \left(\frac{\mathrm{LJ}}{\mathrm{T}}\right)^3 \times \frac{m\rho^2}{36\pi}.$$

The latent heat of evaporation is commonly thought to be the energy due to the attraction of molecules for one another at close ranges in the liquid. In the interior, where the attractions on a particular molecule are balanced, this energy is not apparent or is unavailable. It is apparent only in regions near the surface where these attractions are unequally distributed. The latent heat, then, is the work done in bringing the molecules of a gram mass from the interior through the surface region of unequal attraction and separating them from one another beyond the range of their attraction. Surface energy is regarded as the energy belonging to the molecules in the surface layer only, and, consequently, is only a part of the latent heat. Various attempts to define the relation between these which assume a general form, $E . V^{\frac{2}{3}} = kL$, have so far been unsatisfactory, the experimental data giving no consistent values for the quantities involved. Other attempts to connect the whole internal molecular energy with the latent heat have failed for the same reason. These consist of seeking to verify relations of the form $N . E_m = kL$, where E_m is the energy of one molecule and k is some constant.

A more successful attempt, however, was recently made "to give precision" to the generally recognized relationship between surface energy and internal latent heat by Mr. D. L. Hammick in an article on "Latent Heat and Surface Energy," which he published in the 'Philosophical Magazine,' August 1919 and January 1920. He conceives the true molecular surface energy belonging to all the molecules in a gram molecule to be the potential energy they acquire in reaching their positions in the surface layer. He imagines the gram molecule spread out in a layer one molecular diameter in thickness on a surface of excess of the liquid. This energy is equal to $E . \frac{V}{d} . \frac{1}{J}$ calories, where E is the surface energy per sq. cm., d is the diameter of a molecule, and V is the volume of the liquid gram molecule. His argument to show that this surface energy is one-sixth of the internal latent heat is as follows :—The work done upon a molecule to bring it into the surface layer is not, as is commonly supposed to be the case, one-half of that required to bring it altogether out of its liquid state. For, since the molecule in reaching the surface layer moves perpendicularly to the surface, the work is done against only "one of the three components of internal pressure," and is, therefore, only one-third of this amount, or one-sixth of the work to bring it altogether from the liquid.

It is difficult to conceive, however, that the work done by either of the other two components of internal pressure parallel with the surface can be equal to that done by the one perpendicular to the surface. For in each of the former components the forces which attract the molecule on opposite sides are equal, and the energy is not available, since no work can be done by forces in equilibrium ; whereas in the case of the latter component the forces which attract the molecule towards the interior are greater than those which attract it towards the exterior, and the energy is available, since work may be done by their resultant. Still, from the best available data for d, E, and L for low temperatures, the calculated values of these two expressions for twenty-nine substances bear out his statement that "the relation above fits these facts remarkably well."

Another objection, which applies also to similar reasoning by other writers, lies in the division of the latent heat into two parts. If the external layer of molecules contains potential energy equivalent to only one-sixth of the latent heat, would not the other layers within the range of molecular attraction contain the remaining five-sixths and the surface energy be 6 E ergs per sq. cm. ? Or, is the latent heat composed of two distinct kinds, or does it arise from two distinct sources?

Now it so happens that Mr. Hammick's formula itself suggests a means of completely meeting both these objections. For his expression $\frac{V}{d}$ for the area occupied by the molecules of a gram molecule when they are all arranged in the surface layer, assuming, as it does, that the molecules fill the whole volume, is also the expression for the area occupied by one molecule. This area is thus $\frac{\pi}{6}d^3/d$ or $\frac{1}{6}\cdot\pi d^2$, which is exactly one-sixth of the area of its free surface. If, therefore, we suppose that the surface of the free molecule possesses surface energy of the same intensity as that which he supposes the surface layer to have acquired, the whole surface energy would be six times as much as Mr. Hammick thinks, and his relation should be :

$$E\cdot\frac{6V}{d}\cdot\frac{1}{J}=L_i,$$

which may evidently be written

$$E\cdot N\pi d^2\cdot\frac{1}{J}=L_i,$$

where N is Avogadro's constant. Also, since his molecules occupy all the space in the liquid, we have

$$\frac{\pi}{6} \cdot d^3 = \frac{m}{\rho},$$

where ρ is taken at a low temperature. This enables his relation to be written:

$$\mathrm{E} \cdot \left(\frac{36\,\pi m^2}{\rho^2}\right)^{\frac{1}{3}} \cdot \mathrm{N}^{\frac{1}{3}} \cdot \frac{1}{\mathrm{J}} = m\mathrm{L},$$

where $\mathrm{L}_i = m\mathrm{L}$, which is identical with that we obtained by considering the surface energy given up by the coalescence of liquid spheres.

Thus the carefully compiled data supplied by Hammick's tables really account for the whole of the latent heat, and there is no need to employ the hypothesis of molecular attraction and the doubtful principle of equipartition of energy along the three geometrical axes to obtain a relation that fits in his experimental data.

In this connexion it is interesting to note that Einstein (*Ann. d. Physik*, iv. 3, p. 513 (1901)) has put forward a view that the surface energy of a liquid is of the nature of potential energy expressible by a certain equation, one of whose terms depends on the atomic weight. This is exactly what our formula does, which may be written

$$\mathrm{LJ} = \mathrm{T} \cdot \left(\frac{36\,\pi \mathrm{N}}{m\rho^2}\right)^{\frac{1}{3}},$$

the expression with the brackets being the whole area of the N free molecules of a gram molecule of the gas, and T the energy per sq. cm. which we have supposed to exist in the surface of the free molecules. This energy is potential, as Einstein predicted, for it is stored in the surface of the free molecule incapable of being released for doing work until two such molecules by coming into contact coalesce and lessen their total enveloping area.

Our formula, which was obtained above from the phenomena of the coalescence of liquid spheres and not from any hypothesis concerning molecular attraction, would seem, then, to indicate that this view of the connexion between the latent heat and surface energy is the correct one; and therefore, in addition to the confirmation furnished by Hammick, we shall examine some further experimental evidences in support of this view.

If n, which we may suspect is Avogadro's number, is constant for all substances, we should seek for those values of

T and ρ which are independent of variable conditions of temperature, that is, for those values which they have when the temperature is 0° A. But, as Hammick points out, the data for these are not available, or at least are very incomplete. The value of L, the latent heat at the boiling-point, which is equivalent to the work done in dissociating the molecules from the liquid, that is, in creating the free surface of the molecules of the gas and storing it as potential surface energy, may be taken as constant under all conditions after our determinations have been freed from all external work.

Now, it has been shown experimentally that the surface tension about large masses of any liquid is a linear function of the absolute temperature which holds very approximately for all temperatures of the liquid from its melting-point to the critical temperature θ_c. This result is expressed in the empirical formula

$$T_\theta = A + B\theta,$$

where A and B are constants for any particular liquid. For water the ordinary tables give $T_\theta = 0$ when $\theta = \theta_c = 638°$ A, and $T_\theta = 73{\cdot}3$ as the best available value when $\theta = 288°$ A.

From these we have

$$T_\theta = 133{\cdot}6 - {\cdot}209\,\theta,$$

so that

$$T_0 = 133{\cdot}6.$$

It is evident, however, that only in large spheres or masses can the tension of the envelope obey the law above, for only in these can the heat motion of the molecules next the envelope affect the tension. For spheres containing only a few molecules and for the free molecules themselves an altogether different condition exists, for the heat motion then is the motion of the sphere itself. The tension, therefore, of the envelope about the free molecule will be very approximately the tension about a large mass when the molecules are at rest, that is, the molecular tension T_m will be T_0.

If, then, the law holds through the process of solidification and down to 0° A, we should have for the free water molecule $T_m = 133{\cdot}6$. This seems to be a reasonable supposition when we remember that the forces which cause solidification are in the interior of the molecule or mass and would not, therefore, affect the tension of the envelope on the exterior of the molecule or mass. Moreover (see note at the end), there are experimental indications that the surfaces of solids possess potential energy and consequently are affected by surface tension.

Hence, if we take $L=498$, which is the average of the values given in the tables corrected for external work amounting to 41 calories,

$$\rho_0=\rho_{277}=1, \quad m=18, \quad \text{and} \quad T_m=133{\cdot}6,$$

the formula,

$$n=\left(\frac{L\,.\,J}{T_m}\right)^3 \times \frac{m\rho_0^{\,2}}{36\pi},$$

gives by calculation

$$n=6{\cdot}05\times10^{23}.$$

Millikan's value for this number, obtained by his balanced drop method, is $6{\cdot}065\times10^{23}$; while Perrin's value, obtained by studying the motion of a colloidal sphere in water, is $6{\cdot}86\times10^{23}$.

The result obtained above would seem to furnish an argument in favour of the view that the properties of surface tension can be considered as not depending upon the mutual attraction of molecules. For, if the free molecule has about it this elastic envelope, it is plain that the envelope cannot be material at all. It is simply a force and nothing more.

Since the force of gravity in its relation to potential energy is denoted by $\frac{dE}{dx}$, where x is the distance between the centres of the masses attracting each other, this force will be denoted by $\frac{dE}{dA}$, where A is the area of the mass about which it acts. Also, because of the curvature of this area the force acts to compress the interior of the sphere to a smaller volume; but the action is prevented by another force in the interior which must, therefore, be denoted by $\frac{dE}{dV}$, where V is the volume within the enveloping force. It may be that all physical phenomena may be explained in terms of these three fundamental physical forces, of which beyond these distinguishing characteristics we know but little. What these forces are *per se* we have no idea.

The formula obtained above appears to hold for all spherical masses of liquids, whatever be their size. For, since it gives a value of N practically identical with those we already know, it suggests that the law of coalescence of water spheres does not break down at any point from the coalescence of large spheres to the coalescence of small spheres down to the molecules themselves. The method of obtaining it embodies in its essence discontinuity, involving from the beginning to the end no fewer than p distinct acts of coalescence where $2^p = \frac{\mathrm{N}}{18}$, so that $p=75$.

At first thought it would seem that in the latter stages of condensation, where large spheres coalesce, the decrease in T on account of the heat motion of the molecules next the surface would greatly modify our value of N. Whatever effect this would have, however, must necessarily be very slight. For, suppose the first act of coalescence is to form the molecules into pairs, so that the resulting spheres would each contain two molecules. Then, if we consider only a gram mass, the first act of coalescence may be proved to cause a transference of potential energy into heat amounting to $\mathrm{L}(1-2^{-\frac{1}{3}})$ calories; the second act a transference of $\mathrm{L}(1-2^{-\frac{1}{3}})(2^{-\frac{1}{3}})$ calories; the third act of $\mathrm{L}(1-2^{-\frac{1}{3}})(2^{-\frac{1}{3}})^2$ calories; and so on, the qth act of $\mathrm{L}(1-2^{-\frac{1}{3}})(2^{-\frac{1}{3}})^{q-1}$ calories. These results are calculated in the table below.

Table I.

q.	Energy transferred.
1	102·6 c.
2	81·4 c.
3	64·6 c.
4	51·2 c.
5	40·7 c.
...	....
21	1·0 c.
...	....
75 (final)	0·00000385 c.

At the completion of the fifth act the spheres each consist of 32 molecules, but the gram mass has suffered a change of potential surface energy into heat amounting to 340·5 calories, or of 68 per cent. of the whole. After the 21st act, which causes a change of only 1 calorie, the total surface energy remaining is only 3·91 calories, or considerably less than 1 per cent. Thus we see that the change in T on account of the heat motion in the large spheres towards the latter part of the process of condensation would not affect the value of N materially, causing only a slight increase.

Since the formula obtained above gives the value of N so closely approximating the true value from data experimentally determined in connexion with water, we are led to think that the same value of N would be obtained from similar data in connexion with other substances which are chemically stable under the conditions involved in the determination of these data, though not to the same degree of approximation.

In the Tables II. and III. below are given the values of T_m and N as calculated along with the necessary data for ten of the more common substances, including water. L is the heat energy exclusive of that required for the external work necessary to vaporize a gram mass of the liquid at the boiling-point without change of temperature under atmospheric pressure.

TABLE II.

Substance.	Formula.	m.	θ_c°.	θ°.	T_θ.	$T_0 - B\theta$.
Water	H_2O	18	638	288	73·3	$133·6-·209\theta$
Propyl alcohol...	C_3H_8O	60	536·7	288·4	23·8	$49·5-·0925\theta$
Ethyl alcohol ...	C_2H_6O	46	516·1	293	22	$50·9-·0986\theta$
Methyl alcohol...	CH_4O	32	513	293	23	$53·6-·105\theta$
Ether	$C_4H_{10}O$	74	466·8	293	16·5	$44·3-·095\theta$
Chloroform	$CHCl_3$	119·4	533	288	27·2	$59·3-·111\theta$
Nitrogen	N_2	28	127	77	8·5	$21·6-·170\theta$
Oxygen	O_2	32	155	90	13·1	$30·4-·197\theta$
Benzene...........	C_6H_6	78	561·5	290·5	29·2	$60·0-·107\theta$
Mercury	Hg	200	(1734)*	290·5	547	$657\ -·379\theta$

* Theoretical.

TABLE III.

Substance.	Boil. Pt. °	θ. °	ρ_θ.	$T_m=T_0$.	L.	$N/10^{23}$.
Water	373	277	1·000	133·6	498	6·05
Propyl alcohol...	370·2	293	·804	49·5	152	7·28
Ethyl alcohol ...	351·3	288	·794	50·9	192	10·09
Methyl alcohol...	337·7	288	·796	53·6	246	12·68
Ether	307·6	290	·718	44·3	73·4	1·13
Chloroform	334·2	273	1·526	59·3	52·4	1·24
Nitrogen	77	77	·79	21·6	41·0	·79
Oxygen	90	38	1·27	30·4	47·5	1·28
Benzene...........	353·2	378·4	·879	60·0	86·5	1·19
Mercury	629·7	0	(14·25)	657	61·6	·21

The temperature coefficient of surface tension has been assumed to be constant down to 0° A, and in accordance with the reasons given in the case of water we have regarded the surface tension T_m about a free molecule equal to T_0 in the empirical formula $T_\theta = T_0 - B\theta$. However, the temperature coefficient is only approximately constant, and consequently T_m or T_0 is only approximately determined. These data have been taken from the ordinary tables and are subject to question in some cases. Especially in the case of mercury, where L was determined by Young in 1910, there is a probability that the vapour of mercury involved in the determination was not altogether of dissociated molecules. For example, if the spheres of the mercury vapour contain 32 molecules, the real value of L is seen from Table I. to be $498/(498-340{\cdot}5)$ or 3·16 times the value in the Table III.; and consequently the value of N would be

$$(3{\cdot}16)^3 \times {\cdot}21 \times 10^{23} \quad \text{or} \quad 6{\cdot}64 \times 10^{23},$$

which agrees with that we know.

But the greatest discrepancy is seen in the case of ether, chloroform, nitrogen, oxygen, and benzene. The values of N, however, are very uniform and equal to about one-fifth of the true value. One contributing cause may be in the high vapour pressures common to all of these substances. But the chief cause lies in the use of ρ_θ instead of ρ_0, which in the absence of sufficient data could not be even approximately estimated. So far as we have evidence, ρ_θ increases for the liquid quite rapidly as θ decreases, suddenly increases at solidification, and continues to increase more slowly for the solid down to 0° A. It seems to be quite possible that for these substances the value of ρ_0 would be double the value used; in which case, as N increases with the square of ρ_θ, the values of N would approximate to the true value. For mercury the value 14·25 for ρ_0 was obtained on the supposition that the temperature coefficient of expansion of the liquid remained constant through the solid down to 0° A.

In the case of the alcohols, from considerations in regard to the arrangement of the atoms in the molecule the idea has arisen that the free molecule is not spherical, especially as one of the elements is carbon. If this be so, two such molecules coming into contact would give up less of their enveloping area than they would if they were spherical and coalesced in the same manner as two liquid drops. At first they would resemble two solid particles adhering, and afterwards the combined mass would gradually become spherical as more molecules were included in the enclosure. It would, therefore, take a greater number of molecules in the initial stages of the condensation to produce the same amount of heat, which would account for the larger values of N for these substances. Another cause may be in the long range of temperature for which these substances are liquids.

It would appear, then, that the undetermined elements of these experimental data point to the idea that, if we knew the true values of T_m, ρ_0, and L for all stable substances, the value of N calculated as indicated in our formula would be the true Avogadro's number, which would be a little greater than that obtained in the table for water. Also, the evidence from the data more carefully selected by Hammick points, under the same conditions and with the same limitations, to the same conclusion.

Assuming, therefore, that the conclusion is justifiable, we have, by equating the two expressions for the potential surface energy of the free molecule,

$$\mathrm{LJ}\Big/\frac{\mathrm{N}}{m}=\mathrm{T}_m\times\pi d^2,$$

from which $$d=\left(\frac{m\mathrm{LJ}}{\pi\mathrm{T}_m\mathrm{N}}\right)^{\frac{1}{2}}.$$

This on substituting for LJ the value obtained previously reduces to

$$d=\left(\frac{6m}{\pi\rho_0\mathrm{N}}\right)^{\frac{1}{3}}.$$

The value of d will thus depend upon the value of ρ_0 and will be subject to the same uncertainty as that which belongs to ρ_0. However, as ρ_θ is less than ρ_0, d_θ will be larger than d_0, so that d_θ may be regarded as the average diameter of the space which the actual molecule whose diameter is d_0 occupies at temperature $\theta°$ A. For water we are fairly certain that ρ_0 does not greatly exceed 1 and is not less than 1. The formula gives for water, where $\rho_0=1$, $d_0=3{\cdot}85\times10^{-8}$. For mercury, where ρ_0 is taken equal to 14·25, $d_0=3{\cdot}56\times10^{-8}$. As ρ_θ decreases with rising temperature, d_θ increases, thus making room for the heat vibrations of the real molecule in the solid or liquid.

The following is a brief summary of the views presented in the preceding pages :—

(*a*) Latent heat is the surface energy of the free molecules of the substance in its gaseous state.

(*b*) Surface tension extends to molecular masses causing them to coalesce in the same manner as liquid spheres.

(*c*) Since N is constant for all substances, if any two of the three quantities T_m, ρ_0, and L can be found for any substance, the other may be calculated from the relation

$$N=\left(\frac{LJ}{T_m}\right)^3 \times \frac{m\rho_0{}^2}{36\pi}.$$

(*d*) It is assumed that the empirical law $T_\theta = T_0 - B\theta$ holds down to 0° A, and that $T_m = T_0$.

(*e*) The actual mass diameter of the molecule is

$$d_0=\left(\frac{6m}{\pi\rho_0 N}\right)^{\frac{1}{3}},$$

and the average diameter of the space it occupies in the solid or liquid is

$$d_\theta=\left(\frac{6m}{\pi\rho_\theta N}\right)^{\frac{1}{3}}.$$

Note on the Surface Tension of Solid Bodies.

The idea that solid bodies possess surface tension is believed to be tenable. Van der Mensbrugge (*Anal. Soc. Sci. de Bruxelles,* B. xxix. (1904–5)), from consideration of the forces in the curved meniscus at the side of a vertical glass plate in water, concludes that a glass surface possesses an effective surface tension. M. Petrova (*Jurn. Russk. Fisik. Chimecesk,* xxxvi. Phy. pt. pp. 203 (1905)), from the fact that enlarged photographs of mercury drops on glass taken before and after solidification when superimposed showed no appreciable change, inferred that solid mercury possesses surface tension. And, finally, M. Berggren (*Ann. d. Physik,* xliv. 1, pp. 61–80 (1914)), by measuring the velocity of deformation when amorphous bodies solidify in the case of threads hanging vertically, found that solid amorphous bodies possess surface tension of the same order of magnitude as that of liquids.

LXXXIV. *The Accuracy of the Internally Focussing Telescope in Tacheometry.* *By* T. Smith, *F.Inst.P.*[*]

SINCE its first introduction by Zeiss the internally focussing telescope has come into very general use in surveying instruments. Its adoption appeals to manufacturers since it makes for compactness in other parts of the instrument to which it is fitted, and to surveyors on account of the convenience in an instrument inevitably exposed to all kinds of weather of having all delicate parts permanently sealed up. At first sight it would appear to be unsuited to tacheometrical methods of survey, since the value of a distance determined by the reading on a staff between fixed stadia lines does not differ by a constant length from the actual distance. It is by now well known that with a suitable construction of the telescope the errors involved in its use are not serious over a reasonable range of distances, but the theoretical discussions of these errors hitherto published appear needlessly complex, and the results to which they lead are less general and conclusive than is desirable. Largely as a consequence of the unsatisfactory treatment of the problem, the best procedure in designing and calibrating such telescopes has not been apparent. It is hoped that this discussion avoids these faults.

The staff reading given by the instrument, it must be realized, is essentially a constant divided by the magnification for the image formed by the telescope in the plane of the stadia lines. The distance, apart from an additive constant, is assumed for practical purposes to be a constant multiple of this reading. This consideration forms the key to the following investigation.

Let the staff S be a distance d in front of the first principal focus F of the objective, and the stadia lines S′ a distance x in front of the second principal focus F′ of the focussing lens. Denote the powers of these two lenses by κ and κ' respectively, and the power of the combination for the particular configuration in which the image of S is coplanar with S′ by K. Then since, apart from a constant, x is the separation of the two component lenses,

$$K = A - x\kappa\kappa', \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

where the constant A is the value assumed by K when F′ coincides with S′. Now suppose that the magnification

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[*] Communicated by the Author.

for the staff is m. Since the image is inverted m is negative. Whatever the separation of the lenses may be F and F′ are conjugate points, since both are foci corresponding to a parallel beam of light between the two lenses, and the magnification associated with them is $-\kappa/\kappa'$. Thus the difference between the magnifications at F′ and S′ is $m+\kappa/\kappa'$, and the actual separation of the points being x, the relation between these quantities is

$$x\mathrm{K} = m+\kappa/\kappa'. \quad \ldots\ldots \quad (2)$$

From the ordinary relation between the separations of corresponding pairs of object and image planes

$$x+dm\kappa/\kappa' = 0, \quad \ldots\ldots \quad (3)$$

or alternatively from the relation for the object space analogous to (2)

$$-d\mathrm{K} = 1/m+\kappa'/\kappa \quad \ldots\ldots \quad (4)$$

together with (1) and (2), it follows on eliminating x and K that

$$d = -\frac{\mathrm{A}-\mathrm{M}}{2m\kappa^2}, \quad \ldots\ldots\ldots\ldots \quad (5)$$

where

$$\mathrm{M} = \pm\sqrt{\{\mathrm{A}^2-4\kappa(\kappa+m\kappa')\}}. \quad \ldots \quad (6)$$

A rather subtle point arises in connexion with this ambiguity of sign. It might be thought that A and M must always carry the same sign since d should be zero when $\kappa+m\kappa'$ vanishes. This argument is valid when κ' is positive, but if κ' is negative it is not correct since x must become infinite for some position of the staff between S and F. The correct conclusion is reached by noting that when m increases numerically, that is when the staff approaches the telescope, the power of the combined system must be increased to keep the position of the image stationary. Now d, K, and κ are positive; m is negative. Thus from (3) $x\kappa'$ is positive, and further from (1) A is positive and greater than K. Moreover

$$\mathrm{K} = \tfrac{1}{2}(\mathrm{A}+\mathrm{M}) \quad \ldots\ldots\ldots \quad (7)$$

and therefore M must increase numerically with m. It follows that M must have the same sign as κ'.

In practice m, though never zero, is always small, but this does not justify a solution of the problem by expanding M binomially in a series of ascending powers of m and retaining only the first two terms. The errors resulting from such treatment would not be inappreciable, and would be of the same sign for the whole range of values of m for which the instrument could be used. This conflicts with the fundamental principle on which errors should be treated, which requires them to lie between limits on either side of the approximate result and to reach these limits at each end of the range as well as at the greatest possible number of intermediate points. Any satisfactory discussion must be based on the range of distances for which the instrument is required to give good results. The limits will be determined for near points by the distances it is more convenient to measure with a tape, and for far points by the length of the staff. Let p and q be the limiting magnifications, and P and Q the values of M resulting from the substitution of p and q for m. As m varies continuously from p to q, M will continuously increase or decrease from P to Q. Now if a and b represent any real quantities which at the moment are left arbitrary,

$$\begin{aligned}2(\mathrm{A}-\mathrm{M})(a+b)(a\mathrm{P}+b\mathrm{Q}) &= (a+b)^2(\mathrm{A}^2-\mathrm{M}^2)-\{(a+b)\mathrm{A}-a\mathrm{P}-b\mathrm{Q}\}^2 \\ &\quad +\{(a+b)\mathrm{M}-a\mathrm{P}-b\mathrm{Q}\}^2.\end{aligned}$$

Also

$$\begin{aligned}\{(a+b)\mathrm{M}-a\mathrm{P}-b\mathrm{Q}\}^2 &= \frac{\mathrm{P}-\mathrm{Q}}{\mathrm{P}+\mathrm{Q}}\{(b^2-a^2)\mathrm{M}^2+a^2\mathrm{P}^2-b^2\mathrm{Q}^2\} \\ &\quad +\frac{2(a+b)(a\mathrm{P}+b\mathrm{Q})}{\mathrm{P}+\mathrm{Q}}(\mathrm{M}-\mathrm{P})(\mathrm{M}-\mathrm{Q}).\end{aligned}$$

Thus if, as in all the applications here considered, $a\mathrm{P}+b\mathrm{Q}$ has the same sign as $(a+b)(\mathrm{P}+\mathrm{Q})$, that is as $(a+b)\kappa'$, the last term is negative and

$$0 \leqq \{(a+b)\mathrm{M}-a\mathrm{P}-b\mathrm{Q}\}^2 \leqq \frac{\mathrm{P}-\mathrm{Q}}{\mathrm{P}+\mathrm{Q}}\{(b^2-a^2)\mathrm{M}^2+a^2\mathrm{P}^2-b^2\mathrm{Q}^2\}$$

or

$$\begin{aligned}&\{(a+b)\mathrm{M}-a\mathrm{P}-b\mathrm{Q}\}^2 \\ &\qquad = -2(1+\theta)\kappa\kappa'\frac{\mathrm{P}-\mathrm{Q}}{\mathrm{P}+\mathrm{Q}}\{(b^2-a^2)m+a^2p-b^2q\},\end{aligned}$$

where

$$-1 \leqq \theta \leqq 1.$$

This leads on substitution in (5) to

$$d = -\mathrm{B}-\frac{\mathrm{C}}{m}+\theta\left(\beta-\frac{\gamma}{m}\right), \quad . \quad . \quad . \quad . \quad (8)$$

where

$$\left.\begin{aligned}
\beta &= -\frac{\kappa'(a-b)(\mathrm{P}-\mathrm{Q})}{2\kappa(\mathrm{P}+\mathrm{Q})(a\mathrm{P}+b\mathrm{Q})},\\
\gamma &= -\frac{\kappa'(a^2p-b^2q)(\mathrm{P}-\mathrm{Q})}{2\kappa(a+b)(\mathrm{P}+\mathrm{Q})(a\mathrm{P}+b\mathrm{Q})},\\
\mathrm{B} &= \frac{\kappa'(a+b)}{\kappa(a\mathrm{P}+b\mathrm{Q})}-\beta=\frac{2\kappa'}{\kappa(\mathrm{P}+\mathrm{Q})}+\beta,\\
\mathrm{C} &= \frac{4\kappa^2(a+b)^2-\{(a+b)\mathrm{A}-a\mathrm{P}\quad b\mathrm{Q}\}^2}{4\kappa^2(a+b)(a\mathrm{P}+b\mathrm{Q})}+\gamma\\
&= \frac{4\kappa^2-(\mathrm{A}-\mathrm{P})(\mathrm{A}-\mathrm{Q})}{2\kappa^2(\mathrm{P}+\mathrm{Q})}-\gamma.
\end{aligned}\right\} \quad . \quad (9)$$

It is at once evident that $\mathrm{B}+\beta$ and $\mathrm{B}-\beta$, and hence B, are positive. That C is positive follows at once from its definition in (8). Distances will be obtained from the instrument on the assumption that θ in (8) may in general be given its mean value zero. Thus the quantity known to the surveyor as the constant of the instrument is the distance from the vertical axis of the instrument to F minus **B**. C divided by the multiplier for the staff readings, usually 100, is the correct separation for the stadia lines. β and γ are lengths which specify minimum errors, and it will be apparent in a moment that both have the same sign as κ'.

At the two end points for which $\mathrm{M}=\mathrm{P}$ and $\mathrm{M}=\mathrm{Q}$, θ has the value $+1$, and takes the other extreme value -1 for the intermediate point where $\mathrm{M}=(a\mathrm{P}+b\mathrm{Q})/(a+b)$. Obviously accuracy is lost if either a or b be made zero, since the error does not then rise to a stationary point and subsequently fall. The character of the error, which is determined by the ratio of β to γ, will usually be specified, and two special cases call for particular consideration. In the one case the magnitude of the possible errors is made independent of the distance measured by assigning the value zero to γ; this is the case usually considered, and is secured by giving a and b the values $\sqrt{-q}$ and $\sqrt{-p}$ respectively. A better choice both practically and theoretically is to relate the magnitude of the possible error directly to the distance under measurement. In this case β must vanish and a and b are equal.

Before proceeding further it will be verified that in these two cases β and γ agree in sign with κ'. For when $\gamma=0$, if p is the magnification for the far distance and a and b have the positive values of the square root, $a>b$. On the other hand, since the power is less for the far point than for the near, $P<Q$. The denominator is essentially positive, and thus the sign agrees with that of κ'. When $\beta=0$, γ takes the form $(P-Q)^2/16\kappa^2(P+Q)$ and all factors are positive with the exception of $P+Q$ which has the same sign as κ'.

As would be expected, C is approximately equal to the focal length of the system when focussed for infinity, but the stadia lines will be incorrectly placed if their positions are determined by the aid of collimators making with one another an angle of one-hundredth of a radian. For if R is the value of M for $m=0$, it is easy to throw the second formula for C into the form

$$C=\frac{2}{A+R}-\frac{(P-R)(Q-R)}{2\kappa^2(P+Q)}-\gamma.$$

The two factors $P-R$ and $Q-R$ are positive, and thus both the terms dependent on the range have the same sign. If the focussing lens is positive, the stadia lines must be closer together than the separation determined by the collimator method; if negative, the separation must be greater. The most satisfactory method of spacing the lines is to determine their correct distance by calculation. For this purpose, somewhat greater accuracy than a slide-rule gives is required.

The significance of the expressions found for the constants and errors is more readily appreciated by expressing them in terms of other variables. It has hitherto been necessary to regard M as a function of m defined by equation (6) to enable d to be expressed in the form (8). This is no longer necessary, and equation (7) shows that it may be regarded as a power, and thus be expressed in terms of the powers of the two lenses and of the position of the focussing lens in the telescope. The equations required are

$$\left.\begin{aligned} A &= 2\kappa+\kappa'-l\kappa\kappa', \\ M &= \kappa'(1+\xi\kappa). \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad (10)$$

where l is the constant sum of the separations of the focussing lens from the stadia lines and from the objective, and ξ is the excess of the former distance over the latter. The separations are of course measured from the principal planes, so that the length of the telescope will be rather greater than l plus the space required for the eyepiece. The second formula confirms the result, of which extensive use has already been made, that M and κ' have the same sign. The presence of the objective and graticule at the two ends of the telescope requires ξ to fall within limits somewhat closer than

$$-l < \xi < l, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (11)$$

but it must not be forgotten that if the lens is moved by means of a rack and pinion or similar device the range of movement is restricted to one-half of that implied by these limitations. Substitution from (10) in (9) yields results which are highly suggestive of the principles which should be followed to obtain high accuracy in the estimated distances; but these will not be detailed here, as a method is described later by which a telescope can be designed to give any required degree of accuracy.

The simplest method of applying (9) to any given system is to express M in terms of d, κ, κ', and l or A. The formula required is found at once by squaring (6) and eliminating m by (5), with the result

$$\mathrm{M} = -\kappa'/\kappa d + \surd\{(\mathrm{A}+\kappa'/\kappa d)^2 - 4\kappa^2\}, \quad . \quad . \qquad (12)$$

where the square root agrees in sign with κ'. It should be noted that the values of a and b need not be found. For if L is the value of M for which $\theta = -1$,

$$\left.\begin{aligned} \mathrm{B}+\beta &= \kappa'/\kappa\,\mathrm{L} \\ \text{and} \quad 4\kappa^2(\mathrm{C}-\gamma) &= 2\mathrm{A}-\mathrm{L}-(\mathrm{A}^2-4\kappa^2)/\mathrm{L}. \end{aligned}\right\} \quad . \quad . \qquad (13)$$

Instead of fixing L independently, it will be determined in the two cases here considered by assuming in turn $\beta = 0$ and $\gamma = 0$. In the former case B is known from P and Q, which are found by giving d in (12) the extreme values f and n for the far and near points. Thus L and hence $\mathrm{C}-\gamma$ are known. $\mathrm{C}+\gamma$ is found from the values of P and Q. When $\gamma = 0$, L is given by the solution of (13):

$$\mathrm{L} = \mathrm{A} - 2\kappa^2(\mathrm{C}-\gamma) + 2\kappa\surd\{1 - \mathrm{A}(\mathrm{C}-\gamma) + \kappa^2(\mathrm{C}-\gamma)^2\}, \quad (14)$$

where the square root has the positive sign, and the value of C determined from f and n is used.

Perhaps the clearest insight into the meaning of C is gained by noting that θ becomes unity at each end point, and therefore the power for both these distances satisfies

$$d = -\mathrm{B} + \beta + (\mathrm{C}+\gamma)(d\mathrm{K} + \kappa'/\kappa) \quad . \quad . \quad . \qquad (15)$$

by (4) and (8), and hence, if the powers corresponding to the distances f and n are for the moment denoted by F and N,

$$\mathrm{C}+\gamma=\frac{f-n}{f\mathrm{F}-n\mathrm{N}}$$

and

$$\mathrm{B}-\beta=(\mathrm{C}+\gamma)\frac{\kappa'}{\kappa}-\frac{fn(\mathrm{F}-\mathrm{N})}{f\mathrm{F}-n\mathrm{N}}.$$

These two expressions may if preferred be used in place of the equations in (9) not involving a and b.

The converse problem of designing a telescope to have given constants may now be considered. It will be supposed that f, n, B, C, β, γ are given and that κ, κ', and l are to be found. From (15)

$$\tfrac{1}{2}(\mathrm{A}+\mathrm{P})(\mathrm{C}+\gamma)=1+\{\mathrm{B}-\beta-(\mathrm{C}+\gamma)\kappa'/\kappa\}/f$$

and

$$\tfrac{1}{2}(\mathrm{A}+\mathrm{Q})(\mathrm{C}+\gamma)=1+\{\mathrm{B}-\beta-(\mathrm{C}+\gamma)\kappa'/\kappa\}/n.$$

Also from (9)

$$\tfrac{1}{2}(\mathrm{P}+\mathrm{Q})=\kappa'/\kappa(\mathrm{B}-\beta),$$

and therefore

$$\left.\begin{aligned}
\mathrm{B}-\beta-(\mathrm{C}+\gamma)\kappa'/\kappa &= yg,\\
\tfrac{1}{2}(\mathrm{A}-\mathrm{P})(\mathrm{C}+\gamma) &= y(1-g/f),\\
\tfrac{1}{2}(\mathrm{A}-\mathrm{Q})(\mathrm{C}+\gamma) &= y(1-g/n),\\
\text{where}\qquad y &= \mathrm{A}(\mathrm{C}+\gamma)-1\\
\text{and}\qquad \frac{1}{g} &= \frac{1}{\mathrm{B}-\beta}+\frac{1}{f}+\frac{1}{n}.
\end{aligned}\right\}\quad . \quad . \quad (16)$$

The second equation for C in (9) may be rewritten in the form

$$(\mathrm{A}-\mathrm{P})(\mathrm{A}-\mathrm{Q})=4\kappa^2yg/(\mathrm{B}-\beta),$$

giving by (16)

$$\kappa^2(\mathrm{C}+\gamma)^2=y\{1+(\mathrm{B}-\beta)g/fn\}.\quad .\quad .\quad (17)$$

It is now possible to determine y by substituting from (16) and (17) in the relation implied by (13), with the result

$$\begin{aligned}4y\{yg(\mathrm{C}-\gamma)+2(\mathrm{C}\beta+\mathrm{B}\gamma)\}(\mathrm{B}+\beta)\{1+(\mathrm{B}-\beta)g/fn\}\\
=\{y(\mathrm{B}+\beta+g)+2\beta\}^2(\mathrm{C}+\gamma).\quad (18)\end{aligned}$$

When $\gamma=0$ the solution is

$$\begin{aligned}\frac{\beta}{yg(\mathrm{B}+\beta)}=\frac{1}{2}\left(\frac{1}{g}-\frac{1}{\mathrm{B}+\beta}\right)+\frac{\mathrm{B}-\beta}{fn}\\
-\frac{1}{fn}\sqrt{(\mathrm{B}-\beta)(\mathrm{B}-\beta+fn/g)}\ ;\end{aligned}$$

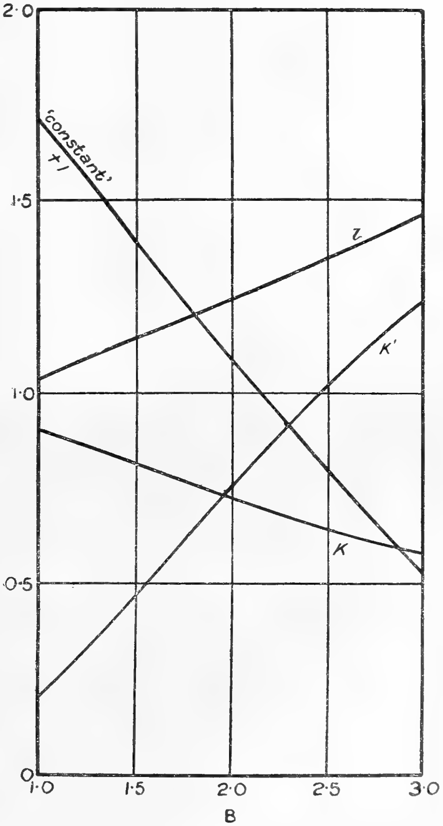
and for $\beta=0$,

$$yg\left\{(C+\gamma)\left(\frac{1}{n}-\frac{1}{f}\right)^2+8\gamma\left(\frac{1}{B}+\frac{1}{n}\right)\left(\frac{1}{B}+\frac{1}{f}\right)\right\}$$
$$=8\gamma B\left(\frac{1}{B}+\frac{1}{n}\right)\left(\frac{1}{B}+\frac{1}{f}\right).$$

It will be of interest to take the latter case in some detail. When a negative focussing lens is employed, since by (17) y is necessarily positive, it is essential that

$$(C+\gamma)\left(\frac{1}{n}-\frac{1}{f}\right)^2+8\gamma\left(\frac{1}{B}+\frac{1}{n}\right)\left(\frac{1}{B}+\frac{1}{f}\right)<0.$$

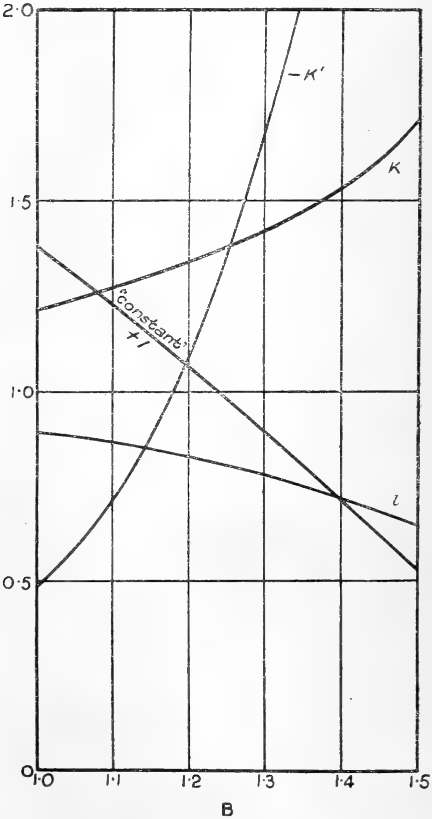
Fig. 1.



Internally-focussing Telescope with positive focussing lens recording distances between 20 and 1000 accurately to 1 part in 1000.

This may be regarded as an equation giving a lower limit to n or an upper limit to B, neither of which may be closely approached. No such restriction occurs when the focussing lens is positive. Fig. 1 illustrates the results obtained when the focussing lens is positive for various values of B with $C=1$, $\gamma=\cdot 001$, $\beta=0$, $f=1000$, $n=20$; and fig. 2 the corresponding results for a negative focussing lens with $\gamma=-\cdot 001$. All the calculations required are easily carried out on a slide-rule. The systems illustrated in the diagrams conflict with no mechanical restraints. The mean position of the focussing lens approaches the stadia lines as B increases in both series of telescopes.

Fig. 2.



Internally-focussing Telescope with negative focussing lens recording distances between 20 and 1000 accurately to 1 part in 1000.

The condition that the constant of the instrument should be zero may be conventionally represented with fair approximation by

$$B = \frac{l}{2} + \frac{1}{\kappa} + \frac{C}{10},$$

and this is satisfied if

$$\kappa = \cdot 7, \qquad \kappa' = \cdot 85, \qquad l = 1\cdot 28,$$

or

$$\kappa = 1\cdot 47, \quad \kappa' = -1\cdot 3, \quad l = \cdot 81.$$

The leading differences between the two series are that when the focussing lens is positive the powers are comparatively small, and therefore larger apertures may be employed ; and, on the other hand, when a negative focussing lens is used the length of the telescope is reduced, the system resembling a tele-photo lens. It is doubtless this feature which has led to the general adoption of the negative lens.

In this investigation no account has been taken of the errors arising through imperfect focussing of the system or inexact reading. The latter introduces an error of the same general type as γ, and a possible error of this type is therefore always present. If care is taken in use, the errors due to inaccurate focussing should not be large, and when the focussing lens is positive it is possible, provided the system is properly corrected for aberration, to adopt a construction which ensures that the possible errors from this cause are quite negligible.

The National Physical Laboratory.
January, 1920.

LXXXV. *The Heat Loss by Convection from Wires in a Stream of Air, and its Relation to the Mechanical Resistance.* *By* A. H. Davis, *B.Sc.*[*]

1. *Introduction.*

It has been shown[†] that some published results for the cooling of cylinders in a stream of air agree excellently with Boussinesq's formula, which may be derived from considerations of similitude[‡], and which may be written in

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[*] Communicated by the Author.

[†] Davis, Phil. Mag. xl. p. 692 (1920).

[‡] Rayleigh, Nature, xcv. p. 66 (1915).

the form

$$H/k = F(lvc/k), \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

where

H = heat lost per second per unit length of the wire, per degree temperature excess above stream temperature.
k = heat conductivity of the fluid.
c = specific heat of the fluid per unit volume.
v = velocity of the fluid stream.
l = diameter of the wire.

It was pointed out that since for a given gas $c\nu/k$ is constant (ν being the kinematical viscosity), the above reduces to

$$H/k = F(vl/\nu). \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

Here vl/ν is the familiar variable in hydrodynamics determining fluid resistance and turbulence, so the equation in this form involves a relation between the thermal and dynamic effects of a fluid stream. The dynamic effect is represented by the formula

$$R/(\rho v^2 l^2) = f(vl/\nu), \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

where R = fluid resistance per diameter length of the wire,
ρ = density of the fluid.

The present paper extends the investigation of the agreement of the similitude equation with published data for heat loss from cylinders in a stream of air, and determines the relation between the thermal and dynamic effects of the stream in this case.

Osborne Reynolds *, considering the heating surface of boilers, pointed out that, from the identity of the two molecular phenomena by means of which convection of heat and of momentum (surface friction) were carried on, their dependence on the conditions of motion would be the same. Formulæ he deduced for cool liquid flowing through a hot pipe indicate that

$\left(\begin{array}{c}\text{Heat loss per}\\ \text{unit area}\end{array}\right) \propto$ Temperature difference between fluid and pipe.
$\propto$ Fluid resistance per unit area.
$\propto$ Reciprocal of the velocity of the fluid in the pipe.

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* Reynolds, Proc. Manchester Lit. & Phil. Soc. 1874. See also Stanton, Report Adv. Committee Aeronautics, p. 45 (1912–13).

Stanton * showed experimentally that for the same pipe and range of temperature the heat transmission varied as v^{n-1}, where n is the index of v in the ordinary resistance formula.

2. *The Convection Constants of Air.*

Below are given the formulæ taken in this paper to represent the variation with temperature of the convection constants of air. Langmuir †, dealing with convection of heat from the point of view of conduction through a stationary film of gas in contact with the hot body, has collected formulæ for the convection constants for various gases. From his paper we have, T being the absolute temperature,

$$\text{viscosity } (\eta)=\frac{(150\times 10^{-7})\mathrm{T}^{\frac{1}{2}}}{1+124/\mathrm{T}}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (4)$$

$$k=4{\cdot}6\times 10^{-6}\mathrm{T}^{\frac{1}{2}}\,\frac{1+0{\cdot}0002\mathrm{T}}{1+124/\mathrm{T}} \text{ cals. per cm. per sec. per }^\circ\text{C.} \quad . \quad . \quad (5)$$

and since for air

$$\rho=0{\cdot}001293\,(273/\mathrm{T}) \text{ gm. per c.c.}$$

it follows that

$$\nu(=\eta/\rho)=\frac{(425\times 10^{-7})\mathrm{T}^{3/2}}{1+124/\mathrm{T}}\text{cm.}^2 \text{ per sec.} \quad . \quad . \quad . \quad . \quad (6)$$

3. *The Thermal Relation.*

For convection loss from thin wires in air, the most complete and satisfactory experimental results are those given by King ‡, and he obtained to represent them a formula which may be written

$$\mathrm{H}/k=\mathrm{B}\sqrt{(vl/\nu)}+\mathrm{C}, \quad . \quad . \quad . \quad . \quad . \quad . \quad (7)$$

where ν has been put for k/c. This agrees with the theory advanced by King and with (2), except that B and C vary slightly with the temperature excess at which the wire works, and that C increases with the diameter of the wire. He gave empirical coefficients which allowed for this, and

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* Stanton, Phil. Trans. cxc. p. 67 (1897).

† Langmuir, Phys. Rev. xxxiv. p. 401 (1912).

‡ King, Phil. Trans. A, ccxiv. p. 373 (1914).

supposed the slight variation in B and C to be due to the change in the values of the physical constants of air.

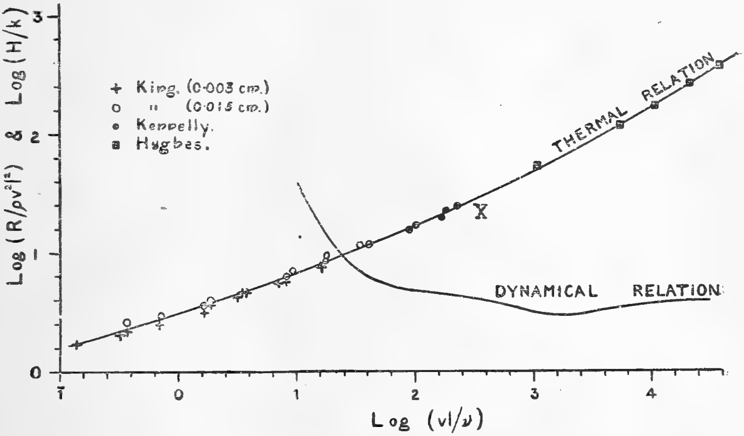
To get a relation between H/k and vl/ν it is necessary to assign values to these physical constants. While throughout this paper the temperature of the air stream is taken as 15° C., the values of k and ν appropriate to any experiment with a hot wire in the stream are neither those for air at the temperature of the wire nor those for air at the temperature of the cold stream. Consequently, as a first approximation, the values taken are those for air at the mean of these temperatures. The relation between H/k and vl/ν is then found to be practically independent of the temperature excess. Fig. 1 shows this and will be referred to later. In section 2 are given the formulæ representing the variation of k and ν with temperature.

With respect to the dependence of C on the diameter l of the wire, it is not possible by rearranging the similitude equation to obtain a term in which l occurs without v, nor is this possible by introducing the temperature coefficients of k and ν. It is possible, however, by taking into account the free convection from the wire, as well as the forced convection. It seems improbable that this is the complete solution, for with the higher wind speeds the effect is still appreciable when variation with diameter of the free convection loss must be negligible. The true explanation may lie in the increasing lack of rigidity in the finer wires, which to eliminate end correction were generally of the same length (23 cm.) as the stouter ones. The former, vibrating in the wind and presumably following the changes in air-pressure more closely, might yield less heat. King himself mentioned the existence of vibration, and that higher air speeds were impossible owing to risk of breakage of the wire at high temperatures.

To cover the whole range of King's experiments, the heat loss has been determined, using his formula, for two wires (0·003 cm. and 0·015 cm. diameter) working at three different temperature excesses (85°, 500°, and 1000° C.) in winds of four speeds (20, 100, 500, and 1000 cm. per sec.). As stated earlier, the values taken for k and ν have been calculated from the formulæ already given, the appropriate temperature for any case being the arithmetical mean of the temperatures of the hot wire and of the ambient cold stream (15° C.). The results have been plotted in fig. 1 and give the lower end of the "Thermal Relation" line.

The upper part of the curve has been obtained from data given by Hughes * for larger cylinders (0·43—5·06 cm. diameter), steam-heated, in air (assumed 15° C.) moving with speeds of 200 to 1400 cm. per sec. These data have been plotted on a graph † having heat loss as ordinate and *vl* as abscissa ; values read off from this curve give points in the upper part of fig. 1.

Fig. 1.



Some values intermediate between the two sets above mentioned were obtained from figures compiled by Langmuir ‡ from a study of results given by Kennelly §. They relate to the high velocity (800–1800 cm. per sec.) values for Kennelly's largest wire (0·0204 cm.) working at temperature excesses of 51° and 128° C.

The graph thus obtained—'Thermal Relation' of fig. 1—is excellent, practically independent of the temperature or diameter of the cylinders involved. The data range over air-streams of from ½ to 50 miles per hour, wires of 0·003 cm. diameter up to 5 cm. piping, and temperatures from that of boiling water to a bright red heat.

The curve may be further examined by comparison with the results of experiments by Kennelly and Sanborn ||, who investigated the heat loss from a wire in air at various pressures, the wire being mounted on a whirling arm in a

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* Hughes, Phil. Mag. xxxi. p. 118 (1916).

† Davis, *loc. cit.*

‡ Langmuir, Trans. A. I. E. E. xxxi. (1) p. 1229 (1912).

§ Kennelly, Trans. A. I. E. E. xxviii. (1) p. 363 (1909).

|| Am. Phil. Soc. Proc. liii. p. 55 (1914).

special chamber. Table I. is calculated from the data given in table iii. of their paper. As before, allowance has been made for variation of k and ν with temperature. It has been assumed that, as indicated by the kinetic theory of gases, both the conductivity k of a gas and its viscosity η are independent of pressure. The kinematical viscosity ν has consequently been taken as inversely proportional to the pressure of the gas. The data of Table I., if plotted on the graph of fig. 1, agree satisfactorily with the curve there given for air at atmospheric pressure. A slight upward displacement above the line occurs for all the points, but this is undoubtedly due to experimental error, since two of the points relate to ordinary atmospheric pressure, and should agree with King's results for the same condition.

TABLE I.

Convection at various air pressures.

Wire diam. 0·0114 cm. $v=970$ cm. per sec.
Values of ν given for a pressure of 1 Atmosphere ($=1·012\times10^6$ bars).

Room temperature about 20° C.

Pressure (bars).	390° C. excess. $\log\nu=\bar{1}·56$ $\log k=\bar{5}·95$		538° C. excess. $\log\nu=\bar{1}·67$. $\log k=\bar{4}·00$.	
$10^6\times$	$\log vl/\nu$.	$\log H/k$.	$\log vl/\nu$.	$\log H/k$.
4·0	—	—	1·97	1·33
3·95	2·07	1·36	—	—
2·02	—	—	1·68	1·18
2·00	1·78	1·20	—	—
1·02	1·48	1·08	1·38	1·05
0·44	1·12	0·92	1·01	0·90

It is interesting to compare the above results for air with data for the heat loss in a stream of other fluid. Worthington and Malone * have shown that the convection of heat from a 0·0256 cm. platinum wire in water follows the same laws as convection in air. Water carries off over 100 times as much

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* J. Frank. Inst. clxxxiv. p. 115 (1917).

heat as air, and they give the following example for the wire:—

Velocity of water relative to wire ...	14·85 cm. per sec.
Temperature excess of the wire	30° C.
Heat loss from the wire	0·88 cal. per cm. per sec.

According to tables by Kaye and Laby

$$c=1, \qquad k=0\cdot00136,$$

hence
$$H/k=21\cdot6, \qquad vlc/k=280.$$

Now for air $k/cv=1\cdot3$ approximately, so in fig. 1 the relation between H/k and vl/ν for air is a relation between H/k and $1\cdot3vlc/k$. Plotting the above value for water on the curve regarded in this manner, gives the point X in fair agreement.

4. *The Dynamical Data.*

The mechanical pressure on smooth wires placed in a stream of air has been given in the 'Technical Report of the Advisory Committee for Aeronautics for 1913–14.' The data relate to 60° F. and a barometric pressure of 760 mm. The wires range from 0·002 in. to $1\frac{1}{4}$ in. diameter, and were placed in winds of 10–50 ft. per second (20–50 ft. per sec. for wires above $\frac{1}{4}$ in. diam.). The results were exhibited in a non-dimensional curve, and this has been re-drawn in the "Dynamical Relation" line of fig. 1, where $R/\rho v^2l^2$ is plotted against vl/ν.

5. *Relation between Thermal and Dynamical Effects.*

Fig. 1 gives for definite values of vl/ν the appropriate values of H/k and of $R/\rho v^2l^2$, and using these fig. 2 has been drawn showing the relation between H/k and $R/\rho v^2l^2$ for corresponding values of vl/ν.

In the above form the curves are non-dimensional and general, but it is instructive to draw a curve which shows for corresponding values of vl the relation of the heat loss H to the mechanical resistance R. Strictly, this cannot be done unless ν is the same in the two cases, and presumably this condition would be satisfied if R were measured not for a cold wire, but for one at the same temperature as that at which H was measured. Failing such measurements, however, the data of Table II. approximately satisfy the condition, for here, for an air stream of 15° C., the mechanical pressure

R of cold wires is compared with the heat loss H from wires at a temperature (100° C) not far removed. The table has been drawn up from data already considered, and the results

Fig. 2.

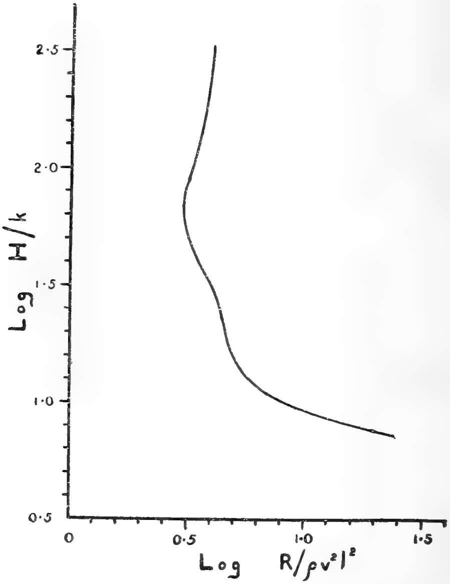
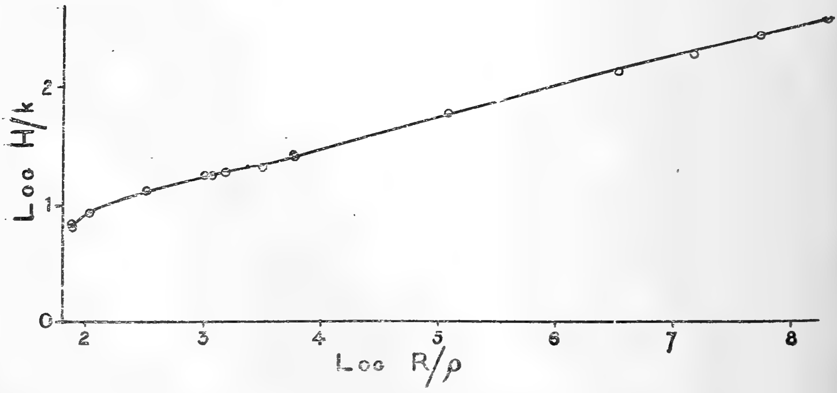


Fig. 3.



are plotted in fig. 3. To facilitate comparison with previous work H/*k* is given instead of H, taking for *k* the value $5{\cdot}77 \times 10^{-5}$ appropriate to the temperature of the ambient air. Similarly, R/ρ is given. This does not alter the shape of the curve. It may be mentioned that a curve between H/k and vl/ν plotted from this table is practically identical with that of fig. 1, where allowance was made for the change of the physical constants with temperature.

Table II.

Relation between R/ρ *and* H/k.

$\nu = 0{\cdot}145$ *. $\log \nu = \bar{1}{\cdot}16$.

$k = 5{\cdot}77 \times 10^{-5}$. $\log k = \bar{5}{\cdot}76$.

$\log \dfrac{vl}{\nu}$.	$\log \dfrac{H}{k}$.	$\log \dfrac{R}{\rho}$.	
1·015	0·81	1·88	King (0·003 cm.).
1·316	0·93	2·03	
1·015	0·84	1·88	(0·015 cm.).
1·714	1·12	2·51	
2·015	1·26	3·01	
3·14	1·78	5·09	Hughes.
3·84	2·13	6·54	
4·14	2·30	7·17	
4·44	2·48	7·79	
4·68	2·64	8·29	
2·05	1·25	3·08	Kennelly.
2·26	1·31	3·50	
2·40	1·41	3·76	
2·10	1·26	3·18	
2·40	1·42	3·76	

As stated earlier, the theory of Osborne Reynolds gives for fluid flow through a given pipe

$$H \propto R/v.$$

This simple relation does not hold for wires in the region considered, for taking the straight line part of the curve of fig. 3 it is seen that

$$H \propto R^{0{\cdot}275} \text{ approx.}, \quad . \quad . \quad . \quad . \quad . \quad . \quad (8)\dagger$$

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* The value taken for ν in the 'Report of the Advisory Committee for Aeronautics' differs slightly but inappreciably from this, and so no allowance has been made for the difference.

† With respect to the reality or otherwise of the unexplained dip in the "Dynamical Relation" of fig. 1, little effect is made in fig. 3 by smoothing out the dip entirely. Smoothing appears, however, to make more uniform the tendency of the relation to take the Reynolds form as vl/ν increases.

and for this region it may be shown from fig. 1 that roughly

$$H \propto v^{0\cdot55} \quad \text{and} \quad R \propto v^{2}.$$

Consequently,

$$H \propto R/v^{1\cdot45}. \qquad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (9)$$

There is, however, most probably a tendency for the relation to take the simpler form given by Reynolds as the size of the cylinders is still further increased. This follows since R tends to be proportional to the square of the velocity, and since Hughes found that the heat loss H tends to be proportional to v.

January 1921.

LXXXVI. *The Annual and Diurnal Periodicity of Earthquakes in Japan.* *By* CHARLES DAVISON, *Sc.D.*[*]

THE annual and diurnal periodicity of earthquakes has been considered by several writers—by Prof. C. G. Knott in 1886, by Prof. F. Omori in 1894 and 1902, and by myself in 1893 and 1896 [†].

Since the year 1896, more complete records have become available. The present paper deals with the earthquakes of a limited area—the empire of Japan [‡]. The method of investigation adopted is the rough form of harmonic analysis

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[*] Communicated by the Author.

[†] C. G. Knott, "Earthquake frequency," Japan Seis. Soc. Trans. vol. ix. pp. 1–20 (1886); C. Davison, "The annual and semi-annual seismic periods," Phil. Trans. 1893, A, pp. 1107–1169; C. Davison, "On the diurnal periodicity of earthquakes," Phil. Mag. vol. xlii. pp. 463–476 (1896); F. Omori, "Annual and diurnal variations of seismic frequency in Japan," Publ. Earthq. Inv. Com. No. 8, pp. 1–94 (1902).

[‡] The records analysed below are contained in the following catalogues:—

Milne, J. (*1*). Japan Seis. Soc. Trans. vol. x. p. 61 (1887); vol. xiii. pt. i. p. 93 (1889); vol. xv. p. 101 (1890); vol. xvi. pp. 56 and 83 (1892).

Milne, J. (*2*). Japan Seis. Soc. Trans. vol. ii. pp. 4–14 and opposite p. 38 (1880); vol. vi. pp. 32–35 (1883); vol. viii. pp. 100–108 (1885); vol. x. pp. 97–99 (1887); vol. xv. pp. 127–134 (1890): Brit. Assoc. Report. 1886, pp. 414–418; 1887, pp, 212–213; 1888, pp. 435–437; 1889, pp. 295–296; 1890, pp. 160–162; 1891, pp. 123–124; 1892, pp. 93–95; 1893, pp. 214–215; 1895, pp. 114–115; 1897, pp. 133–137; 1898, pp. 189–191: 1899, pp. 189–191; Brit. Assoc. Seis. Com. Circulars, vol. i. pp. 29–30, 90–92, 142–144, 223–225.

Milne, J. (*3*). "A Catalogue of 8331 Earthquakes recorded in Japan between 1885 and 1892." Seis. Journ. Japan, vol. iv. pp. 1–367 (1895).

known as the method of overlapping means, which gives results of sufficient accuracy, considering the nature of the materials *.

Annual Periodicity.

In Table I., the results, with certain stated exceptions, refer to the whole country; the maximum epoch occurs about the end of the month given; for the semi-annual period, the earlier of the two epochs is given, the second of course being six months later; in estimating the amplitude, the average monthly number of earthquakes in each case is taken as unity. The amplitude, as Prof. Schuster has shown, should exceed the value of $\sqrt{(\pi/n)}$, where n is the number of earthquakes †, unless the epochs agree in corresponding records.

In Table II., similar results are given for the earthquakes recorded at special stations ‡.

Tables I. and II. lead to the following results:—

(i.) The maximum epoch of the annual period for the whole country occurs from December to March for strong earthquakes, and in September–October for slight earthquakes. In the destructive and slight earthquakes recorded

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Milne, J. (*4*). "A Catalogue of destructive Earthquakes, A.D. 7–1899." Brit. Assoc. Report, 1911, pp. 649–740.

Omori, F. (*1*). "On the After-shocks of Earthquakes." Journ. Coll. Sci., Imp. Univ., Japan, vol. vii. pp. 111–200 (1894).

Omori, F. (*2*). "Notes on the Earthquake Investigation Committee Catalogue of Japanese Earthquakes." Journ. Coll. Sci., Imp. Univ. Japan, vol. xi. pp. 389–437 (1899).

Omori, F. (*3*). "Annual and Diurnal Variations of Seismic Frequency in Japan." Publ. Earthq. Inv. Com. No. 8, pp. 1–94, (1902).

Omori, F. (*4*). Note on the Annual Variation of Seismic Frequency in Tokyo and Kyoto." Bull. Earthq. Inv. Com. vol. ii. pp. 17–20 (1908).

Omori, F. (*5*). "Note on the Annual Variation of the Height of Level of Lake Biwa." *Ibid.*, pp. 51–57.

Omori, F. (*6*). "List of the stronger Japan Earthquakes, 1902–1907." *Ibid.*, pp. 58–88.

* Phil. Trans. 1893 A, pp. 1108–1112. See also Boll. Soc. Sism. Ital. vol. iv. pp. 89–100 (1898). In the present paper, I have not taken account of the different lengths of the months, such refinement being unnecessary.

† A. Schuster, Roy. Soc. Proc. vol. lxi. pp. 455–465 (1897).

‡ The records are instrumental at Tokyo, Hikone, Sapporo, Nagano and Oita; non-instrumental at Numazu, Hamamatsu, and Kyoto; partly instrumental at the other stations, the seismographic record beginning with the year 1888 at Wakayama, 1891 at Ishinomaki, 1894 at Niigata, 1895 at Hakodate, and during the omitted year in the remaining records.

TABLE I.—Annual Seismic Period.

	Duration of Record.	Authority.	No. of Earthquakes (n).	$\surd(\pi/n)$.	Annual		Semi-annual	
					Epoch.	Ampl.	Epoch.	Ampl.
Destructive earthquakes, whole world	1801–1899	Milne (*4*)	2087	·04	Dec.	·10	May	·03
,, ,, , Japan	1001–1899	,,	385	·09	Oct.–Dec.	·08	Mar.	·16
Ordinary earthquakes, ,,	1885–1894	Omori (*2*)	6110	·02	Feb.	·11	Apr.–May	·08
Disturbed area greater than 1000 sq. ri *	1902–1904 1906–1907	Omori (*6*)	489	·08	Feb.–Apr.	·19	Apr.	·20
,, ,, ,, ,,	1885–1890	Milne (*3*)	176	·13	Mar.	·16	Apr.	·12
,, ,, between 100 and 1000 sq. ri	,,	,,	565	·07	Feb.–Mar.	·17	Jun.	·20
,, ,, less than 100 sq. ri	,,	,,	2256	·04	Sept.–Oct.	·16	Mar.	·10
Submarine earthquakes, origin about 5 miles from land	,,	,,	520	·08	Dec.	·37	May	·25
,, ,, ,, 10 ,, ,,	,,	,,	1767	·04	Jan.–Mar.	·17	May	·10
,, ,, ,, 20 or more ,,	,,	,,	194	·13	Jun.	·22	Jun.	·36
,, ,, origin at all distances	,,	,,	2496	·04	Dec.–Jan.	·14	May	·15
Eastern seismic districts, under land	,,	,,	1046	·05	Nov.–Feb.	·17	Apr.–May	·11
,, ,, ,, ,, sea	,,	,,	1321	·05	Feb.	·16	May	·14
Western ,, ,, ,, land	,,	,,	332	·10	May	·52	Feb.	·35
,, ,, ,, ,, sea	,,	,,	97	·18	Apr.	·24	doubtful.	

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* 1 sq. ri = 5·9 sq. miles.

Table II.—Annual Seismic Period.

	Duration of Record.	Authority.	No. of Earthquakes (n).	$\surd(\pi/n)$.	Annual		Semi-Annual	
					Epoch.	Ampl.	Epoch.	Ampl.
Tokyo	1875–1881, 1883–1902	Milne (*2*)	2503	·04	Mar.–Apr.	·08	May	·08
Hikone	1895–1907	Omori (*5*)	351	·09	May	·27	May	·07
Sapporo	1883–1899	Omori (*3*)	134	·15	Dec.	·24	Mar.	·31
Nagano	1890–1899	,,	307	·10	Feb.	·37	Apr.	·31
Oita	1887–1899	,,	264	·11	Dec.–Jan.	·22	Mar.	·13
Hakodate	1873–1899	,,	305	·10	Dec.	·17	May	·20
Niigata	1887-1899	,,	214	·12	Jun.	·51	May	·49
Wakayama	1880–1899	,,	403	·09	May	·17	Apr.	·13
Hiroshima	1885–1891, 1893–1899	,,	101	·18	Dec.	·17	Apr.	·07
Nemuro	1885, 1887–1893	,,	352	·09	Sept.	·16	May-Jun.	·21
Yamagata	1890–1893, 1895–1899	,,	186	·13	Jun.	·76	Jan.	·38
Ishinomaki	1886–1899	,,	1034	·06	Aug.	·38	Apr.	·05
Fukushima	1890-1894, 1896-1899	,,	767	·06	Apr.	·29	Mar.	·10
Utsunomiya	1891-1892, 1894–1899	,,	466	·08	Jul.	·08	Jan.	·15
Kyoto	797–1868	,,	1316	·05	May	·10	May	·05
,, destructive earthquakes	800-1898	Omori (*2*)	228	·12	Sept.	·21	Jan.	·12
,, slight ,,	,,	,,	1088	·05	May.	·11	Apr.–May	·05
Numazu	1885–1899	Omori (*3*)	160	·14	Dec.	·24	Mar.	·24
Hamamatsu	,,	,,	99	·18	Dec.	·75	May	·61

at Kyoto, the reversal of the epoch of the annual period is less clearly marked; the epoch occurs in September with the strong earthquakes and in May with the weak ones.

(ii.) The earlier epoch of the semi-annual period usually occurs in spring (March to May). In the destructive and slight earthquakes recorded at Kyoto, the epochs of the semi-annual period are reversed.

(iii.) As Prof. Omori has pointed out, the maximum seismic frequency occurs during the winter months in some parts of Japan, and during the summer months in others. The latter places are for the most part confined to the eastern portion of the northern half of the country. The smallness of the amplitude of the annual period for the whole country is in part due to this opposition in epoch.

(iv.) Submarine earthquakes with origins not more than five or ten miles from the coast are subject to an annual periodicity with the epoch nearly the same as in land-earthquakes, namely, in winter (Dec.–Mar.); those with origins at a distance of twenty or more miles have the epoch in summer (June).

Diurnal Periodicity of Ordinary Earthquakes.

In considering the diurnal periodicity of earthquakes, seismographic records only are considered, all others being useless for the purpose, owing to the varying conditions of observation which prevail throughout the day. The records for the whole of Japan and its eastern and western districts may to a certain extent be incomplete. They include all earthquakes in Prof. Milne's great catalogue in which the time of occurrence is given in hours, minutes, and seconds. There may, however, be other earthquakes recorded instrumentally in which the time is given in hours and minutes only.

For the diurnal periodicity of ordinary earthquakes, we have the following results:—

(i.) Though the amplitude of the diurnal period is in each case not much above the value of $\sqrt{(\pi/n)}$, the close agreement in epoch throughout is sufficient to establish the reality of the period, with its maximum epoch shortly before or about noon.

(ii.) For the semi-diurnal period, the average amplitude is one-half that for the diurnal period. The maximum epoch occurs at about 8 or 9 A.M. and P.M.

TABLE III.—Diurnal Seismic Period: Ordinary Earthquakes.

	Duration of Record.	Authority.	No. of Earthquakes (n).	$\sqrt{(\pi/n)}$.	Diurnal		Semi-diurnal	
					Epoch.	Ampl.	Epoch. A.M. & P.M.	Ampl.
Japan	1885–1890	Milne (*3*)	1175	·05	12 noon	·11	irreg.	·05
,, Eastern district, land..	,,	,,	268	·11	11 a.	·20	8	·06
,, ,, , sea ...	,,	,,	545	·08	12 noon	·16	irreg.	·05
Tokyo, whole year	1875–1881, 1883–1902	Milne (*2*)	2539	·04	11 a.	·08	9	·10
,, , winter	,,	,,	1290	·05	12 noon	·06	9	·13
,, , summer	,,	,,	1249	·05	11 a.	·10	9	·06
Hikone	1894–1899	Omori (*3*)	245	·11	11 a.	·14	7	·13
Oita	1887–1899	,,	237	·12	11 a.	·39	8	·12

The records at Gifu and Nagoya refer to the after-shocks of the Mino-Owari earthquake of October 28, 1891; at Nemuro to those of the Hokkaido earthquake of March 22, 1894; at Chiran to those of the Kagoshima earthquake of September 7, 1893; and at Kumamoto to those of the Kumamoto earthquake of July 28, 1889.

TABLE IV.—Diurnal Seismic Periods: After-Shocks.

	Duration of Record.	Authority.	No. of Earthquakes (n).	$\sqrt{(\pi/n)}$.	Diurnal		Semi-diurnal	
					Epoch.	Ampl.	Epoch. A.M. & P.M.	Ampl.
Gifu	Oct. 29–Nov. 10, 1891.	Omori (*1*)	1257	·05	12 midn.	·20	6	·11
,, ...	Nov. 11, 1891–Dec. 31, 1899	Omori (*1*) & (*3*)	2856	·03	0½ a.	·13	8½	·06
Nagoya	Oct. 29–Nov. 10, 1891.	Omori (*1*)	572	·07	3 a.	·35	2	·19
,, ...	Nov. 11, 1891–Dec. 31, 1899	Omori (*1*) & (*3*)	1282	·05	11 p. –3 a.	14	3	·11
Nemuro	Mar. 23–31, 1894.	Omori (*3*)	345	·10	1½ a.	·36	7	·25
,, ...	Apr. 1, 1894–Dec. 31, 1899	,,	646	·07	0½ p.	·16	5	·13
Chiran	Sept. 8–21, 1893.	Omori (*1*)	233	·12	4 a.	·22	4	·16
Kumamoto ...	July 31–Aug. 13, 1889.	,,	193	·13	12 midn.	·55	4	·36
Gifu	Nov. 11–Dec. 31, 1891.	,,	839	·06	12 midn.	·32	4	·14
,, ...	1892.	,,	865	·06	12 midn.	·19	8	·21
,, ...	1893.	,,	272	·11	2 p.	·11	11	·11
,, ...	1894–1899.	Omori (*3*)	882	·06	5 p.	·11	9	·14

The following results may be deduced from this Table:—

(i.) In the early after-shocks, the maximum epoch of the diurnal period occurs near midnight, or twelve hours later than for ordinary earthquakes. The epoch of the semi-diurnal period is at first variable.

(ii.) In the later after-shocks—after a month or more at Nemuro and after about two years at Gifu—the epoch of the diurnal period returns to the neighbourhood of noon, and that of the semi-diurnal period at Gifu to about 8 or 9.

(iii.) In the early after-shocks, the amplitudes of both diurnal and semi-diurnal periods are about double those in the later after-shocks.

Origin of Seismic Periodicity.

(i.) *Annual Period.*—The maximum epoch of the annual barometric period occurs almost invariably in December. Prof. Omori has shown that the resultant pressure on the ocean-bed is subject to an annual variation, owing to the presence of a high-pressure system in the neighbourhood of the Aleutian Islands*. The maximum epoch of the annual period occurs in October at Misaki, and in September at Ayukawa, Otaru, Iwasaki, Wajima, and Hamada. The annual periodicity of earthquakes may be due largely to that of barometric pressure, but this is probably not its only cause. The summer maximum in earthquake-frequency at the north-eastern stations may be due in part, as Prof. Omori suggests, to the variation in pressure on the sea-bed. In this part of the country, many earthquakes are of submarine origin.

(ii.) *Semi-Annual Period.*—The first maximum epoch of the semi-annual period occurs from March to May, usually in March or April. That of the mean pressure on the ocean-bed occurs in April, May, or June. As the maximum epoch of the semi-annual period, whether for land or sea earthquakes or both, usually occurs from March to May, it is probable that the semi-annual periodicity of earthquakes is due to that of the barometric pressure, the variation in total pressure on the sea-bed being due to the same cause.

(iii.) *Diurnal Period.*—The maximum epoch of the diurnal period of barometric pressure at Tokyo occurs at about $1\frac{1}{2}$ A.M., that of wind-velocity at Tokyo at about 0.45 P.M., and that of the barometric gradient between Nogano and Tokyo at about 0.45 P.M. The maximum epoch of the diurnal period of

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* Publ. Earthq. Inv. Com., No. 18, pp. 23–26 (1904). Bull. Earthq. Inv. Com., vol. ii. pp. 35–50 (1908).

ordinary earthquakes occurs about or shortly before noon, and that of after-shocks about or shortly after midnight. The diurnal periodicity of ordinary earthquakes is probably due to more than one cause; that of after-shocks mainly to the diurnal variation of barometric pressure.

(iv.) *Semi-diurnal Period.*—The first maximum epoch of the semi-diurnal period of barometric pressure (the amplitude of which does not differ greatly from that of the diurnal period) occurs about 9 A.M., and that of wind-velocity about 2 A.M. As the corresponding epoch for ordinary earthquakes occurs about 8 or 9 A.M., it is probable that the semi-diurnal periodicity of earthquakes is due to that of barometric pressure. In the case of after-shocks, the epoch varies much but, after the lapse of some time, the influence of the pressure variations tends to prevail over other causes.

LXXXVII. *The Effect of an Electric Current on the Photo-Electric Effect.* *By* Allen G. Shenstone, *M.A.*[*]

DURING the course of an experiment to determine whether there was any relation between the Hall effect in bismuth and the photo-electric effect, it was found that the total photo-electric current was so much influenced by the current passing through the bismuth that the original experiment had to be abandoned.

To determine the variation of this apparently new effect, an apparatus, shown in the attached sketch (fig. 1), was constructed, and gave consistent results.

The bismuth plate was kept at a negative potential of 14 volts, which gave practically saturation of the photo-electric current. The electrometer was used with india-ink resistance of $5(10)^8$ ohms to give a sensitivity of, roughly, $(10)^{-12}$ ampere per mm. As only relative variations were being determined, no accurate determination of sensitivity was made.

The work was all carried out at the pressure given by a glass Langmuir diffusion pump, in operation throughout the experiment.

During the first part of the work it was difficult to know what sort of interdependence to expect. This first work was further confused by the fact that the same plate of bismuth was used for a considerable number of runs, and it was afterwards found that one consistent run was all that could

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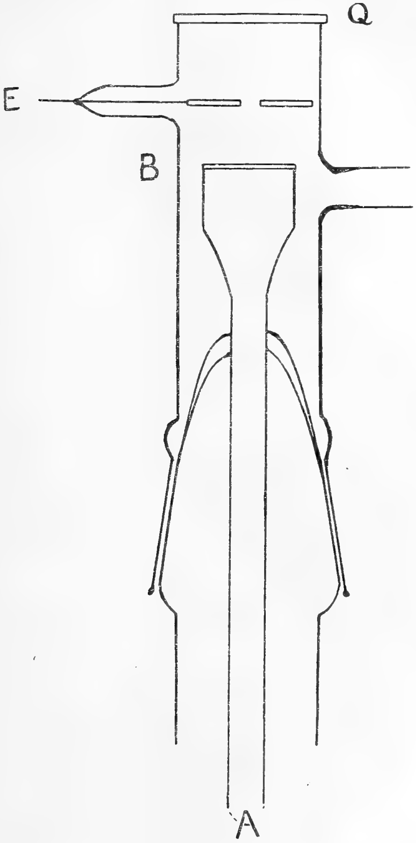
[*] Communicated by Prof. E. P. Adams, Ph.D.

be obtained from a plate. However, it was finally roughly determined that for any given plate current the photo-electric current increased up to a maximum, beyond which it did not go unless a larger plate current was used. With very high currents this maximum was followed by a fairly rapid decrease in the photo-electric current. The rises of the photo-electric current were always very slow, in some cases occupying up to two hours to attain the maximum.

Fig. 1.

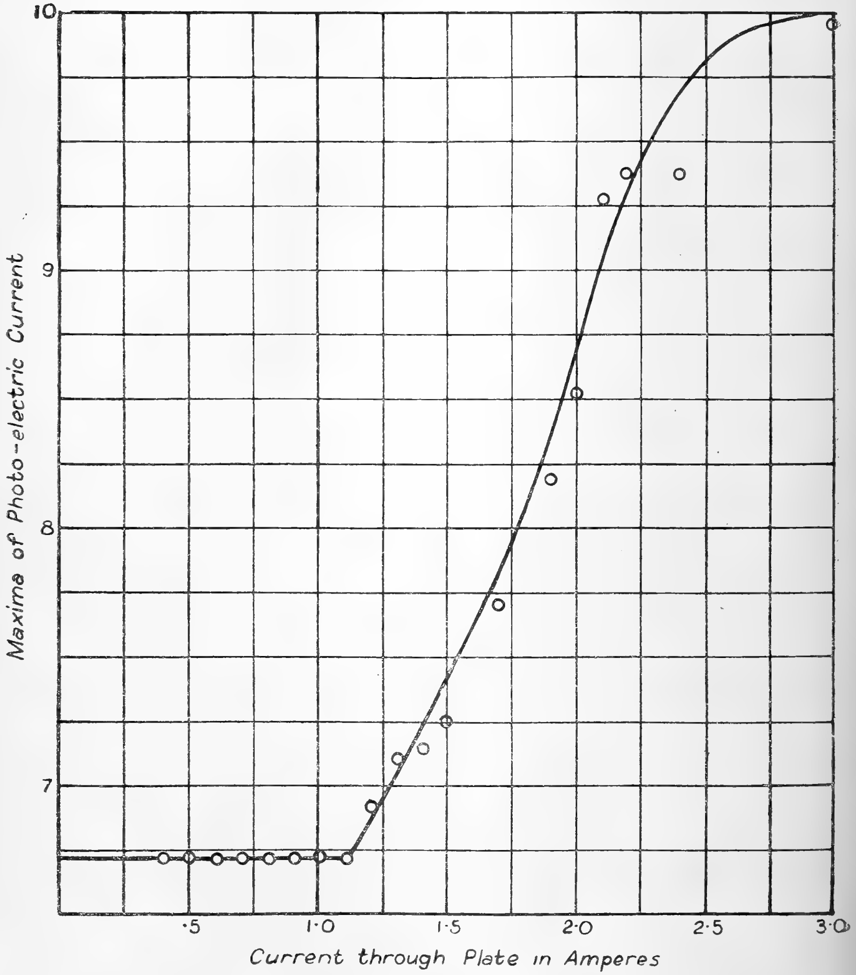


Q, quartz plate; E, copper electrode, with ebonite cover, connected to electrometer; B, bismuth plate: A, battery leads; P, connexion to pump.

When the plate current was stopped, the photo-electric current at once started to decrease, at first quite rapidly, but apparently it never drops to exactly the original zero, usually going beyond it.

It was proved that the effect was not dependent on the light by making a few runs with the plate screened, except when the readings were being taken.

Fig. 2.



Bismuth plate, cross-section 0·112 cm.^{2}

When high currents were being used, readings were always taken with the current shut off, in order to eliminate the effect of the magnetic field.

After the preliminary work it was determined to try to obtain complete curves of the dependence of the photo-electric current maxima on the current through the plate.

This was a matter of some difficulty, because of the large number of factors that had to be kept constant over a long time. The plate was illuminated by the light from a quartz-mercury vapour lamp, the intensity of which is very sensitive to changes in the current caused by other work in the building. A few long sets of readings were obtained, however, and were all similar. A typical curve is shown in fig. 2. This occupied sixteen hours.

To get these results the plate was first thoroughly fatigued by exposure to the light for about twenty-four hours. It was then found to give a very constant photo-electric current. Then a small current was passed through the plate, and readings taken at intervals, until it was determined that no change in the photo-electric current was occurring. In this way by successive increases in plate current, a current was reached where a small increase in the photo-electric current occurred. Beyond this point the increases became greater and greater up to the large current, where the photo-electric current after reaching a maximum began to decrease. Beyond this point the photo-electric current could be further increased by a large rise in the plate current, but consistency vanished at this point (2·2 amperes in fig. 2).

With the current off, the plate rapidly fatigued again, usually going to a point below the original value. If, now, an attempt was made again to go through the process just described, the results were very erratic. The increase started at a slightly lower value of plate current and rose very irregularly. For some increases of plate current there was an abnormally large rise followed by constancy for several further increases, followed further by another abnormal increase. The final value reached in the first run with a given plate reached in some cases 100 per cent. increase.

Sputtered Plates of Bismuth.

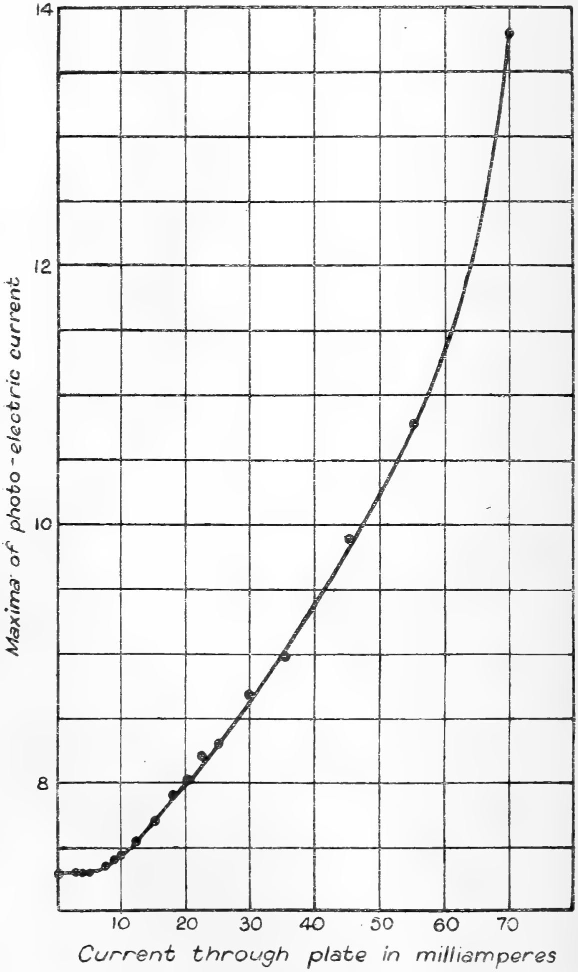
Because of the fact that bismuth behaves very differently in the form of a sputtered film (especially the failure to change resistance in a magnetic field), several sputtered plates of different thicknesses were tried. They all gave results similar to the thick plates, except that they could be run right up to the point at which the films burned out, without any inconsistency appearing. A curve for such a plate is attached (fig. 3).

A zinc plate was also tried and was found to give similar results, but of smaller magnitude.

There seem to be only two ways to account for the effect described :—(1) Temperature effect ; (2) an orientation of the elementary crystals of the metal.

1. The possibility of temperature making such a large change is extremely unlikely. In fact, with pure metals, it is generally accepted that temperature produces no change in the photo-electric effect. Moreover, in this experiment, currents which produced very considerable increases in the photo-electric current caused only a fraction of a degree rise in the temperature.

Fig. 3.



Sputtered bismuth plate (on mica).

2. The hypothesis of the orientation of the elementary crystals seems to be the most probable. The appearance of the effect in sputtered plates cannot be argued against this hypothesis, since the sputtered particles are undoubtedly of more than molecular dimensions.

The fact that no increase of photo-electric current appears until a certain minimum current is reached, indicates that a certain minimum field is necessary to orient an elementary crystal, the further increases being due to the larger numbers so oriented.

Further, the fact that a second series of readings with a plate gives the irregular results described previously, would seem to indicate that the structure of the metal had been in some way weakened. The greater rises under these conditions would likewise support the view.

The validity of the hypothesis could, however, be best tested by working with plates cut perpendicular to the crystal axis of a large bismuth crystal. A further test might be obtained from the measurement of change of contact difference of potential under the conditions existing in this experiment.

Palmer Physical Laboratory,
Princeton, N.J.

LXXXVIII. *Critical Electron Velocities for the Production of Luminosity in Atmospheric Neon.* *By* Frank Horton, *Sc.D., Professor of Physics in the University of London, and* Ann Catherine Davies, *M.Sc., Assistant Lecturer in Physics in the Royal Holloway College, Englefield Green* [*].

In a recent paper [†] the authors have shown that in atmospheric neon there are two critical electron velocities associated with the production of radiation from the gas, and three critical electron velocities associated with the production of ionization, the values of these velocities being :

Minimum radiation velocity . . .	11·8 volts.
Second radiation velocity . . .	17·8 ,,
Minimum ionization velocity . .	16·7 ,,
Second ionization velocity . . .	20·0 ,,
Third ionization velocity	22·8 ,,

In view of the fact that neon is the only gas for which more than one critical ionization velocity has been found under conditions which preclude the possibility of the

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[*] Communicated by the Authors.

[†] Proc. Roy. Soc. A. vol. xcviii. p. 124 (1920).

higher critical velocities being concerned with the ejection of a second electron from an already ionized atom, it seems desirable to mention briefly the precautions taken to ensure the purity of the gas used in these experiments. The neon was purified by Dr. Aston of Cambridge, using the elaborate fractionation apparatus which he devised for this purpose. The process of purification was accompanied by frequent determinations of the density of the gas, and was continued long beyond the stage when no alteration of density could be detected. In admitting the gas to the apparatus adequate precautions were taken to remove any impurities which might have found their way into the neon from the glass walls of the vessel in which it was stored, or from the walls of the tubes through which it passed, and, during most of the experiments, the pure gas was slowly streaming through the ionization-chamber. With these precautions the only possible impurity in the gas used was helium, and in view of the lengthy and careful treatment of Dr. Aston it is unlikely that more than a minute trace of helium was present. A spectroscopic examination of the gas showed no lines but those of the neon spectrum. The presence of helium in sufficient quantity might have accounted for the detection of a critical ionization velocity at 20 volts—approximately the resonance velocity for electrons in helium,—but the experiments about to be described show that this electron velocity is essential for the production of the principal series lines in the neon spectrum.

In view of the results of the positive ray experiments of Sir J. J. Thomson [*] and of the more recent experiments of Dr. Aston [†], which have shown that atmospheric neon contains two constituents of atomic weights 20·00 and 22·00 and possibly a small proportion of a third constituent of atomic weight 21, the detection of three critical electron velocities for ionization of the gas is of particular interest. In order to explain his results, Dr. Aston assumed that the constituents of atmospheric neon are isotopes, an assumption which is justified by the fact that according to Moseley's theory of atomic numbers there can be no unknown element of atomic weight between 20 and 23. The existence of three isotopes would not, however, be expected to lead to three different ionization velocities, since the atoms of isotopes have the same nuclear charge and the same number of surrounding electrons, which would presumably be dis-

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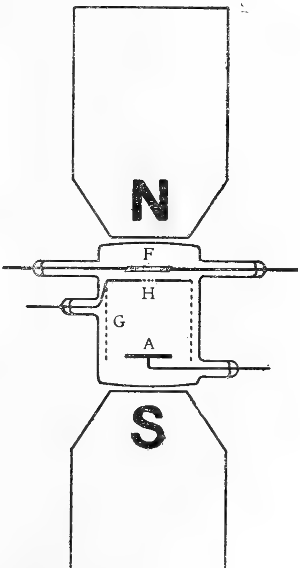
[*] J. J. Thomson. 'Rays of Positive Electricity,' p. 112 (1913).

[†] F. W. Aston, Phil. Mag. vol. xxxix. p. 449 (1920).

tributed in the same way about the nucleus in each case. The existence of the three ionization velocities would be expected if atmospheric neon consisted of three different elements, and in this case it would be possible to excite the spectrum of each element in turn by gradually increasing the speed of the stream of electrons bombarding the gas atoms. It was with the view of obtaining some further information about the three ionizing velocities that the experiments described in the present paper were undertaken. Some preliminary experiments, referred to in our earlier paper, had shown that in order to obtain conclusive evidence it would be necessary to use some arrangement for concentrating the rather feeble glow which appeared in the gas at certain velocities of the electron stream. This concentration of the luminosity was obtained by using a stronger magnetic field in the manner described below.

Description of Apparatus *.

Fig. 1.



The source of electrons was a platinum filament F (fig. 1) covered with a thin coating of lime. This was about 8 mm.

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* I wish to acknowledge my indebtedness to the Government Grant Committee of the Royal Society for the means of purchasing some of the apparatus and materials used in this research.—F. H.

long and 1 mm. wide, and was situated horizontally and with its middle over the hole H (1·5 mm. in diameter) in the centre of a platinum disk which nearly filled the cross-section of the discharge chamber and served to screen off the light of the glowing filament from the lower part of the tube. Round the edges of the platinum disk was fixed a cylindrical piece of fine platinum gauze G, about 1·5 cm. long. The hole H was also covered with a small piece of similar fine gauze so as to improve the electric screening of the space below H. The anode A was a circular platinum plate, 1 cm. in diameter, fixed horizontally just above the level of the open end of the gauze as indicated in the figure. The total length of the ionization chamber was about 2·5 cm., so that it could be easily arranged between the pole pieces N and S of an electromagnet, which, by producing a strong magnetic field along the axis of the tube, concentrated the stream of electrons passing through H into a parallel beam. When the velocity of the electron stream was raised to a suitable value, the luminosity thus concentrated along the axis of the tube was very much brighter than the diffused glow which filled most of the space when no magnetic field was used. The spectrum of the luminosity was viewed by means of a Hilger wave-length spectroscope, the slit of which was arranged so as to be illuminated by the light produced along the axis of the tube between the grid and anode. It was screened by the platinum disk from the direct light of the glowing filament. All the platinum electrodes were boiled for several days in strong nitric acid before being fitted into the apparatus, and the usual precautions were taken for ridding the glass walls of occluded gases.

In the diagram the connexions to the pump &c. are not shown. The arrangements for circulating pure neon through the apparatus during the observations were similar to those described in the paper already referred to. The gas passed from the storage bulb down a fine capillary tube to the discharge chamber, which it entered from a U-tube containing a little carbon and immersed in liquid air. Another tube connected the apparatus to the pumping system.

In some of the experiments which were made with this apparatus, in addition to observing the spectrum of the luminosity produced in the gas under different conditions, the currents between the electrodes were measured at the same time in order to ascertain whether sudden changes in the spectrum of the gas were connected with changes in the gas ionization. The currents were measured by means of a sensitive moving-coil galvanometer which was situated in a room adjoining that in which the neon spectrum was under observation, the latter room being kept in darkness for the better detection of faint lines.

Experimental Results.

For the purpose of determining the limiting electron velocities required to produce particular lines in the neon spectrum, it was necessary that the bombarding electrons should suffer no change of velocity throughout the space which was viewed by the slit of the spectroscope, except such change as results from collisions with gas atoms. To secure this condition, the grid and anode were maintained at the same potential by connecting each to the positive terminal of an insulated battery, the negative end of which was connected to the negative end of the filament. The maximum velocity of any of the electrons passing through the hole H is thus the sum of that due to the applied potential difference and the velocity of emission of the electrons from the filament. If the maximum velocity of emission is determined, the velocity of the swiftest electrons passing through H under any applied potential difference is known. The maximum velocity of the electrons passing through H, when any given line is first seen in the spectroscope, thus gives an upper limit for the critical electron velocity required for the excitation of the particular line in question. The velocity of emission was determined by finding what potential difference it was necessary to apply between the filament and the grid in order to prevent any electrons from the filament reaching the grid. In taking this maximum velocity as the critical velocity, it is assumed that the electrons having this velocity are sufficiently numerous to produce the given line with such intensity as to be visible. In our earlier paper a method was described by means of which the velocity of the electrons actually producing the radiation or ionization current under observation could be obtained. With the present apparatus it was not possible to determine the actual electron velocity in a similar manner; and the electron velocities stated as used in the present research are in all cases the maximum velocities in the electron stream under the conditions of experiment.

The first appearance of luminosity and the alterations in its spectrum were carefully observed as the velocity of the electron stream was gradually raised from about 10 volts to

30 or 40 volts. These observations were repeated with different pressures of neon in the apparatus—up to about 1·5 mm.,—and at each pressure with various intensities of the electron stream. With the filament at a high temperature and with about 40 volts potential difference between the filament and the grid, the spectrum of the glowing gas contained the lines given in the following table. The relative intensities of the lines are also given, together with the series to which they belong according to the recent classification of the lines of the neon spectrum by Paschen *.

Table I.

Lines observed in the spectrum of the glow when the electron velocity was about 40 volts.

λ.	Intensity.	Series.
6717	0·5	1·5, s_2 2, p_5
6678	0·5	1·5, $s_2 - 2, p_4$
6599	0·5	1·5, $s_2 - 2, p_2$
6533	0·5	1·5, $s_3 - 2, p_7$
6507	2	1·5, $s_4 - 2$ p_8
6402	8	1·5, $s_5 - 2, p_9$
6383	1	1·5, $s_4 - 2, p_7$
6305	0·5	1·5, $s_4 - 2, p_6$
6266	2	1·5, $s_3 - 2, p_5$
6217	0·5	1·5, $s_5 - 2, p_7$
6164	1	1·5, $s_3 - 2, p_2$
6143	8	1·5, $s_5 - 2, p_6$
6096	4	1·5, $s_4 - 2, p_4$
6074	4	1·5, $s_4 - 2, p_3$
6030	1	1·5, $s_4 - 2, p_2$
5976	1	1·5, $s_5 - 2, p_5$
5945	3	1·5, $s_5 - 2, p_4$
5852	10	1·5, $s_2 - 2, p_1$
5820	0·5	2, $p_8 - 4, d_4$
5764	1	2, $p_9 - 4, d_4'$
5748	0·5	2, $p_9 - 4, d_1'$
5690	0·5	2, $p_{10} - 3·5, s_5$
5657	0·5	2, $p_7 - 3·5, s_1''''$
5563	0·5	2, $p_8 - 3·5, s_1'''$
5401	3	1·5, $s_4 - 2, p_1$
5341	3	2, $p_{10} - 4, d_5$
5331	3	2, $p_{10} - 4, d_3$
5298	0·5	2, $p_4 - 4·5, s_2$
Some faint lines.		
5222	1	2, $p_8 - 4·5, s_4$
5204	1	2, $p_6 - 5, d_1'$
5189	2	2, $p_9 - 4·5, s_5$
Some blurred lines.		
5145	2	2, $p_4 - 4·5, s_1'''$
5116	2	2, $p_{10} - 3·5, s_1''$
5080	2	2, $p_8 - 5, d_4$
5038	3	2, $p_9 - 5, d_4'$
4710	1 (Blurred.)	

It was found that the minimum electron velocity at which a glow ever appeared in the gas was 20 volts. Over the range of pressures investigated, the point at which the glow appeared was independent of the pressure of neon in the apparatus provided this was not too low (above about 0·05 mm.), but it was found to depend somewhat upon the intensity of the electron stream, and also upon that of the magnetic field used to concentrate the luminosity. In general, as the speed of the electron stream was gradually raised, the brightest lines in the spectrum of the luminosity

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* F. Paschen, *Ann. der Physik*, vol. lx. p. 405 (1919).

first produced in the gas were those given in the table below.

TABLE II.

The brightest lines in the spectrum of the luminosity first produced.

λ.	Series.
6402	$1{\cdot}5,\ s_5-2,\ p_9$
6143	$1{\cdot}5,\ s_5-2,\ p_6$
6096	$1{\cdot}5,\ s_4-2,\ p_4$
6074	$1{\cdot}5,\ s_4-2,\ p_3$
5945	$1{\cdot}5,\ s_5-2,\ p_4$
5852	$1{\cdot}5,\ s_2-2,\ p_1$
5401	$1{\cdot}5,\ s_4-2,\ p_1$

With suitable values of the intensity of the electron stream and gas pressure, slight differences in the minimum voltages at which these bright lines first became visible were detected. In these cases the line λ 5852 was the first to appear, and then the lines of higher wave-length in the table, and finally λ 5401; but the *minimum* electron velocities at which λ 5852 and λ 5401 were ever observed did not differ by more than 0·3 volt. With an intense electron stream, all the lines in Table I. of longer wave-length than λ 5852, and also the line λ 5401, could be identified in the spectrum of the glow a few tenths of a volt after it was first produced, and it was found that the visibility of these lines of higher wave-length than λ 5852 depended solely upon the general brightness of the luminosity produced in the gas. From Table I. it may be seen that all these lines have been classified by Paschen as belonging to sequences of the type $(1{\cdot}5,\ s-m,\ p)$; *i. e.*, they are principal series lines. It was therefore concluded that the spectrum of the glow produced at 20 volts contained all the principal series lines of Table I., the slight differences in the minimum electron velocities at which different lines were first observed being due entirely to the relative intensities of the different lines.

As the speed of the electrons was further raised about 2 volts or more, certain lines in the blue, green, and yellow, of which those given in Table III. were the brightest, suddenly made their appearance.

TABLE III.

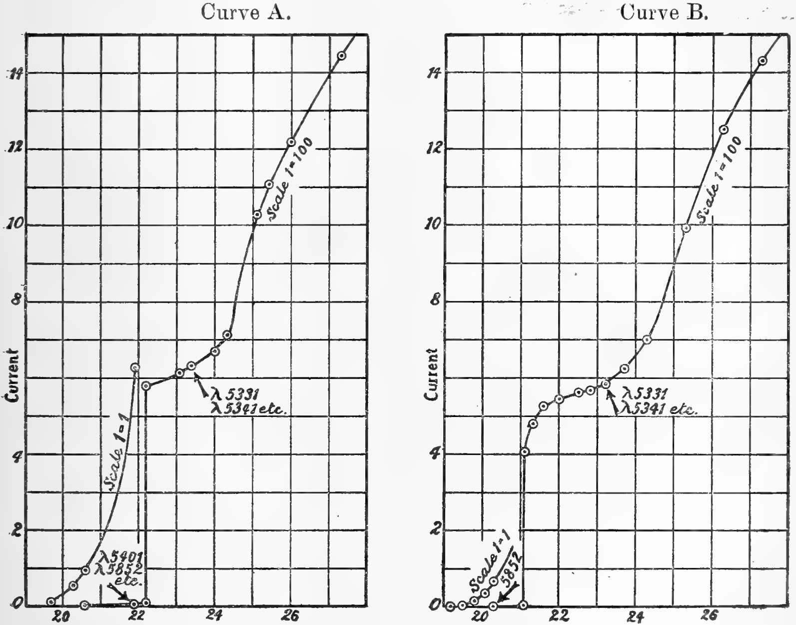
The brightest lines of those which suddenly appeared in the spectrum of the glow at about 23 volts.

λ.	Series.
5764	$2, p_9 -4, d_4'$
5341	$2, p_{10} -4, d_5$
5331	$2, p_{10} -4, d_3$
5189	$2, p_9 -4{\cdot}5, s_5$
5145	$2, p_4 -4{\cdot}5, s_1'''$
5116	$2, p_{10} -3{\cdot}5, s_1''$
5080	$2, p_8 -5, d_4$
5038	$2, p_9 -5, d_4'$

These lines, according to Paschen's classification, all belong to sequences of the type $(2, p-m, s)$ or $(2, p-m, d)$ and are subordinate series lines. A few tenths of a volt after this second set of lines became visible, some of them were distinctly brighter than other lines which were visible at a lower electron velocity and which had gradually brightened as the potential difference was raised. The appearance of the two sets of lines at different electron velocities is therefore not merely due to their production with very different intensities at some definite minimum electron velocity, the same for both, but is due to a genuine difference in the energy required to excite the two sets. Further evidence in support of this view was obtained from a series of measurements of the ionization in the space between the grid and anode taken with the galvanometer while the spectrum was being examined. In obtaining these measurements, a potential difference was maintained between the grid and anode opposite in direction to that between the filament and the grid, and exceeding the latter potential difference by a constant amount of 3 volts, so that none of the electrons from the filament could reach the anode. During the observations the potential difference accelerating the electron stream from the filament was measured by means of a very high resistance voltmeter connected between the grid and the negative end of the filament. In the curve A of fig. 2, which represents the results obtained with gradually increasing electron velocity, it may be observed that there is a sudden large increase of current at 22·2 volts. The lines λ 5852 and λ 5401 and some of the brighter red lines were visible before this increase of current occurred, and the rather feeble glow visible in the tube before 22·2 volts brightened enormously when the current increase occurred.

This increase of luminosity was accompanied by the appearance of the remaining red and orange lines of Table I., which had previously been of too small an intensity to be visible; but the lines of Table III. and, in fact, *all* the lines belonging to subordinate series were absent. The curve shows that after the large increase of ionization, probably caused by a sudden neutralization of the space charge near the filament, the current increased steadily with increasing electron velocity until 23·4 volts, after which it began to increase at a more rapid rate. At the stage where this more rapid increase

Fig. 2.



began the brighter green lines given in Table III. suddenly flashed out, and as the electron velocity was still further increased the remaining subordinate series lines of Table I. gradually made their appearance. The curve B of fig. 2 gives the values of the ionization current corresponding to different values of the electron velocity as this was gradually reduced from about 27 volts. At 23·2 volts, in this curve, the last remaining subordinate series lines suddenly disappeared, the others having gradually faded out earlier as the electron velocity was reduced. After this point the current decreased less rapidly as the velocity was reduced until 21·1 volts, when a sudden large drop in the magnitude of the current occurred. The brighter principal series lines were however still visible, and these did not all disappear until the electron velocity was 20·3 volts, when the yellow line λ 5852 vanished. With an electron velocity of 24 volts the green lines λ 5331 and λ 5341 which are two of the brightest of the subordinate series lines, were as bright as the principal series green line λ 5401. From curve A it may be seen that λ 5401 was visible when the current measured in arbitrary units was 6·28, whereas λ 5331 and λ 5341 were still not visible when the current measured in the same units was 600. The fact that λ 5331 and λ 5341 and the other subordinate series lines require a higher electron velocity for their production than do λ 5401 and the other principal series lines, is thus clearly not due to a mere difference in the relative intensities of the two sets of lines. The results of many series of observations of the changes in the spectrum with increase of the electron velocity seemed to indicate that a minimum velocity of about 23 volts was required to produce the subordinate series lines.

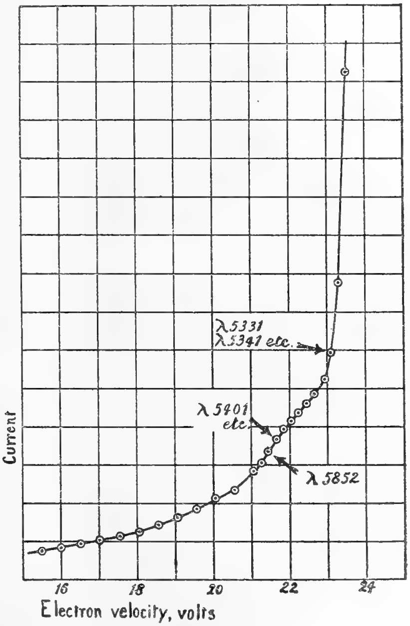
The set of lines which appeared at the higher electron velocity contained, in addition to the brighter lines (given in Table III.), all the other subordinate series lines in Table I. These made their appearance a small fraction of a volt after the brighter lines were visible, when the general brightness of the luminosity was high. Thus, although the principal series lines are produced at 20 volts and the subordinate series lines at 23 volts, the fainter members of these series in each case require for visibility a greater concentration of radiant atoms than exists when only the swiftest members of the electron stream have the necessary minimum velocity.

From the results of these experiments, and from our previously determined values of the ionization velocities for electrons in neon, it seemed probable that the appearance of luminosity in the gas was connected with the 20·0 volt ionization, but that the production of the complete neon spectrum required the third type of ionization, found in our earlier work to begin at 22·8 volts. Some confirmation of this view is provided in the curves already given in fig. 2. In both of these curves the limiting electron velocity for the visibility of any subordinate series lines is practically coincident with the beginning of an alteration in the rate of change of the ionization current with change of electron velocity, such as is usually taken to indicate a change in the type of ionization occurring. Moreover, the bend in the curve occurs at about 23 volts, the velocity at which the third type of ionization begins. In order to test further our view regarding the connexion between the ionization velocities

20·0 volts and 22·8 volts, and the excitation of the lines of the principal and subordinate series respectively, many series of observations of the currents between the different electrodes were made at the same time as observations of the spectrum of the glow when the electron velocity was varied. Examples of some of the curves obtained, with the points at which representatives of the two groups of lines first appeared, are given in figs. 3 and 4.

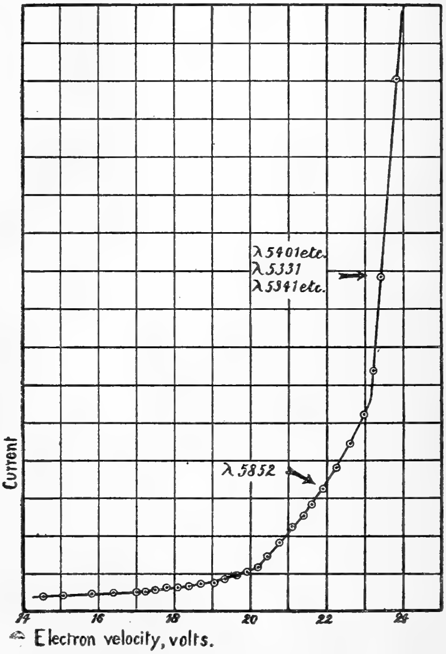
The series of observations represented in fig. 3 was taken with the grid and anode connected together and to one terminal of the galvanometer, the other galvanometer terminal being connected through the battery to the negative end of the filament. Thus the electrons, after being accelerated up to the grid by the applied potential difference, passed into a field-free space and, when the ionization velocity was reached, were able to produce ionization over a considerable distance. Hence, with this arrangement, the luminosity when it first appeared extended over a much bigger length of the distance between the grid and anode than was the case when the electrons were retarded between these electrodes. The current measured by the galvanometer is, with this arrangement, the total current passing between the filament and the other two electrodes. The curve of fig. 3 shows three bends:

Fig. 3.



the first at about 17 volts indicating the beginning of ionization, the second at 20·5 volts indicating an increased production of ionization, and the third and most marked bend at 23·0 volts denoting a still further increased production of ionization. The first appearance of luminosity occurred between 21·2 volts and 21·4 volts, the yellow line λ 5852 being then visible in the spectroscope. At 21·65 volts, λ 5401 and the other lines of Table II. were discernible. At an electron velocity of 23·1 volts the lines λ 5341, λ 5331, and λ 5031 could be seen faintly; and at an electron velocity of 23·5 volts all the lines of Table III. could be seen, and the whole spectrum was much brighter. For this series of observations a fairly strong magnetic field was used for the concentration of the luminosity.

Fig. 4.



In fig. 4 a series of observations of the ionization current and spectrum variations with alteration of the electron velocity, in the absence of a magnetic field, is represented. The arrangement of electric fields was very similar to that used in obtaining the curve of fig. 3. As was to be expected, the luminosity when the magnetic field was not on, extended throughout the whole of the space between the grid and anode instead of being concentrated into a bright central column. The lines seen through the spectroscope were very much less intense in the absence of the magnetic field, and in some cases, even though the luminosity could be seen faintly at electron velocities between 20 volts and 21 volts on observing the apparatus directly, no line but λ 5852 could be seen through the spectroscope until the ionization velocity 22·8 volts was exceeded and the second group of lines appeared; the remaining lines of Table II. then became visible also. This was the case in the series of observations represented in fig. 4. The curve shows ionization beginning at about 17 volts, an increased rate of production of ionization beginning at 20·2 volts and another such increase at 23·1 volts. At 20·8 volts a faint luminosity was observed in the tube, and the yellow line λ 5852 was seen through the spectroscope at 21·9 volts. No other lines appeared until 23·4 volts, at which velocity all the brighter green and red lines of Tables II. and III. could be seen faintly. After the completion of these observations, the spectrum was observed with the magnetic field on and with an electron velocity of 22·5 volts, *i. e.* with the velocity just below that at which the third type of ionization begins. All the brighter principal series lines, including the green line λ 5401, were distinctly visible and were much brighter than they had previously been at 24 volts; but none of the subordinate series lines were present. From these and other similar curves it was concluded that a minimum electron velocity of 22·8 volts is required for the production of the subordinate series lines, but that the principal series lines can be stimulated by electrons having a smaller velocity than this.

The fact that the lines of Table II. are generally first observed at some velocity after the 20·0 volts ionization has been detected, but before the 22·8 volts ionization occurs, combined with the fact that no luminosity was ever observed below 20·0 volts, constitutes fairly conclusive evidence that the stimulation of these principal series lines results from the occurrence of the 20·0 volts type of ionization and not from the 16·7 volts type of ionization. Further evidence on this point was obtained from several series of simultaneous observations of the ionization current and the spectrum of the luminosity taken with different intensities of electron stream and gas pressure. In one of these series the value of the ionization current when there was the first appearance of luminosity was 8·1 arbitrary units, the electron velocity being then slightly greater than 20 volts. In another series of observations, in which the gas pressure was higher and in which a more intense electron stream was used, the ionization current at 19·7 volts was as great as 79 of the same units without there being any trace of luminosity, though as soon as the electron velocity exceeded 20·0 volts a glow appeared in the gas. In neither of these series of observations was a magnetic field used. Since at 20 volts the number of radiating centres is sufficient for luminosity with an ionization current of 8·1 units, the absence of visible light at 19·7 volts with an ionization current of 79 units must be due to the electron velocity being too low, rather than to the glow being produced too faintly to be visible.

The experiments have therefore shown that the lines which we have observed in the neon spectrum make their appearance in two stages, as the velocity of the exciting electron stream is raised. The lines of the principal series type depend for their production upon the presence of the 20·0 volts ionization, while the subordinate series lines require the higher ionization velocity of 22·8 volts. No lines in the visible spectrum result from the ionization which occurs at 16·7 volts. With the exception of the green line λ 5401, all the lines produced by the 20·0 volts ionization lie in the red and orange parts of the spectrum, while the lines which require 22·8 volts for their excitation are in the green and blue parts of the spectrum.

The difference in the velocities needed to excite these two sets of lines is beautifully illustrated if the experiments are carried out at a pressure rather higher than those used in most of the foregoing experiments. With a pressure of about 1 mm., and with the magnetic field applied, a brilliant pencil of light can be obtained along the axis of the discharge tube. If the electric fields are arranged so that the electrons are accelerated from the filament to the grid and then retarded between the grid and the anode by an opposing field sufficiently strong to prevent any of them from reaching the latter electrode, luminosity appears when the velocity of the stream reaches 20 volts; it is then of a blood-red colour, and is confined to a thin layer of gas just under the grid. As the accelerating potential difference is slowly raised, this luminosity gradually extends downwards towards the anode, and, after 23 volts has been passed, the lower end, only, of the column is red, while the rest of it is a brilliant orange colour.

In the red part of the glow none but principal series lines are being produced, and, since the velocity of the electrons from the filament gradually decreases as they travel from the grid towards the anode, the experiment clearly shows that these lines can be stimulated when the velocity of the bombarding electrons has been reduced to a value too small to excite the complete neon spectrum.

An appearance which is the reverse of that just described may be obtained by arranging the electric fields so that they both tend to accelerate the electron stream towards the anode, but with the difference of potential between the filament and the grid considerably less than that necessary for ionization by collisions to take place just below the grid. For instance, in one such experiment, at a pressure of about 1·4 mm., the potential difference between the filament and the grid was maintained at 8 volts and the potential difference between the grid and anode was gradually raised. No luminosity appeared in the gas until the total potential difference between the filament and the anode was about 25 volts, when the central luminous column suddenly appeared, very brilliantly, reaching from near the anode right up to the grid. The lower part of this luminosity was of a bright orange colour and the upper end, nearer the grid, was blood-red. On gradually reducing the potential difference across the tube the column became shorter and no longer reached to the grid, and when the total difference of potential between the filament and the anode had been reduced to about 21·5 volts, the glow occupied only about 5 mm. in the centre of the tube and was all of a crimson colour showing only principal series lines of the neon spectrum. On reducing the maximum electron velocity still further, the glow became smaller in size, and finally disappeared when the potential difference was between 20 volts and 21 volts. It must be remembered that at the relatively high pressure of this experiment the mean free path of an electron in the gas is small (considerably less than 1 mm.), so that very few of the electrons after acquiring the lowest velocity with which they can collide inelastically with neon atoms, viz. 11·8 volts, will travel to the anode without loss of energy. The appearance of luminosity is thus delayed until the potential gradient between the grid and the anode is such that the electrons can acquire 20 volts velocity in a considerably shorter distance. The fact that most of the electrons make inelastic collisions in the gas before they get near to the anode also explains why the glow, when it does appear, does not extend right down to this electrode at this high pressure.

The extension of the glow right up to the grid when it appeared at 25 volts in this experiment, even though the velocity of the electron stream on reaching the grid was not enough to produce ionization, confirms in a striking manner the view that luminosity is produced at the recombination of positive ions and electrons. The positive ions are driven upwards by the electric field between the grid and anode and become neutralized, by electrons from the filament, in positions nearer to the grid than those in which they are formed. As the potential difference between the grid and anode is reduced, the positive ions being now produced nearer the anode recombine with electrons before they get near the grid, and so the luminosity no longer reaches up to that electrode. Similar evidence was obtained from experiments at lower pressures. In all of these the potential difference between the filament and the grid was maintained at a value well below the minimum ionization velocity. The luminosity made its first appearance at the anode, and, as the total potential difference between the filament and the anode was increased, the luminous column gradually became longer and ultimately reached the grid. In one instance the potential difference between the filament and the grid was only 4 volts when a luminous column extending right up to the grid was obtained. Thus the luminosity at the grid could not be due to ionization, but must have resulted from the recombination of the positive ions with electrons from the filament.

Discussion of Results and Conclusion.

The experiments described in this paper lead to the conclusion that there is no luminosity produced in neon as a result of the first type of ionization, shown in our earlier paper to begin at 16·7 volts. Since it is generally accepted that when ionization and subsequent recombination occur, a complete line spectrum corresponding to the particular ionization in question is produced, this absence of luminosity in the case of the 16 7 volts ionization in neon suggests that the corresponding spectrum may consist entirely of lines outside the visible region.

It has also been shown in the course of the paper that the lines contained in the spectrum of the glow when first produced, at 20 volts, are all of the (1·5, $s-m$, p), or principal series, type according to Paschen's classification; while the lines which do not appear until the ionizing velocity 22·8 volts has been reached, are all of the

($2, p-m, s$) and ($2, p-m, d$), or subordinate series, types. On the Bohr-Sommerfeld theory, spectrum lines have their origin in the movements of an electron within an atom from one temporary position of stability (or temporary orbit) to another, in which the energy of the system is smaller. Thus the series of lines ($1{\cdot}5, s-m, p$) result from the return of an electron to the $1{\cdot}5, s$ position of stability after having been displaced to one of the positions designated by $2p$, $3p$, $4p$, &c. It is generally assumed that when an atom is ionized and subsequent recombination takes place, the electron, in returning to the orbit normally occupied, may stop temporarily in any of the intermediate stable orbits. In falling between any two temporary positions it causes the system to emit a radiation, the frequency of which corresponds to the energy differences between these two positions. Thus, when recombination takes place on a considerable scale, a spectrum consisting of lines corresponding to all possible displacements should result. The appearance of one set of lines at the ionization velocity 20 volts, and another set of lines at 22·8 volts, therefore indicates either that two entirely different elements are being ionized at these two velocities, or that displacements between quite different orbits in similar atoms are occurring. In order to distinguish between these two possibilities, it is necessary to ascertain whether there exists any close relationship between the constants of the particular subordinate series of which members appear when the highest ionization velocity is passed, and the constants of the particular principal series of which lines appear at 20 volts. A consideration of Table IV. given below shows that there are instances in which lines of the series ($1{\cdot}5, s-m, p$) appear at 20 volts, and lines of the corresponding ($2, p-m, s$) series do not appear until the electron velocity reaches 22·8 volts.

TABLE IV.

Instances of lines of series $1{\cdot}5, s-m, p$ appearing at 20 volts.		Instances of lines of the corresponding $2, p-m, s$ series appearing at 22·8 volts.	
λ.	Series.	λ.	Series.
6507	$1{\cdot}5, s_4-2, p_8$	5222	$2, p_8-4{\cdot}5, s_4$
6678	$1{\cdot}5, s_2-2, p_4$	5298	$2, p_4-4{\cdot}5, s_2$
6402	$1{\cdot}5, s_5-2, p_9$	5189	$2, p_9-4{\cdot}5, s_5$

Each column of the table includes lines of the two types of series found by Paschen to occur in the neon spectrum, viz. those which follow the Ritz interpolation formula with great exactitude, without any modification of the combination principle, and those which require either a modification of this principle or a modification of the Ritz expression. In the paper already referred to, Paschen gave the extension of the Ritz interpolation formula, in the case of each series of the second type, which was necessary in order that the combination principle might hold. In a more recent paper * he has shown that if the combination principle is modified by the addition of a constant to the combination values of all terms of these series, the values of the terms so obtained can be expressed by formulæ of the exact Ritz type with greater accuracy than by the formulæ given earlier. The series to which λ 6678 and λ 5298 belong are of the type which requires this modification of the combination principle; but the series to which the other lines belong were satisfactorily expressed in the earlier paper. Thus both types of series are represented in the spectrum of the luminosity produced at 20·0 volts and in the additional lines which appear when the ionization velocity of 22·8 volts is passed.

The fact that the combination principle holds without any modification between some of the lines excited at 20·0 volts and some of those excited at 22·8 volts, rules out the possibility of these two ionization velocities corresponding to the ionization of two distinct elements. We are thus forced to the conclusion that the different ionization velocities which have been detected correspond to the removal of differently situated electrons from the neon atom, and hence that lines as closely related as those of a principal and the corresponding subordinate series arise from the return of electrons removed from different positions within the atom.

From the position of argon in the Periodic Table of the elements, its atom would be expected to have an arrangement of external electrons bearing some resemblance to that occurring in neon. We should therefore expect that argon would have more than one ionization velocity corresponding to the removal of a first electron from its atom, and that its spectrum could be excited in stages. We have not yet investigated the minimum electron velocities required for the excitation of spectrum lines in argon, but our experiments on the ionization of this gas gave no indication of the existence of more than one critical electron velocity for the production of ionization, and nothing is recorded in the work of other investigators which suggests that

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* F. Paschen, *Ann. der Physik*, vol. lxiii. p. 201 (1920).

more than one ionization velocity has been detected. In employing the method which we used for the investigation of critical electron velocities in argon and other gases, the detection of a second or third critical velocity for the removal of a first electron from the atom, if more than one critical velocity existed, would depend upon the relative probabilities of ionization resulting from encounters between atoms and electrons having velocities in excess of the critical value, in each case. It is doubtful whether the higher critical velocities would always be detected, unless the corresponding radiation velocities were intermediate to the ionization velocities.

Of the theories of atomic structure which have been put forward, that proposed by Lewis and by Langmuir [*] affords an explanation of the greatest number of different phenomena. This theory, which was originally evolved to account for the facts of chemical combination and valency, has since obtained considerable support from the recent work on X-rays and crystal structure [†], and has been found to be consistent with the results of the investigations on magnetism and atomic structure [‡]. So far, however, it has not been made to supply any explanation of the excitation of spectrum lines and the series relations in spectra. In this direction the Bohr-Sommerfeld theory of the existence within the atom of a series of non-radiating orbits, any of which may be temporarily occupied by an electron, has been found the most satisfactory, but it has not yet adequately explained the existence of the several series of non-radiating orbits necessary to account for all the lines of the complicated spectra of elements whose atoms contain several electrons. The fact that in neon principal and corresponding subordinate series appear to be associated with the transitions of electrons differently situated in the normal atom, seems likely to be of importance in this connexion.

According to the Lewis-Langmuir theory, the electrons in the atom of neon exist in two concentric shells, the inner shell containing two electrons and the outer shell the remaining eight electrons. In this model the electrons are either stationary or execute small oscillations about mean positions. Since the two electrons in the inner shell are closer to the nucleus than the eight outer electrons, it seems reasonable to assume that a much larger quantum of energy

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[*] I. Langmuir, Phys. Rev. vol. xiii. p. 300 (1919).

[†] W. L. Bragg, Phil. Mag. vol. xl. p. 169 (1920).

[‡] A. E. Oxley, Proc. Roy. Soc. A. vol. xcviii. p. 264 (1921).

would be required to remove one of these two electrons than is required to remove one from the outer shell. If, therefore, we interpret our results on the basis of this theory, each of the three ionization velocities found in neon would correspond to the removal of one of the eight outer electrons. Our results therefore indicate that in the outer shell of the neon atom electrons occupy three dissimilar positions with regard to the nucleus and inner shell. This conclusion is in contradiction to the theory of Lewis and Langmuir in its present form, but it is possible that the essential features of the theory might be retained without necessitating absolute similarity of position among the electrons in any particular shell.

LXXXIX. *Ether, Light, and Matter.*

By Sir OLIVER LODGE [*].

IN the October 1913 number of the Phil. Mag. vol. xxvi. pp. 636–673, there is a remarkable though highly speculative paper by Professor Bruce McLaren (killed, alas! in the war) in which, among other things, he attempts to explain gravity by treating matter as a source or sink of ether. The flow of ether which is thus necessarily postulated is supposed to transfer momentum from the destroyed portion to the boundary of its destruction, and thus virtually to exert a stress on the surface of transition, notwithstanding that matter is unlike a foreign body immersed in a stream, but is a peculiarity of the ether itself. He seems to think that physicists will object to locomotion of the ether, as perturbing to the rays of light; but so long as motion is irrotational it can be shown that rays of light are not affected, though the waves are carried along and made excentric. In other words the path of a specified unit of luminous energy is not altered by any irrotational drift, whatever happens to the wave fronts; for in a moving ether the rays, *i. e.* the paths of energy, are not normal to the wave front, and nothing perceptible need happen. That nothing happens is conceded by the Principle of Relativity; though an explanation of why nothing perceptible happens, and the idea of drift of wave fronts, are foreign to that theory.

That nothing happens on ordinary theory, at least to the first order of aberration magnitude, follows thus:—If the

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[*] Communicated by the Author.

drift v is inclined at angle θ to the ray, and if the consequent aberration angle between ray and wave-normal is ϵ, so that

$$\frac{\sin\epsilon}{\sin\theta}=\frac{v}{\mathrm{V}}=\alpha,$$

the resultant velocity is

$$\mathrm{V}'=\mathrm{V}\cos\epsilon+v\cos\theta,$$

and the path of the ray between any two points A and B is determined by

$$\mathrm{T}'=\int_{\mathrm{A}}^{\mathrm{B}}\frac{ds}{\mathrm{V}'}=\text{a minimum,}$$

which can be written, without approximation,

$$\mathrm{T}'=\frac{\mathrm{T}\cos\epsilon}{1-\alpha^2}-\int_{\mathrm{A}}^{\mathrm{B}}\frac{v}{\mathrm{V}^2}\cdot\frac{\cos\theta}{1-\alpha^2}\cdot ds. \quad . \quad . \quad . \quad . \qquad (1)$$

So if $v\cos\theta=d\phi/ds$ (that is if the drift has a velocity potential ϕ) the second term—the only one obviously containing θ—is proportional to $\phi_{\mathrm{B}}-\phi_{\mathrm{A}}$ and is independent of path. In other words, the time is unaffected by the drift, to the first order of v/V.

In another medium, V becomes V/μ, and, if v becomes v/μ^2 in accordance with Fresnel's law, the same velocity potential ϕ still holds good. And this accounts for the negative results of a large number of last century's experiments with prisms and water-filled telescopes, &c., some of which are described in 'Nature,' vol. xlvi. p. 500.

Even proceeding to the second order of drift velocity, only the first term of (1) is effective—at least so long as the drift is constant; because for a closed contour, such as is required for any feasible interference experiment, the second term vanishes. The necessary condition may not be satisfied under the peculiar conditions of destruction of substance and failure of the continuity equation; hence perhaps McLaren's caution. But a second order effect, dependent on cos ϵ, *would* be expected, even in a uniform drift; for the duration of the drifted to-and-fro journey in any direction, compared with the time of the same journey when everything is stagnant, is, by (1)

$$\frac{\mathrm{T}'}{\mathrm{T}}=\frac{\cos\epsilon}{1-\alpha^2}=\frac{\sqrt{(1-\alpha^2\sin^2\theta)}}{1-\alpha^2}; \quad . \quad . \quad . \quad . \qquad (2)$$

an expression which may be regarded as the simplest theoretical summary of Michelson's experiment. For in that experiment the times of a to-and-fro journey are compared, when $\theta = 0$ and when $\theta = 90°$; and, as everyone now knows, the null result of that experiment, combined with my experiment establishing the absence of convective or viscous etherial drift ('Nature,' vol. xlvi. pp. 165 and 501, or Phil. Trans. 1893 and 1897), necessitated some novel explanation, and led to the FitzGerald-Lorentz contraction.

(Parenthetically I take this opportunity of correcting a rather confusing misprint or slip in the summary of my historical survey communicated to the Physical Society in May 1892, which appears on page 165 of 'Nature,' vol. xlvi. Of the two terms in the expression for T', above the middle of the first column, the drift angle θ should appear only in the second term, while in the first term the angle should be the much smaller aberration angle ϵ, which is essentially of the second order in v/V.)

Hence a flow of ether need not be objected to on the ground of a perceptible disturbance to rays of light; but instinctively one feels that a Le Sage-like hypothesis concerning gravity is not likely to be on right lines. There must admittedly be a stress in the ether between two particles of matter, but it should be a static rather than a kinetic stress, and should not be accompanied by locomotion. Locomotion of the ether is objectionable for a variety of reasons: the only motion permissible in a universal medium of infinite extent is a circulatory or vortex motion, such as might occur along lines of magnetic force. Magnetic lines are always closed curves; there is no known way of generating them; they always pre-exist, though they may be of atomic or molecular magnitude, and in a magnetic field are opened out so as to enclose a perceptible area. This is generally admitted to be the process of magnetisation; and when the magnetism ceases the lines shrink up into infinitesimal, or practically infinitesimal, orbits again. That the *quantum* is associated with these ultimate magnetic units is exceedingly likely (*cf.* Dr. H. Stanley Allen, Proc. R. S. Edin. November 1920); and an association of electric units with the magnetic ones is all that is necessary to account for radiation.

In the Phil. Mag. for April 1921, I ventured on a speculation that matter is a sink as well as a source of radiation—a sink not of ether but of the radiation movement of ether. Annulling of the electric component in an ether wave, though the process commonly generates heat, may also under some conditions liberate the magnetic component, which, at a distance of $\lambda/\pi\sqrt{2}$ from a resonating particle, is left behind by the electric component. By analogy it should not be obliterated. The electric component there attains for a moment an infinite speed and gets out of phase with the magnetic component. When the phase difference is 180° the energy-flow is reversed in sign; when the phase difference is 90° there is no energy-flow, and only a stationary vibration existing inside a certain boundary.

The problem is whether part of the magnetic circulation, left stranded, could not cease to be oscillatory and become continuous and permanent; and whether the synchronous electric pulses of myriads of successive waves could not accumulate as a separated pair of opposite electronic charges.

Meanwhile we may roughly estimate that to generate such a pair requires energy comparable to 10^{-6} erg, which would be supplied by a length of a few hundred kilometres of ordinary sunshine if the area of wave contributing to the result were comparable to λ^2/π, as suggested by Lord Rayleigh's "Sound" investigation in the Phil. Mag. for August 1916 (*cf.* 'Nature,' vol. cvii. pp. 169 and 203). But to generate a weighable amount of matter, say a tenth of a milligramme, in the laboratory, a beam of sunlight a square decimetre in section, even if all of it were effective, would have to shine for seventeen years.

P.S.—In the current Proc. Roy. Soc. for February 1921, vol. xcix. No. A 697, p. 118, Professor Eddington shows, in a manner terribly difficult to understand, that it is "impossible to build up matter or electrons from electromagnetic fields alone,—some other form of energy must be present." The other form of energy postulated above is the previously existing particle on which the electromagnetic waves impinge; and an operation, such as Prof. Eddington is probably competent to envisage, is supposed to go on in its immediate neighbourhood—*i. e.* in a region where the electric field is exceedingly intense, and where the product of certain S tensors becomes important. It may be worth just noticing that the ordinary normal surface tension ($2\pi\sigma^2/\kappa$) on an electron, which the structure of the electron is somehow able to resist, approaches to within a twentieth of what I have elsewhere supposed to be its limiting or critical value 10^{33} dynes per square centimetre, or what might be called the weight of a thousand earths per square inch.

XC. *On the Einstein Spectral Shift.*

By Sir OLIVER LODGE *.

REFERRING to the interesting paper by Prof. H. J. Priestley on p. 747 of last month's Phil. Mag., I interpret a sentence in the second paragraph as intending to say that if the Einstein interval ds is transmitted by radiation, instead of the time period dt, then the shift of spectral lines will occur when not the source but the observer is immersed in a strong gravitational field. His further argument is (A) that this transmission of ds is a natural consequence of the principle of equivalence, which makes $ds=0$ along a ray even in a gravitational field, and (B) that the confirmed gravitational deflexion of a ray can be equally well obtained without depending on the principle of least time, or any other pre-relativity physics, and therefore without admitting the constancy of time period which that principle apparently implies.

But, I venture to ask, does Prof. Priestley succeed in establishing proposition B ?

He says truly that, since $ds=0$ along a ray,

$$dr^2+\gamma r^2 d\theta^2=\gamma^2 dt^2 \quad . \quad . \quad . \quad . \quad . \quad . \quad (7)$$

for light ; and he also claims to obtain

$$r^2\frac{d\theta}{dt} \text{ proportional to } \gamma, \quad . \quad . \quad . \quad . \quad . \quad (8)$$

instead of being merely indeterminate as the ratio of two infinities ; wherefore, combining these equations and putting $1/r=u$, he gets

$$\left(\frac{du}{d\theta}\right)^2+\gamma u^2 = \text{constant}.$$

Whence on differentiation (remembering the variability of γ)

$$\frac{d^2u}{d\theta^2}+u=3mu^2, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (9)$$

an equation which may be trusted to behave properly. It is indeed the usual progressing gravitational equation without the constant term responsible for an orbit.

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* Communicated by the Author.

But is his obtaining of (8), by aid of

$$\gamma\frac{dt}{ds}=k \quad \text{and} \quad r^2\frac{d\theta}{ds}=h,$$

legitimate for the case when $ds=0$?

I ask in no controversial spirit. I too thought at one time that it was the observer's field that was effective, but I am now very doubtful; and it would be interesting to have the point settled before a clear experimental verdict is forthcoming.

XCI. *Notices respecting New Books.*

Transactions of the Bose Research Institute, Calcutta, Vol I. parts 1 & 2, 1918; Vol. II. 1919. *Life-movements in Plants,* by Sir J. C. BOSE, Kt., M.A., D.Sc., C.S.I., C.I.E., Professor Emeritus, Presidency College; Director, Bose Research Institute. Published by The Bose Research Institute, Calcutta.

IN these two small volumes a set of papers produced by Sir J. C. Bose and his assistants are collected.

The Bose Institute was opened in 1917 and the work done up to 1919 is here recorded. Owing to the fact that the volumes are published by the Bose Research Institute at Calcutta, they are somewhat inaccessible to English readers. But as they contain detailed descriptions of a long series of experiments on which the various pronouncements of Sir J. C. Bose are founded, a critical evaluation of his work can only be obtained through a study of these volumes. Whatever else may be said, it is entirely evident that new applications of physical methods are being introduced; and it is up to the workers who consider the living organism their own domain, to look into the possibilities with respect to this new school; for the old methods have not proved magical in hurrying on the development of knowledge in this domain, and methods which have produced great increases of knowledge in many different branches of physics might well have some help to offer.

IN *An Outline of Physics* (Methuen, 6/6), L. SOUTHERNS, M.A., B.Sc., Lecturer in Physics, University of Sheffield, attempts a radical rearrangement of the subject-matter of a student's first course at a modern university. He gives first a general sketch or outline of the subject, which is intended to act as a frame on which detailed instruction may be placed. Part II. comprises a course suitable for general purposes, and is planned so as to allow the greatest elasticity and scope for modification.

MAJOR P. A. MACMAHON in his *Introduction to Combinatory Analysis* (Camb. Univ. Press, 7/6) gives an outline of the easier parts of the theory of Combinatory Analysis expounded in his larger volumes of 1915–1916. This small treatise treats the theory of symmetric functions and builds the general results in the theory of distributions on it. It is a valuable and interesting book.

Modern Analysis. Third Edition. By Prof. E. T. WHITTAKER, F.R.S., and Prof. G. N. WATSON, F.R.S. Cambridge University Press. 40*s.*

THERE is no great change in the new edition of *Modern Analysis.* The authors have added an entirely new chapter on Lamé's functions which should prove of great value, and the chapter on Fourier Series has been rearranged in the interests of the large number of applied mathematicians who use this book. It is notoriously difficult to please two masters and to avoid slipping between two stools: but as the book is undoubtedly of great value to applied mathematicians, a small sop of this kind is probably not out of place. There are no new features to remark on beyond these, except the price.

The Experimental Basis of Chemistry. By IDA FREUND. Edited by A. HUTCHINSON and M. BEATRICE THOMAS. Cambridge University Press. Pp. xvi + 408. Price 30*s.* net.

MISS IDA FREUND, who died in 1914, taught chemistry at Newnham College from 1887 to 1912 with well-known success. Her book *The Study of Chemical Composition* caused her name to be familiar to students who had never been to Cambridge, and was received with well-deserved praise. The present work embodies ten chapters left by her in manuscript, which have been admirably edited by two of her friends. We are told that the book was planned to consist of twenty chapters, but the portion published is complete in itself. It shows remarkable originality of treatment, and utters a vigorous protest against the very conventional way in which experiment is dealt with in too many of the laboratory manuals for students. It is an elementary book, but teachers and advanced students will get many valuable suggestions from the perusal of what is a sound, clear, critical, and logical account of the fundamental principles of chemistry. It appears from the sub-title that the book was not primarily intended to be worked right through in the laboratory, but any student with time and patience to do so would acquire a very good knowledge of the principles of the science. The book forms an excellent memorial to a teacher who, as the editors say, "was richly endowed with the critical faculty, keenly sensitive to fallacious reasoning, and quick to detect an unwarrantable assumption."

XCII. *Proceedings of Learned Societies.*

GEOLOGICAL SOCIETY.

[Continued from p. 828.]

June 9th, 1920.—Mr. R. D. Oldham, F.R.S., President, in the Chair.

Mr. Cargill Gilston Knott, D.Sc., LL.D., F.R.S., delivered a Lecture on Earthquake Waves and the Elasticity of the Earth. The Lecturer remarked that the record produced on delicate seismographs of the earth-movements due to distant earthquakes proves that an earthquake is the source of two types of wave-motion which pass through the body of the Earth, and a third type which passes round the surface of the Earth. Before earthquake records were obtained, mathematicians had shown that these three types of wave-motion existed in and over a sphere consisting of elastic solid material. Many volcanic phenomena, however, suggest the quite different conception of a molten interior underlying the solid crust. At first statement these views seem to be antagonistic; but there is no difficulty in reconciling them. Whatever be the nature of the material lying immediately below the accessible crust, it must become at a certain depth a highly-heated fairly-homogeneous substance behaving like an elastic solid, with two kinds of elasticity giving rise to what are called the compressional and distortional waves. The velocities of these waves are markedly different, being at every depth nearly in the ratio of 1·8 to 1. Both increase steadily within the first thousand miles of descent towards the Earth's centre, the compressional wave-velocity ranging from 4·5 miles per second at the surface to 8 miles per second at depths of 1000 miles and more; the corresponding velocities of the distortional wave are 2·5 and 4·3 at the surface and at the 1000-mile depth respectively. At greater depths these high velocities seem to fall off slightly; but the records fail to give us clear information as to velocities at depths greater than about 2500 miles. Down to this depth the Earth behaves towards these waves as a highly-elastic solid. The elastic constants, which at first increase with depth more rapidly than the density, become proportional to the density, for the velocity of propagation becomes practically steady. About half-way down, however, the material seems to lose its rigidity (in the elastic sense of the term), and viscosity possibly takes its place, so that the distortional wave is killed out. In other words, there is a nucleus of about 1600 miles radius which cannot transmit distortional waves. This nucleus is enclosed by a shell of highly-elastic material transmitting both compressional and distortional waves exactly like an elastic solid.

June 23rd.—Mr. R. D. Oldham, F.R.S., President,
in the Chair.

The following communication was read :—

'The Scandinavian Mountain Problem.' By Olaf Holtedahl.

In the introduction a brief account is given of the history of research regarding the Scandinavian mountain problem, which deals with the superposition of highly-metamorphosed, often gneissose rocks upon slightly-altered fossiliferous Cambro-Silurian sediments.

From a consideration of the phenomena in the mountain-belt of deformation, it is inferred that the age of the displaced materials depends upon the angle of inclination of the thrust-planes and their depth. Though the thrusts have extended downwards for a considerable distance, they have not, generally, in the author's opinion, reached below the level of the pre-Cambrian plane of denudation, and no true Archæan rocks could, as a rule, have been tapped.

In support of these conclusions some of the tectonic features of two districts are indicated : (1) Finmarken in Northern Norway, and (2) the southern part of the Sparagmite area near Randsfjord, in South-Central Scandinavia. Brief descriptions are given of the rock-groups in Finmarken and their structural relations. Special attention is directed to the structure of the Alten district, where the main tectonic feature is a highly-undulating thrust which does not intersect the pre-Cambrian floor. Regarding the Randsfjord district, the original order of succession of the strata is indicated, from the *Holmia* Shale to the close of the overlying Cambro-Silurian sediments. Pressure from the north in late Silurian time developed imbricate structure in these sediments, but such displacements are not supposed to have affected the pre-Cambrian floor.

As investigation proceeds, it seems to become increasingly evident (1) that the highly-metamorphic sedimentary rocks of the middle and northern part of the eastern mountain-belt are mainly of earlier Ordovician age, while those west of the Sparagmite region in the south-western mountain district are chiefly of Silurian age; and (2) that the igneous materials associated with these highly-metamorphosed sediments are younger intrusive rocks.

December 1st.—Mr. R. D. Oldham, F.R.S., President,
in the Chair.

The following communication was read :—

'An Æolian Pleistocene Deposit at Clevedon (Somerset).' By Edward Greenly, D.Sc., F.G.S.

Banked up against the craggy hillsides about Clevedon are considerable deposits which contain a terrestrial molluscan and vertebrate fauna of Pleistocene age. Most of the vertebrates were obtained from a small cave, during exploration some years ago by Mr. G. E. Male & Prof. S. H. Reynolds. The deposits consist of sandy breccia, stony sand, loamy sand, and loam. Breccias occur only close to the crags, but the sands may extend all across the valleys. The stones, which are sharply angular, are of exclusively local derivation, the sands of alien derivation. The formation presents a number of unusual characters, among which are a total absence of lamination and a singular vertical cleavage. From its fauna, structures, and composition, but especially from its physiographical relations, it is clear that the formation cannot be of aqueous origin. The breccias are manifestly local talus. The distribution of the sands and loams, and their peculiar structural characters, can, it is argued, be accounted for only by æolian action. The winds, banking them up under the lee of southern slopes, must have been from the north; and, as the land seems then to have stood sufficiently above its present level to lay dry the bed of that part of the Bristol Channel, it is suggested that the alien sand was blown from the older Pleistocene deposits of that hollow.

A comparison is made with the lœss, and it is shown that, out of 20 characteristics, palæontological, structural, and otherwise, the Clevedon drifts have 17 in common with that formation. Those which they have not in common are assignable to the proximity of Clevedon to the sea. The deposits are ascribed, accordingly, to 'lœss-conditions', acting in a region where local circumstances appear to have been especially favourable.

December 15th.—Mr. R. D. Oldham, F.R.S., President,
in the Chair.

The following communications were read:—

1. 'Structure and Stratigraphy of the Tertiary Deposits in North-Western Peru.' By Thomas Owen Bosworth, D.Sc., M.A., F.G.S.

The westernmost ranges of the Andes, in the north of Peru, are of pre-Tertiary age. The Eocene Pacific Ocean lay at the foot of them.

The Tertiary rocks occupy a narrow strip of country between the mountains and the sea. They are exposed in those areas which have been denuded of their horizontal cover of Quaternary deposits. The Tertiary consists of 15,000 to 25,000 feet of clay-shales and sandstones, with innumerable thin seams of beach-pebbles and shells. Thus, during the Tertiary Period, a large subsidence was in progress.

The stratigraphical succession is as follows:—

		Feet.
MIOCENE.	Zorritos Formation	5000+
EOCENE.	Lobitos Formation	5000+
	Negritos Formation.	
	Clavilithes Series *Turritella* Series	7000+

The Tertiary accumulation is greatly broken up by intense block-faulting; between the fault-blocks are differential displacements of many thousand feet. It is inferred that, in the interval between the Tertiary and the Quaternary Periods, an important movement occurred along a great fault-belt parallel with the Andes. The mountains were further upraised, and the sea-floor subsided to a great depth.

The uplifting of the mountains caused a strip of territory along the west side of them, 20 miles wide, to emerge from the sea. This is the littoral: it was part of the crush-belt of the great fault.

2. 'Palæontology of the Tertiary Deposits in North-Western Peru.' By Henry Woods, M.A., F.R.S., T. Wayland Vaughan, Ph.D., J. A. Cushman, Ph.D., and Prof. Herbert Leader Hawkins, D.Sc., F.G.S.

3. 'Geology of the Quaternary Period on a Part of the Pacific Coast of Peru.' By Thomas Owen Bosworth, D.Sc., M.A., F.G.S.

Throughout the Quaternary Period, the littoral has undergone a series of vertical oscillations. It has been lifted up and down repeatedly like a lid, having its edge a few miles out in the Pacific Ocean and its hinge-line in the Andes. During these processes the littoral has several times been alternately overspread with a marine deposit and then raised above the sea.

The ocean-soundings show a steep 2000-foot submarine cliff at the edge of the continental shelf. It follows a fairly direct line, which passes within 5 miles of the land. This cliff is taken to be a submarine fault-scarp, marking the important fracture (Pacific Fault) which was the western boundary of the Quaternary uplifts.

The oldest and highest of the raised sea-floors ('tablazos') now has an elevation of 1100 feet. It extends 20 miles inland, and, within the territory here discussed, it covers an area of 700 square miles. The inland boundary of each 'tablazo' is a raised sea-cliff. The original western limit of each one of them probably was the edge of the continental shelf. Whether there was any oscillation of the deep sea-floor on the west side of the Pacific Fault is not known. The depth, 27 miles from the present coast, is 12,000 feet. The Quaternary deposits formed upon it are presumably deep-sea oozes.

The events on the east side of the Pacific Fault may be grouped into four similar episodes. Each consists of a subsidence accompanied by marine transgression, followed by an uplift causing emergence of new land from the sea. They are as follows:—

(1) The Mancora Episode.
(2) The Talara Episode.
(3) The Lobitos Episode.
(4) The Salina Episode.

Each episode obliterates all trace of any preceding one which was not greater. Four episodes have left their mark; but probably there were many others, of which no evidence remains.

The most substantial of the deposits formed during these marine transgressions is 250 feet thick. The material ranges from shell-limestone to beach-pebbles. The shells have been examined by Col. A. J. Peile, who pronounces them (probably all) to be living species.

On the land, extensive breccia-fans and valley-terraces were produced, under desert conditions, during these oscillations. They are correlated with the marine terraces.

In conclusion, it is considered that not one ten-thousandth part of the Quaternary history, here outlined, can have taken place within the last 500 years.

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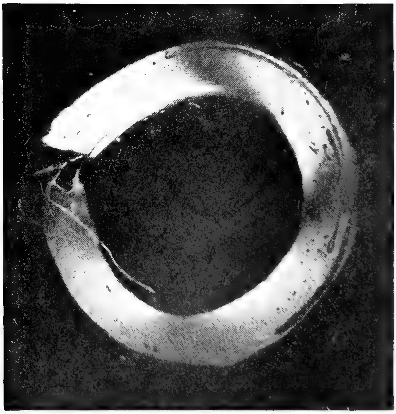

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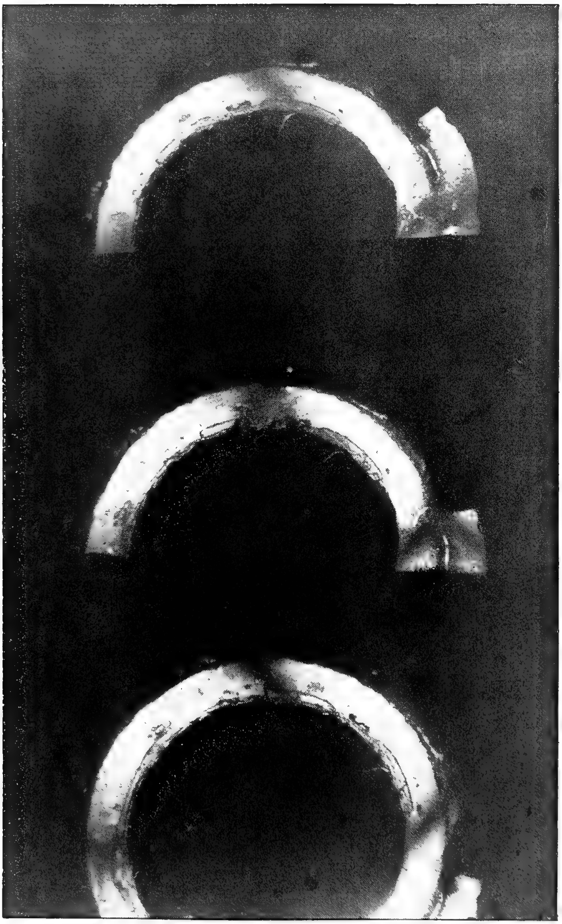

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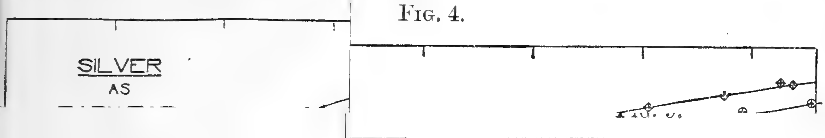
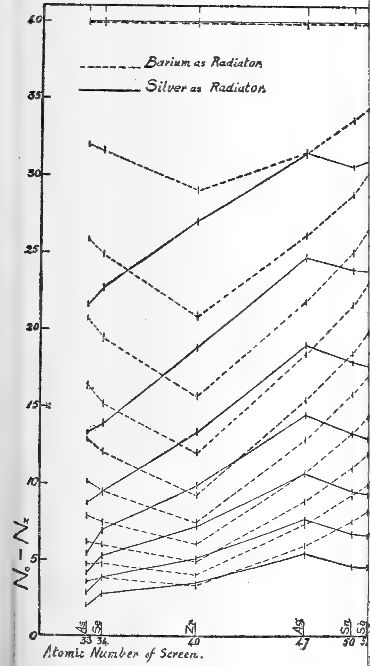
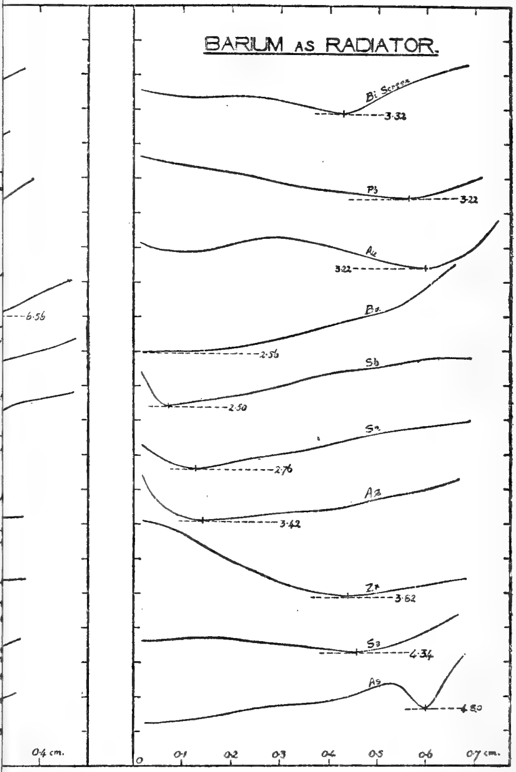






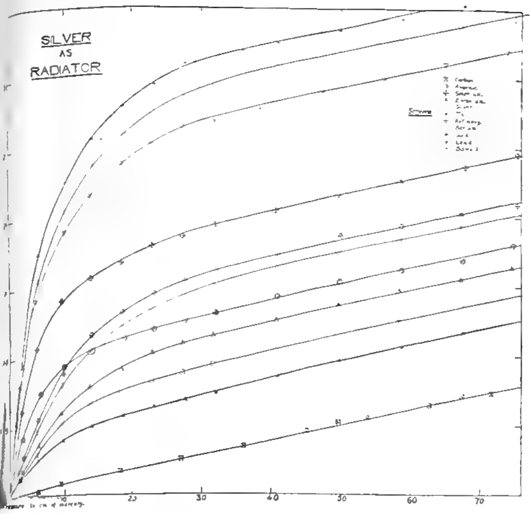
es from the screen as measured in air at 0° C. and

nates, $-\frac{d}{dx}\log_e(N_0-N_x)$, taken from figs. 5 and 6,

FIG. 7.—Ordinates (N_0-N_x) pr|esenting a change of 2·626 in this function and on
planes in air, at 0° C
ated screen as mark|value of $-\frac{d}{dx}\log_e(N_0-N_x)$ is placed on each curve
β-ray energy to be tl

of energy crossing planes successively distant from
change at the distance indicated.

Fig. 3.



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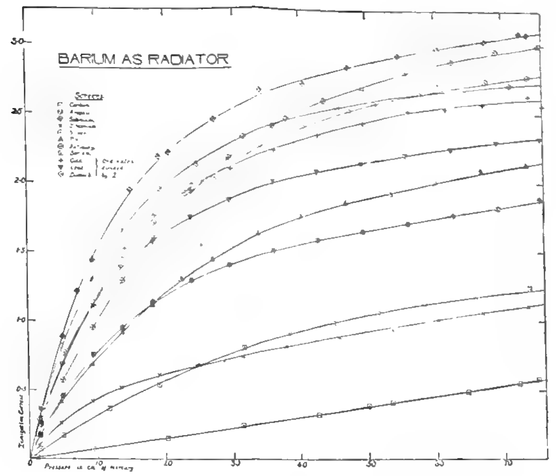

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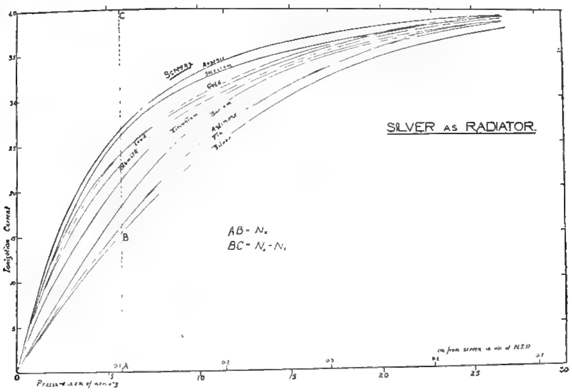

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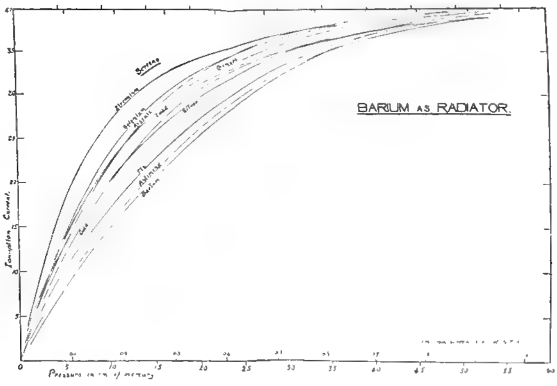

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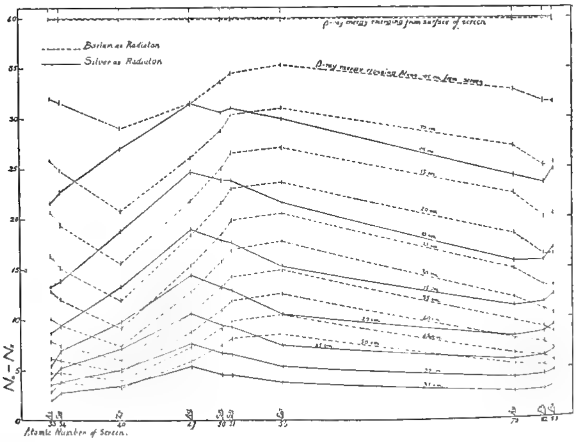
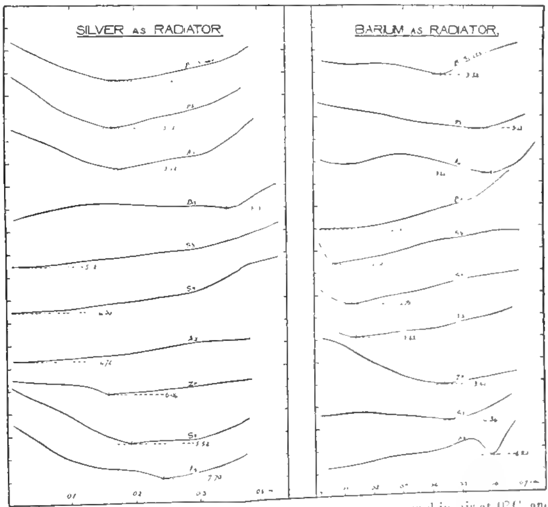


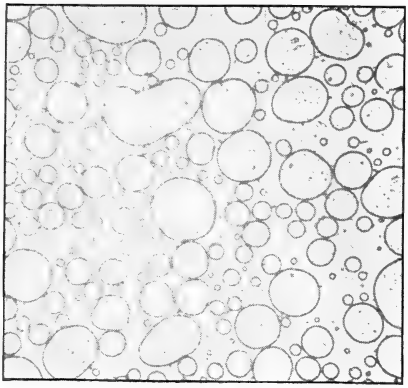
Fig. 7.—Ordinates $(N_o - N_x)$ proportional to the β-ray energy crossing imaginary planes in air, at 0° C. and 76 cm. pressure, at distances from the illuminated screen as marked on the curves, after having adjusted the total β-ray energy to be the same from all the screens.

Fig. 8. Fig. 9.



Figs. 8 & 9.—For abscissæ, distances from the screen as measured in air at 0° C. and 76 cm. pressure. For ordinates, $\frac{d}{dx}\log_e(N - N_x)$, taken from figs. 5 and 6, each division on fig. 8 representing a change of 2·626 in this function and on fig. 9, 1·313. One absolute value of $-\frac{d}{dx}\log_e(N - N_x)$ is placed on each curve for reference.

The rate of diminution of energy crossing planes successively distant from the screens suffers a marked change at the distance indicated



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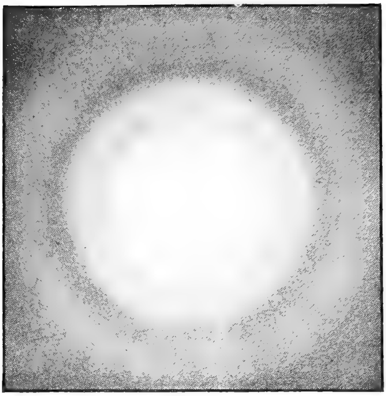

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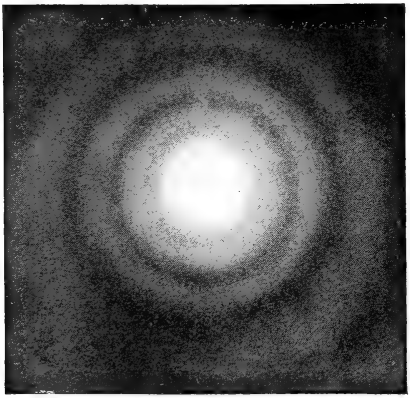

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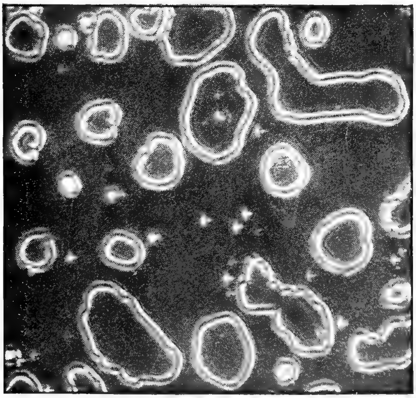

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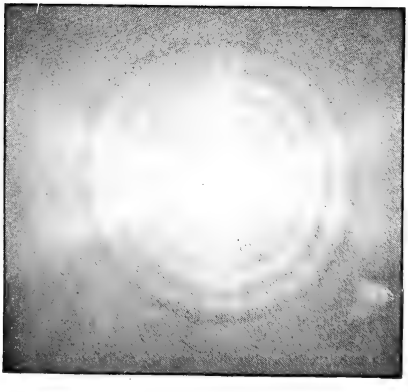

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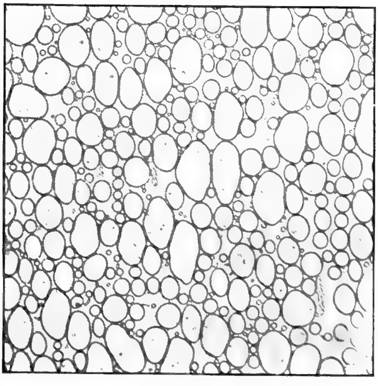

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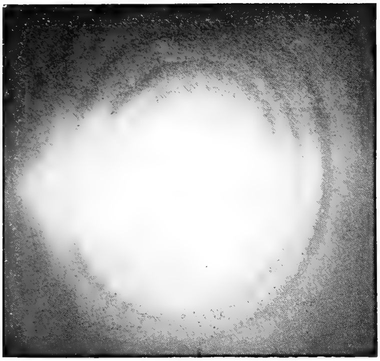

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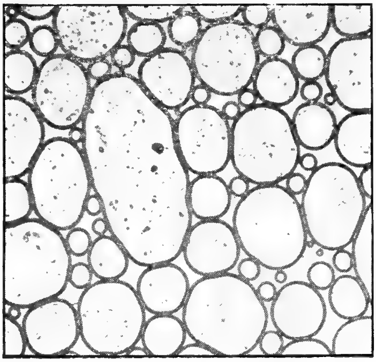

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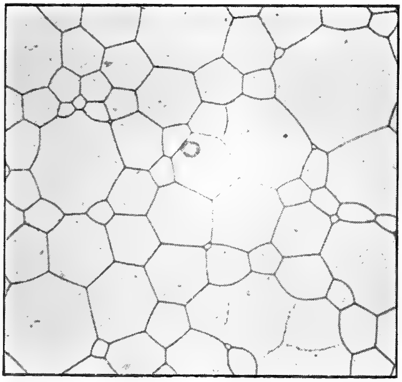

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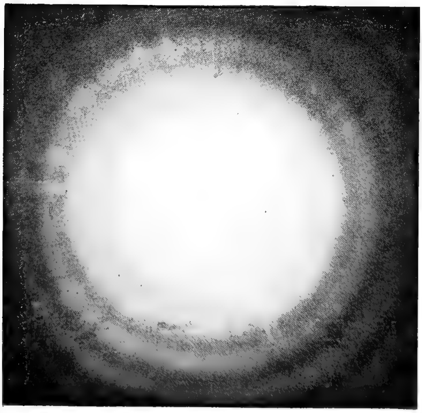

FIG. 11.

FIG. 12.

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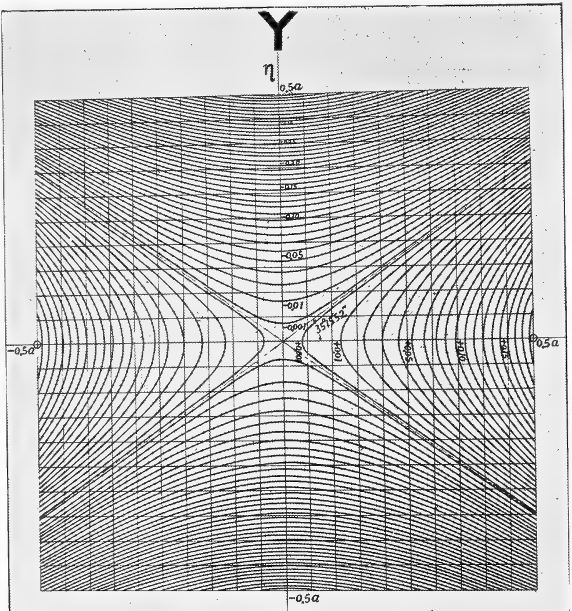
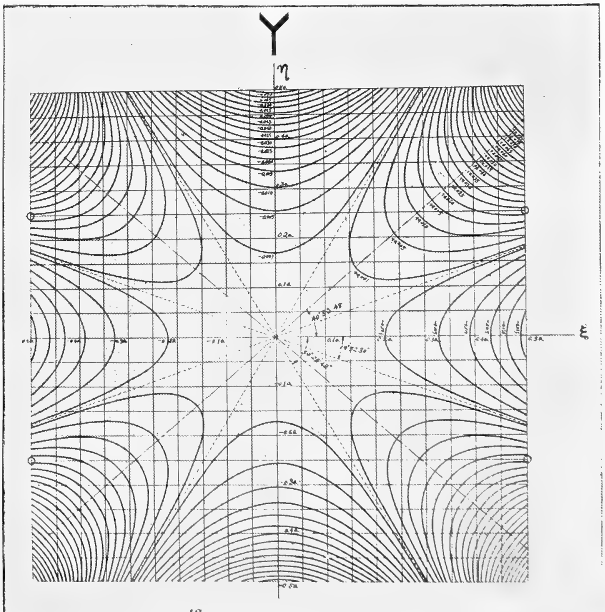


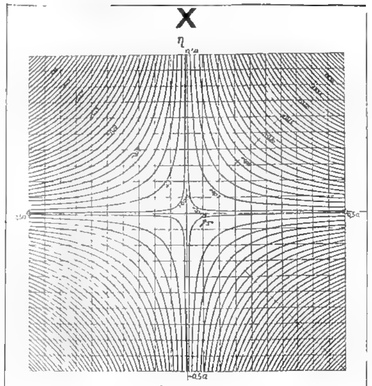
$\frac{3}{4}\{\xi^2-2\eta^2\}=c$

FIG. 8.



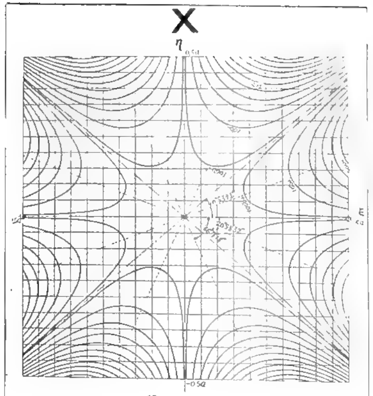
$\frac{18}{125}\{-8\eta^4+24\eta^2\xi^2-3\xi^4\}=c$

Fig. 1.



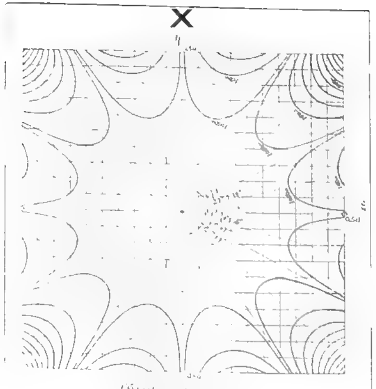
$\frac{3}{2}\xi\eta = c$

Fig. 2.



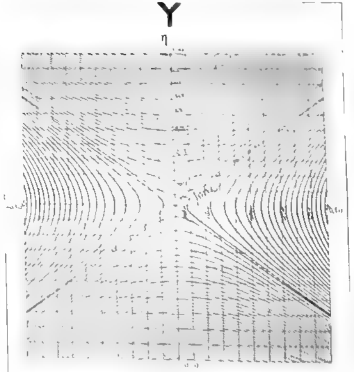
$\frac{15}{16}(3\xi^2\eta - 4\xi\eta^3) = c$

Fig. 3.



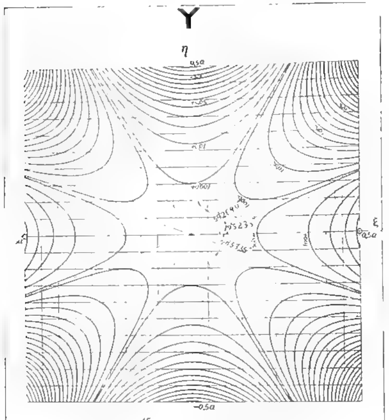
$\frac{15}{4}(\ldots\xi^3\eta \ldots 3\xi^2\eta + 6\xi\eta) = c$

Fig. 4.



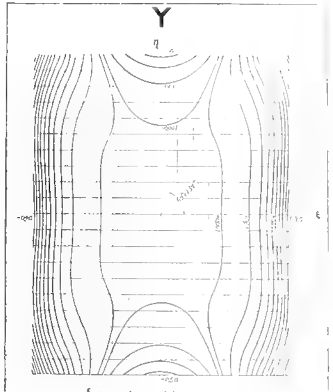
$\frac{3}{4}(\xi^2 - 2\eta^2) = c$

Fig. 5.



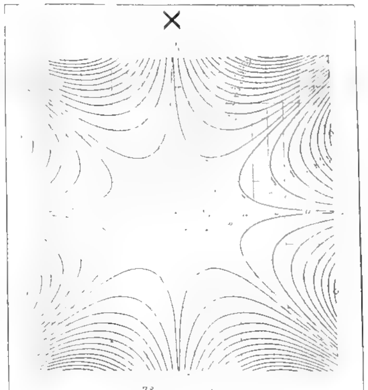
$\frac{15}{64}(3\xi^4 - 24\xi^2\eta^2 + 8\eta^4) = c$

Fig. 6.



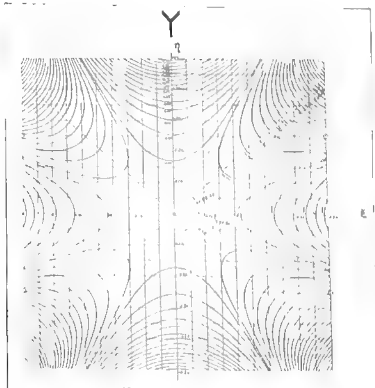
$\frac{5}{256}(1235\xi^6 - 630\xi^4\eta^2 + 840\xi^2\eta^4 - 112\eta^6) = c$

Fig. 7.



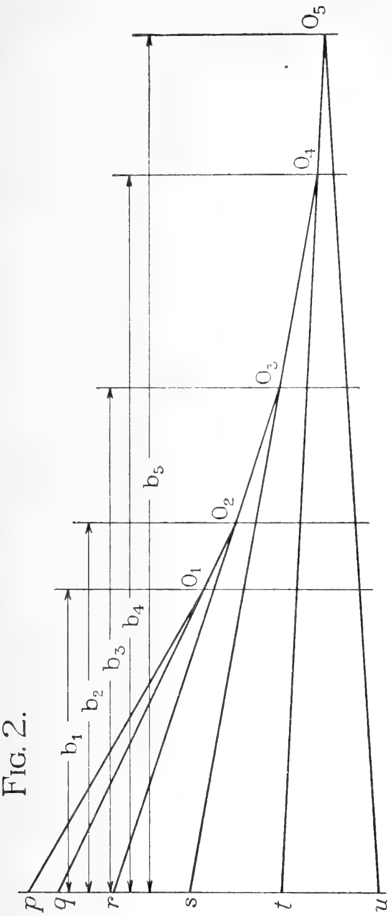
$\frac{72}{125}(+7\xi - 32\xi^3) = c$

Fig. 8.



$\frac{15}{32}(-8\eta^4 + 24\eta^2\xi^2 - 3\xi^4) = c$

FIG. 2.



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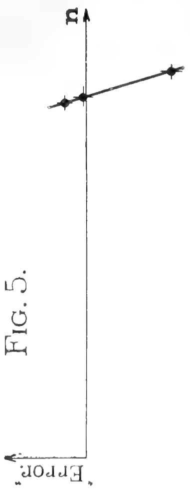

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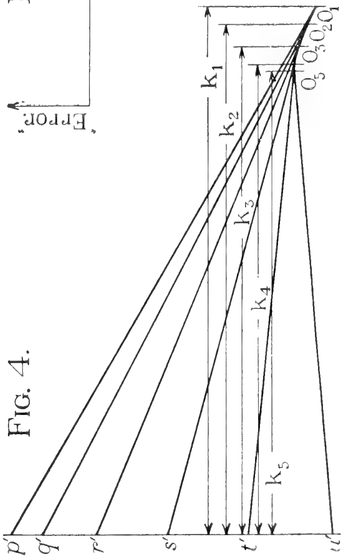

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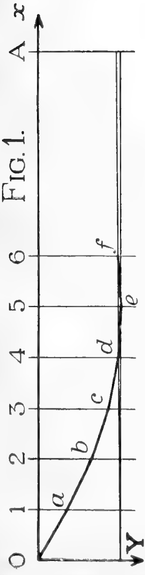

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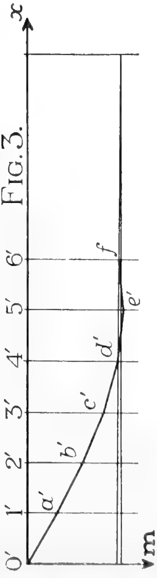

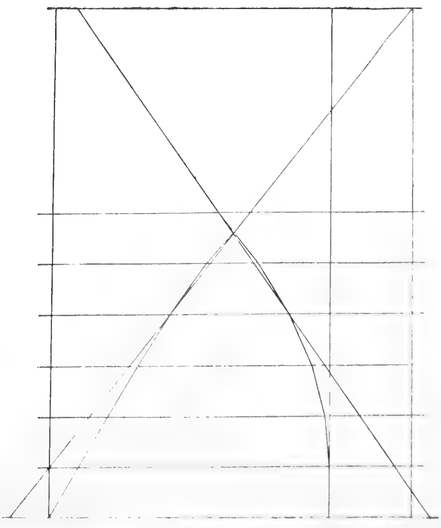

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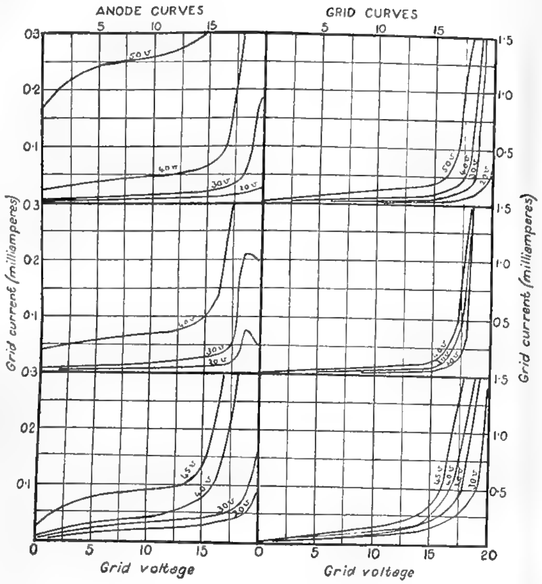

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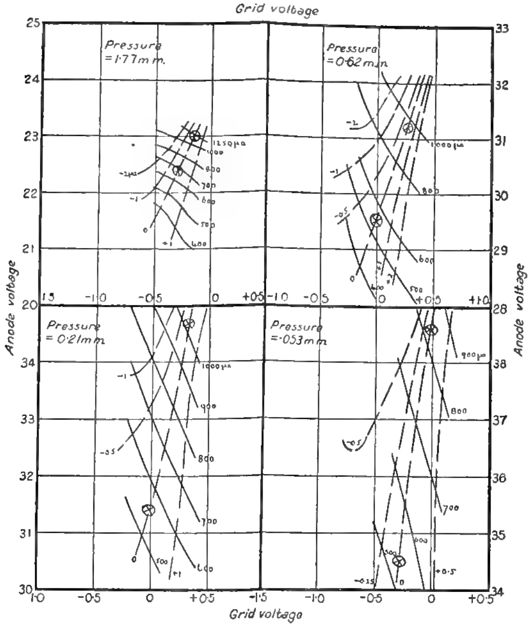
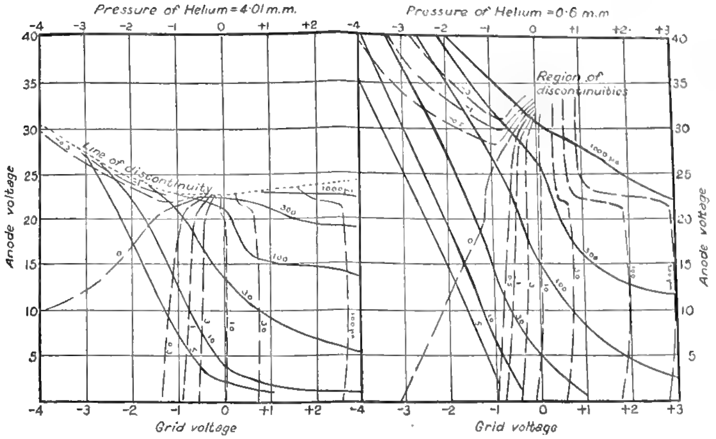


Continuous curves represent anode currents.
Broken „ „ grid currents.
"Oscillating points" shown thus ×.

Fig. 8.



Continuous curves represent anode currents.
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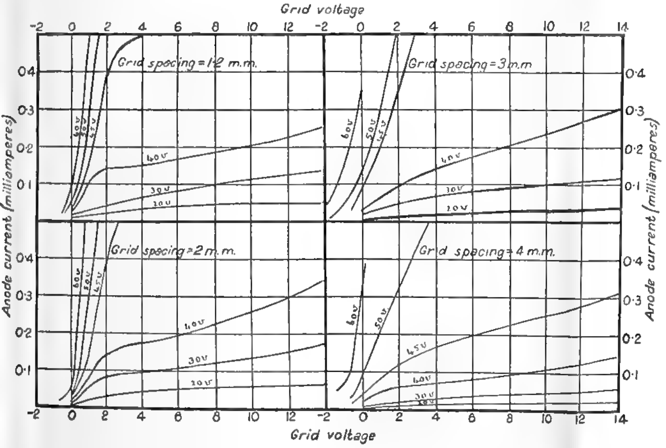
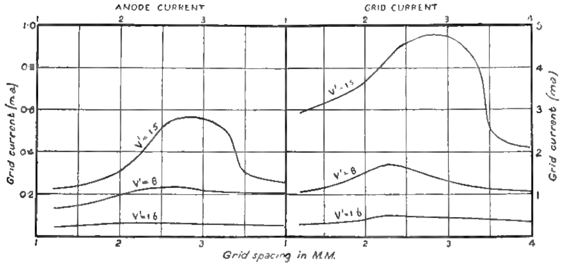
Fig. 4



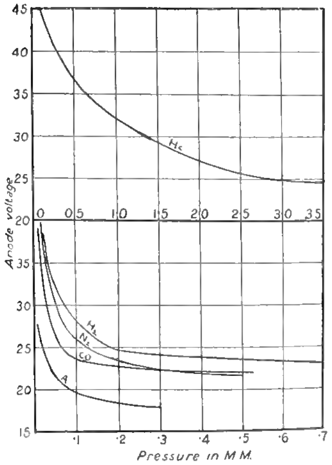
Pressure of hydrogen = ·023 mm.

Fig. 5.



V' = grid potential. Pressure = ·026 mm.

Fig. 9.



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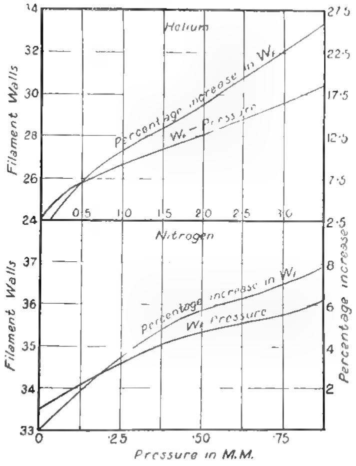

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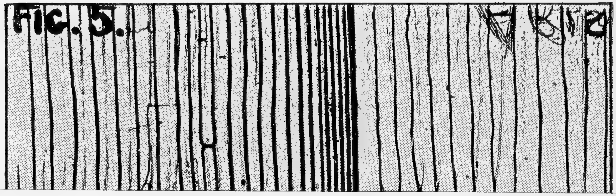

FIG. 5.

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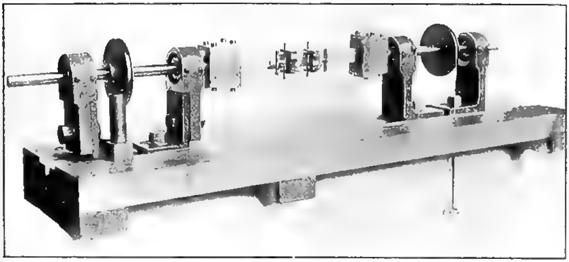

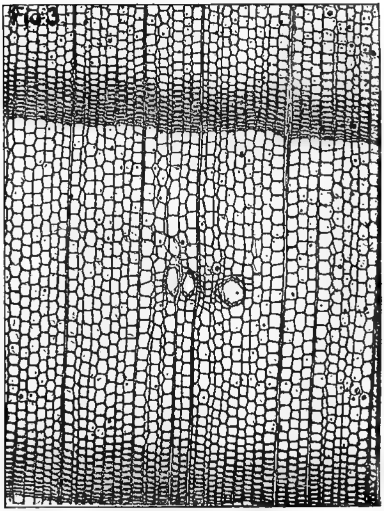

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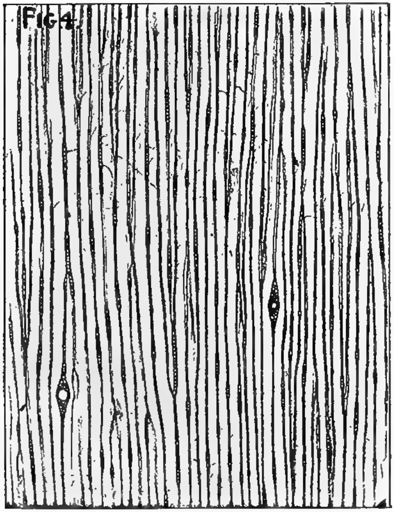

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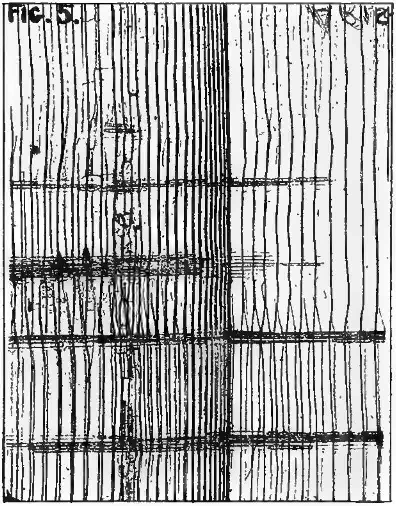

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